TWO NEW COMPUTER BASED RESULTS IN GAME THEORY RELATED TO COMBINATORIAL GAMES AND NASH EQUILIBRIA

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ABSTRACT OF THE DISSERTATION

Two new computer based results in game theory related to combinatorial games and Nash equilibria

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This thesis consists of two chapters.

The first chapter is about the new version of NIM recently introduced by Gurvich together with a generalization of the minimal excludant function (mex). This game NIM(a, b) is played with two piles of matches. Two players alternate turns. By one move, each player can take either "almost the same" number of matches from each pile (the difference of the two numbers is strictly less than a) or any number of matches from one pile and strictly less than b from the other. This game further extends Fraenkel's NIM = NIM$(a, 1)$, which, in its turn, is a generalization of the classic Wythoff NIM = NIM$(1, 1)$.

Gurvich introduced a generalization mex_b of the standard minimum excludant mex = mex_1 defining $\text{mex}_b(S \subset \mathbb{Z}_+) = \min\{n : \forall s \in S \; s \leq n \Rightarrow s + b \leq n\}$. He also showed that P-positions (the kernel) of NIM(a, b) are given by the following recursion:

$$x_n = \text{mex}_b(\{x_i, y_i \mid 0 \leq i < n\}), \;\; y_n = x_n + an; \;\; n \geq 0,$$

and conjectured that for all a, b the limits $\ell(a, b) = x_n(a, b)/n$ exist and are irrational algebraic numbers. Here we prove it showing that $\ell(a, b) = \frac{a}{r-1}$, where $r > 1$ is the

Perron root of the polynomial

$$P(z) = z^{b+1} - z - 1 - \sum_{i=1}^{a-1} z^{\lceil ib/a \rceil},$$

whenever a and b are coprime; furthermore, it is known that $\ell(ka, kb) = k\ell(a, b)$.

In particular, $\ell(a, 1) = \alpha_a = \frac{1}{2}(2 - a + \sqrt{a^2 + 4})$. In 1982, Fraenkel introduced the game NIM(a) = NIM$(a, 1)$, obtained the above recursion and solved it explicitly getting $x_n = \lfloor \alpha_a n \rfloor$, $y_n = x_n + an = \lfloor (\alpha_a + a)n \rfloor$. Here we provide a polynomial time algorithm based on the Perron-Frobenius theory solving game NIM(a, b), although we have no explicit formula for its kernel.

The second chapter of the thesis is about the existence of Nash equilibria (NE) in pure stationary strategies in n-person positional games with no moves of chance, with perfect information, and with the mean or total effective cost function.

We construct a NE-free three-person game with positive local costs, disproving the conjecture suggested by Boros and Gurvich in Math. Soc. Sci. 46 (2003) 207-241.

Still, the following four problems remain open:

Whether NE exist in all *two*-person games with total effective costs such that (I) all local costs are strictly positive or (II) without directed cycles of the cost zero?

If NE exist in all n-person games with the terminal (transition-free) cost functions, provided all directed cycles form a unique outcome c and (III) assuming that c is worse than any terminal outcome or (IV) without this assumption?

For $n = 3$ cases (I) and (II) are answered in the negative, while for $n = 2$ cases (III) and (IV) are proven. We briefly survey other negative and positive results on Nash-solvability in pure stationary strategies for the games under consideration.

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Chapter 1

A polynomial algorithm and linear asymptotic for a two parameter extension of Wythoff NIM

1.1 Introduction

For any positive integer a and b, a game NIM(a, b) was recently introduced by Gurvich (2012) [10] as follows. Two piles contain x and y matches. Two players alternate turns. In one move, it is allowed to take x' and y' matches from these two piles such that

$$0 \le x' \le x, \;\; 0 \le y' \le y, \;\; 0 < x'+y', \;\; \text{and either } |x'-y'| < a \;\; \text{or} \;\; \min(x', y') < b. \quad (1.1)$$

In other words, a player can take "approximately equal" (differing by at most $a-1$) numbers of matches from both piles or any number of matches from one pile but at most $b-1$ from the other. This game, NIM(a, b), extends further the game NIM(a) = NIM$(a, 1)$ considered by Fraenkel (1982, 1984) [4, 5], which, in its turn, is a generalization of the classic game NIM$(1, 1)$ introduced by Wythoff (1907) [18], see also Coxeter (1953) [3].

A position of NIM(a, b) is a non-negative integer pair (x, y). Due to obvious symmetry, positions (x, y) and (y, x) are equivalent. By default, we assume that $x \le y$ unless it is explicitly said otherwise.

Obviously, $(0, 0)$ is a unique terminal position. By definition, the player entering this position is the winner in the *normal* version of the game and (s)he is the loser in its *misère* version.

In each version, (x, y) is called a *P-position* if a player entering it (the *Previous* player) wins and (x, y) is called an *N-position* if a player leaving it (the *Next* player) wins. It is easily seen and well known that

- each move from a P-position leads to an N-position;
- from each N-position there is a move to a P-position;
- $(0, 0)$ is a P-position in the normal version of the game and an N-position in its misère version.

We refer the reader to Berlekamp et. al. (2001–2004) [1] and Conway (1976) [2] for more details of combinatorial game theory.

The normal version of NIM(a, b) was recently solved by Gurvich (2012) [10], by means of the following recursion for the P-positions (x_n, y_n):

$$x_n = \text{mex}_b(\{x_i, y_i \mid 0 \leq i < n\}), \quad y_n = x_n + an; \quad n \geq 0, \tag{1.2}$$

where $x_n \leq y_n$ and the function mex_b is defined as follows:

Given a finite non-empty subset $S \subset \mathbb{Z}_+$ of m non-negative integers, let us order S and extend it by $s_{m+1} = \infty$ and by $s_0 = -b$, to get the sequence $s_0 < s_1 < \cdots < s_m < s_{m+1}$. Obviously there is a unique minimum i such that $s_{i+1} - s_i > b$. By definition, let us set $\text{mex}_b(S) = s_i + b$; in particular, $\text{mex}_b(\emptyset) = 0$.

It is easily seen that mex_b is well-defined and for $b = 1$ it is exactly the classic minimum excludant mex, which assigns to S the (unique) minimum non-negative integer missing in S. Thus, $\text{mex}_1 = \text{mex}$ and (1.2) turns into the recursive solution of NIM$(a, 1)$ given by Fraenkel (1982, 1984) [4, 5].

Furthermore, Fraenkel solved the recursion and got the following explicit formula for (x_n, y_n): Let $\alpha_a = \frac{1}{2}(2 - a + \sqrt{a^2 + 4})$ be the (unique) positive root of the quadratic equation $\widehat{z}^2 + (a-2)\widehat{z} - a = 0$, or equivalently, $\frac{1}{\widehat{z}} + \frac{1}{\widehat{z}+a} = 1$. In particular, $\alpha_1 = \frac{1}{2}(1+\sqrt{5})$ is the *golden section* and $\alpha_2 = \sqrt{2}$. Then, it follows that for all $n \in \mathbb{Z}_+$ we have

$$x_n = \lfloor \alpha_a n \rfloor \quad \text{and} \quad y_n = x_n + an \equiv \lfloor n(\alpha_a + a) \rfloor. \tag{1.3}$$

This recursion implies the asymptotic $\lim\limits_{n\to\infty} \dfrac{x_n(a)}{n} = \alpha_a$ and $\lim\limits_{n\to\infty} \dfrac{y_n(a)}{n} = \alpha_a + a$.

As it was mentioned by Fraenkel (1982) [4], the explicit formula (1.3) solves the game in linear time, in contrast to recursion (1.2), providing only an exponential algorithm.

Yet, it looks too difficult to solve (1.2) explicitly when $b > 1$, because of the following bounds obtained by Gurvich (2012) [10]:

$$b \le x_{n+1} - x_n \le 2b \quad \text{and} \quad b + a \le y_{n+1} - y_n \le 2b + a. \tag{1.4}$$

Hence, for $b = 1$ the difference $x_{n+1} - x_n$ is either 1 or 2, and thus $\alpha_a n$ is a good approximation of x_n. Yet, when $b > 1$, it seems harder to find a similar estimate, since the bound of (1.4) for $x_{n+1} - x_n$ is looser.

Although we are not able to give closed form expressions for x_n and y_n in the case $b > 1$, yet, we can compute these values (and, thus, solve NIM(a, b)) by a polynomial time algorithm.

Theorem 1 *The values x_n and y_n can be computed in $O(g(a, b) \log n)$ iterations (each of which involves arithmetical operations with integers of size $O(n)$) for all $n \in \mathbb{Z}_+$, where $g(a, b)$ is a constant depending only on a and b. Furthermore, given $Z \in \mathbb{R}_+$, we can find the largest index n such that $x_n \le Z$ using $O(g(a, b) \log Z)$ operations.*

The proof will be given in Sections 1.2 and 1.3. This theorem provides a polynomial algorithm to play NIM(a, b). Indeed, given positive integers x and y we can decide in polynomial time whether the position (x, y) is a P-position of NIM(a, b), that is, whether $x = x_n$ and $y = y_n$ for some $n \ge 0$. If yes then there is no winning move from (x, y). If no, we will find in polynomial time a P-position (x_n, y_n) that can be reached from (x, y) in one move, in accordance with the rules of NIM(a, b).

Remark 1 *There are several other impartial games for which polynomial algorithms are obtained but no closed formulas for the kernels are known; see Hadad (2008) [11], Fraenkel and Peled (2013) [7].*

Let us also notice that the number $(x + 1)(y + 1)$ of possible positions in the game beginning in (x, y) is polynomial in x and y but exponential in the size of the input, $\log(xy)$. *In contrast, there are many impartial games in which the number of positions is exponential in the size of the basic parameters and doubly exponential in the size of the input. Several such examples were considered by Fraenkel (2004) [6], who suggested* *a new concept of complexity for the combinatorial games. According to this concept, a game is intractable if it contains doubly exponential plays, even if an optimal move in every position can be computed in polynomial time. Yet, NIM(a, b) remains tractable, even with respect to this new concept.*

For the misère version of NIM(a, b), a recursion very similar to (1.2) was obtained by Gurvich (2011, 2012) [9, 10] and the above polynomial algorithm should be just slightly modified; see Section 1.7.2. For the case $b = 1$ the recursion together with a closed formula for the misère version were obtained earlier by Fraenkel (1984) [5].

The next step is to show that a linear asymptotic still holds for $b > 1$. Gurvich (2012) [10] conjectured that the limits $\ell(a, b) = \lim_{n\to\infty} x_n(a, b)/n$ exist for all positive integer a, b and are irrational algebraic numbers. Here, we prove this conjecture and provide an explicit formula for the limiting values.

Theorem 2 *The limit $\ell(a, b)$ exists for all positive integer a, b and, when they are co-prime, gcd$(a, b) = 1$, it is given by the fraction $\ell(a, b) = \frac{a}{r-1}$, where $r > 1$ is a unique positive real root of the polynomial*

$$P(z) = z^{b+1} - z - 1 - \sum_{i=1}^{a-1} z^{\lceil ib/a \rceil}, \tag{1.5}$$

which is the characteristic polynomial of a non-negative $(b+1) \times (b+1)$ integer matrix associated to game NIM(a, b) and depending only on parameters a and b.

Remark 2 *Note that, by the Perron-Frobenius theorem, we have $|r'| < r$ for any other root r' of $P(z)$.*

For notational convenience, we use a variable transformation $\widehat{z} = a/(z - 1)$. Thus, in the case $b = 1$, Theorem 2 yields the same as the above cited results of Fraenkel (1982, 1984) [4, 5].

The case gcd$(a, b) > 1$ is also covered by Theorem 2, since, as it was shown by Gurvich (2012) [10], $x_n(a, b)$ (and, hence, $y_n(a, b)$ and $\ell(a, b)$ as well) are uniform functions of a and b, that is,

$$x_n(ka, kb) = kx_n(a, b), \quad y_n(ka, kb) = ky_n(a, b), \quad \text{and} \quad \ell(ka, kb) = k\ell(a, b), \qquad (1.6)$$

for all positive integers a, b and k, n. We provide the proof for Theorem 2 in Sections 1.5 and 1.6. It will be derived with the help of the Perron-Frobenius theorem and the Collatz-Wielandt formula for the non-negative matrices; see Chapter 8 of the textbook by Meyer (2000) [16].

Remark 3 *Alternatively, Theorems 1 and 2 could be derived from the Cauchy-Ostrovsky theorem; see theorems 1.1.3 and 1.1.4 in the textbook by Prasolov (2010) [17] and verify that our polynomial $P(z)$ satisfies all condition of the latter.*

1.2 Basic properties

From now on, we assume that a and b are relatively prime positive integers. Then, $a = \alpha b + \beta$, where $\alpha \geq 0$ and $0 < \beta \leq b$ are integers; in particular, if $b = 1$ then $\alpha = a - 1$, $\beta = 1$. Let us introduce the set $B = \{0, 1, ..., b\}$. In our complexity estimates we regard parameters a, b (and α, β) as fixed constants.

We denote by $\mathcal{S} = \mathcal{S}(a, b) = (x_n, y_n \mid n = 0, 1, ...)$ the sequence defined by (1.2), and note that $x_0 = y_0 = 0$ and $x_1 = b$, $y_1 = b + a$, etc. Let us next note that the sequences x_n and y_n are monotone increasing and we have $y_{n+1} - y_n \geq a + b > b$ by (1.4). Thus, every y_i is followed by some x_j, $j > i$ according to (1.4). Let us then introduce $\sigma(i) = j$ denoting the index of the x_j following y_i immediately in the sequence $\mathcal{S}$. Clearly, $\sigma(i)$ is well defined for all $i \in \mathbb{Z}_+$.

The following monotonicity property of the σ operator is immediate from the definitions:

Lemma 1 *If $\sigma^p(i) < j < \sigma^p(i+1)$ for some i, j and p, then, for all $t \in \mathbb{Z}_+$, we have*

$$\sigma^{p+t}(i) < \sigma^t(j) < \sigma^{p+t}(i+1).$$

$\square$

The next statement provides a "local" description of the sequence $\mathcal{S}$, which will be instrumental in our proofs and algorithms.

Lemma 2 *For $i \in \mathbb{Z}_+$ we introduce $j = j(i)$ such that $b + j = x_{i+1} - x_i$. Then, we have*

$$\sigma(i+1) - \sigma(i) = \begin{cases} \alpha + 1 & \text{if} \quad 0 \leq j \leq b - \beta, \\ \alpha + 2 & \text{if} \quad b - \beta < j \leq b. \end{cases} \tag{1.7}$$

Furthermore, the sequence $\mathcal{S} \cap [y_i, x_{\sigma(i+1)}]$ looks like

$$y_i, \; x_{\sigma(i)}, \; x_{\sigma(i)+1}, \; \ldots, \; x_{\sigma(i)+\ell}, \; y_{i+1}, \; x_{\sigma(i+1)},$$

where $\ell = \ell(j) \in \{\alpha, \alpha + 1\}$, as indicated by (1.7), and

$$x_{\sigma(i+1)} - y_{i+1} = x_{\sigma(i)+\ell} - x_{\sigma(i)+\ell-1} = \ldots = x_{\sigma(i)+1} - x_{\sigma(i)} = x_{\sigma(i)} - y_i = b;$$

furthermore, $x_{\sigma(i+1)} - x_{\sigma(i)+\ell} = b + \mu(j)$, where

$$\mu(j) \;=\; \begin{cases} \beta + j & \text{if} \quad 0 \leq j \leq b - \beta, \\ \beta + j - b & \text{if} \quad b - \beta < j \leq b. \end{cases} \tag{1.8}$$

Proof: By (1.4) we know that $y_{i+1} - y_i = x_{i+1} - x_i + a = b + j + a = (\alpha + 1)b + \beta + j$. We also know that there are only x's between y_{i+1} and y_i, and hence, by the mex_b rule we must have some x's b-apart, as long as they fit this interval. This implies that we have $\lfloor \frac{(\alpha+1)b+\beta+j}{b} \rfloor = \alpha + 1 + \lfloor \frac{\beta+j}{b} \rfloor$ many x's between y_{i+1} and y_i. Since we also know that by definition $x_{\sigma(i)} - y_i = b$ for all indices i, the claims follow by elementary calculations. $\square$

By default, all considered vectors are assumed to be column vectors, while all row vectors will be indicated explicitly by the transposition sign.

Let us introduce vectors $\mathbf{e} = (1, 1, ..., 1) \in \mathbb{Z}^B$, $\mathbf{b} = (b, b+1, ..., 2b) \in \mathbb{Z}^B$, and, for an arbitrary pair $i < j$ of indices, let us define the vector $\mathbf{d}(i,j) \in \mathbb{Z}_+^B$, where

$$\mathbf{d}(i,j)_k = |\{s \mid i \leq s < j, \quad x_{s+1} - x_s = b + k\}|$$

is the number of consecutive x's between x_i and x_j the distance between which is exactly $b+k$ for $k \in B$.

Let us denote by $\mathbf{e}^\ell \in \{0,1\}^B$ the ℓ-th unit vector, for $\ell \in B$ and remark that, by definition, for each index $i \in \mathbb{Z}_+$ there is an $\ell = \ell(i) \in B$ such that

$$\mathbf{d}(i, i+1) \;=\; \mathbf{e}^\ell \tag{1.9}$$

The following relations are readily implied by the above definition.

Corollary 1 *The following equalities hold whenever $i < j < k$:*

$$\mathbf{d}(i,k) \;=\; \mathbf{d}(i,j) + \mathbf{d}(j,k); \tag{1.10}$$

$$j - i \;=\; \mathbf{e}^T \mathbf{d}(i,j); \tag{1.11}$$

$$x_j - x_i \;=\; \mathbf{b}^T \mathbf{d}(i,j); \tag{1.12}$$

$$\mathbf{d}(0,1) \;=\; \mathbf{e}^0. \tag{1.13}$$

□

Let us also introduce a non-negative integer matrix $M \in \mathbb{Z}_+^{B \times B}$ by defining

$$M_{i,j} = \begin{cases} \alpha = \lfloor \frac{a+j-1}{b} \rfloor & \text{if } i = 0 \text{ and } 0 \le j \le b - \beta \\ \alpha + 1 = \lfloor \frac{a+j-1}{b} \rfloor & \text{if } i = 0 \text{ and } b - \beta < j \le b \\ 1 & \text{if } i > 0 \text{ and } (j + a - i \bmod b) = 0 \\ 0 & \text{if } i > 0 \text{ and } (j + a - i \bmod b) \neq 0 \end{cases} \tag{1.14}$$

	0	1	$\cdots$	$b-\beta-1$	$b-\beta$	$b-\beta+1$	$\cdots$	$b-1$	b
0	α	α	$\cdots$	α	α	$\alpha+1$	$\cdots$	$\alpha+1$	$\alpha+1$
1	0	0	$\cdots$	0	0	1	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\beta-1$	0	0	$\cdots$	0	0	0	$\cdots$	1	0
β	1	0	$\cdots$	0	0	0	$\cdots$	0	1
$\beta+1$	0	1	$\cdots$	0	0	0	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$b-1$	0	0	$\cdots$	1	0	0	$\cdots$	0	0
b	0	0	$\cdots$	0	1	0	$\cdots$	0	0

Let us notice that if $x_{i+1} - x_i = b + j$, then column j of M provides the distribution of the consecutive x differences between $x_{\sigma(i)}$ and $x_{\sigma(i+1)}$, as shown in Lemma 2. Thus, the next two relations follow readily from Lemma 2.

Corollary 2 *For any $i < j$ we have*

$$\mathbf{d}(\sigma(i), \sigma(j)) = M\mathbf{d}(i, j); \tag{1.15}$$

Let us next note that Lemma 2 and the corollaries above provide us with a computational tool, allowing us to compute indices and x-values, as we move forward by the σ operator.

Corollary 3 *Given a positive integer i, the values x_i, x_{i+1}, and vector $\mathbf{d}(0, i)$, we can compute the index $\sigma(i)$, the values x_k, and vectors $\mathbf{d}(0, k)$ for all $\sigma(i) \le k \le \sigma(i+1)$ in $O((\alpha+1)b^2) = O(1)$ total time.*

Proof: Introduce $j = x_{i+1} - x_i - b$ as in Lemma 2 and set $\mu(j)$ as defined in (1.8). Then, by Corollaries 1 and 2, we can compute first $\sigma(i) = \mathbf{e}^T (\mathbf{d}(0,1) + \mathbf{d}(1, \sigma(i)))$ $= 1 + \mathbf{e}^T M\mathbf{d}(0, i)$ and $\sigma(i+1) = \sigma(i) + \mathbf{e}^T M\mathbf{d}(i, i+1) = \sigma(i) + \mathbf{e}^T M\mathbf{e}^{\mu(j)}$. Then, using Lemma 2, we can compute the vectors

$$\mathbf{d}(0, k) \;=\; \mathbf{d}(0,1) + \mathbf{d}(1, \sigma(i)) + (k - \sigma(i))\mathbf{e}^0 \;=\; (k + 1 - \sigma(i))\mathbf{e}^0 + M\mathbf{d}(0, i) \tag{1.16}$$

for $\sigma(i) \le k < \sigma(i+1)$, and finally,

$$\mathbf{d}(0,\sigma(i+1)) \;=\; \mathbf{d}(0,\sigma(i+1)-1) + \mathbf{e}^{\mu(j)}.$$

Since $x_0 = 0$, by (1.12), we obtain that $\quad x_k \;=\; \mathbf{b}^T\mathbf{d}(0,k) \quad$ for $\;\sigma(i) \le k \le \sigma(i+1)$.

Here, computing $M\mathbf{d}(0,i)$ takes $O(b^2)$ time, computing $M\mathbf{e}^{\mu(j)}$ takes $O(b)$ time, while all other operations are multiplications or additions of vectors of dimension b and, hence, take also $O(b)$ time each. Since we have $O(\alpha+1)$ such operations by Lemma 2, the claim follows. $\square$

To be able to start our algorithms, described in the next section, we need to draw a few more computational consequences of the above basic results.

Corollary 4 *Given positive integers t, i, $\sigma(i)$, x_i, and the corresponding vector $\mathbf{d}(i,\sigma(i))$ we can compute $\sigma^{2^k}(i)$ and $x_{\sigma^{2^k}(i)}$ for all $k = 0,1,...,t$ in $O(t)$ time.*

Proof: Note first that the powers M^{2^j}, $j = 0,1,...,t$ can be computed in $O(t)$ time. Thus, by Corollaries 1 and 2 we get

$$\mathbf{d}(i,\sigma^{2^{k+1}}(i)) \;=\; \mathbf{d}(i,\sigma^{2^k}(i)) + \mathbf{d}(\sigma^{2^k}(i),\sigma^{2^{k+1}}(i)) \;=\; \left(I + M^{2^k}\right)\mathbf{d}(i,\sigma^{2^k}(i))$$

for $k = 0,1,...,t-1$, and hence the claim follows by (1.11) and (1.12). $\square$

Corollary 5 *Given a positive integer N, we can compute the largest integer n such that $\sigma^n(0) \le N$ in $O(\log n)$ time.*

Proof: We compute such a largest n in its binary representation. Just like in Corollary 4, let us compute first M^{2^j} and $\sigma^{2^j}(0)$ for $j = 0,1,...,t{+}1$, where $\sigma^{2^t}(0) \le N < \sigma^{2^{t+1}}(0)$. Note that we also get all the vectors $\mathbf{d}(0,\sigma^{2^j}(0))$ for $j = 0,1,...,t$. As in the previous corollary, we can do all these in $O(t)$ time.

Let us also note that for arbitrary integers m and k we have by (1.11) and (1.15) that

$$\sigma^{m+k}(0) \;=\; \mathbf{e}^T\mathbf{d}(0,\sigma^{m+k}(0)) \;=\; \sigma^m(0) + \mathbf{e}^T M^m \mathbf{d}(0,\sigma^k(0)).$$

Thus, starting with $m = 2^t$ we can find a largest integer $j < t$ for which $\sigma^{m+2^j}(0) \le N$. Then update $m \leftarrow m + 2^j$, compute $M^{m+2^j} = M^m M^{2^j}$, and repeat, until we have $\sigma^m(0) \le N < \sigma^{m+1}(0)$. Then we stop, and output $n = m$.

Note that after the initialization, we have at most t iterations, in which we need to try to add 2^j to m exactly once for all indices $j = 0, 1, ..., t-1$. Since each trial by the above equalities takes $O(1)$ time, the total time is $O(t) = O(\log n)$, as claimed. □

Corollary 6 *Given a positive integer X, we can compute the largest integer n such that $x_{\sigma^n(0)} \le X$ in $O(\log n)$ time.*

Proof: Perfectly analogous to the previous proof. We need to use also equation (1.12). □

1.3 Basic algorithms

We are now ready to describe our main algorithm(s) with which we can answer a number of different questions about NIM(a, b). The precise complexity estimate will follow in the next section, where we show that $\sigma(i)^n$ is an exponential function of n. We will provide three algorithms:

- Compute-X(N): Given a positive integer N, compute x_N;
- Find-N(X): Given a positive integer X, find the maximum N such that $x_N \le X$, return both N and x_N;
- Find-N(Y): Given a positive integer Y, find the maximum N such that $y_N \le Y$, return both N and y_N.

Let us observe first that we can solve the last question by computing $X = Y + b$ and finding the largest integer N such that $x_{\sigma(N)} \le X$. Then, by the definition of the σ operator, the value $y_N = x_N + a$ is the right value to return by Find-N(Y). This observation makes the algorithmic descriptions very similar and allows us to describe all three algorithms simultaneously.

The key idea in our algorithms is to use the σ operator to derive a series of increasingly smaller and smaller windows of upper and lower bounds on the targeted input parameters N or X. We shall describe all three procedures simultaneously, indicating the differences in parentheses.

Initialization: We start with computing a largest integer n such that $\sigma^n(0) \leq N$ (or $x_{\sigma^n(0)} \leq X$) by Corollary 5 (or 6), in $O(\log n)$ time. Then we initialize $i = 0$, and set $\xi_i = 0$, $x_{\xi_i} = 0$, $x_{\xi_i+1} = b$ and $\mathbf{d}(0, \xi_i) = (0, 0, \ldots, 0) \in \mathbb{R}^B$. We also compute the matrices M^n in $O(\lceil \log n \rceil)$ time, and M^{-1} (as rational) in $O(1)$ time.

Parameters: We compute for every index $i = 0, 1, \ldots, n$ a corresponding index ξ_i, the values x_{ξ_i} and x_{ξ_i+1}, and the vector $\mathbf{d}(0, \xi_i)$, in $O(1)$ time for each index i, satisfying the following properties.

Invariant(s): We maintain for every $i = 0, ..., n-1$ that either

$$\sigma^n(0) \leq \sigma^{n-i}(\xi_i) \leq \sigma^{n-i-1}(\xi_{i+1}) \leq N < \sigma^{n-i-1}(\xi_{i+1}+1) \leq \sigma^{n-i}(\xi_i+1) \leq \sigma^{n+1}(0)$$

holds (for the first problem) or

$$x_{\sigma^n(0)} \leq x_{\sigma^{n-i}(\xi_i)} \leq x_{\sigma^{n-i-1}(\xi_{i+1})} \leq X < x_{\sigma^{n-i-1}(\xi_{i+1}+1)} \leq x_{\sigma^{n-i}(\xi_i+1)} \leq x_{\sigma^{n+1}(0)}$$

holds (for the last two problems).

Termination: For $i = n$ we have either $\xi_n = N$ (for the first problem) in which case $x_{\xi_n} = x_N$ is the right output for COMPUTE-X(N), or $x_{\xi_n} \leq X < x_{\xi_n+1}$ (for the last two problems), in which case $N = \xi_n$ and $x_N = x_{\xi_n}$ are the right output for FIND-N(X), and $N = \xi_{n-1}$ and $y_N = a + x_{\xi_{n-1}}$ are the right output for FIND-N(Y).

Main Iteration: If $i = n$, then we go to **Termination**, otherwise we compute, as in Corollary 3, the indices $\sigma(\xi_i)$, $\sigma(\xi_i + 1)$ and the values x_k and vectors $\mathbf{d}(0, k)$ for all indices $\sigma(\xi_i) \leq k \leq \sigma(\xi_i + 1)$ in $O(1)$ time. We also compute the matrix $M^{n-i-1} = M^{-1}M^{n-i}$ in $O(1)$ time.

- If $\sigma(\xi_i + 1) = \sigma(\xi_i) + 1$, then we set $\xi_{i+1} = \sigma(\xi_i)$, $i = i + 1$, and repeat the **Main Iteration**.

- Otherwise, we compute the indices $\sigma^{n-i-1}(k) = \mathbf{e}^T M^{n-i-1}\mathbf{d}(0,k)$ and the values $x_k = \mathbf{b}^T M^{n-i-1}\mathbf{d}(0,k)$ for $\sigma(\xi_i) \le k \le \sigma(\xi_i+1)$ in $O(1)$ time. These indices (in case of the first problem) or values (in case of the other two problems) subdivide the intervals $\left[\sigma^{n-i}(\xi_i), \sigma^{n-i}(\xi_i+1)\right)$ (in case of the first problem) or $\left[x_{\sigma^{n-i}(\xi_i)}, x_{\sigma^{n-i}(\xi_i+1)}\right)$ (in case of the other two problems), and one of these intervals will contain N (or X). Let us then choose index k such that $\sigma^{n-i-1}(k) \le N < \sigma^{n-i-1}(k+1)$ (or $x_{\sigma^{n-i-1}(k)} \le X < x_{\sigma^{n-i-1}(k+1)}$), and set $\xi_{i+1} = k$, $i = i+1$ and return to the **Main Iteration**.

Theorem 3 *The above algorithm(s) correctly compute the answer to all three problems and terminate in $O(n)$ time.*

Proof: The correctness of the computations follow by Lemma 2 and Corollaries 1,2,5 and 6. The complexity then follows since each step in the algorithm takes constant time, and we repeat only the **Main Iteration**, n times.

The correctness follows by the maintained **Invariants(s)**, and by the definition of the σ operator. □

Let us remark finally that to argue that the above procedures are computationally efficient, it is enough to show that $\sigma^n(0)$ is an exponential function of n, which we will prove in the next section.

1.4 Asymptotic distribution of $\mathcal{S}(a,b)$

Let us denote by r_j, $j = 0,1,...,b$ the eigenvalues of M and by $\mathbf{u}(j)$ and $\mathbf{v}(j)$, $j = 0,1,\ldots,b$ the corresponding left and right eigenvectors of M. We label these such that

$$|r_0| \ge |r_1| \ge \cdots \ge |r_b|, \tag{1.17}$$

and we scale the eigenvectors such that $\mathbf{e}^T\mathbf{u}(j) = \mathbf{e}^T\mathbf{v}(j) = 1$ for all $j = 0,1,...,b$. Let us recall from matrix theory that if $r_i \ne r_j$, then we must have

$$\mathbf{u}(i)^T\mathbf{v}(j) = 0, \tag{1.18}$$

as the equalities $r_i\mathbf{u}(i)^T\mathbf{v}(j) = \mathbf{u}(i)^T M\mathbf{v}(j) = r_j\mathbf{u}(i)^T\mathbf{v}(j)$ imply.

Let us next show that M is an irreducible aperiodic matrix, which, for a non-negative matrix, is implied by the fact that a finite power of M has only positive entries.

Lemma 3 *Every entry in M^{2b} is a positive integer.*

Proof: Let us note that the last β entries in the first row are always positive (and the whole first row is positive if $\alpha > 0$). Due to the cyclic arrangement of the 1-s in the columns of M (in rows 1...b), we get that the last 2β entries are positive in M^2, the last 3β entries are positive in M^3, etc. Thus, M^b has its first row positive. Note also that once the first row of a power of M is positive then it remains positive for all higher powers. Let us observe next that row β is positive in M^{b+1}, and it remains positive in all higher powers of M. Then row $(2\beta \mod b) + 1$ is positive in M^{b+2}, and it remains positive in all higher powers of M. Iterating this argument, using the cyclic structure of rows 1...b of M, we get that all rows are positive in M^n for all $n \geq 2b$. □

Corollary 7 *$r(a, b) = r_0 > 1$ is the unique largest eigenvalue of M and the corresponding eigenvectors $\mathbf{u}(0)$ and $\mathbf{v}(0)$ have positive real components.*

Proof: By Lemma 3 matrix M is irreducible, and hence primitive. Thus, we can apply the Perron-Frobenius theorem and conclude that r_0 is a positive real eigenvalue with multiplicity one, the corresponding eigenvectors $\mathbf{u}(0)$ and $\mathbf{v}(0)$ have positive real components and that $r_0 > |r_1|$.

To see that $r_0 > 1$, we apply the Collatz-Wielandt formula claiming that

$$r_0 = \max_{\mathbf{z}\geq 0,\ \mathbf{z}\neq 0}\ \min_{i:z_i\neq 0} \frac{M\mathbf{z}}{z_i}$$

for a non-negative matrix M. Observing then that r_0^{2b} is the largest real eigenvalue of M^{2b} and that M^{2b} has positive integer entries by Lemma 3, we can apply the above formula with $\mathbf{z} = \mathbf{e}$ to M^{2b} and obtain that $r_0^{2b} \geq (b + 1)$, from which we get

$$r_0 \geq (b + 1)^{1/2b} > 1. \tag{1.19}$$

□

Remark 4 *By the above claims it follows that the eigenvalues of M satisfy the inequalities $r_0 > 1$ and $r_0 > |r_1| \geq |r_2| \geq \cdots \geq |r_b|$. Let us also note that all these values depend only on parameters a and b. Then, let us introduce a parameter*

$$1 > \delta \ = \ \delta(a,b) \geq \frac{|r_1|}{r_0} \tag{1.20}$$

such that $\delta r_0 \geq 1$.

A further useful property of these eigenvectors is that they span all unit vectors with a positive real coefficient for $\mathbf{v}(0)$:

Lemma 4 *For every $\ell \in B$ there is a positive real $\gamma_0^\ell = \gamma_0^\ell(a,b) > 0$ and there are complex coefficients γ_j^ℓ, $j = 1, ..., b$ such that*

$$\mathbf{e}^\ell \ = \ \sum_{j \in B} \gamma_j^\ell \mathbf{v}(j). \tag{1.21}$$

Proof: Since the right eigenvectors of M span the space (1.21) holds for some complex coefficients γ_j^ℓ, $j \in B$. By Corollary 7 the left eigenvector $\mathbf{u}(0)$ has positive real components, and by (1.18), we have $\mathbf{u}(0)^T \mathbf{v}(j) = 0$ for all $j \neq 0$. Thus, by (1.21), we get

$$0 \ < \ \mathbf{u}(0)^T \mathbf{e}^\ell \ = \ \gamma_0^\ell \mathbf{u}(0)^T \mathbf{v}(0).$$

Since, by Corollary 7, the right eigenvector $\mathbf{v}(0)$ has positive components, the positivity of γ_0^ℓ follows. □

Using the above, we can prove the following statement, which will be instrumental in our proof of Theorem 2.

Lemma 5 *There is a positive real $C = C(a,b)$ depending only on parameters a and b such that for all integers $n \geq 0$ we have*

$$-C(a,b)(n+1)\delta^n \;\le\; \frac{\sigma^n(1)}{r_0^n} \;-\; \frac{\gamma_0^0 r_0}{r_0-1} \;\le\; C(a,b)(n+1)\delta^n.$$

Proof: Let us first note that, since M depends only on parameters a and b, the same holds for all its eigenvalues and eigenvectors.

Using equations (1.14) and (1.11) iteratively, starting with $(i,j)=(0,1)$, we obtain

$$\sigma^n(1) \;=\; \mathbf{e}^T\left(I+M+M^2+\cdots+M^n\right)\mathbf{d}(0,1). \tag{1.22}$$

Since $\mathbf{d}(0,1)=\mathbf{e}^0$, by Lemma 4, we get

$$\mathbf{d}(0,1) \;=\; \mathbf{e}^0 \;=\; \sum_{j=0}^{b}\gamma_j^0\mathbf{v}(j). \tag{1.23}$$

This, together with (1.22), yields

$$\sigma^n(1) \;=\; \sum_{j=0}^{b}\gamma_j^0(1+r_j+\cdots+r_j^n),$$

implying

$$\sigma^n(1) \;=\; \gamma_0^0\frac{r_0^{n+1}-1}{r_0-1}+\sum_{j=1}^{b}\gamma_j^0\sum_{k=0}^{n}r_j^k. \tag{1.24}$$

Since $|r_j|\le|r_1|<r_0$ for all $j=1,...,b$ by Remark 4 and Lemma 4 we get

$$\begin{aligned}\left|\sum_{j=1}^{b}\gamma_j^0\sum_{k=0}^{n}r_j^k\right| &\le r_0^n\left(\sum_{j=1}^{b}|\gamma_j^0|\right)\left(\sum_{k=0}^{n}\frac{(\delta r_0)^k}{r_0^n}\right)\\ &= r_0^n\left(\sum_{j=1}^{b}|\gamma_j^0|\right)\delta^n\left(\sum_{k=0}^{n}\frac{1}{(\delta r_0)^{n-k}}\right)\\ &\le (n+1)r_0^n\delta^n\left(\sum_{j=1}^{b}|\gamma_j^0|\right)\end{aligned}$$

Since $r_0\delta\ge 1$ by Remark 4, for every $n\ge 0$ we have $(n+1)r_0^n\delta^n\ge 1$. Thus, the claim holds with $C(a,b)=\frac{\gamma_0^0}{r_0-1}+\sum_{j=1}^{b}|\gamma_j^0|$, since the coefficients γ_j^0, $j\in B$ and eigenvalue r_0 depend only on a and b. $\square$

Proof of Theorem 1. It follows directly from Lemma 5 and Theorem 3. $\square$

1.5 Existence of the limiting average value of x_n

Let us first show that the subsequence $x_{\sigma^n(1)}/\sigma^n(1)$ has a limit. For this we prove a claim for the distribution of $x_{\sigma^n(1)}$, analogous to Lemma 5.

Lemma 6 *There is a positive real $D = D(a,b)$ depending only on parameters a and b such that for all integers $n \geq 0$ we have*

$$-D(a,b)(n+1)\delta^n \;\leq\; \frac{x_{\sigma^n(1)}}{r_0^n} \;-\; \frac{\gamma_0^0 r_0(\mathbf{b}^T\mathbf{v}(0))}{r_0-1} \;\leq\; D(a,b)(n+1)\delta^n.$$

Proof: Analogously to the proof of Lemma 5, by using equations (1.14) and (1.12) iteratively, starting with $(i,j) = (0,1)$, we obtain

$$x_{\sigma^n(1)} \;=\; \mathbf{b}^T\left(I + M + M^2 + \cdots + M^n\right)\mathbf{d}(0,1). \tag{1.25}$$

Since $\mathbf{d}(0,1) = \mathbf{e}^0$, by Lemma 4 we get

$$\mathbf{d}(0,1) \;=\; \mathbf{e}^0 \;=\; \sum_{j=0}^{b} \gamma_j^0 \mathbf{v}(j). \tag{1.26}$$

This, together with (1.25), yields

$$x_{\sigma^n(1)} \;=\; \sum_{j=0}^{b} \gamma_j^0 (\mathbf{b}^T\mathbf{v}(j))(1 + r_j + \cdots + r_j^n),$$

implying

$$x_{\sigma^n(1)} \;=\; \gamma_0^0(\mathbf{b}^T\mathbf{v}(0))\frac{r_0^{n+1}-1}{r_0-1} + \sum_{j=1}^{b}\gamma_j^0(\mathbf{b}^T\mathbf{v}(j))\sum_{k=0}^{n} r_j^k. \tag{1.27}$$

Since $|r_j| \leq |r_1| < r_0$ for all $j = 1, ..., b$, by Remark 4 and Lemma 4, we get

$$\begin{aligned}
\left|\sum_{j=1}^{b}\gamma_j^0(\mathbf{b}^T\mathbf{v}(j))\sum_{k=0}^{n} r_j^k\right| \;&\leq r_0^n\left(\sum_{j=1}^{b}|\gamma_j^0(\mathbf{b}^T\mathbf{v}(j))|\right)\left(\sum_{k=0}^{n}\frac{(\delta r_0)^k}{r_0^n}\right)\\
&= r_0^n\left(\sum_{j=1}^{b}|\gamma_j^0(\mathbf{b}^T\mathbf{v}(j))|\right)\delta^n\left(\sum_{k=0}^{n}\frac{1}{(\delta r_0)^{n-k}}\right)\\
&\leq (n+1)r_0^n\delta^n\left(\sum_{j=1}^{b}|\gamma_j^0(\mathbf{b}^T\mathbf{v}(j))|\right)
\end{aligned}$$

Since $r_0\delta \geq 1$ by Remark 4, for every $n \geq 0$ we have $(n+1)r_0^n\delta^n \geq 1$. Thus, the claim holds with $C(a,b) = \frac{\gamma_0^0(\mathbf{b}^T\mathbf{v}(0))}{r_0-1} + \sum_{j=1}^{b} |\gamma_j^0(\mathbf{b}^T\mathbf{v}(j))|$, since the coefficients γ_j^0, $j \in B$, eigenvectors $\mathbf{v}(j)$, $j \in B$, and eigenvalue r_0 depend only on a and b. $\square$

Theorem 4

$$\lim_{n\to\infty} \frac{x_{\sigma^n(1)}}{\sigma^n(1)} \;=\; \mathbf{b}^T\mathbf{v}(0).$$

Proof: Let us denote by $\pm C(a,b)$ and $\pm D(a,b)$ quantities between $-C(a,b)$ and $C(a,b)$ (respectively, between $-D(a,b)$ and $D(a,b)$) which guarantee the equality in Lemma 5 and 6. These lemmas imply

$$\frac{x_{\sigma^n(1)}}{\sigma^n(1)} = \frac{r_0^n\frac{\gamma_0^0 r_0(\mathbf{b}^T\mathbf{v}(0))}{r_0-1} \;\pm\; D(a,b)(n+1)r_0^n\delta^n}{r_0^n\frac{\gamma_0^0 r_0}{r_0-1} \;\pm\; C(a,b)(n+1)r_0^n\delta^n} = \frac{\mathbf{b}^T\mathbf{v}(0) \;\pm\; \frac{D(a,b)(r_0-1)}{\gamma_0^0 r_0}(n+1)\delta^n}{1 \;\pm\; \frac{C(a,b)(r_0-1)}{\gamma_0^0 r_0}(n+1)\delta^n}.$$

Since $\delta < 1$, the factor $(n+1)\delta^n$ goes to zero as $n \to \infty$, and the claim follows. $\square$

Next we show that the range $\sigma^n(i) - \sigma^n(i-1)$ is proportional to r_0^n, if n is large, for all integers $i \geq 1$.

Lemma 7 *For every integer $i \geq 1$ there is an index $\ell = \ell(i) \in B$ and a positive real $E = E(\ell,a,b)$ such that the following bounds hold for all $n \geq 0$:*

$$-E(\ell,a,b)\delta^n \;\leq\; \frac{\sigma^n(i) - \sigma^n(i-1)}{r_0^n} \;-\; \gamma_0^\ell \;\leq\; E(\ell,a,b)\delta^n$$

Proof: As we noted in (1.4), there exists an index $\ell = \ell(i) \in B$ such that $\mathbf{d}(i-1,i) = \mathbf{e}^\ell$. Thus, by (1.11) and Lemma 4, we get

$$\sigma^n(i) - \sigma^n(i-1) \;=\; \mathbf{e}^T M^n \mathbf{d}(i-1,i) \;=\; \mathbf{e}^T M^n \sum_{j\in B} \gamma_j^\ell \mathbf{v}(j) \;=\; \sum_{j\in B} \gamma_j^\ell r_j^n.$$

Since $\mathbf{e}^T\mathbf{v}(j) = 1$ for all $j \in B$, the above implies

$$\frac{\sigma^n(i) - \sigma^n(i-1)}{r_0^n} \;=\; \gamma_0^\ell + \sum_{j=1}^{b} \gamma_j^\ell \left(\frac{r_j}{r_0}\right)^n.$$

Let us note that γ_0^ℓ is a positive real according to Lemma 4. Since $|r_j|/r_0 \leq \delta$ by Remark 4, and since $E(\ell,a,b) = \sum_{j=1}^b |\gamma_j^\ell|$ is a constant depending only on $\ell = \ell(i)$, a and b according to Lemma 4, the claim follows from the above equality. $\Box$

We also prove an analogous claim for the difference $x_{\sigma^n(i)} - x_{\sigma^n(i-1)}$.

Lemma 8 *For every integer $i \in \mathbb{Z}_+$ there is an index $\ell = \ell(i) \in B$ and a positive real $F = F(\ell,a,b)$ such that the following bounds hold for all $n \geq 0$:*

$$-F(\ell,a,b)\delta^n \;\leq\; \frac{x_{\sigma^n(i)} - x_{\sigma^n(i-1)}}{r_0^n} \;-\; \gamma_0^\ell(\mathbf{b}^T\mathbf{v}(0)) \;\leq\; F(\ell,a,b)\delta^n.$$

Proof: As we noted in (1.4), there exists an index $\ell = \ell(i) \in B$ such that $\mathbf{d}(i-1,i) = \mathbf{e}^\ell$. Thus, by (1.11) and Lemma 4 we obtain

$$x_{\sigma^n(i)} - x_{\sigma^n(i-1)} \;=\; \mathbf{b}^T M^n \mathbf{d}(i-1,i) \;=\; \mathbf{b}^T M^n \sum_{j\in B} \gamma_j^\ell \mathbf{v}(j) \;=\; \sum_{j\in B} \gamma_j^\ell(\mathbf{b}^T\mathbf{v}(j)) r_j^n,$$

which in its turn implies

$$\frac{x_{\sigma^n(i)} - x_{\sigma^n(i-1)}}{r_0^n} \;=\; \gamma_0^\ell(\mathbf{b}^T\mathbf{v}(0)) + \sum_{j=1}^b \gamma_j^\ell(\mathbf{b}^T\mathbf{v}(j)) \left(\frac{r_j}{r_0}\right)^n.$$

Let us note that $\gamma_0^\ell(\mathbf{b}^T\mathbf{v}(0))$ is a positive real according to Lemma 4. Since $|r_j|/r_0 \leq \delta$ by Remark 4, and since $F(\ell,a,b) = \sum_{j=1}^b |\gamma_j^\ell(\mathbf{b}^T\mathbf{v}(j))|$ is a constant depending only on $\ell = \ell(i)$, a and b according to Lemma 4, the claim follows from the above equality.

$\Box$

Theorem 5 *For every integer $i \geq 1$ we have*

$$\lim_{n\to\infty} \frac{x_{\sigma^n(i)}}{\sigma^n(i)} \;=\; \mathbf{b}^T\mathbf{v}(0).$$

Proof: By Lemma 7, for each integer $k \geq 1$ there is a real $-E(\ell(k),a,b) \leq \varepsilon(k) \leq E(\ell(k),a,b)$ such that

$$\sigma^n(k) - \sigma^n(k-1) \;=\; \gamma_0^{\ell(k)} r_0^n + \varepsilon(k)\delta^n r_0^n. \tag{1.28}$$

Similarly, by Lemma 8, for every $k \geq 1$, there is a real $-F(\ell(k), a, b) \leq \varphi(k) \leq F(\ell(k), a, b)$ such that

$$x_{\sigma^n(k)} - x_{\sigma^n(k-1)} \;=\; \gamma_0^{\ell(k)}(\mathbf{b}^T\mathbf{v}(0))r_0^n + \varphi(k)\delta^n r_0^n. \tag{1.29}$$

Summing these up for $k = 1, 2, ..., i$ and using Lemmas 5 and 6 for $k = 0$ we get

$$\sigma^n(i) \;=\; \left(\gamma_0^0 \frac{r_0}{r_0 - 1} + \sum_{k=1}^{i} \gamma_0^{\ell(k)}\right) r_0^n \;+\; \left(\sum_{k=0}^{i} \varepsilon(k)\right) r_0^n \delta^n$$

and

$$x_{\sigma^n(i)} \;=\; \left(\gamma_0^0 \frac{r_0}{r_0 - 1} + \sum_{k=1}^{i} \gamma_0^{\ell(k)}\right) (\mathbf{b}^T\mathbf{v}(0))r_0^n \;+\; \left(\sum_{k=0}^{i} \varphi(k)\right) r_0^n \delta^n.$$

Let us notice that $A = \gamma_0^0 \frac{r_0}{r_0-1} + \sum_{k=1}^{i} \gamma_0^{\ell(k)}$ is a positive real, according to Lemma 4. Furthermore, $E(i) = \sum_{k=0}^{i} \varepsilon(k)$ and $F(i) = \sum_{k=0}^{i} \varphi(k)$ are majorized in absolute value by $(i + 1) \max_{\ell \in B} E(\ell, a, b)$ and $(i + 1) \max_{\ell \in B} F(\ell, a, b)$, respectively. Hence, both are linear in i. Consequently, we get

$$\frac{x_{\sigma^n(i)}}{\sigma^n(i)} = \frac{A(\mathbf{b}^T\mathbf{v}(0))r_0^n + E(i)r_0^n\delta^n}{Ar_0^n + F(i)r_0^n\delta^n} = \frac{\mathbf{b}^T\mathbf{v}(0) + \frac{E(i)}{A}\delta^n}{1 + \frac{F(i)}{A}\delta^n}$$

from which the claim follows. □

Now we are ready to prove the existence of the limit.

Theorem 6 *The limiting average value of x_n is given by the formula*

$$\lim_{n \to \infty} \frac{x_n}{n} \;=\; \mathbf{b}^T\mathbf{v}(0).$$

Proof: Let us denote by $m = m(n)$ the largest integer for which $x_n \geq x_{\sigma^m(1)}$, and fix an integer $1 \leq k \leq m$ to be specified later. Note that by Lemma 5 we have $m \approx \log_{r_0} n$ and we will choose $k = o(m)$. Then, there exists an integer $\sigma^k(1) \leq i \leq \sigma^{k+1}(1)$ such that $\sigma^{m-k}(i) \leq n < \sigma^{m-k}(i + 1)$, by the monotonicity of the σ operator. By Lemma 7 we have with $\ell = \ell(i - 1) \in B$

$$\Delta \;=\; n - \sigma^{m-k}(i) \leq \gamma_0^{\ell} r_0^{m-k} + E(\ell, a, b)r_0^{m-k}\delta^{m-k},$$

and consequently

$$x_{\sigma^{m-k}(i)} \le x_n \le x_{\sigma^{m-k}(i)} + 2\Delta b \ \le \ x_{\sigma^{m-k}(i)} + 2br_0^{m-k}\left(\gamma_0^\ell + E(\ell,a,b)\delta^{m-k}\right).$$

Since $\sigma^{m-k}(i) \le n < \sigma^{m-k}(i+1)$ by our choice of i,

$$\sigma^{m-k}(i) \ \le \ \sigma^{m-k}(i) + \Delta \ = \ n \ \le \ \sigma^{m-k}(i) + r_0^{m-k}\left(\gamma_0^\ell + E(\ell,a,b)\delta^{m-k}\right).$$

The above inequalities imply, by using $A = \left(\gamma_0^\ell + E(\ell,a,b)\delta^{m-k}\right)$, that

$$\frac{x_{\sigma^{m-k}(i)}}{\sigma^{m-k}(i) + r_0^{m-k}A} \ \le \ \frac{x_n}{n} \ \le \ \frac{x_{\sigma^{m-k}(i)} + 2br_0^{m-k}A}{\sigma^{m-k}(i)}. \tag{1.30}$$

Let us note that by Lemma 5 and by our choice of i we have

$$\sigma^{m-k}(i) \ \ge \ \sigma^m(1) \ \ge \ \frac{\gamma_0^0 r_0}{r_0 - 1} r_0^m - C(a,b)(m+1)r_0^m\delta^m$$

from which

$$\frac{r_0^{m-k}A}{\sigma^{m-k}(i)} \ \le \ \frac{r_0^{m-k}A}{r_0^m\left(\frac{\gamma_0^0 r_0}{r_0-1} - C(a,b)(m+1)\delta^m\right)} \ = \ O\left(\frac{1}{r_0^k}\right)$$

follows. Thus, for $k \approx m/2$ we get from the above and (1.30) that

$$\frac{\frac{x_{\sigma^{m/2}(i)}}{\sigma^{m/2}(i)}}{1 + O(1/r_0^{m/2})} \ \le \ \frac{x_n}{n} \ \le \ \frac{x_{\sigma^{m/2}(i)}}{\sigma^{m/2}(i)} + O(1/r_0^{m/2}),$$

implying the claim by Theorem 6. $\square$

In what follows, we can provide a simpler expression for the limit $\mathbf{b}^T\mathbf{v}(0)$, by computing precisely the positive real eigenvector $\mathbf{v}(0)$.

1.6 Proof of Theorem 2

We divide the proof of Theorem 2 into two subsections. First we compute the characteristic polynomial of matrix M, and next we determine the eigenvector $\mathbf{v}(0)$ corresponding to the eigenvalue $r = r_0$ as defined in Section 1.4. Using this, we can finally compute the value of $\mathbf{b}^T\mathbf{v}$, which, according to Theorem 6, equals to $\lim_{n\to\infty} \frac{x_n}{n}$.

1.6.1 Characteristic polynomial

To simplify our notations, first we determine the characteristic polynomial of a more general family of matrices. Recall that we assume $\gcd(a,b) = 1$, and thus we have $a = \alpha b + \beta$ for some integer $0 < \beta < b$ satisfying $\gcd(\beta, b) = 1$.

Given real values x_0, x_1, ..., x_b, we define a $(b+1) \times (b+1)$ matrix X by setting

$$X_{i,j} = \begin{cases} x_j & \text{if } i = 0, \\ 1 & \text{if } i > 0 \text{ and } (j + \beta - i \bmod b) = 0, \\ 0 & \text{if } i > 0 \text{ and } (j + \beta - i \bmod b) \neq 0. \end{cases}$$

Lemma 9 *The characteristic polynomial of matrix X is*

$$\det(X - zI) \;=\; (-1)^{b+1}\left(z^{b+1} - (x_b - x_0) - x_0 z^b - z - \sum_{j=1}^{b-1} z^{jc \bmod b} x_j\right), \qquad (1.31)$$

where $c = -\beta^{-1}$ in $\mathbb{Z}/b\mathbb{Z}$.

Proof: Note that since $\gcd(\beta, b) = 1$, β^{-1} is well defined. Let us develop the above determinant by row $i = 0$, and denote by $T(k)$ the submatrix we obtain after the deletion of row 0 and column k, $k = 0, 1, ..., b$. Then we can write

$$\det(X - zI) \;=\; (x_0 - z)\det(T(0)) + \sum_{j=1}^{b} (-1)^j x_j \det(T(j)).$$

Let us also refer to the rows and columns of $T(k)$ by the original index associated to it in $X - zI$. In other words, in matrix $T(k)$ we have rows $i = 1, 2, ..., b$ and columns $j = 0, ..., k-1, k+1, ..., b$.

Let us note first that matrix $\det(T(0))$ is the characteristic polynomial of the permutation matrix corresponding to the cyclic permutation $e_i \to e_{i+\beta \bmod b}$, $i = 1, ..., b$, with $\gcd(\beta, b) = 1$, and we obtain

$$\det(T(0)) \;=\; (-1)^b (z^b - 1).$$

This formula is a special case of the so-called circulant determinant; see, for example, Meyer (2000) [16].

Let us note next that for $j > 0$ matrix $T(j)$ has exactly one non-zero in column 0, namely $T(j)_{\beta,0} = 1$. Thus, we shall continue computing these determinants by column 0. Let us denote by $S(j)$ the submatrix we obtain from $T(j)$ after deleting column 0 and row β:

$$\det(T(j)) \;=\; (-1)^{\beta-1}\det(S(j)).$$

We claim that matrix $S(j)$ has exactly one nonzero permutation in the full development of its determinant. To see this let us first note that for $j = \beta$ we have two nonzeros in every row of $S(\beta)$, except row $\ell = 2\beta \bmod b$, where the single nonzero entry is $S(\beta)_{\ell,\ell} = -z$. Since this entry must belong to all nonzero permutations, the only other nonzero $S(\beta)_{3\beta \bmod b,\ell} = 1$ in its column cannot belong to any, and thus $S(\beta)_{3\beta \bmod b, 3\beta \bmod b} = -z$ must also belong to all nonzero permutations. Since we have a 1 entry in every column, and since $\gcd(\beta, b) = 1$ we can continue this for $b-1$ steps, proving that only the $-z$ entries along the main diagonal can belong to a nonzero permutation, implying

$$\det(S(\beta)) \;=\; (-z)^{b-1}.$$

For $j \neq \beta$, we have exactly two columns, $k = \beta$ and $k = b$, and exactly two rows, $i = j$ and $i = j + \beta \bmod b$ with a single nonzero entry. All other rows and columns have exactly one-one $-z$ and 1 entries. Starting with $S(j)_{2\beta \bmod b,\beta} = 1$ we can mark a series of 1 entries which must belong to all nonzero permutations, each time increasing the row by adding β and taking it $\bmod b$, until we arrive to row j where we cannot continue. In this way we mark exactly ($j\beta^{-1} \bmod b - 1$) entries of 1s in rows $2\beta \bmod b$, $3\beta \bmod b$, ..., j. After deleting the corresponding rows and columns we get a principal submatrix of $S(\beta)$ of size $b - j\beta \bmod b$. It is easy to see that we obtain

$$\det(S(j)) \;=\; (-1)^{\beta+j+j\beta^{-1}-1 \bmod b}(-z)^{b-j\beta^{-1} \bmod b}.$$

Note that this expression is correct also for $j = \beta$.

Putting the above together we get

$$\begin{aligned}
\det(X - zI) \;&=\; (x_0 - z)(-1)^b(z^b - 1) \\
&\quad+\; \textstyle\sum_{j=1}^{b}(-1)^j x_j (-1)^{\beta-1}(-1)^{\beta+j+j\beta^{-1}-1 \bmod b}(-z)^{b-j\beta^{-1} \bmod b} \\
&=\; (x_0 - z)(-1)^b(z^b - 1) + \sum_{j=1}^{b}(-1)^b x_j z^{b-j\beta^{-1} \bmod b},
\end{aligned}$$

from which the statement follows with $c = -\beta^{-1} \bmod b$. $\square$

Theorem 7 *If* $\gcd(a,b) = 1$, $a = \alpha b + \beta$, $0 < \beta < b$, *the following polynomials are identical and represent (up to a factor of* -1*) the characteristic polynomial of matrix* M*:*

$$(-1)^{b+1}\det(M - zI) = z^{b+1} - z - 1 - \alpha\sum_{j=1}^{b} z^j - \sum_{i=1}^{\beta-1} z^{(i\beta^{-1} \bmod b)} \tag{1.32a}$$

$$= z^{b+1} - z - 1 - \sum_{i=1}^{a-1} z^{\lceil \frac{ib}{a} \rceil} \tag{1.32b}$$

Proof: Let us recall that matrix M coincides with X if $x_0 = x_1 = \cdots = x_{b-\beta} = \alpha$ and $x_{b-\beta+1} = \cdots = x_b = \alpha + 1$. Thus, we get (1.32a) from (1.31) after these substitutions:

$$\begin{aligned}(-1)^{b+1}\det(M - zI) &= z^{b+1} - (\alpha + 1 - \alpha) - \alpha z^b - z - \sum_{j=1}^{b-1} \alpha z^{jc \bmod b} - \sum_{j=b-\beta+1}^{b-1} z^{jc \bmod b} \\ &= z^{b+1} - z - 1 - \alpha\sum_{j=1}^{b} z^j - \sum_{j=b+1-\beta}^{b-1} z^{jc \bmod b} \\ &= z^{b+1} - z - 1 - \alpha\sum_{j=1}^{b} z^j - \sum_{j=1}^{\beta-1} z^{j\beta^{-1} \bmod b}\end{aligned}$$

Let us observe next that if either $a = 1$ or $b = 1$, then the polynomials (1.32a) and (1.32b) are identical. For other values of a and b we omit $z^{b+1} - z - 1$ in both expressions, and reformulate the equality of (1.32a) and (1.32b) in the following equivalent way:

Lemma 10 *Let* $\gcd(a,b) = 1$, $a = \alpha b + \beta$, $0 < \beta < b$ *and consider the intervals of integers* $I_k = \{(k-1)a+1, ..., ka\}$ *for* $k = 1, ..., b$, *and the set* $S = \{b, 2b, ..., (a-1)b, ab\}$. *Then we have*

$$|I_k \cap S| = \begin{cases} \alpha + 1 & \textit{if } k = i\beta^{-1} \bmod b \textit{ for some integer } 1 \le i \le \beta - 1, \\ \alpha & \textit{otherwise.} \end{cases}$$

Proof: The distance from $(k-1)a$ to the smallest integer in $I_k \cap S$ is $ka \bmod b = k\beta \bmod b$. Thus, we have $|I_k \cap S| = \alpha + 1$ if and only if this distance is positive and less than β. We can write this as $k\beta = i \bmod b$ for some $i = 1 \ldots \beta - 1$ or equivalently, as $k = i\beta^{-1} \bmod b$. $\square$

Let us finally note that, since the characteristic polynomial of M has all coefficients non-positive, except of the leading term z^{b+1}, we could have used a classical result by Cauchy and Ostrovsky (see theorems 1.1.3, 1.1.4 in the textbook Prasolov (2010) [17]) to show that M has a unique eigenvalue r_0, which is a positive real and strictly larger than the absolute value of any other eigenvalue.

1.6.2 Computing the first eigenvector of M

Due to the structure of matrix M, for $r = r_0$, we have the following equalities for $\mathbf{v} = \mathbf{v}(0)$:

$$r\mathbf{v}_{(\beta \bmod b)} = \mathbf{v}_0 + \mathbf{v}_b \tag{1.33a}$$

$$r\mathbf{v}_{(2\beta \bmod b)} = \mathbf{v}_{(\beta \bmod b)} \tag{1.33b}$$

$$\dots$$

$$r\mathbf{v}_{(i\beta \bmod b)} = \mathbf{v}_{((i-1)\beta \bmod b)} \tag{1.33c}$$

$$\dots$$

$$r\mathbf{v}_b = \mathbf{v}_{(b-1)\beta \bmod b} \;=\; \mathbf{v}_{(b-\beta)}. \tag{1.33d}$$

Let us introduce $\mu = \mathbf{v}_{(\beta \bmod b)}$. From (1.33a–1.33d) it follows that

$$\mathbf{v}_{(j\beta \bmod b)} = \frac{\mu}{r^{j-1}} \text{ for all } j = 1, \dots, b-1, \quad \text{and} \quad \mathbf{v}_b = \frac{\mu}{r^{b-1}}. \tag{1.34}$$

Thus, from (1.33a) and (1.34) we get

$$\mathbf{v}_0 = \frac{\mu(r^b - 1)}{r^{b-1}}. \tag{1.35}$$

Adding together (1.33a–1.33d) we get

$$r\sum_{i=1}^{b} \mathbf{v}_i = \sum_{i=0}^{b} \mathbf{v}_i$$

As $\mathbf{e}^T\mathbf{v} = 1$, we get

$$\mathbf{v}_0 = \frac{r-1}{r} \tag{1.36}$$

Finally, combining (1.35) and (1.36) we obtain

$$\mu = \frac{(r-1)r^{b-2}}{r^b - 1} \tag{1.37}$$

Thus, we can write $\mathbf{v}$ as

$$\begin{aligned} \mathbf{v}_0 &= \tfrac{r-1}{r}, \\ \mathbf{v}_{(k\beta \bmod b)} &= \tfrac{(r-1)r^{b-k-1}}{r^b-1}, \qquad \text{for} \quad k = 1, \dots, b-1, \\ \mathbf{v}_b &= \tfrac{r-1}{(r^b-1)r}. \end{aligned} \tag{1.38}$$

Now we can compute $\mathbf{b}^T\mathbf{v}$. Substitute (1.38) into $\mathbf{b}^T\mathbf{v}$. Since the components of $\mathbf{v}$ sum up to 1 we get

$$\mathbf{b}^T\mathbf{v} = b + \sum_{k=1}^{b-1} \left(\frac{r^{b-k-1}(r-1)}{r^b-1} (k\beta \bmod b) \right) + \frac{b(r-1)}{(r^b-1)r}$$

Using that $k\beta \bmod b = ka - b\left\lfloor \frac{ka}{b} \right\rfloor$, we can see this expression as

$$\begin{aligned} & b + \sum_{k=1}^{b-1} \left(\frac{r^{b-k-1}(r-1)}{r^b-1} \left(ka - b \left\lfloor \frac{ka}{b} \right\rfloor \right) \right) + \frac{b(r-1)}{(r^b-1)r} \\ &= b + \frac{r^{b-2}(r-1)}{r^b-1} \cdot \left(a \sum_{i=1}^{b} \frac{i}{r^{i-1}} - b \sum_{i=1}^{a-1} \sum_{j=\lfloor \frac{ib}{a} \rfloor+1}^{b} \frac{1}{r^{j-1}} \right) \end{aligned} \tag{1.39}$$

Since we have the indentities

$$\sum_{i=1}^{b} \frac{i}{r^{i-1}} = \frac{r^2}{(r-1)^2} - \frac{r^3 - br^2(r-1)}{r^{b+1}(r-1)^2} \tag{1.40}$$

and

$$\sum_{i=1}^{a-1} \sum_{j=\lfloor \frac{ib}{a} \rfloor}^{b} \frac{1}{r^{j-1}} = \frac{\sum_{i=1}^{a-1} r^{1-\lfloor \frac{ib}{a} \rfloor}}{r-1} - \frac{r^2(a-1)}{r^{b+1}(r-1)} \tag{1.41}$$

we can rewrite (1.39) as:

$$\begin{aligned} &= b + \left(\frac{ar^2}{(r-1)^2} \left(1 - \frac{(b+1)r - b}{r^{b+1}} \right) - \frac{br^2}{r-1} \left(\sum_{i=1}^{a-1} \frac{1}{r^{\lfloor \frac{ib}{a} \rfloor + 1}} - \frac{a-1}{r^{b+1}} \right) \right) \frac{r^{b-2}(r-1)}{r^b-1} \\ &= \frac{br(r-1)(r^b-1) + a(r^{b+1} - br - r + b) - b(r-1) \left(\sum_{i=1}^{a-1} r^{b-\lfloor \frac{ib}{a} \rfloor} - (a-1) \right)}{r(r-1)(r^b-1)} \end{aligned} \tag{1.42}$$

Since r is a root of the polynomial (1.32b), we have

$$\sum_{i=1}^{a-1} r^{b-\lfloor \frac{ib}{a} \rfloor} = \sum_{i=1}^{a-1} r^{\lceil \frac{ib}{a} \rceil} = r^{b+1} - r - 1.$$

Therefore, (1.42) can be further rewritten as

$$\begin{aligned}
&= \frac{br(r-1)(r^b-1)+a(r^{b+1}-br-r+b)-b(r-1)\left(r^{b+1}-r-1-(a-1)\right)}{r(r-1)(r^b-1)} \\
&= \frac{br^{b+2}-br^{b+1}-br^2+br+ar^{b+1}-abr-ar+ab-br^{b+2}+br^{b+1}+br^2-br+abr-ab}{r(r-1)(r^b-1)} \\
&= \frac{ar^{b+1}-ar}{r(r-1)(r^b-1)} = \frac{a}{r-1}.
\end{aligned}$$

This completes the proof of Theorem 2 □

1.7 Polynomial algorithms solving NIM(a, b)

1.7.1 Normal version

By definition, a game is solved if for any given position v one can decide whether it is a P-position and if it is not, then one can find a P-position v' that can be reached from v in one move. We will show that one can solve game NIM(a, b) for a position (x^*, y^*) using algorithms Compute-X(N), Find-N(X), and Find-N(Y) not more than once each, that is in $O(\log(\max(x^*, y^*))$ operations.

Solve-Game(x^*, y^*): Given non-negative integers $x^* \leq y^*$, find an index N and P-position (x_N, y_N) such that it is either the same as (x^*, y^*) or is reachable in one move.

1: Compute $N =$Find-N$(X = x^*)$. If $x_N \leq x^* - b$, go to step **2**, otherwise compute $y_N = x_N + aN$. If $y_N > y^*$, go to step **3**, otherwise, either $(x_N, y_N) = (x^*, y^*)$ (**P-position**), or (x_N, y_N) can be reached from (x^*, y^*) in one step by decreasing x^* to x_N and y^* to y_N (since $x^* - x_N < b$). In either case we are done.

2: Since there is no x_i in the interval $(x_N, x^*]$ and this interval is longer than b, we know that there must exists a y_M in this interval. We can compute this index by setting $M =$Find-N$(Y = x^*)$, and by the above we must have $y_M > x^* - b$. Then, the position (x_M, y_M) can be reached from (x^*, y^*) by decreasing x^* to y_M (by less than b) and decreasing y^* to x_M, and thus we are done in this case, too.

3: In this case we have $x^* - b < x_N \leq x^*$ and $y^* < y_N = x_N + aN$. Consequently, $N' = \lfloor \frac{y^* - x^*}{a} \rfloor < N$. Let us then compute the corresponding x-value by $x_{N'}$ =Compute-X($N = N'$) and set $y_{N'} = x_{N'} + aN'$. Then, the position $(x_{N'}, y_{N'})$ can be reached from (x^*, y^*) in one step by decreasing x^* to $x_{N'}$ and decreasing y^* by $(x^* - x_{N'}) + [(y^* - x^*) \bmod a]$.

1.7.2 Misère version

For the joint consideration of the normal and misère versions of an impartial game we refer the reader to the books by Berlekamp et. al. (2001–2004) [1] and Conway (1976) [2] Chapter 12; see also the papers by Fraenkel (1984) [5] and Gurvich (2011, 2012) [9, 10].

In the case $a = 1$, the set of P-positions P^N and P^M (of the normal and misère versions, respectively) "almost coincide". More precisely, their symmetric difference consists of only six positions:

$$P^N \setminus P^M = \{(0,0), (b, b+1), (b+1, b)\}, \text{ while } P^M \setminus P^N = \{(0,1), (1,0), (b+1, b+1)\}.$$

This result was obtained by Fraenkel (1984) [5] for $b = 1$ and extended for any positive integer b by Gurvich (2012) [10]. In the book by Belekamp et. al. (2001–2004) [1], such games, in which P^M and P^N differ "just slightly", are called *tame*.

It is not difficult also to verify (see Gurvich (2011) [9]) that

- (i) from any position of $P^M \setminus P^N$ there is a move to $P^N \setminus P^M$;
- (ii) from any non-terminal position of $P^N \setminus P^M$, that is, from $(b, b+1)$ or $(b+1, b)$, there is a move to $P^N \setminus P^M$;
- (iii) from any position $(x, y) \notin P^N \cup P^M$, either both sets P^N and P^M or none of them can be reached in one move.

Thus, for $a = 1$, the algorithm for the normal version of NIM(a, b) constructed in the previous section is applicable to the misère version, as well; see Gurvich (2011) [9] for more details.

For any integer $a > 1$ (and $b \geq 1$) the kernel of the misère version is defined by the recursion

$$\tilde{x}_n = \text{mex}_b(\{\tilde{x}_i, \tilde{y}_i \mid 0 \le i < n\}), \;\; \tilde{y}_n = \tilde{x}_n + an + 1; \;\; n \in \mathbb{Z}_+. \tag{1.43}$$

This formula was proven by Fraenkel (1984) [5] for $b = 1$ and extended to any positive integer b by Gurvich (2012) [10]. Let us notice that (1.2) and (1.43) differ just slightly. In particular, comparing these two formulas we immediately conclude that for any integer $a > 1$ and $b \ge 1$ the sets of P-positions of the normal and misère versions are disjoint, in contrast to the case $a = 1$; see Gurvich (2011) [9] for more details and, in particular, for the cases $a = 0$ or $b = 0$, which are not considered in this paper.

Moreover, all properties described in Section 1.2 hold, except that (1.10) looks as $\mathbf{d}(0,1) = \mathbf{e}^1$. Hence, we only need to modify the algorithms of Section 1.3, replacing e^0 with e^1 in (1.16) in Corollary 3, getting

$$\mathbf{d}(0,k) \;=\; \mathbf{d}(0,1) + \mathbf{d}(1,\sigma(i)) + (k - \sigma(i))\mathbf{e}^0 \;=\; (k - \sigma(i))\mathbf{e}^0 + \mathbf{e}^1 + M\mathbf{d}(0,i), \tag{1.44}$$

and the initial value of x_{ξ_i+1} with $b + 1$ in the **Initialization** step. After these replacements, the algorithm Solve-Game(x^*, y^*) can be applied without any further modifications.

1.8 Conclusions and problems

Two main recursions (1.2) and (1.43) are deterministic, yet, their solutions (the kernels, or equivalently, the P-positions of the normal and misère versions of NIM(a,b)) behave in a "pseudo-chaotic way" when $b > 1$. For which other combinatorial games, their kernels demonstrate such behavior?

It seems that the four parametric game NIM$(a,b;p,q)$, introduced recently by Gurvich (2010) [8], is a good candidate (this game is a generalization of NIM(a,b) from Gurvich (2012) [10] and NIM(a,p) from Larsson (2006, 2009a) [12, 14]; see also Larsson (2009, 2009b) [13, 15]), yet, the class in question might be much larger.

However, both recursions (1.2) and (1.43) can be solved by a polynomial algorithm based on the Perron-Frobenius theorem. Which other recursions can be solved in such a way?

For $b = 1$, the solutions of both recursions are given by closed formulas, while for $b > 1$ this is unlikely.

Cases $a = 1$ and $a > 1$ also differ substantially. In the first case, the symmetric difference of two kernels, $P^N \Delta P^M$, consists of only six positions, while in the second case these two sets are disjoint, $P^N \cap P^M = \emptyset$; see Section 1.7.2. Gurvich (2011) [9] named such two types of games miserable and strongly miserable, respectively, and obtained simple characterizations for both classes.

This Chapter was published (see the second work in Vita / Publications).

Chapter 2

On the existence of Nash equilibria in pure stationary strategies for the n-person positional games with perfect information, no moves of chance, and mean or total effective cost

2.1 Introduction

2.1.1 Main concepts and definitions

We consider the n-person positional games with no moves of chance, with perfect information, and with the mean or total effective cost function.

Such a game is modeled by a directed graph (digraph) $G = (V, E)$ whose vertex-set V is partitioned into $n+1$ subsets, $V = V_1 \cup \ldots \cup V_n \cup V_T$, where vertices V_i are interpreted as the positions controlled by the player $i \in I = \{1, \ldots, n\}$, while V_T is the set of all terminal (of out-degree 0) positions; V_T might be empty. In fact, we can easily make V_T empty by just adding a loop ℓ_v in each $v \in V_T$. A directed edge (arc) $e = (v, v') \in E$ is interpreted as a move of the player i whenever $v \in V_i$. A move is called *terminal* if $v' \in V_T$. Obviously, a terminal move cannot belong to any directed cycle (dicycle).

Given a local cost function $r : I \times E \to \mathbb{R}$, its value $r(i, e)$ is interpreted as an amount that the player $i \in I$ has to pay for the move $e \in e$. Respectively, $-r(i, e)$ is frequently referred to as the local reward or payoff. Let us remark that all players $i \in I$ pay for each move $e \in E$, not only that i who makes this move. Of course, costs may be 0 or negative.

The local cost function r is called *terminal* [28, 25] (or *transition-free* [38, 39]) if $c(i, e) = 0$ unless e is a terminal move.

Remark 5 *The n-person games with terminal local costs are referred to as the* transition-free [38, 39] *or* Chess-like *[20, 24] games. If $n = 2$, these games are called BW games. In this case $V_1 = V_W$ and $V_2 = V_B$ are called the White and Black positions, respectively. Allowing also Random positions, V_R, we would obtain a much more general class of the* Backgammon-like *or* BWR *games; see [50, 25] for more details. It is not difficult to demonstrate [23, 29] that BWR and the classic Gillette [43] mean cost stochastic games are in fact equivalent.*

In this paper, we restrict the players (and ourselves) by their pure stationary strategies. Such a strategy x_i of a player $i \in I$ is a mapping that assigns a move (v, v') to each position $v \in V_i$. A set of n strategies $x = (x_i \mid i \in I)$ is called a *strategy profile* or *situation*.

A situation x and an initial position $v_0 \in V$ uniquely define a directed path (dipath) $p(v_0, x)$ as follows: Position v_0 is controlled by a player $i \in I$ whose strategy x_i defines the first move (v_0, v'); position v' is controlled by a player $i' \in I$ whose strategy $x_{i'}$ defines the second move (v', v''); etc. The obtained dipath $p(v_0, x)$ is called a *play*. By construction, $p(v_0, x)$ begins in v_0 and either (i) terminates in a $v \in V_T$ or (ii) ends in a cycle that is repeated infinitely. Indeed, as soon as the play comes to a position, where it has already been before, it forms a dicycle that will be repeated infinitely, because all considered strategies are stationary. In case (i) we will call the play *terminal* and in case (ii) a *lasso*.

Remark 6 *By adding a loop ℓ_v to each terminal position $v \in V_T$, we reduce (i) to (ii), since every terminal play becomes a lasso too.*

Given a lasso L that consists of a a dicycle C repeated infinitely and an initial dipath P from v_0 to C, let us introduce the *effective costs* $R(i, L)$ as follows:

If L is a terminal play, $C = \ell_v$, let us set $r(i, \ell_v) = 0$ for each player $i \in I$ and terminal $v \in V_T$, and let $R(i, L) = \sum_{e \in P} r(i, e)$ be the sum of all local costs of P.

If C is not a terminal loop and $r(i, C) = \sum_{e \in C} r(i, e) \neq 0$, let us define $R(i, L) = \infty$ when $r(i, C) > 0$ and $R(i, L) = -\infty$ when $r(i, C) < 0$.

Such effective costs were introduced by Tijsman and Vreze [68, 69] and called *total*.

Remark 7 *Let us note that the case $r(i, C) = 0$ is more sophisticated and we will postpone it till Section 2.4.1, assuming that $r(i, C) \neq 0$ for any $i \in I$ and dicycle C in G, in the rest of the paper. In fact, we will construct our main counterexample in Section 2.2.2, assuming that (j) $r(i, e) > 0$ for all $i \in I, e \in E$, and hence, (jj) $r(i, C) > 0$ for any $i \in I$ and C in G too. (Obviously, (j) implies (jj), but in fact, these two conditions are equivalent; see [28] for a proof based on the Gallai potential transformation [41]; see also [54, 50, 25].)*

Let us repeat that loops ℓ_v, which were artificially added to all terminals $v \in V_T$, are treated differently from the dicycles of G; in particular, in the counterexample of Section 2.2.2 we will set $r(i, ell_v) = 0$ for all $i \in I$ and $v \in V_T$.

In Section 2.3 (only), we will consider also the more traditional *mean* or *average effective cost function*, defined by $R(i, L) = |C|^{-1} \sum_{e \in C} r(i, e)$, where $|C|$ is the length of the cycle C; see for example, [67, 43] and also [36, 37, 60, 61, 54, 50, 55].

The positional structure is a triplet (G, D, v_0), where $G = (V, E)$ is a digraph, the mapping $D : V \setminus V_T \to I$ defines a partition that assigns a player to every non-terminal position, and v_0 is a fixed initial position. Respectively, the *game in positional form* is defined by a positional structure and a local cost function $r : I \times E \to \mathbb{R}$.

The corresponding *normal game form* is a mapping $g : X \to \mathcal{L}$, where $X = \prod_{i \in I} X_i$ and X_i is the set of all (pure stationary) strategies of the player $i \in I$, while $\mathcal{L}$ is the set of lassos of the digraph G. Respectively, the *game in normal form* is a pair (g, R), where $R : I \times \mathcal{L} \to \mathbb{R}$ is the total or mean effective cost function defined above.

We will consider only one concept of solution: the classic *Nash equilibrium (NE)* [63, 64] defined for the normal form game (g, R) as follows: A situation $x \in X$ is called a NE if $R(i, g(x)) \leq R(i, g(x'))$ for every player $i \in I$ and each situation x' that may differ from x only in the ith coordinate. In other words, no player i can reduce his/her effective cost by replacing his/her strategy x_i by another strategy x'_i provided all other players keep their strategies $(x_j \mid j \in I \setminus \{i\})$ unchanged. Furthermore, we say that x is a NE in the positional form game (G, D, v_0, r) if it is a NE in the corresponding normal form game (g, R).

A situation x is called a *uniform* (or *subgame perfect*) NE if it is a NE in (G, D, v_0, r) for every choice of the initial position $v_0 \in V \setminus V_T$.

Remark 8 *The term "subgame perfect" is more frequent in the literature (see, for example, [38, 39]) and this is justified in case of the recursive or acyclic games. Yet, in presence of dicycles, none of the two games (G, D, v_0', r) and (G, D, v_0'', r) is a subgame of the other whenever v' and v'' belong to a dicycle. it seems logical to call properties that hold for all possible initial positions $v_0 \in V \setminus V_T$ "uniform" rather than "subgame perfect".*

2.1.2 Main results and open problems

It is not difficult to construct a NE-free n-person game (G, D, v_0, r) with the total cost function and without zero-dicycles (that is, $r(i, C) = \sum_{e \in C} r(i, e) \neq 0$ for all dicycles C of G and players $i \in I$). Such an example with $n = 4$ was given in [28]. in Section 2.2 we will obtain a much simpler example with $n = 3$. In both these examples, digraph G has a unique dicycle C, yet, it is negative, $r(i, C) < 0$ for a player $i \in I$.

It was conjectured in [28] that such examples fail to exist if we assume additionally that all dicycles are positive, that is, (i): $r(i, C) > 0$ for all $i \in I$ and C in G. (As we already mentioned, (i) is equivalent with a seemingly stronger assumption (ii): $r(i, e) > 0$ for all $i \in I$ and $e \in E$.) In [28] this conjecture was proven for the so-called *play-once* games, in which every player controls only one position, that is, $|V_i| = 1$ for all $i \in I$.

Yet, the general the conjecture fails. In Section 2.2.2 we will give a counterexample.

However, it still remains open if a NE always exists in case of the terminal costs, or in other words, whether the n-person Chess-like (transition free) game structures are NS.

In [19, 25], the following two versions of this problem were considered. Obviously, no terminal move can belong to a dicycle. Hence, $r(i, e) = 0$ for every dicycle C of G, its arc $e \in C$, and player $i \in I$. Standardly [28, 19, 25] we assume that in a terminal game all dicycles form a unique outcome c, in addition to the k terminal outcomes $\{a_1, \ldots, a_k\}$. We might also assume that

(A): the outcome c is the worst (most expensive) one for all players $i \in I$, that is, $R(i,c) < r(i,a_j)$ for all $i \in I$ and $j \in [k] = \{1,\dots,k\}$.

The above NS problem was answered in the positive for $k \leq 2$ [28] and for $k \leq 3$ provided (A) holds [32]. Yet in general, with or without assumption (A), NS of the n-person Chess-like (transition free) game structures remains an open problem [19, 25].

It is solved in the negative if we replace a NE by a uniform (subgame perfect) NE. The latter may not exist even under assumption (A). An example for $n = 3$ was given in [28] and it was strengthened to $n = 2$ in [25]. Moreover, it was shown recently in [31] that in both these examples, a uniform NE fails to exist not only in the pure but even in the mixed strategies.

In contrast, for $n = 2$ the above problem is solved in the positive, even without assumption (A) [28, 30]; see the last section of each paper. The proof is based on an old criterion [45, 46] stating that a game form g is NS if (and only if, of course) a NE exists in the game (g,r) for each zero-sum cost function r taking only the values ± 1. A slightly weaker form of this criterion was obtained earlier by Edmonds and Fulkerson [35] and independently in [44]; see also [49, 22, 30] for more details.

Remark 9 *The problem of NS of the two-person games structures in which every dicycle is a separate outcome (rather than all dicycles form one outcome) was considered in [30]. In this case, the NS criterion of [45, 46] implies partial results.*

Thus, the terminal two-person game structures are NS. In contrast, the following two problems related the non-terminal total costs remain open: whether NS holds for the local cost function r such that for all $i \in I$ and C in G (i): $r(i,C) > 0$ or (iii): $r(i,C) \neq 0$. Obviously, assumption (i) (which is equivalent with (ii)) is stronger than (iii).

Yet, NS certainly fails if the equalities $r(i,C) = 0$ are allowed. Moreover, in Sections 2.3 and 2.4.3 we will construct a two-person NE-free game in which $r(i,C) = 0$ for *all* $i \in I$ and C in G. This example is based on an old example of a two-person NE-free *mean* cost game from [47]; see also [48, 50].

In contrast, the zero-sum two-person games with the total cost function are NS. It was first proven by Tijsman and Vreze in [69], see also [65, 66]. An alternative proof based on the well-known approach of discounted approximation was recently suggested in [26, 27].

Given a lasso L that consists of a dipath P and a zero-dicycle C such that $r(i, C) = 0$ for a player $i \in I$, it seems that the total effective cost function $R(i, L)$ (introduced in [69], see Section 2.4.1 for the definition) is the only one that guarantees NS, at least for the zero-sum two-person games. In Section 2.4.2, we will give an example showing that there may be no saddle point in the game if we "naturally" define for such a lasso:

$R'(i, L) = r'(i, P) = \sum_{e \in L} r(i, e) = \sum_{e \in P} r(i, e)$.

The state of art is summarized by the following Table 2.1, the last four lines of which were announced in the Abstract.

Let us underline that in Table 2.1 the cost is not assumed to be zero-sum. For the latter case, a (much more positive) picture is outlined in the next subsection.

Cost	Dicycles	$n = 2$	$n = 3$
mean	positive	No	No
	arbitrary	No	No
total	non-zero	Open	No
	positive	Open	No
transition-free	worst	Yes	Open
	arbitrary	Yes	Open

Table 2.1: Summary of the results on Nash-solvability

2.1.3 Two-person zero-sum games

Every two-person zero-sum BW game has a saddle point in *pure stationary uniformly optimal strategies*. For the the mean effective costs, $R(i, L) = |C|^{-1} \sum_{e \in C} r(i, e)$, this was proven by Moulin [60, 61] for complete bipartite graphs, by Ehrenfeucht and Mycielski [36, 37] for all bipartite graphs, and by Gurvich, Karzanov, and Khachiyan [50] in general.

In contrast, we will show in Section 2.3 that saddle points may fail to exist in case of the additive but non-averaged cost functions $R'(i, L) = \sum_{e \in C} r(i, e)$.

For the BW games with the total effective cost (see Section 2.3) the above existence result was obtained by Tijsman and Vreze [69]. An alternative simpler proof based on the classic discounted approximation was recently given in [26, 27].

Finally, let us note that the existence of a saddle point in *pure stationary uniformly optimal strategies* for the two-person zero-sum mean cost games holds not only for BW but for a more general BWR [50, 22] model, which includes not only Black and White but also Random positions. This can be easily derived from the classic results of Shapley [67], Gillette [43], Liggett and Lippman [58], since Gillette's and BWR mean cost models are in fact equivalent [23, 29]. Yet, for the BWR games with total effective cost there is no proof.

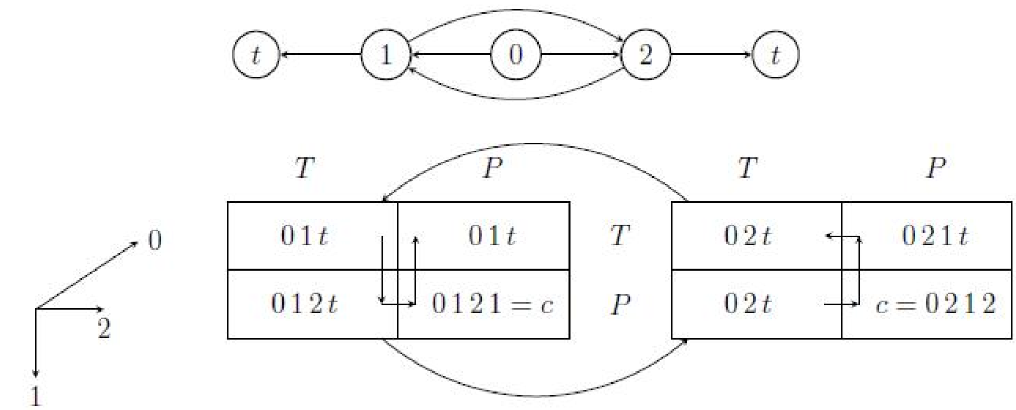
Remark 10 *The case of the terminal (transition free) cost function was considered much earlier. Zermelo [72] was the first who proved solvability of Chess (but in fact, of any two-person Chess-like zero-sum game with perfect information) in pure strategies. Let us notice that already the digraph of Chess has dicycles, since a position can be repeated in the play. However, Zermelo did not restrict the players by their stationary strategies. This was done later, by König [57] and Kalmar [53]; see also [71] and [20] for a recent survey.*

In contrast, for the non-zero-sum case the existence of a NE in pure stationary (but not necessarily uniformly optimal) strategies was proven in [28] only for the two-person Chess-like (transition free) games. Two open problems for $n = 2$ along with several negative results [25, 31] are summarized in the previous section.

2.2 NE-free three-person games with total effective costs

2.2.1 Arbitrary local costs

In the beginning, let us wave the requirement of positivity. Then, a quite simple example can be constructed; see Figure 2.2.1. Players 1 and 2 control respectively positions v_1 and v_2, which form a dicycle of length 2. Each of these two players has two strategies: to terminate (T) or proceed (P). Player 0 controls the initial position v_0 and also has



Three players $I = \{0,1,2\}$ control three non-terminal positions $V \setminus V_T = V \setminus \{v_t\} = \{v_0, v_1, v_2\}$. To save space, we replace the positions just by their indices $0,1,2$ (which denote the corresponding players, too) and t.

Figure 2.1: Not Nash-solvable three-person game form.

two strategies (1) and (2): to move to v_1 or v_2, respectively.

The corresponding normal game form g given by the $2 \times 2 \times 2$ table in Figure 2.2.1. Every its entry is a situation $x \in X$ that defines the play (lasso) $L(x)$, as shown in Figure 2.2.1.

We want to find a local cost function $r : I \times E \to \mathbb{Z}$ such that the obtained game (g, R) is NE-free, where the total effective cost $R(i, L)$ is defined for a given player $i \in I = \{0,1,2\}$ and a lasso $L = L(x)$ as follows: If $L = L(x)$ is a terminal play, that is, a dipath $P = P(x)$ from v_0 to the (unique) terminal position v_t (in which, by convention, we add to P a loop ℓ_{v_t} with $r(i, \ell_{v_t}) = 0$ for all $i \in I$) then $R(i, L) = \sum_{e \in P} r(i, e)$. If $L = L(x)$ is a cyclic play, which ends in a dicycle C, then $R(i, L)$ is ∞ (respectively, $-\infty$) when $r(i, C) = r(i, (v_1, v_2)) + r(i, (v_2, v_1))$ is positive (respectively, negative).

Remark 11 *In the present example, the equality $r(i, C) = 0$ holds for no $i \in I$. In Section 2.4.1, we will recall the definition of the total effective cost for this case and construct an example of a* two-person *NE-free game in which $r(i, C) = 0$ for* all *$i \in I$ and C in G.*

By the definition, a game (g, R) has no NE if and only if for every $x = (x_0, x_1, x_2) \in X$ there is a player $i \in I = \{0,1,2\}$ and a situation $x' = (x'_0, x'_1, x'_2) \in X$ that differs from x only in the coordinate i and such that $R(i, L(x')) < R(i, L(x))$. In Figure 2.2.1, for every $x \in X$ the arrow from the corresponding x' is drawn. Yet, we have to verify that these arrows (strict inequalities) are not contradictory. This should be done for each $i \in I$ separately, since there are no relations between the local costs of different players. It is easily seen that for $i \in I = \{0, 1, 2\}$ we obtain the following three systems:

$$r(0,(v_0,v_2)) + r(0,(v_2,v_t)) < r(0,(v_0,v_1)) + r(0,(v_1,v_t))$$
$$r(0,(v_0,v_1)) + r(0,(v_1,v_2)) < r(0,(v_0,v_2)) + r(0,(v_1,v_t))$$

$$r(1,(v_1,v_t)) < r(1,(v_1,v_2)) + r(1,(v_2,v_t))$$
$$r(1,(v_1,v_2)) + r(1,(v_2,v_1)) < 0$$

$$r(2,(v_2,v_1)) + r(1,(v_1,v_t)) < r(2,(v_2,v_t))$$
$$r(2,(v_1,v_2)) + r(2,(v_2,v_1)) > 0$$

Remark 12 *In this example, the total cost of a cyclic play is either ∞ or $-\infty$. Respectively, in these two cases it is $>$ or $<$ than the total cost of any terminal play, which is finite. Let us also notice that some terms of the corresponding sums are canceled in the above inequalities.*

It is easy to verify that all three systems are feasible. For example, we can set

$\mathbf{r} = (r(i,(v_0,v_1)), r(i,(v_1,v_t)), r(i,(v_0,v_2)), r(i,(v_2,v_t)); r(i,(v_1,v_2)), r(i,(v_2,v_1)))$ to

$(1,4,3,1;1,1), (1,1,1,1;1,-2)$ and $(1,1,1,3;1,1)$ for $i = 0, 1,$ and 2, respectively.

Remark 13 *Let us unite two players 0 and 2 and replace them by the single player 2 getting $I = \{1, 2\}$. It can be verified that the obtained two-person game structure is NS. The corresponding normal 2×4 game form g contains five distinct outcomes (plays): (v_0, v_1, v_t) and (v_0, v_2, v_t) appears twice each, (v_0, v_1, v_2, v_1) and (v_0, v_2, v_1, v_2) form the same outcome c, finally, (v_0, v_1, v_2, v_t) and (v_0, v_2, v_1, v_t) form two separate outcomes. It is easy to verify that the last two outcomes cannot simultaneously be the best responses of the player 2, since the corresponding system of strict linear inequalities is infeasible. It is interesting to notice that the criterion of NS of [45, 46] "a two-person game form* *is NS if and only if it is tight" does not seem to be applicable in this case. Indeed, the considered game form is NS but not tight. The reason is, it is not a game form in the sense of [45, 46], since the outcomes are now the plays and the total payoff being additive cannot take any values, unlike [45, 46]; see [22, 30] for the definition and more details. The above NS criterion might be extendable for such generalized two-person game forms, yet, it should become the subject of a separate research.*

A similar NE-free four-person game was constructed in [28]; see Figure 6 on page 223.

Let us notice that in both these NE-free examples $n > 2$ and there is a dicycle that is negative for at least one player. Whether similar NE-free *two-person* examples (with a positive or arbitrary local cost) exist is still an open problem.

2.2.2 Positive local costs

However, here we will show a computer-generated *three-person* NE-free game (G, D, u_1, r) with positive integer local costs r, thus, disproving the conjecture of [28].

Graph $G = (V, E)$ is given in Figure 2.2; $V = \{u_1, v_1; u_2, v_2; u_3, v_3, t\}$ is its vertex set. For each $i \in I = \{1, 2, 3\}$, the two positions with the subscript i are controlled by the player i. Let us note that this graph contains a unique dicycle $C = (v_1, v_3), (v_3, v_2), (v_2, v_1)$. The initial position is u_1. The integer positive local costs $r(i, e)$ are also given in this figure.

The out-degrees of the vertices are $3, 2; 3, 2; 2, 2$, respectively. Hence, players 1 and 2 have $3 \times 2 = 6$ strategies each, while 3 has only $2 \times 2 = 4$ strategies. The corresponding normal game form and game in normal form are given by the Tables 2.2 and 2.3 respectively. Each of these two tables consists of 4 subtables of the size 6×6 that correspond to the four strategies of the 3d player, while the strategies of the players 1 and 2 correspond to the rows and columns of each subtable, respectively. The strategy of a player $i \in I = \{1, 2, 3\}$ that recommends the moves (u_i, w'_i) and (v_i, w''_i) is denoted by w'_i, w''_i for short, where w'_i, w''_i is a pair of positions of V; see Tables 2.2 and 2.3.

In Table 2.2 each entry is a play, while in Table 2.3 it is the the triplet that defines

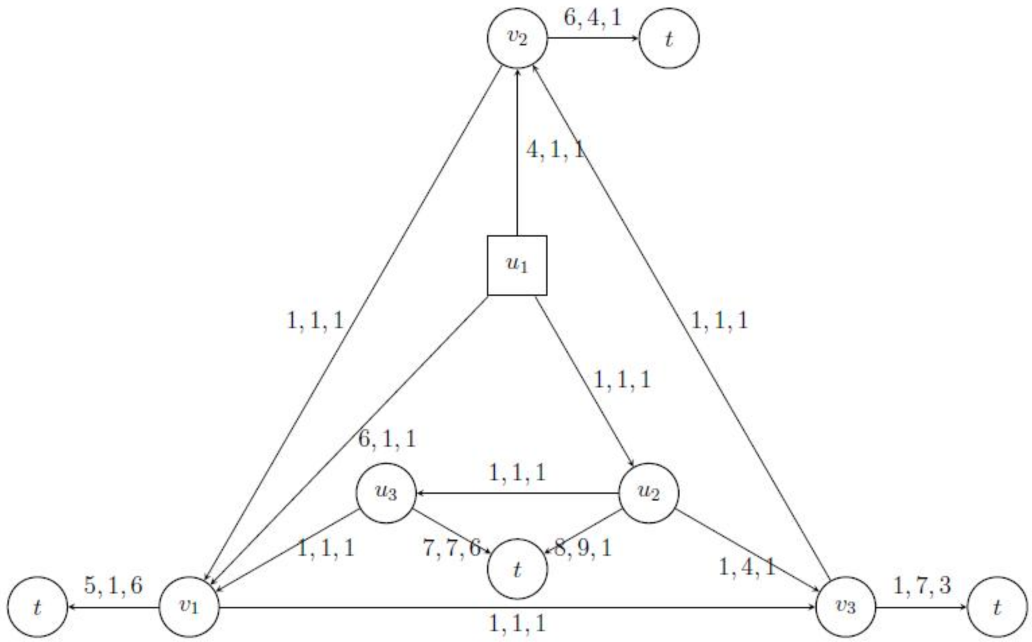


Figure 2.2: A NE-free three-person game with positive integer local costs

the effective total costs of players $1, 2$, and 3. In these tables all cyclic plays, as well as the corresponding total effective costs, (∞, ∞, ∞), are replaced by the word "cycle". In Table 2.3, all minima in each column are given in bold. Hence, by the definition, a NE would be an entry in which all three numbers are bold. Since there exists no such entry, the considered game is NE-free.

2.3 Mean effective costs

Given an infinite play with the sequence of local costs $\mathbf{r} = (r_1, r_2, \ldots,)$ the *mean* effective cost is defined as the Cesaro average $R_M = \frac{1}{k} \lim_{k \to \infty} \sum_{j=1}^{k} r_j$.

In general, the limit may fail to exist but it surely exists when the considered play is a lasso L. In this case, the sequence $\mathbf{r}$ is pseudo-periodical $\mathbf{r} = (r'_1, \ldots, r'_a(r''_1, \ldots, r''_b)^\infty)$ (meaning that the second part, $(r''_1, \ldots, r''_b)$, is repeated infinitely) and $R_M = \frac{1}{b} \sum_{j=1}^{b} r''_j$.

A saddle point (NE, in pure stationary strategies) exists in every two-person zero-sum mean effective cost game [43, 36, 60, 61, 37, 50]; see Introduction for more details.

However, this claim cannot be extended to the two-person but not necessarily zero-sum games. The following BW non-zero-sum game was constructed in [46, 50]; see

3rd player: v_1v_2						1st ↓	3rd player: v_1t					
cycle	$u_1v_1v_3v_2t$	cycle	$u_1v_1v_3v_2t$	cycle	$u_1v_1v_3v_2t$	v_1v_3	$u_1v_1v_3t$	$u_1v_1v_3t$	$u_1v_1v_3t$	$u_1v_1v_3t$	$u_1v_1v_3t$	$u_1v_1v_3t$
u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t	v_1t	u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t
cycle	$u_1u_2u_3v_1v_3v_2t$	cycle	$u_1u_2v_3v_2t$	u_1u_2t	u_1u_2t	u_2v_3	$u_1u_2u_3v_1v_3t$	$u_1u_2u_3v_1v_3t$	$u_1u_2v_3t$	$u_1u_2v_3t$	u_1u_2t	u_1u_2t
$u_1u_2u_3v_1t$	$u_1u_2u_3v_1t$	$u_1u_2v_3v_2v_1t$	$u_1u_2v_3v_2t$	u_1u_2t	u_1u_2t	u_2t	$u_1u_2u_3v_1t$	$u_1u_2u_3v_1t$	$u_1u_2v_3t$	$u_1u_2v_3t$	u_1u_2t	u_1u_2t
cycle	u_1v_2t	cycle	u_1v_2t	cycle	u_1v_2t	v_2v_3	$u_1v_2v_1v_3t$	u_1v_2t	$u_1v_2v_1v_3t$	u_1v_2t	$u_1v_2v_1v_3t$	u_1v_2t
$u_1v_2v_1t$	u_1v_2t	$u_1v_2v_1t$	u_1v_2t	$u_1v_2v_1t$	u_1v_2t	v_2t	$u_1v_2v_1t$	u_1v_2t	$u_1v_2v_1t$	u_1v_2t	$u_1v_2v_1t$	u_1v_2t
u_3v_1	u_3t	v_3v_1	v_3t	tv_1	tt	2nd ↔	u_3v_1	u_3t	v_3v_1	v_3t	tv_1	tt
cycle	$u_1v_1v_3v_2t$	cycle	$u_1v_1v_3v_2t$	cycle	$u_1v_1v_3v_2t$	v_1v_3	$u_1v_1v_3t$	$u_1v_1v_3t$	$u_1v_1v_3t$	$u_1v_1v_3t$	$u_1v_1v_3t$	$u_1v_1v_3t$
u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t	v_1t	u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t	u_1v_1t
$u_1u_2u_3t$	$u_1u_2u_3t$	cycle	$u_1u_2v_3v_2t$	u_1u_2t	u_1u_2t	u_2v_3	$u_1u_2u_3t$	$u_1u_2u_3t$	$u_1u_2v_3t$	$u_1u_2v_3t$	u_1u_2t	u_1u_2t
$u_1u_2u_3t$	$u_1u_2u_3t$	$u_1u_2v_3v_2v_1t$	$u_1u_2v_3v_2t$	u_1u_2t	u_1u_2t	u_2t	$u_1u_2u_3t$	$u_1u_2u_3t$	$u_1u_2v_3t$	$u_1u_2v_3t$	u_1u_2t	u_1u_2t
cycle	u_1v_2t	cycle	u_1v_2t	cycle	u_1v_2t	v_2v_3	$u_1v_2v_1v_3t$	u_1v_2t	$u_1v_2v_1v_3t$	u_1v_2t	$u_1v_2v_1v_3t$	u_1v_2t
$u_1v_2v_1t$	u_1v_2t	$u_1v_2v_1t$	u_1v_2t	$u_1v_2v_1t$	u_1v_2t	v_2t	$u_1v_2v_1t$	u_1v_2t	$u_1v_2v_1t$	u_1v_2t	$u_1v_2v_1t$	u_1v_2t
3rd player: tv_2						1st ↑	3rd player: tt					

Table 2.2: The normal game form corresponding to the game structure of Figure 2.2

3rd player: v_1v_2						1st ↓	3rd player: v_1t					
cycle	14 **7 4**	cycle	14 **7 4**	cycle	14 **7 4**	v_1v_3	8 **9 5**	8 **9** 5	8 **9 5**	8 **9** 5	8 **9 5**	**8 9** 5
11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**	v_1t	11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**
cycle	11 **9 6**	cycle	**9** 10 **4**	**9** 10 **2**	**9** 10 **2**	u_2v_3	**5** 11 **7**	**5** 11 7	**3** 12 **5**	**3** 12 5	9 **10 2**	9 **10 2**
8 4 9	**8 4** 9	**9** 8 10	**9** 10 **4**	**9** 10 **2**	**9** 10 **2**	u_2t	8 **4** 9	8 **4** 9	**3** 12 **5**	**3** 12 5	9 10 **2**	9 10 **2**
cycle	10 **5 2**	cycle	10 **5 2**	cycle	10 **5 2**	v_2v_3	7 10 **6**	10 **5 2**	7 10 **6**	10 **5 2**	**7** 10 **6**	10 **5 2**
10 **3 8**	10 5 **2**	10 **3 8**	10 5 **2**	10 **3 8**	10 5 **2**	v_2t	10 **3 8**	10 5 **2**	10 **3 8**	10 5 **2**	10 **3 8**	10 5 **2**
u_3v_1	u_3t	v_3v_1	v_3t	tv_1	tt	2nd ↔	u_3v_1	u_3t	v_3v_1	v_3t	tv_1	tt
cycle	14 **7 4**	cycle	14 **7 4**	cycle	14 **7 4**	v_1v_3	8 **9 5**	**8 9** 5	8 **9 5**	8 **9 5**	8 **9 5**	**8 9** 5
11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**	v_1t	11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**	11 **2 7**
9 9 8	**9 9** 8	cycle	**9** 10 **4**	**9** 10 **2**	**9** 10 **2**	u_2v_3	9 **9** 8	9 **9** 8	**3** 12 **5**	**3** 12 5	9 10 **2**	9 10 **2**
9 9 **8**	**9** 9 **8**	**9 8** 10	**9** 10 **4**	**9** 10 **2**	**9** 10 **2**	u_2t	9 **9 8**	9 **9 8**	**3** 12 **5**	**3** 12 5	9 10 **2**	9 10 **2**
cycle	10 **5 2**	cycle	10 **5 2**	cycle	10 **5 2**	v_2v_3	**7** 10 **6**	10 **5 2**	7 10 **6**	10 **5 2**	**7** 10 **6**	10 **5 2**
10 **3 8**	10 5 **2**	10 **3 8**	10 5 **2**	10 **3 8**	10 5 **2**	v_2t	10 **3 8**	10 5 **2**	10 **3 8**	10 5 **2**	10 **3 8**	10 5 **2**
3rd player: tv_2						1st ↑	3rd player: tt					

Table 2.3: The corresponding NE-free game in the normal form

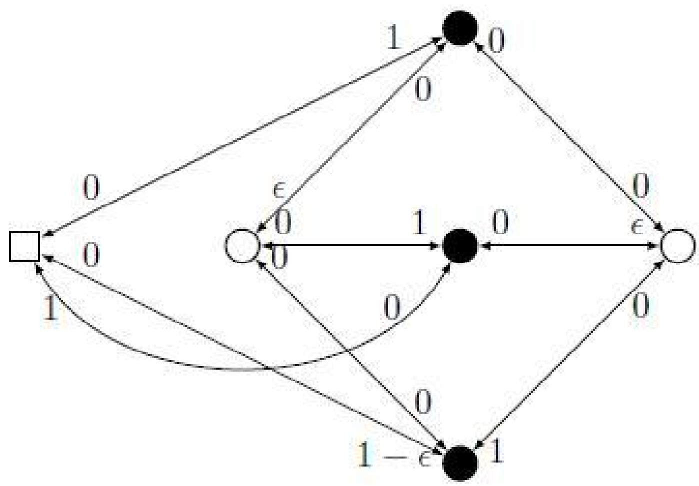


Figure 2.3: A NE-free two-person non-zero-sum mean effective cost game

Figure 2.3. It is defined on the complete bipartite 3×3 digraph $G = (V, E)$, that is, White and Black have three positions each and there is a move from every White to each Black position and vice versa. Every double-arrowed arcs in Figure 2.3 should be replaced by a pair of oppositely directed arcs on which the local costs of each player are equal.

Remark 14 *Such symmetric zero-sum BW games on complete bipartite digraphs were interpreted by Moulin as* ergodic extensions *of matrix games [60, 61]. Since the considered game is not zero-sum, it can be viewed as the ergodic extension of the next* bimatrix *game.*

$$\begin{array}{ccc} 0 & 0 & 1 \\ \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \end{array} \qquad\qquad \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1-\varepsilon & 0 & 1 \end{array}$$

In [48], it was shown that the above example is minimal, since any BW game on a complete bipartite $2 \times k$ digraph has a NE. The proof is based on the criterion of [45, 46].

The obtained game is NE-free for any sufficiently small positive ε, say, $\varepsilon = 0.1$.

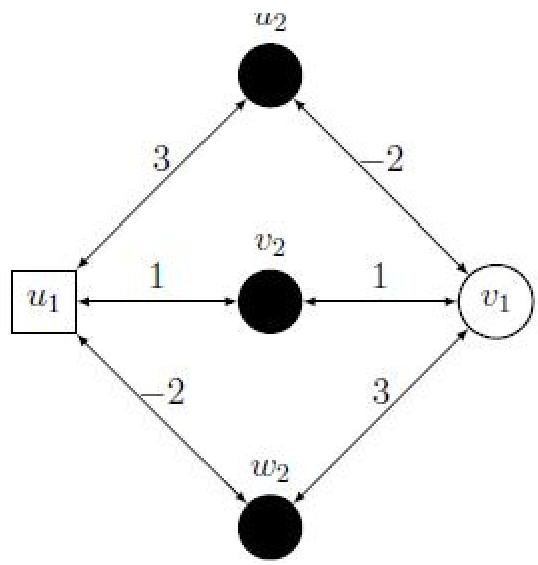


Figure 2.4: A saddle point free zero-sum game with the additive ("pseudo-mean") cost function

One could also try to "simplify" the formula $R_M = \frac{1}{b}\sum_{j=1}^{b} r''_j$ for the mean effective cost of a lasso replacing it by $R'_M = \sum_{j=1}^{b} r''_j$, that is, considering the additive rather than mean effective cost. However, then NS fails already for the two-person zero-sum games. The example is given in Figure 2.3. It represents the ergodic extension of the 2×3 matrix game

$$\begin{array}{ccc} 3 & 1 & -2 \\ -2 & 1 & 3 \end{array}$$

Again, each double-arrowed edge should be replaced by two oppositely directed arcs. White controls 2 positions and has $3^2 = 9$ strategies; Black controls 3 positions and has $2^3 = 8$ strategies; the corresponding 9×8 game form and the matrix game are given by Table 2.4. It is easily seen that this game has no saddle point, since $2 = \max\min < \min\max = 3$. Thus NS fails already in the two-person zero-sum case.

Remark 15 *In fact, the averaged cost function $R(i, L) = |C|^{-1}\sum_{e \in C} r(i, e)$ is in many respects "nicer" than $R'(i, L) = \sum_{e \in C} r(i, e)$. For example, the latter is NP-hard to maximize even for positive local costs, $r(i, e) > 0$, for all $e \in E$ and a given $i \in I$, since this problem generalizes the classic Hamiltonian cycle [42]. In contrast, maximizing*

	$u_2\,u_1$ $v_2\,u_1$ $w_2\,u_1$	$u_2\,u_1$ $v_2\,u_1$ $w_2\,v_1$	$u_2\,u_1$ $v_2\,v_1$ $w_2\,u_1$	$u_2\,u_1$ $v_2\,v_1$ $w_2\,v_1$	$u_2\,v_1$ $v_2\,u_1$ $w_2\,u_1$	$u_2\,v_1$ $v_2\,u_1$ $w_2\,v_1$	$u_2\,v_1$ $v_2\,v_1$ $w_2\,u_1$	$u_2\,v_1$ $v_2\,v_1$ $w_2\,v_1$
$u_1\,u_2$ $v_1\,u_2$	$u_1u_2u_1$ 6	$u_1u_2u_1$ 6	$u_1u_2u_1$ 6	$u_1u_2u_1$ 6	$u_1u_2v_1u_2$ -4	$u_1u_2v_1u_2$ -4	$u_1u_2v_1u_2$ -4	$u_1u_2v_1u_2$ -4
$u_1\,u_2$ $v_1\,v_2$	$u_1u_2u_1$ 6	$u_1u_2u_1$ 6	$u_1u_2u_1$ 6	$u_1u_2u_1$ 6	$u_1u_2v_1v_2u_1$ 3	$u_1u_2v_1v_2u_1$ 3	$u_1u_2v_1v_2v_1$ 2	$u_1u_2v_1v_2v_1$ 2
$u_1\,u_2$ $v_1\,w_2$	$u_1u_2u_1$ 6	$u_1u_2u_1$ 6	$u_1u_2u_1$ 6	$u_1u_2u_1$ 6	$u_1u_2v_1w_2u_1$ 2	$u_1u_2v_1w_2v_1$ 6	$u_1u_2v_1w_2u_1$ 2	$u_1u_2v_1w_2v_1$ 6
$u_1\,v_2$ $v_1\,u_2$	$u_1v_2u_1$ 2	$u_1v_2u_1$ 2	$u_1v_2v_1u_2u_1$ 3	$u_1v_2v_1u_2u_1$ 3	$u_1v_2u_1$ 2	$u_1v_2u_1$ 2	$u_1v_2v_1u_2v_1$ -4	$u_1v_2v_1u_2v_1$ -4
$u_1\,v_2$ $v_1\,v_2$	$u_1v_2u_1$ 2	$u_1v_2u_1$ 2	$u_1v_2v_1v_2$ 2	$u_1v_2v_1v_2$ 2	$u_1v_2u_1$ 2	$u_1v_2u_1$ 2	$u_1v_2v_1v_2$ 2	$u_1v_2v_1v_2$ 2
$u_1\,v_2$ $v_1\,w_2$	$u_1v_2u_1$ 2	$u_1v_2u_1$ 2	$u_1v_2v_1w_2u_1$ 3	$u_1v_2v_1w_2v_1$ 6	$u_1v_2u_1$ 2	$u_1v_2u_1$ 2	$u_1v_2v_1w_2u_1$ 3	$u_1v_2v_1w_2v_1$ 6
$u_1\,w_2$ $v_1\,u_2$	$u_1w_2u_1$ -4	$u_1w_2v_1u_2u_1$ 2	$u_1w_2u_1$ -4	$u_1w_2v_1u_2u_1$ 2	$u_1w_2u_1$ -4	$u_1w_2v_1u_2v_1$ -4	$u_1w_2u_1$ -4	$u_1w_2v_1u_2v_1$ -4
$u_1\,w_2$ $v_1\,v_2$	$u_1w_2u_1$ -4	$u_1w_2v_1v_2u_1$ 3	$u_1w_2u_1$ -4	$u_1w_2v_1v_2v_1$ 2	$u_1w_2u_1$ -4	$u_1w_2v_1v_2u_1$ 3	$u_1w_2u_1$ -4	$u_1w_2v_1v_2v_1$ 2
$u_1\,w_2$ $v_1\,w_2$	$u_1w_2u_1$ -4	$u_1w_2v_1w_2$ 6	$u_1w_2u_1$ -4	$u_1w_2v_1w_2$ 6	$u_1w_2u_1$ -4	$u_1w_2v_1w_2$ 6	$u_1w_2u_1$ -4	$u_1w_2v_1w_2$ 6

Table 2.4: The corresponding normal form; $2 = \max\limits_{col}\min\limits_{row} < \min\limits_{row}\max\limits_{col} = 3$

and minimizing $R(i, L)$ can be easily reduced to LP; In [54], Karp suggested even more efficient procedures.

2.4 Total effective costs

2.4.1 Definitions

Given an infinite play with the sequence of local costs $\mathbf{r} = (r_1, r_2, \ldots,)$ the *total* effective cost is defined as the Cesaro average of the sums $r_1, r_1 + r_2, r_1 + r_2 + r_3, \ldots,$ rather than of the local costs $r_1, r_2, r_3, \ldots$, in other words,

$R_T = \lim_{k\to\infty} \sum_{j=1}^{k} \frac{k-j+1}{k} r_j$; cf. to $R_M = \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} r_j$.

The limit may fail to exist in general but, as we already mentioned, for a lasso L sequence $\mathbf{r}$ is pseudo-periodical, $\mathbf{r} = (r'_1, \ldots, r'_a(r''_1, \ldots, r''_b)^\infty)$. In this case the limit exists and

$$R_T = \sum_{j=1}^{a} r'_j + \sum_{j=1}^{b} \frac{b-j}{b} r''_j. \tag{2.1}$$

In particular, when the play is terminal, the corresponding lasso L ends with the zero-loop ℓ_v for some $v \in V_T$; in this case $b = 1$, $\mathbf{r} = (r'_1, \ldots, r'_a(0)^\infty) = (r'_1, \ldots, r'_a, 0, 0, 0, \ldots)$, and $R_T = r'_1 + \ldots + r'_a = \sum_{j=1}^{a} r'_j$ is just the total cost of the terminal path P of L.

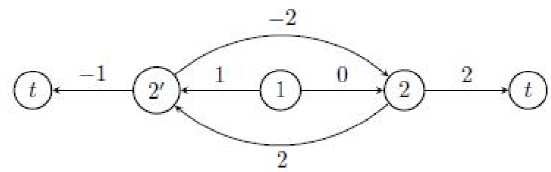


Figure 2.5: A saddle point free zero-sum game with a "pseudo-total" cost function

The same formula holds for any play L that ends in a cycle C in which all local costs are zeros; the terminal zero-loop ℓ_v is just a special case. It is also clear that $R_T(L) = \infty$ (respectively, $-\infty$) when the corresponding cycle C is positive, $\sum_{j=1}^{b} r''_j > 0$ (respectively, negative). Yet, when C is a zero but not identically zero dicycle R_T is defined by (2.1).

For example, the total cost function looks relevant to describe the accumulation of pension contributions. Summation $r_1 + (r_1 + r_2) + \ldots + (r_1 + r_2 + \ldots + r_t) + \ldots$ reflects the fact that the contribution r_i works beginning with the year i. The corresponding play ends with a zero-loop at the year when the individual is retired.

Applications to shortest paths interdiction problems are considered in Section 2.4.4.

2.4.2 A saddle point free game with a pseudo-total costs

In this case, one could try to "simplify" (2.1) replacing it by $R'_T = \sum_{j=1}^{a} r'_j$, since $\sum_{j=1}^{b} r''_j = 0$. In other words, if a lasso L ends in a cycle C such that $\sum_{e \in C} r(i,e) = 0$ for all $i \in I$, it seems logical to define $R'_T(i,L) = \sum_{e \in L} r(i,e) = \sum_{e \in P} r(i,e)$.

However, the following example shows that already a two-person zero-sum game with such a *pseudo-total* cost function may have no saddle points. Such a game is given in Figure 2.5 and Table 2.5 in the positional and normal forms respectively. It is easy to verify that the game has no saddle points, since $0 = \max\min < \min\max = 1$.

In contrast, any two-person zero-sum game with the total cost function defined by (2.1) has a saddle point. First, it was proven in 1998 by Tijsman and Vreze [69]; see also [68].

	$2\,2'$ $2'\,2$	$2\,2'$ $2'\,t$	$2\,t$ $2'\,2$	$2\,t$ $2'\,t$
$1\,2$	$122'2$ 1	$122't$ 1	$12t$ 0	$12t$ 0
$1\,2'$	$12'22'$ 0	$12't$ 2	$12'2t$ 1	$12't$ 2

Table 2.5: The corresponding normal form; $0 = \max_{col}\min_{row} < \min_{row}\max_{col} = 1$

An alternative proof that is based on the well-known approach of the so-called discounted approximation was recently suggested in [26, 27].

2.4.3 Embedding the mean cost games into the total cost ones

Given an infinite play with the sequence of local costs $\mathbf{r} = (r_1, r_2, \ldots,)$ let us replace each local cost r_j by the pair $r_j, -r_j$ getting the sequence $\mathbf{r}' = (r_1, -r_1, r_2, -r_2, \ldots,)$. The partial total sums for the latter are $r_1, 0, r_2, 0, \ldots$. Thus, $2R'_T = R_M$.

Similarly, given an arbitrary mean payoff game $\Gamma = (G, D, v_0, r)$, let subdivide every arc $e \in E$ into e' and e'' and set $r'(i, e') = r(i, e)$, $r'(i, e'') = -r(i, e)$ for all $e \in E$ and $i \in I$. Let us notice that in the obtained game $\Gamma' = (G', D', v_0, r')$ contains only zero-dicycles, $r(i, C) = \sum_{e \in C} r(i, e) = 0$ for all $i \in I$ and C in G'. By the above arguments the total effective cost game Γ' and the mean effective cost game Γ are equivalent. Thus, the mean cost games are embedded into the total cost games that contain only zero-dicycles.

Hence, the former (mean cost) games may be NE-free already for $n = 2$ (see example of Section 2.3 in Figure 2.3), we conclude that the two-person total cost game with only zero-dicycles may have no NE in pure stationary strategies. Yet, as we know, NS becomes an open problem if $r(i, C) \neq 0$ (or even if $r(i, C) > 0$) for all $i \in I = \{1, 2\}$ and C in G'.

2.4.4 Total cost games and the shortest path interdiction problem

The two-person zero-sum total cost games are closely related to the so-called *shortest path interdiction problem* (SPIP) raised by Fulkerson and Harding [40]; see also a short survey by Israely and Wood [52] for more references. The simplest version of SPIP is as follows:

Given a digraph $G = (V, E)$, with weighted arcs $r : E \to \mathbb{R}$, and two vertices $s, t \in V$, eliminate (at most) k arcs of E to maximize the length of a shortest (s, t)-path. This problem is NP-hard; moreover the inapproximability bound $10\sqrt{5} - 21 \approx 1.36$ was derived in [21] (from the same bound for the Minimum Vertex Cover Problem in graphs obtained by Dinur and Safra [34] and improving the previous bound $7/6 \approx 1.17$ given by Håstad [51]).

Unlike the above *total budget* SPIP, the following *node-wise budget* SPIP is more tractable. In this case, we are given a node-wise budget allowing to eliminate (at most) $k(v)$ outgoing arcs from each node $v \in V$.

The case of non-negative weights (local costs) was considered in [56], where an efficient interdiction algorithm was obtained. Given a digraph $G = (V, E)$, a local cost function $r : E \to \mathbb{R}_+$, constraint $k(v)$ in each node $v \in V$, and an initial node s, this algorithm finds in quadratic time an interdiction that maximizes simultaneously the lengths of all shortest paths from s to every node $v \in V$. The execution time is just slightly larger than for the classic Dijkstra shortest path algorithm.

Let us remark that after elimination of the interdicted arcs a dicycle C might be reachable from s. The algorithm of [56] maximizes the total effective cost among all lassos, including the terminal ones, which ends in the artificially added a zero-loop $\ell_v, v \in V_T$.

In case of arbitrary real local costs the SPIP is equivalent (see [56]) with solving the zero-sum mean payoff BW-games. Although the latter problem is known to be in the intersection of NP and co-NP [55], yet, it is not known to be polynomial.

It is also worth noting that the BW mean cost games are a special case of the node-wise interdiction problem corresponding to $k(v) = 0$ for $v \in V_B$ and $k(v) = outdeg(v) - 1$ for $v \in V_W$. Indeed, White, the maximizer, is entitled to choose any move in a position $v \in V_W$ and cannot restrict the choice of Black in any $v \in V_B$.

This Chapter was published (see the first work in Vita / Publications).

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Puplications:

On the existence of Nash equilibria in pure stationary strategies for the n-person positional games with perfect information, no moves of chance, and mean or total effective cost, RUTCOR Research Report (RRR) 26-2012, Rutgers University (With V. Gurvich)

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