YANG-MILLS HEATFLOW ON GAUGED HOLOMORPHIC MAPS

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ABSTRACT OF THE DISSERTATION

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We study the gradient flow lines of a Yang-Mills-type functional on the space of gauged holomorphic maps $\mathcal{H}(P, X)$, where P is a principal bundle on a Riemann surface Σ and X is a Kähler Hamiltonian G-manifold. For compact Σ, possibly with boundary, we prove long time existence of the gradient flow. The flow lines converge to critical points of the functional. So, there is a stratification on $\mathcal{H}(P, X)$ that is invariant under the action of the complexified gauge group.

Symplectic vortices are the zeros of the functional we study. When Σ has boundary, similar to Donaldson's result in [Don92], we show that there is only a single stratum - any element of $\mathcal{H}(P, X)$ can be complex gauge transformed to a symplectic vortex. This is a version of Mundet's Hitchin-Kobayashi result [MiR00] on a surface with boundary.

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Table of Contents

Chapter 1

Introduction

J-holomorphic maps have been important objects in symplectic geometry since they were introduced by Gromov ([Gro85]) in 1985. In an equivariant setting, these generalize to gauged holomorphic maps. Let G be a compact connected Lie group acting on a compact Kähler manifold (X, ω). We assume the action is Hamiltonian and has *moment map* $\Phi : X \to \mathfrak{g}^*$. The G-action preserves the complex structure on X. A *gauged holomorphic map* from a Riemann surface Σ to X is a pair (A, u) consisting of a connection A on a principal G-bundle $P \to \Sigma$ together with a holomorphic section of the associated fiber bundle $P(X) := (P \times X)/G$. The complex structure on $P(X)$ is given by the complex structure on Σ and X and the connection A. The space of gauged holomorphic maps $\mathcal{H}(P, X)$ has a formal Hamiltonian action of the group of gauge transformations $\mathcal{G}(P)$. The moment map is given by $*F_A + \Phi(u)$, where F_A is the curvature of A. The functional

$$\begin{gathered} \mathcal{H}(P, X) \to \mathbb{R} \\ \text{given by } (A, u) \mapsto \|{*F_A} + \Phi(u)\|_{L^2}^2 \end{gathered} \tag{1.1}$$

is the square of the norm of the moment map. In this dissertation, we study the long-time existence and convergence behaviour of the gradient flow trajectories of (1.1)

If Σ is a compact Riemann surface, possibly with boundary, the gradient flow lines of (1.1) exist for all time. If Σ has boundary, the flow equations are solved under the condition $*F_A + \Phi(u) = 0$ on $\partial\Sigma$.

Theorem 1.0.1. (3.0.1) *Suppose Σ is compact. The gradient flow for the functional (1.1) exists for all time. $(A_t, u_t) \in C^0_{loc}([0, \infty), H^1 \times C^0)$. There is a family of gauge transformations $g_t \in H^2(\mathcal{G})$ so that $g_t(A_t, u_t)$ is smooth on $[0, \infty) \times \Sigma$.*

We view this theorem as an infinite-dimensional version of the set-up in [Kir84]. On a compact Kähler manifold with a Hamiltonian G-action, Kirwan describes the Morse strata of the norm square of the moment map. A compact Lie group has a complexification $G_{\mathbb{C}}$, which is a complex reductive group. Since X is Kähler, the action of G extends to an action of $G_{\mathbb{C}}$. The Morse strata are $G_{\mathbb{C}}$ invariant.

When $X \subseteq \mathbb{P}^n$ is a projective variety and $G_{\mathbb{C}}$ acts linearly on it, geometric invariant theory (GIT), developed by Mumford [MFK94], gives a way of defining quotients. Let $A(X)$ be the co-ordinate ring of X, and $A(X)^{G_{\mathbb{C}}}$ be the subring of $G_{\mathbb{C}}$-invariants. $A(X)^{G_{\mathbb{C}}}$ is finitely generated because $G_{\mathbb{C}}$ is reductive. The associated projective variety $X/\!\!/G_{\mathbb{C}}$ is the GIT quotient of X. A point x in X is *semi-stable* if there is a $G_{\mathbb{C}}$-invariant homogeneous polynomial f with positive degree that does not vanish at x. The semistable locus of X^{ss} is Zariski-open in X. An important result of Mumford is that, as a topological space $X/\!\!/G_{\mathbb{C}} \simeq X^{ss}/\sim$, $\sim$ is the orbit-closure relation on X^{ss} given by: $x_1 \sim x_2$ iff $\overline{G_{\mathbb{C}}x_1} \cap \overline{G_{\mathbb{C}}x_2} \cap X^{ss} \neq \phi$. The GIT quotient is equivalent to the symplectic quotient $\phi^{-1}(0)/G$. In the affine case, this is the Kempf-Ness theorem ([KN79]). In the Morse theory picture, [Kir84] shows that the open Morse stratum coincides with X^{ss} and gives an algebraic description of the other Morse strata also.

The work of Atiyah and Bott[AB83] introduces the above ideas in the infinite dimensional setting - on the space of connections $\mathcal{A}$ on a principal bundle over a Riemann surface. This space is equivalent to the space of holomorphic structures on an associated complex vector bundle. There is a stratification of this space by considering the Harder-Narasimhan filtration. The stratification is preserved by the action of the complexified gauge group $\mathcal{G}_{\mathbb{C}}$. The lowest stratum consists of semi-stable bundles. For a bundle E, semi-stability means that for any holomorphic sub-bundle E_1,

$$\frac{c_1(E_1)}{rank(E_1)} \leq \frac{c_1(E)}{rank(E)}.$$

On the differential-geometric side, there is the Morse stratification of the Yang-Mills functional. The Narasimhan-Seshadri theorem ([NS65]) says that, in these two stratifications, the open stratum is the same : 'every stable bundle admits a Yang Mills connection that assumes the minimum value of the functional i.e. $*F_A = 2\pi i c_1(E)/rk(E)$.'

Donaldson gave a diffeo-geometric proof [Don83] of this theorem. Daskalopoulos [Das92] and Råde [Råd92] proved that the algebraic and Morse stratifications agree. Råde's approach is to show that the gradient flow lines of the Yang-Mills functional are continuous and converge as $t \to \infty$.

We use similar techniques as Råde [Råd92] to show the existence of flow for (1.1). The main point of difference is that our flow problem involves u which is a map to a compact Kähler manifold. While solving the flow equations, we assume that $u(t)$ is in C^0, but in the time direction, we assume its regularity is in a Sobolev class. We have to address some issues in defining such a mixed space. The reason why it's necessary to have u_t in C^0, is because the perturbative lower order terms in the parabolic flow equations involve composition of functions, and we need $u_t \in C^0$ to use these results. [Don85] gives a simpler way of obtaining flow lines, albeit modulo gauge. But this approach does not work for us because of the non-linear moment map term. However, after showing the existence of flow, we adapt the technique in [Don85] to show that our flow is smooth in time and space directions modulo gauge.

Similar Morse-theoretic ideas have been applied to the space of holomorphic vector bundles equipped with some extra data. For example, Wilkin[Wil08] studies the space of Higgs pairs (A, ϕ), where A is a connection on a complex vector bundle over a Riemann surface, and $\phi \in \Omega^{1,0}(E)$ such that $\overline{\partial}_A \phi = 0$. With a standard choice of symplectic structure, the action of the gauge group action has moment map $F_A + [\phi, \phi^*]$. This work shows that the Morse stratification of the L^2 norm of the moment map corresponds to a holomorphic stratification. A Higgs pair corresponds to a $GL(n, \mathbb{C})$ connection, so the problem of studying the gradient flow in this case, is reduced to the gradient flow problem on the space of connections.

We prove the following result on the convergence behaviour of the gradient flow lines:

Theorem 1.0.2. *Let* $(A_t, u_t) \in C^\infty_{loc}([0, \infty) \times \Sigma)$ *be the gradient flow (modulo gauge) calculated in theorem 1.0.1. There exists a sequence* $t_i \to \infty$*, a sequence of gauge transformations* $g_i \in \mathcal{G}_{H^2}$ *and a pair* $(A_\infty, u_\infty) \in \mathcal{A}(P)_{H^2} \times \Gamma(\Sigma, P(X))_{C^1}$ *such that,*

a. $g_i(A_{t_i}) \to A_\infty$ *weakly in* H^2

b. *If* Σ *does not have boundary,* $g_i u_{t_i}$ *Gromov converges to a nodal gauged holomorphic map with principal component* u_∞*. Let* $Z \subseteq \Sigma$ *be the finite bubbling set. In compact subsets of* $\Sigma \backslash Z$*,* $g_i u_{t_i} \to u_\infty$ *in* C^1*.*

c. *If* Σ *has boundary,* $g_i u_{t_i} \to u_\infty$ *in* C^1 *- there is no bubbling.*

d. (A_∞, u_∞) *is a critical point of the functional* (1.1)*.*

A stronger result can be obtained in case the Σ has boundary. For this case, we define a sub-group $\mathcal{G}_{\mathbb{C},G}$ of the complexified gauge group consisting of $g \in \mathcal{G}_{\mathbb{C}}$ such that $g|_{\partial\Sigma} \in \mathcal{G}(\partial\Sigma)$

Theorem 1.0.3. *The limit* (A_∞, u_∞) *computed in theorem 1.0.2 lies in the same* $\mathcal{G}_{\mathbb{C},G}$*-orbit as the flow line* (A_t, u_t)*. For a given flow line* (A_t, u_t)*, the limit* (A_∞, u_∞) *is unique up to gauge.*

This result in the case of a surface with boundary can be compared to Donaldson's result [Don92] on Yang-Mills gradient flow. On a two-dimensional base manifold with boundary, it says that any connection can be complex gauge transformed to a flat connection - there is no semi-stability condition involved.

Gauged holomorphic maps that satisfy $F_{A,u} = 0$ are called *symplectic vortices.* These have been studied in [JT80], [Bra90], [CGS00], [CGMiRS02], [Zil06] etc. An important motivation for studying the functional (1.1) is to obtain a holomorphic description of the moduli space of symplectic vortices. [JT80] (theorem 1.1 and 1.2 in chapter 3) gives a classification of vortices on $\mathbb{C}$ with target manifold $X = \mathbb{C}$, with the linear action of S^1. [Bra90] considers the case when Σ is a compact Kähler manifold and $X = \mathbb{C}$. They give a stability condition on the space of gauged holomorphic maps which ensures that there is a vortex in the $\mathcal{G}_{\mathbb{C}}$ orbit - this is a Hitchin-Kobayashi correspondence. Mundet's work [MiR00] generalizes this correspondence to non-linear G-action. He takes X to be a compact Kähler Hamiltonian-G-manifold and gives a stability criterion. Our heat-flow approach to the problem can give a complete stratification of $\mathcal{H}$ - i.e. $\mathcal{H}$ can be decomposed into subsets according to the critical set a point flows to via the gradient flow. Theorem 1.0.3 can be seen as a version of Mundet's result for a Riemann surface with boundary. It shows that the map

$$\begin{aligned}\mathcal{H}(P,X)/\mathcal{G}_{\mathbb{C},G} &\to \text{Space of symplectic vortices from } P \text{ to } X/\mathcal{G}\\ [(A_0,u_0)] &\mapsto [(A_\infty,u_\infty)]\end{aligned}$$

is a bijection.

An application of theorem 1.0.3 is that it can be used to obtain a semi-stability criterion on holomorphic maps on $\mathbb{C}$ to X - this determines which maps on the trivial bundle over $\mathbb{C}$ have a symplectic vortex in their complex gauge orbit. This generalizes the result of [JT80] - in [JT80] the target manifold is $\mathbb{C}$ with a linear S^1-action, whereas our result holds for a compact Kähler manifold with Hamiltonian G-action. This application will be presented elsewhere.

This dissertation is organized as follows: chapter 2 describes connections, gauged holomorphic maps etc. Chapter 3 proves theorem 1.0.1 - the long-time existence of gradient flow and its regularity properties. Chapter 4 discusses the convergence result theorem 1.0.2. Chapters 5 and 6 carefully describe the Sobolev spaces and their properties used in chapter 3.

Chapter 2

Preliminaries

2.1 Hamiltonian actions

Let (X, ω) be a compact Kähler manifold. This means that (X, ω) is a symplectic manifold, alongwith a compatible almost-complex structure $J : TX \to TX$ that is integrable.

Let G be a compact connected Lie group acting on X smoothly. We assume that the action is Hamiltonian, i.e. there is a moment map $\Phi : X \to \mathfrak{g}^*$. Φ is equivariant and satisfies $\iota(\xi_X)\omega = \mathrm{d}\langle \Phi, \xi \rangle, \ \forall \xi \in \mathfrak{g}$, where $\xi_X \in \mathrm{Vect}(X)$ given by the infinitesimal action of ξ on X. Since G is compact, $\mathfrak{g}$ has an Ad-invariant metric. We fix such a metric and identify $\mathfrak{g}$ with $\mathfrak{g}^*$ and so the moment map is $\Phi : X \to \mathfrak{g}$. We also assume that the action of G preserves J.

Consider the gradient flow lines of the functional $f = \frac{1}{2}\langle \Phi, \Phi \rangle$ on X. The Riemannian metric we use here is $g := \omega(\cdot, J\cdot)$.

Proposition 2.1.1. $\operatorname{grad} f(x) = -J\Phi(x)_X$

Proof. For $v \in T_xX$,

$$\langle \operatorname{grad} f, v \rangle_g = \langle d\Phi(v), \Phi \rangle_{\mathfrak{g}} = \iota_{\Phi(x)_X}\omega(v) = \langle v, -J\Phi(x)_X \rangle_g.$$

□

Next we describe $G_{\mathbb{C}}$-*the complexified Lie group* of G. Let $\mathfrak{g}_{\mathbb{C}}$ denote the complexified Lie algebra $\mathfrak{g} \oplus i\mathfrak{g}$. Then,

Proposition 2.1.2. *([Hoc65], p205) For a compact connected Lie group G, there exists a unique connected complex Lie group $G_{\mathbb{C}}$, with the following properties:*

a. *its Lie algebra is* $\mathfrak{g}_{\mathbb{C}}$.

b. G *is a maximal compact subgroup of* $G_{\mathbb{C}}$.

On a Kähler manifold the action of G extends to a unique holomorphic action of $G_{\mathbb{C}}$ (see [GS82]). Since the gradient of $\frac{1}{2}\|\Phi\|^2$ is $J\Phi(x)_X$, the gradient flow preserves the $G_{\mathbb{C}}$ orbit. So, the semistable stratum - which is the open stratum of this gradient flow, is $G_{\mathbb{C}}$ invariant.

Proposition 2.1.3. *A* $G_{\mathbb{C}}$ *orbit has at most one* G*-orbit on which* $\Phi = 0$.

Proof. The map

$$G_{\mathbb{C}} \to G \times \mathfrak{g} \qquad\qquad g \mapsto (k, s) \text{ so that } g = ke^{is} \tag{2.1}$$

is a diffeomorphism. For semi-simple $G_{\mathbb{C}}$, this is shown in [Hel62], p 214. The result is true for any compact Lie group G because of the decomposition $G = Z(G) \times G^s$, where $Z(G)$ is the center of G and G^s is a semisimple subgroup.

Suppose $\Phi(x) = \Phi(gx) = 0$. Since $g = ke^{is}$, where $k \in G$ and $s \in \mathfrak{g}$, we can assume $g = e^{is}$. For $0 \le t \le 1$,

$$\frac{d}{dt}\langle \Phi(e^{its}x), s\rangle = \langle s_X(e^{its}x), s_X(e^{its}x)\rangle \ge 0. \tag{2.2}$$

So, $s_X = 0$ for all points on $t \mapsto e^{its}x$ which means $x = e^{its}x$. $\square$

2.2 The space of connections

Let (Σ, j) be a Riemann surface and $P \to \Sigma$ a principal G-bundle over it. A *connection* A on P is a $\mathfrak{g}$-valued 1-form on P that is G-equivariant and satisfies $A(\xi_P) = \xi$ for all $\xi \in \mathfrak{g}$. Let $\mathcal{A}(P)$ denote the space of all connections. It is an affine space modeled on $\Omega^1(\Sigma, P(\mathfrak{g}))$, where $P(\mathfrak{g}) := (P \times \mathfrak{g})/G$ is the adjoint bundle. A connection A defines an exterior derivative d_A on $\Omega^*(\Sigma, P(\mathfrak{g}))$

$$d_A\xi := d\xi + [A \wedge \xi].$$

Locally this means, in a trivialization of the bundle $P(\mathfrak{g})$ on a neighbourhood $U_\alpha \subseteq \Sigma$, suppose A is given by $d + A_\alpha$, $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$. Then $(d_A\xi)_\alpha = d\xi + [A_\alpha \wedge \xi]$. On the vector bundles $\Omega^k(\Sigma, P(\mathfrak{g}))$, there is an inner product defined using the Hodge star on Σ and the Ad-invariant metric on $\mathfrak{g}$. d_A extends to an exterior derivative $d_A : \Omega^k(\Sigma, P(\mathfrak{g})) \to \Omega^{k+1}(\Sigma, P(\mathfrak{g}))$. Its formal adjoint is $d_A^* = - * d_A *$. Then, the Hodge Laplacian is defined as $\Delta_A = d_A^* d_A + d_A d_A^*$.

The curvature $F_A \in \Omega^2(\Sigma, P(\mathfrak{g}))$ of a connection A is

$$F_A := dA + \frac{1}{2}[A \wedge A].$$

The operator d_A^2 is a tensor on Σ and it satisfies $d_A^2 \xi = [F_A, \xi]$. The curvature varies with the connection as

$$F_{A+ta} = F_A + t d_A a + \frac{t^2}{2}[a \wedge a].$$

A *gauge transformation* is an automorphism of P - it is an equivariant bundle map $P \to P$. It is a section of the bundle $(P \times G)/G$, where G acts on itself by conjugation. Let $\mathcal{G}(P)$ denote the *group of gauge transformations.* $g \in \mathcal{G}(P)$ acts on $\mathcal{A}(P)$ by pullback by g^{-1}. In a local trivialization, a gauge transformation can be seen as a map $U_\alpha \to G$. For a connection $d + A_\alpha$ on U_α, the action of $g \in \mathcal{G}(P)$ is given by

$$g(A)_\alpha = g d g^{-1} + g A_\alpha g^{-1}.$$

Differentiating, we see that the infinitesimal action of $\xi \in \Gamma(\Sigma, P(\mathfrak{g}))$ on A is $-d_A \xi$. Under the action of $g \in \mathcal{G}(P)$, the curvature transforms as $F_{g(A)} = g F_A g^{-1}$. $\mathcal{A}(P)$ can be equipped with a symplectic form - for a, $b \in T_A \mathcal{A} = \Omega^1(\Sigma, P(\mathfrak{g}))$

$$(a, b) \mapsto \int_X \langle a \wedge b \rangle_{\mathfrak{g}}.$$

With this symplectic structure, the action of the gauge group is Hamiltonian.

$\mathcal{A}(P)$ has a complex structure: $J_{\mathcal{A}} : T\mathcal{A} \to T\mathcal{A}$ given by $a \mapsto *a$. $\mathcal{A}(P)$ is an affine space and $J_{\mathcal{A}}$ looks identical at any A, so it is integrable. It is also compatible with the symplectic structure, giving $\mathcal{A}$ a Kähler structure. The action of $\mathcal{G}$ extends to an action of the complexified gauge group $\mathcal{G}_{\mathbb{C}} := \Gamma(P \times_G G_{\mathbb{C}})$, where G acts on $G_{\mathbb{C}}$ by conjugation. (Details in section 3.2.1.)

In this case the L^2-norm square of the moment map is the Yang-Mills functional $A \mapsto \|F_A\|^2_{L^2(\Sigma)}$. Analogous to proposition 2.1.1, the gradient of this functional is

$$\mathrm{grad}(A \mapsto \|F_A\|^2_{L^2(\Sigma)}) = J_{\mathcal{A}}(-d_A(*F_A)) = d_A^* F_A.$$

2.3 Gauged holomorphic maps

A natural way of generalizing J-holomorphic curves to the equivariant setting is to consider G-equivariant maps from P to X, where P is a principal G-bundle over Σ. In this work, we think of an equivariant map from P to X as a section $u : \Sigma \to P(X)$, where $P(X) := (P \times X)/G$ is a bundle over Σ with fibres isomorphic to X. But 'holomorphicity' depends on the choice of connection A on P. A connection A on P gives a splitting $TP(X) = \pi^*T\Sigma \oplus T^{vert}P(X)$. This, and the complex structure J together define an almost complex structure J_A on $P(X) := (P \times X)/G$. This is actually a complex structure because for $\dim \Sigma = 2$, there are no integrability conditions. $\overline{\partial}_A u = 0$ means that $u : \Sigma \to P(X)$ is holomorphic with respect to J_A. That is, $\overline{\partial}_A u := \frac{1}{2}(du + J_A du \circ j)$. A *gauged holomorphic map* is a pair (A, u) that satisfies $\overline{\partial}_A u = 0$. The space of gauged holomorphic maps from $P \to \Sigma$ to X is called $\mathcal{H}(P, X)$:

$$\mathcal{H}(P, X) := \{(A, u) \in \mathcal{A}(P) \times \Gamma(P(X)) : \overline{\partial}_A u = 0\}$$

Remark 2.3.1 (Integrability Conditions). : The Newlander-Nirenberg theorem states that on a vector bundle, an almost-complex structure $\overline{\partial}_A$ is a complex structure if and only if $\overline{\partial}_A^2 = 0$ or $F_A^{0,2} = 0$ (See [AB83] p.555 or [DK90] theorem 2.1.53). On a 2-dimensional base manifold, this condition is automatically satisfied. This result on vector bundles applies to principal bundles and their associated fibre bundles also. This is seen as follows: Suppose $G \subseteq U(n)$. Let $E = P \times_G \mathbb{C}^n$. If $\overline{\partial}_A^2 = 0$, it gives a holomorphic structure on E. The frame bundle $Fr(E)$ of E is a $GL(n, \mathbb{C})$ bundle on Σ. A local holomorphic frame gives a holomorphic section on $Fr(E)$. This makes $Fr(E)$ a holomorphic bundle. Since $P_{\mathbb{C}} := P \times_G G_{\mathbb{C}}$ is an almost-complex submanifold of $Fr(E)$, it is a complex manifold. The fiber bundle $P(X)$ is also holomorphic with respect to $\overline{\partial}_{J_A}$ - $P(X)$ can be written as $P(X) = P_{\mathbb{C}} \times_{G_{\mathbb{C}}} X$. Holomorphic sections on $P_{\mathbb{C}}$ give holomorphic sections of $P(X)$.

$\Gamma(\Sigma, P(X))$ is the space of smooth sections of $P(X)$. It is an infinite dimensional Frechet manifold whose tangent space at u is $T_u = \Gamma(\Sigma, u^*T^{vert}P(X))$. Locally, after a choice of gauge, the bundle $u^*T^{vert}P(X)$ becomes u^*TX. $\Gamma(\Sigma, P(X))$ has a symplectic structure : for $\xi_i \in T_u$, $(\xi_1, \xi_2) \mapsto \int_\Sigma \omega_X(\xi_1, \xi_2)$. The gauge group $\mathcal{G}(P)$ acts component-wise on $\mathcal{A}(P) \times \Gamma(P(X))$ - for any $g \in \mathcal{G}(P)$,

$$g(A, u) \mapsto (g(A), gu) = ((g^{-1})^* A, gu).$$

Define

$$\mathcal{A}(P) \times \Gamma(\Sigma, P(X)) \to \Omega^2(\Sigma, P(\mathfrak{g}))$$
$$(A, u) \mapsto F_{A,u} := F_A + \Phi(u) dvol_\Sigma$$

Since, Φ is G-equivariant, it induces a map $P(X) \to P(\mathfrak{g})$, which is also denoted Φ, so that $\Phi(u)$ in the above definition is a section of $P(\mathfrak{g}) \to \Sigma$. Under the product symplectic structure, the action of the gauge group on $\mathcal{A}(P) \times \Gamma(\Sigma, P(X))$ is Hamiltonian with moment map

$$(A, u) \mapsto *F_A + u^*\Phi = *F_{A,u}.$$

Since both Σ and $P(X)$ are complex, so is the space $\Gamma(\Sigma, P(X))$. The complex structure is given by $\xi \mapsto J_X \xi$, where $\xi \in \Gamma(\Sigma, u^* T^{vert} P(X))$ This is compatible with the symplectic structure and so, $\mathcal{A}(P) \times \Gamma(P(X))$ has a Kähler structure. $\mathcal{G}_{\mathbb{C}}(P)$ acts component-wise on (A, u) and this action is holomorphic. $\mathcal{H}(P, X)$ also has a Kähler structure because it is a subspace of $\mathcal{A}(P) \times \Gamma(P(X))$ whose tangent space is closed under the action of $J_{\mathcal{A} \times \Gamma(P(X))}$:

- Consider $(a, 0) \in T_{A,u}\mathcal{H}(P, X)$ - that means $a_u(jv) = J_X a_u(v)$ for all $v \in Vect(\Sigma)$. The same condition is satisfied for $*a$, since $*a = a \circ j$.

- Consider $(0, \xi) \in T_{A,u}\mathcal{H}(P, X)$. Locally, $u : \mathbb{C} \to \mathbb{C}^n$ is a holomorphic map and $\xi \in u^* T\mathbb{C}^n$. $(0, \xi) \in T_{A,u}\mathcal{H}(P, X)$ translates to $\overline{\partial}\xi = 0$ and this condition will apply to $i\xi$ also.

Notice that holomorphicity of (A, u) is preserved by the action of $\mathcal{G}_{\mathbb{C}}$.

Analogous to the finite-dimensional case and that of $\mathcal{A}(P)$, we consider the norm square of the moment map $(A, u) \mapsto \|F_{A,u}\|^2_{L^2(\Sigma)}$. The gradient at (A, u) is

$$J_{(A,u)}(*F_{A,u})_{\mathcal{A}(P) \times \Gamma(P(X))} = (d_A^* F_{A,u}, J_X(*F_{A,u})_u). \tag{2.3}$$

Recalling notation, given $\xi \in \Gamma(P(\mathfrak{g}))$, $\xi_u \in u^*T^{vert}P(X)$ denotes the action of ξ on the image of u, i.e. for $x \in \Sigma$, $\xi_u(x) = \xi(x)_{u(x)}$. The gradient flow preserves $\mathcal{G}_{\mathbb{C}}$ orbits and so it preserves $\mathcal{H}(P, X)$.

A useful related quantity is the *twisted derivative* $d_A u$. It is the projection of du onto $T^{vert}P(X)$. Locally, this is described as follows - for a neighbourhood $U_\alpha \subseteq \Sigma$, pick a trivialization. Under this, A can be written as $d + A_\alpha$, $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ and u is given by $u_\alpha : U_\alpha \to X$. Then, $d_A u = du_\alpha + (A_\alpha)_{u_\alpha}$.

Chapter 3

Heat flow

In this section, we prove the long-term existence of the gradient flow of (1.1) for the case when Σ is a compact Riemann surface, possibly with boundary. The gradient flow line starting at $(A_0, u_0) \in \mathcal{H}(P, X)$ is given by the system

$$\begin{aligned} &\frac{d}{dt}A = -\mathrm{d}_A^* F_{A,u}, \quad \frac{d}{dt}u = -J(*F_{A,u})_u \\ F_{A,u}|_{\partial\Sigma} &= 0, \\ &A(0) = A_0, \quad u(0) = u_0. \end{aligned} \tag{3.1}$$

The first two equations come from (2.3).

Theorem 3.0.1. (1.0.1) *For any $(A_0, u_0) \in \mathcal{A}(P) \times \Gamma(P(X))$, the gradient flow (A_t, u_t) exists for all time. $(A_t, u_t) \in C^0_{loc}([0,\infty), H^1 \times C^0)$. There is a family of gauge transformations $g_t \in H^2(\mathcal{G})$ so that $g_t(A_t, u_t)$ is smooth on $[0,\infty) \times \Sigma$.*

In section 3.1, we prove the existence of unique flow lines, and in section 3.2, we show that (A_t, u_t) is smooth modulo gauge. In this part of the dissertation, we do not require (A_0, u_0) to be holomorphic.

3.1 Existence of trajectories

3.1.1 Setting up the system of equations for gradient flow

If $(A(t), u(t))$ are solutions of the system (3.1), then $F_{A(t),u(t)}$ satisfies

$$\begin{aligned} \frac{d}{dt}F_{A(t),u(t)} &= \frac{dF_A}{dt} + \frac{d}{dt}u^*\Phi dvol_\Sigma = d_A\frac{dA}{dt} + u^* d\Phi(\frac{du}{dt})dvol_\Sigma \\ &= -\mathrm{d}_A\mathrm{d}_A^* F_{A,u} + u^* d\Phi(-J(*F_{A,u})_u)dvol_\Sigma. \end{aligned}$$

For 0-dimensional forms, $d_A^* d_A = \Delta_A$, which is an elliptic operator. Writing $F_t := *F_{A(t),u(t)}$, the above equation is equivalent to

$$\frac{dF}{dt} = -\Delta_A F - u^* d\Phi(JF_u). \tag{3.2}$$

Except for the non-linear term $u^* d\Phi(JF_u)$, (3.2) is parabolic. Roughly speaking, once we solve this equation in F, $A_t = A_0 - \int_0^t d_A^* F_{A,u}$ and u_t is obtained by integrating the vector field $(*F_{A,u})_{u_t}$. But unfortunately, A occurs in the term $\int_0^t d_A^* F_{A,u}$ and A_t, u_t occur in the equation in F. So, we need to solve the three equations ((3.1) and (3.2)) as a coupled system.

We know, if (A_t, u_t) is a solution of (3.1), then $(A(t), *F_{A(t),u(t)}, u(t))$ is a solution of

$$\frac{d}{dt}A = *\mathrm{d}_A F, \quad \frac{d}{dt}F = -\mathrm{d}_A^* \mathrm{d}_A F - u^* d\Phi J F_u, \quad \frac{d}{dt}u = JF_u. \tag{3.3}$$

with initial data $A(0) = A_0$, $F(0) = *F_{A_0,u_0}$ and $u(0) = u_0$. Note that $F \in \Gamma(\Sigma, P(\mathfrak{g}))$ is an independent variable in this system, whereas F_A denotes the curvature of the connection A and $F_{A,u} = F_A + u^*\Phi dvol_\Sigma \in \Omega^2(\Sigma, P(\mathfrak{g}))$.

Remark 3.1.1. A solution (A, F, u) of (3.3) will satisfy $*F_{A_t,u_t} = F(t)$

We use A_0 as the base connection, and write any connection on A on P as $A_0 + a$, where $a \in \Omega^1(\Sigma, P(\mathfrak{g}))$. Write $u = \exp_{u_0} \xi$, where $\xi \in \Gamma(\Sigma, T_u^{vert} P(X))$. Then, the system (3.3) becomes

$$\begin{aligned} \frac{d}{dt}a - *\mathrm{d}_{A_0} F &= *[a, F] \\ \frac{d}{dt}F + \Delta_{A_0} F &= -u^* d\Phi J F_u - *[a \wedge *\mathrm{d}_{A_0} F] - [d_{A_0}^* a, F] - [a \wedge *[a, F]] \\ \frac{d}{dt}\xi &= -d\exp(\xi)^{-1}(JF_u) \end{aligned} \tag{3.4}$$

with initial conditions $a(0) = 0$, $F(0) = *F_{A_0,u_0}$, $\xi(0) = 0$. The advantage of writing the system this way is that now, a, F and ξ are just sections of vector bundles over Σ.

Remark 3.1.2. For the last equation to make sense, $d\exp : T_{u_0(x)}X \to T_{\exp_{u_0}\xi(x)}X$ has to be invertible, i.e. for any $x \in \Sigma$, $\exp_{u_0(x)}$ is a diffeomorphism in a neighbourhood of $\xi(x)$. So, we ensure

$$\|\xi\|_{C^0} < \mathrm{inj}_X .$$

inj_X is the injectivity radius of X. This is defined as follows: For any $x \in X$, $\mathrm{inj}_X(x) :=$ radius in T_xX for which the exp map is a diffeomorphism. $\mathrm{inj}_X := \inf_{x\in X} \mathrm{inj}_X(x)$. For a compact manifold $\mathrm{inj}_X > 0$.

3.1.2 Description of Sobolev spaces

To show the existence of a solution, we work in Sobolev spaces of sections of vector bundles. In this section, we use Sobolev spaces $H^s = W^{s,2}$, i.e. $p = 2$. For any real r, s, $t_0 > 0$ and a vector bundle E over Σ, $H^{r,s}([0,t_0]\times\Sigma, E)$ (or $H^{r,s}$ or $H^r(H^s)$) denotes the space of time-dependent (equivalence classes of) sections that are in Sobolev class H^r in time and H^s in space. When r and s are non-negative integers, $H^{r,s}$ is the completion of $C^\infty([0,t_0]\times\Sigma, E)$ under the norm

$$\|\sigma\|_{r,s}^2 := \sum_{i=0}^{r}\sum_{j=0}^{s}\|t_0^{-(r-i)}\frac{d^i}{dt^i}\nabla_{A_0}^s\sigma\|_{L^2(\Sigma\times[0,t_0])}^2.$$

For other exponents, the spaces $H^{r,s}$ are defined by interpolation and duality. For negative Sobolev exponents, the elements of $H^{r,s}$, need not be almost-everywhere defined sections, they are just distributions. The norm $\|\cdot\|_{r,s}$ depends on A_0 but is equivalent for any choice of connection, so that the space $H^{r,s}$ is well-defined independent of the connection. A_0 need not be smooth - if $A_0 \in H^1$, then we can define the spaces $H^{r,s}$ for $s \in [-2,2]$. Detailed definitions and properties of these spaces are given in chapter 5. A crucial property is that although the operator norms depend on the choice of connection A, if the curvature satisfies $\|F(A_0)\|_{L^2} < K$, the operator norms are bounded by constants dependent only on K and independent of A. These Sobolev spaces will be used, for example, when $E = \Omega^k(\Sigma, P(\mathfrak{g}))$.

Another type of Sobolev space we use is $H^r([0,t_0], C^0(\Sigma,E))$ - it is the space of (equivalence classes of) sections that are in Sobolev-class r in time and are C^0 in space. This space has norm

$$\|\sigma\|_{r,C^0} := \sup_{x\in\Sigma}\|\sigma_x\|_{H^r([0,t_0],E_x)}.$$

The way this is defined, it is more appropriate to call it $C^0(\Sigma, H^r([0,t_0],E))$, but we call it $H^r(C^0)$ to preserve our convention of having the time-index outside. This space satisfies the expected embedding properties, for example $H^{r,s} \hookrightarrow H^r(C^0)$ for $s > 1$, but that is not obvious because the spaces $H^{r,s} = H^r([0,t_0], H^s(E))$ are defined with the time and space co-ordinates in a different order. These details will be presented in section 5.5. This space is used, for example, when $E = T^{vert}_{u_0}P(X)$, where it is useful to have the norm be independent of the derivatives of u_0.

With initial value $F(0) \in L^2(\Sigma, P(\mathfrak{g}))$, we expect to solve for F in spaces of the type $H^{\frac{1}{2}+r,-2r}([0,t_0]\times\Sigma, P(\mathfrak{g}))$ (see lemma 5.4.9). We choose the following Banach space to solve the system (3.4).

$$\begin{aligned} U(t_0) = \{(a,F,\xi) | a \in H^{1/2+\epsilon}(H^{1-2\epsilon}), \\ F \in H^{1/2+\epsilon}(H^{-2\epsilon}) \cap H^{-1/2+\epsilon}(H^{2-2\epsilon}_\delta), \xi \in H^{1/2+\epsilon}(C^0)\} \end{aligned}$$

$\epsilon \in (0, 1/12)$ is a fixed number in the rest of this section. For $s > 1$, $H^s_\delta(\Sigma, E) \subseteq H^s$ consists of sections that vanish on the boundary of Σ. We will prove:

Proposition 3.1.3. *Let $A_0 \in H^1$ be a connection on P, and $u_0 \in C^0(\Sigma, P(X))$. Then for any $K > 0$ there exists a $t_0 > 0$ such that if $\|F_{A_0}\|_{L^2} < K$ then the initial value problem* (3.4) *has a unique solution $(a,F,\xi) \in U(t_0)$.*

With this proposition, we can prove the existence of a unique solution for the flow equation for all time:

Proof of theorem 3.0.1. By compactness of Σ, $\|u^*\Phi\|_{L^2} \leq \|\Phi\|_{C^0}\operatorname{Vol}(\Sigma) \leq c$. So, $\|F_A\|_{L^2} \leq \|F_{A,u}\|_{L^2} + c$. Applying proposition 3.1.3 with $K = \|F_{A_0,u_0}\|_{L^2} + c$, we get the flow for a time interval $[0,t_0]$, with $(A(t_0), u(t_0)) \in H^1 \times C^0$. (A_t, u_t) are flow lines for the functional $\|F_{A(t),u(t)}\|^2_{L^2}$, with the functional decreasing along the flow. So, $\|F_{A(t_0),u(t_0)}\|_{L^2} < \|F_{A_0,u_0}\|_{L^2}$, and $\|F_{A(t_0)}\|_{L^2} < K$ and so we can get flow for $[t_0, 2t_0]$. The process is repeated to get flow lines for $t \in [0,\infty)$. $\square$

To continue the discussion, we define certain Banach spaces needed to state intermediate results. The first one $U_P(t_0)$ is a subspace of $U(t_0)$ consisting of sections that vanish at $t = 0$.

$$\begin{aligned} U_P(t_0) =& \{(a, F, \xi) | a \in H_P^{1/2+\epsilon}(H^{1-2\epsilon}), \\ & \qquad F \in H_P^{1/2+\epsilon}(H^{-2\epsilon}) \cap H_P^{-1/2+\epsilon}(H_\delta^{2-2\epsilon}), \xi \in H_P^{1/2+\epsilon}(C^0)\} \\ W(t_0) =& \{(a, F, \xi) | a \in H^{-1/2+\epsilon}(H^{1-2\epsilon}), F \in H^{-1/2+\epsilon}(H^{-2\epsilon}), \xi \in H^{-1/2+\epsilon}(C^0)\} \\ \mathcal{X} =& \{(a_0, F_0, \xi_0) | a_0 \in H^1, F_0 \in H^0, \xi_0 \in C^0\} \end{aligned}$$

Notation 3.1.4. We call $x := (a, F, \xi)$ and $x_i := (a_i, F_i, \xi_i)$.

3.1.3 Outline of proof of Proposition 3.1.3

The terms in the system (3.4) can be broken into 2 parts - the leading order terms and the rest. The leading order terms form an operator

$$\begin{aligned} &L : U(t_0) \to W(t_0) \\ &(a, F, \xi) \mapsto (\frac{d}{dt}a - *\mathrm{d}_{A_0}F, (\frac{d}{dt} + \Delta_{A_0})F, \frac{d}{dt}\xi). \end{aligned}$$

When restricted to $U_P(t_0)$, this operator is invertible. (see lemma 3.1.5)

The terms in the right hand side of (3.4) form a non-linear operator $Q : U(t_0) \to W(t_0)$. We break up the solution into 2 parts $x = x_1 + x_2$, where $x_1 \in U(t_0)$ satisfies $Lx_1 = 0$, and $x_1(0) = x_0$. x_1 can be found uniquely (see lemma 3.1.6). $x_2 \in U_P(t_0)$ satisfies

$$Lx_2 = Q(x_1 + x_2).$$

$\|x_2\|_{W(t_0)}$ can be made small by choosing small t_0. Since L is invertible, we find x_2 using an implicit function theorem argument.

Denote $x_0 \mapsto x_1$ by the operator M

$$M : \mathcal{X} \to U(t_0)$$

$$(a_0, F_0, \xi_0) \mapsto (a_1, F_1, \xi_1)$$

where $x_1(0) = x_0$ and $L(a_1, F_1, \xi_1) = 0$.

We define Q_1, Q_2, Q_3 as follows. The terms in Q are split into Q_1, Q_2, Q_3 in a way that they have a linear, quadratic and cubic bound on them respectively. (See lemma

3.1.7.)

$$Q : U(t_0) \to W(t_0) \quad Q = Q_1 + Q_2 + Q_3$$

$$\begin{aligned}
Q_1 : (a, F, \xi) \mapsto & (0, -u_0^* d\Phi - JF_{u_0}, JF_{u_0}) \\
Q_2 : (a, F, \xi) \mapsto & (*[a, F], - * [a \wedge \wedge d_{A_0} F] \\
& - [d_{A_0}^* a, F] - ((\exp_{u_0} \xi)^* d\Phi(J_X F_{\exp_{u_0} \xi}) - u_0^* d\Phi(J_X F_{u_0})), \\
& - ((d \exp \xi)^{-1} (JF_{\exp_{u_0} \xi}) - JF_{u_0}) \\
Q_3 : (a, F, \xi) \mapsto & (0, -[a \wedge *[a, F]], 0)
\end{aligned}$$

3.1.4 Bounds on L, M and Q

The next 3 lemmas prove that L, M, Q are well-defined operators and that they satisfy appropriate bounds, given $\|F(A_0)\|_{L^2} \leq K$. The constants in these bounds, denoted by c_K are independent of (A_0, u_0) and depend only on K.

Lemma 3.1.5. *L is invertible. For any K, there exists a constant c_K such that if $\|F(A_0)\|_{L^2} \leq K$ then*

$$\|L^{-1}\| \leq c_K.$$

Proof. In matrix form,

$$L = \begin{pmatrix} \frac{d}{dt} & - * d_{A_0} & 0 \\ 0 & \frac{d}{dt} + \Delta_{A_0} & 0 \\ 0 & 0 & \frac{d}{dt} \end{pmatrix}.$$

The operators

$$\frac{d}{dt} : H_P^{1/2+\epsilon, 1-2\epsilon} \to H_P^{-1/2+\epsilon, 1-2\epsilon}$$
$$\frac{d}{dt} : H_P^{1/2+\epsilon}(C^0) \to H_P^{-1/2+\epsilon}(C^0)$$

have inverse $\int_0$, which is bounded by c_K using lemma 5.3.6. For 0-forms, $\Delta_{A_0} = \nabla_{A_0}^* \nabla_{A_0}$ and the operator $\frac{d}{dt} + \nabla_{A_0}^* \nabla_{A_0} : H^{1/2+\epsilon, -2\epsilon} \cap H^{-1/2+\epsilon, 2-2\epsilon} \to H^{-1/2+\epsilon, -2\epsilon}$ has an inverse with norm $\leq c_K$ by lemma 5.4.11.

Last, we look at $*d_{A_0}$. The operator $\nabla_{A_0} : H^{2-2\epsilon} \to H^{1-2\epsilon}$ has norm bounded by c_K for all $t \in [0, t_0]$ (using (5.3)). This induces $\nabla_{A_0} : H^{-1/2+\epsilon,2-2\epsilon} \to H^{-1/2+\epsilon,1-2\epsilon}$ with the same bound on the norm (see (5.16)). On 0-forms, $\nabla_{A_0} = d_{A_0}$ and so $\|*d_{A_0}\| \leq c_K$. □

Lemma 3.1.6. *M is a well-defined operator. For any $K > 0$, there exists a constant c_K such that if $\|F(A_0)\|_{L^2} < K$,*

$$\|M\| \leq c_K t_0^{-\epsilon}.$$

Proof. By the lemma (5.4.9), given $F_0 \in L^2$, the system

$$\frac{d}{dt}F_1 + d_{A_0}^* d_{A_0} F_1 = 0, \quad F_1(0) = F_0$$

has a unique solution $F_1 \in H^{1/2+\epsilon,-2\epsilon} \cap H^{-1/2+\epsilon,2-2\epsilon}$ satisfying

$$\|F_1\|_{H^{1/2+\epsilon,-2\epsilon} \cap H^{-1/2+\epsilon,2-2\epsilon}} \leq c_K \|F_0\|_{H^0}.$$

Define $a_1(t) := a_0 + \int_0^t *d_{A_0} F_1$. Then,

$$\|\int_0^t d_{A_0}^* F_1\|_{\frac{1}{2}+\epsilon,1-2\epsilon} \leq c_K \|F_1\|_{-\frac{1}{2}+\epsilon,2-2\epsilon}$$

because $\|d_{A_0}\| \leq 2\|\nabla_{A_0}\| \leq c_K$ by (5.3) and $\int_0^t$ has norm $\leq c$ by lemma 5.3.6. So,

$$\|a_1\|_{\frac{1}{2}+\epsilon,1-2\epsilon} \leq c_K(\|a_0\|_{H^1} + \|F_0\|_{H^0}).$$

Finally, since $\frac{d\xi_1}{dt} = 0$, we set $\xi_1(t) = \xi_0$, and $\|\xi_1\|_{\frac{1}{2}+\epsilon,C^0} \leq c\|\xi_0\|_{C^0}$ for some constant c. □

Lemma 3.1.7. *Let $x = (a, F, \xi)$. Assume $\|\xi\|_{C^0} \leq \mathrm{inj}_X$ (see remark 3.1.2). Then, $Q : U(t_0) \to W(t_0)$ is a well-defined map. It is differentiable so that $dQ(x) : U(t_0) \to W(t_0)$ is a linear map for each $x \in \Sigma$. If $\|F(A_0)\|_{L^2} < K$, there exist constants c_K so that*

$$\|Q_1 x\|_W \leq c_K t_0^{\frac{1}{2}-2\epsilon}\|x\|_U, \quad \|Q_2(x)\|_W \leq c_K t_0^{\frac{1}{2}-2\epsilon}\|x\|_U^2, \quad \|Q_3(x)\|_W \leq c_K t_0^{\frac{1}{2}-2\epsilon}\|x\|_U^3.$$

The derivatives satisfy

$$\|dQ_1(x)\| \leq c_K t_0^{\frac{1}{2}-2\epsilon}, \quad \|dQ_2(x)\| \leq c_K t_0^{\frac{1}{2}-2\epsilon}(1 + \|x\|_U), \quad \|dQ_3(x)\| \leq c_K t_0^{\frac{1}{2}-2\epsilon}\|x\|_U^2.$$

Putting them together,

$$\|Q(x)\|_W \leq c_K t_0^{\frac{1}{2}-2\epsilon}(1 + \|x\|_U^3), \qquad \|dQ(x)\| \leq c_K t_0^{\frac{1}{2}-2\epsilon}(1 + \|x\|_U^2).$$

Proof. $[a,F]$, $[d_{A_0}a,F]$, $[a,d_{A_0}F]$ and $[a,[a,F]]$ are polynomials of a, F and their derivatives. Consider $[a,F]$. $a\in H^{\frac{1}{2}+\epsilon,1-2\epsilon}$ and $F\in H^{0,1}$ by interpolation (see (5.6) and (5.19)). By the multiplication theorem (5.20), $[a,F]\in H^{-\epsilon,-2\epsilon}\hookrightarrow H^{-\frac{1}{2}+\epsilon,-2\epsilon}$. The last inclusion has norm $c_K t_0^{\frac{1}{2}-2\epsilon}$ (5.15) -

$$\begin{aligned}\|[a,F]\|_{-\frac{1}{2}+\epsilon,-2\epsilon} &\le c_K t_0^{\frac{1}{2}-2\epsilon}\|[a,F]\|_{-\epsilon,-2\epsilon}\le c_K t_0^{\frac{1}{2}-2\epsilon}\|a\|_{\frac{1}{2}+\epsilon,1-2\epsilon}\|F\|_{0,1}\\ &\le c_K t_0^{\frac{1}{2}-2\epsilon}\|x\|_U^2.\end{aligned}$$

That the constants depend only on K follows from proposition 5.2.6. The other polynomial terms are bounded the same way. A bound on the derivatives for these terms is obvious : for example,

$$\|d[a,F]\|_{-\frac{1}{2}+\epsilon,-2\epsilon}\le c_K t_0^{\frac{1}{2}-2\epsilon}(\|a\|_{\frac{1}{2}+\epsilon,1-2\epsilon}+\|F\|_{0,1})\le c_K t_0^{\frac{1}{2}-2\epsilon}\|x\|_U.$$

To discuss the other terms, we define a map: for any $u\in C^0(\Sigma,P(X))$, let $X_u:\Gamma(\Sigma,P(\mathfrak{g}))\to\Gamma(\Sigma,T^{vert}_{u_0}P(X))$ be given by $\xi\mapsto J\xi_u$. The terms $u_0^*d\Phi(JF_{u_0})$ and JF_{u_0} are obtained by the action of linear bundle maps on F. The maps are $d\Phi_{u_0}\circ X_{u_0}$ and X_{u_0} respectively. For example, consider the first of these terms. $d\Phi\circ X_{u_0}$ is in $L^2(\Sigma,P(\operatorname{End}\mathfrak{g}))$ and the norm is independent of u_0. $u_0^*d\Phi F_{u_0}$ can be seen as the tensor product of the sections $d\Phi_{u_0}\circ X_{u_0}$ and F. As earlier $F\in H^{0,1}$. Since $d\Phi\circ X_{u_0}$ is constant in the time direction, it is in $H^{1,0}$. By the multiplication theorem (5.20), $u_0^*d\Phi F_{u_0}\in H^{-\epsilon,-2\epsilon}\hookrightarrow H^{-\frac{1}{2}+\epsilon,-2\epsilon}$.

The remaining 2 terms - $((\exp_{u_0}\xi)^*d\Phi(J_XF_{\exp_{u_0}\xi})-u_0^*d\Phi(J_XF_{u_0}))$ and $((d\exp\xi)^{-1}(JF_{\exp_{u_0}\xi})-JF_{u_0})$ require corollary 6.0.2, which is a result on composition of functions in the space $H^{\frac{1}{2}+\epsilon}(C^0)$. Chapter 6 explains this result in detail. Consider the first of these terms. The bundle map $\xi(x)\mapsto(d\Phi\circ X_{\exp_{u_0}\xi(x)}-d\Phi\circ X_{u_0(x)})$ is continuous and by corollary 6.0.2, it induces a map $H^{\frac{1}{2}+\epsilon}(C^0)(\Sigma,P(\mathfrak{g}))\to H^{\frac{1}{2}+\epsilon}(C^0)(\Sigma,P(\operatorname{End}\mathfrak{g}))$. So $(d\Phi\circ X_{\exp_{u_0}\xi}-d\Phi\circ X_{u_0})\in H^{\frac{1}{2}+\epsilon}(C^0)(P(\operatorname{End}\mathfrak{g}))$ and

$$\|d\Phi\circ X_{\exp_{u_0}\xi}-d\Phi\circ X_{u_0}\|_{\frac{1}{2}+\epsilon,C^0}\le c_K\|\xi\|_{\frac{1}{2}+\epsilon,C^0}.$$

c_K is independent of u_0. By compactness of Σ, $d\Phi\circ X_{\exp_{u_0}\xi}-d\Phi\circ X_{u_0}\in H^{\frac{1}{2}+\epsilon,0}$. Multiplying by $F\in H^{0,1}$, we get the result $((\exp_{u_0}\xi)^*d\Phi(J_XF_{\exp_{u_0}\xi})-u_0^*d\Phi(J_XF_{u_0}))\in$

$H^{-\epsilon,-2\epsilon} \hookrightarrow H^{-\frac{1}{2}+\epsilon,-2\epsilon}$ and,

$$\begin{aligned}
&\|((\exp_{u_0}\xi)^* d\Phi(J_X F_{\exp_{u_0}\xi}) - u_0^* d\Phi(J_X F_{u_0}))\|_{-\frac{1}{2}+\epsilon,-2\epsilon} \\
&\leq c_K t_0^{\frac{1}{2}-2\epsilon} \|((\exp_{u_0}\xi)^* d\Phi(J_X F_{\exp_{u_0}\xi}) - u_0^* d\Phi(J_X F_{u_0}))\|_{-\epsilon,-2\epsilon} \\
&\leq c_K t_0^{\frac{1}{2}-2\epsilon} \|d\Phi \circ X_{\exp_{u_0}\xi} - d\Phi \circ X_{u_0}\|_{\frac{1}{2}+\epsilon,C^0} \|F\|_{0,1} \\
&\leq c_K t_0^{\frac{1}{2}-2\epsilon} \|\xi\|_{\frac{1}{2}+\epsilon,C^0} \|F\|_{0,1} \leq c_K t_0^{\frac{1}{2}-2\epsilon} \|x\|_U^2.
\end{aligned}$$

Similarly, for the second term $((d\exp\xi)^{-1}(JF_{\exp_{u_0}\xi}) - JF_{u_0})$,

$$\|(d\exp\xi)^{-1} \circ X_{\exp_{u_0}\xi} - X_{u_0}\|_{\frac{1}{2}+\epsilon,C^0} \leq c_K \|\xi\|_{\frac{1}{2}+\epsilon,C^0}.$$

Applying interpolation, followed by Sobolev embedding (5.7), $F \in H^{-\epsilon,1+2\epsilon} \hookrightarrow H^{-\epsilon}(C^0)$. By multiplication theorem (5.32),

$$((d\exp(\xi))^{-1} \circ X_{\exp_{u_0}\xi} - X_{u_0})F \in H^{-\epsilon}(C^0) \hookrightarrow H^{-\frac{1}{2}+\epsilon}(C^0).$$

Corollary 6.0.2 also gives differentiability and a bound on the derivative for these terms.

□

3.1.5 Proof of Proposition 3.1.3

To show the existence of gradient flow in $[0,t_0]$, we need to solve

$$L(a_2,F_2,\xi_2) = Q(M(a_0,F_0,\xi_0) + (a_2,F_2,\xi_2))$$

for (a_2,F_2,ξ_2). Similar to Råde's proof ([Råd92]), we use the implicit function theorem in the following form:

Proposition 3.1.8. *Suppose that $F : \mathcal{H}_1 \to \mathcal{H}_2$ is a map of Banach spaces, and $F = F_1 + F_2$ with F_1 an invertible linear operator with inverse satisfying a bound $\|F_1^{-1}\| \leq c$, and $\|DF_2\| \leq 1/2c$ on a convex open set $\mathcal{S} \subset \mathcal{H}_1$ Then,*

a. $F_1 + F_2$ is injective on $\mathcal{S}$, and it is a diffeomorphism of $\mathcal{S}$ onto its image.

b. In addition if $B_\delta \subseteq \mathcal{S}$ and $\|F_2(0)\| \leq \delta/4c$, then there exists a unique solution to $F(x) = 0$ on B_δ.

This is same as theorem A.0.2. A proof is given there.

Proof of proposition 3.1.3. The initial data (a_0, F_0, ξ_0) for (3.4) can be taken so that $\|\xi_0\|_{C^0} < inj_X/2$. Then the construction of M gives $\|\xi_1(t)\|_{C^0} = \|\xi_0\|_{C^0} < inj_X/2$. In this proof, x_i will denote (a_i, F_i, ξ_i).

We'll use theorem 3.1.8 to prove the result, taking $\mathcal{S} = \{(a_2, F_2, \xi_2) \in U_P(t_0) : \|\xi_2\|_{C^0} < inj_X/2\}$, $F_1 = -L$, $F_2(x) := (Q_1 + Q_2 + Q_3)(Mx_0 + x)$ and $c = \|L^{-1}\| = c_K$ We have $x_2 \in \mathcal{S} \Rightarrow \|\xi\|_{C^0} \leq inj_X$ and the estimate on Q_2 applies. Since the map $\pi : U_P(t_0) \to \Gamma(\Sigma, P(\mathfrak{g}))_{C^0}$ that takes (a, F, ξ) to ξ is continuous (by Sobolev embedding (5.17)), there exists $\delta > 0$ such that $B_\delta \subset \mathcal{S}$. For $\|x_2\| < \delta$,

$$\|dQ(Mx_0 + x_2)\| \leq c_K t_0^{\frac{1}{2}-2\epsilon}(1 + \|Mx_0 + x_2\|^2) \leq c_K t_0^{\frac{1}{2}-4\epsilon}(1 + \|x_0\|^2)$$

To apply the theorem we need to pick t_0 such that,

- $\|dQ(Mx_0 + x_2)\| < 1/2c$, i.e. $t_0^{\frac{1}{2}-4\epsilon}(\|x_0\| + 1)^2 \leq 1/c_K$ and
- $\|F_2(0)\| = \|Q(Mx_0)\| = c_K t_0^{\frac{1}{2}-3\epsilon}(\|x_0\| + 1) < \delta/4c_K$.

Both these conditions can be met by a small enough value of t_0, that is dependent only on K. This proves the existence part of proposition 3.1.3.

Regularity Next, we prove that there exists a solution with extra regularity $a \in C^0(H^1)$, $F \in C^0(L^2) \cap L^2(H^1)$ and $\xi \in C^0$. This is essential to prove that it is the only solution in $U(t_0)$.

First, we look at (a_1, F_1, ξ_1), using remark (5.4.10), $F_1 \in C^0(L^2) \cap H^{0,1}$. Since F_1 satisfies $(\frac{d}{dt} + \nabla_{A_0}^* \nabla_{A_0})F_1 = 0$, we get $F_1(t) - F_0 = \nabla_{A_0}^* \nabla_{A_0} \int_0 F_1 \in C^0(L^2)$. By elliptic regularity (see proposition 5.4.2), $\int_0 F_1 \in C^0(H^2)$. So, $a_1(t) = a_0 + \int_0 d_{A_0}^* F_1 \in C^0(H^1)$.

It can be checked that lemmas 3.1.5, 3.1.6 and 3.1.7 hold with the following stronger spaces

$$\tilde{U}(t_0) = \{(a, F, \xi) | a \in H^{\frac{1}{2}+\epsilon, 1-2\epsilon} \cap H^{\frac{1}{2},1}, F \in H^{\frac{1}{2}+\epsilon,-2\epsilon} \cap H^{-\frac{1}{2},2}, \xi \in H^{\frac{1}{2}+\epsilon}(C^0)\}$$
$$\tilde{U}_P(t_0) = \{(a, F, \xi) | a \in H_P^{\frac{1}{2}+\epsilon, 1-2\epsilon} \cap H_P^{\frac{1}{2},1}, F \in H_P^{\frac{1}{2}+\epsilon,-2\epsilon} \cap H_P^{-\frac{1}{2},2}, \xi \in H_P^{\frac{1}{2}+\epsilon}(C^0)\}$$
$$\tilde{W}(t_0) = \{(a, F, \xi) | a \in H^{-\frac{1}{2}+\epsilon, 1-2\epsilon} \cap H_P^{-\frac{1}{2},1}, F \in H^{-\frac{1}{2}+\epsilon,-2\epsilon} \cap H_P^{-\frac{1}{2},0},$$
$$\xi \in H^{-\frac{1}{2}+\epsilon}(C^0)\}.$$

So, there exists a solution of (3.4) in $\tilde{U}(t_0)$. Using this, we can get improved estimates for the right hand side of (3.4). For example, $a \in H^{\frac{1}{2},1} \implies \nabla_{A_0} a \in H^{\frac{1}{2},0}$. By interpolation $F \in H^{\frac{1}{4}-\epsilon,\frac{3}{2}+2\epsilon}$. By the multiplication theorem, $[\nabla_{A_0} a, F] \in H^{-\frac{1}{4}-2\epsilon,0}$. Similarly we estimate all terms in the right hand side of (3.4) to get

$$\begin{cases} \frac{d}{dt}a_2 + \mathrm{d}^*_{A_0} F_2 \in H_P^{-\frac{1}{4}-2\epsilon,1} \\ \frac{d}{dt}F_2 + \nabla^*_{A_0}\nabla_{A_0} F_2 \in H_P^{-\frac{1}{4}-2\epsilon,0} \end{cases}$$

By lemma 5.4.11, we know $F_2 \in H_P^{\frac{3}{4}-2\epsilon,0} \cap H_P^{-\frac{1}{4}-2\epsilon,2} \hookrightarrow C^0(L^2)$. So, $F = F_1 + F_2 \in C^0(L^2)$. Also, since $d^*_{A_0} F_2 \in H_P^{-\frac{1}{4}-2\epsilon,1}$, $a_2 \in H_P^{\frac{3}{4}-2\epsilon,1} \hookrightarrow C^0(H^1)$. Therefore, $a = a_1 + a_2 \in C^0(H^1)$.

Uniqueness We now prove uniqueness by contradiction. Suppose $x = (a, F, \xi)$ and $x' = (a', F', \xi')$ are two solutions to (3.4) for some $t_0 > 0$ with the same initial data (a_0, F_0, ξ_0), $a_0 \in L^2$, $F_0 \in H^1$, $\xi_0 \in C^0$.

$$a \in H^{\frac{1}{2}+\epsilon,1-2\epsilon} \cap C^0(H^1), F \in H^{\frac{1}{2}+\epsilon,-2\epsilon} \cap H^{-\frac{1}{2}+\epsilon,2-2\epsilon} \cap C^0(L^2), \xi \in H^{\frac{1}{2}+\epsilon}(C^0)$$

$$a' \in H^{\frac{1}{2}+\epsilon,1-2\epsilon}, F' \in H^{\frac{1}{2}+\epsilon,-2\epsilon} \cap H^{-\frac{1}{2}+\epsilon,2-2\epsilon}, \xi' \in H^{\frac{1}{2}+\epsilon}(C^0)$$

Assume $x \neq x'$. Let t_1 be the largest number such that the restrictions of x and x' to $\Sigma \times [0, t]$ are identical. Since the solutions are in $C^0([0, t_1], H^1 \times L^2 \times C^0)$, $(a(t_1), F(t_1), \xi(t_1))$ is well-defined. Then, (a, F, ξ) and (a', F', ξ') solve the initial value problem (3.4) on $\Sigma \times [t_1, t_0]$ with initial data $(a(t_1), F(t_1), \xi(t_1))$. Therefore, without loss of generality, we may assume that $t_1 = 0$.

We can split $x = x_1 + x_2$, where $Lx_1 = 0$, $x_1(0) = (a_0, F_0, \xi_0)$ and $x_2 \in U_P(t_0)$. Similarly $x' = x_1' + x_2'$. Since M is uniquely defined $x_1 = x_1'$. So, now both x_2 and x_2' are solutions of $Lx = Q(x_1 + x)$ in $U_P(t_0)$. By Sobolev embedding, both $\xi_2, \xi_2' \in H_P^{\frac{1}{2}+\epsilon}(C^0) \hookrightarrow C_P^0([0, t_0], C^0)$. There exists $0 < t < t_0$ such that $\|\xi_2\|_{C_P^0([0,t],C^0)}$ and $\|\xi_2'\|_{C_P^0([0,t],C^0)} < \mathrm{inj}_X$. So, the restrictions of x_2 and x_2' to $U_P(t)$ are in $\mathcal{S}$. But, $-Lx+Q(x_1+x)$ is injective on $\mathcal{S}$, so $x_2 = x_2'$ in $U_P(t)$. This leads to a contradiction. $\square$

We prove another result about the regularity of F which comes in handy in the next section.

Lemma 3.1.9. *In proposition 3.1.3, if* $F(0) \in H^{1+2\epsilon}$*, then* $F \in C^0([0,t_0]\times\Sigma)$

Proof. Recall that $x = (a, F, \xi)$ was obtained as the sum $x = x_1 + x_2$ where $x_1 = Mx_0$ and x_2 is the solution of $Lx_2 = Q(Mx_0 + x_2)$. We have

$$\begin{aligned} Mx_0 + x_2 &\in H^{\frac{1}{2}+\epsilon,1-2\epsilon} \times (H^{\frac{1}{2}+\epsilon,-2\epsilon} \cap H^{-\frac{1}{2}+\epsilon,2-2\epsilon}) \times H^{\frac{1}{2}+\epsilon}(C^0) \\ \implies Lx_2 = Q(Mx_0 + x_2) &\in H^{-\epsilon,1-2\epsilon} \times H^{-\epsilon,-2\epsilon} \times H^{-\epsilon}(C^0) \\ \implies x_2 &= H^{1-\epsilon,1-2\epsilon} \times (H^{1-\epsilon,-2\epsilon} \cap H^{-\epsilon,2-2\epsilon}) \times H^{\frac{1}{2}+\epsilon}(C^0) \end{aligned}$$

The regularity of x_1 can be brought up to the same level and we get

$$\|F_1\|_{1-\epsilon,-2\epsilon\cap-\epsilon,2-2\epsilon} \leq c_K \|F_0\|_{H^{1-4\epsilon}}.$$

$a_1(t) = a_0 + \int_0^t *d_{A_0}F_1 \in H^{1-\epsilon,1-2\epsilon}$. This yields,

$$x = x_1 + x_2 \in H^{1-\epsilon,1-2\epsilon} \times (H^{1-\epsilon,-2\epsilon} \cap H^{-\epsilon,2-2\epsilon}) \times H^{\frac{1}{2}+\epsilon}(C^0)$$

Repeating this process, we get

$$x \in H^{1+2\epsilon,1-2\epsilon} \times (H^{1+2\epsilon,-2\epsilon} \cap H^{2\epsilon,2-2\epsilon}) \times H^{\frac{1}{2}+\epsilon}(C^0).$$

By interpolation, $F \in H^{\frac{1}{2}+\epsilon/2,1+\epsilon} \hookrightarrow C^0(H^{1+\epsilon}) \hookrightarrow C^0([0,t_0]\times\Sigma)$. ☐

3.2 Smooth flow modulo gauge

Since the gradient flow preserves complex gauge orbit, we can write

$$(A_t, u_t) = g_t(A_0, u_0) \quad g_t \in \mathcal{G}_{\mathbb{C}}.$$

Then the system of equations (3.1) can be written as a single equation in g_t.

$$\frac{dg_t}{dt} g_t^{-1} = -iF_t. \tag{3.5}$$

To write F_t in terms of g_t, we need some preliminaries. We follow [Don85].

3.2.1 Action of $\mathcal{G}_{\mathbb{C}}$ on $\mathcal{A}(P)$

First we assume that $G = U(n)$, the arguments naturally extend to general $G \subseteq U(n)$. We work with an associated complex vector bundle of P. The standard linear action of G on $\mathbb{C}^n$ will preserve a Hermitian metric on $\mathbb{C}^n$. That means the associated bundle $E := P \times_G \mathbb{C}^n$ has a Hermitian metric and this is compatible with any connection $A \in \mathcal{A}(P)$. There is a bijection between the space of connections $\mathcal{A}(P)$ and the space of holomorphic structures $\mathcal{C}(E)$ on E. A connection A defines a holomorphic structure on E with Dolbeault operator given by $\overline{\partial}_A := \frac{1}{2}(d_A + J_E d_A \circ j)$. For the reverse direction, a holomorphic structure on E together with the fixed metric determine a unique connection on E (see [GH94]). This corresponds to a connection on P because of the metric compatibility. This correspondence is described in [AB83] and [Don85] - where they show that after choosing a Hermitian metric on a complex vector bundle $E \to \Sigma$, there is a correspondence $\mathcal{A} \to \mathcal{C}$. In our case the Hermitian metric on E is determined by the construction of E.

Locally this corresponds to an isomorphism

$$\begin{aligned} T_A\mathcal{A} = \Omega^1(\mathfrak{g}) \to& \Omega^{0,1}(\mathfrak{g}_{\mathbb{C}}) = T_{\overline{\partial}_A}\mathcal{C} \\ a \mapsto& a^{0,1} = \frac{1}{2}(a + J_E a \circ j). \end{aligned}$$

Note that $\mathcal{C}$ has a complex structure : for $c \in \Omega^{0,1}(\mathfrak{g}_{\mathbb{C}})$, $J_{\mathcal{C}} c = ic = c \circ j = *c$. This pulls back to a complex structure on $\mathcal{A}$: $J_{\mathcal{A}} a = *a$. The complex gauge group acts on $\mathcal{C}(E)$.

$$\overline{\partial}_{g(A)} = g \circ \overline{\partial}_A \circ g^{-1}, \quad g \in \mathcal{G}_{\mathbb{C}}. \tag{3.6}$$

Infinitesimally the action is $\xi \mapsto -\overline{\partial}_A \xi$, where $\xi \in Lie(\mathcal{G}_{\mathbb{C}})$. This is complex-linear - $\overline{\partial}_A(i\xi) = J_E \overline{\partial}\xi$. This action pulls back to a $\mathcal{G}_{\mathbb{C}}$ action on $\mathcal{A}(P)$, that extends the $\mathcal{G}$ action.

The Hermitian metric on E together with the Riemannian metric on Σ give a metric on the spaces $\Omega^k(\Sigma, E)$. For any connection A, let $(\overline{\partial}_A)^*$ denote the formal adjoint of $\overline{\partial}_A$ under this metric. It satifies $\partial_A = *(\overline{\partial}_A)^* *$. Applying this identity to the the connection $g(A)$, we get

$$\partial_{g(A)} = (g^*)^{-1} \circ \partial_A \circ g^*, \quad A \in \mathcal{A}(P),\, g \in \mathcal{G}_{\mathbb{C}}, \tag{3.7}$$

where g^* is the adjoint of g under the metric fixed on $\Gamma(\Sigma, E)$. $d_{g(A)} = \overline{\partial}_{g(A)} + \partial_{g(A)}$ is the unique connection on E that is compatible with the fixed metric and the holomorphic structure $\overline{\partial}_{g(A)}$.

For $g \in \mathcal{G}_{\mathbb{C}}$, define $h(g) := g^*g$. Observe

$$\mathcal{G} = \{g \in \mathcal{G}_{\mathbb{C}} | h(g) = g^*g = \mathrm{Id}\}.$$

From (3.6) and (3.7), $g^{-1} \circ d_{g(A)} \circ g = \overline{\partial}_A + h^{-1} \circ \partial_A \circ h$. On vector bundles, the curvature $F_A = d_A^2$. It transforms as

$$\begin{aligned} g^{-1} \circ F_{g(A)} \circ g &= F_A + \overline{\partial}_A(h^{-1}\partial_A h) \\ &= F_A + h^{-1}(\overline{\partial}_A \partial_A h - (\overline{\partial}_A h)h^{-1}\partial_A h). \end{aligned}$$

$F_{g(A)}$ is also the curvature of the connection $g(A)$ on P.

For a general compact Lie group G, there is a $U(n)$ into which it can be mapped injectively. So, we work on the bundle $P \times_G \mathbb{C}^n$. We look upon the space of G-connections on E as a subset of $U(n)$- connections. The group action preserves G-actions, because the infinitesimal action $-\overline{\partial}_A \xi$ is in $\Omega^{0,1}(\mathfrak{g}_{\mathbb{C}})$. All the relations carry over to the general case.

3.2.2 Gauge-invariant version of the flow equations

If $h_t = g_t^* g_t$, the evolution equation (3.5) can be written as

$$\begin{aligned} \frac{dh_t}{dt} &= -2ig_t^* F_t g_t \\ &= -2ih_t(*F_{A_0} + *\overline{\partial}_0(h_t^{-1}(\partial_0 h_t)) + g_t^{-1}u_t^*\Phi g_t). \end{aligned} \tag{3.8}$$

Note that replacing g_t by $g_t k_t$, $k_t \in \mathcal{G}$ doesn't alter h_t. Also, the term $g_t^{-1}u_t^*\Phi g_t$ is $\mathcal{G}$-invariant - it is unchanged if changing u_t to $k_t u_t$ and g_t to $g_t k_t$. So, solving the above equation solves the gradient flow equation modulo gauge. For example, if we let $g'_t := h_t^{1/2}$, then $g'_t(A_0, u_0)$ differs from (A_t, u_t) by a family $k_t \in \mathcal{G}$. For 0-forms $\Delta_{A_0} = d_{A_0}^* d_{A_0} = *\overline{\partial}_0 \partial_0$ and so, (3.8) can be modified to

$$\begin{aligned} \frac{dh_t}{dt} + \Delta_{A_0} h_t &= -2ih_t\{*F_{A_0} + *(\overline{\partial}_0 h_t)h_t^{-1}(\partial_0 h_t) + g_t^{-1}u_t^*\Phi g_t\} \\ h(0) &= \mathrm{Id}, \\ h|_{\partial\Sigma} &= \mathrm{Id}\,. \end{aligned} \tag{3.9}$$

Proposition 3.2.1. *If (A_0, u_0) are smooth, then the solution of* (3.9) $h_t : [0, t_0] \to \mathcal{G}_{\mathbb{C}}$ *is smooth. Hence the gradient flow (A_t, u_t) computed in proposition 3.1.3 is smooth modulo gauge.*

Proof. From lemma 3.1.9, we know $F_t = *F_{A_t} + u_t^*\Phi$ is in $C^0([0, t_0] \times \Sigma)$. So, the solution g_t of (3.5) is in $C^0([0, t_0] \times \Sigma)$. Hence $h_t = g_t^* g_t$ is also in $C^0([0, t_0] \times \Sigma)$ and by the above discussion, h_t is a weak solution of (3.9). Since (3.9) is a non-linear parabolic equation, the regualarity of h_t can be improved by bootstrapping. $\|h_t\|_{C^0}$ is small enough that its image is contained in $U \subseteq G_{\mathbb{C}}$ for which there is a holomorphic chart $U \to \mathbb{C}^n$. So, we may think of h as a time-dependent section of a vector bundle over Σ. Also, I assume Id in $G_{\mathbb{C}}$ is mapped to 0 in $\mathbb{C}^n$.

For the bootstrapping, we need the following 3 observations

a. If $h_t \in L^2(H^s) \cap H^{s/2}(L^2) \cap C^0$, then the r.h.s of (3.9) is in $L^2(H^{s-1}) \cap H^{(s-1)/2}(L^2)$. This is true because - if u_0 is smooth, $h_t \mapsto g_t^{-1}(u_t^*\Phi)g_t$ is a composition operator (here $u_t = g_t u_0$). By corollary 6.0.2, it is a map from $L^2(H^s) \cap H^{s/2}(L^2) \cap C^0 \to L^2(H^s) \cap H^{s/2}(L^2)$. For the other terms we use multiplication theorem (5.20) etc.

b. If the r.h.s. of (3.9) is in $L^2(H^{2s}) \cap H^s(L^2)$, then $h_t \in L^2(H^{2s+2}) \cap H^{s+1}(L^2)$ by theorem 5.4.13.

c. $h_t \in C^0$ implies r.h.s. of (3.9) is in $L^2(H^{-1})$. By lemma 5.4.11, $h_t \in H^1(H^{-1}) \cap L^2(H^1)$.

The last observation provides the base case for induction. Using, the first 2 observations, we can inductively prove the statement $h_t \in L^2(H^m) \cap H^{m/2}(L^2)$ for any integer $m \geq 0$. So, h_t is a smooth solution of (3.9). Let $g_t := \sqrt{h_t} \in \mathcal{G}_{\mathbb{C}}(\Sigma)$, then $g_t(A_0, u_0)$ is smooth on $[0, \infty) \times \Sigma$ and is gauge equivalent to (A_t, u_t). $\square$

Chapter 4

Convergence

We now consider convergence behaviour of the flow (A_t, u_t). Theorem 4.2.1 is the first result of this section. There is a sequence $\{t_i\}$ so that connections A_{t_i} converge weakly in H^2 $(W^{2,2})$ modulo gauge. u_{t_i} converge in the sense of Gromov, i.e in the limit, sphere bubbles develop in the vertical fibres of $P(X)$. The name 'Gromov convergence' is taken from [MS04], where it refers to a similar notion of convergence for J-holomorphic curves.

In the last section 4.3, we restrict ourselves to the case when Σ has boundary. Then, the limit (A_∞, u_∞) is in the same complex gauge orbit of the gradient flow - and so, there is no bubbling in the limit and the limit is unique up to gauge.

4.1 Gauged holomorphic maps as J-holomorphic curves

As discussed in section 2.3, a connection A determines a complex structure J_A on $P(X)$. If $A_i \to A_\infty$ in C^∞, then $J_{A_i} \to J_{A_\infty}$. If (A_i, u_i) is a gauged holomorphic map, then u_i is a J_{A_i}-holomorphic curve. We may now expect that the Gromov compactness result in [MS04] is applicable on $\{u_i\}$. In this section, we discuss the hypotheses for Gromov compactness in the context of gauged holomorphic maps - we construct a symplectic form on $P(X)$ that tames J_{A_i}, and we describe an energy functional on u_i. This can be compared to [Ott09], where similar ideas are used to study Gromov convergence of vortices.

4.1.1 Symplectic form on $P(X)$

There is a symplectic form ω_X on X and ω_Σ on Σ. Let A be a C^1 connection on P. Consider the following 2-form on $P \times X$,

$$\tilde{\sigma}_A = \pi_2^* \omega_X + d\langle A, \Phi \rangle.$$

$\tilde{\sigma}_A$ is closed, G-invariant and vanishes on the orbits of G. So, it descends to $\sigma_A \in \Omega^2(P(X))$ which is also closed. We write out an explicit expression for σ_A. For a point $[p, x] \in P(X)$,

$$T_{[p,x]} P(X) = (T_p P \times T_x X)/\{(\xi_p, -\xi_x) : \xi \in \mathfrak{g}\}.$$

Using a connection A, a vector in $T_{[p,x]} P(X)$ can be split as a pair $[v, w]$, where $v \in T_p P$ is in $\ker A$ and $w \in T_x X$. We first work with $\tilde{\sigma}_A$,

$$\tilde{\sigma}_A([v_1, w_1], [v_2, w_2]) = \omega(w_1, w_2) + (\langle dA, \Phi \rangle + \langle A, d\Phi \rangle)((v_1, w_1), (v_2, w_2)).$$

The last term vanishes because v_i are horizontal. Further, $dA = F_A - \frac{1}{2}[A \wedge A]$ and $[A(v_1) \wedge A(v_2)] = 0$. So,

$$\sigma_A((v_1, w_1), (v_2, w_2)) = \omega(w_1, w_2) + \langle F_A(v_1, v_2), \Phi \rangle. \tag{4.1}$$

σ_A is non-degenerate in the vertical direction, but not necessarily in the horizontal direction. But, there is a constant c_A (dependent on A and Φ) so that

$$c_A |\omega_\Sigma(v_1, v_2)| > \langle F_A(v_1, v_2), \Phi \rangle \quad \text{for all } v_1,\, v_2 \in T\Sigma.$$

Define

$$\omega_A := \pi_2^* \omega_X + d\langle A, \Phi \rangle + c_A \pi_1^* \omega_\Sigma. \tag{4.2}$$

This is a symplectic form on $P(X)$. So, we have the result:

Lemma 4.1.1. *Let A be a C^1 connection on P, then,*

$$\omega_A := \pi_2^* \omega_X + d\langle A, \Phi \rangle + c_A \pi_1^* \omega_\Sigma$$

is a symplectic form on $P(X)$. c_A is a constant satisfying

$$c_A |\omega_\Sigma(v_1, v_2)| > |\langle F_A(v_1, v_2), \Phi \rangle|$$

for all v_1, $v_2 \in T\Sigma$. If $[v,w]$ is a vector field on $P(X)$ split by A, i.e. $A(v)=0$, then

$$\omega_A([v_1,w_1],[v_2,w_2]) = \omega_X(w_1,w_2) + \langle F_A(v_1,v_2),\Phi\rangle + c_A\omega_\Sigma(v_1,v_2).$$

It is easily seen that J_A is ω_A-tame. Since ω_A-tameness is an open condition, connections C^0-close to A also give ω_A-tame complex structures. This lemma proves it rigorously.

Lemma 4.1.2. *Given a C^1 connection $A_0 \in \mathcal{A}(P)$, there is a constant $c_1(A_0)$, so that for any C^0 connection satisfying $\|A - A_0\|_{C^0} < c_1$, J_A is ω_{A_0}-tame.*

Proof. A connection A can be written as $A = A_0 + a$, where $a \in \Omega^1(\Sigma, P(\mathfrak{g}))$. Let $[v,w] \in \mathrm{Vect}(P(X))$. The splitting is according to A_0 - that is $A_0(v) = 0$. In an A-splitting the vector field is $[v - a(v)_P, w + a(v)_X]$. Using the definition of ω_{A_0},

$$\begin{aligned}\omega_{A_0}([v,w],J_A[v,w]) &= c_{A_0}\omega_\Sigma(\pi_1 v, j_\Sigma(\pi_1 v)) + \langle F_{A_0}(\pi_1 v, j_\Sigma(\pi_1 v)),\Phi\rangle \\ &\quad + \omega_X(w + a(v)_X, J_X(w + a(v)_X)).\end{aligned} \tag{4.3}$$

For a fixed vector field $[v,w]$, $\omega_{A_0}([v,w],J_A[v,w])$ varies continuously with a. Since $\|\omega_{A_0}([v,w],J_{A_0}[v,w])\|_{C^0} > 0$, there is a constant $c_{[v,w]}$ so that $\|\omega_{A_0}([v,w],J_A[v,w])\|_{C^0}$ is positive if $\|a\|_{C^0} < c_{[v,w]}$. $TP(X)$ can be spanned by a finite number of vector fields, so the lemma follows. □

4.1.2 Energy of holomorphic curves

Gromov compactness uses an energy functional defined in [MS04]:

Definition 4.1.3. Let A_0 be a C^1 connection and A a C^0 connection. The *energy* of a J_A-holomorphic curve $u : \Sigma \to P(X)$ is

$$E_{J_A,\omega_{A_0}}(u) := \frac{1}{2}\int_\Sigma |du|^2_{J_A,\omega_{A_0}} dvol_\Sigma.$$

$(\cdot,\cdot)_{J_A,\omega_{A_0}} = \omega_{A_0}(\cdot, J_A\cdot)$ is the Riemannian metric on $P(X)$.

Remark 4.1.4. If J_A is ω_{A_0}-tame and $u : \Sigma \to P(X)$ is J_A-holomorphic, then $E_{J_A,\omega_{A_0}}(u) = \int_\Sigma u^*\omega_{A_0}$. This is proved in [MS04].

In our application, we'll need a bound on $E_{J_A}(u)$ in terms of $\|d_A u\|_{L^2}$, this follows from the lemma:

Lemma 4.1.5. *For a C^1 connection A_0 on P, there is a constant $c_2(A_0)$ so that for any C^0 connection A satisfying $\|A - A_0\|_{C^0} < c_2$,*

$$\omega_{A_0}([v,w], J_A[v,w]) \leq \omega_X(w + a(v)_X, J_X(w + a(v)_X)) + 2c_{A_0}\omega_\Sigma(v, jv) \tag{4.4}$$

for any vector field $[v,w]$ on $P(X)$. $[v,w]$ is split by A_0, that is $A_0(v) = 0$ and $w \in TX$. $A = A_0 + a$.

Proof. First we look at $A = A_0$. By the definition of ω_{A_0},

$$\omega_{A_0}([v,w], J_{A_0}[v,w]) \leq \omega_X(w, J_X w) + 2c_{A_0}\omega_\Sigma(v, jv).$$

By (4.3), we see that both sides of (4.4) vary continuously with a. So, for any vector field $[v,w]$, there is a constant $c_{[v,w]}$ so that the inequality (4.4) is satisfied for $\|a\|_{C^0} < c_{[v,w]}$. By picking a finite set of vector fields that span $TP(X)$, and taking c_2 to be the maximum of all these $c_{[v,w]}$ the lemma is proved. □

Corollary 4.1.6. *If $\|A - A_0\|_{C^0} < c_2$, then for any J_A-holomorphic $u : \Sigma \to P(X)$,*

$$E_{J_A,\omega_{A_0}}(u) \leq \|d_A u\|_{L^2(\Sigma)} + c_{A_0} vol(\Sigma).$$

We assume A_0 is C^1, A is C^0 and u is C^1.

Proof. Let χ be a non-vanishing vector field on Σ. If a non-vanishing vector field does not exist on Σ, we'll work in co-ordinate patches. We may assume that $|\chi(x)| = 1$ for all $x \in \Sigma$. Apply lemma 4.1.5 with $[v,w] := du(\chi)$. Thinking of u as a section of $P(X)$, $du : T\Sigma \to T_u P(X)$ and $d\pi(v) = d\pi(du(\chi) - A_0(du(\chi))_P) = \chi$. The lemma says

$$\begin{aligned}
\omega_{A_0}([v,w], J_A[v,w]) &\leq \omega_X(w + a(v)_X, J_X(w + a(v)_X)) + 2c_{A_0}\omega_\Sigma(\chi, j\chi) \\
&= \omega_X(w + a(v)_X, J_X(w + a(v)_X)) + 2c_{A_0}. \\
\implies |du|^2_{J_A,\omega_{A_0}} &\leq |d_A u|^2 + 2c_{A_0}.
\end{aligned}$$

All the terms are positive. Integrating both sides over Σ proves the corollary. □

4.1.3 Gromov convergence in $P(X)$

As in the case of J-holomorphic curves, bubbling occurs in the space $\mathcal{H}(P, X)$. Since this space consists only of sections $u : \Sigma \to P(X)$, bubbling happens on the fibers of $P(X)$. These structures with bubbles are called *nodal gauged holomorphic maps* and next, we describe them precisely. For this discussion, we assume that Σ is closed, since in our applications, bubbling will be ruled out in the case when Σ has boundary. A *rooted tree* $T = (V, E)$ is a connected acyclic graph, with a distinguished root vertex labeled 0. That is, $V = \{0\} \cup V_s$

Definition 4.1.7. A *nodal Riemann surface* $C = (\Sigma, \underline{z} := \{z_{\alpha\beta}\}_{\alpha E\beta}, T)$ is modeled on a rooted tree $T = (V, E)$. It consists of a closed compact Riemann surface Σ for the vertex 0 and a copy of $\mathbb{P}^1$ for each vertex in V_S up to an equivalence $\sim$. For each edge $(\alpha, \beta) \in E$, $z_{\alpha\beta} \in (\mathbb{P}^1)_\alpha$, $z_{\beta\alpha} \in (\mathbb{P}^1)_\beta$ and $z_{\alpha\beta} \sim z_{\beta\alpha}$. These are the *singular points* of C and we denote $Z_\alpha := \{z_{\alpha\beta}\}_{\alpha E\beta}$.

Notation 4.1.8. If α and β do not share an edge, then $z_{\alpha\beta} \in Z_\alpha$ corresponds to the first edge on the path from α to β.

Definition 4.1.9. A *nodal gauged holomorphic map* from Σ to X consists of the data $(P, A, C, u, \underline{z})$. C is a nodal curve with singular points $\underline{z}$ and principal component Σ. P is a principal G-bundle on Σ with connection A. u is a J_A-holomorphic map from $C \to P(X)$. The principal component of u is a section of $P(X)$ - that is $\pi_{P(X)} \circ u|_\Sigma = \mathrm{Id}_\Sigma$. The other components map to a fibre of $P(X)$ - for any $\alpha \in V_S$, $\pi_{P(X)} \circ u_\alpha = \textit{constt}$. This map satisfies a stability condition : $|Z_\alpha| \geq 3$ if $\alpha \in V_S$ and u_α is constant.

Definition 4.1.10 (Gromov Convergence)**.** Let ω be a symplectic form on $P(X)$ and A_i a sequence of connections on P converging to A_∞ in C^∞ so that each J_{A_i} is ω-tame. Let $u_i : \Sigma \to P(X)$ be J_{A_i}-holomorphic sections of $P(X)$. Then u_i *Gromov converges* to a nodal gauged holomorphic map u_∞ if there exist sequences of rational maps $\phi_i^\alpha : (\mathbb{P}^1)_\alpha \to \Sigma$ for $\alpha \in V_s$ that satisfy the following:

Map
- u_i converges to u_∞^0 in C^∞ on compact subsets of $\Sigma \backslash Z_0$.
- $\forall \alpha \in V_S$, $u_i \circ \phi_i^\alpha$ converges to u_∞^α in C^∞ on compact subsets of $(\mathbb{P}^1)_\alpha \backslash Z_\alpha$.

- For the nodal map u_∞, the connection on the principal component Σ is A_∞.

Rescaling
- $\forall \alpha,\ \beta \in V_s$, $(\phi_i^\beta)^{-1} \circ \phi_i^\alpha : (\mathbb{P}^1)_\alpha \to (\mathbb{P}^1)_\beta$ converges to $z_{\beta\alpha}$ in C^∞ on compact subsets of $\mathbb{P}^1_\alpha \backslash \{z_{\alpha\beta}\}$.
- $\forall \alpha \in V_s$, ϕ_i^α C^∞-converges to $z_{0\alpha}$ on compact subsets of $\mathbb{P}^1_\alpha \backslash \{z_{\alpha 0}\}$.

Energy $\lim_{i\to\infty} E_{\omega,J(A_i)}(u_i) = E_{\omega,J(A_\infty)}(u^0_\infty) + \sum_{\alpha\in V_s} E_{\omega,J_X}(u^\alpha_\infty)$.

The Gromov convergence result in [MS04] is applicable to a sequence of holomorphic sections of $P(X)$. We re-state the result in our context.

Proposition 4.1.11. *Let Σ be a closed compact Riemann surface and (A_i, u_i) a sequence of gauged holomorphic maps. Suppose, $A_i \to A_\infty$ in C^∞, and*

$$sup_i E_{J(A_i),\omega(A_\infty)}(u_i) < \infty.$$

Then a subsequence of u_i Gromov-converges to a nodal gauged holomorphic map.

Remark 4.1.12 (Gromov convergence on Σ with boundary)*.* The above definition of Gromov convergence is applicable on a Riemann surface with boundary with no bubbling at the boundary. Proposition 4.1.11 holds for a Σ with boundary if we impose the additional condition that for any $x \in \partial\Sigma$ there exists a neighbourhood $B_\epsilon(x) \subseteq \Sigma$ such that $sup_i |du_i|_{L^\infty(B_\epsilon(x))} < \infty$. The norm on du_i is taken with respect to the metric $\omega_{A_\infty}(\cdot, J_{A_i}\cdot)$ on $P(X)$. This condition ensures that there is no bubbling on the boundary.

In our application where we have Gromov-type convergence, the convergence of A_i and u_i is not in C^∞. So, we define a weaker notion. Since A_∞ is not in C^1, we do not have the symplectic form ω_{A_∞} and so, we do not talk about energy here.

Definition 4.1.13 (Weak Gromov Convergence)**.** Let A_i a sequence of connections on P converging to A_∞ weakly in H^2. Let $u_i : \Sigma \to P(X)$ be J_{A_i}-holomorphic sections of $P(X)$. Then we say u_i *weakly Gromov converges* to a nodal gauged holomorphic map u_∞ if there exist sequences of rational maps $\phi_i^\alpha : (\mathbb{P}^1)_\alpha \to \Sigma$ for $\alpha \in V_s$ that satisfy the following.

Map

- u_i converges to u^0_∞ in C^1 on compact subsets of $\Sigma\backslash Z_0$. $u^0_\infty \in C^1$.
- $\forall\alpha \in V_S$, $u_i \circ \phi_i^\alpha$ converges to u^α_∞ in C^1 on compact subsets of $(\mathbb{P}^1)_\alpha\backslash Z_\alpha$. u^α_∞ is smooth.
- For the nodal map u_∞, the connection on the principal component Σ is A_∞.

Rescaling

- $\forall\alpha$, $\beta \in V_s$, $(\phi_i^\beta)^{-1} \circ \phi_i^\alpha : (\mathbb{P}^1)_\alpha \to (\mathbb{P}^1)_\beta$ converges to $z_{\beta\alpha}$ in C^∞ on compact subsets of $\mathbb{P}^1_\alpha\backslash\{z_{\alpha\beta}\}$.
- $\forall\alpha \in V_s$, ϕ_i^α C^∞-converges to $z_{0\alpha}$ on compact subsets of $\mathbb{P}^1_\alpha\backslash\{z_{\alpha 0}\}$.

4.2 Convergence of a subsequence modulo gauge

Theorem 4.2.1. *Let $p > 2$ be a constant. Let $(A_t, u_t) \in C^\infty_{loc}([0,\infty)\times\Sigma)$ be the gradient flow (modulo gauge) calculated in theorem 3.0.1. There exists $\{t_i\}_{i=1}^\infty$ with $t_i \to \infty$ as $i \to \infty$ and gauge transformations $g_i \in H^3(\mathcal{G})$ such that*

a. there exists a connection A_∞ in H^2 so that $g_i(A_i) \to A_\infty$ weakly in H^2 and strongly in $W^{1,p}$.

b. there exists a C^1 section $u_\infty : int\Sigma \to P(X)$ so that on compact sets of $\Sigma\backslash\partial\Sigma$, g_iu_i weakly Gromov converges (definition 4.1.13) to a nodal gauged holomorphic map with principal component (A_∞, u_∞). Let Z denote the set of singular points on Σ. Z is finite and $g_iu_i \to u_\infty$ in C^1 on compact subsets of $\Sigma\backslash(Z \cup \partial\Sigma)$.

c. Let $F_i := *F_{A_i} + \Phi(u_i)$, $F_\infty := *F_{A_\infty} + \Phi(u_\infty)$. Then, $\mathrm{Ad}_{g_i} F_i \to F_\infty$ weakly in H^1 and strongly in L^p. If Σ has boundary $F_\infty|_{\partial\Sigma} = 0$ and so $F_\infty = 0$.

d. (A_∞, u_∞) is a critical point of the functional i.e. $d_{A_\infty}F_\infty = 0$ and $(F_\infty)_X = 0$

Remark 4.2.2. The limit (A_∞, u_∞) need not be unique. But, if Σ has boundary, we prove later that the limit is unique modulo gauge.

4.2.1 Convergence of A_i

Definition 4.2.3. The *energy of a gauged holomorphic map* (A, u) on a Riemann surface Σ is defined as

$$E(A,u) := \frac{1}{2}\int_\Sigma |F(A)|^2 + |\Phi \circ u|^2 + |d_A u|^2 dvol_\Sigma$$

Lemma 4.2.4. *([CGS00]) Let Σ be a closed compact Riemann surface and P a principal G-bundle on it. A pair $(A,u) \in \mathcal{A}(P) \times \Gamma(\Sigma, P(X))$ satisfies*

$$\begin{aligned}\frac{1}{2}\int_\Sigma &|F(A)|^2 + |\Phi \circ u|^2 + |d_A u|^2 dvol_\Sigma \\ &= \int_\Sigma |\bar\partial_A u|^2 + \frac{1}{2}| * F_A + \Phi(u)|^2 dvol_\Sigma + \langle \omega_X - \Phi, u\rangle,\end{aligned} \tag{4.5}$$

where $\langle \omega_X - \Phi, u\rangle = \int_\Sigma u^*\omega - d\langle \Phi(u), A\rangle$

Remark 4.2.5.
a. The last term $\langle \omega_X - \Phi, u\rangle$ in (4.5) denotes the pairing of equivariant cohomology and homology. For a closed Σ, the quantity is an invariant of the homotopy class of (A, u). (And so it is independent of the choice of A.) - this is proved in [CGS00].

b. $u^*\omega - d\langle \Phi(u), A\rangle \in \Omega^2(P, \mathfrak{g})$ is equivariant and horizontal, so it descends to a 2-form on Σ. Consider an open set U_α with a trivialization of $P|_{U_\alpha} \simeq U_\alpha \times G$, A can be written as $d + a_\alpha$, $a_\alpha \in \Omega^1(U, \mathfrak{g})$ and $u_\alpha : U \to X$. Here, $u^*\omega - d\langle \Phi(u), A\rangle|_{U_\alpha} = u_\alpha^*\omega - d\langle \Phi(u_\alpha), a_\alpha\rangle$.

c. The identity (4.5) holds for $A \in H^1$ and $u \in C^0 \cap H^1$.

Lemma 4.2.6. *Suppose (A_t, u_t) satisfies the gradient flow equations* (3.1)*. For any t, $E(A_t, u_t) \leq E(A_0, u_0)$.*

Proof. If $\partial\Sigma = \phi$, the result is obvious using the energy identity (4.5). But, for a surface with boundary $\langle \omega - \Phi, (A, u)\rangle$ is not invariant under homotopy (even homotopies fixing u on the boundary). So, we carefully construct a gauged holomorphic map on a closed Riemann surface $\tilde\Sigma$ that is made up of two copies of Σ. For that we first need the result of Step 1. For ease of notation, we assume throughout the proof that $\partial\Sigma$ has one component, the proof works identically for multiple components.

STEP 1: *Close to any boundary component, there is a trivialization* τ *of* P - $\tau : P_{B_\epsilon(\partial\Sigma)} \to \{z \in \mathbb{C} : 1 \le |z| < 1+\epsilon\} \times G$ *so that if* $(\tau^{-1})^* A_0 = d + a_0$, *then* $*a_0 = 0$ *on* $\partial\Sigma$. *This condition is preserved by the flow - i.e. for any* t, *if* $(\tau^{-1})^* A_t = d + a_t$, *then* $*a_t = 0$ *on* $\partial\Sigma$.

Pick a trivialization τ_1 of P close to the boundary : $P|_{B_\epsilon(\partial\Sigma)} \to \{z \in \mathbb{C} : 1 \le |z| < 1+\epsilon\} \times G$. Such a trivialization exists because G is connected and so $P|_{\partial\Sigma}$ is a trivial bundle . Now, $(\tau_1^{-1})^* A_0 = d + a_r dr + a_\theta d\theta$, where $a_r, a_\theta : B_\epsilon(\partial\Sigma) \to \mathfrak{g}$. The radial part of the connection vanishes if we apply the gauge tranformation $g : B_\epsilon(\partial\Sigma) \to G$ given by

$$\frac{dg}{dr} + a_r g = 0, \qquad\qquad g(1,\theta) = \mathrm{Id}$$

The required trivialization is $\tau := g \circ \tau_1$.

Next, we show that this condition is preseved by the flow. For any t, let $(\tau^{-1})^* A(t) = d + a_r(t)dr + a_\theta(t)d\theta$. Under these co-ordinates, the flow equation $\frac{d}{dt}A = *d_A F$ becomes $\frac{d}{dt}a_r = (d_A F)_\theta$. On $\partial\Sigma$, $(d_A F)_\theta = 0$

STEP 2: *Completing the proof*

Consider the bundle $\tilde{P} := P \bigsqcup P/\{(x,x) : x \in P_{\partial\Sigma}\}$ over the Riemann surface $\tilde{\Sigma} := \Sigma \bigsqcup \Sigma/\{(x,x) : x \in \partial\Sigma\}$. At the boundary, the trivialization τ of $P|_{B_\epsilon(\partial\Sigma)}$ extends to a trivialization of $\tilde{P}$: $\tilde{P}|_{B_\epsilon(\partial\Sigma)} \simeq \{z \in \mathbb{C} : \frac{1}{1+\epsilon} < |z| < 1+\epsilon\} \times G$ - this defines the manifold structure of $\tilde{P}$ close to $\partial\Sigma$.

Next, for any t, let the connection on $\tilde{A}_t$ on $\tilde{P}$ be given by the A_t on both copies of P. The trace of these connections on $P|_{\partial\Sigma}$ agree, so this is a H^1 connection on $\tilde{P}$. $\tilde{u}_t : \tilde{\Sigma} \to \tilde{P}(X)$ is defined to be same as u_t on both copies. $(\tilde{A}_t, \tilde{u}_t)$ is homotopically equivalent to $(\tilde{A}_0, \tilde{u}_0)$ and since $\tilde{\Sigma}$ is a closed surface, $\langle \omega - \Phi, (\tilde{A}_t, \tilde{u}_t)\rangle$ is constant for all t. $\|F_{\tilde{A}_t,\tilde{u}_t}\|^2_{L^2(\tilde{\Sigma})} = 2\|F_{A_t,u_t}\|^2_{L^2(\Sigma)}$ decreases with t, so by the energy identity (4.5), $E(\tilde{\Sigma}, (\tilde{A}_t, \tilde{u}_t)) \le E(\tilde{\Sigma}, (\tilde{A}_0, \tilde{u}_0))$ and so, the result follows. $\square$

Proof of theorem 4.2.1 (a). Let $(\tilde{A}_t, \tilde{u}_t) \in C^0_{loc}([0,\infty), H^1 \times C^0)$ be the solution of (3.1). It differs from (A_t, u_t) by a family of H^2 gauge tranformations. Let $\tilde{F}_t = *F(\tilde{A}_t) + \Phi(\tilde{u}_t)$.

By the second equation in the system (3.3),

$$\frac{d}{dt}\|\tilde{F}_t\|_{L^2}^2 = \int_X \langle \tilde{F}_t, d^*_{\tilde{A}_t} d_{\tilde{A}_t}\tilde{F}_t + \tilde{u}_t^* d\Phi(J\tilde{F}_t)_{\tilde{u}(t)}$$
$$= \|d_{\tilde{A}(t)}\tilde{F}_t\|_{L^2}^2 + \int_X g_X((\tilde{F}_t)_{\tilde{u}(t)}, (\tilde{F}_t)_{\tilde{u}(t)})$$

This computation makes sense because $F_t \in H^1$ by lemma 3.1.9. $\|\tilde{F}_t\|_{L^2} \to 0$ as $t \to \infty$. So, one can pick a sequence $\{t_i\}$ ($t_i \to \infty$ as $i \to \infty$), such that

$$\|d_{A(t_i)}F_{t_i}\|_{L^2}, \|(F_{t_i})_{u(t_i)}\|_{L^2} \to 0 \text{ as } i \to \infty.$$

We replace the subscripts t_i by i. By lemma 4.2.6, $\|d_{A_i}u_i\|_{L^2} < c$ for all i and so, $\sup_i \|d_{A_i} * F_{A_i}\|_{L^2} < \infty$. So, we get the result by Uhlenbeck compactness ([Uhl82], [Weh04]) i.e. there exists a sequence of gauge transformations $\{g_i\}$ in $H^3(\mathcal{G})$, such that $g_i(A_i)$ converges weakly to A_∞ in H^2 and strongly in $W^{1,p}$, because of the compact embedding $H^2 \hookrightarrow W^{1,p}$. $\square$

4.2.2 Convergence of u_i

As discussed at the start of this section, to study the convergence behaviour of u_i, we see them as J_{A_i} holomorphic curves. One issue in applying Gromov convergence results in [MS04] is that we do not have smooth convergence of the complex structures J_{A_i}. Our strategy is to apply a sequence of complex gauge transformations to $\{A_i\}$ to obtain smooth convergence.

Lemma 4.2.7. *Let U be a compact Riemann surface with boundary. Let $p > 2$ and A_i be a sequence of connections on a principal bundle P over U. $A_i \to A_\infty$ in $W^{1,p}(U)$ and A_∞ is a flat connection that is smooth on int(U). For large i, there exist complex gauge transformations g_i in $W^{2,p}$ so that $g_i(A_i) - A_\infty \in \ker(d_{A_\infty} \oplus d^*_{A_\infty})$. $g_i \to \mathrm{Id}$ in $W^{2,p}(U')$. For any closed set U' contained in the interior of U, $g_i(A_i)$ is smooth on U' and the sequence converges to A_∞ in $C^\infty(U')$.*

The proof follows ideas in [DK90]. It uses the implicit function theorem, which we state:

Proposition 4.2.8 (Implicit function theorem). *Let E_1, E_2 and F be Banach spaces. $f : E_1 \times E_2 \to F$ a smooth map with partial derivatives $D_1 f$ and $D_2 f$. If the partial derivative $D_2 f$ at (ξ_1, ξ_2) is an isomorphism from E_2 to F, then there is a smooth map h from a neighbourhood of ξ_1 in E_1 to a neighbourhood of ξ_2 in E_2 such that*

$$f(\eta, h(\eta)) = f(\xi_1, \xi_2).$$

.

Proof of lemma 4.2.7. Given a connection $A = A_\infty + a$, we need to find $g = \exp \xi \in \mathcal{G}_{\mathbb{C}}$ so that the function

$$\begin{cases} \Omega^1(P(\mathfrak{g}))_{W^{1,p}} \times \Omega^0(P(\mathfrak{g}_{\mathbb{C}}))_{W^{2,p}_\delta} \to \Omega^2(P(\mathfrak{g}))_{L^p} \times \Omega^0(P(\mathfrak{g}))_{L^p} \\ (a, \xi) \mapsto (d_{A_\infty}(\exp \xi \cdot (A_\infty + a) - A_\infty), d^*_{A_\infty}(\exp \xi \cdot (A_\infty + a) - A_\infty)) \end{cases}$$

vanishes. We use the implicit function theorem (proposition 4.2.8). At $(a, \xi) = (0, 0)$, partial derivative D_ξ is given by $(\xi_1 + i\xi_2) \mapsto (d_{A_\infty} * d_{A_\infty}\xi_2, d^*_{A_\infty} d_{A_\infty}\xi_1)$. The other terms vanish because $d^2_{A_\infty} = F_{A_\infty} = 0$. Write $d_{A_\infty}* = - * d^*_{A_\infty}$, so we need to show that $d^*_{A_\infty} d_{A_\infty} : W^{2,p}_\delta(\Omega^0(P(\mathfrak{g}))) \to L^p(\Omega^0(P(\mathfrak{g})))$ is an isomorphism. This is true because the Dirichlet problem has a unique solution. We also know that g varies continuously with a.

By this argument, for any large i, the required complex gauge transformation g_i exists and $g_i \to \mathrm{Id}$ in $W^{2,p}$. Together with the fact that $A_i \to A_\infty$ in $W^{1,p}$, it implies $g_i(A_i) \to A_\infty$ in $W^{1,p}$.

$d_{A_\infty} \oplus d^*_{A_\infty} : \Omega^1 \to \Omega^0 \oplus \Omega^2$ is an elliptic operator because $d^2_{A_\infty} = F(A_\infty) = 0$, and so, $(d_{A_\infty} \oplus d^*_{A_\infty})^2 = \Delta_{A_\infty}$. For any closed set U' contained in the interior of U, elliptic regularity gives, for $s \geq 0$,

$$\begin{aligned} \|g_i(A_i) - A_\infty\|_{W^{s+2,p}(U')} &\leq c(\|\Delta_{A_\infty}(g_i(A_i) - A_\infty)\|_{W^{s,p}(U)} + \|g_i(A_i) - A_\infty\|_{W^{s+1,p}(U)}) \\ &\leq c\|g_i(A_i) - A_\infty\|_{W^{s+1,p}(U)}. \end{aligned}$$

This leads to the proof of the lemma. □

Remark 4.2.9. In the statement of the lemma, we assume A_∞ is a flat connection that is smooth on $int(U)$. Flatness is enough - because a flat connection is gauge-equivalent to a connection that is smooth in the interior of U. This is seen by an implicit function theorem argument very similar to the proof of the above lemma. If $F_A = 0$ and $A \in H^1$, pick a smooth connection A_0 that is H^1-close to A. Then A can be put into Coulomb gauge with respect to A_0. i.e. there exists $g \in H^2(\mathcal{G})$ such that $d^*_{A_0}(g(A) - A_0) = 0$. Now, we have control on both $d(g(A))$ and $d^*(g(A))$. By elliptic boot-strapping $g(A)$ is smooth in the interior of U.

Proof of theorem 4.2.1 (b). We first work with the case when Σ **has boundary**. By theorem 1 in [Don92], A_∞ can be complex gauge-transformed to a flat connection that is smooth in the interior of Σ. That is, $g(A_\infty)$ is flat for some $g \in W^{2,p}_\delta(\mathcal{G}_\mathbb{C})$. From theorem 4.2.1 (a), $g_i(A_i) \to A_\infty$ in $W^{1,p}(\Sigma)$. So, $gg_i(A_i)$ also converges to $g(A_\infty)$ in $W^{1,p}(\Sigma)$. By lemma 4.2.7, there is a sequence $\{g'_i\} \subseteq W^{2,p}(\mathcal{G}_\mathbb{C})$ converging to Id and such that $(g'_i gg_i(A_i) - g(A_\infty)) \in \ker(d_{g(A_\infty)} \oplus d^*_{g(A_\infty)})$. Let us call $A'_i := g'_i gg_i(A_i)$, $A'_\infty := g(A_\infty)$ and $u'_i := g'_i gg_i u_i$. On any compact set $U \subseteq \Sigma\backslash\partial\Sigma$. A'_i converges to A'_∞ in $C^\infty(U)$ and so $J_{A'_i} \to J_{A'_\infty}$ in $C^\infty(U)$.

Next, we verify that there's an energy bound on u'_i. In the proof of theorem 4.2.1 (a), we showed that $\sup_i \|d_{g_i(A_i)} g_i u_i\|_{L^2(\Sigma)} = \sup_i \|d_{A_i} u_i\|_{L^2(\Sigma)} < \infty$. The quantity $\|d_A u\|_{L^2}$ varies continuously under the action of complex gauge transformations - i.e. the function $g \mapsto \|d_{g(A)} gu\|_{L^2}$ is continuous from $H^2(\mathcal{G}_\mathbb{C})$ to $\mathbb{R}$. So, $\sup_i \|d_{gg_i(A_i)} gg_i u_i\|_{L^2(U)} < \infty$ and since $g_i \to \text{Id}$ in $H^2(\mathcal{G}_\mathbb{C})$, $\sup_i \|d_{A'_i} u'_i\|_{L^2(U)} < \infty$. On any compact set $U \subseteq \Sigma\backslash\partial\Sigma$, A'_i is smooth and $A'_i \to A'_\infty$ in $C^\infty(U)$. Using lemmas 4.1.2 and 4.1.5, for large i, $J_{A'_i}$ is $\omega_{A'_\infty}$-tame and $\sup_i E_{J_{A'_i}, \omega_{A'_\infty}}(u_i, U) < \infty$.

By applying proposition 4.6.1 in [MS04], we see that there is a finite set Z in $\Sigma\backslash\partial\Sigma$ so that for $z_0 \in Z$, there is a sequence $z_i \to z_0$ for which $|du_i(z_i)|$ is unbounded. Consider a compact set $U \subseteq \Sigma\backslash\partial\Sigma$ such that $\partial U \cap Z = \phi$. By remark 4.1.12, a subsequence of u'_i Gromov-converges on U and the principal component is u'_∞. u'_∞ is a secetion of $P(X)$ well-defined on $int(\Sigma)$. Now, we gauge transform back. Let $u_\infty := g^{-1}u'_\infty$. Since $g'_i \to \text{Id}$ in $W^{2,p}(\mathcal{G}_\mathbb{C})$, we get $g_i u_i \to u_\infty$ in $C^1(U')$, where U' is compact subset of $\Sigma\backslash(\partial\Sigma \cup Z)$.

In case, Σ **does not have boundary**, the connection A_∞ cannot be complex gauge transformed to a flat connection. So, we consider closed sets Σ_1 and Σ_2 which have boundary, $\Sigma = \Sigma_1 \cup \Sigma_2$ and $\Sigma_1 \cap \Sigma_2 \neq \phi$. By repeating the arguments above, we see that there is a finite set $Z \subseteq \Sigma$ where bubbling happens. A subsequence of $g_i u_i$ converges on $\Sigma_1 \backslash (Z \cap \partial\Sigma_1)$, and a further subsequence converges on $\Sigma_2 \backslash (Z \cap \partial\Sigma_2)$. The limits agree on the intersection, and so we have u_i weakly Gromov converges to a nodal gauged holomorphic map with principal component u_∞ and connection A_∞. $\square$

4.2.3 (A_∞, u_∞) is a critical point

Proof of theorem 4.2.1 (c) and (d). For this proof, denote $(A_i', u_i') := g_i(A_i, u_i)$ and $F_i' = \mathrm{Ad}_{g_i} F_i = *F(A_i') + \Phi(u_i')$. We consider the sequence $\Phi(u_i')$ in L^p

$$\begin{aligned}\|\Phi(u_i') - \Phi(u_\infty)\|_{L^p(\Sigma)} \leq & \|\Phi(u_i') - \Phi(u_\infty)\|_{L^p(\Sigma \backslash B_\epsilon(Z \cup \partial\Sigma))} \\ & + \|\Phi(u_i')\|_{L^p(B_\epsilon(Z \cup \partial\Sigma))} + \|\Phi(u_\infty)\|_{L^p(B_\epsilon(Z \cup \partial\Sigma))}.\end{aligned}$$

Since, $\|\Phi\|_{L^\infty}$ is bounded, the second and third terms can be made small by taking small enough ϵ. From (b), we have $u_i' \to u_\infty$ in $L^p(\Sigma \backslash B_\epsilon(Z))$. Therefore $\Phi(u_i') \to \Phi(u_\infty)$ in L^p. $F(A_i') \to F(A_\infty)$ in L^p as $A_i' \to A_\infty$ in $W^{1,p}$. Adding, we get $F_i' \to F_\infty$ in L^p.

From the proof of (a), we know, for some constant c, $\|F_i'\|_{H^1} < c$ for all i. So there is a subsequence so that, $F_i' \to \tilde{F}_\infty$ weakly in H^1 and strongly in L^p. Therefore $\tilde{F}_\infty = F_\infty$. In case Σ has boundary:

$$H^1(\Sigma, E) \to H^{1/2}(\partial\Sigma, E|_{\partial\Sigma}) \quad \sigma \mapsto \sigma|_{\partial\Sigma}$$

is a continuous map. Since $F_i' = 0$ on the boundary for all i, F_∞ is also zero on the boundary.

Since $F_i' \rightharpoonup F_\infty$ in H^1, $d_{A_i'} F_i' \rightharpoonup d_{A_\infty} F_\infty$ in L^2 and so,

$$\|d_{A_\infty} F_\infty\|_{L^2} \leq \liminf_i \|d_{A_i'} F_i'\|_{L^2}.$$

We know $\|d_{A_i'} F_i'\|_{L^2} \to 0$. So, $d_{A_\infty} F_\infty = 0$.

To show $(*F_\infty)_{u_\infty} = 0$, we work on a compact set $U \subseteq \Sigma \backslash (Z \cup \partial\Sigma)$ on which there is a trivialization of P. So, u_i' is a map from U to X. Since $u_i' \to u_\infty$ in $C^0(U)$, we can take U small enough that for any i, $u_i'(U) \subseteq V$ and V is a chart of X that is bi-holomorphic to a subset of $\mathbb{C}^n$. So, we may assume $u_i' : U \to \mathbb{C}^n$. Define a map

$$L : U \to \mathrm{Hom}(\mathfrak{g}, \mathbb{C}^n) \qquad\qquad x \mapsto (\xi \mapsto \xi_x). \tag{4.6}$$

L is smooth and $(F_i')_{u_i'} = (L \circ u_i')F_i'$. Since $L \circ u_i' \to L \circ u_\infty$ in C^0 and $F_i' \to F_\infty$ in L^p, we get $(F_i')_{u_i'} \to (F_\infty)_{u_\infty}$ in $L^p(U)$. From the proof of theorem 4.2.1 (a), $(F_i')_{u_i'} \to 0$ in $L^2(\Sigma)$ and so $(F_\infty)_{u_\infty} = 0$ on U and hence on $\Sigma \backslash (Z \cup \partial\Sigma)$. That means $(F_\infty)_{u_\infty} = 0$ almost everywhere on Σ and this proves the result. □

4.3 Unique limit

When Σ has boundary, theorem 4.2.1 says that (A_∞, u_∞), the principal component of the limit, is a vortex i.e. $F_{A_\infty, u_\infty} = 0$. In this section, we show that (A_∞, u_∞) is in the complex gauge orbit of the flow line (A_t, u_t). Since in a complex gauge orbit, a vortex is unique up to gauge transformation (proposition 4.3.2), the limit of the gradient flow is unique up to gauge. The main theorem is

Proposition 4.3.1. *Let $p > 2$. Suppose Σ is a Riemann surface with boundary. Let $(A_i, u_i) \in \mathcal{H}(P, X)$ be a sequence such that $A_i \to A_\infty$ in $W^{1,p}$ and there is a finite set $Z \subseteq \Sigma$ so that $u_i \to u_\infty$ in C^1 on compact subsets of $\Sigma \backslash (Z \cup \partial\Sigma)$. Also, $F_i := *F(A_i) + u_i^*\Phi \to 0$ in L^p. Then, there exist constants C and i_0 so that for $i > i_0$, there is a complex gauge transformation $\exp i\xi_i$, $\xi_i \in W^{2,p}_\delta(\Sigma, P(\mathfrak{g}))$ so that $(\exp i\xi_i)(A_i, u_i)$ is a vortex and satisfies $\|\xi_i\|_{W^{2,p}} < 8C\|F_i\|_{L^p}$.*

Proof. For every (A_i, u_i) in the subsequence found in theorem 4.2.1, define a function,

$$\mathcal{F}_i : \Gamma(\mathfrak{g})_\delta \to \Gamma(\mathfrak{g})$$
$$\xi \mapsto F_{(\exp i\xi)A_i, (\exp i\xi)u_i}.$$

Here $\Gamma(\mathfrak{g})_\delta = \{\xi \in \Gamma(\mathfrak{g}) : \xi|_{\partial\Sigma} = 0\}$. The differential of $\mathcal{F}_i$ at $\xi = 0$ is given by

$$D\mathcal{F}_i(0)\xi_1 = d_{A_i}^* d_{A_i}\xi_1 + u_i^* d\Phi(J(\xi_1)_{u_i}) : W^{2,p}_\delta \to L^p.$$

Step 1: *$D\mathcal{F}_i(0)$ is invertible for all i.*

The operator $\mathrm{Id} + d_{A_i}^* d_{A_i} : W^{2,p}_\delta(\Sigma, \mathfrak{g}) \to L^p(\Sigma, \mathfrak{g})$ is invertible because the Dirichlet problem has a unique solution. So, it has Fredholm index 0. $u_i^* d\Phi(J(\cdot)_u) - \mathrm{Id}$ is a compact perturbation so $D\mathcal{F}_i(0)$ also had Fredholm index 0. It is 1-1 because for any non-zero $\xi_1 \in W^{2,p}_\delta$,

$$\langle d^*_{A_i} d_{A_i}\xi_1 + u_i^* d\Phi(J(\xi_1)_{u_i}), \xi_1\rangle_{\mathfrak{g}} = \|d_{A_i}\xi_1\|^2_{L^2} + \int_X \omega_{u_i}((\xi_1)_{u_i}, J(\xi_1)_{u_i}) > 0,$$

and therefore, it is onto as well.

Step 2: *For large i, $\|D\mathcal{F}_i(0)^{-1}\| < C$ and C is independent of i.*

Denote

$$Q_i := D\mathcal{F}_i(0)^{-1}, \qquad Q_\infty := D\mathcal{F}_\infty(0)^{-1}.$$

On the spaces $W^{s,p}$, we use the norm $\|\cdot\|^{A_\infty}_s$ i.e for $\sigma \in \Gamma(\Sigma, P(\mathfrak{g}))$,

$$\|\sigma\|^{A_\infty}_{s,p} := \sum_{i=0}^{s} \|\nabla^i_{A_\infty}\sigma\|_{L^p}.$$

For notational convenience, we define an operator L_x for every $x \in X$,

$$\begin{aligned} L_x : \mathfrak{g} &\to \mathfrak{g} \\ \xi &\mapsto d\Phi_x(J\xi_x). \end{aligned}$$

We'll proceed by showing that the difference between $D\mathcal{F}_\infty(0)$ and $D\mathcal{F}_i(0)$ is small and so $\|Q_i\|$ can be bounded in terms of $\|Q_\infty\|$. Let $\xi_1 \in W^{2,p}_\delta(\Sigma, \mathfrak{g})$.

$$\begin{aligned} \|(D\mathcal{F}_\infty(0) - D\mathcal{F}_i(0))\xi_1\|_{L^p(\Sigma)} &\le \|d^*_{A_\infty} d_{A_\infty}\xi_1 - d^*_{A_i} d_{A_i}\xi_1\|_{L^p(\Sigma)} \\ &+ \|(L_{u_\infty} - L_{u_i})\xi_1\|_{L^p(\Sigma\setminus B_\epsilon(Z\cup\partial\Sigma))} + \|(L_{u_\infty} - L_{u_i})\xi_1\|_{L^p(B_\epsilon(Z\cup\partial\Sigma))}. \end{aligned} \tag{4.7}$$

$B_\epsilon(Z\cup\partial\Sigma)$ denotes ϵ balls about the points in $Z\cup\partial\Sigma$. The value of ϵ is yet to be fixed. We bound the third term in (4.7) first.

$$\|(L_{u_\infty} - L_{u_i})\xi_1\|_{L^p(B_\epsilon(Z\cup\partial\Sigma))} \le 2\|L\|_{C^0(X)}\|\xi_1\|_{W^{2,p}(\Sigma)} \cdot vol(B_\epsilon(Z\cup\partial\Sigma)).$$

Fix a small enough value of ϵ so that

$$2\|L\|_{C^0(X)} \cdot vol(B_\epsilon(Z\cup\partial\Sigma)) \le \frac{1}{4\|Q_\infty\|}.$$

For the second term - since $u_i \to u_\infty$ in $C^1(\Sigma\setminus B_\epsilon(Z\cup\partial\Sigma))$ for a large enough i,

$$\begin{aligned} \|(L_{u_\infty} - L_{u_i})\xi_1\|_{L^p(\Sigma\setminus B_\epsilon(Z\cup\partial\Sigma))} &\le \|L_{u_\infty} - L_{u_i}\|_{C^0(\Sigma\setminus B_\epsilon(Z\cup\partial\Sigma))}\|\xi_1\|_{W^{2,p}(\Sigma)} \\ &\le \frac{1}{8\|Q_\infty\|}\|\xi_1\|_{W^{2,p}}. \end{aligned}$$

The first term is bounded similarly. For a large enough i,

$$\begin{aligned}\|(d^*_{A_\infty}d_{A_\infty} - d^*_{A_i}d_{A_i})\xi_1\|_{L^p(\Sigma)} &\leq c\|A_\infty - A_i\|_{W^{1,p}(\Sigma)}\|\xi_1\|_{W^{2,p}(\Sigma)} \\ &\leq \frac{1}{4\|Q_\infty\|}\|\xi_1\|_{W^{2,p}(\Sigma)}.\end{aligned}$$

Then,

$$\|D\mathcal{F}_\infty - D\mathcal{F}_i\| \leq \frac{1}{2\|Q_\infty\|} \qquad \text{and so,} \qquad \|Q_i\| \leq 2\|Q_\infty\|.$$

To prove the result we apply the implicit function theorem (in the form of theorem A.0.1) to $\mathcal{F}_i$ with $C := 2\|Q_\infty\|$. We need to find δ for which $\|D\mathcal{F}_i(\xi) - D\mathcal{F}_i(0)\| < \frac{1}{2C}$ for $\|\xi\|_{W^{2,p}} < \delta$.

Step 3: *For large i, $\|\xi\|_{W^{2,p}} < 1$, there is a constant c_1 independent of i such that*

$$\|D\mathcal{F}_i(\xi) - D\mathcal{F}_i(0)\| \leq c_1\|\xi\|_{W^{2,p}}.$$

Proceeding in a similar way as above,

$$\|D\mathcal{F}_i(\xi) - D\mathcal{F}_i(0)\| \leq \|\Delta_{(\exp i\xi)A_i} - \Delta_{A_i}\| + \|L_{(\exp i\xi)u_i} - L_{u_i}\|. \tag{4.8}$$

Consider the first term. Recall that

$$(d^*_{A_i+a}d_{A_i+a} - d^*_{A_i}d_{A_i})\xi_1 = *[a \wedge *d_{A_i}\xi_1] + d^*_{A_i}[a \wedge \xi_1] + *[a \wedge *[a \wedge \xi_1]].$$

We assume that i is large enough so that $\|A_\infty - A_i\|_{W^{1,p}} < \epsilon$ for some fixed ϵ. Then, for any $s \geq 0$, the operator $d_{A_i} : W^{s+1,p} \to W^{s,p}$ has a bound on its norm that is independent of i. Applying multiplication theorem,

$$\|(d^*_{A_i+a}d_{A_i+a} - d^*_{A_i}d_{A_i})\xi_1\|_{L^p} \leq c\|a\|_{1,p}\|\xi_1\|_{2,p}.$$

To bound the first term, we need a bound on $\|(\exp i\xi)A_i - A_i\|_{W^{1,p}}$ in terms of $\|\xi\|_{W^{2,p}}$. At a connection A, the infinitesimal action of $i\xi$ is $*d_A\xi$. So, $(\exp i\xi)A_i$ is given by $A(1)$, where $A(t)$ is the solution of the ODE

$$\frac{dA(t)}{dt} = *d_{A(t)}\xi \quad A(0) = A_i.$$

Write $A(t) = A_\infty + a(t)$ and the equation changes to

$$\frac{da(t)}{dt} = *d_{A_\infty}\xi + *[a(t), \xi] \quad a(0) = A_\infty - A_i.$$

Assume $\|\xi\|_{2,p} \leq 1$. Then,

$$\frac{d}{dt}\|a\|_{1,p} \leq \|\frac{da}{dt}\|_{1,p} \leq c\|\xi\|_{2,p}(1+\|a\|_{1,p}) \leq c(1+\|a\|_{1,p}).$$

Since $\|a(0)\|_{1,p} < \epsilon$, $\|a(t)\|_{1,p} < c = c(\epsilon)$ for $0 \leq t \leq 1$. This means, if $\|\xi\|_{2,p} < 1$,

$$\frac{d}{dt}\|a\|_{1,p} \leq c\|\xi\|_{2,p}.$$

So,

$$\|(\Delta_{(\exp i\xi)A_i} - \Delta_{A_i})\xi_1\|_{L^p} \leq c\|(\exp i\xi)A_i - A_i\|_{1,p}\|\xi_1\|_{2,p} \leq c\|\xi\|_{2,p}\|\xi_1\|_{2,p}.$$

As for the second term in (4.8), $\xi \mapsto (L_{(\exp i\xi)u_i} - L_{u_i})$ is a smooth map. So, $\|L_{(\exp i\xi)u_i} - L_{u_i}\|_{C^0} < c\|\xi\|_{C^0} < c\|\xi\|_{2,p}$. The constants are independent of i, because since X is compact, there is a constant c for which $d_X((\exp i\xi)x, x) < c|\xi|$ for any $x \in X$ and $\xi \in \mathfrak{g}$. Now, by multiplication,

$$\|(L_{(\exp i\xi)u_i} - L_{u_i})\xi_1\|_{L^p} \leq c\|L_{(\exp i\xi)u_i} - L_{u_i}\|_{C^0}\|\xi_1\|_{L^p} \leq c\|\xi\|_{2,p}\|\xi_1\|_{2,p}.$$

Therefore, there is a constant c_1 independent of i, such that for large enough i and $\|\xi\|_{2,p} < 1$,

$$\|D\mathcal{F}_i(\xi) - D\mathcal{F}_i(0)\| \leq c_1\|\xi\|_{W^{2,p}}$$

STEP 4: *Finishing the proof.*

Let $\delta_{max} := 1/2Cc_1$. For i large enough that $\|F_i\|_{L^p} < \frac{\delta_{max}}{4C}$, the implicit function theorem (A.0.1) is applicable on F_i with $\delta = \delta_{max}$.

If we take i such that $\|F_i\|_{L^p} < \frac{\delta_{max}}{8C}$ and apply the implicit function theorem with $\delta = 8C\|F_i\|_{L^p}$, we get $\xi_i \in W^{2,p}_\delta$ so that $\mathcal{F}_i(\xi_i) = 0$ and $\|\xi\|_{2,p} < \delta = 8C\|F_i\|_{L^p}$. $\square$

Proposition 4.3.2. *Let (A,u), $(A',u') \in \mathcal{H}(P,X)$ be vortices related by a complex gauge transformation $g : (A',u') = g(A,u)$ satisfying $g|_{\partial\Sigma} \in \mathcal{G}(\partial\Sigma)$. Then, (A,u) and (A',u') are gauge-equivalent, i.e. $g \in \mathcal{G}$*

Proof. A vortex is a point $(A,u) \in \mathcal{H}(P,X)$ at which the moment map $*F_{A,u}$ vanishes, so the proof is similar to the finite-dimensional case - proposition 2.1.3. The diffeomorphism (2.1) in the proof of proposition 2.1.3, induces a bijection

$$\mathcal{G}_{\mathbb{C}} \to \mathcal{G} \times \Gamma(\Sigma, P(\mathfrak{g})) \qquad\qquad g \mapsto (k,\xi) \text{ so that } g = ke^{i\xi}.$$

So, if (A,u) and (A',u') are vortices that are complex gauge-equivalent, after a gauge transformation, we may assume $(A',u') = e^{i\xi}(A,u)$ where $\xi \in \Gamma(\Sigma, P(\mathfrak{g}))$ and $\xi|_{\partial\Sigma} = 0$.

Let $(A_t,u_t) := e^{it\xi}(A,u)$. We know $F_{A_0,u_0} = F_{A_1,u_1} = 0$. Analogous to (2.2), we have, for $\xi|_{\partial\Sigma} = 0$,

$$\begin{aligned} \frac{d}{dt}\int_\Sigma \langle *F_{A_t,u_t}, \xi\rangle &= \langle d_{A_t}^* d_{A_t}\xi + u_t^* d\Phi(J(\xi)_{u_t}), \xi\rangle_{\mathfrak{g}} \\ &= \|d_{A_t}\xi\|_{L^2}^2 + \int_X \omega_{u_t}((\xi)_u, J(\xi)_u) \geq 0. \end{aligned}$$

The inequality is strict for non-zero ξ. So, $\xi = 0$ and (A,u) and (A',u') are gauge-equivalent. $\square$

Theorem 4.3.3. *Under notations from theorem 4.2.1, let Σ have non-empty boundary. Then,*

a. $u_\infty \in C^1(\Sigma)$ and $g_i u_i \to u_\infty$ in $C^1(\Sigma)$ - i.e. there is no bubbling.

b. (A_∞, u_∞) lies in the same $\mathcal{G}_{\mathbb{C},G}$-orbit as the flow line (A_t,u_t) - i.e. there is a $g \in \mathcal{G}_{\mathbb{C}}$ satisfying $g|_{\partial\Sigma} \subseteq G$ such that $(A_\infty,u_\infty) = g(A_0,u_0)$.

c. For a given flow line (A_t,u_t), the limit (A_∞,u_∞) is unique up to gauge.

Proof. Denote $(A_i',u_i') := g_i(A_i,u_i)$. The outline of the proof is as follows: using the fact that $A_i' \to A_\infty$ and proposition 4.3.1, we show that if $\tilde{g}_i \in \mathcal{G}_{\mathbb{C}}$ is a sequence such that $\tilde{g}_i(A_0') = A_i'$, then $\tilde{g}_i \to g_\infty$. This implies, $u_i' = \tilde{g}_i u_0' \to g_\infty u_0' = u_\infty$.

Apply proposition 4.3.1 to (A_i',u_i'). By dropping a tail of the sequence, we may assume $i_0 = 0$. So, for all i, there exist $\xi_i \in W^{2,p}_\delta$ such that $(A_i'',u_i'') := (\exp i\xi_i)(A_i',u_i')$ is a vortex and $\xi_i \to 0$ in $W^{2,p}$.

The gradient flow preserves the $(\mathcal{G}_{\mathbb{C}})_\delta$ orbit - any (A_t,u_t) is related to (A_0,u_0) by a complex gauge transformation that is identity on $\partial\Sigma$. Also, $\xi_i|_{\partial\Sigma} = 0$. So, using proposition 4.3.2, (A_i'',u_i'') are in the same gauge orbit for all i. We know $A_i'' \to A_\infty$ in $W^{1,p}$. Applying lemma 4.3.4 to the sequence $\{A_i''\}$ gives gauge transformations $\{k_i\} \subseteq W^{2,p}(\mathcal{G})$ such that $k_i \rightharpoonup \mathrm{Id}$ in $W^{2,p}$ and $A_i'' = k_i(A_\infty)$. So, $k_i \to \mathrm{Id}$ in C^1.

Let $\tilde{g}_i := \exp(-i\xi_i)k_i k_0^{-1}\exp(i\xi_0)$. Then, we have $\tilde{g}_i \to k_0^{-1}\exp(i\xi_0)$ in $C^1(\Sigma)$. Since $\tilde{g}_i(A_0') = A_i'$, we get $u_i' = \tilde{g}_i u_0'$ and so $u_i' \to k_0^{-1}\exp(i\xi_0)u_0$ in $C^1(\Sigma)$. So, the limit u_∞

computed in theorem 4.2.1 is same as $k_0^{-1}\exp(i\xi_0)u_0$ - i.e. u_∞ extends to a C^1 map on all of Σ. □

Lemma 4.3.4. *Let $P \to \Sigma$ be a principal G-bundle over a compact Riemann surface. Let $p > 2$ and A_i a sequence of connections converging to A_∞ in $W^{1,p}$. A_i are gauge equivalent, i.e. there exist $g_i \in W^{2,p}(\mathcal{G})$ such that $g_i(A_0) = A_i$. Then, g_i are bounded in $W^{2,p}$ and there exists $g_\infty \in W^{2,p}(\mathcal{G})$ such that $g_i \to g_\infty$ weakly in $W^{2,p}$ and strongly in C^1. $A_\infty = g_\infty(A_0)$ and so is in the same gauge orbit as the sequence.*

Proof. Denote $\Theta_i := g_i(A_0) - A_\infty$. We are given $\Theta_i \to 0$ in $W^{1,p}$ as $i \to \infty$. We work in a neighbourhood U of Σ, over which there is a fixed trivialization of P. So, A_0, A_∞, $\Theta_i \in \Omega^1(U, \mathfrak{g})$ and $g_i : U \to G$ and we know

$$g_i(A_0) = (dg_i)g_i^{-1} + g_i A_0 g_i^{-1}$$

So,

$$dg_i = -g_i A_0 + A_\infty g_i + \Theta_i g_i \tag{4.9}$$

A_0, A_∞ and Θ_i are bounded in L^p. Since the action of G is metric-preserving, the right hand side is bounded in L^p. Since G is compact, $\|g_i\|_{W^{1,p}} < c$ for some constant c. Next, we show that the right hand-side of (4.9) is bounded in $W^{1,p}$. Consider

$$\nabla(\Theta_i g_i) = \Theta_i(\nabla g_i) + (\nabla\Theta_i)g_i.$$
$$\|\nabla(\Theta_i g_i)\|_{L^p} \leq \|\Theta_i\|_{W^{1,p}}\|\nabla g_i\|_{L^p} + \|\nabla\Theta_i\|_{L^p}\|g_i\|_{W^{1,p}}$$

by proposition 5.1.13. So, $\|\nabla(g_i\Theta_i)\|_{W^{1,p}}$ are bounded. Similarly $g_i A_0$ and $A_\infty g_i$ are also bounded in $W^{1,p}$ and so, $\|g_i\|_{W^{2,p}} < c$ for a constant c. Therefore, passing to a subsequence, $g_i \rightharpoonup g_\infty$ in $W^{2,p}$ and the convergence is strong in C^1. Also, $g_i(A_0) \to g_\infty(A_0)$ in L^p and so, $g_\infty(A_0) = A_\infty$. □

Chapter 5

Sobolev Spaces

The goal of this section is to define Sobolev completions of time-dependent sections of vector bundles and prove uniform bounds on certain operators. For a vector bundle $E \to \Sigma$, we define spaces $H^{r,s}([0,t_0] \times \Sigma, E)$ - roughly speaking, when r and s are non-negative integers, this space consists of sections that have r weak derivatives in the time direction, and s in the space direction. This space is defined as $H^r([0,t_0], H^s(\Sigma, E))$, i.e. the space of H^r-regular functions from $[0,t_0]$ to the Hilbert space $H^s(\Sigma, E)$. Subsection 5.1 introduces the spaces $H^s(\Sigma, E)$. The norm on the space $H^s(\Sigma, E)$ is dependent on a choice of connection. Here we'll use a connection A, that satisfies $\|F(A)\|_{L^2} < K$, and then show that the operator norms depend only on K and not on the choice of A. This uses Uhlenbeck compactness and is proved in subsection 5.2. 5.3 describes time dependent sections. Next, in 5.4, we show that in these spaces, the solution of the heat equation has uniformly bounded norm. In the last subsection 5.5, we define the space $H^r(C^0)$ and prove some of its properties.

5.1 Sections of vector bundles

5.1.1 Definition and basic properties

Let Σ be a compact Riemann surface, possibly with boundary. Let $P \to \Sigma$ be a principal G-bundle, where $G \subseteq SO(n)$ is a Lie group. If V is a vector space with a G-action on it, we denote the associated vector bundle $(P \times V)/G$ by $P(V)$. Here, we consider bundles of the form $E = P(V) \otimes \wedge^n T^*\Sigma$. A connection on the principal bundle P and the Levi-Civita connection together determine a connection on E.

Definition 5.1.1. Let A be a smooth connection on the vector bundle $E \to \Sigma$ and s

be a non-negative integer, $\sigma \in \Gamma(\Sigma, E)$

$$\|\sigma\|_s^A := \left(\sum_{i=0}^{s} \|\nabla_A^i \sigma\|_{L^2}^2 \right)^{1/2} \tag{5.1}$$

is a norm. $H^s(E)$ is the completion of $\Gamma(\Sigma, E)$ under this norm.

Remark 5.1.2. $H^s(E)$ can alternately be defined as the equivalence classes of almost-everywhere defined sections σ that satisfy $\nabla_A^i \sigma \in L^2$ for $0 \le i \le s$. The derivatives ∇_A are taken in the distributional sense. [LM72] shows that the space of smooth sections is dense in $H^s(E)$.

The following properties are well known:

For $s_2 < s_1$, the inclusion

$$H^{s_1}(E) \hookrightarrow H^{s_2}(E) \tag{5.2}$$

is continuous.

$$\nabla_A : H^s(E) \longrightarrow H^{s-1}(E \otimes T^*X) \tag{5.3}$$

$$\nabla_A^* : H^s(E \otimes T^*X) \longrightarrow H^{s-1}(E) \tag{5.4}$$

$$\nabla_A^* \nabla_A : H^s(E) \longrightarrow H^{s-2}(E) \tag{5.5}$$

∇_A^* is the same as ∇_A followed by the contraction $T^*X \times T^*X \to \mathbb{R}$. For smooth A, the operators ∇_A, ∇_A^*, $\nabla_A^* \nabla_A$ are continuous by the definition (5.1.1) of $\|\cdot\|_s$.

5.1.2 Interpolation

Definition 5.1.3. If X and Y are Banach spaces such that the inclusion $X \subset Y$ is continuous, an *interpolation space* W is a Banach space, $X \subset W \subset Y$ with the following property: If L is a linear operator from Y into itself, which is continuous from X into itself, then it is also continuous from W into itself. It is an interpolation space of exponent θ if there exists constant C such that

$$\|L\|_W \le C \|L\|_X^{1-\theta} \|L\|_Y^{\theta} \quad \text{for all such operators } L.$$

Further, if $C = 1$, then W is an exact interpolation space.

The complex interpolation functor I_θ produces an exact interpolation space of exponent θ (see [LM72], [Tri95]). We describe this method of obtaining interpolation spaces: Let S be the strip $\{z \in \mathbb{C} : 0 < Re(z) < 1\}$. Let $\mathcal{H}(X,Y)$ denote the space of functions $f : \overline{S} \to Y$ with the following properties:

- f is holomorphic on S
- $f(i\eta) \in X$ and $\eta \mapsto f(i\eta)$ is a bounded continuous function from $\mathbb{R}$ to X
- $\eta \mapsto f(1+i\eta)$ is a bounded continuous function from $\mathbb{R}$ to Y.

$\mathcal{H}(X,Y)$ is equipped with the norm

$$\|f\|_{\mathcal{H}} := \max(\sup_{\eta\in\mathbb{R}}\|f(i\eta)\|_X, \sup_{\eta\in\mathbb{R}}\|f(1+i\eta)\|_Y).$$

Using the *three lines theorem*, theorem 1.9.1 in [Tri95] proves that $\mathcal{H}$ is a Banach space.

Definition 5.1.4 (Complex Interpolation)**.** Let $X \subseteq Y$ be a continuous inclusion of Banach spaces and $0 < \theta < 1$,

$$I_\theta(X,Y) := \{a | \exists f \in \mathcal{H}(X,Y) : f(\theta) = a\}$$

with norm $\|a\|_{I_\theta(X,Y)} = \inf\{\|f\|_{\mathcal{H}} | f(\theta) = a\}$.

We use this construction to define fractional interpolation spaces:

Definition 5.1.5 (Fractional Sobolev spaces)**.** For an integer n and $0 < \theta < 1$, $H^{n+\theta}(E) := I_\theta(H^n(E), H^{n+1}(E))$.

For s_1, $s_2 \geq 0$, and $0 < \theta < 1$, [LM72] proves that the map

$$I_\theta(H^{s_1}, H^{s_2}) \to H^{\theta s_1 + (1-\theta)s_2} \tag{5.6}$$

is an isomorphism. So, (5.2), (5.3), (5.4), (5.5) are bounded maps for all $s > 0$

For $s > \dim \Sigma/2$, there is an embedding ([LM72])

$$H^s \hookrightarrow C^0. \tag{5.7}$$

5.1.3 The spaces H_0^s, H_δ^s

For $s > \frac{1}{2}$,

$$H^s(\Sigma, E) \to H^{s-\frac{1}{2}}(\partial\Sigma, E|_{\delta\Sigma}) \tag{5.8}$$

is well-defined and continuous. $C_0^\infty(\Sigma, E)$ denotes the space of smooth sections supported away from the boundary of Σ. For any $s \geq 0$

Definition 5.1.6. [H_0^s spaces] Let $m \geq 0$ be a non-negative integer

$$H_0^m(\Sigma, E) := \text{closure of } C_0^\infty \text{ in } H^m(\Sigma, E).$$

For $0 < \theta < 1$

$$H_0^{m+\theta}(\Sigma, E) := [H_0^m, H_0^{m+1}]_\theta.$$

Remark 5.1.7 (Alternate characterization of H_0^s). If $s \neq \mu + \frac{1}{2}$, where μ is an integer, the spaces can be directly defined as

$$H_0^s(\Sigma, E) := \text{closure of } C_0^\infty \text{ in } H^s(\Sigma, E).$$

These spaces can be alternately characterized as : $\sigma \in H_0^s$ if and only if $\sigma \in H^s$ and $\frac{\partial^j \sigma}{\partial \nu^j} = 0$ on $\partial\Sigma$ for $j = 0, \ldots, \lfloor s - \frac{1}{2} \rfloor$. So, for $0 < s < \frac{1}{2}$, $H_0^s = H^s$. This makes intuitive sense because for H^s-sections, the boundary trace is not well-defined if $s < \frac{1}{2}$.

However, if $s = \mu + \frac{1}{2}$, $H_0^s(\Sigma, E)$ is a strict subspace of the closure of C_0^∞ in $H^s(\Sigma, E)$, with a finer topology. $H_0^{\mu+1/2}$ is called the Lions-Magenes space and is not closed in $H^{\mu+1/2}$. We'll talk about these spaces more in the 1-dimensional case in section 5.3.

Remark 5.1.8. Our notation here is different from [LM72]. [LM72] defines $H_0^s(\Sigma, E)$ as the closure of C_0^∞ in $H^s(\Sigma, E)$ for all s. $[H^\mu, H^{\mu+1}]_{1/2}$ is called $H_{00}^{\mu+1/2}$.

The H_0^s spaces are well-behaved in terms of interpolation. For $s_1, s_2 \geq 0$ and $0 < \theta < 1$,

$$I_\theta(H_0^{s_1}, H_0^{s_2}) \to H_0^{\theta s_1 + (1-\theta)s_2} \tag{5.9}$$

is an isomorphism.

5.1.4 Defining H^{-s} by duality

Definition 5.1.9. Let $s \geq 0$. $H^{-s}(E) := (H_0^s(E))^*$ i.e. $H^{-s}(E)$ is the completion of $\Gamma(\Sigma, E)$ under the norm

$$\|\sigma\|_{-s} := \sup\{\int_\Sigma (\sigma, \sigma') : \sigma' \in H_0^s(\Sigma, E), \|\sigma'\|_s = 1\}. \tag{5.10}$$

Elements in H^{-s} need not be sections that are defined almost everywhere. H^{-s} is a subspace of the space of distributions.

Notation 5.1.10. We use the notation H_*^s in statements that apply to both H^s and H_0^s.

Using the above duality, we have

Proposition 5.1.11. *The maps* (5.2), (5.3), (5.4), (5.5) *are continuous for all* s, s_1, s_2

By duality, the expected interpolation results also hold for H^{-s} spaces.

Proposition 5.1.12 (Multiplication Theorem)**.** *The map*

$$H_*^{s_1}(E_1) \otimes H_*^{s_2}(E_2) \longrightarrow H_*^{s_3}(E_1 \otimes E_2) \tag{5.11}$$

is continuous if $s_1 + s_2 \geq 0$, $s_3 < \min(s_1, s_2)$ *and* $s_3 \leq s_1 + s_2 - \frac{\dim\Sigma}{2}$.

This is a corollary of the corresponding result on $W^{m,p}$ spaces.

5.1.5 $W^{s,p}$ spaces

Let $p > 1$ and m be a non-negative integer. The space $W^{m,p}(E)$ is a completion of $\Gamma(\Sigma, E)$ under the norm

$$\|\sigma\|_{m,p}^A := \sum_i^m \|\nabla_A^i \sigma\|_{L^p}.$$

By complex interpolation, $W^{s,p}$ can be defined for all non-negative s. $H^s = W^{s,2}$. For $p \neq 2$, $W^{s,p}$ is not a Hilbert space. The negative exponent spaces are defined differently from H^s. For $s \geq 0$, $W^{-s,p} := (W^{s,p^*})^*$ where the pairing is via the L^2-product and p^* is given by $\frac{1}{p} + \frac{1}{p^*} = 1$. We'll need the following embedding results:

$$W^{s_1,p_1} \hookrightarrow W^{s_2,p_2} \qquad \text{if } s_2 < s_1 \text{ and } s_2 - \frac{\dim\Sigma}{p_2} \leq s_1 - \frac{\dim\Sigma}{p_1}, \tag{5.12}$$

$$W^{s,p} \hookrightarrow C^k \qquad \text{if } k < s - \frac{\dim\Sigma}{p}. \tag{5.13}$$

Both the inclusions are compact ([Tri95]).

Proposition 5.1.13. *[Multiplication theorem on $W^{s,p}(E)$ spaces] The multiplication map*

$$W^{s_1,p_1}(E_1) \otimes W^{s_2,p_2}(E_2) \to W^{s_3,p_3}(E_1 \otimes E_2)$$

is continuous if $s_1 + s_2 \geq 0$, $s_3 \leq \min(s_1, s_2)$ and $s_3 - \frac{\dim \Sigma}{p_3} \leq s_1 - \frac{\dim \Sigma}{p_1} + s_2 - \frac{\dim \Sigma}{p_2}$.

This result follows from a corresponding result for functions on bounded Euclidean domains. This result is proved by Hölder's inequality for the case when $s_i = 0$ and then using induction, interpolation and duality.

5.2 Uniform operator bounds

So far, we have used a smooth connection A to define spaces H^s_*, and we have stated some operators between these spaces that have bounded norms. Using a different connection leads to the same spaces, with equivalent norms. Here, we show that one can use a connection $A \in H^{s_0}$, where $s_0 > \frac{\dim \Sigma}{2} - 1$ is an integer, and get an equivalent norm on spaces H^s_* for $s \in [-s_0 - 1, s_0 + 1]$. The norms of operators between these spaces will depend on the choice of connection, but we'll show that if the connection satisfies a curvature bound $\|F(A)\|_{s_0-1} < K$, then the operator norm bounds depend only on K and not on the choice of connection. Constants that depend only on K will be denoted c_K. We'll also use terms like c_K*-bounded*, c_K*-isomorphism* etc. to say that the relevant operator norms are bounded by c_K

Proposition 5.2.1. *Let $s_0 > \frac{\dim \Sigma}{2} - 1$ be an integer and $A \in H^{s_0}$ be a connection on P (and hence E). We assume that B is a smooth connection and that the spaces $H^s(E)$ are Sobolev completions under the norm $\|\cdot\|^B_s$.*

a. For $s \in [-s_0, s_0 + 1]$, the operator $\nabla_A : H^s(E) \to H^{s-1}(E)$ is continuous.

b. For $s \in [-s_0 - 1, s_0 + 1]$, $\|\cdot\|_A$ defines a norm and is equivalent to $\|\cdot\|_B$.

Remark 5.2.2. The above result shows that the space $H^s(E)$ is independent of the connection used to define it.

Proof of proposition 5.2.1. Let $a := A - B \in \Omega^1(X, P(\mathfrak{g}))_{H^{s_0}}$. If $\sigma \in H^s(E)$ ($s \in [s_0, s_0{+}1]$), $\nabla_A \sigma = \nabla_B \sigma + [a, \sigma]$. By the multiplication theorem $\|[a,\sigma]\|^B_{s-1} \le \|a\|^B_{s_0} \|\sigma\|^B_s$. So, $\nabla_A : H^s \to H^{s-1}$ is a bounded operator. This implies that $\|\cdot\|^A_s$ is a norm for $s = 0, 1, \ldots, s_0{+}1$. The definition of the norm can be extended to all $s \in [-s_0{-}1, s_0{+}1]$ by interpolation and duality.

We use induction to prove $\|\cdot\|^A_s \le c\|\cdot\|^B_s$ (when s is a non-negative integer). The result is trivial for $s = 0$, since both norms are just the L^2-norms. Assuming the result for $s - 1$,

$$\begin{aligned}
\|\sigma\|^A_s &\le \|\sigma\|^A_{s-1} + \|\nabla_A \sigma\|^A_{s-1} \\
&\le c(\|\sigma\|^B_{s-1} + \|\nabla_A \sigma\|^B_{s-1}) \\
&\le c(\|\sigma\|^B_{s-1} + \|\nabla_B \sigma\|^B_{s-1} + \|a\|^B_{s_0} \|\sigma\|^B_s) \\
&\le c\|\sigma\|^B_s.
\end{aligned}$$

The other direction $\|\cdot\|^B_s \le c\|\cdot\|^A_s$ can be proved similarly. The result extends to all $s \in [-s_0 - 1, s_0 + 1]$ by duality and interpolation. □

To prove a uniform bound on operator norms, we use local trivializations to give an alternate definition of H^s.

5.2.1 Local trivialization definition of H^s-spaces

It's possible to define the spaces H^s using a local trivialization of the bundle : roughly, $\|\sigma\|_s$ will be the sum of its H^s-norms in each co-ordinate patch. Changing the trivialization would produce equivalent norms. We will pick a trivialization that would produce a norm that is c_K-equivalent to $\|\cdot\|_s$.

A local trivialization data of $E \to \Sigma$ consists of : an open cover $\{\mathcal{U}_\alpha\}_\alpha$ of Σ, with coordinate charts $\tau_\alpha : \mathcal{U}_\alpha \to \mathcal{V}_\alpha \subset \mathbb{R}^n$ (where $n = \dim \Sigma$, and a local section e_α of the principal bundle P. This induces a trivialization of the bundle, $\phi_\alpha : \pi^{-1}\mathcal{U}_\alpha \widetilde{\to} \mathcal{V}_\alpha \times \mathbb{R}^m$, where $\mathbb{R}^m$ is isomorphic to the fibres of E. The trivializations are related by transition functions $g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G$. The following is lemma 3.5 in [Uhl82].

Lemma 5.2.3 (Uhlenbeck Compactness)**.** *Let* $\dim \Sigma = 2, 3$. *Given a* $K > 0$*, there exists a finite cover* $\{\mathcal{U}_\alpha\}_\alpha$ *of* Σ *and constants* c_K *such that for any connection* A *on*

P satisfying $\|F(A)\|_{H^{s_0-1}} < K$, there is a trivialization $\phi_\alpha : \pi^{-1}\mathcal{U}_\alpha \tilde{\to} \mathcal{V}_\alpha \times \mathbb{R}^m$, and if $\tau_\alpha^* A = d + A_\alpha$, $\|A_\alpha\|_{H^{s_0}(\mathcal{V}_\alpha)} \leq c_K$ and $\|g_{\alpha\beta}\tau_\alpha\|_{H^{s_0+1}(\mathcal{V}_\alpha \cap \tau_\alpha^{-1}\mathcal{U}_\beta)} \leq c_K$.

We fix a partition of unity η_α subordinate to this cover.

Let $\sigma_\alpha := \phi_\alpha \circ \sigma \circ \tau_\alpha^{-1}$ represent σ on $\mathcal{V}_\alpha$. Define another norm on $H^s(E)$ as

$$|\sigma|_s := \left(\sum_\alpha [\eta_\alpha^{1/2} \cdot \sigma_\alpha]^2_{H^s(\mathcal{V}_\alpha, \mathbb{R}^m)} \right)^{1/2}. \tag{5.14}$$

We denote the H^s-norm on Euclidean space by $[\cdot]$. The L^2 product corresponding to the norms $|\cdot|$ and $\|\cdot\|$ agree

$$\int_X (\sigma', \sigma) dV = \sum_\alpha \int_{\mathcal{V}_\alpha} \eta_\alpha(\sigma'_\alpha, \sigma_\alpha) dV.$$

The next lemma shows that the norm $|\cdot|_{-s}$ is the dual norm of $|\cdot|$ on H_0^s under the L^2 pairing. This is needed for proving the c_K-equivalence of the norms $\|\cdot\|$ and $|\cdot|$.

Lemma 5.2.4. *For any $0 \leq s \leq s_0 + 1$, there is a constant c_K so that*

$$c_K^{-1}|\sigma|_{-s} \leq |\sigma|_{(H^s)^*} \leq c_K|\sigma|_{-s}.$$

Proof. For the first inequality, we show the existence of $\sigma' \in H^s$ so that

$$c_K^{-1}|\sigma|_{-s} < \frac{(\sigma, \sigma')_{L^2(\Sigma)}}{|\sigma'|_s}.$$

For each α, there exists $\sigma'_\alpha \in H_0^s(\mathcal{V}_\alpha)$ with $|\sigma'_\alpha|_s = 1$ so that

$$(\eta_\alpha^{1/2}\sigma_\alpha, \sigma'_\alpha)_{L^2(\mathcal{V}_\alpha)} \geq \frac{1}{2}|\eta_\alpha^{1/2}\sigma_\alpha|_{H^{-s}(\mathcal{V}_\alpha)}.$$

Define $\sigma' := \sum_\alpha \eta_\alpha^{1/2}\phi_\alpha^{-1} \circ \sigma'_\alpha \circ \tau_\alpha \in H^s(\Sigma, E)$. Then,

$$c_K^{-1}|\sigma|_{H^{-s}} \leq \sum_\alpha |\eta_\alpha^{1/2}\sigma_\alpha|_{H^{-s}(\mathcal{V}_\alpha)} \leq 2\sum_\alpha (\eta_\alpha^{1/2}\sigma_\alpha, \sigma'_\alpha)_{L^2(\mathcal{V}_\alpha)} \leq c_K \frac{(\sigma, \sigma')_{L^2(\Sigma)}}{|\sigma'|_s}.$$

For the last inequality above, we use the fact that the number of co-ordinate patches is bounded by some c_K and so $|\sigma'|_{H^s(\Sigma)} < c_K$.

For the second inequality in the proposition, we need to show that for all $\sigma' \in H^s(\Sigma)$, $\frac{(\sigma,\sigma')_{L^2}}{|\sigma'|_s} \leq |\sigma|_{-s}$.

$$\begin{aligned}(\sigma, \sigma')_{L^2(\Sigma)} &= \sum_\alpha \eta_\alpha(\sigma_\alpha, \sigma'_\alpha) \leq \sum_\alpha |\eta_\alpha^{1/2}\sigma_\alpha|_{H^{-s}(\mathcal{V}_\alpha)} |\eta_\alpha^{1/2}\sigma'_\alpha|_{H^s(\mathcal{V}_\alpha)} \\ &\leq (\sum_\alpha \eta_\alpha|\sigma_\alpha|^2_{H^{-s}(\mathcal{V}_\alpha)})^{1/2} (\sum_\alpha \eta_\alpha|\sigma'_\alpha|^2_{H^s(\mathcal{V}_\alpha)})^{1/2} = |\sigma|_{-s}|\sigma'|_s.\end{aligned}$$

$\square$

Proposition 5.2.5. *Let* $\dim \Sigma = 2, 3$ *and* $s_0 > \frac{n}{2} + 1$ *is an integer. Given* $K > 0$*, there are constants* c_K *so that : if* A *is a connection on* P *satisfying* $\|F_A\|_{s_0-1} < K$*, for any* $s \in [-s_0 - 1, s_0 + 1]$*, the norms* $\|\cdot\|_s^A$ *and* $|\cdot|_s$ *are* c_K*-equivalent on* $H^s(E)$*. The norm* $|\cdot|_s$ *is defined by* (5.14) *and is produced by the trivialization given by lemma 5.2.3.*

Proof. We first prove the result for non-negative integers by induction. For $s = 0$, $|\sigma|_{L^2} = \|\sigma\|_{L^2}$. We assume the estimate is true for $s-1$ and prove $|\sigma|_s \leq c_K \|\sigma\|_s$

$$\begin{aligned}
|\sigma|_s^2 &= |\sigma|_{L^2}^2 + \sum_\alpha [\nabla(\eta_\alpha^{1/2} \cdot \sigma_\alpha)]_{s-1}^2 \\
&\leq \|\sigma\|_{L^2}^2 + \|\nabla_A \sigma\|_{s-1}^2 + \sum_\alpha [A_\alpha \times \sigma_\alpha]_{s-1}^2 + \sum_\alpha [(\nabla \eta_\alpha^{1/2}) \cdot \sigma_\alpha)]_{s-1}^2 \\
&\leq \|\sigma\|_S^2 + c_K |\sigma|_{s-1}^2 \leq c_K \|\sigma\|_s^2.
\end{aligned}$$

The other direction i.e. $\|\sigma\|_s \leq c_K |\sigma|_s$ is similar to the proof of proposition (5.2.1).

We have exact interpolation isomorphisms for both norms $\|\cdot\|$ and $|\cdot|$, so the result extends to all positive s. Using lemma 5.2.4, it extends to negative s by duality. □

In this norm defined using local trivializations, operator norms do not depend on A. So, using the c_K-equivalence, we get

Proposition 5.2.6. *Let* $\dim \Sigma = 2, 3$ *and* $s_0 > \frac{n}{2} + 1$ *is an integer. If*

$$\|F(A)\|_{s_0}^A = \|F(A)\|_{L^2} + \|\nabla_A F(A)\|_{L^2} + ... + \|\nabla_A^{s_0-1} F(A)\|_{L^2} < K,$$

then there exists constants c_K*, depending on* K*, but not on* A*, such that the multiplication operator* (5.11) *and Sobolev embedding* (5.7) *have norm* $\leq c_K$ *and interpolation operators* (5.6)*,* (5.9) *are* c_K*-isomorphisms for sobolev indices in the range* $[-s_0-1, s_0+1]$.

Remark 5.2.7. In the proof of proposition 5.2.6, there is an additional detail in the bound for the multiplication operator: for $(\sigma, \sigma') \mapsto \sigma \otimes \sigma'$, $(\sigma \otimes \sigma')_\alpha$ depends on $g_{\beta\alpha}(\sigma'_\beta|_{\mathcal{V}_\beta \cap \tau_\beta^{-1} \mathcal{U}_\alpha})$. These terms can be bounded using the bound on $g_{\beta\alpha}$.

5.3 Time dependent sections

5.3.1 Sobolev spaces over time intervals

We first define Sobolev completions of functions from a time interval $[0,T]$ to a Hilbert space $\mathcal{K}$. This is very standard and follows the same ideas as the previous section. The only new idea in our definition is - we introduce a T-dependent scaling. This is for technical reasons and its usefulness will be pointed out later.

Definition 5.3.1. Let $m \geq 0$ be an integer. $H^m([0,T],\mathcal{K})$ is the completion of $C^\infty([0,T],\mathcal{K})$ in the norm

$$\|\sigma\|_m := \left(\sum_{i=0}^{m} \|T^{-(m-i)}\frac{d^i}{dt^i}f\|_{L^2}^2\right)^{1/2}.$$

For non-integers, H^r is defined by interpolating between neighbouring integers and $H^{-r}([0,T],\mathcal{K}) := (H_0^r([0,T],\mathcal{K}^*))^*$ with pairing $\langle f,g\rangle \mapsto \int_0^T (f(t),g(T-t))dt$.

Alternately, this norm can be defined by Fourier transform. Our definition differs from the standard one by a T-scaling.

Definition 5.3.2. [Fourier transform definition of $H^s([0,T],\mathcal{K})$]

$$\|f\|_{H^s([0,T],\mathcal{K})} := \inf\|(T^{-2}+\tau^2)^{s/2}\hat{F}(\tau)\|_{L^2},$$

where the infimum is taken over all smooth $F:\mathbb{R}\to\mathcal{K}$ that restrict to f in $[0,T]$.

We need another subspace here

Definition 5.3.3 (H_P^s)**.** Let $C_P^\infty([0,T],\mathcal{K})$ be the subspace of smooth functions that are supported away from $t=0$ (i.e. all derivatives vanish at $t=0$). For a positive integer m, $H_P^m :=$ closure of C_P^∞ in H^s. The definition is extended to non-integers by interpolation and to negative numbers by duality : $H_P^{-s} := (H_P^s)^*$ under the pairing $\langle f,g\rangle \mapsto \int_0^T f(t),g(T-t)dt$.

Remark 5.3.4. $H_0^s = H_P^s = H^s$ if $0 \leq s < \frac{1}{2}$. By duality, $H_P^s = H^s$ for $-\frac{1}{2} < s \leq 0$ also. For $s \leq -\frac{1}{2}$, H_P^s is a formal space - its elements may not correspond to distributions.

Remark 5.3.5. Continuing remark 5.1.7, $H_P^{\mu+1/2}([0,T]) = \{f \in H^{\mu+1/2}([0,T]) : t^{-1/2} f^\mu \in L^2\}$ with norm

$$\|f\|_{H_P^{\mu+1/2}} = \left(\|f\|^2_{H^{\mu+1/2}} + \|t^{-1/2} f\|^2_{L^2}\right)^{\frac{1}{2}}.$$

The topology is finer than that of $H^{\mu+1/2}$, so it is not closed in $H^{\mu+1/2}$. Similarly, the norm of $H_0^{\mu+1/2}([0,T])$ is equivalent to

$$\|f\|_{H_0^{\mu+1/2}} = \left(\|f\|^2_{H^{\mu+1/2}} + \|t^{-1/2} f\|^2_{L^2} + \|(T-t)^{-1/2} f\|^2_{L^2}\right)^{\frac{1}{2}}.$$

We state some important properties

For $s_1 > s_2$, the inclusion

$$\text{(Inclusion)} \quad H_*^{s_1}([0,T],\mathcal{K}) \longrightarrow H_*^{s_2}([0,T],\mathcal{K}) \tag{5.15}$$

has norm $cT^{s_1-s_2}$. This inclusion is compact. The advantage of the scaling is that : by choosing small T, we have a handle on how small a perturbation this operator can cause.

A map $L : \mathcal{K} \to \mathcal{K}'$ between two Hilbert spaces, induces the following continuous operator

$$H_*^{s}([0,T],\mathcal{K}) \longrightarrow H_*^{s}([0,T],\mathcal{K}'). \tag{5.16}$$

Its norm is determined by $\|L\|$.

For $r > \frac{1}{2}$, there is an embedding

$$H^s([0,T],\mathcal{K}) \hookrightarrow C^0([0,T],\mathcal{K}). \tag{5.17}$$

It is a compact operator with norm bounded by $cT^{s-\frac{1}{2}}$.

The multiplication theorem follows from the multiplication theorem for real valued functions,

$$\text{(Multiplication)} \quad H_*^{s_1}([0,T],\mathcal{K}) \otimes H_*^{s_2}([0,T],\mathcal{K}') \longrightarrow H_*^{s_3}([0,T],\mathcal{K}\otimes\mathcal{K}') \tag{5.18}$$

if $s_1+s_2 \geq 0$, $s_3 < s_1+s_2-1/2$ and $s_3 \leq \min(s_1,s_2)$. It has norm $\leq cT^{s_1+s_2-s_3-1/2}$.

If $\mathcal{K}_0$ and $\mathcal{K}_1$ have a common dense subspace k, they form an interpolation pair. Let $\mathcal{K}_\theta = I_\theta(\mathcal{K}_0,\mathcal{K}_1)$. Now, $H^{s_0}([0,T],\mathcal{K}_0)$ and $H^{s_1}([0,T],\mathcal{K}_1)$ form an interpolation pair with common dense subspace $H^{\min(s_0,s_1)}([0,T],k)$ and there is an isomorphism

$$\text{(Interpolation:)} \quad H^{\theta s_0+(1-\theta)s_1}([0,T],\mathcal{K}_\theta) \longrightarrow I_\theta(H^{s_0}([0,T],\mathcal{K}_0), H^{s_1}([0,T],\mathcal{K}_1)). \tag{5.19}$$

Lemma 5.3.6 (Integration). *$\frac{d}{dt} : H_P^{s+1} \to H_P^s$ is invertible, the inverse is given by the integration operator $\int_0$.*

Proof. Integration, $f \mapsto \int_0 f(t)dt$ defines an operator from $C_P^\infty([0,T])$ to itself. It extends to a bounded operator $\int_0 : H_P^n \to H_P^{n+1}$ for integers $n \geq 0$, using the definition 5.3.1 of the norm. The result follows by interpolation and duality. □

Remark 5.3.7. For $s > -\frac{1}{2}$, the integration operator $\int_0 : H_P^s \to H_P^{s+1}$ corresponds to "real integration". Otherwise it is a formal operator. This ties in with the fact that for $f \in H^s$, one can evaluate $f(0)$ only if $s > \frac{1}{2}$.

5.3.2 Mixed spaces

We can define the following mixed spaces to describe time-dependent sections of vector bundles.

Definition 5.3.8. For any real r and s,

$$H^{r,s}(\Sigma \times [0,T], E) = H^r([0,T], H^s(\Sigma, E))$$
$$H^{r,s}_{0,0}(\Sigma \times [0,T], E) = H^r_0([0,T], H^s_0(\Sigma, E))$$
$$H^{r,s}_{P,}(\Sigma \times [0,T], E) = H^r_P([0,T], H^s(\Sigma, E))$$

etc.

The definition is dependent on a choice of a fixed connection A on E, and $A \in H^{s_0}$, where s_0 is a fixed integer $\geq \frac{n}{2}$. So, $H^{r,s}_*$ is well-defined for all r and $s \in [-s_0-1, s_0+1]$. It is assumed that $\|F(A)\|^A_{s_0-1} \leq K$. Putting together the results on $H^s_*(\Sigma, E)$ and $H^r_*([0,T]$, we get all the expected results. c_K will denote a constant independent of A and T, depending only on the manifold, the vector bundle E and the Lie group G. For example, the multiplication map

$$H^{r_1,s_1}_*([0,T], E_1) \otimes H^{r_2,s_2}_*([0,T], E_2) \longrightarrow H^{r_3,s_3}_*([0,T], E_1 \otimes E_2) \tag{5.20}$$

is well-defined and continuous if $r_1 + r_2$, $s_1 + s_2 \geq 0$, $r_3 \leq \min(r_1, r_2, r_1 + r_2 - \frac{1}{2})$ and $s_3 \leq \min(s_1, s_2, s_1 + s_2 - 1)$. It has norm $\leq c_K T^{r_1+r_2-r_3-1/2}$.

5.4 Heat equation

At the center of solving the flow problem, lies the problem of uniformly bounding the solution of a parabolic differential equation. Throughout this section, we fix a connection $A \in H^{s_0}$, where $s_0 \geq dim\Sigma/2$ is an integer. We consider the Laplacian operator $\Delta_A = \nabla_A^* \nabla_A$. We assume that $\|F_A\|_{s_0-1} < K$. The heat equation is solved using standard techniques (see [Eva98]), but we incorporate details for the additional issues - the Laplacian is given by a non-smooth connection, and we need c_K-bounds on the solution. The operator norms we use are : $\|\cdot\|_s := \|\cdot\|_s^A$.

5.4.1 Laplacian equation

Consider the system

$$\begin{cases} (I + \nabla_A^* \nabla_A)\sigma = f \quad \text{on } \Sigma \\ \sigma = 0 \quad \text{on } \partial\Sigma. \end{cases}$$

We denote by $H_\delta^s := \{\sigma \in H^s : \sigma = 0 \text{ on } \partial\Sigma\}$ for $s > \frac{1}{2}$. Note that $H_\delta^s = H_0^s$ for $\frac{1}{2} < s < \frac{3}{2}$.

Proposition 5.4.1. $1 + \nabla_A^* \nabla_A : H_\delta^{s+1} \to H^{s-1}$ *is invertible for* $s \in (-\frac{1}{2}, s_0]$.

Proof. First, we consider $s = 0$, i.e. $\nabla_A^* \nabla_A : H_\delta^1 \to H^{-1}$.

$$\int_X ((1 + \nabla_A^* \nabla_A)\sigma, \sigma') = \int_X ((\sigma, \sigma') + (\nabla_A \sigma, \nabla_A \sigma')) = (\sigma, \sigma')_{H^1} \leq \|\sigma\|_1 \|\sigma'\|_1,$$

i.e. $\|(1 + \nabla_A^* \nabla_A)\sigma\|_{-1} = \|\sigma\|_1$. So, the operator is injective. It is onto by the Riesz representation theorem on H_0^1. For any $\tau \in H^{-1}$, there exists $\sigma \in H^1$, so that $(\tau, \sigma') = (\sigma, \sigma')_{H^1}$ for all $\sigma' \in H^1$. Then, $(1 + \nabla_A^* \nabla_A)\sigma = \tau$.

By injectivity for $s = 0$, it is also injective for $s = 1, .., s_0$. We know that, for any smooth connection B, the operator $1 + \nabla_B^* \nabla_B$ is onto. Let $a = B - A$, then

$$(1 + \nabla_B^* \nabla_B)\sigma - (1 + \nabla_A^* \nabla_A)\sigma = [a, \nabla_A \sigma] + [\nabla_A a, \sigma] + [a, [a, \sigma]].$$

Using multiplication theorem, the right hand side is a compact operator. So, $(1+\nabla_A^*\nabla_A)$ is Fredholm with index 0, and so it is onto.

The result extends to all $s \in [0, s_0]$ by interpolation. Dualizing the map gives $(1+\nabla_A^*\nabla_A)^{-1} : (H_\delta^{s+1})^* \to (H^{s-1})^*$. For $s \in (-\frac{1}{2}, \frac{1}{2})$, $(H_\delta^{s+1})^* = (H_0^{s+1})^* = H^{-s-1}$ and $(H^{s-1})^* = H_0^{-s+1} = H_\delta^{-s+1}$ and the result follows for $s \in (-\frac{1}{2}, s_0]$. □

Proposition 5.4.2. *The norm of* $(I+\nabla_A^*\nabla_A)^{-1} : H^{s-1} \to H_\delta^{s+1}$ *is* $\leq c_K$.

For the proof, we use an elliptic regularity result in Euclidean space (Ch 2, theorem 5.1 [LM72]): Let $V \subseteq \mathbb{R}^n$ be a bounded open set, and L an elliptic operator on V. m is a non-negative integer. Then,

$$\|u\|_{H^{m+2}(V)} \leq c(\|Lu\|_{H^m(V)} + \|ru\|_{H^{m+3/2}(\partial V)} + \|u\|_{m+1}), \tag{5.21}$$

where r denotes restriction of a function to the boundary ∂V.

Proof. We work with local trivializations described by lemma 5.2.3. For a section $\sigma : \Sigma \to E$, on a chart $\mathcal{U}_\alpha$,

$$(\Delta_A\sigma)_\alpha = \Delta\sigma_\alpha + [d\sigma_\alpha, A_\alpha] + [\sigma_\alpha, dA_\alpha] + [A_\alpha, [A_\alpha, \sigma_\alpha]].$$

Assume, $\sigma|_{\partial\Sigma} = 0$. Then, for each α, $(\eta_\alpha^{1/2}\sigma)_\alpha$ vanishes on $\partial\mathcal{V}_\alpha$, using (5.21), we get

$$\begin{aligned}[\eta_\alpha^{1/2}\sigma_\alpha]_s &\leq c([(I+\Delta)\eta_\alpha^{1/2}\sigma_\alpha]_{s-2} + [\eta_\alpha^{1/2}\sigma_\alpha]_{s-1}) \\ &\leq c[((I+\Delta_A)\eta_\alpha^{1/2}\sigma)_\alpha]_{s-2} + c_K[\eta_\alpha^{1/2}\sigma_\alpha]_{s-1}.\end{aligned}$$

Since the norms $\sigma \mapsto \|\sigma\|_s$, $\sigma \mapsto \sum_\alpha [\eta_\alpha^{1/2}\sigma_\alpha]_s$ and $\sigma \mapsto \sum_\alpha [\sigma_\alpha]_s$ are equivalent, we get : if $\sigma|_{\partial\Sigma} = 0$, then

$$\|\sigma\|_s \leq c_K(\|(I+\Delta_A)\sigma\|_{s-2} + \|\sigma\|_{s-1}).$$

The operator $(I+\nabla_A^*\nabla_A)^{-1} : H^{-1} \to H_\delta^1$ has norm 1. By induction, we get the result for all non-negative integers s. As in the proof of proposition 5.4.1, we get the result for all $s \in (-\frac{1}{2}, s_0+1]$. □

$(I+\Delta_A)$ is a positive self-adjoint operator on L^2, so it is possible to define negative and fractional powers of the operator. We'd like to define a family of spaces $\mathcal{F}^s \subseteq H^s$ for which theorem 5.4.4 holds.

Definition 5.4.3. Let $s > -\frac{3}{2}$ and $\sigma \in H^s$. We say, $\sigma \in \mathcal{F}^s$ if $(I + \Delta_A)^j x|_{\delta\Sigma} = 0$ for $j = 0, \ldots, \lfloor \frac{s}{2} - \frac{1}{4} \rfloor$. For $-\frac{3}{2} < s < \frac{1}{2}$, $\mathcal{F}_s = H^s$. We do not define $\mathcal{F}^s$ for the borderline cases $s = 2\mu + \frac{1}{2}$, where μ is an integer.

For our applications we'll use $\mathcal{F}^s$ only for $-\frac{1}{2} < s \leq 2$. So far, we know that the maps $(I + \Delta_A) : \mathcal{F}^s \to \mathcal{F}^{s-2}$ are isomorphisms. We'd like to extend the result to fractional powers of $(I + \Delta_A)$:

Theorem 5.4.4. *Let $s, s + 2r \in (-\frac{1}{2}, s_0]$ and neither is $2\mu + \frac{1}{2}$. Then $(I + \Delta)^r$: $\mathcal{F}^{s+2r} \to \mathcal{F}^s$ is a c_K-isomorphism. r is any real number, not necessarily an integer.*

To prove this theorem, we first need to show that $\mathcal{F}^s$ are a family of interpolation spaces:

Lemma 5.4.5. *Let $0 < \theta < 1$. $\mathcal{F}^{2\theta} = [L^2, H^2_\delta]_\theta$ with c_K-equivalent norms.*

Proof. We need the following result from [LM72] - " Let H be a Hilbert space, whose dual is identified to itself via $\langle \cdot, \cdot \rangle_H$. Let $V \subseteq H$ be a dense subspace, and V' be its dual via $\langle \cdot, \cdot \rangle_H$. (Then, $V \subseteq H \subseteq V'$ are dense inclusions) Then, $[V, V']_{1/2} = H$."

We apply this result with $V = H^2_\delta$, $H = H^1_0 = \mathcal{F}^1$, then, I claim V' is c_K-equivalent to L^2 : we know $\langle u, (I + \Delta_A)v \rangle_{L^2} = \langle u, v \rangle_{H^1}$ for all $u, v \in H^2_\delta$.

$$\|u\|_{V'} = \sup_{v \in H^2_\delta} \frac{\langle u, v \rangle_{H^1}}{\|v\|_{H^2_\delta}} = \sup_{v \in H^2_\delta} \frac{\langle u, (I + \Delta_A)v \rangle_{L^2}}{\|(I + \Delta_A)v\|_{L^2}} = \|u\|_{L^2}$$

and all the equalities mean c_K-equivalences. So, we have $[H^2_\delta, L^2]_{\frac{1}{2}} = H^1_0$.

The result follows because $[H^1_0, L^2]_\theta = H^{1-\theta}_0$ and $[H^2_\delta, H^1_0] = H^{2-\theta}_\delta$, and by the reiteration theorem for interpolation. □

For positive self-adjoint operators, fractional powers of the operator are well-defined and these behave well on interpolation spaces:

Lemma 5.4.6. *Let Λ be an unbounded positive self-adjoint operator on the Hilbert space X. Then, $[X, D(\Lambda)]_\theta = D(\Lambda^\theta)$ with isometry. $D(\Lambda)$ = domain of Λ, with norm $x \mapsto (\|x\|^2_X + \|\Lambda x\|^2_X)^{\frac{1}{2}}$.*

This is infact an equivalent way of defining interpolation spaces which is used in [LM72]. The isometry result can be found in [Yag10], theorem 16.1. This result is required to show a c_K bound on the fractional powers of $I + \Delta_A$.

Proof of theorem 5.4.4. It is enough to show - "Let $0 < \theta < 1$, then $(I + \Delta_A)^\theta : \mathcal{F}^{2\theta} \to L^2$ is a c_K-isomorphism." The theorem is obtained by composing this map with $(I + \Delta_A)^m$, for some integer m.

Denote the operator $(I + \Delta_A)$ by Λ. This is a positive self-adjoint operator. By lemma 5.4.6, $\Lambda^\theta : [L^2, D(\Lambda)]_\theta \to L^2$ is well-defined. Let $x \in H^2_\delta$, using proposition 5.4.2 $\|x\|_{H^2}$ and $\|x\|_{D(\Lambda)}$ are c_K-equivalent. So, $H^2_\delta \hookrightarrow D(\Lambda)$. By interpolation, $\mathcal{F}^s \hookrightarrow D(\Lambda^{s/2})$ for $s \in [0, 2]$.

Now, we have $D(\Lambda) \xrightarrow{\Lambda^{1-\theta}} D(\Lambda^\theta) \xrightarrow{\Lambda^\theta} L^2$, but we need $\mathcal{F}^2 \xrightarrow{A^{1-\theta}} \mathcal{F}^{2\theta} \xrightarrow{\Lambda^\theta} L^2$ for $0 < \theta < 1$. We use the definition 5.1.4 of complex interpolation. Let $a \in H^2_\delta = \mathcal{F}^2$. The function $z \mapsto \Lambda^z a$ is holomorphic from $\{z \in \mathbb{C} : 0 < Re(z) < 1\}$ to L^2. Λ^z is well-defined using the spectral theorem. It is easily verified that this function is in $\mathcal{H}(H^2_\delta, L^2)$, so $\Lambda^{1-\theta} x \in [H^2_\delta, L^2]_{1-\theta} = \mathcal{F}^{2\theta}$ by lemma 5.4.5.

It remains to show that the maps $\Lambda^\theta : \mathcal{F}^{2\theta} \to L^2$ ar isomorphisms. First consider $\frac{1}{2} < s < 2$.

$$\mathcal{F}^2 \longrightarrow \mathcal{F}^s \longrightarrow L^2 \tag{5.22}$$

is an isomorphism, so the second map is onto. Now consider

$$\mathcal{F}^s \longrightarrow L^2 \longrightarrow \mathcal{F}^{s-2}.$$

The right map is obtained by dualizing $\mathcal{F}^{2-s} \to L^2$. The whole map is an isomorphism, so the left map is injective and hence an isomorphism. In (5.22), the left map is also an isomorphism. For $0 < s < \frac{1}{2}$, $\ker \Lambda^{s/2} \cap \mathcal{F}^s \subseteq \ker \Lambda^{s/2} \cap L^2 = \{0\}$.

Finally, the norm of $\mathcal{F}^{2\theta} \xrightarrow{\Lambda^\theta} L^2$ has c_K-bound because of a similar bound on $\Lambda^\theta : D(\Lambda^\theta) \to L^2$. The inverses can be written as $\Lambda^{-\theta} = \Lambda^{-1} \circ \Lambda^{1-\theta}$ and $\|\Lambda^{-1}\| \leq c_K$ from lemma 5.4.2. $\square$

An important consequence of theorem 5.4.4 is :

Proposition 5.4.7. *a. There is a complete orthonormal system* $\{\sigma_i\}_{i\in I}$ *of eigen-sections of the operator* Δ_A *(orthonormalized in* L^2*), such that* $\Delta_A e_i = \lambda_i e_i$*.* $e_i \in H^{s_0+1}$*.*

b. For $s \in (-\frac{3}{2}, s_0+1]$ *and* $\sigma \in \mathcal{F}^s$*,* $c_K^{-1}\|\sigma\|_s \le \|(I+\Delta_A)^{s/2}\sigma\|_{H^0} \le c_K\|\sigma\|_s$*. So,* $\|(I+\Delta_A)^{s/2}\sigma\|_{H^0} = \left(\sum_{i\in I}(1+\lambda_i)^s(\sigma,e_i)_{L^2}^2\right)^{1/2}$ *is a norm on* $\mathcal{F}^s$*, which is* c_K*-equivalent to* $\|\cdot\|_s^A$*.*

Proof. We know $(I+\Delta_A)^{-1} : H^{-1} \to H_0^1$, and the inclusions $L^2 \hookrightarrow H^{-1}$ and $H_0^1 \hookrightarrow L^2$ are compact, $(I+\Delta_A)^{-1}$ is a compact self-adjoint positive operator on L^2. So, it has a complete orthonormal system $\{e_i\}_{i\in I}$ of eigensections. These are eigen-sections for Δ_A also. By bootstrapping, $e_i \in H^{s_0+1}$. (b) easily follows from theorem 5.4.4. □

The eigenvalue norm can also be used for time-dependent sections.

Corollary 5.4.8. *Let* $\sigma \in H^r([0,T],\mathcal{F}^s)$*, then it can be written as* $\sigma = \sum_{i\in I}\sigma_i(t)e_i$*,* $\sigma_i \in H^r([0,T])$ *and*

$$c_K^{-1}\|\sigma\|_{r,s} \le \left(\sum_{i\in I}(1+\lambda_i)^s\|\sigma_i\|_{H^r([0,T])}^2\right)^{\frac{1}{2}} \le c_K\|\sigma\|_{r,s}. \tag{5.23}$$

Proof. $\sigma_i := (\sigma, e_i)$. The operator $\mathcal{F}^s(E) \to \mathbb{R}$ mapping $\eta \mapsto (1+\lambda_i)^s(\eta, e_i)$ is bounded. By (5.16), it is a bounded operator between $H_*^r(\mathcal{F}^s) \to H_*^r$ as well. So, $\sigma_i \in H_*^r([0,T])$. The norm bound (5.23) is straightforward to prove : by using definition 5.3.1 when r is a non-negative integer and then applying interpolation and duality. □

5.4.2 Parabolic equation

Now, we consider the equation

$$\begin{cases} (\dfrac{d}{dt} + \Delta_A)\sigma = f & \text{on } [0,T]\times\Sigma \\ \sigma = 0 & \text{on } [0,T]\times\partial\Sigma \\ \sigma(0) = g & \text{on } \Sigma. \end{cases} \tag{5.24}$$

Here σ is a time-dependent section of E, that is $\sigma : [0,T]\times\Sigma$. $\Delta_A = \nabla_A^*\nabla_A$ is the Laplacian given by a connection $A \in H^{s_0}$ on $P \to \Sigma$. We use standard methods, but get a c_K-bound on the solution.

Lemma 5.4.9. *Let $s, s-2r \in (-3/2, 5/2)$. Given $g \in \mathcal{F}^s$ and $f = 0$, we can find a unique solution $\sigma \in H^{\frac{1}{2}+r, s-2r}$ for* (5.24)*, with bound $\|\sigma\|_{\frac{1}{2}+r,s-2r} \leq c_K T^{-r}\|g\|_s$.*

Proof. Using proposition 5.4.7, g can be written as $g = \sum_{i\in I} g_i(t)e_i$. We aim to find $\sigma = \sum_{i\in I}\sigma_i(t)e_i$. $\forall i \in I$, $\frac{d\sigma_i}{dt} + \lambda_i\sigma_i = 0$ and $\sigma_i(0) = g_i$. This ODE is solved by $\sigma_i(t) = g_i e^{-\lambda_i t}$.

For the norm bound, we use the eigen-value norm in corollary 5.4.8. For each $i \in I$, we need to show

$$\|(1+\lambda_i)^{-r}e^{-\lambda_i t}\|_{H^r([0,T])} \leq cT^{-\frac{1}{2}}|(1+\lambda_i)| \tag{5.25}$$

which holds using $\|e^{-\lambda_i t}\chi_{[0,T]}\|_{H^r([0,T])} \leq c(\lambda_i + T^{-1})^{r-\frac{1}{2}}$ and assuming $T \leq 1$. □

Remark 5.4.10. $g \mapsto \sigma$ is also an operator between $H^s \to C^0(H^s)$ with norm $\leq c$. The proof is similar and follows from $\|e^{-\lambda_i t}\|_{C^0([0,T])} \leq 1$.

Lemma 5.4.11. *Let $-\frac{3}{2} < s < \frac{1}{2}$. Given $f \in H_P^{r,s}$ and $g = 0$,* (5.24) *can be solved uniquely for $\sigma \in H_P^{r+1,s} \cap H_P^{r,s+2}$, with bound $\|\sigma\|_{H^{r+1,s}\cap H^{r,s+2}} \leq c_K\|f\|_{r,s}$.*

Proof. Similar to the proof of lemma 5.4.9, we write f as $f = \sum_{i\in I} f_i(t)e_i$ and solve the ODE $\frac{d\sigma_i}{dt} + \lambda_i\sigma_i = f_i$ and $\sigma_i(0) = 0$. So, $\sigma_i(t) = \int_0^t e^{-\lambda_i(t-s)}f_i(s)ds$. We need, for each $i \in I$,

$$|\sigma_i|_{H_P^{r+1}([0,T])} + (1+\lambda_i)|\sigma_i|_{H_P^r([0,T])} \leq c|f_i|_{H_P^r([0,T])}. \tag{5.26}$$

First assume $r \geq 0$. It is enough to prove the statement for $f_i \in C_P^\infty$. We prove it using the Fourier-transform definition of the norm of $H^r([0,T])$ (see definition 5.3.2). By this, there exists $F_i \in C_0^\infty(\mathbb{R})$ that restricts to f_i on $[0,T]$, vanishes for $t < 0$ and $\|F\|_{H^r(\mathbb{R})} \leq 2\|f\|_{H^r([0,T])}$. Let $S_i = F_i * e^{-\lambda_i t}\chi_{[0,T]}$. Then S_i vanishes for $t < 0$ and restricts to σ_i on $[0,T]$ and $\|(T^{-2}+\tau^2)^{r/2}\hat{S}_i\|_{H^r(\mathbb{R}),T} \leq 2\|\sigma_i\|_{H^r([0,T])}$. So, we need to prove

$$\|(T^{-2}+\tau^2)^{1/2}\hat{S}_i(\tau)\|_{L^2(\mathbb{R})} + |1+\lambda_i| \cdot \|\hat{S}_i(\tau)\|_{L^2(\mathbb{R})} \leq c\|\hat{F}_i(\tau)\|_{L^2(\mathbb{R})},$$

which follows from observing that $\hat{S}_i(\tau) = \hat{F}_i(\tau)\widehat{e^{-\lambda_i t}\chi_{[0,T]}}$ and $\widehat{e^{-\lambda_i t}\chi_{[0,T]}} \leq (T^{-2} + \tau^2)^{-\frac{1}{2}}, \widehat{e^{-\lambda_i t}\chi_{[0,T]}} \leq (1+\lambda_i)^{-1}$.

We've proved that the operator $f_i \mapsto \int_0^t e^{-\lambda_i(t-s)} f_i(s)ds$ is a bounded operator from $H_P^r \to H_P^{r+1}$, and $f_i \mapsto (1+\lambda_i)\int_0^t e^{-\lambda_i(t-s)} f_i(s)ds$ is bounded between $H_P^r \to H_P^r$. These operators are self-adjoint under the H_P^r-H_P^{-r} pairing. So the statement holds for negative r by duality. □

Remark 5.4.12. The spaces $L^2(H^{2s}) \cap H^s(L^2)$ are very natural to solve the heat equation, since the time derivative is order 1 and space derivative is order 2.

The following result gives solutions of the heat equation in higher regularity spaces:

Theorem 5.4.13. *Let $s \geq 0$ and $s \neq \mu + \frac{3}{4}$ where μ is an integer. Suppose $f \in L^2(H^{2s}) \cap H^s(L^2)$ and $g \in H_\delta^{2s+1}$. Assume that the boundary data are compatible, then there is a unique solution $\sigma \in L^2(H^{2s+2}) \cap H^{s+1}(L^2)$ of the system* (5.24) *and*

$$\|\sigma\|_{L^2(H^{2s+2}) \cap H^{s+1,0}} \leq c_K T^{-s-\frac{1}{2}} (\|f\|_{L^2(H^{2s}) \cap H^{s,0}} + \|g\|_{H^{2s+1}}).$$

This can be proved by an induction argument identical to the one in [Eva98], lemmas 5.4.9 and 5.4.11 provide the base case. For our applications, we do not need a c_K bound on the higher regularity solutions.

5.5 Interchanging order of coordinates

We need to define spaces $H^r(C^0)$ - these can be thought of as spaces of sections with r derivatives in the time co-ordinate and continuous in the space co-ordinate. The difficulty here is that C^0 does not have a good dual space, so it is not possible to define such spaces for negative r. To circumvent this problem, we show that the space $H^{r,s}$ can be defined with the order of co-ordinates r, s reversed, as $H^s(\Sigma, H^r([0,T], E))$. Then $H^r(C^0)$ can be defined as $C^0(\Sigma, H^r([0,T], E))$ and this space has relevant properties like $H^{r,1+\epsilon} \hookrightarrow H^r(C^0)$. In this section, the spaces with reversed co-ordinates will be denoted by $\overline{H}^{r,s}$, but this notation will not be used once it is proved equivalent to $H^{r,s}$.

In section 5.2.1, we found that corresponding to a connection A on P satisfying $\|F(A)\|_{s_0-1} < K$, there is a cover $\Sigma = \cup_\alpha \mathcal{U}_\alpha$, $\mathcal{U}_\alpha \simeq \mathcal{V}_\alpha \subseteq \mathbb{R}^2$ and a trivialization of P over these charts such that : the norm $|\cdot|_s$ on $H^s(E)$ defined in terms of the local trivialization is c_K-equivalent to $\|\cdot\|_s^A$ for $s \in [-s_0 - 1, s_0 + 1]$. Here, we fix a connection A and the corresponding trivialization. We use the charts on Σ to define a Hilbert-bundle on Σ. (The connection with the trivialization of P and connection A will become clear later.)

Definition 5.5.1 (Hilbert bundle on Σ)**.** A Hilbert bundle $\pi : \mathcal{H} \to \Sigma$ is a bundle on Σ with fiber-wise inner product, whose fibres are isomorphic to a Hilbert space H. The bundle $\mathcal{H}$ is described by the following data: a local trivialization $\phi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{V}_\alpha \times H$ for all α and transition functions $g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to Aut(H,H)$ where $Aut(H,H)$ is the space of linear isomorphisms from H to H that preserve the inner product.

To define Sobolev completions of $C^\infty(\Sigma, \mathcal{H})$, we recall some notation from section 5.2.1 - given a section $\sigma : \Sigma \to \mathcal{H}$, $\sigma_\alpha := \phi_\alpha \circ \sigma$ represents σ on $\mathcal{V}_\alpha$. η_α is a partition of unity subordinate to the cover $\Sigma = \cup_\alpha \mathcal{U}_\alpha$.

Definition 5.5.2. For real $s \geq 0$, $H^s(\Sigma, \mathcal{H})$ is the completion of $C^\infty(\Sigma, \mathcal{H})$ with norm

$$\|\sigma\|_{H^s(\Sigma,\mathcal{H})} := \left(\sum_\alpha |\eta_\alpha^{1/2} \sigma_\alpha|^2_{H^s_0(\mathcal{V}_\alpha, H)} \right)^{1/2}. \tag{5.27}$$

The definition of the space $H^s(\Sigma, \mathcal{H})$ is dependent on the choice of local trivialization of the bundle $\mathcal{H}$. For

Definition 5.5.3 ($\overline{H}^{r,s}_*$)**.** Let $E := P \times_G \mathbb{R}^m$ denote an associated bundle of P. For any r, the space $H^r_*([0,T], E)$ is a bundle over Σ with fibres $H^r_*([0,T], \mathbb{R}^m)$. Let $s \geq 0$. $\overline{H}^{r,s}_* := H^s(\Sigma, H^r_*([0,T], E))$.

Proposition 5.5.4. *Let $s \geq 0$. $\frac{d}{dt} : H^r_P([0,T], \mathbb{R}^m) \to H^{r-1}_P([0,T], \mathbb{R}^m)$ induces an invertible operator $\frac{d}{dt}^\Sigma : \overline{H}^{r,s}_P \to \overline{H}^{r-1,s}_P$. The inverse is induced by $\int_0$ on the fibres.*

Proof. On any $\mathcal{U}_\alpha$, $\frac{d}{dt}$ induces the map

$$\frac{d}{dt}^{\mathcal{U}_\alpha} : \Gamma(\mathcal{U}_\alpha, H^r_P([0,T], \mathbb{R}^m)) \to \Gamma(\mathcal{U}_\alpha, H^{r-1}_P([0,T], \mathbb{R}^m)). \tag{5.28}$$

For $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$, $g_{\alpha\beta}(x) \in G$ - this just rotates $\mathbb{R}^m$ and is independent of $t \in [0,T]$, We have

$$g_{\alpha\beta}(x)^{-1} \frac{d}{dt}^{\mathcal{U}_\alpha} g_{\alpha\beta}(x) = \frac{d}{dt}^{\mathcal{U}_\beta} \quad \text{on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta.$$

So, the operators $\frac{d}{dt}^{\mathcal{U}_\alpha}$ can be patched up to yield $\frac{d}{dt}^{\Sigma}$.

The operator (5.28) is linear on the fibres with norm $\leq c$ (see lemma 5.3.6). It is also identical on every fibre. So, it induces

$$H^s(\mathcal{U}_\alpha, H^r_P([0,T],\mathbb{R}^m)) \to H^s(\mathcal{U}_\alpha, H^{r-1}_P([0,T],\mathbb{R}^m))$$

and hence,

$$\frac{d}{dt}^{\Sigma} : \overline{H}^{r,s}_P \to \overline{H}^{r-1,s}_P$$

with the same norm. $\int_0$ is the inverse of $\frac{d}{dt}$ fibre-wise, and the result follows for $s \geq 0$. □

Proposition 5.5.5. *For $r \geq 0$ and $s \in [0, s_0+1]$, the identity map*

$$\overline{H}^{r,s} \to H^{r,s} \tag{5.29}$$

is a c_K-isomorphism. For any r and $s \in [0, s_0+1]$, the identity map

$$\overline{H}^{r,s}_P \to H^{r,s}_P \tag{5.30}$$

is a c_K-isomorphism.

Proof. First, we consider the case when r and s are non-negative integers. In the proof of proposition 5.2.6, we showed for $\sigma \in H^s(E)$,

$$c_K^{-1}\|\sigma\|_s \leq |\sigma|_s \leq \|\sigma\|_s.$$

So, it is enough to show that

$$H^s_0(\mathcal{V}_\alpha, H^r_*([0,T],\mathbb{R}^m)) \simeq H^r_*([0,T], H^s_0(\mathcal{V}_\alpha,\mathbb{R}^m)) \tag{5.31}$$

with constants independent of T. These spaces are infact identical when r and s are non-negative integers since both are completions of $C^\infty_{0,P}(\mathcal{V}_\alpha \times [0,T],\mathbb{R}^m)$ under the same norm

$$\left(\sum_{i=0}^{r}\sum_{0\leq|\lambda|\leq s}\|T^{-(r-i)}\frac{d^i}{dt^i}\frac{d^\lambda}{dx^\lambda}\sigma\|^2_{L^2(\mathcal{V}_\alpha\times[0,T])}\right)^{1/2}.$$

The spaces in (5.31) are equivalent for non-integers $r \geq 0$ and $s \geq 0$ by interpolation.

Next, we prove the equivalence of $H_P^{-r,s}$ and $\overline{H}_P^{-r,s}$, for non-negative r and $0 \leq s \leq s_0 + 1$ by induction on r. The result is true for $0 \leq r < 1$. We get an isomorphism between $H_P^{-r,s}$ and $\overline{H}_P^{r,s}$ by

$$H_P^{-r,s} \xrightarrow{\int_0} H_P^{-r+1,s} \xrightarrow{\simeq} \overline{H}_P^{-r+1,s} \xrightarrow{\frac{d}{dt}} \overline{H}_P^{-r,s}$$

Here each of the arrows is an isomorphism with constants $\leq c_K$ - we get the middle arrow from the induction hypothesis. $\square$

Reversing the order of space and time co-ordinates lets us define the space $H^r(C^0)$ for any r.

Definition 5.5.6. For any r, $H^r(C^0) := C^0(X, H^r([0,T], E))$. It is the space of continuous sections of $H^r([0,T], E)$. Its norm is given by

$$\|\sigma\|_{r,C^0} := \sup_{x \in C} \|\sigma(x)\|_{H^r([0,T],E)}.$$

This space satisfies the following properties: for any r, there is an inclusion

$$H^r(C^0) \hookrightarrow H^r(L^2).$$

$\frac{d}{dt}$ is an invertible operator with inverse $\int_0$ between the following spaces

$$H_P^r(C^0) \xrightarrow{\frac{d}{dt}} H_P^{r-1}(C^0).$$

There is a multiplication operator, for $r_3 \leq min(r_1, r_2, r_1 + r_2 - \frac{1}{2})$

$$H_P^{r_1}(C^0) \otimes H_P^{r_2}(C^0) \longrightarrow H_P^{r_3}(C^0). \tag{5.32}$$

For any r, there is an inclusion

$$H_P^{r,1+\epsilon} \hookrightarrow H_P^r(C^0).$$

By the definition of $H^r(C^0)$, it follows that

Proposition 5.5.7. *If $\|F_{A_0}\|_{L^2} < K$, all the above operators have norms bounded by c_K.*

Chapter 6

Composition of functions

Some of the operators we encounter are non-polynomial - they are just smooth maps between sections of bundles. For example,

$$\begin{aligned}
\mathcal{F}_1 : \Gamma(\Sigma, P(\mathfrak{g})) &\to \Gamma(\Sigma, P(\operatorname{End}\mathfrak{g})) \\
\xi &\mapsto (F \mapsto (\exp_{u_0}\xi)^* d\Phi(J_X F_{\exp_{u_0}\xi}) - u_0^* d\Phi(J_X F_{u_0})), \\
\mathcal{F}_2 : \Gamma(\Sigma, P(\mathfrak{g})) &\to \Gamma(\Sigma, P(\operatorname{End}\mathfrak{g})) \\
\xi &\mapsto (F \mapsto (d\exp\xi)^{-1}(JF_{\exp_{u_0}\xi}) - JF_{u_0}).
\end{aligned}$$

We are interested in results of the form

a. $\mathcal{F}_i$ extends to a map from $H^s \cap C^0$ to H^s.

b. Suppose ξ is a time-dependent section and let $r > \frac{1}{2}$. For any $x \in \Sigma$, $\xi_x \to (\mathcal{F}_i\xi)_x$ is a map $H^r([0,T],(E_1)_x) \to H^r([0,T],(E_2)_x)$.

Locally, the above operators can be modeled by something very similar to a composition of functions operator. It is precisely described as follows. Let $U \subseteq \mathbb{R}^n$ and $\Psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^N$ be such that $\Psi(\cdot, 0) = 0$. The operator $\mathcal{F}_\Psi$ is given by $f \mapsto (x \mapsto \Psi(x, f(x)))$.

The results here are very similar to corresponding result on composition of functions. Our presentation treats the slightly different operator arising out of a fibre-preserving smooth map. Also it extends the results to fractional Sobolev indices.

Proposition 6.0.1. *Let $s \geq 0$ and $m := \lceil s \rceil$. If $\Psi \in C^k$, then $\mathcal{F}_\Psi : H^s(U, \mathbb{R}^m) \to H^s(U, \mathbb{R}^N)$ is a continuous map and,*

$$\|\mathcal{F}_\Psi(f)\|_s \leq c\|\Psi\|_{C^k}\|f\|_s(1 + \|f\|_{L^\infty}^{k-1}). \tag{6.1}$$

It is Fréchet-differentiable and satisfies

$$\|d\mathcal{F}_\Psi(f)\| \leq c\|\Psi\|_{C^{k+1}}(1 + \|f\|_s)(1 + \|f\|_{L^\infty}^{k-1}). \tag{6.2}$$

c is independent of Ψ and f.

We recall the definition of Fréchet-differentiability: A map $L : V \to W$ between Banach spaces is Fréchet-differentiable at a point $x \in V$ if there is a linear bounded function $dL_x : V \to W$ such that $\lim_{h\to 0} \frac{\|L(x+h)-L(x)-dL_x(h)\|_W}{\|h\|_V} = 0$. The following are consequences of the above proposition.

Corollary 6.0.2. *Suppose $\Psi : E_1 \to E_2$ is a smooth fibre-preserving map that preserves the zero section, then $\mathcal{F}_\Psi$ extends to a continuous Fréchet-differentiable map between the following spaces of time-dependent sections of E_1 and E_2:*

a. $\mathcal{F} : H^r(C^0) \cap L^\infty \to H^r(C^0)$,

b. $\mathcal{F} : H^r(L^2) \cap L^\infty \to H^r(L^2)$,

c. $\mathcal{F} : L^2(H^s) \cap L^\infty \to L^2(H^s)$,

where r, $s \geq 0$. The bounds on $\|\mathcal{F}\|$ and $\|d\mathcal{F}\|$ are same as in proposition 6.0.1. In (a) and (b), we get stronger bounds.

$$\|\mathcal{F}\xi\|_{r,C^0} \leq c\|\Psi\|_{C^k_{vert}}\|\xi\|_{r,C^0}(1 + \|\xi\|_{L^\infty}^{k-1}),$$

$$\|d\mathcal{F}(\xi)\|_{r,C^0} \leq c\|\Psi\|_{C^{k+1}_{vert}}(1 + \|\xi\|_{r,C^0})(1 + \|\xi\|_{L^\infty}^{k-1}),$$

where $\|\Psi\|_{C^k_{vert}} := \sum_{i=0}^k$ (C^0-norm of i^{th} order vertical derivative of Ψ). Similar inequalities hold for $H^r(L^2)$ spaces also.

Proof of proposition 6.0.1. We carry out the proof for $m = N = 1$. It is identical in other cases. First we assume s **is an integer**, so **s=k**.

Continuity: We start with showing continuity at $f = 0$. To bound $\|\Psi(f)\|_s$, we need to get an L^2-bound on all terms of the form $\frac{\partial^I}{\partial x^I}\mathcal{F}_\Psi(f)$ where I is a multi-index with $|I| \leq m$.

$$\frac{\partial}{\partial x_i}\mathcal{F}_\Psi(f)(x) = \frac{\partial \Psi}{\partial f} \cdot \frac{\partial f}{\partial x_i} + \frac{\partial \Psi}{\partial x_i}.$$

So, $\frac{\partial^I}{\partial x^I}\mathcal{F}_\Psi(\xi)$ is a sum of terms of the form

- $\frac{\partial^{J+j}\Psi}{\partial x^J \partial f^j} \cdot \left(\frac{\partial^{L_1} f}{\partial x^{L_1}}\right) \dots \left(\frac{\partial^{L_N} f}{\partial x^{L_N}}\right)$, where $|J| + j \leq k$ and $|L_1| + \cdots + |L_N| \leq s$.
- $\frac{\partial^I \Psi}{\partial x^I}$, where $|I| \leq k$.

To bound this, we work in $W^{k,p}$ spaces.

$$\begin{aligned}
&\|\frac{\partial^{J+j}\Psi}{\partial x^J \partial f^j} \cdot \left(\frac{\partial^{L_1} f}{\partial x^{L_1}}\right) \dots \left(\frac{\partial^{L_N} f}{\partial x^{L_N}}\right)\|_{L^p} \\
&\leq \|\Psi\|_{C^k} \Pi_{i=1}^N \|\frac{\partial^{L_i} f}{\partial x^{L_i}}\|_{L^{p_i}} \\
&\quad \text{(By Hölder inequality. Here } p_i = kp/l_i,\ l_i = |L_i|) \\
&\leq \|\Psi\|_{C^k} \Pi_{i=1}^N \|f\|_{l_i, p_i} \\
&\leq c\|\Psi\|_{C^k} \Pi_{i=1}^N \|f\|_{k,p}^{j_i/k} \cdot \|f\|_{L^\infty}^{1-j_i/k} \\
&\leq c\|\Psi\|_{C^k} \|f\|_{k,p} \|f\|_{L^\infty}^{k-1}.
\end{aligned}$$

The second-to-last inequality follows from Gagliardo-Nirenberg inequalities (see prop B.1.18, [MS04]). The second term $\frac{\partial^I \Psi}{\partial x^I}$ vanishes for $f = 0$. So, we have

$$\|\frac{\partial^I \Psi}{\partial x^I}(x, f(x))\|_{L^2} \leq \|\frac{\partial}{\partial f}\frac{\partial^I \Psi}{\partial x^I}\|_{L^\infty} \cdot \|f\|_{L^2} = c\|f\|_{L^2}.$$

This proves the inequality (6.1) and continuity of $\mathcal{F}_\Psi$ at $f = 0$. It is continuous at any $f \in H^m$ because the operator $\Delta f \mapsto \mathcal{F}_\Psi(f + \Delta f) - \mathcal{F}_\Psi(f)$ is continuous at $\Delta f = 0$.

Differentiability : For any $f \in H^k$, we claim that $d\mathcal{F}_\Psi(f)|_x := \frac{\partial \Psi}{\partial f}|_{(x,f(x))}$ i.e.

$$d\mathcal{F}_\Psi(f)\Delta f|_x := \frac{\partial \Psi}{\partial f}|_{(x,f(x))} \cdot \Delta f(x).$$

This is the Fréchet-derivative because:

$$\begin{aligned}
&\Psi(f + \Delta f)(x) - \Psi(f)(x) - d\mathcal{F}_\Psi(f)\Delta f|_x \\
&\qquad\qquad = \Delta f(x)^2 \int_0^1 (1-t)\frac{\partial^2 \Psi}{\partial f^2}(x, (f + t\Delta f)(x))dt
\end{aligned}$$

and $\|\int_0^1 (1-t)\frac{\partial^2 \Psi}{\partial f^2}(x, (f + t\Delta f)(x))dt\|_{L^2} < c\|\Psi\|_{C^2}$.

To bound $\|d\mathcal{F}_\Psi(f)\|$,

$$\begin{aligned}
\|\frac{\partial \Psi}{\partial f} \cdot \Delta f\|_m &\leq \|\Delta f\|_m \left(\|\frac{\partial \Psi}{\partial f}(f) - \frac{\partial \Psi}{\partial f}(0)\|_m + \|\frac{\partial \Psi}{\partial f}(0)\|_m\right) \\
&\leq c\|\Psi\|_{C^{m+1}}(1 + \|f\|_m)(1 + \|f\|_{L^\infty}^{m-1})\|\Delta f\|_m.
\end{aligned}$$

The proof extends to the case when s **is not an integer** from the following equivalent norm on $W^{s,p}(\mathbb{R}^n)$: let $s = k + \sigma$, k is an integer and $0 < \sigma < 1$:

$$\|f\|_{s,p}^p \approx \|f\|_{L^p}^p + \|D^k f\|_{L^p}^p + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|D^k f(x) - D^k f(y)|^p}{|x-y|^{n+\sigma p}} dxdy.$$

(remark 4 in p189 [Tri95])

□

Appendix A

Implicit function theorem

The following statement of the implicit function theorem is a part of proposition A.3.4 in [MS04].

Theorem A.0.1. *Let $F : X \to Y$ be a differentiable map between Banach spaces. $DF(0)$ is surjective and has a right inverse Q, with $\|Q\| \leq c$. For all $x \in B_\delta$, $\|DF(x) - DF(0)\| < \frac{1}{2c}$. If $\|F(0)\| < \frac{\delta}{4c}$, then $F(x) = 0$ has a solution in B_δ.*

The next theorem is a slight variation. A complete proof is given for this.

Theorem A.0.2. *Let $F = F_1 + F_2 : X \to Y$ be a differentiable map between Banach spaces. F_1 is a linear invertible map with $\|F_1^{-1}\| \leq c$. In a convex set $\mathcal{S} \subseteq X$, $\|DF_2(x)\| < \frac{1}{2c}$. Then,*

a. F is injective on $\mathcal{S}$.

b. In addition if $B_\delta \subseteq \mathcal{S}$ and $\|F_2(0)\| \leq \delta/4c$, then $F(x) = 0$ has a unique solution on B_δ.

Proof. For any x_1, $x_2 \in \mathcal{S}$, we have

$$\begin{aligned} \|F_1(x_2) - F_1(x_1)\| = \|F_1(x_2 - x_1)\| &\geq \frac{1}{c}\|x_1 - x_0\| \\ \|F_2(x_2) - F_2(x_1)\| &\leq \frac{1}{2c}\|x_2 - x_1\|. \\ \implies \|F(x_2) - F(x_1)\| &\geq \frac{1}{2c}\|x_2 - x_1\|. \end{aligned}$$

which proves (a).

Let $\psi : X \to X$ be given by $x \mapsto F_1^{-1}(F(x) - F(0))$. Then $\|d\psi(x) - \mathrm{Id}\| \leq \frac{1}{2}$ for $x \in \delta$. The theorem follows from lemma A.0.3 - $\|F_1^{-1}F(0)\| < \frac{1}{2}$, so $\psi(x) = -F_1^{-1}F(0)$ has a solution in B_δ. This is a solution of $F(x) = 0$. $\square$

The following is a part of lemma A.3.2 in [MS04]

Lemma A.0.3. *Let $\psi : X \to X$ be a differentiable function between Banach spaces satisfying $\psi(0) = 0$ and for all $x \in B_\delta$, $\|d\psi(x) - \mathrm{Id}\| \leq \frac{1}{2}$. Then ψ is injective on B_δ and $B_{\delta/2} \subseteq \psi(B_\delta) \subseteq B_{3\delta/2}$.*

Proof. Let $\phi = \mathrm{Id} - \psi$. Then in B_δ, $\|d\phi(x)\| < \frac{1}{2}$. So,

$$\|\phi(x_1) - \phi(x_2)\| \leq \frac{1}{2}\|x_1 - x_2\|.$$

Hence,

$$\frac{1}{2}\|x_1 - x_2\| \leq \|\psi(x_1) - \psi(x_2)\| \leq \frac{3}{2}\|x_1 - x_2\|. \tag{A.1}$$

By the second inequality, ψ is injective in B_δ and $\psi(B_\delta) \subseteq B_{3\delta/2}$. To show $B_{\delta/2} \subseteq \psi(B_\delta)$, pick $y \in B_{\delta/2}$. By the first inequality in (A.1), the map $x \mapsto \phi(x) + y$ is a contraction map from $B_{2\|y\|}$ to $B_{2\|y\|}$. So, it has a fixed point x_0 and $\psi(x_0) = y$. $\square$

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