EXISTENCE AND NONEXISTENCE OF SOLUTIONS TO MIXED NONLINEAR BOUNDARY VALUE PROBLEMS

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A dissertation submitted to the

Graduate School—New Brunswick

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Michael Vogelius

and approved by

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New Brunswick, New Jersey

May, 2012

ABSTRACT OF THE DISSERTATION

Existence and Nonexistence of Solutions to Mixed Nonlinear Boundary Value Problems

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We consider solutions to the mixed boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial \nu} = cu + \lambda \left|u\right|^{\alpha-1} u & \text{on } \Gamma \end{cases}$$

on the unit ball $\Omega \in \mathbb{R}^N$, $N \geq 3$, with boundary $\partial\Omega = \Gamma_0 \cup \Gamma$, where Γ_0 and Γ are smooth connected components with $\Gamma_0 \cap \Gamma = \emptyset$. Here $\lambda > 0$, $c \in \mathbb{R}$, and $\alpha > 1$. The question is how the parameters c, λ, α affect existence and behavior of solutions. In particular, we consider the cases when $\alpha + 1$ is less than, greater than, or equal to the critical exponent for the Sobolev trace embedding $\alpha^* = \frac{2(N-1)}{N-2}$. For certain arguments we will need to assume that the Neumann boundary Γ is a subset of the upper hemisphere.

Acknowledgements

I would like to acknowledge Dr. Kurt Bryan for assistance throughout my studies, Dr. YanYan Li and Dr. Richard Falk for helpful consultation, and Dr. Michael Vogelius for his mentorship.

Dedication

This thesis is dedicated to my mother, Barbara Arceneaux, and my father, Jim Trainor, for their constant love and support.

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Chapter 1

Introduction

Let Ω be an open, bounded set in $\mathbb{R}^N$, $N \geq 3$ with smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma$, where Γ_0 and Γ are smooth and connected with $\Gamma_0 \cap \Gamma = \emptyset$. Our problem is

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial \nu} = g(u) & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where g is of the form

$$g(u) = cu + \lambda \left|u\right|^{\alpha-1} u \tag{1.2}$$

for $\lambda > 0, c \in \mathbb{R}$, and $\alpha > 1$. Certain calculations will require specific forms of Ω, Γ_0, and Γ, and so we will make additional assumptions as we need them.

Problem (1.1) arose from considering solutions of problems of the form

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda \sinh(u) & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

This boundary value problem arises from the Butler-Volmer boundary condition used in corrosion modeling (see the introduction in [20]). This problem was studied thoroughly in two dimensions by Kavian and Vogelius in [13], along with Bryan and Vogelius in [6] and Vogelius and Xu in [20]. However, showing the existence of solutions relied on the compactness of the embedding of a Sobolev space into the Orlicz space associated with the function $e^{t^2} - 1$. This embedding holds in dimension two but not in higher dimensions, and so a problem arises trying to carry out the argument for $N \geq 3$ with exponential boundary data. The analogous problem is to have boundary data of the form (1.2), where $\alpha + 1$ is subcritical to the Sobolev embedding $H^1(\Omega)$ into $L^p(\partial\Omega)$.

The argument for existence in two dimension for problem (1.3) presented in [13] carries over quite well to (1.1) in three or more dimensions. This is carried out in Chapter 2, and therein it is explained exactly how the critical exponent arises.

It is not immediately clear that there is not a separate argument to show existence for (1.1) for larger α, or even for $g(u) = \lambda \sinh(u)$ (which is in a sense larger than u^α for any α). Certainly the methods we use in Chapter 2, finding critical points of energy functionals, will fail when α is too large, but there could be other methods to construct solutions. However, the work of Pohozaev in [18] indicates that (1.1) likely does not have solutions if α is too large. Indeed, in [18] Pohozaev shows that the problem

$$\begin{cases} \Delta u + \lambda |u|^p = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

cannot have solutions when p is greater than a critical exponent (as long as Ω is star-shaped). Brezis and Nirenberg ([4]) considered the modified problem

$$\begin{cases} -\Delta u = cu + u^{p^*} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where p^* is the critical exponent. Brezis and Nirenberg showed that existence is retained for certain values of c, depending on the domain and the dimension. At a glance, (1.1) is not much different when $\Gamma_0 = \emptyset$, as we have only moved the data to the boundary. The Pohozaev argument, however, becomes more complicated. Indeed, complications with the Pohozaev argument required adding the linear cu term and having a nonempty Dirichlet boundary Γ_0. These details appear in Chapter 3.

In Chapter 4 we show some numerical plots which provide some evidence for the existence and nonexistence results from Chapters 2 and 3.

There are still some open questions for the nonexistence result. In particular, we only show nonexistence to (1.1) in the case when the exponent is supercritical, $c < -\frac{N-2}{2}$, and the solutions are continuous and satisfy a certain growth condition. Our Pohozaev identity (Theorem 9) is not shown to hold when $c \geq -\frac{N-2}{2}$, and even if it did, a meaningful result is not obtained. It is unclear whether such a result can be found, though some numeric computations in Section 4.4 suggest solutions do not exist.

It is also unclear whether the nonexistence in the supercritical case carries over to boundary data of the form $g(u) = \sinh u$. Even though $\sinh u$ is "larger" than u^α, there is some justification to believing solutions may exist. For instance, Joseph and Lundgreen show in [12] that the problem

$$\begin{cases} \Delta u + \lambda e^u = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the case when Ω is a ball, has solutions even in dimension higher than 2 (in fact, in every dimension there is some range of λ for which solutions exist). However, this result relies on the fact that solutions are radial and the boundary value problem reduces to an ordinary differential equation. It isn't difficult to see that problem (1.1) does not have radial solutions, so it is unclear is there if a relationship between the two problems.

The question of existence to the problem (1.1) has been fairly well studied even when $\Gamma_0 \neq \emptyset$. Our argument in Section 2.1 follows very similarly the arguments presented by Kavian and Vogelius in [13]. The arguments in Section 2.1.1 and Section 2.2 can mostly be found by Adimurthi and Yadava in [1] and [2], though we have in some cases made the arguments simpler by making more restrictive assumptions. For purposes of making this paper self-contained, all the relevant theorems and proofs have been included.

The material in Chapter 3, specifically the Pohozaev Identity in Theorem 9 and corresponding nonexistence result in Corollary 2, is believed to be new. The idea of breaking up the boundary conditions into a Neumann component and a Dirichlet component was aided by [2] and [7]. In the latter paper, Chelbik, Fila, and Reichel consider problem (1.1) where Ω is a half-ball, Γ is the flat component of the boundary, and Γ_0 is the curved component of the boundary. For this domain the authors derive a Pohozaev identity.

Chapter 2

Existence

Let $\Omega \subset \mathbb{R}^N$ be bounded with smooth boundary, $N \geq 3$, and Γ_0, Γ be any two smooth connected subsets of $\partial\Omega$ with $\Gamma_0 \cup \Gamma = \partial\Omega$, $\Gamma_0 \cap \Gamma = \emptyset$. We wish to show that (1.1) has a (weak) solution $u \in H^1(\Omega)$, defined by (2.3) below.

Define $H \subset H^1(\Omega)$ by

$$H = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0\}.$$

For $u, v \in H$, define

$$(u, v)_H = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Gamma uv \, d\sigma \tag{2.1}$$

and

$$\|u\|_H^2 = (u, u)_H. \tag{2.2}$$

We say $u \in H$ is a weak solution (or simply a solution) to (1.1) if

$$\int_\Omega \nabla u \cdot \nabla v \, dx - c \int_\Gamma uv \, d\sigma - \lambda \int_\Gamma u \, |u|^{\alpha-1} \, v \, d\sigma = 0 \tag{2.3}$$

for every $v \in H$. Set

$$\alpha^* = \frac{2(N-1)}{N-2},$$

which is the critical exponent for the embedding of $H^1(\Omega)$ into $L^p(\partial\Omega)$.

We wish to show that there exists a nontrivial $u \in H$ satisfying (2.3). The solution structure will depend on the exponent α. In Section 2.1, we show that (1.1) has a nontrivial solution in H as long as $1 < \alpha < \alpha^* - 1$. In Subsection 2.1.1, we show that there is a solution which may be taken to be nonnegative in Ω. The arguments here rely on the compactness of the embedding of $H^1(\Omega)$ into $L^{\alpha+1}(\partial\Omega)$. In Section 2.2, we show that when $\alpha = \alpha^* - 1$, nontrivial solutions are still obtained under certain conditions on the constant c, even though compactness of the trace embedding is lost.

The ideas in Section 2.1 follow [13] very closely. The main difference is that the embedding $H^1(\Omega)$ into the Orlicz space associated with $e^{t^2}-1$ in $N=2$ dimensions is replaced by the embedding $H^1(\Omega) \hookrightarrow L^p(\partial\Omega)$ in $N \geq 3$ dimensions. We have presented the proofs of all theorems even when there is little to no difference to the corresponding theorems in [13]. The main result is Theorem 2, which states that problem (1.1) admits infinitely many solutions in H.

The ideas in Section 2.1.1 are adapted from [1], [2] and [4]. These papers primarily deal with constructing a solution in the critical case, but the argument works (and is in fact easier) in the subcritical case. The difference between these arguments and the argument presented in Section 2.1 is a different energy functional is used, and the result is showing the existence of only one positive solution, instead of infinitely many solutions with no positivity restriction. The arguments in Section 2.2 follow closely those found in [2]. Due to computational complexity, we state and prove simpler versions of the results, taking the domain Ω to be the unit ball.

2.1 Subcritical Case

Suppose $1 < \alpha < \alpha^* - 1$. We first verify that the integrals in (2.3) are defined for any $u \in H$. Trace and Sobolev embedding tells us that we have

$$H \subset H^1(\Omega) \hookrightarrow H^{1/2}(\Omega) \hookrightarrow L^p(\partial\Omega)$$

for every $1 < p \leq \alpha^*$, and that the embeddings are compact for $1 < p < \alpha^*$. Let $u, v \in H$. As $\alpha^* > 2$, we have $u, v \in L^2(\partial\Omega)$, and so

$$\int_\Gamma uv\, d\sigma = \int_{\partial\Omega} uv\, d\sigma \leq \left(\int_{\partial\Omega} u^2\right)^{\frac{1}{2}} \left(\int_{\partial\Omega} v^2\right)^{\frac{1}{2}} < \infty.$$

Furthermore, $\alpha + 1 < \alpha^*$, so $u, v, \in L^{\alpha+1}(\partial\Omega)$, giving

$$\begin{aligned} \int_\Gamma \left|u\,|u|^{\alpha-1}\, v\right|\, d\sigma &= \int_{\partial\Omega} |u|^\alpha\, |v|\, d\sigma \\ &\leq \left(\int_{\partial\Omega} |u|^{\alpha\frac{\alpha+1}{\alpha}}\right)^{\frac{\alpha}{\alpha+1}} \left(\int_{\partial\Omega} |v|^{\alpha+1}\right)^{\frac{1}{\alpha+1}} \\ &< \infty. \end{aligned}$$

This also shows that the inner product (2.1) is well-defined. Furthermore, the induced norm (2.2) provides a norm on H which is equivalent to the standard norm from $H^1(\Omega)$,

$$\|u\|^2_{H^1(\Omega)} = \int_\Omega \|\nabla u\|^2 \, dx + \int_\Omega u^2 \, dx. \tag{2.4}$$

We first make the following observation. If $u \in H$ is a nontrivial solution to (1.1), we have

$$\int_\Omega \|\nabla u\|^2 \, dx = c \int_\Gamma u^2 \, d\sigma + \lambda \int_\Gamma |u|^{\alpha+1} \, d\sigma.$$

Then we must have

$$0 < c \int_\Gamma u^2 \, d\sigma + \lambda \int_\Gamma |u|^{\alpha+1} \, d\sigma.$$

In particular, if $c \leq 0$ we must have $\lambda > 0$. If $c > 0$, then having $\lambda < 0$ does not provide a contradiction to this inequality, though we do not consider the $\lambda < 0$ case in this paper.

To show the existence of a weak solution we employ a variational argument. For $v \in H$, define

$$E(v) = \frac{1}{2} \int_\Omega \|\nabla v\|^2 \, dx - \frac{c}{2} \int_\Gamma v^2 \, d\sigma - \frac{\lambda}{\alpha+1} \int_\Gamma |v|^{\alpha+1} \, d\sigma. \tag{2.5}$$

The embedding arguments show E is well-defined on H. For $u \neq 0$, set

$$J(u) = \sup_{t>0} E(tu). \tag{2.6}$$

Proposition 1. *The functional J satisfies the following properties.*

(i) $J(v) \geq 0$ for every $v \in H$.

(ii) $J(v) = 0 \Leftrightarrow E(tv) \leq 0$ for every $t > 0$ and $v \in H$.

(ii) $J(v) = \infty \Leftrightarrow v|_\Gamma = 0$ for every $v \in H$.

Proof. From (2.5), $E(0) = 0$, and so $\sup_{t \geq 0} E(tv) \geq 0$ for any $v \in H$. For every fixed $v \in H$, $E(\cdot v)$ is a continuous function on $\mathbb{R}$, and so we must have $J(v) = \sup_{t>0} E(tv) \geq 0$ as well, showing (i). For (ii), let $v \in H$ and suppose first that $J(v) = 0$. Then the supremum of $E(tv)$ over $t > 0$ is zero, and so we must have $E(tv) \leq 0$ for every $t > 0$. On the other hand, if $E(tv) \leq 0$ for every $t > 0$, then $J(v) \leq 0$, and then (i) implies $J(v) = 0$.

To prove (iii), note that

$$E(tv) = \frac{t^2}{2}\left(\int_\Omega \|\nabla v\|^2\, dx - c\int_\Gamma v^2\, d\sigma\right) - \frac{\lambda}{\alpha+1}t^{\alpha+1}\int_\Gamma |v|^{\alpha+1}\, d\sigma.$$

If $v \in H$ is any function that vanishes on Γ, then clearly $E(tv)$ increases without bound as t increases, and $\sup_{t>0} E(tv) = \infty$. Conversely, suppose $J(v) = \infty$ for some $v \in H$. As $\alpha + 1 > 2$ and $\lambda > 0$, we have $\lim_{t\to\infty} E(tv) = -\infty$ as long as

$$\int_\Gamma |v|^{\alpha+1}\, d\sigma$$

is nonzero. Then since $E(tv)$ is bounded for every finite positive t, we cannot have $\sup_{t>0} E(tv) = \infty$ unless $\int_\Gamma |v|^{\alpha+1}\, d\sigma = 0$, which implies $v = 0$ on Γ. □

By Proposition 1, J is a bounded functional away from functions vanishing identically on the boundary, whereas the functional E fails to be bounded. We will follow the ideas in [13], constructing critical points for J and showing that such critical points yield critical points for E, which are then weak solutions to (1.1). Our first theorem relates the critical points of J and E, and is proved through a series of lemmas.

Theorem 1. *Whenever $0 < J(u) < \infty$, there exists a unique number $t = t(u) > 0$ such that*

$$J(u) = E(t(u)u).$$

Furthermore, $t(u)$ is given by

$$t(u)\left[\int_\Omega \|\nabla u\|^2\, dx - c\int_\Gamma u^2\, d\sigma\right] - \lambda t(u)^\alpha \int_\Gamma |u|^{\alpha+1}\, d\sigma = 0.$$

This mapping $t \mapsto t(u)$ is smooth on the set $\{u \in H, 0 < J(u) < \infty\}$, and so J is a C^∞ functional on $\{u \in H, 0 < J(u) < \infty\}$ and continuous from $H\backslash\{0\}$ onto $[0,\infty]$. Lastly, critical points for J on the unit sphere in H yield critical points for E in H by the transformation $u \mapsto t(u)u$.

The proof is broken up into three lemmas.

Lemma 1. *For fixed $u \in H\backslash\{0\}$, define*

$$f(t) = E(tu) = \frac{t^2}{2}\left[\int_\Omega \|\nabla u\|^2\, dx - c\int_\Gamma u^2\, d\sigma\right] - \frac{\lambda}{\alpha+1}|t|^{\alpha+1}\int_\Gamma |u|^{\alpha+1}\, d\sigma$$

for $t \in \mathbb{R}$. Then

(i) f is an even function of t and $f \in C^\infty(0,\infty)$.

(ii) If u is such that

$$0 < \sup_{t>0} f(t) < \infty,$$

then there is a unique $t(u) \in \mathbb{R}$ satisfying

$$f(t(u)) = \sup_{t>0} f(t) = J(u).$$

(iii) The map $w \mapsto t(w)$ is well defined in an H neighborhood of u and is of class C^∞.

Proof. (i) Since t^2 and $|t|^{\alpha+1}$ describe even functions, f is even. Furthermore, restricted to $t \in (0,\infty)$, f is a polynomial in t and is therefore smooth. We note that f is only C^α on all of $\mathbb{R}$.

(ii) Suppose $0 < \sup_{t>0} f(t) < \infty$. Then $0 < J(u) < \infty$, showing that the coefficient of t^2 must be positive and (by Proposition 1) $u \neq 0$ on Γ. Since $\alpha + 1 > 2$ and the coefficient of $t^{\alpha+1}$ is negative, we have

$$\lim_{t\to\infty} f(t) = -\infty.$$

Now f has positive supremum, is bounded above, and approaches $-\infty$ as $t \to \infty$. This implies f is bounded above on some compact interval, so f must obtain its supremum, which is assumed to be positive . Therefore, there exists $t(u) > 0$ such that

$$f(t(u)) = \sup_{t>0} f(t).$$

We compute for $t > 0$,

$$\begin{aligned}
f'(t) &= t\int_\Omega \|\nabla u\|^2 \, dx - ct\int_\Gamma u^2 \, d\sigma - \lambda t^\alpha \int_\Gamma |u|^{\alpha+1} \, d\sigma, \\
f''(t) &= \int_\Omega \|\nabla u\|^2 \, dx - c\int_\Gamma u^2 \, d\sigma - \lambda\alpha t^{\alpha-1} \int_\Gamma |u|^{\alpha+1} \, d\sigma, \\
f'''(t) &= -\lambda\alpha(\alpha-1)t^{\alpha-2}\int_\Gamma |u|^{\alpha+1} \, d\sigma.
\end{aligned}$$

Each of these expressions are defined on $(0,\infty)$, the integrals over Γ are nonzero, and f' is defined for $t = 0$ with $f'(0) = 0$. We also see, since $\alpha > 1$, that $f'(t) \to -\infty$ as $t \to \infty$. As $f(0) = 0 < \sup_{t>0} f(t)$, f' must take on some positive values on $(0,\infty)$.

We then have that f' is a continuous function of t which takes on both positive and negative values on $(0, \infty)$, so there is some $t^* > 0$ so that $f'(t^*) = 0$. Now since $\alpha > 1$, the coefficient of $t^{\alpha-2}$ in $f'''(t)$ is negative, showing f'' is strictly decreasing on $(0, \infty)$. Further, f'' is defined at $t = 0$ as well (again because $\alpha > 1$), and since $J(u) > 0$, the coefficient of t^2 in $f(t)$ must be positive, so $f''(0) > 0$. Then since $f''(t) \to -\infty$ as $t \to \infty$, f'' has a unique root in $(0, \infty)$. Therefore, f' has only one local extrema (a positive maximum), and f' is increasing for positive t less than this point and decreasing for every t beyond this point. This implies that f' can have only one positive root, and so t^* is the only positive critical point of f. Furthermore, $f(0) = 0$ and f' is increasing, and therefore positive, on $(0, t^*)$, showing f is increasing, and therefore positive, on $(0, t^*)$.

We have

$$0 = f'(t^*) = t^* \left(\int_\Omega \|\nabla u\|^2 \, dx - c \int_\Gamma u^2 \, d\sigma \right) - \lambda t^{*\alpha} \int_\Gamma |u|^{\alpha+1} \, d\sigma,$$

so

$$\int_\Omega \|\nabla u\|^2 \, dx - c \int_\Gamma u^2 \, d\sigma = \lambda t^{*\alpha-1} \int_\Gamma |u|^{\alpha+1} \, d\sigma.$$

This gives

$$\begin{aligned} f''(t^*) &= \int_\Omega \|\nabla u\|^2 \, dx - c \int_\Gamma u^2 \, d\sigma - \lambda \alpha t^{*\alpha-1} \int_\Gamma |u|^{\alpha+1} \, d\sigma \\ &= \lambda t^{*\alpha-1} \int_\Gamma |u|^{\alpha+1} \, d\sigma - \lambda \alpha t^{*\alpha-1} \int_\Gamma |u|^{\alpha+1} \, d\sigma \\ &= -\lambda(\alpha - 1) \int_\Gamma |u|^{\alpha+1} \, d\sigma \\ &< 0, \end{aligned}$$

since $\lambda > 0$, $\alpha > 1$, and $t^* > 0$. Thus, f is concave down at t^*, which implies that t^* is a local maximum of f. Furthermore, t^* is the only critical point of f, and $f(0) = 0$, $f(t) \to -\infty$ as $t \to \infty$, so supremum of f must occur at this critical point. Therefore, $t(u) = t^*$ is the unique point in $(0, \infty)$ where

$$f(t(u)) = \sup_{t>0} f(t) = J(u).$$

Moreover, $f'(t(u)) = 0$ gives

$$t(u) \left(\int_\Omega \|\nabla u\|^2 \, dx - c \int_\Gamma u^2 \, d\sigma \right) - \lambda t(u)^\alpha \int_\Gamma |u|^{\alpha+1} \, d\sigma = 0.$$

(iii) Noting that $f''(t(u)) \neq 0$, the equation $f'(t(u)) = 0$ gives an implicit mapping $w \to t(w)$ in an H neighborhood of u from the Implicit Function Theorem, and this mapping is C^∞. □

Lemma 2. *The functional $J : H\backslash\{0\} \to [0, \infty]$ is even and continuous, and J is finite on $H\backslash H_0^1(\Omega)$.*

Proof. For any $u \in H\backslash\{0\}$,

$$\begin{aligned} J(u) &= \sup_{t>0} f(t) \\ &= \sup_{t>0} \left[\frac{t^2}{2} \left(\int_\Omega \|\nabla u\|^2 \, dx - c \int_\Gamma u^2 \, d\sigma \right) - \frac{\lambda}{\alpha+1} t^{\alpha+1} \int_\Gamma |u|^{\alpha+1} \, d\sigma \right], \end{aligned}$$

so $J(u) = J(-u)$. If $u \neq 0$ on $\partial\Omega$, then $u \neq 0$ on Γ, and so from Proposition 1, $J(u) < \infty$.

We wish to show J is continuous on $H\backslash\{0\}$. To this end, suppose $\{u_n\}$ is a sequence converging to some $u_0 \in H\backslash\{0\}$. We aim to show $J(u_n) \to J(u_0)$. There are three cases: (i) $J(u_0) = \infty$, (ii) $J(u_0) = 0$, and (iii) $0 < J(u_0) < \infty$.

Case (i): Suppose $J(u_0) = \infty$. Fix a $t > 0$, so

$$J(u_n) = \sup_{s>0} E(su_n) \geq E(tu_n) \qquad \forall n \in \mathbb{N}.$$

Now E is continuous on H, so $E(tu_n) \to E(tu_0)$ as $n \to \infty$. We then have

$$\begin{aligned} \liminf_{n\to\infty} J(u_n) &\geq \liminf_{n\to\infty} E(tu_n) \\ &= \lim_{n\to\infty} E(tu_n) \\ &= E(tu_0) \end{aligned}$$

Since t was arbitrary, we have

$$\liminf_{n\to\infty} J(u_n) \geq \sup_{s>0} E(su_0) = J(u_0) = \infty.$$

This then gives

$$\liminf_{n\to\infty} J(u_n) = \infty \geq \limsup_{n\to\infty} J(u_n),$$

whence

$$\lim_{n\to\infty} J(u_n) = \infty = J(u_0).$$

Case (ii): Suppose $J(u_0) = 0$. By case (i), the set $A = \{u \in H\backslash\{0\} : J(u) < \infty\}$ has closed complement, and therefore is open. Then as $u_0 \in A$, there must exist an H-ball about u_0 contained in A. Since $u_n \to u_0$ in H, u_n must be contained in this ball for all sufficiently large N, and hence $J(u_n) < \infty$ for large n.

Consider the corresponding parameters $t(u_n)$ (given by Lemma 1), and suppose that $\{t(u_n)\}$ fails to converge to 0 as $n \to \infty$. Then there must be some subsequence $\{t(u_{n_k})\}_{k\geq 0}$ and a positive constant γ such that

$$0 < \gamma < t(u_{n_k})$$

for all k. Now, for every k, if $0 \leq t \leq t(u_{n_k})$, then by definition of $t(u_{n_k})$,

$$E(tu_{n_k}) = f(t) \geq 0$$

for every $0 \leq t \leq t(u_{n_k})$. In particular, $E(tu_{n_k}) \geq 0$ for $0 < t \leq \gamma$, so

$$E(tu_0) = \lim_{k\to\infty} E(tu_{n_k}) \geq 0$$

for $0 \leq t \leq \gamma$. Now,

$$0 = J(u_0) = \sup_{t>0} E(tu_0),$$

so

$$\sup_{0<t\leq\gamma} E(tu_0) \leq \sup_{t>0} E(tu_0) = 0,$$

giving $E(tu_0) \leq 0$ for every $t \in (0, \gamma]$. Thus, $E(tu_0) = 0$ identically on $(0, \gamma]$. We must then have

$$\frac{d^3}{dt^3} E(tu_0) = 0$$

identically on $(0, \infty]$, which contradicts our previous calculation of

$$\frac{d^3}{dt^3} E(tu_0) = -\lambda\alpha(\alpha - 1)t^{\alpha-2} \int_\Gamma |u_0|^{\alpha+1}\, d\sigma,$$

which is strictly negative for every $t > 0$. We must then have $t(u_n) \to 0$ as $n \to \infty$, and so

$$J(u_n) = E(t(u_n)u_n) \to E(0) = 0 = J(u_0).$$

Case (iii): Suppose $0 < J(u_0) < \infty$. The set $\{u \in H\backslash\{0\} : 0 < J(u) < \infty\}$ has closed complement by the previous two cases, and is thus open. Then using the same argument as in case (ii), we have $0 < J(u_n) < \infty$ for all sufficiently large n.

From Lemma 1, the mapping $u \mapsto t(u)$ is C^∞ on $\{0 < J(u) < \infty\}$, which implies $t(u_n) \to t(u_0)$. Then by continuity of E,

$$J(u_n) = E(t(u_n)u_n) \to E(t(u_0)u_0) = J(u_0).$$

Then in every case, $J(u_n) \to J(u_0)$, so J is continuous. $\square$

Lemma 3. *For $u \in H\backslash\{0\}$ with $\sup_{t>0} E(tu) > 0$, let $t(u)$ be as defined in Lemma 1. The functional J is even and C^∞ (in the sense of distributions) on the open set $\{u \in H : 0 < J(u) < \infty\}$. For $0 < J(u) < \infty$, we have*

$$J'(u) = t(u)E'(t(u)u)$$

in H', the dual space of H. Moreover, if $u \in H$ is a critical point for J on the unit sphere

$$\Sigma = \left\{u \in H : \int_\Omega \|\nabla u\|^2 \, dx + \int_\Gamma |u|^2 \, d\sigma = 1\right\}$$

for which $0 < J(u) < \infty$, then $v = t(u)u$ is a critical point for E in H with the same critical value.

Proof. We have already shown J is even. As $J(u) = E(t(u)u)$ when $0 < J(u) < \infty$ and the functional E along with the mapping $u \mapsto t(u)$ are C^∞, J is C^∞ as well.

For any $s > 0$, we have

$$J(su) = \sup_{t>0} E(tsu) = \sup_{\tau>0} E(\tau u) = J(u),$$

so J is constant on the rays $\{su : s > 0\}$. Let μ be the Lagrange multiplier for the critical point u of J on Σ, so

$$J'(u) = \mu q'(u)$$

in H', where

$$q(u) = (u, u)_H = \int_\Omega \|\nabla u\|^2 \, dx + \int_\Gamma |u|^2 \, d\sigma.$$

Note that the equation $J'(u) = \mu q'(u)$ means $\langle J'(u), w\rangle = \langle \mu q'(u), w\rangle$ for all $w \in H$, where $\langle \cdot, \cdot\rangle$ is the pairing between an element in H' on the left and an element in H on the right. We compute

$$\begin{aligned}
\langle q'(u), u\rangle &= \frac{d}{d\epsilon} q(u+\epsilon u)\big|_{\epsilon=0} \\
&= \frac{d}{d\epsilon}\left[\int_\Omega \left(1+2\epsilon+\epsilon^2\right)\|\nabla u\|^2 \, dx + \int_\Gamma (1+\epsilon)^2 u^2 \, d\sigma\right]\Bigg|_{\epsilon=0} \\
&= \left[\int_\Omega (2+2\epsilon)\|\nabla u\|^2 \, dx + \int_\Gamma 2(1+\epsilon) u^2 \, d\sigma\right]\Bigg|_{\epsilon=0} \\
&= 2\int_\Omega \|\nabla u\|^2 \, dx + 2\int_\Gamma u^2 \, d\sigma \\
&= 2q(u).
\end{aligned}$$

For $u \in \Sigma$, $q(u) = 1$, so

$$2\mu = 2\mu q(u) = \mu\langle q'(u), u\rangle = \langle \mu q'(u), u\rangle = \langle J'(u), u\rangle.$$

Since $J(su)$ is constant with respect to s, we have

$$0 = \frac{d}{ds} J(su) = \langle J'(su), u\rangle = s\langle J(u), u\rangle,$$

so $\langle J'(u), u\rangle = 0$, giving $\mu = 0$. This shows that if u is a critical point of J on Σ, then $J'(u) = 0$ in H', as the Lagrange multiplier is zero. Then u is a critical point of J on all of H, and so is su for every $s > 0$. Further, the critical value $\gamma = J(u)$ in Σ must also be a critical value of J in H with

$$\gamma = J(u) = J(su)$$

for every $s > 0$. Thus, critical points on Σ give a ray of critical points in H with the same critical value.

Suppose now that we have a critical point u of J on all of H, and suppose $\gamma = J(u) > 0$. We wish to show $\frac{u}{\|u\|_H}$ is a critical point for J on Σ with the same critical value. Since u is a critical point of J in H, we have $J'(u) = 0$, and so for any fixed

$k > 0$,

$$\begin{aligned}
0 &= \langle J'(u), w\rangle \\
&= \frac{d}{d\epsilon} J(u + \epsilon w)\big|_{\epsilon=0} \\
&= \frac{d}{d\epsilon} J(k(u + \epsilon w))\big|_{\epsilon=0} \\
&= \frac{d}{d\epsilon} J(ku + \epsilon k w)\big|_{\epsilon=0} \\
&= k \frac{d}{d(k\epsilon)} J(ku + (k\epsilon)w)\big|_{k\epsilon=0} \\
&= k\langle J'(ku), w\rangle.
\end{aligned}$$

We therefore have $J'(ku) = 0$ in H' for any $k > 0$, so in particular,

$$J'\left(\frac{u}{\|u\|_H}\right) = 0$$

in H', so $\frac{u}{\|u\|_H} \in \Sigma$ is a critical value for J on Σ with Lagrange multiplier $\mu = 0$. Furthermore

$$J\left(\frac{u}{\|u\|_H}\right) = J\left(\frac{1}{\|u\|_H} u\right) = J(u) = \gamma,$$

so this critical point gives rise to the same critical value.

Finally, for any $w \in H$, set $u_\epsilon = u + \epsilon w$, and note that t is a C^∞ functional on H, so

$$\frac{d}{d\epsilon} t(u_\epsilon)\big|_{\epsilon=0} = \frac{d}{d\epsilon} t(u + \epsilon w)\big|_{\epsilon=0} = \langle t(u), w\rangle.$$

Set

$$\begin{aligned}
h(u) &= |u|^{\alpha-1} u, \\
H(u) &= \frac{1}{\alpha+1} |u|^{\alpha+1}
\end{aligned}$$

and compute for any $w \in H$, noting that $u_\epsilon|_{\epsilon=0} = u$,

$$
\begin{aligned}
\langle J'(u), w\rangle &= \frac{d}{d\epsilon} J(u+\epsilon w)\big|_{\epsilon=0} \\
&= \frac{d}{d\epsilon} E(t(u_\epsilon)u_\epsilon)\big|_{\epsilon=0} \\
&= \frac{d}{d\epsilon}\Bigg[t(u_\epsilon)^2 \left(\int_\Omega \|\nabla u_\epsilon\|^2 \, dx - c\int_\Gamma u_\epsilon^2 \, d\sigma \right) \\
&\qquad - \lambda t(u_\epsilon)^{\alpha+1} \int_\gamma H(u_\epsilon)\, d\sigma \Bigg]_{\epsilon=0} \\
&= 2t(u)\frac{d}{d\epsilon} t(u_\epsilon)\big|_{\epsilon=0} \left(\int_\Omega \|\nabla u\|^2 \, dx - c\int_\Gamma u^2 \, d\sigma \right) \\
&\qquad - \lambda(\alpha+1)t(u)^\alpha \frac{d}{d\epsilon} t(u_\epsilon)\big|_{\epsilon=0} \int_\Gamma H(u)\, d\sigma \\
&\qquad + t(u)^2 \left(2\int_\Omega \nabla u\cdot\nabla w \, dx - 2c\int_\Gamma uw \, d\sigma \right) \\
&\qquad - \lambda t(u)^{\alpha+1} \int_\Gamma h(u) w\, d\sigma \\
&= \langle t'(u), w\rangle \Bigg[2t(u)\int_\Omega \|\nabla u\|^2 \, dx - 2ct(u)\int_\Gamma u^2 \, d\sigma \\
&\qquad - \lambda t(u)^\alpha \int_\Gamma h(u) u \, d\sigma \Bigg] \\
&\qquad + t(u)\Bigg[2t(u)\int_\Omega \nabla u\cdot\nabla w \, dx - 2ct(u)\int_\Gamma uw \, d\sigma \\
&\qquad - \lambda t(u) \int_\Gamma h(u) w\, d\sigma \Bigg] \\
&= \langle t'(u), w\rangle \langle E'(t(u)u), u\rangle + t(u)\langle E'(t(u)u), w\rangle.
\end{aligned}
$$

For the function f defined in Lemma 1, $\langle E'(t(u)u), u\rangle = f'(t(u))$, which must be 0 by definition of $t(u)$ being a maximizer of f. Thus, we have

$$\langle J'(u), w\rangle = t(u)\langle E'(t(u)u), w\rangle$$

for every $w \in H$, and therefore $J'(u) = t(u)E'(t(u)u)$ in H'. Thus, if $0 < J(u) < \infty$, $t(u) > 0$ so $J'(u) = 0$ is equivalent to $E'(v) = E'(t(u)u) = 0$. Then if u is a critical point for J in H, $v = t(u)u$ is a critical point for E in H. But we have already shown that a critical point $u \in \Sigma$ for J is also a critical point for J in all of H with the same critical value. Therefore, if u is a critical value for J on Σ with $0 < J(u) < \infty$, then $v = t(u)u$ is a critical value for E in H, and as $J(u) = E(t(u)u)$, the critical point v yields the same critical value. $\square$

Lemmas 1, 2 and 3 together prove Theorem 1. The next step is to construct the critical points for J. We proceed by a fairly standard technique using a Palais-Smale condition and the Mountain Pass Lemma. This method is attributed to Lyusternik and Shnirelman and can be found in [13]. We start with some definitions.

Recall Σ is the unit sphere in H. If A is a closed subset of Σ, define the genus $\Phi(A)$ to be the smallest integer k for which there exists a continuous odd mapping of A into $\mathbb{R}^k\backslash\{0\}$. This generalizes the idea of dimension for a finite dimensional vector space; the mappings into $\mathbb{R}^k\backslash\{0\}$ are analogous to basis functions.

For $k \geq 1$, define

$$\mathcal{A}_k = \{A \in \Sigma : A \text{ is closed, } A = -A, \Phi(A) \geq k\}.$$

Note that $\mathcal{A}_{k+1} \subset \mathcal{A}_k$ by construction. Now define the numbers

$$\gamma_k = \gamma_k(c, \lambda) = \inf_{A \in \mathcal{A}_k} \sup_{u \in A} J(u).$$

As $\mathcal{A}_{k+1} \subset \mathcal{A}_k$, we have $\gamma_k \leq \gamma_{k+1}$. Furthermore, $\mathcal{A}_k$ is nonempty for any $k \geq 1$. To see this, consider the set

$$A = \left\{ u(x) = \sum_{j=1}^{k} \alpha_j F_j(x) : \left\| \sum_{j=1}^{k} \alpha_j F_j \right\|_H = 1 \right\}$$

for any choice of $\{F_j\} \subset H$ such that the traces $f_j = F_j|_{\partial\Omega}$ are linearly independent. Then $A \subset \Sigma$, A is even, closed, and $\Phi(A) = k$, so $A \in \mathcal{A}_k$. Since $\Phi(A) = k$ for this set, $\gamma_k \leq k$, and so $\gamma_k < \infty$ for all $k \geq 1$. Thus, $\{\gamma_k\}$ is a nondecreasing set of real numbers. We will show that this set is unbounded and forms a set of critical values for J.

Lemma 4. *Let $\{\mu_k\}_{k\geq 1}$ and $\{\phi_k\}_{k\geq 1}$ be the Steklov eigenvalues and (normalized) eigenfunctions defined by*

$$\begin{cases} \Delta\phi_k = 0 & \text{in } \Omega \\ \phi_k = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial\phi_k}{\partial\nu} = \mu_k\phi_k & \text{on } \Gamma \end{cases} \tag{2.7}$$

with

$$(\phi_j, \phi_k)_H = \int_\Omega \nabla\phi_j \cdot \nabla\phi_k \, dx + \int_\Gamma \phi_k\phi_j = \delta_{j,k}.$$

Suppose $c \in \mathbb{R}$ is fixed. Then the following statements hold.

(i) If $c < 0$, there exists constants $R > 0$, $a > 0$ such that

$$E(v) \geq a$$

for every $v \in H$ with $\|v\|_H = R$.

(ii) If $c \geq 0$, set $\mu_0 = 0$ and note that 0 is not an eigenvalue as long as $\Gamma_0 \neq \emptyset$, so $\mu_1 > 0$. For $k_0 \geq 0$ such that $\mu_{k_0} \leq c < \mu_{k_0+1}$, set

$$H_0 = span\{\phi_1, \phi_2, \ldots, \phi_{k_0}\}$$

(set $H_0 = \{0\}$ if $k_0 = 0$). Then there exists constants $R > 0$, $a > 0$ such that

$$E(v) \geq a$$

for every $v \in H_0^\perp$ with $\|v\|_H = R$.

Proof. First, as $\alpha + 1 < \alpha^*$, Sobolev embedding gives (for some constants)

$$\|v\|_{L^{\alpha+1}(\partial\Omega)} \leq C \|v\|_{W^{1/2,2}(\partial\Omega)} \leq C \|v\|_{H^1(\Omega)} \leq C \|v\|_H .$$

Thus,

$$\int_\Gamma |v|^{\alpha+1} \, d\sigma = \int_{\partial\Omega} |v|^{\alpha+1} \, d\sigma = \|v\|^{\alpha+1}_{L^{\alpha+1}(\Omega)} \leq C_1 \|v\|_H^{\alpha+1}$$

for some constant $C_1 > 0$.

Case (i): Assume $c < 0$. Then

$$\begin{aligned}
\frac{1}{2}\int_\Omega \|\nabla v\|^2 \, dx - \frac{c}{2}\int_\Gamma v^2 \, d\sigma &= \frac{1}{2}\left(\|\nabla v\|^2_{L^2(\Omega)} + |c| \|v\|^2_{L^2(\partial\Omega)}\right) \\
&\geq \frac{\min(1, |c|)}{2}\left(\|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\partial\Omega)}\right) \\
&= C_2 \|v\|_H^2
\end{aligned}$$

for a constant $C_2 > 0$. Then we have

$$\begin{aligned}
E(v) &= \frac{1}{2}\int_\Omega \|\nabla v\|^2 \, dx - \frac{c}{2}\int_\Gamma v^2 \, d\sigma - \frac{\lambda}{\alpha+1}\int_\Gamma |v|^{\alpha+1} \, d\sigma \\
&\geq C_2 \|v\|_H^2 - \frac{C_1\lambda}{\alpha+1}\|v\|_H^{\alpha+1}
\end{aligned}$$

Suppose $\|v\|_H = R$ for some $R > 0$ to be chosen. Then

$$E(v) \geq C_2 R^2 - \frac{C_1 \lambda}{\alpha+1} R^{\alpha+1} = R^2 \left(C_2 - \frac{C_1 \lambda}{\alpha+1} R^{\alpha-1} \right).$$

We may now choose R small enough (noting $\alpha - 1 > 0$) so that

$$C_2 - \frac{C_1 \lambda}{\alpha+1} R^{\alpha-1} \geq \delta > 0$$

for some $\delta > 0$. Thus,

$$E(v) \geq R^2 \delta,$$

which completes the proof in this case by setting $a = R^2 \delta$.

Case (ii): Suppose $\mu_{k_0} \leq c < \mu_{k_0+1}$ for some $k_0 \geq 0$. The eigenvalues $\{\mu_k\}$ are defined by the Rayleigh quotient

$$\mu_k = \inf_{v \in \{\phi_1, \ldots, \phi_{k-1}\}^\perp} \frac{\int_\Omega \|\nabla v\|^2 \, dx}{\int_{\partial\Omega} v^2 \, d\sigma}.$$

In particular,

$$\mu_{k_0+1} = \inf_{v \in \{H_0^\perp\}} \frac{\int_\Omega \|\nabla v\|^2 \, dx}{\int_{\partial\Omega} v^2 \, d\sigma}.$$

Thus, for any $v \in H_0^\perp$, we have

$$\mu_{k_0+1} \leq \frac{\int_\Omega \|\nabla v\|^2 \, dx}{\int_{\partial\Omega} v^2 \, d\sigma},$$

so

$$\int_\Gamma v^2 \, d\sigma \leq \frac{1}{\mu_{k_0+1}} \|\nabla v\|^2_{L^2(\Omega)}.$$

Then as $c \geq 0$, we have

$$-\frac{c}{2} \int_\Gamma v^2 \, d\sigma \geq -\frac{c}{2\mu_{k_0+1}} \|\nabla v\|^2_{L^2(\Omega)}.$$

These two inequalities show

$$\|v\|_H^2 = \|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Gamma)} \leq \left(1 + \frac{1}{\mu_{k_0+1}}\right) \|\nabla v\|^2_{L^2(\Omega)}$$

and

$$\frac{1}{2} \|\nabla v\|^2_{L^2(\Omega)} - \frac{c}{2} \|v\|^2_{L^2(\partial\Omega)} \geq \frac{1}{2} \left(1 - \frac{c}{\mu_{k_0+1}}\right) \|\nabla v\|^2_{L^2(\Omega)}.$$

Combining gives

$$\frac{1}{2}\left\|\nabla v\right\|_{L^2(\Omega)}^2 - \frac{c}{2}\int_\Gamma v^2\, d\sigma \geq \frac{1}{2}\left(\frac{\mu_{k_0+1}-c}{\mu_{k_0+1}+1}\right)\left\|v\right\|_H^2,$$

and so

$$E(v) \geq \frac{1}{2}\left(\frac{\mu_{k_0+1}-c}{\mu_{k_0+1}+1}\right)\left\|v\right\|_H^2 - \frac{C_1\lambda}{\alpha+1}\left\|v\right\|_H^{\alpha+1}.$$

As $c < \mu_{k_0+1}$, the coefficient of $\|v\|_H^2$ is positive, and so by the same argument as in case (i), there exists $R > 0$, $a > 0$ so that $E(v) \geq a$ when $\|v\|_H = R$. $\square$

Lemma 5. *Let $c \in \mathbb{R}$ be fixed. If $\mu_{k_0} \leq c < \mu_{k_0+1}$ for some $k_0 \geq 0$, then $0 < \gamma_{k_0+1}$, and so $0 < \gamma_k < \infty$ for every $k \geq k_0 + 1$. If $c < 0$, then $0 < \gamma_k < \infty$ for every $k \geq 1$.*

Proof. Suppose first that $\mu_{k_0} \leq c \leq \mu_{k_0+1}$ for some $k_0 \geq 0$. Fix $k \geq k_0 + 1$ and let $A \in \mathcal{A}_k$ be given. Now $\Phi(A) \geq k > k_0$. Assume, by way of contradiction, that $A \cap H_0^\perp = \emptyset$. Then the orthogonal projection of A onto H_0 yields a continuous odd mapping of A onto $H_0\backslash\{0\}$. But $H_0\backslash\{0\}$ has exactly k_0 linearly independent elements, so this gives a continuous odd mapping onto $\mathbb{R}^{k_0}\backslash\{0\}$, which implies $\Phi(A) \leq k_0$. This is a contradiction, so $A \cap H_0^\perp \neq \emptyset$.

Let $u^* \in A \cap H_0^\perp$, and $a > 0, R > 0$ be as in Lemma 4. Since $u^* \in A \subset \Sigma$, $\|u^*\|_H = 1$. Then $Ru^* \in A$ and $\|Ru^*\|_H = R$, so $E(Ru^*) \geq a$. Thus,

$$J(u^*) = \sup_{t>0} E(tu^*) \geq E(Ru^*) \geq a,$$

and therefore

$$\sup_{u\in A} J(u) \geq J(u^*) \geq a.$$

As $A \in \mathcal{A}_k$ was arbitrary, we have

$$\gamma_{k_0+1} = \inf_{A\in\mathcal{A}_k}\sup_{u\in A} J(u) \geq a > 0.$$

Lastly, the set $\{\gamma_k\}$ is finite and nondecreasing, giving

$$0 < \gamma_{k_0+1} \leq \gamma_k < \infty$$

for every $k \geq k_0 + 1$.

In the case that $c < 0$, Lemma 4 gives the existence of $a > 0, R > 0$ so that $E(v) \geq a$ for every $v \in H$ with $\|v\|_H = R$. Then for any $A \in \mathcal{A}_1$ and for every $u \in A$ we have

$$J(u) \geq E(Ru) \geq a > 0,$$

giving

$$\gamma_1 = \inf_{A \in \mathcal{A}_1} \sup_{u \in A} J(u) \geq a > 0.$$

Therefore, $0 < \gamma_1 \leq \gamma_k < \infty$ for every $k \geq 1$. □

Note that when $0 = \mu_0 \leq c < \mu_1$, the number k_0 is 0, so in this case $0 < \gamma_1 \leq \gamma_k < \infty$ as well. Thus, whenever $c < \mu_1$, $\{\gamma_k\}_{k \geq 1}$ is a positive nondecreasing sequence of finite numbers. When $\mu_{k_0} \leq c < \mu_{k_0+1}$ for $k_0 \geq 1$, $\{\gamma_k\}_{k \geq k_0+1}$ is a positive nondecreasing sequence of finite numbers.

Now to show the values γ_k are critical values for J we need to argue that J satisfies the following Palais-Smale condition. Recall

$$q(u) = (u, u)_H = \|\nabla u\|^2_{L^2(\Omega)} + \int_\Gamma u^2 \, d\sigma.$$

Lemma 6. *(Palais-Smale Condition). Let $0 < \gamma < \infty$ be given. Let $\{u_n\}_{n \geq 1} \subset \Sigma$ and $\{\beta_n\}_{n \geq 1} \subset \mathbb{R}$ be such that*

$$J(u_n) \to \gamma$$

in $\mathbb{R}$ and

$$\epsilon_n = J'(u_n) - \beta_n q'(u_n) \to 0$$

in H'. Then $\beta_n \to 0$ and there is $u \in \Sigma$ and a subsequence $\{u_{n_j}\}$ converging to u in H.

Proof. We will first show that the set $\{v_n\}$ defined by $v_n = t(u_n)u_n$ is bounded in H. Since u_n is in Σ, $\|u_n\|_H = 1$, so $\{v_n\}$ being bounded in H is equivalent to $\{t(u_n)\}$ being bounded in $\mathbb{R}$. From the definition of $t(u)$,

$$t(u)^2 \left(\|\nabla u\|^2_{L^2(\Omega)} - c \int_\Gamma u^2 \, d\sigma \right) = \lambda t(u)^{\alpha+1} \int_\Gamma |u|^{\alpha+1} \, d\sigma$$

for any u with $0 < J(u) < \infty$. Now,

$$\begin{aligned} J(u) &= E(t(u)u) \\ &= \frac{t(u)^2}{2}\left(\|\nabla u\|_{L^2(\Omega)}^2 - c\int_\Gamma u^2\, d\sigma \right) - \frac{\lambda t(u)^{\alpha+1}}{\alpha+1}\int_\Gamma |u|^{\alpha+1}\, d\sigma \\ &= t(u)^2\left(\frac{1}{2} - \frac{1}{\alpha+1}\right)\left(\|\nabla u\|_{L^2(\Omega)}^2 - c\int_\Gamma u^2\, d\sigma \right). \end{aligned}$$

The sequence $\{J(u_n)\}$ converges by assumption, and so $\{J(u_n)\}$ is bounded, and therefore the sequence

$$\left\{ t(u_n)^2 \left(\|\nabla u_n\|_{L^2(\Omega)}^2 - c\int_\Gamma u_n^2\, d\sigma \right) \right\}_{n\geq 1}$$

is bounded as well. Again using the definition of $t(u)$, the sequence

$$\left\{ t(u_n)^{\alpha+1}\int_\Gamma |u_n|^{\alpha+1}\, d\sigma \right\}_{n\geq 1} = \left\{ \int_\Gamma |v_n|^{\alpha+1}\, d\sigma \right\}_{n\geq 1}$$

is therefore bounded. There is then $M > 0$ so that

$$M \geq \int_\Gamma |v_n|^{\alpha+1}\, d\sigma = \int_{\partial\Omega} |v_n|^{\alpha+1}\, d\sigma.$$

By Holder, setting $1/q = 1 - 2/(\alpha+1) = (\alpha-1)/(\alpha+1)$,

$$\begin{aligned} \left\|v_n^2\right\|_{L^1(\partial\Omega)} &\leq \left\|v_n^2\right\|_{L^{\frac{\alpha+1}{2}}(\partial\Omega)} \|1\|_{L^1(\partial\Omega)} \\ &= \left(\int_{\partial\Omega} |v_n|^{\alpha+1}\, d\sigma \right)^{\frac{2}{\alpha+1}} |\partial\Omega|^{\frac{\alpha-1}{\alpha+1}} \\ &\leq M^{\frac{2}{\alpha+1}} |\partial\Omega|^{\frac{\alpha-1}{\alpha+1}}. \end{aligned}$$

Then

$$\|v_n\|_{L^2(\partial\Omega)} = \left\|v_n^2\right\|_{L^1(\partial\Omega)}^{\frac{1}{2}} \leq M^{\frac{1}{\alpha+1}} |\partial\Omega|^{\frac{\alpha-1}{2(\alpha+1)}},$$

and so $\{v_n\} = \{t(u_n)u_n\}$ is bounded in $L^2(\partial\Omega)$. Then the sequence

$$\left\{ t(u_n)^2 \|\nabla u_n\|_{L^2(\Omega)}^2 \right\}_{n\geq 1}$$

is bounded. We further have $\|\nabla u_n\|_{L^2(\Omega)}^2 \leq \|u_n\|_H^2 = 1$, so $\left\{ \|\nabla u_n\|_{L^2(\Omega)}^2 \right\}$ is bounded, implying $\{t(u_n)\}$ is a bounded sequence in $\mathbb{R}$, in turn implying that $\{v_n\}$ is bounded in H-norm.

Now, $J'(w) = t(w)E'(t(w)w)$ for any $w \in H$ with $0 < J(w) < \infty$. Since $J(u_n) \to \gamma > 0$, we have $0 < J(u_n) < \infty$ for sufficiently large n, and using the definition of $t(w)$ gives

$$\langle J'(u_n), u_n \rangle = t(u_n)\langle E'(t(u_n)u_n), u_n \rangle = t(u_n) f'(t(u_n)) = 0.$$

Further, $\langle q'(u_n), u_n \rangle = 2q(u_n) = 2(1) = 2$ since $u_n \in \Sigma$. We then have, for large enough n,

$$\begin{aligned} -2\beta_n &= \langle J'(u_n), u_n \rangle - \beta_n \langle q'(u_n), u_n \rangle \\ &= \langle J'(u_n) - \beta_n, u_n \rangle \\ &= \langle \epsilon_n, u_n \rangle \\ &\to 0, \end{aligned}$$

and so the sequence $\{\beta_n\}$ converges to 0 in $\mathbb{R}$. For any $w \in H$,

$$\begin{aligned} \langle J'(u_n), w \rangle &= \langle t(u_n)E'(t(u_n)u_n), w \rangle \\ &= t(u_n)\langle E'(t(u_n)u_n, w \rangle \\ &= t(u_n)\bigg[\int_\Omega \nabla(t(u_n)u)n) \cdot \nabla w \, dx - c \int_\Gamma t(u_n)u_n w \, d\sigma \\ &\quad - \lambda \int_\Gamma t(u_n)u_n \left| t(u_n)u_n \right|^{\alpha-1} w \, d\sigma \bigg] \\ &= t(u_n)\bigg[\int_\Omega \nabla v_n \cdot \nabla w \, dx - c \int_\Gamma v_n w \, d\sigma \\ &\quad - \lambda \int_\Gamma v_n \left| v_n \right|^{\alpha-1} w \, d\sigma \bigg]. \end{aligned}$$

On the other hand,

$$\begin{aligned} \langle J'(u_n), w \rangle &= \langle \epsilon_n + \beta_n q'(u_n), w \rangle \\ &= \langle \epsilon_n, w \rangle + \beta_n \langle q'(u_n), w \rangle \\ &= \langle \epsilon_n, w \rangle + \beta_n \left[2 \int_\Omega \nabla u_n \cdot \nabla w \, dx + 2 \int_\Gamma u_n w \, d\sigma \right] \\ &= \langle \epsilon_n, w \rangle + 2\beta_n (u_n, w)_H. \end{aligned}$$

We therefore have

$$\begin{aligned} t(u_n) \left[\int_\Omega \nabla v_n \cdot \nabla w \, dx - c \int_\Gamma v_n w \, d\sigma - \lambda \int_\Gamma v_n \left| v_n \right|^{\alpha-1} w \, d\sigma \right] \\ = \langle \epsilon_n, w \rangle + 2\beta_n (u_n, w)_H \quad (2.8) \end{aligned}$$

for every $w \in H$. As $\{v_n\}$ is an H-norm bounded sequence, we can extract a subsequence $\{v_{n_j}\}_{j\geq 1}$ that converges weakly to some $v \in H$. Since H is compactly embedded in $L^2(\Omega)$, we may take (by further extraction of a subsequence and renaming) $\{v_{n_j}\}_{j\geq 1}$ to converge strongly to v in $L^2(\Omega)$. The sequence $\{t(u_n)\}$ is bounded in $\mathbb{R}$, and so has a subsequence converging to some $t \geq 0$. Renaming the index gives a sequence $\{v_{n_j}\}_{j\geq 1}$ such that

$$\begin{aligned} v_{n_j} &\rightharpoonup v \text{ weakly in } H, \\ v_{n_j} &\to v \text{ strongly in } L^2(\Omega), \\ t(u_{n_j}) &\to t \text{ in } \mathbb{R}. \end{aligned}$$

We must actually have $t > 0$, for if $t(u_{n_j}) \to 0$, then $v_{n_j} \to 0$ in H, so $E(v_{n_j}) \to 0$. But $E(v_{n_j}) = E(t(u_{n_j})u_{n_j}) \to J(u_{n_j}) \to \gamma > 0$.

Define $R_{n_j} \in H'$ by

$$R_{n_j}(w) = \int_\Omega v_{n_j} w\, dx + c\int_\Gamma v_{n_j} w\, d\sigma + \lambda \int_\Gamma v_{n_j} \left|v_{n_j}\right|^{\alpha-1} w\, d\sigma \\ + \frac{1}{t(u_{n_j})}\left(\langle e_{n_j}, w\rangle + 2\beta_{n_j}(u_{n_j}, w)_H\right)$$

for $w \in H$. Then by rearranging (2.8), we have

$$\int_\Omega \nabla v_{n_j} \cdot \nabla w\, dx + \int_\Omega v_{n_j} w\, dx = R_{n_j}(w).$$

Now, $v_{n_j} \rightharpoonup v$ weakly in H, and as $H \hookrightarrow H^{1/2}(\partial\Omega) \hookrightarrow L^p(\partial\Omega)$ compactly for every $1 < p < \alpha^*$, we may take $v_{n_j} \to v$ strongly in $L^2(\partial\Omega)$ and in $L^{\alpha+1}(\partial\Omega)$. We have already established $v_{n_j} \to v$ strongly in $L^2(\Omega)$, and so we have that for every $w \in H$,

$$\int_\Omega v_{n_j} w\, dx \to \int_\Omega vw\, dx,$$

$$\int_\Gamma v_{n_j} w\, dx \to \int_\Gamma vw\, dx,$$

and

$$\int_\Gamma v_{n_j} \left|v_{n_j}\right|^{\alpha-1} w\, dx \to \int_\Gamma v\, |v|^{\alpha-1}\, w\, dx.$$

Then as $t(u_{n_j}) \to t > 0$, $\epsilon_{n_j} \to 0$, and $\beta_{n_j} \to 0$, we have

$$R_{n_j}(w) \to R(w),$$

where $R(w)$ is defined by

$$R(w) = \int_\Omega vw\,dx + \lambda \int_\Gamma v\,|v|^{\alpha-1}\,w\,d\sigma + c\int_\Gamma vw\,d\sigma.$$

This convergence is actually uniform in w due to the strong convergence of the functions v_{n_j} in the integrals and applying Holder's inequality. Thus, $R_{n_j} \to R$ in H' norm, given by

$$\|R\|_{H'} = \sup_{w\in H, \|w\|_H=1} |R(w)|\,.$$

Specifically, if $\|w\|_H = 1$, then

$$\begin{aligned}
\left|\int_\Omega (v_{n_j} - v)w\,dx\right| &\le \left\|v_{n_j} - v\right\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \\
&\le C\left\|v_{n_j} - v\right\|_{L^2(\Omega)} \|w\|_H \\
&\le C\left\|v_{n_j} - v\right\|_{L^2(\Omega)} \\
&\to 0.
\end{aligned}$$

The convergence of the other integrals follows similarly.

Now, $v_{n_j} \rightharpoonup v$ weakly in H, so we have

$$\int_\Omega \nabla v_{n_j}\cdot\nabla w\,dx \to \int_\Omega \nabla v\cdot\nabla w\,dx,$$

and so

$$R_{n_j}(w) \to \int_\Omega \nabla v\cdot\nabla w\,dx + \int_\Omega vw\,dx.$$

Thus,

$$\int_\Omega \nabla v\cdot\nabla w\,dx + \int_\Omega vw\,dx = \int_\Omega vw\,dx + \lambda\int_\Gamma v\,|v|^{\alpha-1}\,w\,d\sigma + c\int_\Gamma vw\,d\sigma.$$

Then for every $w \in H$, we have

$$\int_\Omega \nabla v\cdot\nabla w\,dx = \lambda\int_\gamma v\,|v|^{\alpha-1}\,w\,d\sigma + c\int_\Gamma vw\,d\sigma,$$

which shows v is an H solution to

$$\begin{cases}
\Delta v = 0 & \text{in } \Omega \\
v = 0 & \text{on } \Gamma_0 \\
\dfrac{\partial v}{\partial\nu} = cv + \lambda v\,|v|^{\alpha-1} & \text{on } \Gamma.
\end{cases}$$

It remains to show that $v_{n_j} \to v$ strongly in H. To this end, $\{R_{n_j}\}_j$ is an H'-norm convergent sequence, and is thus a Cauchy sequence in H' norm, so

$$\sup_{w \in H, \|w\|=1} \left| R_{n_j}(w) - R_{n_k}(w) \right| \to 0$$

as $j, k \to \infty$. But $R_{n_j}(w) - R_{n_k}(w) = (v_{n_j} - v_{n_k}, w)_{H^1(\Omega)}$, for every $w \in H$, and the convergence above is uniform in w. Thus, for $\epsilon > 0$, there is a J so that for every $j, k > J$, we have

$$\left| (v_{n_j} - v_{n_k}, w)_{H^1(\Omega)} \right| < \epsilon$$

for every $w \in H$. In particular, when $w = v_{n_j} - v_{n_k}$, we have

$$\left\| v_{n_j} - v_{n_k} \right\|_H^2 \le C \left\| v_{n_j} - v_{n_k} \right\|_{H^1(\Omega)}^2 = C \left| (v_{n_j} - v_{n_k}, v_{n_j} - v_{n_k})_{H^1(\Omega)} \right| < C\epsilon,$$

for a constant C depending only on Ω. Therefore, $\{v_{n_j}\}_j$ is Cauchy in H-norm, and thus converges strongly in H. Then by weak limit uniqueness, we have $v_{n_j} \to v$ strongly in H. $\square$

This leads us to our main existence result in the subcritical case.

Theorem 2. *Let $c > 0$, $\lambda > 0$ be fixed, with $\mu_{k_0} \le c < \mu_{k_0+1}$ for some $k_0 \ge 1$. Then $\{\gamma_k\}_{k \ge k_0+1}$ is a non-decreasing sequence of finite positive critical values for J, and $\gamma_k \to \infty$. In particular, problem (1.1) has infinitely many nontrivial solutions $\{v_k\}_k$, with $E(v_k) \to \infty$ and $\|v_k\|_H \to \infty$.*

If $c \le 0$, and $\lambda > 0$, then the same result holds with $k_0 = 0$.

Proof. Lemma 3 guarantees that critical points for J on Σ give critical points for E in H with the same critical value. Each critical point of E is a weak solution to the boundary value problem (1.1). Now for every $k > k_0$,

$$\gamma_k = \inf_{A \in \mathcal{A}_k} \sup_{u \in A} J(u)$$

is positive and finite. We wish to show γ_k is a critical value of J; that is, that there is some $u \in H$ with $J'(u) = 0$ and $J(u) = \gamma_k$.

Fix $k > k_0$, and set $\gamma = \gamma_k$. Assume γ is not a critical value of J, so there is no $u \in H$ with both $J'(u) = 0$ and $J(u) = \gamma$. The idea is that we can start at a value of

J that exceeds γ by a small amount and follow the gradient flow to points where the value of J falls below γ. This will lead to a contradiction to the construction of γ as a saddle point.

Define for $a \in \mathbb{R}$ the sets

$$K_a = \{u \in H : J(u) \leq a\}.$$

We have from the Deformation Theorem ([19, Theorem 3.11], also repeated in Corollary 1 below) that if J satisfies the Palais-Smale condition, then for any sufficiently small $\epsilon > 0$, there exists a constant $0 < \delta < \epsilon$ and a continuous function $\eta : H \to H$ such that $\eta(K_{\gamma+\delta}) \subset K_{\gamma-\delta}$. We may also take η to be an invariant functional on $\mathcal{A}_k$, so that $\overline{\eta(\mathcal{A}_k)} \subset \mathcal{A}_k$.

Now, take $\epsilon > 0$ small enough and choose δ, η from above. By the definition of γ, we may find a set $B \in \mathcal{A}_k$, depending on δ, so that

$$\inf_{A \in \mathcal{A}_k} \sup_{u \in A} J(u) + \delta > \sup_{u \in B} J(u),$$

or

$$\sup_{u \in B} J(u) < \gamma + \delta.$$

Then for every $u \in B$, $J(u) < \gamma + \delta$, so $u \in K_{\gamma+\delta}$. That is, $B \subset K_{\gamma+\delta}$. Then $\eta(B) \subset K_{\gamma-\delta}$. By the continuity of J, we have $\overline{\eta(B)} \subset K_{\gamma-\delta}$ as well. By assumption, we have $\overline{\eta(B)} \in \mathcal{A}_k$, and thus $\overline{\eta(B)}$ is one of the elements taken in the infimum, so

$$\sup_{u \in \overline{\eta(B)}} J(u) \geq \inf_{A \in \mathcal{A}_k} \sup_{u \in A} J(u) = \gamma.$$

But if $u \in \overline{\eta(B)}$, then $u \in K_{\gamma-\delta}$, so $J(u) \leq \gamma - \delta$ for every $u \in \overline{\eta(B)}$. Then

$$\sup_{u \in \overline{\eta(B)}} J(u) \leq \gamma - \delta,$$

which gives

$$\gamma \leq \sup_{u \in \overline{\eta(B)}} J(u) \leq \gamma - \delta,$$

which contradicts $\delta > 0$. Thus, γ must be a critical value for J, so we must have some $u \in H$ with $J'(u) = 0$ and $J(u) = \gamma$. By the preceding discussion, this critical point u yields an H solution to (1.1). $\square$

In closing, we provide the details of the deformation function η used above. The idea will be that if γ is not a critical value of J, we can start at values slightly larger than c and follow the gradient flow "downward" to values smaller than γ in a nice way (without hitting a point where the derivative vanishes). The function η we find will be a solution to a particular ordinary differential equation.

The proof of this theorem and the following corollary follow closely the standard argument which can be found in Chapter 8.5 of Evans ([9]) or Theorem 3.11 in Struwe ([19]).

Recall

$$K_a = \{u \in H : J(u) \leq a\}.$$

Theorem 3. *(Deformation Theorem). Assume J is a C^1 functional on H satisfying the Palais-Smale condition from Lemma 6. Suppose $\gamma \in \mathbb{R}$ is not a critical value of J. Then for every sufficiently small $\epsilon > 0$, there exists a constant $0 < \delta < \epsilon$ and a continuous function $\eta : [0,1] \times H \to H$ satisfying*
(i) $\eta(0,u) = u$ for $u \in H$,
(ii) $\eta(1,u) = u$ for $u \notin J^{-1}[\gamma - \epsilon, \gamma + \epsilon]$,
(iii) $J(\eta(t,u)) \leq J(u)$ for $u \in H$, $0 \leq t \leq 1$,
(iv) $\eta(1, K_{\gamma+\delta}) \subset K_{\gamma-\delta}$.

Proof. First, we claim that there are constants $0 < \sigma < 1$ and $0 < \epsilon < 1$ so that

$$\|J'(u)\| \geq \sigma$$

whenever $u \in K_{\gamma+\epsilon} \backslash K_{\gamma-\epsilon}$. To see this, we proceed by contradiction. If we could not find such constants, then for every $n \in \mathbb{N}$, we could find an element

$$u_n \in K_{\gamma+\frac{1}{n}} \backslash K_{\gamma-\frac{1}{n}}$$

such that

$$\|J'(u_n)\| \leq \frac{1}{n}.$$

But then

$$\gamma - \frac{1}{n} < J(u_n) \leq \gamma + \frac{1}{n},$$

so $\{u_n\}$ is a sequence with $J(u_n) \to \gamma$ and $J'(u_n) \to 0$. The Palais-Smale condition then applies (with $\alpha_n = 0$ for every n), and so there is an element $u \in H$ and a subsequence u_{n_j} converging to u in H. But since J is C^1, this implies $J(u) = \gamma$ and $J'(u) = 0$, whence γ is a critical value of J, a contradiction.

Choose δ so that

$$0 < \delta < \min\left(\epsilon, \frac{\sigma}{2}\right).$$

Define

$$A = \{u \in H : J(u) \leq \gamma - \epsilon \text{ or } J(u) \geq \gamma + \epsilon\},$$

$$B = \{u \in H : \gamma - \delta \leq J(u) \leq \gamma + \delta\}.$$

Define for $u \in H$.

$$g(u) = \frac{\operatorname{dist}(u, A)}{\operatorname{dist}(u, A) + \operatorname{dist}(u, B)}.$$

Note that g is well defined on H, as one of $\operatorname{dist}(u, A)$ or $\operatorname{dist}(u, B)$ must be positive. To see this, suppose u were in A, and note

$$\operatorname{dist}(u, B) = \inf_{v \in B} \|u - v\|_H.$$

For a fixed $v \in B$, we may find by the Mean Value Theorem some $\zeta \in H$ with $|J(u) - J(v)| = \|J'(\zeta)\| \, \|u - v\|$ But $|J(u) - J(v)| \geq \epsilon - \delta$, and J' (being continuous) is bounded on bounded subsets, so $\|u - v\|$ is bounded below by a positive constant, and so $\operatorname{dist}(u, B) > 0$. A similar argument shows that if $u \in B$ then $\operatorname{dist}(u, A) > 0$, and if u is not in A or B, then both distances are positive. It is clear from the definition of g that

$$0 \leq g \leq 1, \; g = 0 \text{ on } A, \; g = 1 \text{ on } B.$$

Define h by $h(t) = 1$ for $t \in [0, 1]$ and $h(t) = 1/t$ for $t \geq 1$. For $u \in H$, define the mapping $V : H \to H$ by

$$V(u) = -g(u) h\left(\left\|J'(u)\right\|\right) J'(u).$$

Now, fix $u \in H$, and let η solve the ODE

$$\frac{d}{dt}\eta(t, u) = V(\eta(t, u))$$

for $t > 0$ with initial condition

$$\eta(0,u) = u.$$

As V is Lipschitz, continuous, there is a unique solution existing for every $t \geq 0$. We restrict the t dependence of η to $[0,1]$. Clearly $\eta(0,u) = u$, so (i) is satisfied. Further, if $u \notin J^{-1}[\gamma - \epsilon, \gamma + \epsilon]$, then $u \in A$, so $g(u) = 0$, and so $V(u) = 0$. Then η is the constant function $\eta(t,u) = t$ for all $0 \leq t \leq 1$, so in particular $\eta(1,u) = u$, and therefore (ii) is satisfied.

We compute for any $u \in H$ and $t \in [0,1]$

$$\begin{aligned}\frac{d}{dt} J(\eta(t,u)) &= \left\langle J'(\eta(t,u)), V(\eta(t,u)) \right\rangle \\ &= -g(\eta(t,u)) h\left(\left\| J'(\eta(t,u)) \right\| \right) \left\| J'(\eta(t,u)) \right\|^2 \\ &\leq 0,\end{aligned}$$

as g and h are always positive. Thus, $J(\eta(t,u))$ is decreasing with t, and so

$$J(u) = J(\eta(0,u)) \geq J(\eta(t,u))$$

for every $0 \leq t \leq 1$. This establishes (iii).

To establish (iv), we need to show that if $u \in K_{\gamma+\delta}$, then $\eta(1,u) \in K_{\gamma-\delta}$. Let $u \in K_{\gamma+\delta}$ be arbitrary, so that $J(u) \leq \gamma + \delta$. We must show

$$J(\eta(1,u)) \leq \gamma - \delta.$$

First, note that $J(\eta(1,u)) \leq J(u) \leq \gamma + \delta$, so $\eta(1,u) \notin B$ is equivalent to $J(\eta(1,u)) < \gamma - \delta$. We then only need to show that $\eta(1,u) \notin B$. Now, in the case that $\eta(t,u) \notin B$ for some $0 \leq t \leq 1$, then $J(\eta(1,u)) \leq J(\eta(t,u)) < \gamma - \delta$, and we are done. Then we only need to consider the case that $\eta(t,u) \in B$ for every $t \in [0,1)$. Assume we are in this case. Then

$$g(\eta(t,u)) = 1$$

for every $t \in [0,1]$, and

$$\frac{d}{dt} J(\eta(t,u)) = -h\left(\left\| J'(\eta(t,u)) \right\| \right) \left\| J'(\eta(t,u)) \right\|^2.$$

There are two cases: either $\|J'(\eta(t,u))\| < 1$ or $\|J'(\eta(t,u))\| \geq 1$. If $\|J'(\eta(t,u))\| < 1$, then

$$h\left(\left\|J'(\eta(t,u))\right\|\right) = 1,$$

so

$$\frac{d}{dt}J(\eta(t,u)) = -\left\|J'(\eta(t,u))\right\|^2.$$

Now $J(\eta(t,u)) \leq \gamma + \delta < \gamma + \epsilon$, so $\eta(t,u) \in K_{\gamma+\epsilon}$. As $\eta(t,u) \in B$, $J(\eta(t,u)) \geq \gamma - \delta > \gamma - \epsilon$, so $\eta(t,u) \notin K_{\gamma-\epsilon}$. Thus, $\eta(t,u) \in K_{\gamma+\epsilon} \backslash K_{\gamma-\epsilon}$, and so

$$\|J(\eta(t,u))\| \geq \sigma.$$

Thus,

$$\frac{d}{dt}J(\eta(t,u)) \leq -\sigma^2.$$

In the case that $\|J'(\eta(t,u))\| \geq 1$, we have

$$h\left(\left\|J'(\eta(t,u))\right\|\right) = \frac{1}{\|J'(\eta(t,u))\|},$$

and (as $0 < \sigma < 1$),

$$\frac{d}{dt}J(\eta(t,u)) = -\left\|J'(\eta(t,u))\right\| \leq -\sigma \leq -\sigma^2.$$

Then in both cases we have

$$\frac{d}{dt}J(\eta(t,u)) \leq -\sigma^2$$

for every $0 \leq t < 1$.

We may find $\xi \in (0,1)$ so that

$$J(\eta(1,u)) - J(\eta(0,u)) = \left.\frac{d}{dt}J(\eta(t,u))\right|_{t=\xi} \leq -\sigma^2,$$

so

$$J(\eta(1,u)) \leq J(\eta(0,u)) - \sigma^2 = J(u) - \sigma^2.$$

As $J(u) \leq \gamma + \delta$, we have

$$J(\eta(1,u)) \leq \gamma + \delta - \sigma^2 < \gamma + \delta - 2\delta = \gamma - \delta.$$

We have then shown $\eta(1,u) \in K_{\gamma-\delta}$, which establishes (iv) and completes the proof. $\square$

Finally, we have the following corollary which gives us the mapping used in Theorem 2.

Corollary 1. *Assume all the conditions of Theorem 3. If γ is not a critical value for J, then for every sufficiently small $\epsilon > 0$, there exists a number $0 < \delta < \epsilon$ and a C^1 mapping $\eta : H \to H$ so that $\eta(K_{\gamma+\delta}) \subset K_{\gamma-\delta}$ and $\overline{\eta(A)} \in \mathcal{A}_k$ for every $A \in \mathcal{A}_k$.*

Proof. We define $\eta : H \to H$ by

$$\eta(u) = \frac{\eta(1,u)}{\|\eta(1,u)\|},$$

where $\eta(1,\cdot)$ is the mapping defined in Theorem 3. Since the map $\eta(1,\cdot)$ is C^1, η is C^1 as well. From Theorem 3 (iv),

$$\eta(1, K_{\gamma+\delta}) \subset K_{\gamma-\delta}.$$

Let $u \in K_{\gamma+\delta}$, so that $J(u) \leq \gamma + \delta$. Then $J(\eta(1,u)) \leq \gamma - \delta$, and so we have

$$J(\eta(u)) = J\left(\frac{\eta(1,u)}{\|\eta(1,u)\|}\right) = J(\eta(1,u)) \leq \gamma - \delta,$$

showing $\eta(u) \subset K_{\gamma-\delta}$.

It remains to show that $\overline{\eta(A)} \in \mathcal{A}_k$ whenever $A \in \mathcal{A}_k$. To this end, recall

$$\mathcal{A}_k = \{A \in \Sigma : A \text{ is closed, } A = -A, \text{ and } \Phi(A) \geq k\}.$$

First, by construction of η we have $\|\eta(u)\| = 1$, and so $\eta(A) \in \Sigma$ for any A, and therefore $\overline{\eta(A)} \in \Sigma$ as well. Clearly, $\overline{\eta(A)}$ is closed for any A. Now, $J(-u) = J(u)$, and so for any $v \in H$,

$$\begin{aligned}
\langle J'(-u), w\rangle &= \frac{d}{d\epsilon} J(-u+\epsilon v)\bigg|_{\epsilon=0} \\
&= \frac{d}{d\epsilon} J(-(u-\epsilon v))\bigg|_{\epsilon=0} \\
&= \frac{d}{d\epsilon} J((u-\epsilon v))\bigg|_{\epsilon=0} \\
&= -\frac{d}{d(-\epsilon)} J((u+(-\epsilon)v))\bigg|_{-\epsilon=0} \\
&= -\langle J'(u), v\rangle \\
&= \langle -J'(u), v\rangle,
\end{aligned}$$

showing $J'(-u) = -J'(u)$. Then the function V constructed in the proof of Theorem 3, being a multiple of J', is odd as well.

Fix a $u \in H$, and suppose $\chi = \chi(t)$ solves

$$\frac{d}{dt}\chi(t) = V(\chi(t))$$

with

$$\chi(0) = u.$$

Then

$$\frac{d}{dt}(\chi(t)) = V(\chi(t)) = -V(-\chi(t)),$$

and so the function χ solves

$$\frac{d}{dt}(-\chi(t)) = V(-\chi(t))$$

with $-\chi(0) = -u$. That is, $-\chi$ is the solution when we change the initial condition from u to $-u$. This shows that for any $0 \leq t \leq 1$, the function $\eta(t, \cdot)$ constructed in Theorem 3 is odd with respect to the initial condition u, and so in particular $\eta(-u) = -\eta(u)$.

From the definition of the genus, if χ is any odd mapping, we have $\gamma(\overline{\chi(A)}) \geq \Phi(A)$ for any set A. Let $A \in \mathcal{A}_k$, so $\Phi(\overline{\eta(A)}) \geq k$. If $u \in \eta(A)$, then there is some $v \in A$ with $u = \eta(v)$, and $so - u = -\eta(v) = \eta(-v) \in \eta(-A)$ as $A = -A$. Then $\eta(A) = -\eta(A)$, and thus $\overline{\eta(A)} = -\overline{\eta(A)}$. We have already verified $\overline{\eta(A)} \in \Sigma$ and $\overline{\eta(A)}$ is closed, and we have therefore shown that $\overline{\eta(A)} \in \mathcal{A}_k$, completing the proof. $\square$

2.1.1 Positive Solutions

Theorem 2 guarantees the existence of a nontrivial solution (in fact, infinitely many) to (1.1). These solutions constructed are enumerated by the solutions of the linearized problem. We know the first Steklov eigenfunction ϕ_1 (corresponding to the smallest eigenvalue μ_1) does not change sign in Ω, and so we may take $\phi_1 \geq 0$. One might expect that if $c < \mu_1$ in (1.1), one of the solutions u for this value of c also does not change sign. Indeed this is the case, though in order to show this we will need to construct the solution in a different way. Our goal is to prove the following theorem.

Theorem 4. *Let $\mu_1 \geq 0$ be the first Steklov eigenvalue and $\phi_1 \in H$ satisfy $\Delta\phi_1 = 0$ in Ω, $\phi_1 = 0$ on Γ_0, $\frac{\partial\phi_1}{\partial\nu} = \mu_1\phi_1$ on Γ, with $\phi_1 \geq 0$. If $c < \mu_1$, then problem (1.1) admits a (weak) solution $u \in H\backslash\{0\}$ with $u \geq 0$ in Ω.*

The idea of the argument will be similar to that of Section 2.1. We will construct a functional Q on H that is bounded below, but instead of finding saddle points like we did with the functional J given in (2.6), Q will have a true minimizer $u \in H$. Since u is a minimizer and we will have $Q(u) = Q(|u|)$, $|u|$ will be a minimizer as well, and hence the minimizer may be taken to be nonnegative. This argument fails to work for the functional J, as we have no guarantee that the absolute value of a saddle point is still a saddle point. The functional Q we construct will not be the unbounded functional E given in (2.5), which is simply the first variation of (1.1), but nevertheless critical points of the functional will give rise (by scaling) to weak solutions of (1.1).

We note that if Ω is the unit sphere and Γ_0 is empty, then we have $\mu_1 = 0$, as any constant would be in H and would solve the eigenvalue problem with zero eigenvalue. If Γ_0 is nonempty, then nonzero constants fail to be in H, and so $\mu_1 > 0$.

To define our functional, we first define a new norm on H. For $c \in \mathbb{R}$, and $u \in H^1(\Omega)$ define

$$\|u\|_c^2 = \int_\Omega \|\nabla u\|^2 \, dx - c\int_{\partial\Omega} |u|^2 \, d\sigma = \|\nabla u\|_{L^2(\Omega)}^2 - c\,\|u\|_{L^2(\partial\Omega)}^2 . \tag{2.9}$$

Note that we have the standard inequalities from Sobolev embedding (for different constants C, $u \in H^1(\Omega)$, and $1 < \alpha \leq \alpha^* - 1$):

$$\begin{cases} \|u\|_{L^2(\Omega)} \leq C\,\|u\|_{H^1(\Omega)} \\ \|u\|_{L^2(\partial\Omega)} \leq C\,\|u\|_{H^1(\Omega)} \\ \|u\|_{L^{\alpha+1}(\partial\Omega)} \leq C\,\|u\|_{H^1(\Omega)} . \end{cases} \tag{2.10}$$

Recall that

$$\|u\|_H^2 = \int_\Omega \|\nabla u\|^2 \, dx + \int_{\partial\Omega} u^2 \, d\sigma.$$

This norm is equivalent to the standard norm $\|u\|_{H^1(\Omega)}$ on $H^1(\Omega)$, and so the right hand side of (2.10) may be replaced by $\|u\|_H$. We first claim that $\|\cdot\|_c$ is equivalent to these norms as well for certain values of c.

Lemma 7. *For $c < \mu_1$, $\|\cdot\|_c$ is an equivalent norm on H.*

Proof. We will show that there are constants $C_1, C_2 > 0$ so that for any $u \in H$,

$$C_1 \|u\|_H^2 \leq \|u\|_c^2 \leq C_2 \|u\|_H^2 .$$

Let $u \in H \backslash \{0\}$ be arbitrary (if $u = 0$ there is nothing to show). We consider the cases when $c < 0$ and when $0 \leq c < \mu_1$ separately (note that if $\mu_1 = 0$ this latter case is vacuous).

Assume first that $c < 0$. Then using (2.10),

$$\begin{aligned}
\|u\|_c^2 &= \|\nabla u\|_{L^2(\Omega)}^2 - c \|u\|_{L^2(\Gamma)}^2 \\
&= \|\nabla u\|_{L^2(\Omega)}^2 + |c| \|u\|_{L^2(\Gamma)}^2 \\
&\leq \|u\|_H^2 + |c| C \|u\|_H^2 \\
&\leq C_2 \|u\|_H^2 .
\end{aligned}$$

For the other direction,

$$\|u\|_c^2 = \|\nabla u\|_{L^2(\Omega)}^2 + |c| \|u\|_{L^2(\Gamma)}^2 \geq \min(1, |c|) \left(\|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Gamma)}^2 \right) = C_1 \|u\|_H^2 ,$$

showing equivalence of the norms in the case when $c < 0$.

Now assume $0 \leq c < \mu_1$. Since $c \geq 0$, we have

$$\|u\|_c^2 = \|\nabla u\|_{L^2(\Omega)}^2 - c \|u\|_{L^2(\Gamma)}^2 \leq \|\nabla u\|_{L^2(\Omega)}^2 \leq \|u\|_H^2 .$$

For the other direction, note first that

$$\mu_1 = \inf_{w \in H \backslash \{0\}} \frac{\|\nabla w\|_{L^2(\Omega)}^2}{\|w\|_{L^2(\Gamma)}^2},$$

and so

$$\mu_1 \leq \frac{\|\nabla u\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Gamma)}^2}.$$

This then implies that

$$\|u\|_{L^2(\Gamma)}^2 \leq \frac{1}{\mu_1} \|\nabla u\|_{L^2(\Omega)}^2$$

and, as $c \geq 0$,

$$-c \|u\|_{L^2(\Gamma)}^2 \geq \frac{-c}{\mu_1} \|\nabla u\|_{L^2(\Omega)}^2 .$$

Then

$$\|u\|_H^2 = \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Gamma)}^2 \leq \left(\frac{\mu_1 + 1}{\mu_1}\right) \|\nabla u\|_{L^2(\Omega)}^2 ,$$

and, using $c < \mu_1$,

$$\|u\|_c^2 = \|\nabla u\|_{L^2(\Omega)}^2 - c\,\|u\|_{L^2(\Gamma)}^2 \geq \left(\frac{\mu_1 - c}{\mu_1}\right) \|\nabla u\|_{L^2(\Omega)}^2 .$$

Combining these two gives

$$\|u\|_H^2 \leq \frac{\mu_1 + 1}{\mu_1 - c} \|u\|_c^2 = C_1 \|u\|_c^2 ,$$

which completes the proof. □

Now, define the function $Q : H^1(\Omega) \to \mathbb{R}$ by

$$Q(u) = \frac{\|u\|_c^2}{\|u\|_{L^{\alpha+1}(\partial\Omega)}^2} = \frac{\|\nabla u\|_{L^2(\Omega)}^2 - c\,\|u\|_{L^2(\partial\Omega)}^2}{\|u\|_{L^{\alpha+1}(\partial\Omega)}^2}.$$

Set

$$S = \inf_{w \in H \setminus \{0\}} Q(w).$$

Lemma 8. *If $\|\cdot\|_c$ is an equivalent norm on H, then $S > 0$.*

Proof. By definition of equivalence, there is $C_1 > 0$ so that for any $w \in H$,

$$\|w\|_c \geq C_1 \|w\|_H$$

and from (2.10) and the equivalence of $\|\cdot\|_H$ and $\|\cdot\|_{H^1(\Omega)}$,

$$\|w\|_{L^{\alpha+1}(\partial\Omega)}^2 \leq C_2 \|w\|_H .$$

for some positive constant C_2. Together these imply that if $w \neq 0$,

$$S(w) \geq \frac{C_1}{C_2},$$

so that

$$S = \inf_{w \in H \setminus \{0\}} Q(w) \geq \frac{C_1}{C_2} > 0.$$

□

The main point here is that Q is bounded below, unlike the first variation E that we usually use. We then employ standard compactness arguments to find a minimizer for Q in $H\backslash\{0\}$. The drawback to using Q is that $Q'(u) = 0$ is not equivalent to (2.3) whereas $E'(u) = 0$ is exactly (2.3). We then have an extra step of showing that a minimizer of Q provides a weak solution of (1.1), which is shown in Lemma 2.11 below.

We first show that Q has a minimizer. This follows a fairly standard argument using the compactness of the Sobolev embeddings and showing Q is lower semicontinuous.

Lemma 9. *Let c be such that $\|\cdot\|_c$ is an equivalent norm on H, and suppose $\{u_n\}$ is a sequence in $H^1(\Omega)$ with $u_n \rightharpoonup v$ weakly in $H^1(\Omega)$. Then there exists a subsequence $\{u_{n_k}\}_k$ so that*

$$Q(v) \leq \liminf_{k\to\infty} Q(u_{n_k}).$$

Proof. As $u_n \rightharpoonup v$ weakly in $H^1(\Omega)$, we may extract a subsequence $\{u_{n_k}\}_k$ so that as $k \to \infty$,

$$u_{n_k} \to v$$

strongly in $L^2(\partial\Omega)$ and $L^{\alpha+1}(\partial\Omega)$. By merit of $\liminf$, we may also assume (by further extraction of a subsequence) that

$$\liminf_{k\to\infty} Q(u_{n_k}) = \lim_{k\to\infty} Q(u_{n_k}).$$

From the homogeneity of Q, we see $Q(tw) = Q(w)$ for any $w \in H^1(\Omega)$ and $t \in \mathbb{R}$, so we may assume

$$\|u_{n_k}\|_{L^{\alpha+1}(\partial\Omega)} = 1$$

for every k.

For v above and for any $w \in H^1(\Omega)$, we have

$$0 \leq \|\nabla v - \nabla w\|^2_{L^2(\Omega)} = \|\nabla v\|^2_{L^2(\Omega)} - 2(\nabla v, \nabla w)_{L^2(\Omega)} + \|\nabla w\|^2_{L^2(\Omega)},$$

so that

$$2(\nabla v, \nabla w)_{L^2(\Omega)} - \|\nabla v\|^2_{L^2(\Omega)} \leq \|\nabla w\|^2_{L^2(\Omega)}.$$

Then

$$\begin{aligned}(2\nabla v, \nabla w - \nabla v)_{L^2(\Omega)} &= (2\nabla v, \nabla w)_{L^2(\Omega)} - (2\nabla v, \nabla v)_{L^2(\Omega)} \\ &= 2(\nabla v, \nabla w)_{L^2(\Omega)} - 2\left\|\nabla v\right\|^2_{L^2(\Omega)} \\ &= 2(\nabla v, \nabla w)_{L^2(\Omega)} - \left\|\nabla v\right\|^2_{L^2(\Omega)} - \left\|\nabla v\right\|^2_{L^2(\Omega)} \\ &\le \left\|\nabla w\right\|^2_{L^2(\Omega)} - \left\|\nabla v\right\|^2_{L^2(\Omega)}.\end{aligned}$$

We therefore have

$$\left\|\nabla w\right\|^2_{L^2(\Omega)} \ge \left\|\nabla v\right\|^2_{L^2(\Omega)} + (2\nabla v, \nabla w - \nabla v)_{L^2(\Omega)}.$$

In particular, letting $w = u_{n_k}$ gives

$$\left\|\nabla u_{n_k}\right\|^2_{L^2(\Omega)} \ge \left\|\nabla v\right\|^2_{L^2(\Omega)} + (2\nabla v, \nabla u_{n_k} - \nabla v)_{L^2(\Omega)}$$

for every k. Since $u_{n_k} \rightharpoonup v$ weakly in $H^1(\Omega)$, we have

$$(w, \nabla u_{n_k} - \nabla v)_{H^1(\Omega)} \to 0$$

for each fixed $w \in H^1(\Omega)$, which implies that

$$(2\nabla v, \nabla u_{n_k} - \nabla v)_{L^2(\Omega)} \to 0$$

as $k \to \infty$. This gives

$$\lim_{k\to\infty} \left\|\nabla u_{n_k}\right\|^2_{L^2(\Omega)} \ge \left\|\nabla v\right\|^2_{L^2(\Omega)}.$$

As $u_{n_k} \to v$ strongly in $L^2(\partial\Omega)$, we have

$$\lim_{k\to\infty} \left\|u_{n_k}\right\|^2_{L^2(\partial\Omega)} = \left\|v\right\|^2_{L^2(\partial\Omega)},$$

giving

$$\begin{aligned}\lim_{k\to\infty} Q(u_{n_k}) &= \lim_{k\to\infty} \left(\left\|\nabla u_{n_k}\right\|^2_{L^2(\Omega)} - c\left\|u_{n_k}\right\|^2_{L^2(\partial\Omega)}\right) \\ &\ge \left\|\nabla v\right\|^2_{L^2(\Omega)} - c\left\|v\right\|^2_{L^2(\partial\Omega)}.\end{aligned}$$

Furthermore, as $u_{n_k} \to v$ in $L^{\alpha+1}(\partial\Omega)$,

$$1 = \lim_{k\to\infty} \left\|u_{n_k}\right\|^2_{L^{\alpha+1}(\partial\Omega)} = \left\|v\right\|^2_{L^{\alpha+1}(\partial\Omega)}.$$

Thus

$$\lim_{k\to\infty} Q(u_{n_k}) \geq Q(v).$$

This finishes the proof as

$$Q(v) \leq \lim_{k\to\infty} Q(u_{n_k}) = \liminf_{k\to\infty} Q(u_{n_k}).$$

□

Lemma 10. *Let c be such that $\|\cdot\|_c$ is an equivalent norm on H. Then there is $v \in H\backslash\{0\}$, $v \geq 0$ in Ω such that $S = Q(v)$.*

Proof. First, if $S = \infty$, then there is nothing to prove as every $w \in H\backslash\{0\}$ would be a minimizer, and so in particular there is an element $v = |w| \geq 0$ with $S = Q(v)$. We may thus assume $S < \infty$.

By the definition of S, we may find a sequence $\{u_n\} \subset H\backslash\{0\}$ so that

$$\lim_{n\to\infty} Q(u_n) = S$$

with

$$\int_{\partial\Omega} |u_n|^{\alpha+1}\, d\sigma = 1.$$

The latter equality follows from $Q(tw) = Q(w)$ for any t. As S is finite and $Q(u_n)$ is equivalent to $\|u_n\|_{H^1(\Omega)}$, we may (extracting a subsequence if necessary) assume $\{u_n\}_n$ is bounded in $H^1(\Omega)$ norm. By extracting subsequences, we may assume $u_n \rightharpoonup v$ weakly in $H^1(\Omega)$ for some $v \in H^1(\Omega)$. We may furthermore assume

$$u_n \to v$$

strongly in $L^{\alpha+1}(\partial\Omega)$ and almost everywhere in $\partial\Omega$. By merit of Lemma 9 we may also assume that

$$Q(v) \leq \liminf_{n\to\infty} Q(u_n).$$

Since $u_n = 0$ on Γ_0 and $u_n \to v$ almost everywhere on $\partial\Omega$, we also have $v = 0$ on Γ_0, so that $v \in H$. We furthermore have v is not identically zero in H, as $u_n \to v$ strongly in $L^{\alpha+1}(\partial\Omega)$, so $\|v\|_{L^{\alpha+1}(\partial\Omega)} = \lim_{n\to\infty} \|u_n\|_{L^{\alpha+1}(\partial\Omega)} = 1$.

We have then that $v \in H \backslash \{0\}$, which implies

$$Q(v) \geq \inf_{w \in H \backslash \{0\}} Q(w) = S.$$

But

$$Q(v) \leq \liminf_{n \to \infty} Q(u_n) \leq \lim_{n \to \infty} Q(u_n) = S,$$

and so $Q(v) = S$. Finally, as $Q(v) = Q(|v|)$, the function $|v| \in H \backslash \{0\}$ is also a minimizer, so we may replace v by $|v|$ and assume $v \geq 0$. □

We now show that a minimizer of Q will solve a particular variational identity.

Lemma 11. *Suppose there is $v \in H \backslash \{0\}$ with $Q(v) = S$, $0 < S < \infty$. Then for every $w \in H$, v satisfies*

$$\int_{\Omega} \nabla v \cdot \nabla w \, dx - c \int_{\partial\Omega} vw \, d\sigma - S \int_{\partial\Omega} |v|^{\alpha-1} vw \, d\sigma = 0. \tag{2.11}$$

Proof. As v is a minimizer of Q, $Q(v) \leq Q(w)$ for every nonzero $w \in H$. This implies

$$\frac{d}{d\epsilon} Q(v + \epsilon w) \bigg|_{\epsilon=0} = 0 \tag{2.12}$$

for every $w \in H$. That is to say, $Q'(v) = 0$ in $H^1(\Omega)$. Calculating,

$$Q(v + \epsilon w) = \frac{\|\nabla v + \epsilon \nabla w\|_{L^2(\Omega)}^2 - c \|v + \epsilon w\|_{L^2(\partial\Omega)}^2}{\|v + \epsilon w\|_{L^{\alpha+1}(\partial\Omega)}^2} \doteq \frac{N(\epsilon)}{D(\epsilon)}.$$

Then differentiating gives

$$\frac{d}{d\epsilon} Q(v + \epsilon w) \bigg|_{\epsilon=0} = \frac{N'(0)D(0) - D'(0)N(0)}{D(0)^2}.$$

Now

$$\begin{aligned}
N(0) &= \|\nabla v\|_{L^2(\Omega)}^2 - c \|v\|_{L^2(\partial\Omega)}^2 = \|v\|_c \\
N'(0) &= 2 \Big((\nabla v, \nabla w)_{L^2(\Omega)} - c\,(v, w)_{L^2(\partial\Omega)} \Big) \\
D(0) &= \|v\|_{L^{\alpha+1}(\partial\Omega)}^2 \\
D'(0) &= \frac{2}{\alpha+1} \left(\int_{\partial\Omega} |v|^{\alpha+1} \, d\sigma \right)^{\frac{2}{\alpha+1}-1} (\alpha+1) \int_{\partial\Omega} |v|^{\alpha-1} vw \, d\sigma \\
&= 2 \|v\|_{L^{\alpha+1}(\partial\Omega)}^{1-\alpha} (|v|^{\alpha-1} v, w)_{L^2(\partial\Omega)}.
\end{aligned}$$

Putting this into the expression for the derivative gives

$$\begin{aligned}\frac{d}{d\epsilon}Q(v+\epsilon w)\bigg|_{\epsilon=0} &= \frac{2\left((\nabla v,\nabla w)_{L^2(\Omega)} - c\,(v,w)_{L^2(\partial\Omega)}\right)\|v\|^2_{L^{\alpha+1}(\partial\Omega)}}{\|v\|^4_{L^{\alpha+1}(\partial\Omega)}} \\ &\qquad - 2\,\|v\|^{1-\alpha}_{L^{\alpha+1}(\partial\Omega)}\,(|v|^{\alpha-1}\,v,w)_{L^2(\partial\Omega)}\frac{\|v\|_c}{\|v\|^4_{L^{\alpha+1}(\partial\Omega)}} \\ &= \frac{2}{\|v\|^2_{L^{\alpha+1}(\partial\Omega)}}A,\end{aligned}$$

where

$$\begin{aligned}A &= (\nabla v,\nabla w)_{L^2(\Omega)} - c\,(v,w)_{L^2(\partial\Omega)} - \frac{\|v\|_c}{\|v\|^{\alpha+1}_{L^{\alpha+1}(\partial\Omega)}}(|v|^{\alpha-1}\,v,w)_{L^2(\partial\Omega)} \\ &= (\nabla v,\nabla w)_{L^2(\Omega)} - c\,(v,w)_{L^2(\partial\Omega)} - Q(v)(|v|^{\alpha-1}\,v,w)_{L^2(\partial\Omega)}.\end{aligned}$$

Then (2.12) and $Q(v) = S$ imply

$$\int_\Omega \nabla v\cdot\nabla w\,dx - c\int_{\partial\Omega} vw\,d\sigma - S\int_{\partial\Omega}|v|^{\alpha-1}\,vw\,d\sigma = 0$$

for every $w \in H$. □

Equation (2.11) suggests that an appropriate scaling of v will give a positive solution u to (1.1), which leads us to the following lemma.

Lemma 12. *Let $c < \mu_1$, $1 < \alpha < \alpha^* - 1$, and $\lambda > 0$. Assume additionally that $S < \infty$. Then there exists an element $u \in H\backslash\{0\}$, $u \geq 0$ solving (1.1) in the weak sense.*

Proof. Since $c < \mu_1$, Lemma 7 verifies the hypothesis of Lemma 10, so we may find $v \in H\backslash\{0\}$, $v \geq 0$, with $Q(v) = S$. Set (noting that $\lambda > 0$ and $S > 0$)

$$u = \beta v,$$

for

$$\beta = \left(\frac{\lambda}{S}\right)^{-\frac{1}{\alpha-1}} > 0.$$

Then $u \in H$, $u \geq 0$, and since v is nonzero, u is nonzero as well. Let $w \in H$ be arbitrary. In light of Lemma 11, we have from (2.11)

$$\beta\left(\int_\Omega \nabla v\cdot\nabla w\,dx - c\int_{\partial\Omega} vw\,d\sigma - S\int_{\partial\Omega}|v|^{\alpha-1}\,vw\,d\sigma\right) = 0.$$

Then as $\beta^{1-\alpha} = \lambda/S$,

$$\begin{aligned} 0 &= \int_\Omega \nabla(\beta v)\cdot\nabla w\,dx - c\int_{\partial\Omega}(\beta v)w\,d\sigma - \beta S\int_{\partial\Omega}|v|^{\alpha-1}\,vw\,d\sigma \\ &= \int_\Omega \nabla u\cdot\nabla w\,dx - c\int_{\partial\Omega} uw\,d\sigma - \beta\beta^{-\alpha}S\int_{\partial\Omega}|u|^{\alpha-1}\,uw\,d\sigma \\ &= \int_\Omega \nabla u\cdot\nabla w\,dx - c\int_{\partial\Omega} uw\,d\sigma - \beta^{1-\alpha}S\int_{\partial\Omega}|u|^{\alpha-1}\,uw\,d\sigma \\ &= \int_\Omega \nabla u\cdot\nabla w\,dx - c\int_{\partial\Omega} uw\,d\sigma - \lambda\int_{\partial\Omega}|u|^{\alpha-1}\,uw\,d\sigma. \end{aligned}$$

As $u = 0$ on Γ_0, this is merely (2.3). Thus, u is a positive H solution to (1.1). $\square$

Finally, the assumption $S < \infty$ is not restrictive, as for every $w \in H$,

$$\|w\|_c \leq C\,\|w\|_H\,,$$

and so choosing an element $w \in H\backslash 0$ with $\|w\|_H < \infty$ and w not identically zero on Γ guarantees

$$Q(w) \leq \frac{C\,\|w\|_H}{\|w\|^2_{L^{\alpha+1}(\Gamma)}} < \infty,$$

and so $S \leq Q(w) < \infty$. This completes the proof of Theorem 4.

2.2 Critical Case

In both arguments for existence in the subcritical case, the compactness of the Sobolev embedding was necessary to construct a minimizer to the functional (see the proofs of Lemma 6 and Lemma 9). Based upon the work by Brezis and Nirenberg in [4], we expect that when $\alpha + 1 = \alpha^*$, we may still be able to construct solutions even though we have lost compactness of the embedding $H^1(\Omega) \hookrightarrow L^{\alpha^*}(\partial\Omega)$, as long as c is in a particular range. We will use the same argument as in Chapter 2.1.1. However, this will require knowing the specific form of the minimizing functions for the functional Q, and so we will need Ω to be the unit sphere. The result holds for more general domains and the arguments are not dissimilar. The argument in full generality can be found in [2]. We will also need c to be in a particular interval, which depends on the domain.

Assume Ω is the unit sphere in $\mathbb{R}^N$, $N \geq 3$, $\lambda > 0$, $-\frac{N-2}{2} < c < \mu_1$, and

$$\alpha + 1 = \alpha^* = \frac{2(N-1)}{N-2}.$$

We note that the bounds on c both depend on Ω. The lower bound $-\frac{N-2}{2}$ will in general depend on the curvature of Ω (at some point) and the dimension (see [2]); this value is constant in our case as the unit sphere has constant curvature. The upper bound μ_1 obviously depends on Ω as well. We wish to prove the following.

Theorem 5. *Let $\mu_1 \geq 0$ be the first Steklov eigenvalue and $\phi_1 \in H$ satisfy $\Delta\phi_1 = 0$ in Ω, $\phi_1 = 0$ on Γ_0, and $\frac{\partial \phi_1}{\partial \nu} = \mu_1 \phi_1$ on Γ. Let $-\frac{N-2}{2} < c < \mu_1$. Then problem (1.1) admits a (weak) solution $u \in H \backslash \{0\}$ with $u \geq 0$ in Ω.*

This theorem is a simplified version of Theorem 2.1 in [2]. It is simpler because we are only taking Ω to be the unit ball, whereas in [2], Adimurthi and Yadava show existence for any domain with smooth enough boundary and which satisfies a convexity condition. However, a number of calculations become more complicated and require estimates which are readily available for the case of the ball, but must be modified for a general domain (in particular, (2.18) below). The condition $c > -\frac{N-2}{2}$ is precisely condition (1) of Theorem 2.1 in [2] in the case that Ω is a ball. The general condition allows for c to be a function of $x \in \partial\Omega$, and the lower bound $-\frac{N-2}{2}$ is replaced by a function depending on the curvature of the boundary. They then require that the function analogous to c compares correctly to the curvature at a particular point on $\partial\Omega$ where the convexity condition is satisfied. The upper bound $c < \mu_1$ is analogous to the requirement that the expression (2.2) in [2] define an equivalent norm on H. We take care of this in Lemma 7 in Section 2.1.1

A straightforward energy argument shows that when $c \geq \mu_1$, no positive solution to (1.1) can exist. When $c < -\frac{N-2}{2}$, a Pohozaev type argument contradicts the existence of any (sufficiently well-behaved) solution. Both these arguments are found in Chapter 3. The exceptional case $c = -\frac{N-2}{2}$ is not included in Chapter 3, though it is expected that the results for $c < -\frac{N-2}{2}$ hold as well.

The difference between this case and the subcritical case is that we have lost compactness of the embedding $H^1(\Omega) \hookrightarrow L^{\alpha^*}(\partial\Omega)$ in (2.10). However, the compactness of embeddings $H^1(\Omega) \hookrightarrow L^2(\Omega)$ and $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ still hold. Recall

$$\|u\|_c^2 = \int_\Omega \|\nabla u\|^2 \, dx - c \int_{\partial\Omega} |u|^2 \, d\sigma = \|\nabla u\|^2_{L^2(\Omega)} - c \, \|u\|^2_{L^2(\partial\Omega)} ,$$

$$Q(u) = \frac{\|u\|_c^2}{\|u\|^2_{L^{\alpha^*}(\partial\Omega)}},$$

and

$$S = \inf_{w \in H \setminus \{0\}} Q(w).$$

The proofs of Lemmas 7 and 8 used only the Sobolev inequalities and not the compactness of the embedding, and so these results still hold in the critical case. In order to proceed along the lines of Lemma 9, we need a preliminary lemma to replace the compactness in $L^{\alpha^*}(\partial\Omega)$. In particular, we have the following lemma due to Brezis and Lieb (see [3]).

Lemma 13. *Let U be any domain and $0 < p < \infty$. Suppose $f_n \to f$ almost everywhere in U, and $\|f_n\|_{L^p(U)} \leq C < \infty$ for every n. Then*

$$\lim_{n\to\infty} \left(\|f_n\|^p_{L^p(U)} - \|f_n - f\|^p_{L^p(U)} \right) = \|f\|^p_{L^p(U)} .$$

Proof. For brevity we will write $\|\cdot\|_p$ for $\|\cdot\|_{L^p(U)}$. First, Fatou's Lemma gives

$$\|f\|_p^p \leq \liminf_{n\to\infty} \|f_n\|_p^p \leq C^p,$$

and so $f \in L^p(U)$ for any p. The proof is easiest when $0 < p \leq 1$, so we will show this case separately. When $0 < p \leq 1$, we have

$$|\alpha + \beta|^p \leq |\alpha|^p + |\beta|^p \tag{2.13}$$

for any real numbers α, β. This follows as the function $\chi(x) = x^p$ is concave for $x > 0$ and $0 < p \leq 1$. Then $\chi(0) = 0$ implies $\chi(x+y) \leq \chi(x) + \chi(y)$, and as χ is increasing this yields $\chi(|\alpha + \beta|) \leq \chi(|\alpha| + |\beta|) \leq \chi(|\alpha|) + \chi(|\beta|)$.

Using (2.13) first with $\alpha = f_n(x) - f(x), \beta = f_n(x)$ and then with $\alpha = f_n(x), \beta = -f(x)$ gives

$$|f_n|^p - |f_n - f|^p \leq |f|^p$$

and

$$|f_n - f|^p - |f_n|^p \le |f|^p \,,$$

respectively. Thus, the function

$$g_n = |f_n|^p - |f_n - f|^p$$

satisfies $g_n \to |f|^p$ a.e. and $|g_n| \le |f|^p$, and as $|f|^p \in L^1(U)$, dominated convergence implies that

$$\int_U g_n \, dx \to \int_U |f|^p \, dx,$$

or

$$\|f_n\|_p^p - \|f_n - f\|_p^p \to \|f\|_p^p \,.$$

The proof when $p > 1$ uses the same idea, but the estimate (2.13) changes. Fix $1 < p < \infty$ and $\epsilon > 0$. Then there is C_ϵ so that for any $\alpha, \beta \in \mathbb{R}$, we have

$$|\alpha + \beta|^p - |\alpha|^p \le \epsilon \, |\alpha|^p + C_\epsilon \, |\beta|^p \,. \tag{2.14}$$

This may be shown by writing $p = k + \delta$ for $k \in \mathbb{N}$, $0 \le \delta < 1$, applying (2.13) to $|\alpha + \beta|^\delta$, expanding $|\alpha + \beta|^k$, and then using Cauchy's inequality with ϵ. Note that when $0 < p \le 1$, estimate (2.14) is trivially implied by (2.13), and the argument for $p > 1$ will apply to $0 < p \le 1$ as well. Now, we apply (2.14) with $\alpha = f_n(x) - f(x)$, $\beta = f(x)$ to obtain

$$|f_n|^p - |f_n - f|^p \le \epsilon \, |f_n - f|^p + C_\epsilon \, |f|^p \,.$$

We also have, using $\alpha = f_n$ and $\beta = -f$,

$$\begin{aligned}
|f_n - f|^p &= \epsilon \, |f_n - f|^p + (1 - \epsilon) \, |f_n - f|^p \\
&\le \epsilon \, |f_n - f|^p + (1 - \epsilon) \left[\epsilon \, |f_n|^p + C_\epsilon \, |f|^p + |f_n|^p \right] \\
&= \epsilon \, |f_n - f|^p + (1 - \epsilon) C_\epsilon \, |f|^p + |f_n|^p - \epsilon^2 \, |f_n|^p \\
&\le \epsilon \, |f_n - f|^p + (1 - \epsilon) C_\epsilon \, |f|^p + |f_n|^p \,,
\end{aligned}$$

giving

$$|f_n - f|^p - |f_n|^p \le \epsilon \, |f_n - f|^p + C'_\epsilon \, |f|^p \,.$$

We have therefore established

$$||f_n|^p - |f_n - f|^p| \le \epsilon \, |f_n - f|^p + C_\epsilon \, |f|^p \,. \tag{2.15}$$

Set

$$Q_{\epsilon,n}(x) = |(|f_n(x)|^p - |f_n(x) - f(x)|^p) - |f(x)|^p| - \epsilon \, |f_n(x) - f(x)|^p$$

and

$$W_{\epsilon,n}(x) = \max\left(Q_{\epsilon,n}(x), 0\right).$$

Applying (2.15),

$$\begin{aligned} |(|f_n|^p - |f_n - f|^p) - |f|^p| &\le ||f_n|^p - |f_n - f|^p| + |f|^p \\ &\le \epsilon \, |f_n - f|^p + (C_\epsilon + 1) \, |f|^p \,, \end{aligned}$$

and so

$$Q_{n,\epsilon} = |(|f_n|^p - |f_n - f|^p) - |f|^p| - \epsilon \, |f_n - f|^p \le (C_\epsilon + 1) \, |f|^p \,.$$

Then

$$|W_{n,\epsilon}(x)| = W_{n,\epsilon}(x) \le Q_{n,\epsilon}(x) \le (C_\epsilon + 1) \, |f(x)|^p$$

and since $W_{n,\epsilon}(x) \to 0$ a.e. and $f \in L^p(U)$, dominated convergence implies that

$$\lim_{n\to\infty} \int_U W_{n,\epsilon}(x) \, dx = 0.$$

Finally, if

$$|(|f_n(x)|^p - |f_n(x) - f(x)|^p) - |f(x)|^p| \le \epsilon \, |f_n(x) - f(x)|^p \,,$$

then $W_{n,\epsilon}(x) = 0$. On the other hand, if

$$|(|f_n(x)|^p - |f_n(x) - f(x)|^p) - |f(x)|^p| > \epsilon \, |f_n(x) - f(x)|^p \,,$$

then

$$W_{n,\epsilon}(x) = |(|f_n(x)|^p - |f_n(x) - f(x)|^p) - |f(x)|^p| - \epsilon \, |f_n(x) - f(x)|^p \,,$$

and so in both cases

$$|(|f_n(x)|^p - |f_n(x) - f(x)|^p) - |f(x)|^p| \le W_{n,\epsilon}(x) + \epsilon \, |f_n(x) - f(x)|^p \,. \tag{2.16}$$

Integrating (2.16) gives

$$\begin{aligned}
I_n &= \int_U |(|f_n(x)|^p - |f_n(x) - f(x)|^p) - |f(x)|^p| \; dx \\
&\leq \int_U W_{n,\epsilon}(x)\, dx + \epsilon \int |f_n(x) - f(x)|^p \; dx \\
&\leq \int_U W_{n,\epsilon}(x)\, dx + \epsilon \, \|f_n - f\|_p \\
&\leq \int_U W_{n,\epsilon}(x)\, dx + \epsilon \left(\|f_n\|_p + \|f\|_p \right) \\
&\leq \int_U W_{n,\epsilon}(x)\, dx + \epsilon C,
\end{aligned}$$

where C is independent of ϵ and n. Then

$$\lim_{n\to\infty} I_n = \epsilon C,$$

which implies that $I_n \to 0$ as $n \to \infty$ by letting $\epsilon \to 0$. Thus, we have

$$\int_U |(|f_n(x)|^p - |f_n(x) - f(x)|^p) - |f(x)|^p| \; dx \to 0,$$

which establishes

$$\lim_{n\to\infty} \left(\|f_n\|_p^p - \|f_n - f\|_p^p \right) = \|f\|_p^p .$$

□

For the remainder, we will assume that Ω is the unit ball in $\mathbb{R}^N$, $N \geq 3$. Much of the argument will work for any $N \geq 3$, though one calculation we will only carry out in the $N = 3$ case. It is worth pointing out that all the results hold in more general settings, and the arguments will rely on transforming the domain locally to either the ball or the half-space (which are conformally equivalent), and then using the arguments presented here.

Let Ω be the unit ball in $\mathbb{R}^N$, $N \geq 3$, and define

$$\begin{aligned}
K_0 &= \inf_{w \in H^1(\Omega)\setminus\{0\}} \frac{\|\nabla w\|_{L^2(\Omega)}^2 + \frac{N-2}{2} \|w\|_{L^2(\partial\Omega)}^2}{\|w\|_{L^{\alpha^*}(\partial\Omega)}^2} \\
&= \inf_{w \in H^1(\mathbb{R}^n_+)} \frac{\|\nabla w\|_{L^2(\mathbb{R}^n_+)}^2}{\|w\|_{L^{\alpha^*}(\partial\mathbb{R}^n_+)}^2} \\
&= \frac{N-2}{2} \omega_N^{\frac{1}{N-1}}, \qquad (2.17)
\end{aligned}$$

where ω_N is the surface volume of the $N-1$ sphere in $\mathbb{R}^N$. The fact that these expressions are equivalent is due to Escobar in [8]. In this paper, Escobar also shows the infimum is achieved by constants in the case of the ball, and we can readily see that when w is any constant function, K_0 has the value given in (2.17).

We note that from (2.17), we immediately have

$$\|u\|^2_{L^{\alpha^*}(\partial\Omega)} \leq \frac{1}{K_0} \|\nabla u\|^2_{L^2(\Omega)} + C \|u\|^2_{L^2(\partial\Omega)} . \tag{2.18}$$

for any $u \in H^1(\Omega)$ and a constant $C = \frac{N-2}{2K_0}$. In the case of a general Ω, we have a generalized result due to Cherrier (see [5]). For any $\epsilon > 0$, there is a constant $C(\epsilon)$ so that

$$\|u\|^2_{L^{\alpha^*}(\partial\Omega)} \leq \left(\frac{1}{K_0} + \epsilon\right) \|\nabla u\|^2_{L^2(\Omega)} + C(\epsilon) \|u\|^2_{L^2(\Omega)} .$$

for every $u \in H^1(\Omega)$. This would be used in place of (2.18) for a general domain.

We are now ready to prove the main existence lemma for the case of the ball. This lemma follows very similarly Lemma 3.2 in [1]. There are some modifications to the argument, because in [1] they consider homogeneous Neumann data on the entire boundary and nonlinear data for Δu in the interior. The key to being able to prove Lemma 14 in the critical case is the result from Brezis and Lieb, Lemma 13 above. This allows us to get around the lack of compactness which was key in proving Lemma 10 in Section 2.1.1.

Lemma 14. *If $S < K_0$, then there is $v \in H\backslash\{0\}$, $v \geq 0$, with $Q(v) = S$.*

Proof. Let $\{u_k\} \subset H\backslash\{0\}$ be such that

$$Q(u_k) \to S,$$

$$\|u_k\|_{L^{\alpha^*}(\partial\Omega)} = 1.$$

The sequence $\{u_k\}$ is then bounded in $H^1(\Omega)$, and so by moving to subsequences, we may assume $u_k \rightharpoonup v$ weakly in $H^1(\Omega)$, $u_k \to v$ strongly in $L^2(\partial\Omega)$, and $u_k \to v$ almost everywhere in $\overline{\Omega}$ for some $v \in H^1(\Omega)$. We first show that $v \neq 0$.

Suppose that v is the zero function in $H^1(\Omega)$. Then $u_k \to 0$ in $L^2(\partial\Omega)$, so

$$\lim_{k\to\infty} \|\nabla u_k\|^2_{L^2(\Omega)} = \lim_{k\to\infty} \|u_k\|^2_c = S.$$

From 2.18, since $u_k \in H \subset H^1(\Omega)$, we have

$$1 \le \frac{1}{K_0} \|\nabla u_k\|^2_{L^2(\Omega)} + C \|u_k\|^2_{L^2(\partial\Omega)},$$

and so letting $k \to \infty$ gives

$$1 \le \frac{S}{K_0},$$

which is a contradiction to $S < K_0$, and so we must have $v \neq 0$.

Define $v_k = u_k - v$. Then $v_k \rightharpoonup 0$ weakly in $H^1(\Omega)$, $v_k \to 0$ strongly in $L^2(\partial\Omega)$, and $v_k \to 0$ almost everywhere in $\overline{\Omega}$.

First note that for any $w \in H$, we may write

$$\begin{aligned} \|w\|^2_c &= \|\nabla w\|^2_{L^2(\Omega)} - c\|w\|^2_{L^2(\partial\Omega)} \\ &= (\nabla w, \nabla w)_{L^2(\Omega)} - c(w,w)_{L^2(\partial\Omega)} \\ &= (w,w)_c. \end{aligned}$$

Then for any $w, w' \in H$,

$$\|w + w'\|^2_c = (w + w', w + w')_c = \|w\|^2_c + \|w'\|^2_c + 2(w, w')_c.$$

Therefore,

$$\|u_k\|^2_c = \|v_k + v\|^2_c = \|v\|^2_c + \|v_k\|^2_c + 2(v_k, v)_c.$$

Since $v_k \rightharpoonup 0$ weakly in $H^1(\Omega)$ and strongly in $L^2(\partial\Omega)$, we have $2(v_k, v)_c \to 0$. We may thus write

$$\|u_k\|^2_c = \|v\|^2_c + \|v_k\|^2_c + \delta^{(1)}_k,$$

where $\delta^{(1)}_k \to 0$. Since $\|u_k\|^2_c \to S$, we have

$$\begin{aligned} S &= \|u_k\|^2_c + (S - \|u_k\|^2_c) \\ &= \|v\|^2_c + \|\nabla v_k\|^2_{L^2(\Omega)} + (S - \|u_k\|^2_c) + \delta^{(1)}_k, \end{aligned}$$

and so

$$S = \|v\|^2_c + \|\nabla v_k\|^2_{L^2(\Omega)} + \delta^{(2)}_k \tag{2.19}$$

for $\delta^{(2)}_k \to 0$.

From the Sobolev inequalities, $\{u_k\}$ is bounded in $L^{\alpha+1}(\partial\Omega)$, and so by Lemma 13,

$$\|u_k\|_{L^{\alpha^*}(\partial\Omega)}^{\alpha^*} = \|v\|_{L^{\alpha^*}(\partial\Omega)}^{\alpha^*} + \|u_k - v\|_{L^{\alpha^*}(\partial\Omega)}^{\alpha^*} + \delta_k^{(3)}$$

for $\delta_k^{(3)} \to 0$. We thus have

$$\begin{aligned}\|u_k\|_{L^{\alpha^*}(\partial\Omega)}^{2} &= \left(\|u_k\|_{L^{\alpha^*}(\partial\Omega)}^{\alpha^*}\right)^{\frac{2}{\alpha^*}} \\ &\leq \left(\|v\|_{L^{\alpha^*}(\partial\Omega)}^{\alpha^*} + \|u_k - v\|_{L^{\alpha^*}(\partial\Omega)}^{\alpha^*} + \delta_k^{(3)}\right)^{\frac{2}{\alpha^*}} \\ &\leq \|v\|_{L^{\alpha^*}(\partial\Omega)}^{2} + \|u_k - v\|_{L^{\alpha^*}(\partial\Omega)}^{2} + \delta_k^{(4)}\end{aligned}$$

for $\delta_k^{(4)} \to 0$. Note that we're using $0 < \frac{2}{\alpha^*} < 1$. We then have, again using Lemma 2.18 and noting $\|v_k\|_{L^2(\partial\Omega)} \to 0$,

$$\begin{aligned}1 &= \|u_k\|_{L^{\alpha^*}(\partial\Omega)}^{2} \\ &\leq \|v\|_{L^{\alpha^*}(\partial\Omega)}^{2} + \|v_k\|_{L^{\alpha^*}(\partial\Omega)}^{2} + \delta_k^{(4)} \\ &\leq \|v\|_{L^{\alpha^*}(\partial\Omega)}^{2} + \frac{1}{K_0}\|\nabla v_k\|_{L^2(\partial\Omega)}^{2} + \delta_k^{(5)}\end{aligned}$$

for $\delta_k^{(5)} \to 0$. Multiplying by S gives

$$\begin{aligned}S &\leq S\,\|v\|_{L^{\alpha^*}(\partial\Omega)}^{2} + \frac{S}{K_0}\|\nabla v_k\|_{L^2(\partial\Omega)}^{2} + S\delta_k^{(5)} \\ &< S\,\|v\|_{L^{\alpha^*}(\partial\Omega)}^{2} + \|\nabla v_k\|_{L^2(\partial\Omega)}^{2} + \delta_k^{(6)}\end{aligned}$$

as $S < K_0$.

Using (2.19),

$$\|v\|_c^2 + \|\nabla v_k\|_{L^2(\Omega)}^2 + \delta_k^{(2)} \leq S\,\|v\|_{L^{\alpha^*}(\partial\Omega)}^{2} + \|\nabla v_k\|_{L^2(\partial\Omega)}^{2} + \delta_k^{(6)},$$

or

$$\|v\|_c^2 \leq S\,\|v\|_{L^{\alpha^*}(\partial\Omega)}^{2} + \delta_k^{(7)}.$$

Letting $k \to \infty$, and recalling $v \neq 0$, gives

$$Q(v) = \frac{\|v\|_c^2}{\|v\|_{L^{\alpha^*}(\partial\Omega)}^{2}} \leq S.$$

It remains to show that $v \in H$. We know $v \in H^1(\Omega)$, and as $u_k \to v$ almost everywhere on $\partial\Omega$, $u_k = 0$ on Γ_0, we have $v = 0$ on Γ_0 as well, so $v \in H$. Thus, v is one of the elements taken in the infimum defining S, so

$$Q(v) \geq S,$$

showing $Q(v) = S$. Lastly, we may replace v with $|v|$ and assume that $v \geq 0$. $\square$

The following lemma follows very similarly Lemma 3.3 of [2]. We will only prove the lemma in the case when $N = 3$, though it remains true when $N > 3$. The difference is the cases $N = 3$, $N = 4$, and $N \geq 5$ need to be treated separately due to to the integrands that arise when estimating S.

Lemma 15. *Let K_0 be given by (2.17). If $-\frac{N-2}{2} < c < \mu_1$, then $S < K_0$.*

Proof. As our problem is invariant under shifts and rotation, we may assume that $\Omega = \{(x', x_N) \in \mathbb{R}^N : x'^2 + (x_N - 1)^2 < 1\}$ and that $0 \in \Gamma$. The idea is to evaluate $Q(u)$ for a specific $u \in H$ that is in some sense close to the minimizing functions.

Choose a fixed $R > 0$ small enough so that $B(R) \cap \Gamma_0 = \emptyset$, where $B(R)$ is the ball of radius R centered at the origin. Let ϕ be a compactly supported $C^\infty(\mathbb{R})$ function with $\phi(\rho) = 1$ for $\rho < R/2$, $\phi(\rho) = 0$ for $\rho > R$. By the choice of R, $\phi(|x|)$ is identically zero for points $x \in \Gamma_0$.

We will proceed only in the case when $N = 3$. The argument for $N > 3$ is the same, except the integrals become more complicated and the estimates differ slightly. For the argument in full generality, see [2].

For any $\epsilon > 0$, set

$$u_\epsilon(x, y, z) = \frac{\phi(\rho)}{(x^2 + y^2 + (z + \epsilon)^2)^{1/2}},$$

where $\rho = \sqrt{x^2 + y^2 + z^2}$. For each fixed $\epsilon > 0$, $u_\epsilon \in H$. We wish to compute $Q(u_\epsilon)$ and show that for sufficiently small $\epsilon > 0$,

$$Q(u_\epsilon) < K_0 = \frac{1}{2}\omega_2^{1/2} = \sqrt{\pi}.$$

The first observation to make is that any integral of u_ϵ over a set away from the origin will be $O(1)$ as $\epsilon \to 0$. This is clear as u_ϵ and ∇u_ϵ only have a singularity at the origin as $\epsilon \to 0$. Indeed, suppose $x^2 + y^2 + z^2 \geq a$ for some $a > 0$. Then as $x^2 + y^2 + (z+\epsilon)^2$ is exactly the distance squared from (x, y, z) to $(0, 0, -\epsilon)$, we have

$$(x^2 + y^2 + (z+\epsilon)^2)^k \geq (x^2 + y^2 + z^2)^k \geq a^k$$

for any $k > 0$. It follows that if U is any set so that $\mathbb{R}^3_+ \backslash U$ contains a compact set V with $0 \in V$, we have

$$\int_U \frac{1}{(x^2 + y^2 + (z+\epsilon)^2)^k}\, dx \leq a^{-k} vol(U) = O(1).$$

The same estimate applies to the corresponding surface integrals. This observation will be used frequently in the computations to follow.

In order to estimate $Q(u_\epsilon)$, we need to compute (noting when $N = 3$, $\alpha^* = 4$),

$$\begin{aligned} \|\nabla u_\epsilon\|^2_{L^2(\Omega)} &= \int_\Omega \left\| \nabla \left(\frac{\phi(\rho)}{(x^2 + y^2 + (z+\epsilon)^2)^{\frac{1}{2}}} \right) \right\|^2 dV, \\ \|u_\epsilon\|^2_{L^2(\partial\Omega)} &= \int_{\partial\Omega} \frac{\phi(\rho)^2}{x^2 + y^2 + (z+\epsilon)^2}\, d\sigma, \\ \|u_\epsilon\|^4_{L^4(\partial\Omega)} &= \int_{\partial\Omega} \frac{\phi(\rho)^4}{(x^2 + y^2 + (z+\epsilon)^2)^2}\, d\sigma, \end{aligned}$$

where $\rho = \sqrt{x^2 + y^2 + z^2}$. We compute

$$\nabla \phi(\rho) = \frac{\phi'(\rho)}{\rho} \langle x, y, z \rangle,$$

giving

$$\begin{aligned} \|\nabla \phi(\rho)\|^2 &= \phi'(\rho)^2, \\ \nabla \phi(\rho) \cdot \langle x, y, z+\epsilon \rangle &= \phi'(\rho) \frac{x^2 + y^2 + z(z+\epsilon)}{\rho}. \end{aligned}$$

Both these terms are a multiple of $\phi'(\rho)$, which is supported only when $\langle x, y, z \rangle$ is outside $B(R)$. By the previous discussion, the integrals of these two terms over Ω will be $O(1)$.

Now

$$\begin{aligned} \nabla u_\epsilon(x, y, z) &= \frac{\nabla \phi(\rho)}{(x^2 + y^2 + (z+\epsilon)^2)^{\frac{1}{2}}} - \frac{\phi(\rho) \langle x, y, z+\epsilon \rangle}{(x^2 + y^2 + (z+\epsilon)^2)^{\frac{3}{2}}}, \\ \|\nabla u_\epsilon(x, y, z)\|^2 &= \frac{\|\nabla \phi(\rho)\|^2}{x^2 + y^2 + (z+\epsilon)^2} + \frac{2\phi(\rho) \nabla \phi(\rho) \cdot \langle x, y, z+\epsilon \rangle}{(x^2 + y^2 + (z+\epsilon)^2)^2} \\ &\quad + \frac{\phi(\rho)^2}{(x^2 + y^2 + (z+\epsilon)^2)^2}. \end{aligned}$$

The integrals over Ω of the first two terms are $O(1)$. Furthermore, $\phi(\rho)^2 - 1$ is supported compactly away from the origin, and so

$$\begin{aligned}
\|\nabla u_\epsilon\|^2_{L^2(\Omega)} &= \int_\Omega \frac{\phi(\rho)^2}{(x^2+y^2+(z+\epsilon)^2)^2}\,dV + O(1) \\
&= \int_{\Omega \bigcap B(R)^+} \frac{\phi(\rho)^2}{(x^2+y^2+(z+\epsilon)^2)^2}\,dV + O(1) \\
&= \int_{\Omega \bigcap B(R)^+} \frac{1+\phi(\rho)^2-1}{(x^2+y^2+(z+\epsilon)^2)^2}\,dV + O(1) \\
&= \int_{\Omega \bigcap B(R)^+} \frac{1}{(x^2+y^2+(z+\epsilon)^2)^2}\,dV + O(1) \\
&= \int_{B(R)^+} \frac{1}{(x^2+y^2+(z+\epsilon)^2)^2}\,dV \\
&\quad - \int_\Sigma \frac{1}{x^2+y^2+(z+\epsilon)^2)^2}\,dV + O(1) \\
&= I_1 - I_2 + O(1),
\end{aligned}$$

where

$$B(R)^+ = \{(x,y,z) : x^2+y^2+z^2 < R^2, z > 0\},$$

$$\Sigma = B(R)^+ \backslash \Omega = \left\{(x,y,z) \in B(R)^+ : z \le 1 - \sqrt{1-x^2-y^2}\right\}.$$

We compute I_1 by replacing $B(R)^+$ by the cylinder $\{(x,y,z) : x^2+y^2 < R, 0 < z < R\}$. This adds on an integral over the difference between the cylinder and $B(R)^+$, but as this region is away from the origin, the integral is $O(1)$. Setting $D(R) = \{(x,y) :$

$x^2 + y^2 < R^2\}$, we have

$$
\begin{aligned}
I_1 &= \int_{B(R)^+} \frac{1}{(x^2+y^2+(z+\epsilon)^2)^2}\, dV \\
&= \int_0^R \int_{D(R)} \frac{1}{(r^2+(z+\epsilon)^2)^2}\, dA dz + O(1) \\
&= 2\pi \int_0^R \int_0^R \frac{r}{(r^2+(z+\epsilon)^2)^2}\, dr dz + O(1) \\
&= \frac{2\pi}{\epsilon} \int_0^{R/\epsilon} \int_0^{R/\epsilon} \frac{s}{(s^2+(\xi+1)^2)^2}\, ds d\xi + O(1) \\
&= \frac{2\pi}{\epsilon} \int_0^{\infty} \int_0^{\infty} \frac{s}{(s^2+(\xi+1)^2)^2}\, ds d\xi + O(1) \\
&= \frac{2\pi}{\epsilon} \int_0^{\infty} \int_{(1+\xi)^2}^{\infty} \frac{1}{2\eta^2}\, d\eta d\xi + O(1) \\
&= \frac{\pi}{\epsilon} \int_0^{\infty} \frac{1}{(1+\xi)^2}\, d\xi + O(1) \\
&= \frac{\pi}{\epsilon} + O(1).
\end{aligned}
$$

To compute I_2, we set $p(r) = 1 - \sqrt{1-r^2}$ and again use cylindrical coordinates.

$$
\begin{aligned}
I_2 &= \int_{\Sigma} \frac{1}{(x^2+y^2+(z+\epsilon)^2)^2}\, dV \\
&= \int_{D(R)} \int_0^{p(r)} \frac{dz}{(r^2+(z+\epsilon)^2)^2}\, dA \\
&= 2\pi \int_0^R \int_0^{p(r)} \frac{r}{(r^2+(z+\epsilon)^2)^2}\, dz dr.
\end{aligned}
$$

We switch the order of integration to obtain

$$\begin{aligned}
I_2 &= 2\pi \int_0^{p(R)} \int_{(2z-z^2)^{1/2}}^{R} \frac{r}{(r^2+(z+\epsilon)^2)^2}\, drdz \\
&= -\frac{2\pi}{2} \int_0^{p(R)} \left(\frac{1}{R^2+(z+\epsilon)^2} - \frac{1}{2z-z^2+(z+\epsilon)^2} \right) dz \\
&= O(1) + \pi \int_0^{p(R)} \frac{1}{2z-z^2+(z+\epsilon)^2}\, dz \\
&= \pi \int_0^{p(R)} \frac{1}{2(1+\epsilon)z+\epsilon^2}\, dz + O(1) \\
&= \frac{\pi}{2(1+\epsilon)} \left[\log(2(1+\epsilon)p(R)+\epsilon^2) - \log(\epsilon^2)\right] + O(1) \\
&= \frac{\pi}{2(1+\epsilon)} \log\left(\frac{1}{\epsilon^2}\right) + O(1) \\
&= \frac{\pi}{1+\epsilon} \log\left(\frac{1}{\epsilon}\right) + O(1) \\
&= (\pi + O(\epsilon)) \log\left(\frac{1}{\epsilon}\right) + O(1) \\
&= \pi \log\left(\frac{1}{\epsilon}\right) + O\left(\epsilon \log\left(\frac{1}{\epsilon}\right)\right) + O(1).
\end{aligned}$$

We therefore have

$$\|\nabla u_\epsilon\|^2_{L^2(\Omega)} = \frac{\pi}{\epsilon} - \pi \log\left(\frac{1}{\epsilon}\right) + O\left(\epsilon \log\left(\frac{1}{\epsilon}\right)\right) + O(1). \tag{2.20}$$

The next step is to compute $\|u_\epsilon\|^2_{L^2(\partial\Omega)}$ and $\|u_\epsilon\|^2_{L^4(\partial\Omega)}$. We parameterize the surface $\partial\Omega \cap \{z < 1\}$ with polar coordinates:

$$\langle x, y, z\rangle = \left\langle r\cos\theta, r\sin\theta, 1 - \sqrt{1-r^2}\right\rangle.$$

A parameterization of $\partial\Omega \cap \{z \geq 1\}$ is not needed as ϕ is only supported near the origin. The normal vector for this parameterization is

$$n = -r\left\langle \frac{r}{\sqrt{1-r^2}}\cos\theta, \frac{r}{\sqrt{1-r^2}}\cos\theta, -1\right\rangle$$

with magnitude

$$|n| = \frac{r}{\sqrt{1-r^2}}.$$

Utilizing the same ideas preceding the calculation of $\|\nabla u_\epsilon\|^2_{L^2(\Omega)}$, we compute

$$\begin{aligned}
\|u_\epsilon\|^2_{L^2(\partial\Omega)} &= \int_{\partial\Omega} \frac{\phi(\rho)^2}{x^2+y^2+(z+\epsilon)^2}\, d\sigma \\
&= \int_{B(R)^+} \frac{1}{x^2+y^2+(z+\epsilon)^2}\, d\sigma + \int_{B(R)^+} \frac{\phi(\rho)^2-1}{x^2+y^2+(z+\epsilon)^2}\, d\sigma \\
&= \int_{B(R)^+} \frac{1}{x^2+y^2+(z+\epsilon)^2}\, d\sigma + O(1) \\
&= \int_0^{2\pi}\int_0^R \frac{|n|}{r^2+(z+\epsilon)^2}\, dr d\theta + O(1) \\
&= 2\pi \int_0^R \frac{r/\sqrt{1-r^2}}{r^2+(1-\sqrt{1-r^2}+\epsilon)^2}\, dr + O(1) \\
&= 2\pi I_3 + O(1).
\end{aligned}$$

We compute

$$\begin{aligned}
I_3 &= \int_0^R \frac{r/\sqrt{1-r^2}}{r^2+(1-\sqrt{1-r^2}+\epsilon)^2}\, dr \\
&= \int_0^R \frac{r/\sqrt{1-r^2}}{r^2+(1+\epsilon)^2-2(1+\epsilon)\sqrt{1-r^2}+1-r^2}\, dr \\
&= \int_0^R \frac{r/\sqrt{1-r^2}}{1+(1+\epsilon)^2-2(1+\epsilon)\sqrt{1-r^2}}\, dr \\
&= \frac{1}{2(1+\epsilon)}\int_{\epsilon^2}^{\eta_R} \frac{1}{\eta}\, d\eta \\
&= \frac{1}{2(1+\epsilon)}\left[\log(\eta_R)-\log(\epsilon^2)\right] \\
&= \frac{1}{2(1+\epsilon)} 2\log\left(\frac{1}{\epsilon}\right) + \frac{1}{2(1+\epsilon)}\log(\eta_R) \\
&= \log\left(\frac{1}{\epsilon}\right) + O\left(\epsilon\log\left(\frac{1}{\epsilon}\right)\right) + \frac{1}{2(1+\epsilon)}\log(\eta_R),
\end{aligned}$$

where we used the substitution

$$\eta = 1+(1+\epsilon)^2-2(1+\epsilon)\sqrt{1-r^2}$$

and set

$$\eta_R = 1+(1+\epsilon)^2-2(1+\epsilon)\sqrt{1-R^2} = O(1),$$

noting that R is chosen sufficiently small so that $\eta_R > 0$. Thus,

$$I_3 = \log\left(\frac{1}{\epsilon}\right) + O\left(\epsilon\log\left(\frac{1}{\epsilon}\right)\right) + O(1),$$

and so

$$\|u_\epsilon\|^2_{L^2(\partial\Omega)} = 2\pi \log\left(\frac{1}{\epsilon}\right) + O\left(\epsilon \log\left(\frac{1}{\epsilon}\right)\right) + O(1) \tag{2.21}$$

A very similar calculation applies to $\|u_\epsilon\|^2_{L^4(\partial\Omega)}$. First,

$$\begin{aligned}
\|u_\epsilon\|^4_{L^4(\partial\Omega)} &= \int_{\partial\Omega} \frac{\phi(\rho)^4}{(x^2+y^2+(z+\epsilon)^2)^2}\, d\sigma \\
&= \int_{B(R)^+} \frac{1}{(x^2+y^2+(z+\epsilon)^2)^2}\, d\sigma + \int_{B(R)^+} \frac{\phi(\rho)^4 - 1}{(x^2+y^2+(z+\epsilon)^2)^2}\, d\sigma \\
&= \int_{B(R)^+} \frac{1}{(x^2+y^2+(z+\epsilon)^2)^2}\, d\sigma + O(1) \\
&= \int_0^{2\pi}\int_0^R \frac{|n|}{(r^2+(z+\epsilon)^2)^2}\, dr d\theta + O(1) \\
&= 2\pi \int_0^R \frac{r/\sqrt{1-r^2}}{(r^2+(1-\sqrt{1-r^2}+\epsilon)^2)^2}\, dr + O(1) \\
&= \frac{2\pi}{2(1+\epsilon)} \int_{\epsilon^2}^{\eta_R} \frac{1}{\eta^2}\, d\eta + O(1) \\
&= \frac{2\pi}{2(1+\epsilon)} \left(\frac{1}{\epsilon^2} - \frac{1}{\eta_R}\right) + O(1) \\
&= \frac{\pi}{1+\epsilon}\frac{1}{\epsilon^2} + O(1) \\
&= \frac{\pi}{\epsilon^2} - \frac{\pi}{\epsilon} + O(1),
\end{aligned}$$

utilizing

$$\frac{1}{1+\epsilon} = 1 - \epsilon + O(\epsilon^2).$$

We thus have

$$\|u_\epsilon\|^4_{L^4(\partial\Omega)} = \frac{\pi}{\epsilon^2} - \frac{\pi}{\epsilon} + O(1). \tag{2.22}$$

Finally, we put (2.20), (2.21), and (2.22) together to compute

$$\begin{aligned}
Q(u_\epsilon) &= \frac{\|\nabla u_\epsilon\|^2_{L^2(\Omega)} - c\,\|u_\epsilon\|^2_{L^2(\partial\Omega)}}{\|u_\epsilon\|^2_{L^4(\partial\Omega)}} \\
&= \frac{\frac{\pi}{\epsilon} - \pi\log\left(\frac{1}{\epsilon}\right) + -c2\pi\log\left(\frac{1}{\epsilon}\right) + O\left(\epsilon\log\left(\frac{1}{\epsilon}\right)\right) + O(1)}{\left(\frac{\pi}{\epsilon^2} - \frac{\pi}{\epsilon} + O(1)\right)^{1/2}} \\
&= \sqrt{\pi}\left(\frac{1 - \epsilon\log\left(\frac{1}{\epsilon}\right) - 2c\epsilon\log\left(\frac{1}{\epsilon}\right) + O\left(\epsilon^2\log\left(\frac{1}{\epsilon}\right)\right) + O(\epsilon)}{(1-\epsilon+O(\epsilon^2))^{1/2}}\right) \\
&= \sqrt{\pi}\left(\frac{1 - \epsilon\log\left(\frac{1}{\epsilon}\right)(1+2c) + O\left(\epsilon^2\log\left(\frac{1}{\epsilon}\right)\right) + O(\epsilon)}{(1-\epsilon+O(\epsilon^2))^{1/2}}\right).
\end{aligned}$$

Expanding the denominator gives

$$Q(u_\epsilon) = \sqrt{\pi}\left[1 - \epsilon \log\left(\frac{1}{\epsilon}\right)(1+2c)\right] + O\left(\epsilon^3 \log\left(\frac{1}{\epsilon}\right)\right)$$
$$+ O\left(\epsilon^2 \log\left(\frac{1}{\epsilon}\right)\right) + O(\epsilon).$$

Then if $1 + 2c > 0$, we may take $\epsilon > 0$ sufficiently small so that

$$1 - \epsilon \log\left(\frac{1}{\epsilon}\right)(1+2c) < 1$$

and

$$Q(u_\epsilon) < \sqrt{\pi}.$$

Then we have

$$S \leq Q(u_\epsilon) < \sqrt{\pi} = K_0.$$

□

The proof of Lemma 11 from Section 2.1.1 follows identically in the critical case, and so as long as there is $v \in H \backslash \{0\}$ with $Q(v) = S$, $0 < S < \infty$, then v satisfies

$$\int_\Omega \nabla v \cdot \nabla w \, dx - c \int_{\partial\Omega} vw \, d\sigma - S \int_{\partial\Omega} |v|^{\alpha-1} vw \, d\sigma = 0$$

for every $w \in H$. Furthermore, the same scaling argument in Lemma 12 holds, and the function

$$u = \left(\frac{\lambda}{S}\right)^{-\frac{1}{\alpha-1}} v$$

solves (1.1) in the weak sense. Combining Lemmas 14, 15, and the above discussion proves Theorem 5.

2.3 Behavior of Solutions

In this section we study the behavior of the solutions to (1.1) as $\lambda > 0$ varies. In particular, we will show that for every fixed $c \in \mathbb{R}$ (for which (1.1) yields a solution), there is a sense in which the solution blows up in H norm as $\lambda \to 0^+$. We prove the following theorem.

Theorem 6. *Fix $c \in \mathbb{R}$ and $\alpha > 1$. Suppose $v \in H\backslash\{0\}$ is a solution to*

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ v = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial v}{\partial \nu} = cv + |v|^{\alpha-1} v & \text{on } \Gamma. \end{cases} \tag{2.23}$$

Then for any $\lambda > 0$, there exists a solution $u_\lambda \in H$ of (1.1) with

$$u_\lambda(x) = \lambda^{\frac{-1}{\alpha-1}} v(x).$$

In particular, there exists a solution to (1.1) with arbitrarily large H norm. Moreover, if $u_\lambda \in H\backslash\{0\}$ is a solution to (1.1) for some $\lambda > 0$, then

$$v = \lambda^{\frac{1}{\alpha-1}} u_\lambda$$

is a solution to (2.23).

Proof. Suppose $v \in H$ is any nontrivial solution to (2.23). For $\lambda > 0$, define

$$u_\lambda = \lambda^{\frac{-1}{\alpha-1}} v.$$

Then clearly $\Delta u_\lambda = 0$ in Ω and $u_\lambda = 0$ on Γ_0. On Γ,

$$\begin{aligned} \frac{\partial u_\lambda}{\partial \nu} &= \lambda^{\frac{-1}{\alpha-1}} \frac{\partial v}{\partial \nu} \\ &= \lambda^{\frac{-1}{\alpha-1}} \left(cv + |v|^{\alpha-1} v \right) \\ &= cu_\lambda + \lambda^{\frac{-1}{\alpha-1} + \frac{\alpha}{\alpha-1}} |u_\lambda|^{\alpha-1} u_\lambda \\ &= cu_\lambda + \lambda |u_\lambda|^{\alpha-1} u_\lambda, \end{aligned}$$

and so u_λ solves (1.1). Then as long as (2.23) has one solution, (1.1) has a solution for any $\lambda > 0$ with

$$\|u_\lambda\|_H = \lambda^{\frac{-1}{\alpha-1}} \|v\|_H ,$$

and so (noting $\alpha > 1$), $\|u_\lambda\|_H$ can be taken to be arbitrarily large for sufficiently small $\lambda > 0$.

The completely analogous argument shows that $v = \lambda^{-1/(\alpha-1)} u_\lambda$ solves (2.23) whenever u_λ solves (1.1). $\square$

It is important to note that when $\alpha < \alpha^* - 1$ (or when $\alpha = \alpha^* - 1$ and $-\frac{N-2}{2} < c < \mu_1$), (2.23) is guaranteed to have a solution $v \in H \backslash \{0\}$. However, this solution need not be unique. In fact, we constructed infinitely many solutions in the subcritical case, and even if we restrict ourselves to only positive solutions, we do not have a uniqueness theorem. As such, Theorem 6 does not say explicitly that any solution to (1.1) blows up as $\lambda \to 0^+$, because for each λ we can compare u_λ to a solution v of (2.23), but as this solution is not unique, we don't know that we are comparing the same solution for different λ. It is likely possible to show that we may follow a solution upon a particular "branch" as $\lambda \to 0^+$, and each such solution does indeed blow up. Figure 4.4 in Chapter 4.2 shows that solutions increase as $\lambda \to 0^+$.

Chapter 3

Nonexistence

We have seen that when $c < \mu_1$, there exists a positive solution to (1.1). It also holds that when $c \geq \mu_1$, there cannot exist a positive solution. This is a standard energy argument, relying on the fact that $\phi_1 \geq 0$.

Theorem 7. *Let $\phi_1 \geq 0$ be the first Steklov eigenfunction with eigenvalue μ_1. Then if $c \geq \mu_1$, there can exist no positive solution to (1.1).*

Proof. Fix $c \geq \mu_1$, $\lambda > 0$, $\alpha > 1$, and suppose $u \in H\backslash\{0\}$ solves (1.1) with $u \geq 0$ in Ω. Then using $v = \phi_1$ in (2.3), we have

$$\int_\Omega \nabla u \cdot \nabla \phi_1 \, dx - c \int_\Gamma u \phi_1 \, d\sigma - \lambda \int_\Gamma u^\alpha \phi_1 \, d\sigma = 0.$$

On the other hand, by definition of ϕ_1, we have

$$\int_\Omega \nabla \phi_1 \cdot \nabla u \, dx = \mu_1 \int_\Gamma \phi_1 u \, d\sigma,$$

and so

$$\mu_1 \int_\Gamma \phi_1 u \, d\sigma - c \int_\Gamma u \phi_1 \, d\sigma - \lambda \int_\Gamma u^\alpha \phi_1 \, d\sigma = 0.$$

Rearranging gives

$$(\mu_1 - c) \int_\Gamma u \phi_1 \, d\sigma = \lambda \int_\Gamma u^\alpha \phi_1 \, d\sigma,$$

which implies

$$\int_\Gamma u^\alpha \phi_1 \, d\sigma \leq 0.$$

This integral must be strictly positive, however, as neither ϕ_1 nor u can be identically zero on Γ. Thus, we cannot have the existence of such a u. $\square$

Note that this argument did not depend on the criticality of the exponent. In the subcritical case, we have existence for every c, and existence of a positive solution only when $c < \mu_1$. For the critical case, we have only proven existence of a positive solution (when $-\frac{N-2}{2} < c < \mu_1$). When $c \geq \mu_1$, Theorem 7 in Chapter 3.1 shows there cannot exist a positive solution. We may still expect, however, that other (sign-changing) solutions may exist, though we do not show this. In fact, there is numerical evidence for the existence of sign-changing solutions when $c \geq \mu_1$ and the exponent is critical. See Figure 4.6b in Chapter 4.3.

Corollary 2 in Chapter 3.1 shows that when $c < -\frac{N-2}{2}$ and the exponent is critical or supercritical, no nontrivial solution to (1.1) that is continuous up to the boundary and satisfies a growth constraint (condition (3.43) in Section 3.1) may exist.

3.1 Nonexistence in the Supercritical Case

In this section, we show that if (1.1) has a solution, it will have to satisfy a Pohozaev-type identity, similar to those established in [18] and [7]. This will imply constraints on the exponent α and the parameter c.

We wish to apply the following identity. For any smooth vector field h and any H^2 function u, there holds

$$\operatorname{div}\left((h\cdot\nabla u)\nabla u - \frac{1}{2}\|\nabla u\|^2 h\right) = (Dh\nabla u)\cdot\nabla u - \frac{1}{2}\|\nabla u\|^2 \operatorname{div} h + (h\cdot\nabla u)\Delta u. \quad (3.1)$$

This is a straightforward computation, and integrating (3.1) for $u \in H^2$ solving (1.1) and applying the divergence theorem is equivalent to multiplying (1.1) by $h \cdot \nabla u$ and integrating by parts.

The problem with applying (3.1) is that our solution u is only in $H \subset H^1(\Omega)$. Because we have a mixed boundary value problem, standard regularity arguments do not apply and we can not guarantee $u \in H^2(\Omega)$. In fact, the arguments in this section will imply that a nontrivial solution to (1.1) for $c > -\frac{N-2}{2}$ must fail to be in $H^2(\Omega)$. As such, we cannot directly apply (3.1). Nevertheless, our solution will be H^2 in a subdomain $\Omega\backslash U$ for any open set U containing the interface between Γ_0 and Γ. Our idea is then to apply (3.1) in a smaller domain with the boundary interface removed, and then examine the contributions as the removed portion becomes smaller.

We first consider the argument for full $H^2(\Omega)$ solutions. This computation will indicate why we consider a mixed boundary problem instead of a standard Neumann problem (see the discussion following Proposition 2). We consider the specific case when Ω is the unit ball in $\mathbb{R}^N$, $\{x \in \mathbb{R}^N, |x| < 1\}$. We make the following choice for the field h. This field is very special to the geometry of the ball (h will be a contained in the tangent plane of Ω at each boundary point), and if our domain were different, we would have to attempt to construct a different h with analogous properties. In fact, the field h was constructed by considering the conformal transformation of the unit ball to the half-space $\mathbb{R}^N_+$.

Define

$$h(x) = x_N x - \frac{1}{2}\left(1 + |x|^2\right) e_N. \tag{3.2}$$

Note that on $\partial\Omega$, $|x| = 1$ and $\nu = x$, so for $x \in \partial\Omega$,

$$h(x) \cdot \nu(x) = x_N x \cdot x - \frac{1}{2}\left(1 + |x|^2\right) e_N \cdot x = x_N - x_N = 0.$$

Thus, h is tangent to Ω. A direct computation shows

$$\operatorname{div} h = (N-1)x_N + 2x_N - x_N = N x_n, \tag{3.3}$$

$$Dh = x_N I + M,$$

where M is an anti-symmetric matrix. Then

$$(Dh\nabla u) \cdot \nabla u = x_N \|\nabla u\|^2. \tag{3.4}$$

We begin by applying (3.1) to a potential solution $u \in H^2(\Omega)$ to (1.1).

Proposition 2. *Let Ω be the unit ball in $\mathbb{R}^N$, $\alpha > 1$, $\lambda > 0$, and $c \in \mathbb{R}$. If $u \in H^2(\Omega) \cap C(\overline{\Omega})$ is a solution to (1.1), then*

$$\left(\frac{N-2}{2} - \frac{N-1}{\alpha+1}\right) \lambda \int_\Gamma x_N |u|^{\alpha+1}\, d\sigma = \left(\frac{N-2}{4} + \frac{c}{2}\right) \int_\Gamma x_N u^2\, d\sigma. \tag{3.5}$$

Proof. Since $u \in H^2(\Omega)$, we have $\Delta u = 0$ (in the sense of distributions). Then using (3.3) and (3.4), we have

$$\begin{aligned}\int_\Omega \left[(Dh\nabla u)\cdot\nabla u - \frac{1}{2}\|\nabla u\|^2 \operatorname{div} h + (h\cdot\nabla u)\Delta u\right] dx &= \int_\Omega \left(1 - \frac{N}{2}\right) x_N \|\nabla u\|^2\, dx \\ &= \frac{2-N}{2}\int_\Omega x_N \|\nabla u\|^2\, dx.\end{aligned}$$

We may integrate this expression by parts to obtain

$$\begin{aligned}
\frac{2-N}{2}\int_{\Omega} x_N \left\|\nabla u\right\|^2\, dx &= \frac{2-N}{2}\left[\int_{\partial\Omega} x_N u\frac{\partial u}{\partial \nu}\, d\sigma - \int_{\Omega} u \operatorname{div}\left(x_N \nabla u\right) dx\right] \\
&= \frac{2-N}{2}\left[\int_{\partial\Omega} x_N u\frac{\partial u}{\partial \nu}\, d\sigma - \int_{\Omega} u\frac{\partial u}{\partial x_N}\, dx\right] \\
&= \frac{2-N}{2}\left[\int_{\partial\Omega} x_N u\frac{\partial u}{\partial \nu}\, d\sigma - \frac{1}{2}\int_{\Omega} \frac{\partial u^2}{\partial x_N}\, dx\right] \\
&= \frac{2-N}{2}\left[\int_{\partial\Omega} x_N u\frac{\partial u}{\partial \nu}\, d\sigma - \frac{1}{2}\int_{\partial\Omega} u^2 \nu_N\, d\sigma\right] \\
&= \frac{2-N}{2}\left[\int_{\partial\Omega} x_N u\frac{\partial u}{\partial \nu}\, d\sigma - \frac{1}{2}\int_{\partial\Omega} x_N u^2\, d\sigma\right].
\end{aligned}$$

Recall $g(u) = cu + \lambda u \left|u\right|^{\alpha-1}$, and set

$$G(u) = \int_0^u g(t)\, dt.$$

Since $u = 0$ on Γ_0, the boundary integrals are only over Γ, so we have

$$\frac{2-N}{2}\int_{\Omega} x_N \left\|\nabla u\right\|^2\, dx = \frac{2-N}{2}\int_{\Gamma} x_N u g(u)\, d\sigma + \frac{N-2}{4}\int_{\Gamma} x_N u^2\, d\sigma.$$

Integrating the right-hand side of (3.1) gives

$$\int_{\Omega} \operatorname{div}\left((h\cdot\nabla u)\nabla u - \frac{1}{2}\left\|\nabla u\right\|^2 h\right) dx = \int_{\partial\Omega}\left((h\cdot\nabla u)\nabla u\cdot\nu - \frac{1}{2}\left\|\nabla u\right\|^2 h\cdot\nu\right) d\sigma.$$

Now, $h\cdot\nu = 0$ on $\partial\Omega$. Furthermore, as $u = 0$ on Γ_0, ∇u must be in the normal direction on Γ_0, and so $h\cdot\nabla u = 0$ on Γ_0. Thus, the boundary integral is only over Γ, and we have

$$\int_{\Omega} \operatorname{div}\left((h\cdot\nabla u)\nabla u - \frac{1}{2}\left\|\nabla u\right\|^2 h\right) dx = \int_{\Gamma} h\cdot\nabla G\, d\sigma,$$

where we have used $(h\cdot\nabla u)g(u) = h\cdot\nabla G(u)$.

We express $h\cdot\nabla G$ in polar coordinates and integrate by parts. Define $\rho = |x|$, and

$$x_N = \rho\cos\phi_1,\;\; x_{N-1} = \rho\sin\phi_1\cos\phi_2,\;\; x_{N-2} = \rho\sin\phi_1\sin\phi_2\cos\phi_3,\; \ldots,$$

where $0 \le \phi_1 \le \pi$, $0 \le \phi_k < 2\pi$, $k = 2\ldots N-1$. The chain rule gives

$$x\cdot\nabla G = \sum_{i=1}^{N} x_i\frac{\partial G}{\partial x_i} = \sum_{i=1}^{N}\rho\frac{x_i}{\rho}\frac{\partial G}{\partial x_i} = \rho\sum_{i=1}^{N}\frac{\partial x_i}{\partial \rho}\frac{\partial G}{\partial x_i} = \rho\frac{\partial G}{\partial \rho}$$

and, since $\phi_2 \dots \phi_{N-1}$ are independent of x_N,

$$\begin{aligned}\frac{\partial G}{\partial x_N} &= \frac{\partial G}{\partial \rho}\frac{\partial \rho}{\partial x_N} + \frac{\partial G}{\partial \phi_1}\frac{\partial \phi_1}{\partial x_N} \\ &= \frac{x_N}{\rho}\frac{\partial G}{\partial \rho} - \frac{\sin\phi_1}{\rho}\frac{\partial G}{\partial \phi_1}\end{aligned}$$

Then on $\partial\Omega$, $\rho = |x| = 1$, and

$$\begin{aligned}h \cdot \nabla G &= x_N x \cdot \nabla G - \frac{1}{2}\left(1 + |x|^2\right) e_N \cdot \nabla G \\ &= x_N \frac{\partial G}{\partial \rho} - \left(x_N \frac{\partial G}{\partial \rho} - \sin\phi_1 \frac{\partial G}{\partial \phi_1}\right) \\ &= \sin\phi_1 \frac{\partial G}{\partial \phi_1}.\end{aligned}$$

Assume for simplicity that Γ is parameterized by $\rho = 1$, $0 \le \phi_1 < \phi^*$, $\phi_k \in [0, 2\pi)$ for $k \ge 2$, where $\phi^* = \phi^*(\phi_2, \dots, \phi_{N-1})$. Letting θ denote $\phi_2\phi_3 \dots \phi_{N-1}$ and $J(\theta) = \sin^{N-3}\phi_2 \dots \sin\phi_{N-2}$, we have

$$\begin{aligned}\int_\Gamma h \cdot \nabla G \, d\sigma &= \int_{[0,2\pi)^{N-1}} \int_0^{\phi^*} \sin\phi_1 \frac{\partial G}{\partial \phi_1} \sin^{N-2}\phi_1 \, |J(\theta)| \, d\phi_1 d\theta \\ &= \int_{[0,2\pi)^{N-1}} \int_0^{\phi^*} \sin^{N-1}\phi_1 \frac{\partial G}{\partial \phi_1} \, d\phi_1 \, |J(\theta)| \, d\theta \\ &= -(N-1) \int_{[0,2\pi)^{N-1}} \int_0^{\phi^*} \cos\phi_1 \sin^{N-2}\phi_1 G \, d\phi_1 \, |J(\theta)| \, d\theta \\ &= -(N-1) \int_\Gamma x_N G(u) \, d\sigma.\end{aligned}$$

Here we have used that $u = 0$ when $\phi_1 = \phi^*$, and $G(0) = 0$. Combining our calculations, (3.1) becomes

$$-(N-1)\int_\Gamma x_N G(u) \, d\sigma + \frac{N-2}{2}\int_\Gamma x_N u g(u) \, d\sigma = \frac{N-2}{4}\int_\Gamma x_N u^2 \, d\sigma.$$

Now $g(u) = cu + \lambda u \, |u|^{\alpha-1}$ and

$$G(u) = \frac{c}{2}u^2 + \frac{\lambda}{\alpha+1}|u|^{\alpha+1}.$$

This then gives

$$\begin{aligned}\left(\frac{N-2}{2} - \frac{N-1}{\alpha+1}\right)\lambda \int_\Gamma x_N |u|^{\alpha+1} \, d\sigma + \left(\frac{N-2}{2} - \frac{N-1}{2}\right) c \int_\Gamma x_N u^2 \, d\sigma \\ = \frac{N-2}{4}\int_\Gamma x_N u^2 \, d\sigma,\end{aligned}$$

or

$$\left(\frac{N-2}{2} - \frac{N-1}{\alpha+1}\right)\lambda \int_\Gamma x_N\,|u|^{\alpha+1}\,d\sigma = \left(\frac{N-2}{4} + \frac{c}{2}\right)\int_\Gamma x_N u^2\,d\sigma,$$

establishing (3.5). □

First notice that if $\Gamma_0 = \emptyset$, the boundary integrals are over all of $\partial\Omega$ and the term x_N changes sign. We cannot conclude that the integrals are positive. In fact, if we were to take $\alpha = 1$, $c = 0$ and $N = 3$, then we would have

$$\left(\frac{1}{4} + \frac{\lambda}{2}\right)\int_{\partial\Omega} x_3 u^2\,d\sigma = 0,$$

implying that $\lambda < 0$ or $\int_{\partial\Omega} x_3 u^2\,d\sigma = 0$. As the eigenvalues of the linear problem are nonnegative, we must have $\int_{\partial\Omega} x_3 u^2\,d\sigma = 0$. This does give a constraint on u (for instance, this would be satisfied if u were symmetric), but it also implies that we cannot expect the integrals appearing in (3.5) to be positive. We can not, therefore, expect to get much meaning from (3.5) if $\Gamma_0 = \emptyset$.

In order to conclude something meaningful from (3.5), we consider the case when Γ is contained in the upper hemisphere, so $x_N > 0$. Then the integrals are positive for nontrivial solutions u. In the subcritical or critical case, when $\alpha \leq \alpha^* - 1$, we have

$$\alpha + 1 \leq \alpha^* = \frac{2(N-1)}{N-2},$$

giving

$$\frac{N-2}{2} - \frac{N-1}{\alpha+1} \leq 0.$$

Then as $\lambda > 0$, the left hand side of (3.5) is nonpositive. But this is a contradiction as long as $c > -\frac{N-2}{2}$. We conclude that the solutions guaranteed by theorems 2 and 5 cannot be in H^2 as long as $c > -\frac{N-2}{2}$. We actually expect that the solutions will not be H^2 for any value of c, though this is not implied by this argument. The lack of regularity of a solution under mixed boundary conditions is examined by Grisvard in [11] and Mghazli in [16], and a similar result appears in Theorem 8 below.

Then the mixed boundary condition seems to give rise to a lack of regularity in the solution, so the Pohozaev identity does not apply as is. If we consider a pure Neumann problem, then the Pohozaev identity does not provide useful information. Some effort was made to choose a different vector field h in order to get some positive function $p(x_N)$ appearing in the integrals instead of x_N. The field h from (3.2) arises from considering the derivative of the conformal map between the unit ball and $\mathbb{R}^N_+$, and we considered examining the second derivative instead, though no progress was made along these lines.

Another option would be to consider Ω to be a half-ball instead of a ball, for then x_N would always be positive. Similar problems arise when taking Neumann data on the entire boundary, or when taking Dirichlet data on the flat portion and Neumann data on the curved portion. When the flat boundary has Neumann data and the curved boundary has Dirichlet data, a Pohozaev identity can be derived, as was done by Adimurthi and Yadava in [1].

Our approach is then to find a way to apply (3.1) when we have a mixed boundary and our solution is only in H^1. The idea is to remove a region surrounding the interface between Γ_0 and Γ. In the domain with this region removed, there is no jump in boundary condition, and therefore no singularity and we may apply (3.1). Going through the argument in Proposition 2 would then pick up integrals over the boundary of the region removed. This calculation will include the results from Proposition 2 as a special case where the extra contribution is zero. We then expect to obtain an identity like (3.5) with an additional term arises from the singularity. The hope is that this term will have a particular sign, which will imply nonexistence when α is supercritical and provide no contradiction when α is subcritical.

In order to proceed, we will transform our domain to the half-space $\mathbb{R}^N_+$ through a conformal transformation. Then on $\mathbb{R}^N_+$ we will remove a region surrounding the interface between Neumann and Dirichlet boundary conditions and apply (3.1) to this domain. Most of the argument will be done in a general case when Ω is conformally equivalent to $\mathbb{R}^N_+$, though at some point we will need to again assume that Ω is the unit ball in order to calculate with a specific conformal factor. The argument could be done entirely in the ball, but moving to the half-space has the advantage of a simpler geometry, and also indicates how we would proceed for a more general domain.

Let Ω be conformally equivalent to $\mathbb{R}^N_+$ and $\Phi : \Omega \to \mathbb{R}^N_+$ be a conformal mapping.

As Φ is conformal, we know

$$D\Phi = \beta U,$$

$$D\left(\Phi^{-1}\right) = \beta^{-1} U^T$$

for a unitary matrix U and conformal factor β. In the case when Ω is the unit ball, we have

$$\Phi(x) = \frac{x + e_N}{|x + e_N|^2} - \frac{1}{2} e_N, \tag{3.6}$$

with inverse

$$\Phi^{-1}(y) = \frac{y + \frac{1}{2} e_N}{\left|y + \frac{1}{2} e_N\right|^2} - e_N \tag{3.7}$$

and conformal factor

$$\beta = \left|y + \frac{1}{2} e_n\right|^2 = \frac{1}{|x + e_n|^2} \tag{3.8}$$

for $x \in \Omega$ and $y = \Phi(x) \in \mathbb{R}^N_+$.

The idea is to take a solution to (1.1) on a domain Ω conformal to $\mathbb{R}^N_+$ and construct a solution to an analogous boundary value problem on $\mathbb{R}^N_+$. We have the following proposition.

Proposition 3. *Let Ω be conformally equivalent to $\mathbb{R}^N_+$. A function $u \in H$ is a solution to (1.1) if and only if the function $v \in H^1_{loc}(\mathbb{R}^N_+)$ given by $v(y) = u(\Phi^{-1}(y))$ is a solution to*

$$\begin{cases} div\left(\beta^{2-N} \nabla v\right) = 0 & in\ \mathbb{R}^N_+ \\ v = 0 & on\ \Phi(\Gamma_0) \\ \dfrac{\partial v}{\partial \nu} = \dfrac{1}{\beta} g(v) & on\ \Phi(\Gamma). \end{cases} \tag{3.9}$$

Proof. We simply change variables from $x \in \Omega$ to $y \in \mathbb{R}^N_+$ and carry out the change of variables in the integrals. Suppose first $u \in H$ solves (1.1). Then for any $\phi \in H$, we have

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx - \int_\Gamma \phi g(u) \, d\sigma = 0.$$

Set $y = \Phi(x)$, $v(y) = u(\Phi^{-1}(y))$. Let $\psi \in H^1_{loc}(\mathbb{R}^N_+)$ be any function with $\psi = 0$ on $\Phi(\Gamma_0)$. Define $\phi(x) = \psi(\Phi(x))$, so that $\phi \in H$. Now, $\nabla_x = D\Phi \nabla_y = \beta U \nabla_y$ for some U with $UU^T = U^T U = I$. The Jacobian for the change of variables $y = \Phi(x)$ is $\det D\Phi = \pm\beta^N$, and so $|\det D\Phi| = \beta^N$, giving

$$\begin{aligned}\int_\Omega \nabla u \cdot \nabla \phi \, dx &= \int_\Omega (\nabla \phi)^T \, \nabla u \, dx \\ &= \int_{\mathbb{R}^N_+} (\beta U \nabla \psi)^T \, \beta U \nabla v \, \beta^{-N} dy \\ &= \int_{\mathbb{R}^N_+} \beta^{2-N} \, (\nabla \psi)^T \, U^T U \nabla v \, dy \\ &= \int_{\mathbb{R}^N_+} \beta^{2-N} \nabla v \cdot \nabla \psi \, dy,\end{aligned}$$

and

$$\begin{aligned}\int_\Gamma \phi g(u) \, d\sigma(x) &= \int_{\Phi(\Gamma)} \psi g(v) \beta^{-(N-1)} \, d\sigma(y) \\ &= \int_{\Phi(\Gamma)} \beta^{-1} \psi \beta^{2-N} g(v) \, d\sigma(y).\end{aligned}$$

Then we have

$$\int_{\mathbb{R}^N_+} \beta^{2-N} \nabla v \cdot \nabla \psi \, dy - \int_{\Phi(\Gamma)} \beta^{-1} \psi \beta^{2-N} g(v) \, d\sigma(y) = 0$$

for any $\psi \in H^1_{loc}(\mathbb{R}^N_+)$ with $\psi = 0$ on $\Phi(\Gamma_0)$. This is exactly the weak formulation of problem (3.9) on the space $\{v \in H^1_{loc}(\mathbb{R}^N_+) : v = 0 \text{ on } \Phi(\Gamma_0)\}$. The argument for the converse is identical. $\square$

For ease of calculation, we transform problem (3.9) to a problem with no first order derivative on the function v.

Proposition 4. *A function $v \in H^1_{loc}(\mathbb{R}^N_+)$ with $v = 0$ on $\Phi(\Gamma_0)$ is a solution to (3.9) if and only if the function $w = \beta^{\frac{2-N}{2}} v$ is a solution to*

$$\begin{cases} \Delta w = \dfrac{2-N}{4} \left(\dfrac{2\beta\Delta\beta - N \, |\nabla \beta|^2}{\beta^2} \right) w & \text{in } \mathbb{R}^N_+ \\ w = 0 & \text{on } \Phi(\Gamma_0) \\ \dfrac{\partial w}{\partial \nu} = \dfrac{2-N}{2\beta} \dfrac{\partial \beta}{\partial \nu} w + \beta^{-\frac{N}{2}} g\left(\beta^{\frac{N-2}{2}} w\right) & \text{on } \Phi(\Gamma). \end{cases} \tag{3.10}$$

Proof. Suppose v solves (3.9) in the weak sense, and let $\psi \in H^1_{loc}(\mathbb{R}^N_+)$, $\psi = 0$ on $\Phi(\Gamma_0)$ be arbitrary. For notational ease, set $f = \log\left(\beta^{2-N}\right)$ and define $w = e^{f/2} v$. Since v is a weak solution to (3.9), using the test function $e^{-f/2}\psi$ and noting $\beta^{2-N} = e^f$,

$$\int_{\mathbb{R}^N_+} e^f \nabla v \cdot \nabla (e^{-f/2}\psi)\, dy = \int_{\Phi(\Gamma)} \frac{e^f}{\beta} g(v) e^{-f/2}\psi\, d\sigma.$$

Then as $v = e^{-f/2}w$,

$$\begin{aligned}\int_{\mathbb{R}^N_+} e^f \left[e^{-f}\nabla w \cdot \nabla \psi\right] dy + \int_{\mathbb{R}^N_+} e^f \left|\nabla e^{-f/2}\right|^2 w\psi\, dy \\ + \int_{\mathbb{R}^N_+} e^f \left[e^{-f/2} w \nabla\psi \cdot \nabla e^{-f/2} + e^{-f/2}\psi \nabla w \cdot \nabla e^{-f/2}\right] dy \\ = \int_{\Phi(\Gamma)} \frac{e^{f/2}}{\beta} g(e^{-f/2}w)\psi\, d\sigma,\end{aligned}$$

which simplifies to

$$\begin{aligned}\int_{\mathbb{R}^N_+} \nabla w \cdot \nabla\psi\, dy + \frac{1}{4}\int_{\mathbb{R}^N_+} |\nabla f|^2\ dy - \frac{1}{2}\int_{\mathbb{R}^N_+} \nabla(w\psi)\cdot\nabla f\, dy \\ = \int_{\Phi(\Gamma)} \frac{e^{f/2}}{\beta} g(e^{-f/2}w)\psi\, d\sigma.\end{aligned}$$

Applying the Divergence Theorem,

$$\begin{aligned}\int_{\mathbb{R}^N_+} \nabla w \cdot \nabla\psi\, dy + \frac{1}{4}\int_{\mathbb{R}^N_+} |\nabla f|^2 w\psi\, dy - \frac{1}{2}\int_{\Phi(\Gamma)} w\psi \frac{\partial f}{\partial \nu}\, d\sigma \\ + \frac{1}{2}\int_{\mathbb{R}^N_+} w\psi \Delta f\, dy = \int_{\Phi(\Gamma)} \frac{e^{f/2}}{\beta} g(e^{-f/2}w)\psi\, d\sigma.\end{aligned}$$

Rearranging, we see that w satisfies

$$\begin{aligned}\int_{\mathbb{R}^N_+} \nabla w \cdot \nabla\psi\, dy + \int_{\mathbb{R}^N_+} \left(\frac{\Delta f}{2} + \frac{|\nabla f|^2}{4}\right) w\psi\, dy \\ = \int_{\Phi(\Gamma)} \left(\frac{1}{2}\frac{\partial f}{\partial \nu} w + \frac{e^{f/2}}{\beta} g(e^{-f/2}w)\right)\psi\, d\sigma,\end{aligned}$$

or, expressing in terms of β,

$$\begin{aligned}\int_{\mathbb{R}^N_+} \nabla w \cdot \nabla\psi\, dy + \int_{\mathbb{R}^N_+} \frac{2-N}{4}\left(\frac{2\beta\Delta\beta - N|\nabla\beta|^2}{\beta^2}\right) w\psi\, dy \\ = \int_{\Phi(\Gamma)} \left(\frac{2-N}{2\beta}\frac{\partial \beta}{\partial \nu} w + \beta^{-\frac{N}{2}} g\left(\beta^{\frac{2-N}{2}} w\right)\right)\psi\, d\sigma.\end{aligned}$$

Therefore, w is a weak solution to (3.10). For the converse, suppose $w \in H^1_{loc}(\mathbb{R}^N_+)$, $w = 0$ on $\Phi(\Gamma_0)$, and is a weak solution to (3.10). Let ψ be any function in $H^1_{loc}(\mathbb{R}^N_+)$

with $\psi = 0$ on $\Phi(\Gamma_0)$, and define $v = e^{-f/2}w$ for the same f as before. Then by definition of w being a weak solution to (3.10) with the test function $e^{f/2}\psi$, we have

$$\int_{\mathbb{R}^N_+} \nabla w \cdot \nabla(e^{f/2}\psi)\, dy + \int_{\mathbb{R}^N_+} \left(\frac{\Delta f}{2} + \frac{|\nabla f|^2}{4} \right) w e^{f/2} \psi \, dy$$
$$= \int_{\Phi(\Gamma)} \left[\frac{1}{2}\frac{\partial f}{\partial \nu} w + \frac{e^{f/2}}{\beta} g\left(e^{-f/2}w\right) \right] e^{f/2}\psi \, d\sigma.$$

We now write $w = e^{f/2}v$ and expand to obtain

$$\int_{\mathbb{R}^N_+} e^f \nabla v \cdot \nabla \psi \, dy + \int_{\mathbb{R}^N_+} \frac{e^f}{2} \nabla f \cdot \nabla(v\psi)\, dy + \int_{\mathbb{R}^N_+} \frac{e^f}{4} |\nabla f|^2 v\psi \, dy$$
$$+ \int_{\mathbb{R}^N_+} e^f \left(\frac{\Delta f}{2} + \frac{|\nabla f|^2}{4} \right) v\psi \, dy = \int_{\Phi(\Gamma)} e^f \left[\frac{1}{2}\frac{\partial f}{\partial \nu} v + \frac{1}{\beta} g(v) \right] \psi \, d\sigma.$$

Integrating by parts gives

$$\int_{\mathbb{R}^N_+} \frac{e^f}{2} \nabla f \cdot \nabla(v\psi)\, dy = \int_{\Phi(\Gamma)} \frac{e^f}{2}\frac{\partial f}{\partial \nu} v\psi \, d\sigma - \int_{\mathbb{R}^N_+} \frac{e^f}{2} |\nabla f|^2 v\psi \, dy$$
$$- \int_{\mathbb{R}^N_+} \frac{e^f \Delta f}{2} v\psi \, dy,$$

and entering this into the previous equation, we have

$$\int_{\mathbb{R}^N_+} e^f \nabla v \cdot \nabla \psi \, dy = \int_{\Phi(\Gamma)} e^f \frac{1}{\beta} g(v)\psi \, d\sigma.$$

As $e^f = \beta^{2-N}$ and ψ was arbitrary, we see v is a weak solution to (3.9). □

In the case when Ω is the unit sphere and β is given by (3.8), problem (3.10) simplifies to $\Delta w = 0$ in $\mathbb{R}^N_+$. Indeed, we see that when β is given by (3.8),

$$\nabla \beta = 2y + e_N \tag{3.11}$$

and

$$|\nabla \beta|^2 = 4\left| y + \frac{1}{2} e_n \right|^2 = 4\beta,$$
$$\Delta \beta = \operatorname{div} \nabla \beta = 2N.$$

Then

$$\frac{2\beta\Delta\beta - N|\nabla\beta|^2}{\beta^2} = \frac{2\beta(2N) - N(4\beta)}{\beta^2} = 0.$$

Moreover, on the boundary $\partial\mathbb{R}^N_+$, we have $y_N = 0$ and $\nu = -e_N$, giving

$$\nabla\beta \cdot \nu = (2y + e_N) \cdot (-e_N) = -1.$$

We have thus shown the following proposition.

Proposition 5. *Suppose Ω is the unit ball in $\mathbb{R}^N$. A function $v \in H^1_{loc}(\mathbb{R}^N_+)$ with $v = 0$ on $\Phi(\Gamma_0)$ is a solution to (3.9) if and only if the function $w = \beta^{\frac{2-N}{2}} v$ is a solution to*

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^N_+ \\ \quad w = 0 & \text{on } \Phi(\Gamma_0) \\ \dfrac{\partial w}{\partial \nu} = \dfrac{N-2}{2}\beta^{-1} w + \beta^{-\frac{N}{2}} g\left(\beta^{\frac{N-2}{2}} w\right) & \text{on } \Phi(\Gamma). \end{cases}$$

For $g(u) = cu + \lambda\, |u|^{\alpha-1}\, u$, this becomes

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^N_+ \\ \quad w = 0 & \text{on } \Phi(\Gamma_0) \\ \dfrac{\partial w}{\partial \nu} = \beta^{-1}\left(c + \dfrac{N-2}{2}\right) w + \lambda \beta^{\frac{N-2}{2}\alpha - \frac{N}{2}}\, |w|^{\alpha-1}\, w & \text{on } \Phi(\Gamma). \end{cases} \tag{3.12}$$

To (3.12) we will make the Pohozaev argument (after removing a portion of the domain around the boundary interface). Some of the calculation will still apply to (3.10), though to obtain the full Pohozaev identity we will need to use the specific form of β arising from the case when Ω is a ball.

Before we establish the Pohozaev identity, we need a couple of lemmas dealing with the behavior of a (weak) solution to (3.12) near the boundary interface. As in the discussion following Proposition 2, the solutions which exist in the subcritical case cannot be in H^2. The goal of these lemmas is to show that a solution still has some amount of regularity (beyond H^1) and can be expanded in local coordinates.

Lemma 16. *For $x \in \mathbb{R}^N$ define cylindrical coordinates (x', r, θ) by*

$$x' = (x_1, \ldots, x_{N-2})$$

$$r\cos\theta = x_{N-1}$$

$$r\sin\theta = x_N.$$

For any $R > 0$, let $U_R \subset \mathbb{R}^N$ be the domain $U_R = \{(x', r, \theta) : |x'|^2 + r^2 < R^2, \, 0 < \theta < \pi\}$. Set $Q_0 = \{x \in \partial U_R : \theta = 0\}, Q = \{x \in \partial U_R : \theta = \pi\}, P = \{x \in \partial U_R : \left|x'^2\right| + r^2 = R^2\}$. Let $f \in L^2(P), \mu \in C^\infty(L)$ with $\mu \leq M < 0$. Let A be a symmetric positive definite matrix with bounded eigenvalues. Assume further that A has the form

$$A = \begin{pmatrix} A' & 0 \\ 0 & I_2 \end{pmatrix},$$

where I_2 is the 2×2 identity, so $div\,(A\nabla) = div'(A'\nabla') + \Delta_2$, where ∂' denotes derivatives with respect to the $N-2$ dimensional x' and ∂_2 denotes derivatives with respect to x_{N-1} and x_N. Let $u \in H^1(U_R)$ solve

$$\begin{cases} div\,(A\nabla u) = 0 & \text{in } U_R \\ u = 0 & \text{on } Q_0 \\ A\nabla u \cdot \nu = \mu(x) u & \text{on } Q \\ u = f & \text{on } P. \end{cases} \tag{3.13}$$

Then there is a smooth function C of x' and some $R_0 > 0$ so that the expansions

$$\begin{cases} u(x', r, \theta) = C(x') r^{1/2} \sin(\theta/2) + O(r^{3/4-\epsilon}) \\ u_r(x', r, \theta) = \frac{1}{2} C(x') r^{-1/2} \sin(\theta/2) + O(r^{-1/4-\epsilon}) \\ u_\theta(x', r, \theta) = \frac{1}{2} C(x') r^{1/2} \cos(\theta/2) + O(r^{3/4-\epsilon}) \\ u_{x_i}(x', r, \theta) = C_{x_i}(x') r^{1/2} \sin(\theta/2) + O(r^{3/4-\epsilon}), \quad 1 \leq i \leq N-2 \end{cases} \tag{3.14}$$

hold in U_{R_0} for any $\epsilon > 0$.

Proof. The idea here is that the singular structure will be two dimensional, and the $N-2$ dimensional variable x' acts as a parameter for the problem. The reason for this is that the jump between boundary conditions happens at $r = 0$, and there is no break as we move in the x' direction. We are thus motivated by the two dimensional argument.

In the two dimensional case, our domain is the upper half-plane. The argument is to first contract our domain 90 degrees to the first quadrant, so that we have Dirichlet boundary conditions on the horizontal axis and Neumann boundary conditions on

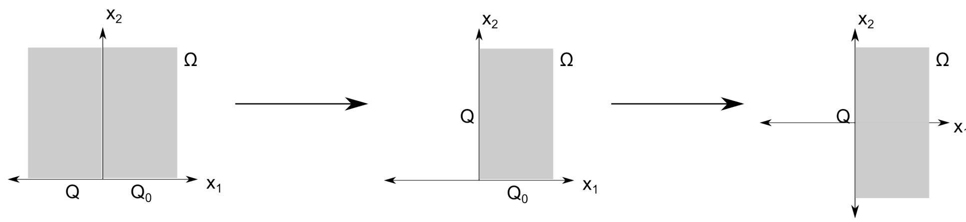


Figure 3.1: Rotation and reflection of $\Omega = \mathbb{R}^N_+$

the vertical axis. We then extend the solution oddly about the horizontal axis, to get a problem defined on the entire right half-plane with Neumann boundary conditions. Since this problem has no mixed boundary conditions, we will be able to apply elliptic regularity results to get a certain amount of smoothness, and therefore a Taylor expansion ([17]). Following back to our original domain will then provide (3.14). Figure 3.1 shows first the domain contraction given by $(r, \theta) \mapsto \left(r^{\frac{1}{2}}, \frac{\theta}{2}\right)$ and then the reflection about the x_1 axis.

In order to make this argument work, we first need to establish that our problem is regular in the x' direction. We follow a usual argument of taking difference quotients and estimating the H^1 norm of the difference quotients. Fix any $1 \leq l \leq N-2$. Let $D^h\zeta = D^h_l\zeta = \frac{\zeta(x+he_l)-\zeta(x)}{h}$, and $\zeta^h(x) = \zeta(x+he_l)$.

Let η be smooth cutoff function where $\eta = 1$ on $V = B_{R/2} \cap \mathbb{R}^N_+$, $\eta = 0$ on $\mathbb{R}^N \backslash V$, $0 \leq \eta \leq 1$. We now take a weak solution u to (3.13) and take as a test function $v = -D^{-h}\eta^2 D^h u$. Set $U = U_R$. We have

$$\sum_{i,j=1}^{n} \int_U a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx = \int_{\partial U} A\nabla u \cdot \nu v \, d\sigma.$$

Write this expression as

$$I_1 = I_2. \tag{3.15}$$

For I_1 we use the fact that we have a constant $\Lambda > 0$ with $\Lambda\, |\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}\xi_j\xi_i$. Then we can show

$$I_1 \geq \frac{\Lambda}{2} \int_U \eta^2 \left\| \nabla D^h u \right\| \, dx - C \int_U \left\| \nabla u \right\|^2 \, dx. \tag{3.16}$$

Establishing (3.16) is a fairly standard argument. We first have

$$\begin{aligned}
I_1 &= \sum_{i,j=1}^{n} \int_U a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx \\
&= \sum_{i,j=1}^{n} \int_U a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial}{\partial x_j} \left(-D^{-h} \eta^2 D^h u\right) dx \\
&= \sum_{i,j=1}^{n} \int_U D^h \left(a_{ij} \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_j} \left(\eta^2 D^h u\right) dx \\
&= \sum_{i,j=1}^{n} \int_U a_{ij}^h \frac{\partial D^h u}{\partial x_i} \frac{\partial \eta^2 D^h u}{\partial x_j}\, dx \\
&\quad + \sum_{i,j=1}^{n} \int_U D^h a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \eta^2 D^h u}{\partial x_j}\, dx \\
&= \sum_{i,j=1}^{n} \int_U \eta^2 a_{ij}^h \frac{\partial D^h u}{\partial x_i} \frac{\partial D^h u}{\partial x_j}\, dx \\
&\quad + \sum_{i,j=1}^{n} \int_U \left(D^h a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \eta^2 D^h u}{\partial x_j} + a_{ij}^h \frac{\partial D^h u}{\partial x_i} 2\eta \frac{\partial \eta}{\partial x_j} D^h u \right) dx \\
&= I_1^{(1)} + I_1^{(2)}.
\end{aligned}$$

Uniform ellipticity implies

$$I_1^{(1)} \geq \Lambda \int_U \eta^2 \left\| \nabla D^h u \right\|^2 dx. \tag{3.17}$$

For $I_1^{(2)}$, we have

$$I_1^{(2)} = \sum_{i,j=1}^{n} \int_U \left(\eta^2 D^h a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial D^h u}{\partial x_j} + 2\eta \frac{\partial \eta}{\partial x_j} \left(D^h a_{ij} \frac{\partial u}{\partial x_i} D^h u + a_{ij}^h \frac{\partial D^h u}{\partial x_i} D^h u \right) \right) dx.$$

Then

$$\left| I_1^{(2)} \right| \leq C \int_U \left(\eta \left|\nabla u\right| \left|\nabla D^h u\right| + \eta \left|\nabla u\right| \left|D^h u\right| + \eta \left|\nabla D^h u\right| \left|D^h u\right| \right) dx$$

where C depends on A, η, U, V. We now use Cauchy's inequality with ϵ to isolate the $\left|\nabla D^h u\right|$ term. Then

$$\begin{aligned}
\left| I_1^{(2)} \right| &\leq \epsilon \int_U \eta^2 \left|\nabla D^h u\right|^2 dx + C(\epsilon) \int_U \eta^2 \left(\left|\nabla u\right|^2 + \left|D^h u\right|^2 + \left|\nabla u\right| \left|D^h u\right| \right) dx \\
&\leq \epsilon \int_U \eta^2 \left|\nabla D^h u\right|^2 dx + C(\epsilon) \int_U \left|\nabla u\right|^2 dx.
\end{aligned}$$

We now choose $\epsilon = \frac{\Lambda}{2}$ and absorb the first term into $I_1^{(1)}$ to obtain (3.16).

For I_2, we first realize that v vanishes on the boundary portions P and Q_0 (for sufficiently small h), and so the boundary integral is only over Q. Then

$$\begin{aligned}
I_2 &= \int_Q \mu(x) uv \, d\sigma \\
&= \int_Q \mu(x) u \left(-D^{-h} \eta^2 D^h u \right) \, d\sigma \\
&= \int_Q D^h (\mu(x) u) \eta^2 D^h u \, d\sigma \\
&= \int_Q \mu^h(x) \eta^2 \left(D^h u \right)^2 \, d\sigma + \int_Q D^h \mu(x) \eta^2 u D^h u \, d\sigma. \\
&= I_2^{(1)} + I_2^{(2)}.
\end{aligned}$$

Note that by assumption $\mu(x) < 0$ so that $I_2^{(1)} < 0$. Keeping this in mind we examine $I_2^{(2)}$. Applying Holder and then Euler's inequality gives

$$\begin{aligned}
\left| I_2^{(2)} \right| &\leq \left(\int_Q \left(D^h \mu(x) \right)^2 u^2 \, dx \right)^{1/2} \left(\int_Q \eta^2 \left(D^h u \right)^2 \, d\sigma \right)^{1/2} \\
&\leq C(\epsilon) \int_Q \left(D^h \mu(x) \right)^2 u^2 \, d\sigma + \epsilon \int_Q \eta^2 \left(D^h u \right)^2 \, d\sigma \\
&\leq C \left\| u \right\|_{L^2(\partial U)}^2 + \epsilon \int_Q \eta^2 \left(D^h u \right)^2 \, d\sigma.
\end{aligned}$$

The second term can be made arbitrarily small (the constant C in the first term growing very large) and absorbed into $I_2^{(1)}$. That is,

$$I_2 \leq C \left\| u \right\|_{L^2(\partial U)}^2 + \int_Q \left(\mu^h(x) + \epsilon \right) \eta^2 \left(D^h u \right)^2 \, d\sigma,$$

and by taking ϵ sufficiently small we have

$$I_2 < C \left\| u \right\|_{L^2(\partial U)}^2 . \tag{3.18}$$

Combining (3.15), (3.16), and (3.18) gives

$$\frac{\Lambda}{2} \int_U \eta^2 \left\| \nabla D^h u \right\| \, dx - C \int_U \left\| \nabla u \right\|^2 \, dx < C \left\| u \right\|_{L^2(\partial U)}^2 ,$$

or

$$\left\| \nabla D^h u \right\|_{L^2(V)}^2 \leq C \left(\left\| u \right\|_{H^1(U)}^2 + \left\| u \right\|_{L^2(\partial U)}^2 \right). \tag{3.19}$$

We therefore have $\frac{\partial u}{\partial x_l} \in H^1(V)$ for $1 \leq l \leq N-2$. We may now repeat this argument to u_{x_l} instead of u, picking up lower order terms in the estimate (3.19) and obtaining $\frac{\partial^k u}{\partial x_l^k} \in H^1(V)$ for $1 \le l \le N-2$. Thus, u is smooth in the x' variable (in some smaller set).

We are now prepared to isolate the x_{N-1} and x_N variables from the remaining dimensions, carrying on the extra variables as a parameter. First, by replacing u with ηu for a smooth cutoff function η, we may assume $f = 0$ (possibly shrinking R if needed). We may then extend the solution u to the entire half-space $\mathbb{R}^N_+$, and so we may assume u is compactly supported and solves

$$\begin{cases} \operatorname{div}'(A'\nabla' u) + \Delta_2 u = e & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } Q_0 \\ A\nabla u \cdot \nu = \mu(x) u & \text{on } Q, \end{cases} \tag{3.20}$$

where $e \in L^2(\mathbb{R}^N_+)$. Note that ν is the vector $-e_N$, and so by our assumption of A, $A\nabla u \cdot \nu$ is simply $\frac{\partial_2 u}{\partial \nu_2}$, where $\nu_2 = \langle 0, -1\rangle$.

We now do the contraction of coordinates described Figure 3.1. Define

$$v(x', r, \theta) = u(x', r^2, 2\theta), \quad B(x', r, \theta) = A(x', r^2, 2\theta).$$

Set $V_1 = \{x \in \mathbb{R}^N : x_N > 0, x_{N-1} > 0\}$, which is the contracted domain of $\mathbb{R}^N_+$ through $\theta = 90°$. Using ∂' to denote derivatives in the x' variable and ∂_2 to denote derivatives in the $\{r, \theta\}$ variables, we have

$$\operatorname{div}'(A\nabla' u)(r^2, 2\theta) + \Delta_2 u(r^2, 2\theta) = \operatorname{div}'(B\nabla' v)(r, \theta) + \frac{1}{4r^2}\Delta_2 v(r, \theta).$$

Then v satisfies

$$\begin{cases} \Delta_2 v = 4r^2 e(x', r^2, 2\theta) + 4r^2 \operatorname{div}'(B\nabla' v) & \text{in } V_1 \\ v = 0 & \text{on } Q_0 \\ \dfrac{\partial_2 v}{\partial \nu} = 2r\mu(x', r^2, 2\theta) v & \text{on } \{x \in \partial V_1 : x_{N-1} = 0\}. \end{cases}$$

Letting $\tilde{v}$ be the odd extension of v along $x_N = 0$, $\tilde{B}, \tilde{\mu}$ be even extensions, and $V_2 = \{x \in \mathbb{R}^N : x_{N-1} > 0\}$, we have

$$\begin{cases} \Delta_2 \tilde{v} = 4(x_{N-1}^2 + x_N^2)\left(\tilde{e}(x', r^2, 2\theta) + \operatorname{div}'(\tilde{B}\nabla' \tilde{v})\right) & \text{in } V_2 \\ \dfrac{\partial_2 v}{\partial \nu} = 2\,|x_N|\,\tilde{\mu}(x', r^2, 2\theta)\tilde{v} & \text{on } \partial V_2. \end{cases}$$

Now, $\tilde{v} \in H^1(V_2) = H^1(E \times F)$, where $E = \mathbb{R}^{N-2}$, $F = \mathbb{R}^+ \times \mathbb{R}$ (here $x_{N-1} \in \mathbb{R}^+$ and $x_N \in \mathbb{R}$). We have already established that u, and therefore $\tilde{v}$ is smooth in the x' variable, and so we may consider $\tilde{v}$ to be a function in $H^1(F)$ with values in $C^\infty(E)$. The idea now is that we may perform a bootstrapping argument on the two dimensional space F, and eventually we will be able to embed into a Holder continuous space as we have L^2 estimates and the dimension is two.

In the following we will say $\tilde{v} \in X$ to mean $\tilde{v} \in X$ with values in $C^\infty(E)$. Trace embedding gives $\tilde{v} \in H^{1/2}(\partial F) \hookrightarrow L^2(\partial F)$. Then we have $\partial\tilde{v}/\partial\nu \in L^2(\partial F)$ (recalling $\tilde{v}$ has compact support so $|x_N|\,\tilde{v}$ is still in $L^2(\partial F)$). Then using the elliptic regularity of the operator Δ_2, we have $\tilde{v} \in H^{3/2}(F)$. We may apply the argument once more to obtain $\tilde{v} \in H^1(\partial F)$, so $|x_N|\,\tilde{v} \in H^1(\partial F)$, giving $\partial\tilde{v}/\partial\nu \in H^1(\partial F)$ and $\tilde{v} \in H^{5/2}(F)$. If we attempt to go one step further, we would have $\tilde{v} \in H^2(\partial F)$, but the absolute value function is not in H^2, so we cannot conclude $\frac{\partial\tilde{v}}{\partial\nu} \in H^2(\partial F)$. However, we can go as high as $H^{3/2-\epsilon'}(\partial F)$ for $\epsilon' > 0$, as the absolute value function in one dimension just misses being in $H^{3/2}$. We conclude then that $\frac{\partial\tilde{v}}{\partial\nu} \in H^{3/2-\epsilon'}(\partial F)$ for any $\epsilon' > 0$ (noting that $\tilde{v}$ is bounded on ∂F as $\tilde{v} \in H^{3/2}(\partial F)$), and so $\tilde{v} \in H^{3-\epsilon'}(F)$ for any $\epsilon' > 0$.

Because the dimension of F is two, we may embed into a Holder space. In particular $H^{3-\epsilon'}(F) \hookrightarrow C^{1,1-\epsilon}(F)$ for any sufficiently small ϵ, ϵ'. We therefore conclude $\tilde{v} \in H^{3-\epsilon}(F)$ with values in $C^\infty(E)$. As such, $\tilde{v}$ has a fractional Taylor series expansion (see [17]),

$$\begin{aligned}\tilde{v}(x', x_{N-1}, x_N) = \tilde{v}(x', 0, 0) + \frac{\partial\tilde{v}}{\partial x_{N-1}}(x', 0, 0)x_{N-1} \\ + \frac{\partial\tilde{v}}{\partial x_N}(x', 0, 0)x_N + O\left(\left(x_{N-1}^2 + x_N^2\right)^{3/4-\epsilon}\right).\end{aligned} \tag{3.21}$$

As $\tilde{v} = 0$ when $x_N = 0$, $\partial\tilde{v}/\partial x_{N-1} = 0$ when $x_N = 0$, and (3.21) reduces to

$$\tilde{v}(x', x_{N-1}, x_N) = C(x')x_N + O\left(\left(x_{N-1}^2 + x_N^2\right)^{3/4-\epsilon}\right),$$

or

$$\tilde{v}(x', r, \theta) = C(x')r\sin\theta + O\left(r^{3/2-\epsilon}\right),$$

where $C(x') = \frac{\partial \tilde{v}}{\partial x_N}(x', 0, 0)$ is smooth. This expansion holds in all of $\tilde{V}$, and in particular it holds in V where $v = \tilde{v}$, and as $u(x', r, \theta) = v\left(x', r^{1/2}, \theta/2\right)$, this implies

$$u(x', r, \theta) = C(x') r^{1/2} \sin(\theta/2) + O\left(r^{3/4-\epsilon}\right)$$

in U_{R_0} for some sufficiently small R_0.

In a similar manner we obtain the expansion for u_r. We expand $\tilde{v}_{x_{N-1}}$ and $\tilde{v}_{x_N}$ in a fractional Taylor polynomial,

$$\begin{aligned}
\frac{\partial \tilde{v}}{\partial x_{N-1}}(x', x_{N-1}, x_N) &= \frac{\partial \tilde{v}}{\partial x_{N-1}}(x', 0, 0) + O\left((x_N^2 + x_{N-1}^2)^{1/4-\epsilon}\right), \\
\frac{\partial \tilde{v}}{\partial x_N}(x', x_{N-1}, x_N) &= \frac{\partial \tilde{v}}{\partial x_N}(x', 0, 0) + O\left((x_N^2 + x_{N-1}^2)^{1/4-\epsilon}\right).
\end{aligned}$$

Again $\frac{\partial \tilde{v}}{\partial x_{N-1}}(x', 0, 0) = 0$, and so we have

$$\begin{aligned}
\frac{\partial \tilde{v}}{\partial r}(x', x_{N-1}, x_N) &= \cos\theta \frac{\partial \tilde{v}}{\partial x_{N-1}}(x', x_{N-1}, x_N) + \sin\theta \frac{\partial \tilde{v}}{\partial x_N}(x, x_{N-1}, x_N) \\
&= \sin\theta \frac{\partial \tilde{v}}{\partial x_N}(x', 0, 0) + O(r^{1/2-\epsilon}) \\
&= C(x') \sin\theta + O(r^{1/2-\epsilon}),
\end{aligned}$$

using $C(x') = \frac{\partial \tilde{v}}{\partial x_N}(x', 0, 0)$. Then v has the same expansion, and since $v_r(x', r, \theta) = 2r u_r(x', r^2, 2\theta)$, this gives

$$2r \frac{\partial u}{\partial r}(x', r^2, 2\theta) = C(x') \sin\theta + O(r^{1/2-\epsilon}),$$

or

$$2r^{1/2} \frac{\partial u}{\partial r}(x', r, \theta) = C(x') \sin(\theta/2) + O(r^{1/4-\epsilon})$$

in U_{R_0}. Expressing $\tilde{v}_\theta$ in the same manner and returning to the function u and r, θ provides

$$\frac{\partial u}{\partial \theta}(x', r, \theta) = \frac{1}{2} C(x') \cos(\theta/2) + O(r^{1/4-\epsilon})$$

Lastly, we expand for $1 \le i \le N-2$

$$\begin{aligned}
\frac{\partial \tilde{v}}{\partial x_i}(x', x_{N-1}, x_N) = \frac{\partial \tilde{v}}{\partial x_i}(x', 0, 0) &+ \frac{\partial}{\partial x_{N-1}} \frac{\partial \tilde{v}}{\partial x_i}(x', 0, 0) x_{N-1} \\
&+ \frac{\partial}{\partial x_N} \frac{\partial \tilde{v}}{\partial x_i}(x', 0, 0) x_N + O\left((x_N^2 + x_{N-1}^2)^{3/4-\epsilon}\right).
\end{aligned}$$

Equating mixed partials (which are continuous) yields

$$\frac{\partial \tilde{v}}{\partial x_i}(x', x_{N-1}, x_N) = \frac{\partial}{\partial x_i} C(x') x_N + O\left((x_N^2 + x_{N-1}^2)^{3/4-\epsilon} \right),$$

and returning to the function u and r, θ completes the proof. $\square$

We remark that the leading term $r^{1/2}$ in (3.14) suggests that $u \in H^{3/2-\epsilon}(\Omega)$, as r is two dimensional and u is smooth in the x' variable. We therefore suspect that $H^{3/2}$ is the best Sobolev space we could hope to be in, so in particular we suspect the solutions to the nonlinear problem (3.12) will not be H^2.

We proceed in the following manner. Having established that the linearized problem to (3.12) satisfies a particular expansion, we compare a solution w of (3.12) to the linearized solution u. The idea is that if w is small then (3.12) is a small perturbation of the linearized problem, and so the expansions should agree to leading order. For this we need to assume w is continuous so that $w = 0$ on $\partial\Phi(\Gamma)$ and w is small close to the boundary interface.

Lemma 17. *Let w be a weak solution to (3.12) that is continuous up to the boundary, and fix $y^0 \in \partial\Phi(\Gamma)$. Let z_N be the coordinate from y^0 in the direction of $\hat{y_N}$, and let z_{N-1} be in the direction of the outward normal to $\partial\Phi(\Gamma)$ at y^0. Let $z' = (z_1, \ldots, z_{N-2})$ be local coordinates about y^0 in the manifold $\partial\Phi(\Gamma)$. Choose r, θ to be standard polar coordinates with $z_{N-1} = r\cos\theta$, $z_N = r\sin\theta$. Suppose $c + \frac{N-2}{2} < 0$. Then there exists a function M of z', a neighborhood U of y^0 such that*

$$\left| w(z', r, \theta) \right| \le M(z') r^{1/2} \sin(\theta/2) \tag{3.22}$$

for $\{z', r, \theta\} \in U$.

Proof. We may assume without loss of generality that $\lambda = 1$. Let $L_0 = \Phi(\Gamma_0)$, $L = \Phi(\Gamma)$, and

$$\mu(z) = \frac{1}{\beta(z)} \left(c + \frac{N-2}{2} \right),$$
$$q = \frac{N-2}{2}\alpha - \frac{N}{2},$$

where β is given by 3.8. Now $\beta(z) > 0$ and is bounded above and below in a neighborhood of y^0, and so $\mu(z) < 0$ and μ is bounded above and below by a positive constant (depending on c). Construct the function

$$\chi(z', r, \theta) = w(z', r, \theta) + M(z')r^{1/2}\sin(\theta/2),$$

where M is some positive number (depending on z') to be chosen. Now, as $w(y^0) = 0$ and w is continuous, we may find a sufficiently small neighborhood $U \subset \overline{\Omega}$ of y^0 so that

$$|w|^{\alpha-1} < \inf\left|\beta^{-q}\mu\right|$$

in U. Let $f = w|_{\partial U}$, and parameterize ∂U by $r = r(z')$. Then χ satisfies

$$\begin{cases} \Delta\chi = M''(z')r^{1/2}\sin(\theta/2) & \text{in } U \\ \chi = 0 & \text{on } \partial U \cap L_0 \\ \dfrac{\partial\chi}{\partial\nu} = \mu(z)w + \beta^q(z)\,|w|^{\alpha-1}\,w & \text{on } \partial U \cap L \\ \chi = f + M(z')r(z')^{1/2}\sin(\theta/2) & \text{on } \partial U \backslash (L_0 \cup L). \end{cases}$$

We now require $M(z') \geq 0$, $M''(z') \leq 0$ in U, and

$$f(z', \theta) + M(z')r(z')^{1/2}\sin(\theta/2) \geq 0$$

on $\partial U \backslash (L_0 \cup L)$. With these conditions satisfied, $\Delta\chi \leq 0$ in U, so χ must take on its minimum on ∂U. If this minimum were negative and occurring on $\partial U \cap L$, then there would be a point $\hat{z}$ where $\chi(\hat{z})$ and $\frac{\partial\chi}{\partial\nu}(\hat{z})$ are both negative. Since

$$\frac{\partial\chi}{\partial\nu} = w\left(\mu(z) + \beta^q(z)\,|w|^{\alpha-1}\right)$$

and $\beta^q(z)\,|w|^{\alpha-1} < \inf|\mu|$, this implies $w(\hat{z}) > 0$. But

$$\chi(z) = w(z) + M(z')r^{1/2}$$

for any $z \in L$, and $M(z') \geq 0$ implies that $\chi(\hat{z}) > 0$, a contradiction. Then the minimum of χ must be nonnegative, which implies $\chi \geq 0$ in U. This gives

$$w(z', r, \theta) \geq -M(z')r^{1/2}\sin(\theta/2)$$

in U. Repeating the argument with $\chi(z)$ replaced by $\chi^{-}(z) = w(z) - M(z')r^{1/2}\sin(\theta/2)$ gives

$$w(z', r, \theta) \leq M(z')r^{1/2}\sin(\theta/2)$$

in U, and so

$$\left|w(z', r, \theta)\right| \leq M(z')r^{1/2}\sin(\theta/2).$$

It remains to show that we can find such a function M. Set $D = \operatorname{diam} U$, so that $\left|z - y^0\right| < D$ for $z \in U$. Consider

$$m(z') = e^{D^2} - e^{|z'|^2}.$$

Then $m(z') > 0$ and $m''(z') \leq 0$. Since w is bounded in U, f is bounded. Moreover, the function $r(z')^{1/2}\sin(\theta/2)$ only vanishes when $\theta = 0$, or when $r(z')$ vanishes. However, $\theta = 0$ corresponds to points in L_0, where w, and hence f, vanishes. Similarly r vanishes only along points in ∂L, where w vanishes as well. Thus, we may take M to be a sufficiently large multiple of m to guarantee

$$f(z', \theta) + M(z')r(z')^{1/2}\sin(\theta/2) \geq 0.$$

□

Lemma 18. *Let w be a weak solution to (3.12), and fix $y^0 \in \partial\Phi(\Gamma)$. Let (z', r, θ) be the coordinates described in Lemma 17, and suppose $c + \frac{N-2}{2} < 0$. Then there exists a smooth function C of z', a neighborhood U of y^0 and a real number $\gamma > 1/2$ such that the expansion*

$$w(z', r, \theta) = C(z')r^{1/2}\sin(\theta/2) + O\left(r^{\gamma}\right) \tag{3.23}$$

holds for $\{z', r, \theta\} \in U$.

Proof. The idea is to apply Lemma 16 to get an expansion in the form of (3.23) for a solution u of the linearized problem, and then use a barrier argument to compare u and w. Lemma 17 will guarantee the nonlinear component is small in comparison to the linear term, and this will allow us to show that w has the same expansion to leading order as u. Again we may assume $\lambda = 1$.

From Lemma 17, we may find a neighborhood U of y^0 in which

$$\left|w(z', r, \theta)\right|^{\alpha} \leq M(z')r^{\alpha/2}$$

holds for some smooth function M of z'. Now, in this neighborhood we perform a straightening of the boundary. By choosing an appropriate open subset of U and translating, we may assume that upon straightening we are in the domain U_R with Q_0, Q, P described in Lemma 16, and R is some fixed value (depending on y^0). Let x be the coordinates in the straightened domain (so the curved coordinates z' map to straight coordinates x' and r, θ remain the same). For sake of notation, let w also denote the function defined on this new domain, so w solves

$$\begin{cases} \operatorname{div}(A\nabla w) = 0 & \text{in } U_R \\ w = 0 & \text{on } Q_0 \\ A\nabla w \cdot \nu = \mu(x)w + \beta^q(x)\left|w\right|^{\alpha-1} w & \text{on } Q \\ w = f & \text{on } P, \end{cases}$$

where f is smooth, $\mu(x) = \beta(x)(c+(N-2)/2) < 0$, and A is a smooth, positive definite matrix with uniformly bounded eigenvalues. We can further assume that $\det(A) = 1$ and that A leaves the derivatives in the x_{N-1} and x_N direction unchanged, so that A satisfies all the hypotheses of Lemma 16.

Construct the function u satisfying

$$\begin{cases} \operatorname{div}(A\nabla u) = 0 & \text{in } U_R \\ u = 0 & \text{on } Q_0 \\ A\nabla u \cdot \nu = \mu(x)u & \text{on } Q \\ u = f & \text{on } P. \end{cases}$$

Applying Lemma 16 (noting that β is bounded above and below on a small enough set, so μ is negative and away from 0), we have the existence of a neighborhood U_{R_0} of y^0, $0 < R_0 \leq R$, in which (3.14) holds, and so by shrinking R if necessary we may assume (3.14) holds in U_R.

Define $\phi = w - u$ and $\zeta = |w|^{\alpha-1} w|_L$. Then ϕ satisfies

$$\begin{cases} \operatorname{div}(A\nabla\phi) = 0 & \text{in } U_R \\ \phi = 0 & \text{on } Q_0 \\ A\nabla\phi\cdot\nu = \mu(x)\phi + \zeta & \text{on } Q \\ \phi = 0 & \text{on } P. \end{cases}$$

Now construct the function $v \in H^1(\Omega)$ satisfying

$$\begin{cases} \operatorname{div}(A\nabla v) = 0 & \text{in } U_R \\ v = 0 & \text{on } Q_0 \\ A\nabla v\cdot\nu - \mu(x)v = 1 & \text{on } Q \\ v = 0 & \text{on } P. \end{cases}$$

Such a function exists because the eigenvalue problem $A\nabla v \cdot \nu = \mu v$ on Q with zero data everywhere else has positive eigenvalues (due to the positive definiteness of A), and so when $\mu < 0$ this problem has only the trivial solution. Set

$$M = \sup_Q |\zeta|,$$

and $h_\pm = \phi \pm Mv$. Then $h_\pm$ solves

$$\begin{cases} \operatorname{div}(A\nabla h_\pm) = 0 & \text{in } U_R \\ h_\pm = 0 & \text{on } Q_0 \\ A\nabla h_\pm\cdot\nu - \mu(x)h_\pm = \zeta \pm M & \text{on } Q \\ h_\pm = 0 & \text{on } P. \end{cases}$$

Then the maximum principle implies $h_- \leq 0$ in $\overline{U_R}$, for if h_- obtained a positive maximum on Q, $\mu(x) < 0$ implies $\zeta - M > 0$, which contradicts the definition of M. Similarly, a minimum principle applied to h_+ implies $h_+ \geq 0$ in $\overline{U_R}$. We then have

$$|\phi| \leq |v|\, M$$

in $\overline{U_R}$. Finally, the same argument used in Lemma 16 that showed the solution u to the linearized problem was bounded will imply that $v \in L^\infty(\overline{U_R})$, and so we have

$$\left|\phi(x', R, \theta)\right| \leq \|v\|_{L^\infty(\overline{U_R})} \leq C \sup_Q |\zeta| \leq C R^{\alpha/2},$$

where the constant C depends on everything but R. Repeating this bound for any $r < R$ gives

$$\left|\phi(x', r, \theta)\right| \le Cr^{\alpha/2}$$

for $(x', r, \theta) \in U_R$. Therefore

$$\begin{aligned} w(x', r, \theta) &= u(x', r, \theta) + \phi(x', r, \theta) \\ &= C(x')r^{1/2}\sin(\theta/2) + O(r^{3/4-\epsilon}) + O(r^{\alpha/2}) \\ &= C(x')r^{1/2}\sin(\theta/2) + O(r^{\gamma}), \end{aligned}$$

where $\gamma = \min(3/4 - \epsilon, \alpha/2) > 1/2$. Returning to the original coordinates completes the proof. $\square$

It remains to show that the derivatives of w (with respect to the appropriate coordinates) follow the same expansion as those of the linear case. This is not immediately implied by Lemma 3.23 as we only have a bound for w and we cannot simply differentiate the expression. Such a bound would not be possible if w were to oscillate highly as $r \to 0$, and so we exclude this possibility. In particular, we assume $|w_r| \le Cr^{\delta-1}$ for some $\delta > 0$ as $r \to 0$.

Theorem 8. *Let w be a weak solution to (3.12), and fix $y^0 \in \partial\Phi(\Gamma)$. Let (z', r, θ) be the coordinates described in Lemma 17, and let $c < -\frac{N-2}{2}$. Assume w is continuous up to the boundary and $|rw_r| \le C_0 r^{\delta}$ for some $\delta > 0$ and constant C_0 (depending possibly on z' and θ). Then there exists a function C of z' and a neighborhood U of y^0 such that the expansions*

$$\begin{cases} w(z', r, \theta) = C(z')r^{1/2}\sin(\theta/2) + O\left(r^{\gamma}\right) \\ w_r(z', r, \theta) = \dfrac{1}{2}C(z')r^{-1/2}\sin(\theta/2) + O\left(r^{\gamma-1}\right) \\ w_\theta(z', r, \theta) = \dfrac{1}{2}C(z')r^{1/2}\cos(\theta/2) + O\left(r^{\gamma}\right) \\ w_{z_i}(z', r, \theta) = C_{z_i}(z')r^{1/2}\sin(\theta/2) + O\left(r^{\gamma}\right) \end{cases} \tag{3.24}$$

hold for $\{z', r, \theta\} \in U$, $1 \le i \le N-2$ and some $\gamma > 1/2$.

Proof. The idea here is to form the same function $\phi = w - u$ as in Lemma 18 and then consider the equation satisfied by $\dot{\phi} = \frac{\partial \phi}{\partial x_i}$, $1 \le i \le N$. The derivatives in the x_{N-1} and

x_N direction will have to be handled with more care, as x_{N-1} is in the normal direction and the x_N direction contains a change in boundary condition. The argument for each will be similar to the barrier argument used in Lemma 18.

As in the previous proof, suppose we have straightened the boundary curve $\partial\Phi(\Gamma)$ locally. Let $U_R = B_R \cap \mathbb{R}^N_+$. Then w satisfies

$$\begin{cases} \operatorname{div}(A\nabla w) = e & \text{in } \mathbb{R}^N_+ \\ w = 0 & \text{on } Q_0 \\ \dfrac{\partial_2 w}{\partial \nu} = \mu(x)w + \beta^q(x)\,|w|^{\alpha-1}\,w & \text{on } Q \end{cases}$$

where the support of w lies in U_R and e is some function depending on w in at most first order derivatives. Again because A is the identity matrix in the 2×2 block for the x_{N-1} and x_N variables, the boundary term $A\nabla w \cdot \nu$ simply becomes $\frac{\partial_2 w}{\partial \nu} = -\frac{\partial w}{\partial x_N}$. Here we have written $Q_0 = \{x \in \mathbb{R}^N : x_N = 0, x_{N-1} \geq 0\}$, $Q = \{x \in \mathbb{R}^N : x_N = 0, x_{N-1} < 0\}$.

Construct u solving

$$\begin{cases} \operatorname{div}(A\nabla u) = e & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } Q_0 \\ \dfrac{\partial_2 u}{\partial \nu} = \mu(x)u & \text{on } Q \end{cases}$$

and define $\phi = w - u$. Then ϕ satisfies

$$\begin{cases} \operatorname{div}(A\nabla \phi) = 0 & \text{in } \mathbb{R}^N_+ \\ \phi = 0 & \text{on } Q_0 \\ \dfrac{\partial_2 \phi}{\partial \nu} = \mu(x)\phi + \beta^q(x)\,|w|^{\alpha-1}\,w & \text{on } Q. \end{cases}$$

Set $\dot{\phi} = \frac{\partial \phi}{\partial x_i}$ for $1 \leq i \leq N-2$. Then $\dot{\phi}$ satisfies the boundary value problem

$$\begin{cases} \operatorname{div}(A\nabla \dot{\phi}) = -\operatorname{div}(\dot{A}\nabla \phi) & \text{in } \mathbb{R}^N_+ \\ \dot{\phi} = 0 & \text{on } Q_0 \\ \dfrac{\partial_2 \dot{\phi}}{\partial \nu} = \dot{\mu}\phi + \mu\dot{\phi} + \dot{B}\,|w|^{\alpha-1}\,w + \alpha B\,|w|^{\alpha-1}\,\dot{w} & \text{on } Q, \end{cases}$$

where we have set $B = \beta^q$. Writing $\dot{w} = \dot{\phi} + \dot{u}$ and rearranging, we may write the boundary condition on Q as

$$\frac{\partial_2 \dot{\phi}}{\partial \nu} - \left(\mu + \alpha B\,|w|^{\alpha-1}\right)\dot{\phi} = \dot{\mu}\phi + |w|^{\alpha-1}\left(\dot{B}w + \alpha B\dot{u}\right).$$

From Lemma 17, and because μ is bounded above by a strictly negative constant, we may select R small enough so that

$$\mu + \alpha B \left|w\right|^{\alpha-1} \leq K < 0,$$

and so we can repeat the barrier argument from Lemma 18 to obtain

$$\left\|\dot{\phi}\right\|_{L^\infty(\overline{\mathbb{R}^N_+})} \leq C \left\|\dot{\mu}\phi + \left|w\right|^{\alpha-1}(\dot{B}w + \alpha B\dot{u})\right\|_{L^\infty(Q)}.$$

We have from lemmas 16, 17, and 18 that $|\phi| \leq Cr^\gamma$, $|w|^{\alpha-1} \leq Cr^{(\alpha-1)/2}$, $|w| \leq Cr^{1/2}$, and $|\dot{u}| \leq Cr^{1/2}$ for some $\gamma > 1/2$. Combining all of this gives

$$\left\|\dot{\phi}\right\|_{L^\infty(\overline{U_R})} \leq CR^\gamma,$$

for $\gamma > 1/2$, which completes the bound for $\frac{\partial w}{\partial x_i}$, $1 \leq i \leq N-2$.

To bound $\frac{\partial \phi}{\partial x_N}$ requires a different argument. This is in the normal direction to the flat boundary, and so we cannot differentiate the boundary condition by passing through the limits in difference quotients. However, we realize that

$$\left.\frac{\partial}{\partial \nu}\right|_Q = -\left.\frac{\partial}{\partial x_N}\right|_Q,$$

and so we may write the boundary value problem for ϕ as

$$\begin{cases} \operatorname{div}(A\nabla\phi) = 0 & \text{in } \mathbb{R}^N_+ \\ \phi = 0 & \text{on } Q_0 \\ \dfrac{\partial \phi}{\partial x_N} = -g & \text{on } Q, \end{cases}$$

where $g = \mu(x)\phi + \beta^q(x)\left|w\right|^{\alpha-1} w$.

This can be viewed as a Dirichlet problem for $\frac{\partial \phi}{\partial x_N}$. However, along $x_{N-1} = 0$, $x_N = r$, and under our assumptions this derivative may blow up, and so the value of $\frac{\partial \phi}{\partial x_N}$ may not be bounded on Q_0 as $x_{N-1} \to 0$. We therefore consider the function

$$\chi = x_N \phi.$$

In $\mathbb{R}^+_N$, $\operatorname{div}(A\nabla\chi) = e$ for some function e depending on first order derivatives of ϕ. Then χ satisfies

$$\begin{cases} \operatorname{div}(A\nabla\chi) = e & \text{in } \mathbb{R}^N_+ \\ \chi = 0 & \text{on } Q_0 \\ \dfrac{\partial \chi}{\partial x_N} = -x_N g + \phi & \text{on } Q, \end{cases}$$

Set $\dot{\chi} = \frac{\partial \chi}{\partial x_N}$. Then $\dot{\chi}$ satisfies (recalling A does not depend on x_N),

$$\begin{cases} \operatorname{div}(A\nabla\dot{\chi}) = \dot{e} & \text{in } \mathbb{R}^N_+ \\ \dot{\chi} = f & \text{on } Q_0 \\ \dot{\chi} = -x_N g + \phi & \text{on } Q, \end{cases}$$

where we have set $f = \frac{\partial \chi}{\partial x_N}$ on Q_0.

Set $M = \max\left(\sup_Q |-x_N g + \phi|, \sup_{Q_0} |f|\right) > 0$. We need to bound $f = \frac{\partial \chi}{\partial x_N}$ on Q_0. Note first that on Q_0, $x_{N-1} > 0$ and $x_N = 0$. Then since $\chi = 0$ on Q_0,

$$\left|\frac{\chi(x', x_{N-1}, h) - \chi(x', x_{N-1}, 0)}{h}\right| = \left|\frac{\chi(x', x_{N-1}, h)}{h}\right| = \left|\phi(x', x_{N-1}, h)\right| \le R^\gamma.$$

Therefore, $|f|$ is bounded by R^γ, and $|-x_N g + \phi|$ is bounded by R^γ as well, so $M \le CR^\gamma$.

Construct $v_\pm$ satisfying

$$\begin{cases} \operatorname{div}(A\nabla v_\pm) = \pm\dot{e}/M & \text{in } \mathbb{R}^N_+ \\ v_\pm = 1 & \text{on } Q_0 \\ v_\pm = 1 & \text{on } Q. \end{cases}$$

Set $h_\pm = \dot{\chi} \pm M v_\mp$. Then $h_- \le 0$ on the boundary, and so $\dot{\chi} \le Mv$ in $\mathbb{R}^N_+$. Similarly $h_+ \ge 0$, so $\dot{\phi} \ge -Mv_\mp$. Then as $v_\mp$ is bounded, we have $|\dot{\chi}| \le CM$, and M is bounded by R^γ, and the same argument as before gives

$$\left|\frac{\partial \chi}{\partial x_N}\right| \le Cr^\gamma,$$

yielding

$$\left|x_N \frac{\partial \phi}{\partial x_N}\right| \le Cr^\gamma.$$

Note that this argument of constructing a Dirichlet problem for the derivative could also be applied to $\frac{\partial \phi}{\partial \theta}$, as $\frac{\partial \phi}{\partial \theta} = \pm r \frac{\partial \phi}{\partial x_N}$ on the boundary. This yields

$$\left|\frac{\partial \phi}{\partial \theta}\right| \le Cr^\gamma.$$

To bound $\frac{\partial \phi}{\partial x_{N-1}}$, we apply the same idea as the previous case. The difference now is that we are differentiating in the tangent direction, so we will have a Neumann problem for $\frac{\partial \phi}{\partial x_{N-1}}$ on Q. However, as before we expect this derivative to blow up on the order of $r^{-1/2}$. We therefore consider the function

$$\chi = x_{N-1}\phi.$$

Set $\dot{\chi} = \frac{\partial \chi}{\partial x_{N-1}}$. There are no issues differentiating inside, so $\operatorname{div}(A\nabla\dot{\chi}) = \dot{e}$ in $\mathbb{R}^N_+$. Along Q_0, ϕ is identically zero, and so is χ, and so $\dot{\chi} = 0$ on Q_0. Note that when $x_{N-1} = 0$, we have

$$\dot{\chi}\big|_{x_{N-1}=0} = \lim_{h\to 0} \frac{\chi(x', h, 0) - \chi(x', 0, 0)}{h} = \lim_{h\to 0} \frac{h\phi(x', h, 0)}{h} = \lim_{h\to 0} \phi(x', h, 0) = 0,$$

where we have used the assumption that w, and therefore ϕ, is continuous from both sides at $x_{N-1} = 0$. Therefore, $\dot{\chi} = 0$ when $x_{N-1} = 0$. We need $\dot{\chi}$ to be continuous at $x_{N-1} = 0$, and so we need $\lim_{x_{N-1}\to 0} \dot{\chi}(z', x_{N-1}, 0) = 0$. The right-hand limit is zero since $\dot{\chi}$ is identically zero for $x_{N-1} > 0$. To examine the left-hand limit, first realize when $x_{N-1} < 0$ and $x_N = 0$, we have $x_{N-1} = -r$. Then for $x_{N-1} < 0$ and $x_N = 0$,

$$\begin{aligned}
\left|\dot{\chi}(x', x_{N-1}, 0)\right| &= \left|x_{N-1}\dot{\phi}(x', x_{N-1}, 0) + \phi(x', x_{N-1}, 0)\right| \\
&= \left|-r\dot{\phi}(x', -r, 0) + \phi(x', -r, 0)\right| \\
&\leq r\left|\dot{\phi}(x', -r, 0)\right| + \left|\phi(x', -r, 0)\right| \\
&\leq C\left(r\left|w_r\right| + r\left|u_r\right| + r^{\gamma}\right) \\
&\leq C\left(r^{\delta} + r^{1/2} + r^{\gamma}\right),
\end{aligned}$$

which goes to 0 as $r \to 0$. Therefore $\dot{\chi}$ is continuous as $x_{N-1} \to 0$.

Turning to the Neumann boundary condition, along Q we have

$$\begin{aligned}
\frac{\partial \dot\chi}{\partial \nu} &= \frac{\partial}{\partial x_{N-1}} \frac{\partial \chi}{\partial \nu} \\
&= \frac{\partial}{\partial x_{N-1}} \left(- \frac{\partial x_{N-1}\phi}{\partial x_N} \right) \\
&= \frac{\partial}{\partial x_{N-1}} \left(x_{N-1} \frac{\partial \phi}{\partial \nu} \right) \\
&= x_{N-1} \frac{\partial}{\partial x_{N-1}} \frac{\partial \phi}{\partial \nu} + \frac{\partial \phi}{\partial \nu} \\
&= x_{N-1} \left(\dot\mu \phi + \mu \dot\phi + \dot B \left|w\right|^{\alpha-1} w + \alpha B \left|w\right|^{\alpha-1} \dot w \right) + \mu\phi + B \left|w\right|^{\alpha-1} w \\
&= \mu(x_{N-1}\dot\phi + \phi) + \alpha B \left|w\right|^{\alpha-1} (x_{N-1}\dot\phi + \phi) - \alpha B \left|w\right|^{\alpha-1} \phi \\
&\qquad + x_{N-1} \left(\dot\mu \phi + \dot B \left|w\right|^{\alpha-1} w + \alpha B \left|w\right|^{\alpha-1} \dot u \right) + B \left|w\right|^{\alpha-1} w \\
&= \left(\mu + \alpha B \left|w\right|^{\alpha-1} \right) \dot\chi + B \left|w\right|^{\alpha-1} (w - \alpha\phi) \\
&\qquad + x_{N-1} \left|w\right|^{\alpha-1} \left(\dot\mu\phi + \dot B w + \alpha B \dot u \right).
\end{aligned}$$

We rewrite this as

$$\frac{\partial \dot\chi}{\partial \nu} - \left(\mu + \alpha B \left|w\right|^{\alpha-1} \right) \dot\chi = B \left|w\right|^{\alpha-1} (w - \alpha\phi) + x_{N-1} \left|w\right|^{\alpha-1} \left(\dot\mu\phi + \dot B w + \alpha B \dot u \right). \tag{3.25}$$

Then we have $\dot\chi$ satisfies

$$\begin{cases}
\operatorname{div}(A\nabla\dot\chi) = \dot e & \text{in } \mathbb{R}^N_+ \\
\dot\chi = 0 & \text{on } Q_0 \\
\dfrac{\partial \dot\chi}{\partial \nu} - \left(\mu + \alpha B \left|w\right|^{\alpha-1} \right) \dot\chi = g, & \text{on } Q
\end{cases}$$

where g is the right-hand side of (3.25) and satisfies $|g| \le |x_{N-1}|^\gamma$. We then construct $v_\pm$ satisfying

$$\begin{cases}
\operatorname{div}(A\nabla v_\pm) = \pm \dfrac{\dot e}{M} & \text{in } \mathbb{R}^N_+ \\
v_\pm = 0 & \text{on } Q_0 \\
\dfrac{\partial v_\pm}{\partial \nu} - \left(\mu + \alpha B \left|w\right|^{\alpha-1} \right) v_\pm = 1, & \text{on } Q,
\end{cases}$$

and again consider $h_\pm = \dot\chi \pm M v_\mp$. The maximum principle argument gives

$$\left|\dot\chi\right| \le C \left|x_{N-1}\right|^\gamma,$$

and so

$$\left| x_{N-1}\dot\phi + \phi \right| \le C \left|x_{N-1}\right|^\gamma,$$

which yields

$$\left|x_{N-1}\dot{\phi}\right| \leq C\left|x_{N-1}\right|^{\gamma} \leq Cr^{\gamma}.$$

Finally, we use the bounds for derivatives with respect to x_{N-1} and x_N to obtain the bound for ϕ_r. We may write

$$\frac{\partial \phi}{\partial r} = \frac{x_{N-1}}{r}\frac{\partial \phi}{\partial x_{N-1}} + \frac{x_N}{r}\frac{\partial \phi}{\partial x_N},$$

giving

$$\left|r\frac{\partial \phi}{\partial r}\right| \leq Cr^{\gamma},$$

which, along with the expansion we already have for u_r, establishes the bound for w_r. □

Having established an expansion of a potential solution to (3.12), we are prepared to derive a Pohozaev identity.

Theorem 9. *(A Pohozaev Identity). Suppose* $w \in H^1_{loc}(\mathbb{R}^N_+)$ *is a weak solution to (3.12) satisfying all the hypotheses of Theorem 8. Set* $L_0 = \Phi(\Gamma_0)$, $L = \Phi(\Gamma)$ *for* Φ *given by (3.6). Suppose the function* g *is of the form*

$$g(u) = cu + \lambda\left|u\right|^{\alpha-1}u$$

for $\lambda > 0$ *and* $c < -\frac{N-2}{2}$. *Then* w *satisfies*

$$\begin{aligned}\left(\frac{N-2}{2} - \frac{N-1}{\alpha+1}\right)\lambda\int_L \beta^{\frac{(N-2)\alpha-N-2}{2}}\left(\frac{1}{2}-\beta\right)\left|w\right|^{\alpha+1}\,d\sigma = -C^2 \\ +\frac{1}{2}\left(c+\frac{N-2}{2}\right)\int_L \beta^{-2}\left(\frac{1}{2}-\beta\right)w^2\,d\sigma\end{aligned} \tag{3.26}$$

where β *is given by (3.8) and* C *is a real number depending on* w *and all the parameters of the problem.*

Proof. Fix $\epsilon > 0$, and let T_ϵ be the neighborhood of ∂L

$$T_\epsilon = \{y \in \mathbb{R}^N_+ : \operatorname{dist}(y, \partial L) = \epsilon\}.$$

Set

$$U_\epsilon = \mathbb{R}^N_+ \backslash T_\epsilon,$$

$$L_{0,\epsilon} = L_0 \cap \partial U_\epsilon,$$

$$L_\epsilon = L \cap \partial U_\epsilon,$$

$$R_\epsilon = \partial U_\epsilon \backslash \left(L_{0,\epsilon} \cup L_\epsilon \right).$$

Suppose a solution $w \in H^1_{loc}(\mathbb{R}^N_+)$ to (3.12) exists with $w \neq 0$. We now apply identity (3.1) to w on the reduced domain U_ϵ. Since the interface between the Neumann and Dirichlet boundary has been removed, Δw will exist on U_ϵ in the sense of distributions, and so (3.1) makes sense in the sense of distributions. We choose $h = y$, integrate (3.1) over U_ϵ and use the Divergence Theorem to obtain

$$\int_{\partial\Omega_\epsilon} \left((y \cdot \nabla w)(\nabla w \cdot \nu) - \frac{1}{2}(y \cdot \nu) \left|\nabla w\right|^2 \right) d\sigma = \frac{2-N}{2} \int_{\Omega_\epsilon} \left|\nabla w\right|^2 \, dy. \tag{3.27}$$

Now, the left hand side of (3.27) has three components: an integral over L_ϵ, an integral over $L_{0,\epsilon}$, and an integral over R_ϵ. First, as $u = 0$ on $L_{0,\epsilon}$, ∇u is parallel to ν, and as $y \cdot \nu = 0$ on $L_{0,\epsilon}$, the integral over $L_{0,\epsilon}$ vanishes. We also have $y \cdot \nu = 0$ on L_ϵ, and so (3.27) becomes

$$A(\epsilon) + \int_{L_\epsilon} (y \cdot \nabla w) \frac{\partial w}{\partial \nu} \, d\sigma = \frac{2-N}{2} \int_{\Omega_\epsilon} \left|\nabla w\right|^2 \, dy, \tag{3.28}$$

where

$$A(\epsilon) = \int_{R_\epsilon} \left((y \cdot \nabla w)(\nabla w \cdot \nu) - \frac{1}{2}(y \cdot \nu) \left|\nabla w\right|^2 \right) d\sigma. \tag{3.29}$$

Integrating by parts and using $w = 0$ on $\Gamma_{0,\epsilon}$, we have

$$\begin{aligned} \frac{2-N}{2} \int_{\Omega_\epsilon} \left|\nabla w\right|^2 \, dy &= \frac{2-N}{2} \int_{\partial\Omega_\epsilon} w \frac{\partial w}{\partial \nu} \, d\sigma \\ &= \frac{2-N}{2} \int_{L_\epsilon} w \frac{\partial w}{\partial \nu} \, d\sigma + B(\epsilon) \\ &= \frac{2-N}{2} \left(\frac{N-2}{2} + c \right) \int_{L_\epsilon} \beta^{-1} w^2 \, d\sigma \\ &\quad + \frac{2-N}{2} \lambda \int_{L_\epsilon} \beta^q \left|w\right|^{\alpha+1} \, d\sigma + B(\epsilon), \end{aligned} \tag{3.30}$$

where

$$B(\epsilon) = \frac{2-N}{2} \int_{R_\epsilon} w \frac{\partial w}{\partial \nu} \, d\sigma \tag{3.31}$$

and we have set $q = ((N-2)\alpha - N)/2$.

Define

$$G_1(w) = \frac{w^2}{2}, \quad G_2(w) = \frac{|w|^{\alpha+1}}{\alpha+1}.$$

Then on L_ϵ, we have

$$\begin{aligned}(y\cdot\nabla w)\frac{\partial w}{\partial\nu} &= (y\cdot\nabla w)\left[\left(\frac{N-2}{2}+c\right)\beta^{-1}w + \lambda\beta^q |w|^{\alpha-1}w\right] \\ &= \left(\frac{N-2}{2}+c\right)\beta^{-1}(y\cdot\nabla w)w + \lambda\beta^q(y\cdot\nabla w)|w|^{\alpha-1}w \\ &= \left(\frac{N-2}{2}+c\right)\beta^{-1}(y\cdot\nabla G_1) + \lambda\beta^q(y\cdot\nabla G_2).\end{aligned}$$

The boundary component L_ϵ is contained in the flat space $\mathbb{R}^{N-1}$, where $y_N = 0$. Using $'$ to denote the variables and operators in $N-1$ dimensions, we observe $y\cdot\nabla\eta = y'\cdot\nabla'\eta$ for any function η. We then need to integrate

$$\int_{L_\epsilon} \beta^p(y'\cdot\nabla'\eta)\,dy'$$

for the pairs $p=-1, \eta = G_1$ and $p = q, \eta = G_2$, recalling that $\beta = \beta(y')$ and noting $d\sigma = dy'$ on Ł$_\epsilon$. Integrating by parts,

$$\begin{aligned}\int_{L_\epsilon}\beta^p(y'\cdot\nabla'\eta)\,dy' &= \int_{\partial L_\epsilon}\beta^p(y'\cdot\nu)\eta\,dl - \int_{L_\epsilon}\beta^p\mathrm{div}'y'\eta\,dy' - \int_{L_\epsilon}\nabla'\beta^p\cdot y'\eta\,dy' \\ &= C(\epsilon) - (N-1)\int_{L_\epsilon}\beta^p\eta(w)\,d\sigma - p\int_{L_\epsilon}\beta^{p-1}\nabla'\beta\cdot y'\eta\,dy'.\end{aligned}$$

When $y_N = 0$, $\beta = |y'|^2 + 1/4$, and so $\nabla'\beta = 2y'$, giving

$$\nabla'\beta\cdot y' = 2\left|y'\right|^2 = 2(\beta - 1/4).$$

Then

$$\int_{L_\epsilon}\beta^p(y'\cdot\nabla'\eta)\,dy' = C(\epsilon) - (N-1+2p)\int_{L_\epsilon}\beta^p\eta(w)\,d\sigma + \frac{p}{2}\int_{L_\epsilon}\beta^{p-1}\eta(w)\,d\sigma. \quad (3.32)$$

Applying (3.32) with $p=-1$ and $\eta(w) = G_1(w) = w^2/2$ gives

$$\begin{aligned}\left(\frac{N-2}{2}+c\right)\int_{L_\epsilon}\beta^{-1}(y\cdot\nabla G_1)\,d\sigma = C_1(\epsilon) &- \left(\frac{N-2}{2}+c\right)\frac{N-3}{2}\int_{L_\epsilon}\beta^{-1}w^2\,d\sigma \\ &- \left(\frac{N-2}{2}+c\right)\frac{1}{4}\int_{L_\epsilon}\beta^{-2}w^2\,d\sigma, \quad (3.33)\end{aligned}$$

where

$$C_1(\epsilon) = \left(\frac{N-2}{2} + c\right) \int_{\partial L_\epsilon} \beta^{-1}(y' \cdot \nu') \frac{w^2}{2}\, dl. \tag{3.34}$$

Applying (3.32) with $p = q$ and $\eta(w) = G_2(w) = |w|^{\alpha+1}/(\alpha+1)$ gives

$$\begin{aligned} \lambda \int_{L_\epsilon} \beta^q (y \cdot \nabla G_2)\, d\sigma = C_2(\epsilon) - \frac{N-1+2q}{\alpha+1} \lambda \int_{L_\epsilon} \beta^q |w|^{\alpha+1}\, d\sigma \\ + \frac{q}{2(\alpha+1)} \lambda \int_{L_\epsilon} \beta^{q-1} |w|^{\alpha+1}\, d\sigma, \end{aligned} \tag{3.35}$$

where

$$C_2(\epsilon) = \frac{\lambda}{\alpha+1} \int_{\partial L_\epsilon} \beta^q (y' \cdot \nu') |w|^{\alpha+1}\, dl. \tag{3.36}$$

Putting (3.30), (3.33), and (3.35) into (3.28),

$$\begin{aligned} A(\epsilon) + C_1(\epsilon) - \left(\frac{N-2}{2} + c\right) \frac{N-3}{2} \int_{L_\epsilon} \beta^{-1} w^2\, d\sigma - \left(\frac{N-2}{2} + c\right) \frac{1}{4} \int_{L_\epsilon} \beta^{-2} w^2\, d\sigma \\ + C_2(\epsilon) - \frac{N-1+2q}{\alpha+1} \lambda \int_{L_\epsilon} \beta^q |w|^{\alpha+1}\, d\sigma + \frac{q}{2(\alpha+1)} \lambda \int_{L_\epsilon} \beta^{q-1} |w|^{\alpha+1}\, d\sigma \\ = \frac{2-N}{2} \left(\frac{N-2}{2} + c\right) \int_{L_\epsilon} \beta^{-1} w^2\, d\sigma + \frac{2-N}{2} \lambda \int_{L_\epsilon} \beta^q |w|^{\alpha+1}\, d\sigma + B(\epsilon). \end{aligned}$$

Combining all the like terms and recalling $q = ((N-2)\alpha - N)/2$, this simplifies to

$$\begin{aligned} \left(\frac{N-2}{2} - \frac{N-1}{\alpha+1}\right) \lambda \int_{L_\epsilon} \beta^{q-1} \left(\frac{1}{2} - \beta\right) |w|^{\alpha+1}\, d\sigma \\ = \frac{1}{2} \left(\frac{N-2}{2} + c\right) \int_{L_\epsilon} \beta^{-2} \left(\frac{1}{2} - \beta\right) w^2\, d\sigma \\ - A(\epsilon) + B(\epsilon) - C_1(\epsilon) - C_2(\epsilon). \end{aligned} \tag{3.37}$$

Finally, we claim

$$\int_{L_\epsilon} \beta^{q-1} \left(\frac{1}{2} - \beta\right) |w|^{\alpha+1}\, d\sigma = \int_{L} \beta^{q-1} \left(\frac{1}{2} - \beta\right) |w|^{\alpha+1}\, d\sigma + O(\epsilon), \tag{3.38}$$

$$\int_{L_\epsilon} \beta^{-2} \left(\frac{1}{2} - \beta\right) w^2\, d\sigma = \int_{L} \beta^{-2} \left(\frac{1}{2} - \beta\right) w^2\, d\sigma + O(\epsilon) \tag{3.39}$$

$$C_1(\epsilon) = O(\epsilon), \quad C_2(\epsilon) = O(\epsilon). \tag{3.40}$$

$$B(\epsilon) = O(\epsilon) \tag{3.41}$$

$$A(\epsilon) = C^2 + O(\epsilon) \tag{3.42}$$

Then (3.37) becomes

$$\left(\frac{N-2}{2}-\frac{N-1}{\alpha+1}\right)\lambda\int_L \beta^{\frac{(N-2)\alpha-N-2}{2}}\left(\frac{1}{2}-\beta\right)|w|^{\alpha+1}\,d\sigma=-C^2$$
$$+\frac{1}{2}\left(c+\frac{N-2}{2}\right)\int_L \beta^{-2}\left(\frac{1}{2}-\beta\right)w^2\,d\sigma+O(\epsilon),$$

and letting $\epsilon\to 0$ gives (3.26). It remains only to show claims (3.38) through (3.42).

Claims (3.38) and (3.39) follow since w is continuous on L and the measure of $L\backslash L_\epsilon$ is bounded by $O(\epsilon)$. The same argument applies to (3.40). To show (3.41) and (3.42), we need to appeal to Theorem 8.

Estimate (3.41) follows directly from the expansion (3.24). Along R_ϵ, ν is parallel to the r direction vector, and so $w\frac{\partial w}{\partial \nu}$ is $O(1)$ in a neighborhood about each point on R. As the size of R_ϵ is bounded by $O(\epsilon)$, this gives that $B(\epsilon)$ is bounded by $O(\epsilon)$.

To establish (3.42), we use (3.24) and directly compute $A(\epsilon)$. About each point $y\in\partial L$ there is a neighborhood $U(y)$ in which (3.23) holds. Choose $U_1,\ldots,U_k$ to be a finite subcover of the covering $\cup_{y\in\partial L}U(y)$. Let $\{\rho_j\}$ be a partition of unity subordinate to the subcovering $\{U_j\}$, and set $E_j=R_\epsilon\cap U_j$. For computational purposes, we will make sure that for sufficiently small ϵ, E_j has the following geometry. Let $Y_j=\partial L\cap U_j$, and assume $E_j=\{y\in\mathbb{R}^N_+ : \operatorname{dist}(y,Y_j)=\epsilon,\}$, so that E_j is a portion of a tube whose core is the manifold ∂L. This ensures that the parameterization of the integral is natural in the coordinates $\{z',r,\theta\}$ used in Lemma 18. We can assume such a form for E_j because R_ϵ was chosen to be equidistant from ∂L, and we may choose the neighborhoods $\{U_j\}$ to have tubular geometry.

Let

$$\iota(w)=(y\cdot\nabla w)(\nabla w\cdot\nu)-\frac{1}{2}(y\cdot\nu)\left\|\nabla w\right\|^2$$

denote the integrand for $A(\epsilon)$. We have

$$
\begin{aligned}
A(\epsilon) &= \int_{R_\epsilon} \iota(w)\, d\sigma \\
&= \int_{R_\epsilon} \iota(w) \sum_{j=1}^{k} \rho_j \, d\sigma \\
&= \sum_{j=1}^{k} \int_{R_\epsilon \cap U_j} \iota(w) \rho_j \, d\sigma \\
&= \sum_{j=1}^{k} \int_{E_j} \iota(w) \rho_j \, d\sigma.
\end{aligned}
$$

In order to compute the integrals over E_j, we express $\iota(w)$ in terms of $\{z', r, \theta\}$. Fix $y^0 \in Y_j$, and choose $\{z', r, \theta\}$ as in Lemma 17 (these coordinates depend on y^0 and the shape of ∂L). Let e_N be the unit vector in the y_N direction (which is also the z_N direction, and choose $\hat{n}$ to be the outward normal to ∂L at the point y^0 (that is, $\hat{n}$ is the unit vector in the direction of z_{N-1}). Recall z' is a coordinate lying on the manifold ∂L.

For $y \in E_j$, the vector $y - y^0$ lies in the $\hat{n} \times e_N$ plane and is parallel to the normal vector ν. We express

$$
\begin{aligned}
y - y^0 &= ((y - y^0) \cdot \hat{n})\hat{n} + ((y - y^0) \cdot e_N) e_N \\
&= z_{N-1}\hat{n} + z_N e_N \\
&= r \cos\theta \hat{n} + r \sin\theta e_N.
\end{aligned}
$$

By the definition of $\hat{n}$, we have $y^0 = \left|y^0\right| \hat{n}$, so

$$
y = \left(r \cos\theta + \left|y^0\right|\right) \hat{n} + r \sin\theta e_N.
$$

The normal vector ν is pointing outward to Ω_ϵ and therefore inward to R_ϵ, and so ν is a unit vector directed opposite of $y - y^0$,

$$
\nu = -\cos\theta \hat{n} - \sin\theta e_N.
$$

We similarly express ∇w

$$
\begin{aligned}
\nabla w &= \nabla' w + \frac{\partial w}{\partial z_{N-1}} \hat{n} + \frac{\partial w}{\partial z_N} e_N \\
&= \nabla' w + \left(\cos\theta \frac{\partial w}{\partial r} - \frac{\sin\theta}{r} \frac{\partial w}{\partial \theta}\right) \hat{n} + \left(\sin\theta \frac{\partial w}{\partial r} + \frac{\cos\theta}{r} \frac{\partial w}{\partial \theta}\right) e_N,
\end{aligned}
$$

where $\nabla' w$ is the vector $\text{proj}_T \nabla w$ where T is the tangent plane of ∂L at y^0. This gives

$$\|\nabla w\|^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 + \left\|\nabla' w\right\|^2.$$

We then compute

$$\begin{aligned}
y \cdot \nabla w &= \left(r + \left|y^0\right| \cos\theta\right) \frac{\partial w}{\partial r} - \frac{\sin\theta \left|y^0\right|}{r} \frac{\partial w}{\partial \theta} \\
\nabla w \cdot \nu &= -\frac{\partial w}{\partial r} \\
y \cdot \nu &= -r.
\end{aligned}$$

Then the integrand becomes

$$\begin{aligned}
-\frac{r + \left|y^0\right|\cos\theta}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{\left|y^0\right|\sin\theta}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta} + \frac{r + \left|y^0\right|\cos\theta}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 \\
- \frac{-r + \left|y^0\right|\cos\theta}{2}\left\|\nabla' w\right\|^2.
\end{aligned}$$

We then plug in the expansions for w. From (3.24),

$$\begin{aligned}
w(z', r, \theta) &= C(z') r^{1/2} \sin(\theta/2) + O(r^\gamma) \\
w_r(z', r, \theta) &= \frac{1}{2} C(z') r^{-1/2} \sin(\theta/2) + O(r^{\gamma - 1}) \\
w_\theta(z', r, \theta) &= \frac{1}{2} C(z') r^{1/2} \cos(\theta/2) + O(r^\gamma) \\
\nabla' w(z', r, \theta) &= \nabla' C(z') r^{1/2} \sin(\theta/2) + O(r^\gamma),
\end{aligned}$$

where $\gamma > 1/2$. The integrand will be evaluated at $r = \epsilon$, and will be multiplied by an additional factor of ϵ from the surface area element $(r d\theta)$. Therefore we are only interested in the terms of order $1/r$ in the integrand. Putting in the expansions gives

$$\begin{aligned}
\frac{C(z')^2 \left|y^0\right|}{r} &\left(-\frac{1}{8}\cos\theta \sin^2(\theta/2) + \frac{1}{8}\cos\theta\cos^2(\theta/2) + \frac{1}{4}\sin(\theta/2)\cos(\theta/2)\sin\theta\right) + O(1) \\
&= \frac{C(z')^2 \left|y^0\right|}{r}\left(\frac{1}{8}\cos\theta\cos\theta + \frac{1}{8}\sin\theta\sin\theta\right) + O(1) \\
&= \frac{C(z')^2 \left|y^0\right|}{8r} + O(1).
\end{aligned}$$

Then

$$\begin{aligned}\int_{E_j} &= \int_{Y_j}\int_0^{\pi}\left(\frac{C(z')^2\left|y^0\right|}{r} + O(1)\right)\rho_j\, \epsilon d\theta dz' \\ &= \int_{Y_j}\frac{C(z')^2\left|y^0\right|\pi}{8}\rho_j\, dz' + O(\epsilon) \\ &= C_j^2 + O(\epsilon).\end{aligned}$$

Note that z' depends on the point y^0, but we do not need to know the nature of this dependence since we only care about the sign of the integral. Note also that $\rho_j \geq 0$. Summing over j gives

$$A(\epsilon) = C^2 + O(\epsilon)$$

for some C independent of ϵ, establishing (3.42) and completing the proof. □

Theorem 9 implies our main nonexistence result. The Pohozaev Identity (3.26) will contradict supercritical solutions to the problem (3.12), which by Proposition 3 contradicts supercritical solutions to (1.1). The details are in the following Corollary.

Recall that we only having a solution $w \in H^1_{loc}$ to (3.12) is not enough to establish the Pohozaev identity, as we are not able to establish the expansions (3.24) only knowing $w \in H^1_{loc}$. We need to further assume, as in the hypotheses of Theorem 8,

$$\begin{cases} w \in H^1_{loc}(\mathbb{R}^N_+) \cap C\left(\overline{\mathbb{R}^N_+}\right) \\ \qquad\qquad |w_r| \leq Cr^{\delta-1}, \end{cases} \tag{3.43}$$

where $\delta > 0$ and r is the two-dimensional coordinate used in Theorem 8.

Corollary 2. *Suppose Ω is the unit ball in $\mathbb{R}^N$, and Γ is contained in the upper hemisphere $\{x \in \partial\Omega, x_N > 0\}$. If $\alpha \geq \alpha^* - 1$ and $c < -\frac{N-2}{2}$, then problem (1.1) has no nontrivial solution $u \in H$ whose transformation w described in Proposition 3 and Proposition 5 satisfies (3.43).*

Proof. We proceed by contradiction. Suppose there were a function $u \neq 0$ satisfying (3.43) and solving (1.1). Then by Proposition (5) we have a function $w \in H^1_{loc}(\mathbb{R}^N_+)$ solving (3.12). By Theorem 9, w satisfies (3.26). We observe

$$\Phi\left(\{x \in \mathbb{R}^N : |x| = 1, x_N > 0\}\right) = \left\{y \in \mathbb{R}^N : y_N = 0, \left|y'\right|^2 < \frac{1}{4}\right\}.$$

Then $L = \Phi(\Gamma)$ is contained in $\{y \in \mathbb{R}^N : y_N = 0, |y'|^2 < 1/4\}$, and so we have

$$\frac{1}{2} - \beta = \frac{1}{2} - \left(|y'|^2 + \frac{1}{4} \right) = \frac{1}{4} - |y'|^2 > 0$$

on L. Since $\lambda > 0$ and $\alpha + 1 \geq \alpha^* = \frac{2(N-1)}{N-2}$, the left-hand side of (3.26),

$$\begin{aligned} \left(\frac{N-2}{2} - \frac{N-1}{\alpha+1} \right) \lambda \int_L \beta^{\frac{(N-2)\alpha - N - 2}{2}} \left(\frac{1}{2} - \beta \right) |w|^{\alpha+1} \, d\sigma = -C^2 \\ + \frac{1}{2} \left(c + \frac{N-2}{2} \right) \int_L \beta^{-2} \left(\frac{1}{2} - \beta \right) w^2 \, d\sigma, \end{aligned}$$

is nonnegative. As $c < -\frac{N-2}{2}$, and w is nonzero, the right-hand side of (3.26) is strictly negative, giving a contradiction. $\square$

We remark that we have only established (3.26) when $c < -\frac{N-2}{2}$. It is still expected that the identity holds for $c \geq -\frac{N-2}{2}$, though to show this one would have to establish (3.41) and (3.42) without resorting to the expansions established in Theorem 8, or show those expansions hold for $c \geq -\frac{N-2}{2}$.

We have also only contradicted the existence of solutions which are continuous up to the boundary and whose derivative does not behave too poorly. It still remains to show that there are solutions to (1.1) that satisfy conditions (3.43). We believe that this is the case, though establishing regularity results for nonlinear mixed boundary value problems is tricky. We do believe, however, that the solutions constructed in Chapter 2 do satisfy (3.43), though showing this is still an open question. The plots constructed in Chapter 4 do suggest the solutions have the desired amount of regularity.

Chapter 4

Numerical Results

In this section, we provide some plots of numerical approximations to solutions of (1.1) in the case when Ω is the unit sphere in $\mathbb{R}^3$, Γ is the upper hemisphere ($x_3 > 0$) and Γ_0 is the lower hemisphere. These approximations help to illustrate the dependence of the solution structure of (1.1) on the parameters α, λ, c.

In order to generate an approximation to a solution of (1.1), we consider a spherical harmonic expansion of a potential solution,

$$u(\rho, \theta, \phi) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} A_{k,m} \rho^k P_k^m(\cos\theta) e^{im\phi}, \tag{4.1}$$

where P_k^m are the associated Legendre polynomials, and ρ, θ, ϕ are standard spherical coordinates with azimuthal angle $\theta \in [0, \pi]$ and polar angle $\phi \in [0, 2\pi)$. In order to construct a solution, we terminate the expansion (4.1) at some finite $M \in \mathbb{N}$,

$$u_M(\rho, \theta, \phi) = \sum_{k=0}^{M} \sum_{m=-k}^{k} A_{k,m} \rho^k P_k^m(\cos\theta) e^{im\phi}, \tag{4.2}$$

and find the $(M+1)^2$ unknowns $\{A_{k,m}\}$ such that $u_M = 0$ on the lower hemisphere and $\partial u_M / \partial \nu = c u_M + \lambda |u_M|^{\alpha-1} u_M$ on the upper hemisphere. As we have $(M+1)^2$ unknowns, we must construct $(M+1)^2$ independent constraints based on the boundary data. We construct these constraints through a collocation method: we select $(M+1)^2$ points along the unit sphere, and require the boundary conditions are satisfied at each collocation point. We then have $(M+1)^2$ equations, which we solve using Newton's Method.

The collocation points were selected through the following rule. As the points lie on the unit sphere, it is unwise to select evenly distributed points in $[0, \pi] \times [0, 2\pi)$, which would cause bunching towards the poles. Instead we select points that are distributed fairly evenly with respect to surface area. To accomplish this, we selected θ_j by

$$\theta_j = \arccos(z_j)$$

where $\{z_j\}$ are the $(M+1)$ roots of the $(M+1)$ order Legendre polynomial on the interval $[-1,1]$. We then select $\{\phi_l\}$ in a way to evenly distribute the collocation points with respect to surface area (so more points are at the equator than the poles). For each j, the circumference of the circle at constant angle θ_j is $2\pi\sin(\theta_j)$. The number of points at the angle θ_j should then be the same proportion of the total number of points as the ratio of the circumference at θ_j to the sum of all the circumferences (that is, the discrete approximation of the surface area). This is given by

$$n_j = (M+1)^2 \frac{\sin(\theta_j)}{\sum_{l=1}^{M+1} \sin(\theta_l)},$$

which must then be rounded and corrected for a total of $(M+1)^2$ points. We choose n_j evenly spaced points in $[0, 2\pi)$ offset by some irrational number (in our case, we chose $1/\sqrt{1+j}$) to avoid evaluating at points where the sine or cosine would vanish. Doing this for all j provides $(M+1)^2$ points for $\{\phi_l\}$. Finally, we repeat each θ_j n_j times so that $\{\theta_j\}$ contains $(M+1)^2$ points as well, and select the collocation points

$$\{(\theta_j, \phi_j)\}_{j=1}^{(M+1)^2}. \tag{4.3}$$

We define our constraining function $F_j^{\alpha,\lambda,c,M}$ as follows

$$F_j^{\alpha,\lambda,c,M}(X) = \begin{cases} \sum_{k=1}^{M} \sum_{m=-k}^{k} k A_{k,m} P_k^m(\cos\theta_j) e^{im\phi_j} - c u_M(1,\theta_j,\phi_j) & \\ \qquad -\lambda \left| u_M(1,\theta_j,\phi_j) \right|^{\alpha-1} u_M(1,\theta_j,\phi_j) & \text{if } 0 \le \theta_j < \frac{\pi}{2} \\ u_M(1,\theta_j,\phi_j) & \text{if } \frac{\pi}{2} \le \theta_j \le \pi, \end{cases} \tag{4.4}$$

where u_M is given by 4.2 and X represents the $(M+1)^2$ component array $\{A_{k,m}\}_{k=0,m=-k}^{k=M,m=k}$. We then apply Newton's Method to find a collection of coefficients X so that

$$F_j^{\alpha,\lambda,c,M}(X) = 0, \tag{4.5}$$

which is equivalent to u_M satisfying the boundary conditions.

In order to express the problem in such a way to apply Newton's Method, we reindex $X_1 = A_{0,0}, X_2 = A_{1,-1}, X_3 = A_{1,0}, X_4 = A_{1,1}, \ldots X_{(M+1)^2} = A_{M,M}$. That is, we have a new index $s(k,m) = k(k+1)+m+1$. The derivative matrix $DF^{\alpha,\lambda,c,M}$ used in each iteration of Newton's Method is given by

$$\begin{aligned}\left[DF^{\alpha,\lambda,c,M}\right]_{j,s} &= \frac{\partial F_j^{\alpha,\lambda,c,M}}{\partial X_s} \\ &= \begin{cases} (k(s)-c)P_{k(s)}^{m(s)}(\cos\theta_j)e^{im(s)\phi_j} \\ -\alpha\lambda\left|u_M(1,\theta_j,\phi_j)\right|^{\alpha-1} P_{k(s)}^{m(s)}(\cos\theta_j)e^{im(s)\phi_j} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } 0 \le \theta_j < \frac{\pi}{2} \\ P_{k(s)}^{m(s)}(\cos\theta_j)e^{im(s)\phi_j} \qquad\qquad \text{if } \frac{\pi}{2} \le \theta_j \le \pi, \end{cases}\end{aligned} \tag{4.6}$$

where $k(s) = \lfloor\sqrt{s-1}\rfloor$ and $m(s) = s - k(s)(k(s)+1) - 1$. We then proceed by the standard multivariate Newton's Method. We choose an initial guess $X^0 = \left(X_1^0, \ldots, X_{(M+1)^2}^0\right)$ and define

$$X^t = X^{t-1} - \left[DF^{\alpha,\lambda,c,M}(X^{t-1})\right]^{-1} F^{\alpha,\lambda,c,M}(X^{t-1}) \tag{4.7}$$

for $t \ge 1$. We terminate when either t exceeds some arbitrary maximum t_{max} (we usually take $t_{max} = 30$), or when the value of the constraining function (4.4) falls below some tolerance, usually taken to be 10^{-8}. For many choices of initial guess the value of (4.4) increases very rapidly and the algorithm diverges, so there is some manual searching for a good choice of initial guess so that the value of (4.4) falls below the tolerance within a few (less than 10) iterations.

We construct solutions using the approximation scheme described above, and plot these solutions on the unit sphere $\rho = 1$. For simplicity in plotting, we only consider the dependence on the azimuthal angle θ. Many of the solutions constructed are independent of the polar angle ϕ anyway (mostly due to the choice of initial guess), and so there is nothing lost by only considering such a cross-section. Every plot is plotting the value of the function $u(1,\theta,0)$ on the vertical axis against the azimuthal angle $0 \le \theta \le \pi$ on the horizontal axis.

This section is broken up into four subsections. First, we use this method to estimate the first Steklov eigenvalue μ_1. The following subsections consider the value of the exponent α: subcritical α (we expect solutions for all values of parameters), critical α (we expect solutions for certain values of parameters), and supercritical α (we do not expect solutions).

All computations were done using MATLAB and the plots were done in Maple.

4.1 Estimation of First Steklov Eigenvalue

Recall the Steklov eigenfunction ϕ_1 and eigenvalue μ_1 are defined by

$$\begin{cases} \Delta\phi_1 = 0 & \text{in } \Omega \\ \phi_1 = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial \phi_1}{\partial \nu} = \mu_1 \phi_1 & \text{on } \Gamma. \end{cases}$$

If ϕ_1 were a constant then ϕ_1 would have to be identically zero, as Γ_0 is nonempty. Then $\mu_1 > 0$, but the value of μ_1 is not immediately obvious. This value is of particular interest because if $c > \mu_1$, then no positive solution to (1.1) may exist (Theorem 7).

We can use a similar numeric idea as in the prequel to estimate μ_1. We express the boundary condition $\frac{\partial u}{\partial \nu} = \mu u$ on Γ, $u = 0$ on Γ_0, as an eigenvalue problem

$$AX = \mu BX,$$

where $A = (a_{rs})$ and $B = (b_{rs})$ are given by

$$a_{rs} = \begin{cases} k(s) P_{k(s)}^{m(s)}(\cos\theta_j) e^{im(s)\phi_j} & \text{if } 0 \le \theta \le \frac{\pi}{2} \\ P_{k(s)}^{m(s)}(\cos\theta_j) e^{im(s)\phi_j} & \text{if } \frac{\pi}{2} < \theta \le \pi \end{cases}$$

and

$$b_{rs} = P_{k(s)}^{m(s)}(\cos\theta_j) e^{im(s)\phi_j},$$

and $X = (X_s)$. The points (θ_j, ϕ_j) are the collocation points from (4.3). Then the j^{th} component of BX is exactly $u_M(1, \theta_j, \phi_j)$, and the j^{th} component of AX is $\frac{\partial u_M}{\partial \nu}(1, \theta_j, \phi_j)$ if (θ_j, ϕ_j) lies in the upper hemisphere and $u_M(1, \theta_j, \phi_j)$ if (θ_j, ϕ_j) lies in the lower hemisphere.

If $\mu \neq 1$ satisfies $AX = \mu BX$ for some X, then the vector X must give a solution which vanishes on the lower hemisphere, because the $AX = \mu BX$ implies $u_M = \mu u_M$

at points in the lower hemisphere. We therefore find the eigenvalues for $AX = \mu BX$, given by $\det(A - \mu B) = 0$. The smallest nonzero eigenvalue, if it is not equal to 1, will be an approximation of μ_1. Using MATLAB to compute the eigenvalues, this gives $\mu_1 \approx 0.67$ (taking $M = 14$).

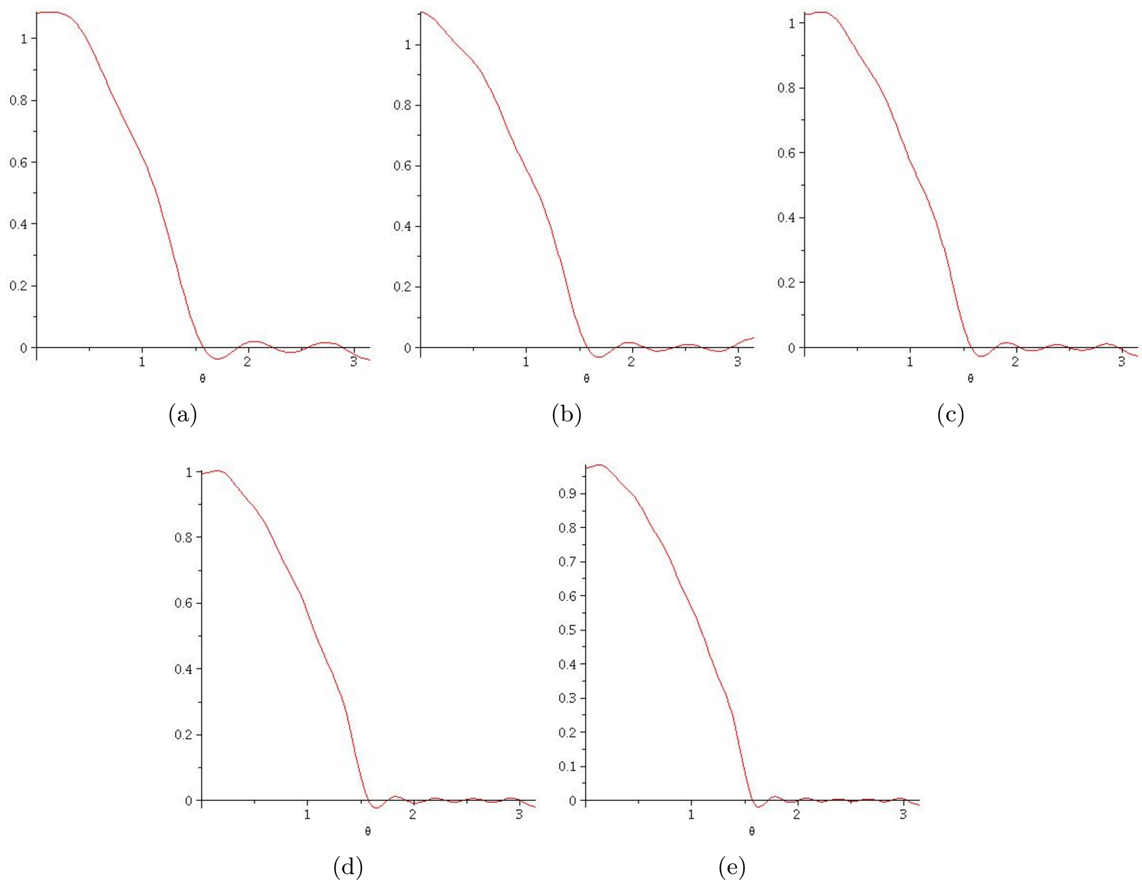
4.2 Numerical Results in the Subcritical Case

In Section 2.1, we showed that problem (1.1) admits nontrivial solutions for any $1 < \alpha < \alpha^* - 1$ ($\alpha^* = 4$ in three dimensions), any $c \in \mathbb{R}$, and any $\lambda > 0$. Using the method described in the introduction, we construct approximate solutions to (1.1) for various values of the parameters.

We first look at the solutions for the particular case $\alpha = 1.5$, $\lambda = 0.9$, and $c = 0$. We plot the solutions using the values $M = 8, 10, 12, 16, 20$ in (4.2). This corresponds to $81, 121, 169, 289$, and 400 terms in the spherical harmonic expansion, respectively. Using M to be much larger proves to be problematic for two reasons. First, as the number of terms in the expansion is $(M+1)^2$, the running time for the algorithm becomes large fairly quickly. Secondly, we must use $(M+1)^2$ collocation points chosen on the unit sphere according to (4.3). As this number gets larger, the derivative matrix (4.6) becomes more rank deficient, presumably because the collocation points begin to cluster. The plots are shown in Figure 4.1.

We do not perform a rigorous analysis of the convergence of algorithm, but rather assuming heuristically that if Newton's Method converges (in the sense of (4.4) falling below the tolerance) for some initial guess X^0 and some M, and if as M gets large and the figures vary less and less, then the series $\{u_M\}$ is converging in $H^1(\Omega)$. If we cannot find an initial guess X^0 for which Newton's Method converges, or if Newton's Method does converge but as we increase M and the plots do not seem to converge, then we take this to imply that the algorithm is not providing a spherical harmonic expansion that converges in H^1, and therefore does not give a solution to (1.1).

The plots in Figure 4.1 appear to converge to a solution which does not change sign, the existence of which was shown in Section 2.1.1. This is primarily due to the



(a) (b) (c) (d) (e)

Figure 4.1: $\alpha = 1.5,\ \lambda = 0.9,\ c = 0,\ (a) : M = 8, (b) : M = 10, (c) : M = 12, (d) : M = 16, (e) : M = 20.$

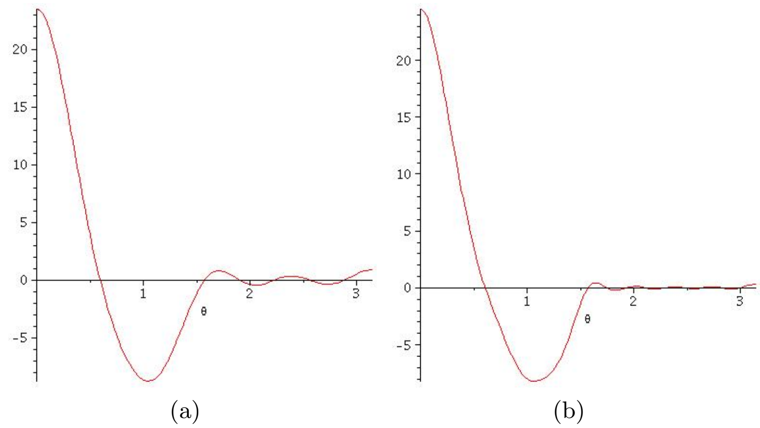
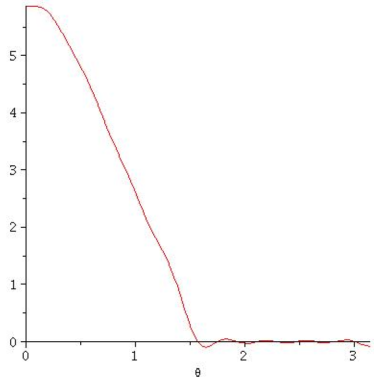


Figure 4.2: $\alpha = 1.5$, $\lambda = 0.9$, $c = 0$, $(a) : M = 8, (b) : M = 16$.

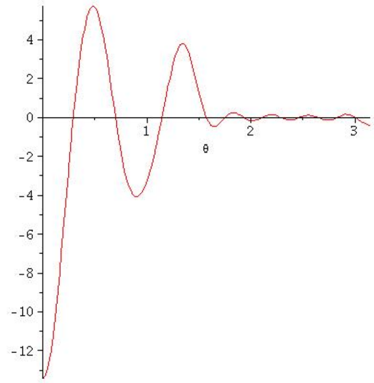
choice of initial guess, as we have chosen the initial guess to have $X_3 = 1$ and the other components zero. This corresponds to the first nonconstant spherical harmonic, which does not change sign. Moreover, as the positive solution constructed in Section 2.1.1 corresponds to the first Steklov eigenfunction (which does not change sign) with smallest eigenvalue μ_1, it seems reasonable that this solution is the most stable, and thus would be found by the algorithm in most cases. We can, however, find other (sign-changing) solutions with the same parameters by varying the choice of initial guess. We select $X_7 = 1$ instead of X_3, which corresponds to the next spherical harmonic. These are shown in Figure 4.2.

From Theorem 7 we know that when $c > \mu_1$, no positive solution should exist, and so if our algorithm converges, the solution found should be sign-changing. Indeed this is the case. We estimate $\mu_1 \approx 0.67$, and plot solutions for different c: $c = -1, c = 0.6, c = 5$. In the case when $c = 5 > \mu_1$, we are unable to find an initial guess for which our algorithm converges to a solution which does not change sign. The algorithm does converge to a sign-changing solution. When $c = 0.6$, $c < \mu_1$ but is close to μ_1, and the solution seems to be relatively small in size compared to the solution when $c = 0$. This is reasonable, as when $c = \mu_1$ the only solution which does not change sign is the trivial solution (Theorem 7), and so we may expect that as $c \to \mu_1^-$ the positive solution is very small. These plots are shown in Figure 4.3

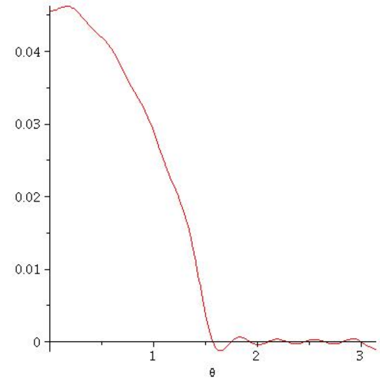
From Section 2.3, we see that the parameter $\lambda > 0$ is only a scaling factor, and as



(a)



(b)



(c)

Figure 4.3: $\alpha = 1.5$, $\lambda = 0.9$, $M = 12$, $(a) : c = -1, (b) : c = 5, (c) : c = 0.6$.

$\lambda \to 0^+$, there is a solution which becomes larger and larger on the order of $\lambda^{\frac{1}{1-\alpha}}$. The following plots using values of λ decreasing to 0 exemplify the results of Theorem 6. We see from the relative sizes of the solution that the scaling $\lambda^{\frac{1}{1-\alpha}}$ is consistent. In particular, the solution when $\alpha = 1.5$ and $\lambda = 0.1$ should be $(0.1/0.9)^{1/-0.5} = 81$ times the solution when $\alpha = 1.5$ and $\lambda = 0.9$. Indeed, the solution in Figure 4.4d appears to be 81 times the size of that of Figure 4.1c.

Lastly, we examine what happens as α increases to the critical value 3 (that is, $\alpha+1$ approaches $\alpha^* = 4$). The solutions appear to grow in value, which we may expect as the boundary data is larger whenever $|u| > 1$. These are showin in Figure 4.5.

4.3 Numerical Results in the Critical Case

Here we take $\alpha = 3$, so $\alpha+1 = 4 = \alpha^*$. According to Theorem 5, there is a (nonnegative) solution to (1.1) whenever $-1/2 < c < \mu_1$, and $\lambda > 0$. As such we expect our algorithm to converge to an apparent solution, and indeed this is the case. Note that we did not prove the existence of sign-changing solutions, but there is no reason to believe these do not exist (in fact, the construction in Section 2.2 could have provided a solution that changes sign; no part of the argument excluded that possibility). Solutions are shown for $\alpha = 3$, $\lambda = 0.9$, $c = 0.4$, and $M = 12$ in Figure 4.6a.

When $c > \mu_1$, we have shown (1.1) cannot have positive solutions, though we have no information regarding the existence of solutions which could change sign. Sign-changing

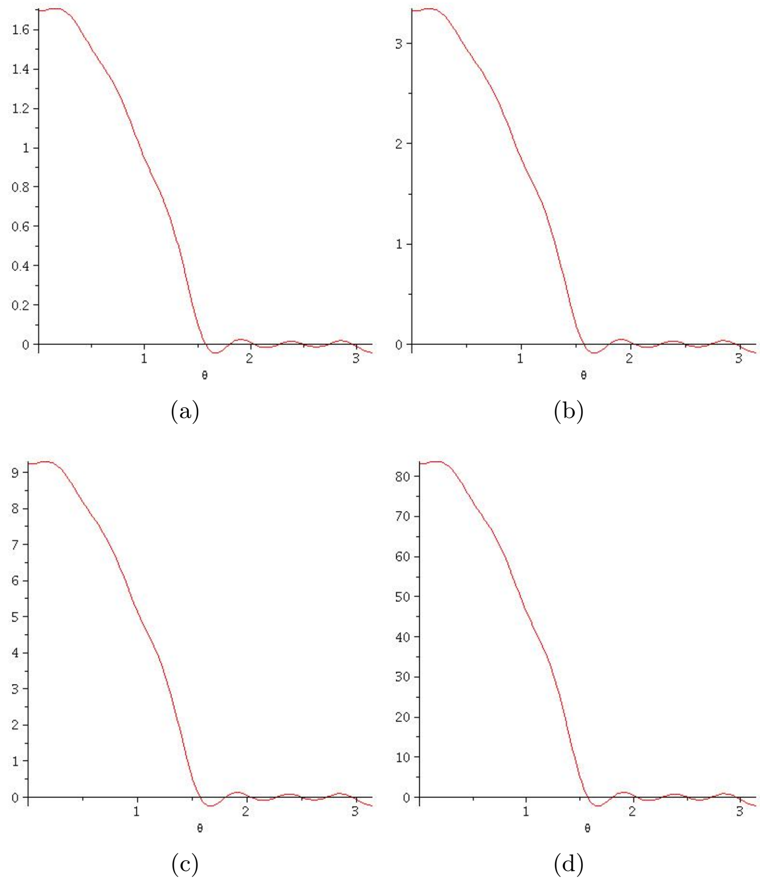


Figure 4.4: $\alpha = 1.5,\ M = 12,\ c = 0\ (a) : \lambda = 0.7, (b) : \lambda = 0.5, (c) : \lambda = 0.3, (d) : \lambda = 0.1$.

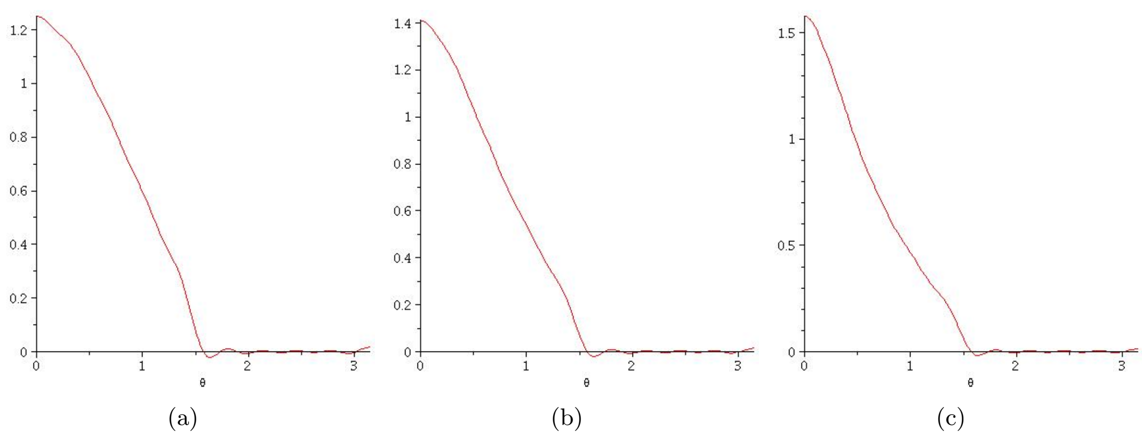
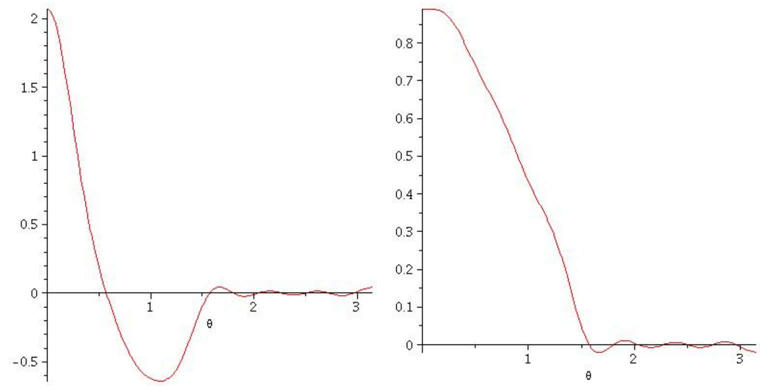


Figure 4.5: $M = 18$, $c = 0$, $\lambda = 0.9$, (a): α 2, (b): $\alpha = 2.5$, (c): $\alpha = 2.9$.



(a) (b)

Figure 4.6: $\alpha = 1.5$, $\lambda = 0.9$, $M = 12$, $(a) : c = 2, (b) : c = 0.4$.

solutions appear to exist, and such a solution for $c = 2$ is Figure 4.6b.

When $c < -1/2$, Corollary 2 disallows the existence of any solution in the critical or supercritical case, and so we expect our algorithm to fails to provide a solution. Taking $c = -1$ (for $\alpha = 3, \lambda = 0.9$) fails to find a solution for a number of choices of initial guess and various M.

4.4 Numerical Results in the Supercritical Case

When $\alpha > 3$, Corollary 2 disallows the existence of any solution when $c < -1/2$. Taking $\alpha = 5$, $\lambda = 0.9$, and $c = -1$ fails to give a solution for many choices of initial conditions and various M.

When $c > -1/2$, Theorem 9 does not provide a contradiction to the existence of a nontrivial solution. It is unclear whether there should be solutions or if there is some separate argument for nonexistence. The possible existence of solutions may be motivated by the fact that the problem

$$\begin{aligned} -\Delta u = \lambda f(u) \quad & \text{in } \Omega \\ u > 0 \quad & \text{in } \Omega \\ u = 0 \quad & \text{on } \partial\Omega \end{aligned} \tag{4.8}$$

admits radial solutions for $f(u) = (1+u)^p$ for large p and even for $f(u) = e^u$, even in dimensions higher than two, where λ is in some range (see [12]). This result may be unexpected as one may consider e^u to be supercritical to the critical Sobolev Embedding exponent. However, $e^u = 1 + u + O(u^2)$, and so problem (4.8) could be seen as a (subcritical) perturbation of the linear problem, and so the expectation of solutions is not entirely unjustified. Or this phenomenon could be because when considering radial solutions, (4.8) is in some sense a one-dimensional problem and as such any exponent would be subcritical. The methods of [12] rely on analyzing the corresponding ordinary differential equation coming from radial solutions to (4.8). These ideas do not carry over to (1.1).

It is still unclear whether we should expect to be able to find supercritical solutions to (1.1) when $c > -1/2$ or if we should expect to find a nonexistence result. We then look to see what happens in our numerical method in this case.

There is a curious numeric behavior in the supercritical case when $c > -1/2$. Figure 4.7 shows plots for $\alpha = 5, \lambda = 0.9, c = 0$ and $M = 8, 10, 12, 14, 16$. For $M = 8$ this appears to be a solution, but as we increase M the solutions appears to increase (albeit slowly), indicating possible divergence. The previous plots (Figure 4.1 for example) decreased in size as more terms were added. Moreover, the derivative with respect to θ at $\theta = 0$ appears to be increasing as well, and if the spherical harmonic expansion converged in H^1, this derivative would have to be zero. This indicates that the expansion (4.2) is diverging, and these are not H^1 solutions to our problem.

We have not been able to find a choice of initial conditions or parameters (with $c > -1/2$, $\alpha > 3$) so that the plots constructed when the algorithm converges behave in the same manner as those in Chapters 4.2 and 4.3. This is some justification for being able to find a nonexistence result for $c > -1/2$, but is by no means definitive, especially because our numerical method depends greatly on the initial guess.

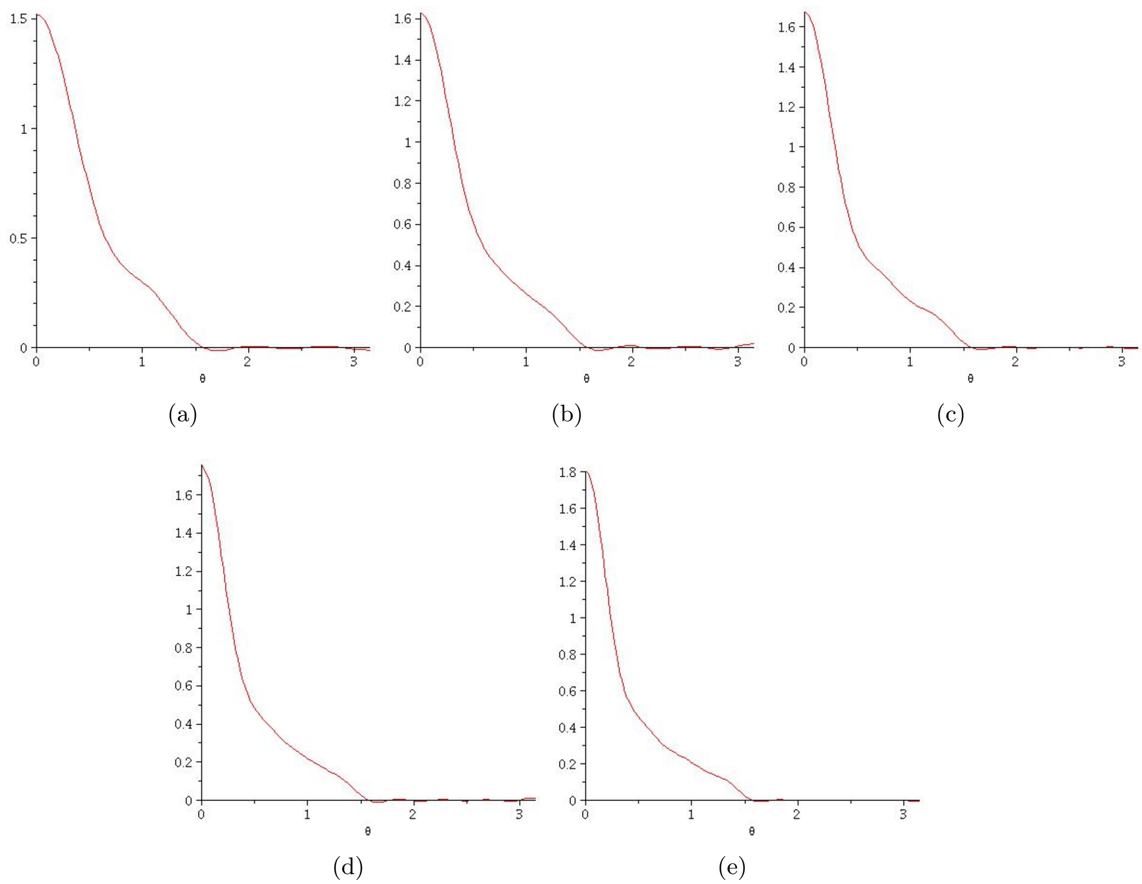


Figure 4.7: $\alpha = 5,\ \lambda = 0.9,\ c = 0,\ (a) : M = 8, (b) : M = 10, (c) : M = 12, (d) : M = 14, (e) : M = 16.$

Chapter 5

Conclusions

We have seen the existence of solutions to (1.1) depends on the parameters α and c. The behavior changes depending on whether α is subcritical, critical, or supercritical. We note that the Pohozaev identity (3.26) has only been shown to be valid when $c < -\frac{N-2}{2}$ and for solutions satisfying (3.43). We suspect, however, that the identity remains true for every c, though it is not clear how to remove this restriction from our proofs. The difficulty arises in estimating the integral (3.29) by considering an expansion of the solution in local coordinates. However, we only ever use the sign of this integral to derive (3.26), and there may be a way to conclude this integral has the proper sign without resorting to coordinates.

When $\alpha < \alpha^* - 1$ (the subcritical case), the left-hand side of (3.26) is negative, and the right-hand side is negative as well when $c < -\frac{N-2}{2}$. Thus there is no contradiction, which is consistent as we have shown existence in Chapter 2.1. Note that (3.5) implies the solutions for $c \geq -\frac{N-2}{2}$ cannot be in $H^2(\Omega)$.

When $\alpha = \alpha^* - 1$ (the critical case), the left-hand side of (3.26) is zero. Then if $c < -\frac{N-2}{2}$ there can be no nontrivial solutions. If $c > -\frac{N-2}{2}$, we do not know that (3.26) applies (and if it did, it would not provide a contradiction). In fact we have shown in Chapter 2.2 that solutions do exist for $-\frac{N-2}{2} < c < \mu_1$. For $c \geq \mu_1$, positive solutions cannot exist, though numerics suggest that sign-changing solutions still exist (see Chapter 4.3). When $c = -\frac{N-2}{2}$, (3.26) was not shown to be valid, though if it were it would imply $C = 0$ in (3.26), and therefore if solutions do exist they have more regularity (from expansion (3.24). The numerical algorithm described in Chapter 4 does not provide a solution when $c = -\frac{1}{2}$ (N=3) for various choices of M and initial guesses, and this suggests that solutions do not exist when $c = -\frac{N-2}{2}$ as well.

If $\alpha > \alpha^* - 1$, then as the left-hand side of (3.26) is positive, there can be no solution when $c < -(N-2)/2$. For $c \geq -\frac{N-2}{2}$, there may exist solutions, though the numerics in Chapter 4.4 suggest there are not.

Proving nonexistence for the region $c \geq -\frac{N-2}{2}$ is still open, as is an analogous nonexistence result if $\Gamma_0 = \emptyset$, as well as the question of developing nonexistence results when Ω is a more general domain (for instance, not conformal to $\mathbb{R}^N_+$).

The condition (3.43) which we have to assume to prove nonexistence is expected to not be restrictive. That is, we expect the solutions which we have shown to exist actually satisfy (3.43), but we do not currently know how to show this.

We summarize the results below. These results hold for any $\lambda > 0$.

$c\backslash\alpha$	α subcritical	α critical	α supercritical
$c < -\frac{N-2}{2}$	solutions exist, and positive solutions exist	no solutions satisfying (3.43) exist	no solutions satisfying (3.43) exist
$c = -\frac{N-2}{2}$	solutions exist, and positive solutions exist	unknown, but we suspect no solutions satisfying (3.43) exist	unknown, but we suspect no solutions satisfying (3.43) exist
$-\frac{N-2}{2} < c < \mu_1$	solutions exist, and positive solutions exist	positive solutions exist	solutions expected not to exist from numerics
$c \geq \mu_1$	solutions exist; positive solutions cannot exist	positive solutions cannot exist; existence of sign-changing solutions expected from numerics	positive solutions cannot exist; solutions suspected not to exist from numerics

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