ON THE DYNAMICS OF RANDOM NETWORK CODING

ANALYZING RANDOM NETWORK CODING WITH DIFFERENTIAL EQUATIONS

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ABSTRACT OF THE DISSERTATION

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As a network transport scheme, an important quantity for random network coding (RNC) is the non-redundant information a node or a set of nodes possess over time, which is called the rank. In general, the rank is a random process that is affected by a number of factors including the random coding operation, the channel and the randomness built in the MAC and PHY. Due to its complicated nature, previous approaches to random network coding chose to focus on the asymptotic behavior of the rank processes only. For the first time, we develop a dynamical system framework for analyzing RNC in a wireless network based on differential equations (DE). It turns out under the fluid approximation, ranks of different nodes and sets are intertwined in the form of a system of differential equations. The system of DEs allow us to focus on the transient behavior of RNC, rendering a more complete picture of RNC dynamics. The DE framework can be used to reveal the throughput of RNC, or numerically

solved to evaluate various performance measures. Many aspects of the network, such as topology, source and destination configuration, inter-session coding, PHY and MAC technologies that are employed, are all captured in the system as initial conditions or parameters of the equations. Consequently, the dynamical system framework proves to be powerful in modeling RNC with fine details present in the channel, in the MAC/PHY or in the topology. We will show its versatility by first discussing the theoretical application proving the throughput achievability theorem. We will then expand on its practical application in cross layer design, especially in terms of resource allocation with a throughput dependent objective.

Acknowledgements

I want to thank my advisor, Prof. Narayan B. Mandayam, for over the five years teaching me how to think, criticize and present my work in a scholastic way. I would also thank Prof. Roy Yates from Rutgers, Dr. Shyam Parekh and Dr. Kenneth Budka from Alcatel-Lucent, for giving me the opportunity of a wonderful summer internship at the Bell Labs where the main idea of this thesis started to brew. My thanks also go to Prof. Daniel Ocone from the Math department at Rutgers, who carefully reviewed the part covered in Chapter 3–5 and shared with me Lemma 5.2. Fig. 7.8–7.10 are composed by Kai Su from WINLAB.

I am much indebted to my wife, who never loses her faith in me, and to my daughter, whose arrival magically sweetens my life.

Dedication

To *Lijun*, *Liz* and *my parents*.

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Chapter 1

Introduction

Since the pioneering work by Ahlswede *et al.* [1] that established the benefits of coding in routers and provided theoretical bounds on the capacity of such networks, the breadth of areas that have been touched by network coding is vast and includes not only the traditional disciplines of information theory, coding theory and networking, but also topics such as routing algorithms [2], distributed storage [3, 4], network monitoring, content delivery [5, 6], and security [7]. Among other variants, random network coding (RNC) [8,9] has received extensive interest in particular. By allowing routers to perform random linear operations, RNC is shown to be capacity achieving and fault tolerant. In spite of the excellent progress previous studies have made in the area of RNC, what is still missing is a simple framework that can be used to describe the evolution of state in a wireless network where RNC is employed. In this thesis we present a framework based on differential equations (DE). The DE framework serves as a powerful numerical and analytical tool to study RNC. We demonstrate this by presenting theoretical analysis of information flows with RNC as well as numerical examples. We will give proofs based on the DE framework to the well known result that RNC achieves the min-cut bound in the context of a general lossy wireless network. The flexibility of the DE framework in performance analysis will also be shown via illustrative examples of networks with multiple multicast sessions, user cooperation and arbitrary topologies.

Using the DE framework, we present results similar to Ho *et al.* [10] which for the first time characterized the achievable throughput of RNC by analyzing the codes algebraically; and also to Lun *et al.* [9] which later studied the same problem with a Jackson network approach, analyzing the achievable capacity by treating the propagation of innovative packets through the network as concatenated queuing systems. While the coding strategy considered in this thesis is similar to [9], we will take a rather different approach to the analysis of RNC by treating the internal mechanics of RNC as a dynamical system. We show that the multicast capacity achievability of RNC is a direct consequence of the convergence of the fluid model in this particular case, which does not hold in general. In [9], the fluid approximation is also used to characterize a fictitious queueing system whose throughput lower bounds the real process of innovative packet propagation. In our work, however, the fluid approximation is used directly on the rank evolution processes in the course of innovative packet propagation. Concentration to the differential equation solution is shown with the assistance of a lemma (Lemma 5) that is simple but deep in its nature.

Since we use a system of DEs to directly describe the rank evolution processes, there is a lot to offer in terms of computability. We can numerically solve these DEs to obtain the estimated rank, the instantaneous throughput and the expected time before starting decoding. The computability can be exploited in a number of ways for engineering purposes, such as cross layer design. We can explicitly formulate the interaction between RNC and the lower MAC/PHY layer using a program with differential equation constraints. This type of problems fall in the category of optimal dynamical system control. We present a universal solution that adaptively searches for the best lower layer parameters using a state feedback. We will discuss in detail its general design and present an example to show its application to power control on the PHY layer in a CDMA type wireless network.

In what follows, Chapter 2 discusses the basic operation of RNC and related works. Chapter 3 introduces the hypergraph model for a wireless network first proposed in [9], concepts such as cut sets, the min cut and connectivity for a hypergraph. It is also the place where we set up the DE framework for RNC using a fluid approximation. In Chapter 4 we apply the framework to characterize the average throughput of RNC under the fluid approximation. In Chapter 5 we give an achievability proof of RNC that makes no assumption on field size using the DE framework described here. Chapter 6 presents a dynamical system approach to the cross layer design problems with RNC. Chapter 7 discusses extensive numerical examples to illustrate the application of the DE framework to situations of multiple multicast sessions, complex topology and joint reception. It also includes a detailed RNC aware power control example to show the effectiveness of the proposed cross layer design method. We conclude in Chapter 8.

Chapter 2

Preliminary and Related Works

2.1 Linear Random Network Coding Principles

A formal description of RNC in the language of coding theory can be found in [11]. For the purpose of this work, a simplified description that avoids coding theory terminology, yet equally rigorous, seems to serve the purpose even better. The description has also been adopted in previous works under the name of "coded packets" [9]. We will adopt this term in what follows.

A coded packet is a (column) vector of fixed length. The exact length is immaterial. Each component of the vector is an element from GF(q), i.e., a finite field of size q. To describe the coding operation, assume there is a set of m source packets denote by

$$\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m.$$

The coding operations on the network level guarantees that at any time, any packet is a linear combination of the m source packets. Suppose at time t, a node has n (n is not necessarily equal to m, the number of source packets) packets

$$\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n.$$

Then $\forall i \in \{1, 2, \ldots, n\}$,

$$\mathbf{c}_i = \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \cdots & \mathbf{s}_m \end{bmatrix} \boldsymbol{\beta}_i, \tag{2.1}$$

where $\boldsymbol{\beta}_i \in \mathrm{GF}(q)^m$ is the coding coefficient vector for $\mathbf{c}_i$,

$$\boldsymbol{\beta}_i = \begin{bmatrix} \beta_{i,1} \\ \beta_{i,2} \\ \vdots \\ \beta_{i,m} \end{bmatrix}. \tag{2.2}$$

A coded packet generated at this node at time t will be a linear combination of the n packets. Let the coded packet be $\mathbf{c}$, then

$$\mathbf{c} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \tag{2.3}$$

where the scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathrm{GF}(q)$ are randomly generated. Consequently,

$$\begin{aligned} \mathbf{c} &= \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \cdots \mathbf{s}_m \end{bmatrix} \begin{bmatrix} \beta_{1,1} & \beta_{2,1} & \cdots & \beta_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{1,m} & \beta_{2,m} & \cdots & \beta_{n,m} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \\ &= \begin{bmatrix} [\mathbf{s}_1 & \mathbf{s}_2 & \cdots \mathbf{s}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 & \cdots & \boldsymbol{\beta}_m \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}. \end{aligned} \tag{2.4}$$

Note the coding vector

$$\boldsymbol{\beta} = \begin{bmatrix} \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 & \cdots & \boldsymbol{\beta}_m \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

can be iteratively computed through the locally generated coefficients α_i's.

Whenever a node gets to transmit using its MAC protocol, it generates a coded packet $\mathbf{c}$ and sends it along with the code coefficient vector $\boldsymbol{\beta}$. By doing so, any node that receives the coded packet also receives the coefficient vector that enables the iterative computation of coefficient vectors. When a node has collected $n = m$ coded packets whose coefficient vectors are linearly independent, the source packets can be recovered by a linear inverse

$$\begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \cdots & \mathbf{s}_m \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 & \cdots \boldsymbol{\beta}_m \end{bmatrix}^{-1}. \tag{2.5}$$

As (2.5) indicates, the most important quantity to track with RNC is the number of linearly independent coefficient vectors a node has collected over time. This number will be called the *rank* of the node. It is a direct measure of the amount of information a node has obtained about the source. For any node i, we use $N_i(t)$ to denote the rank of i. In other words, if we let $\mathcal{S}_i(t)$ be the vector space spanned by the coefficient vectors node i has at time t, then $\dim \mathcal{S}_i(t) = N_i(t)$. It will also be necessary to define the rank for an arbitrary set as well. For a set of nodes $\mathcal{K}$, define its rank $N_{\mathcal{K}}(t)$ in the following way

$$\mathcal{S}_{\mathcal{K}}(t) = \sum_{i \in \mathcal{K}} \mathcal{S}_i(t), \quad N_{\mathcal{K}}(t) = \dim_{\mathrm{GF}(q)} \mathcal{S}_{\mathcal{K}}(t). \tag{2.6}$$

Please be reminded that the sum of vector spaces is not the same as their union, which is in general no longer a vector space. Rather, given k vector spaces $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k$ over the field GF(q), their sum is defined as follows:

$$\sum_{i=1}^{k} \mathcal{S}_i \triangleq \{\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k : \forall \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathrm{GF}(q) \text{ and } \mathbf{v}_i \in \mathcal{S}_i, i = 1, 2, \ldots, k\}. \tag{2.7}$$

Users can check that (2.7) indeed defines a vector space that contains $\cup_{i=1}^{k} \mathcal{S}_i$. Alternatively, the sum can be defined as

$$\sum_{i=1}^{k} \mathcal{S}_i \triangleq \mathrm{span}\left\{\bigcup_{i=1}^{k} \mathcal{S}_i\right\}. \tag{2.8}$$

Both $N_i(t)$ and $N_{\mathcal{K}}(t)$ are random processes as the coding operation, the channel conditions and the probabilistic MAC/PHY all contribute to their randomness. Tracking $N_i(t)$ or $N_{\mathcal{K}}(t)$ thus entails certain prediction on overall effect of all the aforementioned factors.

It is also clear from the description above that RNC is relatively decoupled from the underlying MAC and PHY. The coding operation of RNC remains the same regardless of the types of MAC and PHY. In fact, we assume in this work that all nodes in the network perform random network coding. Whenever the MAC of the wireless node is capable of admitting another coded packet, one is generated from the RNC layer and handed down to the MAC layer. It is entirely up to the MAC and PHY to send out that coded packet. The timing of transmission and the exact method of transmission are transparent to the RNC layer. On the receiver side, the MAC pops a new successfully received coded packet to the RNC layer at random. We should point out that although the operation of RNC is independent of the MAC and PHY, its performance is affected by the underlying MAC and PHY as will be seen later in this thesis.

The coding vector $\boldsymbol{\beta}$ of the newly arrived (at node i and at time t) coded packet is examined to see if $\boldsymbol{\beta} \in \mathcal{S}_i(t)$. If so, the packet is simply dropped because it contains no innovative information. Otherwise, the packet is called an *innovative* packet and

$$\mathcal{S}_i(t+) = \text{span}\left\{\mathcal{S}_i(t) \cup \{\boldsymbol{\beta}\}\right\} \quad \text{and} \quad N_i(t+) = N_i(t) + 1. \tag{2.9}$$

The main theme of this work is to present a method of tracking $N_i(t)$ and $N_{\mathcal{K}}(t)$ at an arbitrary time t. In Chapter 3 and 5, we will rigorously define in what sense we track these random processes and how we do it. But we note that for a class of problems about RNC, tracking $N_i(t)$ and $N_{\mathcal{K}}(t)$ for any t is not necessary. Certain bounds on $N_i(t)$ when $t \to \infty$ is sufficient to claim a number of properties of RNC, as the next section will show. This is largely the approach all the previous works adopted.

2.2 Related Works on RNC Performance

Deterministic network coding was initially described and its achievability of multicast capacity was proven in [1]. In the case of linear network coding, the major distinction in deterministic coding, as opposed to RNC, is that the coefficient vector $\boldsymbol{\beta}$ for a coded packet $\mathbf{c}$ is the result of a deliberate design. Consequently, its analysis is different from that of RNC and its application is severely limited in dynamical environment like wireless.

In [9], Lun *et al.* introduced the concept of coded packets and the same random coding operation as described in Section 2.1. Their paper adopted a queueing theoretic approach to the analysis of RNC performance, assuming a Poisson transmission schedule at each node. To obtain the average behavior of RNC, they first defined a network of *virtual queues* consisting of innovative packets at each node. The key idea is that the innovative packet propagation can be abstracted as propagation through those queues with known arrival rates from upstream queues. Channel erasures are included in the arrival rates. The authors then solved the Skorohod equations [12] for tandem queues under fluid approximation. The result shows that the bottleneck link in the tandem queue would determine the throughput. This result, combined with conformal decomposition from graph theory, shows that the throughput of a multicast is determined by the min cut of the source destination pair. The authors then extended the results to a wireless network by mapping it into a wired network. To finalize the achievability proof, a concentration argument is necessary. To this end, Lun *et al.* first argued that the arrival of innovative packets at any node is Poisson, from which they proved the concentration result with the large deviation theory [13].

In [10], Ho *et al.* took a different approach to RNC. Starting with a wired network, they modeled the multicast problem with RNC as a natural extension to the Slepian-Wolf problem. The decodability is examined as a problem of kernel design in algebraic coding theory. The kernel and its analysis is not different from the deterministic network coding. After the kernel is computed, a probabilistic analysis is carried out to show that the randomly generated kernel is very likely to guarantee decodability. Since we already know that deterministic network coding's capacity achievability, the achievability of RNC thus follows.

2.3 Related Works on Cross Layer Design with RNC

Following their respective works on the performance analysis of RNC, Lun *et al.* and Ho *et al.* also examined the problem of cross layer design with RNC that naturally arises. The achievability proof showed that given MAC and PHY, RNC always achieves the best possible throughput, but it does not tell how to design MAC and PHY to further optimize RNC throughput. Moreover, as RNC's universal coding operation is insensitive to the dynamic environment, a method that allows dynamical cross layer design is especially attractive to wireless networking. To that end, it would be desirable to explore the so called multicast advantage [14] inherent to wireless. Previous cross layer design approaches differ in their ways of exploiting multicast advantage.

Lun *et al.* [15] attacked the problem of cross layer design using their queueing framework. After mapping a wireless network to a wired network, the min cut is identified by a linear program. The cross layer design problem is therefore formulated as an optimization program with linear constraints, which serve to identify the min cut, plus nonlinear constraints that map the optimization variables from the MAC or PHY layers to link capacities. They exploited the multicast advantage by carefully mapping the wireless network into a wired network that conserves the multicast advantage. This is complicated by the fact that, although during a wireless transmission (broadcasting) any other node has a chance to receive the signal, in reality the reception may be unsuccessful due to a variety of reasons. Consequently, the set of nodes that receive the signal successfully varies. However, the basic model of a wired network requires prior knowledge of nodes capable of successful reception. To cope with this difficulty, a different set of virtual links has to be assigned to each possible set of reception nodes.

Ho *et al.* [14] addressed the same problem from a different angle. Because their analysis of RNC is centered around a wired network not to be extended to wireless in an obvious way, they have to make compromises in cross layer design for wireless. Specifically, they trim the wireless (broadcasting) links to identify a subgraph of routes to be used. Only transmission in the subgraph will be considered. Opportunistic reception/transmission outside the subgraph that can help information delivery is thus ignored, leading to some loss of multicast advantage. With the subgraph approach, the cross layer design also boils down to a traditional optimization problem with linear constraints that identify the min cut.

Other works on cross layer design with RNC are mostly derived from either [14] or [15], with specialized network topology or MAC/PHY technology to simplify the original formulations.

Chapter 3

The Differential Equation Framework for RNC

3.1 A Review of the Hypergraph Model and RNC

A generic wireless network is modeled as a hypergraph $G = (\mathcal{N}, \mathcal{E})$ consisting of N nodes $\mathcal{N} = \{1, 2, \ldots, N\}$ and hyperarcs $\mathcal{E} = \{(i, \mathcal{K}) | i \in \mathcal{N}, \mathcal{K} \subset \mathcal{N}\}$. Each hyperarc captures the fact that, as any wireless transmission is inherently a broadcast, a packet sent from node i can be received by some or all the nodes in a set $\mathcal{K} \subset \mathcal{N}$. This idea is shown in Fig. 3.1 where the hypergraph of a four-node network is shown. The transmission from node 1 can be overheard by node 2 and 3, while the transmission from node 3 can only be overheard by node 4, all with a probability. This relationship between nodes can be conveniently represented with arrows. One should not, however, confuse the arrow representation with the digraph of a wired network. Assume some underlying MAC is operating in its steady state such that each node i is transmitting according to an independent Poisson process with the intensity of λ_i packets per second. We say that a

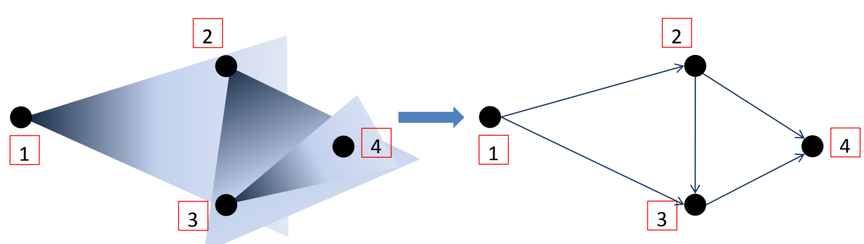

Figure 3.1: Dots-arrows representation of a hypergraph.

packet is successfully received by a set $\mathcal{K}$ of nodes if the packet is successfully received by at least one node in $\mathcal{K}$, which happens with a probability $P_{i,\mathcal{K}}$. Note the definition of $P_{i,\mathcal{K}}$ is general and does not assume independent receptions among the nodes in $\mathcal{K}$. This generality allows channel correlation or user cooperation (e.g., joint detection) to be analyzed in a unified framework. We define the effective transmission rate $z_{i,\mathcal{K}}$ for $(i, \mathcal{K})$ (i.e., from i to $\mathcal{K}$) as

$$z_{i,\mathcal{K}} = \lambda_i P_{i,\mathcal{K}}. \tag{3.1}$$

which is the intensity of the Poisson process of packets from node i successfully arriving/being received by $\mathcal{K}$. $z_{i,\mathcal{K}}$ also can be regarded as the extended concept of link capacity from node i to the set $\mathcal{K}$. When $\mathcal{K} \subset \mathcal{T} \subset \mathcal{N}$, we must have

$$z_{i,\mathcal{K}} \leq z_{i,\mathcal{T}} \tag{3.2}$$

because $P_{i,\mathcal{K}} \leq P_{i,\mathcal{T}}$. Suppose $\mathcal{S}, \mathcal{K} \subset \mathcal{N}$ and $\mathcal{S} \cap \mathcal{K} = \emptyset$. Define a cut for the pair $(\mathcal{S}, \mathcal{K})$ as a set $\mathcal{T}$ satisfying $\mathcal{K} \subset \mathcal{T} \subset \mathcal{S}^c$. Let $C(\mathcal{S}, \mathcal{K})$ denote the collection of all cuts for $(\mathcal{S}, \mathcal{K})$. The size of $\mathcal{T}$ is defined as

$$c(\mathcal{T}) = \sum_{i \in \mathcal{T}^c} z_{i,\mathcal{T}}. \tag{3.3}$$

A min cut $\mathcal{T}_{\min}$ for $(\mathcal{S}, \mathcal{K})$, whose size is denoted as $c_{\min}(\mathcal{S}, \mathcal{K})$ is a cut satisfying

$$c(\mathcal{T}_{\min}) = \min_{\mathcal{T}' \in C(\mathcal{S}, \mathcal{K})} c(\mathcal{T}'). \tag{3.4}$$

We denote the collection of cuts for $(\mathcal{S}, \mathcal{K})$ that satisfy (3.4) as $C_{\min}(\mathcal{S}, \mathcal{K})$. Conventionally, we have

$$C_{\min}(\emptyset, \mathcal{K}) = \{\mathcal{T} | \mathcal{K} \subset \mathcal{T} \subset \mathcal{N}\} \quad \text{and} \quad c_{\min}(\emptyset, \mathcal{K}) = 0. \tag{3.5}$$

We say G is connected if, for any $\emptyset \neq \mathcal{T} \subsetneq \mathcal{N}$, $c(\mathcal{T}) > 0$.

3.2 Rank Evolution Modeled with DE

The DE framework begins with the following lemma that describes the mean of $N_{\mathcal{K}}(t)$:

Lemma 3.1.

$$\mathrm{d}E[N_{\mathcal{K}}(t)]/\mathrm{d}t = \sum_{i\in\mathcal{K}^c} z_{i,\mathcal{K}} E[1 - q^{N_{\mathcal{K}}(t) - N_{\{i\}\cup\mathcal{K}}(t)}] \tag{3.6}$$

Proof. Let $\Delta_{\mathcal{K}}(t)$ denote the increment in the number of innovative packets in $(t, t+\Delta t)$, then

$$N_{\mathcal{K}}(t + \Delta t) = N_{\mathcal{K}}(t) + \Delta_{\mathcal{K}}(t). \tag{3.7}$$

Since every node i sends packets according to an independent Poisson process with intensity λ_i,

$$E[\Delta_{\mathcal{K}}(t)] = \sum_{i\in\mathcal{K}^c} E[\Delta_{i,\mathcal{K}}(t)], \tag{3.8}$$

where $\Delta_{i,\mathcal{K}}(t)$ is the number of innovative packets (either 0 or 1) sent from node $i \in \mathcal{K}^c$ in $[t, t + \Delta t)$. Using the chain rule, we have

$$E[\Delta_{\mathcal{K}}(t)] = \sum_{i\in\mathcal{K}^c} \lambda_i \Delta t E[\Delta_{i,\mathcal{K}}(t)|1 \text{ transmission from } i] \tag{3.9}$$

$$= \sum_{i\in\mathcal{K}^c} \lambda_i \Delta t P_{i,\mathcal{K}} E[\Delta_{i,\mathcal{K}}(t)|1 \text{ reception from } i]. \tag{3.10}$$

A packet sent from $i \in \mathcal{K}^c$ being innovative (i.e., $\Delta_{i,\mathcal{K}} = 1$) if and only if it comes from $S_i \setminus (S_i \cap S_{\mathcal{K}})$. Since

$$|S_i \cap S_{\mathcal{K}}| = q^{\dim S_i \cap S_{\mathcal{K}}} = q^{N_i + N_{\mathcal{K}} - N_{\mathcal{K}\cup\{i\}}}, \tag{3.11}$$

$$\text{and} \quad |S_i| = q^{\dim S_i} = q^{N_i}, \tag{3.12}$$

it follows that the probability of the incoming packet being innovative to $\mathcal{K}$ is given by

$$(|S_i| - |S_i \cap S_{\mathcal{K}}|)/|S_i| = 1 - q^{N_{\mathcal{K}} - N_{\mathcal{K}\cup\{i\}}}. \tag{3.13}$$

Averaged over all possible values of $(N_{\{i\}\cup\mathcal{K}}, N_{\mathcal{K}})$, we have

$$E[\Delta_{\mathcal{K}}(t)] = \sum_{i\in\mathcal{K}^c} \lambda_i \Delta t P_{i,\mathcal{K}} E[1 - q^{N_{\mathcal{K}}(t) - N_{\{i\}\cup\mathcal{K}}(t)}]. \tag{3.14}$$

Therefore we have a precise differential equation for $E[N_{\mathcal{K}}(t)]$ as follows:

$$\frac{\mathrm{d}E[N_{\mathcal{K}}(t)]}{\mathrm{d}t} = \sum_{i\in\mathcal{K}^c} z_{i,\mathcal{K}} E[1 - q^{N_{\mathcal{K}}(t) - N_{\{i\}\cup\mathcal{K}}(t)}]. \tag{3.15}$$

□

Let $V_i(t) = E[N_i(t)]$ and $V_{\mathcal{K}}(t) = E[N_{\mathcal{K}}(t)]$. We want to build a system of differential equations that (approximately) describe $V_i(t)$ and $V_{\mathcal{K}}(t)$. Though Lemma 3.1 does not precisely provide the equations we want (the right-hand sides are not a function of the unknowns), we can turn them into such via a fluid approximation argument: when m is large, the stochastic process $N_{\mathcal{K}}(t)$ behaves on a macro scale like a deterministic function which is $V_{\mathcal{K}}(t)$. This leads us to make the following approximation

$$E[1 - q^{N_{\mathcal{K}}(t) - N_{\{i\}\cup\mathcal{K}}(t)}] \approx 1 - q^{V_{\mathcal{K}}(t) - V_{\{i\}\cup\mathcal{K}}(t)}, \tag{3.16}$$

and consequently we have

$$\dot{V}_{\mathcal{K}} \approx \sum_{i\notin\mathcal{K}} z_{i\mathcal{K}} (1 - q^{V_{\mathcal{K}} - V_{\mathcal{K}\cup\{i\}}}). \tag{3.17}$$

The solution of (3.17) gives the expectation of the rank of a set $\mathcal{K}$ at any given time instant t. It actually stands for a system of $2^N - 1$ equations, each for an nonempty $\mathcal{K} \subset \mathcal{N}$. They collectively give a complete description of rank evolution in the system. Note $V_{\mathcal{K}}$ is solely determined by $\{V_{\mathcal{K}\cup\{i\}}\}_{i\notin\mathcal{K}}$. This dependency can be explored to arrange (3.17) into a partial order "$\lesssim$" such that $V_{\mathcal{K}} \lesssim V_{\mathcal{L}}$ if and only if $\mathcal{K} \subset \mathcal{L}$. This partial order can be pictorially represented as a layered structure, for which an example is shown in Fig. 3.2 for $N = 3$. To determine a quantity on any particular layer, one

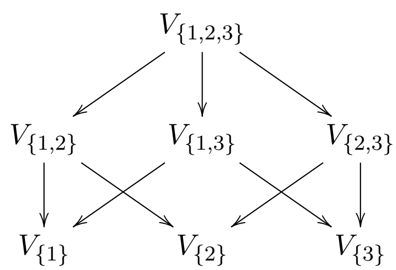

Figure 3.2: Layered structure for the rank evolution of a 3-node network

only needs to know the the quantities on the layer immediately above indicated by arrows. The layered structured will be exploited in Chapter 4 to facilitate the proofs.

Theoretically, with appropriate boundary conditions, (3.17) can be solved. The instantaneous throughput is then obtained as $\dot{V}_{\mathcal{K}}$ or $\dot{V}_i$. For example, assuming node 1 is the unique source with m packets to deliver, the boundary conditions (B.C.) for this systems of DEs are

$$V_{\mathcal{K}}(0) = \begin{cases} m, & 1 \in \mathcal{K}, \\ 0, & \text{o.w.} \end{cases} \tag{3.18}$$

If only a subset of the nodes, say $\mathcal{L} \subset \mathcal{N}$, participate in carrying the flow, (3.17) still holds, except that we should replace $\mathcal{K}$ with $\mathcal{K} \cap \mathcal{L}$ and the top layer in the layered structure consists of $V_{\mathcal{L}}$ alone.

In practice, q is usually chosen to be an integral power of 2, not only because their arithmetic is particularly amenable to machines, but also because they are the natural granularity used in storage and communication, e.g., bits, bytes, words, etc. As $V_{\mathcal{K}}$ can never exceed $V_{\mathcal{K}\cup\{i\}}$, $V_{\mathcal{K}} - V_{\mathcal{K}\cup\{i\}}$ is nonpositive and in this case we may approximate $1 - q^{V_{\mathcal{K}} - V_{\mathcal{K}\cup\{i\}}}$ by

$$1 - q^{V_{\mathcal{K}} - V_{\mathcal{K}\cup\{i\}}} \approx \begin{cases} 1, & V_{\mathcal{K}} < V_{\mathcal{K}\cup\{i\}}, \\ 0, & V_{\mathcal{K}} = V_{\mathcal{K}\cup\{i\}}. \end{cases} \tag{3.19}$$

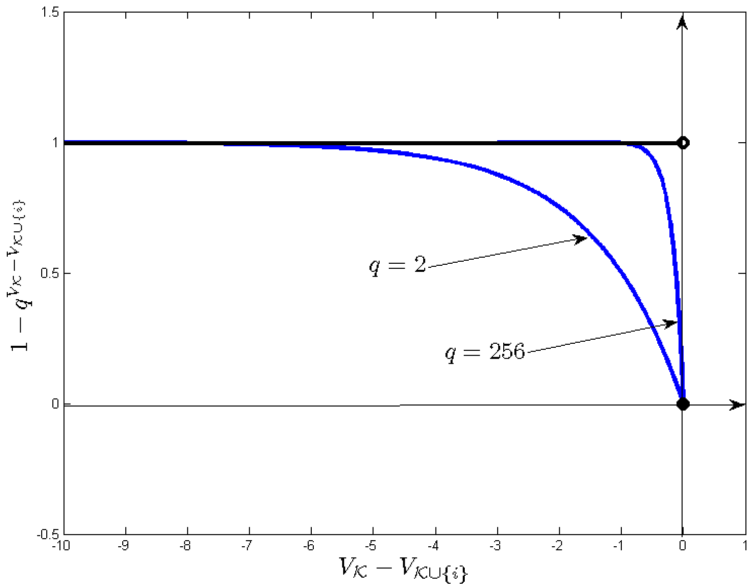

Figure 3.3: Approximate $1 - q^{V_\mathcal{K} - V_{\mathcal{K}\cup\{i\}}}$ with $V_{\mathcal{K}\cup\{i\}} \ominus V_\mathcal{K}$ when $q = 2$ and $q = 256$.

The approximation for different values of q is shown in Fig. 3.3. It is evident that, when $q = 256$ the approximation is very close for every nonpositive integer. Even when $q = 2$, the approximation is very good when $V_\mathcal{K} - V_{\mathcal{K}\cup\{i\}} < -6$ or $V_\mathcal{K} - V_{\mathcal{K}\cup\{i\}} = 0$. For other nonpositive integer values, the approximation has an error bounded by 1. When $q \to \infty$, (3.19) becomes more accurate. However, as will be shown in numerical examples, when the total rank is large the approximation rarely fails even for $q = 2$. Consequently we may rewrite (3.17) as

$$\dot{V}_\mathcal{K} = \sum_{i \notin \mathcal{K}} z_{i\mathcal{K}} (V_{\mathcal{K}\cup\{i\}} \ominus V_\mathcal{K}), \quad \forall \mathcal{K} \subset \mathcal{N} \tag{3.20}$$

with the same boundary conditions as in (3.18). The binary operation $\ominus$ is defined as

$$x \ominus y = \begin{cases} 1, & x > y, \\ 0, & \text{o.w.} \end{cases} \tag{3.21}$$

Though the simplified DEs shown in (3.20) have discontinuous right-hand sides due

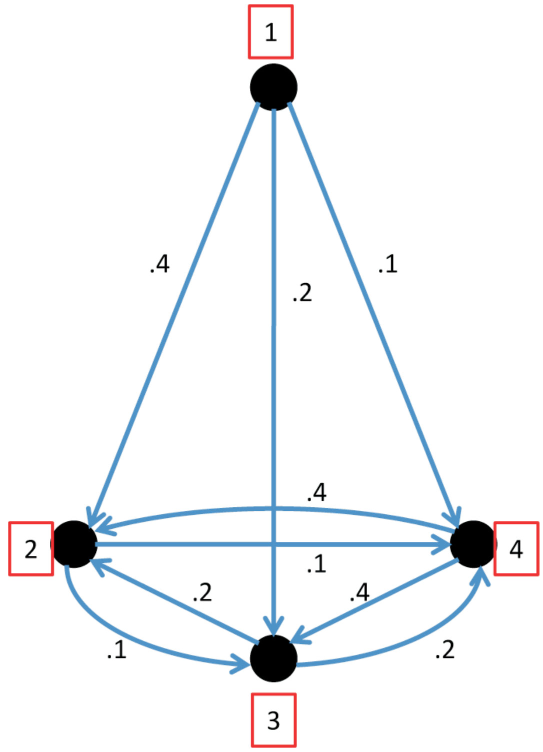

Figure 3.4: A simplistic wireless P2P network.

to the $\ominus$ operation, they are no longer subject to the same precision problem. Numerical solution of (3.20) can be obtained by any DE solvers fairly efficiently. To demonstrate this, consider the following example illustrating a simplistic wireless network that employs RNC in a P2P-like transmission scheme, shown in Fig. 3.4. Assume $\lambda_i = 1$ sec^{-1}. The labels attached to the arrows show reception probabilities, which are independent to each other. This means, for example, $P_{1,2} = 0.4$, but $P_{1,\{2,3\}} = 1 - (1 - 0.4)(1 - 0.2) = 0.52$. We assume that node 1 is the server which has 400 packets to be downloaded to node 2, 3 and 4 with RNC. Like a typical wired P2P network, node 2, 3 and 4 broadcast to each other to enhance efficiency. Fig. 3.5 shows the rank evolution at the four nodes, through both simulation and the solution to the corresponding simplified DEs. It is evident that the DE solution fits the simulated curves nicely.

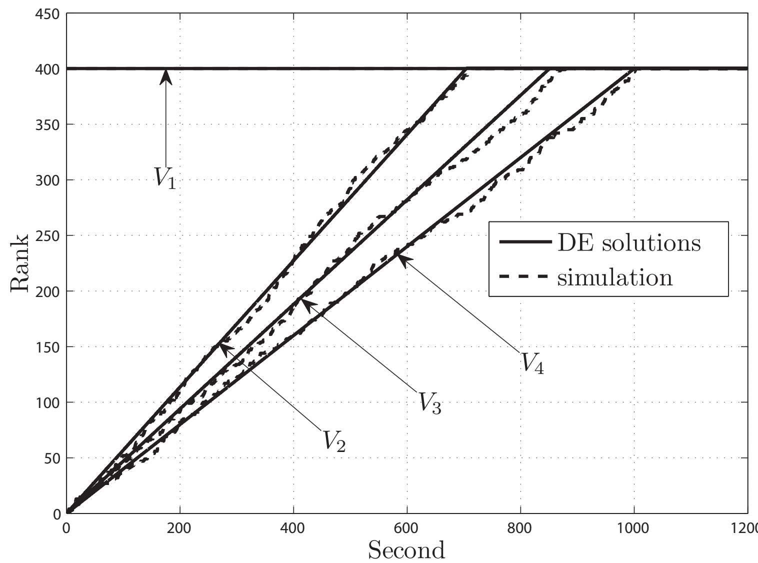

Figure 3.5: Rank evolution of the simplistic wireless P2P network is obtained through simulation as well as solution to the corresponding DEs.

3.3 Rank Evolution Modeled with DI

While (3.20) can be numerically evaluated with any DE solver, it is not amenable to analysis due to the discontinuous right-hand sides. Besides, the approximation shown in (3.19) becomes most inaccurate when $V_{\{i\}\cup\mathcal{K}} - V_{\mathcal{K}} \to 0^+$. In fact, no matter what q is, if $V_{\{i\}\cup\mathcal{K}} - V_{\mathcal{K}}$ is sufficiently small, $1 - q^{V_{\mathcal{K}} - V_{\mathcal{K}\cup\{i\}}}$ can take on any value in $[0, 1)$. This discrepancy prompts us to modify the right-hand side of (3.20) to incorporate semicontinuity [16], which allows a range of values for $V_{\{i\}\cup\mathcal{K}}(t) \ominus V_{\mathcal{K}}(t)$ to choose when $V_{\mathcal{K}}(t) = V_{\{i\}\cup\mathcal{K}}(t)$. Specifically, we define an upper semicontinuous function $\mathrm{Sgn}^+ : \mathbb{R} \to 2^{\mathbb{R}}$

$$\mathrm{Sgn}^+(x) = \begin{cases} \{0\}, & x < 0 \\ [0, 1], & x = 0 \\ \{1\}, & x > 0 \end{cases} \tag{3.22}$$

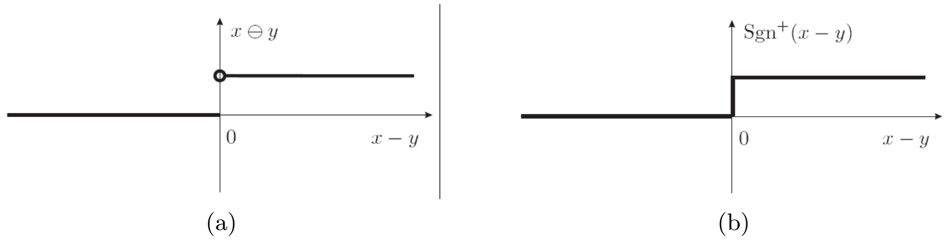

Figure 3.6: (a) Plot of $x \ominus y$ as a function of $x - y$. (b) Plot of Sgn^+ as a set-valued function of $x - y$.

to replace the "$\ominus$" operation:

$$\dot{V}_{\mathcal{K}} \in \sum_{i \notin \mathcal{K}} z_{i\mathcal{K}} \mathrm{Sgn}^+(V_{\mathcal{K}\cup\{i\}} - V_{\mathcal{K}}), \quad \forall \mathcal{K} \subset \mathcal{N}. \tag{3.23}$$

Fig. 3.6 illustrates the conversion, where the Sgn^+ function shown in 3.6(b) has apparently re-acquired certain continuity compared to the jump discontinuity shown in 3.6(a). The same boundary condition in (3.18) still holds. To be compatible with (3.20), when $\mathcal{K} = \mathcal{N}$, we define the right-hand side of (3.23) to be $\{0\}$ instead of $\emptyset$. In the mathematical literature, the system of inclusions in (3.23) plus the same boundary condition in (3.18) is called differential inclusions (DI), first systematically studied by A. F. Filippov [17]. DI is a generalization of the dynamical system described by DEs, allowing them, in particular, to have discontinuous right-hand sides, which is exactly the case in (3.20). Such dynamical systems with derivative discontinuities arise extensively in mechanics, electronics and biology. For example, an initial value problem on a time interval $[0, \infty)$ for DI takes the following form

$$\dot{\underline{x}} \in F(t, \underline{x}), \quad \underline{x}(0) = \underline{x}_0. \tag{3.24}$$

where $\underline{x}(t) \in \mathbb{R}^d$ is the state vector, $F : [0, \infty) \times \mathbb{R}^d \to 2^{\mathbb{R}^d}$ is a set-valued function and d is the dimension of the dynamical system. Its solution is defined to be an absolutely continuous function $\underline{y}(t)$ such that $\underline{y}(0) = \underline{x}_0$ and $\dot{\underline{y}}(t) \in F(t, \underline{y}(t))$ almost everywhere

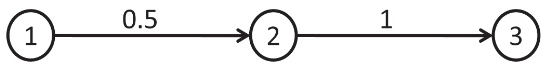

Figure 3.7: A three node network with linear topology. Node 1 tries to deliver 100 packets to node 2 and node 3.

in $[0, \infty)$. In this article, however, owing to the particular form of the Sgn^+ function, we will be dealing with a special collection of DI's such that the solutions only need to be continuous functions satisfying the inclusion at all but finitely many points in $[0, \infty)$.It is clear that any solutions to (3.20) are necessarily solutions to (3.23). It is possible that the reformulation via DI's could enlarge the set of solutions. However, as we will see, for our specific problem, (3.23) turns out to have a unique solution in our discussion.

The generalization from (3.20) to (3.23) not only paves the way for easy analysis of RNC, but also furnishes a better interpretation to the solution of (3.20), which is illustrated by the example shown in Fig. 3.7. Suppose we wish to use RNC in a network consisting of three nodes 1, 2 and 3 to deliver $m = 100$ packets from node 1 to node 2 and 3. The network has a linear topology shown in Fig. 3.7. Based on the underlying MAC, node 1 transmits at 0.5 packets/second to node 2 which transmits at 1 packets/second to node 3, i.e. $z_{12} = z_{1,\{23\}} = 0.5 \text{ sec}^{-1}$, $z_{23} = 1 \text{ sec}^{-1}$. We wish to know at what rates $V_2(t)$ and $V_3(t)$ increase by solving the corresponding system of DEs as given in (3.20):

$$\dot{V}_2 = z_{12}(m \ominus V_2), \tag{3.25}$$

$$\dot{V}_3 = z_{23}(V_{\{23\}} \ominus V_3), \tag{3.26}$$

$$\dot{V}_{\{23\}} = z_{1,\{23\}}(m \ominus V_{\{23\}}), \tag{3.27}$$

$$\text{B.C.} \quad V_2 = V_3 = V_{\{23\}} = 0, \tag{3.28}$$

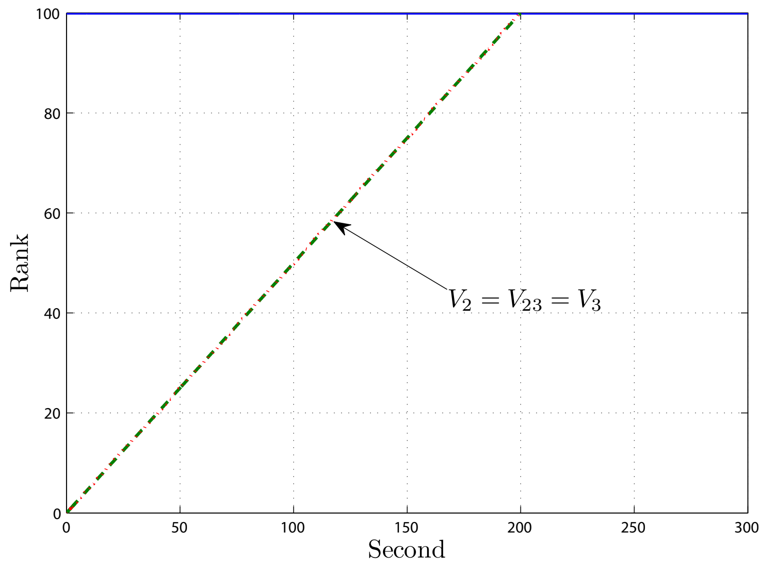

Figure 3.8: Bottle neck phenomenon: rank at node 3 follows rank at node 2. Numerical solver exhibits fluctuation of V_3 around V_2.

for which the solutions obtained by a numerical DE solver are shown in Fig. 3.8. We are not particularly interested in $V_{\{23\}}$ per se but by comparing (3.25) and (3.27) we observe that they have the same solutions, i.e., $V_2(t) = V_{\{23\}}(t)$, $\forall t$. In fact, Fig. 3.8 shows $V_2(t) = V_{\{23\}}(t) = V_3(t)$, $\forall t \geq 0$ and $\dot{V}_2(t) = \dot{V}_{\{23\}}(t) = \dot{V}_3(t) = 0.5$, $\forall t \in [0, 200)$. However, if we plug the solution back into (3.26), we get $\dot{V}_3(t) = 0$, $\forall t \in [0, 200)$. This discrepancy arises due to the discontinuous right-hand sides of the system of DEs in (3.25)–(3.27). This can be explained if we recast (3.25)–(3.27) into differential inclusions as follows.

$$\dot{V}_2 \in z_{12}\mathrm{Sgn}^+(m - V_2) = 0.5\mathrm{Sgn}^+(100 - V_2),$$

$$\dot{V}_3 \in z_{23}\mathrm{Sgn}^+(V_{\{23\}} - V_3) = \mathrm{Sgn}^+(V_{\{23\}} - V_3),$$

$$\dot{V}_{\{23\}} \in z_{1,\{23\}}\mathrm{Sgn}^+(m - V_{\{23\}}) = 0.5\mathrm{Sgn}^+(100 - V_{\{23\}}),$$

$$\text{B.C.} \quad V_2 = V_3 = V_{\{23\}} = 0.$$

By doing so, it is trivial to see that $V_2(t) = V_{\{23\}}(t) = V_3(t) = 0.5t$ is a solution for the system of differential inclusions for $t \in [0, 200)$.

Chapter 4

Analyzing Information Flows with Differential Equations – The Average Case

In this section, using a fluid approximation and the DE framework, we first show that the average throughput of RNC is determined by the min-cut bound. The concentration behavior is presented in the next section.

For the average case, we begin by explicitly solving the deterministic DE (3.23). We will directly deal with multiple multicast sessions and the general topology. Then we will specialize the results to show that the average throughput of a single multicast session is determined by the min-cut bound. In general, suppose we have a wireless network $G = (\mathcal{N}, \mathcal{E})$ and J independent multicast sessions and session j originates from a set of source nodes

$$\mathcal{S}_j = \{s_{j,1}, s_{j,2}, \ldots, s_{j,n_j}\}, \quad j = 1, 2, \ldots, J, \tag{4.1}$$

where each node in $\mathcal{S}_j$ contains the same set of m_j packets to be delivered to the rest of the network or part of it. Though it is not usual to have multiple source nodes in a multicast session in the traditional store-and-forward fashion due to coordination difficulties, using multiple source nodes when RNC is employed adds no coordination cost to the network because the coding operation at every source node, as well as other nodes in the network, remains independent and fully distributed. In addition, using multiple source nodes improves the throughput performance by providing diversity from the source side. As will be clear shortly, the throughput is determined by the min cut between the source nodes and destination nodes. Using multiple source nodes essentially enlarges the min cut. The definition in (4.1) also allows that a source node serves more than one multicast session and it contains as many sets of packets. To identify the source for any nonempty $\mathcal{K} \subset \mathcal{N}$, define

$$\mathrm{Src}(\mathcal{K}) = \{j | \mathcal{S}_j \cap \mathcal{K} = \emptyset, j = 1, 2, \ldots, J\}. \tag{4.2}$$

For the coding scheme, we let each node generate a coded packet by randomly linearly mixing all the packets it holds, regardless which multicast sessions these packets belong to. Suppose all the multicast sessions start synchronously from time 0 as an integral information flow. This scenario is captured by the following system of DI's:

$$\begin{aligned} &\dot{V}_{\mathcal{K}} \in \sum_{i \notin \mathcal{K}} z_{i,\mathcal{K}} \mathrm{Sgn}^{+}(V_{\{i\}\cup\mathcal{K}} - V_{\mathcal{K}}), \\ \text{B.C.} \quad & V_{\mathcal{K}}(0) = \sum_{\substack{1 \le j \le J \\ j \notin \mathrm{Src}(\mathcal{K})}} m_j. \end{aligned} \tag{4.3}$$

Recall that $c(\mathcal{K})$ is defined as the cut size of $\mathcal{K}$ (cf. (3.3)), we now state the following theorem which provides an explicit solution to (4.3):

Theorem 4.1. *The solution to* (4.3) *is given recursively as*

$$V_{\mathcal{K}}(t) = \min\{V_{\mathcal{K}}(0) + c(\mathcal{K})t, \min_{\ell \notin \mathcal{K}}\{V_{\{\ell\}\cup\mathcal{K}}(t)\}\}, \tag{4.4}$$

$$= \min\{V_{\mathcal{K}}(0) + c(\mathcal{K})t, \min_{\mathcal{K}' \supset \mathcal{K}}\{V_{\mathcal{K}'}(t)\}\}, \tag{4.5}$$

$$= \min_{\mathcal{K}' \supset \mathcal{K}}\{V'_{\mathcal{K}}(0) + c(\mathcal{K}')t\} \tag{4.6}$$

and

$$V_{\mathcal{N}}(t) = \sum_{j=1}^{J} m_j. \tag{4.7}$$

Besides, for each $\mathcal{K}$, there is a sequence

$$0 = t_0 < t_1 < \cdots < t_{n_{\mathcal{K}}-1} < t_{n_{\mathcal{K}}} = \infty \tag{4.8}$$

such that over $[t_p, t_{p+1})$, $p = 0, 1, \ldots, n_{\mathcal{K}} - 1$, $V_{\mathcal{K}}$ is affine:

$$V_{\mathcal{K}}(t) = V_{\mathcal{K}'}(0) + c(\mathcal{K}')t, \quad t \in [t_p, t_{p+1}), \tag{4.9}$$

where $\mathcal{K}'$ satisfies

$$\mathcal{K}' \in C_{\min}(\cup_{j \in \mathrm{Src}(\mathcal{K}')} \mathcal{S}_j, \mathcal{K}). \tag{4.10}$$

We need a few preliminaries for the proof of Theorem 4.1. We begin with Lemma 4.2 which gives a solution to (4.3) on an interval.

Lemma 4.2. *Suppose $V_{\mathcal{K}}(t_1)$ is known, (4.3) has a solution on $[t_1, t_2)$ given as*

$$V_{\mathcal{K}}(t) = V_{\mathcal{K}}(t_1) + z(t - t_1) \tag{4.11}$$

if there is a set of nodes $\mathcal{P}$ such that $\mathcal{P} \cap \mathcal{K} = \emptyset$ and $z \geq 0$ satisfying

1. *$V_{\{i\} \cup \mathcal{K}}(t) = V_{\mathcal{K}}(t_1) + z(t - t_1)$, $\forall t \in [t_1, t_2)$, $\forall i \in \mathcal{P}$;*

2. *$V_{\{j\} \cup \mathcal{K}} \geq V_{\mathcal{K}}(t_1) + z(t - t_1)$, $\forall t \in [t_1, t_2)$, $\forall j \notin \mathcal{K} \cup \mathcal{P}$;*

3. *$\sum_{j \notin \mathcal{K} \cup \mathcal{P}} z_{j,\mathcal{K}} \leq z \leq \sum_{i \notin \mathcal{K}} z_{i,\mathcal{K}}$.*

Proof. Suppose this is not true such that there is $t'' \in [t_1, t_2)$ satisfying

$$V_{\mathcal{K}}(t'') \neq V_{\mathcal{K}}(t_1) + z(t - t_1).$$

Let

$$t' = \sup_{t_1 \leq t \leq t''} \{V_{\mathcal{K}}(t) = V_{\mathcal{K}}(t_1) + z(t - t_1)\}, \tag{4.12}$$

then t' exists, $t_1 \le t' < t'' < t_2$ and $V_{\mathcal{K}}(t) \neq V_{\mathcal{K}}(t_1) + z(t - t_1)\ \forall t \in (t', t'']$ by definition. Because $V_{\mathcal{K}}(t)$ is continuous, either

$$V_{\mathcal{K}}(t) > V_{\mathcal{K}}(t_1) + z(t - t_1), \quad \forall t \in (t', t''], \tag{4.13}$$

or

$$V_{\mathcal{K}}(t) < V_{\mathcal{K}}(t_1) + z(t - t_1), \quad \forall t \in (t', t'']. \tag{4.14}$$

If (4.13) holds, by assumption 1, $V_{\{i\}\cup\mathcal{K}}(t) < V_K(t)$, $\forall t \in (t', t'']$, $\forall i \in \mathcal{P}$. So, with assumption 2,

$$\dot{V}_{\mathcal{K}}(t) = \sum_{i \notin \mathcal{K}} z_{i,\mathcal{K}} \mathrm{Sgn}^+(V_{\{i\}\cup\mathcal{K}} - V_{\mathcal{K}}) \le \sum_{j \notin \mathcal{K}\cup\mathcal{P}} z_{j,\mathcal{K}}, \tag{4.15}$$

thus

$$\begin{aligned}
V_{\mathcal{K}}(t'') &= V_{\mathcal{K}}(t') + \int_{t'}^{t''} \dot{V}_{\mathcal{K}}(t)dt \\
&= V_{\mathcal{K}}(t_1) + z(t' - t_1) + \int_{t'}^{t''} \dot{V}_{\mathcal{K}}(t)dt \\
&\le V_{\mathcal{K}}(t_1) + (t' - t_1) + \sum_{j \notin \mathcal{K}\cup\mathcal{P}} z_{j,\mathcal{K}}(t'' - t') \\
&\le V_{\mathcal{K}}(t_1) + z(t' - t_1) + z(t'' - t') \quad \text{(assumption 3)} \\
&= V_{\mathcal{K}}(t_1) + z(t'' - t_1),
\end{aligned} \tag{4.16}$$

which is a contradiction to (4.13). If (4.14) holds, by assumption 1 and 2,

$$\dot{V}_{\mathcal{K}} = \sum_{i \notin \mathcal{K}} z_{i,\mathcal{K}} \mathrm{Sgn}^+(V_{\{i\}\cup\mathcal{K}} - V_{\mathcal{K}}) = \sum_{i \notin \mathcal{K}} z_{i,\mathcal{K}}. \tag{4.17}$$

Then by assumption 3,

$$
\begin{aligned}
V_{\mathcal{K}}(t'') &= V_{\mathcal{K}}(t_1) + z(t' - t_1) + \int_{t'}^{t''} \dot{V}_{\mathcal{K}}(t)dt \\
&= V_{\mathcal{K}}(t_1) + z(t' - t_1) + \sum_{i \notin \mathcal{K}} z_{i,\mathcal{K}}(t'' - t') \\
&\geq V_{\mathcal{K}}(t_1) + z(t' - t_1) + z(t'' - t') \quad \text{(assumption 3)} \\
&= V_{\mathcal{K}}(t_1) + z(t'' - t_1), \qquad (4.18)
\end{aligned}
$$

which is a contradiction to (4.14). □

Proof to Theorem 4.1. Since (4.6) is the expansion of (4.4) and (4.5), they are equivalent, hence it suffices to prove (4.6). We prove this via induction on $|\mathcal{K}^c|$. When $|\mathcal{K}^c| = 0$, $\mathcal{K} = \mathcal{N}$, $V_{\mathcal{N}}(0) = \sum_{j=1}^{J} m_j$, $\forall t \geq 0$. Eq. (4.4)–(4.6) are trivially true. Assume it is true when $|\mathcal{K}^c| \leq k - 1$, we prove it is also true for $|\mathcal{K}^c| = k$. Let

$$U_{\mathcal{K}}(t) = \min_{\mathcal{K}' \supset \mathcal{K}} \{V_{\mathcal{K}'}(0) + c(\mathcal{K}')t\}, \qquad (4.19)$$

then $U_{\mathcal{K}}(t)$ is piecewise linear (since it is the minimum of a finitely many affine functions) and there is a sequence

$$0 = t_0 < t_1 < \cdots < t_{n_{\mathcal{K}}-1} < t_{n_{\mathcal{K}}} = \infty \qquad (4.20)$$

such that for each $p = 0, 1, \ldots, n_{\mathcal{K}} - 1$,

$$U_{\mathcal{K}}(t) = V_{\mathcal{K}'}(0) + c(\mathcal{K}')t, \quad t \in [t_p, t_{p+1}), \qquad (4.21)$$

for some $\mathcal{K}'$. We claim $\mathcal{K}' \in C_{\min}(\cup_{j \in \mathrm{Src}(\mathcal{K}')} \mathcal{S}_j, \mathcal{K})$. Otherwise, let $\mathcal{K}'' \in C_{\min}(\cup_{j \in \mathrm{Src}(\mathcal{K}')} \mathcal{S}_j, \mathcal{K})$ but $\mathcal{K}'' \neq \mathcal{K}$. By definition of min cut for the hypergraph model, we have

$$c(\mathcal{K}'') < c(\mathcal{K}'). \qquad (4.22)$$

Since $(\cup_{j\in \mathrm{Src}(\mathcal{K}')}\mathcal{S}_j)\cap \mathcal{K}'' = \emptyset$, $\mathrm{Src}(\mathcal{K}')\subset \mathrm{Src}(\mathcal{K}'')$, so

$$V_{\mathcal{K}''}(0) \leq V_{\mathcal{K}'}(0). \tag{4.23}$$

Therefore $\forall t \in (t_p, t_{p+1})$,

$$V_{\mathcal{K}''}(0) + c(\mathcal{K}'')t < V_{\mathcal{K}'}(0) + c(\mathcal{K}')t, \tag{4.24}$$

which is a contradiction to (4.19).

We want to show that $V_{\mathcal{K}}(t) = U_{\mathcal{K}}(t)$, $\forall t \in [t_p, t_{p+1})$, using Lemma 4.2, which amounts to checking three conditions. Let $\mathcal{P} = \mathcal{K}' \setminus \mathcal{K}$, $z = \sum_{i\notin \mathcal{K}'} z_{i,\mathcal{K}'} = c(\mathcal{K}')$. First note

$$V_{\{i\}\cup\mathcal{K}}(t) = U_{\mathcal{K}}(t), \quad \forall i \in \mathcal{P}, \forall t \in [t_p, t_{p+1}). \tag{4.25}$$

This is because, on one hand, $V_{\{i\}\cup\mathcal{K}}(t) \geq U_{\mathcal{K}}(t)$ by (4.19), while on the other hand, by induction assumption ($|\mathcal{N} \setminus (\{i\}\cup\mathcal{K})| = k-1$)

$$\begin{aligned} V_{\{i\}\cup\mathcal{K}}(t) &= \min_{(\{i\}\cup\mathcal{K})\subset\mathcal{K}''}\{V_{\mathcal{K}''}(0) + c(\mathcal{K}'')t\} \\ &\leq V_{\mathcal{K}'}(0) + c(\mathcal{K}')t \quad (\text{since } \{i\}\cup\mathcal{K}\subset\mathcal{K}') \\ &= U_{\mathcal{K}}(t). \end{aligned} \tag{4.26}$$

Meanwhile, by (4.19) we have

$$V_{\{j\}\cup\mathcal{K}}(t) \geq U_{\mathcal{K}}(t), \quad \forall t \in [t_p, t_{p+1}), \forall j \notin \mathcal{K}\cup\mathcal{P}. \tag{4.27}$$

Because $\mathcal{K}' = C_{\min}(\cup_{j\in\mathrm{Src}(\mathcal{K}')}\mathcal{S}_j, \mathcal{K})$, $z \leq \sum_{i\notin\mathcal{K}} z_{i,\mathcal{K}}$. Because $\mathcal{K}\subset\mathcal{K}'$,

$$\sum_{i\notin\mathcal{K}'} z_{i,\mathcal{K}} \leq \sum_{i\notin\mathcal{K}'} z_{i,\mathcal{K}'} = z. \tag{4.28}$$

Thus assumption 3 of Lemma 4.2 is checked for all t. We then check assumptions 1 and 2 piecewise. When $p = 0$,

$$U_{\mathcal{K}}(t_0) = U_{\mathcal{K}}(0) = \min_{\mathcal{K}' \supset \mathcal{K}} \{V_{\mathcal{K}'}(0)\} = V_{\mathcal{K}}(0) \tag{4.29}$$

Let $I(\cdot)$ be the indicator function that takes 1 when the predicate inside is true or 0 otherwise, then we have, for any $\mathcal{K}' \supset \mathcal{K}$,

$$\begin{aligned} V_{\mathcal{K}'}(0) &= \sum_{j=1}^{J} I(\mathcal{S}_j \cap \mathcal{K}' \neq \emptyset) m_j \\ &\geq \sum_{j=1}^{J} I(\mathcal{S}_j \cap \mathcal{K} \neq \emptyset) m_j = V_{\mathcal{K}}(0). \end{aligned} \tag{4.30}$$

From (4.25) and (4.30),

$$V_{\{i\} \cup \mathcal{K}}(t) = V_{\mathcal{K}'}(0) + c(\mathcal{K}')t = V_{\mathcal{K}}(0) + c(\mathcal{K}')t, \quad \forall t \in [t_0, t_1), \forall i \in \mathcal{P}, \tag{4.31}$$

Hence assumption 1 is checked for $[t_0, t_1)$. From (4.27) and (4.30),

$$V_{\{j\} \cup \mathcal{K}}(t) \geq V_{\mathcal{K}'}(0) + c(\mathcal{K}')t = V_{\mathcal{K}}(0) + c(\mathcal{K}')t, \quad \forall t \in [t_0, t_1), \forall j \notin \mathcal{K} \cup \mathcal{P}. \tag{4.32}$$

Hence assumption 2 is checked for $[t_0, t_1)$. Therefore $V_{\mathcal{K}}(t) = U_{\mathcal{K}}(t)$, $\forall t \in [t_0, t_1)$. But this in turn implies that $V_{\mathcal{K}}(t_1) = U_{\mathcal{K}}(t_1)$ by continuity (cf. 3.3), which implies that assumption 1 and 2 are also checked for $[t_1, t_2)$ (same argument as for $[t_0, t_1)$). Therefore $V_{\mathcal{K}}(t) = U_{\mathcal{K}}(t)$, $\forall t \in [t_1, t_2)$. Repeat this argument $n_{\mathcal{K}}$ times, we conclude that $V_{\mathcal{K}}(t) = U_{\mathcal{K}}(t)$, $\forall t \geq 0$. This shows the validity of (4.6) for $|\mathcal{K}^c| = k$.

□

Essentially, Theorem 4.1 (cf. Eq. (4.6)) states that $V_{\mathcal{K}}(t)$ is the min-envelope of $2^{|\mathcal{K}^c|}$ affine functions corresponding to the subsets of nodes that contain $\mathcal{K}$. The partial order "$\lesssim$" illustrated by the layered structure also implies the usual linear order "$\leq$", i.e.,

$$\mathcal{K} \lesssim \mathcal{K}' \Rightarrow V_{\mathcal{K}}(t) \leq V'_{\mathcal{K}}(t), \quad \forall t \geq 0. \tag{4.33}$$

Therefore it is always true

$$V_{\mathcal{K}} \leq V_{\mathcal{N}} = \sum_{j=1}^{J} m_j. \tag{4.34}$$

A stronger statement than (4.34) can be made when G is connected, i.e.,

Corollary 4.3. *If $G = (\mathcal{N}, \mathcal{E})$ is connected, then $\forall \mathcal{K} \neq \emptyset$*

$$V_{\mathcal{K}}(t) = \sum_{j=1}^{J} m_j, \quad t \in [t_{n_{\mathcal{K}}-1}, \infty). \tag{4.35}$$

Proof. Because G is connected, $\forall \mathcal{K} \subset \mathcal{K}' \subsetneq \mathcal{N}$, $c(\mathcal{K}') > 0$. Therefore when t is sufficiently large,

$$V'_{\mathcal{K}}(0) + c(\mathcal{K}')t > \sum_{j=1}^{J} m_j = V_{\mathcal{N}}(t).$$

Hence we have the conclusion from (4.6) of Theorem 4.1. □

Corollary 4.3 implies that with the RNC scheme for multiple flows as described here, a node may have to wait until its rank reaches $\sum_{j=1}^{J} m_j$ to start decoding. This time is denoted as $T_{\mathcal{K}}^{\text{total}}$. Though there could be fairly large decoding delay for nodes only interested in one or few sessions, the intersession coding is optimal in the sense of min cut bound. Applying (4.9) in Theorem 4.1 to $[t_{n_{\mathcal{K}}-1}, t_{n_{\mathcal{K}}})$, it is clear that $T_{\mathcal{K}}^{\text{total}}$ is determined by one of the min cut bounds that $\mathcal{K}$ has to take into consideration. The min cut that determines the finish time can be regarded as the worst bottleneck for $\mathcal{K}$. More precisely, we have

Corollary 4.4. *If $G = (\mathcal{N}, \mathcal{E})$ is connected, then*

$$T_{\mathcal{K}}^{total} = \max_{S \subset \mathrm{Src}(\mathcal{K})} \{\sum_{j \in S} m_j / c_{\min}(\cup_{i \in S} \mathcal{S}_i, \mathcal{K}))\}. \tag{4.36}$$

Proof. Clearly $T_{\mathcal{K}}^{\text{total}} = t_{n_{\mathcal{K}}-1}$. By Theorem 4.1, there is $\mathcal{K}' \supset \mathcal{K}$ such that $\forall t \in [t_{n_{\mathcal{K}}-2}, t_{n_{\mathcal{K}}-1})$,

$$\begin{aligned} V_{\mathcal{K}}(t) &= V_{\mathcal{K}'}(0) + c(\mathcal{K}')t \\ &= \sum_{j \notin \mathrm{Src}(\mathcal{K}')} m_j + c_{\min}(\cup_{i \in \mathrm{Src}(\mathcal{K}')} \mathcal{S}_i, \mathcal{K})t, \end{aligned} \tag{4.37}$$

and by setting $V_{\mathcal{K}}(t_{n_{\mathcal{K}}-1}) = \sum_{j=1}^{J} m_j$, we get

$$\begin{aligned} T_{\mathcal{K}}^{\text{total}} &= t_{n_{\mathcal{K}}-1} \\ &= \left(\sum_{j=1}^{J} m_j - \sum_{\substack{1 \le j' \le J \\ j' \notin \mathrm{Src}(\mathcal{K}')}} m_{j'} \right) \Bigg/ c_{\min}(\cup_{i \in \mathrm{Src}(\mathcal{K}')} \mathcal{S}_i, \mathcal{K}) \\ &= \sum_{j \in \mathrm{Src}(\mathcal{K}')} m_j / c_{\min}(\cup_{i \in \mathrm{Src}(\mathcal{K}')} \mathcal{S}_i, \mathcal{K}) \\ &\le \max_{S \subset \mathrm{Src}(\mathcal{K})} \{ \sum_{j \in S} m_j / c_{\min}(\cup_{i \in S} \mathcal{S}_i, \mathcal{K}) \}, \end{aligned} \tag{4.38}$$

where the last inequality holds because $\mathcal{K}' \supset \mathcal{K}$, hence $\mathrm{Src}(\mathcal{K}') \subset \mathrm{Src}(\mathcal{K})$. However, if there is $S' \subset \mathrm{Src}(\mathcal{K})$, such that

$$\sum_{j \in S'} m_j / c_{\min}(\cup_{i \in S'} \mathcal{S}_i, \mathcal{K}) > T_{\mathcal{K}}^{\text{total}}, \tag{4.39}$$

let $\mathcal{K}'' = C_{\min}(\cup_{i \in S'} \mathcal{S}_i, \mathcal{K})$, then we have

$$V_{\mathcal{K}''}(0) \le \sum_{\substack{1 \le j'' \le J \\ j'' \notin S'}} m_{j''} \tag{4.40}$$

because $S' \subset \mathrm{Src}(\mathcal{K}'')$, and

$$\begin{aligned} V_{\mathcal{K}''}(0) + c(\mathcal{K}'')T_{\mathcal{K}}^{\text{total}} &= V_{\mathcal{K}''}(0) + c_{\min}(\cup_{i \in S'} \mathcal{S}_i, \mathcal{K}) T_{\mathcal{K}}^{\text{total}} \\ &< \sum_{\substack{1 \le j'' \le J \\ j'' \notin S'}} m_{j''} + \sum_{j \in S'} m_j = \sum_{j=1}^{J} m_j = V_{\mathcal{K}}(T_{\mathcal{K}}^{\text{total}}), \end{aligned} \tag{4.41}$$

which contradicts (4.33). So

$$T_{\mathcal{K}}^{\text{total}} \geq \max_{S \subset \text{Src}(\mathcal{K})} \{\sum_{j \in S} m_j / c_{\min}(\cup_{i \in S} \mathcal{S}_i, \mathcal{K}))\}. \tag{4.42}$$

Combine (4.38) with (4.42), we get (4.36). □

From Theorem 4.1 and Corollary 4.3 we can readily show that the average throughput under the fluid approximation is given by the min-cut bound, i.e., we have

Theorem 4.5. *If $G = (\mathcal{N}, \mathcal{E})$ is connected and node 1 is the single source of a multicast session, the solution to (3.23) with B.C. (3.18) that describes this scenario is given as*

1. $\forall \mathcal{K} \subset \mathcal{N}$ and $1 \in \mathcal{K}$,

$$V_{\mathcal{K}}(t) = m, \quad \forall t \in [0, \infty); \tag{4.43}$$

2. $\forall \mathcal{K} \subset \mathcal{N}$ and $1 \notin \mathcal{K}$,

$$V_{\mathcal{K}}(t) = \begin{cases} c_{\min}(1, \mathcal{K})t, & \forall t \in [0, m/c_{\min}(1, \mathcal{K})), \\ m, & \forall t \in [m/c_{\min}(1, \mathcal{K}), \infty). \end{cases} \tag{4.44}$$

Proof. By Theorem 4.1, the solution to (3.23) for each $\mathcal{K}$ is piecewise linear. If $1 \in \mathcal{K}$, for any $\mathcal{K}'$ that satisfy (4.9) and (4.10), we must have $V_{\mathcal{K}'}(0) = m$ and $c(\mathcal{K}') = 0$. This implies $V_{\mathcal{K}}(t)$ is a constant. By Corollary 4.3, we have (4.43). If $1 \notin \mathcal{K}$, there are two possibilities:

1. $\mathcal{K}' \in (1, \mathcal{K})$ and $c(\mathcal{K}') = c_{\min}(1, \mathcal{K})$;

2. $\mathcal{K}' \in (\emptyset, \mathcal{K})$ and $c(\mathcal{K}') = 0$.

By Corollary 4.3, the second case applies to $t \in [t_1, \infty)$, hence the first case applies to $t \in [t_0, t_1)$ as we know $V_{\mathcal{K}}(0) = 0$. We therefore have (4.44). □

Theorem 4.5 states that the rank of $\mathcal{K}$ increases until it reaches m at the rate allowed by the min cut that separates $\mathcal{K}$ from the source.

Corollary 4.6. *For* $1 \notin \mathcal{K}$, $\dot{V}_{\mathcal{K}} = c_{\min}(1, \mathcal{K})$, *when* $V_{\mathcal{K}} < m$.

Specializing Corollary 4.6 to an arbitrary destination node i, we obtain the same result shown in [9]:

Corollary 4.7. *For* $i \neq 1$, $\dot{V}_i = c_{\min}(1, i)$, *when* $V_i < m$.

Corollary 4.7 shows that if a unicast at average rate R exists for each destination i separately, i.e., $c_{\min}(1, i) \geq R$, then the proposed coding scheme is capable to implement a multicast at average rate R.

Chapter 5

Concentration Behavior of Differential Equation Solution – The Asymptotic Case

Chapter 4 presented an average analysis of RNC throughput based on the fluid approximation. In this section we show that this throughput can be achieved asymptotically with increasing number of source packets m. This asymptotic result was previously proven in [9] using a queueing approach and graph decomposition. In this thesis, we begin with (3.6) and solely work with differential equations to show the same result. This perspective on RNC is new.

To motivate the achievability problem, we first prove a weak[1] version of min-cut max-flow theorem for RNC.

Theorem 5.1. *Assume node* 1 *is the only source in the network and the transmission begins at* $t = 0$*. Let* $N_{\mathcal{K}}(t)$ *be the incremental process of innovative packets at* $\mathcal{K} \subset \mathcal{N} \setminus \{1\}$*, then*

$$E[N_{\mathcal{K}}(t)] \leq c_{\min}(1, \mathcal{K})t. \tag{5.1}$$

[1] A stronger version would say $\mathrm{d}E[N_{\mathcal{K}}(t)]/\mathrm{d}t \leq c_{\min}(1, \mathcal{K})$.

Proof. Suppose $\mathcal{T} \in C_{\min}(1, \mathcal{K})$. We have, from Lemma 3.1,

$$\frac{\mathrm{d}E[N_{\mathcal{T}}(t)]}{\mathrm{d}t} = \sum_{i \in \mathcal{T}^c} z_{i,\mathcal{T}} E[1 - q^{N_{\mathcal{T}}(t) - N_{\{i\} \cup \mathcal{T}}(t)}] \tag{5.2}$$

$$\leq \sum_{i \in \mathcal{T}^c} z_{i,\mathcal{T}} = c_{\min}(1, \mathcal{K}). \tag{5.3}$$

Since $N_{\mathcal{K}}(0) = 0$, we have

$$E[N_{\mathcal{K}}(t)] \leq E[N_{\mathcal{T}}(t)] \leq c_{\min}(1, \mathcal{K})\, t. \tag{5.4}$$

□

Though Theorem 5.1 indicates that the time average throughput of RNC is governed by the min-cut bound, we will show that the min-cut bound can be asymptotically achieved. In this section, we assume the hypergraph is connected and we give the asymptotic achievability proof of RNC within the DE framework. For any $\mathcal{K} \subset \mathcal{N}$, we let $D_{\mathcal{K}}(m_1, m_2)$ denote the time taken for $N_{\mathcal{K}}$ to increase from m_1 to m_2. We will prove $\forall 1 > \epsilon > 0$

$$\lim_{m \to \infty} P\left(m / D_{\mathcal{K}}(0, m) > \epsilon c_{\min}(1, \mathcal{K})\right) = 1. \tag{5.5}$$

In order to prove this, we will follow the strategy outlined as below: We begin with Lemma 5.2 that is fundamental for the argument. Lemma 5.3 builds a system of DEs for $\mathrm{var}[N_{\mathcal{K}}(t)]$, from which we will prove Theorem 5.4 with the help of Lemma 5.2 that says the standard deviation of $N_{\mathcal{K}}(t)$ is upper bounded by a sublinear function. Then we scale $N_{\mathcal{K}}(t)$ to produce a new process $M_{\mathcal{K}}(t)$ and show in Theorem 5.6 that $M_{\mathcal{K}}(t)$ converges in probability to $c_{\min}(1, \mathcal{K})t$. While the following Corollary 5.7 implies that the throughput is given by $c_{\min}(1, \mathcal{K})$ for the entire course of transmission except the last few packets, Proposition 5.8 and Corollary 5.9 show that the last few packets take bounded time to get transmitted. Theorem 5.10 combines Corollary 5.7 and Corollary 5.9 to complete the achievability proof.

Lemma 5.2. *Let X be an r.v. and let $f : \mathbb{R} \to \mathbb{R}$ be a decreasing function, then*

$$E[Xf(X)] \leq E[X]E[f(X)]. \tag{5.6}$$

In other words,

$$\mathrm{cov}[X, f(X)] \leq 0. \tag{5.7}$$

Proof. Let X' be an independent copy of X. Since f is decreasing, we have

$$(X - X')(f(X) - f(X')) \leq 0 \quad \text{a.s.}, \tag{5.8}$$

so

$$E[(X - X')(f(X) - f(X')] \leq 0. \tag{5.9}$$

Expanding (5.9) we have (5.6). □

It should be pointed out that a more general statement of Lemma 5.2 can be found in [18].

Lemma 5.3. *For any set $\mathcal{K}$ of nodes, we have*

$$\frac{\mathrm{dvar}[N_{\mathcal{K}}(t)]}{\mathrm{d}t} = \frac{\mathrm{d}E[N_{\mathcal{K}}(t)]}{\mathrm{d}t} + 2\sum_{i \in \mathcal{K}^c} z_{i,\mathcal{K}} \mathrm{cov}[N_{\mathcal{K}}(t), 1 - q^{N_{\mathcal{K}}(t) - N_{\{i\} \cup \mathcal{K}}(t)}]. \tag{5.10}$$

Proof. The proof is manifest in the following computation.

$$N_{\mathcal{K}}(t + \Delta t) = N_{\mathcal{K}}(t) + \Delta_{\mathcal{K}}(t), \tag{5.11}$$

so

$$N_{\mathcal{K}}^2(t) = N_{\mathcal{K}}^2(t) + \Delta_{\mathcal{K}}^2(t) + 2N_{\mathcal{K}}(t)\Delta_{\mathcal{K}}(t). \tag{5.12}$$

We have

$$E[\Delta_{\mathcal{K}}^2(t)] = \sum_{i\in\mathcal{K}^2} \lambda_i \Delta t P_{i,\mathcal{K}} E[1 - q^{N_{\mathcal{K}}(t) - N_{\{i\}\cup\mathcal{K}}(t)}] + o(\Delta t), \tag{5.13}$$

and

$$\begin{aligned}&E[N_{\mathcal{K}}(t)\Delta_{\mathcal{K}}(t)] \\ =&\sum_{i\in\mathcal{K}^c} \lambda_i \Delta t P_{i,\mathcal{K}} E[N_{\mathcal{K}}(t)(1 - q^{N_{\mathcal{K}}(t) - N_{\{i\}\cup\mathcal{K}}(t)})] + o(\Delta t).\end{aligned} \tag{5.14}$$

Therefore

$$\begin{aligned}&\frac{\mathrm{d}E[N_{\mathcal{K}}^2(t)]}{\mathrm{d}t} \\ =&\sum_{i\in\mathcal{K}^2} \lambda_i P_{i,\mathcal{K}} E[1 - q^{N_{\mathcal{K}}(t) - N_{\{i\}\cup\mathcal{K}}(t)}] \\ &+ \sum_{i\in\mathcal{K}^c} \lambda_i P_{i,\mathcal{K}} E[N_{\mathcal{K}}(t)(1 - q^{N_{\mathcal{K}}(t) - N_{\{i\}\cup\mathcal{K}}(t)})] \\ =&\frac{\mathrm{d}E[N_{\mathcal{K}}(t)]}{\mathrm{d}t} + \sum_{i\in\mathcal{K}^c} z_{i,\mathcal{K}} E[N_{\mathcal{K}}(t)(1 - q^{N_{\mathcal{K}}(t) - N_{\{i\}\cup\mathcal{K}}(t)})],\end{aligned} \tag{5.15}$$

where the last equality follows from Lemma 3.1. We also have

$$\begin{aligned}&\frac{\mathrm{d}E^2[N_{\mathcal{K}}(t)]}{\mathrm{d}t} \\ =&2E[N_{\mathcal{K}}(t)]\frac{\mathrm{d}E[N_{\mathcal{K}}(t)]}{\mathrm{d}t} \\ =&2\sum_{i\in\mathcal{K}^c} z_{i,\mathcal{K}} E[N_{\mathcal{K}}(t)] E[1 - q^{N_{\mathcal{K}}(t) - N_{\{i\}\cup\mathcal{K}}(t)}],\end{aligned} \tag{5.16}$$

so

$$\begin{aligned}&\frac{\mathrm{dvar}[N_{\mathcal{K}}(t)]}{\mathrm{d}t} \\ =&\frac{\mathrm{d}E[N_{\mathcal{K}}^2(t)]}{\mathrm{d}t} - \frac{\mathrm{d}E^2[N_{\mathcal{K}}(t)]}{\mathrm{d}t} \\ =&\frac{\mathrm{d}E[N_{\mathcal{K}}(t)]}{\mathrm{d}t} + 2\sum_{i\in\mathcal{K}^c} z_{i,\mathcal{K}} \mathrm{cov}[N_{\mathcal{K}}(t), 1 - q^{N_{\mathcal{K}}(t) - N_{\{i\}\cup\mathcal{K}}(t)}].\end{aligned} \tag{5.17}$$

□

Theorem 5.4 bounds $\sqrt{\text{var}[N_{\mathcal{K}}(t)]}$ with a sublinear function $d_{\mathcal{K}}t^{1-\delta_{\mathcal{K}}/2}$ in time. Since we know $E[N_{\mathcal{K}}(t)] \approx c_{\min}(1,\mathcal{K})t$ from Chapter 4, it later enables us to use Chebyshev inequality to show concentration of $N_{\mathcal{K}}(t)$.

Theorem 5.4. *There exists constants $d_{\mathcal{K}} > 0$ and $1 \geq \delta_{\mathcal{K}} > 0$ for each $\mathcal{K}$ independent of q, m, such that*

$$\limsup_{t\to\infty} \frac{\text{var}[N_{\mathcal{K}}(t)]}{t^{2-\delta_{\mathcal{K}}}} \leq d_{\mathcal{K}}. \tag{5.18}$$

Proof. We first make a trivial observation that, if we let

$$a_{\mathcal{K}} = \sum_{i\in\mathcal{K}^c} z_{i,\mathcal{K}}, \tag{5.19}$$

then

$$\frac{\mathrm{d}E[N_{\mathcal{K}}(t)]}{\mathrm{d}t} \leq a_{\mathcal{K}} \tag{5.20}$$

by Lemma 3.1. The rest of the proof is by induction using the partial order. When $\mathcal{K} = \{1\}^c$, we have

$$\frac{\mathrm{d}\text{var}[N_{\mathcal{K}}(t)]}{\mathrm{d}t} = \frac{\mathrm{d}E[N_{\mathcal{K}}(t)]}{\mathrm{d}t} + 2\sum_{i\in\mathcal{K}^c} z_{i,\mathcal{K}}\text{cov}[N_{\mathcal{K}}(t), 1-q^{N_{\mathcal{K}}(t)-m}]. \tag{5.21}$$

Since $1-q^{N_{\mathcal{K}}(t)-m}$ is decreasing and concave in $N_{\mathcal{K}}(t)$, by Lemma 5.2, we know

$$\frac{\mathrm{d}\text{var}[N_{\mathcal{K}}(t)]}{\mathrm{d}t} \leq \frac{\mathrm{d}E[N_{\mathcal{K}}(t)]}{\mathrm{d}t}, \tag{5.22}$$

so

$$\text{var}[N_{\mathcal{K}}(t)] \leq E[N_{\mathcal{K}}(t)] \leq a_{\mathcal{K}}t \tag{5.23}$$

by (5.20), as $\text{var}[N_{\mathcal{K}}(0)] = E[N_{\mathcal{K}}(0)] = 0$. Therefore we can assign $d_{\mathcal{K}} = a_{\mathcal{K}}$ and $\delta_{\mathcal{K}} = 1$. Now suppose the statement is true $\forall\mathcal{K}' \supset \mathcal{K}$, $\mathcal{K}' \neq \mathcal{K}$, we prove it is also true for $\mathcal{K}$.

First note

$$\begin{aligned}
&\text{cov}[N_{\mathcal{K}}(t), 1 - q^{N_{\mathcal{K}}(t) - N_{\{i\}\cup\mathcal{K}}(t)}] \\
=&\text{cov}[N_{\mathcal{K}}(t) - N_{\{i\}\cup\mathcal{K}}(t), 1 - q^{N_{\mathcal{K}}(t) - N_{\{i\}\cup\mathcal{K}}(t)}] \\
&+ \text{cov}[N_{\{i\}\cup\mathcal{K}}(t), 1 - q^{N_{\mathcal{K}}(t) - N_{\{i\}\cup\mathcal{K}}(t)}]. \qquad (5.24)
\end{aligned}$$

The first term on the right-hand side of (5.24) is non-positive by Lemma 5.2. By Cauchy-Schwarz inequality, the second term can be upper bounded as

$$\begin{aligned}
&\text{cov}[N_{\{i\}\cup\mathcal{K}}(t), 1 - q^{N_{\mathcal{K}}(t) - N_{\{i\}\cup\mathcal{K}}(t)}] \\
\leq &\sqrt{\text{var}[N_{\{i\}\cup\mathcal{K}}(t)]}\sqrt{\text{var}[1 - q^{N_{\mathcal{K}}(t) - N_{\{i\}\cup\mathcal{K}}(t)}]} \\
\leq &\sqrt{\text{var}[N_{\{i\}\cup\mathcal{K}}(t)]} \\
\leq &\sqrt{d_{\{i\}\cup\mathcal{K}}}\, t^{1-\delta_{\{i\}\cup\mathcal{K}}/2}, \qquad (5.25)
\end{aligned}$$

so

$$\frac{\text{dvar}[N_{\mathcal{K}}(t)]}{\text{d}t} \leq \frac{\text{d}E[N_{\mathcal{K}}(t)]}{\text{d}t} + 2\sum_{i\in\mathcal{K}^c} z_{i,\mathcal{K}}\sqrt{d_{\{i\}\cup\mathcal{K}}}\, t^{1-\delta_{\mathcal{K}}/2}. \qquad (5.26)$$

By induction, we have

$$\text{var}[N_{\mathcal{K}}(t)] \leq a_{\mathcal{K}} t + 2\sum_{i\in\mathcal{K}^c} \frac{z_{i,\mathcal{K}}\sqrt{d_{\{i\}\cup\mathcal{K}}}}{2 - \delta_{\{i\}\cup\mathcal{K}}/2}\, t^{2-\delta_{\{i\}\cup\mathcal{K}}/2}, \qquad (5.27)$$

as $\text{var}[N_{\mathcal{K}}(0)] = E[N_{\mathcal{K}}(t)] = 0$. Therefore we can pick

$$d_{\mathcal{K}} = a_{\mathcal{K}} + 2\sum_{i\in\mathcal{K}^c} \frac{z_{i,\mathcal{K}}\sqrt{d_{\{i\}\cup\mathcal{K}}}}{2 - \delta_{\{i\}\cup\mathcal{K}}/2} \qquad (5.28)$$

and

$$\delta_{\mathcal{K}} = \min\{1, \min_{i\in\mathcal{K}^c}\{\delta_{\{i\}\cup\mathcal{K}}/2\}\}. \qquad (5.29)$$

□

The next result is more conveniently discussed in terms of the scaled process defined as

$$M_{\mathcal{K}}^{(m)}(t) \triangleq \frac{1}{m} N_{\mathcal{K}}(mt). \tag{5.30}$$

The scaled process $M_{\mathcal{K}}^{(m)}(t)$ contains essentially the same information as $N_{\mathcal{K}}(t)$ except its graph is scaled down with a fixed aspect ratio. We list some obvious properties of $M_{\mathcal{K}}^{(m)}(t)$.

Proposition 5.5. *$M_{\mathcal{K}}^{(m)}(t)$ has the following properties:*

1. *$E[M_{\mathcal{K}}^{(m)}(t)]$ is increasing from 0 to 1 for $1 \notin \mathcal{K}$.*

2. *$M_{\mathcal{K}}^{(m)}(t) = 1$ for $1 \in \mathcal{K}$. For $1 \notin \mathcal{K}$, $M_{\mathcal{K}}^{(m)}(t)$ satisfies*
$$\frac{\mathrm{d}E[M_{\mathcal{K}}^{(m)}(t)]}{\mathrm{d}t} = \sum_{i \in \mathcal{K}^c} z_{i,\mathcal{K}} E\left[1 - (q^m)^{M_{\mathcal{K}}^{(m)}(t) - M_{\{i\}\cup\mathcal{K}}^{(m)}(t)}\right]. \tag{5.31}$$

3.
$$E[M_{\mathcal{K}}^{(m)}(t)] = \frac{1}{m} E[N_{\mathcal{K}}(mt)] \tag{5.32}$$
$$\leq c_{\min}(1,\mathcal{K})t, \quad \forall t \in [0, m/c_{\min}(1,\mathcal{K})), \forall \mathcal{K} \not\ni 1.$$

4. *$\{E[M_{\mathcal{K}}^{(m)}(t)]\}_m$ are uniformly Lipschitz continuous.*

5. *$\exists d_{\mathcal{K}} > 0$ and $\delta_{\mathcal{K}} > 0$ independent of m, such that*
$$\limsup_{m\to\infty} \frac{\mathrm{var}[M_{\mathcal{K}}^{(m)}(t)]}{m^{-\delta_{\mathcal{K}}}} \leq d_{\mathcal{K}} t^{2-\delta_{\mathcal{K}}} \tag{5.33}$$
for fixed t.

Proof. Property 1) is obvious. Property 2) follows from Lemma 3.1. Property 3) follows from Theorem 5.1. Property 4) follows from

$$\frac{\mathrm{d}E[N_{\mathcal{K}}(t)]}{\mathrm{d}t} \leq \sum_{i \in \mathcal{K}^c} z_{i,\mathcal{K}} \tag{5.34}$$

and

$$\frac{\mathrm{d}E[M_{\mathcal{K}}^{(m)}(t)]}{\mathrm{d}t} = \left.\frac{E[N_{\mathcal{K}}(\tau)]}{\mathrm{d}\tau}\right|_{\tau=mt}. \tag{5.35}$$

Property 5) follows from Lemma 5.3:

$$\begin{aligned}&\limsup_{m\to\infty} \frac{\mathrm{var}[M_{\mathcal{K}}^{(m)}(t)]}{m^{-\delta_{\mathcal{K}}}}\\ =&\limsup_{m\to\infty} \frac{\mathrm{var}\left[\frac{1}{m}N_{\mathcal{K}}(mt)\right]}{m^{-\delta_{\mathcal{K}}}} \le d_{\mathcal{K}} t^{2-\delta_{\mathcal{K}}}.\end{aligned} \tag{5.36}$$

□

Note that property 3) of Proposition 5.5 provides an upper bound on $E[M_{\mathcal{K}}^{(m)}(t)]$, we will show that in fact $M_{\mathcal{K}}^{(m)}(t)$ asymptotically concentrates to this upper bound, i.e.,

Theorem 5.6. *Given $\mathcal{K}$, $M_{\mathcal{K}}^{(m)}(t) \xrightarrow{p} c_{\min}(1,\mathcal{K})t$, $\forall t \in (0, 1/c_{\min}(1,\mathcal{K})]$ as $m \to \infty$.*

Proof. Due to property 4) of Proposition 5.5, $\{E[M_{\mathcal{K}}^{(m)}(t)]\}_m$ are Lipschitz continuous hence equicontinuous. They are also uniformly bounded by constant 1. By the Arzelà-Ascoli Theorem, $\forall \mathcal{K} \subset \mathcal{N}$, a subsequence $\{E[M_{\mathcal{K}}^{(m_\ell)}(t)]\}_{\ell=1}^{\infty}$ converges uniformly to some continuous function $M_{\mathcal{K}}(t)$ over $[0, 1/c_{\min}(1,\mathcal{K})]$. In fact, we can pick a single sequence $\{E[M_{\mathcal{K}}^{(m_\ell)}(t)]\}$ that converges uniformly to $M_{\mathcal{K}}(t)$ for all $\mathcal{K}$ because there are only a finite number of $\mathcal{K} \subset \mathcal{N}$. So we make this assumption and in what follows we prove $M_{\mathcal{K}}(t) = c_{\min}(1,\mathcal{K})\, t$ by contradiction and via induction on the partial order of $\mathcal{K}$.

When $\mathcal{K} = \{1\}^c$. Suppose at $t_2 \in (0, 1/c_{\min}(1,\mathcal{K})]$, $c_{\min}(1,\mathcal{K})t_2 - M_{\mathcal{K}}(t_2) = h > 0$. Because $M_{\mathcal{K}}(0) = 0$, by continuity, $\exists 0 < t_1 < t_2$, such that $c_{\min}(1,\mathcal{K})t_1 - M_{\mathcal{K}}(t_1) = h/2$ and

$$c_{\min}(1,\mathcal{K})t - M_{\mathcal{K}}(t) \ge h/2, \quad \forall t \in [t_1, t_2]. \tag{5.37}$$

Let

$$\lambda_{\mathcal{K}}^{\ell} = m_{\ell}^{\delta_{\mathcal{K}}/4}, \quad \text{then} \quad \lim_{\ell\to\infty} \frac{1}{(\lambda_{\mathcal{K}}^{\ell})^2} = 0,. \tag{5.38}$$

From property 5) of Proposition 5.5 we know

$$\lim_{\ell\to\infty} \lambda_{\mathcal{K}}^{\ell} \sqrt{\mathrm{var}[M_{\mathcal{K}}^{(m_\ell)}(t)]} = 0, \text{ uniformly } \forall t \in [t_1, t_2]. \tag{5.39}$$

Because $E[M_{\mathcal{K}}^{(m_\ell)}(t)] \stackrel{u}{\to} M_{\mathcal{K}}(t)$ on $[0, 1/c_{\min}(1,\mathcal{K})]$, $\exists L_1$ such that $\forall \ell > L_1$

$$|E[M_{\mathcal{K}}^{(m_\ell)}(t)] - M_{\mathcal{K}}(t)| < h/8, \quad \forall t \in [t_1, t_2], \tag{5.40}$$

so

$$E[M_{\mathcal{K}}^{(m_\ell)}(t)] < M_{\mathcal{K}}(t) + h/8, \quad \forall t \in [t_1, t_2], \forall \ell > L_1, \tag{5.41}$$

and in particular $\forall \ell > L_1$,

$$E[M_{\mathcal{K}}^{(m_\ell)}(t_1)] > c_{\min}(1,\mathcal{K})t_1 - 5h/8, \tag{5.42}$$

$$E[M_{\mathcal{K}}^{(m_\ell)}(t_2)] < c_{\min}(1,\mathcal{K})t_2 - 7h/8. \tag{5.43}$$

Let $p(\gamma)$ be the distribution function of $M_{\mathcal{K}}^{(m_\ell)}(t)$ and let A be the interval

$$A \triangleq \left[0, E[M_{\mathcal{K}}^{(m_\ell)}(t)] + \lambda_{\mathcal{K}}^{\ell} \sqrt{\mathrm{var}[M_{\mathcal{K}}^{(m_\ell)}(t)]}\right], \tag{5.44}$$

then we may calculate

$$\begin{aligned}
&\frac{\mathrm{d}E[M_{\mathcal{K}}^{(m_\ell)}(t)]}{\mathrm{d}t} \qquad (5.45)\\
&= z_{1,\mathcal{K}} E[1-(q^{m_\ell})^{M_{\mathcal{K}}^{(m_\ell)}(t)-1}]\\
&\geq \int_A p(\gamma) z_{1,\mathcal{K}}(1-(q^{m_\ell})^{\gamma-1})\mathrm{d}\gamma\\
&\geq \int_A p(\gamma) z_{1,\mathcal{K}}(1-(q^{m_\ell})^{E[M_{\mathcal{K}}^{(m_\ell)}(t)]+\lambda_{\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\mathcal{K}}^{(m_\ell)}(t)]}-1})\mathrm{d}\gamma\\
&> \int_A p(\gamma) z_{1,\mathcal{K}}(1-(q^{m_\ell})^{M_{\mathcal{K}}(t)+h/8+\lambda_{\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\mathcal{K}}^{(m_\ell)}(t)]}-1})\mathrm{d}\gamma\\
&= z_{1,\mathcal{K}}(1-(q^{m_\ell})^{M_{\mathcal{K}}(t)+h/8+\lambda_{\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\mathcal{K}}^{(m_\ell)}(t)]}-1})\\
&\cdot P\left(M_{\mathcal{K}}^{(m_\ell)}(t) < E[M_{\mathcal{K}}^{(m_\ell)}(t)] + \lambda_{\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\mathcal{K}}^{(m_\ell)}(t)]}\right)
\end{aligned}$$

But,

$$P\left(M_{\mathcal{K}}^{(m_\ell)}(t) < E[M_{\mathcal{K}}^{(m_\ell)}(t)] + \lambda_{\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\mathcal{K}}^{(m_\ell)}(t)]}\right) \geq 1 - \frac{1}{(\lambda_{\mathcal{K}}^{\ell})^2} \to 1, \quad (\ell \to \infty) \qquad (5.46)$$

by Chebyshev inequality, which is true $\forall t \in [t_1, t_2]$ uniformly. Using (5.37),(5.39) and noticing that $t \leq t_2 \leq 1/c_{\min}(1,\mathcal{K})$, we have

$$\limsup_{\ell\to\infty} M_{\mathcal{K}}(t) + h/8 + \lambda_{\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\mathcal{K}}^{(m_\ell)}(t)]} - 1 \leq c_{\min}(1,\mathcal{K})t - 3h/8 - 1 \leq -3h/8, \qquad (5.47)$$

so,

$$\lim_{\ell\to\infty} 1 - (q^{m_\ell})^{M_{\mathcal{K}}(t)+h/8+\lambda_{\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\mathcal{K}}^{(m_\ell)}(t)]}-1} = 1, \qquad (5.48)$$

which is also true $\forall t \in [t_1, t_2]$ uniformly. Therefore, given

$$\epsilon \in \left(0, \frac{h/4}{c_{\min}(1,\mathcal{K})(t_2-t_1)}\right), \qquad (5.49)$$

$\exists L_2 > L_1 > 0$, such that $\forall \ell > L_2$,

$$\frac{\mathrm{d}E[M_{\mathcal{K}}^{(m_\ell)}(t)]}{\mathrm{d}t} > (1-\epsilon)z_{1,\mathcal{K}} = (1-\epsilon)c_{\min}(1,\mathcal{K}), \quad \forall t \in [t_1, t_2]. \tag{5.50}$$

Therefore

$$\begin{aligned} & E[M_{\mathcal{K}}^{(m_\ell)}(t_2)] \\ \geq & E[M_{\mathcal{K}}^{(m_\ell)}(t_1)] + \int_{t_1}^{t_2} (1-\epsilon)c_{\min}(1,\mathcal{K})\mathrm{d} \\ > & c_{\min}(1,\mathcal{K})t_1 - 5h/8 + (1-\epsilon)c_{\min}(1,\mathcal{K})(t_2 - t_1) \\ = & c_{\min}(1,\mathcal{K})t_2 - 5h/8 - \epsilon c_{\min}(1,\mathcal{K})(t_2 - t_1) \\ > & c_{\min}(1,\mathcal{K})t_2 - 7h/8, \end{aligned} \tag{5.51}$$

which is a contradiction to (5.43). This shows $M_{\mathcal{K}}(t) = c_{\min}(1,\mathcal{K})\, t$ for $t \in [0, 1/c_{\min}(1,\mathcal{K})]$ when $\mathcal{K} = \{1\}^c$. Suppose this statement is true $\forall \mathcal{K}' \supset \mathcal{K}$, $\mathcal{K}' \neq \mathcal{K}$. We claim that we still have $M_{\mathcal{K}}(t) = c_{\min}(1,\mathcal{K})t$. Otherwise, suppose at $t_2 \in (0, 1/c_{\min}(1,\mathcal{K})]$ we have

$$h = c_{\min}(1,\mathcal{K})t_2 - M_{\mathcal{K}}(t_2) > 0, \tag{5.52}$$

then similarly, we can find $t_1 \in (0, t_2)$ such that

$$c_{\min}(1,\mathcal{K})t_1 - M_{\mathcal{K}}(t_1) = h/2 \tag{5.53}$$

and

$$c_{\min}(1,\mathcal{K})t - M_{\mathcal{K}}(t) \geq h/2, \quad \forall t \in [t_1, t_2]. \tag{5.54}$$

Reuse the definition of $\lambda_{\mathcal{K}}^{\ell}$ such that (5.38) and (5.39) still apply. Define $\lambda_{\{i\}\cup\mathcal{K}}^{\ell}$ in the similar way with respect to $\delta_{\{i\}\cup\mathcal{K}}$ and $M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)$. Because $E[M_{\mathcal{K}}^{(m_\ell)}(t)] \xrightarrow{u} M_{\mathcal{K}}(t)$ and $E[M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)] \xrightarrow{u} M_{\{i\}\cup\mathcal{K}}(t)$ on $[0, 1/c_{\min}(1,\mathcal{K})]$, $\exists L_1$ such that $\forall \ell > L_1$ and $\forall t \in [t_1, t_2]$,

$$\begin{aligned} |E[M_{\mathcal{K}}^{(m_\ell)}(t)] - M_{\mathcal{K}}(t)| < h/8, \\ |E[M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)] - M_{\{i\}\cup\mathcal{K}}(t)| < h/8. \end{aligned} \tag{5.55}$$

Let $p(\gamma_{\mathcal{K}}, \gamma_{\{i\}\cup\mathcal{K}})$ be the joint distribution function of $M_{\mathcal{K}}^{(m_\ell)}(t)$ and $M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)$. Let

$$A \triangleq \left[0, E[M_{\mathcal{K}}^{(m_\ell)}(t)] + \lambda_{\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\mathcal{K}}^{(m_\ell)}(t)]}\right] \times \left[E[M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)] - \lambda_{\{i\}\cup\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)]}, 1\right]. \quad (5.56)$$

Then we may calculate as follows

$$E[1 - (q^{m_\ell})^{M_{\mathcal{K}}^{(m_\ell)}(t) - M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)}] \geq \int_A p(\gamma_{\mathcal{K}}, \gamma_{\{i\}\cup\mathcal{K}})(1 - (q^{m_\ell})^{\gamma_{\mathcal{K}} - \gamma_{\{i\}\cup\mathcal{K}}}). \quad (5.57)$$

But when $(\gamma_{\mathcal{K}}, \gamma_{\{i\}\cup\mathcal{K}}) \in A$, we have

$$\gamma_{\mathcal{K}} - \gamma_{\{i\}\cup\mathcal{K}} \leq E[M_{\mathcal{K}}^{(m_\ell)}(t)] + \lambda_{\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\mathcal{K}}^{(m_\ell)}(t)]} - E[M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)] + \lambda_{\{i\}\cup\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)]}. \quad (5.58)$$

Using (5.54), (5.55) and the induction assumption $M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t) = c_{\min}(1, \{i\} \cup \mathcal{K})t$, we have

$$\begin{aligned}
\gamma_{\mathcal{K}} - \gamma_{\{i\}\cup\mathcal{K}} &< M_{\mathcal{K}}(t) + h/8 + \lambda_{\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\mathcal{K}}^{(m_\ell)}(t)]} \\
&\quad - M_{\{i\}\cup\mathcal{K}}(t) + h/8 + \lambda_{\{i\}\cup\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)]} \\
&\leq c_{\min}(1, \mathcal{K})t - h/2 + h/8 + \lambda_{\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\mathcal{K}}^{(m_\ell)}(t)]} \\
&\quad - c_{\min}(1, \{i\} \cup \mathcal{K})t + h/8 + \lambda_{\{i\}\cup\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)]} \\
&\leq -h/4 + \lambda_{\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\mathcal{K}}^{(m_\ell)}(t)]} + \lambda_{\{i\}\cup\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)]} \\
&\to -h/4, \quad (\ell \to \infty), \quad \text{uniformly } \forall t \in [t_1, t_2].
\end{aligned} \quad (5.59)$$

By Inclusion-Exclusion Principle, we have for any two events A and B

$$P(A \wedge B) \geq P(A) + P(B) - 1. \quad (5.60)$$

So

$$P\left(M_{\mathcal{K}}^{(m_\ell)}(t) < E[M_{\mathcal{K}}^{(m_\ell)}(t)] + \lambda_{\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\mathcal{K}}^{(m_\ell)}(t)]}\wedge\right.$$

$$\left. M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t) > E[M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)] - \lambda_{\{i\}\cup\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)]}\right)$$

$$\geq P\left(M_{\mathcal{K}}^{(m_\ell)}(t) < E[M_{\mathcal{K}}^{(m_\ell)}(t)] + \lambda_{\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\mathcal{K}}^{(m_\ell)}(t)]}\right)$$

$$+ P\left(M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t) > E[M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)] - \lambda_{\{i\}\cup\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)]}\right) - 1$$

$$> \left(1 - \frac{1}{(\lambda_{\mathcal{K}}^{\ell})^2}\right) + \left(1 - \frac{1}{(\lambda_{\{i\}\cup\mathcal{K}}^{\ell})^2}\right) - 1$$

$$\to 1, \quad (\ell \to \infty), \quad \text{uniformly } \forall t \in [t_1, t_2], \quad (5.61)$$

by Chebyshev Inequality and (5.38).

Consequently,

$$E[1 - (q^{m_\ell})^{M_{\mathcal{K}}^{(m_\ell)}(t) - M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)}]$$

$$> P\left(M_{\mathcal{K}}^{(m_\ell)}(t) < E[M_{\mathcal{K}}^{(m_\ell)}(t)] + \lambda_{\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\mathcal{K}}^{(m_\ell)}(t)]}\wedge\right.$$

$$\left. M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t) > E[M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)] - \lambda_{\{i\}\cup\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)]}\right)$$

$$\cdot (1 - (q^{m_\ell})^{-h/4 + \lambda_{\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\mathcal{K}}^{(m_\ell)}(t)]} + \lambda_{\{i\}\cup\mathcal{K}}^{\ell}\sqrt{\mathrm{var}[M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)]}}$$

$$\to 1, \quad (\ell \to \infty), \quad \text{uniformly } \forall t \in [t_1, t_2].$$

Therefore, given ϵ as defined in (5.49), $\exists L_2 > L_1$, such that $\forall \ell > L_2$,

$$\frac{\mathrm{d}E[M_{\mathcal{K}}^{(m_\ell)}(t)]}{\mathrm{d}t} = \sum_{i \in \mathcal{K}^c} z_{i,\mathcal{K}} E\left[1 - (q^{m_\ell})^{M_{\mathcal{K}}^{(m_\ell)}(t) - M_{\{i\}\cup\mathcal{K}}^{(m_\ell)}(t)}\right]$$

$$\geq (1 - \epsilon) \sum_{i \in \mathcal{K}^c} z_{i,\mathcal{K}}$$

$$\geq (1 - \epsilon) c_{\min}(1, \mathcal{K}), \quad (5.62)$$

from which we reach the same contradiction as in (5.51). This shows that $E[M_{\mathcal{K}}^{(m_\ell)}(t)] \xrightarrow{u} c_{\min}(1,\mathcal{K})t$ over $[0, 1/c_{\min}(1,\mathcal{K})]$. Now we claim that, in fact, $E[M_{\mathcal{K}}^{(m)}(t)] \to c_{\min}(1,\mathcal{K})t$ over $[0, 1/c_{\min}(1,\mathcal{K})]$. For otherwise $\exists t \in [0, 1/c_{\min}(1,\mathcal{K})]$, $\exists \epsilon > 0$ and $\exists \{m_\ell\}_\ell$ which is a subsequence of $m = 1, 2, \ldots$, such that $c_{\min}(1,\mathcal{K})t - E[M_{\mathcal{K}}^{(m_\ell)}(t)] > \epsilon$. Apply the previous set of induction arguments to $\{E[M_{\mathcal{K}}^{(m_\ell)}(t)]\}_\ell$, we know that a subsequence of $\{E[M_{\mathcal{K}}^{(m_\ell)}(t)]\}_\ell$ converges to $c_{\min}(1,\mathcal{K})t$ uniformly on $[0, 1/c_{\min}(1,\mathcal{K})]$, which is a contradiction.

Because $\forall t \in (0, 1/c_{\min}(1,\mathcal{K})]$, $E[M_{\mathcal{K}}^{(m)}(t)] \to c_{\min}(1,\mathcal{K})t$ and, from Property 5) of Proposition 5.5, $\text{var}[M_{\mathcal{K}}^{(m)}(t)] \to 0$, the conclusion follows from Chebyshev Inequality.

□

Since $M_{\mathcal{K}}(t) \to c_{\min}(1,\mathcal{K})t$, we know that except for the last few packets, the transmission happens at the rate arbitrarily close to $c_{\min}(1,\mathcal{K})$, i.e., we have

Corollary 5.7. *Given* $\alpha \in (0,1)$*, we have*

$$\lim_{m\to\infty} P\left(D(0, \alpha^2 m) \le \frac{\alpha m}{c_{\min}(1,\mathcal{K})}\right) = 1. \tag{5.63}$$

Proof.

$$\begin{aligned}
&\lim_{m\to\infty} P\left(D_{\mathcal{K}}(0, \alpha^2 m) \le \frac{\alpha m}{c_{\min}(1,\mathcal{K})}\right) \\
=& \lim_{m\to\infty} P\left(N_{\mathcal{K}}\left(\frac{\alpha m}{c_{\min}(1,\mathcal{K})}\right) \ge \alpha^2 m\right) \\
=& \lim_{m\to\infty} P\left(\frac{1}{m} N_{\mathcal{K}}\left(\frac{\alpha m}{c_{\min}(1,\mathcal{K})}\right) \ge \alpha^2\right) \\
=& \lim_{m\to\infty} P\left(M_{\mathcal{K}}^{(m)}\left(\frac{\alpha}{c_{\min}(1,\mathcal{K})}\right) \ge \alpha c_{\min}(1,\mathcal{K}) \cdot \frac{\alpha}{c_{\min}(1,\mathcal{K})}\right) \to 1
\end{aligned} \tag{5.64}$$

according to Theorem 5.6 with

$$t = \frac{\alpha}{c_{\min}(1,\mathcal{K})}. \tag{5.65}$$

□

However, the last few packets do not really affect the ensemble transmission rate because we can put a bound on the time it takes to transmit them. This bound is proportional to the amount of "slow" packets at the end of transmission and can be made negligible compared to the bulk of the transmission. Specifically, we have

Proposition 5.8. *Given q and $1 > \alpha > 0$, there exists b_i for each node i, such that*

$$\lim_{m\to\infty} P(D_i(\alpha m, m) \leq b_i(1-\alpha)m) = 1. \tag{5.66}$$

Proof. First consider the special case when $z_{1,i} > 0$ and we ignore packets received at i from nodes other than 1. When $N_i(t) < m$, any incoming packet is innovative with probability of $1 - q^{N_i(t)-m} \geq 1 - q^{-1}$. So let

$$b_i = \frac{3}{z_{1,i}(1-q^{-1})} \tag{5.67}$$

and apply Chebyshev inequality we have

$$P(D_i(\alpha m, m) \leq b_i(1-\alpha)m) \to 1. \tag{5.68}$$

If there is a path, say (WLOG), $1, 2, \ldots, i-1$, such that $z_{j,j+1} > 0$ ($j = 1, 2, \ldots, i-1$) (This must be true since we assume G is connected), we focus on the transmission along this path and ignore other transmissions. We use $D'_j(m_1, m_2)$ to denote the time $N_j(t)$ takes to increase from m_1 to m_2, assuming during that time $N_{j-1}(t) = m$. Let

$$b_i = \sum_{j=1}^{i-1} \frac{3}{z_{j,j+1}(1-q^{-1})}, \tag{5.69}$$

then

$$
\begin{aligned}
&P(D_i(\alpha m, m) \le b_i(1-\alpha)m) \\
\ge& P\left(\sum_{j=1}^{i-1} D'_{j+1}(\alpha m, m) \le b_i(1-\alpha)m\right) \\
\ge& P\left(\bigwedge_{j=1}^{i-1} D'_{j+1}(\alpha m, m) \le \frac{3(1-\alpha)m}{z_{j,j+1}(1-q^{-1})}\right) \\
\to& 1,
\end{aligned} \tag{5.70}
$$

where the first inequality follows from the fact that for $j = 2, 3, \ldots, i$, $N_{j-1}(t)$ reaches m before $N_j(t)$ reaches m; and the limit follows from the inclusion-exclusion principle and (5.68). □

The following corollary trivially extends the conclusion to an arbitrary set $\mathcal{K}$.

Corollary 5.9. *For any nonempty set $\mathcal{K} \subset \mathcal{N}$, $\exists b_{\mathcal{K}} > 0$ such that*

$$\lim_{m\to\infty} P(D_{\mathcal{K}}(\alpha m, m) \le b_{\mathcal{K}}(1-\alpha)m) = 1. \tag{5.71}$$

Proof. Assume node $i \in \mathcal{K}$ and apply Proposition 5.8. □

Now we are in a position to finish the achievability proof by combining Corollary 5.7 and Corollary 5.9.

Theorem 5.10. $\forall \mathcal{K} \subset \mathcal{N}$ *and* $\forall 0 < \epsilon < 1$,

$$\lim_{m\to\infty} P\left(\frac{m}{D_{\mathcal{K}}(0,m)} > \epsilon c_{\min}(1, \mathcal{K})\right) = 1. \tag{5.72}$$

Proof. We note $\forall \alpha \in (0,1)$,

$$\begin{aligned}
&P\left(\frac{m}{D_{\mathcal{K}}(0,m)} > \epsilon c_{min}(1,\mathcal{K})\right) \\
=&P\left(D_{\mathcal{K}}(0,m) < \frac{m}{\epsilon c_{\min}(1,\mathcal{K})}\right) \\
\geq &P\left(D(0,\alpha^2 m) \leq \frac{\alpha m}{c_{\min}(1,\mathcal{K})}\right. \\
&\left.\wedge D(\alpha^2 m, m) \leq \frac{m}{\epsilon c_{\min}(1,\mathcal{K})} - \frac{\alpha m}{c_{\min}(1,\mathcal{K})}\right) \\
\geq &P\left(D(0,\alpha^2 m) \leq \frac{\alpha m}{c_{\min}(1,\mathcal{K})}\right) \\
&+ P\left(D(\alpha^2 m, m) \leq \frac{m}{\epsilon c_{\min}(1,\mathcal{K})} - \frac{\alpha m}{c_{\min}(1,\mathcal{K})}\right) - 1.
\end{aligned} \tag{5.73}$$

Pick $\alpha \in (0,1)$ such that

$$\alpha > 1 - \frac{1/\epsilon - 1}{2 b_{\mathcal{K}} c_{\min}(1,\mathcal{K})}, \tag{5.74}$$

then we have

$$\begin{aligned}
\frac{1}{\epsilon c_{\min}(1,\mathcal{K})} &> \frac{1}{c_{\min}(1,\mathcal{K})} + b_{\mathcal{K}}(2-2\alpha) \\
&> \frac{\alpha}{c_{\min}(1,\mathcal{K})} + b_{\mathcal{K}}(2-2\alpha) \\
&\geq \frac{\alpha}{c_{\min}(1,\mathcal{K})} + b_{\mathcal{K}}(1-\alpha^2).
\end{aligned} \tag{5.75}$$

By Corollary 5.9, we know

$$\lim_{m\to\infty} P\left(D(\alpha^2 m, m) \leq \frac{m}{\epsilon c_{\min}(1,\mathcal{K})} - \frac{\alpha m}{c_{\min}(1,\mathcal{K})}\right) = 1. \tag{5.76}$$

By Corollary 5.7

$$\lim_{m\to\infty} P\left(D(0,\alpha^2 m) \leq \frac{\alpha m}{c_{\min}(1,\mathcal{K})}\right) = 1. \tag{5.77}$$

Pluging (5.76) and (5.77) back in (5.73) and taking the limit $m \to \infty$, we get (5.72). □

Theorem 5.10 is a little more general than the statements made in [9] and [19] since it reveals that, not only the rank at a single node, but also the rank at any subset $\mathcal{K} \subset \mathcal{N}$ increases at the rate determined by the min-cut bound $c_{\min}(1, \mathcal{K})$. It should also be pointed out that in the proof to Theorem 5.10, typical difficulties with cycles in the network topology do not arise due to the layered structure of the DI's that has encoded all topological information.

Chapter 6

Cross Layer Design with RNC

6.1 Dynamical System View of Random Network Coding

In this chapter we consider the problem of cross layer design with RNC. The wireless network is still modeled as a hypergraph [20] $G = (\mathcal{N}, \mathcal{E})$ where $\mathcal{N}$ is the set of transceiver nodes and $\mathcal{E}$ are a collection of hyperarcs in the form of $(i, \mathcal{K})$ pairs where $i \in \mathcal{N}$ and $\mathcal{K} \subset \mathcal{N}$. We assume every node in the network performs RNC on the finite field GF(q). Whenever a node gets to transmit, it transmits a coded packet that is a random linear combination of all the packets available to it, including the packets it originates and the coded packets it has received. Because RNC takes packets as the atomic object to process, it represents a transport strategy that is decoupled from the underlying MAC or PHY technologies, whose operations are considered independent of RNC. Take an arbitrary node i for example, typically the MAC layer of node i determines the transmission (broadcast) rate measured in packets per second. The PHY layer of node i determines the error probability that a transmitted packet can be correctly decoded at any receiver node j. In fact, according to [20], the respective effect of MAC and PHY in a wireless network can be shown through two variables:

- λ_i: the transmission rate at node i;
- $P_{i,\mathcal{K}}$: the probability that a packet sent from node i can be correctly decoded by

at least one node in $\mathcal{K}$.

It is possible to summarize the overall effect of MAC and PHY by defining $z_{i,\mathcal{K}} = \lambda_i P_{i,\mathcal{K}}$, which can be conceived as the capacity of the hyperarc $(i, \mathcal{K})$. Given $i \in \mathcal{K}^c$, we further define the min cut capacity for $(i, \mathcal{K})$ as

$$c_{\min}(i, \mathcal{K}) = \min_{\mathcal{K} \subset \mathcal{T}, i \notin \mathcal{T}} \sum_{j \in \mathcal{T}^c} z_{j,\mathcal{T}}. \tag{6.1}$$

As decodability of RNC entirely depends on the rank, and as there is no explicit routing, it is important to keep track of the rank at any node. Again, we use V_i to denote the average rank at any node i and $V_\mathcal{K}$ the average rank of $\mathcal{K}$. As we discussed in Chapter 3, V_i and $V_\mathcal{K}$ may be modeled as a system of differential equations. When MAC and PHY are fixed such that $z_{i,\mathcal{K}}$ are known parameters, these equations are explicitly written down as

$$\dot{V}_\mathcal{K} = \sum_{i \in \mathcal{K}^c} z_{i,\mathcal{K}} (1 - q^{V_\mathcal{K} - V_{\{i\} \cup \mathcal{K}}}), \quad \forall \mathcal{K} \neq \emptyset, \mathcal{K} \subset \mathcal{N}. \tag{6.2}$$

If we stack $V_\mathcal{K}$ into a column vector $\mathbf{V}$ and stack all $z_{i,\mathcal{K}}$ in $\mathbf{z}$, we may write (6.2) in a more compact form

$$\dot{\mathbf{V}} = \mathbf{f}(\mathbf{V}, \mathbf{z}) \tag{6.3}$$

where $\mathbf{f}$ is the stacking of the right-hand sides of (6.2). Note the instantaneous throughput of set $\mathcal{K}$ is given by $\dot{V}_\mathcal{K}$. Suppose node 1 is the only source node. When $m \to \infty$, we have shown in Chapter 5 that

$$\dot{V}_\mathcal{K}(t) = c_{\min}(1, \mathcal{K}), \quad \text{when } V_\mathcal{K}(t) < m, \tag{6.4}$$

i.e., the instantaneous throughput is almost a constant if m is sufficiently large. The fact that when $\mathbf{z}$ is fixed and m is large, $\dot{\mathbf{V}}$ is constant can be written in a more compact form

$$J_{\mathbf{V}}\mathbf{f}(\mathbf{V},\mathbf{z}) = 0, \quad \text{(for large } m\text{)} \tag{6.5}$$

where $J_{\mathbf{V}}\mathbf{f}(\mathbf{V},\mathbf{z})$ is the Jacobian of $\mathbf{f}(\mathbf{V},\mathbf{z})$. Another useful observation from (6.2) is that $\dot{V}_{\mathcal{K}}$ only linearly depends on $z_{i,\mathcal{K}}$. Mathematically, this implies

$$(J_{\mathbf{z}}\mathbf{f})(\mathbf{V},\mathbf{z}) = (J_{\mathbf{z}}\mathbf{f})(\mathbf{V}). \tag{6.6}$$

While (6.6) is precise, (6.5) is a good approximation with large m. Equations (6.5) and (6.6) constitute the basis for the resource allocation strategy to be discussed.

6.2 Resource Allocation with the Dynamical System Model

Since RNC is decoupled from MAC and PHY layers, it is particularly easy to discuss resource allocation and its effect on RNC at an abstract level. When there is a single resource to be allocated to N nodes, the allocation scheme is represented by a N-dimensional column vector $\mathbf{r}$, whose i-th component represents the allocation to node i. Consequently, the effect of resource allocation is captured in the mapping

$$\lambda_i = \lambda_i(\mathbf{r}) \quad \text{and} \quad P_{i,\mathcal{K}} = P_{i,\mathcal{K}}(\mathbf{r}), \tag{6.7}$$

or more concisely, $z_{i,\mathcal{K}} = z_{i,\mathcal{K}}(\mathbf{r})$. In vector form, we may write $\mathbf{z} = \mathbf{z}(\mathbf{r})$.

An allocation scheme is invariably a deliberate design to achieve a certain objective. In this article, we assume the objective is a function of throughputs, which can be solved with the system of differential equations in (6.2). Specifically, we assume the objective to be maximized can be written as $T(\dot{\mathbf{V}})$ where T can be any continuous function in the first quadrant with partial derivatives defined almost everywhere. As a result, we wish to solve

$$\begin{aligned} \text{maximize} \quad & T(\dot{\mathbf{V}}) && (6.8) \\ \text{subject to} \quad & \dot{\mathbf{V}} = \mathbf{f}(\mathbf{V}, \mathbf{z}) && (6.9) \\ & \mathbf{z} = \mathbf{z}(\mathbf{r}) && (6.10) \end{aligned}$$

Equation (6.8) takes a distinct form from previous resource allocation problems for RNC. Rather than treating RNC as a static transport strategy, (6.8) looks at it from the dynamical system point of view, which means the allocation may also be dynamic and changes over time. Moreover, it can take the form of state feedback

$$\dot{\mathbf{r}} = \mathbf{g}(\dot{\mathbf{V}}) \tag{6.11}$$

where $\mathbf{g}$ is some function of $\mathbf{V}$. The second distinction is that (6.8) is the first formulation that takes full multicast advantage of a wireless network, while previous formulations inevitably break down the wireless transmission into independent links to approximately convert the wireless network to a wired network. Doing so, they lose a portion of the multicast advantage.

6.3 Gradient Based Resource Allocation

While a problem of the form of (6.8) falls in the traditional category of optimal control and is usually solved using calculus of variations, there are a few reasons that this approach may not be appropriate. First, calculus of variations is as computationally intensive as expensive, which can be practically prohibitive in wireless networks. Second, the wireless communication environment can be dynamic (e.g., due to fading or mobility), and may require certain adaptivity in the devices. Fortunately, when we restrict our objective T to be a function of throughputs, the special problem structure of RNC can be utilized to derive a simple gradient based control which is adaptive.

Consider the objective function $T(\dot{\mathbf{V}})$ which can be any function that has a gradient almost anywhere in the first quadrant. Our choice of feedback (see (6.11)) is $\mathbf{g} = \nabla_{\mathbf{r}} T(\dot{\mathbf{V}})$. Note without (6.13), it is not clear how to compute $\nabla_{\mathbf{r}} T(\dot{\mathbf{V}})$ because $\dot{\mathbf{V}}$ is apparently affected by a number of factors including topology, transmission rates and the coding operation. By applying (6.2), the effect of the resource vector $\mathbf{r}$ is abstracted in (6.10), the topology information encoded in the partially ordered index $\mathcal{K} \subset \mathcal{N}$ and the dynamism represented as the differential operator. The system of equations in (6.2) thus allow us to find an expression for the feedback as[1]

$$\dot{\mathbf{r}} = a J_{\mathbf{r}}^{\top} \mathbf{z} J_{\mathbf{z}}^{\top} \mathbf{f} \nabla_{\dot{\mathbf{V}}} T \tag{6.12}$$

where $\nabla_{\dot{\mathbf{V}}} T$ is the gradient of $T(\dot{\mathbf{V}})$, $J_{\mathbf{z}}\mathbf{f}$ the Jacobian of $\mathbf{f}(\mathbf{V}, \mathbf{z})$ with respect to $\mathbf{z}$, $J_{\mathbf{r}}(\mathbf{z})$ the Jacobian of $\mathbf{z}(\mathbf{r})$. Note we have introduced $a > 0$ as a feedback gain parameter. Combining (6.9) and (6.12) we have a closed-loop system that describes the rank evolution with a dynamic resource allocation strategy

$$\begin{aligned} \dot{\mathbf{V}} &= \mathbf{f}(\mathbf{V}, \mathbf{z}), \\ \dot{\mathbf{r}} &= a J_{\mathbf{r}}^{\top} \mathbf{z} J_{\mathbf{z}}^{\top} \mathbf{f} \nabla_{\dot{\mathbf{V}}} T. \end{aligned} \tag{6.13}$$

Strictly speaking, the feedback in (6.12) is not a gradient of the objective function T. It is nevertheless capable of continuously computing the instantaneous resource vector $\mathbf{r}$ in the direction that T improves.

---

[1] The superscript $\top$ stands for matrix transpose. Detailed discussion of the chosen feedback is presented in Section 6.4.

6.4 Analysis of Resource Allocation

In this section we will show that choosing the particular feedback in the form of (6.12) most effectively improves the objective function T. As we will see, the system of differential equations in (6.2) not only facilitates design, but also analysis. The main theorem is the following

Theorem 6.1. *By introducing the feedback* (6.12) *in* (6.13)*,* $\dot{T} \geq 0$*. Among all possible feedback* $\dot{\mathbf{r}} = \mathbf{g}$ *such that* $\|\mathbf{g}\| = \|aJ_{\mathbf{r}}^{\top}\mathbf{z}J_{\mathbf{z}}^{\top}\mathbf{f}\nabla_{\dot{\mathbf{V}}}T\|$*,* $\dot{T}$ *is maximized by* (6.12)*.*

Proof. $\dot{T} \geq 0$ can be seen from the following computation

$$\begin{aligned}
\dot{T} &= (\nabla_{\dot{\mathbf{V}}}T)^{\top}\ddot{\mathbf{V}} = (\nabla_{\dot{\mathbf{V}}}T)^{\top}\dot{\mathbf{f}}(\mathbf{V};\mathbf{z}) \\
&= (\nabla_{\dot{\mathbf{V}}}T)^{\top}\left(J_{\mathbf{V}}\mathbf{f}\dot{\mathbf{V}} + J_{\mathbf{z}}\mathbf{f}\dot{\mathbf{z}}\right) \\
&= (\nabla_{\dot{\mathbf{V}}}T)^{\top}J_{\mathbf{z}}\mathbf{f}J_{\mathbf{r}}\mathbf{z}\dot{\mathbf{r}} \quad \text{(due to (6.5))} \\
&= (\nabla_{\dot{\mathbf{V}}}T)^{\top}aJ_{\mathbf{z}}\mathbf{f}J_{\mathbf{r}}\mathbf{z}J_{\mathbf{r}}^{\top}\mathbf{z}J_{\mathbf{z}}^{\top}\mathbf{f}\nabla_{\dot{\mathbf{V}}}T \\
&= a(\nabla_{\dot{\mathbf{V}}}T)^{\top}J_{\mathbf{z}}\mathbf{f}J_{\mathbf{r}}\mathbf{z}J_{\mathbf{r}}^{\top}\mathbf{z}J_{\mathbf{z}}^{\top}\mathbf{f}\nabla_{\dot{\mathbf{V}}}T \\
&= a\|(\nabla_{\dot{\mathbf{V}}}T)^{\top}J_{\mathbf{z}}\mathbf{f}J_{\mathbf{r}}\mathbf{z}\|^2 \geq 0.
\end{aligned} \tag{6.14}$$

Given any $\mathbf{g}$ such that $\|\mathbf{g}\| = \|aJ_{\mathbf{r}}^{\top}\mathbf{z}J_{\mathbf{z}}^{\top}f\nabla_{\dot{\mathbf{V}}}T\|$, repeating the computation as shown in (6.14), we get

$$\dot{T} = a(\nabla_{\dot{\mathbf{V}}}T)^{\top}J_{\mathbf{z}}\mathbf{f}J_{\mathbf{r}}\mathbf{z}\mathbf{g}, \tag{6.15}$$

so

$$\begin{aligned}
|\dot{T}| &\leq \|a(\nabla_{\dot{\mathbf{V}}}T)^{\top}J_{\mathbf{z}}\mathbf{f}J_{\mathbf{r}}\mathbf{z}\|\|\mathbf{g}\| \\
&= a\|(\nabla_{\dot{\mathbf{V}}}T)^{\top}J_{\mathbf{z}}\mathbf{f}J_{\mathbf{r}}\mathbf{z}\|^2.
\end{aligned} \tag{6.16}$$

Note (6.16) achieves equality when the feedback is chosen as shown in (6.12). □

Note Theorem 6.1 is true regardless whether T is concave or whether T is monotonic. It is also true regardless whether $\mathbf{z} = \mathbf{z}(\mathbf{r})$ is concave or monotonic. Therefore it can be applied to a wide range of problems. Theorem 6.1 is also true when T is non-differentiable on a null set since what really matters is $\int \dot{T} \mathrm{d}t$. As a result, when we have multiple interested destinations $d_1, d_2, \ldots \in \mathcal{D}$, an objective of the form

$$T(\dot{\mathbf{V}}) = \min_{d \in \mathcal{D}} V_d \tag{6.17}$$

can still be maximized with the chosen feedback. With Theorem 6.1, we immediately have

Corollary 6.2. *If* $\mathbf{r}$ *is contained in a bounded set (i.e., finite resource allocation),* $\mathbf{z}(\mathbf{r})$ *is continuous and* T *is continuous, with the feedback in* (6.12), T *converges.*

Proof. We have

$$\dot{V}_{\mathcal{K}} = \sum_{i \in \mathcal{K}^c} z_{i,\mathcal{K}} (1 - q^{V_{\mathcal{K}} - V_{\mathcal{K} \cup \{i\}}}) \leq \sum_{i \in \mathcal{K}^c} z_{i,\mathcal{K}}. \tag{6.18}$$

Because $\mathbf{r}$ is bounded, it is contained in a compact set. Therefore $\mathbf{z}$ is contained in a compact set and from (6.18) $\dot{\mathbf{V}}$ is contained in a compact set. Since T is continuous on this compact set, T is also bounded. This implies that, if T is monotonically increasing, it must converge. □

From (6.14) we see that $\dot{T}$ (the rate of convergence) can be calculated if an expression is known for $J_{\mathbf{r}}(\mathbf{z})$. In the absence of the exact knowledge of $J_{\mathbf{r}}(\mathbf{z})$, we also see the feedback gain a can be used to speed up convergence. However, the tradeoff is, with a too large we are at the risk of changing $\mathbf{z}$ too fast thus invalidating (6.5). It remains to be answered how different the solution and optimality will be if we vary a without invalidating (6.5). The next theorem shows that the solutions for $\mathbf{V}$ and $\mathbf{r}$ are basically the same except for a proper scaling and the optimality is insensitive for a range of a. This is generally not true when dealing with other dynamical systems where different feedback gains lead to very different trajectories. Before we proceed to the theorem, it would be convenient to define $\mathbf{f}^a$ to be the same as $\mathbf{f}$, except that the field size q is replaced by q^a.

Theorem 6.3. *Let $\mathbf{V}^a$, $\mathbf{r}^a$, $\mathbf{z}^a$ satisfy (6.13) with an arbitrary $a > 0$. Let $\mathbf{V}(t) = a\mathbf{V}^a(t/a)$, $\mathbf{r}(t) = \mathbf{r}^a(t/a)$ and $\mathbf{z}(t) = \mathbf{z}^a(t/a)$. Then $\mathbf{V}$, $\mathbf{r}$ and $\mathbf{z}$ satisfy*

$$\begin{aligned} \dot{\mathbf{V}}(t) &= \mathbf{f}^{1/a}(\mathbf{V}, \mathbf{r}), \\ \dot{\mathbf{r}}(t) &= J_{\mathbf{r}}^{\top} \mathbf{z} J_{\mathbf{z}}^{\top} \mathbf{f}^a \nabla_{\dot{\mathbf{V}}} T(\dot{\mathbf{V}}). \end{aligned} \tag{6.19}$$

Proof. For rank evolution, we have

$$\begin{aligned} \dot{\mathbf{V}}(t) &= a\dot{\mathbf{V}}^a(t/a)/a = \mathbf{f}(\mathbf{V}^a(t/a), \mathbf{z}^a(t/a)) \\ &= \mathbf{f}\left(\frac{1}{a}\mathbf{V}(t), \mathbf{z}(t)\right) = \mathbf{f}^{1/a}(\mathbf{V}(t), \mathbf{z}(t)). \end{aligned} \tag{6.20}$$

The last step follows from the fact

$$1 - q^{\frac{1}{a}V_{\mathcal{K}} - \frac{1}{a}V_{\mathcal{K}\cup\{i\}}} = 1 - \left(q^{1/a}\right)^{V_{\mathcal{K}} - V_{\mathcal{K}\cup\{i\}}}. \tag{6.21}$$

To verify the feedback equation in (6.19), we assume z, r, V are an arbitrary component of $\mathbf{z}$, $\mathbf{r}$, $\mathbf{V}$, respectively. Likewise, assume z^a, r^a, V^a are an arbitrary component of $\mathbf{z}^a$, $\mathbf{r}^a$, $\mathbf{V}^a$, respectively. Then we have

$$\frac{\partial z}{\partial r}(t) = \frac{\dot{z}}{\dot{r}}(t) = \frac{\dot{z^a}}{\dot{r^a}}(t/a) = \frac{\partial z^a}{\partial r^a}(t/a), \tag{6.22}$$

$$\frac{\partial f}{\partial z}(\mathbf{V}, \mathbf{z}) = \frac{\partial f}{\partial z}(a\mathbf{V}^a(t/a)) = \frac{\partial f^a}{\partial z^a}(t/a), \quad \text{(by (6.6))} \tag{6.23}$$

$$\begin{aligned} \frac{\partial T(\dot{\mathbf{V}})}{\partial \dot{V}} &= \lim_{\Delta t \to 0} \frac{T(\dot{\mathbf{V}}^a(t/a + \Delta t/a)) - T(\dot{\mathbf{V}}^a(t/a))}{\dot{V}^a(t/a + \Delta t/a) - \dot{V}^a(t/a)} \\ &= \frac{\partial T}{\partial \dot{V}^a}\left(\frac{t}{a}\right). \end{aligned} \tag{6.24}$$

From (6.22) – (6.24) and

$$\dot{\mathbf{r}}^a(t) = aJ_{\mathbf{r}^a}^\top \mathbf{z}^a J_{\mathbf{z}^a}^\top \mathbf{f} \nabla_{\dot{\mathbf{V}}^a} T(\mathbf{V}^a), \tag{6.25}$$

we know

$$\begin{aligned}\dot{\mathbf{r}}(t) &= \frac{1}{a}\dot{\mathbf{r}}^a\left(\frac{t}{a}\right) = J_{\mathbf{r}^a}^\top \mathbf{z}^a\left(\frac{t}{a}\right) J_{\mathbf{z}^a}^\top \mathbf{f}\left(\frac{t}{a}\right)\nabla_{\dot{\mathbf{V}}^a} T(\dot{\mathbf{V}}^a)\left(\frac{t}{a}\right)\\ &= J_{\mathbf{r}}^\top \mathbf{z}(t) J_{\mathbf{z}}^\top \mathbf{f}^a(t)\nabla_{\dot{\mathbf{V}}} T(\dot{\mathbf{V}})(t).\end{aligned} \tag{6.26}$$

□

In [20], a concentration result has been obtained regardless of q as long as q is sufficiently large. In practice, the effect of finite q on the concentration speed is minimal if $q \geq 2$, because in this case $q^{-x} \to 0$ quickly as $x > 0$ increases. Rank evolution is therefore insensitive to a wide range of q, with which it concentrates to the same solution. As a result, as long as $q^a \gg 1$ and $q^{1/a} \gg 1$, different a's in (6.13) yield the same system trajectory given by

$$\begin{aligned}\dot{\mathbf{V}} &= f(\mathbf{V}, \mathbf{r}),\\ \dot{\mathbf{r}} &= J_{\mathbf{r}}^\top \mathbf{z} J_{\mathbf{z}}^\top \mathbf{f} \nabla_{\dot{\mathbf{V}}} T(\dot{\mathbf{V}}),\end{aligned} \tag{6.27}$$

after proper scaling as described in Theorem 6.3. Hence we have

Corollary 6.4. *If a is chosen such that $\mathbf{f} \approx \mathbf{f}^a \approx \mathbf{f}^{1/a}$, T converges to the same value.*

Proof. This is true because after scaling $\mathbf{V}$, $\mathbf{z}$, $\mathbf{r}$ satisfy (6.27). But $\dot{\mathbf{V}}(t) = \dot{\mathbf{V}}^a(t/a)$, so

$$\lim_{t\to\infty} T(\dot{\mathbf{V}}^a(t)) = \lim_{t\to\infty} T(\dot{\mathbf{V}}^a(t/a)) = \lim_{t\to\infty} T(\dot{\mathbf{V}}(t)). \tag{6.28}$$

□

When choosing a, we want it to be large for fast convergence, but not so large that $q^{1/a} \approx 1$ in (6.20). On the other hand, we want to avoid making a so small that $q^a \approx 1$

in (6.23). Both extremes would undermine the concentration assumption. In general, concentration is essential for RNC to achieve the throughput, and it is also the key assumption for what we have discussed so far to remain valid. To do this, given the real field size q used, we may set a threshold $q^{\mathrm{th}} < q$ below which the concern for concentration will arise. Then pick a such that

$$q^a, q^{1/a} \geq q^{\mathrm{th}}, \tag{6.29}$$

i.e., from the interval $[\log q^{\mathrm{th}}/\log q, \log q/\log q^{\mathrm{th}}]$.

If we wish to know how badly the algorithm would perform with a so large that (6.5) becomes invalid (but the concentration assumption still holds), Theorem 6.5 indicates that (6.13) may still work.

Theorem 6.5. *Assume a is so large that* (6.5) *is not valid. If T is continuously differentiable in the first quadrant and $\mathbf{r}$ is constrained in a bounded set, $\dot{T} \geq 0$ until $\|J_{\mathbf{r}}^{\top}\mathbf{z}J_{\mathbf{z}}^{\top}\mathbf{f}^a\nabla_{\dot{\mathbf{V}}}T(\dot{\mathbf{V}})\| < c/a$ for some constant c independent of a.*

Proof. Since we do not have (6.5) any more, we must write

$$\begin{aligned}
\dot{T} &= (\nabla_{\dot{\mathbf{V}}}T)^{\top}\ddot{\mathbf{V}} \\
&= (\nabla_{\dot{\mathbf{V}}}T)^{\top}\left(J_{\mathbf{V}}\mathbf{f}\dot{\mathbf{V}} + J_{\mathbf{z}}\mathbf{f}\dot{\mathbf{z}}\right) \\
&= (\nabla_{\dot{\mathbf{V}}}T)^{\top}J_{\mathbf{V}}\mathbf{f}\dot{\mathbf{V}} + a\|(\nabla_{\dot{\mathbf{V}}}T)^{\top}J_{\mathbf{z}}\mathbf{f}J_{\mathbf{r}}\mathbf{z}\|^2.
\end{aligned} \tag{6.30}$$

From (6.18) we already know $\dot{\mathbf{V}}$ is contained in a compact set. $\nabla_{\dot{\mathbf{V}}}T$ is thus bounded as T is continuously differentiable. Moreover, we can bound $\|J_{\mathbf{V}}f\|$. To see this, note for arbitrary row $\mathcal{K}$ of $J_{\mathbf{V}}\mathbf{f}$, all entries are zero except for column $\mathcal{K}$ and column $\mathcal{K}\cup\{i\}$

$(i \in \mathcal{K}^c)$, whose entries are

$$\frac{\partial \dot{V}_{\mathcal{K}}}{\partial V_{\mathcal{K}}} = -\log q \sum_{i \in \mathcal{K}^c} z_{i,\mathcal{K}} q^{V_{\mathcal{K}} - V_{\mathcal{K} \cup \{i\}}} \geq -\log q \sum_{i \in \mathcal{K}^c} z_{i,\mathcal{K}}; \tag{6.31}$$

$$\frac{\partial \dot{V}_{\mathcal{K}}}{\partial V_{\mathcal{K} \cup \{i\}}} = \log q z_{i,\mathcal{K}} q^{V_{\mathcal{K}} - V_{\mathcal{K} \cup \{i\}}} \leq \log q z_{i,\mathcal{K}}. \tag{6.32}$$

Therefore, $J_{\mathbf{V}}\mathbf{f}$ is also bounded. Hence there exists a constant c independent of a, such that

$$\|(\nabla_{\dot{\mathbf{V}}} T)^{\top} J_{\mathbf{V}} \mathbf{f} \dot{V}\| \leq c. \tag{6.33}$$

It is clear that, as long as $a\|(\nabla_{\dot{\mathbf{V}}} T)^{\top} J_{\mathbf{z}} \mathbf{f} J_{\mathbf{r}} z\|^2 \geq c$, $\dot{T} \geq 0$. □

Note when $\dot{T} < 0$, it must be true that $a\|(\nabla_{\dot{\mathbf{V}}} T)^{\top} J_{\mathbf{z}} \mathbf{f} J_{\mathbf{r}} z\|^2 < c$, which can be regarded as the first order stopping criteria. Although Theorem 6.5 makes the choice of a even more liberal, it is nevertheless advisable to choose a reasonably small, otherwise additional nonlinearities may start to enter the picture. This is especially true when we move from the continuous system (6.13) to a discretized system that adjusts resource allocation only at time instants it is triggered.

Chapter 7

Numerical Results of the DE Framework

In this section we present extensive numerical examples of the DE framework for RNC. We would use $q = 2$ for all the RNC examples to show that, even for the small field size, the DE framework provides desirable accuracy. We give the simulation results based on different network topologies, shown as dots and arrows (cf. Chapter 3.1 and Fig. 3.1). We remind the readers that the dots represent the nodes whose transmissions are according to independent Poisson processes; the arrows represent the reachability. The intensity is set uniformly to $\lambda_i = 1$ packet/second for the Poisson transmission process at every node i. If multiple arrows emanate from the same node, it means when this node transmits, all the nodes on the other end of the arrows have a chance to receive this packet. In our simulation, unless specifically indicated, the receptions are independent. Their independent reception probabilities are shown as the numbers attached to the arrows. We will demonstrate the ability of the DE framework to handle multiple sessions, complex network topology and correlated receptions by comparing the rank processes at different nodes obtained from simulation with the DE solution. We assume all transmissions begin from $t = 0$.

Because the convergence of the fluid approximation to $E[N_{\mathcal{K}}(t)]$ is extremely fast, we choose to show sample paths rather than the ensemble average to demonstrate the accuracy and versatility of the DE framework.

7.1 Two Multicast Sessions

Consider a three node wireless network shown in Fig. 7.1. Assume the information flow is comprised of two multicast sessions originating from node 1 and node 3, respectively. Node 1 has 200 packets to deliver to node 2 and 3, while node 3 has 300 packets to deliver to node 1 and 2. We may write out the DI's that describe this scenario:

$$
\begin{aligned}
\dot{V}_1 &= z_{2,1}\mathrm{Sgn}^+(V_{\{12\}} - V_1),\\
\dot{V}_2 &= z_{1,2}\mathrm{Sgn}^+(V_{\{12\}} - V_2) + z_{3,2}\mathrm{Sgn}^+(V_{\{23\}} - V_2),\\
\dot{V}_3 &= z_{2,3}\mathrm{Sgn}^+(V_{\{23\}} - V_3),\\
\dot{V}_{\{12\}} &= z_{2,3}\mathrm{Sgn}^+(V_{\{123\}} - V_{\{12\}}),\\
\dot{V}_{\{23\}} &= z_{1,2}\mathrm{Sgn}^+(V_{\{123\}} - V_{\{23\}}),\\
\dot{V}_{\{123\}} &= 0,
\end{aligned}
\tag{7.1}
$$

with the B.C.

$$
V_{\mathcal{K}}(0) = \begin{cases} 300, & 1 \in \mathcal{K}, 3 \notin \mathcal{K},\\ 200, & 3 \in \mathcal{K}, 1 \notin \mathcal{K},\\ 500, & \{1,3\} \subset \mathcal{K},\\ 0, & \text{o.w..} \end{cases}
\tag{7.2}
$$

Fig. 7.2 shows the analytical solution to (7.1) as well as the simulation results. The analysis matches the simulations closely. Clearly the rank increase at node 1 should be subject to its min cut bound $c_{\min}(3,1) = 0.5$ sec^{-1} and node 3 subject to $c_{\min}(1,3) = 0.7$ sec^{-1}. Consequently, $T_1^{\text{total}} = 300/0.5 = 600$ (sec) and $T_3^{\text{total}} = 200/.7 = 285.7$ (sec). For node 2, the flow from node 1 cannot exceed $c_{\min}(1,2)$; the flow from node 3 cannot exceed $c_{\min}(3,2)$; and the flow from the ensemble of node 1, 3 cannot exceed $c_{\min}(\{1,3\},2)$. Therefore,

$$T_2^{\text{total}} = \max\{m_1/c_{\min}(1,2), m_2/c_{\min}(3,2), (m_1+m_2)/c_{\min}(\{1,3\},2)\} = 500 \text{ (sec)}.$$

These calculations are readily verified in Fig. 7.2.

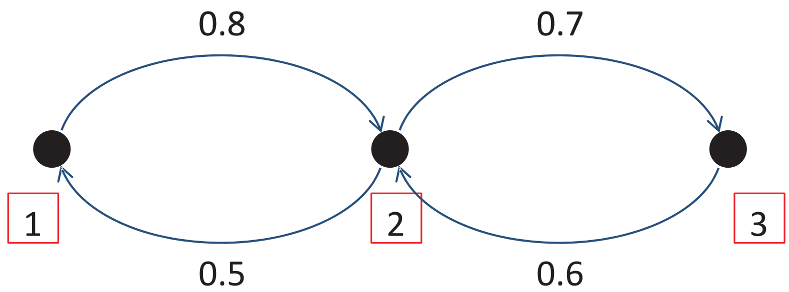

Figure 7.1: A three node wireless network.

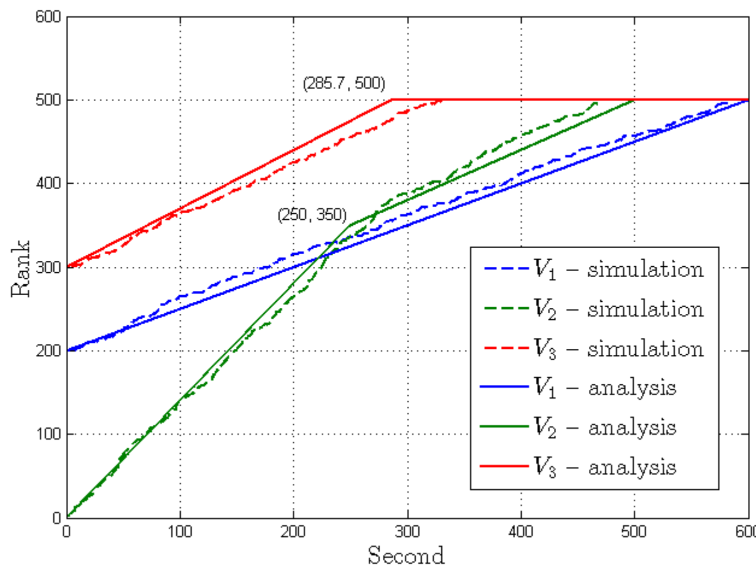

Figure 7.2: Two multicast sessions with two sources using RNC.

7.2 A Complex Topology

This example is intended to illustrate that the DE framework is capable of handling complex networks. The wireless network in Fig. 7.3 has 10 nodes and a fairly intricate connectivity. While the unidirectional arrows have the same meaning as in an arrow-dot representation of the hypergraph, the bidirectional arrows simply represent two unidirectional arrows whose reception probabilities are equal and as labeled. In this example we again assume independent reception at each node and the transmission rate $\lambda_i = 1$ sec^{-1}, $i = 1, 2, \ldots, 10$. Node 1 is the only source node that has 100 packets to deliver. Fig. 7.4 shows the rank evolution at node $2, 4, 7$ and 10. For node 7, the min cut is shown to be $0.1 + 0.1 = 0.2$ (sec^{-1}). For node 10, it is shown to be $0.2 + 0.3 = 0.5$ (sec^{-1}). For the other nodes, the min cut is $1 - (1 - 0.9) \times (1 - 0.8) = 0.98$ (sec^{-1}). These facts are reflected as the slope of the rank evolution curves on Fig. 7.4 where the simulated curves well match the analytical solutions.

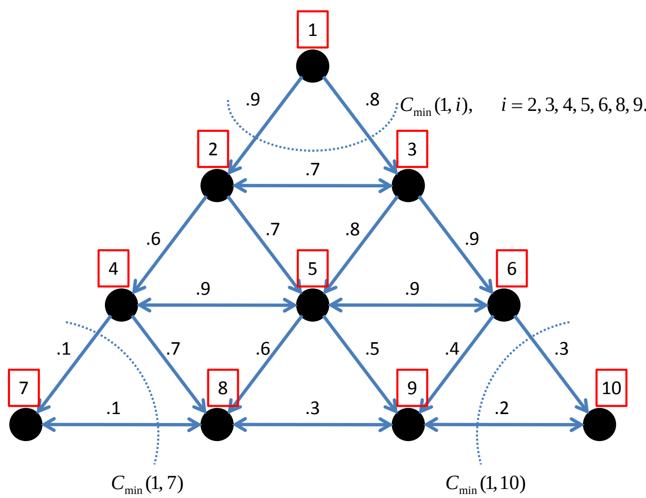

Figure 7.3: A 10-node wireless network with node 1 being the unique source.

7.3 Correlated Reception

The DE framework allows the analysis of rank evolution when receptions are not independent. The lack of independence could be due to correlated channels or joint

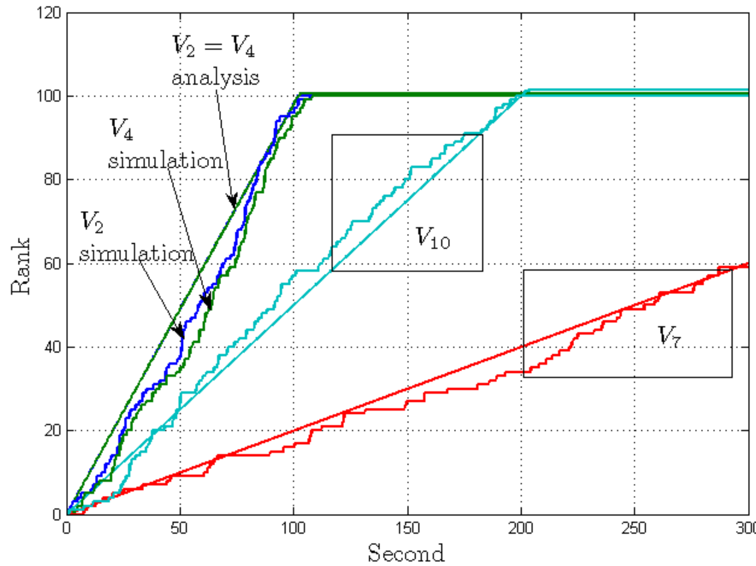

Figure 7.4: Rank evolution for the network shown in Fig. 7.3.

reception by design and they are not uncommon in wireless communications. Being capable of modeling correlated reception is an advantage of the DE framework. Consider the four-node network shown in Fig. 7.5 where as usual the point-to-point reception probabilities are shown as labeled. Node 1 is the only source node that has 400 packets to deliver. However, assume node 2 and node 3 are in cooperation or the channels from node 1 are highly correlated such that the receptions are not independent:

$$P_{1,2} = 0.49, \quad P_{1,3} = 0.49, \quad P_{1,\{2,3\}} = 0.5, \tag{7.3}$$

the rank evolution can still be accurately predicted by the DE framework as shown in Fig. 7.6. In this case, V_2 and V_3 increase at the same rate of $c_{\min}(1,2) = c_{\min}(1,3) = 0.49$ (sec^{-1}) while V_4 increases at

$$c_{\min}(1,4) = z_{1,\{2,3\}} = \lambda_1 P_{1,\{2,3\}} = 0.5 \text{ (sec}^{-1}\text{)}.$$

As a contrast, the results for independent receptions are shown in Fig. 7.7, where V_2 and V_3 increase at the same rate of $c_{\min}(1,2) = c_{\min}(1,3) = 0.49$ (sec^{-1}) while V_4 increases at

$$c_{\min}(1,4) = z_{1,\{2,3\}} = \lambda_1 P_{1,\{2,3\}} = \lambda_1(1-(1-P_{1,2})(1-P_{1,3})) = 0.74 \text{ (sec}^{-1}\text{)}.$$

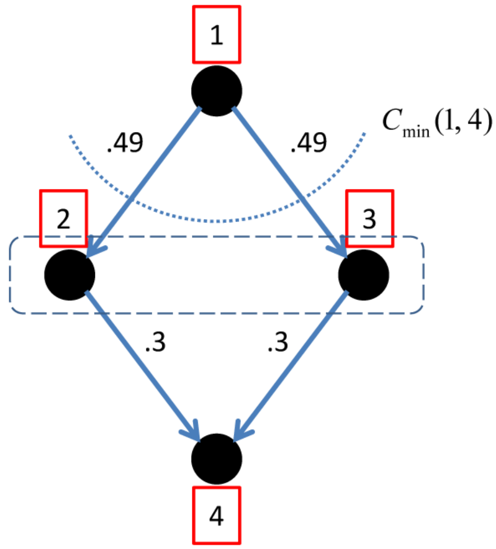

Figure 7.5: Modeling correlated reception with DE in a four node wireless network.

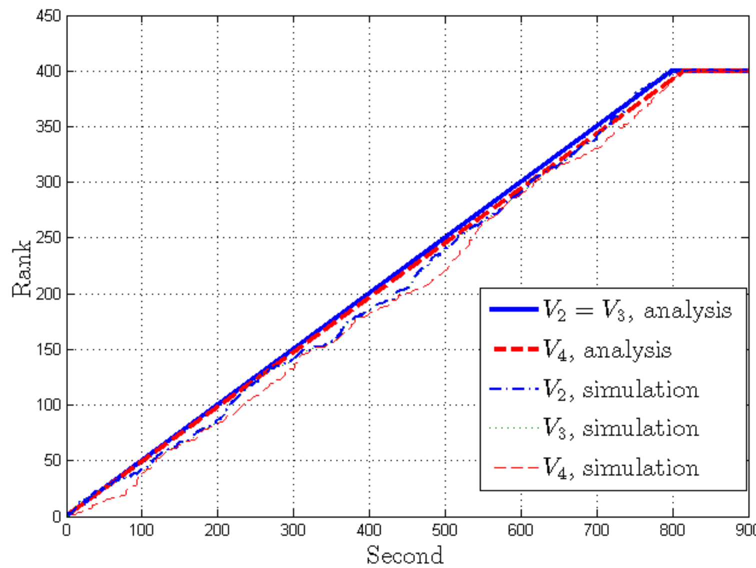

Figure 7.6: Rank evolution if node 2 and 3 have correlated reception.

7.4 Power Control with RNC

In this section, we use the network coding aware power control as an example to demonstrate the performance of the gradient based algorithm as shown in (6.13). Consider the 6-node network shown in Fig. 7.8 where nodes perform RNC on GF(2). Node 1 attempts to multicast to nodes in $\mathcal{D} = \{2, 3, 4, 5, 6\}$ and we wish to maximize the worst-case throughput in the network, i.e., maximize the minimum throughput given in (6.17). We assume the MAC has been fixed so that node i ($i = 1, \ldots, 6$) transmits at $\lambda_i = 1$ packet per millisecond. However, the transmit power of each node can be flexibly adjusted from 0 to 15dBm. The initial transmit powers are set uniformly to 13dBm. Let P_i^{Tx} denote the transmit powers at node i, h_{ij} denote the link gains from node i to node j, $\sigma^2 = -101$dBm denote the thermal noise power at every receiver. We also associate with each node i a processing gain $g_i = 8$. Consequently, we can model the signal-to-interference-noise ratio (SINR) at node j when j attempts to decode the

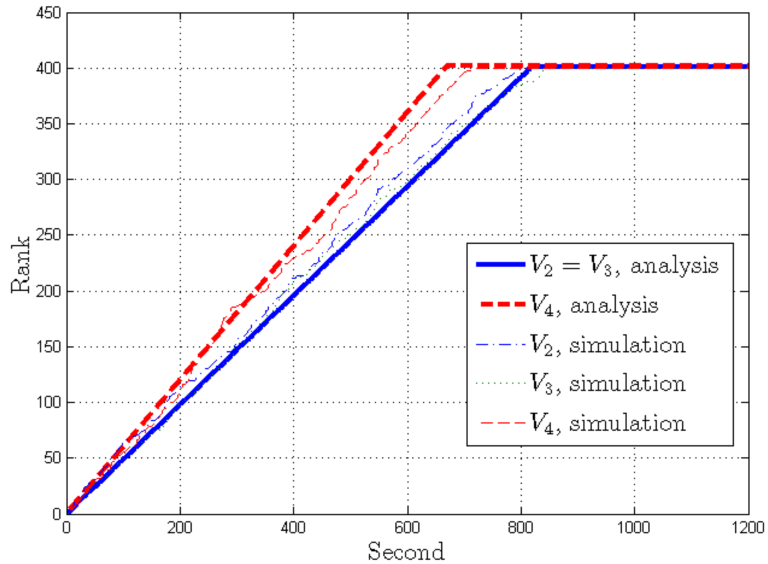

Figure 7.7: Rank evolution if node 2 and 3 have independent reception.

transmission from i as

$$\mathrm{SINR}_{ij} = \frac{P_i^{\mathrm{Tx}} h_{ij}}{\sum_{k \neq i,j} P_k^{\mathrm{Tx}} h_{kj}/g_j + \sigma^2}. \tag{7.4}$$

For the link gains, we use the ITU indoor attenuation model [21] based on distance. The path loss PL over distance d is given by

$$\mathrm{PL} = 20 \log f_c + 10n \log d + P_f(n) - 28 \tag{7.5}$$

where f_c is the center frequency, n the path loss exponent and $P_f(n)$ the floor penetration factor. Consequently the link gain is given by $h = 10^{\mathrm{PL}/10}$. In the simulation, we choose $f_c = 2.4\mathrm{GHz}$, $n = 3$, $P_f(n) = 11$. We use uncoded BPSK modulation with a codeword/packet length of $L = 160$ bits. Assuming a Gaussian distribution for the interference, the bit error rate for j decoding i is given by

$$P_{ij}^{\mathrm{bit}} = Q\left(\sqrt{\mathrm{SINR}_{ij}}\right). \tag{7.6}$$

The packet error rate is given by

$$P_{ij} = 1 - (1 - P_{ij}^{\text{bit}})^L. \tag{7.7}$$

Assuming independent reception, we thus have

$$P_{i,\mathcal{K}} = 1 - \prod_{j \in \mathcal{K}} P_{ij} \quad \text{and} \quad z_{i,\mathcal{K}} = \lambda_i P_{i,\mathcal{K}}. \tag{7.8}$$

Clearly in this example the resource to be allocated is the transmit power, i.e.,

$$\mathbf{r} = \begin{bmatrix} P_1^{\text{Tx}} & P_2^{\text{Tx}} & P_3^{\text{Tx}} & P_4^{\text{Tx}} & P_5^{\text{Tx}} & P_6^{\text{Tx}} \end{bmatrix}^\top .$$

The objective T is given by

$$T = \min\{\dot{V}_2, \dot{V}_3, \dot{V}_4, \dot{V}_5, \dot{V}_6\}.$$

Using the model specified by (7.4) – (7.8), we can execute (6.13) to demonstrate our power control algorithm. However, in practice, we probably never truly evaluate (6.12) in the matrix form because $J_{\mathbf{z}}^\top \mathbf{f}$ is often sparse[1]. In what follows, we break down the computations of the feedback for the readers. In general, it is an iterative application of the chain rule. First, we identify the label of the node receiving the minimum throughput

$$d^* = \text{argmin}_{d \in \{2,3,4,5,6\}} \dot{V}_d, \tag{7.9}$$

then

$$T = \dot{V}_{d^*}. \tag{7.10}$$

We compute $\nabla_{\mathbf{r}} T$ componentwise. Using chain rule,

$$\frac{\partial T}{\partial P_i^{\text{Tx}}} = \frac{\partial \dot{V}_{d^*}}{\partial P_i^{\text{Tx}}} = \sum_{j \neq d^*} \frac{\partial \dot{V}_{d^*}}{\partial z_{j,d^*}} \frac{\partial z_{j,d^*}}{\partial P_i^{\text{Tx}}}. \tag{7.11}$$

---

[1] $\forall i, \mathcal{K}, \mathcal{T}, \partial \dot{V}_{\mathcal{K}} / \partial z_{i,\mathcal{T}} = 0$ if $i \in \mathcal{K}$ or $\mathcal{T} \neq \mathcal{K}$.

Using the DE framework

$$\dot{V}_{d^*} = \sum_{j \neq d^*} z_{j,d^*} (1 - q^{V_{d^*} - V_{\{j,d^*\}}}), \tag{7.12}$$

we have

$$\frac{\partial \dot{V}_{d^*}}{\partial z_{j,d^*}} = 1 - q^{V_{d^*} - V_{\{j,d^*\}}}. \tag{7.13}$$

For $\partial z_{j,d^*} / \partial P_i^{\mathrm{Tx}}$, we apply the chain rule

$$\frac{\partial z_{j,d^*}}{\partial P_i^{\mathrm{Tx}}} = \lambda_j \frac{\partial P_{j,d^*}}{\partial P_i^{\mathrm{Tx}}} = \lambda_j \frac{\partial P_{j,d^*}}{\partial P_{j,d^*}^{\mathrm{bit}}} \frac{\partial P_{j,d^*}^{\mathrm{bit}}}{\partial P_i^{\mathrm{Tx}}}. \tag{7.14}$$

From (7.7) we get

$$\frac{\partial P_{j,d^*}}{\partial P_{j,d^*}^{\mathrm{bit}}} = L(1 - P_{j,d^*})^{L-1}. \tag{7.15}$$

For $\partial P_{j,d^*}^{\mathrm{bit}} / \partial P_i^{\mathrm{Tx}}$ we apply the chain rule

$$\frac{\partial P_{j,d^*}^{\mathrm{bit}}}{\partial P_i^{\mathrm{Tx}}} = \frac{\partial P_{j,d^*}^{\mathrm{bit}}}{\partial \mathrm{SINR}_{j,d^*}} \frac{\partial \mathrm{SINR}_{j,d^*}}{\partial P_i^{\mathrm{Tx}}}. \tag{7.16}$$

From (7.6) we get

$$\frac{\partial P_{j,d^*}^{\mathrm{bit}}}{\partial \mathrm{SINR}_{j,d^*}} = \frac{1}{2\sqrt{\pi \mathrm{SINR}_{j,d^*}}} e^{-\mathrm{SINR}_{j,d^*}/2}. \tag{7.17}$$

For $\partial \mathrm{SINR}_{j,d^*} / \partial P_i^{\mathrm{Tx}}$ we have

$$\frac{\partial \mathrm{SINR}_{j,d^*}}{\partial P_i^{\mathrm{Tx}}} = \begin{cases} \frac{h_{j,d^*}}{\sum_{k \neq j,d^*} P_k^{\mathrm{Tx}} h_{k,d^*}/g_{d^*} + \sigma^2}, & i = j, \\ -\mathrm{SINR}_{j,d^*} \frac{h_{i,d^*}/g_{d^*}}{\sum_{k \neq j,d^*} P_k^{\mathrm{Tx}} h_{k,d^*}/g_{d^*} + \sigma^2}, & i \neq j, d^*, \\ 0, & i = d^*. \end{cases} \tag{7.18}$$

At this point, we have all the formulas necessary to compute the feedback.

Fig. 7.9 – 7.10 show the throughputs as functions of time with $a = 10$, and $a = 0.1$, respectively, in addition to throughputs achieved without power control. With $a = 10$,

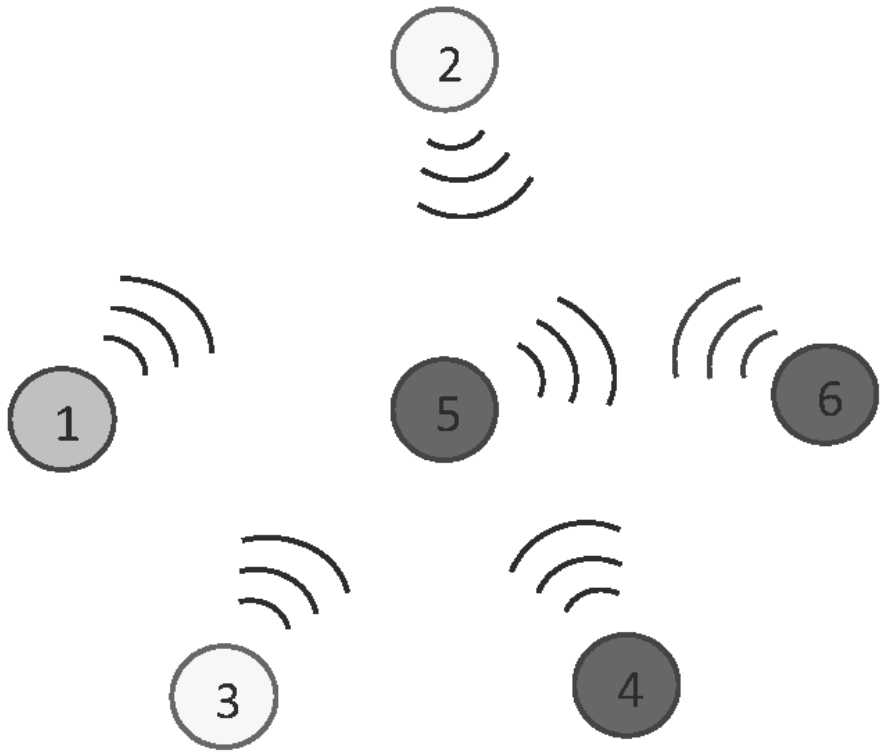

Figure 7.8: A 6-node wireless network.

the minimum throughput T (the lower envelop of all the curves) increases quickly. But as $\mathbf{z}$ changes too fast, (6.5) becomes invalid and leads to oscillation of the trajectory of T. Lowering a results in a smoother trajectory with slower convergence. However, in both settings, T eventually achieves the the same value of 1 packet per millisecond. Considering the transmit rate of node 1 (the source) is also 1 packet per millisecond, the gradient based resource allocation has achieved the maximum possible objective without giving up any multicast advantage and also showed significant improvement over no power control.

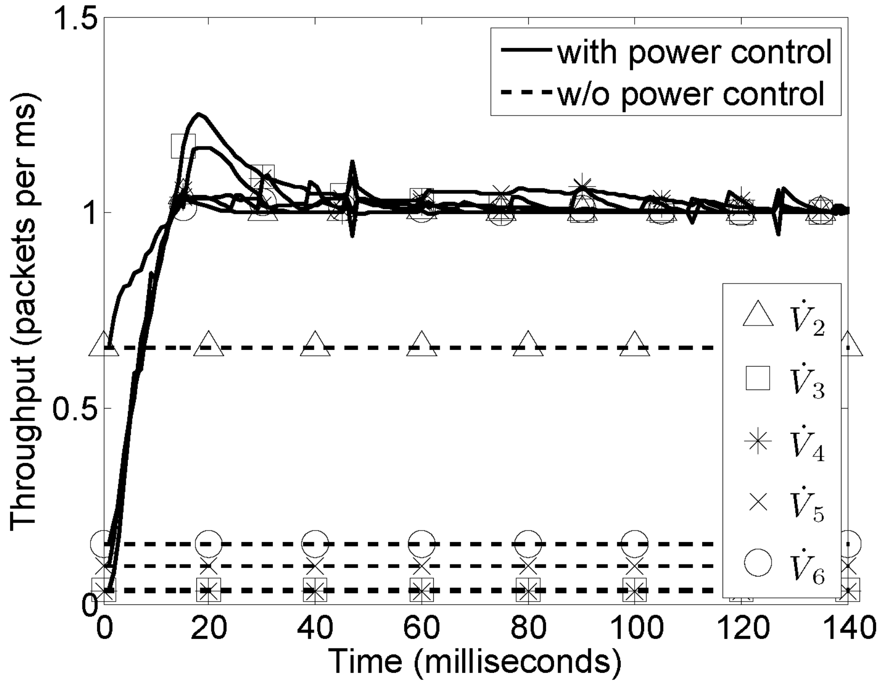

Figure 7.9: Throughput evolution with power control, $a = 10$.

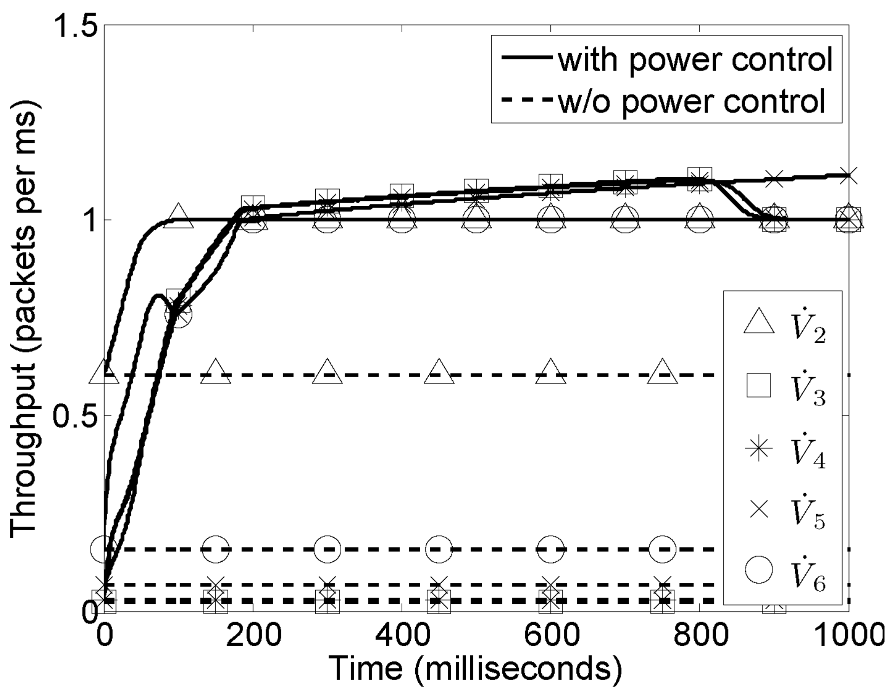

Figure 7.10: Throughput evolution with power control, $a = 0.1$.

Chapter 8

Concluding Remarks

We presented a new framework, based on differential equations, for analyzing the rank dynamics of RNC in a wireless network. We showed that by adopting the fluid approximation, we can derive a system of deterministic DEs, the solution of which shows that the average throughput is given by the min-cut bound. We next showed that the min-cut bound can be asymptotically achieved, by analyzing the exact system of DEs that characterizes the mean and the variance of the rank evolution process. We demonstrated the versatility of the DE framework by presenting a systematic approach to the cross layer design problems that involve RNC. We numerically evaluated different RNC use scenarios including multiple multicast sessions, complex topology and correlated reception to show the versatility of our framework. We also discussed a cross layer design example on power control in an interference limited wireless network with RNC.

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