RANDOM COVERING IN HIGH DIMENSION BY A UNION OF SCALED CONVEX SETS

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ABSTRACT OF THE DISSERTATION

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This PhD thesis is devoted to random covering theory; we study the covering property of a set by a union of randomly placed sets, and focus mainly on the condition for almost sure coverage of every point of the set. A. Dvoretzky initiated this direction of research by proving that covering every *fixed point* with probability 1 does not necessarily imply that *every point* is covered with probability 1 when the set to be covered is uncountable, by giving an example where covering every point in a unit circumference circle almost surely does not imply covering the whole circle [6]. Since then, to study this phenomenon, several settings have been proposed; we concentrate on two of these, the Dvoretzky problem and the Mandelbrot problem.

For the Dvoretzky problem, let $\mathcal{C}$ be a convex set and let $\{v_n\}$ be a sequence of volumes of scaled copies of $\mathcal{C}$ that are placed uniformly on the d-dimensional torus. We find a necessary condition and also a sufficient condition for the union of the sets to cover a fixed k-dimensional hyperplane, $k > 0$. Furthermore, a necessary and sufficient condition is also obtained for the special case when $k = 1$.

For the Mandelbrot problem, let $\mathcal{C}$ be a convex set with volume 1 in $\mathbb{R}^d$, and let each point (x, z), where $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^+$ be associated with a convex set $x + z\mathcal{C}$. Let Φ be a Poisson point process in $\mathbb{R}^d \times \mathbb{R}^+$ with intensity $\lambda \otimes \mu$, where λ is a Lebesgue measure and μ is a σ-finite measure. We give a necessary condition and also a sufficient condition on μ for the union of all convex sets associated with points in Φ to cover any k-dimensional hyperplane in $\mathbb{R}^d$. Furthermore, a necessary and sufficient condition is also obtained for the special case when $k = 1$.

We also consider covering a more general set. In particular, we derive a necessary condition and also a sufficient condition for covering a Cantor set and its generalized version in the one-dimensional Mandelbrot problem setting.

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Table of Contents

Chapter 1

Introduction

1.1 The Dvoretzky problem

Consider the d-dimensional torus $\mathbb{T}^d$, where $\mathbb{T} = \mathbb{R}\backslash\mathbb{Z}$ and a sequence of open convex sets $g_1, g_2, \ldots$ with volumes $v_1, v_2, \ldots$, respectively. We suppose that $v_1 > v_2 > \cdots > v_n > \cdots$, approaches zero. Let $G_1, G_2, \cdots$ be translated copies of convex sets $g_1, g_2, \ldots$, placed independently and randomly on $\mathbb{T}^d$; that is, $G_k = \omega_k + g_k$, where ω_k's are independently uniformly distributed random variables on $\mathbb{T}^d$. A random covering U is defined as a union of all convex sets G_k, i.e., $U = \bigcup G_k$. It is of interest to find a necessary and sufficient condition on the sequence $\{v_k\}$ for almost sure covering; that is, $P(A \subset U) = 1$, for a set of interest A.

The problem was first posed by A. Dvoretzky [6] for $d = 1$, where G_k's are random arcs on a unit circumference circle. It is easy to see that a fixed point in $\mathbb{T}$ is covered almost surely if $\sum v_k = \infty$, using the Borel-Catelli lemma. Furthermore, when $\sum v_k = \infty$, the Lebesgue measure, m, of the uncovered part is zero almost surely. Indeed, m is non-negative and its expected value is zero. That is,

$$Em = E\int_0^1 \mathbb{I}\{x \text{ not covered by } U\}dx = \int_0^1 P(x \notin U)dx = 0.$$

However, A. Dvoretzky, in [6], constructed a sequence $\{v_k\}_{k=1}^{\infty}$ such that the union of those corresponding arcs covers every fixed point almost surely but fails to cover the whole circle almost surely. Since then, the question of finding a necessary and sufficient condition attracted attentions of many people, including P. Lévy, J. P. Kahane [15,16], P. Erdö [8], P. Billard [3], B. Mandelbrot [20]. Finally, it was settled in 1972 by L.

Shepp [25], who showed that $\mathbb{T}$ is covered almost surely if and only if

$$\sum_{n=1}^{\infty} n^{-2} \exp(v_1 + \cdots + v_n) = \infty. \tag{1.1}$$

Yet, several related questions remained open, and some were considered afterwards. J-P. Kahane [16] and J. Hawkes [10] considered conditions on $\{v_k\}$ for covering an α Hausdorff dimensional set almost surely. In an attempt to generalize Shepp's condition, Wschebor [30] considered a sequence of random measurable sets with Lebesgue measure v_k's, instead of random arcs. He concluded that if a random covering with arcs can not cover the circle, a random covering with measurable sets can not do any better. In fact, Huffer and Shepp [11] showed that the probability of covering the circle by n arcs is Schur-convex. It suggests that replacing arcs by measurable sets with the same length make it harder to cover the circle. However, whether (1.1) is a necessary and sufficient condition for covering with open measurable sets remains open. A. Durand [5] computed the Hausdorff measure of the almost sure covered set in the Dvoretzky setting. Recently, Jonasson and Steif [14] considered two interesting dynamical models of the Dvoretzky problem, where arcs are allowed to move along the circle according to a Brownian motion and to be updated according to a Poisson process, looking for conditions when covering fails at some time even when (1.1) holds.

In the high dimensional setting, as given above, no complete solution has been found yet. Most of recent results are mainly due to J-P. Kahane [17–19]. Let A be a compact set of $\mathbb{T}^d$ and σ be a probability measure on A. Let χ_k be the characteristic function of G_k, and define

$$\xi_k(x) = \int_{\mathbb{T}^d} \chi_k(x+y)\chi_k(y)dy,$$

that is, $\xi_k(x)$ is the volume of the intersection between G_k and its translation $G_k + x$. We also define

$$\int f(x)d\tau(x) = \int\int f(x-y)d\sigma(x)d\sigma(y).$$

Kahane [17] gave a necessary condition for almost sure covering

Proposition 1.1. *If* $\sum v_k^2 < \infty$ *and*

$$\int \prod \big(1 + \xi_k(x)\big) d\tau(x) < \infty,$$

then A is not covered almost surely.

The proposition leads to several more straightforward conditions

Proposition 1.2. *If A is a Lebesgue measurable set with positive measure and g_k's are convex with*

$$\sum v_k^2 \exp(v_1 + \cdots + v_k) < \infty,$$

then A is not covered almost surely.

Proposition 1.3. *If A has positive α-dimensional Hausdorff measure, $\sum v_k^2 < \infty$, and*

$$\sum v_k d_k^\alpha \exp(v_1 + \cdots + v_k) < \infty,$$

where d_k is the diameter of g_k, then A is not covered almost surely.

Remark 1.4. *The condition $\sum v_k^2 < \infty$ can be relaxed, as one can show that when $\sum v_k^2 = \infty$, the whole torus $\mathbb{T}^d$ is covered almost surely. In fact, we provide a proof for a more general result, lemma 4.1 in chapter 4, which says that if $\sum v_k^{1+\varepsilon} = \infty$, for some $\varepsilon > 0$, then the torus $\mathbb{T}^d$ is covered almost surely.*

Kahane also gave a sufficient condition in [17].

Proposition 1.5. *If g_k's are convex and*

$$\overline{\lim_{n\to\infty}} \left(\sum_{k=1}^{n} v_k - d \log n \right) = \infty,$$

then $\mathbb{T}^d$ is covered almost surely.

Since then, there has not been any breakthrough on this general problem. Several people have tackled this problem with some additional assumptions. The most common one is that all random sets are homothetic to a pre-specified convex set $\mathcal{C}$ with volume 1 and center of gravity at the origin. Y. El Hélou [7] shows

Theorem 1.6. *Suppose*

$$D = \overline{\lim_{n\to\infty}} \left(\frac{1}{\log(1/v_n)} \sum_{k=1}^{n} v_k \right),$$

and the box dimension of A is α.

1. If $\alpha/d < D$, then A is covered almost surely.

2. If $\alpha/d > D$, then A is not covered almost surely.

3. The random subset of A which is not covered infinitely often has Hausdorff dimension $\alpha - Dd$ a.s.

Corollary 1.7. *The torus $\mathbb{T}^d$ is covered almost surely if $D > 1$ and not covered almost surely if $D < 1$.*

However, the corollary 1.7 is inconclusive about the coverage of $\mathbb{T}^d$ when $D = 1$. To close the gap between the necessary condition and the sufficient condition, Kahane [19] considered a more restricted family of convex sets, which he called "simplex-like" convex sets. Precisely, a convex set $\mathcal{C}$ belongs to this family if it satisfies

(a) Convex set $\mathcal{C}$ has a $(d-1)$-dimensional face, denoted by π.

(b) There exist a cone γ with vertex at the origin such that $\gamma + x$ intersects $\mathcal{C}$ when $x \in \pi$ and is disjoint from $\mathcal{C}$ when $x \in \partial\mathcal{C}\backslash\pi$, where $\partial\mathcal{C}$ is the boundary of $\mathcal{C}$.

For example, simplexes, cones, or frustums belong to this family. In 2 dimension, any quadrilateral that is not a parallelogram also belongs to this family. Another example is a spherical cap that is strictly contained in half of a sphere. However, squares, cubes, cylinders, circles, balls, etc. are not members of this family. He showed [19] that with the convex set $\mathcal{C}$ belonging to this family, the necessary and sufficient condition for covering $\mathbb{T}^d$ almost surely is

$$\int_0^1 \exp\left(\sum_{n=1}^{\infty} v_n\big(1 - (s/v_n)^{1/d}\big)^+\right) ds = \infty.$$

Recently, S-Y Shiu [28] found a necessary and sufficient condition for covering a connected curve (i.e. a one-dimensional cross-section) on the 2-dimensional torus with random squares. He showed that

Proposition 1.8. *A necessary and sufficient condition for covering the connected curve $\{0\} \times \mathbb{T}$ is*

$$\sum_{n=1}^{\infty} \frac{l_n}{\left(\sum_{i=1}^n l_i\right)^2} \exp\left(\sum_{i=1}^n l_i^2\right) = \infty,$$

where $l_1, l_2, \ldots$ are side lengths of those squares.

In this thesis, one of our main results provides a necessary condition and a sufficient condition for covering any k-dimensional cross-section of $\mathbb{T}^d$ almost surely, which improves the result of El Hélou, but fails to obtain the necessary and sufficient condition as Kahane's result (although we consider a general convex set instead of Kahane's restricted convex set). Furthermore, we also generalize Shiu's result, proposition 1.8, and proves a necessary and sufficient condition for covering a connected curve in $\mathbb{T}^d$ by any general convex set.

Theorem A. *Let $\{g_n\}$ be a sequence of open convex sets in $\mathbb{T}^d$ that are homothetic to a pre-specified convex set with volume 1 and center of gravity at the origin, and $\{v_n\}$ for $n = 1, 2, \cdots$ be their volumes. Then*

(i) (Sufficient condition) A fixed k-dimensional cross-section of $\mathbb{T}^d$ is covered almost surely, if

$$\int_0^1 \exp\left\{(1-\varepsilon)\sum_{n=1}^{\infty} v_n \left(1 - \frac{s^{1/k}}{v_n^{1/d}}\right)^+\right\} ds = \infty, \qquad \textit{for some } \varepsilon > 0.$$

(ii) (Necessary condition) A fixed k-dimensional cross-section of $\mathbb{T}^d$ is not covered almost surely, if

$$\int_0^1 \exp\left\{\sum_{n=1}^{\infty} v_n \left(1 - \frac{s^{1/k}}{v_n^{1/d}}\right)^+\right\} ds < \infty$$

(iii) Moreover, a necessary and sufficient condition for covering a connected curve $\mathbb{T} \times \{0\}^{d-1}$, i.e., $k = 1$ almost surely is

$$\sum_{n=1}^{\infty} \frac{v_n^{(d-1)/d}}{a_n^2} \exp\{v_1 + \cdots + v_n\} = \infty,$$

where

$$a_n = \sum_{k=1}^{n} v_k^{(d-1)/d}.$$

Remark 1.9. *In case of $d = 2$ and the convex set is a square, our necessary and sufficient condition for covering a connected curve recovers Shiu's result [28].*

Remark 1.10. *The theorem A strongly suggests that the necessary and sufficient condition on $\{v_k\}$ for covering any k-dimensional cross-section of $\mathbb{T}^d$ almost surely is*

$$\int_0^1 \exp\left\{\sum_{n=1}^{\infty} v_n \left(1 - \frac{s^{1/k}}{v_n^{1/d}}\right)^+\right\} ds = \infty.$$

Remark 1.11. *In the expository paper [27], L. Shepp conjectured that there exists a convex set $\mathcal{C}$ and a scaling sequence $\{v_n\}$ such that the corresponding union of scaled convex sets covers $(k-1)$-dimensional hyperplane with probability 1 but does not cover k-dimensional hyperplane almost surely. The theorem A effectively confirms the conjecture. In fact, consider any convex set $\mathcal{C}$ and a sequence $\{v_k = \frac{\lambda}{k+1}, k = 1, 2, 3, \ldots\}$. The theorem A implies that*

(a) When $\lambda > 1$, the whole torus $\mathbb{T}^d$ **is covered** *almost surely.*

(b) When $\lambda < 1$, the whole torus $\mathbb{T}^d$ **is not covered** *almost surely.*

(c) When $\lambda > k/d$, the k-dimensional hyperplane **is covered** *almost surely.*

(d) When $\lambda < k/d$, the k-dimensional hyperplane **is not covered** *almost surely.*

1.2 The Mandelbrot problem

Let $\mathcal{C}$ be a convex set in $\mathbb{R}^d$ with volume 1 and center of gravity at the origin. We define a random covering as the union of random convex sets which are translated and scaled copies of $\mathcal{C}$ in $\mathbb{R}^d$. More precisely, let Φ be a Poisson point process on $\mathbb{R}^d \times (0, \infty)$, with intensity $\lambda_d \times \mu$, where λ_d is a Lebesgue measure in $\mathbb{R}^d$ and μ is a σ-finite measure on $(0, \infty)$. For each point $(x_k, z_k) \in \Phi$, where $x_k \in \mathbb{R}^d$ and $z_k \in (0, \infty)$, we define a random convex set as $\mathcal{C}_k = x_k + z_k \cdot \mathcal{C}$, and a random covering U as $U = \bigcup \mathcal{C}_k$. Similar to the Dvoretzky problem, the question of interest is to find a necessary and sufficient condition on the measure μ for almost sure covering. That is, set A is said to be covered almost surely if $P(A \subset U) = 1$. It is easy to show that a condition for covering any arbitrary point in $\mathbb{R}^d$ is

$$\int_0^\infty z^d d\mu(z) = \infty. \tag{1.2}$$

Indeed, due to the translation invariant property of Poisson point processes, the probability of almost sure covering of any point is equivalent to that of the origin. Also, random convex set $\mathcal{C}_k$ covers the origin if $-\frac{x_k}{z_k} \in \mathcal{C}$. Let A be a cone such that

$$A = \{(x, z) : x \in \mathbb{R}^d, z \in (0, \infty), -\frac{x}{z} \in \mathcal{C}\}. \tag{1.3}$$

The origin is covered by the union of random convex sets almost surely if A contains at least one point of Φ with probability 1. That is, $P(N(A) > 0) = 1$ or $P(N(A) = 0) = 0$, where $N(\cdot)$ is the total number of points of Φ in a given region. It follows that

$$P(N(A) = 0) = \exp(-\lambda \otimes \mu(A)) = 0$$

which implies (1.2). Furthermore, when (1.2) holds, P. Hall [9] showed that the Lebesgue measure of the uncovered set is zero almost surely. In other words, the random covering covers the whole space almost everywhere almost surely. Hence, it is natural to ask whether we can replace "almost everywhere" by "everywhere". That is, what is a necessary and sufficient condition on the measure μ such that a given set is covered with probability 1?

The problem was first raised by B. Mandelbrot [21] for the case when $d = 1$, i.e., random intervals on the real line. He remarked that his problem is closely related to the Dvoretzky problem; however, his setting is more natural to Dvoretzky's setting, and could provide an insight to the solution of this problem. In fact, by using a similar approach as in [25] for the Dvoretzky problem, L.A. Shepp [26] gave a necessary and sufficient condition on μ for covering the whole real line a.s.,

$$\int_0^1 \exp\left\{\int_x^\infty (z - x) d\mu(z)\right\} dx = \infty. \tag{1.4}$$

It is easy to see that the Mandelbrot problem's condition (1.4) is identical to that of the Dvoretzky problem (1.1) when

$$d\mu(z) = \sum \delta_{z_i}(z), \qquad (z_i^d = v_i)$$

where δ is a Dirac measure. Similar to the generalized version of the Dvoretzky problem in the one dimension, Wschebor [31] showed that if (1.4) does not hold, the random covering, constructed by replacing intervals with Lebesgue measurable sets, can not cover the whole real line with probability 1. Later on, when tackling the Dvoretzky problem in the high dimension, the main approach is to consider the equivalent Mandelbrot problem, obtaining results and then "converting" those to the Dvoretzky problem, as have been done in [1,13,19]. Therefore, one can generally view the Mandelbrot problem as a non-compact version of the Dvoretzky problem.

Despite a close relation, there is a distinct difference between these two problems. In the Dvoretzky problem, the size of all convex sets is naturally less that 1 (otherwise, it is a trivial case); meanwhile, there is no restriction on those sizes in the Mandelbrot problem. Due to this, Mandelbrot defined two types of almost sure covering: (1) by a few very large volume convex sets and (2) by a large number of very small convex sets. It leads to the following definition about the measure μ with respect to the covering behavior,

Definition 1.12. *Consider a random covering constructed from a Poisson point process with intensity $\lambda \otimes \mu$.*

(i) A measure μ is said to give a high frequency coverage if the whole space is covered almost surely by convex sets whose volume is less than 1.

(ii) A measure μ is said to give a low frequency coverage if the whole space is covered almost surely by convex sets whose volume greater than or equal to 1.

To study these two types of coverage separately, we restrict the support of the measure μ on $(1, \infty)$ for the low frequency covering case and on $(0, 1]$ for the high frequency covering case. Note that the cutoff value of 1 is arbitrary and can be set to any positive real value. L. Shepp [26] showed that when $d = 1$, the necessary and sufficient condition on μ to give a low frequency coverage is identical to (1.2), and high frequency coverage is (1.4).

The problem extends to the higher dimensional setting, as given here, with the goal of finding a necessary and sufficient condition on μ for almost sure coverage. When restricting the convex set $\mathcal{C}$ to be a ball and the measure μ to be finite, Roy and Meester [23] showed that the condition (1.2) is a necessary and sufficient condition for covering the whole space almost surely. Biermé and Estrada [2] extended Roy and Meester result, and showed that a necessary and sufficient condition on a σ-finite measure μ for a low frequency coverage when $\mathcal{C}$ is a ball is

$$\int_1^\infty r^d d\mu(r) = \infty.$$

Furthermore, Biermé and Estrada also showed that a necessary condition and a sufficient condition for covering $\mathbb{R}^d$ by random balls is

(i) A necessary condition for high frequency covering by random Poisson balls is

$$\int_0^1 u^{d-1} \exp\left(\int_u^1 r^{d-1}(r-u)d\mu(r)\right) du = \infty.$$

(ii) A sufficient condition for high frequency covering by random Poisson balls is

$$\limsup_{u\to 0} u^d \exp\left(\int_u^1 (r-u)^d d\mu(r)\right) = \infty.$$

Up to now, the only necessary and sufficient condition available for a high frequency coverage was obtained by J-P. Kahane, when he consider a restricted family of convex sets, as mentioned in the Dvoretzky problem subsection. He showed that the whole $\mathbb{R}^d$ is covered almost surely if and only if

$$\int_0^1 \exp\left(\int_x^1 (z-x)z^{d-1}d\mu(z)\right) x^{d-1}dx = \infty. \tag{1.5}$$

Hence, it is tempting to conjecture that the condition (1.5) holds for any convex set. However, we fail to relax the restriction on the shape of the convex set. Alternatively, we find a necessary condition and a sufficient condition separately, whose gap is very narrow. In chapter 3, we show that

Theorem B.1. ***(High frequency coverage)****Let μ be a σ-finite non-negative measure on $(0,1]$. Then,*

(i) (Sufficient condition) A union of random convex sets covers a k-dimensional hyperplane in $\mathbb{R}^d$ with probability 1, if

$$\int_0^1 \exp\left((1-\varepsilon)\int_x^1 (z-x)z^{d-1}d\mu(z)\right) x^{k-1}dx = \infty, \qquad \textit{for some } \varepsilon > 0.$$

(ii) (Necessary condition) If a union of random convex sets covers a k-dimensional hyperplane in $\mathbb{R}^d$ with probability 1, then

$$\int_0^1 \exp\left(\int_x^1 (z-x)z^{d-1}d\mu(z)\right) x^{k-1}dx = \infty.$$

(iii) *Moreover, when* $k = 1$*, i.e., an arbitrary line in* $\mathbb{R}^d$*, a necessary and sufficient condition on* μ *for almost sure covering is*

$$\int_0^1 \exp\left(\int_x^\infty (z-x)z^{d-1}d\mu(z)\right)dx = \infty.$$

Remark 1.13. *The theorem B.1 is a strong indication that (1.5) is indeed a general necessary and sufficient condition on* μ *for covering the whole space, regardless the shape of the convex set.*

Theorem B.2. ***(Low frequency coverage)*** *A union of random convex sets covers the whole space with probability one if and only if*

$$\int_1^\infty z^d d\mu(z) = \infty.$$

Remark 1.14. *The theorem B.2 implies that the whole space is covered almost surely by a low frequency measure if and only if any arbitrary subset is covered almost surely. It indicates that the low frequency coverage is easier to occur than the high frequency coverage. For example, consider the measure* $d\mu(z) = \lambda z^{-(d+1)}dz$*,* $\lambda > 0$*. From theorem B.1 and theorem B.2, we have that*

(a) *When* $\lambda > d$*, the measure* μ *gives a high frequency covering.*

(b) *When* $\lambda < d$*, the measure* μ *does not give a high frequency covering.*

(c) *Regardless of the value of* λ*, the measure* μ *gives a low frequency covering.*

Chapter 2

Background and preliminary results

2.1 General notations

- We use $\mathbb{R}^d$ to denote the d-dimensional Euclidean space, $\mathcal{B}(\mathbb{R}^d)$ for the σ-field on the Borel sets. The notation $\mathbb{R}^k \times \{0\}^{d-k}$ is denoted the k-dimensional hyperplane where the $(k+1)^{th}$ to the d^{th} coordinate are all zero. The notation $\mathbb{Z}^d$ and $\mathbb{Q}^d$ is used to denote the set in $\mathbb{R}^d$ whose coordinates are integer and rational number, respectively.

- In $\mathbb{R}^d$, two set operator addition and multiplication by a real number is defined as following: For set A and B in $\mathbb{R}^d$, and a real number λ

$$A + B = \{x + y : x \in A, y \in B\}$$
$$\lambda A = \{\lambda x : x \in A\}.$$

  A translate of A is defined as $x + A = \{x + y : y \in A\}$.

- The Lebesgue measure in $\mathbb{R}^d$ is denoted by λ_d. For a set A in $\mathbb{R}^d$, we write $|A|$ for the d-dimensional volume of A. When integrating with respect to the Lebesgue measure, we use dx instead of $\lambda_d(dx)$.

- We use $\|\cdot\|$ to denote the Euclidean norm, $\lfloor x \rfloor$ to denote the large integer that smaller or equal x, and $(x)^+ = \max(x, 0)$.

2.2 A volume inequality concerning the intersection of convex set and its translation

A set A is said to be convex if a line segment connecting any two points of A is contained in A. The diameter, $diamA$, of A is the supremum of the distance between two points of the set. The circumradius of a set A is the minimum radius of a Euclidean ball which contains the set. The minimum ball is called the circumball of A. Opposite to the circumradius is the inradius, the maximum radius of a Euclidean ball which is contained in the set.

One of the useful theorem in convex geometry is the Minkowski theorem on mixed volumes

Theorem 2.1 (Minkowski theorem). *Let $C_1, \dots, C_m$ be convex sets in $\mathbb{R}^d$. Then there are coefficients $V(C_{i_1}, \dots, C_{i_d})$, $1 \le i_1, \dots, i_d \le m$, called mixed volumes, which are symmetric in the indices and such that*

$$|\lambda_1 C_1 + \cdots + \lambda_m C_m| = \sum_{i_1,\dots,i_d=1}^{m} V(C_{i_1}, \dots, C_{i_d}) \lambda_{i_1} \cdots \lambda_{i_d} \textit{ for } \lambda_1, \dots, \lambda_m \ge 0.$$

Definition 2.2. *Let C is a convex set and B is a unit ball in $\mathbb{R}^d$, the mixed volume*

$$W_k(C) = V(\underbrace{C, C, \dots, C}_{d-k \textit{ times}}, \underbrace{B, B, \dots, B}_{k \textit{ times}})$$

is called k-th quermassintegral of C.

Remark 2.3. *For a convex set C, we denote $|C|$ and $S(C)$ be its volume and its surface area. Then*

$$\begin{aligned} W_0(C) &= |C|_d \\ W_1(C) &= \frac{1}{d} S(C). \end{aligned}$$

An important special case of the Minkowski theorem is Steiner formula for parallel bodies.

Theorem 2.4 (Steiner formula). *Let C be a convex set and B^d be an unit ball in $\mathbb{R}^d$. Then*

$$|C + \lambda B^d| = \sum_{k=0}^{d} \binom{d}{k} W_k(C) \lambda^k \textit{ for } \lambda \ge 0,$$

where $W_k(C)$ is a k-th quermassintegral of C.

Finally, we are ready to prove the following inequality regarding the volume of a convex set and its translate. The inequality was first mentioned without a proof in [19].

Lemma 2.5. *Let C be a convex body in $\mathbb{R}^d$ with volume v and center of gravity at the origin. Then, there exist two positive numbers a and b such that*

$$v(1 - a\|x\|)^+ \leq |C \cap (C + x)| \leq v(1 - b\|x\|)^+, \text{ for all } x \in \mathbb{R}^d. \tag{2.1}$$

Proof. Consider the lower bound in (2.1). Intuitively, for any x away from the origin, we can pick a large enough to make $(1 - a\|x\|)^+ = 0$, which guarantees the lower bound. So, we only need to prove the lower bound when x is close to the origin. We observe that $(C \cup (C + x)) \subset (C + \|x\|B)$, where B is a unit ball in $\mathbb{R}^d$. Hence, by applying the Steiner formula, we obtain

$$\begin{aligned}
|C \cap (C + x)| &= |C| + |(C + x)| - |C \cup (C + x)| \\
&\geq 2v - \big|C + \|x\|B\big| \\
&= 2v - \left(v + \sum_{k=1}^{d} \binom{d}{k} W_k(C)\|x\|^k\right) \\
&\geq v - \sum_{k=1}^{d} \binom{d}{k} W_k(C)\|x\| \\
&= v\left(1 - \frac{\sum_{k=1}^{d} \binom{d}{k} W_k(C)}{v}\|x\|\right).
\end{aligned}$$

Hence, it proves that $|C \cap (C + x)| \geq v(1 - a\|x\|)^+$, where

$$a = \max\left(1, \frac{\sum_{k=1}^{d} \binom{d}{k} W_k(C)}{v}\right)$$

To prove the upper bound in (2.1), we consider 2 separate cases: when x is close to the origin and when it is away from the origin. First, consider the upper bound when $\|x\| \geq r/2$, where r is the inradius of C. Now, since $|C \cap (C + x)|$ is bounded above by v, we denote the largest volume of the intersection by K, i.e.,

$$K = \sup\{|C \cap (C + x)| : \|x\| \geq r/2\}.$$

We also denote L be the furthest distance of x such that the volume of the intersection is still nonzero,

$$L = \sup\{\|x\| : C \cap (C + x) \neq \emptyset\}.$$

It is easy to see that for any $x \geq r/2$, with $b = \frac{v-K}{vL}$,

$$v(1 - b\|x\|) = v - \frac{(v-K)\|x\|}{L} \geq v - \frac{(v-K)\|x\|}{\|x\|} = K \geq |C \cap (C+x)|.$$

Secondly, let $proj_x(\cdot)$ be the projection onto the hyperplane, having normal vector x. We define

$$K' = \inf \left\{ \left| \mathrm{proj}_x\big(C \cap (C+x)\big) \right| : \|x\| < r/2 \right\}$$

It follows that for all $x < r/2$,

$$\begin{aligned} |C \backslash (C+x)| &\geq \|x\|K' \\ |C| - |C \cap (C+x)| &\geq \|x\|K' \\ |C \cap (C+x)| &\leq |C| - \|x\|K' = v\left(1 - \frac{K'}{v}\|x\|\right) \end{aligned}$$

The desired inequality is obtained by pick $b = \min(\frac{v-K}{vL}, \frac{K'}{v})$. $\square$

2.3 Poisson point processes

In this section, we briefly review the theory of Poisson point processes from the random measure point of view. The systematic development of the general theory of point processes has been carried out in Daley and Vere-Jones [4].

Definition 2.6. *For $x \in \mathbb{R}^d$, a Dirac measure is a probability measure, such that for any Borel set A,*

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Definition 2.7. *A counting measure on $\mathbb{R}^d$ is a σ-finite integer-valued measure such that*

$$N = \sum_{i=1}^{\infty} \delta_{x_i}.$$

A counting measure is said to be simple if $N\big(\{x\}\big) = 0$ or 1 for all $x \in \mathbb{R}^d$. We also denote $\mathcal{N}_{\mathbb{R}^d}$ and $\mathcal{N}^*_{\mathbb{R}^d}$ be the family of all counting measures and all simple counting measures, respectively.

Definition 2.8. *(i) A point process N on space $\mathbb{R}^d$ is a measurable mapping from a probability space $(\Omega, \mathcal{E}, \mathcal{P})$ into $(\mathcal{N}_{\mathbb{R}^d}, \mathcal{B}(\mathcal{N}_{\mathbb{R}^d}))$.*

(ii) A point process N is simple when $P(N \in \mathcal{N}^*_{\mathbb{R}^d}) = 1$.

Definition 2.9. *A σ-finite measure Λ is a intensity measure for a point process N if for any Borel set A in $\mathbb{R}^d$,*

$$\Lambda(A) = EN(A),$$

i.e., $\Lambda(A)$ is the average number of points of N falling in A.

Definition 2.10. *A Poisson point process N on space $\mathbb{R}^d$ with intensity measure Λ is a simple point process possessing the two following properties:*

(i) The number of points $N(A)$ has a Poisson distribution with mean $\Lambda(A)$, for any Borel set A.

(ii) For a finite family of bounded, disjoint Borel sets $\{A_i, i = 1, \cdots, k\}$ in $\mathcal{B}_{\mathbb{R}^d}$, the random variable $N(A_1), \cdots, N(A_k)$ are mutually independent.

Definition 2.11. *The avoidance probability of a point process is the probability of there being no point of the process in the given test set A,*

$$P_0(A) = P(N(A) = 0).$$

Theorem 2.12. *Let μ be a non-atomic measure on $\mathbb{R}^d$, finite on bounded sets, and suppose that the simple point process N is such that for any set A,*

$$P_0(A) = P(N(A) = 0) = \exp(-\mu(A)).$$

Then N is a Poisson point process with mean $\mu(A)$.

Definition 2.13. ***(Stationary and isotropic point process)***

(i) A point process X is stationary if its distribution is invariant under any translation. That is, $X + s = \{x + s, x \in X\}$ has the same distribution as X, for all $s \in \mathbb{R}^d$.

(ii) A point process X is isotropic if its distribution is invariant under rotations around the origin. That is, $gX = \{g \cdot x, x \in X\}$ has the same distribution as X, for any rotation g.

Theorem 2.14. *A Poisson point process on $\mathbb{R}^d$, having a constant intensity with respect to Lebesgue measure on $\mathbb{R}^d$ is stationary and isotropic.*

Proof. The theorem is a simple application of theorem 2.12, and the fact that Lebesgue measure is invariant under translation and rotation. □

2.4 Theory of random sets

Let $\mathcal{G}$ and $\mathcal{K}$ be a family of open sets and a family of compact sets in $\mathbb{R}^d$, respectively and $(\Omega, \mathscr{F}, \mathcal{P})$ be the usual probability space.

Definition 2.15. *A map $X : \Omega \to \mathcal{G}$ is called a random open set if, for every compact set K in $\mathbb{R}^d$,*

$$\{\omega : K \cap X \neq \emptyset\} \in \mathscr{F}$$

Definition 2.16. ***(Capacity functional)*** *A functional $T : \mathcal{K} \to [0, 1]$ given by*

$$T_X(K) = P(X \cap K \neq \emptyset) \qquad K \in \mathcal{K}$$

is said to be capacity functional of X. The notation $T(K)$ is used instead of $T_X(K)$ when no ambiguous occurs.

Definition 2.17. *A functional $T : \mathcal{K} \to [0, 1]$ is said to be upper semicontinuous if*

$$T(K_n) \searrow T(K) \text{ as } K_n \searrow K, \text{ in } \mathcal{K}.$$

With every functional T defined on family of compact sets, we define the following successive differences

$$\begin{aligned} \Delta_{K_1} T(K) &= T(K) - T(K \cup K_1) \\ \Delta_{K_n \cdots K_1} T(K) &= \Delta_{K_{n-1} \cdots K_1} T(K) - \Delta_{K_{n-1} \cdots K_1} T(K \cup K_n) \end{aligned}$$

Definition 2.18. *A functional $T : \mathcal{K} \to [0,1]$ is said to be complete alternating if*

$$\Delta_{K_n \cdots K_1} T(K) \leq 0,$$

for all $n \geq 2$ and all $K_1, \ldots, K_n$ in $\mathcal{K}$.

Theorem 2.19. ***(Choquet theorem)*** *A functional $T : \mathcal{K} \to [0,1]$ such that $T(\emptyset) = 0$ is the capacity functional of a necessarily unique random open sets in $\mathbb{R}^d$ if and only if T is upper semicontinuous and complete alternating.*

Further details on theory of random sets as well as proofs and related problems could be found in [24].

2.5 Preliminary results

In this section, we consider the invariant property under transformations of a random covering and its zero-one law regarding almost sure covering. From the setting of the Dvoretzky problem, it is straightforward that the random covering is invariant under translation, due to the invariant property of the uniform distribution on the torus $\mathbb{T}^d$. Under rotation transformation, if $\mathcal{C}$ is the shape of the convex set, to which g_n's are homothetic, the rotated random covering is identical with the original random covering with a rotated convex set. That is, if g is the rotation operator, then

$$g \cdot U = g \bigcup (g_n + \omega_n) = \bigcup (g \cdot g_n + \omega_n),$$

where ω_n's are uniformly distributed random variables. Hence, covering of any arbitrary k-dimensional cross-section of torus $\mathbb{T}^d$ is equivalent to covering $\mathbb{T}^k \times \{0\}^{d-k}$, for $k = 1, \ldots, d$. However, such property is not trivial for the setting in the Mandelbrot problem, although similar results still hold. The next lemmas describe the behavior of a random covering under a translation and a rotation in $\mathbb{R}^d$.

Lemma 2.20. *The random covering U is invariant under translation, i.e., the distribution of $U + a = \{\mathcal{C}_k + a\}$ is the same as the distribution of U for any $a \in \mathbb{R}^d$.*

Proof. For a fixed $a \in \mathbb{R}^d$, we have

$$U + a = \{(x_k + a) + z_k \cdot \mathcal{C} : (x_k, z_k) \in \Phi\}.$$

From theorem 2.14, the distribution of the center of convex sets $\mathcal{C}_k$ is invariant under any translation. Furthermore, the intensity measure corresponding the location and the scaling are independent. Hence, the distribution of the random covering is invariant under translation. $\square$

Lemma 2.21. *For a rotation g in $\mathbb{R}^d$, random covering gU with convex set $\mathcal{C}$ has the same distribution with rotated random covering U with convex set $g\mathcal{C}$.*

Proof. For a rotation operator g around the origin, we have

$$\begin{aligned} gU &= \{g \cdot (x_k + z_k \cdot \mathcal{C}) : (x_k, z_k) \in \Phi\} \\ &= \{g \cdot x_k + z_k(g \cdot \mathcal{C}) : (x_k, z_k) \in \Phi\} \end{aligned}$$

From theorem 2.14, the distribution of the center of convex sets $\mathcal{C}_k$ is invariant under any rotation around the origin. Furthermore, the intensity measure corresponding the location and the scaling are independent. Hence, the distribution of the rotated random covering is the same as the distribution of the random covering with the rotated convex set. $\square$

The following lemma states a zero-one law for probability of covering the subspace in $\mathbb{R}^d$, which is similar to that in [2]. It allows us to work primarily on any convenient compact set instead of on the whole space.

Lemma 2.22. *Let X be the k-dimensional coordinate hyperplane in $\mathbb{R}^d$, that is $X = \mathbb{R}^k \times \{0\}^{d-k}$, and U be a random covering, constructed as described in previous section.*

(i) $P(X \subset U) =$ *zero or one.*

(ii) If there exists K a compact set of X with non-empty interior such that $P(K \subset U) = 1$ then $P(X \subset U) = 1$.

(iii) If there exists K a compact set of X such that $P(K \subset U) < 1$ then $P(X \subset U) = 0$.

Proof. (i) The proof for this part is adapted from [26]. Let $p_n = P([n, \infty)^k \times \{0\}^{d-k} \subset U)$. It is easy to see that $p_n \nearrow p_\infty$, where

$$p_\infty = P([n, \infty)^k \times \{0\}^{d-k} \subset U \text{ for some } n).$$

The latter event has probability zero or one because it is a tail event, depending only on the behavior of Φ outside an arbitrarily large rectangle in $\mathbb{R}^d \times (0, \infty)$. Thus, $p_\infty =$ zero or one. On the other hand p_n does not depend in n by translational invariance of U. Therefore, letting $n \to -\infty$, we have

$$P(X \subset U) = \text{zero or one}$$

(ii) Let Ω be the set of all points in X, where their coordinates are rational numbers; that is, $\Omega = \mathbb{Q}^d \cap X$. Since K is compact with non-empty interior, $X = \bigcup_{x\in\Omega}(K + x)$. It follows

$$P(X \not\subset U) = P\left(\bigcup_{x\in\Omega}(K + x) \not\subset U\right) \leq \sum_{x\in\Omega} P(K \not\subset U) = 0,$$

Therefore, $P(X \subset U) = 1$.

(iii) Since $P(X \subset U) \leq P(K \subset U) < 1$, and $P(X \subset U)$ only has value of zero or one, we must have $P(X \subset U) = 0$.

$\square$

Chapter 3

Covering the subspace in $\mathbb{R}^d$

3.1 The outline of the proof

Throughout the history of the random covering problem, a necessary condition is relatively easier to prove than a sufficient condition. For the necessary condition, the main approach that could be effectively applied to almost all situations, as was done in [16, 25, 26], is due to P. Billard [3], akin to the second moment method in combinatorics. Another approach has been carried out by J-P. Kahane in [19] based on the martingale theory. However, in this thesis, we opt to use Billard's idea, due to its elegance and its ease to understand. On the other end, a sufficient condition, as showed in some special cases [19,25], has been proven using the idea of the "first uncovered point" argument (or its variant, the stopping time). Although the high dimensional space lacks of the complete order system to define "the first uncovered point", Kahane [19] argued that, by restricting the shape of convex sets, one can find a point that could be treated as "the first uncovered point". Without the shape restriction, the proof completely breaks down. If one just wants to obtain a sufficient condition only (without being able to match with a necessary condition) for any convex set, one can approximate the given convex set $\mathcal{C}$ by a sequence of convex sets whose shape belongs to Kahane's restricted family. For some special shapes like squares, rectangles or semi-circles, this strategy works fine; however, it does not work for any arbitrary convex set (e.g., circle, ball, ellipse, etc.). Our main idea to get around this issue is to embed the current setting into a one dimensional higher space, by "transforming" our convex set $\mathcal{C}$ into a new convex set that satisfies two conditions. First, a sufficient condition for covering the new space using new convex sets is the same as that in the original setting. Secondly, the new convex set can be approximated by a sequence of Kahane's special convex sets.

This can be achieved by constructing a cylinder with the original shape as the base and the height is fixed at 1. Although not belonging to the Kahane's family of convex sets, the cylinder can be approximated by a sequence of frustums (chopped cones), one of Kahane's restricted convex sets. Adopting Kahane's argument in [19], we are able to obtain a sufficient condition. However, the drawback of this approach is that we cannot obtain a necessary and sufficient condition due to the fact that the frustum must be inside the cylinder and must be different from the cylinder.

3.2 A necessary condition of covering a subspace

For a fixed $0 < \epsilon < 1$, we denote U_ϵ be a union of convex sets associated with those points of Φ that $\epsilon < z < 1/\epsilon$. Let m be a measure of a part of the unit ball B in the k-dimensional subspace which is left uncovered by U_ϵ. That is,

$$m = \int_B \chi(x)dx$$

where

$$\chi(x) = \begin{cases} 1 & \text{if} \quad x \notin U_\epsilon \\ 0 & \text{if} \quad x \in U_\epsilon \end{cases}$$

Let φ denote an event that $B \not\subset U_\epsilon$.

$$\varphi = \begin{cases} 1 & \text{if} \quad B \notin U_\epsilon \\ 0 & \text{if} \quad B \in U_\epsilon \end{cases}$$

Since $\varphi = 0$ implies $m = 0$, we have $m = m\varphi$. Applying Schwarz's inequality, we have

$$(Em)^2 \leq Em^2 E\varphi^2$$

Since $E\varphi^2 = E\varphi = P(B \not\subset U_\epsilon)$, we have

$$P(B \not\subset U_\epsilon) \geq \frac{(Em)^2}{Em^2} \tag{3.1}$$

The first moment is easy to calculate thanks to the translation invariance property of the random covering.

$$\begin{aligned} Em = E\int_B \chi(x)dx = \int_B E\chi(x)dx = v_B P(0 \notin U_\epsilon) \\ = v_B \exp\bigl(-\lambda \otimes \mu(A)\bigr) \end{aligned} \tag{3.2}$$

where v_B is the volume of the unit ball B, and A is defined as

$$A = \{(x,z) : x \in \mathbb{R}^d, z \in (\epsilon, 1/\epsilon), -\frac{x}{z} \in \mathcal{C}\}. \tag{3.3}$$

The second moment is calculated as

$$\begin{aligned} Em^2 &= \int_B \int_B P(x_1 \notin U_\epsilon, x_2 \notin U_\epsilon) dx_1 dx_2 \\ &= \int_B \int_B P(0 \notin U_\epsilon, x_2 - x_1 \notin U_\epsilon) dx_1 dx_2. \end{aligned} \tag{3.4}$$

Let $u = x_1$ and $v = x_2 - x_1$, (3.4) becomes

$$\begin{aligned} Em^2 &= \int_{2B} \int_{S(v)} P(0 \notin U_\epsilon, v \notin U_\epsilon) du dv \\ &= \int_{2B} |S(v)| P(0 \notin U_\epsilon, v \notin U_\epsilon) dv, \end{aligned}$$

where $S(v) = \{u \in \mathbb{R}^k : \|u\| \le 1\} \cap \{u \in \mathbb{R}^k : \|u + v\| \le 1\}$, the intersection of two unit balls center at the origin and at v, respectively. Since $|S(v)| \le v_B$, we obtain

$$\begin{aligned} Em^2 &\le v_B \int_{2B} P(0 \notin U_\epsilon, x \notin U_\epsilon) dx \\ &= v_B \int_{2B} P(N(A \cup A_x) = 0) = v_B \int_{2B} \exp(-\lambda \otimes \mu(A \cup A_x)) dx \end{aligned} \tag{3.5}$$

where $A_x = A + x$, and A as defined in (3.3). We observe that

$$\begin{aligned} \lambda \otimes \mu(A \cup A_x) &= (\lambda \otimes \mu)(A) + (\lambda \otimes \mu)(A_x) - (\lambda \otimes \mu)(A \cap A_x) \\ &= 2(\lambda \otimes \mu)(A) - (\lambda \otimes \mu)(A \cap A_x). \end{aligned} \tag{3.6}$$

Furthermore, by applying the lemma 2.5, we obtain

$$\begin{aligned} (\lambda \otimes \mu)(A \cap A_x) &= \int_\epsilon^{1/\epsilon} |z \cdot C \cap (z \cdot C + x)| \, d\mu(z) \\ &= \int_\epsilon^{1/\epsilon} z^d \left| C \cap \big(C + (x/z)\big) \right| d\mu(z) \\ &\le \int_\epsilon^{1/\epsilon} z^d \left(1 - \frac{b\|x\|}{z}\right)^+ d\mu(z), \end{aligned} \tag{3.7}$$

for some constant b which only depends on $\mathcal{C}$. Hence, combining (3.5), (3.6), and (3.7), we can bound the second moment by

$$Em^2 \le v_B \big(P(0 \notin U_\epsilon)\big)^2 \int_{2B} \exp\left\{ \int_\epsilon^{1/\epsilon} z^d \left(1 - \frac{b\|x\|}{z}\right)^+ d\mu(z) \right\} dx \tag{3.8}$$

We notice that the integrand in (3.8) only depends the term $\|x\|$. By changing (3.8) to the polar coordinates and integrating out all variables but the radius, we obtain

$$Em^2 \leq c \cdot \big(P(0 \notin U_\epsilon)\big)^2 \int_0^2 \exp\left\{\int_\epsilon^{1/\epsilon} z^d \left(1-\frac{br}{z}\right)^+ d\mu(z)\right\} r^{k-1}dr, \tag{3.9}$$

where c is a constant. Therefore, putting (3.2) and (3.9) back into (3.1), we have

$$\begin{aligned} P(B \not\subset U_\epsilon) &\geq \frac{c_1 \cdot P(0 \notin U_\epsilon)^2}{P(0 \notin U_\epsilon)^2 \int_0^2 \exp\left\{\int_\epsilon^{1/\epsilon} z^d \left(1-\frac{br}{z}\right)^+ d\mu(z)\right\} r^{k-1}dr} \\ &= \frac{c_1}{\int_0^2 \exp\left\{\int_\epsilon^{1/\epsilon} z^d \left(1-\frac{br}{z}\right)^+ d\mu(z)\right\} r^{k-1}dr}, \end{aligned} \tag{3.10}$$

where c_1 is another constant. Hence, the probability $P(B \not\subset U) > 0$ if the denominator in (3.10) is finite when ϵ approaches to infinity. That is,

$$\int_0^2 \exp\left\{\int_0^\infty z^d \left(1-\frac{br}{z}\right)^+ d\mu(z)\right\} r^{k-1}dr < \infty. \tag{3.11}$$

Based on lemma 2.22, if $P(B \not\subset U) > 0$, then the k-dimensional subspace is not covered almost surely. Furthermore, for the high frequency coverage case, i.e., $\mu((1,\infty)) = 0$, by a simple substitution, we can derive a necessary condition as

$$\int_0^1 \exp\left\{\int_0^x z^d \left(1-\frac{x}{z}\right)^+ d\mu(z)\right\} x^{k-1}dr < \infty.$$

For the low frequency covering case, i.e., when $\mu\big((0,1]\big) = 0$, the whole space $\mathbb{R}^d$ is covered with probability 1, if

$$\int_1^\infty z^d d\mu(z) < \infty.$$

3.3 A sufficient condition of covering a subspace

3.3.1 A frustum model

As we see in the previous section, the necessary condition can be obtained through the probability of covering an arbitrary and covering two distinct points. Hence, in this section, we consider a random covering with frustums with an interest only in these two quantities. First, let Ξ^* be a Poisson point process in $\mathbb{R}^d \times \mathbb{R} \times (0,\infty)$, with intensity measure $\lambda_d \otimes \lambda \otimes \mu$. For each point $(x_k, y_k, z_k) \in \Xi^*$, $x_k \in \mathbb{R}^d$, $y_k \in \mathbb{R}$, and $z_k \in (0, \infty)$, and for a fixed $\kappa > 0$, we define a new convex set ξ_k^* in $\mathbb{R}^{(d+1)}$ as

$$\xi_k^* = (x_k, y_k) + \mathcal{C}(z_k),$$

where

$$\mathcal{C}(z) = \{(u, v) : u \in \mathbb{R}^d, v \in \mathbb{R}, v \in (0, 1), u \in (1 - \kappa v)z \cdot \mathcal{C}\}.$$

In other words, $\mathcal{C}(z)$ is a frustum whose top base is $(1 - \kappa)z \cdot \mathcal{C}$, bottom base is $z \cdot \mathcal{C}$, and height is always 1, regardless the value of z. For $0 < \epsilon < 1$, let U_ϵ^* be a union of random frustums associated with those points of Ξ^* for which $\epsilon \leq z_k \leq 1/\epsilon$. Secondly, the probability of covering any arbitrary point can be computed as following. Since the intensity of Poisson point process corresponding to the position of the center of convex sets are invariant under translation, we have

$$P(x \in U_\epsilon^*) = 1 - P(x \notin U_\epsilon^*) = 1 - P(0 \notin U_\epsilon^*).$$

Let A be a region in $\mathbb{R}^d \times \mathbb{R} \times (0, \infty)$, such that a random frustum covers the origin if its associated point in Ξ^* is in this region. That is,

$$\begin{aligned} A &= \big\{(x, y, z) \in \mathbb{R}^d \times \mathbb{R} \times (0, \infty) : 0 \in (x, y) + \mathcal{C}(z)\big\} \\ &= \big\{(x, y, z) \in \mathbb{R}^d \times \mathbb{R} \times (0, \infty) : (x, y) \in -C(z)\big\} \\ &= \big\{(x, y, z) \in \mathbb{R}^d \times \mathbb{R} \times (0, \infty) : y \in (-1, 0) \text{ and } x \in -(1 + \kappa y)z \cdot \mathcal{C}\big\} \end{aligned} \tag{3.12}$$

Hence, the probability of not covering the origin is

$$P(0 \notin U_\epsilon^*) = \exp\big\{-\lambda_{d+1} \otimes \mu(A)\big\} \tag{3.13}$$

The exponential in (3.13) can be further reduced as

$$\begin{aligned} \lambda_{d+1} \otimes \mu(A) &= \int_\epsilon^{1/\epsilon} \int_{-1}^0 \int_{\mathbb{R}^d} 1_{\{x \in -(1+\kappa y)z \cdot \mathcal{C}\}} dx dy d\mu(z) \\ &= \int_\epsilon^{1/\epsilon} \int_{-1}^0 (1 + \kappa y)^d z^d dy d\mu(z) \\ &= \frac{1 - (1 - \kappa)^{d+1}}{\kappa(d + 1)} \int_\epsilon^{1/\epsilon} z^d d\mu(z) \end{aligned}$$

Therefore, the probability of not covering the origin is

$$P(0 \notin U_\epsilon^*) = \exp\left\{-\frac{1-(1-\kappa)^{d+1}}{\kappa(d+1)}\int_\epsilon^{1/\epsilon} z^d d\mu(z)\right\}.$$

Next, we consider the probability of not covering two distinct points.We observe that due to the translation invariance property of a random covering, the probability is the same as that of not covering the origin and the other point in the upper half-space. That is,

$$P(0 \notin U_\epsilon^*, r \notin U_\epsilon^*),$$

where $r = (r_1, r_2) \in \mathbb{R}^d \times \mathbb{R}$ and $r_2 \geq 0$. Let A_x be the translation of A, defined in (3.12), by vector x for any $x \in \mathbb{R}^{d+1}$. We have

$$\begin{aligned}
P(0 \notin U_\epsilon^*, r \notin U_\epsilon^*) &= P\big(N(A \cup A_r) = 0\big) \\
&= \exp\big\{-\lambda_{d+1} \otimes \mu(A \cup A_r)\big\} \\
&= \exp\big\{-\lambda_{d+1} \otimes \mu(A) - \lambda_{d+1} \otimes \mu(A_r) + \lambda_{d+1} \otimes \mu(A \cap A_r)\big\} \\
&= \exp\big\{-2\lambda_{d+1} \otimes \mu(A)\big\} \exp\big\{\lambda_{d+1} \otimes \mu(A \cap A_r)\big\}, \qquad (3.14)
\end{aligned}$$

where $N(\cdot)$, a counting measure, returns the number of points of Ξ^* in a given region. We recognize that the first term in (3.14) is the square of the probability of not covering the origin, i.e., $P(0 \notin U_\epsilon^*)$. Hence, we only focus on the exponential in the second term of (3.14), i.e., $\lambda_{d+1} \otimes \mu(A \cap A_r)$. Consider the region $A \cap A_r$,

$$\begin{aligned}
(x, y, z) \in A &\Longrightarrow y \in (-1, 0), \text{ and } x \in -(1+\kappa y)z \cdot \mathcal{C} \\
(x, y, z) \in A_r &\Longrightarrow y \in (r_2 - 1, r_2) \text{ and } x \in r_1 - (1 + \kappa(y - r_2))z \cdot \mathcal{C}
\end{aligned}$$

Therefore, if $r_2 > 1$, then $A \cap A_r = \emptyset$. If $0 \leq r_2 \leq 1$, then $(x, y, z) \in A \cap A_r$ when $y \in (r_2 - 1, 0)$, and $x \in S(y, z)$ where

$$S(y, z) = -\Big((1+\kappa y)z \cdot \mathcal{C}\Big) \bigcap \Big(r_1 - \big(1 + \kappa(y - r_2)\big)z \cdot \mathcal{C}\Big).$$

The lower bound for the volume of $S(y, z)$ is obtained as

$$\begin{aligned}
\big|S(y, z)\big| &= \big(1 + \kappa(y - r_2)\big)^d \cdot z^d \left|\frac{1+\kappa y}{1 + \kappa(y - r_2)}\mathcal{C} \bigcap \left(\frac{-r_1}{(1 + \kappa(y - r_2)) \cdot z} + \mathcal{C}\right)\right| \\
&\geq \big(1 + \kappa(y - r_2)\big)^d \cdot z^d \left|\mathcal{C} \bigcap \left(\frac{-r_1}{(1 + \kappa(y - r_2)) \cdot z} + \mathcal{C}\right)\right|
\end{aligned}$$

since, for $0 \leq r_2 \leq 1$, $\frac{1+\kappa y}{1+\kappa(y-r_2)} > 1$. Applying the lemma 2.5 regarding the volume of the intersection between a convex set and its translation, we obtain

$$|S(y,z)| \quad \geq \quad \big(1+\kappa(y-r_2)\big)^d \cdot z^d \cdot \left(1 - \frac{a\|r_1\|}{\big(1+\kappa(y-r_2)\big)\cdot z}\right)^+,$$

for some a that only depends on $\mathcal{C}$. Therefore,

$$\begin{aligned}
\lambda \otimes \mu(A \cap A_r) &= \int_\epsilon^{1/\epsilon} \int_{r_2-1}^0 |S(y,z)| dy d\mu(z) \\
&\geq \int_\epsilon^{1/\epsilon} \int_{r_2-1}^0 \big(1+\kappa(y-r_2)\big)^d z^d \left(1 - \frac{a\|r_1\|}{\big(1+\kappa(y-r_2)\big)\cdot z}\right)^+ dy d\mu(z) \\
&\geq \int_\epsilon^{1/\epsilon} \int_{r_2-1}^0 \big(1+\kappa(y-r_2)\big)^d z^d \left(1 - \frac{a\|r_1\|}{(1-\kappa)z}\right)^+ dy d\mu(z) \\
&= \int_\epsilon^{1/\epsilon} z^d \left(1 - \frac{a\|r_1\|}{(1-\kappa)z}\right)^+ d\mu(z) \int_{r_2-1}^0 \big(1+\kappa(y-r_2)\big)^d dy \\
&= \frac{(1-\kappa r_2)^{d+1} - (1-\kappa)^{d+1}}{\kappa(d+1)} \int_\epsilon^{1/\epsilon} z^d \left(1 - \frac{a\|r_1\|}{(1-\kappa)z}\right)^+ d\mu(z).
\end{aligned}$$

Putting everything together, (3.14) becomes

$$\begin{aligned}
&\frac{P(0 \notin U_\epsilon^*, r \notin U_\epsilon^*)}{P(0 \notin U_\epsilon^*)^2} \geq \\
&\qquad \exp\left\{ \frac{(1-\kappa r_2)^{d+1} - (1-\kappa)^{d+1}}{\kappa(d+1)} \int_\epsilon^{1/\epsilon} z^d \left(1 - \frac{a\|r_1\|}{(1-\kappa)z}\right)^+ d\mu(z) \right\}. \qquad (3.15)
\end{aligned}$$

While it is possible to obtain the upper bound formula for the probability of not covering any two arbitrary points, the lower bound is all we need for the proof of a sufficient condition in the next section.

3.3.2 A upper bound for the probability of not covering by random frustums

In this section, we consider the probability of not covering a compact set with non-empty interior on the k-dimensional hyperplane $\Omega_k = \mathbb{R}^k \times \{0\}^{d-k}$, for $k = 1, \ldots, d$. We denote any point in $\Omega_k \times \mathbb{R}$ by (x, y), where $x \in \Omega_k$ and $y \in \mathbb{R}$. Let Γ be a cone in $\Omega_k \times \mathbb{R}$ with vertex at the origin, base is a $\mathbb{R}^k$-ball in the hyperplane $\{y = 1\} \subset \Omega_k \times \mathbb{R}$, that satisfies the Kahane's condition. That is, the cone $\Gamma + x$ does not intersect the frustum if x is on the boundary of the frustum, but not on the bottom base. We also denote that $\Gamma(u,v)$ is part of the cone Γ between two hyperplane $\{y=u\}$ and $\{y=v\}$, and $\Gamma(t)=\Gamma(0,t)$. Furthermore, we denote $m(u,v)$ be the Lebesgue measure of the uncovered part of $\Gamma(u,v)$, i.e.,

$$m(u,v)=\int_{\Gamma(u,v)}\chi(x)dx,$$

where $\chi(\cdot)$ is the indicator of an event that x is not covered by U^*_ϵ, and $m(t)=m(0,t)$.

We consider a regular grid in Ω_k; that is, $\Omega_k\times\mathbb{R}\cap\delta\mathbb{Z}^{k+1}$ for fixed δ. Let $\tau=(\tau_1,\tau_2)$ be the first vertex of the grid in $\Gamma(t)$, that is left uncovered by random covering U^*_ϵ. Here, t is the fixed value which will be chosen later in our proof. The order system is the lexicographic order; that is, we say $(x_1,y_1)\prec(x_2,y_2)$ when $x_1<x_2$ or $x_1=x_2$ and $y_1<y_2$. We have

$$\begin{aligned}
Em(t) &= \sum_{\theta\in\Gamma(t)\cap\delta\mathbb{Z}^{k+1}} P(\tau=\theta)E[m(\theta,t)|\tau=\theta]\\
&\geq \sum_{\theta\in\Gamma(t/2)\cap\delta\mathbb{Z}^{k+1}} P(\tau=\theta)E[m(\theta,t)|\tau=\theta]\\
&\geq \sum_{\theta\in\Gamma(t/2)\cap\delta\mathbb{Z}^{k+1}} P(\tau=\theta)E[m(\theta,\theta+t/2)|\tau=\theta],
\end{aligned}$$

where $\theta=(\theta_1,\theta_2)$ denotes an vertex of the grid. So, when $(\tau_1,\tau_2)=(\theta_1,\theta_2)$, the cone $\Gamma(\theta_2)$ is covered by the random covering U^*_ϵ, while the point (θ_1,θ_2) is not covered by U^*_ϵ. We observe that any frustum intersects the cone $\Gamma(\theta_2)$ cannot intersect the cone $\Gamma(t)+(\theta_1,\theta_2)$, due to the condition of Kahane's convex shape family. It means that given the point (θ_1,θ_2) is not covered by U^*_ϵ, we can determine two disjoint regions in $\mathbb{R}^d\times\mathbb{R}$. Any points of Ξ^* associated with frustums that intersect $\Gamma(\theta_2)$ belong to the first region; while points associated with frustum that intersect $\Gamma(t)+(\theta_1,\theta_2)$ lie inside the second region. By definition of a Poisson point process, the event of frustum intersecting $\Gamma(\theta_2)$ is independent of the event of frustum intersecting $\Gamma(t)+(\theta_1,\theta_2)$, conditioning on $\theta\notin U^*_\epsilon$. Therefore,

$$E(m(\theta,\theta+t/2)|\tau=\theta)=E(m(\theta,\theta+t/2)|\theta\notin U^*_\epsilon).$$

Furthermore,using the fact that the cone $\Gamma(t/2)+\theta \subset \Gamma(\theta,t)$ and the translation invariant property, we have

$$E(m(\theta,\theta+t/2)|\theta \notin U_\epsilon^*) \geq E(m(t/2)|0 \notin U_\epsilon^*)$$

Hence, we obtain

$$\begin{aligned} Em(t) &\geq \sum_{\theta\in\Gamma(t/2)\cap\delta\mathbb{Z}^{k+1}} P(\tau=\theta)E(m(t/2)|0\notin U_\epsilon^*) \\ &= E(m(t/2)|0\notin U_\epsilon^*) \sum_{\theta\in\Gamma(t/2)\cap\delta\mathbb{Z}^{k+1}} P(\tau=\theta) \qquad (3.16) \end{aligned}$$

The sum in (3.16) is simply $P(\tau \in \Gamma(t/2)\cap\delta\mathbb{Z}^{k+1})$ and the event $\{\tau \in \Gamma(t/2)\cap\delta\mathbb{Z}^{k+1}\}$ only happens when $\theta \notin U_\epsilon^*$ for some $\theta \in \Gamma(t/2)\cap\delta\mathbb{Z}^{k+1}$. Therefore,

$$P(\theta \notin U_\epsilon^* \text{ for some } \theta \in \Gamma(t/2)\cap\delta\mathbb{Z}^{k+1}) \leq \frac{Em(t)}{E(m(t/2)|0\notin U_\epsilon^*)} \qquad (3.17)$$

As $\delta \to 0$ through power of 1/2, the left side of (3.17) increases to

$$P(\theta \notin U_\epsilon^* \text{ for some } \theta,\text{ a binary rational in } \Omega_k\times\mathbb{R}, \text{ and } \theta\in\Gamma(t/2))$$

Because U_ϵ^* is a finite union of open sets, the probability that it covers all binary rational in $\Gamma(t/2)$ but not all points is zero and so

$$\begin{aligned} P(\Gamma(t/2)\not\subset U_\epsilon^*) &\leq \frac{Em(t)}{E(m(t/2)|0\notin U_\epsilon^*)} \\ &= c\cdot\frac{P(0\notin U_\epsilon^*)}{\int_{\Gamma(t/2)} P(r\notin U_\epsilon^*|0\notin U_\epsilon^*)dr} \\ &= c\cdot\left(\int_{\Gamma(t/2)} \frac{P(r\notin U_\epsilon^*,0\notin U_\epsilon^*)}{P(0\notin U_\epsilon^*)^2}dr\right)^{-1}, \end{aligned}$$

where c is the volume of the cone $\Gamma(t)$. From the inequality in (3.15), we obtain the upper bound for the probability of not covering a cone.

$$P(\Gamma(t/2)\not\subset U_\epsilon^*) \leq c\cdot\left(\int_{\Gamma(t/2)} \exp\left\{I(r_2)\int_\epsilon^{1/\epsilon} z^d\left(1-\frac{a\|r_1\|}{(1-\kappa)z}\right)^+ d\mu(z)\right\}dr\right)^{-1} \qquad (3.18)$$

where

$$I(r_2)=\frac{(1-\kappa r_2)^{d+1}-(1-\kappa)^{d+1}}{\kappa(d+1)}.$$

We also denote that

$$M = \int_{\Gamma(t/2)} \exp\left\{ I(r_2) \int_{\epsilon}^{1/\epsilon} z^d \left(1 - \frac{a\|r_1\|}{(1-\kappa)z}\right)^+ d\mu(z) \right\} dr.$$

We have

$$M = \int_0^{t/2} \int_{B(r_2)} \exp\left\{ I(r_2) \int_{\epsilon}^{1/\epsilon} z^d \left(1 - \frac{a\|r_1\|}{(1-\kappa)z}\right)^+ d\mu(z) \right\} dr_1 dr_2,$$

where $B(r_2) = \Gamma(t) \cap \{y = r_2\}$, a cross-section of the cone $\Gamma(t)$. By transforming variable r_1 into the polar coordinate system, (3.3.2) becomes

$$\begin{aligned} M &= \int_0^{t/2} \int_0^{b \cdot r_2} c_1 \cdot \exp\left\{ I(r_2) \int_{\epsilon}^{1/\epsilon} z^d \left(1 - \frac{a \cdot x}{(1-\kappa)z}\right)^+ d\mu(z) \right\} x^{k-1} dx dr_2 \\ &= \int_0^{b \cdot t/2} \int_{x/b}^{t/2} c_1 \cdot \exp\left\{ I(r_2) \int_{\epsilon}^{1/\epsilon} z^d \left(1 - \frac{a \cdot x}{(1-\kappa)z}\right)^+ d\mu(z) \right\} dr_2 x^{k-1} dx, \end{aligned} \quad (3.19)$$

where b is a constant, derived based on the relationship between the height r_2 and the radius of the cross-section of $\Gamma(t)$ at the height r_2, c_1 is another constant, obtained after integrating out all integral resulting from the coordinate transformation, except the radius. Since $r_2 \leq t/2$, and κ is small, we have $I(r_2) \geq I(t/2)$. Hence, from (3.19), we obtain

$$\begin{aligned} M &\geq c_1 \int_0^{bt/2} \int_{x/b}^{t/2} dr_2 \exp\left\{ I(t/2) \int_{\epsilon}^{1/\epsilon} z^d \left(1 - \frac{a \cdot x}{(1-\kappa)z}\right)^+ d\mu(z) \right\} x^{k-1} dx \\ &= c_1 \int_0^{bt/2} (t/2 - x/b) \exp\left\{ I(t/2) \int_{\epsilon}^{1/\epsilon} z^d \left(1 - \frac{a \cdot x}{(1-\kappa)z}\right)^+ d\mu(z) \right\} x^{k-1} dx \end{aligned} \quad (3.20)$$

If $t = 2\kappa$, we easily see that

$$I(t/2) = \frac{(1-\kappa^2)^{d+1} - (1-\kappa)^{d+1}}{\kappa(d+1)} \longrightarrow 1 \text{ as } \kappa \longrightarrow 0.$$

Hence, for any fixed $\rho > 0$, there exist κ_0 such that,

$$\frac{(1-\kappa_0^2)^{d+1} - (1-\kappa_0)^{d+1}}{\kappa_0(d+1)} \geq 1 - \rho$$

Therefore, by using $t = 2\kappa_0$, we obtain

$$
\begin{aligned}
M &\geq c_1 \int_0^{b\kappa_0} (\kappa_0 - x/b) x^{k-1} \exp\left\{ (1-\rho) \int_\epsilon^{1/\epsilon} z^d \left(1 - \frac{a \cdot x}{(1-\kappa)z}\right)^+ d\mu(z) \right\} dx \\
&\geq c_1 \cdot \int_0^{b\kappa_0/2} (\kappa_0 - x) x^{k-1} \exp\left\{ (1-\rho) \int_\epsilon^{1/\epsilon} z^d \left(1 - \frac{a \cdot x}{(1-\kappa)z}\right)^+ d\mu(z) \right\} dx
\end{aligned}
$$

Since $(\kappa_0 - x/b) \geq \kappa_0/2$ for all $x \leq b\kappa_0/2$, we have

$$
M \geq c_2 \cdot \int_0^{\kappa_0/2} \exp\left\{ (1-\rho) \int_\epsilon^{1/\epsilon} z^d \left(1 - \frac{a \cdot x}{(1-\kappa_0)z}\right)^+ d\mu(z) \right\} x^{k-1} dx, \tag{3.21}
$$

where c_2 is a constant. By a substitution in (3.21), and combining it with (3.18), we have

$$
P(\Gamma(\kappa_0) \not\subset U_\epsilon^*) \quad \leq \quad c_3 \cdot \left(\int_0^{c_4} \exp\left\{ (1-\rho) \int_\epsilon^{1/\epsilon} z^d \left(1 - \frac{x}{z}\right)^+ d\mu(z) \right\} x^{k-1} dx \right)^{-1},
$$

where c_3, c_4 are constants.

3.3.3 A sufficient condition for covering a subspace

Let Ξ^* be a Poisson point process in $\mathbb{R}^d \times \mathbb{R} \times (0, \infty)$ with intensity $\lambda_d \times \lambda \times \mu$, as defined in the random frustums. However, instead of having $\mathcal{C}(z)$ be a frustum, we define $\mathcal{C}^*(z)$ as a cylinder, i.e.,

$$
\mathcal{C}^*(z) = \{(u, v) \in \mathbb{R}^d \times \mathbb{R}, u \in z \cdot \mathcal{C}, v \in (0, 1)\}.
$$

A random covering with cylinders, denoted by U^{**}, is a union of translated $\mathcal{C}^*(z_k)$. That is,

$$
U^{**} = \bigcup_{(x_k, y_k, z_k) \in \Xi^*} (x_k, y_k) + \mathcal{C}^*(z_k)
$$

Lemma 3.1. *Random convex sets, constructed by intersecting random covering with cylinders U^{**} and a hyperplane $\{y = c\}$ for any arbitrary c, has the same distribution with the original random covering with $\mathcal{C}$ in $\mathbb{R}^d$.*

Proof. To prove this lemma, we show that the capacity functionals in both constructions are identical, which implies that they have the same distribution, by theorem 2.19. For a fixed compact set K in $\mathbb{R}^d$, the capacity functional for the original random covering with $\mathcal{C}$ is

$$T_U(K) = 1 - \exp\left\{-\int |z \cdot \mathcal{C} \oplus \check{K}| d\mu(z)\right\}.$$

The capacity functional for the random covering U^{**} is

$$\begin{aligned} T_{U^{**}}(K) &= P(U^{**} \cap K \neq \emptyset) = 1 - P(U^{**} \cap K = \emptyset) \\ &= 1 - P(x \notin U^{**}, \text{ for all } x \in K). \end{aligned}$$

Let A^* be a cone such that any point of Ξ^* in A^* corresponds to a random cylinder covering the origin.

$$A^* = \{(x, y, z) \in \mathbb{R}^d \times \mathbb{R} \times (0, \infty) : -\frac{x}{z} \in \mathcal{C}, y \in (-1, 0)\}.$$

So, we obtain

$$\begin{aligned} T_{U^{**}}(K) &= 1 - \exp\left\{-\int_{c-1}^{c} \int |z \cdot \mathcal{C} \oplus \check{K}| d\mu(z) dy\right\} \\ &= 1 - \exp\left\{-\int |z \cdot \mathcal{C} \oplus \check{K}| d\mu(z)\right\}, \end{aligned}$$

which proves the lemma. □

Furthermore, using the coupling argument, we conclude that the probability of not covering a given set by random frustums is larger than that by random cylinders. That is, each points of a Poisson point process Ξ^* is the center for both a frustum and a cylinder at the same time. Since the frustum is always smaller that the cylinder, any point that is covered by random frustums is covered by random cylinders. Therefore, it follows that

$$P(\Gamma(\kappa_0) \not\subset U_\epsilon^{**}) \quad \leq \quad P(\Gamma(\kappa_0) \not\subset U_\epsilon^{*})$$

Since the distribution of random covering U and U^{**} in $\mathbb{R}^d$ is identical, by lemma 3.1, we have

$$\begin{aligned} P(\Gamma(\kappa_0) \not\subset U_\epsilon) &= P(\Gamma(\kappa_0) \not\subset U_\epsilon^{**}) \\ &\leq c_1 \cdot \left(\int_0^{c_2} \exp\left\{(1-\rho)\int_\epsilon^{1/\epsilon} z^d \left(1 - \frac{x}{z}\right)^+ d\mu(z)\right\} x^{d-1} dx\right)^{-1} \end{aligned}$$

for some constant c_1, c_2. Based on lemma 2.22, if $P(\Gamma(\kappa_0) \not\subset U^{**}) = 0$, then the hyperplane $\mathbb{R}^k \times \{0\}^{d-k} \times \mathbb{R}$ is covered almost surely. This is equivalent to

$$\int_0^1 \exp\left\{(1-\rho)\int_0^\infty z^d \left(1-\frac{x}{z}\right)^+ d\mu(z)\right\} x^{k-1}dx = \infty, \tag{3.22}$$

when ϵ approaches to 0.

Furthermore, for any k-dimensional subspace X of $\mathbb{R}^d$, there is a rotation g such that $gX = \mathbb{R}^k \times \{0\}^{d-k}$. Hence, by theorem 2.21, covering the subspace X is equivalent to covering the k-dimensional hyperplane with the convex set $g\mathcal{C}$. Since the sufficient condition is independent of the shape of the convex set, we conclude that (3.22) is a sufficient condition for covering any hyperplane almost surely.

3.4 The necessary and sufficient condition of covering a line

Since the random covering is invariant under translation (theorem 2.20), covering any arbitrary line is equivalent to covering a line that goes through the origin. Now, we first consider the first coordinate axis in $\mathbb{R}^d$, and determine the condition for covering it. The necessary condition for covering a line is a special case of covering a subspace in $\mathbb{R}^d$. From previous section, we obtain the necessary condition by setting $k = 1$, i.e.,

$$\int_0^1 \exp\left\{\int_x^1 z^d \left(1-\frac{x}{z}\right) d\mu(z)\right\} dx = \infty.$$

So, we'll show that this condition is also the sufficient condition, using the virtually identical argument to §3 in [26]. In this section, a point t is referred to point $(t, 0, \ldots, 0) \in \mathbb{R}^d$.

Lemma 3.2. *If $t_1 < \cdots < t_k < t < x$ then*

$$P(x \notin U_\epsilon | t \notin U_\epsilon, t_i \in U_\epsilon \textit{ for all } i) = P(x \notin U_\epsilon | t \notin U_\epsilon).$$

Proof. It is easy to see that t divides any random convex set ξ that intersects the first coordinate axis into 2 disjoint groups: either $\xi \cap (-\infty, t) \times \{0\}^{d-1} \neq \emptyset$ or $\xi \cap (t, +\infty) \times \{0\}^{d-1} \neq \emptyset$. Hence, given $t \notin U_\epsilon$, the event $\{x \notin U_\epsilon\}$ and $\{t_i \in U_\epsilon \text{ for all } i\}$ are independent. Therefore, this proves the lemma. $\square$

For fixed $k > 0$, let τ is the first uncovered point of the sequence of points on the axis $\{j/k\}, j = 1, 2, \ldots$ (with the convention $\tau = \infty$ if all points in the sequence are covered). Define $m(a, b)$ to be the measure of the uncovered part of $(a, b) \times \{0\}^{d-1}$. We have

$$
\begin{aligned}
Em(0,2) &= \sum_j E\big(m(0,2)|\tau = j/k\big)P(\tau = j/k) \\
&\geq \sum_{j=1}^{k} E\Big(m\big(j/k, (j/k)+1\big)|\tau = j/k\Big)P(\tau = j/k)
\end{aligned}
$$

Using the lemma 3.2, we have

$$
\begin{aligned}
E\Big(m\big(j/k, (j/k)+1\big)\Big) &= \int_{j/k}^{(j/k)+1} P(t \notin U_\epsilon|\tau = j/k)dt \\
&= \int_{j/k}^{(j/k)+1} P(t \notin U_\epsilon|(j/k) \notin U_\epsilon, (i/k) \in U_\epsilon \text{ for all } i)dt \\
&= \int_{j/k}^{(j/k)+1} P(t \notin U_\epsilon|(j/k) \notin U_\epsilon)dt \\
&= \int_0^1 P(t \notin U_\epsilon|0 \notin U_\epsilon)dt,
\end{aligned}
$$

for $j = 1, \ldots, k$, since random covering is invariant under translation. We obtain

$$
Em(0,2) \geq \int_0^1 P(t \notin U_\epsilon|0 \notin U_\epsilon)dt \sum_{j=1}^{k} P(\tau = j/k)
$$

Therefore,

$$
\begin{aligned}
P(j/k \notin U_\epsilon \text{ for some } j \leq k) &\geq \frac{Em(0,2)}{\int_0^1 P(t \notin U_\epsilon|0 \notin U_\epsilon)dt} \\
&= \frac{2P(0 \notin U_\epsilon)}{\int_0^1 P(t \notin U_\epsilon|0 \notin U_\epsilon)dt}.
\end{aligned}
$$

As $k \to \infty$, the left side of (3.23) increases to

$$
P(t \notin U_\epsilon \text{ for some } t \text{ a rational number between 0 and 1})
$$

Since U_ϵ is a finite union of open convex sets, the probability that U_ϵ covers all rational number in $[0, 1] \times \{0\}^{d-1}$ but not $[0, 1] \times \{0\}^{d-1}$ is zero and so

$$P([0,1] \times \{0\}^{d-1} \not\subset U_\epsilon) \leq \frac{2P(0 \notin U_\epsilon)}{\int_0^1 P(t \notin U_\epsilon | 0 \notin U_\epsilon) dt}. \tag{3.23}$$

The upper bound in (3.23) can be computed explicitly. First, we consider

$$\begin{aligned} P(t \notin U_\epsilon, 0 \notin U_\epsilon) &= P(N(A \cup A_t) = 0) \\ &= \exp\{-\lambda \otimes \mu(A \cup A_t)\} \\ &= \exp\{-2\lambda \otimes \mu(A)\} \exp\{\lambda \otimes \mu(A \cap A_t)\} \\ &= P(0 \notin U_\epsilon)^2 \exp\left\{\int_\epsilon^1 z^d \left(1 - \frac{at}{z}\right)^+ d\mu(z)\right\} \end{aligned}$$

where A is given by (1.3). Hence, we can rewrite the upper bound in (3.23) as

$$P([0,1] \times \{0\}^{d-1} \not\subset U_\epsilon) \leq \frac{2}{\int_0^1 \exp\left\{\int_\epsilon^1 z^d \left(1 - \frac{at}{z}\right)^+ d\mu(z)\right\} dt}.$$

As $\epsilon \to 0$, $P([0,1] \times \{0\}^{d-1} \not\subset U) \leq 0$ if

$$\int_0^1 \exp\left\{\int_t^1 z^d \left(1 - \frac{t}{z}\right) d\mu(z)\right\} dt, \tag{3.24}$$

which proves the sufficient condition for covering the first coordinate axis.

Furthermore, for any line L goes through the origin, there exists a rotation g such that $gL = \mathbb{R} \times \{0\}^{d-1}$. By lemma 2.21, the covering of the line L with convex set $\mathcal{C}$ is equivalent to covering $\mathbb{R} \times \{0\}^{d-1}$ with the convex set $g\mathcal{C}$. Due to the fact that the sufficient condition does not depend on the shape of the convex set $\mathcal{C}$, the condition (3.24) is the sufficient condition for covering any arbitrary line in $\mathbb{R}^d$.

Chapter 4

Covering the cross-section in $\mathbb{T}^d$

4.1 Covering the cross-section of $\mathbb{T}^d$

In this chapter, we prove the theorem A, regarding a necessary and a sufficient condition for the Dvoretzky problem. We follows the same approach as Kahane's in [19], which he made a connection between these conditions and Mandelbrot problem's conditions. We begin the proof of theorem A by showing the following lemma

Lemma 4.1. *The torus $\mathbb{T}^d$ is covering almost surely if*

$$\sum_{j=1}^{\infty} v_j^{1+\varepsilon} = \infty, \tag{4.1}$$

for some $\varepsilon > 0$.

Proof. The lemma can be obtained easily from proposition 1.5, which states that if g_n's are convex set, then a sufficient condition for covering the torus $\mathbb{T}^d$ is

$$\limsup_{n\to\infty} \left(\sum_{j=1}^{\infty} v_j - d \log n \right) = \infty \tag{4.2}$$

We prove that (4.1) implies (4.2) by contrapositive. If (4.2) fails, there is a $M < \infty$ for which

$$\sum_{j=1}^{n} v_j \leq d \log(n) + M, \qquad n \geq 1.$$

Since $\{v_n\}$ is a non-increasing sequence,

$$v_n \leq 1/n \sum_{j=1}^{n} v_j \leq \frac{d \log(n) + M}{n}, \qquad n \geq 1.$$

Therefore,

$$\sum_{j=1}^{\infty} v_j^{1+\varepsilon} \leq \sum_{j=1}^{\infty} \left(\frac{d \log(n) + M}{n} \right)^{1+\varepsilon} \tag{4.3}$$

We observe that the infinite series in (4.3) converges or diverges together with

$$\sum_{j=1}^{\infty}\left(\frac{\log(n)}{n}\right)^{1+\varepsilon}. \tag{4.4}$$

However, the series in (4.4) converges by the integral test, which proves the lemma. □

Proof of part (i) in Theorem A

A sufficient condition for covering a k-dimensional cross-section of torus, i.e., $\mathbb{T}^k \times \{0\}^{d-k}$, is

$$\int_0^1 \exp\left\{(1-\rho)\sum_{n=1}^{\infty} v_n\left(1-\frac{s^{1/k}}{v_n^{1/d}}\right)^+\right\} ds = \infty, \tag{4.5}$$

where $\rho > 0$ when $k = 2, \ldots, d$, and $\rho = 0$, when $k = 1$. It is easy to recognize that (4.5) is also a sufficient condition for the Mandelbrot problem when

$$d\mu(z) = \sum_{n=1}^{\infty} \delta_{z_n} \qquad (z_n^d = v_n),$$

where δ is a Dirac measure. Suppose that (4.5) holds and we show that $\mathbb{T}^k \times \{0\}^{d-k}$ is covered almost surely. Suppose that

$$\begin{cases} v_1' &= v_1 &= u_1, & & \\ v_2' &= v_3' &= v_3 &= u_2, & \\ v_4' &= v_5' &= v_6' &= v_6 &= u_3, \\ \cdots & & & & \end{cases}$$

The sequence $\{v_n'\}$ repeats the value u_m m times and are less than or equal to $\{v_n\}$. From the lemma 4.1, if $\sum v_n^{1+\varepsilon} = \infty$, then the torus is covered almost surely; hence, so is any cross-section of $\mathbb{T}^d$. Hence, we assume that $\sum v_n^{1+\varepsilon} < \infty$ for $\varepsilon < 1/5$. It follows that $\sum m u_m^{1+\varepsilon} < \infty$, since $v_n \geq v_n'$, and the sequence $\{v_n'\}$ repeats u_m m times. By lemma 4.3, a simple application of Hölder inequality, it follows that $\sum u_m < \infty$. By a summation by part, we have $\sum m(u_{m-1}-u_m) < \infty$, which implies that $\sum(v_n - v_n') < \infty$. We claim that

Lemma 4.2. *If two non-increasing sequences $\{v_n\}$ and $\{v_n'\}$, $v_n \geq v_n'$ for all n, and $\sum(v_n - v_n') < \infty$, then*

$$\sum v_n\left(1-\frac{s^{1/k}}{v_n^{1/d}}\right)^+ - \sum v_n'\left(1-\frac{s^{1/k}}{{v_n'}^{1/d}}\right)^+ = O(1), \tag{4.6}$$

for $0 < s < 1$ and $k = 1, \ldots, d$.

Proof. See the appendix at the end of this chapter. □

The lemma 4.2 implies that the condition (4.5) still holds if we replace $\{v_n\}$ by $\{v'_n\}$. Let $\{v''_n\}$ be the subsequence of $\{v'_n\}$, obtained by removing $\lfloor m^{2/3} \rfloor$ terms which are equal to u_m, for each m. Since the sequence $\{v''_n\}$ is non-increasing, we have $v''_n \leq v'_n$ and

$$\sum (v'_n - v''_n) \leq \sum m^{2/3} u_m \leq (\sum m^{-a})^{\alpha} (\sum m u_m^{1+\varepsilon})^{\beta}$$

when $\beta - a\alpha = 2/3$, $(1+\varepsilon)\beta = 1$, and $\alpha + \beta = 1$. As $\varepsilon < 1/5$, and $\beta > 5/6$, $\alpha < 1/6$, $a > 1$, the series $\sum (v'_n - v''_n)$ converges, which implies that the condition (4.5) also holds when replaced $\{v_n\}$ by $\{v''_n\}$, thanks to lemma 4.2. We define a measure $\mu''(dz)$ as

$$\mu''(dz) = \sum \delta_{z''_n} \qquad \left((z''_n)^d = v''_n \right)$$

and consider a Poisson point process $\Xi = \{(x_i, z_i)\}$ on $\mathbb{R}^d \times (0, \infty)$ with intensity $\lambda_d \otimes \mu''$. Due to the discreteness of the measure μ'', we can consider Ξ as a final result after superimposing multiple independent Poisson point processes, indexed by the value of z. For each m, the collection of x_i under $z_i^d = u_m$ is a Poisson point process with intensity $m - \lfloor m^{2/3} \rfloor$ in $\mathbb{R}^d$. Let N_m be a number of x_i's such that $x_i \in [0,1]^d$ for each m. It follows that N_m's are independent Poisson distributed random variables with mean $m - [m^{2/3}]$. We observe that, for m large enough, $N_m \leq m$ almost surely. Now, using the Poisson point process Ξ, we construct the Mandelbrot problem's random covering U'' in $\mathbb{R}^d$. Since a sufficient condition for the Mandelbrot problem is identical to (4.5), random covering U'' covers the hyperplane $\mathbb{R}^k \times \{0\}^{d-k}$ almost surely. It follows that the cube $[1/3, 2/3]^k \times \{0\}^{d-k}$ is covered almost surely. Furthermore, random covering U'' only has N_m convex sets of size u_m, while random covering U has m convex sets of size v_n that larger than u_m. Hence, by the coupling argument, the Dvoretzky problem random covering U covers $[1/3, 2/3]^k \times \{0\}^{d-k}$ almost surely. This implies that $\mathbb{T}^k \times \{0\}^{d-k}$ is covered almost surely.

Proof of part (ii) in Theorem A

A necessary condition for covering a k-dimensional cross-section of torus, i.e., $\mathbb{T}^k \times$

$\{0\}^{d-k}$ is

$$\int_0^1 \exp\left\{\sum_{n=1}^{\infty} v_n \left(1 - \frac{s^{1/k}}{v_n^{1/d}}\right)^+\right\} ds = \infty. \tag{4.7}$$

It is easy to show that the condition (4.7) is an application of proposition 1.1, mentioned in chapter 1. Let B be a ball with radius 1/4 in the k-dimensional cross-section $\mathbb{T}^k \times \{0\}^{d-k}$, v_B be the volume of B and $\xi_k(x)$ be the volume of the intersection of g_k and its translation. Denote that

$$\int f(x) d\tau(x) = \int\int f(x-y) dx dy.$$

The proposition 1.1 states that B is not covered almost surely if $\sum v_k^2 < \infty$ and

$$\int_B \prod_{k=1}^{\infty} \big(1 + \xi_k(x)\big) d\tau(x) < \infty. \tag{4.8}$$

We denote the left hand side of (4.8) by M. That is,

$$M = \int_B \prod_{k=1}^{\infty} \big(1 + \xi_k(x)\big) d\tau(x).$$

By lemma 2.5, we have

$$\xi_k(x) \leq v_k \left(1 - \frac{b\|x\|}{v_k^{1/d}}\right)^+, \tag{4.9}$$

for some constant $b > 0$. With (4.8) and (4.9), we have

$$\begin{aligned} M &\leq \int_B \int_B \prod_{k=1}^{\infty} \left(1 + v_k \left(1 - \frac{b\|x-y\|}{v_k^{1/d}}\right)^+\right) dx dy \\ &\leq \int_B \int_B \exp\left(\sum_{k=1}^{\infty} v_k \left(1 - \frac{b\|x-y\|}{v_k^{1/d}}\right)^+\right) dx dy, \end{aligned} \tag{4.10}$$

By changing variable $u = y$ and $v = x - y$, we have

$$\begin{aligned} M &\leq \int_{2B} \int_{S(v)} \exp\left(\sum_{k=1}^{\infty} v_k \left(1 - \frac{b\|v\|}{v_k^{1/d}}\right)^+\right) du dv \\ &= \int_{2B} |S(v)| \exp\left(\sum_{k=1}^{\infty} v_k \left(1 - \frac{b\|v\|}{v_k^{1/d}}\right)^+\right) dv, \end{aligned} \tag{4.11}$$

where $S(v) = \{u \in \mathbb{R}^k : \|u\| \leq 1/4\} \cap \{u \in \mathbb{R}^k : \|u+v\| \leq 1/4\}$, the intersection of two balls center at the origin and at v, respectively. Since $|S(v)| \leq v_B$, (4.11) becomes

$$M \leq v_B \int_{2B} \exp\left(\sum_{k=1}^{\infty} v_k \left(1 - \frac{b\|v\|}{v_k^{1/d}}\right)^+\right) dv \tag{4.12}$$

From (4.12), by changing to the polar coordinates and integrating out all variables except the radius, we obtain

$$M \leq c \int_0^{1/2} \exp\left(\sum_{k=1}^{\infty} v_k \left(1 - \frac{br}{v_k^{1/d}} \right)^+ \right) r^{k-1} dr, \tag{4.13}$$

where c is a constant. With a simple substitution $s = (br)^k$, (4.13) becomes

$$M \leq c \int \exp\left(\sum_{k=1}^{\infty} v_k \left(1 - \frac{s^{1/k}}{v_k^{1/d}} \right)^+ \right) ds,$$

which proves the necessary condition.

4.2 Covering 1-dimensional cross-section of $\mathbb{T}^d$

In the previous section, we show that a necessary and sufficient condition for covering a one-dimensional cross-section of $\mathbb{T}^d$ is

$$\int_0^1 \exp\left\{ \sum_{n=1}^{\infty} v_n \left(1 - \frac{s}{v_n^{1/d}} \right)^+ \right\} ds = \infty. \tag{4.14}$$

Now, we show that (4.14) is equivalent to

$$\sum_{n=1}^{\infty} \frac{v_n^{(d-1)/d}}{a_n^2} \exp\left(v_1 + \cdots + v_n \right) = \infty \tag{4.15}$$

where

$$a_n = \sum_{k=1}^{n} v_k^{(d-1)/d}.$$

Let M be the left hand side of (4.14); that is,

$$M = \int_0^1 \exp\left\{ \sum_{n=1}^{\infty} v_n \left(1 - \frac{s}{v_n^{1/d}} \right)^+ \right\} ds.$$

We have that

$$
\begin{aligned}
M &= \sum_{k=1}^{\infty} \int_{v_{k+1}^{1/d}}^{v_k^{1/d}} \exp\left\{ \sum_{n=1}^{k} v_n \left(1 - \frac{s}{v_n^{1/d}} \right) \right\} ds \\
&= \sum_{k=1}^{\infty} \int_{v_{k+1}^{1/d}}^{v_k^{1/d}} \exp\left(\sum_{n=1}^{k} v_n \right) \exp(-a_k s) ds \\
&= \sum_{k=1}^{\infty} \exp\left(\sum_{n=1}^{k} v_n \right) \frac{1}{a_k} \left(\exp(-a_k v_{k+1}^{1/d}) - \exp(-a_k v_k^{1/d}) \right) \\
&= \sum_{k=1}^{\infty} \exp\left(\sum_{n=1}^{k+1} v_n \right) \frac{1}{a_k} \exp(-a_{k+1} v_{k+1}^{1/d}) - \sum_{k=1}^{\infty} \exp\left(\sum_{n=1}^{k} v_n \right) \frac{1}{a_k} \exp(-a_k v_k^{1/d}) \\
&= -\frac{1}{v_1^{(d-1)/d}} + \sum_{k=2}^{\infty} \frac{v_k^{(d-1)/d}}{a_{k-1} a_k} \exp(v_1 + \cdots + v_k - a_k v_k^{1/d}).
\end{aligned} \tag{4.16}
$$

Since $a_n / a_{n+1} \to 1$ as n approaches infinity, (4.16) converges or diverges together with

$$
\sum_{k=1}^{\infty} \frac{v_k^{(d-1)/d}}{a_k^2} \exp(v_1 + \cdots + v_k - a_k v_k^{1/d}). \tag{4.17}
$$

Therefore, (4.17) is also a necessary and sufficient condition. Next, we show that (4.17) is equivalent to (4.15). It is easy to see that (4.17) is less or equal to (4.15), which implies that if (4.17) diverges, then so does (4.15). Hence, we only need to show that if (4.17) converges, then (4.15) also converges.

Define for all $n \geq 1$,

$$
\xi_n = v_1 + \cdots + v_n - v_n^{1/d} \cdot \sum_{k=1}^{n} v_n^{(d-1)/d} = z_1^d + \cdots + z_n^d - z_n \cdot \sum_{k=1}^{n} z_k^{d-1}.
$$

We observe that the sequence $\{\xi_n\}$ is an increasing sequence, because

$$
\xi_{n+1} - \xi_n = a_n (z_n - z_{n+1}).
$$

Dividing by a_n, summing over all $n \geq k$, and using the fact that $\{z_n\}$ is decreasing and approach 0 as n goes to infinity, we obtain that for all $k \geq 1$,

$$
z_k = \sum_{n=k}^{\infty} \frac{1}{a_n} (\xi_{n+1} - \xi_n).
$$

By multiplying a_k on both side and applying summation by part, we have

$$\begin{aligned}
a_k z_k &= a_k \sum_{n=k}^{\infty} \frac{1}{a_n}(\xi_{n+1} - \xi_n) \\
&= a_k \sum_{n=k+1}^{\infty} \left(\frac{1}{a_{n-1}} - \frac{1}{a_n} \right) \xi_n - \xi_k \\
&= a_k \sum_{n=k+1}^{\infty} \left(\frac{1}{a_{n-1}} - \frac{1}{a_n} \right) \xi_n - z_1^d - \cdots - z_k^d + z_k a_k
\end{aligned}$$

Therefore,

$$v_1 + \cdots + v_k = z_1^d + \cdots z_k^d = a_k \sum_{n=k+1}^{\infty} \left(\frac{1}{a_{n-1}} - \frac{1}{a_n} \right) \xi_n.$$

We denote that

$$\phi_n = \frac{z_n^{d-1}}{a_n^2} \exp(\xi_n).$$

Since $a_k \sum_{n>k}(1/a_{n-1} - 1/a_n) = 1$, and the exponential function is convex, we obtain

$$\exp\left\{ a_k \sum_{n=k+1}^{\infty} \log(\phi_n) \left(\frac{1}{a_{n-1}} - \frac{1}{a_n} \right) \right\} \le a_k \sum_{n=k+1}^{\infty} \phi_n \left(\frac{1}{a_{n-1}} - \frac{1}{a_n} \right).$$

Therefore, since $\xi_n = \log(\phi_n a_n^2 / z_n^{d-1})$,

$$\begin{aligned}
\exp(v_1 + \cdots + v_k) &= \exp\left\{ a_k \sum_{n=k+1}^{\infty} \left(\frac{1}{a_{n-1}} - \frac{1}{a_n} \right) \log(\phi_n a_n^2 / z_n^{d-1}) \right\} \\
&\le \left[a_k \sum_{n=k+1}^{\infty} \phi_n \left(\frac{1}{a_{n-1}} - \frac{1}{a_n} \right) \right] \exp\left\{ 2a_k \sum_{n=k+1}^{\infty} \log(a_n) \left(\frac{1}{a_{n-1}} - \frac{1}{a_n} \right) \right\} \cdot \\
&\quad \cdot \exp\left\{ -(d-1)a_k \sum_{n=k+1}^{\infty} \log(z_n) \left(\frac{1}{a_{n-1}} - \frac{1}{a_n} \right) \right\} \\
&= \left[a_k \sum_{n=k+1}^{\infty} \phi_n \left(\frac{1}{a_{n-1}} - \frac{1}{a_n} \right) \right] \exp(M_1) \exp(M_2),
\end{aligned}$$

where

$$\begin{aligned}
M_1 &= 2a_k \sum_{n=k+1}^{\infty} \log(a_n) \left(\frac{1}{a_{n-1}} - \frac{1}{a_n} \right) \\
M_2 &= -(d-1)a_k \sum_{n=k+1}^{\infty} \log(z_n) \left(\frac{1}{a_{n-1}} - \frac{1}{a_n} \right)
\end{aligned}$$

Next, we deal with each term separately. First, we consider the term M_1, and apply the summation by part.

$$
\begin{aligned}
M_1 &= 2\log(a_{k+1}) + 2a_k \sum_{n>k} \frac{1}{a_n} \log\left(\frac{a_{n+1}}{a_n}\right) \\
&= 2\log(a_{k+1}) + 2a_k \sum_{n>k} \frac{1}{a_n} \log\left(1 + \frac{z_{n+1}^{d+1}}{a_n}\right)
\end{aligned}
$$

Using the fact that $\log(1+x) \leq x$ for $x < 1$, we have

$$
\begin{aligned}
M_1 &\leq 2\log(a_{k+1}) + 2a_k \sum_{n>k} \frac{1}{a_n} \cdot \frac{z_{n+1}^{d-1}}{a_n} \\
&\leq 2\log(a_{k+1}) + 2a_k \sum_{n>k} \frac{z_n^{d-1}}{a_{n-1}a_n} \\
&= 2\log(a_{k+1}) + 2a_k \sum_{n>k} \left(\frac{1}{a_{n-1}} - \frac{1}{a_n}\right) \\
&= 2\log(a_{k+1}) + 2.
\end{aligned}
$$

Therefore, $\exp(M_1) \leq \exp(2)a_{k+1}^2$. Now, we consider the term M_2, and apply the summation by part.

$$
\begin{aligned}
M_2 &= -(d-1)\log(z_{k+1}) - (d-1)a_k \sum_{n>k} \frac{1}{a_n}(\log(z_{n+1}) - \log(z_n))) \\
&= -(d-1)\log(z_{k+1}) + (d-1)a_k \sum_{n>k} \frac{1}{a_n} \log\left(\frac{z_n}{z_{n+1}}\right)
\end{aligned}
$$

Using the fact that a_n is a increasing sequence, we have

$$
\begin{aligned}
M_2 &\leq -(d-1)\log(z_{k+1}) + (d-1)\sum_{n>k} \log\left(\frac{z_n}{z_{n+1}}\right) \\
&= -(d-1)\log(z_{k+1}) + (d-1)\log\left(\prod_{n>k} \frac{z_n}{z_{n+1}}\right) \\
&= -(d-1)\log(z_{k+1}) + (d-1)\log(z_k) \\
&\leq -(d-1)\log(z_{k+1})
\end{aligned}
$$

Therefore, $\exp(M_2) \leq 1/z_{k+1}^{d-1}$. Hence,

$$
\exp(v_1 + \cdots + v_k) \leq e^2 \frac{a_{k+1}^2}{z_{k+1}^{d-1}} a_k \sum_{n>k} \phi_n \left(\frac{1}{a_{n-1}} - \frac{1}{a_n}\right)
$$

Thus, we have

$$\frac{z_{k+1}^{d-1}}{a_{k+1}^{2}} \exp(v_1 + \cdots + v_k) \leq (e^2) a_k \sum_{n>k} \phi_n \left(\frac{1}{a_{n-1}} - \frac{1}{a_n} \right).$$

We obtain

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{z_{k+1}^{d-1}}{a_{k+1}^{2}} \exp(v_1 + \cdots + v_k) &\leq (e^2) \sum_{k=1}^{\infty} a_k \sum_{n>k} \phi_n \left(\frac{1}{a_{n-1}} - \frac{1}{a_n} \right) \\
&= (e^2) \sum_{n=2}^{\infty} \left(\sum_{k<n} a_k \right) \phi_n \left(\frac{1}{a_{n-1}} - \frac{1}{a_n} \right) \\
&\leq (e^2) \sum_{n=2}^{\infty} n a_n \frac{z_n^{d-1}}{a_{n-1} a_n} \phi_n \\
&\leq (e^2) \sum_{n=2}^{\infty} \frac{a_n}{a_{n-1}} \phi_n.
\end{aligned}$$

Since $\frac{a_n}{a_{n-1}} \to 1$ as n approaches to infinity, the series $\sum \frac{a_n}{a_{n-1}} \phi_n$ and $\sum \phi_n$ converge or diverge together. Therefore, if $\sum \phi_n < \infty$ then so is

$$\sum_{k=1}^{\infty} \frac{z_k^{d-1}}{a_k^2} \exp(v_1 + \cdots + v_k),$$

due to the fact that $v_k < 1$ for all $k \leq 1$.

4.3 Appendix

Lemma 4.3. *For a positive sequence u_n, and $0 < \varepsilon < 1$, if $\sum u_m = \infty$ then $\sum m u_m^{1+\varepsilon} = \infty$.*

Proof. Suppose that $\sum m u_m^{1+\varepsilon} < \infty$. By Holder inequality with $p = 1 + \varepsilon$ and $q = (1 + \varepsilon)/\varepsilon$, we obtain

$$\sum u_m \leq \left[\sum \left(m^{1/(1+\varepsilon)} u_m \right)^{1+\varepsilon} \right]^{1/(1+\varepsilon)} \left[\sum \frac{1}{m^{1/\varepsilon}} \right]^{\varepsilon/(1+\varepsilon)}$$

Since series in the second square brake converges for all $0 < \varepsilon < 1$, $\sum u_m$ converges if $\sum m u_m^{1+\varepsilon}$ converges. $\square$

Proof of lemma 4.2. Fixed $0 < s < 1$, and $k = 1, \cdots, d$. Let n_0 and n_0' be the largest integers such that $v_n^{1/d} \leq s^{1/k}$, for all $n > n_0$ and ${v'_n}^{1/d} \leq s^{1/k}$, for all $n > n_0'$, respectively. Since $v_n \geq v'_n$, $n_0 > n'_0$. We notice that

$$\sum_{n=n_0}^{\infty} v_n \left(1 - \frac{s^{1/k}}{v_n^{1/d}}\right)^+ - \sum_{n=n_0}^{\infty} v'_n \left(1 - \frac{s^{1/k}}{{v'_n}^{1/d}}\right)^+ = 0.$$

So, the difference only depends on the first n_0 terms. Let $u_n = v'_n$, for $n = 1, \cdots, n'_0 - 1$, and $u_n = s$, for $n = n'_0, \cdots, n_0$. We have

$$\begin{aligned}
&\sum v_n \left(1 - \frac{s^{1/k}}{v_n^{1/d}}\right)^+ - \sum v'_n \left(1 - \frac{s^{1/k}}{{v'_n}^{1/d}}\right)^+ \\
&\quad = \sum_{n=1}^{n_0-1} v_n \left(1 - \frac{s^{1/k}}{v_n^{1/d}}\right) - \sum u_n \left(1 - \frac{s^{1/k}}{u_n^{1/d}}\right) \\
&\quad = \sum_{n=1}^{n_0-1} (v_n - u_n) + s^{1/k} \sum_{n=1}^{n_0-1} (u_n^{(d-1)/d} - v_n^{(d-1)/d}) \\
&\quad \leq \sum (v_n - v'_n) + s^{1/k} \sum_{n=1}^{n_0-1} (u_n^{(d-1)/d} - v_n^{(d-1)/d}) \qquad (4.18)
\end{aligned}$$

Let $f(x) = x^{(d-1)/d}$. By the mean value theorem, we have

$$|f(u_n) - f(v_n)| = f'(c)|u_n - v_n|,$$

where c is a value between u_n and v_n. It follows that

$$\begin{aligned}
s^{1/k}(u_n^{(d-1)/d} - v_n^{(d-1)/d}) &= \frac{s^{1/k}}{c^{1/d}}(v_n - u_n) \\
&\leq s^{1/k-1/d}(v_n - u_n) \leq (v_n - u_n) \qquad (4.19)
\end{aligned}$$

Therefore, by (4.18), (4.19), and $\sum (v_n - v'_n) < \infty$, we obtain

$$\sum_{n=n_0}^{\infty} v_n \left(1 - \frac{s^{1/k}}{v_n^{1/d}}\right)^+ - \sum_{n=n_0}^{\infty} v'_n \left(1 - \frac{s^{1/k}}{{v'_n}^{1/d}}\right)^+ = O(1).$$

□

Chapter 5

Covering the Cantor set

5.1 Introduction

In the previous chapters, we have showed a necessary condition and a sufficient condition for covering a k-dimensional hyperplane in the Mandelbrot problem, a torus $\mathbb{T}^d$ or a k-dimensional cross-section in the Dvoretzky problem. It is natural to raise a question regarding the condition for covering a more general set. In the Dvoretzky problem, Y. El Helou [7] and J.P. Kahane [17] considered this problem. Let $v_1 > v_2 > \cdots$, approaching 0, be volumes of a sequence of convex sets and a set A with Minkowski dimension s, and suppose that

$$D = \limsup_{n\to\infty} \frac{v_1 + \cdots + v_n}{\log(n)}. \tag{5.1}$$

(i) If $s/d < D$, then A is covered almost surely.

(ii) If $s/d > D$, then A is not covered almost surely.

The case when $s/d = D$ is still open. In the Mandelbrot problem, there are very few results regarding this question. Recently, Biermé and Estrade [2] showed that for random balls in $\mathbb{R}^d$, governing by a Poisson point process with intensity $\lambda_d \otimes \mu$, denote

$$l(\mu) = \limsup_{\epsilon\to\infty} \left(|\log(\epsilon)|^{-1} \int_\epsilon^1 z^d d\mu(z) \right) \in [0, +\infty], \tag{5.2}$$

and A is a compact set in $\mathbb{R}^d$ with Hausdorff dimension s. If $l(\mu) < s$, then A is not covered almost surely. If $l(\mu) > s$, then A is covered almost surely. The case $l(\mu) = s$ still remains unsolved. It is noted that (5.2) recovers (5.1) when $\epsilon = 1/n$ and

$$d\mu(z) = \sum_{k=1}^{\infty} \delta_{z_k} \qquad (z_k^d = v_k),$$

where δ is a Dirac measure.

In this chapter, we restrict ourselves to study the condition on μ for covering the Cantor sets and their generalized versions in the one dimensional framework (the same setting that was considered in [26]). Let Φ be a Poisson point process in $\mathbb{R} \times (0, 1]$ with intensity $\lambda \otimes \mu$. For each point $(x_k, y_k) \in \Phi$, a random interval is defined as $(x_k, x_k + y_k)$, and a random covering is $U = \bigcup (x_k, x_k + y_k)$. We also denote by U_ϵ the union of all random intervals of length greater or equal to ϵ.

First, a regular Cantor set is constructed by the following procedure. Let $0 < \theta < 1/2$. We begin with the unit interval $C_0 = [0, 1]$ and let C_1 be the set of two intervals of length θ, created by deleting the open interval $(\theta, 1 - \theta)$, i.e.,

$$C_1 = [0, \theta] \cup [1 - \theta, 1]$$

Next, for each interval of C_1, we remove the middle open interval so that the remaining intervals have length θ^2. At this stage, we get

$$C_2 = [0, \theta^2] \cup [\theta - \theta^2, \theta] \cup [1 - \theta, 1 - \theta + \theta^2] \cup [1 - \theta^2, 1]$$

We repeat the process for each interval of C_2, and so on. The procedure gives us a sequence $\{C_k\}, k = 0, 1, 2, \cdots$ of closed intervals, such that

$$C_0 \supset C_1 \supset C_2 \supset \cdots \supset C_k \supset \cdots .$$

We define a Cantor set as

$$C(\theta) = \bigcap_{k=0}^{\infty} C_k.$$

Then $C(\theta)$ is an uncountable compact set without interior points, with zero Lebesgue measure. The traditional Cantor set corresponds to $\theta = 1/3$. For any value of θ, the Hausdorff dimension of $C(\theta)$ is $\log 2 / \log(1/\theta)$ (see [22] for the proof and more details).

Furthermore, we can also construct a generalized Cantor sets by varying the value of θ at each of the stages. Let $h : [0, \infty) \to [0, \infty)$ be a continuous increasing function such that $h(0) = 0$ and $h(2r) < 2h(r)$ for all $0 < r < \infty$. We inductively select $\theta_1, \theta_2, \ldots$ such that $h(\theta_1 \cdots \theta_k) = 2^{-k}$. From the sequence $\{\theta_i\}$, we construct a generalized Cantor set $C(h)$ in the manner as described above. Note that, if $h(r) = r^s$, we recover the regular Cantor set.

For a technical reason, we assume additional conditions on the function $h(r)$:

1. $\sup\{\theta_k\} < 1/2$, i.e., $\sup\Big\{h^{-1}(2^{-k-1})/h^{-1}(2^{-k})\Big\} < 1/2$.

2. The function h is differentiable.

For example, $h(r) = r^s$, for traditional Cantor set, satisfies these condition. For $h(r) = r^s \log(1/r)$, for small r, these conditions hold for all $0 < s < 1$, but not when $s = 1$. Another example is $h(r) = r^s/\log(1/r)$ for small r, which satisfies these conditions for all $0 < s < 1$, but not for $s = 0$.

For generalized Cantor set, constructed with $h(r)$ satisfying the above conditions, a necessary condition and a sufficient condition on the measure μ for covering almost surely is

Theorem 5.1. *Let μ be a non-negative σ-finite measure on $(0, 1]$.*

(i) A generalized Cantor set $C(h)$, with $h(r)$ satisfying the above assumptions, is covered almost surely if

$$\limsup_{\epsilon \searrow 0} h\left(\left(\int_\epsilon^1 d\mu(y)\right)^{-1}\right) \exp\left\{\int_\epsilon^1 y d\mu(y)\right\} = \infty.$$

(ii) The generalized Cantor set $C(h)$ is not covered with a positive probability, if

$$\int_0^1 \exp\left\{\int_x^1 (y - x) d\mu(y)\right\} h'(x) dx < \infty,$$

where $h'(x)$ is the derivative of h.

Corollary 5.2. *Let μ be a non-negative σ-finite measure on $(0, \infty)$.*

(i) The regular Cantor set with Hausdorff dimension s is covered almost surely if

$$\limsup_{\epsilon \searrow 0} \frac{\exp\left(\int_\epsilon^1 y d\mu(y)\right)}{\left(\int_\epsilon^1 d\mu(y)\right)^s} = \infty.$$

(ii) The regular Cantor set with Hausdorff dimension s is not covered almost surely if

$$\int_0^1 \exp\left(\int_x^1 (y - x) d\mu(y)\right) x^{s-1} dx < \infty.$$

Comparing to Biermé and Estrada results, our theorem gives a better result. However, it is expected since we know and take advantage of the structure of the Cantor set. Let us demonstrate this via couple examples.

Example 1. Consider a power law measure μ that is $d\mu(y) = \beta y^{-\alpha} dy$ and a regular Cantor set C with Hausdorff dimension s. Both Biermé and Estrada condition and our corollary 5.2 give the following results.

(i) For $0 < \alpha < 2$, the Cantor set C is not covered almost surely.

(ii) For $\alpha > 2$, the Cantor set C is covered almost surely.

(iii) For $\alpha = 2$,

- If $0 < \beta < s$, then the Cantor set C is not covered a.s.
- If $\beta > s$, then the Cantor set C is covered a.s.
- When $\beta = s$, no answer is available.

Example 2. Consider the measure $d\mu(y) = \frac{s}{y^2}\left(1 + \frac{k}{|\ln(y)|}\right) 1_{(0,e^{-1})}(y)dy$, for $s > 0$, and $k > 0$. Corollary 5.2 immediately yields the following results

(i) If $k < 0$, then the Cantor set C is not covered almost surely.

(ii) If $k > 0$, then the Cantor set C is covered almost surely.

Meanwhile, Biermé and Estrada condition remains inconclusive for this measure.

5.2 A necessary condition for covering a Cantor set

Our proof of a necessary condition on μ for covering the generalized Cantor set almost surely relies on the following lemma, which is essentially a variation of the second moment method in combinatorics.

Lemma 5.3. *The probability of union of event $E_1, E_2, \cdots, E_n$ is bounded by*

$$P\left(\bigcup E_i\right) \geq \frac{\left(\sum P(E_i)\right)^2}{\sum_{i,j} P(E_i E_j)}$$

Proof. Let φ_i be the indicator of event E_i happening for $i = 1, 2, \cdots, n$ and φ be the indicator of the event $\{\sum \varphi_i > 0\}$. Since $\varphi = 0$ implies that $\sum \varphi_i = 0$, we have $\sum \varphi_i = \varphi \sum \varphi_i$. Applying Schwarz's inequality, we have

$$E\left(\sum \varphi_i\right) \le E\left(\sum \varphi_i\right)^2 E\varphi^2$$

Since $E\varphi^2 = E\varphi = P\big(\bigcup E_i\big)$, we have

$$P\left(\bigcup E_i\right) \ge \frac{\left(E\sum \varphi_i\right)^2}{E(\sum \varphi_i)^2} = \frac{\left(\sum_{i=1}^n P(E_i)\right)^2}{\sum_{i,j} P(E_i E_j)}.$$

□

Let B_i be the collection of endpoints of all intervals in C_i, as defined in the construction of a Cantor set, and $D = \bigcup_{i=1}^\infty B_i$. Since D is dense in $C(h)$, and U_ϵ is a finite union of open sets, the probability of covering D but not covering $C(h)$ is zero. Hence, we have

$$P\big(C(h) \not\subset U_\epsilon\big) = P\big(\lim_{i\to\infty} B_i \not\subset U_\epsilon\big) = \lim_{i\to\infty} P(B_i \not\subset U_\epsilon).$$

For a fixed i, we apply the lemma 5.3, where E_k is the event that the k$^{\text{th}}$ point in B_i is not covered by U_ϵ. Hence, we obtain that

$$P(B_i \not\subset U_\epsilon) \ge \frac{\left(\sum_{x\in B_i} P(x \notin U_\epsilon)\right)^2}{\sum_{x,y\in B_i} P(x \notin U_\epsilon, y \notin U_\epsilon)} \tag{5.3}$$

First, consider the denominator in (5.3), $\sum_{x,y\in B_i} P(x \notin U_\epsilon, y \notin U_\epsilon)$. Due to the translation invariance property of the Poisson point process, the probability $P(x \notin U_\epsilon, y \notin U_\epsilon)$ only depends on the distance between two points x and y. That is,

$$P(x \notin U_\epsilon, y \notin U_\epsilon) = P\big(0 \notin U_\epsilon, |x-y| \notin U_\epsilon\big) = \exp\left\{-\lambda \otimes \mu\big(A(0) \cup A(t)\big)\right\}, \tag{5.4}$$

where $t = |x - y|$, and $A(t)$ is a wedge-shape region in $\mathbb{R} \times (0, 1]$, defined as

$$A(t) = \big\{(u, v) \in \mathbb{R} \times (\epsilon, 1], u < t, t - u < v\big\}.$$

It follows from (5.4) that the probability $P(x \notin U_\epsilon, y \notin U_\epsilon)$ decreases as t increases. Hence, we use the following scheme to bound the denominator in (5.3). There are 2^{i+1} points in B_i, and half of them are on the left hand side of $(\theta_1, 1 - \theta_1)$, the interval removed in the first stage of Cantor set construction, and the other half on the right hand side. Hence, there are 2^{2i} pairs of points where their distance is more than $1-2\theta_1$. In the second stage of the construction, the interval $\big(\theta_1\theta_2, \theta_1(1-\theta_2)\big)$ and $\big(1-\theta_1(1-\theta_2), 1-\theta_1\theta_2\big)$ have been removed, which helps to identify 2^{2i-1} pair of points in B_i whose distance between them is more than $\theta_1(1-2\theta_2)$. Hence, in the j^{th} stage, where $j \le i$, the removed intervals helps us to identify 2^{2i-j+1} pairs of points whose distance is more than $\theta_1\cdots\theta_{j-1}(1-2\theta_j)$. At the end of the i^{th} stage, we have 2^i intervals of length $\theta_1\cdots\theta_i$, which also contribute to the sum in the denominator of (5.3). Therefore, we have

$$\begin{aligned}
\sum_{x,y\in B_i} P(x \notin U_\epsilon, y \notin U_\epsilon) &\le \sum_{k=1}^{i} 2^{2i-k+1} P(0 \notin U_\epsilon, \theta_1\cdots\theta_{k-1}(1-2\theta_k) \notin U_\epsilon) \\
&\quad + 2^i P(0 \notin U_\epsilon, \theta_1\cdots\theta_i \notin U_\epsilon) \\
&\le \sum_{k=1}^{i+1} 2^{2i-k+1} P\Big(0 \notin U_\epsilon, \theta_1\cdots\theta_{k-1}(1-2\theta_k) \notin U_\epsilon\Big) \\
&\le \sum_{k=0}^{i} 2^{2i-k} \exp\Big\{-\lambda\otimes\mu\Big(A(0)\cup A\big(\theta_1\cdots\theta_k(1-2\theta_{k+1})\big)\Big)\Big\} \\
&\le \sum_{k=0}^{i} 2^{2i-k} \exp\Big\{-\lambda\otimes\mu\Big(A(0)\cup A\big(\theta_1\cdots\theta_k(1-2\theta)\big)\Big)\Big\},
\end{aligned} \tag{5.5}$$

where $\theta = \sup_k\{\theta_k\}$. It is easy to see that for any value of t,

$$\lambda\otimes\mu\big(A(0)\cup A(t)\big) = 2\lambda\otimes\mu\big(A(0)\big) - \lambda\otimes\mu\big(A(0)\cap A(t)\big)$$

and

$$\lambda\otimes\mu\big(A(0)\cap A(t)\big) = \int_\epsilon^1 (y-t)^+ d\mu(y),$$

where $(x)^+ = \max(x,0)$. Hence, the exponential in (5.5) becomes

$$\exp\big\{-\lambda\otimes\mu\big(A(0)\cup A(t)\big)\big\} = \exp\big\{-2\lambda\otimes\mu\big(A(0)\big)\big\}\exp\left\{\int_\epsilon^1 (y-t)^+ d\mu(y)\right\} \tag{5.6}$$

$$= P(0 \notin U_\epsilon)^2 \exp\left\{\int_\epsilon^1 (y-t)^+ d\mu(y)\right\}. \tag{5.7}$$

Therefore, combining (5.5) and (5.6), we obtain

$$\begin{aligned}\frac{\sum_{x,y\in B_i} P(x\notin U_\epsilon, y\notin U_\epsilon)}{2^{2i}P(0\notin U_\epsilon)^2} &\le \sum_{k=0}^{i} 2^{-k}\exp\left\{\int_\epsilon^1 \Big(y-\theta_1\cdots\theta_k(1-2\theta)\Big)^+ d\mu(y)\right\}\\ &= \sum_{k=0}^{i} 2^{-k}\exp\left\{\int_\epsilon^1 \Big(y-h^{-1}(2^{-k})(1-2\theta)\Big)^+ d\mu(y)\right\},\end{aligned} \tag{5.8}$$

where h is a pre-specified function, used in the construction of a Cantor set, and h^{-1} is the inverse function of h. By expanding each term of the sum in (5.8), we have

$$\begin{aligned}\frac{\sum_{x,y\in B_i} P(x\notin U_\epsilon, y\notin U_\epsilon)}{2^{2i}P(0\notin U_\epsilon)^2} &\le \sum_{k=0}^{2^i} \frac{1}{2^i}\exp\left\{\int_\epsilon^1 \left(y-h^{-1}\left(\frac{2^{\lfloor\log_2 k\rfloor}}{2^i}\right)(1-2\theta)\right)^+ d\mu(y)\right\}\\ &\le \sum_{k=0}^{2^i} \frac{1}{2^i}\exp\left\{\int_\epsilon^1 \left(y-h^{-1}\left(\frac{2^{\log_2 k}}{2\cdot 2^i}\right)(1-2\theta)\right)^+ d\mu(y)\right\}\\ &= \sum_{k=0}^{2^i} \frac{1}{2^i}\exp\left\{\int_\epsilon^1 \left(y-h^{-1}\left(\frac{k}{2\cdot 2^i}\right)(1-2\theta)\right)^+ d\mu(y)\right\}\end{aligned} \tag{5.9}$$

We recognize that, when i approaches to infinity,

$$\begin{aligned}&\sum_{k=0}^{2^i} \frac{1}{2^i}\exp\left\{\int_\epsilon^1 \left(y-h^{-1}\left(\frac{k}{2\cdot 2^i}\right)(1-2\theta)\right)^+ d\mu(y)\right\}\\ &\qquad \longrightarrow \int_0^1 \exp\left\{\int_\epsilon^1 (y-(1-2\theta)h^{-1}(x/2))^+ d\mu(y)\right\}.\end{aligned} \tag{5.10}$$

Hence, combining (5.3) and (5.10), taking the limit when i approaches to infinity as in (5.10), and then doing a simple substitution, we have

$$\lim_{i\to\infty} P(B_i \not\subset U_\epsilon) \ge c\left(\int_0^1 \exp\left\{\int_\epsilon^1 \Big(y-(1-2\theta)h^{-1}(x)\Big)^+ d\mu(y)\right\}dx\right)^{-1}, \tag{5.11}$$

where c is a constant. Therefore, the lower bound for the probability of not covering a Cantor set is

$$\begin{aligned}P\big(C(h)\not\subset U_\epsilon\big) &\ge c\left(\int_0^1 \exp\left\{\int_\epsilon^1 \Big(y-(1-2\theta)h^{-1}(x)\Big)^+ d\mu(y)\right\}dx\right)^{-1}\\ &= c\left(\int_0^1 \exp\left\{\int_\epsilon^1 \Big(y-x(1-2\theta)\Big)^+ d\mu(y)\right\}h'(x)dx\right)^{-1}\end{aligned}$$

Hence, $P(C(h)\not\subset U) > 0$ if

$$\int_0^1 \exp\left\{\int_x^1 (y-x)d\mu(y)\right\} h'(x)dx < \infty,$$

obtained by simple substitution and the fact that $\mu((a,b)) < \infty$ for all $a, b > 0$.

Remark 5.4. *From (5.11), it is easy to see that without the differentiability of h, a necessary condition is*

$$\int_0^1 \exp\left\{\int_0^1 \left(y - h^{-1}(x)\right)^+ d\mu(y)\right\} dx = \infty.$$

5.3 A sufficient condition for covering a Cantor set

The main tool in this proof is the Hunter-Worsley inequality [12, 29],

Lemma 5.5. *Represent events $E_1, E_2, \cdots, E_n$ by vertices $v_1, v_2, \cdots, v_n$ of a graph G where there is an edge between vertex v_i and v_j if and only if E_i and E_j are not mutually exclusive. Let $\mathcal{T}$ be a subgraph of G. Then*

$$P(\cup_{i=1}^n E_i) \le \sum_{i=1}^n P(E_i) - \sum_{(i,j)\in\mathcal{T}} P(E_i E_j),$$

if and only if $\mathcal{T}$ is a tree. Equality happens when $G = \mathcal{T}$.

Let B_i be the collection of endpoints of all intervals in C_i, as defined in the construction of a Cantor set, and $D = \bigcup_{i=1}^\infty B_i$. Since D is dense in $C(h)$, and U_ϵ is a finite union of open sets, the probability of covering D but not covering $C(h)$ is zero. Hence, we have

$$P\big(C(h) \not\subset U_\epsilon\big) = P\big(\lim_{i\to\infty} B_i \not\subset U_\epsilon\big) = \lim_{i\to\infty} P(B_i \not\subset U_\epsilon).$$

For a fixed i, we apply the lemma 5.5, where E_k is the event of not covering the k$^{\text{th}}$ point in B_i by U_ϵ. The graph G is constructed in the same manner as described in lemma 5.5. It is easy to see that G is a complete graph (since not covering a point in B_i depends on the event of not covering another point). We construct the tree $\mathcal{T}$, a subgraph of G, by the following procedure. If two point $x, y \in B_i$ are the endpoints of any interval that was removed up to the i$^{\text{th}}$ stage in the Cantor set construction, we put an edge between two vertices representing the event that x, y are not covered by U_ϵ. To complete the tree $\mathcal{T}$, we put an edge on any pair of points that are the endpoints of all remaining intervals in C_i, the subset of $[0,1]$ obtained after i^{th} stage of Cantor set construction. Hence, we have

$$P(B_i \not\subset U_\epsilon) \le \sum_{x\in B_i} P(x \notin U_\epsilon) - \sum_{(x,y)\in\mathcal{T}} P(x \notin U_\epsilon, y \notin U_\epsilon) \tag{5.12}$$

Due to the translation invariant property of Poisson point processes, the second sum in (5.12) only depends on the distance between those two points. From the construction of the tree $\mathcal{T}$, there are 1 edge of length $1-2\theta_1$, 2 edges of length $\theta_1(1-2\theta_2)$, 4 edges of length $\theta_1\theta_2(1-2\theta_3)$, etc. Hence, we have

$$\begin{aligned}
\sum_{(x,y)\in\mathcal{T}} P(x \notin U_\epsilon, y \notin U_\epsilon) &= \sum_{(x,y)\in\mathcal{T}} P\big(0 \notin U_\epsilon, |x-y| \notin U_\epsilon\big) \\
&= \sum_{k=0}^{i-1} 2^k P\big(0 \notin U_\epsilon, \theta_1\cdots\theta_k(1-2\theta_{k+1}) \notin U_\epsilon\big) \\
&\qquad + 2^i P(0 \notin U_\epsilon, \theta_1\cdots\theta_i \notin U_\epsilon) \\
&\ge \sum_{k=0}^{i} 2^k P(0 \notin U_\epsilon, \theta_1\cdots\theta_k \notin U_\epsilon) \\
&= \sum_{k=0}^{i} 2^k P\left(0 \notin U_\epsilon, h^{-1}(2^{-k}) \notin U_\epsilon\right),
\end{aligned} \tag{5.13}$$

where h^{-1} is the inverse function of h. Combining the bound in (5.13) and (5.12), we have

$$\begin{aligned}
P(B_i \not\subset U_\epsilon) &\le 2^{i+1} P(0 \notin U_\epsilon) - \sum_{k=0}^{i} 2^k P(0 \notin U_\epsilon)^2 \exp\left\{\int_\epsilon^1 \left(y - h^{-1}(2^{-k})\right)^+ d\mu(y)\right\} \\
&= P(0 \notin U_\epsilon)\left(2^{i+1} - \sum_{k=0}^{i} 2^k P(0 \notin U_\epsilon) \exp\left\{\int_\epsilon^1 \left(y - h^{-1}(2^{-k})\right)^+ d\mu(y)\right\}\right) \\
&= P(0 \notin U_\epsilon)\left(1 + \sum_{k=0}^{i} 2^k - \sum_{k=0}^{i} 2^k P(0 \notin U_\epsilon) \exp\left\{\int_\epsilon^1 \left(y - h^{-1}(2^{-k})\right)^+ d\mu(y)\right\}\right) \\
&= P(0 \notin U_\epsilon)\left(1 + \sum_{k=0}^{i} 2^k \left(1 - P(0 \notin U_\epsilon) \exp\left\{\int_\epsilon^1 \left(y - h^{-1}(2^{-k})\right)^+ d\mu(y)\right\}\right)\right)
\end{aligned} \tag{5.14}$$

Denote the sum in (5.14) as M; that is,

$$M = \sum_{k=0}^{i} 2^k \left(1 - P(0 \notin U_\epsilon) \exp\left\{\int_\epsilon^1 (y - h^{-1}(2^{-k}))^+ d\mu(y)\right\}\right).$$

We have

$$M = \sum_{k=0}^{i} 2^k \left(1 - \exp\left\{-\int_\epsilon^1 y d\mu(y) + \int_\epsilon^1 \left(y - h^{-1}(2^{-k})\right)^+ d\mu(y)\right\}\right)$$
$$= \sum_{k=0}^{k_0} 2^k \left(1 - \exp\Big\{-\int_\epsilon^{h^{-1}(2^{-k})} y d\mu(y) - \int_{h^{-1}(2^{-k})}^1 h^{-1}(2^{-k}) d\mu(y)\Big\}\right) +$$
$$+ \sum_{k=k_0+1}^{i} 2^k \left(1 - \exp\left\{-\int_\epsilon^1 h^{-1}(2^{-k}) d\mu(y)\right\}\right),$$

where k_0 is the largest integer k such that $\epsilon < h^{-1}(2^{-k})$. Furthermore, we can bound M as

$$M \leq \sum_{k=0}^{i} 2^k \left(1 - \exp\left\{-\int_\epsilon^1 h^{-1}(2^{-k}) d\mu(y)\right\}\right). \tag{5.15}$$

Consider a function $f : (0, \infty) \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x > 1 \\ x & \text{if } x \leq 1 \end{cases}$$

It is easy to see that $1 - \exp(-x) \leq f(x)$ for all $x \geq 0$. We denote that

$$k_1 = \max\left\{k : \int_\epsilon^1 h^{-1}(2^{-k}) d\mu(y) > 1\right\}. \tag{5.16}$$

From (5.15), we have

$$M \leq \sum_{k=0}^{k_1} 2^k + \sum_{k=k_1+1}^{i} 2^k \int_\epsilon^1 h^{-1}(2^{-k}) d\mu(y)$$
$$= 2^{k_1} - 1 + \int_\epsilon^1 d\mu(y) \sum_{k=k_1+1}^{i} 2^k \theta_1 \cdots \theta_k$$
$$= 2^{k_1} - 1 + 2^{k_1+1}\theta_1 \cdots \theta_{k_1+1} \int_\epsilon^1 d\mu(y) \sum_{k=0}^{i-k_1-1} 2^k \theta_{k_1+2} \cdots \theta_{k_1+1+k} \tag{5.17}$$

From definition of k_1, we have

$$\int_\epsilon^1 h^{-1}(2^{-k_1}) d\mu(y) > 1 \Rightarrow 2^{k_1} < \frac{1}{h\Big(\Big(\int_\epsilon^1 d\mu(y)\Big)^{-1}\Big)} \tag{5.18}$$

$$\int_\epsilon^1 h^{-1}(2^{-k_1-1}) d\mu(y) < 1 \Rightarrow \theta_1 \cdots \theta_{k_1+1} \int_\epsilon^1 d\mu(y) < 1 \tag{5.19}$$

Furthermore, from the condition on h, we denote $\theta = \sup\{\theta_k\} < 1/2$. Hence,

$$\sum_{k=0}^{i-k_1-1} 2^k \theta_{k_1+2} \cdots \theta_{k_1+1+k} \leq \sum_{k=0}^{i-k_1-1} 2^k \theta^k$$
$$= \frac{1 - (2\theta)^{i-k_1-1}}{1 - 2\theta} \tag{5.20}$$

Combining (5.17), (5.18), and (5.20), we obtain

$$M \leq -1 + \frac{1}{h\Big(\Big(\int_\epsilon^1 d\mu(y)\Big)^{-1}\Big)} + \frac{2}{h\Big(\Big(\int_\epsilon^1 d\mu(y)\Big)^{-1}\Big)} \frac{1-(2\theta)^{i-k_1-1}}{1-2\theta} \tag{5.21}$$

From (5.14) and (5.21), we obtain

$$P(B_i \not\subset U_\epsilon) \quad \leq \quad \frac{P(0 \notin U_\epsilon)}{h\Big(\Big(\int_\epsilon^1 d\mu(y)\Big)^{-1}\Big)} \left(1 + \frac{2\big(1-(2\theta)^{i-k_1-1}\big)}{1-2\theta}\right)$$

To approximate the Cantor set, we take the limit to infinite.

$$\begin{aligned} P(\lim_{i\to\infty} B_i \not\subset U_\epsilon) &= \lim_{i\to\infty} P(B_i \not\subset U_\epsilon) \\ &\leq c\frac{P(0 \notin U_\epsilon)}{h\Big(\Big(\int_\epsilon^1 d\mu(y)\Big)^{-1}\Big)}, \end{aligned} \tag{5.22}$$

where c is a constant. Therefore, $P\big(C(h) \not\subset U\big) = 0$ if

$$\limsup_{\epsilon\to 0} h\left(\left(\int_\epsilon^1 d\mu(y)\right)^{-1}\right) \exp\left\{\int_\epsilon^1 y d\mu(y)\right\} = \infty.$$

References

[1] J. Barral and A. Fan, *Covering numbers of different points in dvoretzky covering*, Bulletin des sciences mathematiques, 129 (2005), pp. 275–317.

[2] H. Biermé and A. Estrade, *Covering the whole space with poisson random balls*, ALEA Lat. Am. J. Probab. Math. Stat., 9 (2012), pp. 213–229.

[3] P. Billard, *Séries de Fourier aléatoirement bornées, continues, uniformément convergentes*, Ann. Sci. École Norm. Sup. (3), 82 (1965), pp. 131–179.

[4] D. J. Daley and D. Vere-Jones, *An introduction to the theory of point processes. Vol. II*, Probability and its Applications (New York), Springer, New York, second ed., 2008. General theory and structure.

[5] A. Durand, *On randomly placed arcs on the circle*, in Recent developments in fractals and related fields, Appl. Numer. Harmon. Anal., Birkhäuser Boston Inc., Boston, MA, 2010, pp. 343–351.

[6] A. Dvoretzky, *On covering a circle by randomly placed arcs*, Proc. Nat. Acad. Sci. U.S.A., 42 (1956), pp. 199–203.

[7] Y. El Hélou, *Recouvrement du tore T^q par des ouverts aléatoires et dimension de Hausdorff de l'ensemble non recouvert*, C. R. Acad. Sci. Paris Sér. A-B, 287 (1978), pp. A815–A818.

[8] P. Erdős, *Some unsolved problems*, Magyar Tud. Akad. Mat. Kutató Int. Közl., 6 (1961), pp. 221–254.

[9] P. Hall, *Introduction to the theory of coverage processes*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons Inc., New York, 1988.

[10] J. Hawkes, *On the covering of small sets by random intervals*, Quart. J. Math. Oxford Ser. (2), 24 (1973), pp. 427–432.

[11] F. W. Huffer and L. A. Shepp, *On the probability of covering the circle by random arcs*, J. Appl. Probab., 24 (1987), pp. 422–429.

[12] D. Hunter, *An upper bound for the probability of a union*, J. Appl. Probability, 13 (1976), pp. 597–603.

[13] S. Janson, *Random coverings in several dimensions*, Acta Mathematica, 156 (1986), pp. 83–118.

[14] J. Jonasson and J. E. Steif, *Dynamical models for circle covering: Brownian motion and Poisson updating*, Ann. Probab., 36 (2008), pp. 739–764.

[15] J.-P. Kahane, *Sur le recouvrement d'un cercle par des arcs disposés au hasard*, C. R. Acad. Sci. Paris, 248 (1959), pp. 184–186.

[16] ——, *Some random series of functions*, D. C. Heath and Co. Raytheon Education Co., Lexington, Mass., 1968.

[17] ——, *Some random series of functions*, D. C. Heath and Co. Raytheon Education Co., Lexington, Mass., 2nd ed., 1985.

[18] ——, *Recouvrement par des simplexes homothétiques aléatoires*, C. R. Acad. Sci. Paris Sér. I Math., 310 (1990), pp. 419–423.

[19] ——, *Recouvrements aléatoires et théorie du potentiel*, Colloq. Math., 60/61 (1990), pp. 387–411.

[20] B. B. Mandelbrot, *On Dvoretzky coverings for the circle*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 22 (1972), pp. 158–160.

[21] B. B. Mandelbrot, *Renewal sets and random cutouts*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 22 (1972), pp. 145–157.

[22] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, vol. 44 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[23] R. Meester and R. Roy, *Continuum percolation*, vol. 119 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1996.

[24] I. Molchanov, *Theory of random sets*, Probability and its Applications (New York), Springer-Verlag London Ltd., London, 2005.

[25] L. A. Shepp, *Covering the circle with random arcs*, Israel J. Math., 11 (1972), pp. 328–345.

[26] ——, *Covering the line with random intervals*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 23 (1972), pp. 163–170.

[27] ——, *Probability problems of Dobrushin type*, Mosc. Math. J., 10 (2010), pp. 807–810, 840.

[28] S. Shiu, *Covering a connected curve on the torus with squares*, Journal of Theoretical Probability, (2011), pp. 1–9.

[29] K. J. Worsley, *An improved Bonferroni inequality and applications*, Biometrika, 69 (1982), pp. 297–302.

[30] M. Wschebor, *Sur le recouvrement du cercle par des ensembles placés au hasard*, Israel J. Math., 15 (1973), pp. 1–11.

[31] ——, *Sur un théorème de Léonard Shepp*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 27 (1973), pp. 179–184.

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