Description: Investigating an algebraic generalization for building Unifix-cube towers n cubes tall with exactly 2 red cubes from towers with exactly 1 red cube **Parent Tape: Early Algebra Ideas About Binomial Expansion, Stephanie's Interview Seven of Seven** **Date: 1996-04-17** **Location: Union Catholic** **Researcher: Professor Carolyn Maher**	**Transcriber(s): Aboelnaga, Eman** **Verifier(s): DeLeon, Christina** **Date Transcribed: Spring 2009** **Page: 1 of 20**

1	R1	Now. Maybe what we should do is write out the cases that work. We wrote out one of them really very clearly here. [*R1 takes out some papers.*] Right? You did the one here um where you started with um exactly one when it was green and blue in this case, right?
2	Stephanie	Um.
3	R1	And all those that had one green, and you said there were four. And then you said the next step was to build all possible towers with two green. All right. And that's what you did here. And you showed how you had to keep it.
4	Stephanie	Um hm.
5	R1	The one green in the same place so either of these could be green.
6	Stephanie	Um hm.
7	R1	I like that. Okay. So then you said so you multiply by four. What's this? Let's see if I'm reading this right. You say multiply four by one and divide by four.
8	Stephanie	There's stuff on the back. [*R1 turns over each of the papers she is holding.*] No, on the other back.
9	R1	On this. Oh. That helps. Now let's see which one am I doing on this? After doing this with all four of the towers we had twelve towers with two green. That you built here. [*R1 indicates the towers with two red cubes and two yellow cubes.*]
10	Stephanie	Um hm.
11	R1	[*reading*] 'There were some duplications. Each new tower came in a pair so there was only really six new towers. We took the six new towers and from each one, using the same method, created towers of four high with three green.' And that's this stack. *[R1 indicates towers on the table.]* 'We created

		twelve towers, but like before, there were duplicates. This time they came in groups of three.' Here's your groups of three. [*R1 indicates the appropriate towers on the table. She continues reading Stephanie's synopsis.*] 'four new towers and we started to see a pattern.' And this is where you wrote 'the formula looked like this: $\frac{4\ x\ 3}{2} = 6$ Right? Towers four high'
12	Stephanie	Hm.
13	R1	Four and now this is where – maybe it would be helpful to go over again. Let's talk about the formula. Okay? Explain that to us. – The four
14	Stephanie	Wait.
15	R1	Times three over two equals six. – That must be
16	Stephanie	For this one, I guess?
17	R1	for
18	Stephanie	I – uh – for this one? With um – these. And it would be
19	R1	(inaudible) a number one, two, three, four, five, six. So - - How would that work?
20	Stephanie	That was wrong.
21	R1	Is it? Why do you say it's wrong?
22	Stephanie	Wouldn't it be um six times two? Because um
23	R1	Well. Think about what you did. Remember you reorganized them. When

		you first had this one here [*R1 points to* $\begin{bmatrix} R \\ Y \\ Y \\ Y \end{bmatrix}$.] right?
24	Stephanie	Um hm.
25	R1	Okay.
26	Stephanie	Um hm.
27	R1	When you were making two's out of them
28	Stephanie	Um hm.
29	R1	Right? - You had – you kept this [*points to the top red cube of the tower*] the same so you had here and here [*indicates the top and second positions*]. That was one of them, right?
30	Stephanie	Yeah.
31	R1	Then you had here and here [*the top and third positions*]. That was another one. Then you had here and here. [*top and bottom positions*] That was another one. So you got three, didn't you?
32	Stephanie	Yes.
33	R1	You got three.
34	Stephanie	Oh. Okay.
35	R1	Right?
36	Stephanie	Yeah.

37	R1	Isn't that right? And the same thing here. This was the same. [*points to the red cube in the* $\begin{bmatrix}Y\\R\\Y\\Y\end{bmatrix}$ *tower*] But you went here and here. [*points to the top and second positions*] One. Here and here. [*second and third positions*] Two. Here and here. [*second and bottom positions*] Three.
38	Stephanie	Yeah.
39	R1	So for each of these four, you got three.
40	Stephanie	Yeah.
41	R1	Now, can you predict by looking at this row [*pointing to the row of towers four high with one red*] that there would be three that would come when you go from one to two?
42	Stephanie	Yes.
43	R1	What helps you predict?
44	Stephanie	Because there's three places where you can put the second one.
45	R1	Three places. Okay. So -
46	Stephanie	Yes. I think it should -
47	R1	Well. Let's write this down, so we can – so we don't – 'cause it's easy -
48	Stephanie	Alright.
49	R1	It's easy for me when you disassemble it to remember, you know what I'm saying,

50	Stephanie	Um hm.
51	R1	what happened before it was assembled.
52	Stephanie	So what do you want me to write?
53	R1	So let's think. Well, let's think – what makes sense to you? Um. You started with – right? – four towers
54	Stephanie	Four towers.
55	R1	with exactly one red, if you want to say there, right?
56	Stephanie	Okay.
57	R1	And you told from each of these, right?
58	Stephanie	Um hm.
59	R1	from each one of these
60	Stephanie	You can get three.
61	R1	Three. Because there are three positions
62	Stephanie	Um hm.
63	R1	to place a red.
64	Stephanie	Yes.
65	R1	Isn't that right? So it's - - so you have three positions by four towers. So four is the number of towers. Just happens to be that they're height four. Or is that…?
66	Stephanie	Oh. No no no. I think what it should've been is four towers.

67	R1	Four towers?
68	Stephanie	That's what I should've wrote down.
69	R1	But – but – well. If the – if the towers are height three, would there be four towers?
70	Stephanie	No-o.
71	R1	How many would there've been?
72	Stephanie	Two.
73	R1	If it's height three, can you tell me why?
74	Stephanie	Because you only have two other places where you can put a red. If it's height
75	R1	Yes, but how many would you start up with if it were height three, where you had one in each position? One red in each position?
76	Stephanie	Three.
77	R1	You'd start with three. – If it was height
78	Stephanie	Yeah. It would start with – Oh. Like that it would work, but I think it – it – that what I should've wrote was four towers.
79	R1	Right, but but if they're four high -
80	Stephanie	Yeah. It – it really
81	R1	it would be four. If they're three high
82	Stephanie	Um hm
83	R1	If they're two high

84	Stephanie	Yeah. It would be two.
85	R1	Does that make sense?
86	Stephanie	Yeah.
87	R1	So if they're *n* high?
88	Stephanie	It would be *n*.
89	R1	Okay. So. So this would be
90	Stephanie	It – it – it still works.
91	R1	Actually it works that it's the number of towers, doesn't it?
92	Stephanie	Yeah.
93	R1	It's a way to think about it. So it's – you can think of it as the number of towers.
94	Stephanie	Um hm.
95	R1	Right.
96	Stephanie	Um hm.
97	R1	'Cause that always is that way, isn't it? So if the towers are *n* high,
98	Stephanie	It would be *n*.
99	R1	With exactly one, wouldn't you have anything, you could have – 'cause there are *n* positions, right?
100	Stephanie	Um hm.
101	R1	Okay. So. So. Towers *n* high, there'd be *n*. That helps me. If they're *n*

		high, how many positions are there to place that second one?
102	Stephanie	*n* minus one?
103	R1	Interesting? So it would be *n* times *n* minus one. Why don't we just write that down while we have it in our heads?
104	Stephanie	You want me to write *n* times *n* minus one?
105	R1	What do you think? Do you want to? If we talk about towers *n* high, this – since we – you're doing algebra let's try to connect algebra.
106	Stephanie	Do you want me to write – ' For towers *n* high…'
107	R1	For towers – yeah – why not? Let's think about these simple cases and see what we can do with *n* high.
108	Stephanie	For towers *n* high, um, like, if you're putting
109	R1	If you're moving from – what – what's the question here? If you're moving from um *n* things taken one at a time to *n* things taken two at a time
110	Stephanie	Okay.
111	R1	Right? Isn't that right?
112	Stephanie	Um hm. [*Stephanie is writing down all the information as it is being said.*]
113	R1	If you're thinking of selecting those exactly one red
114	Stephanie	From
115	R1	to selecting those with exactly two red.
116	Stephanie	So if…
117	R1	Could you imagine this in your head?

118	Stephanie	Yeah.
119	R1	as we're talking
120	Stephanie	Yeah.
121	R1	about this?
122	Stephanie	I know what you're saying.
123	R1	That's – that's very impressive. I couldn't have done that at your age, Stephanie.
124	Stephanie	[*chuckles*] (inaudible)
125	R1	Could you have, Donna? - - So I'm I'm not – I really – if you don't – if you can't imagine it, tell me 'cause I have to think.
126	Stephanie	Towers one *n* at a time [*writes this as she speaks*]
127	R1	Well, the way – um – I don't know – the way I like to write it
128	Stephanie	Okay
129	R1	is this way. [*It isn't possible from this tape to see what R1 writes.*] But my son tells me you can write it this way. And he told me another way you could write it. - - This way: 'C' 'n" 'one'. Isn't that right? There are three ways you can write it. You have your choice.
130	Stephanie	Um. Yeah.
131	R1	This means *n* high. This means exactly one red.
132	Stephanie	Um hm.
133	R1	Right? How many – all – this tells you that there are *n* of them and what you are supposed to imagine in your mind – these *n* high towers with

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134	Stephanie	Um hm,
135	R1	Right?
136	Stephanie	Yeah.
137	R1	One position at *n* minus one. If you can imagine – if you can begin to think like that and imagine these things.
138	Stephanie	You want me to write that instead?
139	R1	Well. Any way you – it makes sense to you.
140	Stephanie	'Cause I'll just
141	R1	If you want to start trying
142	Stephanie	move from towers
143	R1	to use the notation, this is good practice, you know.
144	Stephanie	*n* – um - one at a time [*Stephanie writes* $\binom{n}{1}$.] to *n* two at a time. Um. It would be *n* times *n* minus one.
145	R1	Okay. So *n* would be
146	Stephanie	*n* would be like the tower – like the, the number of towers.
147	R1	Okay. So write that. *n* is the number total number of towers.
148	Stephanie	Okay. Well, it would be
149	R1	So what's n minus one?
150	Stephanie	be *n* times *n* minus one. Um <u>*n*</u> be – *n*'s the number of towers. Ooooh.

		Wait. *n*'s the number of towers in like
151	R1	Do you know what just occurred to me? You can think of *n* as the number of towers. That's true. That will always work. But it is also the total number of positions.
152	Stephanie	Yeah.
153	R1	Alright? So with towers four high, you'd have four possible positions. Right? It just so happened
154	Stephanie	So. Should I write *n* being the number of towers and the number of positions?
155	R1	Yeah. I'm just trying to think what's useful here.
156	Stephanie	Because it – the number of towers is useful, but the number of positions is useful when you're talking about *n* minus one.
157	R1	Maybe so. Yeah.
158	Stephanie	Okay.
159	R1	Okay. That's a good idea.
160	Stephanie	[*Stephanie speaks as she writes.*] The position – um – *n* minus one being the number of positions – minus one?
161	R1	Let's try to think what the *n* minus one means. The *n* minus one in this problem is what?
162	Stephanie	The *n* minus one in this problem is?
163	R1	Um hm.
164	Stephanie	Is – like -

165	R1	What's *n*?
166	Stephanie	Well, *n* is this [*Stephanie indicates the entire tower.*] or
167	R1	What number is it?
168	Stephanie	*n* is four?
169	R1	Four. And so what's *n* minus one?
170	Stephanie	The red, I guess?
171	R1	Well, if *n* is four -
172	Stephanie	Yes.
173	R1	We know what *n* minus one is.
174	Stephanie	Oh! It's three. [*laughs*]
175	R1	That's what happens after a while, by the way. It really is 'cause you're thinking of something else. Now. Okay. So *n* is four. *n* minus one is three.
176	Stephanie	is three.
177	R1	But what does it mean in terms of moving from here to here? [*Dr. Maher moves from the towers with one red to towers with two reds.*]
178	Stephanie	That means that you're taking away – like – well, we're talking about – like – aren't you talking about like *n* minus one being like yellow minus space like yellow being like – replaced by red?
179	R1	Yeah. Right. That's exactly what I'm thinking about.
180	Stephanie	So like *n* minus one would be like this [*Stephanie points to the tower of all*

		yellow.] being replaced by this [*Stephanie indicates* $\begin{bmatrix} R \\ Y \\ Y \\ Y \end{bmatrix}$*.*] Right? 'Cause like – it's not taking away - -
181	R1	Okay. So wait a minute. I thought we were going from here to here. [*R1 indicates the towers with one red and then the towers with two reds.*] Let's do that one.
182	Stephanie	Yeah. Right.
183	R1	Let's go from here to here.
184	Stephanie	Alright.
185	R1	But what we did is, remember – this one belonged here. [*R1 moves* $\begin{bmatrix} R \\ R \\ Y \\ Y \end{bmatrix}$ *to beside* $\begin{bmatrix} R \\ Y \\ Y \\ Y \end{bmatrix}$*.*]
186	Stephanie	Um hm.
187	R1	Right?
188	Stephanie	Yeah,

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189	R1	And then – uh – this one belonged here [*moves the* $\begin{bmatrix} R \\ Y \\ Y \\ R \end{bmatrix}$]
190	Stephanie	And this one [*Stephanie moves* $\begin{bmatrix} R \\ Y \\ R \\ Y \end{bmatrix}$.]
191	R1	This one belonged here. Right?
192	Stephanie	Um hm.
193	R1	Okay. So – when we moved from one to two, right?
194	Stephanie	Um hm.
195	R1	we ended up with three
196	Stephanie	Yes.
197	R1	Why three? Because – because why?
198	Stephanie	Well. If that's *n* – well – because we're putting – there's three places where you can put it.
199	R1	Right.
200	Stephanie	Yeah.
201	R1	Isn't that right?
202	Stephanie	Yeah.

203	R1	So you could put it here. [*R1 points to the second position.*] You could put it here. [*She points to the third position.*]
204	Stephanie	You could put it there.
205	R1	You could've put it there. [*R1 indicates the bottom position.*]
206	Stephanie	Um hm.
207	R1	Here, here, or here. So if you have *n* positions. Right?
208	Stephanie	Um hm.
209	R1	And really what you have here *n* minus one of them are yellow if one is red -
210	Stephanie	Um hm.
211	R1	Right. Because *n* minus one plus one is *n*.
212	Stephanie	Um hm.
213	R1	Does that make sense? *n* minus one plus one
214	Stephanie	[*laughs*] is just *n*.
215	R1	Isn't that right?
216	Stephanie	Yeah.
217	R1	So we have *n* positions and one red and *n* minus one yellow.
218	Stephanie	Okay.
219	R1	Okay. So *n* minus one in this case – four positions – three yellow and of those three positions we could've put a red.

220	Stephanie	Um hm.
221	R1	And that's where the three came in. The three is the *n* minus one.
222	Stephanie	Okay.
223	R1	Alright. But that could've happened not just one time. It could have happened [*R1 taps the table in front of each of the four towers with one red and three yellows.*]
224	Stephanie	Four times.
225	R1	four times or *n* times.
226	Stephanie	Okay. – So like
227	R1	So it was *n* times *n* minus one.
228	Stephanie	Um hm.
229	R1	That's how we got twelve.
230	Stephanie	Yes.
231	R1	But we got duplicates.
232	Stephanie	Yes.
233	R1	Alright. So we know we know we ended up with *n*, right?
234	Stephanie	Um hm.
235	R1	*n* positions times *n* minus one – how many choices we have for red?
236	Stephanie	How many choices do you have for red? You have – um – I guess – *n* minus one number of choices?

237	R1	*n* minus one. Okay. So we have *n* times *n* minus one.
238	Stephanie	Um hm.
239	R1	If you're writing your formula down. But that's too many. Alright.
240	Stephanie	So *n* times.
241	R1	Four times three is too many.
242	Stephanie	Yeah. Divided by um
243	R1	In that case, what did you have to divide by?
244	Stephanie	Well, these number of positions.
245	R1	You actually ended up dividing by two.
246	Stephanie	Oh.
247	R1	When you found your duplicates.
248	Stephanie	Well. Oh. That's right. But I don't know how many -
249	R1	Hmmm. That's the part. That's the tricky part. That's the part we haven't really worked out yet.
250	R2	Yeah.
251	Stephanie	[*in the background*] (inaudible)
252	R1	Why do we divide by two. Maybe we don't know that yet. Maybe that's something to keep in the back of our minds as something we're trying to figure out, right?
253	Stephanie	Um hm.

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254	R1	But, we know we had to divide by two just by sheer working it out.
255	Stephanie	Yes.
256	R1	Isn't that true?
257	Stephanie	Yes.
258	R1	'Cause you found duplicates. So, so that time it was four times three divided by two. Which is what you wrote here, by the way.
259	Stephanie	Yeah.
260	R1	Four, right?
261	Stephanie	Yes.
262	R1	Four positions times three
263	Stephanie	Um hm.
264	R1	available postions to choose red.
265	Stephanie	Um hm.
266	R1	Right? Divided by the number of duplicates.
267	Stephanie	Yeah.
268	R1	Let's write down what these mean again so we don't lose track of that.
269	Stephanie	Here?
270	R1	So this number – the four – is like your *n* – the number of positions -
271	Stephanie	Alright.

272	R1	or the height of the tower – either way. How tall it is. However you want to think
273	Stephanie	Okay. Four is [*writing*] number of positions
274	R1	or height of tower. Right?
275	Stephanie	Um hm. - - or height of tower. Um. Three is um the number of spaces you can put a red.
276	R1	Very good. So – that's very good. So the spaces available for red. [*Stephanie continues to write.*]
277	Stephanie	And two is 'cause they came in pairs.
278	R1	Number of duplicates. Right. And that what we we know how we know this is n and we know this is *n* minus one. [*R1 points to the four and the three.*] Right?
279	Stephanie	Um hm.
280	R1	That's very helpful. But. What is that two? …Hmmm…Right?
281	R2	That's a tough one (inaudible)
282	R1	I mean I studied probability at college and they told me you had to divide by these numbers - -
283	Stephanie	Um hm.
284	R1	I didn't know why. Did you know why when they just told you to divide by these numbers? No. Don't tell me. I'm not going to ask you. But – But do you see? That's – that's part of the problem. It would be nice to think about, is there a nice explanation that we can see as we generate this – that division by two. So let's keep that in back of our minds and go to the next step. Then we'll come to this one.

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