CONSTRUCTING AND CLASSIFYING FULLY IRREDUCIBLE OUTER AUTOMORPHISMS OF FREE GROUPS

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ABSTRACT OF THE DISSERTATION

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The main theorem of this document emulates, in the context of $Out(F_r)$ theory, a mapping class group theorem (by H. Masur and J. Smillie) that determines precisely which index lists arise from pseudo-Anosov mapping classes. Since the ideal Whitehead graph gives a finer invariant in the analogous setting of a fully irreducible $\phi \in Out(F_r)$, we instead focus on determining which of the twenty-one connected, loop-free, five-vertex graphs are ideal Whitehead graphs of ageometric, fully irreducible $\phi \in Out(F_3)$. Our main theorem accomplishes this by showing that there are precisely eighteen graphs arising as such. We also give a method for identifying certain complications called periodic Nielsen paths, prove the existence of conveniently decomposed representatives of ageometric, fully irreducible $\phi \in Out(F_r)$ having connected, $(2r-1)$-vertex ideal Whitehead graphs, and prove a criterion for identifying representatives of ageometric, fully irreducible $\phi \in Out(F_r)$. The strategies we use for constructing fully irreducible outer automorphisms of free groups, as well as our identification and decomposition techniques, can be used to extend our main theorem, as they are valid in any rank. Our methods of proof rely primarily on Bestvina-Feighn-Handel train track theory and the theory of attracting laminations.

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Dedication

To those who loved me, believed in me, and supported me through the rough times as well as the glorious times. This is dedicated to my beloved parents, my incredibly large and wonderful family, and my very dear and special friends.

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Chapter 1

Introduction

The main theorem of this document (Theorem 14.1) is motivated by a theorem in mapping class group theory. The *mapping class group* $MCG(S)$ of a compact surface S is the group of homotopy classes of homeomorphisms $h : S \to S$. The most common mapping classes are called *pseudo-Anosov*. (One characterization of a pseudo-Anosov mapping class is that some representative of a pseudo-Anosov mapping class expands and contracts a pair of transverse singular measured foliations on the surface). Because of their fundamental importance in topology and geometry, both mapping class groups and pseudo-Anosov mapping classes have been objects of extensive research. The list of singularity indices associated to a pseudo-Anosov mapping class is an important invariant of the class. (Each foliation singularity for the pair of transverse singular measured foliations has an associated index). H. Masur and J. Smillie (see [MS93]) proved precisely which lists of singularity indices arise from pseudo-Anosov mapping classes. This document is the first step to proving an analogous theorem for outer automorphism groups of free groups.

For a free group of rank r, F_r, the *outer automorphism group*, $Out(F_r)$, consists of equivalence classes of automorphisms $\Phi : F_r \to F_r$, where two automorphisms are equivalent when they differ by an inner automorphism, i.e. a map Φ_b defined by $\Phi_b(a) = b^{-1}ab$ for all $a \in F_r$. Outer automorphisms can be described geometrically as follows. Let R_r be the r-petaled rose (graph having r edges and a single vertex v) with fundamental group identified with F_r. Given a graph Γ with no valence-one vertices, we can assign to Γ a *marking* (identification of the fundamental group with the free group F_r) via a homotopy equivalence $R_r \to \Gamma$. We call such a graph Γ, together with its marking $R_r \to \Gamma$, a *marked graph*. Each element ϕ of $Out(F_r)$ can be represented geometrically by a homotopy equivalence $g : \Gamma \to \Gamma$ of a marked graph, where $\phi = g_*$ is the induced outer automorphism of fundamental groups. For the proof of our analogue to the mapping class group theorem, we will focus on constructing representatives of ageometric, fully irreducible outer automorphisms with particular ideal Whitehead graphs. An ideal Whitehead graph is a strictly finer invariant than a singularity index list and encodes information about the attracting lamination for a fully irreducible outer automorphism.

There is potential for an $Out(F_r)$ analogue to the Masur-Smillie theorem because of deep connections between outer automorphism groups of free groups and mapping class groups of surfaces. When $r = 2$, we have $Out(F_2) \cong Out(\Pi_1(\Sigma_{1,1})) \cong MCG(\Sigma_{1,1})$, where $\Sigma_{1,1}$ denotes a genus-1 torus with a single puncture. Furthermore, elements $\phi \in Out(F_2)$ are induced by homeomorphisms of $\Sigma_{1,1}$ and "fully irreducible outer automorphisms" (Section 2.3) are induced by pseudo-Anosov homeomorphisms. While we do not have such exact correspondences for $r > 2$, there are still strong similarities between all of the outer automorphism groups $Out(F_r)$ and mapping class groups $MCG(S)$, as well as between the fully irreducible $\phi \in Out(F_r)$ and pseudo-Anosov $\psi \in MCG(S)$. In fact, some $\phi \in Out(F_r)$ with $r > 2$ are even still induced by homeomorphisms of a compact surface with boundary (such ϕ are called *geometric*).

There is a large group of mathematicians exploring the parallel properties between the $Out(F_r)$ groups and the $MCG(S)$ groups. They have made significant progress to this affect. We use some of their definitions and machinery (including the definitions of singularities, indices, and ideal Whitehead graphs for outer automorphisms of free groups, as defined in [GJLL98] and [HM11]), in order to understand an appropriate $Out(F_r)$-analogue to the Masur-Smillie theorem for mapping class groups (as described in the next chapter).

1.1 The Question

Let $i(\phi)$ denote the sum of the singularity indices for a fully irreducible $\phi \in Out(F_r)$ as defined in [GJLL98] and in Section 2.9 below. An index can in some sense be thought of as recording the number of germs of initial edge segments (*directions*) emanating from a vertex that are fixed by a given geometric representative of ϕ. [GJLL98] gives an inequality $i(\phi) \geq 1 - r$ bounding the index sum for a fully irreducible outer automorphism $\phi \in Out(F_r)$, in contrast with the equality $i(\psi) = \chi(S)$, for a pseudo-Anosov ψ on a surface S, dictated by the Poincare-Hopf Theorem. With this in mind, one can ask whether every index list who's sum satisfies this inequality is achieved. M. Handel and L. Mosher pose the question in [HM11]:

Question 1.1. *What possible index types, which satisfy the index inequality $i(\phi) \geq 1-r$, are achieved by a nongeometric, fully irreducible element of $Out(F_r)$?*

What we present here focuses on several constructions that may eventually allow us to attack this question directly and that, in the meantime, give us a stronger result for the rank-3 case when restricting to "ageometric" fully irreducible outer automorphisms and connected, $(2r-1)$-vertex ideal Whitehead graphs with no single-vertex edges (see Theorem 14.1). For an ageometric, fully irreducible $\phi \in Out(F_r)$ and TT representative $g: \Gamma \to \Gamma$ on a rose having $2r-1$ fixed directions at the unique vertex and no *periodic Nielsen paths (PNPs)*, i.e. paths ρ in Γ such that $g^k(\rho) \cong \rho$ for some k, the ideal Whitehead graph $IW(\phi)$ is the graph with one vertex for each fixed direction of Γ and an edge between two such vertices when there exists some $k > 0$ and edge e of Γ such that $g^k(e)$ crosses over the turn formed by the directions corresponding to the vertices.

Having $2r-1$ vertices is maximal when we refine our search to ageometric, fully irreducible outer automorphisms and connected, loop-free graphs. We focus on ageometric, fully irreducible outer automorphisms, as they are far more common and better understood than parageometrics, the only other kind of nongeometric, fully irreducible outer automorphism (geometric outer automorphisms are induced by surface homeomorphisms, thus are already understood). We focus on connected graphs, as this allows us to only look at homotopy equivalences of roses (see Proposition 3.3).

In the mapping class group case, one only sees circular ideal Whitehead graphs, making singularity index lists the best possible invariant. However, this is not true for fully irreducible outer automorphisms. Thus, a better analogue to the Masur-Smillie theorem would record possible "ideal Whitehead graphs," instead of just singularity indices. For an ageometric, fully irreducible $\phi \in Out(F_r)$, the index of a component in $IW(\phi)$ is simply $1 - \frac{k}{2}$, where k is the number of vertices of the component. Ideal Whitehead graphs for outer automorphisms of free groups are defined in [HM11] and discussed in Section 2.9 below. In the spirit of focusing on ideal Whitehead graphs, the question we give a partial answer to in this document (see Theorem 14.1) is that also posed by L. Mosher and M. Handel in [HM11]:

Question 1.2. *Which ideal Whitehead graphs arise from ageometric, fully irreducible outer automorphisms of rank-3 free groups?*

We call graphs with no single-vertex edges (i.e. no edges are loops) *loop-free* and a connected, $(2r-1)$-vertex, loop-free graph a *Type (*) potential ideal Whitehead graph* or *Type (*) pIW graph (pIWG).* The partial answer (Theorem 14.1) we give completely answers the following subquestion posed in person by L. Mosher and M. Feighn:

Question 1.3. *Which of the twenty-one five-vertex Type (*) pIW graphs are ideal Whitehead graphs for ageometric, fully irreducible $\phi \in Out(F_3)$?*

What we state here is Theorem 14.1 and is the complete answer to Questions 1.1.

Theorem 1.4. *Precisely eighteen of the twenty-one five-vertex Type (*) pIW graphs are ideal Whitehead graphs for ageometric, fully irreducible $\phi \in Out(F_3)$.*

As mentioned above, we chose to look at 5-vertex graphs because, with the restriction that $\phi \in Out(F_3)$ must be ageometric and fully irreducible and the restriction that $IW(\phi)$ is loop-free and connected, five vertices is maximal. We focused on connected graphs as this allowed us to focus on representatives on the rose (see Proposition 3.3).

As they will be used throughout, we list now the 21 5-vertex Type (*) pIWGs.

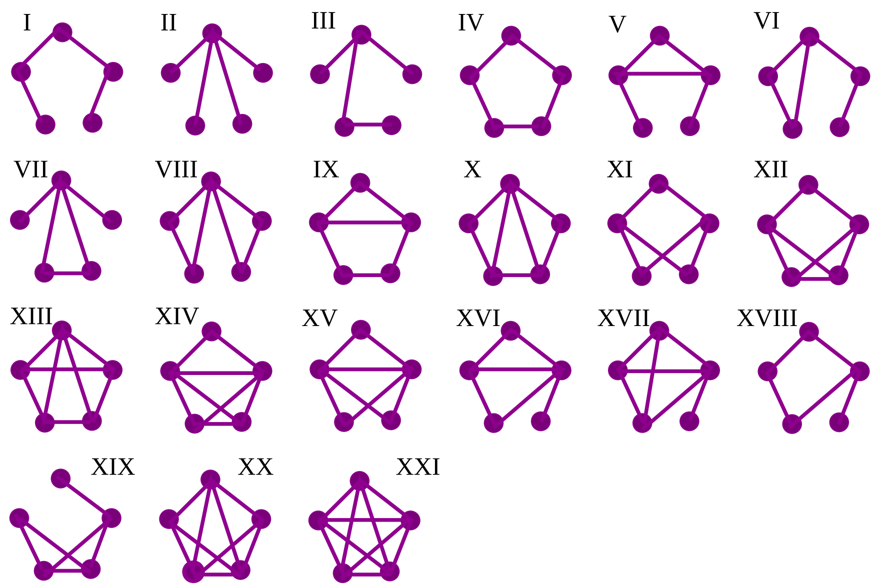


Figure 1.1: *The twenty-one* 5*-vertex Type (*) pIWGs (up to graph isomorphism) (See [CP84])*

Remark 1.5. The 5-vertex Type (*) pIWGs that are not ideal Whitehead graphs for ageometric, fully irreducible $\phi \in Out(F_3)$ are:

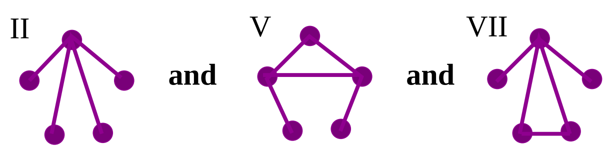


Figure 1.2: *Unachievable Graphs*

1.2 Outline of Document

The first step to proving the main theorem, Theorem 14.1, is Proposition 3.3:

Proposition 1.6. *Let an ageometric, fully irreducible $\phi \in Out(F_r)$ be such that $IW(\phi)$ is a Type (*) pIWG. Then there exists a PNP-free, rotationless representative of a power $\psi = \phi^R$ of ϕ on the rose. Furthermore, the representative can be decomposed as a sequence of proper full folds of roses.*

(The point of ψ being rotationless is its representative fixing the periodic directions.)

The significance, for our purposes, of the proposition is its refining our search for representatives with desired ideal Whitehead graphs to those "ideally decomposed," as in the proposition.

The next step to proving Theorem 14.1 is to define, as we do in Chapter 4, "lamination train track structures " (LTT structures). We "build" or "construct" portions of desired ideal Whitehead graphs by determining "construction compositions" from smooth paths in LTT structures. "Construction compositions" are in ways analogues to Dehn twists (mapping class group elements used in pseudo-Anosov construction methods, including those of Penner in [P88]). We appropriately compose construction compositions to construct (for the Theorem 14.1 proof) the representatives of outer automorphisms with particular ideal Whitehead graphs. On the other hand, we use that LTT structures for ageometric, fully irreducible outer automorphisms are "birecurrent" to show in Proposition 5.4 that the ideal Whitehead graph of an ageometric, fully irreducible outer automorphism cannot be of a certain type. While stated in the restricted form it is used for (and we give definitions for) in this document, what the Proposition 5.4 proof really says is:

Proposition 1.7. *The LTT structure for a train track representative of a fully irreducible outer automorphism is birecurrent.*

However, Proposition 5.4 only explains one of the three graphs Theorem 14.1 deems unachievable. We prove all three graphs are unachievable in Chapter 12.

To determine which construction compositions to use and how to appropriately compose them, we define "AM Diagrams" in Chapter 9. If there is an ageometric, fully irreducible $\phi \in Out(F_r)$ with a particular Type (*) pIWG, $\mathcal{G}$, as its ideal Whitehead graph, then there is a loop in the AM Diagram for $\mathcal{G}$ that corresponds to an "ideal decomposition" (defined in Chapter 3) of a representative g for a power of ϕ. This fact is proved in Proposition 10.11 and helps us rule out unobtainable ideal Whitehead graphs. Additionally, Chapters 9 and 13 tell us how to construct representatives yielding the obtainable Type (*) pIWGs.

Finally, in order to prove that our representatives are representatives of ageometric fully irreducible $\phi \in Out(F_r)$, we proved in Section 10 the "Full Irreducibility Criterion" or "FIC" (Lemma 10.9). We need three conditions to apply the criterion. First, it requires a representative be PNP-free. Proposition 11.2 of Section 11 offers a method for identifying PNPs for an ideally decomposed train track representative g. Second, the FIC includes a condition that the local Whitehead graphs be connected (a condition satisfied in our case by the ideal Whitehead graph being connected). A local Whitehead graph records how images of edges enter and exit a particular vertex and, for a representative g, the local Whitehead graph at a vertex x will be denoted $LW(g;x)$. Finally, the FIC includes a condition on the transition matrix for $g : \Gamma \to \Gamma$ satisfied when there exists some $k > 0$ such that g^k maps each edge of Γ over each other edge of Γ.

Lemma 1.8. *(The Full Irreducibility Criterion) Let $g : \Gamma \to \Gamma$ be an irreducible train track representative of $\phi \in Out(F_r)$. Suppose that g has no PNPs, that the transition matrix for g is Perron-Frobenius, and that all $LW(g;x)$ for g are connected. Then ϕ is fully irreducible.*

For our proof of the criterion we appeal to the train track machinery of M. Bestvina, M. Feighn, and M. Handel. The proof uses several different revised versions (defined in [BFH00] and [FH09]) of "relative train track representatives." Definitions of the relative train track representatives relevant to our situation are also given in Chapter 10. Outside of Chapter 10 we restrict our discussions to train track representatives.

As a final note, we comment that, while our methods have been designed for constructing ageometric, fully irreducible outer automorphisms with particular ideal Whitehead graphs, and thus are particularly well-suited for this purpose, there are other methods for constructing fully irreducible outer automorphisms, such as the recent one described in [CP10].

1.3 Summary of Results

The main theorem of this document (Theorem 14.1) lists precisely which Type (*) pI-WGs arise as ideal Whitehead graphs for ageometric, fully irreducible $\phi \in Out(F_3)$. Of independent interest and use are two propositions and a lemma used in the proof of the main theorem. The propositions and lemma show the existence of a useful decomposition of a certain class of fully irreducible $\phi \in Out(F_r)$ (Proposition 3.3), a method for identifying PNPs (Proposition 11.2), and a criterion for identifying representatives of ageometric fully irreducible $\phi \in Out(F_r)$ (Lemma 10.9). Finally, our Theorem 14.1 proof outlines fully irreducible representative construction strategies that have use beyond proving Theorem 14.1, as they can, for example, be used in any rank and include, for example, the techniques for constructing AM Digrams, yielding infinitely many representatives when a single representative exists.

Chapter 2

Preliminary Definitions

In this chapter we give some definitions used throughout the document. We continue with the notation established in the introduction.

2.1 Markings

Again we let R_r be the r-petaled rose (graph having r edges and a single vertex v) with fundamental group identified with F_r and, again, for a finite graph Γ with no valence-one vertices, we define a *marking* to be a homotopy equivalence $R_r \to \Gamma$. We call such a graph Γ, together with its marking $R_r \to \Gamma$, a *marked graph.* Since a homotopy equivalence has an inverse (rel homotopy), we can talk about an *inverse marking* $\Gamma \to R_r$, and we sometimes do. Two marked graphs $m : R_r \to \Gamma$ and $m : R_r \to \Gamma'$ are considered equivalent when there exists a homeomorphism $h : \Gamma \to \Gamma'$ such that $h \circ m$ is homotopic to m'.

2.2 Train Track Representatives

Thurston called a homotopy equivalence $g : \Gamma \to \Gamma$ of marked graphs a *train track map* when, for all $k > 0$, the restriction of g^k to the interior of each edge of Γ were locally injective (having no "backtracking" in edge images). If g induces a $\phi \in Out(F_r)$ (as a map of fundamental groups) and $g(\mathcal{V}) \subset \mathcal{V}$ (where $\mathcal{V}$ is the vertex set of Γ), then g is called a *topological (or train track) representative* for ϕ [BH92]. Train track representatives are in many ways the most natural representatives to work with and [BH92] gives an algorithm for finding a train track representative of any irreducible $\phi \in Out(F_r)$. In this document we focus on train track representatives and several versions of their more general "relative train track" representatives (see Section 10). **Unless otherwise stated, one should assume throughout that an outer automorphism representative is a train track representative (abbreviated "train track").**

2.3 Reducibility

"Fully irreducible" outer automorphisms of free groups ("iwips") are induced by pseudo-Anosov surface homeomorphisms in rank 2 and still resemble pseudo-Anosovs in higher ranks. They are our main focus and have both algebraic and geometric definitions. We algebraically define fully irreducible outer automorphisms, but geometrically define representative reducibility and irreducibility.

An outer automorphism $\phi \in Out(F_r)$ is *reducible* if there are proper free factors $F^1, \dots, F^k$ of F_r such that ϕ permutes the conjugacy classes of the F^i's and $F^1 * \cdots * F^k$ is a free factor of F_r (i.e. there is a free group F_l such that $(F^1 * \cdots * F^k) * F_l = F_r$). If ϕ is not reducible, we call ϕ *irreducible*. If every power of ϕ is irreducible, we call ϕ *fully irreducible*.

A train track representative $g : \Gamma \to \Gamma$ of a $\phi \in Out(F_r)$ is *reducible* if it has a nontrivial invariant subgraph Γ_0 (meaning $g(\Gamma_0) \subset \Gamma_0$) with at least one noncontractible component. The representative g is otherwise called *irreducible*. [BH92, BFH97]

In other words, an outer automorphism is reducible if and only if some topological representative is reducible; irreducible if and only if every topological representative is irreducible; and fully irreducible if and only if every topological representative of every power is irreducible.

It is important to note that a reducible outer automorphism may have irreducible representatives. It only needs one reducible representative to be reducible. Thus, a fully irreducible outer automorphism is an outer automorphism such that no power has a reducible representative.

2.4 Turns, Paths, Circuits, and Tightening

We remind the reader here of a few definitions from [BH92], establish notation, and defined the notion of an "edge-indexed graph". These definitions are important in establishing notions of "legality," prevalent in train track theory, and are needed to define ideal Whitehead graphs, the outer automorphism invariant finer than a singularity index list. First we establish notation.

Let $g : \Gamma \to \Gamma$ be a train track representative of $\phi \in Out(F_r)$ and $\mathcal{E}^+(\Gamma) = \{E_1, \dots, E_n\}$ the set of edges in Γ with some prescribed orientation. For any edge $E \in \mathcal{E}^+(\Gamma)$, let $\overline{E}$ denote E oriented oppositely as E and then let $\mathcal{E}(\Gamma)$
$= \{E_1, \overline{E_1}, \dots, E_n, \overline{E_n}\} = \{e_1, e_1, \dots, e_{2n-1}, e_{2n}\}$. If the indexing $\{E_1, \dots, E_n\}$ of the edges (thus the indexing $\{e_1, e_1, \dots, e_{2n-1}, e_{2n}\}$) is prescribed, then we call the graph Γ an *edge-indexed* graph. A $2r$-element set of the form $\{x_1, \overline{x_1}, \dots, x_r, \overline{x_r}\}$, where elements are paired into *edge pairs* $\{x_i, \overline{x_i}\}$, will be called an *edge pair labeling set* of *rank* r. In such a case it will be standard to say $\overline{\overline{x_i}} = x_i$. A graph where the vertices are labeled by an edge pair labeling set will be called a *pair labeled* graph. If an indexing is prescribed to the pairs, it will be called an *indexed pair-labeled* graph. Such graphs will be prevalent starting in Chapter 4.

Suppose Γ_i, with inverse marking π_i, and Γ_j, with inverse marking π_j, are edge-indexed marked graphs with edge indices such that $\mathcal{E}_i = \mathcal{E}(\Gamma_i) = \{e_{(i,1)}, e_{(i,2)}, \dots,$
$e_{(i,2n-1)}, e_{(i,2n)}\}$ and $\mathcal{E}_j = \mathcal{E}(\Gamma_j) = \{e_{(j,1)}, e_{(j,2)}, \dots, e_{(j,2n-1)}, e_{(j,2n)}\}$ (here the first index in the elements of $\mathcal{E}_i$ and $\mathcal{E}_j$ just lets us know which indexing we are in, while the second index gives us the actual index prescribed to the edge). We will say Γ_i and Γ_j are *equivalent* if they are equivalent as marked graphs (differ by a change of marking) and if $\pi_i(E_{i,k})$ is homotopic to $\pi_i(E_{j,k})$ for each k.

Finally, let $\mathcal{V}(\Gamma)$ denote the set of vertices of Γ (or just $\mathcal{V}$, when Γ is clear). We continue with this notation throughout the document.

Next we state the definition versions used here for paths and circuits. These notions are important for analyzing images of edges under train tracks and in discussions of RTTs and laminations.

Let Γ be a marked graph with universal cover $\tilde{\Gamma}$ and projection map $p : \tilde{\Gamma} \to \Gamma$. A *path* in $\tilde{\Gamma}$ is either a proper embedding $\tilde{\gamma} : I \to \tilde{\Gamma}$, where I is a (possibly infinite) interval, or a map of a point $\tilde{\gamma} : x \to \tilde{\Gamma}$. *Paths* in Γ are projections $p \circ \tilde{\gamma}$, where $\tilde{\gamma}$ is a path in $\tilde{\Gamma}$. Paths differing by an orientation-preserving change of parametrization are considered to be the same path. [BFH00]

Let $\tilde{\gamma}$ be a path in $\tilde{\Gamma}$. Then $\tilde{\gamma}$ can be written as a concatenation of subpaths, each of which is an oriented edge of $\tilde{\Gamma}$ (with the exception that the first and last subpaths of $\tilde{\gamma}$ may actually only be partial edges). We call this sequence of oriented edges (and partial edges) the *edge path associated to* $\tilde{\gamma}$. Its projection gives a decomposition of γ as a concatenation of oriented edges in Γ. We will call this sequence of oriented edges in Γ the *edge path associated to* γ. [BFH00]

A *circuit* in Γ is an immersion $\alpha : S^1 \to \Gamma$ of the circle into Γ. Edge paths for circuits in Γ are defined as with paths in Γ, but with the edges for only one period of the edge path are listed. [BFH00] This will mean that there can be multiple edge paths representing the same circuit.

Directions will be important for defining ideal Whitehead graphs and are prevalent throughout the proofs of this document.

For a point $x \in \Gamma$, $\mathcal{D}(x)$ will denote the set of *directions* at x, i.e. germs of initial segments of edges emanating from x, and $\mathcal{D}(\Gamma) = \underset{v \in \mathcal{V}(\Gamma)}{\cup} \mathcal{D}(v)$. [BH92] For an edge $e \in \mathcal{E}(\Gamma)$, $D_0(e)$ will denote the initial direction of e (germ of initial segments of e). It will be standard to write $D_0(\bar{e})$ as $\bar{d}$ when $D_0(e) = d$. If Γ is an edge-indexed graph, then $\mathcal{D}(\Gamma)$ will inherit an indexing $\mathcal{D}(\Gamma) = \{d_1, d_2, \dots, d_{2n-1}, d_{2n}\}$ from $\mathcal{E}(\Gamma) = \{e_1, e_2, \dots, e_{2n-1}, e_{2n}\}$ where $D_0(e_i) = d_i$ for each i.

For a path $\gamma = e_1 \dots e_k$, define $D_0\gamma = D_0(e_1)$. We denote the map of directions g induces by Dg, i.e. $Dg(d) = D_0(g(e))$ for $d = D_0(e)$. (Note that $D(f \circ g) = Df \circ Dg$). If Γ is denoted Γ_k with the edge-indexing such that $\mathcal{E}_k = \mathcal{E}(\Gamma_k) = \{e_{(k,1)}, e_{(k,2)}, \dots, e_{(k,2n-1)}, e_{(k,2n)}\}$ and denoted Γ_{k-1} with the edge-indexing such that $\mathcal{E}_{k-1} = \mathcal{E}(\Gamma_{k-1}) = \{e_{(k-1,1)}, e_{(k-1,2)}, \dots, e_{(k-1,2n-1)}, e_{(k-1,2n)}\}$, then $g : \Gamma_{k-1} \to \Gamma_k$ induces a map of the index set, which we call the *index map induced by* g.

$d \in \mathcal{D}(\Gamma)$ is *periodic* if $Dg^k(d) = d$ for some $k > 0$ and *fixed* if $k = 1$. We denote the set of periodic directions at an $x \in \Gamma$ by $Per(x)$ and the fixed point set by $Fix(x)$. [BH92]

Turns, legality, and tightening are used for stating properties of the different RTT variants and laminations and are prevalent throughout this document's proofs.

A *turn* in Γ is defined as an unordered pair of directions $\{d_1, d_2\}$ at a vertex $v \in \Gamma$. Let $\mathcal{T}(v)$ denote the set of turns at v. For a vertex $v \in \Gamma$, Dg induces a map of turns $D^t g$ on $\mathcal{T}(v)$, defined by $D^t g(\{d_1, d_2\}) = \{Dg(d_1), Dg(d_2)\}$ for each $\{d_1, d_2\} \in \mathcal{T}(v)$. A turn $\{d_i, d_j\}$ is *degenerate* if $d_i = d_j$ and *nondegenerate* otherwise. The turn is *illegal* with respect to g: $\Gamma \to \Gamma$ if some $D^t g^k(\{d_1, d_2\})$ is degenerate and is otherwise *legal*. [BH92]

It is an important property of any train track representative $g : \Gamma \to \Gamma$ that one never has $g^k(e) = \dots \overline{e_i} e_j \dots$, where $D_0(e_i) = d_i$, $D_0(e_j) = d_j$, $\{d_i, d_j\}$ is an illegal turn for g, and $e, e_i, e_j \in \mathcal{E}(\Gamma)$. (In other words, for a train track representative g, no iterate of g maps an edge over an illegal turn).

The set of *gates* with respect to g at a vertex $v \in \Gamma$ is the set of equivalence classes in $\mathcal{D}(v)$ where $d \sim d'$ if and only if $(Dg)^k(d) = (Dg)^k(d')$ for some $k \geq 1$. In other words, pairs of directions in the same gate form illegal turns and pairs of directions in different gates form legal turns. [BH92]

For an edge path $e_1 e_2 \dots e_{k-1} e_k$ associated to a path γ in Γ, we say that γ *contains (or crosses over)* the turn $\{\overline{e_i}, e_{i+1}\}$ for each $1 \leq i < k$. A path $\gamma \in \Gamma$ is called *legal* if it does not contain any illegal turns and *illegal* if it contains at least one illegal turn. [BH92]

Every map of the unit interval $\tilde{\alpha} : I \to \tilde{\Gamma}$ is homotopic rel endpoints to a unique path in $\tilde{\Gamma}$, which we denote by $[\tilde{\alpha}]$. We then say that $[\tilde{\alpha}]$ is obtained from $\tilde{\alpha}$ by *tightening*. ($[\tilde{\alpha}]$ is obtained from $\tilde{\alpha}$ by removing all "backtracking"). If α is the projection to Γ of $\tilde{\alpha}$, then the projection $[\alpha]$ of $[\tilde{\alpha}]$ is said to be obtained from α by *tightening*. [BH92]

A homotopy equivalence $g : \Gamma \to \Gamma$ is *tight* if, for each edge $e \in \mathcal{E}(\Gamma)$, either $g(e) \in \mathcal{V}$ or g is locally injective on $int(e)$. Any homotopy equivalence can be tightened to a unique tight homotopy equivalence by a homotopy rel $\mathcal{V}$. For a train track representative $g : \Gamma \to \Gamma$, we define $g_\#$ by $g_\#(\alpha) = [g(\alpha)]$ for each path α in Γ. [BH92]

2.5 Lines

We give here several definitions from [BFH00]. These definitions will be important for defining the laminations analogous to the attracting laminations for pseudo-Anosovs that we use in the proofs of Proposition 5.4 and Lemma 10.9.

We start by establishing the notion of a line in a marked graph and in its universal cover. Again let Γ be a marked graph with universal cover $\tilde{\Gamma}$ and projection map $p : \tilde{\Gamma} \to \Gamma$. A *line* in $\tilde{\Gamma}$ is the image of a proper embedding of the real line $\tilde{\lambda} : \mathbf{R} \to \tilde{\Gamma}$. We denote by $\tilde{\mathcal{B}}(\Gamma)$ the space of lines in $\tilde{\Gamma}$ with the compact-open topology (one can define a basis for $\tilde{\mathcal{B}}(\Gamma)$ where an open set consists of all lines sharing a given line segment). A *line* in Γ is the image of a projection $p \circ \tilde{\lambda}$ of a line $\tilde{\lambda}$ in $\tilde{\Gamma}$. We denote by $\mathcal{B}(\Gamma)$ the space of lines in Γ with the quotient topology induced by the natural projection map from $\tilde{\mathcal{B}}(\tilde{\Gamma})$ to $\mathcal{B}(\Gamma)$.

The appropriate notion of convergence in $\mathcal{B}$ is "weak convergence." Suppose that $g : \Gamma \to \Gamma$ represents ϕ. If $\gamma' \in \Gamma$ realizes $\beta' \in \mathcal{B}$ and $\gamma \in \Gamma$ realizes $\beta \in \mathcal{B}$, then β' is *weakly attracted* to β if, for each subpath $\alpha \in \gamma$, $\alpha \subset g^k_{\#}(\gamma')$ for all sufficiently large k.

Now we give a characterization of lines as pairs of points in the space of ends of $\tilde{\Gamma}$ (viewed as ∂F_r). We then relate this characterization back to the definitions just given. The characterization of lines as pairs of points in the space of ends of $\tilde{\Gamma}$ is used to discuss laminations. Let Δ be the diagonal in ∂F_r x ∂F_r. $\tilde{\mathcal{B}}$ is obtained from (∂F_r x ∂F_r) - Δ by quotienting out by the action that interchanges the factors of ∂F_r x ∂F_r. We denote by $\mathcal{B}$ the quotient of $\tilde{\mathcal{B}}$ under the diagonal action of F_r on ∂F_r x ∂F_r. We can identify the Cantor Set ∂F_r with the space of ends of $\tilde{\Gamma}$. In particular, if $(b_1, b_2) \in \partial F_r$ x ∂F_r is an unordered pair of distinct elements of ∂F_r, then there exists a unique line $\tilde{\gamma} \in \tilde{\Gamma}$ with endpoints corresponding to b_1 and b_2. This defines a homeomorphism between $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{B}}(\tilde{\Gamma})$ that projects to a homeomorphism between $\mathcal{B}$ and $\mathcal{B}(\Gamma)$ (see [BFH00]). For a path $\beta \in \mathcal{B}$, we say that $\gamma \in \mathcal{B}(\Gamma)$ *realizes* β *in* Γ if γ corresponds to β under the projection of the homeomorphism between $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{B}}(\Gamma)$.

As we use it in Section 2.6, we give one last definition here. For a marked graph Γ, we say that a line $\tilde{\gamma}$ in $\tilde{\Gamma}$ is *birecurrent* if every finite subpath of $\tilde{\gamma}$ occurs infinitely often as an unoriented subpath in each end of $\tilde{\gamma}$. A line γ in Γ representing a birecurrent line $\tilde{\gamma} \in \tilde{\Gamma}$ (with either choice of orientation) is called *birecurrent*. [BFH00]

2.6 Laminations

The following two definitions are required to state the attracting lamination definition for a $\phi \in Out(F_r)$. Attracting laminations are used in Chapter 5 to prove a necessary condition for a Type (*) pIWG to be the ideal Whitehead graph of a fully irreducible $\phi \in Out(F_r)$ and are used in the the Full Irreducibility Criterion proof (Chapter 10). To avoid reading about laminations, one can simply skip the proofs requiring them in Chapters 5 and 10. All definitions in this section are from [BFH00].

Definition 2.1. An *attracting neighborhood* of $\beta \in \mathcal{B}$ for the action of ϕ is a subset $U \subset \mathcal{B}$ such that $\phi_{\#}(U) \subset U$ and $\{\phi_{\#}^k(U) : k \geq 0\}$ is a neighborhood basis for β in $\mathcal{B}$.

Definition 2.2. For a free factor F^i of F_r, $[[F^i]]$ will denote the conjugacy class of F^i. Consider the set of circuits in $\mathcal{B}$ determined by the conjugacy classes in F_r of F^i. Lines $\beta \in \mathcal{B}$ in the closure of this set of circuits are said to be *carried by* $[[F^i]]$.

We are now ready to give the definition of an attracting lamination.

Definition 2.3. An *attracting lamination* Λ for $\phi \in Out(F_r)$ is a closed subset of $\mathcal{B}$ that is the closure of a single point λ which:

(1) is birecurrent,

(2) has an attracting neighborhood for the action of some ϕ^k, and

(3) is not carried by a ϕ-periodic rank-1 free factor.

In such a circumstance we say that γ is *generic* for Λ or Λ-*generic*. $\mathcal{L}(\phi)$ will denote the set of attracting laminations for ϕ.

Remark 2.4. It is proved in [BFH00] that a fully irreducible outer automorphism $\phi \in Out(F_r)$ has a unique attracting lamination associated to it (in fact, any irreducible train track representative having a Perron-Frobenius transition matrix, as defined below, has a unique attracting lamination associated with it).

The notation in the literature for this unique attracting lamination varies in ways possibly confusing to the reader unaware of this fact. For example, in [BFH97] and [BFH00] it is denoted by Λ^+_ϕ, or just Λ^+, while the authors of [HM11] used the notation Λ_-, more consistent with the terminology of dynamical systems (Λ_- turns out to be the dual lamination to the tree T_-). To avoid confusion, we simply denote the unique attracting lamination associated to the fully irreducible outer automorphism ϕ by $\Lambda(\phi)$ (or just Λ when we believe ϕ to be clear).

In addition to the notational variance, there is also variance in the name assigned to $\Lambda(\phi)$. An attracting lamination is called a *stable lamination* in [BFH97]. It is also referred to in the literature, at times, as an *expanding lamination.*

2.7 Periodic Nielsen Paths and Geometric, Parageometric, and Ageometric Fully Irreducible Outer Automorphisms

Recall that "periodic Nielsen paths" are important for determining fully irreducibility (see the Full Irreducibility Criterion) and are used to identify ageometric outer automorphisms, the type of outer automorphisms we focus on.

Definition 2.5. A nontrivial path ρ between fixed points $x, y \in \Gamma$ is called a *periodic Nielsen Path (PNP)* if, for some k, $g^k(\rho) \simeq \rho$ rel endpoints. If $k = 1$, then ρ is called a *Nielsen Path (NP)*. ρ is called an *indivisible Nielsen Path (iNP)* if it cannot be written as a nontrivial concatenation $\rho = \rho_1 \cdot \rho_2$, where ρ_1 and ρ_2 are NPs. A particularly nice property of an iNP for an irreducible train track representative [Lemma 3.4, BH97] is that there exist unique, nontrivial, legal paths α, β, and τ in Γ such that $\rho = \bar{\alpha}\beta$, $g(\alpha) = \tau\alpha$, and $g(\beta) = \tau\beta$.

In [BF94], immersed paths $\alpha_1, \dots \alpha_k \in \Gamma$ are said to form an *orbit of periodic Nielsen paths* if $g^k(\alpha_i) \simeq \alpha_{\mathrm{i+1 \ mod \ k}}$ rel endpoints, for all $1 \leq i \leq k$. The orbit is called *indivisible* if α_1 is not a concatenation of subpaths belonging to orbits of PNPs. We call each α_i in an indivisible orbit an *indivisible periodic Nielsen path* (*iPNP*).

In order to define geometric, ageometric, and parageometric fully irreducible outer automorphisms, we first remind the reader of the following definitions.

Definition 2.6. Let CV_r denote *Outerspace*, defined in [CV86] to be the set of projective equivalence classes of marked graphs (where the equivalence is up to marking-preserving isometry). We remind the reader that Outerspace can also be defined in terms of free, simplicial F_r-trees up to isometric conjugacy and that elements of the compactification are represented by equivalence classes of actions of F_r on $\mathbf{R}$-trees (sometimes called *F_r-trees*) that are:

(1) minimal: there exists no proper, nonempty, F_r-invariant subtree and

(2)very small:

(a) the stabilizer of every nondegenerate arc is either trivial or a cyclic subgroup generated by a primitive element of F_r and

(b) the stabilizer of every triod is trivial.

In the tree definition, elements in an equivalence class differ by F_r-equivariant bijections that multiply their metrics by a constant.

Let $\phi \in Out(F_r)$ be fully irreducible. T_+ is defined as the unique point in ∂CV_r which is the attracting point for every forward orbit of ϕ in CV_r. The point's uniqueness is proved in [LL03].

It is proved in [BF94, Theorem 3.2] that, for a fully irreducible $\phi \in Out(F_r)$, T_+ is a geometric $\mathbf{R}$-tree if and only if every TT representative of every positive power of ϕ has at least one iPNP. Recall that a fully irreducible $\phi \in Out(F_r)$ is called *geometric* if it is induced by a homeomorphism of a compact surface with boundary. A defining characteristic of geometric fully irreducible outer automorphisms is that they have a power with a representative having only a single closed iPNP (and no other iPNPs) [BH92]. In fact, such a ϕ can be realized as a pseudo-Anosov homeomorphism of the surface obtained from Γ by gluing a boundary component of an annulus along this loop [BFH, Proposition 4.5]. In the remaining circumstances where T_+ is a geometric $\mathbf{R}$-tree, but ϕ is not geometric, every representative of a positive power of ϕ has at least one indivisible periodic Nielsen path that is not closed. This type of outer automorphism was defined by M. Lustig and is called *parageometric* [GJLL98].

In the case where T_+ is nongeometric, ϕ is called *ageometric.* In other words, a fully irreducible $\phi \in Out(F_r)$ is ageometric if and only if there exists a representative of a power of ϕ having no PNPs (closed or otherwise).

Since our question was answered in the geometric case by the work of H. Masur and J. Smillie, we do not focus on geometric outer automorphisms in this document. We also ignore the parageometric case and instead focus on ageometric fully irreducible outer automorphisms.

2.8 Rotationlessness

M. Feighn and M. Handel defined rotationless outer automorphisms and rotationless train track representatives in [FH09]. The following (from [HM11]) is the description of a rotationless train track map that we will use. A vertex is called *principal* if it is either an endpoint of an iPNP or has at least three periodic directions.

Definition 2.7. An irreducible train track map $g : \Gamma \to \Gamma$ is called *(forward) rotationless* if it satisfies:

(1) every principal vertex is fixed and

(2) every periodic direction at a principal vertex is fixed.

The property of being rotationless is an outer automorphism invariant and so it suffices to have a definition of a rotationless representative, as above. That is, ϕ is rotationless if and only if some (every) RTT representative is rotationless [FH09, Proposition 3.29].

Remark 2.8. An important fact proved in [FH09, Lemma 4.43] is that there exists a $K_r > 0$, depending only on r, such that ϕ^{K_r} is forward rotationless for all $\phi \in Out(F_r)$. (Thus all representatives of a given $\phi \in Out(F_r)$ have a rotationless power).

2.9 Local Whitehead Graphs, Local Stable Whitehead Graphs, Ideal Whitehead Graphs, and Singularity Indices

In order to define singularity indices (the weaker outer automorphism invariant), we first give a special case definition for ideal Whitehead graphs (the finer outer automorphism invariant). It is important to notice that these ideal Whitehead graphs, local Whitehead graphs, and local stable Whitehead graphs given here are as defined in [HM11] differ from the Whitehead graphs mentioned elsewhere in the literature. As this has been a reoccurring point of confusion, we clarify a difference here. In general, Whitehead graphs come from looking at the turns taken by immersions of 1-manifolds into graphs. In our case the 1-manifold is a set of lines, the attracting lamination. In much of the literature the 1-manifolds are circuits representing conjugacy classes of free group elements. For example, for the Whitehead graphs referred to in [CV86], the images of edges are viewed as cyclic words. This is not the case for local Whitehead graphs, local stable Whitehead graphs, or ideal Whitehead graphs, as we define them.

The following set of definitions is taken from [HM11], though it is not their original source. We start by defining the Whitehead graph variants in a way more user-friendly for the purposes of our document and then give the definitions, involving singular leaves and points in ∂F_r, found in [HM11]. The definitions we begin with involve turns taken by a given representative of $\phi \in Out(F_r)$.

Definition 2.9. Let $v \in \Gamma$ be a vertex of the connected marked graph Γ and let $g : \Gamma \to \Gamma$ be a train track representative of $\phi \in Out(F_r)$. Then the *local Whitehead graph* for g at v (denoted $LW(g; v)$) has:

(1) a vertex for each direction $d \in \mathcal{D}(v)$ and

(2) an edge connecting vertices corresponding to directions $d_1, d_2 \in \mathcal{D}(v)$ if the turn $\{d_1, d_2\} \in \mathcal{T}(v)$ is taken by some $g^k(e)$, where $e \in \mathcal{E}(\Gamma)$.

The *local Stable Whitehead graph* for g at v, $SW(g; v)$, is the subgraph of $LW(g; v)$ obtained by restricting to precisely the vertices with labels $d \in Per(v)$, i.e. vertices corresponding to periodic directions at v. If Γ is a rose with vertex v, then we denote the single local stable Whitehead graph $SW(g; v)$ by $SW(g)$ and the single local Whitehead graph $LW(g; v)$ by $LW(g)$.

If g has no PNPs (which is the only case we consider in this document), then the ideal *Whitehead graph of* ϕ, $IW(\phi)$, is isomorphic to $\bigsqcup_{\text{singularities } v \in \Gamma} SW(g; v)$, where a *singularity* for g in Γ is a vertex with at least three periodic directions. In particular, when Γ has only one vertex v (and no PNPs), $IW(\phi) = SW(g; v)$.

Let $g : \Gamma \to \Gamma$ be a PNP-free representative of an ageometric, fully irreducible

$\phi \in Out(F_r)$. We take from [HM11] the definition of the index for a singularity v to be $i(g;v) = 1 - \frac{k}{2}$, where k is the number of vertices of $SW(g;v)$. The index of ϕ is then the sum $i(\phi) = \sum_{\text{singularities } v \in \Gamma} i(g;v)$. When Γ has only a single vertex v, $i(\phi) = i(g;v)$. The *index type* of ϕ is the list of indices of the components of $IW(\phi)$, written in increasing order. Since the index type can be determined by counting the vertices in the components of the ideal Whitehead graph, one can ascertain that the ideal Whitehead graph is indeed a finer invariant than the index type for a fully irreducible outer automorphism.

While we took the definition from [HM11], the index sum of a fully irreducible $\phi \in Out(F_r)$ was studied much before [HM11], in papers including [GJLL98]. The papers written by D. Gaboriau, A. Jaeger, G. Levitt, and M. Lustig take a perspective of studying outer automorphisms via **R**-trees. We focus instead on train track representatives.

Example 2.10. Let $g : \Gamma \to \Gamma$, where Γ is a 3-petalled rose and g, defined by

$$g = \begin{cases} a \mapsto abacbaba\bar{c}abacbaba \\ b \mapsto ba\bar{c} \\ c \mapsto c\bar{a}\bar{b}\bar{a}\bar{b}\bar{a}\bar{b}\bar{c}\bar{a}\bar{b}\bar{a}c \end{cases},$$

is a train track representative of an ageometric, fully irreducible $\phi \in Out(F_r)$. We will see in Chapter 11 that g has no PNPs and in Chapter 13 that ϕ is fully irreducible.

In Chapter 4 we officially define the lamination train track structure (LTT structure) $G(g)$ for a Type (*) representative g. We will end this example by giving the LTT structure for g as a warm up for the definitions of Chapter 4. Since $G(g)$ will encapsulate the information of $SW(g)$ and $LW(g)$ into a single graph (along with information about the marked graph Γ), we first determine $SW(g)$ and $LW(g)$.

The periodic (actually fixed) directions for g are $\{a, \bar{a}, b, c, \bar{c}\}$. $\bar{b}$ is not periodic since $Dg(\bar{b}) = c$, which is a fixed direction, meaning that $Dg^k(\bar{b}) = c$ for all $k \geq 1$ and thus $Dg^k(\bar{b})$ does NOT equal $\bar{b}$ for any $k \geq 1$. The vertices for $LW(g)$ are $\{a, \bar{a}, b, \bar{b}, c, \bar{c}\}$ and the vertices of $SW(g)$ are $\{a, \bar{a}, b, c, \bar{c}\}$.

The turns taken by the $g^k(E)$ where $E \in \mathcal{E}(\Gamma)$ are $\{a, \bar{b}\}$, $\{\bar{a}, \bar{c}\}$, $\{b, \bar{a}\}$, $\{b, \bar{c}\}$, $\{c, \bar{a}\}$, and $\{a, c\}$. Since $\{a, \bar{b}\}$ contains the nonperiodic direction $\bar{b}$, this turn is not represented by an edge in $SW(g)$, though is represented by an edge in $LW(g)$. All of the other turns listed are represented by edges in both $SW(g)$ and $LW(g)$.

There will be a vertex in $G(g)$ and $LW(g)$ for each of the directions $a, \bar{a}, b, \bar{b}, c, \bar{c}$. The vertex in $G(g)$ corresponding to $\bar{b}$ is red and all others are purple. There are purple edges in $G(g)$ for each edge in $SW(g)$. And $G(g)$ has a single red edge for the turn $\{a, \bar{b}\}$ (the turn represented by an edge in $LW(g)$, but not in $SW(g)$. $G(g)$ is obtained from $LW(g)$ by adding black edges connecting the pairs of vertices $\{a, \bar{a}\}$, $\{b, \bar{b}\}$, and $\{c, \bar{c}\}$ (these black edges correspond precisely to the edges a, b, and c of Γ).

$SW(g)$, $LW(g)$, and $G(g)$ respectively look like (in these figures, A will be used to denote $\bar{a}$, B will be used to denote $\bar{b}$, and C will be used to denote $\bar{c}$):

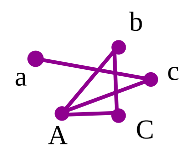


Figure 2.1: *Stable Whitehead Graph $SW(g)$ for g*

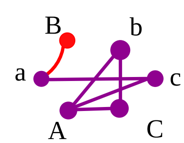


Figure 2.2: *Local Whitehead Graph $LW(g)$ for g*

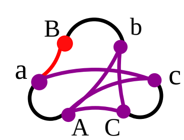


Figure 2.3: *LTT Structure $G(g)$ for g*

We now relate the definitions of ideal Whitehead graphs, etc, given above to those only relying on the attracting lamination for a fully irreducible outer automorphism. The purpose will be to show that an ideal Whitehead graph is indeed an outer automorphism invariant. Each of the following definitions can be found in [HM11].

Definition 2.11. A fixed point x is *repelling* for the action of f if it is an attracting fixed point for the action of f^{-1}, i.e. if there exists a neighborhood U of x such that, for each neighborhood $V \subset U$ of x, there exists an $N > 0$ such that $f^{-k}(y) \in V$ for all $y \in U$ and $k \geq N$.

Let $g : \Gamma \to \Gamma$ be a rotationless irreducible train track representative of $\phi \in Out(F_r)$ and let $\tilde{g} : \tilde{\Gamma} \to \tilde{\Gamma}$ be a *principal lift* of g, i.e. a lift to the universal cover such that the boundary extension has at least three nonrepelling fixed points. We denote the boundary extension of g by $\hat{g}$. $\tilde{\Lambda}(\phi)$ will denote the lift of the attracting lamination to the universal cover $\tilde{\Gamma}$ of Γ. The ideal Whitehead graph, $W(\tilde{g})$, for $\tilde{g}$ is defined to be the graph where:

(1) Vertices correspond to nonrepelling fixed points of $\hat{g}$.

(2) Edges connect vertices corresponding to P_1 and P_2 precisely when P_1 and P_2 are the ideal (boundary) endpoints of some leaf in $\tilde{\Lambda}(\phi)$.

We then define $W(g) = \sqcup W(\tilde{g})$, leaving out components with two or fewer vertices. $IW(g)$ is the quotient of $W(g)$ by conjugation by covering transformations of $\tilde{\Gamma}$.

Since the attracting lamination is an outer automorphism invariant (and, in particular, the properties of leaves having nonrepelling fixed point endpoints and sharing an endpoint are invariant), the definition we just gave does not rely on the choice of representative g for a given $\phi \in Out(F_r)$. Thus, once we establish equivalence between this definition and that given at the beginning of this section, it should be clear that the ideal Whitehead graph is an outer automorphism invariant.

Corollary 2.12 below is Corollary 3.2 of [HM11]. It relates the definition of an ideal Whitehead graph that we gave above Example 2.10 to that given in Definition 2.11.

For Corollary 2.12 to actually make sense, one needs the following definitions and identification from [HM11]. A *cut vertex* of a graph is a vertex separating a component of the graph into two components. $SW(\tilde{v}; \tilde{\Gamma})$ denotes the lift of $SW(v; \Gamma)$ to the universal cover $\tilde{\Gamma}$ of Γ (having countably many disjoint copies of $SW(v; \Gamma)$, one for each lift of v).

Let $g : \Gamma \to \Gamma$ be an irreducible train track representative of an iterate of $\phi \in Out(F_r)$ such that:

(1) each periodic vertex $v \in \Gamma$ is fixed and

(2) each periodic direction at such a v is fixed.

Choose one of these fixed vertices v. Suppose $\tilde{v} \in \tilde{\Gamma}$ is a lift of v to the universal cover, $\tilde{g} : \tilde{\Gamma} \to \tilde{\Gamma}$ is a lift of g fixing $\tilde{v}$, and d is a direction at $\tilde{v}$ fixed by $D\tilde{g}$. Furthermore, let $\tilde{E}$ be the edge at $\tilde{v}$ whose initial direction is d. The *ray determined by* d (or *by* $\tilde{E}$) is defined as $\tilde{R} = \bigcup_{j=0}^{j=\infty} \tilde{g}^j(\tilde{E})$. This is a ray in $\tilde{\Gamma}$ converging to a nonrepelling fixed point for $\hat{g}$. Such a ray is called *singular* if the vertex $\tilde{v}$ it originates at is principal (i.e. v is principal). With these definitions:

(1) the vertices of $SW(\tilde{v}; \tilde{\Gamma})$ correspond to singular rays $\tilde{R}$ based at $\tilde{v}$ and

(2) directions d_1 and d_2 represent endpoints of an edge in $SW(\tilde{v}; \tilde{\Gamma})$ if and only if $\tilde{l} = \tilde{R}_1 \cup \tilde{R}_2$ is a singular leaf of $\tilde{\Lambda}$ realized in $\tilde{\Gamma}$, where $\tilde{R}_1$ and $\tilde{R}_2$ are the rays determined by d_1 and d_2 respectively.

Noticing that the ideal (boundary) endpoints of singular rays are precisely the nonrepelling fixed points at infinity for the action of $\tilde{g}$, combining this with what has already been said, as well as Corollary 2.12 and what follows, we have the correspondence proving ideal Whitehead graph invariance.

Corollary 2.12. *[HM11] Let $\tilde{g}$ be a principal lift of g. Then:*

(1) $W(\tilde{g})$ is connected.

(2) $W(\tilde{g}) = \bigcup_{\tilde{v} \in Fix(\tilde{g}) \in \Gamma} SW(\tilde{v})$.

(3) For $i \neq j$, $SW(\tilde{v}_i)$ and $SW(\tilde{v}_j)$ intersect in at most one vertex. If they do intersect at a vertex P, then P is a cut point of $W(\tilde{g})$, in fact P separates $SW(\tilde{v}_i)$ and $SW(\tilde{v}_j)$ in $W(\tilde{g})$.

By [Lemma 3.1, HM11], in our case (where there are no PNPs), there is in fact only one $\tilde{v} \in Fix(\tilde{g})$ and so the above corollary gives that $W(\tilde{g}) = SW(\tilde{v})$.

This concludes our justification of how an ideal Whitehead graph is an outer automorphism invariant. Consult [HM11] for clarification of the relationship between ideal Whitehead graphs and **R**-trees or for other ideal Whitehead graph characterizations.

2.10 Folds, Decompositions, and Generators

2.10.1 Folds

John Stallings introduced "folds" in [St83]. Bestvina and Handel use in [BH92] several versions of folds in their construction of TT representatives of irreducible $\phi \in Out(F_r)$. We use folds in Chapter 3 for defining and proving ideal decomposition existence.

Let $g : \Gamma \to \Gamma$ be a homotopy equivalence of marked graphs. Suppose that $g(e_1) = g(e_2)$ as edge paths, where the edges $e_1, e_2 \in \mathcal{E}(\Gamma)$ emanate from a common vertex $v \in \mathcal{V}(\Gamma)$. One can obtain a graph Γ_1 by identifying e_1 and e_2 in such a way that $g : \Gamma \to \Gamma$ projects to $g_1 : \Gamma_1 \to \Gamma_1$ under the quotient map induced by the identification of e_1 and e_2. g_1 is also a homotopy equivalence and one says that g_1 and Γ_1 are obtained from $g : \Gamma \to \Gamma$ by an *elementary fold* of e_1 and e_2. [St83, BH92]

One can generalize this definition by only requiring that $e_1' \subset e_1$ and $e_2' \subset e_2$ be maximal, initial, nontrivial subsegments of edges emanating from a common vertex $v \in \mathcal{V}(\Gamma)$ such that $g(e_1') = g(e_2')$ as edge paths and such that the terminal endpoints of e_1 and e_2 are in $g^{-1}(\mathcal{V}(\Gamma))$. Possibly redefining Γ to have vertices at the endpoints of e_1' and e_2', one can fold e_1' and e_2' as e_1 and e_2 were folded above. If both e_1' and e_2' are proper subedges then we say that $g_1 : \Gamma_1 \to \Gamma_1$ is obtained by a *partial fold* of e_1 and e_2. If only one of e_1' and e_2' is a proper subedge (and the other is a full edge), then we call the fold a *proper full fold* of e_1 and e_2. In the remaining case where e_1' and e_2' are both full edges, we call the fold an *improper full fold*. [St83, BH92]

2.10.2 Nielsen Generators

Now let $S =< x_1, \dots, x_r >$ be a free basis for the free group F_r. From [N86] we know that any $\Phi \in Aut(F_r)$ can be written as a composition of "Nielsen generators" having one of the following two forms (Nielsen gave a longer list, but these suffice):

(1) $\Phi(x) = xy$ for some $x, y \in S \cup S^{-1}$ (and $\Phi(z) = z$ for all $z \in S \cup S^{-1}$ with $z \neq x^{\pm 1}$)

(2) a permutation σ of $S \cup S^{-1}$ preserving inverses (if $\sigma(x) = y$, then $\sigma(x^{-1}) = y^{-1}$).

Definition 2.13. In general, we will call an automorphism such as Φ in (1) the *Nielsen generator* (or just *generator*) $x \mapsto xy$.

Consider two metric roses R_r and R'_r with respective edge-labelings $\{a_1, a_2, a_3, \dots\}$ and $\{A_1, A_2, A_3, \dots\}$ and markings where the homotopy class of each a_i in $\pi_1(R_r)$ and each A_i in $\pi_1(R'_r)$ are identified with the free basis element x_i under the respective markings. Consider a homotopy equivalence $g : R_r \to R'_r$ that linearly maps a_i over $A_i \cup A_j$ and, for each $k \neq i$, linearly maps a_k over A_k. Let $a_i = a'_i \cup a''_i$ where a'_i is mapped by g over A_i and a''_i is mapped by g over A_j. Now consider a quotient map (a "proper full fold") $q : R_r \to R^q_r$ identifying a''_i with a_j. There exists a homeomorphism $h : R^q_r \to R'_r$ such that $g = h \circ q$, i.e. g and $h \circ q$ give the same induced map of fundamental groups. In fact, the homeomorphism linearly maps a'_i over A_i and linearly maps each other a_k over A_k. Sometimes we instead call g the proper full fold.

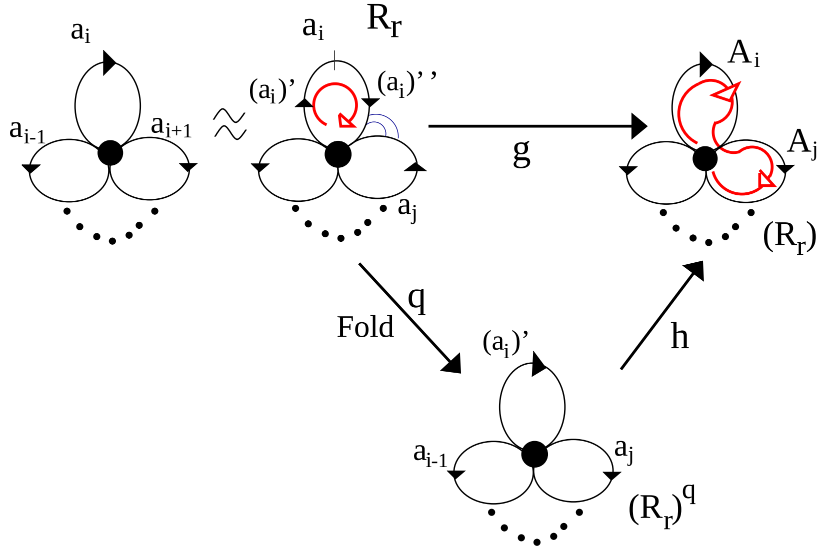


Figure 2.4: *Proper full fold*

Under the identification, for each i, of the homotopy classes of the a_i and A_i with the free basis element x_i, the *induced* automorphism $\Phi \in Aut(F_r)$ is the automorphism where $\Phi(x_i) = x_i x_j$ and $\Phi(x_k) = x_k$ for all $k \neq i$. We say that g in the pervious paragraph *corresponds* to the Nielsen generator $x_i \to x_i x_j$. We have similar situations for cases where $g : R_r \to R'_r$ maps a_i linearly over $A_i \cup \overline{A_j}$ and $\Phi(x_i) = x_i(x_j)^{-1}$, or A_i

is replaced by its inverse, or both A_i and A_j are replaced by their inverses.

2.10.3 Stallings Fold Decompositions

Stallings showed in [St83] that one can decompose a tight (in the sense defined in Section 2.4 above) homotopy equivalence of graphs as a composition of elementary folds together with a final homeomorphism. We will call such a decomposition a *Stallings Fold Decomposition*.

One description of a Stallings Fold Decomposition can be found in [S89], where Skora described a Stallings Fold Decomposition for some $g : \Gamma \to \Gamma'$ as a sequence of folds performed continuously. Consider a lift $\tilde{g} : \tilde{\Gamma} \to \tilde{\Gamma}'$, where here $\tilde{\Gamma}'$ is viewed as having the path metric. Foliate $\tilde{\Gamma}$ x $\tilde{\Gamma}'$ with the leaves $\tilde{\Gamma}$ x $\{x'\}$ for $x' \in \Gamma'$. Define the *vertical t-neighborhood* of the graph $Gr(\tilde{g}) = \{(x, x') \in \tilde{\Gamma} \text{ x } \tilde{\Gamma}' \mid \tilde{g}(x) = x'\}$ by $N_t(\tilde{g}) = \{(x, x') \in \tilde{\Gamma} \text{ x } \tilde{\Gamma}' \mid d(\tilde{g}(x), x') \leq t\}$. For each t, by restricting the foliation to N_t and collapsing all leaf components, one obtains a tree Γ_t. If one quotients by the F_r-action, they see the sequence of folds being performed on the graphs below over time.

Alternatively, one could, at an illegal turn for $g : \Gamma \to \Gamma$, fold maximal initial segments having the same image in $\tilde{\Gamma}'$ to obtain a map $g^1 : \Gamma_1 \to \Gamma'$ of the quotient graph Γ_1. Then they could do the same for g^1. If some g^k did not have an illegal turn, then it would have to be a homeomorphism and the fold sequence would be complete. By using this description, we can assume that only the final element of the decomposition is a homeomorphism. Thus, a Stallings fold decomposition of $g : \Gamma \to \Gamma$ can be written $\Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n$ where each g_k, with $1 \leq k \leq n-1$, is a fold and g_n is a homeomorphism. If each Γ_k is edge-indexed with $\mathcal{E}(\Gamma_k) = \{e_{k,1}, e_{k,2}, \dots, e_{k,2r-1}, e_{k,2r}\}$, then the index permutation σ, where $g_n(e_{n-1,i}) = e_{n,\sigma(i)}$ for each $1 \leq i \leq 2r$, will be called the *edge index permutation associated to the homeomorphism in the Stallings Fold Decomposition.* **If the homeomorphism's associated edge index permutation is trivial, we will sometimes leave it out when writing the decomposition.**

We will say that Γ_0 and Γ_n are *equivalently edge-indexed* if it is possible to edge-index Γ with $\mathcal{E}(\Gamma) = \{e_1, e_2, \dots, e_{2r-1}, e_{2r}\}$ such that, for each t with $1 \leq t \leq 2r$, $g(e_t) = e_{i_1} \dots e_{i_s}$ where $(g_n \circ \cdots \circ g_1)(e_{0,t}) = e_{n,i_1} \dots e_{n,i_s}$.

In the circumstance where the elementary folds are proper full folds of roses, the elements of this decomposition have induced Nielsen generators, as described in the paragraphs following Definition 2.13. Ideally, the Nielsen generators in a decomposition $\Phi = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1 \circ \phi_0$ of $\Phi \in Aut(F_r)$ would all be of form (1) above and there would be a representative $g : \Gamma \to \Gamma$ of ϕ with a Stallings fold decomposition $\Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n$ such that:

- The Stallings fold decomposition *corresponds* to the Nielsen generator decomposition in the sense that $\phi_k = \pi_k^{-1} \circ g_k \circ \pi_k$ (where π_k is the marking on Γ_k) for each $1 \leq k \leq n$, and

- Each g_k is a proper full fold of a rose. In particular, each Γ_k is an edge-indexed rose with an indexing $\{e_{(k,1)}, e_{(k,2)}, \dots, e_{(k,2r-1)}, e_{(k,2r)}\}$ such that:

  (I) The edge index permutation associated to the homeomorphism in the Stallings fold decomposition is trivial, so the homeomorphism was not included in the decomposition.

  (II) Γ_0 and Γ_n are equivalently edge-indexed;

  (III) for some i_k and j_k with $e_{k,i_k} \neq (e_{k,j_k})^{\pm 1}$

$$g_k(e_{k-1,t}) := \begin{cases} e_{k,i_k} e_{k,t} \text{ for } t = j_k \\ e_{k,t} \text{ for all } e_{k-1,t} \neq e_{k-1,j_k}^{\pm 1}\text{; and} \end{cases} .$$

In Chapter 3 we prove that such is the case in the scenario we want it for.

Chapter 3

Ideal Decompositions

At the end of Subsection 2.10.3, we described a representative with an "ideal" Stallings fold decomposition consisting entirely of proper full folds of roses, as well as satisfying a list of other properties. In Proposition 3.3 of this chapter we show that, for an ageometric, fully irreducible $\phi \in Out(F_r)$ such that $IW(\phi)$ is a Type (*) pIWG, we have a representative of a rotationless power with the desired decomposition that is additionally PNP-free. Our methods and theory surround these representatives, as these are the representatives we will construct. We show in Chapter 7 that the decompositions of these representatives will consist of elements of only two types, "switches" and "extensions" and, after filtering out certain switches and extensions and trimming off some loose ends, we get a diagram containing as loops all such representatives for an ageometric, fully irreducible ϕ such that $IW(\phi)$ is a Type (*) pIWG.

For Proposition 3.3, we need the following from [HM11]: Let an ageometric, fully irreducible $\phi \in Out(F_r)$ be such that $IW(\phi)$ is a Type (*) pIWG $\mathcal{G}$, then ϕ is *rotationless* if and only if the vertices of $IW(\phi)$ are fixed by the action of ϕ.

We will also need the following lemmas:

Lemma 3.1. *Let $g : \Gamma \to \Gamma$ be a PNP-free train track representative of a fully irreducible $\phi \in Out(F_r)$ and let $\Gamma = \Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n = \Gamma$ be a decomposition of g into homotopy equivalences of marked graphs. Let $f_k : \Gamma_k \to \Gamma_k$ denote the composition $\Gamma_k \xrightarrow{g_{k+1}} \Gamma_{k+1} \xrightarrow{g_{k+2}} \cdots \xrightarrow{g_{k-1}} \Gamma_{k-1} \xrightarrow{g_k} \Gamma_k$. Then f_k is also a PNP-free train track representative of ϕ (and, in particular, $IW(f_k) \cong IW(g)$).*

Proof: Suppose that $h = f_k$ had a PNP ρ and let h^p be such that the path is fixed (up to homotopy rel endpoints), i.e. $h^p(\rho) \simeq \rho$ rel endpoints. Let $\rho_1 = g_n \circ \cdots \circ g_{k+1}(\rho)$. First notice that ρ_1 cannot be trivial or $h^p(\rho) = (g_k \circ \cdots \circ g_1 \circ g^{p-1})(g_n \circ \cdots \circ g_{k+1}(\rho)) =$

$(g_k \circ \cdots \circ g_1 \circ g^{p-1})(\rho_1)$ would be trivial, contradicting that ρ is a PNP.

$g^p(\rho_1) = g^p((g_k \circ \cdots \circ g_1)(\rho)) = (g_n \circ \cdots \circ g_{k+1}) \circ h^p(\rho)$. Now, $h^p(\rho) \simeq \rho$ rel endpoints and so $(g_n \circ \cdots \circ g_{k+1}) \circ h^p(\rho) \simeq (g_n \circ \cdots \circ g_{k+1})(\rho)$ rel endpoints (just compose the homotopy with $g_n \circ \cdots \circ g_{k+1}$) But that means that $g^p(\rho_1) = g^p((g_k \circ \cdots \circ g_1)(\rho)) = (g_n \circ \cdots \circ g_{k+1}) \circ h^p(\rho)$ is homotopic to $(g_n \circ \cdots \circ g_{k+1})(\rho) = \rho_1$ relative endpoints. This makes ρ_1 a PNP for g, contradicting that g is PNP-free. Thus, $h = f_k$ must be PNP-free, as desired.

Let $\pi : R_r \to \Gamma$ be the marking on Γ_1. Since g_1 is a homotopy equivalence, $g_1 \circ \pi$ gives a marking on Γ and g and h just differ by a change of marking. Thus, g and h are representatives of the same outer automorphism ϕ. We are thus only left to show that h is also a train track representative.

For contradiction's sake suppose $h(e)$ crossed over an illegal turn $\{d_1, d_2\}$. Since each h_j is necessarily surjective, some $(g_k \circ \cdots \circ g_1)(e_i)$ would transverse e. So $(g_k \circ \cdots \circ g_1)(e_i)$ would cross $\{d_1, d_2\}$. And $g^2(e_i) = (g_n \circ \cdots \circ g_{k+1}) \circ h \circ (g_k \circ \cdots \circ g_1)(e_i)$ would cross the turn $\{D(g_n \circ \cdots \circ g_{k+1})(d_1), D(g_n \circ \cdots \circ g_{k+1})(d_2)\}$, which will either be illegal or degenerate (making $g^2(e_i)$ contain back-tracking). This would contradict g's being a train track map. So h must be a train track representative, as desired.

QED.

Lemma 3.2. *Let $h : \Gamma \to \Gamma$ be a PNP-free train track representative of a fully irreducible $\phi \in Out(F_r)$ such that h has $2r - 1$ fixed directions. Let* $\Gamma = \Gamma_0 \xrightarrow{h_1} \Gamma_1 \xrightarrow{h_2} \cdots \xrightarrow{h_{n-1}} \Gamma_{n-1} \xrightarrow{h_n} \Gamma_n = \Gamma$ *be the Stallings fold decomposition for h. Let h^i be such that $h = h^i \circ h_i \circ \cdots \circ h_1$. Let $d_{(1,1)}, \dots, d_{(1,2r-1)}$ be the fixed directions for Dh and let $d_{j,k} = D(h_j \circ \cdots \circ h_1)(d_{1,k})$ for each $1 \leq j \leq n$ and $1 \leq k \leq 2r - 1$. Then $D(h^i)$ cannot identify any of the directions $d_{(i,1)}, \dots, d_{(i,2r-1)}$.*

Proof: Let $d_{(1,1)}, \dots, d_{(1,2r-1)}$ be the fixed directions for Df and let $d_{j,k} = D(h_j \circ \cdots \circ h_1)(d_{1,k})$ for each $1 \leq j \leq n$ and $1 \leq k \leq 2r - 1$. Suppose that $D(h^i)$ identified any of the directions $d_{(i,1)}, \dots, d_{(i,2r-1)}$, then we would have that Df had fewer than $2r - 1$ directions in its image, contradicting that it has $2r - 1$ fixed directions.

QED.

In the following proposition we prove the existence of our representative with the desired decomposition. That the permutation of indices induced by the final homeomorphism is trivial is implied by the homeomorphism being left out all together in (III). Notice that Proposition 3.3 also gives that the representative can be chosen to be PNP-free.

Proposition 3.3. *Let an ageometric, fully irreducible $\phi \in Out(F_r)$ be such that $IW(\phi)$ is a Type (*) pIWG. Then there exists a train track representative of a power $\psi = \phi^R$ of ϕ that is:*

1. *on the rose,*
2. *rotationless,*
3. *PNP-free, and*
4. *decomposable as a sequence of proper full folds of roses.*

In particular, it decomposes as $\Gamma = \Gamma_0 \xrightarrow{h_1} \Gamma_1 \xrightarrow{h_2} \cdots \xrightarrow{h_{n-1}} \Gamma_{n-1} \xrightarrow{h_n} \Gamma_n = \Gamma$*, where:*
(I) the index set $\{1, \dots, n\}$ *is viewed as the set* $\mathbf{Z}/n\mathbf{Z}$ *with its natural cyclic ordering;*
(II) each Γ_k *is an edge-indexed rose with an indexing* $\{e_{(k,1)}, e_{(k,2)}, \dots, e_{(k,2r-1)}, e_{(k,2r)}\}$ *such that:*

(a) Γ_0 *and* Γ_n *are equivalently edge-indexed;*

(b) *for some* i_k, j_k *with* $e_{k,i_k} \neq (e_{k,j_k})^{\pm 1}$

$$h_k(e_{k-1,t}) := \begin{cases} e_{k,t}e_{k,j_k} \text{ for } t = i_k \\ e_{k,t} \text{ for all } e_{k-1,t} \neq e_{k-1,j_k}^{\pm 1}\text{; and} \end{cases}.$$

(The edge index permutation associated to the homeomorphism in the Stallings fold decomposition is trivial and so is left out).

(c) *for each* $e_t \in \mathcal{E}(\Gamma)$ *such that* $t \neq j_n$*, we have* $Dh(d_t) = d_t$*, where* $d_t = D_0(e_t)$*. (g fixes every direction except for* d_{j_n}*).*

Proof: Suppose ϕ is as described in the proposition. Since ϕ is ageometric, there exists a PNP-free train track representative g of a power of ϕ. Let $h = g^k : \Gamma \to \Gamma$ be such that h fixes all of g's periodic directions (h is rotationless). Then h is also a PNP-free train track representative of some ϕ^l and h (and all powers of h) satisfy (2)-(3). Since h has no PNPs (meaning $IW(\phi^R) \cong \bigsqcup_{\text{singularities } v \in \Gamma} SW(h; v)$), since h fixes all of its periodic directions (in particular $SW(h; v) \cong IW(\phi^R)$ if Γ is the rose), and since the ideal Whitehead graph of ϕ (hence ϕ^R) is a Type (*) pIW graph, Γ must have a vertex with $2r - 1$ fixed directions. Thus, Γ must be one of the only three graphs of rank r with a valence $2r - 1$ or higher vertex:

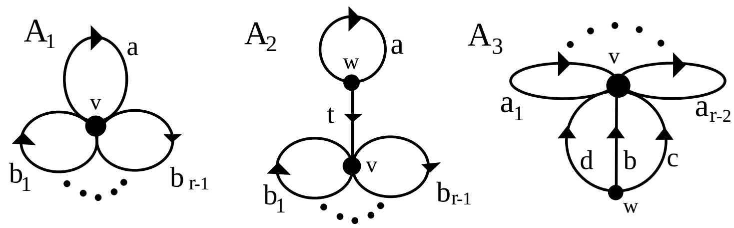


Figure 3.1: *Graphs of rank r with a valence $2r - 1$ or higher vertex*

If $\Gamma = A_1$, then h will satisfy (3). We will show that, in this case we also have the decomposition for (4). However, first we show that Γ cannot be A_2 or A_3 by ruling out all possibilities for folds in the Stallings fold decomposition for h in the cases of $\Gamma = A_2$ and $\Gamma = A_3$.

If we had $\Gamma = A_2$, then the vertex with $2r - 1$ fixed directions could only be v. h must have an illegal turn unless it were a homeomorphism, which it could not be and still be irreducible. Notice that w could not be mapped to v in a way not forcing an illegal turn at w, as this would mean that either we would have an illegal turn at v (if t were wrapped around some b_i) or we would have backtracking on t (contradicting that g is a train track map, so must be locally injective on edge interiors). Because all $2r-1$ of the directions at v are fixed by h, if h had an illegal turn, it would have to occur at w (as no two of the $2r - 1$ fixed directions can share a gate).

The turns at w are $\{a, \bar{a}\}$, $\{a, t\}$, and $\{\bar{a}, t\}$. However, we only need to rule out illegal turns at $\{a, \bar{a}\}$ and $\{a, t\}$, as the situations with $\{\bar{a}, t\}$ and $\{a, t\}$ are identical.

First, suppose that $\{a, \bar{a}\}$ were an illegal turn and that the first fold in the Stallings decomposition were of $\{a, \bar{a}\}$. Fold $\{a, \bar{a}\}$ maximally to obtain $(A_2)_1$. The fold cannot completely collapse a, as this would change the homotopy type of A_2.

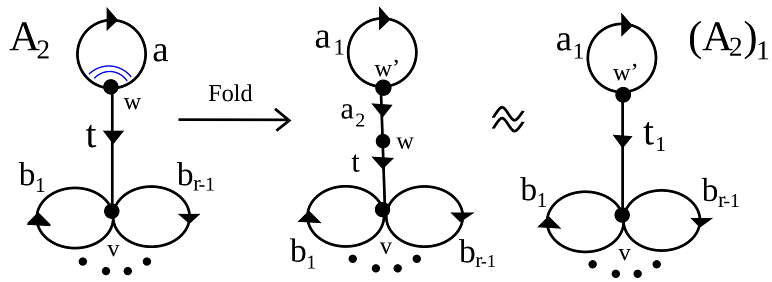


Figure 3.2: *a_1 is the portion of a not folded, a_2 is the edge created by the fold, w' is the vertex created by the fold, and t_1 is $a_2 \cup t$ without the (now unnecessary) vertex w*

Let $h_1 : (A_2)_1 \to (A_2)_1$ be the induced map, as in [BH92] and explained in Section 2.10 above. Since the fold of $\{a, \bar{a}\}$ was maximal, $\{a_1, \overline{a_1}\}$ must be legal. Since h was a train track map, and thus was locally injective on edges of Γ (on the edge a in particular), $\{t_1, a_1\}$ and $\{t_1, \overline{a_1}\}$ would also be legal. But then h_1 would fix all directions at both vertices of Γ_1 (since it still would need to fix all directions at v). This would make h_1 a homeomorphism, again contradicting irreducibility. Thus, $\{a, \bar{a}\}$ could not have been the first turn folded. We are left to rule out $\{a, t\}$.

Suppose the first turn folded in the Stallings decomposition were $\{a, t\}$. Fold $\{a, t\}$ maximally. Let $h_1' : (A_2)_1' \to (A_2)_1'$ be the induced map of [BH92] and Section 2.10. Either

A. all of t is folded with a full power of a;

B. all of t is folded with a partial power of a; or

C. part of t is folded with either a full or partial power of a.

If (A) or (B) held, $(A_2)_1'$ would be a rose and h_1' would give a representative on the rose, returning us to the case of A_1. So we just need to analyze (C).

Consider first (C), i.e. suppose that part of t is folded with either a full or partial power of a:

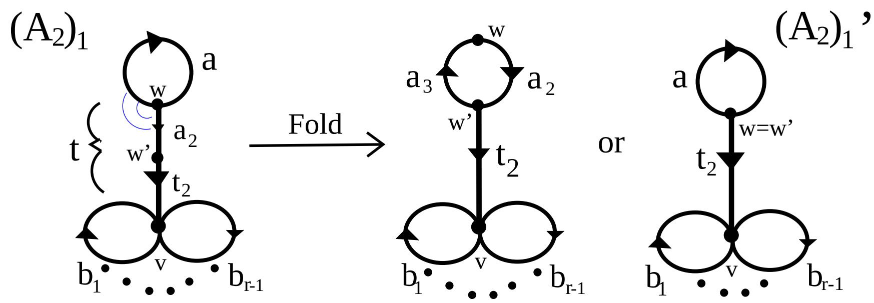


Figure 3.3: *Of the two scenarios on the right, the leftmost is where the fold ends in the middle of a. a_2 is a possible portion of a folded with the portion of t, a_3 would be the portion of a not folded with t, and t_2 would be the portion of t not folded with a*

Let h^1 be such that $h = h^1 \circ g_1$, where g_1 is the single fold performed thus far. h^1 could not identify any of the directions at w': h^1 could not identify a_2 and t_2 or h would have had back-tracking on t; h^1 could not identify t_2 and $\bar{a}$ or h would have had back-tracking on a; and h^1 could not identify t_2 and $\overline{a_3}$ because the fold was maximal. But then all the directions of $(A_2)_1'$ would be fixed by h^1, making h^1 a homeomorphism and the Stallings decomposition complete. However, this would make h consist of the single fold g_1 and a homeomorphism, contradicting h's irreducibility. We have thus shown all cases where $\Gamma = A_2$ either impossible or yielding the representative on the rose for (1).

We are left to analyze when $\Gamma = A_3$. In this case, v must be the vertex with $2r-1$ fixed directions. As with A_2, because h must fix all $2r-1$ directions at v, if h had an illegal turn (which it still has to) the turn would be at w. Without loss of generality assume $\{b, d\}$ is an illegal turn and that the first Stallings fold maximally folds $\{b, d\}$. If all of b and d were folded, this would change the homotopy type. Thus also assume (again without losing generality) that either:

- all of b is folded with part of d or
- only proper initial segments of b and d are folded with each other.

If all of b is folded with part of d, we get a PNP-free train track representative on the rose. So suppose only proper initial segments of b and d are folded with each other.

Let $h_1 : (A_3)_1 \to (A_3)_1$ be the induced map of [BH92]. The fold and $(A_3)_1$ look like:

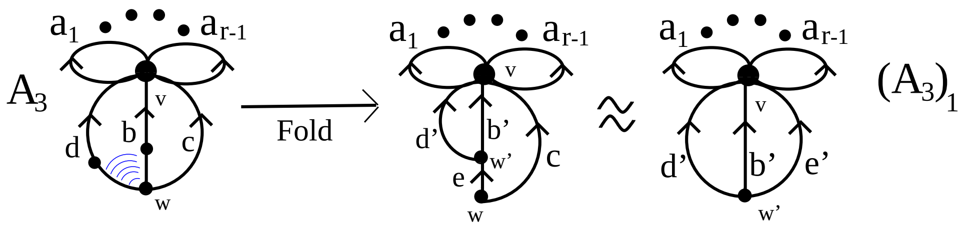


Figure 3.4: *e is the edge created by the fold and e' is $\bar{e} \cup c$ without the (now unnecessary) vertex w*

The new vertex w' has 3 distinct gates ($\{b', d'\}$ is legal since the fold was maximal and $\{b', \bar{e}\}$ and $\{d', \bar{e}\}$ must also be legal or h would have had back-tracking on b or d, respectively). This leaves the situation where the entire Stallings decomposition is a single fold with a homeomorphism, again leading to the contradiction of h being reducible.

Having ruled out all cases, we have completed the analysis of A_3 and thus proved for (1) that we have a PNP-free representative of a power $\psi = \phi^R$ of ϕ on the rose. We now prove (4).

Let $h : \Gamma \to \Gamma$ be the PNP-free train track representative of ϕ^R on the rose and let $\Gamma = \Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n = \Gamma$ be the Stallings decomposition for h. Each g_i is either an elementary fold or locally injective (in which case it would be a homeomorphism). We can assume that g_n is the only homeomorphism. Let $h^i = g_n \circ \cdots \circ g_{i+1}$. Since h has precisely $2r - 1$ gates, h has precisely one illegal turn. We first determine what g_1 could be. g_1 cannot be a homeomorphism or we would have $h = g_1$, making h reducible. So g_1 must maximally fold the illegal turn. Suppose first that the fold is a proper full fold. (If the fold is not a proper full fold, then see the analysis below about what would happen with an improper or partial fold.)

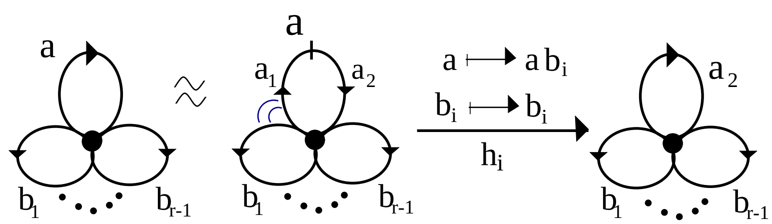


Figure 3.5: *Proper Full Fold*

By Lemma 3.2, h^1 cannot have more than one turn $\{d_1, d_2\}$ such that $Dh^1(\{d_1, d_2\})$ is degenerate (we will call such a turn an *order-one illegal turn* for h^1). If it has no order-one illegal turn, then h^1 must be a homeomorphism and we have determined the entire decomposition. So suppose that h^1 has an order-one illegal turn (it cannot have more than one or h could not have $2r-1$ distinct gates) The next Stallings fold must thus maximally fold this illegal turn. Similar logic dictates that we can can continue as such until either h is obtained, in which case the desired decomposition has been found, or until the next fold is not a proper full fold. The next fold cannot be an improper full fold because this would change the homotopy type of the rose. Suppose after the last proper full fold we have:

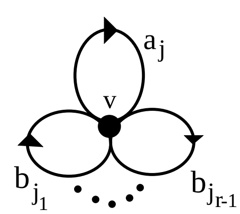


Figure 3.6: *After Last Proper Full Fold*

Suppose the illegal turn is $\{a_j, \overline{a_j}\}$ (the same argument holds for any $\{b_{j_i}, \overline{b_{j_i}}\}$). Maximally folding $\{a_j, \overline{a_j}\}$ would yield A_2, as above. This cannot be the final fold in the decomposition since A_1 is not homeomorphic to A_2. By Lemma 3.1, the illegal turn must be at w. The fold of Figure 3.4 cannot be performed, as our fold was maximal. If the fold of Figure 3.5 were performed, there would be backtracking on a.

Now suppose, without loss of generality, that the first Stallings fold that is not a proper full fold is a partial fold of b' and c', as in the following figure.

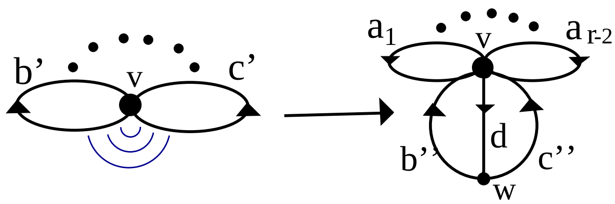


Figure 3.7: *d is the edge created by folding initial segments of b' and c', b'' is the terminal segment of b' not folded, and c'' is the terminal segment of c' not folded*

As in the analysis of the case of $\Gamma = A_3$ above, the next fold has to be at w or the next generator would be a homeomorphism, which does not make sense since A_3 is not a rose and the image of h is a rose. Since the previous fold was maximal, the next fold cannot be of $\{b'', c''\}$. Also, $\{b'', \bar{d}\}$ and $\{c'', \bar{d}\}$ cannot be illegal turns or h would have had backtracking on edges, contradicting that h is a train track. Thus, h_i was not possible in the first place, meaning that all of the folds in the Stallings decomposition must be proper full folds between roses, proving (4).

Since all folds in the Stallings decomposition are proper full folds of roses, it is possible to index the edge sets $\mathcal{E}_k = \mathcal{E}(\Gamma_k)$ as $\{E_{(k,1)}, \overline{E_{(k,1)}}, E_{(k,2)}, \overline{E_{(k,2)}}, \dots, E_{(k,r)}, \overline{E_{(k,r)}}\} = \{e_{(k,1)}, e_{(k,2)}, \dots, e_{(k,2r-1)}, e_{(k,2r)}\}$ so that, for each $1 \leq k \leq n-1$,

(a) $g_k : e_{k-1,j_k} \mapsto e_{k,i_k} e_{k,j_k}$ where $e_{k-1,j_k} \in \mathcal{E}_{k-1}$, $e_{k,i_k}, e_{k,j_k} \in \mathcal{E}_k$, and

(b) $g_k(e_{k-1,i}) = e_{k,i}$ for all $e_{k-1,i} \neq e^{\pm 1}_{k-1,j_k}$.

Suppose we similarly index the directions $D(e_{k,i}) = d_{k,i}$.

Let $g_n = h'$ be the homeomorphism in the Stalling's decomposition and suppose that Dh' its edge index permutation was nontrivial. Some power p of the permutation would be trivial. Replace h by h^p, rewriting the decomposition of h^p as follows. Let σ denote the permutation of second indices defined by Dh', i.e. $h'(e_{n-1,i}) = e_{n-1,\sigma(i)}$. Then, for $n \leq k \leq 2n-p$ define g_k by $g_k : e_{k-1,\sigma^{-s+1}(j_t)} \mapsto e_{k,\sigma^{-s+1}(i_t)} e_{k,\sigma^{-s+1}(j_t)}$ where $k = sp + t$ and $0 \leq t \leq p$. Adjust the corresponding proper full folds accordingly. This decomposition still gives h, but now the homeomorphism's edge index permutation is trivial, making it unnecessary for the decomposition.

This concludes the proof of the proposition.

QED

Representatives with a decomposition satisfying (I)-(II) of Proposition 3.3 will be called *ideally decomposable* with an *ideal decomposition*. We establish here notation used for discussing ideally decomposed representatives.

Standard Notation and Terminology 3.4. ((Semi)-Ideal Decompositions) For an ideally decomposable representative of a $\phi \in Out(F_r)$, we will consider the notation of the proposition standard for an ideal decomposition. Additionally,

- We denote e_{k-1,j_k} by e^{pu}_{k-1}, e_{k,j_k} by e^u_k, e_{k,i_k} by e^a_k, and $e_{k-1,i_{k-1}}$ by e^{pa}_{k-1};
- $\mathcal{D}_k$ will denote the set of directions corresponding to $\mathcal{E}_k$.
- $f_k := h_k \circ \cdots \circ h_1 \circ h_n \circ \cdots \circ h_{k+1} : \Gamma_k \to \Gamma_k$.
-
$$h_{k,i} := \begin{cases} h_k \circ \cdots \circ h_i : \Gamma_{i-1} \to \Gamma_k \text{ if } k > i \\ \text{and } h_{k,i} = h_k \circ \cdots \circ h_1 \circ h_n \circ \cdots \circ h_i \text{ if } k < i \end{cases} .$$
- G_k will denote $G(f_k)$
- $G_{k,l}$ will denote the subgraph of G_k containing
  (1) all black edges and vertices of G_k (given the same colors and labels as in G_k) and
  (2) all colored edges representing turns in $h_{k,l}(e)$ taken by $e \in \mathcal{E}_{l-1}$.
- d^u_k will denote $D_0(e^u_k)$, which we will sometimes call the ***u**nachieved direction* for h_k because it is not in the image of Dh_k.
- d^a_k will denote $D_0(e^a_k)$ and sometimes be called the *twice-**a**chieved direction* for h_k, as it is the image of both d^{pu}_{k-1} ($= D_0(e_{k-1,j_k})$) and d^{pa}_{k-1} ($= D_0(e_{k-1,i_k})$) under Dh_k. d^{pu}_{k-1} will sometimes call the ***p**re-unachieved direction* for h_k and d^{pa}_{k-1} the ***p**re-twice-achieved direction* for h_k.

- If we additionally require $\phi \in Out(F_r)$ be ageometric and fully irreducible and that $IW(\phi)$ be a Type (*) pIWG, then we will say g is of *Type (*)*. (By saying h is of Type (*), it will be implicit that, not only is ϕ ageometric and fully irreducible, but ϕ is ideally decomposed, or at least ideally decomposable).

- We use the same notation for a decomposition satisfying only conditions (I-IIb) of the proposition. A representative with such a decomposition will be instead be called *semi-ideally decomposed*. Saying that g is of Type (*) will still mean that $IW(\phi)$ is a Type (*) pIWG.

Remark 3.5. We refer to $E_{k,i}$ as E_i for all k in circumstances where we believe it will not cause confusion. In these circumstances we may also refer to Γ_k as Γ.

While we may abuse notation by writing E_i instead of $E_{(j,i)}$, unless otherwise specified, e_i will always denote an element of $\mathcal{E}(\Gamma)$ (or $\mathcal{E}(\Gamma_k)$ when specified), where the index of e_i will not necessarily match the index of the corresponding element of $\mathcal{E}(\Gamma)$ (or $\mathcal{E}(\Gamma_k)$). d_i will still denote $D_0(e_i)$.

Additionally, for reasons of typographical clarity we sometimes put parantheses around the subscripts.

Chapter 4

Lamination Train Track (LTT) Structures

This chapter contains our definitions for several different abstract and specific notions of "lamination train track (LTT) structures." M. Bestvina, M. Feighn, and M. Handel discussed in their papers slightly different notions of train track structures than the notions we describe here. However, those we describe in this chapter contain as smooth paths realizations of leaves of the attracting lamination for the outer automorphism. This fact makes them useful for ruling out the achievability of particular ideal Whitehead graphs and for constructing the particular representatives we seek. The need for all of the properties included in the LTT definitions should become clear in Chapter 5 when we prove the necessity of the "Admissible Map Properties."

4.1 Abstract Lamination Train Track Structures

An example of the following definition can be found in Figure 2.3.

Definition 4.1. A *train track graph* is a finite graph G satisfying:

STTG1: G has no valence-1 vertices;

STTG2: each edge of G has 2 distinct vertices (single edges are never loops); and

STTG3: the set of edges of G are partitioned into two subsets, $\mathcal{E}_b$ (the "black" edges) and $\mathcal{E}_c$ (the "colored" edges), such that each vertex is incident to at least one $E_b \in \mathcal{E}_b$ and at least one $E_c \in \mathcal{E}_c$.

Two train track graphs will be considered *equivalent* if they are isomorphic as graphs via an isomorphism preserving the partition of the edges into $\mathcal{E}_b$ and $\mathcal{E}_c$.

Remark 4.2. In the circumstances we will deal with, the train track graphs will be colored with black, purple, and red edges. All black edges will be in $\mathcal{E}_b$ and all purple and red edges will be in $\mathcal{E}_c$.

Definition 4.3. A path in a train track graph is *smooth* if no two consecutive edges of the path are in $\mathcal{E}_b$ and no two consecutive edges of the path are in $\mathcal{E}_c$.

An important property of a train track graph G will be its being *birecurrent*, by which we will mean that there exists a smooth line in G such that each edge of G occurs infinitely often in each end of the line.

We now give our first abstract notion of a lamination train track (LTT) structure. An example of this definition can also be found in Figure 2.3.

Definition 4.4. A *Lamination Train Track (LTT) Structure* G is a pair-labeled colored train track graph (black edges will be included, but not considered colored) satisfying each of the following:

LTT1: Each edge of $\mathcal{E}_c$ and each vertex is either purple or red. The interiors of edges of $\mathcal{E}_b$ are black.

LTT2: No pair of vertices is connected by two distinct colored edges.

LTT3: Edges of G are of the following 3 types:

> **(Black Edges):** A single black edge connects each (edge-pair)-labeled vertices. There are no other black edges. In particular, for each vertex v in G, there is a unique black edge containing v.
>
> **(Red Edges):** A colored edge is red if and only if at least one of its endpoint vertices is red.
>
> **(Purple Edges):** A colored edge is purple if and only if both endpoint vertices are purple.

G_A is an *augmented* LTT structure with *legal structure* G if it is obtained from G by adding some green edges to the colored edges of G and the green edges satisfy:

LTT4: At least one vertex of each green edge is red.

LTT5: A green edge and a nongreen edge never connect the same vertex pair.

Definition 4.5. Two LTT structures differing by an ornamentation-preserving (label and color preserving), vertex-preserving homeomorphism will be considered *equivalent.*

Standard Notation and Terminology 4.6. (LTT Structures) In the context of an LTT Structure G:

- An edge connecting a vertex pair $\{d_i, d_j\}$ will be denoted $[d_i, d_j]$.
- The interior of $[d_i, d_j]$ will be denoted (d_i, d_j).
  (While the notation $[d_i, d_j]$ may be ambiguous when there is more than one edge connecting the vertex pair $\{d_i, d_j\}$, we will be clear in such cases as to which edge we refer to.)
- $[e_i]$ will denote $[d_i, \overline{d_i}]$
- Red vertices will be called *nonperiodic (direction) vertices.*
- Red edges will be called *nonperiodic (turn) edges.*
- Purple vertices will be called *periodic (direction) vertices.*
- Purple edges will be called *periodic (turn) edges.*
- The purple subgraph of an LTT structure G will be called the *potential ideal Whitehead graph associated to* G and will be denoted $PI(G)$. For a finite graph $\mathcal{G} \cong PI(G)$, we will say that G *is an LTT Structure for* $\mathcal{G}$.
- $C(G)$ will denote the colored subgraph of the LTT structure G and will be called the *colored subgraph associated to* (or *of*) G.
- We say that the LTT structure G is *admissible* if G is additionally birecurrent as a train track graph.

- For an augmented LTT structure G_A with legal structure G, we denote the set of green edges of G_A by $\mathcal{E}_g(G_A)$ and call elements of $\mathcal{E}_g(G_A)$ *green illegal turn edges* (or say that they *correspond to illegal turns*).

4.1.1 Type (*) LTT Structures for Type (*) pIWGs

The following specialized abstract LTT structure is tailored for the case of a fully irreducible, ageometric $\phi \in Out(F_r)$ such that $IW(\phi) \cong \mathcal{G}$ is a Type (*) pIWG. For this definition, a (potential) ideal Whitehead graph must be designated, but the structure does not use or record any other information about $\phi \in Out(F_r)$. For an example, once again see Figure 2.3, which is an LTT structure for the Type (*) pIWG depicted in Figure 2.1 (Graph VI in Figure 1.1).

Definition 4.7. A *Type (*) Lamination Train Track Structure* is an LTT structure G for a Type (*) pIW graph $\mathcal{G}$ such that:

LTT(*)7: G has only a single red vertex (all other vertices are purple), and hence has a unique red edge.

The following lemma gives a precondition for a Type (*) LTT structure to be birecurrent (thus admissible). By a *valence*-1 *edge*, we will mean an edge with a valence-1 vertex.

Lemma 4.8. *If the colored subgraph of a Type (*) LTT structure G has at least one valence-*1 *edge of the form* $[x_i, \overline{x_i}]$*, then G will not be birecurrent.*

Proof: Suppose, for the sake of contradiction, that G were birecurrent with some birecurrent line l. Without loss of generality assume that x_i were the valence-l vertex in $C(G)$. Since l must be birecurrent with both orientations, we can focus on the situation where l passes over $[x_i, \overline{x_i}]$ oriented from x_i to $\overline{x_i}$. Since l is smooth, it must pass over a black edge after $[x_i, \overline{x_i}]$, but the only black edge at $\overline{x_i}$ is the black edge $[\overline{x_i}, x_i]$. After passing over the black edge $[\overline{x_i}, x_i]$, it must pass over a colored edge containing x_i. Since x_i was of valence-one in $C(G)$, this would mean that l would have to pass over $[x_i, \overline{x_i}]$ again. Inductively, one sees that l will get caught in this loop formed by the colored edge $[x_i, \overline{x_i}]$ and black edge $[\overline{x_i}, x_i]$ and never again pass over a colored edge other than $[x_i, \overline{x_i}]$ as it heads toward this end, violating birecurrency.

QED.

Definition 4.9. For a Type (*) LTT structure to be considered *indexed (edge-pair)-labeled*, we will require that it is index pair-labeled (of rank r) as a graph and that the vertices of the black edges are indexed by edge pairs. Two index pair-labeled Type (*) LTT structures G and G' will be considered *equivalent* if they are equivalent as LTT structures and if this equivalence preserves the indexing of the set $\{x_1, x_2, \dots, x_{2r-1}, x_{2r}\}$ where, for each $1 \leq i \leq r$, we have $X_i = x_{2i-1}$ and $\overline{X_i} = x_{2i}$.

Definition 4.10. A Type (*) pIW graph $\mathcal{G}$ will be called *edge-pair (index)-labeled* if its vertices are labeled by a $2r-1$ element subset of the rank r (indexed) edge pair labeling set. The edge-pair labeling will be considered *preadmissible* if $\mathcal{G}$ contains no more than one (edge-pair)-labeled edge.

The reason for this conditions is that, otherwise, no matter how one attaches the red edge, the colored graph of a Type (*) LTT structure for it would have at least one (edge-pair)-labeled edge, violating birecurrency (see Lemma 4.8).

Standard Notation and Terminology 4.11. (Type (*)) In the context of a Type (*) LTT structure G for $\mathcal{G}$:

- The label on the unique red vertex will sometimes be written d^u and called the *unachieved direction.*
- e^R, or $[t^R]$, will denote the unique red edge and $\overline{d^a}$ will denote the label on its purple vertex. So $t^R = \{d^u, \overline{d^a}\}$ and $e^R = [d^u, \overline{d^a}]$.
- $\overline{d^a}$ is contained in a unique black edge, which we call the *twice-achieved edge.*
- The other twice-achieved edge vertex will be labeled by d^a and called the *twice-achieved direction.*

- For a Type (*) LTT structure to be *admissible*, we will require that it is admissible as an LTT structure and, in particular:

(LTT(*)8)***:** $C(G)$ has no valence-1 vertices contained in purple or red edges of the form $[d, \bar{d}]$ (see Lemma 4.8).

- If G has a subscript, the subscript carries over to all relevant notation. So, for example, in G_k, d_k^u will label the red vertex and e_k^R the red edge.

4.1.2 Based Lamination Train Track Structures

Instead of relying on information about $IW(\phi)$ for $\phi \in Out(F_r)$, the following LTT structure focuses on the marked graph Γ for a representative $g : \Gamma \to \Gamma$ of ϕ.

Definition 4.12. Let Γ be a connected marked graph with no valence-one vertices. A *Lamination Train Track Structure* with *Base Graph* Γ is an LTT structure G such that:

LTT(Based)1: For each vertex $v \in \Gamma$ and direction $d \in \mathcal{D}(v)$, there exists a vertex in G labeled by d. In particular, the vertices of G are in one-to-one correspondence with $\mathcal{D}(\Gamma)$

LTT(Based)2: Edges of G are of the following 3 types:

(Purple Edges) connect the purple vertices in G corresponding to certain distinct pairs $\{d_1, d_2\}$ of directions at a common vertex v of Γ (we will call such pairs of directions *periodic turns*);

(Red Edges) connect the vertices in G corresponding to certain distinct pairs $\{d_1, d_2\}$ of directions at a common vertex v of Γ such that at least one direction in the pair is represented by a red vertex in G. (We will call such pairs of directions *nonperiodic turns*);

(Black Edges) connect precisely vertex pairs $\{D_0(e), D_0(\bar{e})\}$ where $e \in \mathcal{E}(\Gamma)$.

Standard Notation and Terminology 4.13. (Based LTT Structures) In the context of Definition 4.12, for a given vertex $v \in \Gamma$:

- We call the union of the purple edges $[d_1, d_2]$, where $d_1, d_2 \in \mathcal{D}(v)$, the *stable Whitehead graph* $SW(v, \Gamma, G)$ *at* v.

- We call the union of the purple and red edges $[d_1, d_2]$ corresponding to turns $\{d_1, d_2\}$, where $d_1, d_2 \in \mathcal{D}(v)$, the *local Whitehead graph* $LW(v, \Gamma, G)$ *at* v.

- $[e]$ will denote $[D_0(e), D_0(\overline{e})] = [d, \overline{d}]$ for each edge $e \in \mathcal{E}(\Gamma)$

- As with the Type (*) LTT structure Notation, a subscript on Γ will carry over to all notation and, for example, we will have $[e_{k,i}] = [d_{k,i}, \overline{d_{k,i}}]$ for all $e_{k,i} \in \mathcal{E}_k$.

Definition 4.14. Let Γ be an r-petaled rose with vertex v. A *Type (*) LTT Structure* G with *base graph* Γ is an LTT structure with base graph Γ additionally satisfying:

LTT(*)(Based)1: $PI(G)$ is a Type (*) pIWG.

An *Augmented Type (*) Lamination Train Track Structure for* G *with Base Graph* Γ is an augmented LTT structure G_A with legal structure G additionally satisfying:

LTT(*)(Based)2: $\overline{G_A - \mathcal{E}_G}$ is a Type (*) LTT structure with base graph Γ.

LTT(*)(Based)3: $\mathcal{E}_G$ contains only a single edge, which we denote $T(G)$, or just T, and call the *green illegal turn edge* of G or *edge corresponding to the illegal turn*;

We describe here what it means for two based LTT structures to be marked-graph equivalent.

Definition 4.15. Suppose G and G' are LTT structures with respective base graphs Γ and Γ'. A vertex-preserving, marked graph equivalence $H : \Gamma_i \to \Gamma'_i$ *extends* to an ornamentation-preserving homeomorphism $H^T : G_i \to G'_i$ if, for each edge $e \in \mathcal{E}(\Gamma)$, there exists a homeomorphism $i_e : \text{int}(e) \to \text{int}([e])$ and, for each edge $e' = H(e) \in \mathcal{E}(\Gamma)$, there exists a homeomorphism $i'_e : H(\text{int}(e)) \to H(\text{int}([e]))$ such that the following commutes:

$$
\begin{array}{ccc}
\text{int}([e]) & \xrightarrow{H^T_{\text{int}([e])}} & G'_i \\
i_e \big\uparrow & & \big\uparrow i'_e \\
\text{int}(e) & \xrightarrow{H_{\text{int}(e)}} & H(\text{int}(e))
\end{array}
$$

One would also say in such a circumstance that H^T *restricts* to H.

Suppose that G and G' are LTT structures with respective base graphs Γ and Γ'. If there exists a marked graph equivalence $H : \Gamma \to \Gamma'$ extending to an ornamentation-preserving homeomorphism $H^T : G \to G'$ then we say that $H : \Gamma \to \Gamma'$ *induces a marked-graph equivalence of G and G'* and that G and G' are *marked-graph equivalent* based LTT structures.

Definition 4.16. An indexed (edge-pair)-labeled Type (*) LTT structure G can be considered to be based at a rank-r edge-indexed rose. In such a case it will be standard to use the notation $\{D_1, \overline{D_1}, \dots, D_r, \overline{D_r}\} = \{d_1, d_2, \dots, d_{2r-1}, d_{2r}\}$ for the vertex labels (instead of $\{X_1, \overline{X_1}, \dots, X_r, \overline{X_r}\} = \{x_1, x_2, \dots, x_{2r-1}, x_{2r}\}$). Then *indexed based graph equivalence* of G_i and G_j, based at Γ_i and Γ_j respectively, will consist of

1. a homeomorphism $H : \Gamma_i \to \Gamma_j$ sending $e_{i,k}$ to $e_{j,k}$ (preserving orientation) for each $1 \leq k \leq r$ and
2. an equivalence of G_i and G_j as indexed (edge-pair)-labeled Type (*) LTT structures.

4.1.3 Maps of Based Lamination Train Track Structures

Let G and G' be LTT structures with vertex-preserving homeomorphic respective base graphs Γ and Γ'. Let $g : \Gamma \to \Gamma'$ be a tight homotopy equivalence taking each edge to a nondegenerate edge-path. Recall that Dg induces a map of turns $D^t g : \{a, b\} \mapsto \{Dg(a), Dg(b)\}$. Dg additionally induces a map on the corresponding edges of $C(G)$ and $C(G')$ if the appropriate edges exist in $C(G')$:

Definition 4.17. Let G and G' be LTT structures with vertex-preserving homeomorphic respective base graphs Γ and Γ'. Let $g : \Gamma \to \Gamma'$ be a tight homotopy equivalence taking each edge to a nondegenerate edge-path. When the map sending

1. the vertex labeled d in G to that labeled by $Dg(d)$ in G' and

2. the edge $[d_i, d_j]$ in $C(G)$ to the edge $[Dg(d_i), Dg(d_j)]$ in $C(G')$

also satisfies that

3. each $LW(\Gamma, v)$ is mapped into $LW(\Gamma', g(v))$ and

4. each $SW(\Gamma, v)$ is mapped isomorphically onto $SW(\Gamma', g(v))$,

then we call it the *a map of colored subgraphs induced by* g and denote it $D^C(g) : C(G) \to C(G')$.

We now describe what it means for such a $g : \Gamma \to \Gamma'$ to extend to a map of based LTT structures.

Definition 4.18. Let G and G' be LTT structures with vertex-preserving homeomorphic respective base graphs Γ and Γ'. Let $g : \Gamma \to \Gamma'$ be a tight homotopy equivalence taking each edge to a nondegenerate edge-path. . When it exists, the map $D^T(g) : G \to G'$ *induced by* g is the extension of $D^C(g) : C(G) \to C(G')$ taking the interior of the black edge of G corresponding to the edge $E \in \mathcal{E}(\Gamma)$ to the interior of the smooth path in G' corresponding to $g(E)$. In this case we say that $g^T = D^T(g) : G \to G'$ is the *extension* of $g : \Gamma \to \Gamma'$ to the map of based LTT structures and that $g : \Gamma \to \Gamma'$ *extends* to $D^T(g)$.

Remark 4.19. The same definitions work when G and G' be indexed (edge-pair)-labeled Type (*) LTT structures based respectively at the rank-r edge-indexed roses Γ and Γ'.

Example 4.20. We describe here an induced map of LTT structures for the map $g_2 : x \mapsto xz$ of the base roses.

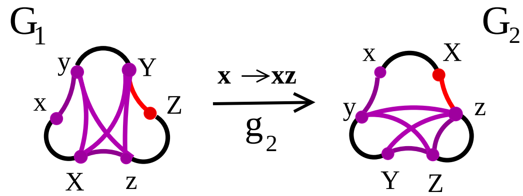


Figure 4.1: *The induced map of LTT structures for $g_2 : x \mapsto xz$ would send the vertex labeled $\bar{x}$ in G_1 to the vertex labeled $\bar{z}$ in G_2 and send every other vertex in G_1 to the identically labeled vertex in G_2. [y] in G_1 would map to [y] in G_2, [z] in G_1 would map to [z] in G_2, and [x] in G_1 would map to $[x] \cup [\bar{x}, z] \cup [z]$ in G_2. The purple edge $[\bar{x}, y]$ in G_1 would map to the purple edge $[\bar{z}, y]$ in G_2, the purple edge $[\bar{x}, \bar{y}]$ in G_1 would map to the purple edge $[\bar{z}, \bar{y}]$ in G_2, $[\bar{x}, z]$ in G_1 would map to the purple edge $[\bar{z}, z]$ in G_2, and each other purple edge in G_1 would be sent to the identically labeled purple edge in G_2. Finally, the red edge $[\bar{z}, \bar{y}]$ in G_1 would be sent to the purple edge $[\bar{z}, \bar{y}]$ in G_2.*

4.1.4 Indexed Generating Triples

Since we deal with representatives decomposed into Nielsen generators, we will use an abstract notion of an "indexed generating triple" (see Figures 7.5 and or 7.6 Example 4.25).

Definition 4.21. By a *triple* (g_k, G_{k-1}, G_k), we will mean an ordered set of three objects where $g_k : \Gamma_{k-1} \to \Gamma_k$ is a proper full fold of roses, and, for each $i = k-1, k$, G_i is an LTT structure with base graph Γ_i.

Definition 4.22. An *indexed generating triple* is a triple (g_k, G_{k-1}, G_k) where

(GTI) $g_k : \Gamma_{k-1} \to \Gamma_k$ is a proper full fold of edge-indexed roses defined by

a. $g_k(e_{k-1,j_k}) = e_{k,i_k} e_{k,j_k}$ where $d^a_k = D_0(e_{k,i_k})$, $d^u_k = D_0(e_{k,j_k})$, and $e_{k,i_k} \neq (e_{k,j_k})^{\pm 1}$ and

b. $g_k(e_{k-1,t}) = e_{k,t}$ for all $e_{k-1,t} \neq (e_{k,j_k})^{\pm 1}$;

(GTII) G_i is an indexed (edge-pair)-labeled Type (*) LTT structure with base graph Γ_i for $i = k-1, k$; and

(GTIII) The induced map of based LTT structures $D^T(g_k): G_{k-1} \to G_k$ exists and, in particular, restricts to an isomorphism from $PI(G_{k-1})$ to $PI(G_k)$.

Standard Notation and Terminology 4.23. (Indexed Generating Triples) In the context of Definition 4.12 and an indexed generating triple (g_k, G_{k-1}, G_k):

1. The triple will be called *admissible* if G_k and G_{k-1} are both birecurrent (and thus are actually indexed (edge-pair)-labeled Type (*) admissible LTT structures) and if either $d^u_{k-1} = d_{k-1,j_k}$ or $d^u_{k-1} = d_{k-1,i_k}$. In this case g_k will also be considered *admissible.*

2. We call G_{k-1} the *source LTT structure* and G_k the *destination LTT structure.*

3. g_k will be called the *(ingoing) generator* and will sometimes be written $g_k : e^{pu}_{k-1} \mapsto e^a_k e^u_k$ ("p" is for "pre"). Thus, d_{k-1,j_k} will sometimes be written d^{pu}_{k-1}.

4. e^{pa}_{k-1} denotes e_{k-1,i_k} (again "p" is for "pre").

5. If G_k and G_{k-1} are additionally both indexed (edge-pair)-labeled Type (*) LTT structures for a given Type (*) PIW graph $\mathcal{G}$, then the indexed generating triple (g_k, G_{k-1}, G_k) will be called an *indexed generating triple for* $\mathcal{G}$.

6. T_{k-1} will denote the turn $\{d^{pu}_{k-1}, d^{pa}_{k-1}\}$.

Remark 4.24. An important distinction to make here notationally is that, while each d^u_i is determined by the red vertex of G_i (and thus does not rely on other information in the triple), d^{pu}_{k-1} and d^{pa}_{k-1} actually rely on information in the triple, cannot be determined by knowing only G_{k-1}.

Example 4.25. The triple (g_2, G_1, G_2) of Example 4.20 is an example of an indexed generating triple where x is being used to denoted both $E_{1,1}$ and $E_{2,1}$,y is being used to denoted both $E_{1,2}$ and $E_{2,2}$, and z is being used to denoted both $E_{1,3}$ and $E_{2,3}$. Another possible indexed generating triple (g_2, G'_1, G_2) for the same g_2 and G_2 is given by:

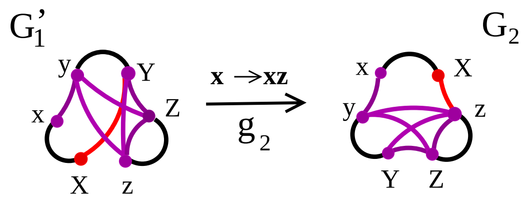


Figure 4.2: *The induced map of LTT structures for $g_2 : x \mapsto xz$ would send the vertex labeled $\bar{x}$ in G_1' to the vertex labeled $\bar{z}$ in G_2 and send every other vertex in G_1' to the identically labeled vertex in G_2. [y] in G_1' would map to [y] in G_2, [z] in G_1' would map to [z] in G_2, and [x] in G_1' would map to $[x] \cup [\bar{x}, z] \cup [z]$ in G_2. The red edge $[\bar{x}, \bar{y}]$ in G_1' would map to the purple edge $[\bar{z}, \bar{y}]$ in G_2 and each purple edge in G_1' would be sent to the identically labeled purple edge in G_2.*

The following establishes equivalences for indexed generating triples.

Definition 4.26. Suppose that (g_i, G_{i-1}, G_i) and $(g_i', G_{i-1}', G_i)'$ are indexed generating triples. Let $g_i^T : G_{i-1} \to G_i$ be the map of LTT structures induced by $g_i : \Gamma_{i-1} \to \Gamma_i$ and let $g_i^T : G_{i-1}' \to G_i'$ be the map of LTT structures induced by $g_i : \Gamma_{i-1}' \to \Gamma_i'$.

We say that (g_i, G_{i-1}, G_i) and (g_i', G_{i-1}', G_i') are *equivalent* indexed generating triples if there exist indexed (edge-pair)-labeled graph equivalences $H_{i-1} : \Gamma_{i-1} \to \Gamma_{i-1}'$ and $H_i : \Gamma_i \to \Gamma_i'$ such that

- $H_i : \Gamma_i \to \Gamma_i'$ induces an equivalence of G_i and G_i' as indexed (edge-pair)-labeled LTT structures,
- $H_{i-1} : \Gamma_{i-1} \to \Gamma_{i-1}'$ induces an equivalence of G_{i-1} and G_{i-1}' as indexed (edge-pair)-labeled LTT structures,
- and the following diagram commutes:

$$\begin{tikzcd}
\Gamma_i \arrow[r, "H_i"] & \Gamma_i' \\
\Gamma_{i-1} \arrow[u, "g_i"] \arrow[r, "H_{i-1}"] & \Gamma_{i-1}' \arrow[u, "g_i'"']
\end{tikzcd}$$

4.2 LTT Structures of Type (*) Representatives

We now give a few definitions that will enable us to apply the abstract definitions given earlier to the setting of Type (*) representatives, as defined in Chapter 3.

Definition 4.27. Let $g : \Gamma \to \Gamma$ be ideally decomposable with the standard notation 3.4, except that the $2r-1$ periodic directions may not actually be fixed.

The *Colored local Whitehead graph at the vertex* $v \in \Gamma$, $CW(g; v)$, is the uncolored graph $LW(g; v)$ but with the subgraph $SW(g; v)$ colored purple and $LW(g; v) - SW(g; v)$ colored red (including the nonperiodic vertices).

Let Γ_N be the graph obtained from Γ by removing a contractible neighborhood, $N(v)$, of the vertex v of Γ and adding vertices d_i and $\overline{d_i}$ at the corresponding boundary points of each partial edge $E_i - (N(v) \cap E_i)$, for each $E_i \in \mathcal{E}^+$. A *Lamination Train Track Structure* $G(g)$ for g is formed from $\Gamma_N \bigsqcup CW(g; v)$ by identifying the vertex labeled d_i in Γ_N with the vertex labeled d_i in $CW(g; v)$. The vertices for nonperiodic directions are red, the edges of Γ_N remain black, and all periodic vertices remain purple.

An LTT structure $G(g)$ is given a *smooth structure* via a partition of the edges at each vertex into the two sets: $\mathcal{E}_b$ (containing all black edges of $G(g)$) and $\mathcal{E}_c$ (containing all colored edges of $G(g)$). A *smooth path* will be a path alternating between colored and black edges.

The *Augmented Lamination Train Track Structure* for g, $G_A(g)$, is formed from $G(g)$ by adding a green edge for each illegal turn of g.

In Chapter 6 we show that, for a Type (*) representative $g : \Gamma \to \Gamma$, $G(g)$ is a Type (*) LTT structure with base graph Γ.

Remark 4.28. We record here the following remarks about LTT structures:

(1) $G(g)$ could also be built from $\bigsqcup_{\text{vertices } v \in \Gamma} CW(g; v)$ by adding a black edge connecting each vertex pair $\{D_0(e_i), \overline{D_0(e_i)}\}$.

(2) The train track structures we define are not quite the same as those in [BH97].

Each edge image path $g(e_i) = e_{j_1} \dots e_{j_k}$ determines a smooth path in $G(g)$ that transverses the black edge $[d_{j_1}, \overline{d_{j_1}}]$, then the colored edge $[\overline{d_{j_1}}, d_{j_2}]$, then the black edge

$[d_{j_2}, \overline{d_{j_2}}]$, and so on, until it ends with the black edge $[d_{j_k}, \overline{d_{j_k}}]$. This observation is related to one of the most important properties of LTT structures for fully irreducible representatives, i.e. they contain leaves of the attracting lamination as locally smoothly embedded lines. (Lemma 5.7).

Definition 4.29. Let $g : \Gamma \to \Gamma$ be a train track representative of a fully irreducible, ageometric $\phi \in Out(F_r)$ and γ a smooth (possibly infinite) path in $G(g)$. The *path (or line) in G corresponding to* γ is $\dots e_{-j}e_{-j+1}\dots e_{-1}e_0e_1\dots e_j\dots$, with

$$\gamma = \dots [d_{-j}, \overline{d_{-j}}][\overline{d_{-j}}, d_{-j+1}]\dots[\overline{d_{-1}}, d_0][d_0, \overline{d_0}][\overline{d_0}, d_1]\dots[d_j, \overline{d_j}]\dots,$$

where each $d_i = D_0(e_i)$, each $[d_i, \overline{d_i}] = [e_i]$ is the black edge of G corresponding to the edge $e_i \in \mathcal{E}(\Gamma)$, and each $[d_i, \overline{d_{i+1}}]$ is a colored edge. We denote such a path

$$\gamma = \dots [d_{-j}, \overline{d_{-j}}, d_{-j+1}, \dots, \overline{d_{-1}}, d_0, \overline{d_0}, d_1, \dots, d_j, \overline{d_j}\dots].$$

Definition 4.30. Let $g : \Gamma \to \Gamma$ be an ideally decomposed Type (*) representative of $\phi \in Out(F_r)$ with the standard 3.4 notation. Then G_k will denote the LTT structure $G(f_k)$ and $G_{k,l}$ will denote the subgraph of G_l containing

(1) all black edges and vertices (given the same colors and labels as in G_l) and

(2) all colored edges representing turns in $g_{k,l}(e)$ for some $e \in \mathcal{E}_{k-1}$.

For any k, l, we have a direction map $Dg_{k,l}$ and an induced map of turns $Dg^t_{k,l}$. The *induced map of LTT Structures* $Dg^T_{k,l} : G_{l-1} \mapsto G_k$ is such that

(1) the vertex corresponding to a direction d is mapped to the vertex corresponding to the direction $Dg_{k,l}(d)$,

(2) the colored edge $[d_1, d_2]$ is mapped linearly as an extension of the vertex map to the edge $[Dg^t_{k,l}(\{d_1, d_2\})] = [Dg_{k,l}(d_1), Dg_{k,l}(d_2)]$, and

(3) the interior of the black edge of G_{l-1} corresponding to the edge $E \in \mathcal{E}(\Gamma_{l-1})$ to the interior of the smooth path in G_k corresponding to $g(E)$.

Remark 4.31. It still makes sense to define G_k when ϕ is only irreducible (not fully irreducible) and possibly even is not ageometric. The difference will be that, while the purple subgraph will be $SW(g)$, it will not necessarily be $IW(g)$. If Γ had more than one vertex, one would define $G(g)$ by creating a colored graph $CW(g;v)$ for each vertex, removing an open neighborhood of each vertex when forming Γ_N, and then continuing with the identifications as above in $\Gamma_N \bigsqcup (\cup CW(g;v))$. Dropping the condition on g having $2r-1$ fixed directions more drastically changes what definitions actually make sense or what they look like if they do make sense.

Chapter 5

Admissible Map Properties

The aim of this chapter is establishing additional properties held by any Type (*) representative. In particular, we determine several necessary characteristics of LTT structures G_k arising in an ideal decomposition of a Type (*) representative and give the background to identify (as described in Chapter 7) the only two possible types of (fold/peel) relationships between any LTT structures G_{k-1} and G_k in an ideal decomposition. The properties proved necessary in this chapter will be called "Admissible Map Properties." They are summarized in the final section, Section 5.10.

In subsequent chapters, we will define and outline a method for associating, a diagram (the "AM Diagram") to a Type (*) pIW graph $\mathcal{G}$. This diagram will contain a loop for each map having the Admissible Map Properties we establish in this chapter. Thus, in particular, the diagram contains a loop for any Type (*) representative g with $IW(g) = \mathcal{G}$. If no loop in the diagram gives an irreducible, PNP-free representative g with $IW(g) = \mathcal{G}$, then we know that $\mathcal{G}$ does not occur as $IW(\phi)$ for any ageometric, fully irreducible $\phi \in Out(F_r)$. We will use this fact to rule out the possibility of achieving certain graphs in Subsection 12.2 of Chapter 12. We also give in subsequent chapters strategies for constructing the Type (*) representatives, if they do exist.

The conditions of an ideal decomposition will be relaxed slightly for many of the sections of this chapter in order to highlight the necessity of certain properties (we will make it clear when representatives will also be required to be ideally decomposed of Type (*)). *For each of these sections,* $g : \Gamma \to \Gamma$ *will be an irreducible train track representative of* $\phi \in Out(F_r)$ semi-ideally decomposed *as:* $\Gamma = \Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n = \Gamma$. *We will use the standard (semi)-ideal decomposition notation 3.4.*

We begin this chapter by proving a preliminary lemma that will be used later in this chapter to prove the necessity of the "Admissible Map Properties."

5.1 Cancellation and a Preliminary Lemma

Before stating the lemma, we clarify for the reader what is meant by "cancellation."

Definition 5.1. We say that an edge path $\gamma = e_1 \dots e_k$ in Γ has *cancellation* if $\overline{e_i} = e_{i+1}$ for some $1 \leq i \leq k-1$. We say that g has *no cancellation on edges* if for no $l > 0$ and edge $e \in \mathcal{E}(\Gamma)$ does the edge path $g^l(e)$ have cancellation.

We are now ready to state and prove the lemma.

Lemma 5.2. *Suppose that $g : \Gamma \to \Gamma$ is a semi-ideally decomposed train track representative of $\phi \in Out(F_r)$ with the standard 3.4 notation. For this lemma we index the generators in the decomposition of all powers g^p of g so that $g^p = g_{pn} \circ g_{pn-1} \circ \cdots \circ g_{(p-1)n} \circ \cdots \circ g_{(p-2)n} \circ \cdots \circ g_{n+1} \circ g_n \circ \cdots \circ g_1$ ($g_{mn+i} = g_i$, but we want to use the indices to keep track of a generator's place in the decomposition of g^p). With this notation, $g_{k,l}$ will mean $g_k \circ \cdots \circ g_l$. Then:*

(1) for each $e \in \mathcal{E}(\Gamma_{l-1})$, no $g_{k,l}(e)$ has cancellation;

(2) for each $0 \leq l \leq k$ and each edge $E_{l-1,i} \in \mathcal{E}^+(\Gamma_{l-1})$, the edge $E_{k,i}$ is contained in the edge path $g_{k,l}(E_{l-1,i})$; and

(3) if $e^u_k = e_{k,j}$, then the turn $\{\overline{d^a_k}, d^u_k\}$ is in the edge path $g_{k,l}(e_{l-1,j})$, for all $0 \leq l \leq k$.

Proof: Let s be minimal so that some $g_{s,t}(e_{t-1,j})$ has cancellation. Before continuing with our proof of (1), we first proceed by induction on $k-l$ to show that (2) holds for $k < s$. For the base case observe that $g_{l+1}(e_{l,j}) = e_{l+1,j}$ for all $e_{l+1,j} \neq (e^{pu}_l)^{\pm 1}$. Thus, if $e_{l,j} \neq e^{pu}_l$ and $e_{l,j} \neq \overline{e^{pu}_l}$ then $g_{l+1}(e_{l,j})$ is precisely the path $e_{l+1,j}$ and so we are only left for the base case to consider when $e_{l,j} = (e^{pu}_l)^{\pm 1}$. If $e_{l,j} = e^{pu}_l$, then $g_{l+1}(e_{l,j}) = e^a_{l+1} e_{l+1,j}$ and so the edge path $g_{l+1}(e_{l,j})$ contains $e_{l+1,j}$, as desired. If $e_{l,j} = \overline{e^{pu}_l}$, then $g_{l+1}(e_{l,j}) = e_{l+1,j}\overline{e^a_{l+1}}$ and so the edge path $g_{l+1}(e_{l,j})$ also contains $e_{l+1,j}$ in this case. Having considered all possibilities, the base case is proved.

For the inductive step, we assume that $g_{k-1,l+1}(e_{l,j})$ contains $e_{k-1,j}$ and show that $e_{k,j}$ is in the edge path $g_{k,l+1}(e_{l,j})$. Let $g_{k-1,l+1}(e_{l,j}) = e_{i_1} \dots e_{i_{q-1}} e_{k-1,j} e_{i_{q+1}} \dots e_{i_r}$ for some edges $e_i \in \mathcal{E}_{k-1}$. As in the base case, for all $e_{k-1,j} \neq (e_k^u)^{\pm 1}$, $g_k(e_{k-1,j})$ is precisely the edge path $e_{k,j}$. Thus (since g_k is an automorphism and since there is no cancellation in $g_{j_1,j_2}(e_{j_1,j_2})$ for $1 \leq j_1 \leq j_2 \leq k$), $g_{k,l+1}(e_{l,j}) = \gamma_1 ... \gamma_{q-1}(e_{k,j})\gamma_{q+1}...\gamma_m$ where each $\gamma_{i_j} = g_l(e_{i_j})$ and where no $\{\overline{\gamma_i}, \gamma_{i+1}\}$, $\{\overline{e_{k,j}}, \gamma_{q+1}\}$, or $\{\overline{\gamma_{q-1}}, e_{k,j}\}$ is an illegal turn. So each $e_{k,j}$ is in $g_{k,l+1}(e_{l,j})$, as desired. We are only left to consider for the inductive step the cases where $e_{k-1,j} = e_k^{pu}$ and where $e_{k-1,j} = \overline{e_k^{pu}}$.

If $e_{k-1,j} = e_k^{pu}$, then $g_k(e_{k-1,j}) = e_k^a e_{k,j}$, and so $g_{k,l+1}(e_{l,j}) = \gamma_1 ... \gamma_{q-1} e_k^a e_{k,j} \gamma_{q+1} \dots \gamma_m$ (where no $\{\overline{\gamma_i}, \gamma_{i+1}\}$, $\{\overline{e_{k,j}}, \gamma_{q+1}\}$, or $\{\overline{\gamma_{q-1}}, e_k^a\}$ is an illegal turn), which contains $e_{k,j}$, as desired. If instead $e_{k-1,j} = \overline{e_k^{pu}}$, then $g_k(e_{k-1,j}) = e_{k,j}\overline{e_k^a}$ and so $g_{k,l+1}(e_{l,j}) = \gamma_1 \dots \gamma_{q-1} e_{k,j} \overline{e_k^a} \gamma_{q+1} \dots \gamma_m$, which also contains $e_{k,j}$. Having considered all possibilities, the inductive step is now also proven and the proof is complete for (2) in the case of $k < s$.

We now finish our proof of (1). We are still assuming that s is minimal so that $g_{s,t}(e_{t-1,j})$ has cancellation for some $e_{t-1,j} \in \mathcal{E}_j$. Let t be such that $g_{s,t}(e_{t-1,j})$ has cancellation. Let α_j, for $1 \leq j \leq m$, be edges in Γ_{s-1} so that $g_{s-1,t}(e_{t-1,j}) = \alpha_1 \cdots \alpha_m$. Since s was minimal, either $g_s(\alpha_i)$ has cancellation for some $1 \leq i \leq m$ or $Dg_s(\overline{\alpha_i}) = Dg_s(\alpha_{i+1})$ for some $1 \leq i < m$. Since each g_s is a generator, no $g_s(\alpha_i)$ has cancellation. Thus, there exists an i such that $Dg_s(\overline{\alpha_i}) = Dg_s(\alpha_{i+1})$. Since we have already proved (1) for all $k < s$, we know that the edge path $g_{t-1,1}(e_{0,j})$ contains $e_{t-1,j}$. Then $g_{s,1}(e_{0,j}) = g_{s,t}(g_{t-1,1}(e_{0,j}))$ contains cancellation, which implies that $g^p(e_{0,j}) = g_{pn,s+1}(g_{s,1}(e_{0,j})) = g_{s,t}(\dots e_{t-1,j} \dots)$ for some p (with $pn > s+1$) contains cancellation, which contradicts that g is a train track map.

We now prove (3). Let $e_k^u = e_{k,l}$. By (2) we know that the edge path $g_{k-1,l}(e_{l-1,j})$ contains $e_{k-1,j}$. Let $e_1, \dots e_m \in \mathcal{E}_{k-1}$ be such that $g_{k-1,l}(e_{l-1,j}) = e_1 \dots e_{q-1} e_{k-1,j} e_{q+1} \dots e_m$. Then $g_{k,l}(e_{l-1,j}) = \gamma_1 \dots \gamma_{q-1} e_k^a e_k^u \gamma_{q+1} \dots \gamma_r$ where $\gamma_j = g_k(e_j)$ for all j. Thus $g_{k,l}(e_{k-1}^{pu})$ contains $\{\overline{d_k^a}, d_k^u\}$, as desired.

QED.

5.2 LTT Structures, Birecurrency, and AM Property I

LTT structures were defined in Chapter 4 and the "birecurrency"(defined below) of each LTT structure G_k in a semi-ideal decomposition is the first property we will prove necessary for a Type (*) representative, i.e. *AM Property I.*

Definition 5.3. We will say that a train track graph G is *birecurrent* if there exists a locally smoothly embedded line in G that crosses each edge of G infinitely many times as $\mathbf{R} \to \infty$ and as $\mathbf{R} \to -\infty$.

Proposition 5.4. *Let $g : \Gamma \to \Gamma$ be a Type (*) representative of $\phi \in Out(F_r)$. Then $G(g)$ is birecurrent.*

Our proof of this proposition will require the following lemmas recording the relationship between the local Whitehead graph for g, $LW(g)$, and the realization of the leaves of the attracting lamination, Λ_ϕ, for ϕ. The proofs will use facts about laminations that can be found in [BFH97] and [HM11], but will not be recorded here.

Lemma 5.5. *Let $g : \Gamma \to \Gamma$ be a Type (*) representative of $\phi \in Out(F_r)$. The only turns possible in the realization in Γ of a leaf of the attracting lamination Λ_ϕ for ϕ are those corresponding to edges in $LW(g)$. Conversely, each turn represented by an edge of $LW(g)$ is a turn of some (hence all) leaves of Λ_ϕ (as realized in Γ).*

Proof: To prove the forward direction, we first notice, as follows, that each edge $E_i \in \mathcal{E}(\Gamma)$ has a fixed point in its interior. Since g is irreducible, some $g^k(E_i)$ contains a path with at least three edges (some $g^k(E_i)$ contains at least two edges of Γ, including E_i and then $g^{2k}(E_i)$ contains at least three edges). Let $g^k(E_i) = e_1e_2...e_m$, with each $e_i \in \mathcal{E}(\Gamma)$. Again, since g is irreducible, for some l, the edge path $g^l(e_2)$ contains either E_i or $\overline{E_i}$. Thus, $g^{k+l}(E_i)$ contains either E_i or $\overline{E_i}$ in its interior, implying that E_i has a fixed point in its interior. This then tells us that, for each edge $E_i \in \mathcal{E}(\Gamma)$, there is a periodic leaf of Λ_ϕ obtained by iterating a neighborhood of a fixed point of E_i.

Consider any turn $\{d_1, d_2\}$ taken by the realization in Γ of a leaf L of Λ_ϕ. Since periodic leaves are dense in the lamination, either $\overline{e_1}e_2$ or $\overline{e_2}e_1$ (where $D_0(e_1) = d_1$ or

$D_0(e_2) = d_2$) is a subpath of any periodic leaf of the lamination. In particular, either $\overline{e_1}e_2$ or $\overline{e_2}e_1$ is a subpath of the leaf obtained by iterating a neighborhood of a fixed point of e for any $e \in \mathcal{E}(\Gamma)$, so $\overline{e_1}e_2$ is contained in some $g^k(e)$, for each $e \in \mathcal{E}(\Gamma)$. Thus, $\{d_1, d_2\}$ is represented by an edge in $LW(g)$, as desired. This concludes the forward direction.

We now prove the converse. The presence of the turn $\{d_1, d_2\}$ as an edge of $LW(g)$ indicates that, for some i and k, $\overline{e_1}e_2$ is a subpath of $g^k(E_i)$. We showed above that each $E_i \in \mathcal{E}(\Gamma)$ has a fixed point in its interior and hence that there is a periodic leaf of Λ_ϕ obtained by iterating a neighborhood of the fixed point of E_i. $g^k(E_i)$ is a subpath of this periodic leaf and (since periodic leaves are dense) of every leaf of Λ_ϕ. Since the leaves contain $g^k(E_i)$ as a subpath, they contain $\overline{e_1}e_2$ as a subpath, and thus the turn $\{d_1, d_2\}$. This concludes the proof of the converse, and hence lemma.
QED.

We will need one more definition for the proof of the second lemma.

Definition 5.6. Let $g : \Gamma \to \Gamma$ be a train track representative of a fully irreducible, ageometric $\phi \in Out(F_r)$. Let γ be a smooth (possibly infinite) path in $G(g)$. The *path (or line) in* Γ *corresponding to* γ is $\dots e_{-j}e_{-j+1}\dots e_{-1}e_0e_1\dots e_j\dots$, where

$$\gamma = \dots[d_{-j}, \overline{d_{-j}}][\overline{d_{-j}}, d_{-j+1}]\dots[d_{-1}, \overline{d_{-1}}][\overline{d_{-1}}, d_0][d_0, \overline{d_0}][\overline{d_0}, d_1][d_1, \overline{d_1}]\dots[d_j, \overline{d_j}]\dots,$$

where each $d_i = D_0(e_i)$, each $[d_i, \overline{d_i}] = [e_i]$ is the black edge of G corresponding to the edge $e_i \in \mathcal{E}(\Gamma)$, and each $[d_i, \overline{d_{i+1}}]$ is a colored edge.

Lemma 5.7. *Let* $g : \Gamma \to \Gamma$ *be a TT representative of a fully irreducible, ageometric* $\phi \in Out(F_r)$*. Then* $G(g)$ *contains smooth paths corresponding to the realizations in* Γ *of the leaves of* Λ_ϕ*.*

Proof of Lemma: Consider the realization λ of a leaf of Λ_ϕ and any single subpath $\sigma = e_1e_2e_3$ in λ. If it exists, the representation in $G(g)$ of σ would be by the path $[d_1, \overline{d_1}][\overline{d_1}, d_2][d_2, \overline{d_2}][\overline{d_2}, d_3][d_3, \overline{d_3}]$, as above. Lemma 5.5 above tells us that $[\overline{d}_1, d_2]$ and $[\overline{d_2}, d_3]$ are edges of $LW(g)$ and hence are colored edges in $G(g)$. The path representing σ in $G(g)$ thus exists and alternates between colored and black edges. By looking at overlapping subpaths, we can see that the path in $G(g)$ corresponding to λ has no

consecutive colored or black edges and so is smooth. We have proved the lemma.
QED.

We are now ready for the proof of the proposition.

Proof of Proposition 5.4: We need that $G(g)$ contains a locally smoothly embedded line crossing over each edge of $G(g)$ infinitely many times as $\mathbf{R} \to \infty$ and as $\mathbf{R} \to -\infty$. We will show that the path γ corresponding to the realization λ of a leaf of Λ_ϕ is such a line. We first consider any colored edge $[d_i, d_j]$ in $G(g)$. By Lemma 5.5, λ must contain either $\overline{e_i}e_j$ or $\overline{e_2}e_1$ as a subpath. Birecurrency of the lamination leaves of a fully irreducible $\phi \in Out(F_r)$ implies that γ must cross the subpath $\overline{e_i}e_j$ or $\overline{e_j}e_i$ infinitely many times as $\mathbf{R} \to \infty$ and as $\mathbf{R} \to -\infty$. We showed in Lemma 5.7 above that this means that λ contains either $\overline{e_i}e_j$ or $\overline{e_j}e_i$ infinitely many times as $\mathbf{R} \to \infty$ and as $\mathbf{R} \to -\infty$. This concludes the proof for a colored edge.

Now consider a black edge $[d_l, \overline{d_l}] = [e_l]$. Each vertex is shared with a colored edge. Let $[d_l, \overline{d_m}]$ be such an edge. As shown above, $\overline{e_l}e_m$ or $\overline{e_m}e_l$ occur in realizations λ infinitely many times as $\mathbf{R} \to \infty$ and as $\mathbf{R} \to -\infty$. In particular, it crosses over e_l infinitely many times as $\mathbf{R} \to \infty$ and as $\mathbf{R} \to -\infty$. And so γ crosses over $[d_l, \overline{d_l}] = [e_l]$ infinitely many times as $\mathbf{R} \to \infty$ and as $\mathbf{R} \to -\infty$. This concludes the proof.
QED.

In combination with Proposition 5.4, the second of the following two lemmas proves the necessity of AM Property I. The first (Lemma 5.8) is used in the proof of the second (Lemma 5.10) .

Lemma 5.8. *Let g be a semi-ideally decomposed train track representative. Each f_k has the same number of gates (and thus periodic directions).*

Proof: Suppose, for the sake of contradiction, that f_k had more gates than f_l. Let p_k be such that $D(f_k^{p_k})$ maps each gate of f_k to a single direction and let p_l be such that $D(f_l^{p_l})$ maps each gate of f_l to a single direction. Let $\{\mathcal{G}_1, \dots, \mathcal{G}_s\}$ be the set of gates for f_k, let α_i be the periodic direction of $\mathcal{G}_i$ for each $1 \leq i \leq s$, let $\{\mathcal{G}'_1, \dots, \mathcal{G}'_{s'}\}$ be the set of gates for f_l, and let α'_i be the periodic direction of $\mathcal{G}'_i$ for each $1 \leq i \leq s'$.

Consider $f_k^{p_k+p_l+1} = f_{k,l+1} \circ f_l^{p_l} \circ f_{l,k+1} \circ f_k^{p_k}$. Let $\{d_1, \dots, d_t\} = D(f_{l,k+1} \circ f_k^{p_k})(\mathcal{D}_k)$. Then $\{d_1, \dots, d_t\}$ is mapped by $D(f_l^{p_l})$ into $\{\alpha'_1 \dots \alpha'_{s'}\}$ and, consequently, $D(f_l^{p_l} \circ f_{l,k+1} \circ f_k^{p_k})(\mathcal{D}_k) \subset \{\alpha'_1 \dots \alpha'_{s'}\}$. This implies that $D(f_{k,l+1})(D(f_l^{p_l} \circ f_{l,k+1} \circ f_k^{p_k})(\mathcal{D}_k)) = D(f_k^{p_k+p_l+1})(\mathcal{D}_k) \subset D(f_{k,l+1})(\{\alpha'_1 \dots \alpha'_{s'}\})$, which has at most s' elements. But this contradicts f_k having more gates that f_l. Thus, all f_k have the same number of gates. QED.

Remark 5.9. If g is an ideally decomposed Type (*) representative, then the above lemma shows that each G_k has the same number of purple periodic vertices.

Lemma 5.10. *If $g : \Gamma \to \Gamma$ satisfies (I)-(III) of Proposition 3.4, then so does each f_k. If g is ideally decomposed, then so is each f_k. If g represents $\phi \in Out(F_r)$, then each f_k represents the same ϕ. In particular, if g is of Type (*), then so is each f_k.*

Proof of Lemma: If $\Gamma = \Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n = \Gamma$ is an ideal decomposition of g, then f_k can be decomposed as $\Gamma_k \xrightarrow{g_{k+1}} \Gamma_{k+1} \xrightarrow{g_{k+2}} \cdots \xrightarrow{g_{k-1}} \Gamma_{k-1} \xrightarrow{g_k} \Gamma_k$.

What we need to show is that this decomposition of f_k is an ideal decomposition and that f_k is a representative of ϕ (we already know that ϕ is ageometric and fully irreducible, as well as that $IW(\phi)$ is a Type (*) pIWG). Properties (I)-(III) of an ideal decomposition hold for the decomposition of f_k because they hold for the decomposition of f and the decompositions have the same Γ_i and g_i (just renumbered). By the previous lemma we know that f_k also has $2r-1$ gates. Thus, Df_k fixes $2r-1$ periodic directions. Suppose, for the sake of contradiction, that the directions were not fixed.

Since d_k^u is not in the image of Dg_k, it cannot be in the image of Df_k and thus is the unique nonfixed direction. We thus also know that (IV) holds for the decomposition of f_k and we are only left to prove that f_k is a representative of ϕ. Now, g is a representative of ϕ and $g = (g_{1,k})^{-1} f_k g_{1,k}$. Let $\pi : R_r \to \Gamma$ be the marking on Γ. Since $g_{1,k}$ is a homotopy equivalence, $g_{1,k} \circ \pi$ gives a marking on Γ_k and g and f_k just differ by a change of marking. Thus, g and f_k are representatives of the same outer automorphism, i.e. ϕ. This concludes the proof.
QED.

We have thus shown that every ideally decomposed Type (*) representative satisfies

AM Property I: Each LTT structure G_k is birecurrent.

5.3 Periodic Directions and AM Property II

The main goal of this section is Proposition 5.13, giving *AM Property II* for a Type (*) representative of $\phi \in Out(F_r)$ such that $IW(\phi) = \mathcal{G}$.

We add to the notation already established that $t_k^R = \{\overline{d_k^a}, d_k^u\}$ *and* $T_k = \{d_k^{pa}, d_k^{pu}\}$.

The following lemma is used in the proof of Proposition 5.13.

Lemma 5.11. *T_k is an illegal turn for g_{k+1} and, thus, also for f_k.*

Proof: Recall that $T_k = \{d_k^{pa}, d_k^{pu}\}$. Since

$D^t g_{k+1}(\{d_k^{pa}, d_k^{pu}\}) = \{Dg_{k+1}(d_k^{pa}), Dg_k(d_k^{pu})\} = \{d_{k+1}^a, d_{k+1}^a\}$,

$D^t f_k(\{d_k^{pa}, d_k^{pu}\}) = D^t(g_{k,k+2} \circ g_{k+1})(\{d_k^{pa}, d_k^{pu}\}) =$

$D^t(g_{k,k+2})(D^t g_{k+1}(\{d_k^{pa}, d_k^{pu}\})) = D^t g_{k,k+2}(\{d_{k+1}^a, d_{k+1}^a\}) =$

$\{D^t g_{k,k+2}(d_{k+1}^a), D^t g_{k,k+2}(d_{k+1}^a)\}$, which is degenerate. So T_k is an illegal turn for f_k, as desired.

QED.

Definition 5.12. As a result of the previous lemma, for the generator $g_{k+1} : e_k^{pu} \mapsto e_{k+1}^a e_{k+1}^u$ (or f_k), we sometimes call $T_k = \{d_k^{pa}, d_k^{pu}\}$ the *green illegal turn in G_k* (even though T_k does not technically live in G_k, but in the augmented LTT structure).

We are now ready to prove the proposition.

Proposition 5.13. *Let $g : \Gamma \to \Gamma$ be semi-ideally decomposed (though not necessarily irreducible). g has $2r-1$ periodic directions if and only if, for each k, the illegal turn $T_k = \{d_k^{pa}, d_k^{pu}\}$ contains d_k^u, ie, either $d_k^{pu} = d_k^u$ or $d_k^{pa} = d_k^u$. In fact, if each T_k contains d_k^u, the image of Dg contains all directions in $\mathcal{D}(\Gamma)$ except d_n^u.*

Proof: We start by proving the forward direction. Suppose that our map g has $2r-1$ periodic directions and, for the sake of contradiction, that the illegal turn $T_k = \{d_k^{pa}, d_k^{pu}\}$ does not contain $d_k^u = d_{k,i}$. Let $d_{k+1}^u = d_{k+1,s}$ and $d_{k+1}^a = d_{k+1,t}$. Then $Dg_k(d_{k-1,s}) = d_{k,s}$ and $Dg_k(d_{k-1,t}) = d_{k,t}$, which means that

$D^t(g_{k+1} \circ g_k)(\{d_{(k-1,s)}, d_{(k-1,t)}\}) = \{D(g_{k+1} \circ g_k)(d_{(k-1,s)}), D(g_{k+1} \circ g_k)(d_{(k-1,t)})\} = \{Dg_{k+1}(d_{k,s} = d_k^{pu}), Dg_{k+1}(d_{k,t} = d_k^{pa})\} = \{d_{k+1}^a, d_{k+1}^a\}$ and so $d_{k-1,s}$ and $d_{k-1,t}$ share a gate. But $d_{k-1,i}$ is already in a gate with more than one element and we already established that $d_{k-1,i} \neq d_{k-1,s}$ and $d_{k-1,i} \neq d_{k-1,t}$. So f_{k-1} has a maximum of $2r-2$ gates. Since each f_k has the same number of gates, this would imply that g has a maximum of $2r-2$ gates, giving a contradiction. The forward direction is thus proved.

Now suppose that, for each $1 \leq k \leq n$, the illegal turn T_k for the generator g_{k+1} always contained the unachieved direction d_k^u for the generator g_k. We will proceed by induction to prove that g would then have $2r-1$ distinct gates. In fact, we will show that the image of Dg is missing precisely d_n^u, where $g = g_n \circ \cdots \circ g_1$.

For the base case we need that g_1 has $2r-1$ distinct gates. By our assumptions, $g_1 : e_0^{pu} \mapsto e_1^a e_1^u$. The direction map for g_1, Dg_1, is defined by $Dg_1(d_0^{pu}) = d_1^a$ and $Dg_1(d_{0,t}) = d_{1,t}$ for all t with $d_{0,t} \neq d_0^{pu}$. Thus, the image of Dg_1 includes $2r-1$ distinct directions and is missing precisely d_1^u. Also, the only direction with two preimages is d_1^a. This concludes the proof of the entire base case.

For the inductive step assume that $g_{k-1,1}$ has $2r-1$ distinct gates (there are $2r-1$ distinct directions (and second indices) in the image of $Dg_{k-1,1}$) and that d_{k-1}^u is the only direction not in the image of $Dg_{k-1,1}$. We also assume that g_k is defined by $g_k : e_{k-1}^{pu} \mapsto e_k^a e_k^u$ where either (1) $d_{k-1}^u = e_{k-1}^{pu}$ or (2) $d_{k-1}^u = e_{k-1}^{pa}$.

Consider Case (1) where $d_{(k-1,i_1)}, d_{(k-1,i_2)}, \dots, d_{(k-1,i_{2r-1})}$ are the $2r-1$ directions in the image of $Dg_{k-1,1}$ (none of which is $d_{k-1}^{pu} = d_{k-1}^u = d_{k-1,j}$). $Dg_k : d_{k-1}^u = d_{k-1}^{pu} \mapsto d_k^a$ and preserves the second indices of all of other directions. Since none of the $d_{(k-1,i_1)}, d_{(k-1,i_2)}, \dots, d_{(k-1,i_{2r-1})}$ are $d_{k-1}^{pu} = d_{k-1}^u = d_{k-1,j}$, Dg_k acts as the identity on the second indices of $d_{(k-1,i_1)}, d_{(k-1,i_2)}, \dots, d_{(k-1,i_{2r-1})}$, leaving $d_{(k,i_1)}, d_{(k,i_2)}, \dots, d_{(k,i_{2r-1})}$ as $2r-1$ distinct directions in the image of Dg_k (still none of which is equal to $d_k^u = d_{k,j}$) and the second indices in the image of $Dg_{k,1}$ and $Dg_{k-1,1}$ are the same. Since $d_{k-1,i_1}, d_{k-1,i_2}, \dots, d_{k-1,i_{2r-1}}$ were the only directions in the image of $Dg_{k-1,1}$, their images are the only directions in the image of $Dg_{k,1}$, meaning that $d_k^u = d_{k,j}$ is also not in the image of $Dg_{k,1}$. Thus, $g_{k,1}$ has precisely $2r-1$ distinct gates and $d_k^u = d_{k,j}$ is not in the image of $Dg_{k,1}$, which were our two desired conclusions.

Now, consider Case (2), i.e $d^u_{k-1} = d^{pa}_{k-1} (= d_{k-1,j})$, where $d^{pu}_{k-1} = d_{(k-1,i_1)}$, $d_{(k-1,i_2)}, \dots, d_{(k-1,i_{2r-1})}$ are the $2r-1$ directions in the image of $Dg_{k-1,1}$ (none of which is $d^u_{k-1} = d^{pa}_{k-1} = d_{k-1,j}$). Dg_k is defined by $Dg_k(d^{pu}_{k-1}) = d^a_k (= d_{k,j})$, mapping d^{pu}_{k-1} to $d^a_k = d_{k,j}$ and d_{k-1,i_t} to d_{k,i_t} for $2 \leq t \leq 2r-1$ (replacing the index i_1 with the previously absent index j and fixing all other indices). Since $i_1 \neq i_t$ for $1 \neq t$, since i_1 is replaced by j, and since $d^u_k = d_{k,i_1}$, we can conclude that d^u_k is not in the image of $Dg_{k,1}$. Thus we have shown our two desired conclusions in this case also, i.e. that $g_{k,1}$ has $2r-1$ distinct gates and d^u_k is not in the image of $Dg_{k,1}$, as desired.

We have thus completed the inductive step and consequently inductively proved the backward direction, completing the proof of the entire proposition.

QED.

Corollary 5.14. ***(of Proposition 5.13)*** *For each k, $t^R_k = \{\overline{d^a_k}, d^u_k\}$, must contain either d^{pu}_k or d^{pa}_k.*

Proof: We showed that, for each $1 \leq k \leq n$, the illegal turn $T_k = \{d^{pa}_k, d^{pu}_k\}$ always contains d^u_k. At the same time, we know that $t^R_k = \{\overline{d^a_k}, d^u_k\}$, implying t^R_k contains d^u_k and thus either d^{pa}_k or d^{pu}_k.

QED.

We have shown that every ideally decomposed Type (*) representative satisfies

AM Property II: At each graph G_k, the illegal turn T_k for the generator g_{k+1} exiting G_k always contains the unachieved direction d^u_k for the generator g_k entering the graph G_k, i.e. either $d^u_k = d^{pa}_k$ or $d^u_k = d^{pu}_k$.

5.4 The Nonperiodic Red Direction and AM Property III

The conditions for this section are the same as described in the start of the chapter.

The following corollary of the proof of Proposition 5.13 gives *AM Property III.*

Corollary 5.15. ***(of Proof of Proposition 5.13)*** *For each $1 \leq k \leq n$, d^u_k is not a periodic direction for f_k. In particular, the vertex labeled by d^u_k in G_k is red and $[t^R_k] = [\overline{d^a_k}, d^u_k]$ is a red edge in G_k.*

Proof: Since Dg_k is defined by $d^{pu}_{k-1} \mapsto d^a_k$ (and $Dg_k(d_{k-1,j}) = d_{k-1,j}$ for all j such that $e^{pu}_{k-1} \neq e_{k-1,j}$ and $e^{pu}_{k-1} \neq \overline{e_{k,j}}$), d^u_k is not in the image of Dg_k. Suppose for the sake of contradiction, that d^u_k is a periodic direction for f_k. Let N be a sufficiently high power for f_k so that $D(f^N_k)$ fixes all periodic directions of f_k. Then g_k would still be the final generator in the decomposition of f_k and thus d^u_k would also not be in the image of $D(f^N_k)$. This contradicts d^u_k being a periodic direction for f_k. So d^u_k is not a periodic direction for f_k and hence labels a red vertex in G_k.

Before we can identify that $[t^R_k]$ is a red edge, we first need to show that $[t^R_k]$ is an edge of $LW(f_k)$. In order to show that $[t^R_k]$ is an edge of $LW(f_k)$, it suffices to show that $[t^R_k] = [\overline{d^a_k}, d^u_k]$ is in $f_k(e^u_k)$. Let $e^u_k = e_{k,l}$. By Lemma 5.2 we know that the edge path $g_{k-1,k+1}(e^u_k = e_{k,l})$ contains $e_{k-1,l}$. Let e_j be edges in Γ_{l-1} such that $g_{k-1,k+1}(e^u_k) = e_1 \dots e_{q-1} e_{k-1,l} e_{q+1} \dots e_m$. Then $f_k(e^u_k) = g_{k,k+1}(e^u_k) = \gamma_1 \dots \gamma_{q-1} e^a_k e^u_k \gamma_{q+1} \dots \gamma_m$ where $\gamma_j = g_k(e_{i_j})$ for all j. Thus $f_k(e^u_k)$ contains $\{\bar{d}^a_k, d^u_k\}$, as desired and $[t^R_k]$ is an edge of $LW(f_k)$. Since $[\overline{d^a_k}, d^u_k]$ contains the red vertex d^u_k, $[\overline{d^a_k}, d^u_k]$ is a red edge in G_k.

QED.

Definition 5.16. As a consequence of the proof of Corollary 5.15, we will say that g_k *creates the edge* $[\overline{d^a_k}, d^u_k]$ in G_k (in the sense that $\{\bar{d}^a_k, d^u_k\}$ is a turn in the image of $g_{k,l}(e^u_k)$ for any $1 \leq l \leq n$ and $[\overline{d^a_k}, d^u_k]$ is in G_k (and is, in fact, the red edge of G_k, as it is not in the image of $d^C(g_k)$, but in the image of the black edge $[e^u_k]$ in G_{k-1} under dg^T_k)). Further details are discussed in Section 5.6.

As a consequence of Corollary 5.15, we will also henceforth sometimes refer to d^u_k as the *(red) unachieved direction in* G_k *(and* $G_{k,l}$*)*, $t^R_k = \{\overline{d^a_k}, d^u_k\}$ as the *new red turn in* G_k *(and* $G_{k,l}$*)*, and e^R_k =$[t^R_k]$ as the *red edge in* G_k *(and* $G_{k,l}$*)*. We will sometimes call d^a_k the *twice-achieved direction in* G_k*(and* $G_{k,l}$*)* for reasons ascertainable by analyzing the proof of Proposition 5.13.

Remark 5.17. Visually what we established in Corollary 5.15 for a Type (*) representative is that, in each augmented LTT structure $G_A(f_k)$, the interchapter of the red edge and green edge is the red vertex.

We have now shown that every ideally decomposed Type (*) representative satisfies

AM Property III: The vertex labeled by d_k^u is red in G_k and $[t_k^R] = [d_k^u, \overline{d_k^a}]$ is a red edge in G_k.

Since we have what is necessary to do so at this point, and it will be used later, we will prove a final lemma about periodic directions here before shifting our focus.

Lemma 5.18. *Suppose that $g : \Gamma \to \Gamma$ is semi-ideally decomposed and has $2r-1$ periodic directions. Then the image under Dg of the $2r$ directions at the vertex $v \in \Gamma$ is precisely the set of the $2r-1$ periodic directions for g.*

Proof: Since Dg_n's image is missing d_n^u, it is clear that the image of Dg has at most $2r-1$ directions. So we are left to show that Dg's image cannot be missing a periodic direction for g.

For the sake of contradiction, let d_k be a periodic direction not in the image of Dg. Then d_k is also not in the image of any Dg^n since $Dg^n = Dg \circ Dg^{n-1}$. Let N be such that Dg^N fixes every periodic direction. Then d_k is still not in the image of Dg^N, so it cannot be one of the periodic directions. This is a contradiction, meaning that the image of Dg cannot be missing a periodic direction for g. The lemma is proved.
QED.

5.5 $D^C g_{k,l}$ Edge Images and AM Property IV

The following lemma gives *AM Property IV*.

Lemma 5.19. *Let $g : \Gamma \to \Gamma$ be a semi-ideally decomposed representative of $\phi \in Out(F_r)$ with the standard 3.4 notation. If $[d_{(l,i)}, d_{(l,j)}]$ is a purple or red edge in G_l, then $[D^t g_{k,l+1}(\{d_{(l,i)}, d_{(l,j)}\})]$ is a purple edge in G_k.*

Proof: It suffices to show two things:

(1) $D^t g_{k,l+1}(\{d_{(l,i)}, d_{(l,j)}\})$ is a turn in some edge path $f_l^p(e_{l,m})$ with $p \geq 1$ and

(2) $Dg_{k,l+1}(d_{l,i})$ and $Dg_{k,l+1}(d_{l,j})$ are periodic directions for f_l.

We will proceed by induction and start with (1). For the base case of (1) assume that the turn $\{d_{(k-1,i)}, d_{(k-1,j)}\}$ is represented by a purple or red edge in G_{k-1}. Then $f_{k-1}^p(e_{k-1,t}) = s_1 \dots \overline{e_{(k-1,i)}} e_{(k-1,j)} \dots s_m$ for some edges $e_{(k-1,t)}, s_1, \dots s_m \in \mathcal{E}_{k-1}$ and $p \geq 1$. By Lemma 5.2, $e_{k-1,t}$ is contained in the edge path $g_{k-1} \circ \cdots \circ g_1 \circ g_n \circ \cdots \circ g_{k+1}(e_{k,t})$. Thus, since $f_{k-1}^p(e_{k-1,t}) = s_1 \dots \overline{e_{(k-1,i)}} e_{(k-1,j)} \dots s_m$ and no $g_{i,j}(e_{j-1,t})$ can have cancellation, $f_{k-1}^p \circ g_{k-1} \circ \cdots \circ g_1 \circ g_n \circ \cdots \circ g_{k+1}(e_{k,t})$ contains $s_1 \dots \overline{e_{(k-1,i)}} e_{(k-1,j)} \dots s_m$ as a subpath. Applying g_k to $f_{k-1}^p \circ g_{k-1} \circ \cdots \circ g_1 \circ g_n \circ \cdots \circ g_k(e_{k-1,t})$, we get $f_k^{p+1}(e_{k,t})$.

Suppose first that $Dg_k(e_{k-1,i}) = e_{k,i}$ and $Dg_k(e_{k-1,j}) = e_{k,j}$. Then $g_k(\dots \overline{e_{(k-1,i)}} e_{(k-1,j)} \dots) = \dots \overline{e_{(k,i)}} e_{(k,j)} \dots$, with possibly different edges before and after $\overline{e_{k,i}}$ and $e_{k,j}$ than before and after $\overline{e_{k-1,i}}$ and $e_{k-1,j}$. Thus, in this case, $f_k^{p+1}(\dots \overline{e_{(k-1,i)}} e_{(k-1,j)} \dots)$ contains the turn $\{d_{(k,i)}, d_{(k,j)}\}$, which in this case is $D^t g_k(\{d_{(k-1,i)}, d_{(k-1,j)}\})$. So $D^t g_k(\{d_{(k-1,i)}, d_{(k-1,j)}\})$ is represented by an edge in G_k.

Now suppose that $g_k : e_{k-1,j} \mapsto e_{k,l} e_{k,j}$. Then $g_k(\dots \overline{e_{k-1,i}} e_{k-1,j} \dots) = \dots \overline{e_{k,i}} e_{k,l} e_{k,j} \dots$, (again with possibly different edges before and after $\overline{e_{k,i}}$ and $e_{k,j}$). So $g_k(\dots \overline{e_{(k-1,i)}} e_{(k-1,j)} \dots)$ contains the turn $\{\overline{d_{(k,l)}}, d_{(k,j)}\}$, which in this case is $D^t g_k(\{d_{(k-1,i)}, d_{(k-1,j)}\})$, so $D^t g_k(\{d_{(k-1,i)}, d_{(k-1,j)}\})$ is again represented by an edge in G_k.

Finally, suppose that g_k is defined by $g_k : e_{k-1,j} \mapsto e_{k,j} e_{k,l}$. Unless $\overline{e_{k-1,i}} = e_{(k-1,j)}$, $g_k(\dots \overline{e_{(k-1,i)}} e_{(k-1,j)} \dots) = \dots \overline{e_{(k,i)}} e_{(k,j)} e_{(k,l)} \dots$, which contains the turn $\{d_{(k,i)}, d_{(k,j)}\}$ $= D^t g_k(\{d_{(k-1,i)}, d_{(k-1,j)}\})$, implying that $D^t g_k(\{d_{(k-1,i)}, d_{(k-1,j)}\})$ is represented by an edge in G_k in this case also.

If $\overline{e_{k-1,i}} = e_{k-1,j}$, then we are actually in a reflection of the previous case. The other cases ($g_k : \overline{e_{k-1,i}} \mapsto \overline{e_{k,i}} e_{k,l}$ and $g_k : \overline{e_{k-1,i}} \mapsto e_{k,l} \overline{e_{k,i}}$) follow similarly by symmetry. We have thus completed the base case for our proof of (1).

We now must prove the base case for (2). Since $[D^t g_k(\{d_{(k-1,i)}, d_{(k-1,j)}\})] = [Dg_k(d_{(k-1,i)}), Dg_k(d_{(k-1,j)})]$, both vertices of $[D^t g_k(\{d_{(k-1,i)}, d_{(k-1,j)}\})]$ are directions in the image of Dg_k. By Lemma 5.18, combined with Lemma 5.10, this means that both vertices represent periodic directions. Thus, $[D^t g_k(\{d_{(k-1,i)}, d_{(k-1,j)}\})]$ is actually a purple edge in G_k, concluding our proof of the base case.

Now suppose inductively that $[d_{(l,i)}, d_{(l,j)}]$ is a purple or red edge in G_l and $[D^t g_{k-1,l+1}(\{d_{(l,i)}, d_{(l,j)}\})]$ is a purple edge in G_{k-1}. Then the base case implies that

$[D^t g_k(D^t g_{k-1,l+1}(\{d_{(l,i)}, d_{(l,j)}\})]$ is a purple edge in G_k. But
$D^t g_k(D^t g_{k-1,l+1}(\{d_{(l,i)}, d_{(l,j)}\})) = D^t g_{k,l+1}(\{d_{(l,i)}, d_{(l,j)}\})$. So the lemma is proved.
QED.

We have now shown that every ideally decomposed Type (*) representative satisfies

AM Property IV: If $[d_{(l,i)}, d_{(l,j)}]$ is a purple or red edge in G_l, then $[D^C g_{k,l+1}(\{d_{(l,i)}, d_{(l,j)}\})]$ is a purple edge in G_k.

5.6 The Red Turn and AM Property V

The aim of this section is to better understand red edges, their properties, and how they are "created." This section should also begin to shed light on how generic edges in an ideal Whitehead graph are "created" by generators in an ideal decomposition.

For this section, $g : \Gamma \to \Gamma$ is an ideally decomposed Type (*) representative of $\phi \in Out(F_r)$ with the standard ideal decomposition 3.4 notation.

As above, we say that g_k creates the edge $e = [d_{(k,i)}, d_{(k,j)}]$ of G_k if g_k is defined by either $e_{k-1,i} \mapsto \overline{e_{k,j}} e_{k,i}$ or $e_{k-1,j} \mapsto \overline{e_{k,i}} e_{k,j}$. The first and second of the following lemmas, together with Lemma 5.24, tell us that g_k "creating" $\{\overline{d_k^a}, d_k^u\}$ means what we intuitively want for it to mean.

Lemma 5.20. *For each $1 \leq l, k \leq n$, $[D^t g_{l,k}(\{\overline{d_{k-1}^a}, d_{k-1}^u\})]$ is a purple edge in G_l.*

Proof: By Property IV proved in Lemma 5.19, it suffices to show that $[\overline{d_{k-1}^a}, d_{k-1}^u]$ is a colored edge of G_{k-1}. This was shown in Corollary 5.15.
QED.

Lemma 5.21. *$[\overline{d_k^a}, d_k^u]$ is not in $D^C g_k(G_{k-1})$.*

Proof: By Lemma 5.19, all purple and red edges of G_{k-1} are mapped to purple edges in G_k. On the other hand, $[\overline{d_k^a}, d_k^u]$ is a red edge in G_k. Thus, $[\overline{d_k^a}, d_k^u]$ is not in $D^C g_k(G_{k-1})$.
QED.

Remark 5.22. Notice that the above lemmas also show the uniqueness of g_k once the red edge and red nonperiodic direction vertex of G_k are known. This is explained further in the next chapter.

The following Lemma (together with Corollary 5.15) gives *AM Property V.*

Lemma 5.23. *$LW(g)$ can have at most 1 edge segment connecting the nonperiodic red direction vertex to the set of purple periodic direction vertices.*

Proof: First notice that the nonperiodic direction vertex is the red vertex d_k^u in G_k. If $g_k(e_{k-1,i}) = e_{k,i}e_{k,j}$, then the red direction in G_k is $\overline{d_{k,i}}$ (where $d_{k,i} = D_0(e_{k,i})$ and $d_{k,j} = D_0(e_{k,j})$). If g_k is the final generator in the decomposition, then the vertex $\overline{d_{k,i}}$ will be adjoined to the vertex for $d_{k,j}$ and only $d_{k,j}$, as every occurrence of $e_{k-1,i}$ in the image under $g_{k-1,1}$ of any edge has been replaced by $e_{k,i}e_{k,j}$ and every occurrence of $\overline{e_{k,i}}$ has been replaced by $\overline{e_{k,i}e_{k,j}}$, ie, there are no copies of $e_{k,j}$ without $e_{k,i}$ following them and no copies of $\overline{e_{k,i}}$ without $\overline{e_{k,j}}$ preceding them.
QED.

We have now shown that every ideally decomposed Type (*) representative satisfies

AM Property V: Each $C(G_k)$ can have at most one edge segment connecting the red (nonperiodic) vertex of G_k to the set of purple (periodic) vertices of G_k. This single edge is red and is in fact the edge $[t_k^R] = [d_k^u, \overline{d_k^a}]$.

5.7 The Ingoing Nielsen Generator and AM Property VI

Given an LTT structure G_k in a Type (*) representative ideal decomposition (or even just given the red vertex or red edge), there is only one possibility for the generator g_k entering G_k. We will use this fact when constructing representatives yielding our desired ideal Whitehead graphs.

We continue to assume that $g : \Gamma \to \Gamma$ is an ideally decomposed Type (*) representative of $\phi \in Out(F_r)$ with the standard ideal decomposition 3.4 notation.

The following lemma gives AM Property VI.

Lemma 5.24. *Let $g : \Gamma \to \Gamma$ be an ideally decomposed Type (*) representative of $\phi \in Out(F_r)$ with the standard 3.4 notation. Suppose that the unique red edge in G_k is $[t_k^R] = [d_{(k,j)}, \overline{d_{(k,i)}}]$ and that the vertex representing $d_{k,j}$ is red. Then $g_k(e_{k-1,j}) = e_{k,i}e_{k,j}$ and*

$g_k(e_{k-1,t}) = e_{k,t}$ for $e_{k-1,t} \neq (e_{k-1,j})^{\pm 1}$, where $D_0(e_{s,t}) = d_{s,t}$ and $D_0(\overline{e_{s,t}}) = \overline{d_{s,t}}$ for all s, t.

Proof: By the definition of an ideal decomposition, g_k must be of the form $g_k : e_{k-1,j} \mapsto e_{k,i}e_{k,j}$ ($g_k(e_{k-1,i}) = e_{k,i}$ for $e_{k-1,i} \neq (e_{k-1,j})^{\pm 1}$ and $e_{k,i} \neq (e_{k,j})^{\pm 1}$). Corollary 5.15 indicates that $D_0(e_{k,j}) = d_k^u$, i.e. the direction associated to the red vertex of G_k. In other words, the second index of d_k^u uniquely determines the index j and so $e_{k-1,j} = e_{k-1}^{pu}$ and $e_{k,i} = e_k^a$. Additionally, the proof of Corollary 5.15 indicates that $[\overline{d_{(k,i)}}, d_{(k,j)}]$ is the red edge of G_k. This means that we must have $e_{k,i} = e_k^a$. g_k has thus been determined to be $g_k : e_{k-1}^{pu} \mapsto e_k^a e_k^u$, i.e, $e_{k-1,j} \mapsto e_{k,i}e_{k,j}$, as desired.
QED.

Definition 5.25. The g_k in Lemma 5.24 will be called the *ingoing Nielsen generator* for G_k.

We have now shown that every ideally decomposed Type (*) representative satisfies

AM Property VI: Given that $[t_k^R] = [d_k^u, \overline{d_k^a}]$ is the red edge of G_k and d_k^u labels the single red vertex of G_k, g_k is defined by $g_k(e_{k-1}^{pu}) = e_k^a e_k^u$ and $g_k(e_{k-1,i}) = e_{k,i}$ for $e_{k-1,i} \neq (e_{k-1}^{pu})^{\pm 1}$, where $D_0(e_k^u) = d_k^u$, $D_0(\overline{e_k^a}) = \overline{d_k^a}$, $e_{k-1}^{pu} = e_{(k-1,j)}$, and $e_k^u = e_{k,j}$.

5.8 Isomorphic Ideal Whitehead Graphs and AM Property VII

The aim of this section is *AM Property VII* (stated in Proposition 5.26), giving that representatives of the same outer automorphism have isomorphic ideal whitehead graphs.

Proposition 5.26. *Let $g : \Gamma \to \Gamma$ be an ideally decomposed Type (*) representative of $\phi \in Out(F_r)$ with the standard 3.4 notation. For each $0 \leq l, k \leq n$, $Dg_{l,k+1}$ induces an isomorphism from SW(f_k) onto SW(f_l).*

The proof of the proposition will come after the following two lemmas used in the proof. Notice that Lemma 5.10 implies that SW(f_k) and SW(f_l) are isomorphic and so the key point of the proposition is that this isomorphism is induced by $Dg_{l,k+1}$.

Lemma 5.27. *Each $D^C f_k$ maps the purple subgraph $PI(G_k)$ of G_k isomorphically (as a graph) onto itself. Further, the graph isomorphism preserves the vertex and edge labels.*

Proof: Lemma 5.19 implies that $D^C f_k$ maps the purple subgraph of G_k into itself. However, Df_k fixes all directions corresponding to vertices of the purple graph. Thus, $D^C f_k$ restricted to $PI(G_k)$ is a label-preserving graph isomorphism onto its image.
QED.

Lemma 5.28. *The set of purple edges of G_{k-1} is mapped by $D^C g_k$ injectively into the set of purple edges of G_k.*

Proof: Since d_k^a is the only direction with more than one preimage of Dg_k and these two preimages are d_{k-1}^{pa} and d_{k-1}^{pu}, the only edges in G_k with more than one preimage under $D^C g_k$ are those of the form $[d_{(k,i)}, d_k^a]$ and the two preimages are the edges $[d_{(k-1,i)}, d_{k-1}^{pa}]$ and $[d_{(k-1,i)}, d_{k-1}^{pu}]$ in G_{k-1}. However, by Proposition 5.13, either $e_{k-1}^u = e_{k-1}^{pu}$ or $e_{k-1}^u = e_{k-1}^{pa}$, meaning that one of the preimages of d_k^a is actually d_{k-1}^u, i.e. one of the preimage edges is actually $[d_{(k-1,i)}, d_{k-1}^u]$. Since $[t_{k-1}^R]$ is the only purple or red edge of G_{k-1} containing d_{k-1}^u, one of the preimages of $[d_{(k,i)}, d_k^a]$ must be $[e_{k-1}^R]$, leaving only one possible purple preimage.
QED.

Proof of Proposition 5.26: Since compositions of injective maps are injective, by Lemma 5.28, the set of purple edges of G_k is mapped injectively by $D^C g_{l,k+1}$ into the set of purple edges of G_l. Likewise, the set of purple edges of G_l is mapped injectively by $D^C g_{k,l+1}$ into G_k. Additionally, by the first lemma proved above, $D^C f_k = (D^C g_{k,l+1}) \circ (D^C g_{l,k+1})$ is a bijection. Thus, since each of these sets of edges is a finite set, the map that $D^C g_{l,k+1}$ induces on the set of purple edges of G_k is a bijection. It is only left to show that two purple edges share a vertex in G_k if and only if their images under $D^C g_{l,k+1}$ share a vertex in G_l.

Suppose that we have two purple edges $[x, d_1]$ and $[x, d_2]$ in G_k sharing the vertex x. Then $D^C g_{l,k+1}([x, d_1]) = [Dg_{l,k+1}(x), Dg_{l,k+1}(d_1)]$ and $D^t g_{l,k+1}([x, d_2]) =$

$[Dg_{l,k+1}(x), Dg_{l,k+1}(d_2)]$ share the vertex $Dg_{l,k+1}(x)$. This proves the forward direction. To prove the other direction, observe that, if two purple edges $[w, d_3]$ and $[w, d_4]$ in G_l share the vertex w, then $[D^t g_{k,l+1}(\{w, d_3\})] = [Dg_{k,l+1}(w), Dg_{k,l+1}(d_3)]$ and $[D^t g_{k,l+1}(\{w, d_4\})] = [Dg_{k,l+1}(w), Dg_{k,l+1}(d_4)]$ share the vertex $Dg_{k,l+1}(w)$ in G. Since $D^C f_l$ is an isomorphism on $PI(G_l)$, $D^C g_{l,k+1}$ and $D^C g_{k,l+1}$ act on inverses. So the preimages of $[w, d_3]$ and $[w, d_4]$ under $D^C g_{l,k+1}$ share a vertex in G_l.

QED.

Corollary 5.29. ***(of Proposition 5.26)*** *Purple edges of G_k are images under $D^C g_k$ of purple edges of G_{k-1}.*

Proof of Corollary: From the proposition, we know that $D^C g_k$ gives a bijection on the set of purple edges of G_{k-1}. In particular, it is surjective, meaning that the purple edges of G_k are all images under Dg_k of purple edges of G_{k-1}, as desired.

QED.

We have now shown that every ideally decomposed Type (*) representative satisfies:

AM Property VII: $Dg_{l,k+1}$ induces an isomorphism from $SW(f_k)$ onto $SW(f_l)$ for $0 \leq l, k \leq n$.

5.9 Irreducibility and AM Property VIII

In order for a train track map to represent a fully irreducible outer automorphism, it certainly needs to be irreducible. We begin this section with several definitions.

Definition 5.30. The *transition matrix* for an irreducible TT representative g is the square matrix such that, for each i and j, the ij^{th} entry is the number of times $g(E_j)$ crosses E_i in either direction. A matrix $A = [a_{ij}]$ is an *irreducible matrix* if each entry $a_{ij} \geq 0$ and if, for each i and j, there exists a $k > 0$ so that the ij^{th} entry of A^k is strictly positive. If the same k works for each index pair $\{i, j\}$, then the matrix is called *aperiodic*. If each sufficiently high k works for all index pairs $\{i, j\}$, then the matrix is called *Perron-Frobenius (PF)*. [BH92]

Remark 5.31. PF matrices are part of the Full Irreducibility Criterion. We collect here the following facts about transition matrices and PF matrices:

(1) Any power of a Perron-Frobenius matrix is Perron-Frobenius and irreducible.

(2) A power of an irreducible matrix need not be irreducible.

(3) While aperiodic matrices are irreducible, the converse is not always true.

(4) A topological representative is irreducible if and only if its transition matrix is irreducible [BH92].

The following three lemmas give properties stemming from irreducibility (though not proving irreducibility). Together they comprise AM Property VIII.

We will assume that $g : \Gamma \to \Gamma$ is a semi-ideally decomposed train track representative of $\phi \in Out(F_r)$ with the standard 3.4 notation.

Lemma 5.32. *For each $1 \leq j \leq r$, there exists a k such that either $e_k^u = E_{k,j}$ or $e_k^u = \overline{E_{k,j}}$.*

Proof: Suppose, for the sake of contradiction, that, if there is some j so that $e_k^u \neq E_{k,j}$ and $e_k^u \neq \overline{E_{k,j}}$ for all k. We will proceed by induction to show that $g(E_{0,j}) = E_{0,j}$ and so g is certainly reducible. Induction will be on the k in $g_{k-1,1}$.

For the base case, we need to show that $g_1(E_{0,j}) = E_{1,j}$ if $e_1^u \neq E_{1,j}$ and $e_1^u \neq \bar{E}_{1,j}$. g_1 is defined by $e_0^{pu} \mapsto e_1^a e_1^u$ and $g_1(e_{0,1}) = e_{1,l}$ for all $e_{0,1} \neq (e_0^{pu})^{\pm 1}$. Since $e_1^u \neq E_{1,j}$ and $\overline{e_1^u} \neq \overline{E_{(1,j)}}$, $e_0^{pu} \neq E_{(0,j)}$ and $e_0^{pu} \neq \overline{E_{(0,j)}}$. Thus, $g_1(E_{0,j}) = E_{(1,j)}$, as desired. Now suppose inductively that $g_{k-1,1}(E_{0,j}) = E_{k-1,j}$ and that neither $e_k^u = E_{k,j}$ nor $\overline{e_k^u} = E_{k,j}$. Then $e_{k-1}^{pu} \neq E_{k-1,j}$ and $e_{k-1}^{pu} \neq \overline{E_{k-1,j}}$. Thus, since g_k is defined by $e_{k-1}^{pu} \mapsto e_k^a e_k^u$ and $g_k(e_{k-1,l}) = e_{k,l}$ for all $e_{k-1,l} \neq (e_{k-1}^{pu})^{\pm 1}$, $g_k(E_{k-1,j}) = E_{k,j}$. So $g_{k,1}(E_{0,j}) = g_k(g_{k-1,1}(E_{0,j})) = g_k(E_{k-1,j}) = E_{(k,j)}$, as desired. Inductively, this proves that $g(E_{0,j}) = E_{0,j}$, we have our contradiction and the lemma is proved.

QED

Lemma 5.33. *For each $1 \leq j \leq r$, there exists a k such that either $e_k^a = E_{k,j}$ or $e_k^a = \overline{E_{k,j}}$.*

Proof: For the sake of contradiction, suppose that, for some $1 \leq j \leq r$, $e_k^a \neq E_{k,j}$ and $e_k^a \neq \overline{E_{k,j}}$ for each k. The goal will be to inductively show that, for each $E_{0,i}$ with $E_{0,i} \neq E_{0,j}$ and $E_{0,i} \neq \overline{E_{(0,j)}}$, $g(E_{0,i})$ does not contain $E_{0,j}$ and does not contain $\overline{E_{0,j}}$ (contradicting irreducibility).

We start with the base case. g_1 is defined by $e_0^{pu} \mapsto e_1^a e_1^u$ (and $g_1(e_{0,l}) = e_{1,l}$ for all $e_{0,l} \neq (e_0^{pu})^{\pm 1}$). First suppose that either $E_{0,j} = e_0^{pu}$ or $E_{0,j} = \overline{e_0^{pu}}$. Then $e_0^{pu} \neq E_{0,i}$ and $e_0^{pu} \neq \overline{E_{0,i}}$ (since $E_{0,i} \neq E_{0,j}$ and $E_{0,i} \neq \overline{E_{(0,j)}}$) and so $g_1(E_{0,i}) = E_{1,i}$, which does not contain $E_{1,j}$ or $\overline{E_{1,j}}$. Now suppose that $E_{0,j} \neq e_0^{pu}$ and $E_{0,j} \neq \overline{e_0^{pu}}$ Then $e_1^a e_1^u$ does not contain $E_{1,j}$ or $\overline{E_{1,j}}$ (since $e_k^a \neq (E_{k,j})^{\pm 1}$ by the assumption), which means that $E_{1,j}$ and $\overline{E_{1,j}}$ are not in the image of $E_{0,i}$ if $E_{0,i} = e_0^{pu}$ (since the image is of $E_{0,i}$ is then $e_1^a e_1^u$) and are not in the image of $\overline{E_{0,i}}$ (since the image is $\overline{e_1^u e_1^a}$) and are not in the image $E_{0,i}$ if $E_{0,i} \neq e_0^{pu}$ and $E_{0,i} \neq \overline{e_0^{pu}}$ (since the image is $E_{1,i}$, which does not equal $E_{1,j}$ or $\overline{E_{1,j}}$). So the base case is proved.

Now inductively suppose that $g_{k-1,1}(E_{0,i})$ does not contain $E_{k-1,j}$ or $\overline{E_{k-1,j}}$. A similar analysis to the above shows that $g_k(E_{k-1,i})$ does not contain $E_{k,j}$ or $\overline{E_{k,j}}$ for any $E_{k,i} \neq E_{k,j}$ and $E_{k,i} \neq \overline{E_{k,j}}$. Since $g_{k-1,1}(E_{k-1,i})$ does not contain $E_{k-1,j}$ or $\overline{E_{k-1,j}}$, $g_{k-1,1}(E_{0,i}) = e_1 \dots e_m$ with each $e_i \neq E_{k-1,j}$ and $e_i \neq \overline{E_{k-1,j}}$. Thus, no $g_k(e_i)$ contains $E_{k,j}$ or $\overline{E_{k,j}}$, which means that $g_{k,1}(E_{0,i}) = g_k(g_{k-1,1}(E_{0,i})) = g_k(e_1) \dots g_k(e_m)$ does not contain $E_{k,j}$ or $\overline{E_{k,j}}$. This completes the inductive step and thus proves the lemma. QED.

Remark 5.34. While the above lemmas are necessary for g to be irreducible, they are not sufficient to prove the irreducibility of a semi-ideally decomposed representative. For example, the composition of $a \mapsto ab$, $b \mapsto ba$, $c \mapsto cd$, and $d \mapsto dc$ would satisfy these lemmas, but is clearly reducible. On the other hand, Lemma 6.1 below gives a necessary and sufficient condition for irreducibility.

Definition 5.35. Let $g = g_n \circ \cdots \circ g_1 : \Gamma \to \Gamma$ be a semi-ideally decomposed Type (*) representative of $\phi \in Out(F_r)$ with the standard 3.4 notation, except that we return to the convention of Lemma 5.2 and index the generators in the decomposition of all powers g^p of g so that $g^p = g_{pn} \circ g_{pn-1} \circ \cdots \circ g_{(p-1)n} \circ \cdots \circ g_{(p-2)n} \circ \cdots \circ g_{n+1} \circ g_n \circ \cdots \circ g_1$

($g_{mn+i} = g_i$, but we want to use the indices to keep track of a generator's place in the decomposition of g^p). Again, with this notation, $g_{k,l}$ will mean $g_k \circ \cdots \circ g_l$. We recursively define the *edge containment sequence* for an edge $E_{0,j}$ of Γ (or just for j). For $1 \leq j \leq r$, the *level-1 edge containment set for* j, denoted $\mathcal{C}_j^1$, contains each index i such that, for some k, $e_k^{pu} = (E_{k,j})^{\pm 1}$ and $e_{k+1}^a = (E_{k+1,i})^{\pm 1}$. Recursively define the *level* k *edge containment set for* j, denoted $\mathcal{C}_j^k$, as $\underset{i \in \mathcal{C}_j^{k-1}}{\cup} \mathcal{C}_i^1$ with duplicates of indices removed. The *edge containment sequence* for $E_{j,0}$ (or just j) is $\{\mathcal{C}_j^1, \mathcal{C}_j^2, \dots\}$.

Lemma 5.36. *g has a Perron-Frobenius transition matrix if and only if for each $1 \leq k, l \leq r$, we have $l \in \mathcal{C}_k^i$ for some i.*

Proof: Suppose that for some $1 \leq k, l \leq r$, we have that l is not in $\mathcal{C}_k^i$ for any i. Let H be the subgraph of Γ that includes precisely the edges E_t where $t \in \mathcal{C}_k^i$ for some i. Then it is not too difficult to see that H is a proper invariant subgraph (proper since it does not contain E_l). This proves the that g is not irreducible and, in particular, does not have a Perron-Frobenius transition matrix.

Now suppose that for each $1 \leq k, l \leq r$, we have that $l \in \mathcal{C}_k^i$ for some i. This means that for each $1 \leq k, l \leq r$ some $g^{p(k,l)}(E_k)$ passes over E_l (in some direction). Let p be the least common multiple of the $p(k, l)$. Then $M^{p(k,l)}$ is strictly positive where M is the transition matrix for g. And, in fact, M^N is strictly positive for any $N \geq p(k, l)$, since g maps each E_l over itself. This proves that g has a Perron-Frobenius transition matrix and thus proves the reverse direction.

QED.

Remark 5.37. It will be relevant later that a semi-ideally decomposed train track representative satisfies that, for each $1 \leq k, l \leq r$, we have $l \in \mathcal{C}_k^i$ for some i, is not just irreducible, but actually has a Perron-Frobenius transition matrix. Since this is a condition in the FIC, it is useful to have this way to check the condition.

We have now shown that every ideally decomposed Type (*) representative satisfies:

AM Property VIII: g is irreducible (and, in fact, has a PF transition matrix), i.e.

(a) for each $1 \leq j \leq r$, there exists a k such that either $e_k^u = E_{k,j}$ or $e_k^u = \overline{E_{k,j}}$;

(b) for each $1 \leq j \leq r$, there exists a k such that either $e_k^a = E_{k,j}$ or $e_k^a = \overline{E_{k,j}}$; and

(c) for each $1 \leq k, l \leq r$, we have that $l \in \mathcal{C}_k^i$ for all i.

5.10 Admissible Map Properties Summarized

We proved in this chapter that a list of properties hold for any ideally decomposed Type (*) representative of a $\phi \in Out(F_r)$. However, one can at least analyze whether they hold in any situation where $\Gamma = \Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n = \Gamma$ is an ideal decomposition of a TT representative g such that $SW(g)$ is a Type (*) pIWG.

For the sake of clarity we list here the properties we proved hold for ideally decomposed Type (*) representatives and call them "Admissible Map (AM) Properties". We use the standard ideal decomposition 3.4 notation.

Definition 5.38. Let $\mathcal{G}$ be a Type (*) pIWG. Let $(g_{(i-k,i)}, G_{i-k-1}, G_{i-k}, \dots, G_{i-1}, G_i)$, with $k \geq 0$, be such that $g_{i-k,i}$ can be decomposed as $\Gamma_{i-k-1} \xrightarrow{g_{i-k}} \Gamma_{i-k} \xrightarrow{g_{i-k+1}} \cdots \xrightarrow{g_{i-1}} \Gamma_{i-1} \xrightarrow{g_i} \Gamma_i$, and each G_j is an LTT structure for $\mathcal{G}$. We say $(g_{(i-k,i)}, G_{i-k-1}, G_{i-k}, \dots, G_{i-1}, G_i)$ satisfies the *Admissible Map (AM) Properties* if it satisfies:

AM Property I: Each LTT structure G_j, with $i-k-1 \leq j \leq i$, is birecurrent.

AM Property II: For each LTT structure G_j, with $i-k-1 \leq j \leq i$, the illegal turn T_j for the generator g_{j+1} exiting G_j contains the unachieved direction d_j^u for the generator g_j entering the graph G_j, i.e. either $d_j^u = d_j^{pa}$ or $d_j^u = d_j^{pu}$.

AM Property III: In each LTT structure G_j, with $i-k-1 \leq j \leq i$, the vertex labeled d_j^u and the edge $[t_j^R] = [d_j^u, \overline{d_j^a}]$ are both red.

AM Property IV: For all $i-k-1 \leq j < m \leq i$, if $[d_{(j,i)}, d_{(j,l)}]$ is a purple or red edge in G_j, then $D^C g_{m,j+1}([d_{(j,i)}, d_{(j,l)}])$ is a purple edge in G_m.

AM Property V: For each $i-k-1 \leq j \leq i$, $C(G_j)$ has precisely one edge segment containing the red (nonperiodic) vertex d_j^u of G_j. This single edge is red and is in fact $[t_j^R] = [d_j^u, \overline{d_j^a}]$.

AM Property VI: For each $i-k \le j \le i$, the generator g_j is defined by $g_j : e_{j-1}^{pu} \mapsto e_j^a e_j^u$ (where $e_j^u = e_{j,m}$, $D_0(e_j^u) = d_j^u$, $D_0(\overline{e_j^a}) = \overline{d_j^a}$, and $e_{j-1}^{pu} = e_{j-1,m}$).

AM Property VII: $Dg_{l,j+1}$ induces an isomorphism from $SW(f_j)$ onto $SW(f_l)$ for all $i-k-1 \le j < l \le i$.

AM Property VIII: g is irreducible (and, in fact, has a Perron-Frobenius transition matrix), i.e.

(a) for each $1 \le j \le r$, there exists a k such that either $e_k^u = E_{k,j}$ or $e_k^u = \overline{E_{k,j}}$;

(b) for each $1 \le j \le r$, there exists a k such that either $e_k^a = E_{k,j}$ or $e_k^a = \overline{E_{k,j}}$; and

(c) for each $1 \le k, l \le r$, we have that $l \in \mathcal{C}_k^i$ for all i.

Chapter 6

Lamination Train Track Structures are Lamination Train Track Structures

In this chapter we simply show that the lamination train track structures defined in Section 4.2 are indeed abstract lamination train track structures in the sense of Definition 4.7.

Lemma 6.1. *Let $g : \Gamma \to \Gamma$ be a Type (*) representative of $\phi \in Out(F_r)$ such that $IW(g) \cong \mathcal{G}$. Then $G(g)$ is a Type (*) LTT structure with base graph Γ. Furthermore,*

1. *$PI(G(g)) \cong \mathcal{G}$ and*

2. *if $\Gamma = \Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n = \Gamma$ is an ideal decomposition of g with the standard 3.4 notation, then each $G_j = G(f_j)$ is a Type (*) LTT structure with base graph Γ_j such that*

   a. *$PI(G_j) \cong \mathcal{G}$,*

   b. *the vertex labeled d^u_j is the red vertex of G_j, and*

   c. *the red edge of G_j is $[t^R_j] = [d^u_j, \overline{d^a_j}]$.*

Proof: We first need that each G_j is a Type (*) LTT structure with base graph Γ_j. However, since each f_j is also an ideally decomposed Type (*) representative with the same ideal Whitehead graph as $G(g)$ (and even the same ideal decomposition with simply a shifting of indices), it suffices to show that $G(g)$ is a Type (*) LTT structure with base graph Γ

For STTG1 to hold, we need that $G(g)$ has a colored edge containing each vertex, since each vertex labeled d_i or $\overline{d_i}$ is contained in the black edge $[e_i]$. Note that this would also prove STTG3. Since $\mathcal{G}$ must have $2r-1$ vertices and $PI(G(g)) \cong \mathcal{G}$, there is at most one vertex without a colored edge containing it. However, this vertex must be the red vertex contained in the red edge $[t_n^R] = [d_n^u, \overline{d_n^a}]$ by AM Property V, combined with Corollary 5.15. We now prove STTG2. Colored edges of $G(g)$ contain distinct vertices because they correspond to turns taken by images of edges. The black edges contain distinct vertices because they connect the directions corresponding to the initial and terminal directions in each edge of Γ, which are distinct. This proves STTG2 and $G(g)$ is a train track graph.

LTT1-LTT2 hold by construction (in the definition of $G(g)$). That the edges of $G(g)$ are either black, purple, or red follows from the construction of the definition. LTT3(Black Edges) holds by construction (in the definition of $G(g)$). If an edge is red in $G(g)$, by how $G(g)$ is constructed, it means that the edge is in $LW(g)$, but not in $SW(g)$. For this to be true, it must have a nonperiodic vertex, i.e. a red vertex. This implies both LTT3(Red Edges) and LTT3(Purple Edges). We are left to show that Γ is a base for $G(g)$, but this can also easily be seen to be true by construction in the definition of $G(g)$.

LTT(*)7 holds because the fact that each Γ_j is a rose means that each G_j has $2r$ vertices and because AM Property VII implies that each G_j has precisely $2r-1$ purple vertices.

(1) is true by construction. (2a) is true by (1) combined with AM Property VII. (2b) and (2c) are true by AM Property III.

QED

Chapter 7

Peels, Extensions, and Switches

Let $\mathcal{G}$ be a Type (*) pIWG. We saw in Chapter 3 that, if there is an ageometric, fully irreducible $\phi \in Out(F_r)$ with $IW(\phi) \cong \mathcal{G}$, then there is a Type (*) representative g of a power of ϕ. By Chapter 5, such a representative would satisfy AM Properties I-VIII. Thus, if we can show that a representative satisfying all these properties does not exist, then we have shown that the Type (*) representative cannot exist, and thus that there is no ageometric, fully irreducible $\phi \in Out(F_r)$ with $IW(\phi) \cong \mathcal{G}$ (we use this fact in Chapter 12). On the other hand, this fact provides us with a strategy for finding representatives of the achievable graphs.

We will show how to construct all ideally decomposed representatives satisfying the AM properties by working backward (for decreasing k) finding possibilities for the generating triples (g_k, G_{k-1}, G_k) one triple at a time. We determine, given knowledge of an LTT structure G_k in the decomposition, all possibilities for the generating triple (g_k, G_{k-1}, G_k) respecting the AM properties. We prove in Proposition 7.12 that, if the structure G_k and a purple edge $[d, d^a_k]$ in G_k are set, then there is only one g_k possibility and at most two G_{k-1} possibilities (one generating triple possibility will be called a "switch" and the other an "extension").

In Chapter 9, we construct the "PreAdmissible Map (PreAM) Diagram" for $\mathcal{G}$ (denoted $PreAM(\mathcal{G})$) from all "admissible switches" and "admissible extensions." Then, from $PreAM(\mathcal{G})$, we construct the "Admissible Map (AM) Diagram" for $\mathcal{G}$ (denoted $AM(\mathcal{G})$), in which any Type (*) representative g with $IW(g) \cong \mathcal{G}$ will be "realizable" as a loop. Thus, as a consequence of the above, if no loop in the $AM(\mathcal{G})$ satisfies all the AM properties, then $\mathcal{G}$ is "unachievable." The simplest "unachievable" examples arise when all loops in the $AM(\mathcal{G})$ represent reducible maps.

One should note that, while we do not restrict the rank r, we only consider ageometric, fully irreducible $\phi \in Out(F_r)$ such that $IW(\phi)$ is a Type (*) pIWG. The definitions below would need to be tailored for any other circumstance.

7.1 Peels

As a warm-up for the following sections, we describe here a geometric method for visualizing "switches" and "extensions" as moves, called "peels," that are an extension of a fold inverse to the LTT structure, transforming an LTT structure G_i into an LTT structure G_{i-1}.

Each peel of an LTT structure G_i involves three directed edges of G_i (see Figure 7.1):

- The *First Edge of the Peel* (*New Red Edge* in G_i): the red edge from d_i^u to $\overline{d_i^a}$.
- The *Second Edge of the Peel* (*Twice-Achieved Edge* in G_i): the black edge from $\overline{d_i^a}$ to d_i^a.
- The *Third Edge of the Peel* (*Determining Edge* for the peel): a purple edge from d_i^a to d. (In G_{i-1}, this vertex d will be the red edge's attaching vertex, labeled $\overline{d_{i-1}^a}$).

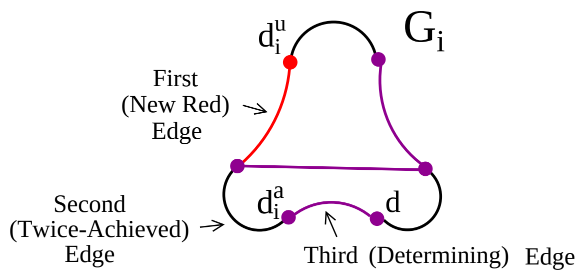


Figure 7.1: *There are three important edges involved in a peel.*

For each choice of a determining edge $[d_i^a, d]$ in G_i, we arrive at one "peel switch" (see Figure 7.3) and one "peel extension" (see Figure 7.2). When G_i has only a single purple edge at d_i^a, the peel and switch differ only by a switch of the color of two edges and two vertices. We will start by explaining this case. Afterward, we explain the preliminary step that must first be performed as part of any switch where d_i^a has more than one purple edge containing it in G_i.

The following describes how the two peels determined by $[d_i^a, d]$ transform G_i into G_{i-1} when G_i has only a single purple edge at d_i^a. Starting at the vertex $\overline{d_i^a}$, peel off the black edge $[\overline{d_i^a}, d_i^a]$ and the third edge $[d_i^a, d]$, while keeping d fixed, keeping copies of $[\overline{d_i^a}, d_i^a]$ and $[d_i^a, d]$, while also creating a new edge $[d_i^u, d]$ from the concatenation of the first, second, and third edges of the peel (see Figure 7.2 or 7.3).

In the case of a *peel extension*, $[d_i^u, \overline{d_i^a}]$ disappears into the concatenation and does not exist in G_{i-1}, the copy of $[\overline{d_i^a}, d_i^a]$ left behind stays black in G_{i-1}, the copy of $[d_i^a, d]$ left behind stays purple in G_{i-1}, the edge $[d_i^u, d]$ formed from the concatenation is red in G_{i-1}, and nothing else changes from G_i to G_{i-1} (if once ignores the first indices of the vertex labels). The triple (g_i, G_{i-1}, G_i), with g_i as in AM Property VI, will be called the *extension determined by* $[d_i^a, d]$.

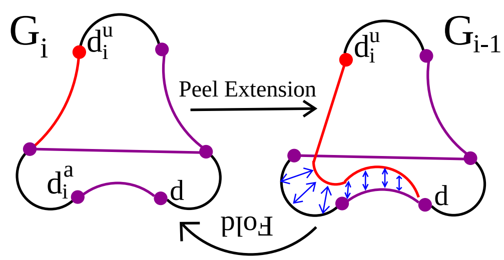


Figure 7.2: *Peel Extension: Note that the first, second, and third edges of the peel concatenate to form the red edge* $[d_i^u, d]$ *in* G_{i-1} *and that copies of* $[\overline{d_i^a}, d_i^a]$ *and* $[d_i^a, d]$ *remain in* G_{i-1}.

In the case of a *peel switch* (where $[d_i^a, d]$ was the only purple edge in G_i containing d_i^a), again $[d_i^u, \overline{d_i^a}]$ has disappeared into the concatenation and the copy of $[\overline{d_i^a}, d_i^a]$ left behind stays black in G_{i-1}, but now the edge $[d_i^u, d]$ formed from the concatenation is purple in G_{i-1}, the copy of $[d_i^a, d]$ left behind and the vertex d_i^a are both red in G_{i-1} (so that d_i^a is now actually d_{i-1}^u), and the vertex d_i^u is purple in G_{i-1}. The triple (g_i, G_{i-1}, G_i), with g_i as in AM Property VI, will be called the *switch determined by* $[d_i^a, d]$.

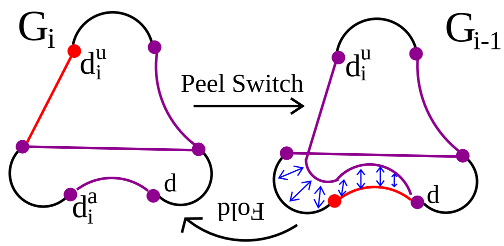


Figure 7.3: *Peel Switch (when d_i^a only belongs to one purple edge in G_{i-1}): The first, second, and third edges of the peel concatenate to form a purple edge $[d_i^u, d]$ in G_{i-1}. The determining edge $[d_i^a, d]$ is the red edge of G_{i-1}, with red vertex d_i^a.*

The following is the preliminary step necessary for a switch where purple edges other than the determining edge $[d_i^a, d]$ contain the vertex d_i^a in G_i. For each purple edge $[d_i^a, d']$ in G_i where $d \neq d'$, form a purple concatenated edge $[d', d_i^u]$ in G_{i-1} by concatenating $[d', d_i^a]$ with a copy of $[d_i^a, \overline{d_i^a}, d_i^u]$, created by splitting open, as in Figure 7.4, $[d_i^a, \overline{d_i^a}]$ from d_i^a to $\overline{d_i^a}$ and $[\overline{d_i^a}, d_i^u]$ from $\overline{d_i^a}$ to d_i^u.

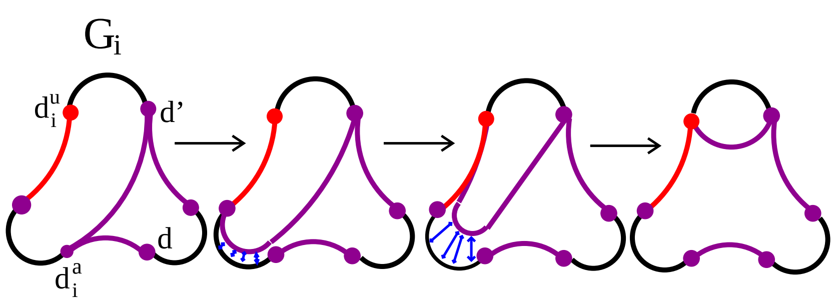


Figure 7.4: *Peel Switch Preliminary Step: For each purple edge $[d_i^a, d']$ in G_i, the peeler peels a copy of $[d_i^a, \overline{d_i^a}, d_i^u]$ off to concatenate with $[d_i^a, d']$ and form the purple edge $[d_i^u, d']$.*

To check that the peel switch was performed correctly, one can simply remove the red edge of G_i, then pick up the vertex d_i^a (with the purple edges containing it dangling from one's fingers) and drop it in the spot of the vertex d_i^u, while leaving behind a copy of $[d_i^a, d]$ to become the new red edge of G_{i-1} (with d_{i-1}^{pa} as the red vertex).

Remark 7.1. A composition of extensions peels open an LTT structure along a path formed by the black twice-achieved edges and purple determining edges of the extensions in the composition. In Section 8.2 we see that the composition, in the right circumstance, can actually ensure the existence of these purple edges in the ideal Whitehead graph, a fact we use heavily in our construction strategies.

7.2 Extensions and Switches

Let $\mathcal{G}$ be a Type (*) pIWG. In this section we define extensions and switches "entering" an indexed (edge-pair)-labeled Type (*) admissible LTT structure G_k for $\mathcal{G}$. Extensions will be important because, when composed, they give a smooth path in an LTT structure such that. The colored edges in the image of the path are "constructed" in $\mathcal{G}$ (see Proposition 8.13). "Switches" change the LTT structure more dramatically, start the construction of a new path, and are necessary for reducibility. Our goal in building train track representatives will be to use compositions of extensions to construct $\mathcal{G}$, with only the minimal necessary number of switches to piece the compositions together and to ensure irreducibility. The fewer switches we use, the easier it will be to track our progress in construction $\mathcal{G}$.

Throughout this section we use the Standard LTT Structure Notation 4.6, Standard Based LTT Structure Notation 4.13, Standard Type (*) Notation 8.13, and Standard Generating Triple Notation 4.23.

Lemma 7.2. *Let G_k be an indexed (edge-pair)-labeled Type (*) LTT structure for a Type (*) pIW graph $\mathcal{G}$ with rose base graph Γ_k and the standard 3.4 notation. There exists a colored edge having an endpoint at d^a_k, so that it may be written $[d^a_k, d_{k,l}]$. This edge must be purple.*

Proof: If d^a_k were red, the red edge would be $[d^a_k, \overline{d^a_k}]$, violating that it is an LTT structure for $\mathcal{G}$ and hence $PI(G) \cong \mathcal{G}$. d^a_k must be contained in an edge $[d^a_k, d_{k,l}]$ or $\mathcal{G}$ would not have $2r-1$ vertices (so would not be a Type (*) pIWG). If $d_{k,l}$ were red, i.e. $d_{k,l} = d^u_k$, then both $[d^u_k, \overline{d^a_k}]$ and $[d^u_k, d^a_k]$ would be red, violating [LTT(*)7]. So the edge must be purple.

QED.

Definition 7.3. (See Figure 7.5) Let G_k be an indexed (edge-pair)-labeled Type (*) admissible LTT structure for a Type (*) pIW graph $\mathcal{G}$ with indexed rose base graph Γ_k and the standard notation. For a purple edge $[d^a_k, d_{k,l}]$ in G_k, the *extension determined by* $[d^a_k, d_{k,l}]$, is the indexed generating triple (g_k, G_{k-1}, G_k) for $\mathcal{G}$, with the notation of Definition 4.22 and the Standard Notations 4.6, 8.13, 4.23, and 4.13, satisfying additionally each of the following:

(EXTI): The restriction of $D^T(g_k)$ to $PI(G_{k-1})$ is defined by sending, for each j, the vertex labeled $d_{k-1,j}$ to the vertex labeled $d_{k,j}$ and extending linearly over edges.

(EXTII): $d^u_{k-1} = d^{pu}_{k-1}$, i.e. $d^{pu}_{k-1} = d_{k-1,j_k}$ labels the single red vertex in G_{k-1}.

(EXTIII): $\overline{d^a_{k-1}} = d_{k-1,l}$.

Remark 7.4. (EXTIII) implies that the single red edge $e^R_{k-1} = [d^u_{k-1}, \overline{d^a_{k-1}}]$ of G_{k-1} can be written, among other ways, as $[d^{pu}_{k-1}, d_{(k-1,l)}]$.

As explained in Section 7.1, but with this section's notation, an extension transforms LTT structures as follows:

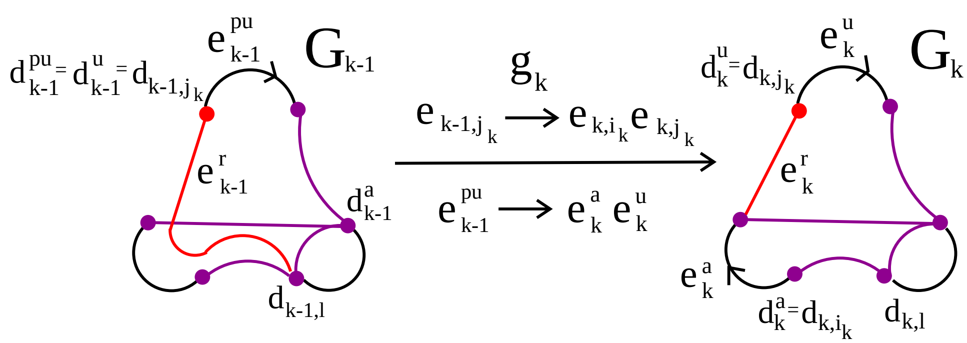


Figure 7.5: *Extension*

Lemma 7.5. *Given an indexed (edge-pair)-labeled Type (*) admissible LTT structure G_k for a Type (*) PIWG $\mathcal{G}$ and purple edge $[d^a_k, d_{k,l}]$ in G_k, with rose base graph Γ_k, the extension (g_k, G_{k-1}, G_k) determined by $[d^a_k, d_{k,l}]$ is unique.*

I. *G_{k-1} can be obtained from G_k by the following steps:*

1. *removing the interior of the red edge from* G_k*;*
2. *replacing each vertex label* $d_{k,i}$ *with* $d_{k-1,i}$ *and each vertex label* $\overline{d_{k,i}}$ *with* $\overline{d_{k-1,i}}$*; and*
3. *adding a red edge* e^R_{k-1} *connecting the red vertex to* $d_{k-1,l}$*.*

II. *The fold is such that the corresponding homotopy equivalence maps the oriented edge* e_{k-1,j_k} *in* Γ_{k-1} *over the edge path* $e_{k,i_k}e_{k,j_k}$ *in* Γ_k *and then each oriented edge* $e_{k-1,t}$ *in* Γ_{k-1} *with* $e_{k-1,t} \neq e^{\pm 1}_{k-1,j_k}$ *over* $e_{k,t}$*.*

Proof: We start by proving uniqueness. $PI(G_{k-1})$ is uniquely determined by the isomorphism in (EXTI) to differ from $PI(G_k)$ by the relabeling of vertices described in (I2), so (I2) is both necessary and unique to give the correct definition of $PI(G_{k-1})$ in an extension (it determines the labels on $2r-1$ vertices and so the final label must be the only label left). Since G_{k-1} must be a Type (*) LTT structure for (GTII) to hold, [LTT(*)7] implies that G_{k-1} both has precisely one red vertex and one unique red edge. (I3) is necessary and unique since the label on the red vertex is dictated by (EXTII) to be $d_{(k-1,j_k)}$, where $d^u_k = d_{(k,j_k)}$, and the red edge is dictated to be $[d_{(k-1,j_k)}, d_{(k-1,l)}]$ by (EXTIII). Since the black edges of an LTT structure connect precisely (edge-pair)-labeled vertices (which (II) indicates to be the same second index-wise for G_{k-1} as for G_k), and we have already determined the colored edges and vertex labels, the LTT structure G_{k-1} is uniquely determined by (I1)-(I3). g_k is uniquely determined by (GTI). It is clear that the procedure gives us the structure G_{k-1} described and so we are left to show that (g_k, G_{k-1}, G_k) is indeed an extension.

First, we need that G_{k-1} is a Type (*) LTT structure for $\mathcal{G}$. [LTT1] and [LTT3] hold by construction. Since [LTT2] holds for G_k and since (other than vertex label first indices), the only difference between $C(G_{k-1})$ and $C(G_k)$ is where the red edge is attached, we only need to be concerned that there is not a purple edge sharing both its vertices with the red edge. But this cannot happen because d^u_{k-1} shares it second index with d^u_k, $PI(G_{k-1})$ and $PI(G_k)$ are isomorphic via an isomorphism preserving second indices, and the only colored edge at d^u_k was the red edge e^R_k. By construction G_{k-1} satisfies [LTT(*)7].

$PI(G_k) \cong \mathcal{G}$, since G_k is a Type (*) LTT structure for $\mathcal{G}$, and $PI(G_k) \cong PI(G_{k-1})$ by (I2). So $PI(G_{k-1}) \cong \mathcal{G}$ and G_{k-1} is a Type (*) LTT structure for $\mathcal{G}$.

Since the fold in (II) gives $dg_k(d_{k-1,j_k}) = d_{k-1,t}$ and $dg_k(d_{k-1,t}) = d_{k-1,t}$ for all $t \neq j_k$, our fold is consistent with (GTI). (GTII) holds by the above paragraphs and by construction. Since $dg_k(d_{k-1,j_k}) = d_{k-1,t}$ for all $t \neq j_k$ and this precisely undoes the relabeling we did of $PI(G_k)$ in (I2), (GTIII) holds. (EXTII) holds because the red vertex of G_k was d_{k,j_k} and we just replaced the label d_{k,j_k} with d_{k-1,j_k} and added the red edge between that vertex and $d_{k-1,l}$. (EXTI) and (EXTIII) hold by construction. QED.

Remark 7.6. In addition to being able to use Lemma 7.5 to construct G_{k-1} from G_k, one could use the peel extension described in Section 7.1 for this purpose.

Remark 7.7. It can be noted that we cannot have $d_{k,l} = \overline{d^u_k}$ (similarly $e_{k,i_k} \neq (e_{k,j_k})^{\pm 1}$). If $d_{k,l} = \overline{d^u_k}$, then the red edge of G_{k-1} would be $[d^{pu}_{k-1}, \overline{d^{pu}_{k-1}}] = [d^u_{k-1}, \overline{d^u_{k-1}}]$, which would cause two problems. First, it would contradict [(LTT(*)8)*]. Second, $e^R_{k-1} = [d^u_{k-1}, \overline{d^u_{k-1}}]$ would force the ingoing generator for G_{k-1} to be $e^{pu}_{k-2} \mapsto e^u_{k-1} e^u_{k-1}$, which is not a generator.

Definition 7.8. (See Figure 7.6) Let G_k be an indexed (edge-pair)-labeled Type (*) admissible LTT structure for a Type (*) pIWG $\mathcal{G}$ with indexed rose base graph Γ_k with the standard notation. The *switch* determined by a purple edge $[d^a_k, d_{(k,l)}]$ in G_k is the indexed generating triple (g_k, G_{k-1}, G_k) for $\mathcal{G}$, with the notation of Definition 4.22 and the Standard Notation 4.6, 8.13, 4.23, and 4.13, satisfying:

(SWITCHI): $D^T(g_k)$ restricts to an isomorphism from $PI(G_{k-1})$ to $PI(G_k)$ defined by

$$PI(G_{k-1}) \xrightarrow{d^{pu}_{k-1} \mapsto d^a_k = d_{k,i_k}} PI(G_k)$$

$(d_{k-1,t} \mapsto d_{k,t}$ for $d_{k-1,t} \neq d^{pu}_{k-1})$ and extended linearly over edges.

(SWITCHII): $d^{pa}_{k-1} = d^u_{k-1}$.

(SWITCHIII): $\overline{d^a_{k-1}} = d_{k-1,l}$.

Remark 7.9. (SWITCHII) implies that the red edge $e^R_{k-1} = [d^u_{k-1}, d^a_{k-1}]$ of G_{k-1} can be written $[d^{pa}_{k-1}, \overline{d^a_{k-1}}]$, among other ways. (SWITCHIII) implies that e^R_{k-1} can be written $[d_{(k-1,i_k)}, d_{(k-1,l)}]$.

As explained in Section 7.1, but with this section's notation, a switch transforms LTT structures as follows:

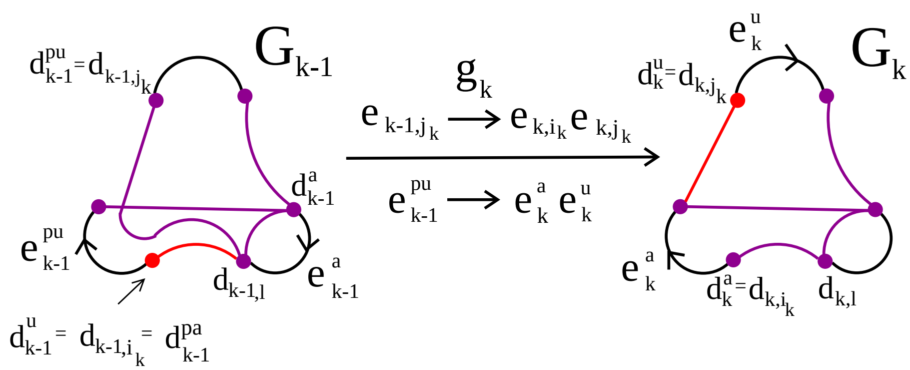


Figure 7.6: Switch

Lemma 7.10. *Given an indexed (edge-pair)-labeled Type (*) admissible LTT structure G_k for a Type (*) PIWG $\mathcal{G}$, with indexed rose base graph Γ_k and purple edge $[d^a_k, d_{k,l}]$ in G_k, the switch (g_k, G_{k-1}, G_k) determined by $[d^a_k, d_{k,l}]$ is unique.*

I. *G_{k-1} can be obtained from G_k by the following steps:*

1. *Start with $PI(G_k)$.*
2. *Replace each vertex label $d_{k,i}$ with $d_{k-1,i}$.*
3. *Switch the attaching (purple) vertex of the red edge to be $d_{k-1,l}$.*
4. *Switch the labels $d_{(k-1,j_k)}$ and $d_{(k-1,i_k)}$, so that the red vertex of G_{k-1} will be d_{k-1,i_k} and the red edge of G_{k-1} will be $[d_{(k-1,i_k)}, d_{(k-1,l)}]$.*
5. *Include black edges connecting inverse pair labeled vertices (there is a black edge $[d_{(k-1,i)}, d_{(k-1,j)}]$ in G_{k-1} if and only if there is a black edge $[d_{(k,i)}, d_{(k,j)}]$ in G_k).*

II. *The fold is such that the corresponding homotopy equivalence maps the oriented edge e_{k-1,j_k} in Γ_{k-1} over the edge path $e_{k,i_k}e_{k,j_k}$ in Γ_k and then each oriented edge $e_{k-1,t}$ in Γ_{k-1} with $e_{k-1,t} \neq e^{\pm 1}_{k-1,j_k}$ over $e_{k,t}$.*

Proof: We must show both that the triple (g_k, G_{k-1}, G_k) determined by Steps 1-5 is indeed a switch and that this switch is the unique switch determined by $[d^a_k, d_{k,l}]$.

The isomorphism in (SWITCHI) uniquely determines $PI(G_{k-1})$. Since G_{k-1} must be a Type (*) LTT structure, [LTT(*)7] implies it has only a single red vertex and hence also a unique red edge. The label on the red vertex is dictated by (SWITCHII) to be $d^{pa}_{k-1} = d_{k-1,i_k}$, as does (I4) above. The attaching vertex of the red edge is dictated by (SWITCHIII) to be $d_{(k-1,l)}$, as it also is dictated to be by (I3). For a triple (g_k, G_{k-1}, G_k) to be a switch, e^R_{k-1} would have to be $[d_{(k-1,i_k)}, d_{(k-1,l)}]$, precisely as the steps dictate. (GTI) indicates how one uniquely determines g_k from e^R_k, which is the same as is dictated by the last sentence in the lemma statement.

Since $g_k(e_{k-1,t}) = e_{k,t}$ for all $e_{k-1,t} \neq e^{\pm 1}_{k-1,j_k}$, Dg_k creates a one to one correspondence between an $(r-1)$-element subset of the set of inverse direction pairs in $\mathcal{D}_{k-1}$ and an $(r-1)$-element subset of the set of inverse direction pairs in $\mathcal{D}_k$. Thus, $D^T(g_k)$ dictates all but one of the inverse direction pairs in G_{k-1}. But then the last pair must just be the remaining two directions in $\mathcal{D}_{k-1}$. Since the black edges of an LTT structure connect precisely vertices with inverse labels, the black edges of G_{k-1} are uniquely determined by $D^T(g_k)$ to be as in Step 5. Since we have already determined the colored edges and vertex labels, the LTT structure G_{k-1} is uniquely determined by Definition 7.8. g_k is uniquely determined by (GTI) to be as in the lemma statement.

Since (SWITCHII), (SWITCHIII), and(GTI) are shown above, we are just left to show that the triple given by (I) and (II) satisfies (GTII)-(GTIII).

We must show for (GTII) that G_{k-1} is an indexed (edge-pair)-labeled Type (*) LTT structure with base graph Γ_{k-1}. [LTT1] and [LTT3] hold by construction.

Since [LTT2] holds for G_k and since the only differences between $C(G_{k-1})$ and $C(G_k)$ are vertex labels and where the red edge is attached, we only need to be concerned that there is not a purple edge sharing both its vertices with the red edge. Before the labels d_{k-1,j_k} and d_{k-1,i_k} were switched, $PI(G_k)$ and $PI(G_{k-1})$ were isomorphic via the isomorphism preserving second indices, which means that d_{k-1,j_k} was not contained in any purple edge and so now d_{k-1,i_k} cannot be contained in any purple edge and, since d_{k-1,i_k} is the red vertex of the red edge, this means that the red edge cannot share both vertices with the same purple edge of G_{k-1}. We thus have [LTT2]. By construction G_{k-1} satisfies [LTT(*)7].

As in Lemma 7.5, $PI(G_k) \cong \mathcal{G}$ since G_k is a Type (*) LTT structure for $\mathcal{G}$, and $PI(G_k) \cong PI(G_{k-1})$ by (I2). So $PI(G_{k-1}) \cong \mathcal{G}$ and G_{k-1} is a Type (*) LTT structure for $\mathcal{G}$. [LTT1(Based)1] and [LTT1(Based)2] hold by construction.

Notice that the direction map from (II) sends d_{k-1,j_k} to d_{k,i_k}, which is precisely the vertex map of (SWITCHI), giving (GTIII).
QED.

Remark 7.11. For similar reasons as to why we could not have $d_{k,l} = \overline{d_k^u}$ for an extension (see Remark 7.7), we cannot have $d_{k,l} = d_k^a$ for a switch.

Proposition 7.12. *Suppose that (g_k, G_{k-1}, G_k) is a triple for $\mathcal{G}$ such that:*

1. *$\mathcal{G}$ is a Type (*) pIWG and*

2. *G_{k-1} and G_k are indexed (edge-pair)-labeled Type (*) LTT structures for $\mathcal{G}$ with respective base graphs Γ_{k-1} and Γ_k.*

Then (g_k, G_{k-1}, G_k) satisfies AM Properties I-VII if and only if it is either an admissible switch or an admissible extension.

In particular, in the circumstance where $d_{k-1}^u = d_{k-1}^{pa}$, the triple is a switch and, in the circumstance where $d_{k-1}^u = d_{k-1}^{pu}$, the triple is an extension.

Proof: We start with the forward direction. Consider a triple (g_k, G_{k-1}, G_k) satisfying AM Properties I-VII, as well as (1)-(2) in the proposition statement. We will first show that the triple is either a switch or an extension, as G_{k-1} and G_k are birecurrent by AM Property I.

Assumption (1) in the proposition statement implies (GTII).

By AM Property VI, g_k is defined by $g_k(e^{pu}_{k-1}) = e^a_k e^u_k$ and $g_k(e_{k-1,i}) = e_{k,i}$ for $e_{k-1,i} \neq (e^{pu}_{k-1})^{\pm 1}$, $D_0(e^u_k) = d^u_k$, $D_0(\overline{e^a_k}) = \overline{d^a_k}$, and $e^{pu}_{k-1} = e_{(k-1,j)}$, where $e^u_k = e_{k,j}$). This gives us (GTI).

By AM Property VII, Dg_k induces on isomorphism from $SW(G_{k-1})$ to $SW(G_k)$. Since the only direction whose second index is not fixed by Dg_k is d^{pu}_{k-1}, the only vertex label of $SW(G_{k-1})$ that is not determined by this isomorphism is the preimage of d^a_k (which AM Property IV dictates to be either d^{pu}_{k-1} or d^{pa}_{k-1}). When the preimage is d^{pa}_{k-1}, this gives us (EXTI). When the preimage is d^{pu}_{k-1}, this gives (SWITCHI). For the isomorphism to extend linearly over edges, we need that the image of an edge in G_{k-1} is an edge in G_k, i.e. $[Dg_k(d_{k-1,i}), Dg_k(d_{k-1,j})]$ is an edge in G_k for each edge $[d_{(k-1,i)}, d_{(k-1,j)}]$ in G_{k-1}. This follows from AM Property IV. So we have (GTIII).

AM Property II tells us that either $d^u_{k-1} = d^{pa}_{k-1}$ or $d^u_{k-1} = d^{pu}_{k-1}$. When we are in the switch case, the above arguments tell us that d^{pu}_{k-1} labels a purple periodic vertex, so we must have that $d^u_{k-1} = d^{pa}_{k-1}$ (since AM Property III tells us d^u_{k-1} is red). This gives us (SWITCHII) once one appropriately coordinates the notation. In the extension case, the above arguments tell us that instead d^{pa}_{k-1} labels a purple periodic vertex, meaning that $d^u_{k-1} = d^{pu}_{k-1}$ (again since AM Property III tells us d^u_{k-1} is red). This gives us (EXTII). We are now only left with (EXTIII) and (SWITCHIII), with what we need being, that $[d^a_k, d_{k,l}]$ is a purple edge in G_k where $\overline{d^a_{k-1}} = d_{k-1,l}$.

By AM Property V, G_{k-1} has a single red edge $[t^R_{k-1}] = [\overline{d^a_{k-1}}, d^u_{k-1}]$. By AM Property IV, the image of $[t^R_{k-1}]$ is a purple edge in G_k. First consider what we established is the switch case, i.e. assume $d^u_{k-1} = d^{pa}_{k-1}$. The goal is to determine that $[t^R_{k-1}]$ is $[d_{(k-1,i_k)}, d_{(k-1,l)}]$, where $d^a_k = d_{k,i_k}$ ($d_{k-1,i_k} = d^{pa}_{k-1}$) and $[d^a_k, d_{k,l}]$ is a purple edge in G_k (making (g_k, G_{k-1}, G_k) the switch determined by $[d^a_k, d_{k,l}]$). Since $d^u_{k-1} = d^{pa}_{k-1}$, we know $[t^R_{k-1}] = [\overline{d^a_{k-1}}, d^u_{k-1}] = [\overline{d^a_{k-1}}, d^{pa}_{k-1}]$. We know $\overline{d^a_{k-1}} \neq d^{pa}_{k-1}$ (since (STTG2) implies $\overline{d^a_{k-1}} \neq d^u_{k-1}$, which equals d^{pa}_{k-1}). Thus, AM Property VI says $D^C([t^R_{k-1}]) = D^C([\overline{d^a_{k-1}}, d^{pa}_{k-1}]) = [d_{(k,l)}, d^a_k]$ where $\overline{d^a_{k-1}} = e_{k-1,l}$. So $[d_{(k,l)}, d^a_k]$ is a purple edge in G_k. We thus have (SWITCHIII). Now consider what we established is the extension case, i.e. assume $d^u_{k-1} = d^{pu}_{k-1}$. We need that the red edge $[t^R_{k-1}]$ is $[d_{(k-1,j_k)}, d_{(k-1,l)}]$, where $d^u_{k-1} = d_{k-1,j_k}$ and $[d^a_k, d_{k,l}]$ is a purple edge in G_k

(making (g_k, G_{k-1}, G_k) the extension determined by $[d^a_k, d_{k,l}]$). Since $d^u_{k-1} = d^{pu}_{k-1}$, we know $[t^R_{k-1}] = [\overline{d^a_{k-1}}, d^u_{k-1}] = [\overline{d^a_{k-1}}, d^{pu}_{k-1}]$. We know $\overline{d^a_{k-1}} \neq d^{pu}_{k-1}$ (since (STTG2) implies $\overline{d^a_{k-1}} \neq d^u_{k-1}$, which equals d^{pu}_{k-1}). Thus, by AM Property VI, $D^C([t^R_{k-1}]) = D^C([\overline{d^a_{k-1}}, d^{pu}_{k-1}]) = [d_{(k,l)}, d^a_k]$, where $\overline{d^a_{k-1}} = e_{k-1,l}$. We thus have (EXTIII) and have completed our proof of the forward direction.

For the converse we assume that (g_k, G_{k-1}, G_k) is either an admissible switch or an admissible extension and show that it satisfies AM Properties I-VII.

Since we have required that the extensions and switches be admissible, G_{k-1} and G_k are both birecurrent. This gives us AM Property I.

Notice that the first and second parts of AM Property II are equivalent and that the second part holds by (EXTII) for an extension and (SWITCHII) for a switch. For AM Property (III) notice that there is only a single red unachieved direction vertex labeled d^u_k in G_k and that there is only a single red unachieved direction vertex labeled d^u_{k-1} in G_{k-1} follows from the requirement in (GTII) that G_k and G_{k-1} are Type (*) LTT structures (see the standard notation for why this is notationally consistent with that in the AM properties).

What is left of AM Property III is that the edge $[t^R_k] = [d^u_k, \overline{d^a_k}]$ in G_k and the edge $[t^R_{k-1}] = [d^u_{k-1}, \overline{d^a_{k-1}}]$ in G_{k-1} are both red. This follows from (GTI) combined with (EXTII) for an extension and (SWITCHII) for a switch.

AM Property IV follows from (GTIII). For AM Property V, notice that AM Property III implies that e^R_k is a red edge containing the red vertex d^u_k. (LTT(*)7) implies the uniqueness of both the red edge and red direction.

Since AM Property VI follows from (GTI), combined with (EXTII) for an extension and (SWITCHII) for a switch, and AM Property VII follows from (GTIII), we have proved the converse.

QED.

In light of Proposition 7.12, by an *admissible map* (g_k, G_{k-1}, G_k), we will mean a triple for a Type (*) pIWG $\mathcal{G}$ that is either an admissible switch or admissible extension or (equivalently) satisfies AM Properties I-VII.

Chapter 8

Compositions of Extensions and Switches

8.1 Admissible Compositions

We will call a composition of switches and extensions an "admissible composition":

Definition 8.1. An *admissible composition* for a Type (*) pIW graph $\mathcal{G}$ is a homotopy equivalence $g_{i-k,i}$ (with $k \geq 0$) decomposed into a sequence of proper full folds of (edge-pair)-indexed roses,

$$\Gamma_{i-k-1} \xrightarrow{g_{i-k}} \Gamma_{i-k} \xrightarrow{g_{i-k+1}} \cdots \xrightarrow{g_{i-1}} \Gamma_{i-1} \xrightarrow{g_i} \Gamma_i,$$

together with the data of an *associated sequence of Type (*) LTT structures* for $\mathcal{G}$,

$$G_{i-k-1} \xrightarrow{D^T(g_{i-k})} G_{i-k} \xrightarrow{D^T(g_{i-k+1})} \cdots \xrightarrow{D^T(g_{i-1})} G_{i-1} \xrightarrow{D^T(g_i)} G_i,$$

where, for each $i-k-1 \leq j < i$, (g_{j+1}, G_j, G_{j+1}) is an admissible extension or switch for $\mathcal{G}$.

This admissible composition will be written $(g_{i-k}, \dots, g_i, G_{i-k-1}, \dots, G_i)$.

Standard Notation and Terminology 8.2. (Admissible Compositions) For the admissible composition $(g_{i-k}, \dots, g_i, G_{i-k-1}, \dots, G_i)$ of Definition 8.1, the notation of the definition statement will be standard. Additionally:

- We call the composition of generators $g_{i,i-k} = g_i \circ \cdots \circ g_{i-k}$ the *associated automorphism.*

- G_{i-k-1} will be called the *source LTT structure* and G_k the *destination LTT structure.*

- $(g_{i-k}, \dots, g_i; G_{i-k-1}, \dots, G_i)$ is said to be *realized* if there exists an ideally decomposed Type (*) representative $g : \Gamma' \to \Gamma'$ of $\phi \in Out(F_r)$ decomposed as $\Gamma' = \Gamma'_0 \xrightarrow{g'_1} \Gamma'_1 \xrightarrow{g'_2} \cdots \xrightarrow{g'_{n-1}} \Gamma'_{n-1} \xrightarrow{g'_n} \Gamma'_n = \Gamma'$ with the sequence of LTT structures $G'_0 \xrightarrow{D^T(g'_1)} G'_1 \xrightarrow{D^T(g'_2)} \cdots \xrightarrow{D^T(g'_{n-1})} G'_{n-1} \xrightarrow{D^T(g'_n)} G'_n$, such that $g_j \cong g'_j$ for all $i-k \leq j \leq i$ and $G_j \cong G'_j$ for all $i-k-1 \leq j \leq i$.

8.2 Construction Compositions

When constructing representatives for Theorem 14.1 for a representative g to be composed of switches and extensions, for it to start and end with the same LTT structure, and for its transition matrix to be Perron-Frobenius. We will also need that $IW(g) \cong \mathcal{G}$. To ensure that $IW(g) \cong \mathcal{G}$, we use "building block" compositions of extensions called "construction compositions:"

Definition 8.3. An *admissible construction composition* for a Type (*) pIW graph $\mathcal{G}$ is an admissible composition $(g_{i-k}, \dots, g_i; G_{i-k-1}, \dots, G_i)$ for $\mathcal{G}$ with the standard notation such that

1. each (g_j, G_j, G_{j+1}) with $i-k \leq j \leq i$ is an admissible extension,
2. $(g_{i-k}, G_{i-k-1}, G_{i-k})$ is an admissible switch, and
3. $PI(G_j) = \mathcal{G}$ for each $i-k-1 \leq j \leq i$.

The associated composition of generators will be called a *construction automorphism.*

If $k=1$, the composition is simply an admissible switch. If we leave out the switch, we get a *purified construction automorphism* $g_p = g_i \circ \cdots \circ g_{i-k+1}$ and admissible composition $(g_{i-k+1}, \dots, g_i; G_{i-k}, \dots, G_i)$ called a *purified construction composition.*

A construction automorphism will always have the form of a Dehn twist automorphism $e^{pu}_{i-k-1} \mapsto w e^{u}_{i-k}$, where $w = e^{a}_{i-k} \dots e^{a}_{i}$, and one can view the composition as twisting the edge corresponding to e^{pu}_{i-k-1} around the path corresponding to w in the destination LTT structure. Since construction compositions are in ways analogous to Dehn twist mapping class group representatives and since many construction methods for pseudo-Anosov mapping classes (including those of Penner in [P88]) used Dehn twists, it was natural to look into properties of construction compositions. Their properties and connections to Dehn Twists certainly warrant further investigation and Clay and Pettet use them in their fully irreducible construction methods. However, our methods here utilize a single special construction compositions property, i.e. that they, in some sense, "construct" smooth paths in the destination LTT structures (see Proposition 8.13). Since we use the paths in our procedure for constructing (ideal Whitehead graph)-yielding representatives, we include their definition.

However, before giving the definition, we first note that we abuse notation throughout this section, and the following section, by dropping indices. While not necessary, this abuse may make the visual aspects of the properties and procedures much clearer, as well as reduce the potential for confusion over indices.

Definition 8.4. A *construction path* associated to a construction composition $(g_1, \dots, g_k; G_1, \dots, G_k)$ is a smooth path in G_k starting with the red vertex d^u_k, transversing the red edge $[d^u_k, \overline{d^a_k}]$ from the red vertex d^u_k to the vertex $\bar{d}^a_k$, transversing the black edge $[\overline{d^a_k}, d^a_k]$ from the vertex $\overline{d^a_k}$ to the vertex d^a_k, transversing the extension determining purple edge $[d^a_k, d_k] = [d^a_k, \overline{d^a_{k-1}}]$ from d^a_k to $d_k = \overline{d^a_{k-1}}$, transversing the black edge $[\overline{d^a_{k-1}}, d^a_{k-1}]$ from the vertex $\overline{d^a_{k-1}}$ to the vertex d^a_{k-1}, transversing the extension determining purple edge $[d^a_{k-1}, d_{k-1}] = [d^a_{k-1}, \overline{d^a_{k-2}}]$ from the vertex d^a_{k-1} to the vertex $d_{k-1} = \overline{d^a_{k-2}}$, transversing the black edge $[\overline{d^a_{k-2}}, d^a_{k-2}]$ from the vertex $\overline{d^a_{k-2}}$ to the vertex d^a_{k-2}, continues as such through the purple edges determining each g_i (inserting black edges between), and finally ending by transversing $[d^a_2, d_2] = [d^a_2, \overline{d^a_1}]$ and the black edge $[\overline{d^a_1}, d^a_1]$ from the vertex $\overline{d^a_1}$ to the vertex d^a_1.

In short-hand, a construction path can be written $[d^u_k, \overline{d^a_k}, d^a_k, d_k, d^a_{k-1}, d_{k-1}, \dots,$ $d^a_2, d_2, d^a_1]$ or $[d^u_k, \overline{d^a_k}, d^a_k, \overline{d^a_{k-1}}, d^a_{k-1}, \dots, \overline{d^a_1}, d^a_1]$. The following lemma proves that these construction paths are indeed smooth paths in the destination LTT structures for the compositions.

Lemma 8.5. *The construction path associated to a realized construction composition $(g_{i-k}, \dots, g_i; G_{i-k-1}, \dots, G_i)$ for a Type (*) pIW graph $\mathcal{G}$, with*

- *decomposition* $\Gamma_{i-k-1} \xrightarrow{g_{i-k}} \Gamma_{i-k} \xrightarrow{g_{i-k+1}} \cdots \xrightarrow{g_{i-1}} \Gamma_{i-1} \xrightarrow{g_i} \Gamma_i$ *and*
- *LTT structures* $G_{i-k-1} \xrightarrow{D^T(g_{i-k})} G_{i-k} \xrightarrow{D^T(g_{i-k+1})} \cdots \xrightarrow{D^T(g_{i-1})} G_{i-1} \xrightarrow{D^T(g_i)} G_i$

is a smooth path in the LTT structure G_i *and can be written:*

$$[d_i^u, \overline{d_i^a}, d_i^a, \overline{d_{i-1}^a}, d_{i-1}^a, \dots, d_{s+1}^a, \overline{d_{i-k}^a}, d_{i-k}^a].$$

Proof: We will proceed by induction for decreasing s values. Since nothing in the proof will rely on G_{i-k-1} (which is the only thing that distinguishes that $(g_{i-k}, G_{i-k-1}, G_{i-k})$ is a switch instead of an extension), proof by induction is valid here.

For the base case realize that $[d_i^u, \overline{d_i^a}]$ is the red edge in G_i. So $[d_i^u, \overline{d_i^a}, d_i^a]$ is a path in G_i and is smooth because it alternates between colored and black edges ($[d_i^u, \overline{d_i^a}]$ is colored and $[\overline{d_i^a}, d_i^a]$ is black). For the sake of induction assume that, for $i > s > i-k$, $[d_i^u, \overline{d_i^a}, d_i^a, \overline{d_{i-1}^a}, d_{i-1}^a, \dots, d_{s+1}^a, \overline{d_s^a}, d_s^a]$ is a smooth path in G_i (ending with the black edge $[\overline{d_s^a}, d_s^a]$).

The red edge that g_{s-1} creates in G_{s-1} is $[d_{s-1}^u, \overline{d_{s-1}^a}]$ (see Definition 5.16 the proof of Corollary 5.15). By Lemma 5.28, $D^C g_s([d_{s-1}^u, \overline{d_{s-1}^a}]) = [d_s^a, \overline{d_{s-1}^a}]$ is a purple edge in G_s. Since purple edges are always mapped to themselves by extensions (in the sense that D^C preserves the second index of their vertex labels) and $D^C g_s([d_s^u, \overline{d_{s-1}^a}]) = [d_s^a, \overline{d_{s-1}^a}]$ is a purple edge in G_s, $D^C g_{n,s}(\{d_{s-1}^u, \overline{d_{s-1}^a}\}) = D^C g_{n,s+1}(D^C g_s([d_{s-1}^u, \overline{d_{s-1}^a}]))$ $= D^C g_{n,s+1}([d_s^a, \overline{d_{s-1}^a}]) = [d_s^a, \overline{d_{s-1}^a}]$ is a purple edge in G_i. Thus, including the purple edge $[d_s^a, \overline{d_{s-1}^a}]$ in the smooth path $[d_i^u, \overline{d_i^a}, d_i^a, \overline{d_{i-1}^a}, d_{i-1}^a, \dots, d_{s+1}^a, \overline{d_s^a}, d_s^a]$ gives the smooth path $[d_i^u, \overline{d_i^a}, d_i^a, \overline{d_{i-1}^a}, d_{i-1}^a, \dots, d_{s+1}^a, \overline{d_s^a}, d_s^a, \overline{d_{s-1}^a}]$. (This path is smooth because we added a colored edge to a path with edges alternating between colored and black that ended with a black edge). By including the black edge $[\overline{d_{s-1}^a}, d_{s-1}^a]$ we get the construction path $[d_i^u, \overline{d_i^a}, d_i^a, \overline{d_{i-1}^a}, d_{i-1}^a, \dots, d_s^a, \overline{d_{s-1}^a}, d_{s-1}^a]$. (Again this path is smooth because we added a black edge to a path with edges alternating between colored and black that ended with a colored edge).

This concludes the inductive step and hence the proof.

QED.

Definition 8.6. Let G be an admissible Type (*) LTT structure with the standard notation. The *construction subgraph* G_C is constructed from G via the following procedure:

- Start by removing the interior of the black edge $[e^u]$, the purple vertex $\overline{d^u}$, and the interior of any purple edges containing the vertex $\overline{d^u}$. Call the graph with these edges and vertices removed G^1.
- Given G^{j-1}, recursively define G^j as follows: Let $\{\alpha_{j-1,i}\}$ be the set of vertices in G^{j-1} not contained in any colored edge of G^{j-1}. G^j will be the subgraph of G^{j-1} obtained by removing all black edges containing a vertex $\alpha_{j-1,i} \in \{\alpha_{j-1,i}\}$, as well as the interior of each purple edge containing a vertex of the form $\overline{\alpha_{j-1,i}}$.
- $G_C = \bigcap_j G^j$.

Example 8.7. We find the construction subgraph G_C for a Graph XIII LTT structure G.

Start with the following LTT Structure for Graph XIII:

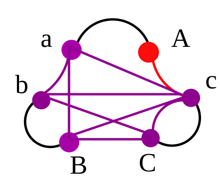


Figure 8.1: *LTT Structure for Graph XIII*

Remove the interior of the black edge $[\bar{a}, a]$:

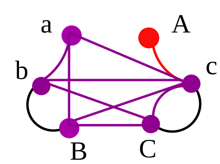


Figure 8.2: *Having removed the black edge $[\bar{a}, a]$ interior from the Graph XIII LTT Structure of Figure 8.1*

Remove a and the interior of all purple edges containing a:

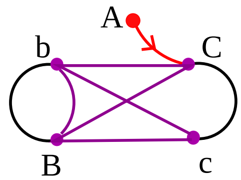


Figure 8.3: *Construction Subgraph for the LTT Structure in Figure 8.1*

A construction path will always live in the construction subgraph of its destination LTT structure. The following lemma gives some conditions under which a path in an admissible (edge-pair)-indexed Type (*) LTT structure G is guaranteed to be the construction path for a construction composition with the destination LTT structure G. It also explains how to find such a construction composition.

Lemma 8.8. *Let G be an admissible Type (*) LTT structure. Consider a smooth path $\gamma = [d^u, \overline{x_1}, x_1, \overline{x_2}, x_2, \dots, x_{k+1}, \overline{x_{k+1}}]$ in the construction subgraph G_C starting with e^R (oriented from d^u to $\overline{d^a}$) and ending with the black edge $[x_{k+1}, \overline{x_{k+1}}]$.*

Edge-index r-petaled roses $\Gamma_{i-k-1}, \dots, \Gamma_i$ and define the homotopy equivalences $\Gamma_{i-k-1} \xrightarrow{g_{i-k}} \Gamma_{i-k} \xrightarrow{g_{i-k+1}} \cdots \xrightarrow{g_{i-1}} \Gamma_{i-1} \xrightarrow{g_i} \Gamma_i$ by $g_l : e_{l-1,s} \mapsto e_{l,t_l} e_{l,s}$, where $D_0(e_{l,t_l}) = \overline{x_{i-l+1}}$, and $g_l(e_{l-1,j}) = e_{l,j}$ for $e_{l-1,j} \neq e^{\pm 1}_{l-1,s}$.

Define the LTT structures with respective base graphs Γ_j (and maps between) $G_{i-k-1} \xrightarrow{D^T(g_{i-k})} G_{i-k} \xrightarrow{D^T(g_{i-k+1})} \cdots \xrightarrow{D^T(g_{i-1})} G_{i-1} \xrightarrow{D^T(g_i)} G_i$ by having

1. *each $PI(G_l)$ isomorphic to $PI(G_i)$ via an isomorphism preserving the second indices of the vertex labels,*
2. *the second index of the vertex label on the single red vertex in each G_l be "s" (the same as in G_i), and*
3. *the single red edge in G_l be $[d_{l,s}, \overline{d_{l,t_l}}]$.*

If each G_j is a Type (*) LTT structure for $\mathcal{G}$ with base graph Γ_j, then $(g_{i-k}, \dots, g_i; G_{i-k-1}, \dots, G_i)$ is a purified construction composition. In fact, it is the *unique realized purified construction composition with γ as its construction path and, for each $i-k+1 \leq l \leq i$, (g_l, G_{l-1}, G_l) is the extension determined by $[\overline{x_{i-l+1}}, x_{i-l+2}]$.*

Proof: We need to show that $(g_{i-k}, \dots, g_i; G_{i-k-1}, \dots, G_i)$ is indeed a construction composition, that its construction path is $[d_i^u, \overline{d_i^a}, d_i^a, \overline{d_{i-1}^a}, d_{i-1}^a, \dots, d_{i-k+1}^a, \overline{d_{i-k}^a}, d_{i-k}^a]$, and that it is the unique construction composition with that path.

We first show that each (g_l, G_{l-1}, G_l) is the extension determined by $[\overline{x_{i-l+1}}, x_{i-l+2}]$. (EXT I) is ensured by our requirement that each G_j is a Type (*) LTT structure with rose base graph. The G_l are all Type (*) LTT structures for $PI(G)$ by (1)-(3) in the lemma statement. This, together with how we defined our notation, implies (GTIII) and (EXTI). The second index of the single red vertex label is the same in each G_l as in G_i, giving (EXT II). To see that (EXTIII) holds by (1), notice that $[\overline{x_{i-l+1}}, x_{i-l+2}]$ is a purple edge in G_l (since it is in G and $PI(G) \cong PI(G_l)$) and would be the determining edge for the extension.

The construction path is $[d_i^u, \overline{d_i^a}, d_i^a, \overline{d_{i-1}^a}, d_{i-1}^a, \dots, d_{i-k+1}^a, \overline{d_{i-k}^a}, d_{i-k}^a]$ by Lemma 8.5.

That the G_l must be as stated follows from

1. each $PI(G_l)$ being isomorphic to $PI(G_i)$ via an isomorphism preserving the second indices of the vertex labels in order for the (g_l, G_{l-1}, G_l) to be extensions
2. the G_l being Type (*) LTT structures,
3. the second index of the red vertices being the same, making each (g_l, G_{l-1}, G_l) an extension, and
4. knowing, by Lemma 8.5, that the attaching vertex for e_l^R in G_l must be x_{i-l+1}.

Once each G_l is determined, that g_l must be as stated follows from AM Property VI.

QED.

Definition 8.9. We call $\Gamma_{i-k-1} \xrightarrow{g_{i-k}} \Gamma_{i-k} \xrightarrow{g_{i-k+1}} \cdots \xrightarrow{g_{i-1}} \Gamma_{i-1} \xrightarrow{g_i} \Gamma_i$, together with its sequence of LTT structures $G_{i-k-1} \xrightarrow{D^T(g_{i-k})} G_{i-k} \xrightarrow{D^T(g_{i-k+1})} \cdots \xrightarrow{D^T(g_{i-1})} G_{i-1} \xrightarrow{D^T(g_i)} G_i$, as in the lemma, the construction composition *determined by (or corresponding to)* the path $\gamma = [d^u, \overline{x_1}, x_1, \overline{x_2}, x_2, \dots, x_{k+1}, \overline{x_{k+1}}]$.

Example 8.10. In the following LTT structure, G, for Graph XX, the numbered edges give a construction path determined by the construction automorphism $a \mapsto ab\bar{c}\bar{c}bbcb$ (all other edges are fixed by the automorphism).

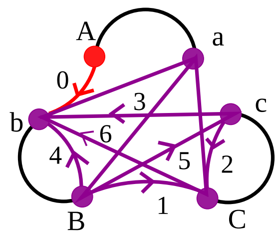


Figure 8.4: *The construction path associated to the construction automorphism $a \mapsto ab\bar{c}\bar{c}bbcb$*

In Figure 8.5 we show the construction composition determined by the construction path of Figure 8.4. The source LTT structure of the switch is left out to highlight the fact that it does not affect the construction path.

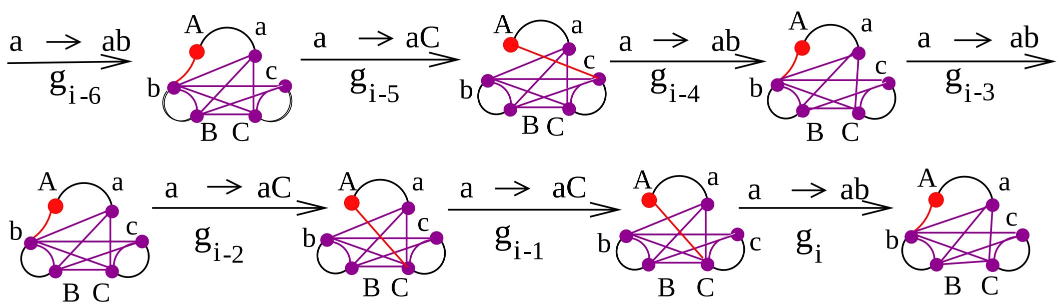


Figure 8.5: *Construction composition associated to the construction path of Figure 8.4*

We determine the sequence of LTT structures in the construction composition by attaching the red edge in G_{i-k} to the terminal vertex of edge k in the construction path. The generator can be determined by the red edge in its destination LTT structure: If the red vertex of G_j is d_s and the red edge is $[d_s, d_t]$, then g_j is defined by $e_s \mapsto \bar{e}_t e_s$.

Definition 8.11. In light of Lemma 8.8, and for the purposes of Chapter 13, we define a *potential construction path* in the composition subgraph G_C of an admissible Type (*) LTT structure G with the standard notation to be a smooth oriented path $[d^u, \overline{d^a}, d^a, \overline{x_2}, x_2, \ldots, x_{n-1}, \overline{x_n}, x_n]$ in G that:

1. starts with the red edge of G (oriented from d^u to $\overline{d^a}$);

2. is entirely contained in G_C after the initial red edge and subsequent black edge;

3. and satisfies the following: Each G_t is an LTT structure (and, in particular, is birecurrent), where G_t is obtained from G by moving the red edge of G to be attached at $\overline{x_t}$.

Example 8.12. A potential composition construction path in the construction subgraph of Figure 8.3 is given by the numbered colored edges and black edges between in:

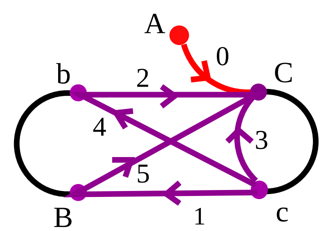


Figure 8.6: *A potential composition construction path in the construction subgraph of Figure 8.3*

The following proposition tells us that construction paths are "built" by construction compositions. We upgraded this proposition from a lemma to a proposition because it is so heavily used in our construction methods (see Chapter 13). By saying a turn is *taken by* $g_{(k,l)}$, we will mean that the turn is taken by $g_{k,l}(e_{l-1,i})$ for some $e_{l-1,i} \in \mathcal{E}_{l-1}$.

Proposition 8.13. *Let $g : \Gamma \to \Gamma$ be an ideally decomposed Type (*) representative of $\phi \in Out(F_r)$ with $IW(\phi) = \mathcal{G}$. Suppose that g is decomposable as $\Gamma = \Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n = \Gamma$, with the sequence of LTT structures for $\mathcal{G}$:*

$$G_{i-k-1} \xrightarrow{D^T(g_{i-k})} G_{i-k} \xrightarrow{D^T(g_{i-k+1})} \cdots \xrightarrow{D^T(g_{i-1})} G_{i-1} \xrightarrow{D^T(g_i)} G_i.$$

Assume the Standard Notation for a Type (*) LTT Structure. If $g' = g_n \circ \cdots \circ g_{k+1}$ is a construction composition, then $\mathcal{G}$ contains as a subgraph the purple edges in the construction path for g'.

Proof: We will proceed by induction for decreasing k values. Since nothing in the proof will rely on G_k (which is the only thing that distinguishes that (g_k, G_k, G_{k+1}) is a switch instead of an extension), proof by induction is valid here.

For the base case consider $g_n \circ g_{n-1}$. By the Corollary 5.15 proof, g_{n-1} creates the red edge $[d^u_{n-1}, \overline{d^a_{n-1}}]$ in G_{n-1}. We know that g_n is defined by g_n:$e^{pu}_{n-1} \mapsto e^a_n e^u_n$ and $g_n(e_{n-1,l}) = e_{n,l}$ for all $e_{n-1,l} \neq (e^{pu}_{n-1})^{\pm 1}$. Thus, since $d^{pu}_{n-1} = d^u_{n-1} \neq \overline{d^a_{n-1}}$, we know that $Dg_n(\overline{d^a_{n-1}}) = \overline{d^a_{n-1}}$. So $D^C g_n([d^u_{n-1}, \overline{d^a_{n-1}}]) = D^C g_n([d^{pu}_{n-1}, \overline{d^a_{n-1}}]) = [d^a_n, \overline{d^a_{n-1}}]$ and, since $D^C g_n$ images of purple and red edges of G_{n-1} are purple edges of G_n, $[d^a_n, \overline{d^a_{n-1}}]$ is a purple edge in G_n. The base case is proved.

For the inductive step assume that, for $n > s > k+1$, G_n contains as a subgraph the purple edges in the construction path associated to $g_{n,s}$. The red edge that g_{s-1} creates in G_{s-1} is $[d^u_{s-1}, \overline{d^a_{s-1}}]$ (see the proof of Corollary 5.15). As above, $D^C g_s([d^u_{s-1}, \overline{d^a_{s-1}}]) = [d^a_s, \overline{d^a_{s-1}}]$ is represented by a purple edge in G_s. Since purple edges are always mapped to themselves by extensions and $D^C g_s([d^u_s, \overline{d^a_{s-1}}]) = [d^a_s, \overline{d^a_{s-1}}]$, $D^C g_{n,s}([d^u_{s-1}, \overline{d^a_{s-1}}]) = D^C g_{n,s+1}(D^t g_s([d^u_{s-1}, \overline{d^a_{s-1}}])) = D^C g_{n,s+1}([d^a_s, \overline{d^a_{s-1}}]) = [d^a_s, \overline{d^a_{s-1}}]$, proving the inductive step. The proposition is proved.

QED.

8.3 Switch Paths

While they do not give insight into the progress of building $\mathcal{G}$ and have more restrictions, switch sequences also have associated paths. The usefulness of switch paths lies in the aid they give in constructing the switch sequences required in our strategies below. This section focuses on switch sequences and their associated switch paths.

We will continue in this chapter with the abuse of notation from the previous section (mainly consisting of ignoring second indices).

Definition 8.14. An admissible *switch sequence* for a Type (*) pIW graph $\mathcal{G}$ is an admissible composition $(g_{i-k}, \dots, g_i; G_{i-k-1}, \dots, G_i)$ for $\mathcal{G}$ such that

(SS1) each (g_j, G_{j-1}, G_j) with $i - k \leq j \leq i$ is a switch and

(SS2) $d^a_{n+1} = d^u_n \neq d^u_l = d^a_{l+1}$ and $\overline{d^a_l} \neq d^u_n = d^a_{n+1}$ for all $i \geq n > l \geq i - k$.

Sometimes we will just call $\Gamma_{i-k-1} \xrightarrow{g_{i-k}} \Gamma_{i-k} \xrightarrow{g_{i-k+1}} \cdots \xrightarrow{g_{i-1}} \Gamma_{i-1} \xrightarrow{g_i} \Gamma_i$ a switch sequence when either the LTT structures should be clear from the decomposition or are unnecessary for discussion.

We call the associated automorphism $g_{i,i-k} = g_i \circ \cdots \circ g_{i-k}$ a *switch sequence automorphism.*

Remark 8.15. (SS2) is not implied by (SS1) and is necessary for a switch path to indeed be a path. Certain statements in the proof of Lemma 8.18 below (where we show that the switch path corresponding to a switch sequence is realized as a smooth path in the destination LTT structure) would be incorrect without (SS2).

Definition 8.16. A *switch path* associated to an admissible switch sequence $(g_j, \dots, g_k; G_{j-1}, \dots, G_k)$ is a path in the destination LTT structure G_k for g_k that starts with the red vertex d^u_k, transverses the red edge $[d^u_k, \overline{d^a_k}]$ for g_k from the red vertex d^u_k to the vertex $\overline{d^a_k}$, transverses the black edge $[\overline{d^a_k}, d^a_k]$ from the vertex $\overline{d^a_k}$ to the vertex d^a_k, transverses what is the red edge $[d^u_{k-1}, \overline{d^a_{k-1}}] = [d^a_k, \overline{d^a_{k-1}}]$ in G_{k-1} (and a purple edge in G_k) from $d^a_k = d^u_{k-1}$ to $\overline{d^a_{k-1}}$, transverses the black edge $[\overline{d^a_{k-1}}, d^a_{k-1}]$ from the vertex $\overline{d^a_{k-1}}$ to the vertex d^a_{k-1}, continues as such through all of the new red edges for the g_i with $j \leq i \leq j$ (inserting black edges between), and ends by transversing the black edge $[\overline{d^a_{j+1}}, d^a_{j+1}]$ from the vertex $\overline{d^a_{j+1}}$ to the vertex d^a_{j+1}, what is the red edge $[d^u_j, \overline{d^a_j}] = [d^a_{j+1}, \overline{d^a_j}]$ in G_j (purple edge in G_k), and then the black edge $[\overline{d^a_j}, d^a_j]$ from the vertex $\overline{d^a_j}$ to the vertex d^a_j.

In other words, a switch path alternates between the red edges (oriented from the unachieved direction d^u_j to $\overline{d^a_j}$) for the G_j (for descending j) and the black edges between.

Remark 8.17. We clarify here some ways in which switch paths and construction paths differ.

(1) Switch paths look like construction paths but, while the purple edges in the construction path for a construction composition $(g_{i-k}, \dots, g_i; G_{i-k-1}, \dots, G_i)$ are purple in each G_l with $i - l \leq l < i$, for a switch path, they will be red edges in the structure G_l they are created in and then will not exist at all in the structures G_m with $m < l$. The change of color of red edges and then disappearance of edges is the reason for (SS2) in the switch sequence definition.

(2) Unlike constructions paths, switch paths do not give subpaths of lamination leaves.

The following lemma proves that switch paths are indeed smooth paths in destination LTT structures. It is important to note that this only holds when (SS1) and (SS2) hold.

Lemma 8.18. *The switch path associated to a realized switch sequence* $(g_j, \dots, g_k; G_{j-1}, \dots, G_k)$ *forms a smooth path in the LTT structure* G_k*.*

Proof: The red edge in G_k is $[d_k^u, \overline{d_k^a}]$. We are left to show (by induction) that:

(1) For each $1 \leq l < k$, $[d_l^u, \overline{d_l^a}] = [d_{l+1}^a, \overline{d_l^a}]$ is a purple edge of G_k and

(2) the purple edges $[d_{l+1}^a, \overline{d_l^a}]$ (together with the black edges in the switch sequence) form a smooth path in G_k.

We start with the base case. By the switch properties, the red edge in G_{k-1} is $[d_{k-1}^u, \overline{d_{k-1}^a}] = [d_k^a, \overline{d_{k-1}^a}]$. Since $d_k^a \neq d_k^u$ and $\overline{d_{k-1}^a} \neq d_k^u$ (by the switch sequence definition), $D^t g_k(\{d_k^a, \overline{d_{k-1}^a}\}) = \{d_k^a, \overline{d_{k-1}^a}\}$. Thus, by Lemma 5.20, $[d_k^a, \overline{d_{k-1}^a}]$, is a purple edge in G_k. The red edge in G_k is $[d_k^u, \overline{d_k^a}]$. So, by including the black edge $[\overline{d_k^a}, d_k^a]$, we have a path $[d_k^u, \overline{d_k^a}, d_k^a, \overline{d_{k-1}^a}]$ in G_k. This path is smooth since it alternates between colored and black edges. So our proof of the base case is complete.

We now prove the inductive step. By the inductive hypothesis we assume that the sequence of switches associated to $g_k, \dots, g_{k-i}$ gives us a smooth path $[d_k^u, \dots, \overline{d_{k-i}^a}]$ in G_k ending with a purple edge with "free" vertex $\overline{d_{k-i-1}^a}$. We know that the red edge in G_{k-i-1} is $[d_{k-i-1}^u, \overline{d_{k-i-1}^a}] = [d_{k-i}^a, \overline{d_{k-i-1}^a}]$. As long as we do not have $d_l^u = d_{k-i}^a$ or $d_l^u = \overline{d_{k-i-1}^a}$ for $k-i \leq l \leq k$ (which holds again by the definition of a switch sequence), $D^t g_{k,k-i}(\{d_{(k-i-1)}^u, \overline{d_{(k-i-1)}^a}\}) = D^t g_{k,k-i}(\{d_{(k-i)}^a, \overline{d_{(k-i-1)}^a}\}) = \{d_{(k-i)}^a, \overline{d_{(k-i-1)}^a}\}$. This, as above, makes $[d_{(k-i)}^a, \overline{d_{(k-i-1)}^a}]$ a purple edge in G_k by Lemma 5.20.

Since $[d_k^u, \dots, \overline{d_{k-i}^a}]$ is a smooth path in G_k ending with a black edge, $[d_k^u, \dots, \overline{d_{k-i}^a}, d_{k-i}^a, \overline{d_{k-i-1}^a}]$ is also a smooth path in G_k, $[\overline{d_{k-i}^a}, d_{k-i}^a]$ is a black edge in G_k and as $[d_{k-i}^a, \overline{d_{k-i-1}^a}]$ is a purple edge in G_k.

QED.

Example 8.19. We return to the LTT structure G (for Graph XX) of Example 8.10 and number below the colored edges of a switch path.

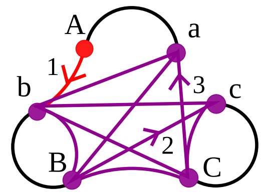


Figure 8.7: *Switch path in G with colored edges numbered*

The switch sequence for G constructed from the switch path is:

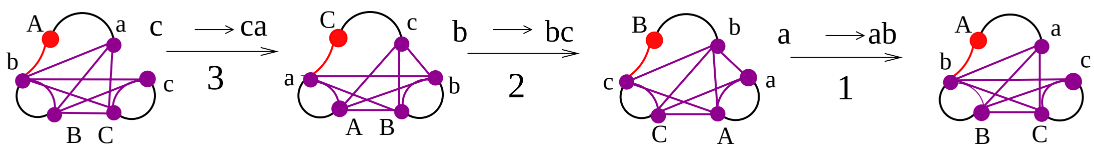


Figure 8.8: *Switch sequence constructed from the switch path in Figure 8.7*

Notice how the red edge in the destination LTT structure for the generator labeled by (1) is the first colored edge in the switch path, the red edge in the destination LTT structure for (2) is the second colored edge in the switch path, etc.

Chapter 9

AM Diagrams

In this chapter we describe how to construct the "AM Diagram" for a Type (*) pIWG, as well as prove that Type (*) representatives are realized as loops in these diagrams.

9.1 AM Diagrams Defined

Definition 9.1. Let $\mathcal{G}$ be a given Type (*) pIW graph. A *PreAdmissible Map Diagram (PreAM Diagram) for* $\mathcal{G}$ *(or* $PreAM(\mathcal{G})$*)* is the directed graph where

1. the nodes correspond to equivalence classes of admissible index pair-labeled Type (*) LTT structures for $\mathcal{G}$ and

2. for each admissible indexed generator triple (g_i, G_{i-1}, G_i) for $\mathcal{G}$ equivalence class, there exists a directed edge $E(g_i, G_{i-1}, G_i)$ from the node $[G_{i-1}]$ to the node $[G_i]$.

The disjoint union of the maximal strongly connected subgraphs of $PreAM(\mathcal{G})$ will be called the *Admissible Map (AM) Diagram for* $\mathcal{G}$ (or $AM(\mathcal{G})$).

The following proposition shows that we can restrict our search for representatives to loops in AM Diagrams. We say that an ideal decomposition $\Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{k-1}} \Gamma_{k-1} \xrightarrow{g_k} \Gamma_k$ of a train track g with known indexed Type (*) LTT structures $G_0 \xrightarrow{D^T g_1} G_1 \xrightarrow{D^T g_2} \cdots \xrightarrow{D^T g_{k-1}} G_{k-1} \xrightarrow{D^T g_k} G_k$ for a Type (*) pIW graph $\mathcal{G}$ is *realized* by $E(g_1, G_0, G_1) * \cdots * E(g_k, G_{k-1}, G_k)$ in $AM(\mathcal{G})$, where $E(g_1, G_0, G_1) * \cdots * E(g_k, G_{k-1}, G_k)$ denotes the oriented path in $AM(\mathcal{G})$ from $[G_0]$ to $[G_k]$ transversing the edges corresponding to the generators in the order they are composed (starting with the edge $E(g_1, G_0, G_1)$ and concluding with the edge $E(g_k, G_{k-1}, G_k)$).

Proposition 9.2. *If $g = g_k \circ \cdots \circ g_1 : \Gamma \to \Gamma$ is an ideally decomposed Type (*) representative of $\phi \in Out(F_r)$ such that $IW(\phi) = \mathcal{G}$, with corresponding sequence of indexed LTT structures $G_0 \xrightarrow{D^T(g_1)} G_1 \xrightarrow{D^T(g_2)} \cdots \xrightarrow{D^T(g_{k-1})} G_{k-1} \xrightarrow{D^T(g_k)} G_k$, then $E(g_1, G_0, G_1) * \cdots * E(g_k, G_{k-1}, G_k)$ forms an oriented loop in $AM(\mathcal{G})$.*

Proof: Suppose that g is such a representative. We showed in Proposition 7.12 that g is a composition of admissible maps (g_i, G_{i-1}, G_i). This tells us that $E(g_1, G_0, G_1) * \cdots * E(g_k, G_{k-1}, G_k)$ forms an oriented path in $PreAM(\mathcal{G})$. We know that this, in fact, is a loop because $G_0 = G_k$. Since all loops of a graph are contained in the union of the maximal strongly connected subgraphs of the graph, we know that $E(g_1, G_0, G_1) * \cdots * E(g_k, G_{k-1}, G_k)$ is actually in $AM(\mathcal{G})$, proving the proposition.
QED.

Remark 9.3. If one is skeptical that we do not speak nonsense when we say $G_0 = G_k$, realize that ϕ^2 is an ideally decomposed Type (*) representative of ϕ^2 with ideal decomposition $\Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_k} \Gamma_k = \Gamma_0 \xrightarrow{g_1 = g_{k+1}} \Gamma_1 = \Gamma_{k+1} \xrightarrow{g_2 = g_{k+2}} \cdots \xrightarrow{g_k = g_{2k}} \Gamma_k = \Gamma_{2k}$.

Definition 9.4. We denote the loop $E(g_1, G_0, G_1) * \cdots * E(g_k, G_{k-1}, G_k)$ by $L(g_1, \dots, g_k; G_0, G_1, \dots, G_{k-1}, G_k)$ and call it the *representative loop* for $g_{k,1}$ or the loop *realizing* $g_{k,1}$ in $AM(\mathcal{G})$.

Corollary 9.5. ***(of Proposition 9.2)*** *If no loop in $AM(\mathcal{G})$ gives a Type (*) representative of an ageometric, fully irreducible outer automorphism $\phi \in Out(F_r)$ such that $IW(\phi) = \mathcal{G}$, then such a ϕ does not exist. In particular, any of the following properties of an AM Diagram would prove that such a representative does not exist:*

(1) There is at least one edge direction pair $\{d_i, \overline{d_i}\}$, where $e_i \in \mathcal{E}(\Gamma)$, such that no red vertex in $AM(\mathcal{G})$ is labeled by either d_i or $\overline{d_i}$.

(2) The representative corresponding to each loop in $AM(\mathcal{G})$ has a PNP.

Proof: Proposition 3.3 says that such a ϕ would have an ideally decomposed Type (*) representative g and Proposition 9.2 shows that any ideally decomposed Type

(*) representative would be realized by a loop in $AM(\mathcal{G})$. Thus, g has a realization $L(g_1, \dots, g_m; G_0, G_1, \dots, G_{m-1}, G_m)$ in $AM(\mathcal{G})$.

If, for some $1 \leq i \leq r$, $L(g_1, \dots, g_m; G_0, G_1, \dots, G_{m-1}, G_m)$ did not contain an LTT structure G_k where either $d^u_k = d^i$ or $d^u_k = \overline{d^i}$, then the corresponding automorphism would fix the generator of F_r corresponding to E_i, which would make g reducible, contradicting that $\phi \in Out(F_r)$ is fully irreducible. So (1) is proved.

Since Type (*) representatives must be PNP-free, if no loop in $AM(\mathcal{G})$ realizes a PNP-free automorphism, then no Type (*) representative exists. This proves (2) and thus the entire corollary since the first sentence is a direct consequence of the Proposition 9.2.

QED.

Definition 9.6. In the case of Corollary 9.5(1) we will say that the AM Diagram *lacks irreducibility potential.*

9.2 Constructing AM Diagrams

In this section we indexed edge-pair label the vertices of LTT structures with a $2r$ element set of paired labels $\{x_1, \overline{x_1}, \dots, x_{2r}, \overline{x_{2r}}\}$, where each pair $\{x_i, \overline{x_i}\}$ labels the vertices of a different black edge. As usual, pairs of labels of the form $\{x_i, \overline{x_i}\}$ will be called edge pairs. Additionally, the vertices of pIW graphs will be labeled by $2r-1$ element subsets of the $2r$ element set $\{x_1, \overline{x_1}, \dots, x_{2r}, \overline{x_{2r}}\}$. Through Step 4, to avoid unnecessary work, we view graphs only up to edge pair permutation (EPP) isomorphism:

Definition 9.7. We will call a permutation of the indices $1 \leq i \leq 2r$ combined with a permutation of the two elements of each pair $\{x_i, \overline{x_i}\}$ an *Edge Pair (EP) Permutation.* Edge-indexed graphs will be considered *Edge Pair Permutation (EPP) Isomorphic* if there is an EP permutation making the labelings identical (this still holds even if only a subset of $\{x_1, \overline{x_1}, \dots, x_{2r}, \overline{x_{2r}}\}$ were used to label the vertices, as in the case of a pIWG).

Example 9.8. The following are EPP-isomorphic (here, and in following examples, z denotes x_1, Z denotes $\overline{x_1}$, y denotes x_2, Y denotes $\overline{x_2}$, x denotes x_3, and X denotes

$\overline{x_3}$):

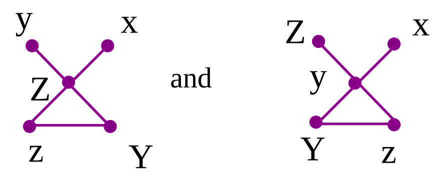


Figure 9.1: *Graphs EPP Isomorphic by the permutation taking y to $\overline{z}$ and $\overline{y}$ to z*

Remark 9.9. For Type (*) LTT structures, equivalence with respect to EP permutation is the same as unmarked, unlabeled equivalence of Type (*) LTT structures.

In Step 5 it will become necessary to distinguish between different indexed edge-pair vertex labelings in both pIWGs and LTT structures. Consequently, the notion of equivalence will shift to be as in Definition 4.16 or 4.9.

Let $\mathcal{G}$ be a Type (*) pIW graph. We explain in Steps 1-6 below a procedure for constructing the admissible map diagram $AM(\mathcal{G})$. As shown in Section 10.2, any ideally decomposed Type (*) train track g with $IW(g) \cong \mathcal{G}$ would give a loop in $AM(\mathcal{G})$. We describe strategies for finding such loops in Chapter 13.

STEP 1: *CONSTRUCTING AN LTT CHART FOR $\mathcal{G}$*

A. An LTT Chart for $\mathcal{G}$ will contain precisely one column for each EPP-isomorphism class of $\mathcal{G}$, preadmissibly vertex-labeled with $\{x_1, \overline{x_1}, x_2, \overline{x_2}, \dots, x_{r-1}, \overline{x_{r-1}}, x_r\}$.

Example 9.10. This graph is not preadmissibly vertex-labeled as it has two distinct pairs of (edge-pair)-labeled vertices, each connected by a valence-1 edge.

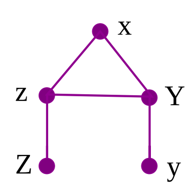


Figure 9.2: *Not in LTT Chart since $[y, \overline{y}]$ and $[z, \overline{z}]$ are valence-one edges*

B. The columns of the LTT chart will be labeled with a representative (without $\overline{x_r}$ as a vertex label) from each preadmissibly labeled EPP-isomorphism classes of $\mathcal{G}$ (any column labeling choice will lead to the same diagram $AM(\mathcal{G})$). For notational simplicity, throughout the following, we assume We will call these graphs the *Determining SW-Graphs* for the columns and color them purple. In our examples, we label them from left to right by SW_I, SW_{II}, etc.

Example 9.11. For Graph VII, one possible choice of Determining SW-Graphs is:

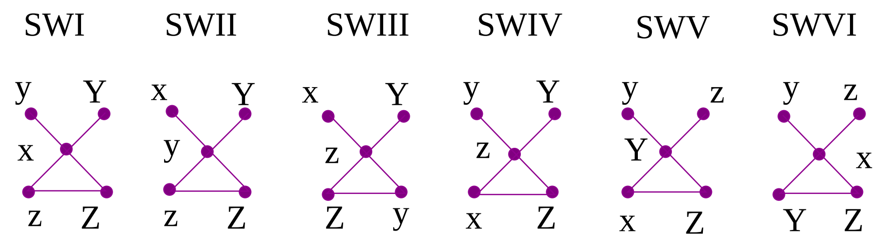


Figure 9.3: *Determining SW-Graphs for Graph VII*

C. Each column of the LTT Chart will contain a graph for each way of attaching (at a single vertex, called the *attaching vertex*) a red edge to the column's determining SW-graph so that:

1. the red edge's valence-1 vertex (colored red and called the *free vertex*) is labeled by $\overline{x_r}$,
2. the graph is preadmissible, and
3. the graph is not EPP isomorphic to another graph already in the chart.

Example 9.12. The following graphs are not included in the LTT Chart because they are not preadmissible, as they have a valence-1 edge connecting edge-pair vertices.

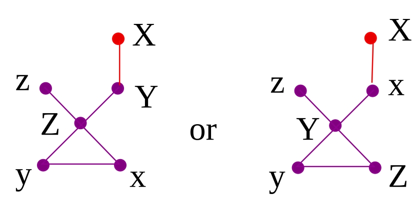


Figure 9.4: *Graphs not in LTT Chart because both* $[z, \overline{z}]$ *and* $[\overline{x}, x]$ *are valence-*1 *edges*

(In what follows we will label graphs that would be in the first (SW_I) column, were they preadmissible and not EPP-isomorphic to another graph in the chart, from top to bottom by I_a, I_b, I_c, $\ldots$; graphs in the second (SW_{II}) column from top to bottom by II_a, II_b, II_c, $\ldots$; etc.)

Example 9.13. *An LTT Chart for Graph VII*

Graphs left out are either EPP-isomorphic to one among those already included (see Figure 9.1) or violate preadmissibility (as in Figure 9.4).

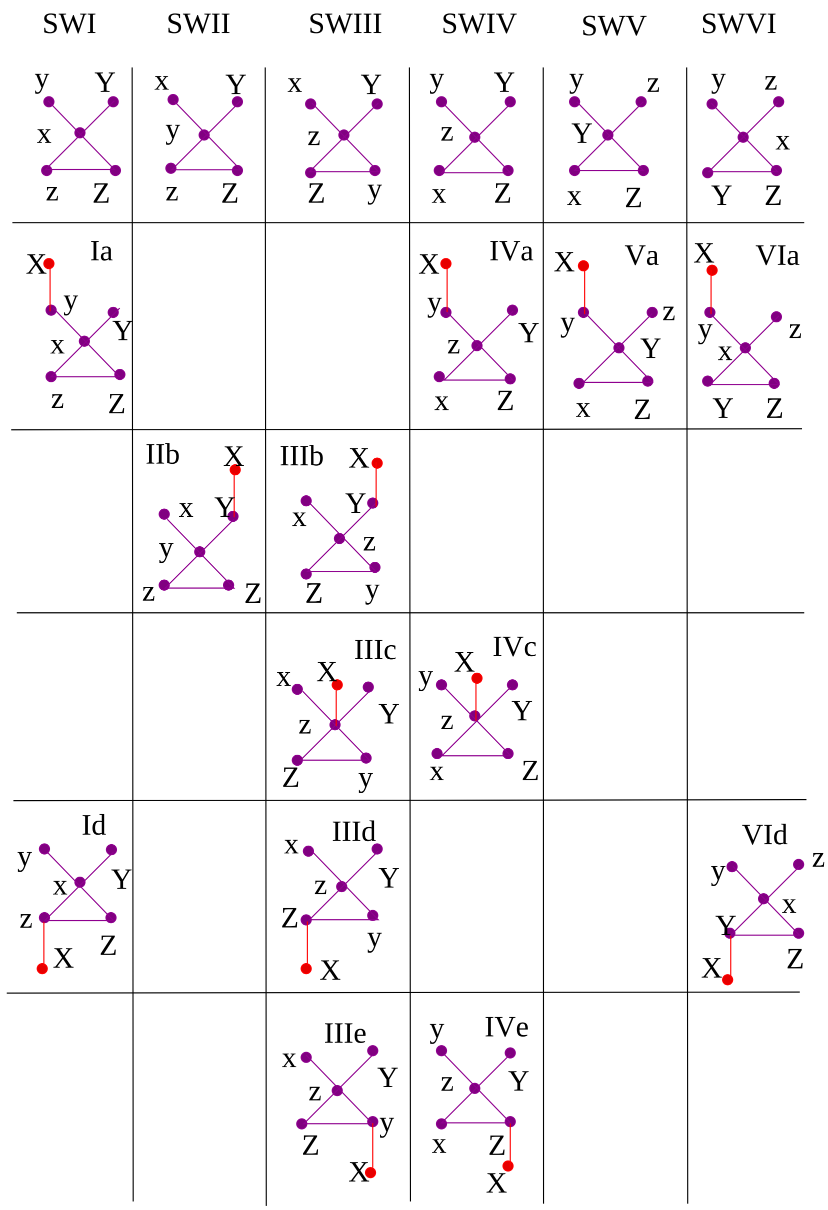


Figure 9.5: *The LTT Chart for Graph VII*

STEP 2: *ADMISSIBILITY (BIRECURRENCY)*

For each graph G in the LTT Chart for $\mathcal{G}$, we obtain an LTT structure $L(G)$ by adding a black edge $[x_i, \overline{x_i}]$ connecting x_i and $\overline{x_i}$ for each $1 \leq i \leq r$ (the colored subgraph $C(L(G)) = G$ will be called the *CLW-graph* for $L(G)$). Each $L(G)$ should be checked for admissibility (by checking it for birecurrency). If $L(G)$ is admissible, both G and $L(G)$ will be called *preAM-included*, as they will be present in the preAM Diagram. In what follows we put a box in the LTT chart around preAM-included G and cross out G that are not preAM-included.

Example 9.14. *Birecurrency for Graph VII*

We leave out LTT structure vertex labels to highlight that LTT structure birecurrency is EPP isomorphism invariant.

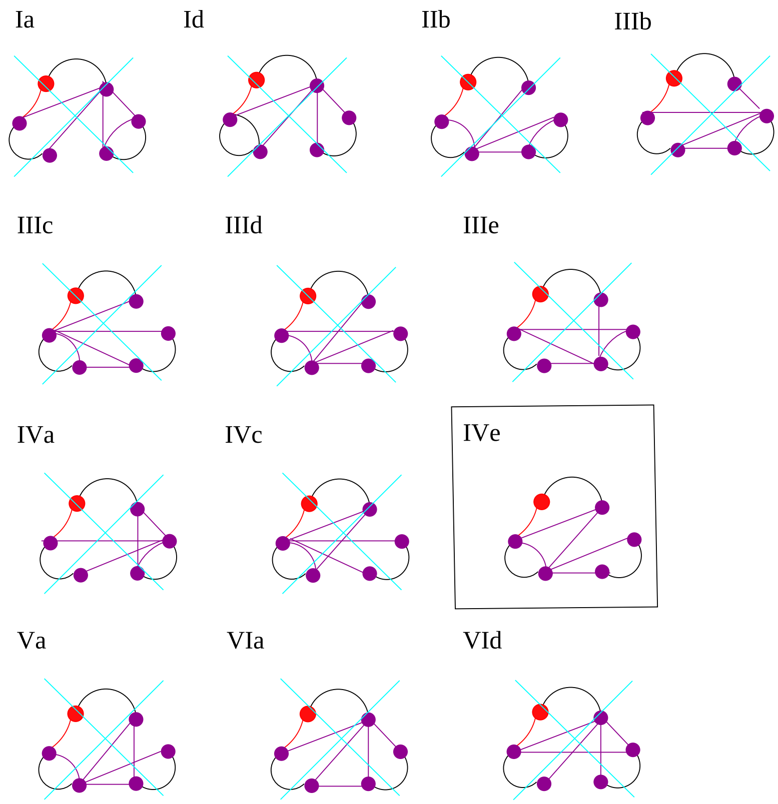


Figure 9.6: *Birecurrency Analysis for Graph VII: We drew the LTT structure for each labeled graph in the LTT Chart and checked the structures for birecurrency. The only birecurrent LTT structure was that for IVe.*

By eliminating graphs G where $L(G)$ was not birecurrent, our LTT Chart is reduced down to:

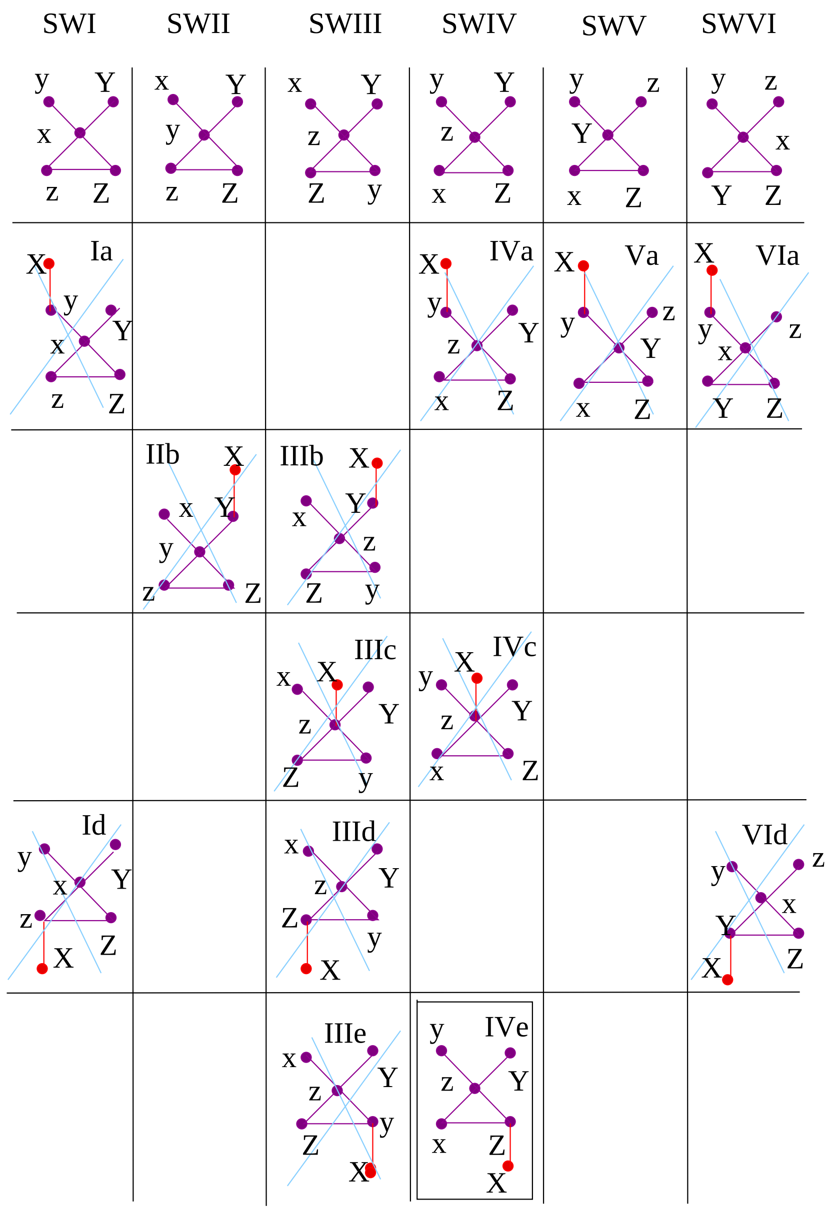


Figure 9.7: *We crossed out and "boxed" the graphs in the LTT Chart for Graph VII that correspond to the LTT structures crossed out and "boxed" in Figure 9.6*

STEP 3: ***A PERMITTED EXTENSION/SWITCH WEB*** $\mathcal{W}''(\mathcal{G})$

A. Choose a preAM-included graph G from the LTT chart to start with.

In what follows, for each $1 \leq i \leq r$*, we denote by* v_i *both* $d_{k,i}$ *and* $d_{k-1,i}$*. The* *attaching vertex of G (the purple vertex of the red edge of G) will be denoted by v_a (the "a" here is for "attaching").*

B. Determine all admissible maps (g_k, G_{k-1}, G_k) with $L(G)$ as their destination LTT structure (i.e. $L(G) = G_k$) as follows:

1. For each vertex v_s distance-1 in G from $\overline{v_a}$ determine, as below, two potential "ingoing LTT Structures" (one for a switch and one for an extension). In LTT structure notation (where $L(G) = G_k$), d_k^a denotes $\overline{v_a}$ and d_k^u labels the free vertex.

   (a) *Potential Ingoing Extension Graph $G^e_{v_s}$ determined by v_s* (Reference Lemma 7.5):

   - the same purple subgraph as that of G (i.e. $PI(G^e_{v_s}) \cong PI(G)$)
   - the second index of the label on the free vertex as in G, i.e. $d_{k-1}^{pu} = d_{k-1}^u$, and, if we use the same letters to label the vertices in G_k and G_{k-1}, the same letter remains on the free vertex
   - the red edge is now attached at v_s, instead of at v_a

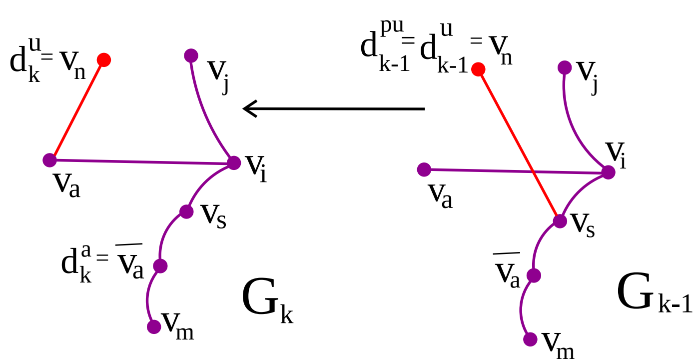


Figure 9.8: *Potential Ingoing Extension Graph for v_s: Notice that the label on the free vertex remains the same (v_n) but that the attaching vertex of the red edge changes from $v_a (= \overline{d_k^a})$ to the distance-one vertex v_s.*

   (b) *Potential Ingoing Switch Graph $G^s_{v_s}$ determined by v_s* (Reference Lemma 7.10):

$PI(G^s_{v_s}) \cong PI(G)$, via an isomorphism preserving almost all vertex with the exception being that the second index of d^u_k in G_k is the second index of the vertex in G_{k-1} mapped by the isomorphism to the vertex labeled with d^a_k. (If we use the same letters to label the vertices in G_k and G_{k-1}, or equivalently drop first indices, it will look as if the label on the red vertex in G has moved to the position in $G^s_{v_s}$ of what had been d^a_k in G (the inverse of the attaching vertex)).

- The red vertex of $G^s_{v_s}$ is labeled, d^a_k
- $G^s_{v_s}$'s red edge is attached at v_s, and
- and the free vertex's label is now d^a_k.

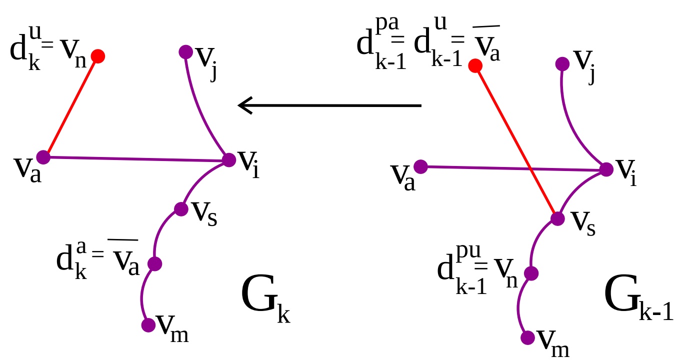


Figure 9.9: *Potential Ingoing Switch Graph for v_s: Notice how the purple graph with its labels looks the same except for movement of V_a and v_n. Also notice how the red edge shifts.*

2. The admissible maps (g_k, G_{k-1}, G_k) with $G_k = L(G)$ are precisely the triples where either $G_{k-1} = G^e_{v_s}$ or $G_{k-1} = G^s_{v_s}$, determined in (1) and having $L(G_{k-1})$ birecurrent, and g_k as in Lemma 7.5 or Lemma 7.10, respectively. We will call each CLW-graph $G^s_{v_s}$ and $G^e_{v_s}$, by the name of the LTT Chart EPP-isomorphic graph (in our examples labels will be of the form $(RomanNumeral)_{Letter}$).

C. $\mathcal{W}''(\mathcal{G})$ is determined so far to include:

- a node for each preAM-included potential ingoing LTT structure CLW-graph, H_j (i.e. we include a node when $L(H_j)$ is actually an admissible

Type (*) LTT structure) and

- a directed edge from each such node H_j to the node for G.

(This can be achieved by determining all potential ingoing LTT structures through the process in (B1), and then eliminating from consideration any CLW-graph not EPP-isomorphic to a graph in the LTT Chart (i.e. if it is not preadmissible) or EPP-isomorphic to a graph G' in the LTT chart such that $L(G')$ was not admissible).

D. We now have a node for a graph G with a number of edges directed into it, originating at nodes for preAM-included potential ingoing LTT structure CLW-graphs H_j. We identify which H_j are EPP isomorphic to G and call them *branch ends.*

Example 9.15. *Determining preAM-included potential ingoing LTT structures in the process of determining $\mathcal{W}''(\mathcal{G})$ where $\mathcal{G}$ is Graph VII*

Above we saw that the only LTT chart graph with a birecurrent LTT structure is IVe. $\overline{v_a} = z$, so the distance-1 vertices are those labeled $y, \overline{y}, \overline{x}$, and $\overline{z}$. We get 4 potential ingoing extension graphs by attaching the red edge to each of these vertices. However, the only one with a birecurrent LTT structure is that where the red edge was not moved (and is still attached at z). Thus, the only permitted ingoing extension is the self-map of IVe. To get the 4 potential ingoing switch graphs for IVe, we again attach the red edge at each of $y, \overline{y}, \overline{x}$, and $\overline{z}$, but this time also switch the labels so that the free vertex is labeled z and what was labeled z before is now labeled x. The only of these potential ingoing switch graphs having a birecurrent LTT structure is that EPP-isomorphic to IVe, i.e. that with attaching vertex $\overline{x}$.

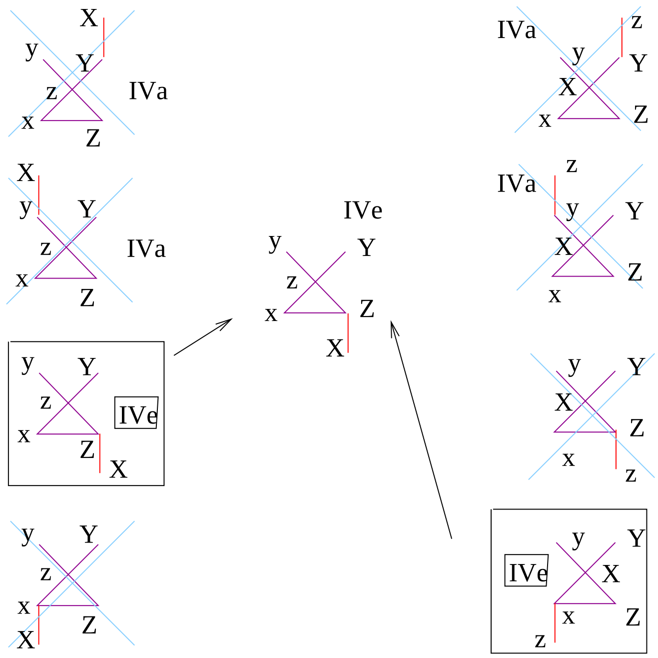


Figure 9.10: *Determining Permitted Extension/Switch Web for Graph VII (the permitted ingoing extensions are on the left and the permitted ingoing switches are on the right)*

Thus far, we have that $\mathcal{W}''(\mathcal{G})$ will look like:

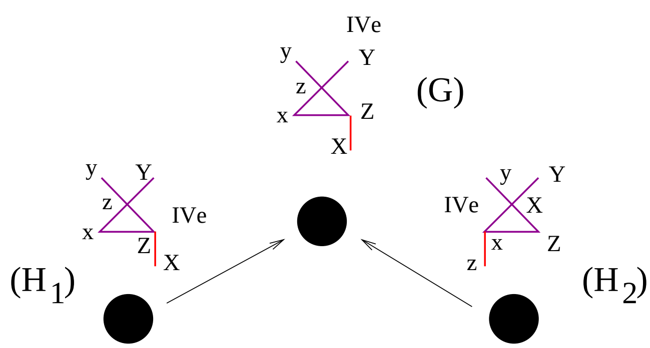


Figure 9.11: *Permitted Extension/Switch Web for Graph VII Thus Far (the only permitted ingoing extension is on the left and the only permitted ingoing switch is on the right)*

E. For each $G_{v_s}^s$ or $G_{v_s}^e$ giving a node in [C], that is not a branch end, carry out the same procedure, as was carried out with G in [B], to find all of the potential ingoing preAM-included LTT structure CLW-graphs, their labels, and the directed edges originating at them.

- In this (and all subsequent) stages, potential ingoing preAM-included LTT structure nodes EPP-isomorphic to graphs already determined in $\mathcal{W}''(\mathcal{G})$ will again be called *branch ends* and do not need their potential ingoing preAM-included LTT structures determined.

F. Continue this process recursively.

Example 9.16. In Example 9.15, since the initial vertices of both directed edges into G are branch ends, this will be all of $\mathcal{W}''(\mathcal{G})$. Otherwise, we would have had to have found all permitted ingoing extensions and switches for H_1 and H_2 and continued from there.

G. When the recursion ends (see Remark 9.17 for why it must end), one portion of $\mathcal{W}''(\mathcal{G})$ is complete. Repeat process to create further portions:

(1) If some preAM-included graph G_1 in the LTT chart is not EPP-isomorphic to any graph in the portion of $\mathcal{W}''(\mathcal{G})$ obtained by starting with G, then use any such G_1 to start the construction of another portion of the web in the same way we constructed the portion starting with G.

(2) Keep constructing portions of $\mathcal{W}''(\mathcal{G})$ as such until all preAM-included graphs in the LTT chart appear in at least one portion of the web constructed.

Remark 9.17. The recursion process ends, as the LTT chart is finite (so there are a finite number of "portions" of the web) and a branch can only contain each LTT Chart element once.

STEP 4: THE *(REFINED) SCHEMATIC EXTENSION/SWITCH WEB* $\mathcal{W}(\mathcal{G})$

A. A directed graph $\mathcal{W}'(\mathcal{G})$ is obtained from $\mathcal{W}''(\mathcal{G})$ by identifying nodes with labels in the same EPP-isomorphism class.

B. The *Refined Schematic Extension/Switch Web* $\mathcal{W}(\mathcal{G})$ is the union of the maximal strongly connected components of $\mathcal{W}'(\mathcal{G})$.

Remark 9.18. The schematic web misses information about EPPs in its loops, but we do not yet need that information and can/will retrieve it from $\mathcal{W}''(\mathcal{G})$ as necessary.

Example 9.19. *(Refined) Schematic Extension/Switch Web $\mathcal{W}(\mathcal{G})$ where $\mathcal{G}$ is Graph VII*

Since all nodes in $\mathcal{W}''(\mathcal{G})$ (see Figure 9.11) are labeled with EPP isomorphic graphs, they are all glued togetherin forming $\mathcal{W}(\mathcal{G})$. We have precisely one extension and one switch mapping IVe to itself. So $\mathcal{W}(\mathcal{G})$ looks like:

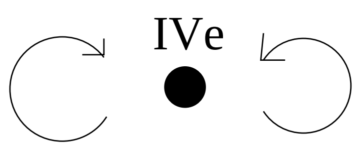


Figure 9.12: *Schematic Permitted Extension/Switch Web for Graph VII*

Since no trimming was necessary, here $\mathcal{W}(\mathcal{G}) \cong \mathcal{W}'(\mathcal{G})$.

STEP 5: *AM DIAGRAM* $AM(\mathcal{G})$

In this step we add to $\mathcal{W}(\mathcal{G})$ information about EPPs, including EPP isomorphic graphs as separate nodes.

A. Choose a node V_i in $\mathcal{W}(\mathcal{G})$. Let $G(V_i)$ be the graph in the LTT chart EPP-isomorphic to V_i ($L(V_i)$ will denote $L(G(V_i))$).

B. For each directed edge entering V_i in $\mathcal{W}(\mathcal{G})$ include a directed edge in $AM(\mathcal{G})$ directed toward $L(V_i)$.

C. Vertices at the initial ends of the edges directed toward the vertex V_i will correspond to the LTT structures determined as follows:

1. Find the corresponding arrow in $\mathcal{W}(\mathcal{G})$.

2. If the terminal LTT structure $L(V_i)'$ in $\mathcal{W}(\mathcal{G})$ differs from $L(V_i)$ by an EPP-isomorphism, in order to determine the initial structure for the directed edge in $AM(\mathcal{G})$, apply the vertex label permutation realizing $L(V_i)$ from $L(V_i)'$ to the directed edge's initial LTT structure in $\mathcal{W}(\mathcal{G})$.

3. If $L(V_i) = L(V_i)'$, include a single $L(V_i)$ node, drawing the directed edge as a loop.

D. Label directed edges terminating at $L(V_i)$ by $x \mapsto xy$ where x labels the red vertex of $L(V_i)$ and $[x, \overline{y}]$ is the red edge of $L(V_i)$ (Reference Lemmas 7.5 and 7.10).

E. Repeat (B) through (D) for each initial LTT structure node for each directed edge constructed thus far. However, if the initial LTT structure for a directed edge is already in the diagram, just start the directed edge at the vertex corresponding to the copy of the LTT structure already in the diagram (the directed edge will form a loop in the diagram).

F. Recursively follow the process until every node in the diagram has the same number of directed edges entering it as the corresponding vertex in $\mathcal{W}(\mathcal{G})$. If $\mathcal{W}(\mathcal{G})$ had more than one component, the whole process must be carried out for each component.

G. The final step in constructing $AM(\mathcal{G})$ will be to apply all possible EPPs to the vertices of the LTT structures labeling the nodes in what has been created thus far (for each component, the same permutation should be applied to the LTT structures labeling all nodes in the component at the same time) and eliminating duplicate components so that there will be precisely one node for each representative in the equivalence class of each graph labeling a vertex in $\mathcal{W}(\mathcal{G})$.

Example 9.20. *A Component of the AM Diagram $AM(\mathcal{G})$ where $\mathcal{G}$ is Graph VII*

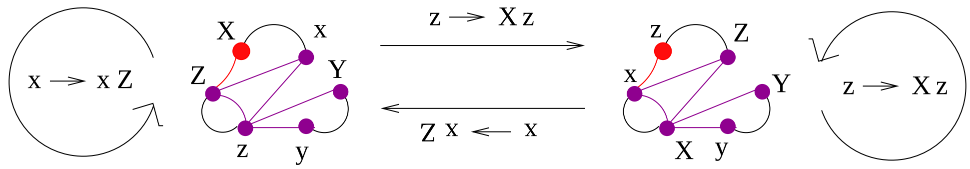


Figure 9.13: *A Component of the AM Diagram for Graph VII (all other components are EPP isomorphic to this component)*

Remark 9.21. To look for ideally decomposed Type (*) representatives or show that they cannot exist, one only needs one component of $AM(\mathcal{G})$ for each EPP isomorphism class and so we generally only include one component for each EPP isomorphism class of components when we actually write down an AM diagram.

Chapter 10

Full Irreducibility Criterion

The main goal of this chapter is the proof of a "Folk Lemma" giving a criterion, the "Full Irreducibility Criterion (FIC)," for an irreducible train track map to represent a fully irreducible outer automorphism. Our original approach to proving the criterion involved the "Weak Attraction Theorem," several notions of train tracks, laminations, and the basin of attraction for a lamination. However, Michael Handel graciously provided a way to finish the proof making much of our initial work unnecessary. The proof of the criterion we give here uses Michael Handel's recommendation.

10.1 Free Factor Systems, Filtrations, and RTTs

The following definitions are necessary to understand the definition of a relative train track representative for an outer automorphism. While [BH92] gives that we always have train track representatives for irreducible outer automorphisms, this is not true for reducible outer automorphisms. Relative train tracks were invented by Bestvina and Handel to approximate train tracks as best possible in this circumstance. We use relative train tracks in our proof of the Full Irreducibility Criterion.

We begin by defining a "free factor system" for a free group, F_r, of rank r.

Definition 10.1 (BFH00)**.** $\mathcal{F} = \{[[F^1]], \dots, [[F^k]]\}$ is a *free factor system* for F_r if $F^1 * F^2 * \cdots * F^k$ is a free factor of F_r and each F^i is nontrivial. For free factor systems $\mathcal{F}_1$ and $\mathcal{F}_2$, we say that $\mathcal{F}_1 \sqsubset \mathcal{F}_2$ when, for each $[[F^i]] \in \mathcal{F}_1$, there exists some $[[F^j]] \in \mathcal{F}_2$ such that $[[F^i]] \sqsubset [[F^j]]$, i.e. F^i is conjugate to a free factor of F^j.

A distinguishing characteristic of reducible outer automorphism representatives is the existence of proper nontrivial invariant subgraphs. A relative train track representative of such an outer automorphism will have a "filtration" of invariant subgraphs "realizing" a nested sequence of free factors. Over the course of the next few definitions we describe what this means.

Definition 10.2 (BH92)**.** For a topological representative $g : \Gamma \to \Gamma$ of $\phi \in Out(F_r)$, an increasing sequence of g-invariant subgraphs $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k = \Gamma$ such that each component of each subgraph contains at least one edge is called a *filtration*. For such a filtration, the closure H_t of $\Gamma_t - \Gamma_{t-1}$ is called the t^{th} *stratum*. Let $\mathcal{E}_t^+ = \{E_1^t, \dots, E_{n_t}^t\}$ denote the set of edges of H_t with some prescribed orientation and let $\mathcal{E}_t = \{E_1^t, \overline{E_1^t}, \dots, E_{n_t}^t, \overline{E_{n_t}^t}\}$. The *transition submatrix* for the stratum H_t is the square matrix M_t such that, for each i and j, the ij^{th} entry is the number of times $g(E_j^t)$ crosses over E_i^t in either direction. Strata with zero matrices as their transition submatrices are called *zero strata*. Let λ_t denote the *Perron-Frobenius eigenvalue* for M_t, i.e. the real eigenvalue of largest norm. Then H_t is called *exponentially growing (EG)* if $\lambda_t > 1$ and a *nonexponentially growing (NEG)* if $\lambda_t = 1$.

We are now ready for the relative train track representative definition, as defined in [BH92].

Definition 10.3 (BH92)**.** A *Relative Train Track (RTT) Representative* of an outer automorphism $\phi \in Out(F_r)$ is a topological representative $g : \Gamma \to \Gamma$ and filtration $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k = \Gamma$ such that:

(1) Each M_t is either the zero matrix or is irreducible;

(2) all vertices have valence greater than one; and

(3) each EG-stratum satisfies:

(a) for each edge $E \in \mathcal{E}_t$, the first edge of $g(E)$ is in $\mathcal{E}_t$,

(b) $g_\#(\beta)$ is nontrivial for each nontrivial path $\beta \subset \Gamma_{t-1}$ having endpoints in $\Gamma_{t-1} \cap H_t$, and

(c) $g(\gamma) \subset \Gamma_r$ is a *t-legal path* (i.e. Γ_{t-1} contains each of its illegal turns) for each legal path $\gamma \subset H_t$.

The following are needed for understanding the revised versions of RTTs used in proving the Full Irreducibility Criterion.

Definition 10.4. Suppose Γ is a marked graph and Γ_i is a subgraph with non-contractible components $C_1, \ldots, C_k$. Then $\mathcal{F}(\Gamma_i) = \{\pi(C_1), \ldots, \pi(C_k)\}$ is called the free factor system $\mathcal{F}(\Gamma_i)$ *realized* by Γ_i. A nested sequence of free factor systems $\mathcal{F}^1 \sqsubset \mathcal{F}^2 \sqsubset \cdots \sqsubset \mathcal{F}^m$ is said to be *realized* by an RTT $g : \Gamma \to \Gamma$ and filtration $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k = \Gamma$ if each $\mathcal{F}^j$ is realized by some F_{i_j}. [BH92]

A topological representative $g : \Gamma \to \Gamma$ and filtration $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k = \Gamma$ are called *reduced* if each H_t satisfies: For any $l > 0$ and each ϕ^l-invariant free factor system $\mathcal{F}$ such that $\mathcal{F}(\Gamma_{i-1}) \sqsubset \mathcal{F} \sqsubset \mathcal{F}(\Gamma_i)$, either $\mathcal{F} = \mathcal{F}(\Gamma_{i-1})$ or $\mathcal{F} = \mathcal{F}(\Gamma_i)$. [BFH00]

We will use a correspondence proved in [BFH00] between attracting laminations for an outer automorphism $\phi \in Out(F_r)$ and the EG-strata of an RTT representative $g : \Gamma \to \Gamma$ of ϕ: For each EG stratum H_t of g, there exists a unique attracting lamination (denoted by Λ_t) having H_t as the highest stratum crossed by the realization $\lambda \subset \Gamma$ of a Λ_t-generic line. H_t is called the *EG-stratum determined by* $\Lambda_t \in \mathcal{L}(\phi)$.

We will remind the reader of the definition of a revised version of a relative train track called a "complete split relative train track (CT)." These train tracks are defined by M. Feighn and M. Handel in [FH09]. However, we first give the definition of a "complete splitting." Both these definitions are specialized definitions for the case of ageometric outer automorphisms (where there are, in particular, no closed PNPs).

Definition 10.5 (FH09)**.** *A* nontrivial path or circuit γ is *completely split* if it has a *complete splitting*, i.e. can be written as $\gamma = \ldots \gamma_{l-1}\gamma_l \ldots$ where

(1) each γ_i is either a single edge in an irreducible stratum, an iNP, or a taken connecting path in a zero stratum and

(2) $g^k_\#(\gamma) = \ldots g^k_\#(\gamma_{l-1}) g^k_\#(\gamma_l) \ldots$.

In the case of a complete splitting we write $\gamma = \ldots \gamma_{l-1} \bullet \gamma_l \ldots$.

Definition 10.6 (FH09)**.** An RTT representative $g : \Gamma \to \Gamma$ of $\phi \in Out(F_r)$, together with its filtration $\mathcal{F}$ given by $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k = \Gamma$, is a *Completely Split Relative Train Track (CT)* if it satisfies all of the following:

(CT1) g is rotationless.

(CT2) g is completely split, that is:

(a) $g(E)$ is completely split for each edge E in each irreducible stratum and

(b) $g_\#(\sigma)$ is completely split for each taken connecting path σ in a zero stratum.

(CT3) $\mathcal{F}$ is reduced and the cores of the Γ_i are also filtration elements. (Recall that the *core* of a finite graph K is the subgraph consisting of all edges of K crossed by some circuit in K).

(CT4) The endpoints of all iNPs are vertices (necessarily principal). For each NEG-stratum H_i and nonfixed edge of H_i, the terminal endpoint is principal (and hence fixed).

(CT5) Periodic edges are fixed and the endpoints of fixed edges are principal. For a fixed stratum H_t with unique edge E_t, either E_t is a loop or each end of E_t is in Γ_{t-1} and Γ_{t-1} is a core graph.

(CT6) For each zero stratum H_i, there is an EG-stratum H_t (with $t > i$) such that:

(a) H_i is enveloped by H_t, i.e.

(i) for some $u < i < t$, H_u is irreducible,

(ii) no component of G_t is contractible,

(iii) H_i is a component of G_{t-1}, and

(iv) each H_i-vertex is of valence greater than one in G_t.

(b) Each edge of H_i is t-*taken* (i.e. is a maximal subpath of $g_\#^k(E)$ in H_i for some $k > 0$ and edge E in H_t)

(c) H_t contains every vertex of H_i and the link of each vertex of H_i is contained in $H_i \cup H_t$.

(CT7) There are no *linear edges* (i.e. edges E_i of NEG-strata H_i such that $g(E_i) = E_i u_i$ for some nontrivial NP $u_i \in \Gamma_{i-1}$).

(CT8) The highest edges of iNPs do not belong to NEG-strata.

(CT9) Suppose that H_t is an EG-stratum and that ρ is a height-t iNP. Then the restriction of g to Γ_t is $\theta \circ g_{t-1} \circ g_t$ where:

(a) $g_t : \Gamma_t \to \Gamma^1$ can be decomposed into proper extended folds defined by iteratively folding ρ,

(b) $g_{t-1} : \Gamma^1 \to \Gamma^2$ can be decomposed into folds involving edges in Γ_{t-1}, and

(c) $\theta : \Gamma^2 \to \Gamma_t$ is a homomorphism.

Remark 10.7. If $\phi \in Out(F_r)$ is forward rotationless and $\mathcal{C}$ is a nested sequence of ϕ-invariant free factor systems, then ϕ is represented by a CT $g : \Gamma \to \Gamma$ and filtration $\mathcal{F}$ that realizes $\mathcal{C}$ [FH09, Theorem 4.29].

Before we can finally give our Fully Irreducibility Criterion proof, we need to remind the reader of the following.

Definition 10.8 (BFH00)**.** The *complexity* of the free factor system $\mathcal{F} = \{[[F^1]], \dots, [[F^k]]\}$ is defined to be zero if $\mathcal{F}$ is trivial and is otherwise defined to be the non-increasing sequence of positive integers obtained by rearranging the set $\{rank(F^1), \dots, rank(F^k)\}$. The set of all complexities is given a lexicographic ordering.

The *free factor support* for a subset $B \subset \mathcal{B}$ is defined in [BFH00, Corollary 2.6.5] to be the unique free factor system of minimal complexity carrying every element of $\mathcal{B}$.

The only relevant information about the free factor support for our proof of the FIC is that, if a lamination is carried by a proper free factor, then its support is a proper free factor. If this were not the case, then the free factor support would have to have rank r (and thus complexity $\{r\}$), while the free factor carrying the lamination (because it is proper) must be of rank less than r, giving it complexity less than the free factor support and hence contradicting that a free factor support is of minimal complexity.

We would like to credit Michael Handel for his contributions to the proof of the following "Folk lemma," which we will call the *Full Irreducibility Criterion (FIC).*

Lemma 10.9. *(The Full Irreducibility Criterion) Let $g : \Gamma \to \Gamma$ be an irreducible train track representative of $\phi \in Out(F_r)$. Suppose that*

(I) g has no PNPs,

(II) the transition matrix for g is Perron-Frobenius, and

(III) all $LW(x; g)$ for g are connected.

Then ϕ is a fully irreducible outer automorphism.

Proof: Suppose that $g : \Gamma \to \Gamma$ is an irreducible TT representative of $\phi \in Out(F_r)$ with Perron-Frobenius transition matrix, with connected local Whitehead graphs, and with no PNPs. Since g has a Perron-Frobenius transition matrix, as an RTT, it has precisely one stratum and that stratum is EG. Hence, it has precisely one attracting lamination [BFH00]. Since the number of attracting laminations belonging to a TT representative of ϕ is independent of the choice of representative, any representative of ϕ would also have precisely one attracting lamination.

Suppose, for the sake of contradiction, that ϕ were not fully irreducible. Then some power ϕ^k of ϕ would be reducible. If necessary, take an even higher power so that ϕ will also be rotationless (this does not change the reducibility). Notice that, since $\mathcal{L}(\phi)$ is ϕ-invariant, any representative of ϕ^k would also have precisely one attracting lamination.

Since ϕ^k is reducible (and rotationless), there exists a completely split train track representative $h : \Gamma' \to \Gamma'$ of ϕ^k with more than one stratum [FH09, Theorem 4.29]. Since ϕ^k has precisely one attracting lamination, h will have precisely one EG-stratum H_t. Each stratum H_i, other than H_t and any zero strata (if they exist), would be an NEG-stratum consisting of a single edge E_i [FH09, Lemma 4.22]. We will consider separately the cases where $t = 1$ and where $t > 1$.

Notice that, since any zero stratum has zero transition matrix (and thus must have every edge mapped to a lower filtration element by h), a zero stratum could not be H_1. Thus, if $t > 1$, then H_1 is NEG and must consist of a single edge E_1. Since H_1 is bottom-most, it would have to be fixed, as there are no lower strata for its edge to be mapped into. But then, according to (CT5), E_1 would have to be an invariant loop, which would mean that ϕ^k would have a rank-1 invariant free factor. However, g was PNP-free, which means that g^k was PNP-free and thus that ϕ^k should not be able to have a rank-1 invariant free factor. We have thus reached a contradiction for the case where $t > 1$.

Now assume that $t = 1$. This would imply that $\Lambda(\phi^k)(= \Lambda(\phi))$ is carried by a proper free factor. Proposition 2.4 of [BFH97] states that, if a finitely generated subgroup $A \subset F_r$ carries Λ_ϕ, then A has finite index in F_r. The necessary conditions for this proposition are actually only: (1) the transition matrix of g is irreducible and (2) at each vertex of Γ, the local Whitehead graph is connected. (Up to the contradiction in the proof of Proposition 2.4 of [BFH97], the only properties used in the proof are that the support is finitely generated, proper, and carries the lamination. The contradiction uses Lemma 2.1 of [BFH97], which simply proves that properties (1) and (2) carry over to lifts of g to finite-sheeted covering spaces, using no properties other than properties (1) and (2).) The assumptions (1) and (2) are assumptions in the hypotheses of our criteria and Λ is still the attracting lamination for g and so we can apply the proposition to create a contradiction with the fact that Λ has proper free factor support. Applying the proposition, since proper free factors have infinite index, the support must be the whole group. This contradicts that the EG-stratum is H_1 and that there must be more than one stratum.

We have thus shown that we cannot have more than one stratum with $t = 1$ or $t > 1$. So all powers of ϕ must be irreducible and thus ϕ is fully irreducible, as desired. QED.

Remark 10.10. To apply this lemma, we need a procedure for proving the nonexistence of PNPs, as stated in Proposition 11.2 of the following chapter.

10.2 Representative Loops

The goal of this section is to prove, for a Type (*) pIW graph $\mathcal{G}$, that representatives coming from the loops in the $AM(\mathcal{G})$ and satisfying certain prescribed properties, as mentioned before, are indeed train track representatives of ageometric, fully irreducible $\phi \in Out(F_r)$ such that $IW(\phi) = \mathcal{G}$.

Proposition 10.11. *Suppose that $\mathcal{G}$ is a Type (*) pIW graph and that $L(g_1, \dots, g_k; G_0, G_1 \dots, G_{k-1}, G_k) = E(g_1, G_0, G_1) * \cdots * E(g_k, G_{k-1}, G_k)$ is a loop in $AM(\mathcal{G})$ satisfying:*

1. *Each purple edge of $G(g)$ correspond to a turn taken by some $g^k(E_j)$ where $E_j \in \mathcal{E}(\Gamma)$;*
2. *for each $1 \leq i, j \leq q$, there exists some $k \geq 1$ such that $g^k(E_j)$ contains either E_i or $\bar{E}_i$; and*
3. *g has no periodic Nielsen paths.*

Then $g : \Gamma \to \Gamma$ is a train track representative of an ageometric fully irreducible $\phi \in Out(F_r)$ such that $IW(\phi) = \mathcal{G}$.

Proof of Proposition: By the FIC, we only need to show that g is a train track map, the transition matrix of g is Perron-Frobenius, and $IW(\phi) = \mathcal{G}$. Property (2) of this proposition is the same as AM Property (VIc) and so that the transition matrix is Perron-Frobenius follows from Lemma 6.1. That g is a train track map follows from Lemma 6.1. Since g has no PNPs, $IW(g) = SW(g)$, where $SW(g) = \bigcup_{\text{vertices v} \in \Gamma} LSW(v; g)$. By the definition of $LSW(v; g)$, $\bigcup_{\text{vertices v} \in \Gamma} LSW(v; g)$ edges correspond precisely with turns crossed over by some $g^k(E_j)$ were $E_j \in \mathcal{E}(\Gamma)$. By the definition of $G(g)$ the edges of $\bigcup_{\text{vertices v} \in \Gamma} LSW(v; g)$ correspond precisely with the purple edges of $G(g)$.

This completes the proof.

QED.

Chapter 11

Nielsen Path Identification

In this chapter we give a method for finding all iPNPs, thus PNPs, for ideally decomposed train track representatives $g : \Gamma \to \Gamma$ (with the standard 3.4 notation and decomposition $\Gamma = \Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n = \Gamma$), additionally satisfying AM Property II:

For each $0 \leq k \leq n-1$, the illegal turn $T_k = \{d_k^{pu}, d_k^{pa}\}$ satisfies that either $d_k^u = d_k^{pa}$ or $d_k^u = d_k^{pu}$.

Notice that Lemma 5.8 and Lemma 5.10 still apply and so, for each k, $T_k = \{d_k^{pu}, d_k^{pa}\}$ is the unique illegal turn for $f_k = g_k \circ \cdots \circ g_{k+1} : \Gamma_k \to \Gamma_k$. Also, notice that AM Property II guarantees that each f_k is also a train track map.

While we have not yet proved the method's finiteness, its application ended quickly in all examples we used it for.

As a warm-up to the procedure, we show in Example 11.1 its application to the ideal decomposition of the representative g for Graph XIII shown in Figure 11.1. We then explain the steps of the procedure and how we used them in Example 11.1. Finally, we conclude this chapter with a proof of the procedure's validity.

Example 11.1. In this example we apply the procedure to show that the ideally decomposed Type (*) representative for Graph XIII shown in Figure 11.1 has no iPNPs (thus no PNPs). We will show in Chapter 13 how this example was constructed.

We abbreviate the generators $g_i : \Gamma_{i-1} \to \Gamma_i$ in Figure 11.1 by only showing, for each i, how g_i maps the edge e_{i-1}^{pu} (all other edges are mapped trivially). Also, for simplicity's sake, the LTT structures $G_i = G(f_i)$ are shown without the black edges $[e_j]$ connecting the vertex pairs $\{d_j, \overline{d_j}\}$. Underneath each LTT structure G_i we included the green illegal turn T_i from the augmented LTT structure. We will often abuse notation by writing e for $D_0(e)$ where $e \in \{a, \bar{a}, b, \bar{b}, c, \bar{c}\}$.

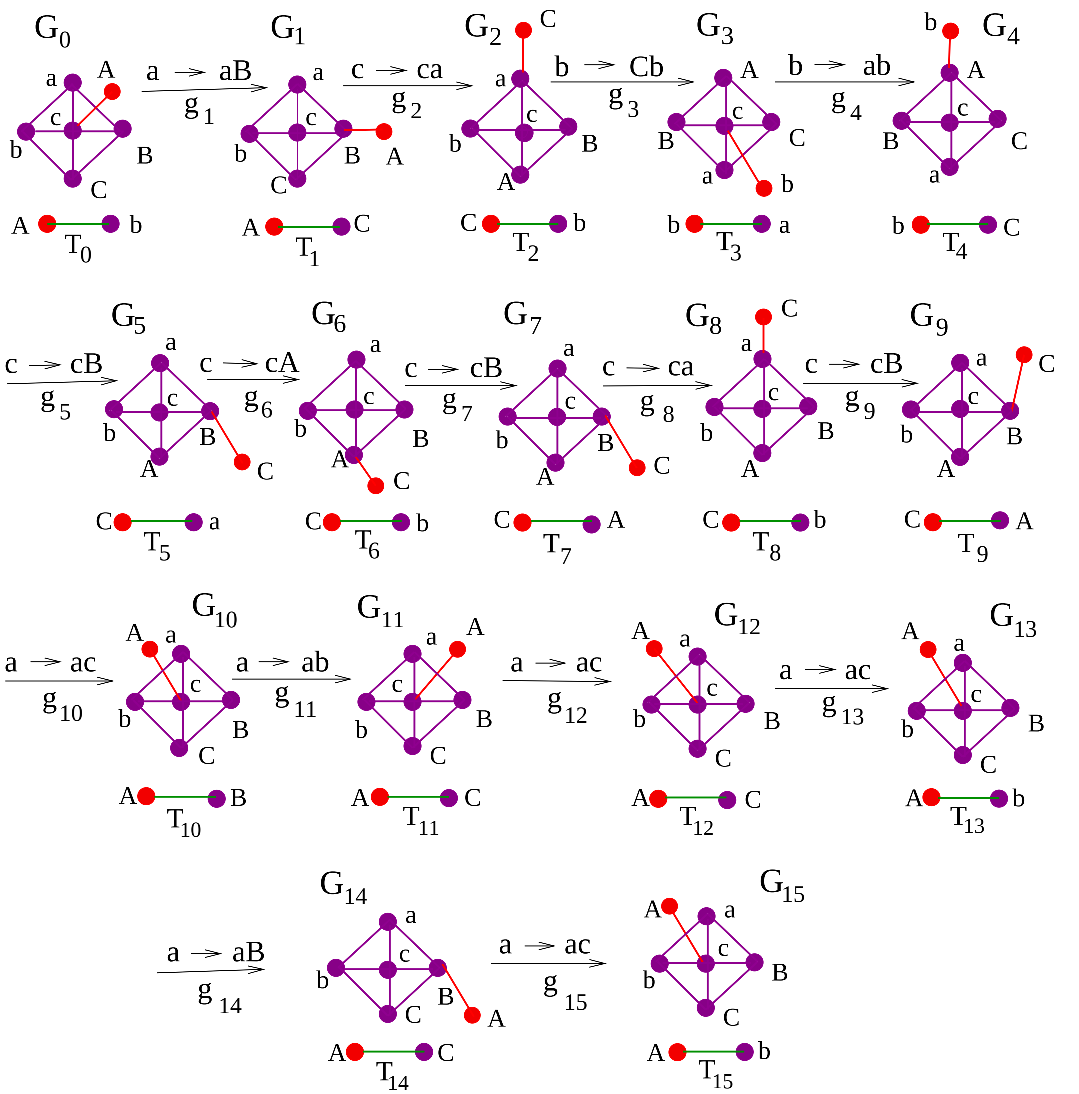


Figure 11.1: *The ideal decomposition for g with the green illegal turns from the augmented LTT structures below*

Since $\{\overline{a}, b\}$ is the only green illegal turn for g, any iPNP would necessarily contain this as its unique illegal turn. We thus try to build an iPNP $\overline{\rho_1}\rho_2$ where $\rho_1 = \overline{a}\dots$ and $\rho_2 = b\dots$ are legal paths. For convenience we use the notation $\rho_1 = \overline{a}e_2e_3\dots$ and $\rho_2 = be_2'e_3'\dots$ where all $e_i, e_i' \in \mathcal{E}(\Gamma)$, except that the final edge in either ρ_1 or ρ_2 may be partial.

Since $g_1(b) = b$ is the initial subpath of $g_1(\overline{a}) = b\overline{a}$, proper cancelation would force

ρ_2 to contain another edge e'_2 after b (see Proposition 11.2 (I)). And, since $\overline{a}$ labels the red vertex in G_1 (i.e. $D_0(\overline{a})$ is the unachieved direction d^u_1), the edge e'_2 would have to be such that $D_0(e'_2)$ is a preimage under Dg_1 of $D_0(\overline{c})$ (proper cancelation requires $Dg_{2,1}(e'_2) = Dg_2(\overline{a}) = D_0(\overline{c})$ but, since $Dg_1(e'_2)$ cannot be the unachieved direction $\overline{a}$, and the only other preimage under Dg_2 of the twice-achieved direction $\overline{c}$ is the other direction in the illegal turn $T_1 = \{\overline{a}, \overline{c}\}$, namely the twice-achieved direction $\overline{c}$, we must have $Dg_1(e'_2) = \overline{c}$). The only preimage under Dg_1 of $\overline{c}$ is $\overline{c}$. Thus, e'_2 would also have to be $\overline{c}$. So, thus far, we have that we would need $\rho_1 = \overline{a}\dots$ and $\rho_2 = b\overline{c}\dots$

Since $g_{2,1}(\overline{a}) = g_2(b)\overline{a}$ is the initial subpath of $g_{2,1}(b\overline{c}) = g_2(b)\overline{ac}$, we know that ρ_1 would have to contain another edge e_2 after $\overline{a}$ (see Proposition 11.2 (III)). Since $\overline{c}$ labels the red unachieved direction vertex in G_2, we know that $D_0(e_2)$ cannot be a preimage under $Dg_{2,1}$ of $\overline{c}$, so our best hope is that $Dg_{3,1}(e_2) = Dg_3(\overline{c})$. This can only happen if $Dg_{2,1}(e_2)$ is a preimage under Dg_3 of the other direction in the illegal turn $T_2 = \{\overline{c}, b\}$, namely the twice-achieved direction b. There are two such preimages under $Dg_{2,1}$ (the two direction in the illegal turn $T_0 = \{\overline{a}, b\}$):

Case 1 will be where $e_2 = \overline{a}$.

Case 2 will be where $e_2 = b$.

We analyze what would happen in the circumstance of each of these being e_2.

For Case 1, suppose that the next edge e_2 of ρ_1 were $\overline{a}$. Since $g_{3,1}(b\overline{c}) = g_{3,2}(b)g_3(\overline{a})\overline{c}$ is the initial subpath of $g_{3,1}(\bar{a}\bar{a}) = g_{3,2}(b)g_3(\overline{a})\overline{c}b\overline{a}$, it follows that ρ_2 would have to contain another edge e'_3 after $\overline{c}$. Since b labels the red unachieved direction vertex in G_3, we would need that $D_0(e'_3)$ were a preimage of a under Dg_3 (since $T_3 = \{a, b\}$, this follows as above). The only such preimage is a. So e'_3 would have to be a, giving $\rho_1 = \bar{a}\bar{a}\dots$ and $\rho_2 = b\overline{c}a\dots$.

Notice that $g_{4,1}(b\overline{c}a) = g_{4,2}(b)g_{4,3}(\overline{a})g_4(\overline{c})a\overline{b}\overline{a}c$ and $g_{4,1}(\bar{a}\bar{a}) = g_{4,2}(b)g_{4,3}(\overline{a})g_4(\overline{c})ab\overline{a}$. So, since $\{\overline{b}, b\} \neq T_4$ (and $b \neq \overline{b}$), we could not have $\rho_1 = \bar{a}\bar{a}$ and $\rho_2 = b\overline{c}a$ (see Proposition 11.2 IVc). This tells us that Case 1 could not have occurred.

For Case 2, suppose that the next edge e_2 of ρ_1 were b. Since $g_{3,1}(b\overline{c}) = g_{3,2}(b)g_3(\overline{a})\overline{c}$ is the initial subpath of $g_{3,1}(\overline{a}b) = g_{3,2}(b)g_3(\overline{a})\overline{c}b$, it follows that ρ_2 would have to contain another edge e'_3 after $\overline{c}$. Since b labels the red unachieved direction vertex in G_3, we can follow the logic above and see that e_3' would have to be a, giving $\rho_1 = \overline{a}b\dots$ and $\rho_2 = b\overline{c}a\dots$.

Since $g_{4,1}(b\overline{c}a) = g_{4,2}(b)g_{4,3}(\overline{a})g_4(\overline{c})a\overline{b}\overline{a}c$ and $g_{4,1}(\overline{a}b) = g_{4,2}(b)g_{4,3}(\overline{a})g_4(\overline{c})ab$, cancelation again leaves us with $\{\overline{b}, b\}$. So, as above, we could not have that $\rho_1 = \overline{a}b\dots$ and $\rho_2 = b\overline{c}a\dots$.

This rules out all possibilities for $\overline{\rho_1}\rho_2$ and so g has no iPNPs, and thus no PNPs, as desired.

Let $g : \Gamma \to \Gamma$ be an ideally decomposed train track representative (with the standard 3.4 notation and decomposition $\Gamma = \Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n = \Gamma$), additionally satisfying AM Property II. We now give the general procedure for finding any iPNPs $\rho = \overline{\rho_1}\rho_2$ for g, where $\rho_1 = e_1 \dots e_m$; $\rho_2 = e_1' \dots e_{m'}'$; $e_1, \dots, e_m, e_1', \dots, e_{m'}' \in \mathcal{E}(\Gamma)$; and $\{D_0(e_1), D_0(e_1')\} = \{d_1, d_1'\}$ is the unique illegal turn of ρ. We let $\rho_{1,k} = e_1 \dots e_k$ and $\rho_{2,l} = e_1' \dots e_l'$ throughout the procedure. After each step is explained in italics, we show its use in Example 11.1.

(I) *Apply generators g_1, g_2, etc, to e_1 and to e_1' until $Dg_{j,1}(e_1') = Dg_{j,1}(e_1)$. Either $g_{j,1}(e_1)$ is the initial subpath of $g_{j,1}(e_1')$ or vice versa. Without loss of generality assume that $g_{j,1}(e_1')$ is the subpath of $g_{j,1}(e_1)$ so that $g_{j,1}(e_1) = g_{j,1}(e_1')t_2\dots$, for some edge t_2. (Otherwise just switch all e_i and e_i', ρ_1 and ρ_2, etc, in the following arguments). Then, ρ_2 must contain another edge e_2'.*

In Example 11.1, we only needed to apply g_1 because $g_1(b) = b$ was the initial subpath of $g_1(\overline{a}) = b\overline{a}$. This told us that ρ_2 necessarily contained another edge after b.

(II) *Inductively, assume that $g_{j,1}(\rho_{1,k}) = g_{j,1}(\rho_{2,s})t_{s+1}\dots$ (or again switch e_i for e_i', ρ_1 for ρ_2, and so on).*

In Example 11.1, since $\rho_{1,1} = b$, $\rho_{2,1} = \overline{a}$, and $g_{2,1}(\overline{a}) = g_2(b)\overline{a}$ is the initial subpath of $g_{2,1}(b\overline{c}) = g_2(b)\overline{ac}$ we assumed $g_{2,1}(\rho_{1,1}) = g_{2,1}(\rho_{2,1})t_2\dots$.

(III) *ρ_2 must have another edge e'_{s+1} after $\rho_{2,s}$. There are two cases to consider (either $D_0(t_{s+1}) = d^u_j$ or $D_0(t_{s+1}) \neq d^u_j$):*

(a) *If $D_0(t_{s+1}) = d^u_j$, then the different possibilities for e'_{s+1} are determined by the directions d'_{s+1} such that $T_j = \{Dg_{j,1}(d'_{s+1}), D_0(t_{s+1})\}$ where $D_0(e'_{s+1}) = d'_{s+1}$.*

In Example 11.1, since $D_0(\overline{a})$ labeled the red vertex in G_1, we knew $D_0(\overline{a}) = d^u_1$ and we were in this case. $T_1 = \{\overline{a}, \overline{c}\}$ implied $Dg_1(d'_2) = \overline{c}$. Since the only preimage of $\overline{c}$ under Dg_1 was $\overline{c}$, this told us that we would have to have $d'_2 = D_0(\overline{c})$, $e'_2 = \overline{c}$, and $\rho_{2,2} = b\overline{c}$.

We again hit this case in Example 11.1 when determining possibilities for e_2.

(b) *If $D_0(t_{s+1}) \neq d^u_j$, then the different possibilities for e'_{s+1} are all edges e'_{s+1} such that $Dg_{j,1}(d'_{s+1}) = D_0(t_{s+1})$ where $D_0(e'_{s+1}) = d'_{s+1}$.*

We did not encounter this circumstance in Example 11.1, but we explain here what would have happened if instead $D_0(\overline{a})$ did not label the red vertex in G_1. Then we would have to look for edges e'_2 with $Dg_1(d'_2) = \overline{a}$. We could not otherwise have $Dg_{2,1}(d'_2) = \overline{a}$ because the other direction of T_1 would be the red unachieved direction and thus would not have a preimage under Dg_1.

Since ρ_2 must be legal, throw out any choices for d'_{s+1} where $T_0 = \{\overline{d'_s}, d'_{s+1}\}$ is the green illegal turn for g. Each remaining d'_{s+1} in (a) or (b) gives another prospective iPNP that we must continue applying the steps to.

For finding e'_2 in Example 11.1, we only had to examine the single possibility, $\overline{c}$, with $Dg_1(\overline{c}) = \overline{c}$. However, both $\overline{a}$ and b (referred to as Case 1 and Case 2) gave prospective directions to be analyzed in finding e_2. For Case 1 and Case 2 we had to separately proceed through the following steps.

(IV) *Continue composing generators g_i until either:*

(a) *We composed with enough g_i to obtain a $g^{p'}$ such that $g^{p'}(\rho_{2,k}) = \tau' e'_1 \dots$ and $g^{p'}(\rho_{1,s}) = \tau' e_1 \dots$ for some legal path τ' (in this case proceed to (V)),*

We did not encounter this circumstance in Example 11.1. This circumstance, when it occurs, makes an iPNP look promising and so was generally not encountered in our proofs that particular representatives are PNP-free. (Va) may identify an iPNP, if it is there, as does (Vc), except that (Vc) involves a little "trimming," as it deals with the case where the initial and final edges of the path are actually only partial edges. (Vb) and (Vd) direct one to possibly find an iPNP by continuing to add edges.

(b) *$g_{j,l}(\rho_{2,k})$ is a subpath of $g_{j,l}(\rho_{1,s})$ or vice versa (in this case, return to (II) and continue with the steps as before)*

We encounter this situation in both Case 1 and Case 2 of Example 11.1. In Case 1, (IIIa) is used to obtain $\rho_{1,2} = \bar{a}\bar{a}$ and $\rho_{2,3} = b\bar{c}a$. In Case 2, (IIIa) yields $\rho_{1,2} = \bar{a}b$ and $\rho_{2,3} = b\bar{c}a$.

(c) *or some $g_{l,1} \circ g^{p'}(\rho_{2,k}) = \tau' \gamma_{2,k}$ and $g_{l,1} \circ g^{p'}(\rho_{1,s}) = \tau' \gamma_{1,s}$ where $\{D_0(\gamma_{2,k}), D_0(\gamma_{1,s})\}$ is a legal turn in G_l, i.e. $T_l \neq \{D_0(\gamma_{2,k}), D_0(\gamma_{1,s})\}$. In this case there cannot be an iPNP with $\overline{\rho_{2,k}}\rho_{1,s}$ as a subpath (proceed to (VIII)).*

We encounter this situation with the legal turn $\{D_0(\bar{b}), D_0(b)\} \neq T_4$ in both Case 1 and Case 2 of Example 11.1 after applying IIIa to obtain $\rho_{1,2}$ and $\rho_{2,3}$. In both Case 1 and Case 2, $\tau = g_{4,2}(b) g_{4,3}(\bar{a}) g_4(\bar{c}) a$. In Case 1, $\{D_0(\gamma_{2,k}), D_0(\gamma_{1,s})\} = \{D_0(\bar{b}\bar{a}c), D_0(b\bar{a})\}$ $= \{D_0(\bar{b}), D_0(b)\}$. In Case 2 $\{D_0(\gamma_{2,k}), D_0(\gamma_{1,s})\} = \{D_0(\bar{b}\bar{a}c), D_0(b)\} = \{D_0(\bar{b}), D_0(b)\}$.

(V) *For each $1 \leq p'$ such that $g^{p'}(\rho_{2,m}) = \tau' e_1 \dots$ and $g^{p'}(\rho_{1,n}) = \tau' e'_1 \dots$ for some legal path τ' (for the appropriate m and n), check if $g^{p'}_{\#}(\overline{\rho_{1,n}}\rho_{2,m}) \subset \overline{\rho_{1,n}}\rho'_{2,m}$ or vice versa and follow the appropriate step (among (a)-(d) below).*

(a) *If, for some* $1 \leq p'$, $g_{\#}^{p'}(\overline{\rho_{1,n}}\rho_{2,m}) = \overline{\rho_{1,n}}\rho_{2,m}$, *then* $\overline{\rho_{1,n}}\rho_{2,m}$ *is the only possible iPNP for* g.

(b) *For each* $1 \leq p'$ *such that* $g_{\#}^{p'}(\overline{\rho_{1,n}}\rho_{2,m}) \subset \overline{\rho_{1,n}}\rho_{2,m}$ *(where containment is proper), proceed to (VI).*

(c) *If* $\overline{\rho_{1,n}}\rho_{2,m} \subset g_{\#}^{p'}(\overline{\rho_{1,n}}\rho_{2,m})$ *(where containment is proper), consider the final occurrence of* e_n *in the copy of* $\rho_{1,n}$ *in* $g^{p'}(\overline{\rho_{1,n}}\rho_{2,m})$ *and the final occurrence of* e'_m *in the copy of* $\rho_{2,m}$ *in* $g^{p'}(\overline{\rho_{1,n}}\rho_{2,m})$. *This final occurrence of* e_n *must have come from* $g^{p'}(e_n)$ *and this final occurrence of* e'_m *must have come from* $g^{p'}(e'_m)$. *This means that we have fixed points in* e_n *and* e'_m. *Replace* $\overline{\rho_{1,n}}\rho_{2,m}$ *with* $\overline{\rho'_{1,n}}\rho'_{2,m}$ *where* $\overline{\rho'_{1,n}}\rho'_{2,m}$ *is the same as* $\overline{\rho_{1,n}}\rho_{2,m}$, *except that* e_n *and* e'_m *are replaced with some partial edges ending at the fixed points. Repeat this process until some* $\overline{\rho'_{1,n}}\rho'_{2,m}$ *is an iPNP.*

(d) *If we do not have* $g_{\#}^{p'}(\overline{\rho_{1,n}}\rho_{2,m}) \subset \overline{\rho_{1,n}}\rho'_{2,m}$ *or vice versa for any* $1 \leq p' \leq b$, *then there is only one circumstance where we can possibly have an iPNP with* $\overline{\rho_{2,m}}\rho_{1,n}$ *as a subpath. This is the case where* $g_{\#}^{p'}(\overline{\rho_{1,n}}\rho_{2,m}) = \overline{\gamma_{1,n}}\gamma_{2,m}$ *where either* $\gamma_{1,n} \subset \rho_{1,n}$ *and* $\rho_{2,m} \subset \gamma_{2,m}$ *or* $\gamma_{2,m} \subset \rho_{2,m}$ *and* $\rho_{1,n} \subset \gamma_{1,n}$. *In this case, apply (VI) to the side that is too short. Otherwise, there cannot be an iPNP with* $\overline{\rho_{2,m}}\rho_{1,n}$ *as a subpath, so proceed to (VII).*

(VI) *We assume here that* $g^{p'}(\bar{\rho}_{1,n}\rho_{2,m}) \subset \overline{\rho_{1,n}}\rho_{2,m}$ *(where containment is proper). Without loss of generality, assume that there exists a* t_{m+1} *such that* $g^{p'}(\overline{\rho_{1,n}}\rho_{2,m})t_{m+1} \subset \overline{\rho_{1,n}}\rho_{2,m}$. *For each direction* d_i *such that* $Dg^{p'}(d_i) = D_0(t_{m+1})$ *and such that* $\{D_0(\overline{e_{i-1}}), D_0(e_i)\}$ *is not the green illegal turn* $\{D_0(e_1), D_0(e'_1)\}$ *for* g, *return to (V) with* $\rho_{2,m+1}$ *where* $D_0(e_{m+1}) = d_i$.

(VII) *Continue to rule out the other possible subpaths that arose via this procedure (by different choices of* d_i, *as in (III) or (VI)). If there are no other possible subpaths, then we have shown there are no iPNPs, thus no PNPs, for* g.

In Example 11.1, in discovering options for e_2, after analyzing Case 1, this step was what set us back to analyzing Case 2.

Proposition 11.2. *Let $g : \Gamma \to \Gamma$ be an ideally decomposed train track representative with the standard 3.4 notation and decomposition $\Gamma = \Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} \Gamma_{n-1} \xrightarrow{g_n} \Gamma_n = \Gamma$, and additionally satisfying AM Property II:*

For each $0 \leq k \leq n-1$, the illegal turn $T_k = \{d_k^{pu}, d_k^{pa}\}$ satisfies that either $d_k^u = d_k^{pa}$ or $d_k^u = d_k^{pu}$.

Then the procedure described in steps (I)–(VII) determines all iPNPs for g.

We will need the following Lemma(s) for the proof of this proposition.

Lemma 11.3. *Subpaths of legal paths are legal.*

Proof of Lemma: The set of turns of the subpath is a subset of the set of turns of the path. Since all turns of the path are legal, all turns of the subpath must also be legal. So the subpath must also be a legal path.
QED.

Lemma 11.4. *For train track maps, images of legal paths and turns are legal.*

Proof of Lemma: Suppose that γ is a legal path and suppose that g is a train track map. Since g is a train track map, the image under g of any edge of γ is legal. Thus, we only need to be concerned about what happens with the turns of γ. Since γ is legal, all turns of γ are legal. Since images of legal turns are legal, the images of all turns of γ are legal. Thus, the image of γ is legal, as desired.
QED.

We also remind the reader that:

- Since d_k^u will be one vertex of the green illegal turn T_k of G_k, T_k cannot also be a purple edge in G_k. Also, d_k^u must be a vertex of the red edge $[t_k^R]$ of G_k.
- For each k, T_k is not the red edge in G_k, so is not represented by any edge in G_k.

Proof of Proposition: We begin with an argument that will be used throughout the proof. Since $\rho = \overline{\rho_1}\rho_2$ is an iPNP, ρ_1 and ρ_2 are both legal paths. Since subpaths of legal paths are legal and since the images under $g_{k,1}$ of legal paths are legal, the paths $g_{k,1}(e_1 \dots e_l)$ and $g_{k,1}(e'_1 \dots e'_{l'})$ are legal for each $1 \leq k \leq n$, $1 \leq l \leq m$, and $1 \leq l' \leq n'$.

Since $\{D_0(e_1), D_0(e'_1)\}$ is an illegal turn, for some j, $Dg_{j,1}(e'_1) = Dg_{j,1}(e_1)$. We need to prove that either $g_{j,1}(e_1)$ is a subpath of $g_{l,1}(e'_1)$ or vice versa. Let $d_1 = D_0(e_1)$ and $d'_1 = D_0(e'_1)$. Since $\{D_0(e_1), D_0(e'_1)\}$ is an illegal turn for g and the only illegal turn for g is $T_0 = \{d_0^{pu}, d_0^{pa}\}$, $\{d_1, d'_1\} = \{d_0^{pu}, d_0^{pa}\}$. Without loss of generality suppose that $d_1 = d_0^{pu}$ (and $d'_1 = d_0^{pa}$). Since g_1 is defined by $e_0^{pu} \mapsto e_1^a e_1^u$, it follows that $g_1(e_1) = g_1(e_0^{pu}) = e_1^a e_1^u$ and $g_1(e'_1) = g_1(e_0^{pa}) = e_1^a$. So $g_1(e'_1)$ is a subpath of $g_1(e_1)$. Since $g_{j,2}$ is an automorphism and also takes legal paths to legal paths, $g_{j,2}(g_1(e'_1)) = g_{j,2}(e_1^a)$ is a subpath of $g_{j,2}(g_1(e_1)) = g_{j,2}(e_1^a e_1^u) = g_{j,2}(e_1^a) g_{j,2}(e_1^u)$, as desired.

The final thing to show for (I) is that ρ_2 must contain a second edge e'_2. Suppose ρ_2 did not contain a second edge. Then tightening would cancel out all of $g_{j,1}(\rho_2)$ with an initial subpath of $g_{j,1}(\rho_1)$ and so certainly would cancel out all of $g_{j,1}(\rho_2)$ with an initial subpath of $g_{j,1}(\rho_1)$. Thus, $(g_{j,1})_\#(\rho) = (g_{j,1})_\#(\overline{\rho_1}\rho_2)$ would be a subpath of $g_{j,1}(\rho_2)$ and hence would be legal. So $g^p_\#(\rho) = (g^{p-1} \circ g_{n,j+1})_\#((g_{j,1})_\#(\rho))$ is legal for all p, contradicting that some $g^p_\#(\rho) = \rho$, which has an illegal turn.

To start off proving (III) we need a similar argument, as in the previous paragraph, to show that ρ_2 would have to have another edge e'_{s+1}. For the sake of contradiction, suppose that ρ_2 ended with e'_s. Then $(g_{j,1})_\#(\rho) = (g_{j,1})_\#(\overline{\rho_1}\rho_2)$ would be a subpath of $g_{j,1}(\rho_1)$ (for similar reasons as above), which leads to a contradiction as in the argument above.

We now prove the claims of (IIIa) and (IIIb). Since $g_{j,1}(\rho_{1,k}) = g_{j,1}(\rho_{2,s}) t_{s+1} \dots$, $(g_{j,1})_\#(\overline{\rho_{1,k}}\rho_{2,s}) = \dots \overline{t_{s+1}} = \overline{\gamma}$ for some legal path γ (γ is legal because it is a subpath of the image of the legal path $\rho_{1,k}$). Additionally, $g_{j,1}(e'_{s+1})$ will be legal since e'_{s+1} is a legal path. Thus, $(g_{j,1})_\#(\overline{\rho_{1,k}}\rho_{2,s}) = (\overline{\gamma} g_{j,1}(e'_{s+1}))_\#$, which is just $\overline{\gamma} g_{j,1}(e'_{s+1})$ unless $Dg_{j,1}(d'_{s+1}) = D_0(t_{s+1})$, and then is legal unless $\{D_0(\gamma), D_0(g_{j,1}(e'_{s+1}))\}$ is an illegal turn, i.e. $T_j = \{Dg_{j,1}(d'_{s+1}), D_0(t_{s+1})\}$ with $D_0(e'_{s+1}) = d'_{s+1}$.

Suppose first that $D_0(t_{s+1}) = d_j^u$ (as in (III)(a)). Notice that, in this case, $D_0(t_{s+1})$ is not in the image of Dg_j and thus is not in the image of $Dg_{j,1} = D(g_j \circ g_{j-1,1})$ and so $Dg_{j,1}(d'_{s+1}) \neq D_0(t_{s+1})$. This tells us that $(\overline{\gamma} g_{j,1}(e'_{s+1}))_\# = \overline{\gamma} g_{j,1}(e'_{s+1})$, which will be a legal path unless $T_j = \{Dg_{j,1}(d'_{s+1}), D_0(t_{s+1})\}$. However, if $(g_{j,1})_{\#}(\overline{\rho_{1,k}}\rho_{2,s})$ $= \overline{\gamma} g_{j,1}(e'_{s+1})$, then

$$(g_{j,1})_{\#}(\rho) = (g_{j,1})_{\#}(\overline{\rho_1}\rho_2) = (g_{j,1})_{\#}(\overline{e_m}\dots\overline{e_{k+1}})\overline{\gamma} g_{j,1}(e'_{s+1})(g_{j,1})_{\#}(e'_{s+2}\dots e'_m) =$$
$$g_{j,1}(\overline{e_m}\dots\overline{e_{k+1}})\overline{\gamma} g_{j,1}(e'_{s+1})g_{j,1}(e'_{s+2}\dots e'_m),$$

since ρ_1 and ρ_2 are legal paths and the images of edges are legal. But $g_{j,1}(\overline{e_m}\dots\overline{e_{k+1}})\overline{\gamma}$ is a subpath of $g_{j,1}(\overline{\rho_1})$, so is legal, and $g_{j,1}(e'_{s+1})g_{j,1}(e'_{s+2}\dots e'_m)$ is a subpath of $g_{j,1}(\rho_2)$, so is legal, and we still have that $\overline{\gamma} g_{j,1}(e'_{s+1})$ is legal, which together would make $(g_{j,1})_{\#}(\rho)$ legal. This contradicts that some $g^p_{\#}(\rho) = (g^{p-1} \circ g_{j,n+1})_{\#}(g_{j,1})_{\#}(\rho))$ must be ρ, which has an illegal turn. So, $T_j = \{Dg_{j,1}(d'_{s+1}), D_0(t_{s+1})\}$, as desired.

Suppose now (as in (III)(b)) that $D_0(t_{s+1}) \neq d^u_j$. For the sake of contradiction suppose that $Dg_{j,1}(d'_{s+1}) \neq D_0(t_{s+1})$, where $D_0(e'_{s+1}) = d'_{s+1}$. First off, notice that this means that again $(\overline{\gamma} g_{j,1}(e'_{s+1}))_{\#} = \overline{\gamma} g_{j,1}(e'_{s+1})$. Also, since $Dg_{j,1}(d'_{s+1})$ cannot be d^u_j (see reasoning above) and $D_0(t_{s+1}) \neq d^u_j$, we cannot have $T_j = \{Dg_{j,1}(d'_{s+1}), D_0(t_{s+1})\}$. This would make $\overline{\gamma} g_{j,1}(e'_{s+1})$ legal, which leads to a contradiction as above. So $Dg_{j,1}(d'_{s+1}) =$ $D_0(t_{s+1})$, as desired.

The final observation about (III) is that choices for e'_{s+1} such that $T_0 = \{D_0(e'_s), D_0(e'_{s+1})\}$ must be thrown out since ρ_2 must be a legal path.

We need to show for (IVc) that, if $g_{l,1} \circ g^{p'}(\rho_{2,k}) = \tau'\gamma_{2,k}$ and $g_{l,1} \circ g^{p'}(\rho_{1,s}) = \tau'\gamma_{1,s}$ where $\{D_0(\gamma_{1,s}), D_0(\gamma_{2,k})\}$ is a legal turn in G_l, then there cannot be an iPNP with $\overline{\rho_{2,k}}\rho_{1,s}$ as a subpath. We prove this now. Under the stated conditions, $g_{l,1} \circ g^{p'}(\rho_2) =$ $\tau'\gamma_2$ and $g_{l,1} \circ g^{p'}(\rho_1) = \tau'\gamma_1$ where $\gamma_{2,k}$ is an initial subpath of γ_2 (both of which are legal) and $\gamma_{1,s}$ is an initial subpath of γ_1 (both of which are legal). Since $\{D_0(\gamma_1), D_0(\gamma_2)\} =$ $\{D_0(\gamma_{1,s}), D_0(\gamma_{2,k})\}$ is a legal turn, $(g_{l,1} \circ g^{p'})_{\#}(\rho) = (g_{l,1} \circ g^{p'})_{\#}(\overline{\rho_1}\rho_2) = \bar{\gamma}_1\gamma_2$, which is a legal path. Let p be such that $g^p_{\#}(\rho)) = \rho$. (Without loss of generality we can assume that $p > p'$ by replacing p by a multiple of p if necessary). Then, $g^p_{\#}(\rho) =$ $((g^{p-p'-1} \circ g_{n,l+1}) \circ (g_{l,1} \circ g^{p'}))_{\#}(\rho) = (g^{p-p'-1} \circ g_{n,l+1})_{\#}((g_{l,1} \circ g^{p'})_{\#}(\rho)) = (g^{p-p'-1} \circ$ $g_{n,l})_{\#}(\overline{\gamma_1}\gamma_2) = (g^{p-p'-1} \circ g_{n,l})(\overline{\gamma_1}\gamma_2)$, since $\overline{\gamma_1}\gamma_2$ is a legal path. This makes $g^p_{\#}(\rho)$ legal since images under admissible compositions of legal paths are legal. And this contradicts that $g^p_{\#}(\rho) = \rho$, which is not a legal path. We have now verified everything needing verification in (IV).

As in (V), suppose that $g_{l,1} \circ g^{p'}(\rho_{2,m}) = \tau' e'_1 \dots$ and $g_{l,1} \circ g^{p'}(\rho_{1,n}) = \tau' e_1 \dots$ for some legal path τ' (for the appropriate m and n). (Va) is true by definition, (Vb) just refers us to a later step, the verification of (Vc) is left to the reader, so we focus on (Vd) for now. The first thing that we need to prove for (Vd) is that there is only one circumstance where we can possibly have an iPNP with $\overline{\rho_{1,n}}\rho_{2,m}$ as a subpath. Suppose that, for no power p' do we ever have $g^{p'}_{\#}(\overline{\rho_{1,n}}\rho_{2,m}) = \overline{\rho_{1,n}}\rho_{2,m}$, $g^{p'}_{\#}(\overline{\rho_{1,n}}\rho_{2,m}) \subset \overline{\rho_{1,n}}\rho_{2,m}$, $\overline{\rho_{1,n}}\rho_{2,m} \subset g^{p'}_{\#}(\overline{\rho_{1,n}}\rho_{2,m})$, or $g^{p'}_{\#}(\overline{\rho_{1,n}}\rho_{2,m}) = \overline{\gamma_{1,n}}\gamma_{2,m}$ where either $\gamma_{1,n} \subset \rho_{1,n}$ and $\rho_{2,m} \subset \gamma_{2,m}$ or $\gamma_{2,m} \subset \rho_{2,m}$ and $\rho_{1,n} \subset \gamma_{1,n}$. Now, for the sake of contradiction, suppose that some $\overline{\rho_{1,n+k}}\rho_{2,m+l}$ containing $\overline{\rho_{1,n}}\rho_{2,m}$ is an iPNP of period p. Since $\overline{\rho_{1,n+k}}\rho_{2,m+l}$ is an iPNP, $\rho_{1,n+k}$ and $\rho_{2,m+l}$ are both legal paths (as are the subpath $\rho_{1,n}$ and $\rho_{2,m}$). This tells us that $g^{p}_{\#}(\overline{\rho_{1,n+k}}\rho_{2,m+l}) = g^{p}(\overline{e'_{n+k}} \dots \overline{e_{n+1}}) g^{p'}_{\#}(\overline{\rho_{1,n}}\rho_{2,m}) g^{p}(e_{m+1} \dots e_{m+l})$. So $g^{p'}_{\#}(\overline{\rho_{1,n}}\rho_{2,m}) \subset g^{p}_{\#}(\overline{\rho_{1,n+k}}\rho_{2,m+l}) = \overline{\rho_{1,n+k}}\rho_{2,m+l}$. But this lands us in one of the situations we said could not occur, which is a contradiction.

There is nothing really to prove in (VI) since the conditions for (V) still hold.

QED.

Chapter 12

Unachievable Type (*) pIWGs in Rank 3 graph

12.1 Proving Unachievability Using the Birecurrency Condition: Unachievability of Graph II (Four Edges Sharing a Vertex)

The first 5-vertex Type (*) pIW graph $\mathcal{G}$ that we show to be unachievable, is that consisting of four edges adjoined at a single vertex (Graph II). For this graph we use the birecurrency condition given in Proposition 5.4. Chapter 6 tells us that what we need is that every Type (*) Admissible LTT Structure for $\mathcal{G}$ is not birecurrent. Up to EPP-isomorphism, there are two such LTT structures to consider neither of which is birecurrent):

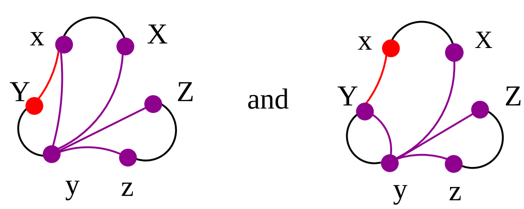


Figure 12.1: *Potential LTT structures for four edges adjoined at a single vertex*

These are the only structures worth considering as follows:

Either three of the valence-1 vertices of $\mathcal{G}$ belong to different edge pairs or the valence-1 vertices are labeled with two sets of edge pairs. First consider the case where the valence-1 vertices are labeled with two sets of edge pairs. The red edge cannot be attached in such a way that it is labeled with an edge pair and all the other resulting LTT structures are the same as the first structure up to EPP-isomorphism. Now consider the case where three of the valence-1 vertices of $\mathcal{G}$ belong to different edge pairs. One of these three have the label of the inverse of the valence-4 vertex. The red edge can only be attached at one vertex choice and without causing an edge pair labeled vertex set connected by a valence-1 edge in the colored subgraph of the LTT structure. Up to EPP-isomorphism, this just leaves us with the second structure.

12.2 Proving Unachievability by Showing $AM(\mathcal{G})$ Lacks Irreducibility Potential: Unachievability of Graph V and Graph VII

If the Type (*) pIW graph $\mathcal{G}$ were unachievable, the ideally decomposed train track, existing by Proposition 3.3, would be represented by a loop in $AM(\mathcal{G})$. By proving that such a loop does not exist, we are able to prove the unachievability of Graph V and Graph VII. To prove the unachievability of Graph V and Graph VII, we are able to show that, for each of them, no loop in their AM Diagrams could represent an irreducible element (see Corollary 9.5). We use the following:

Definition 12.1. Irreducibility Potential Test: Check whether, in each connected component of $AM(\mathcal{G})$, for each edge vertex pair $\{d_i, \overline{d_i}\}$, there is a node N in the component such that either d_i or $\overline{d_i}$ labels the red vertex in the structure N. For irreducibility, there must be at least one such component and we only need to check for $L(g_1, \dots, g_k; G_0, G_1 \dots, G_{k-1}, G_k)$ in such components. If it holds for no component, then $\mathcal{G}$ is unachievable.

12.2.1 Unachievability of Graph VII

We showed in Section 9.2 that the following is $AM(\mathcal{G})$ where $\mathcal{G}$ Graph VII:

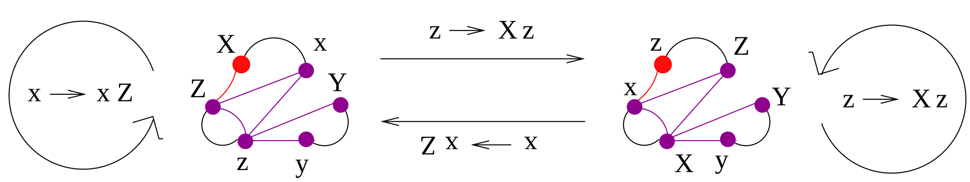


Figure 12.2: *AM Diagram for Graph VII*

Since $AM(\mathcal{G})$ contains only red vertices labeled z and $\bar{x}$ (leaving out the edge vertex pair $\{y, \overline{y}\}$) unless some other component contains all 3 edge vertex pairs ($\{x, \overline{x}\}$, $\{y, \overline{y}\}$, and $\{z, \overline{z}\}$), Graph VII would be unachievable. Since no other component does contain all 3 edge vertex pairs (all components are EPP-isomorphic), Graph VII is indeed unachievable.

12.2.2 Unachievability of Graph V

As we did for Graph VII, we show that the AM Diagram lacks irreducibility potential. We draw the AM diagram here without labels on the edges because it is clear even from this much that no map represented by a loop in this diagram would be irreducible, as the only edge pairs labeling red vertices are $\{x, \overline{x}\}$ and $\{z, \overline{z}\}$:

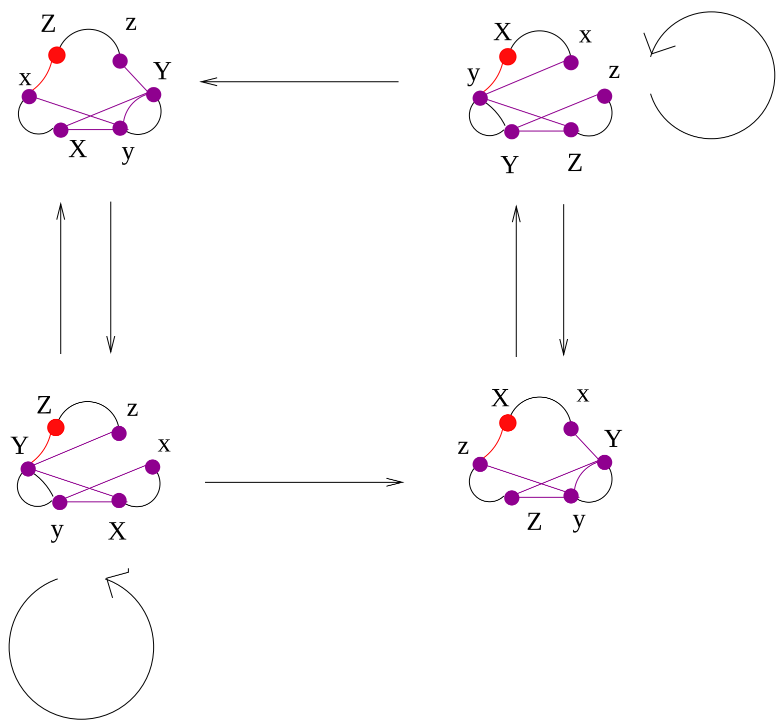


Figure 12.3: *A component of the AM Diagram (all others are the same up to EPP-isomorphism)*

Chapter 13

Representative Construction Strategies

In this chapter we give three different strategies for constructing train track representatives that have the potential to be of Type (*) with a Type (*) pIW ideal Whitehead graph $\mathcal{G}$. Different strategies work better in different circumstances. For example, if most of the LTT structures G with $PI(G) = \mathcal{G}$ are birecurrent, then Strategies II and III are better suited (as $AM(\mathcal{G})$ may be very large and impractical to construct). On the other hand, if only a few of the LTT structures G with $PI(G) = \mathcal{G}$ are birecurrent, then constructing $AM(\mathcal{G})$ is much simpler than using "guess and check" strategies and so the Category I strategies generally prove more practical.

In all figures of this chapter we continue with the convention that Y denotes $\bar{y}$, etc.

13.1 Preliminary Definitions and Tools (Tracking Progress)

The different strategies that we describe in this chapter will frequently require that we track our progress in ensuring that all edges of $\mathcal{G}$ are actually in the ideal Whitehead graph for a given representative $g_{\mathcal{G}}$. In this section we give methods and terminology we use for this purpose.

We establish first the notion of a "preimage subgraph."

Definition 13.1. For an admissible map $(g_{(k,m)}; G_{m-1}, \dots, G_k)$, the *preimage subgraph* under $(g_{(k,m)}; G_{m-1}, \dots, G_k)$ for a subgraph $H \subset PI(G_i)$ will be denoted $H^{-g_{k,m}}$ and is obtained from H by replacing each edge of H with its preimage under the isomorphism from $PI(G_{m-1})$ to $PI(G_k)$.

Example 13.2. Consider a subgraph H of an LTT structure G_i:

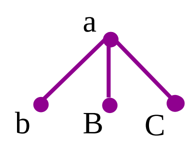


Figure 13.1: *The subgraph H of an LTT structure G_i that we will take the preimage of*

We find the preimage subgraph H^{-g_i} under the direction map $Dg_i : \bar{a} \mapsto \bar{b}$ for $g_i : \bar{a} \mapsto \bar{b}\bar{a}$:

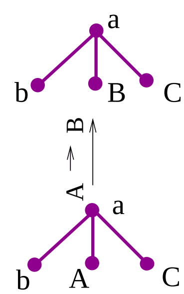


Figure 13.2: *On the bottom is the preimage subgraph H^{-g_i} for the graph H in Figure*

Remark 13.3. Recall that, for an extension, (g_i, G_{i-1}, G_i), there exists an isomorphism from $PI(G_{i-1})$ to $PI(G_i)$ fixing the second index of the labels of each vertex of $PI(G_{i-1})$ (it sends the vertex labeled $d_{i-1,j}$ in $PI(G_{i-1})$ to the vertex labeled $d_{i,j}$ in $PI(G_i)$ for all $d_{i-1,j}$). The isomorphism extends naturally from vertices to edges. Thus, for each edge $[d_{(i,j)}, d_{(i,j')}]$ in H, there is an edge $[d_{(i-1,j)}, d_{(i-1,j')}]$ in H^{-g_i}. And, for a purified extension $(g_{(k,m)}; G_{m-1}, \ldots, G_k)$, $H^{-g_{k,m}}$ is obtained is obtained from H by changing the first indices of all vertex labels from k to $m-1$. If instead (g_i, G_{i-1}, G_i) is a switch, the isomorphism from $PI(G_{i-1})$ to $PI(G_i)$ sends the vertex labeled d^{pu}_{i-1} to the vertex labeled d^a_i and fixes the second index of the labels of all other vertices of $PI(G_{i-1})$ (it sends the vertex labeled $d_{i-1,j}$ in $PI(G_{i-1})$ to the vertex labeled $d_{i,j}$ in $PI(G_i)$ for all $d_{i-1,j} \neq d^{pu}_{i-1}$). Again the isomorphism extends naturally from vertices to edges and so, for each edge $[d_{(i,j)}, d^a_i]$ in H there is an edge $[d_{(i-1,j)}, d^{pu}_i]$ in H^{-g_i} and for each edge $[d_{(i,j)}, d_{(i,j')}]$ in H, where $d^a_i \neq d_{i,j}$ and $d^a_i \neq d_{(i,j')}$, there is an edge $[d_{(i-1,j)}, d_{(i-1,j')}]$ in H^{-g_i}. Consequently, if $(g_{(k,m)}; G_{m-1}, \ldots, G_k)$ is a construction composition, for each edge $[d_{(m,j)}, d^a_m]$ in H, there is an edge $[d_{(m-1,j)}, d^{pu}_m]$ in $H^{-g_{k,m}}$ and for each edge

$[d_{(k,j)}, d_{(k,j')}]$ in H, where $d^a_k \neq d_{k,j}$ and $d^a_k \neq d_{(k,j')}$, there is an edge $[d_{(m-1,j)}, d_{(m-1,j')}]$ in $H^{-g_{k,m}}$.

We define here further notation that will also be used to track our progress in ensuring that all edges of $\mathcal{G}$ are actually in the ideal Whitehead graph for $g_{\mathcal{G}}$. What we define is a graph G^a_k that lets us know what edges have been "constructed" thus far.

Definition 13.4. Let $g_{\mathcal{G}} = s_n \circ h^p_n \circ s_{n-1} \circ h^p_{n-1} \circ \cdots \circ s_1 \circ h^p_1$ where each h^p_k is a purified construction composition with destination LTT structure G_{i_k} and each s_k is a switch. We define G^a_1 as the subgraph of G_{i_1} consisting of precisely the purple edges in the construction path for $h = s_1 \circ h^p_1$. Let $P(\gamma_{h_k})$ denoted the set of purple edges in the construction path γ_{h_k} for $h_k = s_k \circ h^p_k$. Then $G^a_1 = P(\gamma_{h_1})$ and we inductively define G^a_k as the subgraph $P(\gamma_{h_k}) \cup (G^a_{k-1})^{-s_{k-1}}$ of G_{i_k}.

In addition to tracing subgraphs, one can check that the entire graph is built by taking images of the red edges created by g_i, as in the following example:

Example 13.5. We show here an example of how to check that the entire Type (*) pIW graph is "built" (we iteratively take the image under each Dg_k of the edges "created" thus far):

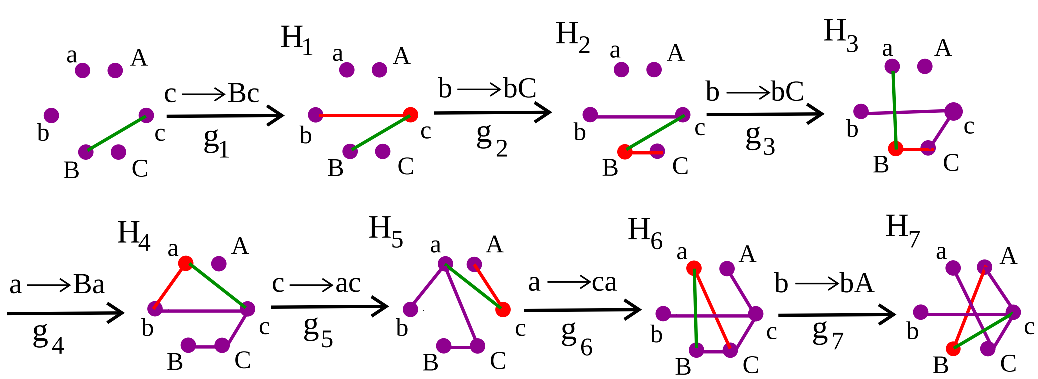


Figure 13.3: *"Graph Building"*

We include subgraphs H_i of the LTT structures G_i to track how edges are "built." g_1 is defined by $c \mapsto \bar{b}c$. Thus, the red edge e^R_1 in the destination LTT structure G_1 for g_1 will be $[c, b]$, where the red periodic vertex is labeled c. g_2 is defined by $b \mapsto b\bar{c}$.

Thus, the red edge e_2^R in the destination LTT structure G_2 for g_2 will be $[\bar{b}, \bar{c}]$, where the red direction vertex is labeled $\bar{c}$. This LTT structure will also contain the image $[c, b]$ of the red edge $[c, b]$ under $Dg_2 : \bar{b} \mapsto c$. g_3 is defined again by $b \mapsto b\bar{c}$. Thus, the red edge e_3^R in the destination LTT structure G_3 for g_3 will be again $[\bar{b}, \bar{c}]$, where the red periodic vertex is labeled $\bar{c}$. This LTT structure will also contain the image $[c, \bar{c}]$ of the red edge $[\bar{b}, \bar{c}]$ and the image $[c, b]$ of the purple edge $[c, b]$ under $Dg_3 : \bar{b} \mapsto c$. g_4 is defined by $a \mapsto \bar{b}a$. Thus, the red edge e_4^R in the destination LTT structure G_4 for g_4 will be $[a, b]$, where the red periodic vertex is labeled a. This LTT structure will also contain the image $[\bar{b}, \bar{c}]$ of the red edge $[\bar{b}, \bar{c}]$ and the images $[c, b]$ and $[c, \bar{c}]$ of the purple edges $[c, b]$ and $[c, \bar{c}]$ under $Dg_4 : a \mapsto \bar{b}$. The remaining H_i are constructed similarly.

13.2 Loop-Finding Methods

In Chapter 9 we showed a method for constructing $AM(\mathcal{G})$. Proposition 3.3 tells us that, if a Type (*) pIWG $\mathcal{G}$ is unachievable, then it has an ideally decomposed train track representative. Proposition 10.11 tells us that this representative has an associated representative loop in $AM(\mathcal{G})$. This section provides guidance on finding these loops. However, before even attempting to find a loop, it is advisable to check the irreducibility potential of $AM(\mathcal{G})$ (see the Irreducibility Potential Test of Section 12.2). And then, once one finds a loop, they still must test the representative constructed from the loop to ensure that it is PNP-free (see the procedure of Chapter 11 for identifying PNPs), that the transition matrix is Perron-Frobenius, and that $IW(g) \cong \mathcal{G}$. These issues are addressed in Section 13.3 and, because these tests are not included before Section 13.3, we will call the loops we find in this section “Test Loops”.

13.2.1 Category I Strategies: Finding “Test” Loops when the Entire AM Diagram is Known

Let $\mathcal{G}$ be a Type (*) pIWG. There are multiple techniques for finding the desired representative loop $L(g_1, \dots, g_k; G_0, G_1 \dots, G_{k-1}, G_k)$ when $AM(\mathcal{G})$ is known. We describe here two such strategies (Strategy Ia and Ib).

(Ia) In this strategy we use potential composition paths to build subgraphs of $\mathcal{G}$ (following progress using preimage subgraphs). We show in this subsection only how to find the paths. One can reference Strategy III for how to piece them together. To find the paths, we identify the subdiagrams of $AM(\mathcal{G})$ where paths for construction compositions would have to live (the "Extension Subdiagram"). We then find a subgraph (the "Potential Composition Subgraph") of the LTT structures in a component of the subdiagram their construction paths would live in that would actually contain the construction paths.

(1) Each directed edge in $AM(\mathcal{G})$ corresponds to either a switch or an extension. The *extension subdiagram* $(AM(\mathcal{G}))_e$ of $AM(\mathcal{G})$ consists precisely of the directed edges (including their nodes) for extensions.

Example 13.6. *Extension Subdiagram,* $(AM(\mathcal{G}))_e$ *where* $\mathcal{G}$ *is Graph III:*

The following is a component of $AM(\mathcal{G})$, where $\mathcal{G}$ is Graph III. There are two components to the extension subdiagram living in this component of $AM(\mathcal{G})$.

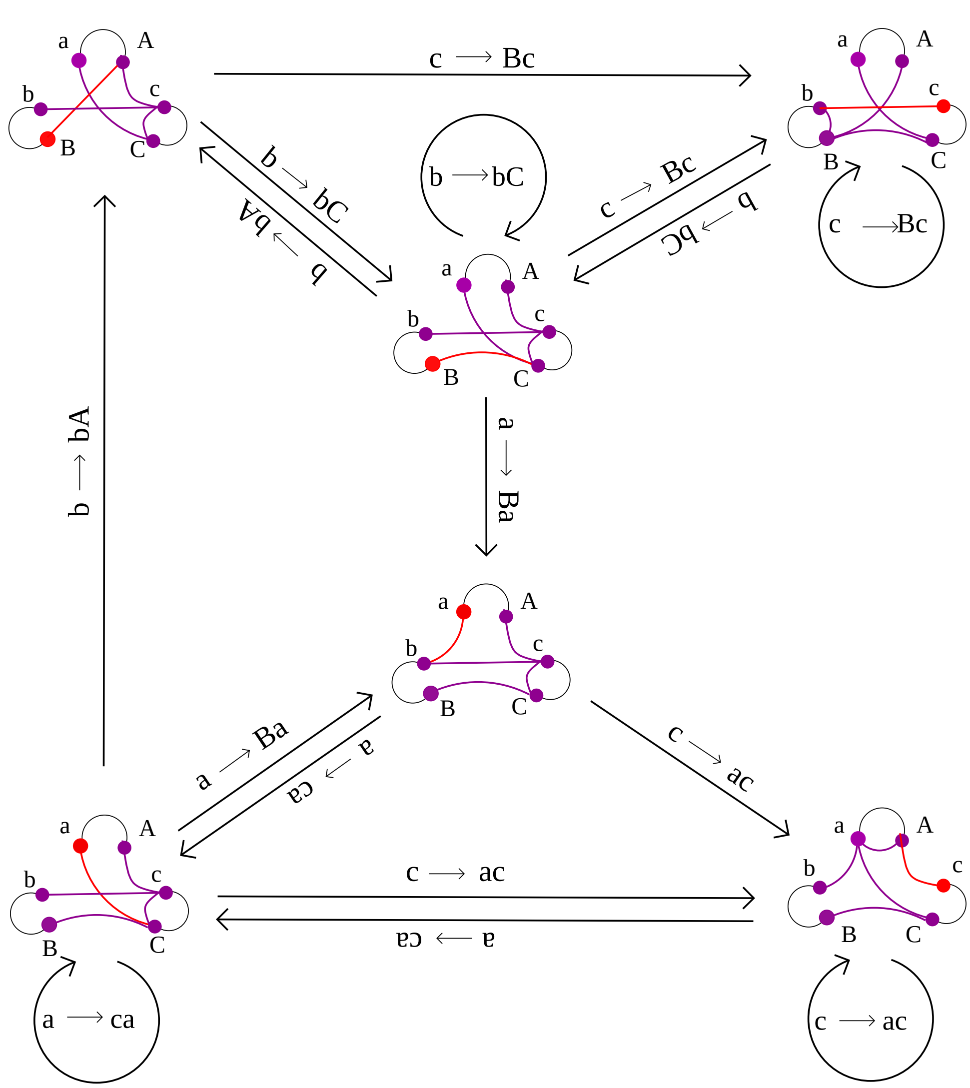


Figure 13.4: *A component of $AM(\mathcal{G})$ where $\mathcal{G}$ is Graph III*

The components of $(AM(\mathcal{G}))_e$ living in the component of $AM(\mathcal{G})$ given in Figure 13.4 above (notice how the LTT structures in each component of $AM(\mathcal{G})$ have the same purple subgraph):

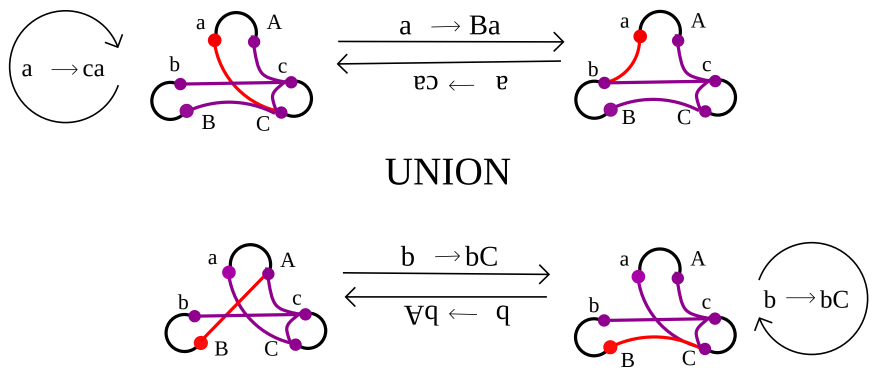


Figure 13.5: *The components of $(AM(\mathcal{G}))_e$ living in the component of $AM(\mathcal{G})$ given in Figure 13.4*

(2) Find the potential composition PI subgraph for each component of $(AM(\mathcal{G}))_e$: All LTT structures (extension source and destination LTT structures) labeling nodes in a connected component of $(AM(\mathcal{G}))_e$ share a purple (edge-pair)-indexed subgraph, the *potential composition PI subgraph* for the component.

Example 13.7. *Potential Composition PI Subgraphs for the components of $(AM(\mathcal{G}))_e$ given in Figure 13.5.* (The left-hand graph is for the top component and the right-hand graph is for the bottom component.)

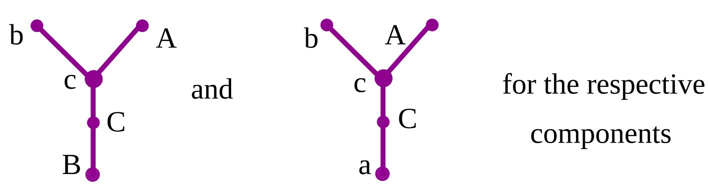


Figure 13.6: *Two Graph III Potential Composition PI Subgraphs*

(3) Find the *Potential Composition Subgraph* for an $(AM(\mathcal{G}))_e$ Connected Component for C:
Add black edges connecting edge-pair vertices in the potential composition PI subgraph, then recursively remove valence-1 edges (leaving the larger valence vertex each time a valence-1 edge is removed).

Example 13.8. The composition subgraph for the graph on the left in Figure 13.6 is obtained by first adding the black edges $[a, \bar{a}]$, $[b, \bar{b}]$, and $[c, \bar{c}]$ to obtain the graph on the left in Figure 13.7 below and then removing both $[a, \bar{a})$, $(\bar{a}, c]$ to obtain the graph on the right now containing any valence-1 vertices.

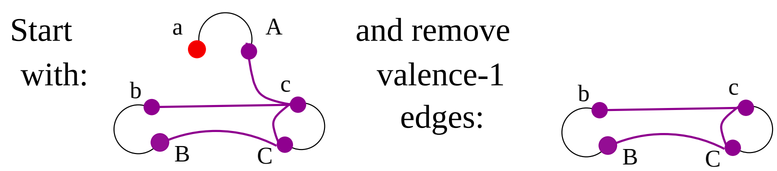


Figure 13.7: *The Potential Composition Subgraph for the graph on the left in Figure 13.6*

(4) Find a potential composition paths in the potential composition subgraph for C:

Find a directed smooth path $[d_i^a, d_{i,j_1}, \overline{d_{i,j_1}}, \dots, d_{i,j_n}, \overline{d_{i,j_n}}]$ in a potential composition subgraph (where the potential composition subgraph is viewed as a subgraph of some LTT structure G_i in $(AM(\mathcal{G}))_e$).

Example 13.9. A potential composition path in the potential composition subgraph of Figure 13.7.

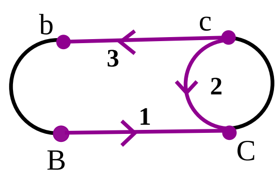


Figure 13.8: *The numbered colored edges, combined with the black edges between give a Graph III potential composition path. (Note: This path is not used to compute the representative below.)*

(5) Check that the construction composition for the potential composition path of (4) (see Lemma 8.8) is actually realized in $AM(\mathcal{G})$. For example, it may be that the destination LTT structure for one of the extensions in the decomposition of the construction composition was not birecurrent (so the extension was not admissible) or even just that the directed edge in $preAM(\mathcal{G})$ labeled

by one of the extensions was not in a maximal strongly connected component of $preAM(\mathcal{G})$.

If the construction composition is realized by a path in $AM(\mathcal{G})$, the path may give the final segment in the loop realizing a representative. Including this path in $AM(\mathcal{G})$ as the final segment of a loop in $AM(\mathcal{G})$ will guarantee that the purple edges of its construction path are in the ideal Whitehead graph (see Proposition 8.13).

(6) One way to continue with this strategy is:

(a) Choose a node V_i in C such that $\overline{d_i^a}$ is the attaching red vertex in the LTT structure labeling V_i.

(b) The final segment of your loop in $AM(\mathcal{G})$ will be the path in $AM(\mathcal{G})$ realizing the construction composition for the potential composition path of (4).

(c) Determine what edges of the ideal Whitehead graph are still missing (not contained in the construction path).

(d) Trace those edges via their preimages to another component of $(AM(\mathcal{G}))_e$ where the PI subgraph contains (at least some) of the preimage edges (see Strategy III).

(e) Find a path in the potential composition subgraph containing those preimage edges.

(f) Continue as such until the entire ideal Whitehead graph is built.

(f) Finish off the loop with a path returing to V_i.

(Ib) "Guess and Add" with PreTest Loops:

In this strategy we find a "pretest" loop in $AM(\mathcal{G})$ such that, for each vertex edge pair $\{d_i, \overline{d_i}\}$, either d_i nor $\overline{d_i}$ is the red vertex for some LTT structure labeling a node in the loop. Add small loops until the entire graph is built (see Example 13.5 for how to check this).

Example 13.10. Finding $L(g_1, \dots, g_k; G_0, \dots, G_k)$, where $\mathcal{G}$ is Graph I:

The following is a component of $AM(\mathcal{G})$ (we left out the LTT structure black edges for simplicity's sake):

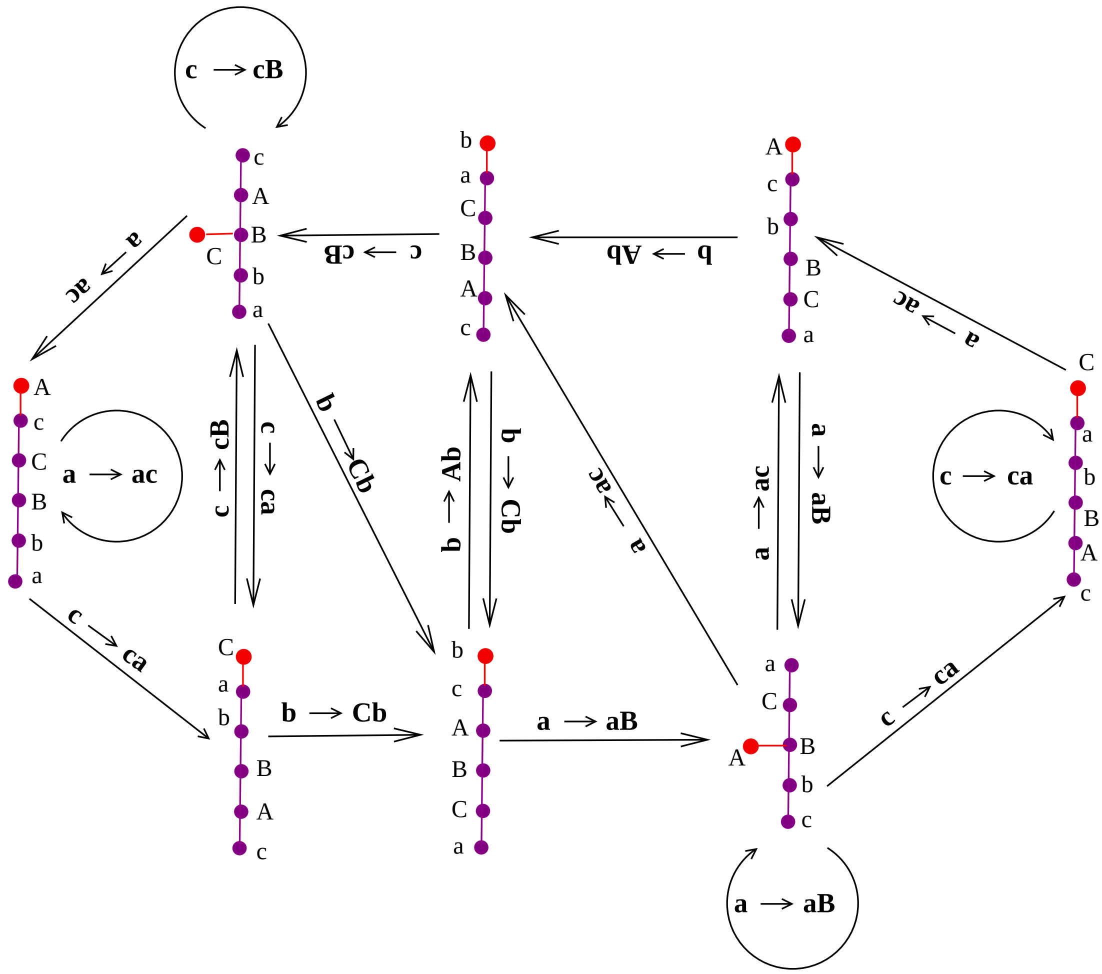


Figure 13.9: *A component of $AM(\mathcal{G})$, where $\mathcal{G}$ is Graph I (black edges in the LTT structures are left out for simplicity's sake).*

We find a pretest loop in $AM(\mathcal{G})$ (the blue directed edges together give the loop):

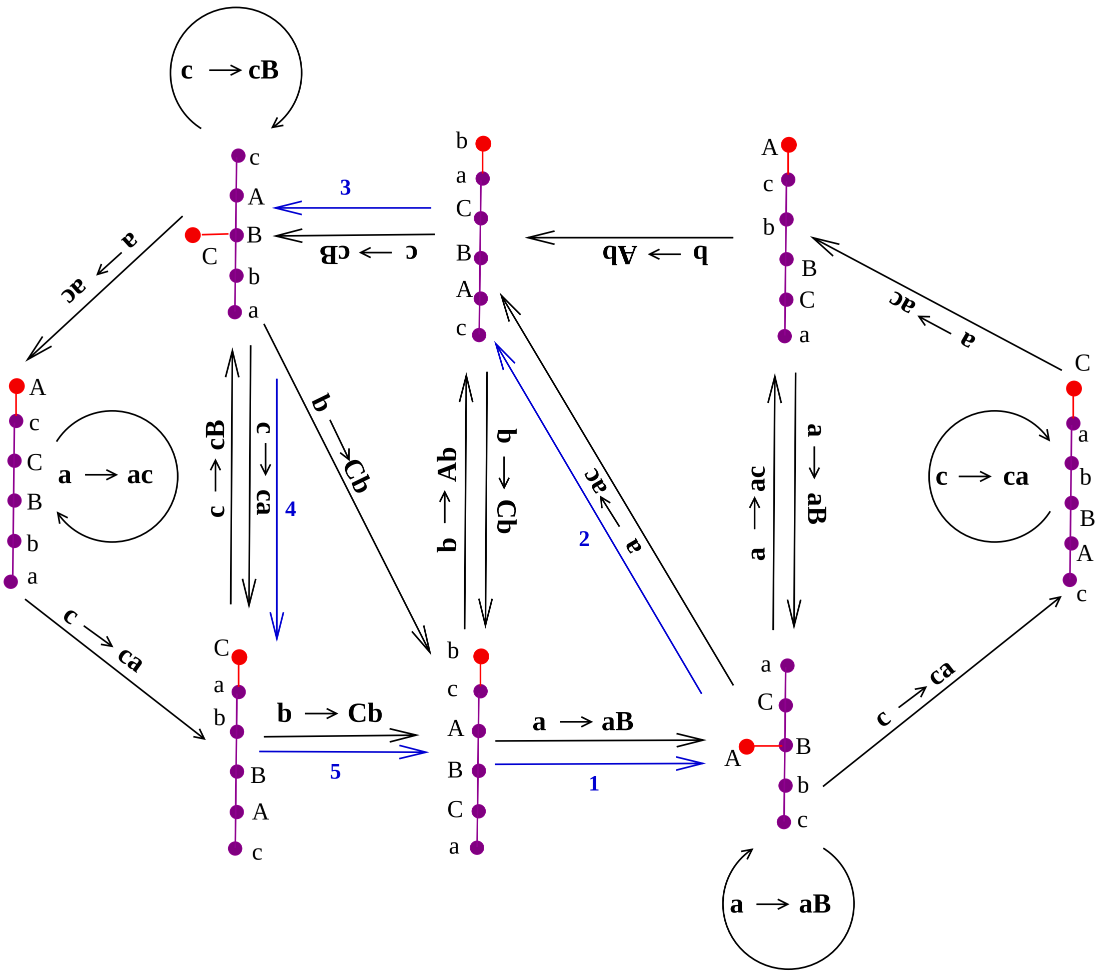


Figure 13.10: *The blue directed edges together give the (preTest) loop in $AM(\mathcal{G})$ that we will test for ideal Whitehead graph "building" and then possibly add small loops to.*

The preTest loop realizes:

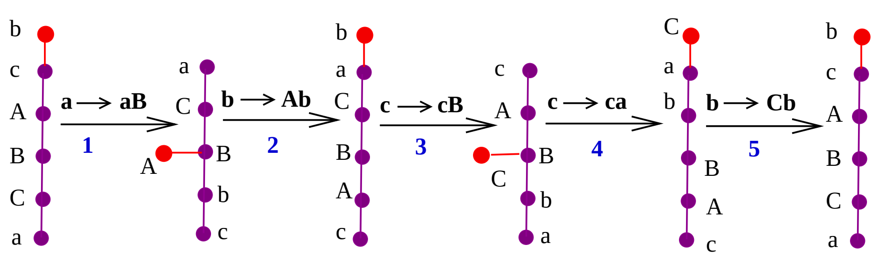


Figure 13.11: *Corresponding map for preTest loop in $AM(\mathcal{G})$*

By the procedure illustrated in Example 13.5 we see that we do not get all of $\mathcal{G}$ (we do not get the edge $[\bar{b}, \bar{c}]$), so we must add a second loop to the pretest loop of Figure 13.10:

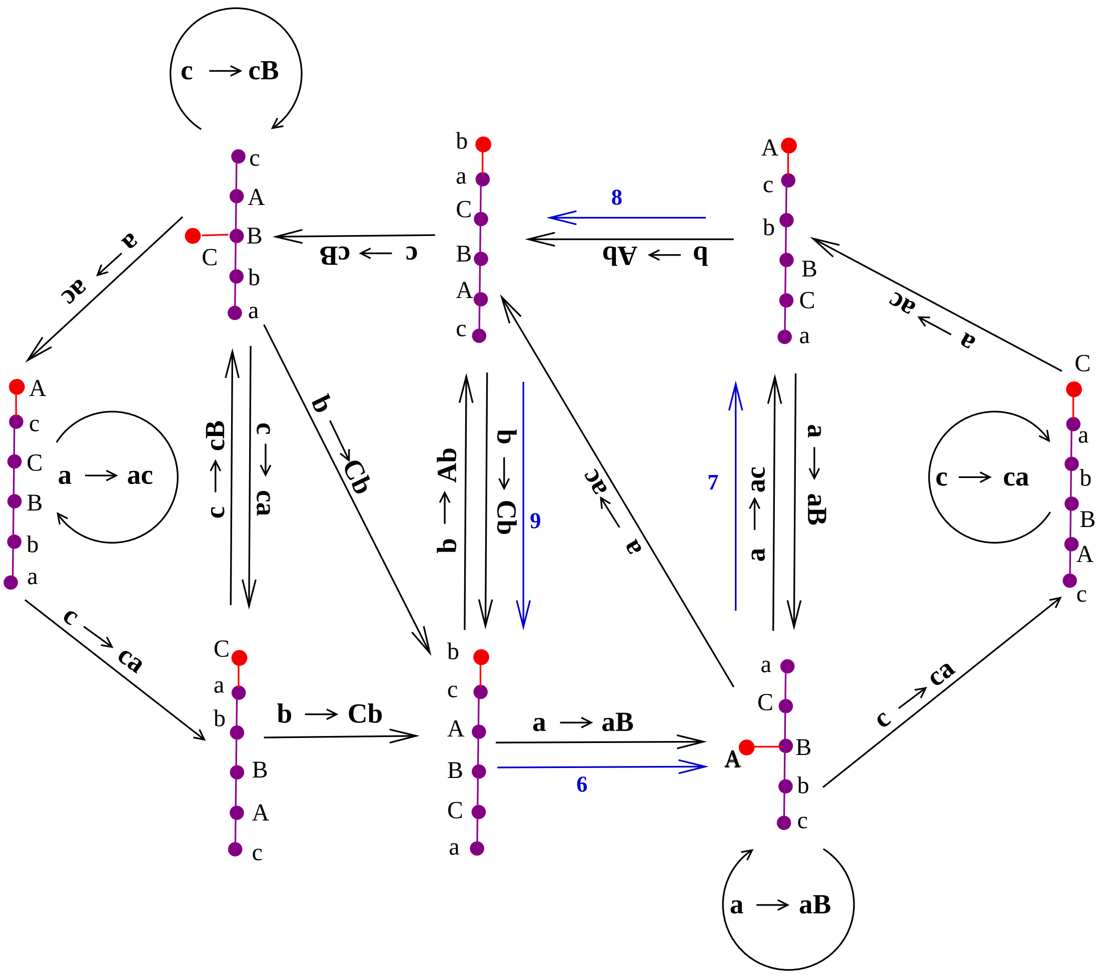


Figure 13.12: *The blue edges give a small loop to add to the pretest loop of Figure 13.10 in an attempt to fill in the missing edges of $\mathcal{G}$.*

The loop gives:

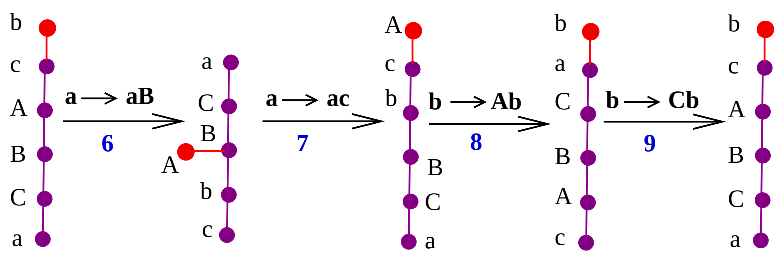


Figure 13.13: *What the second loop gives*

Combining the two loops we get the representative yielding Graph I (the line):

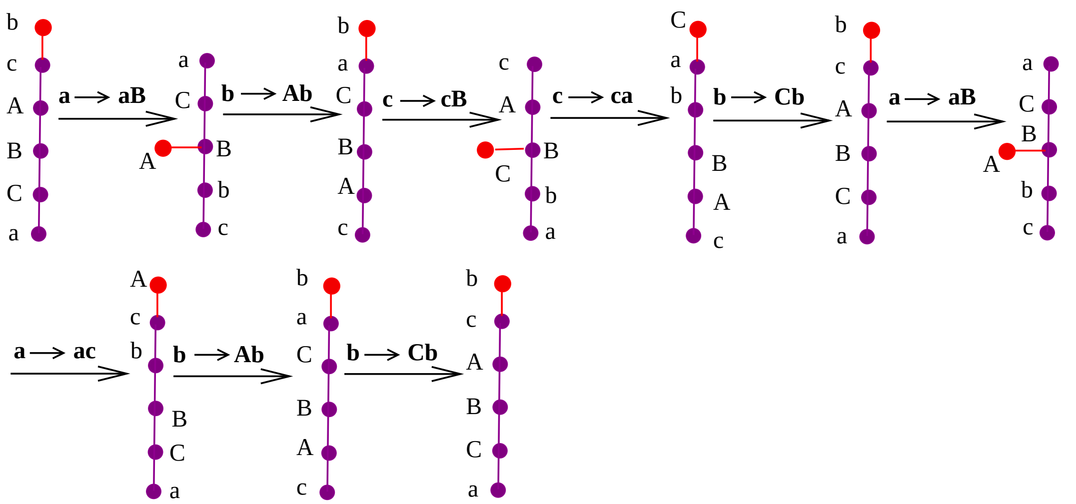


Figure 13.14: *Ideal Decomposition for the representative yielding Graph I (the line)*

13.2.2 Strategy II: Piecing Together Construction Compositions

Again let $\mathcal{G}$ be a Type (*) pIWG and let G be a Type (*) admissible LTT structure with $PI(G) = \mathcal{G}$ and the standard Type (*) admissible LTT structure notation. Strategy II will be similar to Strategy Ia except that here we do not have $AM(\mathcal{G})$ to work with. Instead we alternate between using construction paths to find construction compositions and using "guess and check" to find admissible switches and extensions that lead us to the next admissible LTT structure we find a construction path in.

<u>STEP 1</u>: First Building Subgraph

The first step we take in "building" a representative $g_{\mathcal{G}}$ with $IW(g_{\mathcal{G}}) = \mathcal{G}$ will be to determine the construction subgraph G_C of G, as in Definition 8.6. We will call this the "first building subgraph" for our Test loop.

Example 13.11. Recall from 8.7 the Construction Subgraph for the LTT structure in Figure 8.1:

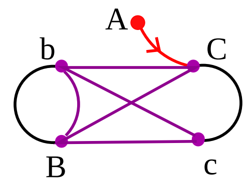


Figure 13.15: *Construction Subgraph for the LTT Structure in Figure 8.1*

<u>STEP 2</u>: Finding a potential construction path in G_C and Purified Construction Composition h_1^p

The next step is to find a potential construction path $\gamma = [d^u, \overline{d^a}, d^a, \overline{x_2}, x_2, \dots, x_{k+1}, \overline{x_{k+1}}]$ in G_C from which we will obtain a purified construction composition, h_1^p, as in Lemma 8.8, (a good choice would be one of minimal length among all potential construction paths transversing the maximum number of edges of G'_{ep}). If none can be found, construct $AM(\mathcal{G})$ and determine whether $g_{\mathcal{G}}$ exists at all.

$\Gamma_{i-k} \xrightarrow{g_{i-k+1}} \cdots \xrightarrow{g_{i-1}} \Gamma_{i-1} \xrightarrow{g_i} \Gamma_i$ will denote the decomposition of h_1^p and $G_{i-k} \xrightarrow{D^T(g_{i-k+1})} \cdots \xrightarrow{D^T(g_{i-1})} G_{i-1} \xrightarrow{D^T(g_i)} G_i$ the corresponding sequence of LTT structures.

Example 13.12. A potential composition construction path in the construction subgraph of Figure 8.3 (and 13.15) is given by the numbered colored edges and black edges between in:

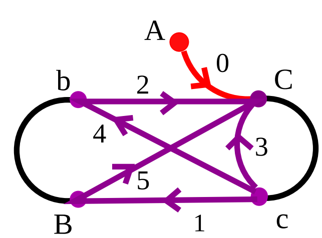


Figure 13.16: *A potential composition construction path in the construction subgraph of Figure 8.3 (and 13.15)*

The purified construction composition corresponding to the potential composition construction path in Figure 13.16 (black edges are left out):

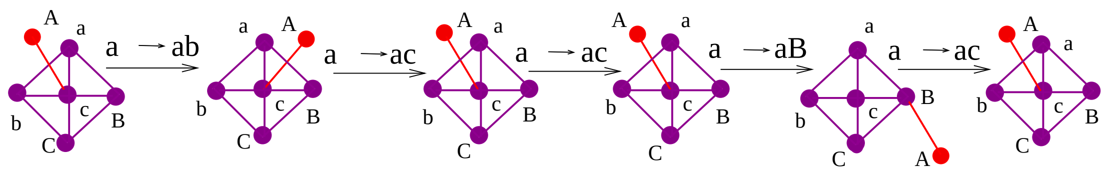


Figure 13.17: *Purified construction composition corresponding to the potential composition construction path in Figure 13.16: We left out the black edges in the LTT structures, as they are not necessary to understand what is going on.*

Note that all relevant LTT structures are Type (*) admissible LTT structures for $\mathcal{G}$ (and are birecurrent, in particular).

<u>STEP 3</u>: Determine the purple edges of $\mathcal{G}$ missed by the construction path in Step 2.

One just looks at the purple subgraph of G and looks at what edges are not hit by the construction path. These are the remaining edges that still need to be "built" in the ideal Whitehead graph.

Example 13.13. The purple edges left after the construction path in Figure 13.16 are:

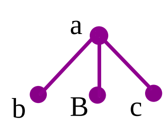


Figure 13.18: *Purple edges left after construction path in Figure 13.16*

<u>STEP 4</u>: SWITCH s_1

In this step we determine a switch $(g_{i-k}, G_{i-k-1}, G_{i-k})$ to precede h_1^p in the decomposition of $g_{\mathcal{G}}$. To determine choices that may give the switch, one has to look at the source LTT structure $G_{j_1} = G_{i-k}$ for the first generator in the purified construction composition. There is one switch for each purple edge $[d^a_{j_1}, d]$ of G_{j_1} such that $d \neq \overline{d^u_{k_1}}$ in G_{j_1}. Disregard switches that are not admissible (in particular, those whose source LTT structure is not birecurrent). Choose one of the remaining switches and call it s_1. Denote the source LTT structure G_{i-k-1} by $G_{j_1'}$.

Example 13.14. The two options for the switch proceeding the pure construction composition of Example 13.12 can be identified by giving their source LTT structures (as the generator is determined to be $a \mapsto ac$ by the red edge $[\bar{a}, c]$ in G_{i-k}. The two source LTT structures are (the black edges in the LTT structures are left out, as they are easily ascertained):

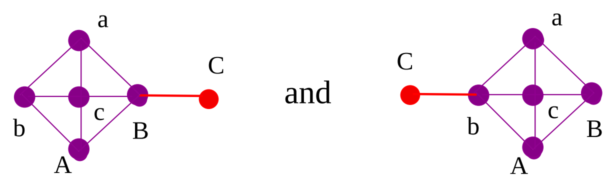


Figure 13.19: *Options for source LTT structures for switch proceeding pure construction composition of Example 13.12 (in short hand)*

Both of the LTT structures are admissible Type (*) LTT structures, so are options.

<u>STEP 5</u>: RECURSIVE CONSTRUCTION COMPOSITION BUILDING

Recursive Process of Construction Composition Building:

Steps I-IV below are repeated recursively with the following assumptions until $G_N^a = PI(G_{i_N})$ for some N. The assumptions are that $s_{n-1} \circ h_{n-1}^p \circ s_{n-2} \circ h_{n-2}^p \circ \cdots \circ s_1 \circ h_1^p$ where each h_k^p is a purified construction composition with source LTT structure G_{j_k} and destination LTT structure G_{i_k} and each s_k is a switch with source LTT structure $G_{j_k'} = G_{i_k - 1}$ and destination LTT structure G_{j_k}. (Notice that, for each $1 \leq k \leq n-1$, $G_{i_{k+1}} = G_{j_k'}$.)

I. Determine the first building subgraph $(G_{j_{n-1}})_C$ for $G_{j_{n-1}}$.

II. Find a potential construction path in $(G'_{j_{n-1}})_C$ (an "optimal strategy," similar to that in Step 2, may involve choosing the path to be of minimal length among all potential construction paths transversing the maximum number of colored edges of $(G'_{j_{n-1}})_C - G_n^a$). Call the corresponding purified construction composition h_n^p and the construction path γ_{h_n}. If no valid construction composition can be found via this method, one can try using different construction compositions in the previous steps. If this does not work, one can find $AM(\mathcal{G})$ and determine whether $g_{\mathcal{G}}$ exists at all.

$\Gamma_{j_n} = \Gamma_{(i_n-k_n)} \xrightarrow{g_{(i_n-k_n+1)}} \cdots \xrightarrow{g_{(i_n-1)}} \Gamma_{(i_n-1)} \xrightarrow{g_{i_n}} \Gamma_{i_n}$ will denote the decomposition of h_n^p and $G_{j_n} = G_{(i_n-k_n)} \xrightarrow{D^T(g_{(i_n-k_n+1)})} \cdots \xrightarrow{D^T(g_{(i_n-1)})} G_{(i_n-1)} \xrightarrow{D^T(g_{i_n})} G_{i_n}$ will denote the corresponding sequence of LTT structures.

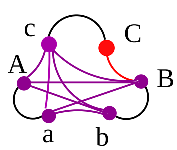
III. Determine s_n:

There is one switch for each purple edge $[d^a_{j_n}, d]$ of $G_{j_n} = G_{i_n-k_n}$. Choose an admissible switches and call it s_n. Denote the source LTT structure for s_n by $G_{j'_n}$.

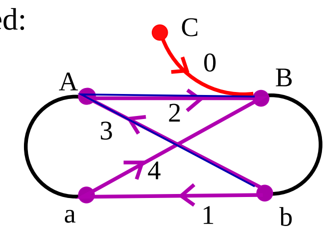
IV. Repeat (I)-(III) recursively until $G^a_N = PI(G_{j_N})$ for some N.

Example 13.15. We continue with the example for Graph XIII

We chose the source LTT structure:

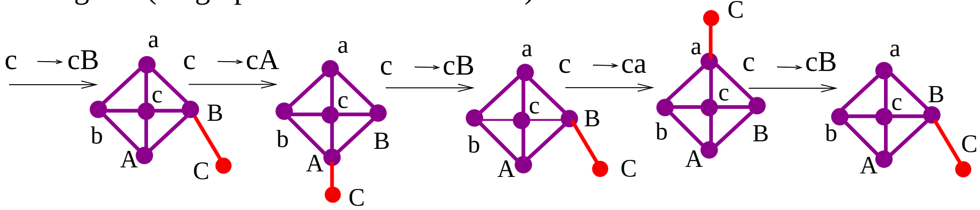


Construct path maximizing blue edges crossed:



(We are still left with a  c)

This gives (all graphs here are birecurrent):

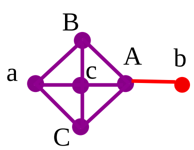


The preimage of  under the direction map for

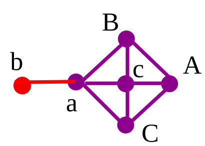
$c \longrightarrow cB \quad (C \longrightarrow b)$ is 

The choices for the source LTT structure

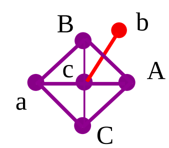
for the switch starting the composition are (in short-hand):



and



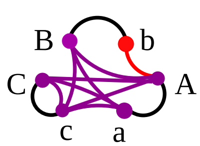
and



All are birecurrent.

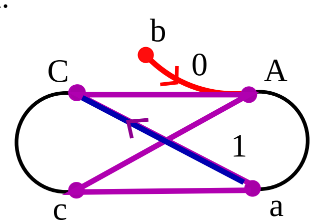
We decide to continue with the Left-hand graph.

The LTT structure is:

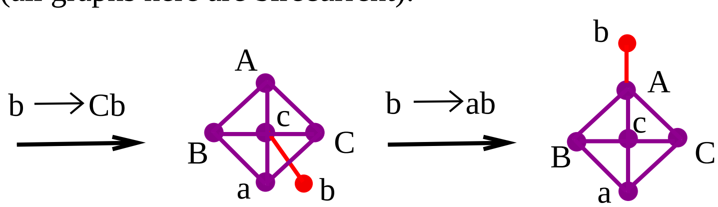


Construct path maximizing

blue edges crossed:



This gives (all graphs here are birecurrent):



<u>STEP 6</u>: CONCLUDING SWITCH SEQUENCE

Once we have $G_N^a = PI(G_{j_N})$, we find the shortest possible admissible switch sequence
$G = G_{(i_N-k_N)} \xrightarrow{g_{(i_N-k_N+1)}} \cdots \xrightarrow{g_{(i_N-1)}} G_{(i_N-1)} \xrightarrow{g_{i_N}} G_{i_N} = G_{j_N}$ with G as the source LTT structure and G_{j_N} as the destination LTT structure. A switch path in G_{j_N} may be used for this purpose, though it will be necessary to check that the corresponding switch sequence is indeed an admissible switch sequence (in particular that each G_j with $i_N - k_N \leq j \leq i_N = j_N$ is an admissible Type (*) LTT structure for $\mathcal{G}$).

If it is not possible to get a pure sequences of switches, then one can try any admissible composition with G as its source LTT structure (and G_{j_N} its destination LTT structure) or, if necessary, find a path in $AM(\mathcal{G})$ from G to G_{j_N} (see Chapter 9.2 for how to construct $AM(\mathcal{G})$). It may be possible to find the path in $AM(\mathcal{G})$ without actually building the entire diagram by instead just looking at the portion of the permitted extension/switch web constructed starting with G_{j_N} (see Chapter 9.2).

Example 13.16. We continue with the example for Graph XIII.

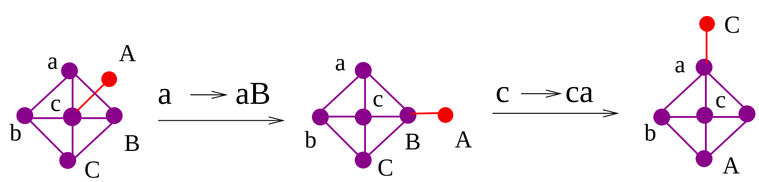


Figure 13.20: *Concluding sequence of generators for Graph XIII example*

We have the final map and get the entire representative for Graph XIII:

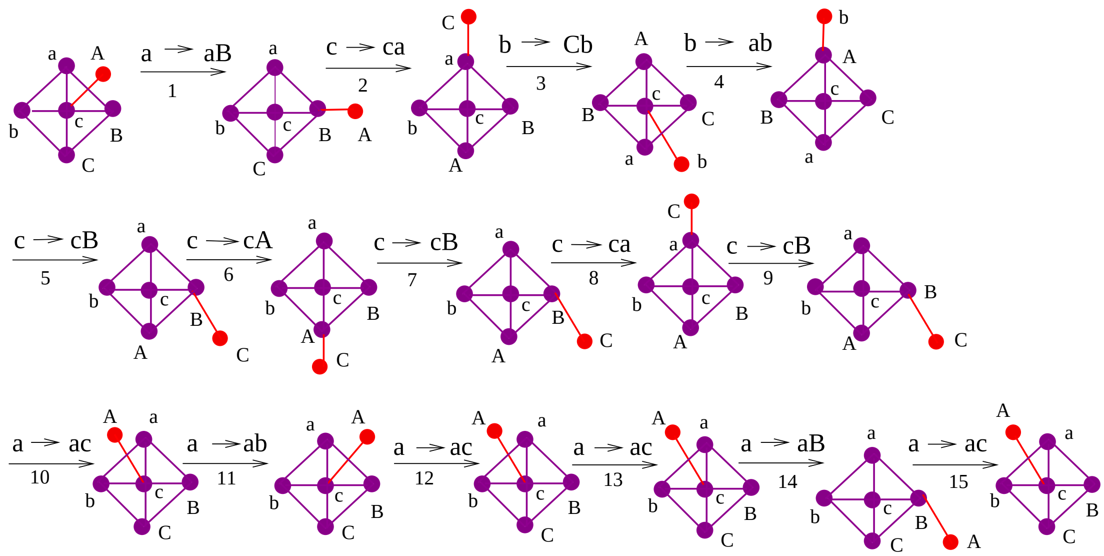


Figure 13.21: *The entire representative for Graph XIII*

We showed that this map does not have any PNPs in Section 11.

13.2.3 Strategy III: Inserting Construction Compositions from Construction Loops into Switch Sequences

(A) Find a switch sequence $(g_{(i,i-k)}, G_{i-k-1}, G_i)$ with $G_{i-k-1} = G_i$ such that, for each vertex pair $\{d_i, \overline{d_i}\}$, either d_i or $\overline{d_i}$ is the red vertex in some LTT structure in the sequence. (Such a composition would be represented by a loop in $AM(\mathcal{G})$ and can be found as a loop in $AM(\mathcal{G})$, if not by switch paths or trial and error. It would also work to use a loop in $AM(\mathcal{G})$ that does not represent a switch sequence, but the condition on vertex pairs still holds.)

(B) As in Strategy II, find a construction path in $(G_i)'_{ep}$ transversing as many edges of $(G_i)'_{ep}$ as possible, except that we now have the added condition that the corresponding purified construction composition must start and end with the same LTT structure.

(C) Proceed as in Strategy II with the added condition of (B) and with the condition that the switches between the purified construction compositions are determined by the switch sequence $(g_{(i,i-k)}, G_{i-k-1}, \ldots, G_i)$.

Example 13.17. We return to Graph XX:

A switch sequence for this graph is given in Example 8.19. Our first construction composition was given in Example 8.10. What was still needed after that composition was:

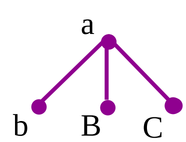


Figure 13.22: *Edges still needed after the composition given in Example 8.10 (in Figure 8.5)*

We take the preimage under the direction map for the final switch and get:

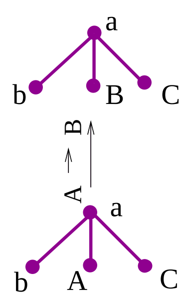


Figure 13.23: *Preimage of edges left (under the direction map for the switch in Example 8.19)*

Since we could not obtain all of these edges from a single construction composition, we take another preimage:

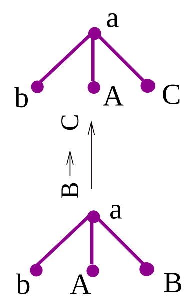


Figure 13.24: *Preimage of edges left (under the direction map of a second switch in the switch sequences of Example 8.19)*

We use the construction composition for the following construction path to obtain these edges:

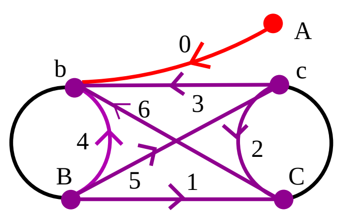


Figure 13.25: *Construction path in the Graph XX LTT structure used to obtain the remaining edges (given in Figure 13.24)*

When composed we get:

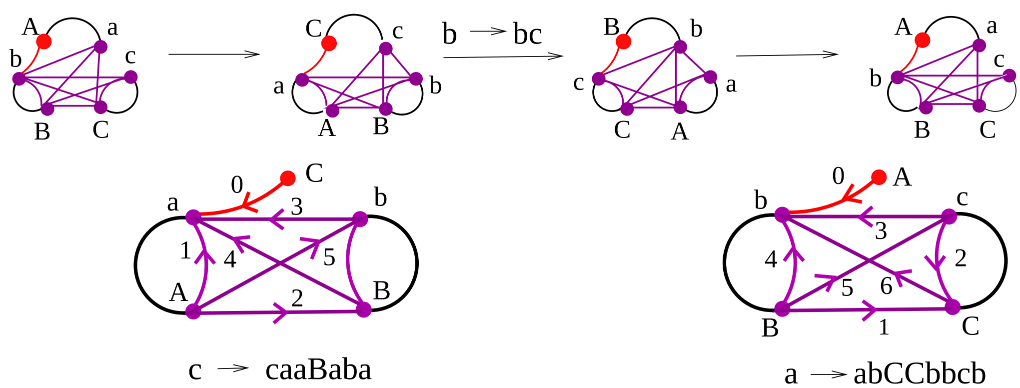


Figure 13.26: *The combination of everything we have so far in realizing Graph XX*

The automorphism we have obtained is:

$$h = \begin{cases} a \mapsto ab\bar{c}\bar{c}bbcb \\ b \mapsto bc \\ c \mapsto cab\bar{c}\bar{c}bbcbab\bar{c}\bar{c}bbcb\bar{c}\bar{c}\bar{b}ab\bar{c}\bar{c}bbcbbccab\bar{c}\bar{c}bbcb \end{cases},$$

Since the periodic directions for this map are not fixed, we compose h with itself to get $g = h^2$.

13.3 Final Checks

As mentioned before, the loops we find are only test loops and still have properties they must satisfy. The map is not acceptable if any of the following holds:

(1) For some vertex edge pair $\{d_i, \overline{d_i}\}$, neither d_i nor $\overline{d_i}$ is the red vertex in any LTT structure in the decomposition.

One can check (1) visually. If (1) fails in Strategy I, "attach" small loops to the initial loop in $AM(\mathcal{G})$, where the red vertices of the added small loops include labels from each of the pairs $\{d_i, \overline{d_i}\}$ not yet included. If (1) fails in Strategy II or III, one can try finding an alternative concluding switch sequence (or tacking on a concluding sequence) resolving the problem.

(2) There are not $2r - 1$ fixed directions.

Notice first that there would still be $2r - 1$ periodic directions, since we are dealing with admissible compositions. One can check (2) by composing generator direction maps. If (2) fails, take a power of the map fixing all periodic directions.

(3) The map constructed is PNP-free.

One can check (3) via the procedure in Section 11. (See Example 11.1 for the procedure applied to show that the map we gave for Graph XIII in Example 13.16 was PNP-free).

(4) All of $\mathcal{G}$ is "built."

One can check (4) by looking at the union of the $[Dg_{k+1,n}(t_k^R)]$ (See Example 13.5). If (4) fails in Strategy I, again one can "attach" small loops to the initial loop in $AM(\mathcal{G})$ until the entire graph is built. This can be done strategically by using the potential composition PI subgraphs (determine potential composition paths to ensure inclusion of necessary remaining edges, keeping in mind that direction maps map purple edges of the construction path into the destination LTT structure). If this fails in Strategy II or Strategy III, one can add extra construction compositions or try using an alternative route to the current final sequence of admissible maps.

Chapter 14

Achievable 5-Vertex Type (*) pIW Ideal Whitehead Graphs in Rank 3

This chapter includes the main theorem of this document. For the theorem we use our strategies to determine which 5-vertex Type (*) pIW graphs arise as $IW(\phi)$ for ageometric, fully irreducible $\phi \in Out(F_3)$. Since there are precisely twenty-one 5-vertex Type (*) pIW graphs (see Figure 1.1 for a complete list), we can handle them on a case-by-case basis. *For all figures of this chapter we continue with the convention that A notates $\bar{a}$.*

Theorem 14.1. *Precisely eighteen of the twenty-one 5-vertex Type (*) pIW graphs are ideal Whitehead graphs for ageometric, fully irreducible $\phi \in Out(F_3)$.*

Proof: The unachievable graphs (Graph II, Graph V, and Graph VII) were already handled in Chapter 12. We now give representatives for the remaining graphs, leaving it to the reader to prove that these representatives are PNP-free (using the procedure of Chapter 11), have Perron-Frobenius transition matrices, and have the appropriate ideal Whitehead graph. Proposition 10.11 then gives that they are representatives of ageometric, fully irreducible $\phi \in Out(F_r)$ with the desired ideal Whitehead graphs.

GRAPH I (The Line):

We give here the representative g whose ideal whitehead graph, $\mathcal{G}$, is GRAPH I:

$$g = \begin{cases} a \mapsto ac\bar{b}ca\bar{b}cacac\bar{b}ca \\ b \mapsto \bar{a}\bar{c}b\bar{c}\bar{a}\bar{c}\bar{a}\bar{c}b \\ c \mapsto cac\bar{b}ca\bar{b}cac \end{cases}$$

Our ideal decomposition for g is described by the following figure:

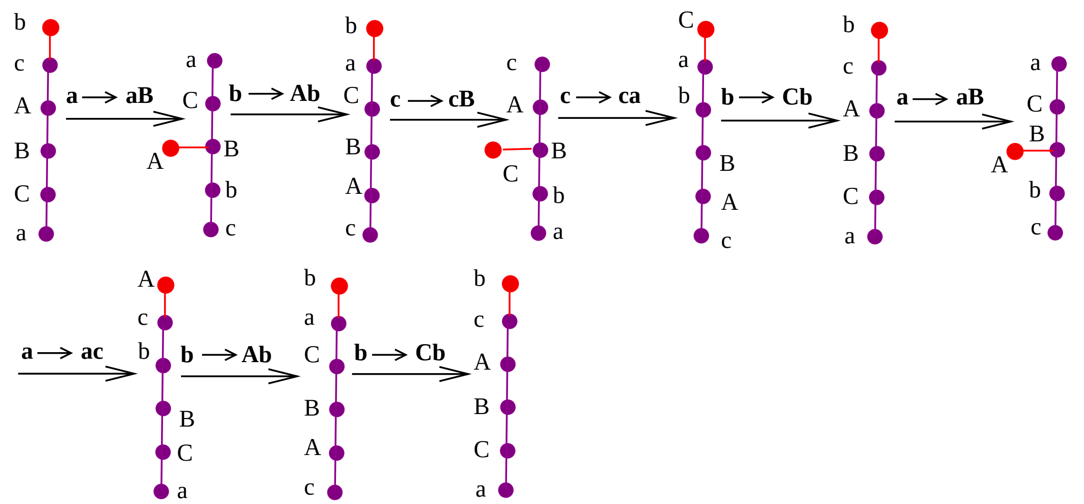


Figure 14.1: *Ideal Decomposition of Representative whose Ideal Whitehead Graph is GRAPH I*

For this we constructed a component of $AM(\mathcal{G})$ and used Strategy I. This strategy made the most sense here as there were only a few birecurrent LTT structures to be included in $AM(\mathcal{G})$.

GRAPH III:

The representative g whose ideal Whitehead graph, $\mathcal{G}$, is GRAPH III was again constructed using Strategy I. We started with a path in $AM(\mathcal{G})$.

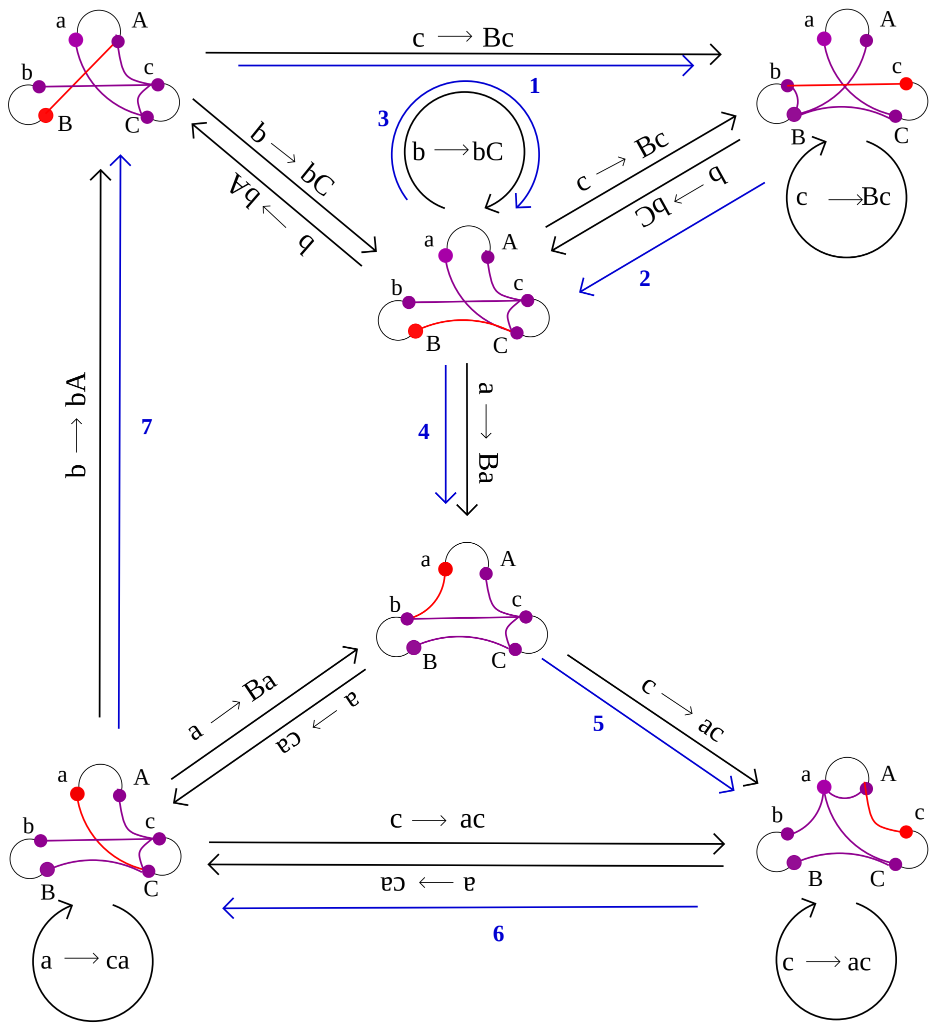


Figure 14.2: *The blue path in* $AM(\mathcal{G})$ *gives an ideal decomposition* g.

The path in $AM(\mathcal{G})$ corresponds to the ideal decomposition:

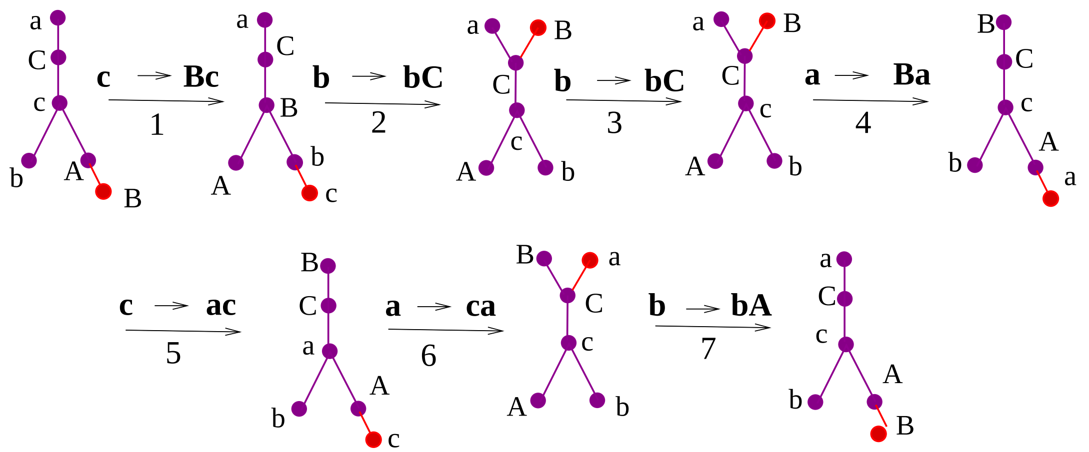


Figure 14.3: *Ideal Decomposition of Representative whose Ideal Whitehead Graph is GRAPH III*

And our representative is:

$$g = \begin{cases} a \mapsto a\bar{b}ca \\ b \mapsto b\bar{a}\bar{c}\bar{a}\bar{c}\bar{c}\bar{a}\bar{c} \\ c \mapsto caccaca\bar{b}cac \end{cases}$$

GRAPH IV:

The representative g whose ideal whitehead graph, $\mathcal{G}$, is GRAPH IV is:

$$g = \begin{cases} a \mapsto c\bar{b}a \\ b \mapsto bc\bar{a}b \\ c \mapsto c\bar{b}abc\bar{a}bc \end{cases}$$

We used Strategy I here. Constructing $AM(\mathcal{G})$ was exceptionally easy because of the symmetry in the graph. Our ideal decomposition for g is:

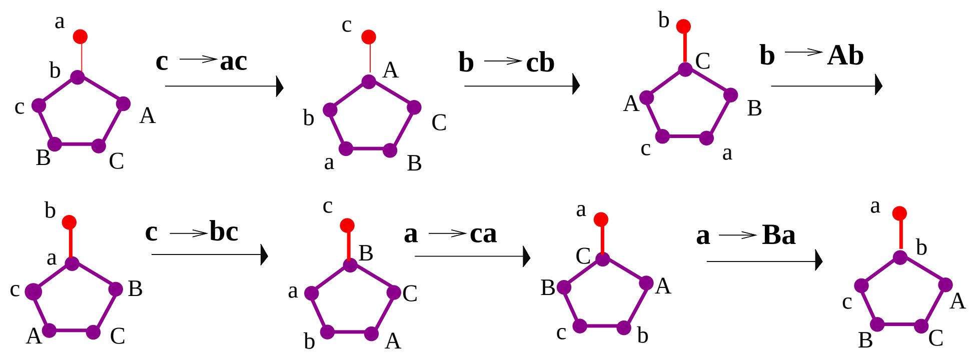


Figure 14.4: *Ideal Decomposition of Representative whose Ideal Whitehead Graph is GRAPH IV*

GRAPH VI:

The representative g whose ideal whitehead graph is GRAPH VI is:

$$g = \begin{cases} a \mapsto abacbaba\bar{c}abacbaba \\ b \mapsto ba\bar{c} \\ c \mapsto c\bar{a}\bar{b}\bar{a}\bar{b}\bar{a}\bar{b}\bar{c}\bar{a}\bar{b}\bar{a}c \end{cases}$$

Our ideal decomposition for g is described by the following figure:

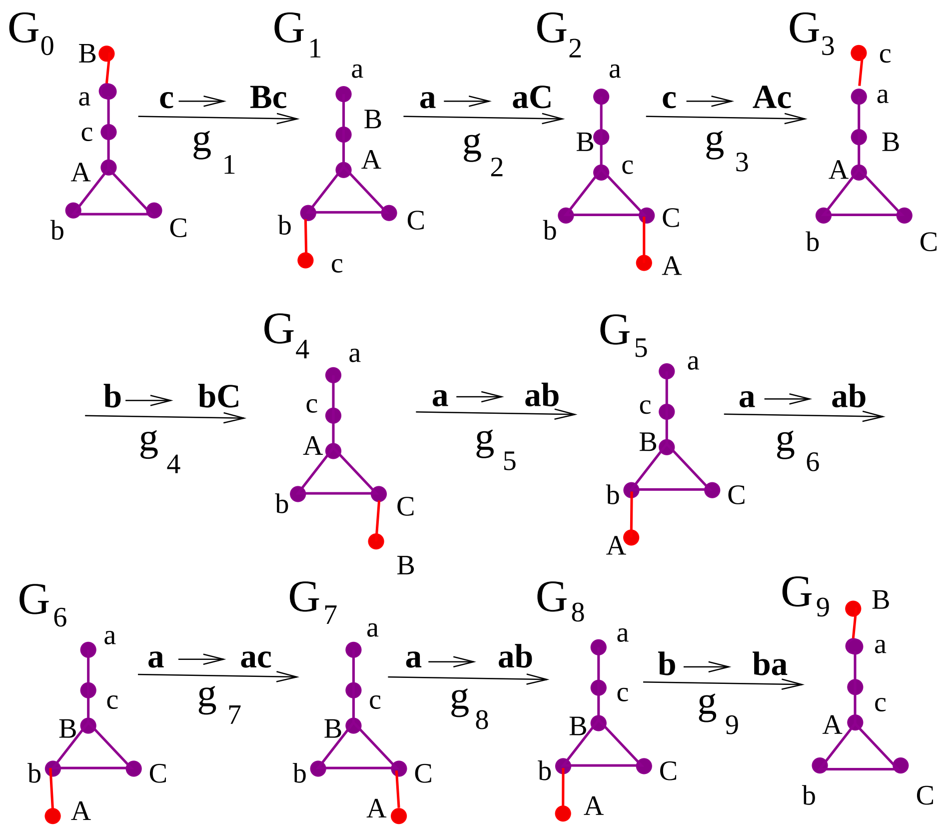


Figure 14.5: *Ideal Decomposition of Representative whose Ideal Whitehead Graph is GRAPH VI*

For this example we used a combination of Strategy II and Strategy III with a construction loop in G_8. Originally we had a construction composition stemming from the construction path at G_9 that we realized we did not need:

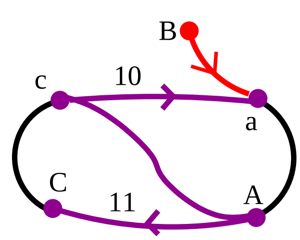


Figure 14.6: *Unused Construction Path for GRAPH VI*

The construction composition that we did keep came from the construction path in G_8 where edges are labeled by the extension they are the purple edge for:

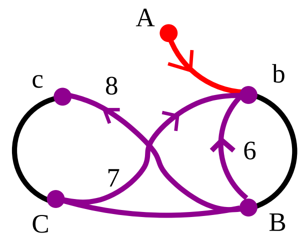


Figure 14.7: *Construction Path for GRAPH VI*

GRAPH VIII:

The representative g whose ideal whitehead graph, $\mathcal{G}$, is GRAPH VIII is:

$$g = \begin{cases} a \mapsto a\bar{c}aa\bar{b}a\bar{c}b\bar{a}\bar{a}ca\bar{c}aa\bar{b}a\bar{c}a \\ \\ b \mapsto b\bar{a}\bar{a}c \\ \\ c \mapsto c\bar{a}b\bar{a}\bar{a}c\bar{a}b\bar{a}\bar{a}c \end{cases}$$

Our ideal decomposition for g is described by the following figure:

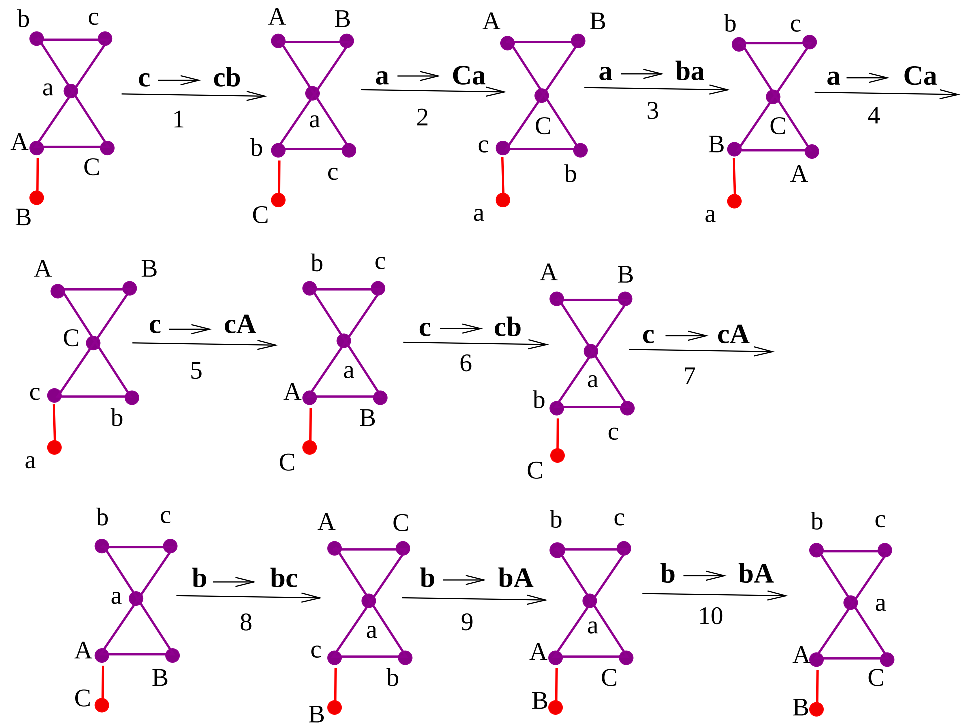


Figure 14.8: *Ideal Decomposition of Representative whose Ideal Whitehead Graph is GRAPH VIII*

For this we constructed a component of $AM(\mathcal{G})$ and used Strategy I. This strategy made the most sense here as there were only a few birecurrent LTT structures to be included in $AM(\mathcal{G})$.

GRAPH IX:

The representative g whose ideal whitehead graph, $\mathcal{G}$, is GRAPH IX is:

$$g = \begin{cases} a \mapsto ab\bar{c}b\bar{c}ab\bar{c}b\bar{c}\bar{b}c\bar{b}\bar{a}\bar{c}\bar{b}ab\bar{c}b\bar{c} \\ b \mapsto bcab\bar{c}bc\bar{b}c\bar{b}\bar{a}cab\bar{c}b \\ c \mapsto c\bar{b}c\bar{b}\bar{a} \end{cases}$$

Our ideal decomposition for g is described by the following figure:

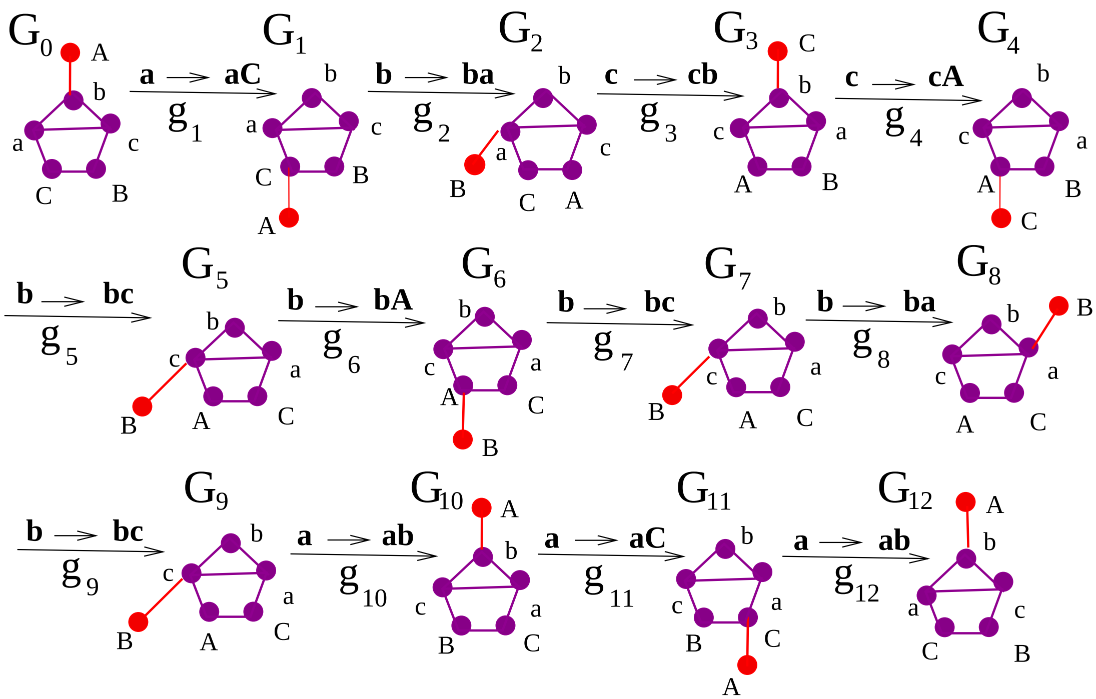


Figure 14.9: *Ideal Decomposition of Representative whose Ideal Whitehead Graph is GRAPH IX*

For this example we used a combination of Strategy II and Strategy III with a construction loops in $G_5 \cong G_9$ and $G_{10} \cong G_{12}$.

GRAPH X:

The representative g whose ideal whitehead graph, $\mathcal{G}$, is GRAPH X is:

$$g = \begin{cases} a \mapsto abacbabac\bar{a}b\bar{c}\bar{a}\bar{b}\bar{a}\bar{c}\bar{a}\bar{b}\bar{a}\bar{b}\bar{c}\bar{a}\bar{b}\bar{a}babac\bar{b}abacbabacabac\bar{b}a \\ b \mapsto babac\bar{a}b\bar{c}\bar{a}\bar{b}\bar{a}\bar{c}\bar{a}\bar{b}\bar{a}\bar{b}\bar{c}\bar{a}\bar{b}\bar{a}babac\bar{a}b\bar{c}\bar{a}\bar{b}\bar{a}\bar{c}\bar{a}\bar{b}\bar{a}\bar{b}\bar{c}\bar{a}\bar{b}\bar{a}b \\ c \mapsto babac\bar{a}b\bar{c}\bar{a}\bar{b}\bar{a}\bar{c}\bar{a}\bar{b}\bar{a}\bar{b}\bar{c}\bar{a}\bar{b}\bar{a}babac\bar{a}b\bar{c}\bar{a}\bar{b}\bar{a}\bar{c}\bar{a}\bar{b}\bar{a}\bar{b}\bar{c}\bar{a}\bar{b}\bar{a}babacbabac\bar{a}b\bar{c}\bar{a}\bar{b}\bar{a}\bar{c}\bar{a}\bar{b}\bar{a}\bar{b}\bar{c}\bar{a}\bar{b}\bar{a}babac \end{cases}$$

Instead of giving our entire ideal decomposition here, we give a condensed decomposition where construction compositions starting and ending at a graph are shown as paths below it. This is an example of a case where Strategy III is used to find our desired representative. If you leave out the initial generator (the upper left-most) and the pure construction compositions corresponding to the paths indicated, we have a switch sequence.

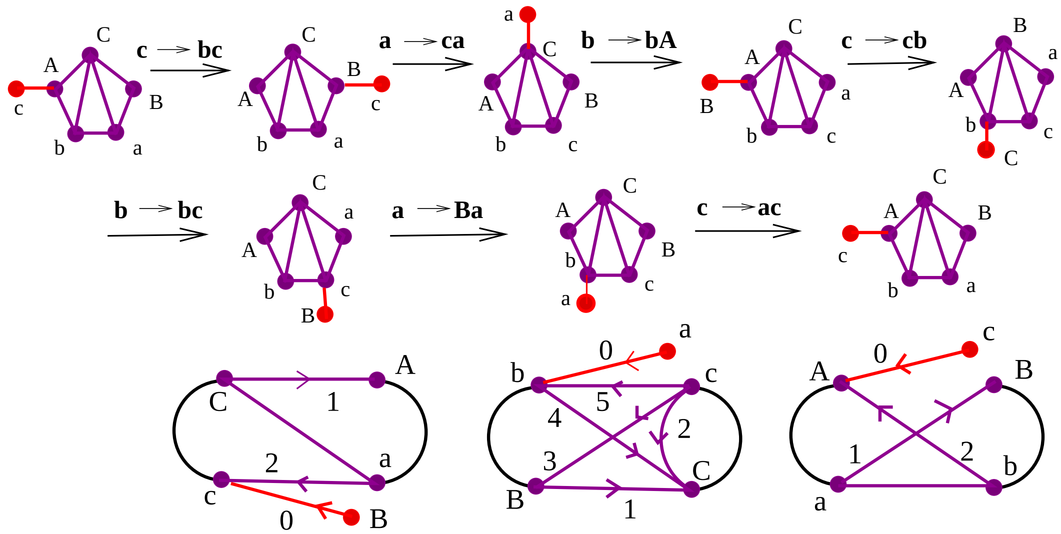


Figure 14.10: *Ideal Decomposition of Representative whose Ideal Whitehead Graph is GRAPH X*

GRAPH XI:

The representative g whose ideal whitehead graph, $\mathcal{G}$, is GRAPH XI is:

$$g = \begin{cases} a \mapsto a\bar{b}\bar{b}\bar{c}b\bar{c}\bar{b}c\bar{b}cbc\bar{a}bc\bar{b}cbc\bar{a}\bar{b}c\bar{b} \\ b \mapsto b\bar{c}ba\bar{c}\bar{b}\bar{c}b\bar{c}\bar{b}a\bar{c}\bar{b}\bar{c}b\bar{c}b \\ c \mapsto c\bar{b}cbc\bar{a}\bar{b}c\bar{b}cbc\bar{a}bc\bar{b}cbc\bar{a}\bar{b}c \end{cases}$$

Again, instead of giving our entire ideal decomposition here, we give a condensed decomposition where construction compositions starting and ending at a graph are shown as paths below it. However, the graph in the lower right actually gives a construction composition with source LTT structure above it to the left and destination LTT structure above it to the right. This representative thus actually uses a variant of Strategy

III where we allow this. (We in fact use a combination of Strategy II and Strategy III).

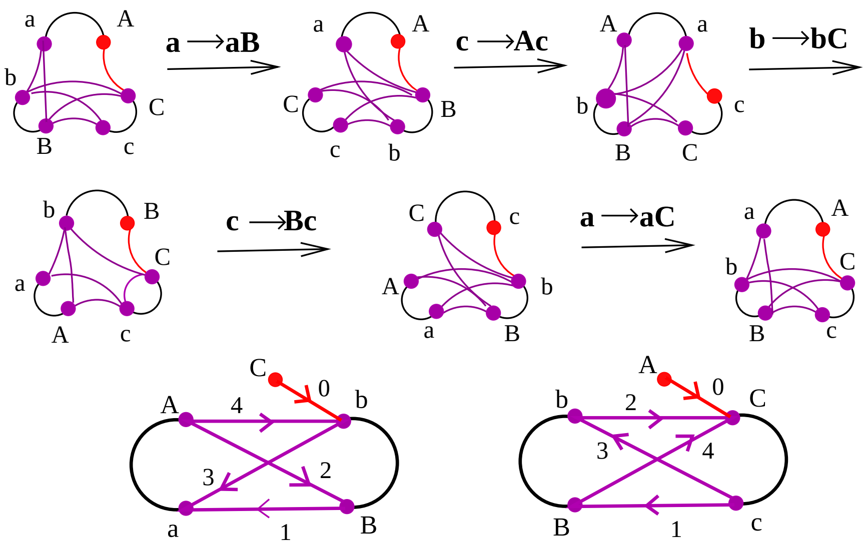


Figure 14.11: *Ideal Decomposition of Representative whose Ideal Whitehead Graph is GRAPH XI*

GRAPH XII:

The representative g whose ideal whitehead graph is GRAPH XII is:

$$g = \begin{cases} a \mapsto a\bar{c}\bar{b}\bar{b}\bar{c}b\bar{c}\bar{b}c\bar{b}cbcc\bar{a}bc\bar{b}cbcc\bar{a}\bar{b}c\bar{b} \\ b \mapsto b\bar{c}ba\bar{c}\bar{c}\bar{b}\bar{c}b\bar{c}\bar{b}a\bar{c}\bar{c}\bar{b}\bar{c}b\bar{c}b \\ c \mapsto c\bar{b}cbcc\bar{a}\bar{b}c\bar{b}cbcc\bar{a}bc\bar{b}cbcc\bar{a}\bar{b}c \end{cases}$$

Again, instead of giving our entire ideal decomposition here, we give a condensed decomposition where construction compositions starting and ending at a graph are shown as paths below it.

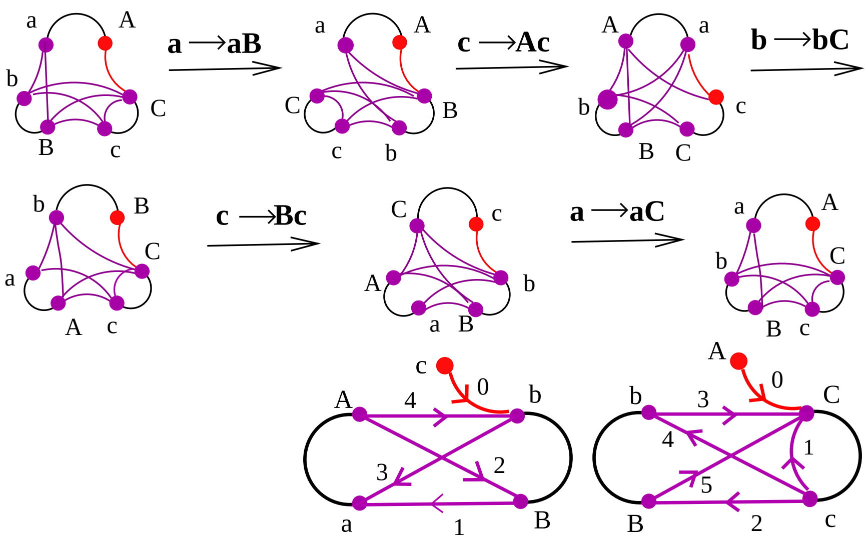


Figure 14.12: *Ideal Decomposition of Representative whose Ideal Whitehead Graph is GRAPH XII*

GRAPH XIII:

The representative g whose ideal whitehead graph is GRAPH XIII is:

$$g = \begin{cases} a \mapsto ac\bar{b}ccbc\bar{b}\bar{c}\bar{b}\bar{c}\bar{c}b\bar{c}\bar{a}c\bar{b}ac\bar{b}ccbc\bar{b}\bar{c}\bar{b}\bar{c}\bar{c}b\bar{c}\bar{a}\bar{b} \\ b \mapsto bac\bar{b}ccbcb\bar{c}\bar{b}\bar{c}\bar{c}b\bar{c}\bar{a}b\bar{c}ac\bar{b}ccbcb \\ c \mapsto c\bar{b}ac\bar{b}ccbc\bar{b}\bar{c}\bar{b}\bar{c}\bar{c}b\bar{c}\bar{a}\bar{b}ac\bar{b}ccbc \end{cases}$$

Our ideal decomposition for this representative and further explanation were given in Example 13.15.

GRAPH XIV:

The representative g whose ideal whitehead graph is GRAPH XIV is:

$$g = \begin{cases} a \mapsto acabaabcabaa \\ b \mapsto \bar{a}\bar{a}\bar{b}\bar{a}\bar{c}\bar{b}\bar{a}\bar{a}\bar{b}\bar{a}\bar{c}\bar{a}b\bar{a}\bar{a}\bar{b}\bar{a}\bar{c}acabaabcabaab\bar{a}\bar{a}\bar{b}\bar{a}\bar{c}\bar{a}\bar{a}\bar{b}\bar{a}\bar{c}\bar{b}\bar{a}\bar{a}\bar{b}\bar{a}\bar{c}\bar{a}b \\ c \mapsto cabaa\bar{b} \end{cases}$$

Our ideal decomposition for g is described by the following figure:

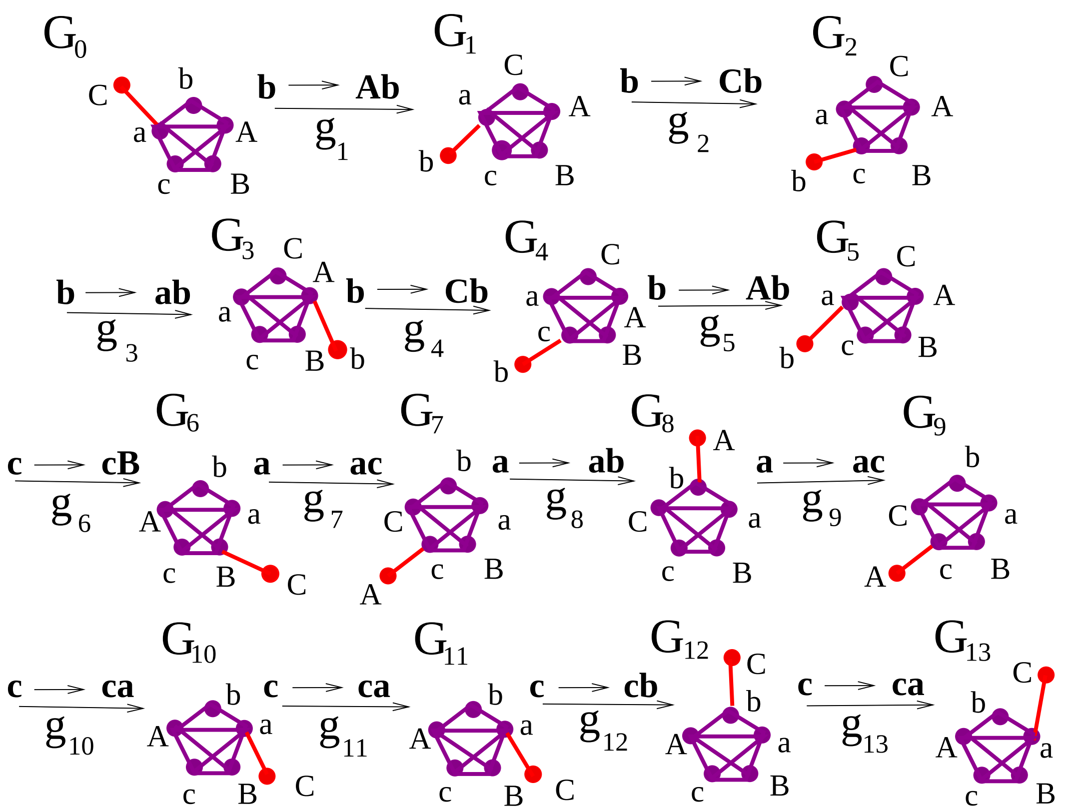


Figure 14.13: *Ideal Decomposition of Representative whose Ideal Whitehead Graph is GRAPH XIV*

This example was constructed using Strategy III with construction loops at $G_1 \cong G_5$, $G_7 \cong G_9$, and $G_{10} \cong G_{13}$.

GRAPH XV:

The representative g whose ideal whitehead graph, $\mathcal{G}$, is GRAPH XV is:

$$g = \begin{cases} a \mapsto a\bar{c}\bar{b}\bar{b}\bar{c}b\bar{c}\bar{b}c\bar{b}cbbc\bar{a}bc\bar{b}cbbc\bar{a}\bar{b}c\bar{b} \\ b \mapsto b\bar{c}ba\bar{c}\bar{b}\bar{b}\bar{c}b\bar{c}\bar{b}a\bar{c}\bar{b}\bar{b}\bar{c}b\bar{c}b \\ c \mapsto c\bar{b}cbbc\bar{a}\bar{b}c\bar{b}cbbc\bar{a}bc\bar{b}cbbc\bar{a}bc \end{cases}$$

Again, instead of giving our entire ideal decomposition here, we give a condensed decomposition where construction compositions starting and ending at a graph are shown as paths below it. The similarities between this construction and that of Graph XI are not a coincidence. Since XI is a subgraph of XV missing only a single edge, once the representative for XI was constructed, we could alter the representative by adding the edge $[b, \bar{b}]$ to the final (right-most) construction path to add that edge to $\mathcal{G}$. It must also be checked, however, that, if we add the preimages of this edge into the previous LTT structures that they are still birecurrent, that we still have a composition of switches and extensions, that we still have no PNPs, and that our initial and terminal LTT structures for the entire train track are the same.

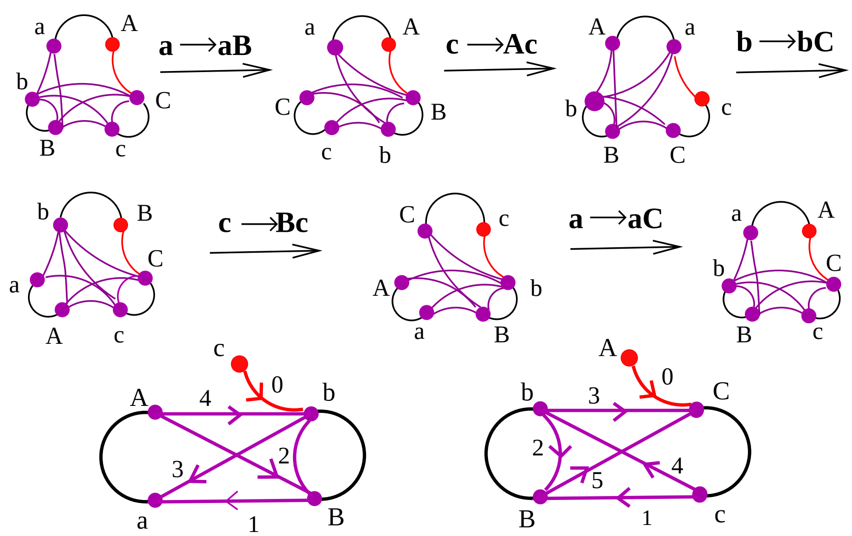


Figure 14.14: *Ideal Decomposition of Representative whose Ideal Whitehead Graph is GRAPH XV*

GRAPH XVI:

The representative g whose ideal whitehead graph is GRAPH XVI is:

$$g = \begin{cases} a \mapsto a\bar{b}ccbc\bar{b}c \\ b \mapsto b\bar{c}\bar{b}\bar{c}\bar{c}b\bar{a}\bar{c}b \\ c \mapsto ca\bar{b}ccbc\bar{b}c \end{cases}$$

Our ideal decomposition for g is described by the following figure:

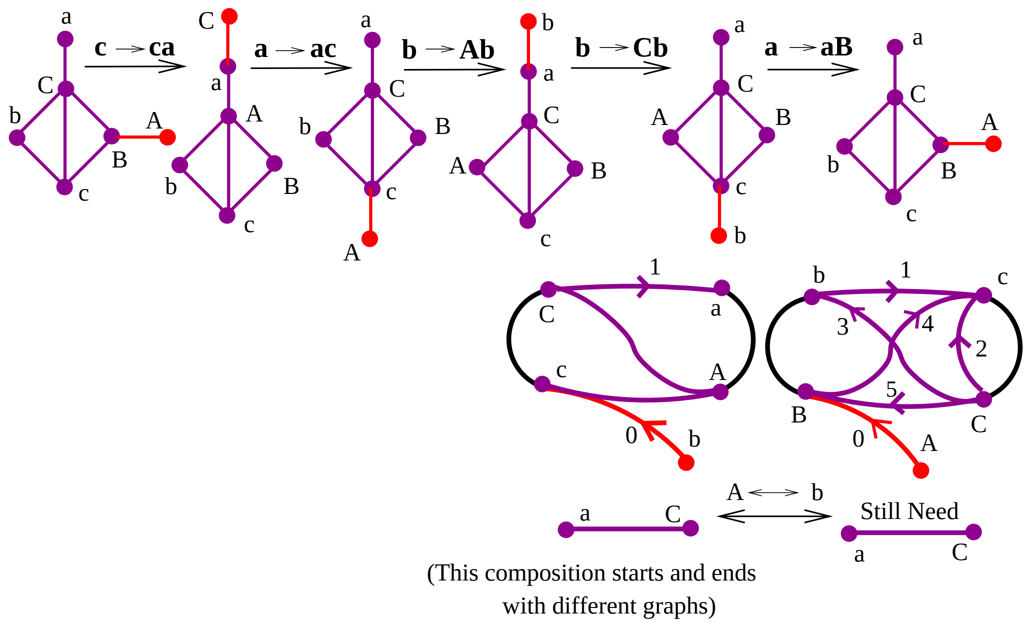


Figure 14.15: *Ideal Decomposition of Representative whose Ideal Whitehead Graph is GRAPH XVI*

GRAPH XVII:

The representative g whose ideal whitehead graph is GRAPH XVII is:

$$g = \begin{cases} a \mapsto acbcc\bar{b}c\bar{b}acbcc\bar{b}acbcc \\ b \mapsto b\bar{c}\bar{c}\bar{b}\bar{c}\bar{a}b\bar{c}\bar{c}\bar{c}\bar{b}\bar{c}\bar{a}b\bar{c}\bar{c}\bar{b}\bar{c}\bar{a}b\bar{c}b \\ c \mapsto c\bar{b}acbcc\bar{b}\bar{b}c\bar{b}acbcc\bar{b}acbccc\bar{b}acbcc\bar{b}acbcc\bar{b}c\bar{b}acbcc\bar{b}acbcc \end{cases}$$

Our ideal decomposition for g is described by the following figure:

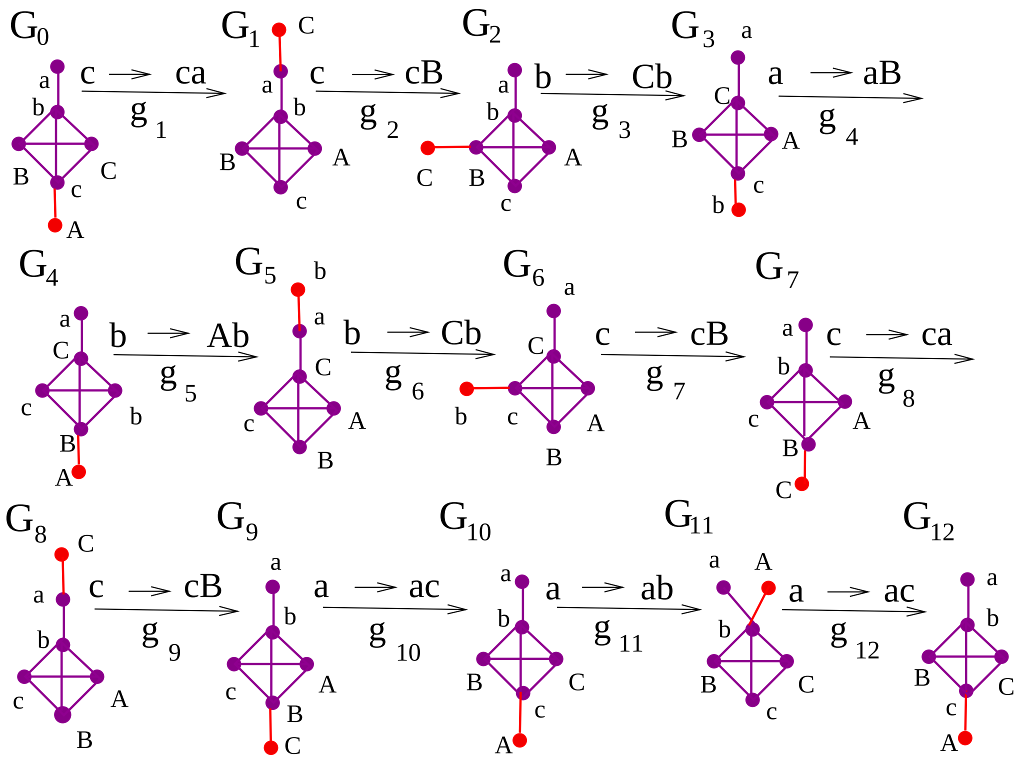


Figure 14.16: *Ideal Decomposition of Representative whose Ideal Whitehead Graph is GRAPH XVII*

For this we used Strategy I, though did not need to construct the entire AM Diagram in order to obtain the map we needed (using Guess and Check). We did use a single small construction loop in G_{12}.

GRAPH XVIII:

The representative g whose ideal whitehead graph is GRAPH XVIII is:

$$g = \begin{cases} a \mapsto a\bar{b}c\bar{b}a\bar{b}\bar{c} \\ b \mapsto b\bar{a}b\bar{c}b\bar{a}b\bar{a}b \\ c \mapsto cb\bar{a}b\bar{c}b\bar{a}b\bar{a}b\bar{c}b\bar{a}b\bar{a}bc \end{cases}$$

Our ideal decomposition for g is described by the following figure:

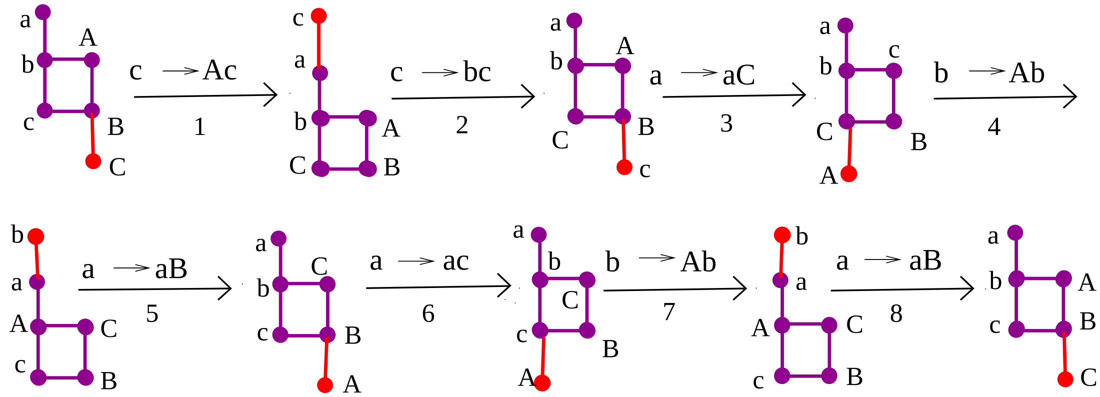


Figure 14.17: *Ideal Decomposition of Representative whose Ideal Whitehead Graph is GRAPH XVIII*

For this example we used Strategy I and constructed the entire AM Diagram.

GRAPH XIX:

The representative g whose ideal whitehead graph is GRAPH XIX is:

$$g = \begin{cases} a \mapsto acc\bar{b}cbc \\ b \mapsto b\bar{c}\bar{b}\bar{c}b\bar{c}\bar{c}\bar{a}b\bar{c}b \\ c \mapsto c\bar{b}acc\bar{b}cbc\bar{b}acc\bar{b}cbc \end{cases}$$

Again, instead of giving our entire ideal decomposition here, we give a condensed decomposition where construction compositions starting and ending at a graph are shown as paths below.

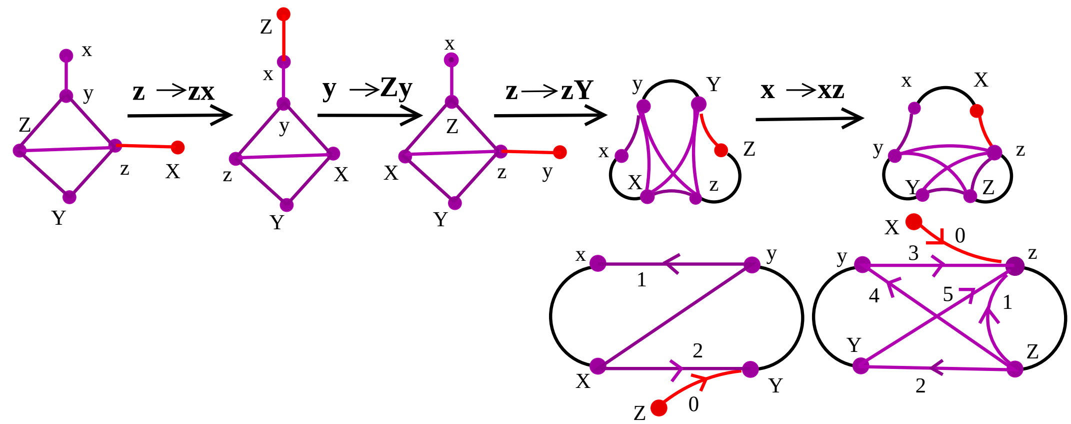


Figure 14.18: *Ideal Decomposition of Representative whose Ideal Whitehead Graph is GRAPH XIX*

GRAPH XX:

The representative $g = h^2$ having ideal Whitehead graph GRAPH XX, where

$$h = \begin{cases} a \mapsto ab\bar{c}\bar{c}bbcb \\ b \mapsto bc \\ c \mapsto cab\bar{c}\bar{c}bbcbab\bar{c}\bar{c}bbcb\bar{c}\bar{c}\bar{b}ab\bar{c}\bar{c}bbcbbccab\bar{c}\bar{c}bbcb \end{cases},$$

was constructed in the examples above.

GRAPH XXI (Complete Graph):

The representative g whose ideal whitehead graph is GRAPH XXI is:

$$g = \begin{cases} a \mapsto aba\bar{b}aac\bar{b}aba\bar{b}aacbaba\bar{b}aacaba\bar{b}aac\bar{b}a \\ b \mapsto baba\bar{b}aac\bar{a}b\bar{c}\bar{a}\bar{a}b\bar{a}\bar{b}\bar{a}\bar{c}\bar{a}\bar{a}b\bar{a}\bar{b}\bar{a}\bar{b}\bar{c}\bar{a}\bar{a}b\bar{a}\bar{b}\bar{a}b \\ c \mapsto aba\bar{b}aac \end{cases}$$

Below we include an ideal decomposition of the representative. We only label the vertices of the red edge in each graph of this example since the remainder of the graph is completely symmetric and so any permutation of the remaining labels gives exactly the same graph. This is an example of a case where Strategy I would have been extremely impractical and Strategy II was particularly easy to apply. Similar strategies as used to construct this representative could also be used to construct the representative whose ideal Whitehead graph is the complete graph in any odd rank greater than five.

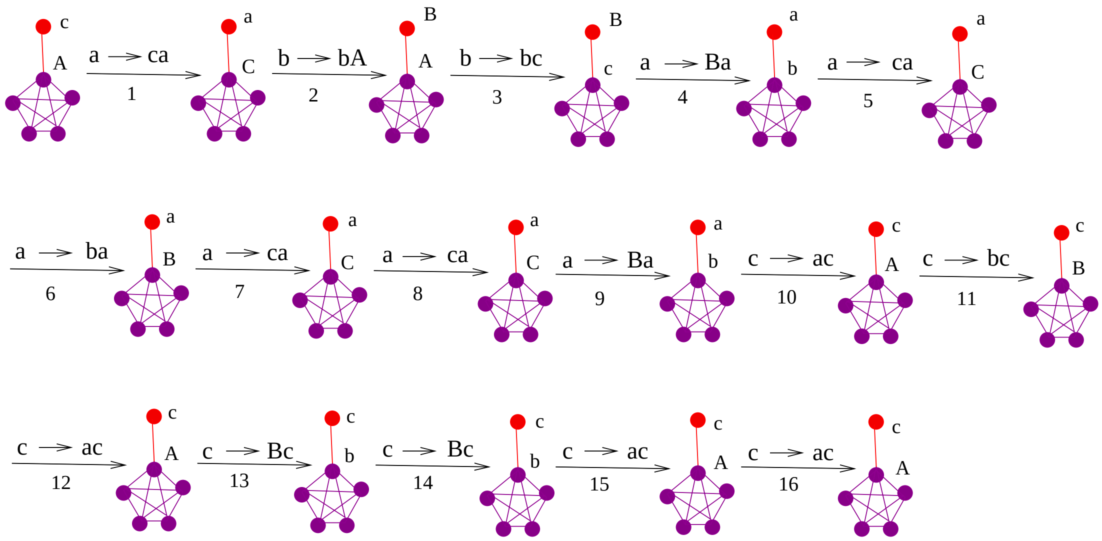


Figure 14.19: *Ideal Decomposition of Representative whose Ideal Whitehead Graph is GRAPH XXI*

Since we have either given representatives yielding or shown that they cannot exist for all twenty-one Type (*) pIW graphs with five vertices, we have completed the proof. QED.

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