LINKING AND DISCRETENESS IN HYPERBOLIC 4-SPACE

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ABSTRACT OF THE DISSERTATION

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A pair of isometries of the 4-dimensional hyperbolic space is called linked if they can be expressed as compositions of two involutions, one of which is common to both isometries. While every pair of isometries of hyperbolic space in dimensions 2 and 3 is linked, not all pairs of isometries of hyperbolic 4-space are linked. One type of such an involution is called half-turn which is an orientation preserving elliptic isometry with a 2-dimensional fixed point set. We provide some geometric conditions for such a pair to be linked by half-turns. Here we develop a theory of pencils, twisting planes and half-turn banks that gives results about each of the pair-types of isometries and their simultaneous factorization. In order to provide conditions under which a given pair is linked via a half-turn, sets of hyperplanes in hyperbolic 4-space are defined for each orientation preserving isometry that enables one to locate the half-turns for which linking is possible. Once a pair is linked, known conditions about discreteness of the group, generated by a pair of isometries, in lower dimensional hyperbolic spaces can be generalized to some linked pairs in dimension 4. If a pair has a common invariant hyperplane or plane, the known conditions such as compact-core-geodesic-intersection and non-separating-disjoint-circles apply.

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Chapter 1

Introduction

A pair of isometries A, B of the hyperbolic space $\mathbb{H}^n$ is said to be linked if there are involutions α, β and γ such that $A = \alpha\beta$ and $B = \beta\gamma$. If furthermore α, β and γ have $(n-2)$-dimensional fixed point set, then the pair is called linked by a half-turn. In dimensions 2 and 3, every pair of isometries are linked. This type of factoring is used to determine the discreteness of the group $\langle A, B\rangle$ using the Gilman-Maskit algorithm in dimension 2, the non-separating-disjoint-circles condition and the compact-core-geodesic-intersection condition in dimension 3. Unfortunately in dimension 4, not all pairs are linked. If a pair has this property, procedures and conditions that are known in lower dimensions work with minor modifications. One of the goals here is to find conditions for which a pair of isometries of $\mathbb{H}^4$ is linked. Furthermore, some of these conditions imply that the group generated by the pair leave a lower dimensional subspace invariant, answering the discreteness question with a known results in lower dimension. The deeper motivation is to determine the discreteness of two-generator subgroup of the isometry group of $\mathbb{H}^4$ by the intersections of axes of non-parabolic palindromes with a fixed plane or line.

The first step in approaching this problem is to break the pair of isometries into compositions of orientation preserving involutions.

In particular, the concept of pencils derived from hyperbolic 3-space or conformal 2-sphere is reformulated to find a space of reflections that factorize an atomic type isometry (type I elliptic, pure hyperbolic or pure parabolic). The other types of isometries can be factored into two half-turns but can be broken canonically into other factors of atomic type. More definitions of pencils (invariant and twisting) are formed in addition to classical ones. The pencils have several indispensable properties that enable one to find a common half-turn factor of certain pairs of isometries.

The pencils contain and locate the planes in which a half-turn factoring is possible. The planes with these properties are bounded by the circles of the half-turn bank of an isometry. There is a geometric criterion for a plane to be located in a half-turn bank related to the fixed point set and other invariants of an isometry, and this criterion along with the properties of the pencils is exploited to state geometric conditions for which a pair of isometries are linked by half-turns. From a far perspective, a plane is in the half-turn bank of an isometry γ if it is orthogonal to the axis, direction and twisting plane of γ whichever are applicable. Equivalently, a plane is in the half-turn bank of γ if it is the intersection of an element in the permuted pencil and and element of the invariant/twisting pencil of γ.

The pencils and hence the half-turn bank of an isometry can be parametrized by real numbers so finding a common element in the half-turn banks of isometries A and B is enough for the pair (A, B) to be linked by a half-turn. Moreover every half-turn factorization of an isometry comes from its half-turn bank. Therefore, the linking of a pair by a half-turn is a matter of studying their half-turn banks.

Theorem 1.0.1. *A pair of orientation preserving isometries of $\mathbb{H}^4$ is linked by half-turns if and only if they have a common element in their half-turn banks.*

The geometric conditions for linking thus reduce to finding a common element in the half-turn banks. The fixed points, axes, directions, invariant planes and pencils help locating the common element that links a pair.

While the main topic of this thesis is linking, we also include a chapter on the enumeration of primitive words in a rank two free group and make some modifications in different well known enumeration schemes. We produce a faster enumeration algorithm for primitive words. We apply results from the modified enumeration schemes to modify the Gilman-Maskit discreteness algorithm.

1.1 Organization

The outline of this thesis is as follows. In Chapter 2 we review necessary information about hyperbolic 4-space including classifications of orientation preserving isometries, reflections, hyperplanes and intersections of varying dimensions of subplanes. The importance of orthogonal subplanes, complements and correspondence to Euclidean space are emphasized.

In Chapter 3 the permuted, invariant, dual and twisting pencils are used to construct the so-called half-turn banks. In the elliptic case, the half-turn is defined and used in other cases. For each case, sufficient conditions are stated for a circle or a plane to be in a half-turn bank. The properties of pencils used in linking and factoring are enumerated. In the isometries with rotational parts, the definition, existence and uniqueness of twisting planes are stated and proved.

The results in Chapter 3 imply some results in [2,16] and [1] but stronger statements are necessary for proving results in Chapter 4. The pencils defined in Chapter 3 can be generalized to higher dimensions but are not stated in a general manner since the chapter is only intended to be a machinery for Chapter 4.

The main theorems about linking are stated in Chapter 4. Like in Chapter 3, the conditions are shown in a case-by-case basis. The pairs considered do not have elliptic elements in them but can have rotational parts. There are a total of ten cases listed but each case may have a few conditions for a pair to be linked. Some conditions imply that the pair has invariant plane or hyperplane where the discreteness question can be passed to lower dimensions with known results.

In Chapter 5, discreteness conditions found in [6] and [8] are extended to hyperbolic 4-space. For pairs that can be linked, the non-separating-disjoint-spheres condition found in [2,16] can be reduced to three spheres containing the elements half-turn banks of the pair of isometries that links it. For the compact-core-geodesic-intersection condition to extend immediately to dimension 4, the pair has to have an invariant lower dimensional plane implied by some of the conditions in Chapter 4. The rest of the conditions need further investigations for answering the discreteness question.

Finally in Chapter 6 we discuss enumeration schemes for primitive words in a rank two free group and apply this very detailed discussion to produce a faster algorithm for enumerating the primitive words and a modification of the Gilman-Maskit discreteness algorithm.

Chapter 2

Expository Background for Hyperbolic Space

In this chapter, the construction of the hyperbolic space of different models are covered. Lines, planes, angles and distances are defined from one model, and the maps among other models carry the geometric structures along the spaces. In section 2.1, the Lorentz space is introduced and how the hyperboloid model is embedded. The Lorentz inner product relate with lengths and dihedral angles in the hyperboloid. Its comparison with the Euclidean inner product shows common properties of Euclidean and hyperbolic geometries. Lorentz orthogonal matrices form a group as a subset of the endomorphisms of Lorentz vector space.

In section 2.2, the inversive geometry of the Euclidean space is extended from reflections across spheres and planes into Möbius maps of the compactified space. The Möbius maps extend to a map of one higher dimension, and the result is called Poincaré extension that serve as an isometry of the upper-half space model of hyperbolic space. A special Möbius map called stereographic projection conjugates the isometries of the upper-half space model into isometries of the conformal ball model.

In section 2.3, the isometries of hyperbolic space are classified using the properties of Möbius transformations of the conformal ball model. In section 2.4, the formulas of some maps among the hyperboloid, upper-half and conformal ball models are shown. In section 2.5 the metric, Riemannian structure, and subplanes are described and compared.

In section 2.6, hyperbolic geometry is restricted to dimensions 2 and 3 to highlight the connection to complex analysis and holomorphic maps. The field structure of the complex plane is utilized to define a different cross-ratio which provides a way to construct the common perpendicular line between ultra-parallel lines.

In section 2.7, isometries of hyperbolic space in dimension 4 are further classified into type I elliptic, type II elliptic, pure parabolic, screw parabolic, pure hyperbolic and pure loxodromic. The details of the special classification is deferred to the next chapter. This chapter serve as background information about hyperbolic geometry.

2.1 The Hyperboloid Model

Consider the *Lorentz space* $\mathbb{R}^{n,1}$ which is the set $\mathbb{R}^{n+1}$ equipped with a pseudo-Riemannian inner product defined by

$$\langle x, y\rangle_L = x_1y_1 + x_2y_2 + \cdots + x_ny_n - x_{n+1}y_{n+1}.$$

From the inner product comes the classification of vectors: time-like; space-like; and light-like. The *norm* $\|x\|_L$ of a vector $x \in \mathbb{R}^{n,1}$ is the principal square root of $\langle x, x\rangle_L$. A *time-like vector* is an element with negative norm squared. The vectors with positive norm are called *space-like*, and those with zero norm are called *light-like*.

The Lorentz space inherits the vector space structure of $\mathbb{R}^{n+1}$ which is finite dimensional. A vector subspace with a time-like vector is called a *time-like subspace*. A subspace whose all nontrivial vectors are space-like is called *space-like*. The rest of the subspaces are called *light-like*. The set of light-like vectors does not form a vector subspace but is called the *light cone*. It separates the set of time-like vectors into two: those with positive $(n+1)$th coordinate; and those with negative $(n+1)$th coordinate. The *hyperbolic space* $\mathbb{H}^n$ embeds in the Lorentz space as the vectors with norm squared equal to -1 but positive $(n+1)$th coordinate. This embedding is called the *hyperboloid model*.

A pair of vectors $x, y \in \mathbb{R}^{n,1}$ are called *Lorentz orthogonal* or in this section, simply orthogonal if $\langle x, y\rangle_L = 0$. The orthogonal complement $\langle x\rangle^L$ of x is the set of vectors orthogonal to x. If V is a subspace of $\mathbb{R}^{n,1}$, then the *Lorentz orthogonal complement* V^L is the set of vectors that are orthogonal to every $x \in V$.

2.1.1 Relating Euclidean and Lorentzian inner products

The Euclidean inner product of $x, y \in \mathbb{R}^{n+1}$ is denoted $\langle x, y\rangle_E$ and defined

$$\langle x, y\rangle_E = x_1y_1 + x_2y_2 + \cdots + x_ny_n + x_{n+1}y_{n+1}$$

where $x = (x_1, x_2, \ldots, x_{n+1})$ and $y = (y_1, y_2, \ldots, y_{n+1})$. It is similar to the Lorentzian inner product in $\mathbb{R}^{n,1}$ but the orthogonal complements are related by the $(n+1)\times(n+1)$ diagonal matrix J with entries $(1, 1, \ldots, 1, -1)$. For any $x, y \in \mathbb{R}^{n+1} = \mathbb{R}^{n,1}$, the inner products are related by

$$\langle x, y\rangle_L = \langle Jx, y\rangle_E = \langle x, Jy\rangle_E. \tag{2.1}$$

Denote the Euclidean orthogonal complement of a vector subspace V of $\mathbb{R}^{n+1}$ with V^E. Then V can be a subspace of both $\mathbb{R}^{n+1}$ and $\mathbb{R}^{n,1}$. The matrix J is a linear isomorphism from any vector subspace V onto JV. Since $J^2 = \mathrm{Id}_{n+1}$, it is also an isomorphism from JV to V. Equation (2.1) implies that $V^L = (JV)^E$.

To show that $J\left(V^E\right) = (JV)^E$, let $x \in (JV)^E$. Then $\langle x, y\rangle_E = 0$ for all $y \in JV$. Let $v_y = Jy \in V$ so $Jv_y = y$. Then $\langle x, y\rangle_E = \langle x, Jv_y\rangle_E = \langle Jx, v_y\rangle_E = 0$ for each $y \in JV$. This shows that $Jx \in J(V^E)$ and so $(JV)^E \subseteq J(V^E)$. Conversely, let $x \in J(V^E)$. Then $x = Jw$ for some $w \in V^E$. By definition, $\langle w, v\rangle_E = 0$ for all $v \in V$. But $w = Jx$ so $\langle Jx, v\rangle_E = \langle x, Jv\rangle_E = 0$ for all $v \in V$. It follows that $x \in (JV)^E$. In conclusion, $V^L = (JV)^E = J(V^E)$.

From these, we have

$$\begin{aligned}\left(V^L\right)^L &= J\big(\left(V^L\right)^E\big) = J\Big(\left(J\left(V^E\right)\right)^E\Big)\\ &= \Big(J\big(J\left(V^E\right)\big)\Big)^E\\ &= \left((JJ)\left(V^E\right)\right)^E\\ &= \left(V^E\right)^E = V.\end{aligned}$$

Now that $\left(V^L\right)^L = V$ and $V^L = J\left(V^E\right)$, the properties of Euclidean orthogonal complements of subspaces of $\mathbb{R}^{n+1}$ are inherited by the Lorentzian orthogonal complements of subspaces of $\mathbb{R}^{n,1}$. In particular, the dimension of V^L is equal to $(n+1) - \dim(V)$.

2.1.2 Orthonormal bases and orthogonal transformations

A basis $\{v_1, v_2, \ldots, v_{n+1}\}$ for $\mathbb{R}^{n,1}$ is called *Lorentz orthonormal* if $\|v_{n+1}\|_L^2 = -1$, $\langle v_i, v_j\rangle_L = 0$ for $i \neq j$ and $\langle v_i, v_i\rangle_L = 1$ for $i = 1, 2, \ldots, n$. The standard basis $\{e_1, e_2, \ldots, e_{n+1}\}$ is an example of a Lorentz orthonormal basis. A subspace V of $\mathbb{R}^{n,1}$ has a basis but if it is space-like, any basis fails to be Lorentz orthonormal as it lacks a time-like vector. If V is time-like, it is possible to construct a Lorentz orthonormal basis and it is a goal of this section.

A linear transformation or endomorphism f of $\mathbb{R}^{n,1}$ is called a *Lorentz transformation* if it preserves the Lorentz inner product. That is, $\langle v, w\rangle_L = \langle f(v), f(w)\rangle_L$ for all pairs $v, w \in \mathbb{R}^{n,1}$. From this definition, a Lorentz transformation maps a Lorentz orthonormal basis into another. Conversely, a linear transformation that maps the standard basis into a Lorentz orthonormal basis is a Lorentz transformation [17, Theorem 3.1.3]. It follows that f is bijective and so the set of Lorentz transformations forms a group under composition called the *Lorentz group*. This group is denoted $\mathrm{O}(n, 1)$.

Using the standard basis, an endomorphism of $\mathbb{R}^{n,1}$ can be expressed as an $(n+1) \times (n+1)$ matrix. Other equivalent definitions of a Lorentz transformation are stated in terms of its matrix form.

Theorem 2.1.1 ([17, Theorem 3.1.4])**.** *Let A be an $(n+1) \times (n+1)$ matrix with real coefficients and J be the $(n+1) \times (n+1)$ diagonal matrix with entries $(1, 1, \ldots, 1, -1)$. Then the following are equivalent.*

1. *As an endomorphism, A is Lorentzian.*
2. *The columns of A form a Lorentz orthonormal basis for $\mathbb{R}^{n,1}$.*
3. *The matrix A satisfies the equation $A^tJA = J$.*
4. *The matrix A satisfies the equation $AJA^t = J$.*
5. *The rows of A form a Lorentz orthonormal basis for $\mathbb{R}^{n,1}$.*

A subgroup of $\mathrm{O}(n, 1)$ that preserves the sign of the $(n+1)$th coordinate of each time-like vector is called *positive Lorentz group* and is denoted $\mathrm{PO}(n, 1)$. Since a Lorentz transformation preserves Lorentz orthonormal bases, a time-like subspace is mapped onto another time-like subspace. Moreover for every pair of time-like subspaces V, W of the same dimension, there is a positive Lorentz transformation that restricts to an isomorphism from V to W.

Theorem 2.1.2 ([17, Theorem 3.1.5]). *For each dimension m, the natural action of $\mathrm{PO}(n,1)$ on the set of m-dimensional time-like subspaces of $\mathbb{R}^{n,1}$ is transitive.*

2.1.3 Orthogonal Complements

By combining subsets of the standard basis, their spans form time-like subspaces of varying dimensions. Define the following subspaces of $\mathbb{R}^{4,1}$.

$$\begin{aligned} V_0 &= \mathrm{span}\{e_5\} & V_{23} &= \mathrm{span}\{e_3, e_4, e_5\} \\ V_1 &= \mathrm{span}\{e_1, e_5\} & V_{31} &= \mathrm{span}\{e_1, e_2, e_3, e_5\} \\ V_{21} &= \mathrm{span}\{e_1, e_2, e_5\} & V_{32} &= \mathrm{span}\{e_2, e_3, e_4, e_5\} \end{aligned}$$

Then their corresponding orthogonal complements are as follows.

$$\begin{aligned} V_0^L &= \mathrm{span}\{e_1, e_2, e_3, e_4\} & V_{23}^L &= \mathrm{span}\{e_1, e_2\} \\ V_1^L &= \mathrm{span}\{e_2, e_3, e_4\} & V_{31}^L &= \mathrm{span}\{e_4\} \\ V_{21}^L &= \mathrm{span}\{e_3, e_4\} & V_{32}^L &= \mathrm{span}\{e_1\} \end{aligned}$$

Notice that the orthogonal complement of a time-like subspace is space-like or trivial. When pairing two time-like subspaces V, W in the examples above, the span of $V^L \cup W^L$ is space-like. Furthermore, $V \cap W$ is still time-like. The following theorem states the more general results.

Theorem 2.1.3 ([17, Theorem 3.2.6]). *Let x and y be linearly independent space-like vectors in $\mathbb{R}^{n,1}$. Then $\mathrm{span}\{x, y\}$ is space-like if and only if $(\mathrm{span}\{x\})^L \cap (\mathrm{span}\{y\})^L$ is a time-like subspace.*

While this theorem deals with n-dimensional subspaces of $\mathbb{R}^{n,1}$, the examples above suggest that a similar statement works for lower and varying dimension subspaces.

The following lemma is immediate from previous theorems but are necessary here and not stated in the references.

Lemma 2.1.1. *Let V be a time-like subspace of $\mathbb{R}^{n,1}$ and $v \in V$ be a time-like vector. Then there is an orthonormal basis $\{v_1, v_2, \ldots, v_k, v_{k+1}\}$ for V where $k+1 = \dim(V)$ and v_{k+1} is linearly dependent with v.*

Proof. The span of $e_1, e_2, \ldots, e_k$ and e_{n+1} is isomorphic to $\mathbb{R}^{k,1}$ in both vector space and Lorentz structures. The group $\mathrm{PO}(k,1)$ embeds into $\mathrm{PO}(n,1)$ by filling an $(n+1) \times (n+1)$ matrix with a $(k+1) \times (k+1)$ matrix from an element of $\mathrm{PO}(k,1)$ to the lower right entries, the upper right entries with I_{n-k} identity matrix, and the rest with zeros. If $\sigma \in \mathrm{PO}(k,1)$, denote its extension to $\mathrm{PO}(n,1)$ with $\hat{\sigma}$.

There is an $f_v \in \mathrm{PO}(n,1)$ that maps V onto $\mathrm{span}\{e_1, e_2, \ldots, e_k, e_{n+1}\}$. The vector $f_v(v)$ must be time-like so there is $\sigma_v \in \mathrm{PO}(k,1)$ that maps $\mathrm{span}\{f_v(v)\}$ to $\mathrm{span}\{e_{n+1}\}$. Then $(\hat{\sigma}_v \circ f_v)^{-1} \in \mathrm{PO}(n,1)$ sends the set $\{e_1, e_2, \ldots, e_k, e_{n+1}\}$ into an orthonormal basis of V, one vector of which is linearly dependent with v. $\square$

The following lemma is necessary here but not stated in the references.

Lemma 2.1.2. *Let P be a time-like subspace of a time-like subspace V of $\mathbb{R}^{n,1}$. Suppose $k = \dim(P)$ and $m = \dim(V)$. Then there is a Lorentz orthonormal basis $\{v_1, v_2, \ldots, v_m\}$ for V such that $\{v_1, v_2, \ldots, v_{k-1}, v_m\}$ is an orthonormal basis for P.*

Proof. Let $\{u_1, u_2, \ldots, u_{k-1}, u_m\}$ be an orthonormal basis of V where $u_m \in P$. Then there is $f \in \mathrm{PO}(n,1)$ that sends P to $\mathrm{span}\{u_1, u_2, \ldots, u_{k-1}, u_m\}$ while leaving V invariant. The existence of f is implied by the proof the lemma above. Let $v_i = f^{-1}(u_i)$ for $i = 1, 2, \ldots, m$. Then $\{v_1, v_2, \ldots, v_m\}$ is an orthonormal basis for V while $\{v_1, v_2, \ldots, v_{k-1}, v_m\}$ is an orthonormal basis for P. $\square$

The following is used in succeeding sections to show that intersecting subplanes of the hyperbolic space behave much like Euclidean subplanes. Specifically the subplanes can intersect orthogonally although the intersections can be of different dimensions.

Theorem 2.1.4. *Let P be a time-like proper vector subspace of another time-like proper subspace V of $\mathbb{R}^{n,1}$. Then there is a unique time-like vector subspace $V^\perp$ containing P such that*

1. $(V^\perp)^L$ *is Lorentz orthogonal to* V^L,
2. $\dim(V^\perp) = n - \dim(V) + \dim(P) + 1$,
3. $\operatorname{span}\left(V^\perp \cup V\right) = \mathbb{R}^{n,1}$ *and*
4. $P = V^\perp \cap V$.

Proof. Let $\{v_1, v_2, \ldots, v_{n+1}\}$ be a Lorentz orthonormal basis of $\mathbb{R}^{n,1}$ where P is the span of $\{v_1, v_2, \ldots, v_{k-1}, v_{n+1}\}$ and $V = \operatorname{span}\{v_1, v_2, \ldots, v_{m-1}, v_{n+1}\}$. Define $V^\perp$ to be the span of $\{v_1, v_2, \ldots, v_{k-1}, v_m, v_{m+1}, \ldots, v_n, v_{n+1}\}$. Then $V^L = \operatorname{span}\{v_m, v_{m+1}, \ldots, v_n\}$ and $(V^\perp)^L = \operatorname{span}\{v_k, v_{k+1}, \ldots, v_{m-1}\}$.

From construction, V^L is Lorentz orthogonal to $(V^\perp)^L$. Since $V^\perp$ has the maximum dimension that contains P, it is unique. $\square$

2.1.4 Metric and subplanes of $\mathbb{H}^n$

The subset $\mathbb{H}^n$ is equipped with a metric induced by the Lorentz inner product. The metric on $\mathbb{H}^n$ is given by $d_{\mathcal{H}^n}(x, y) = \cosh^{-1}\big(-\langle x, y\rangle_L\big)$ for $x, y \in \mathbb{H}^n$. Every isometry of $\mathbb{H}^n$ extends uniquely into an element of $\mathrm{PO}(n, 1)$, and every element of $\mathrm{PO}(n, 1)$ restricts to an isometry of $\mathbb{H}^n$. Hence $\mathrm{Isom}\big(\mathbb{H}^n\big)$ is isomorphic to $\mathrm{PO}(n, 1)$. A *geodesic line* $\mathbb{R} \to \mathbb{H}^n$ is an isometry between $\mathbb{R}$ onto its image. A function $\lambda : \mathbb{R} \to \mathbb{H}^n$ is a geodesic line if and only if there are $x \in \mathbb{H}^n$ and $y \in \mathbb{R}^{n,1}$ with $\langle x, y\rangle_L = 0$ and $\|y\|_L = 1$ such that $\lambda(t) = (\cosh t)x + (\sinh t)y$ for all $t \in \mathbb{R}$ [17, Theorem 3.2.5]. The span of $\{x, y\}$ from the formula is a 2-dimensional time-like vector subspace so geodesics in $\mathbb{H}^n$ are intersections of 2-dimensional time-like subspaces with $\mathbb{H}^n$.

The subplanes of $\mathbb{H}^n$ inside $\mathbb{R}^{n,1}$ are the intersections of time-like subspaces with $\mathbb{H}^n$ [17]. More precisely, $V \cap \mathbb{H}^n$ is an m-dimensional *subplane* of $\mathbb{H}^n$ if and only if V is an $(m + 1)$-dimensional time-like vector subspace of $\mathbb{R}^{n,1}$. A 1-dimensional subplane is called a *hyperbolic line* or simply *line*. A point is also called 0-dimensional subplane.

An $(n-1)$-dimensional subplane is called a *hyperplane*. Using the Lorentz space, the dihedral angles between intersecting hyperplanes can be described without dealing with the geodesics passing through the intersections. Furthermore the minimum distance between hyperplanes and the common perpendicular line can be computed or located.

Translating Theorem 2.1.4 to $\mathbb{H}^n$, it follows that the hyperbolic subplanes passing through a given point behave much like the Euclidean space.

The following corollary is essential for the succeeding chapters. It is not found in references so the statement and proof are provided.

Corollary 2.1.1. *Let h be a proper subplane of $\mathbb{H}^n$ of dimension m and P a proper subplane of h with dimension k. Allow P to be a single point in h which has dimension 0. Then there is a unique subplane $h_P^{\perp}$ (hyperbolic) orthogonal to h through P and of dimension $n-m+k$.*

Proof. Embed $\mathbb{H}^n$ into $\mathbb{R}^{n,1}$ so the image is the hyperboloid model. Then $h = \mathbb{H}^n \cap V$ and $P = \mathbb{H}^n \cap W$ for some time-like subspaces V and W. Since $P \subset h$, then $W \subset V$. The hypothesis shows that $W \subset V \subset \mathbb{R}^{n,1}$ are proper subsets so there is a unique $V_W^{\perp}$ containing W so that $(V_W^{\perp})^L$ and is Lorentz orthogonal to V^L. The Lorentz orthogonal relation between $(V_W^{\perp})^L$ and V^L translates to hyperbolic orthogonal relation between $\mathbb{H}^n \cap V^L$ and $\mathbb{H}^n \cap (V_W^{\perp})^L$. $\square$

2.1.5 Dropping a perpendicular

Let h be an n-dimensional time-like subspace of $\mathbb{R}^{n,1}$. Let v be time-like vector not in h. Then h^L is 1-dimensional space-like subspace of $\mathbb{R}^{n,1}$. Let $P = \operatorname{span}\left(\{v\} \cup h^L\right)$, so the dimension of P is 2. By dimension count, $\operatorname{span}(h \cup P) = \mathbb{R}^{n,1}$ and $h \cap P$ is a 1-dimensional subspace. Moreover, it is interesting to show that $h \cap P$ is time-like. Let x be a nontrivial vector in h^L. Then for each $t \in \mathbb{R}$, the linear combination $tx + v$ is in P.

Since $(\operatorname{span}\{x\})^L = h$, one can solve the equation $\langle tx + v, x\rangle_L = 0$ for t and the resulting solution t_0 makes $t_0 x + v$ a vector in h. The equation yields to a unique solution

$$t_0 = -\frac{\langle v, x\rangle_L}{\langle x, x\rangle_L}.$$

To check that $t_0 x + v$ is time-like, we can compute $\|t_0 x + v\|_L^2$.

$$\begin{aligned}
\|t_0 x + v\|_L^2 &= \langle t_0 x + v, t_0 x + v\rangle_L \\
&= t_0^2 \|x\|_L^2 + 2t_0 \langle v, x\rangle_L + \|v\|_L^2 \\
&= \left(-\frac{\langle v, x\rangle_L}{\langle x, x\rangle_L}\right)^2 \|x\|_L^2 + 2\left(-\frac{\langle v, x\rangle_L}{\langle x, x\rangle_L}\right)\langle v, x\rangle_L + \|v\|_L^2 \\
&= \frac{\langle v, x\rangle_L^2}{\langle x, x\rangle_L} - \frac{2\langle v, x\rangle_L^2}{\langle x, x\rangle_L} + \|v\|_L^2 \\
&= -\frac{\langle v, x\rangle_L^2}{\langle x, x\rangle_L} + \|v\|_L^2.
\end{aligned}$$

Since x is space-like and v is time-like, $\|t_0 x + v\|^2$ is negative so both $t_0 x + v$ and $h \cap P$ are time-like. The angular relation between h and P is likewise an important property of the construction of P. In particular, h and P project to $\mathbb{H}^n$ as orthogonal hyperplane and line.

Theorem 2.1.5 ([17, Theorem 3.2.7])**.** *Let h be an n-dimensional time-like subspace of $\mathbb{R}^{n,1}$ and v a time-like vector not in h. Then there exists a unique 2-dimensional time-like subspace P such that*

1. *P contains v,*

2. *$h \cap P$ is 1-dimensional and time-like, and*

3. *P^L and h^L are Lorentz orthogonal.*

Proof. The construction of P is described above. It is the span of $h^L \cup \{v\}$. What is left to show is that P^L and h^L are Lorentz orthogonal.

By definition, $h^L \subset P$. Let $w \in h^L$. Then $w \in P$. For each $y \in P^L$, the equation $\langle w, y\rangle_L = 0$ follows. Likewise if $w_1 \in P^L$, then $\langle w_1, y_1\rangle_L = 0$ for all $y_1 \in P$ which includes h^L. Hence h^L is Lorentz orthogonal to P^L. $\square$

The following corollary allows dropping a perpendicular line from a point to a disjoint subplane of any dimension. It is immediate from the well-known theorem that drops a perpendicular line from a point to a hyperplane. It is used several times in the next chapters so the statement and proof are provided.

Corollary 2.1.2. *For each subplane H of $\mathbb{H}^n$ and each $p \in \mathbb{H}^n \setminus H$, there is a unique line P passing through p such that P is (hyperbolic) orthogonal to H.*

Proof. Let $k = \dim(H)$. Embed $\mathbb{H}^n$ into $\mathbb{R}^{n,1}$. Then H and p can be extended uniquely into time-like vector subspaces W_H and V_p respectively. The span of $W_H \cup V_p$ is isomorphic to $\mathbb{R}^{k+1,1}$ so there is a unique 2-dimensional subspace Q in span $(W_H \cup V_p)$ that contains V_p such that Q^L is Lorentz orthogonal to W_H^L. Let $P = Q \cap \mathbb{H}^n$. Then P is the unique line orthogonal to H and passing through p. □

If v and w are linearly independent space-like vectors then $\text{span}\{v, w\}$ is time-like if and only if $\text{span}\{v\}^L \cap \text{span}\{w\}^L$ is space-like. The intersections of $\mathbb{H}^n$ with either $\text{span}\{v\}^L$ or $\text{span}\{w\}^L$ is an $(n-1)$-dimensional subplane of $\mathbb{H}^n$ called hyperplane. It follows that $\mathbb{H}^n \cap \text{span}\{v\}^L$ and $\mathbb{H}^n \cap \text{span}\{w\}^L$ are disjoint if $\text{span}\{v, w\}$ is time-like.

Conversely, if $\mathbb{H}^n \cap \text{span}\{v\}^L$ and $\mathbb{H}^n \cap \text{span}\{w\}^L$ are disjoint, then either $\text{span}\{v, w\}$ is time-like or light-like.

Definition 1. *Let P and Q be suplanes of $\mathbb{H}^n$ with possibly different or equal dimensions. Then P and Q are called* ultra-parallel *if* $\text{span}(P) \cap \text{span}(Q) \subset \mathbb{R}^{n,1}$ *is space-like. If* $\text{span}(P) \cap \text{span}(Q)$ *is light-like, then P and Q are said to be* tangent at infinity*.*

Theorem 2.1.6 ([17, Theorem 2.3.7])**.** *If P and Q are ultra-parallel hyperplanes of $\mathbb{H}^n$, then there is a unique line orthogonal to both P and Q.*

2.2 Reflections across spheres and planes

The vector space $\mathbb{R}^n$ can be equipped with the Euclidean inner product structure denoted in this section as dot-product. If $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, the *Euclidean inner product* of x and y is

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

This inner product induces a norm and a distance structures of $\mathbb{R}^n$ given by

$$\|x\| = \sqrt{x \cdot x} \text{ and } d_E(x, y) = \|x - y\|.$$

A sphere $S(a, r)$ where $a \in \mathbb{R}^n$, $r > 0$ is the set of points equidistant to a of length r. If $a \in \mathbb{R}^n$ has $\|a\| = 1$ and $t \in \mathbb{R}$, a Euclidean plane $P(b, t)$ can be uniquely determined by a and t. Specifically, $S(a, r)$ and $P(b, t)$ are defined as follows.

$$S(a, r) = \{x \in \mathbb{R}^n : \|x - a\| = r\}$$
$$P(b, t) = \{x \in \mathbb{R}^n : b \cdot x = t\}$$

Note that a in $S(a, r)$ can be of any norm whereas b in $P(b, t)$ is assumed to be of unit length. From the definition, $P(b, t) = P(-b, -t)$. The *reflections* across $P(b, t)$ and $S(a, r)$ are defined as follows.

$$\sigma(x) = a + \left(\frac{r}{\|x - a\|}\right)^2 (x - a)$$
$$\rho(x) = x + 2(t - b \cdot x)b$$

By computing their compositions, $\sigma\sigma$ and $\rho\rho$ are identity map on $\mathbb{R}^n$. The fixed point set of σ is $S(a, r)$ and that of ρ is $P(b, t)$ [17, Theorem 4.1.3]. These reflections preserve the Euclidean inner product, that is $\rho(x) \cdot \rho(y) = \sigma(x) \cdot \sigma(y) = x \cdot y$ for all $x, y \in \mathbb{R}^n$ not equal to a for σ. They also reverse the orientation of $\mathbb{R}^n$.

By extending $\mathbb{R}^n$ into $\widehat{\mathbb{R}^n}$ defined as $\mathbb{R}^n \sqcup \{\infty\}$, $\widehat{\mathbb{R}^n}$ is a one point compactification of $\mathbb{R}^n$. The reflections σ and ρ also extend to $\widehat{\mathbb{R}^n}$ via $\sigma(a) = \infty$, $\sigma(\infty) = a$ and $\rho(\infty) = \infty$. A hyperplane $P(b, t)$ naturally extends into a topological sphere by adjoining ∞. Henceforth, a reflection in $\widehat{\mathbb{R}^n}$ is considered to be across a sphere, whether it is a Euclidean sphere of an extended Euclidean plane. The composition of a finite number of reflections is called a *Möbius transformation*. The set of Möbius transformations forms a group under composition called the *Möbius group* and is denoted $\text{Möb}\big(\widehat{\mathbb{R}^n}\big)$.

Let e_n be the point in $\mathbb{R}^n$ whose nth coordinate is 1 and other coordinates are 0. Let B_n be the set $\{x \in \mathbb{R}^n : \|x\| < 1\}$ and U_n be the set $\{x \in \mathbb{R}^n : x \cdot e_n > 0\} \cup \{\infty\}$. The reflection across the sphere $S(e_n, \sqrt{2})$ maps the lower half space $-U_n = \{-x : x \in U_n\}$ onto B_n. It can be precomposed with the reflection across the hyperplane

$\mathbb{R}^{n-1} \times \{0\} \cup \{\infty\}$ and the resulting Möbius transformation maps U_n onto B_n. This composition s_π, called *stereographic projection*, restricts to a homeomorphism from $\widehat{\mathbb{R}^{n-1}} = \mathbb{R}^{n-1} \times \{0\} \cup \{\infty\}$ onto $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$, and has an explicit formula

$$s_\pi(x) = \begin{cases} e_n & \text{if } x = \infty\,; \\ \left(\frac{2x_1}{\|x\|^2+1}, \frac{2x_2}{\|x\|^2+1}, \ldots, \frac{2x_{n-1}}{\|x\|^2+1}, \frac{\|x\|^2-1}{\|x\|^2+1}\right) & \text{if } x \neq \infty. \end{cases}$$

The inverse of the restriction has a formula expressed as

$$s_\pi^{-1}(y) = \left(\frac{y_1}{1-y_n}, \frac{y_2}{1-y_n}, \ldots, \frac{y_{n-1}}{1-y_n}, 0\right).$$

A reflection within $\widehat{\mathbb{R}^n}$ can be extended to a reflection in $\widehat{\mathbb{R}^{n+1}}$ using the same formula. The extension leaves the upper half-space U_{n+1} invariant, and hence any Möbius transformation of $\widehat{\mathbb{R}^n}$ extends to a unique Möbius transformation of $\widehat{\mathbb{R}^{n+1}}$ called *Poincaré extension*. It can be shown that a Möbius transformation of $\widehat{\mathbb{R}^{n+1}}$ is a Poincaré extension if and only if it leaves U_{n+1} invariant. The set of Poincaré extensions of $\text{Möb}\big(\widehat{\mathbb{R}^n}\big)$ forms a subgroup of $\text{Möb}\big(\widehat{\mathbb{R}^{n+1}}\big)$ and is denoted $\text{Möb}\big(U_{n+1}\big)$. It follows that $\text{Möb}\big(U_{n+1}\big)$ is isomorphic to $\text{Möb}\big(\widehat{\mathbb{R}^n}\big)$. The stereographic projection therefore conjugates $\text{Möb}\big(U_{n+1}\big)$ into a subgroup of $\text{Möb}\big(\widehat{\mathbb{R}^{n+1}}\big)$ that leaves S^n and B_{n+1} invariant. This subgroup is denoted $\text{Möb}\big(S^n\big)$ and is also isomorphic to $\text{Möb}\big(U_{n+1}\big)$ and $\text{Möb}\big(\widehat{\mathbb{R}^n}\big)$.

2.2.1 Cross Ratio

Let $u, v, x, y \in \widehat{\mathbb{R}^n}$ with $u \neq v$ and $x \neq y$. The *cross ratio* of an ordered quadruple (u, v, x, y) denoted by $[u, v, x, y]$ is defined given by

$$[u, v, x, y] = \frac{\|s_\pi(u) - s_\pi(x)\|\|s_\pi(v) - s_\pi(y)\|}{\|s_\pi(u) - s_\pi(v)\|\|s_\pi(x) - s_\pi(y)\|}.$$

If one of $\{u, v\}$ and one of $\{x, y\}$ are ∞, then the factors of the numerator and denominator equal to ∞ in the definition can be ignored. If one of $\{u, v\}$ but none of $\{x, y\}$ is ∞, the cross ratio is 0. It can be shown that a function from $\widehat{\mathbb{R}^n}$ to $\widehat{\mathbb{R}^n}$ is a Möbius transformation if and only if it preserves cross ratios of all qualified quadruples.

Then a Möbius transformation preserves (extended) spheres of $\widehat{\mathbb{R}^n}$ of any dimension lower than n. Moreover, $\text{Möb}\big(\widehat{\mathbb{R}^n}\big)$ is transitive on the set of k-dimensional spheres of $\widehat{\mathbb{R}^n}$. If a Möbius transformation fixes $(n-1)$-dimensional sphere, it is either a reflection or the identity map.

2.2.2 Boundaries at infinity

The upper-half and conformal ball models are subsets of $\mathbb{R}^n$. They have natural boundaries in $\widehat{\mathbb{R}^n}$ that are homeomorphic to S^{n-1}. Their Möbius groups actions extend to $\widehat{\mathbb{R}^n}$ and leave their respective boundaries invariant. The boundary of U_n is $\mathbb{R}^{n-1}\times\{0\}\cup\{\infty\}$ and that of B_n is S^{n-1}. As models of hyperbolic space, U_n and B_n are compactified by their boundaries called *boundary at infinity*, *visual boundary* or *conformal sphere at infinity*. These boundaries inherit the conformal structure of $\widehat{\mathbb{R}^n}$ and the Möbius groups act on them as conformal bijections.

The hyperboloid model $\mathcal{H}^n$ does not have a boundary within $\mathbb{R}^{n,1}$ but its boundary at infinity correspond to the set of nontrivial light-like vectors of $\mathbb{R}^{n,1}$. Hence a pair of subplanes P and Q of B_n extend to S^{n-1} and they are tangent at infinity if and only if their extensions in S^{n-1} intersect at a unique point. Likewise P and Q are ultra-parallel if and only if their extensions to S^{n-1} have empty intersection. In the succeeding chapters, the boundary at infinity is denoted $\partial\mathbb{H}^n$ or $\widehat{\mathbb{R}^{n-1}}$.

2.3 Classification of Möbius transformations

The homeomorphism property from S^n to $\widehat{\mathbb{R}^n}$ and the Brouwer fixed point theorem [10] force a Möbius transformation of S^n or $\widehat{\mathbb{R}^n}$ to have a fixed point. Therefore the classification of Möbius transformations can be determined by the function's number of fixed points. Let $f \in \text{Möb}\,(S^n)$ so that f restricts to a function from $B_{n+1} \cup S^n$ to $B_{n+1} \cup S^n \subset \widehat{\mathbb{R}^{n+1}}$. Then f is

1. *elliptic* if f fixes a point of B_{n+1};

2. *parabolic* if f fixes a no point of B_{n+1} but fixes a unique point in S^n;

3. *hyperbolic* if f fixes no point of B_{n+1} but fixes two points on S^n.

By extending f to a function $\widehat{\mathbb{R}^{n+1}} \to \widehat{\mathbb{R}^{n+1}}$ and conjugating it within $\text{Möb}\big(\widehat{\mathbb{R}^{n+1}}\big)$, a hyperbolic Möbius transformation f is conjugate to an element of $\text{Möb}\big(\widehat{\mathbb{R}^{n+1}}\big)$ with the form $x \mapsto kAx$ where $k > 1$ and $A \in \mathrm{O}(n+1)$ which has exactly two fixed points, namely the origin and ∞. Hence a hyperbolic transformation has exactly two fixed points.

2.4 Maps between models of hyperbolic space

Let $\mathcal{H}^n$ be the hyperboloid model of the n-dimensional hyperbolic space. That is, $\mathcal{H}^n$ is the subset of $\mathbb{R}^{n,1}$ consisting of vectors with Lorentz norm equal to i, the "imaginary number." A homeomorphism $\zeta : B_n \to \mathcal{H}^n$ is defined given by

$$x \mapsto \left(\frac{2x_1}{1-\|x\|^2}, \frac{2x_2}{1-\|x\|^2}, \ldots, \frac{2x_n}{1-\|x\|^2}, \frac{1+\|x\|^2}{1-\|x\|^2}\right).$$

The inverse of ζ has a formula given by

$$y \mapsto \left(\frac{y_1}{1+y_{n+1}}, \frac{y_2}{1+y_{n+1}}, \ldots, \frac{y_n}{1+y_{n+1}}\right).$$

Recall that the stereographic projection s_π is a homeomorphism from $\widehat{\mathbb{R}^{n-1}}$ onto S^{n-1}. It extends uniquely to a Möbius transformation of $\widehat{\mathbb{R}^n}$ that sends U_n to B_n. Thus the composition ζs_π is a homeomorphism from U_n to $\mathcal{H}^n$. Using these homeomorphisms, U_n and B_n inherit the metric structure of $\mathcal{H}^n$ that is compatible with their inherent conformal structure. Moreover, these homeomorphisms conjugate the groups $\text{Möb}\left(S^{n-1}\right)$ and $\text{Möb}\left(U_n\right)$ into $\mathrm{PO}(n,1)$, the set of positive Lorentz transformations of $\mathbb{R}^{n,1}$.

The set U_n is called the *upper-half space model* and the set B_n is called the *conformal ball model*. There is another model called the Klein disc model but it is covered in a separate document.

2.5 Comparison of models

The conformal maps s_π, ζs_π, ζ among the models of hyperbolic space inherit the metric structure of $\mathcal{H}^n$. The metric on B_n is explicitly given by

$$d_{B_n}(x,y) = \cosh^{-1}\left(1 + \frac{2\|x-y\|_E^2}{\left(1-\|x\|_E^2\right)\left(1-\|y\|_E^2\right)}\right)$$

whereas the metric on on U_n is given by

$$d_{U_n}(x,y) = \cosh^{-1}\left(1 + \frac{2\|x-y\|_E^2}{2x_n y_n}\right).$$

The Riemannian structure on B_n is

$$\frac{dx_1 \otimes dy_1 + dx_2 \otimes dy_2 + \cdots + dx_n \otimes dy_n}{(1 - \langle x, y\rangle_E)}$$

while that of U_n is

$$\frac{dx_1 \otimes dy_1 + dx_2 \otimes dy_2 + \cdots + dx_n \otimes dy_n}{x_n y_n}.$$

The m-dimensional subplanes of $\mathcal{H}^n$ are the intersections of $\mathcal{H}^n$ with $(m+1)$-dimensional time-like subspaces of $\mathbb{R}^{n,1}$. Those of B_n are the intersections of B_n with m-dimensional Euclidean spheres or planes of $\mathbb{R}^n$ orthogonal to S^{n-1}. In the upper-half space model U_n, the m-dimensional subplanes are the intersections of U_n with Euclidean spheres or planes of $\mathbb{R}^n$ orthogonal to $\mathbb{R}^{n-1} \times \{0\}$.

The isometry space of $\mathcal{H}^n$ is $\mathrm{PO}(n,1)$ with a subgroup of orientation preserving maps denoted $\mathrm{PSO}(n,1)$ consisting of matrices of positive determinant. The isometry space of U_n and B_n are $\text{Möb}\big(U_n\big)$ and $\text{Möb}\big(S^{n-1}\big)$ respectively. Their orientation preserving subgroups consist of those compositions of even number of reflections.

2.6 Dimensions 2 and 3

The Euclidean plane modeled with $\mathbb{R}^2$ has a natural identification with the field of complex numbers $\mathbb{C}$. Adjoining ∞ to $\mathbb{C}$ yields to $\widehat{\mathbb{C}}$ called the Riemann sphere. The calculus of complex numbers show that holomorphic functions give rise to conformal functions. In particular, the set of conformal automorphisms of $\widehat{\mathbb{C}}$ can be expressed by fractional linear transformations which are functions of the form

$$z \mapsto \frac{az+b}{cz+d}$$

where $a,b,c,d \in \mathbb{C}$ with $ac - bd \neq 0$ extending to $\widehat{\mathbb{C}}$ via $\infty \mapsto a/c$ and $-d/c \mapsto \infty$. This set, denoted $\mathrm{Aut}\big(\widehat{\mathbb{C}}\big)$, has the same defining properties as the set of orientation preserving subgroup of $\text{Möb}\big(\widehat{\mathbb{R}^2}\big)$, namely cross ratio preservation and unique determination by its action on 3 distinct points. The rest of $\text{Möb}\big(\widehat{\mathbb{R}^2}\big)$ can be identified with the complex maps of the form

$$z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}.$$

The group $\mathrm{Aut}\big(\widehat{\mathbb{C}}\big)$ is homomorphic to a 2×2 matrix multiplication but changing the coefficients a,b,c,d into $-a,-b,-c,-d$ yields to the same function so $\mathrm{Aut}\big(\widehat{\mathbb{C}}\big)$ is isomorphic to $\mathrm{PGL}_2\mathbb{C}$. Hence $\text{Möb}(U_3)$, $\text{Möb}(S^2)$ and $\mathrm{PO}(3,1)$ are isomorphic to the two copies of $\mathrm{PGL}_2\mathbb{C}$.

Restricting the group $\mathrm{Aut}\big(\widehat{\mathbb{C}}\big)$ to those that preserve $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, called the extended real line, defines a subgroup isomorphic to the group of orientation preserving Möbius transformation of U_2 and B_2. An element of $\mathrm{Aut}\big(\widehat{\mathbb{C}}\big)$ of the form $z \mapsto \frac{az+b}{cz+d}$ preserves the extended real line if and only if $a,b,c,d \in \mathbb{R} \subset \mathbb{C}$ so the group of orientation preserving Möbius transformations of U_2 and $\mathrm{PSO}(2,1)$ are isomorphic to $\mathrm{PGL}_2\mathbb{R}$.

The cross ratio in $\widehat{\mathbb{R}^2}$ can be modified to encode the field structure of $\mathbb{C}$, defined as follows. Let $a,b,c,d \in \widehat{\mathbb{C}}$ where no three of them coincide. The *(complex) cross-ratio* of the ordered quadruple (a,b,c,d) is $\not{R}(a,b,c,d)$ defined given by

$$\not{R}(a,b,c,d) = \frac{(c-a)(d-b)}{(c-b)(d-a)}.$$

The case where one or two of them are ∞ is handled similarly as the previous cross ratio.

One advantage of using this complex cross-ratio $\not{R}$ is the possibility of having a negative or non-real value. In particular, having a -1 complex cross-ratio corresponds to perpendicular lines described as follows.

Definition 2 ([3])**.** *Let $a,b,c,d \in \widehat{\mathbb{C}}$. The pair of pairs (a,b), (c,d) is called* harmonic *if either exactly three of $\{a,b,c,d\}$ coincide or a, b, c and d are distinct and $\not{R}(a,b,c,d) =$*

-1.

Theorem 2.6.1 ([3]). *For every two non-ordered pairs of points (a, b) and (c, d) which do not consist of the same points and such that no three of $\{a, b, c, d\}$ coincide, there is a unique pair of non-ordered points (x, y) which is harmonic with both (a, b) and (c, d).*

Proof. The points a, b with $a \neq b$ bound a unique hyperbolic line in U_3. Use the transitivity of $\mathrm{PGL}_2\mathbb{C}$ acting on U_3 on hyperbolic lines to assume that $a = \infty$ and $b = 0$. Let $x = -\sqrt{cd}$ and $y = \sqrt{cd}$. Then,

$$\begin{aligned}
\not{R}(c, d, x, y) &= \frac{(-\sqrt{cd} - c)(\sqrt{cd} - d)}{(-\sqrt{cd} - d)(\sqrt{cd} - c)} \\
&= \frac{(\sqrt{cd} + c)(\sqrt{cd} - d)}{(\sqrt{cd} + d)(\sqrt{cd} - c)} \\
&= \frac{cd - d\sqrt{cd} + c\sqrt{cd} - cd}{cd + d\sqrt{cd} - c\sqrt{cd} - cd} \\
&= \frac{\sqrt{cd}(c - d)}{\sqrt{cd}(d - c)} = -1
\end{aligned}$$

$$\begin{aligned}
\not{R}(a, b, x, y) &= \frac{(x - \infty)(y - 0)}{(x - 0)(y - \infty)} \\
&= \frac{y}{x} \\
&= \frac{\sqrt{cd}}{-\sqrt{cd}} \\
&= -1
\end{aligned}$$

□

The theorem can be stated in a more geometric terms: Given two ultra-parallel lines in $\mathbb{H}^3$, there is a unique line perpendicular to both of them. The points $a, b, c, d \in \mathbb{C}$ serve as boundary points of the lines joining a to b and c to d. The line bounded by x and y is their common perpendicular. By alternately switching between upper-half and hyperboloid models, the theorem can be generalized into higher dimensions.

The following corollary is well-known in hyperbolic geometry but the proofs are not stated in the references.

Corollary 2.6.1. *For every two ultra-parallel lines in $\mathbb{H}^n$, there is a unique line orthogonal to both of them.*

Proof. Let ℓ_1 and ℓ_2 be ultra-parallel lines in $\mathbb{H}^n$. Using the hyperboloid model $\mathcal{H}^n$, ℓ_1 and ℓ_2 span a vector space V in $\mathbb{R}^{n,1}$. If V is 3-dimensional, then ℓ_1 and ℓ_2 are hyperplanes of V so there is a unique line in $\mathcal{H}^n \cap V$ perpendicular to both of them.

If V is 4-dimensional, then $\mathcal{H}^n \cap V$ is isometric to $\mathbb{H}^3$ that can be modeled with the upper-half space U_3. Then ℓ_1 and ℓ_2 are bounded by 4 distinct points $a, b, c, d \in \widehat{\mathbb{C}}$. Suppose $\partial \ell_1 = \{a, b\}$ and $\ell_2 = \{c, d\}$. Then there is a unique line N perpendicular to both ℓ_1 and ℓ_2. Any other line connecting a point from ℓ_1 to ℓ_2 must be a subset of $\mathcal{H}^n \cap V$ so N is the unique in all $\mathcal{H}^n$. □

2.7 Subclassifications in dimension 4

The Möbius transformations acting as isometries of hyperbolic space are classified into elliptic, parabolic and hyperbolic in dimensions 2 or higher. An elliptic isometry has a set of fixed points with a dimension that can range from 0 to n. The dimension of the fixed point set is one way to classify elliptic elements further. In dimension 4, an orientation preserving elliptic isometry has either one point, a plane or the whole $\mathbb{H}^4$ of fixed point set. If it fixes a plane pointwise, it is called type I elliptic isometry or Möbius transformation. If it fixes only one point, it is called type II elliptic.

A parabolic Möbius transformation, acting on $\widehat{\mathbb{R}^3}$ via $x \mapsto Ax+b$ where $A \in \mathrm{O}(3)$ and $b \in \mathbb{R}^3 \setminus \{0\}$, either has $A = \mathrm{Id}_3$ or A nontrivial. Any orientation preserving parabolic isometry is conjugate to this form so orientation preserving parabolic isometries can be classified further into whether A is trivial or not. If A is trivial, the parabolic isometry is called pure parabolic or screw parabolic if A is nontrivial.

A hyperbolic Möbius transformation acting on $\widehat{\mathbb{R}^3}$ is conjugate to another that is of the form $x \mapsto \lambda A x$ where $1 \neq \lambda > 0$ and $A \in \mathrm{O}(3)$. Thus an orientation preserving hyperbolic isometry is classified into pure hyperbolic if A is trivial or pure loxodromic if A nontrivial.

More information about subclassification of isometries of $\mathbb{H}^4$ are provided in Chapter 3. The classes are listed as follows.

1. elliptic (see section 3.1)
   (a) type I elliptic
   (b) type II elliptic (see section 3.1.2)

The type II elliptic is further classified into two: those with unique pair of invariant planes; and involutions.

   (c) identity map

2. parabolic

   (a) pure parabolic or parabolic translation (see section 3.3)

   (b) screw parabolic (see section 3.5)

3. hyperbolic

   (a) pure hyperbolic or hyperbolic translation (see section 3.2)

   (b) pure loxodromic or loxodromic transformation (see section 3.4)

Chapter 3

Pencils and Half-Turn Banks

In this chapter, pencils, half-turn, half-turn banks and twisting planes are defined. It is shown that isometries that can be expressed as a composition of two reflections have an invariant pencil; those with twisting planes have a twisting pencil instead. All types have half-turn banks. Hyperbolic, parabolic and type I elliptic types have permuted pencils. Detailed definitions are given for each type of isometry. Theorems about half-turn factoring are exactly the same for all types, but their proofs, while similar to each other, differ in details.

The pencils have a few properties that can be used for constructing half-turn banks, factoring isometries and linking pairs of them. These properties are also utilized for constructing and proving the uniqueness of a twisting plane which is a feature of dimension 4. The pencils and half-turn banks saturate the hyperbolic space and its boundary, reducing the difficulty in finding a common orthogonal plane essential for linking. Sufficient conditions for a plane or circle to be in a half-turn bank are stated for each type of isometry.

With the exception of type-II elliptic isometry, the model of $\mathbb{H}^4$ used in this chapter is the upper-half space embedded in the ambient space $\mathbb{R}^4$. The boundary at infinity is therefore $\mathbb{R}^3 \times \{0\} \cup \{\infty\}$ but denoted simply with $\widehat{\mathbb{R}^3}$.

Theorems 3.1.4, 3.2.5, 3.3.4, 3.4.3, and 3.5.3 serve as geometric classification for a plane to be involved in a half-turn factoring. In dimension 3, the analog of these planes are the lines perpendicular to the axis, which is extended by Fenchel [3] to allow "improper lines." In dimension 4, these planes are also described as intersections of special hyperplanes. Theorems 3.1.5, 3.1.4, 3.2.6, 3.3.5, 3.4.4 and 3.5.4 are statements with the same conclusion that differ only on the isometry types or classes. They allow linking of pairs of isometries to be unordered in cases where the involutions are half-turns.

In section 3.6, the half-turn bank is proven to be the only source of half-turn factorization of an isometry. The conclusion is that two isometries of $\mathbb{H}^4$ have a common half-turn factor if and only if their half-turn banks have an element in common.

3.1 Elliptic

Let h and h' be hyperplanes in $\mathbb{H}^4$. If they intersect in a point they automatically intersect in a plane P. Denote the reflections across h and h' by R_h and $R_{h'}$ respectively. The compositions $R_h R_{h'}$ and $R_{h'} R_h$ fix P pointwise so they are elliptic isometries of $\mathbb{H}^4$. Suppose further that h and h' are orthogonal. Then both $R_{h'}(h)$ and h are orthogonal to h' through P since $R_{h'}$ is conformal. There is only one hyperplane orthogonal to h' through P so $R_{h'}(h) = h$. Conjugating R_h with $R_{h'}$ yield to another reflection $R_{h'} R_h R_{h'}$. Both R_h and $R_{h'}$ are involutions so for each $x \in h$, it follows that $R_{h'}(x) \in h$ and $R_{h'} R_h R_{h'}(x)= R_{h'} R_{h'}(x) = x$. Hence $R_{h'} R_h R_{h'}$ is the reflection across h. It follows that $R_{h'}$ and R_h commute and their composition is both elliptic and an involution. There is only one elliptic isometry fixing P and of angle π so $R_h R_{h'}$ is the unique elliptic involution that fixes P.

Definition 3. *Let P be a plane in $\mathbb{H}^4$. The* half-turn about P *is the elliptic isometry H_P that is a composition of reflections across an orthogonal pair of hyperplanes that intersect in P.*

If h and h' do not meet orthogonally, then R_h and $R_{h'}$ do not commute. However if g is another hyperplane that contains $h \cap h'$, it can be shown that $R_g R_h R_{h'}$ and $R_h R_{h'} R_g$ are reflections across hyperplanes that also contain $h \cap h'$. To see this fact, consider first the hyperplanes bounded by Euclidean planes $P, Q \subset \mathbb{R}^3 \times \{0\} \subset \partial\mathbb{H}^4$ such that the origin is both in P and Q. Let $a, b \in \mathbb{R}^3$ be unit vectors normal to P and Q respectively. Suppose $P \neq Q$ so that $a \neq b$. Then the span of a and b is a Euclidean plane orthogonal to both P and Q. Hence R_P, R_Q and $R_P R_Q$ leave span$\{a, b\}$ invariant. If P' is another Euclidean plane containing $P \cap Q$, then $R_{P'}$, $R_{P'} R_P R_Q$ and $R_P R_Q R_{P'}$

also leave span$\{a, b\}$ invariant. The action of reflections across planes containing $P \cap Q$ is therefore completely determined by reflections across Euclidean lines in span$\{a, b\}$. As a result, the stabilizer of a point of S^2 in O(3) is isomorphic to O(2).

Consider $\mathbb{R}^2$ and the reflections across Euclidean lines through the origin. The formula for reflections in $\mathbb{R}^n$ still works so if a is a unit vector normal to the line P, then

$$\begin{aligned} R_P(x) &= x + 2(0 - a \cdot x)a \\ &= x - 2(a \cdot x)a. \end{aligned}$$

By identifying $\mathbb{R}^2$ with $\mathbb{C}$ which has multiplication and complex conjugation, the formula for R_P can be simpler.

$$R_P(x) = -a^2\bar{x}$$

Let Q be another Euclidean line through $(0, 0)$ in $\mathbb{R}^2$. If b is a unit vector normal to Q, then $R_P R_Q$ can be expressed as

$$R_P R_Q = a^2\bar{b}^2 x.$$

Let P' be another Euclidean line through $(0, 0)$ which has a unit normal vector c. Then,

$$\begin{aligned} R_{P'} R_P R_Q &= -c^2\bar{a}^2 b^2\bar{x} \\ &= -(c\bar{a}b)^2\bar{x} \\ R_P R_Q R_{P'} &= a^2\bar{b}^2(-c^2\bar{x}) \\ &= -(a\bar{b}c)^2\bar{x}. \end{aligned}$$

Since $c\bar{a}b$ and $a\bar{b}c$ are both unit vectors, $R_{P'} R_P R_Q$ and $R_P R_Q R_{P'}$ are reflections across the orthogonal complements of $c\bar{a}b$ and $a\bar{b}c$ respectively. Thus the composition of an odd number of reflections across lines that meet at one point is another reflection across another line that passes through the same point. By extension to $\mathbb{R}^3$, it is true for reflections across planes meeting in a line. It can be extended further to reflections in $\mathbb{H}^4$ except that reflections across spheres need to be investigated. However one does not worry about them as any circle in $\widehat{\mathbb{R}^n}$ can be mapped by a Möbius transformation into a Euclidean line passing through the origin, and the following theorem can be used.

Theorem 3.1.1 ([17, Theorem 4.7.1])**.** *A Möbius transformation ϕ of $\widehat{\mathbb{R}^n}$ is elliptic if and only if ϕ is conjugate to an orthogonal transformation of the Euclidean space $\mathbb{R}^n$.*

Here, this theorem is used when $n = 4$. A Möbius transformation of $\widehat{\mathbb{R}^4}$ extends to an isometry of $\mathbb{H}^5$ but the main idea is that the stabilizer of a point in $\mathbb{H}^4$ is isomorphic to $\mathrm{O}(4)$.

Corollary 3.1.1. *The composition of an odd number of reflections across hyperplanes in $\mathbb{H}^4$ intersecting in a common plane P is also a reflection across a hyperplane that contain P.*

Definition 4. *A* type I elliptic isometry of $\mathbb{H}^4$ *is a composition of two reflections across distinct hyperplanes that intersect in a plane.*

Let $\rho = R_h R_{h'}$ for some distinct hyperplanes $h, h' \in \mathbb{H}^4$ that intersect in a plane P. Then P is fixed pointwise by ρ. Since $h \neq h'$, ρ does not fix any point outside of P. Thus the fixed point set of ρ forms a plane.

Definition 5. *Let ρ be a type I elliptic isometry of $\mathbb{H}^4$. The* twisting plane of ρ *is the set of fixed points of ρ in $\mathbb{H}^4$.*

Remark 3.1.1. *In* [2,16]*, it is called* axial plane*. However axial planes are defined only for a type I elliptic isometry. Here, twisting planes are defined also for pure loxodromic and screw parabolic.*

Definition 6. *Let ρ be a type I elliptic isometry of $\mathbb{H}^4$ with a twisting plane P. The (elliptic)* permuted pencil of ρ *is the set*

$$\mathcal{F}_\rho = \left\{\partial h \subset \partial\mathbb{H}^4 : h \textit{ is a hyperplane containing } P\right\}.$$

The elements of $\mathcal{F}_\rho$ fill up $\partial\mathbb{H}^4$ and the hyperplanes they bound fill up $\mathbb{H}^4$.

Theorem 3.1.2. *Let ρ be a type I elliptic isometry of $\mathbb{H}^4$ with twisting plane P. Then for each $x \in \left(\partial\mathbb{H}^4\right) \setminus (\partial P)$, there is a unique $t_x \in \mathcal{F}_\rho$ such that $x \in t_x$.*

Proof. There is a Möbius transformation g of $\widehat{\mathbb{R}^4}$ so that $g\rho g^{-1} \in \mathrm{O}(4)$. Since ρ is orientation preserving, $g\rho g^{-1} \in \mathrm{SO}(4)$. By precomposition, we may assume that g

sends P to a plane bounded by ∞ and 0. Then $g\rho g^{-1} \in \mathrm{SO}(3)$ that fixes $g(\partial P)$ pointwise. Since $x \notin \partial P$, $g(x) \notin g(\partial P)$ there is a unique Euclidean plane P_x spanned by $g(x)$ and $g(\partial P)$. Then $g^{-1}(P_x)$ contains ∂P so $g^{-1}(P_x) \in \mathcal{F}_\rho$. $\square$

Corollary 3.1.2. *Let ρ be a type I elliptic isometry of $\mathbb{H}^4$ with a twisting plane P. Then for each $x \in \mathbb{H}^4 \setminus P$ there is a unique hyperplane h_x containing x such that $\partial h_x \in \mathcal{F}_\rho$*

Proof. There is a line ℓ connecting x and $H_P(x)$. It is orthogonal to P through a point since H_P leaves ℓ invariant. Then $P \cup \ell$ spans a unique hyperplane h_x. Since $P \subset h_x$, $\partial h_x \in \mathcal{F}_\rho$. $\square$

Definition 7. *Let ρ be a type I elliptic isometry of $\mathbb{H}^4$ with a twisting plane P. The (elliptic)* invariant pencil of *ρ is the set*

$$\mathcal{T}_\rho = \left\{\partial h \subset \partial\mathbb{H}^4 : h \text{ is a hyperplane orthogonal to } P\right\}.$$

Definition 8. *Let ρ be a type I elliptic isometry of $\mathbb{H}^4$ with a twisting plane P. The* dual pencil of *ρ is the set*

$$\begin{aligned}\mathcal{D}_\rho = \{\partial Q \subset \partial\mathbb{H}^4 : \ & Q \text{ is a plane orthogonally intersecting} \\ & P \text{ in a unique point}\}.\end{aligned}$$

The set $\mathcal{F}_\rho$ serves as a source for reflection factoring of ρ. If s is a sphere in $\partial\mathbb{H}^4$, let R_s denote the reflection across the hyperplane bounded by s. Likewise, if c is a circle in $\partial\mathbb{H}^4$, H_c denotes the half-turn about the plane bounded by c.

Definition 9. *Let s be a sphere in $\partial\mathbb{H}^4$. The reflection across the hyperplane bounded by s is denoted R_s. If c is a circle in $\partial\mathbb{H}^4$, the half-turn about the plane bounded by c is denoted H_c.*

Theorem 3.1.3. *Let ρ be a type I elliptic isometry of $\mathbb{H}^4$. Then for every $h \in \mathcal{F}_\rho$, there exist $h_1, h_2 \in \mathcal{F}_\rho$ such that $\rho = R_{h_1}R_h$ and $\rho = R_hR_{h_2}$.*

Proof. If ρ is type I elliptic, there are hyperplanes P and P' such that $\rho = R_PR_{P'}$. Then $P \cap P'$ must be the twisting plane of ρ since otherwise ρ would fix two different planes.

It follows that $\rho R_h = R_P R_{P'} R_h$ and $R_h \rho = R_h R_P R_{P'}$ are both reflections across some hyperplanes h_1 and h_2 respectively such that $P \cap P' \subset h, h'$. Then $\partial h_1, \partial h_2 \in \mathcal{F}_\rho$ and

$$\rho R_h = R_{h_1} \qquad\qquad R_h \rho = R_{h_2}.$$

They imply that

$$\rho = R_{h_1} R_h \qquad\qquad \rho = R_h R_{h_2}.$$

$\square$

3.1.1 Properties of the invariant pencil

Let ρ be a type I elliptic isometry.

1. For each $t \in \mathcal{T}_\rho$, $\rho(t) = t$ and t is orthogonal to each $s \in \mathcal{F}_\rho$.
2. If $t, u \in \mathcal{T}_\rho$ such that $u \cap t$ has more than one point but $u \neq t$, then $u \cap t \in \mathcal{D}_\rho$.
3. If a sphere $s \subset \widehat{\mathbb{R}^3}$ contains some $c \in \mathcal{D}_\rho$, then $s \in \mathcal{T}_\rho$.
4. For each $d \in \mathcal{D}_\rho$, there are $t, u \in \mathcal{T}_\rho$ such that $d = t \cap u$.
5. For each pair $p, q \in \mathcal{D}_\rho$ with $p \neq q$, there is a unique $t \in \mathcal{T}_\rho$ containing $p \cup q$.

Definition 10. *Let ρ be a type I elliptic isometry of $\mathbb{H}^4$. The* half-turn bank of ρ *is the set*

$$\mathcal{K}_\rho = \left\{ s \cap t \subset \partial\mathbb{H}^4 : s \in \mathcal{F}_\rho \text{ and } t \in \mathcal{T}_\rho \right\}.$$

The half-turn bank of the identity map of $\mathbb{H}^4$ *is the set $\mathcal{K}_{\mathrm{id}}$ consisting of all 2-planes in $\mathbb{H}^4$.*

Recall that if two planes P and Q intersect in a line ℓ, $P \cup Q$ form a unique hyperplane h in $\mathbb{H}^4$. If for each $x \in \ell$ the line $\ell_x^P \subset P$ perpendicular to ℓ through x and the line $\ell_x^Q \subset Q$ perpendicular to ℓ through x are perpendicular (i.e. $\ell_x^P \perp \ell$, $\ell_x^Q \perp \ell$ and $\ell_x^P \perp \ell_x^Q$), then P and Q are called orthogonal through ℓ. Alternatively, if $(\mathrm{span} P)^L$ and $(\mathrm{span} Q)^L$ are Lorentz orthogonal in h, then P and Q are orthogonal in $\mathbb{H}^4$.

Theorem 3.1.4. *Let ρ be a type I elliptic isometry of $\mathbb{H}^4$ with twisting plane P. Then Q is a plane orthogonal to P through a line if and only if $\partial Q \in \mathcal{K}_\rho$.*

Proof. If Q intersects P in a line ℓ, $P \cup Q$ forms a unique hyperplane h. There is a hyperplane t orthogonal to h through Q. Then P is orthogonal to t through ℓ since $P \subset h$. It follows that $\partial t \in \mathcal{T}_\rho$ and $\partial h \in \mathcal{F}_\rho$. Since $Q = t \cap h$, $\partial Q \in \mathcal{K}\rho$.

Conversely if $\partial Q \in \mathcal{K}_\rho$, let $h \in \mathcal{F}_\rho$ and $t \in \mathcal{T}_\rho$ such that $\partial Q = h \cap t$. Let $\hat{h}$ and $\hat{t}$ be the hyperplane bounded by h and t respectively. The hyperplane $\hat{t}$ and the plane P are orthogonal through a line. Then $(\partial P) \cap t$ is a set of two points contained in ∂Q since $\partial P \subset h$. Both P and Q are planes in the hyperplane bounded by h; they intersect in the line $P \cap \hat{t}$. Since $\hat{t}$ is orthogonal to P and $Q \subset \hat{h}$, then P and Q are orthogonal through the line $P \cap \hat{t}$. □

Theorem 3.1.5. *Let ρ be a type I elliptic isometry of $\mathbb{H}^4$. Then for each $k \in \mathcal{K}_\rho$, there exist $k_1, k_2 \in \mathcal{K}_\rho$ such that $\rho = H_{k_1} H_k = H_k H_{k_2}$.*

Proof. If $k \in \mathcal{K}_\rho$, there are $s \in \mathcal{F}_\rho$ and $t \in \mathcal{T}_\rho$ such that $k = s \cap t$. Then there are $s_1, s_2 \in \mathcal{F}_\rho$ such that $\rho = R_{s_1} R_s = R_s R_{s_2}$. Let $k_1 = s_1 \cap t$ and $k_2 = s_2 \cap t$. Since s, s_1 and s_2 are orthogonal to t, $H_{k_1} = R_{s_1} R_t = R_t R_{s_1}$ and $H_{k_2} = R_{s_2} R_t = R_t R_{s_2}$. Then,

$$\begin{aligned} \rho &= R_{s_1} R_s \\ &= R_{s_1} R_t R_t R_s \\ &= H_{k_1} H_k \end{aligned} \qquad\qquad \begin{aligned} \rho &= R_s R_{s_2} \\ &= R_s R_t R_t R_{s_2} \\ &= H_k H_{k_2}. \end{aligned}$$

□

3.1.2 Miscellaneous Elliptic Results

Let $\rho_1 = R_{h_{11}} R_{h_{12}}$ and $\rho_2 = R_{h_{21}} R_{h_{22}}$ be type I elliptic isometries so that $h_{11} \cap h_{12}$ and $h_{21} \cap h_{22}$ are orthogonal planes intersecting in a unique point x. Then both ρ_1 and ρ_2 leave $h_{11} \cap h_{12}$ and $h_{21} \cap h_{22}$ invariant. The composition $\rho_1 \rho_2$ also leaves them invariant, but it can be shown that it fixes only x (See Lemma 3.6.4). Hence $\rho_1 \rho_2$ is an elliptic isometry fixing only x.

Definition 11. *An elliptic isometry ρ of $\mathbb{H}^4$ is called* of type II *if ρ fixes a unique point in $\mathbb{H}^4$.*

A type II elliptic isometry can be orientation reversing and hence can not be expressed as a composition of two type I elliptic isometries. However, it can be shown that an orientation preserving type II elliptic isometry is a composition of two commuting type I elliptic isometries whose twisting planes intersect orthogonally in the fixed point. In order to prove it, we give definitions and theorems about maximal tori of $\mathrm{GL}_n\mathbb{R}$. The next two definitions and two theorems can be found in a book [18] of Kristoffer Tapp.

Definition 12 ([18, Definition 9.7])**.** *Let Γ be a subgroup of $\mathrm{GL}_n\mathbb{R}$. A* torus in Γ *is a subgroup of Γ that is isomorphic to a torus group $\mathbb{R}^k/\mathbb{Z}^k$ for some $k \leq n$. A torus in Γ is* maximal *if it is not contained in a higher dimensional torus in Γ.*

Definition 13. *The* standard maximal torus *of* $\mathrm{SO}(4)$ *is*

$$M^{\mathrm{SO}(4)} = \left\{ \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & \sin\beta & \cos\beta \end{pmatrix} : \alpha, \beta \in [0, 2\pi) \right\}.$$

Theorem 3.1.6 ([18, Theorem 9.7])**.** *The defined $M^{\mathrm{SO}(4)}$ above is a maximal torus of* $\mathrm{SO}(4)$*.*

Theorem 3.1.7 ([18, Theorem 9.18])**.** *For each $\rho \in \mathrm{SO}(4)$, there exists $g \in \mathrm{SO}(4)$ such that $g\rho g^{-1} \in M^{\mathrm{SO}(4)}$.*

This theorem shows that for each $\rho \in \mathrm{SO}(4)$, there is an unordered pair of angles α and β associated to ρ. These angles identify 2-dimensional planes that are left invariant by ρ. If one of them, say α, is not an integer multiple of π, then the plane P_α associated to α has no lines left invariant by ρ. Then the orthogonal complement of P_α is the plane P_β associated to β. Although β can be an integer multiple of π, the plane P_β is uniquely identified by α. Hence, if either α or β is not an integer multiple of π, there are unique unordered pair of planes P_α and P_β that is left invariant by ρ.

If α is an integer multiple of 2π but β is an odd multiple of π, then planes P_α and P_β can still be associated to α and β canonically. Furthermore, P_α is fixed pointwise by ρ and is its corresponding twisting plane, which is a canonical choice for an invariant plane of ρ. There are other planes left invariant by ρ but ρ reverses their orientation.

If both α and β are odd multiples of π, then ρ is an involution with four linearly independent eigenvectors. The span of any two linearly independent vectors is rotated and left invariant by ρ. Thus the angles α and β do not determine a canonical pair of orthogonal planes left invariant by ρ. This can happen if and only if ρ is conjugate to a diagonal matrix with -1 in its entries.

In summary, a $\rho \in \mathrm{SO}(4)$ either has associated pairs of angles and planes or is an involution. Going back to hyperbolic geometry, a type II elliptic isometry either has a canonical pair of invariant planes or reflects every line passing through its fixed point into opposite direction. Then an orientation preserving type II elliptic isometry can be classified further into those with canonical pair of invariant planes and those that are involutions.

Corollary 3.1.3. *If ρ is an orientation preserving type II elliptic isometry of $\mathbb{H}^4$ with fixed point x and an associated angle that is not an integer multiple of π, then there are unique planes up to order $P_1, P_2 \subset \mathbb{H}^4$ orthogonally intersecting at the single point x and a unique pair up to order of type I elliptic isometries ρ_1, ρ_2 with respective twisting planes P_1, P_2 such that $\rho = \rho_1\rho_2$.*

Proof. Using Theorem 3.1.1, there is a Möbius transformation s of $\widehat{\mathbb{R}^4}$ that conjugates ρ into an orthogonal 4×4 matrix with $s(0,0,0,0) = x$. Since ρ is orientation preserving, $s\rho s^{-1} \in \mathrm{SO}(4)$. By a theorem of Tapp, it is further conjugate to an element

$$\rho_{\alpha\beta} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & \sin\beta & \cos\beta \end{pmatrix}$$

of $M^{\mathrm{SO}(4)}$ via an element $g \in \mathrm{SO}(4)$. Since ρ fixes only one point, neither α nor β is an integer multiple of 2π. Let ρ_α and ρ_β be defined as follows.

$$\rho_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \rho_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & \sin\beta & \cos\beta \end{pmatrix}$$

Then ρ_α and ρ_β are rotations about orthogonal planes P_α and P_β respectively intersecting in the origin. Since they fix their respective planes pointwise, they are products of reflections across 3-dimensional subspaces. Define ρ_1, ρ_2, P_1 and P_2 as follows.

$$\rho_1 = g^{-1}s^{-1}\rho_\alpha sg \qquad \rho_2 = g^{-1}s^{-1}\rho_\beta sg$$

$$P_1 = g^{-1}s^{-1}(P_\alpha) \qquad P_2 = g^{-1}s^{-1}(P_\beta)$$

Then ρ_1 and ρ_2 are type I elliptic isometries and $\rho = \rho_1\rho_2$. Suppose α is not an integer multiple of π. Then the choice of P_1 is unique to ρ. The orthogonal complement of P_1 through x is also unique and is equal to P_2. Hence P_1 and P_2 are the canonical invariant planes of ρ. ◻

Definition 14. *Let $\rho = \rho_1\rho_2$ be an orientation preserving type II elliptic isometry of $\mathbb{H}^4$ with an associated angle not an integer multiple of π so that ρ_1 and ρ_2 are the unique type I elliptic elements whose twisting planes intersect orthogonally in the fixed point. The* half-turn bank of ρ *is the set*

$$\mathcal{K}_\rho = \left\{ s \cap t \subset \partial\mathbb{H}^4 : s \in \mathcal{F}_{\rho_1} \text{ and } t \in \mathcal{F}_{\rho_2} \right\}.$$

Definition 15. *Let ρ be an orientation preserving type II elliptic isometry of $\mathbb{H}^4$ such that ρ^2 is the identity map. The* half-turn bank of ρ *is the set*

$$\mathcal{K}_\rho = \left\{ P \subset \partial\mathbb{H}^4 : P \text{ is a plane containing } x \right\}.$$

Corollary 3.1.4. *Let ρ be an orientation preserving type II elliptic isometry of $\mathbb{H}^4$. Then for each $k \in \mathcal{K}_\rho$, there are $k_1, k_2 \in \mathcal{K}_\rho$ such that $\rho = H_{k_1}H_k = H_kH_{k_2}$.*

Proof. Suppose first that ρ is an involution with fixed point x. Then there is a unique plane k_1 orthogonal to k through only x. Since k_1 and k are orthogonal, they are left invariant by both H_k and H_{k_1}. The action of these half-turns on $k \cup k_1$ is therefore an involution and so do $H_k H_{k_1}$ and $H_{k_1} H_k$. The compositions $H_{k_1} H_k$ and $H_k H_{k_1}$ are conjugate to elements of SO(4) where x correspond to the origin and (k, k_1) is a pair of orthogonal discs that can contain a basis for $\mathbb{R}^4$. An element $x \in B_4 = \mathbb{H}^4$ outside $k \cup k_1$ can be expressed as a linear combination of this basis, so $\left(H_{k_1} H_k\right)^2$ and $\left(H_k H_{k_1}\right)^2$ map x back to itself (See Corollary 3.6.1).

As matrices in SO(4), H_{k_1}, H_k and ρ are involutions with inverses equal to their transpose. They are thus diagonalizable with -1 entries since they all fix x (See Lemma 3.6.4). Therefore ρ, $H_{k_1} H_k$ and $H_k H_{k_1}$ are equal isometries.

Suppose next that ρ has an associated angle not an integer multiple of π. Then there are unique type I elliptic elements ρ_1 and ρ_2 whose twisting planes are left invariant by ρ. If $k \in \mathcal{K}_\rho$, there are $h_1 \in \mathcal{F}_{\rho_1}$ and $h_2 \in \mathcal{F}_{\rho_2}$ such that $k = h_1 \cap h_2$. Then there are $h_{11}, h_{12} \in \mathcal{F}_{\rho_1}$ and $h_{21}, h_{22} \in \mathcal{F}_{\rho_2}$ such that $\rho_1 = R_{h_{11}} R_{h_1} = R_{h_1} R_{h_{12}}$ and $\rho_2 = R_{h_{21}} R_{h_2} = R_{h_2} R_{h_{22}}$. Let P_1 and P_2 be the twisting planes of ρ_1 and ρ_2 respectively. Since P_1 and P_2 are orthogonal in a unique point, $\partial P_2 \in \mathcal{D}_{\rho_1}$ and $\partial P_1 \in \mathcal{D}_{\rho_2}$. Then h_1, h_{11} and h_{12} are orthogonal to ∂P_2 since they contain ∂P_1. It follows that $h_1, h_{11}, h_{12} \in \mathcal{T}_{\rho_2}$ and similarly, $h_2, h_{21}, h_{22} \in \mathcal{T}_{\rho_1}$. Each sphere of $\{h_1, h_{11}, h_{12}\}$ is orthogonal to every sphere of $\{h_2, h_{21}, h_{22}\}$. The following relations hold.

$$
\begin{aligned}
R_{h_1} R_{h_{21}} &= R_{h_{21}} R_{h_1} & \qquad H_k &= R_{h_2} R_{h_1} \\
R_{h_2} R_{h_{12}} &= R_{h_{12}} R_{h_2} & \qquad H_k &= R_{h_1} R_{h_2}
\end{aligned}
$$

Let $k_1 = h_{11} \cap h_{21}$ and $k_2 = h_{12} \cap h_{22}$ so that $H_{k_1} = R_{h_{11}} R_{h_{21}}$ and $H_{k_2} = R_{h_{12}} R_{h_{22}}$. The desired conclusion can be computed as follows.

$$
\begin{aligned}
\rho_1 \rho_2 &= R_{h_{11}} R_{h_1} R_{h_{21}} R_{h_2} & \qquad \rho_1 \rho_2 &= R_{h_1} R_{h_{12}} R_{h_2} R_{h_{22}} \\
&= R_{h_{11}} R_{h_{21}} R_{h_1} R_{h_2} & &= R_{h_1} R_{h_2} R_{h_{12}} R_{h_{22}} \\
&= H_{k_1} H_k & &= H_k H_{k_2}
\end{aligned}
$$

Since $h_{11}, h_{12} \in \mathcal{F}_{\rho_1}$ and $h_{21}, h_{22} \in \mathcal{F}_{\rho_2}$, $k_1, k_2 \in \mathcal{K}_\rho$. $\square$

3.2 Pure hyperbolic

Let δ be a pure hyperbolic isometry of $\mathbb{H}^4$. It fixes two points $a_1, a_2 \in \partial\mathbb{H}^4$ which bound a unique line L. The image of L under δ is also L so it is the unique line that is left invariant of δ. Define L as the axis of δ. For each $x \in L$, there is a unique hyperplane h_x orthogonal to L through x. If $y \in L$ is not equal to $x \in L$, h_x and h_y are disjoint and the minimum distance between them is equal to the distance between x and y. Since δ preserves angles, δ sends h_x to $h_{\delta(x)}$ for every $x \in L$. The action of δ extends to $\partial\mathbb{H}^4$ so $\partial h_x \cap \partial h_{\delta(x)} = \emptyset$ for all $x \in L$.

The following results help in simplifying the succeeding theorems.

Theorem 3.2.1 ([17, Theorem 4.7.5]). *An isometry ϕ of $\mathbb{H}^4$ is hyperbolic if and only if ϕ is conjugate in* $\mathrm{Isom}\mathbb{H}^n$ *to an isometry of the form $\psi(x) = kAx$ where $k > 1$ and A is an orthogonal transformation of $\mathbb{R}^{n-1}$.*

Corollary 3.2.1. *A pure hyperbolic isometry of $\mathbb{H}^4$ is conjugate in* $\mathrm{Isom}\mathbb{H}^4$ *to an isometry of the form $\psi(x) = kx$ where $k > 1$.*

Definition 16. *Let γ be a hyperbolic isometry of $\mathbb{H}^4$ with axis L. The* permuted pencil $\mathcal{F}_\gamma$ of γ *is the set*

$$\mathcal{F}_\gamma = \left\{\partial h_x \subset \partial\mathbb{H}^4 : x \in L \text{ and } h_x \text{ is the orthogonal}\right.$$
$$\left.\text{complement of } L \text{ through } x\right\}.$$

For example, define $\delta : \mathbb{H}^4 \to \mathbb{H}^4$ given by

$$(x_1, x_2, x_3, t) \mapsto (\lambda x_1, \lambda x_2, \lambda x_3, \lambda t)$$

for a positive λ not equal to 1. The extension $\hat{\delta}$ of δ to $\partial\mathbb{H}^4$ is expressed as follows.

$$\hat{\delta}(x_1, x_2, x_3, 0) = (\lambda x_1, \lambda x_2, \lambda x_3, 0)$$
$$\hat{\delta}(\infty) = \infty$$

Then δ does not fix any point in $\mathbb{H}^4$, but $\hat{\delta}$ fixes only the points $(0, 0, 0, 0)$ and ∞. The axis of δ is

$$L = \left\{(0, 0, 0, t) \in \mathbb{R}^4 : t > 0\right\}.$$

For each $(0,0,0,t) \in L$, the orthogonal complement of L through $(0,0,0,t)$ is

$$h_t = \left\{(x_1, x_2, x_3, u) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + u^2 = t^2\right\}.$$

To show that δ is pure hyperbolic, one must express δ as a composition of two reflections. Consider the reflections across h_1 and $h_{\sqrt{t}}$. Define σ_1 and σ_2 as follows.

$$\sigma_1(x) = \frac{\lambda}{\|x\|^2}x \qquad\qquad \sigma_2(x) = \frac{1}{\|x\|^2}x$$

Then σ_2 is the reflection across h_1, and σ_1 is the reflection across $h_{\sqrt{\lambda}}$. The composition $\sigma_1 \circ \sigma_2$ simplifies to the formula of δ.

$$\begin{aligned}
\sigma_1 \circ \sigma_2 &= \frac{\lambda}{\|\sigma_2(x)\|^2}\sigma_2(x) \\
&= \frac{\lambda}{\left\|\frac{1}{\|x\|^2}x\right\|^2}\left(\frac{1}{\|x\|^2}x\right) \\
&= \lambda x \\
&= \delta(x)
\end{aligned}$$

Notice that the hyperplanes in which reflection-factoring of δ are bounded by elements of $\mathcal{F}_\delta$. Moreover, $\mathcal{F}_\delta$ serves as a reflection bank for factoring δ. The following theorem states it more precisely.

If h is a hyperplane, let R_h be the reflection across h. By abuse of notation, let $R_{\partial h}$ be equal to R_h.

Theorem 3.2.2. *Let δ be pure hyperbolic isometry of $\mathbb{H}^4$. Then for every $h \in \mathcal{F}_\delta$, there exist $h_1, h_2 \in \mathcal{F}_\delta$ such that $\delta = R_{h_1}R_h$ and $\delta = R_hR_{h_2}$.*

Proof. Let L be the axis of δ and x be the intersection of h and L. The midpoint m_1 of x and $\delta(x)$ must be along L. Then there is a hyperplane bounded by $h_1 \in \mathcal{F}_\delta$ that intersects L through m_1. We must show $\delta = R_{h_1}R_h$. The common orthogonal line between h_1 and h is L so δ and $R_{h_1}R_h$ have the same axis. Since δ is pure hyperbolic, it is enough to show that $R_{h_1}R_h$ sends x to $\delta(x)$ since δ is conjugate to a Möbius map of the form $x \mapsto \lambda x$, $(\lambda > 0)$.

$$\begin{aligned}
R_{h_1} \circ R_h(x) &= R_{h_1}(x) \\
&= \delta(x)
\end{aligned}$$

Let $\hat{h}$ be the hyperplane bounded by h. By construction, $x \in \hat{h}$ so R_h fixes it. Since L is orthogonal to $\hat{h}$, $R_h(x) \in L$. The midpoint between $R_{h_1}(x)$ and x is $m_1 \in h_1 \cap L$. It follows that $\delta(x) = R_{h_1}(x)$ since they both sit in L.

There is also a hyperplane bounded by $h_2 \in \mathcal{F}_\delta$ that intersects L through $R_h(m_1)$. Then the composition $R_h R_{h_2}$ sends x to $\delta(x)$ since x is the midpoint between m_1 and $R_h(m_1)$. Similarly, $\delta = R_h R_{h_2}$. $\square$

It is important to note that the elements of $\mathcal{F}_\delta$ fill up most of $\partial\mathbb{H}^4$. Moreover, the hyperplanes bounded by them fill up $\mathbb{H}^4$.

Theorem 3.2.3. *Let γ be a hyperbolic isometry of $\mathbb{H}^4$ with fixed points $v, w \in \partial\mathbb{H}^4$. Then for each $x \in \partial\mathbb{H}^4 \setminus \{v, w\}$, there is a unique $h_x \in \mathcal{F}_\gamma$ containing x.*

Proof. If x is neither v nor w, there is a unique line L perpendicular to the axis of γ and bounded by x. Let p be the intersection point of L with the axis of γ. There is a unique $h_x \in \mathcal{F}_\gamma$ that bounds a hyperplane h_p containing p. Then $L \subset h_p$ since it is orthogonal to the axis of γ. Consequently, $x \in h_x$ for it bounds L. If there were two elements $h, g \in \mathcal{F}_\gamma$ containing x, the hyperplanes they bound would be tangent or equal. The former case would contradict that the hyperplanes bounded by g and h have the axis as their common perpendicular line. $\square$

Corollary 3.2.2. *Let γ be a hyperbolic isometry of $\mathbb{H}^4$ with axis L. Then,*

$$\mathbb{H}^4 = \bigcup_{x \in L} h_x$$

where h_x is the hyperplane bounded by $s_x \in \mathcal{F}_\gamma$ and containing x.

Proof. Let p be a point in $\mathbb{H}^4$. If $p \in L$, there is a unique hyperplane h_p orthogonal to L through p. The boundary of h_p at infinity is an element of $\mathcal{F}_\gamma$. If $p \notin L$, there is a unique line L_p perpendicular to L through p. The intersection point $p_r \in L_p \cap L$ and p are contained in the hyperplane h_{p_r} that is orthogonal to L through p_r. The boundary of h_{p_r} at infinity is in $\mathcal{F}_\gamma$. $\square$

Going back to the previous example, let δ be a pure hyperbolic isometry of $\mathbb{H}^4$ defined by

$$\delta(x_1, x_2, x_3, t) = (\lambda x_1, \lambda x_2, \lambda x_3, \lambda t)$$

where λ is a positive real number not equal to 1. The axis is $L = \{(0,0,0,t) \in \mathbb{R}^4 : t > 0\}$. Recall that if $(0,0,0,t) \in L$, the orthogonal complement of L through $(0,0,0,t)$ is

$$h_t = \left\{(x_1, x_2, x_3, u) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + u^2 = t^2\right\}.$$

The boundary of h_t at infinity is s_t defined by

$$s_t = \left\{(x_1, x_2, x_3, 0) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 = t^2\right\}.$$

Hence, $\mathcal{F}_\delta$ is parametrized by positive real numbers.

For the moment, concentrate on the boundary of $\mathbb{H}^4$ at infinity. Identify the subplane $\{(x_1, x_2, x_3, 0) : x_1, x_2, x_3 \in \mathbb{R}\}$ of $\mathbb{R}^4$ with $\mathbb{R}^3$ so that the boundary of $\mathbb{H}^4$ is also identified with $\widehat{\mathbb{R}^3} = \mathbb{R}^3 \cup \{\infty\}$.

Consider the Euclidean lines through $0 \in \mathbb{R}^3 \subset \widehat{\mathbb{R}^3}$. They contain the radii of every $s_t \in \mathcal{F}_\delta$ so they are orthogonal to every s_t. If ℓ is a Euclidean line through 0, it intersects each s_t in two points $v, -v \in \ell$ where v is one of the two points in ℓ with Euclidean norm equal to t. By adjoining ∞ to ℓ, $\ell \cup \{\infty\}$ forms a circle that bound a plane P_ℓ in $\mathbb{H}^4$. This plane contains the axis L of δ since ℓ passes through 0. Let h_t be the hyperplane bounded by s_t. Then $h_t \cap P_\ell$ is a line bounded by v and $-v$. Hence, h_t and P_ℓ intersect orthogonally through a line.

Definition 17. *Let γ be a hyperbolic isometry of $\mathbb{H}^4$ with axis L. The* dual pencil of δ *or* pencil dual to $\mathcal{F}_\gamma$ *is the set*

$$\mathcal{D}_\gamma = \left\{\partial P \subset \widehat{\mathbb{R}^3} : P \textit{ is a plane containing } L\right\}.$$

Similar to $\mathcal{F}_\gamma$, the elements of $\mathcal{D}_\gamma$ fill up $\widehat{\mathbb{R}^3}$ and the planes they bound fill up $\mathbb{H}^4$.

Theorem 3.2.4. *Let γ be a hyperbolic isometry of $\mathbb{H}^4$ with fixed points $v, w \in \widehat{\mathbb{R}^3}$. Then for each $x \in \widehat{\mathbb{R}^3} \setminus \{v, w\}$, there is a unique $p_x \in \mathcal{D}_\gamma$ containing x.*

Proof. Let L be the axis of γ. Since x does not bound L, there is a unique line ℓ_x perpendicular to L and bounded by x. Both L and ℓ_x are contained in a unique plane P_x. Let p_x be the boundary of P_x at infinity. Then $p_x \in \mathcal{D}_\gamma$. Since $\ell_x \subset P_x$ and x bounds ℓ_x, $x \in p_x$. If there were another $m \in \mathcal{D}_\gamma$ containing x, then x, v and w would uniquely determine m so $m = p_x$. □

Corollary 3.2.3. *Let γ be a hyperbolic isometry of $\mathbb{H}^4$ with axis L. Then for each $x \in \mathbb{H}^4 \setminus L$, there is a unique $p_x \in \mathcal{D}_\gamma$ such that the plane bounded by p_x contains x.*

Proof. If $x \notin L$, there is a unique line ℓ_x perpendicular to L through x. There is a unique plane P_x containing L and ℓ_x. The boundary p_x of P_x is therefore an element of $\mathcal{D}_\gamma$. It is unique since ℓ_x and P_x are unique. □

Combining a pair of distinct elements of $\mathcal{D}_\gamma$ forms a sphere in $\widehat{\mathbb{R}^3}$. In the example where δ sends (x_1, x_2, x_3, t) to $(\lambda x_1, \lambda x_2, \lambda x_3, \lambda t)$, the elements of $\mathcal{D}_\delta$ are exactly the Euclidean lines through the origin. Any two distinct elements of $\mathcal{D}_\delta$ span a Euclidean plane. Let P be a Euclidean plane in $\widehat{\mathbb{R}^3}$ through the origin. The image of P under δ is itself so δ leaves the hyperplane bounded by P invariant. For each $s \in \mathcal{F}_\delta$, s is orthogonal to P through the circle $s \cap P$. It is interesting to collect the spheres $P \subset \widehat{\mathbb{R}^3}$ containing the fixed points of δ.

Definition 18. *Let δ be a pure hyperbolic isometry of $\mathbb{H}^4$ with axis L. The* invariant pencil of δ *is the set*

$$\mathcal{T}_\delta = \left\{ \partial t \subset \widehat{\mathbb{R}^3} : t \textit{ is a hyperplane containing } L \right\}.$$

It follows that any sphere in $\widehat{\mathbb{R}^3}$ containing the fixed points of δ is an element of $\mathcal{T}_\delta$. Any hyperbolic isometry γ has both $\mathcal{F}_\gamma$ and $\mathcal{D}_\gamma$; hence it is possible to define a $\mathcal{T}_\gamma$. However, γ does not necessarily leave every element of $\mathcal{T}_\gamma$ invariant.

3.2.1 Properties of the invariant pencil

Let δ be a pure hyperbolic isometry fixing v and w in $\widehat{\mathbb{R}^3}$.

1. For each $t \in \mathcal{T}_\delta$, $\delta(t) = t$ and t is orthogonal to each $s \in \mathcal{F}_\delta$.

2. If $t, u \in \mathcal{T}_\delta$ with $u \neq t$, then $u \cap t \in \mathcal{D}_\delta$.

3. If a sphere $s \subset \widehat{\mathbb{R}^3}$ contains some $c \in \mathcal{D}_\delta$, then $s \in \mathcal{T}_\delta$.

4. For each pair $p, q \in \mathcal{D}_\delta$ with $p \neq q$, there is a unique $t \in \mathcal{T}_\delta$ containing $p \cup q$.

5. For each $d \in \mathcal{D}_\delta$, there are $t, u \in \mathcal{T}_\delta$ such that $d = t \cap u$.

Also interesting to collect are the intersections of elements of $\mathcal{F}_\delta$ with those of $\mathcal{T}_\delta$.

Definition 19. *Let δ be a pure hyperbolic isometry of $\mathbb{H}^4$. The* half-turn bank of δ *is the set*

$$\mathcal{K}_\delta = \left\{ s \cap t \subset \widehat{\mathbb{R}^3} : s \in \mathcal{F}_\delta \text{ and } t \in \mathcal{T}_\delta \right\}.$$

Theorem 3.2.5. *Let δ be a pure hyperbolic isometry of $\mathbb{H}^4$ with axis L. Then P is a plane orthogonal to L if and only if $\partial P \in \mathcal{K}_\delta$.*

Proof. Suppose P is a plane orthogonal to L. Let b be the intersection point of P and L. There are orthogonal lines $v_P, w_P \subset P$ through b. Let y be the unique line orthogonal to v_P, w_P and L through b. There is a unique hyperplane h_b orthogonal to L through b. Then $y \subset h_b$. Let $\hat{t}$ be the unique hyperplane spanned by y, w_P and v_P. Then $h_b \cap \hat{t} = P$ since $w_P, v_P \subset h_b \cap \hat{t}$. Let $t = \partial \hat{t}$ and $s = \partial h_b$. Then $t \cap s = \partial P$. Since h_b is orthogonal to L, $s \in \mathcal{F}_\delta$. Similarly, the fact that $\hat{t}$ contains L implies that $t \in \mathcal{T}_\delta$.

Conversely if $\partial P \in \mathcal{K}_\delta$, there are $h \in \mathcal{F}_\delta$ and $t \in \mathcal{T}_\delta$ such that $\partial P = h \cap t$. Let $\hat{h}$ and $\hat{t}$ be the hyperplanes bounded by h and t respectively. Then $\hat{h}$ is orthogonal to L through a point x. Since $\hat{t}$ contains L and P, then $x \in L \cap P$. As a subset of $\hat{h}$ containing x, P is orthogonal to L. ☐

Theorem 3.2.6. *Let δ be a pure hyperbolic isometry of $\mathbb{H}^4$. Then for each $k \in \mathcal{K}_\delta$, there exist $k_1, k_2 \in \mathcal{K}_\delta$ such that $\delta = H_{k_1} H_k = H_k H_{k_2}$.*

Proof. If $k \in \mathcal{K}_\delta$, there are $s \in \mathcal{F}_\delta$ and $t \in \mathcal{T}_\delta$ such that $k = s \cap t$. Then there are $s_1, s_2 \in \mathcal{F}_\delta$ such that $\delta = R_{s_1} R_s = R_s R_{s_2}$. Let $k_1 = s_1 \cap t$ and $k_2 = s_2 \cap t$. Since s, s_1

and s_2 are orthogonal to t, $H_{k_1} = R_{s_1}R_t = R_tR_{s_1}$ and $H_{k_2} = R_{s_2}R_t = R_tR_{s_2}$. Then,

$$\begin{aligned} \delta &= R_{s_1}R_s & \qquad \delta &= R_sR_{s_2} \\ &= R_{s_1}R_tR_tR_s & &= R_sR_tR_tR_{s_2} \\ &= H_{k_1}H_k & &= H_kH_{k_2}. \end{aligned}$$

□

3.3 Pure parabolic

A pure parabolic isometry τ of $\mathbb{H}^4$ is a composition of exactly two reflections across spheres that are tangent to a unique point. One way to locate the invariant planes and hyperplanes of τ is to study the properties of a reflection. Henceforth, the boundary of $\mathbb{H}^4$ at infinity is identified with $\widehat{\mathbb{R}^3}$ using the upper-half space model.

A hyperplane P bounded by a Euclidean plane can be defined using a point $b \in \mathbb{R}^3$ with Euclidean norm 1 and a real number t, so that

$$P = \left\{ x \in \mathbb{H}^4 : b \cdot x = t \right\}.$$

Then the reflection R_P across P is given by the formula

$$R_P(x) = x + 2(t - b \cdot x)b.$$

Using the same formula, R_P extends to $\widehat{\mathbb{R}^4}$ with $R_P(\infty) = \infty$ and ∂P is a Euclidean plane fixed pointwise by R_P. Consider a Euclidean line $\ell \subset \mathbb{R}^3$ orthogonal to ∂P. Since R_P is a conformal map of $\widehat{\mathbb{R}^3}$ fixing a point of ℓ, $R_P(\ell)$ is equal to ℓ.

If a hyperplane S is bounded by a sphere, it can be uniquely defined using a point $a \in \mathbb{R}^3$ and a radius $r > 0$ so that

$$S = \left\{ x \in \mathbb{H}^4 : \|x - a\| = r \right\}.$$

the reflection R_S across S is given by the formula

$$R_S(x) = a + \frac{r^2}{\|x - a\|^2}(x - a).$$

It also extends to $\widehat{\mathbb{R}^4}$, with the same formula and still switching a and ∞.

A Euclidean line $L \subset \mathbb{R}^3$ passing through a is orthogonal to S and is left invariant by R_S.

Let P and Q be Euclidean planes in $\mathbb{R}^3$. If P and Q are not parallel, they meet in a Euclidean line that has more than one point. Otherwise, there is a unique Euclidean line ℓ orthogonal to P and Q passing through the origin. The composition $R_P R_Q$ then leaves ℓ invariant and is a pure parabolic isometry of $\mathbb{H}^4$ fixing ∞. Not all pure parabolic isometries fixing ∞ leave ℓ invariant so ℓ has a valuable information about $R_P R_Q$. Specifically, there are $\lambda > 0$ and $b \in \ell$ such that $\|b\| = 1$ and

$$R_P R_Q(x) = x + \lambda b \quad \text{for all } x \in \mathbb{H}^4 \cup \widehat{\mathbb{R}^3}.$$

This formula can be verified by computing the composition $R_P R_Q$ directly. The planes P and Q uniquely determine λ and b but b can be determined solely by $R_P R_Q$ since the computation of $R_P R_Q$ recorded the common normal vector of P and Q without their exact locations.

Lemma 3.3.1. *Let τ be a pure parabolic isometry of $\mathbb{H}^4$ fixing ∞ of $\widehat{\mathbb{R}^3}$. Then there is a unit vector $b \in \mathbb{R}^3$ such that the Euclidean span of b is left invariant by τ. It is unique up to taking signs.*

Let S be a sphere in $\mathbb{R}^3$ centered at a. If T is another sphere of Euclidean plane in $\mathbb{R}^3$ tangent to a point c in S, then the composition $R_S R_T$ is a pure parabolic isometry fixing c. There is a unique Euclidean line L connecting c to a. If T is also a sphere, L passes through the center of T as well. Therefore L is orthogonal to both S and T. It follows that $R_S R_T$ leave $L \cup \{\infty\}$ invariant.

If $R_S R_T$ can be expressed as another composition $R_{S'} R_{T'}$, $S' \cap T'$ must contain only c. Both spheres or planes S' and T' must be orthogonal to L through c. Otherwise, there would be another Euclidean line L' such that $R_S R_T$ leave $L' \cup \{\infty\}$ invariant. This implies that $R_S R_T(\infty)$ are both in L and L'. But $L \cap L' = \{c\}$ so $R_S R_T(\infty) = c$ contradicts the assumption that $R_S R_T$ fixes c. Thus, the construction of L is independent of S or T and it depends only on the pure parabolic isometry $R_S R_T$.

Definition 20. *Let τ be a pure parabolic isometry of $\mathbb{H}^4$ with fixed point $v \in \widehat{\mathbb{R}^3}$. Then* the direction of τ *is the Euclidean line*

$$L\tau = \begin{cases} \operatorname{span}\{\tau(0)\} & \textit{if } v = \infty \\ \{\lambda b_\tau + v : \lambda \in \mathbb{R}\} & \textit{if } v \neq \infty \end{cases}$$

where

$$b_\tau = \frac{1}{\|\tau(\infty) - v\|}\big(\tau(\infty) - v\big).$$

The direction L_τ of τ, by construction, is left invariant by τ. If v is the fixed point of τ and S is a sphere in $\widehat{\mathbb{R}^3}$ orthogonal to L_τ through v, then $\tau(S)$ is also a sphere in $\widehat{\mathbb{R}^3}$ that is orthogonal to L_τ through v. It is interesting to collect such a sphere.

Definition 21. *Let τ be pure parabolic isometry of $\mathbb{H}^4$ with fixed point $v \in \widehat{\mathbb{R}^3}$ and direction L_τ. The (parabolic)* permuted pencil of τ *is the set*

$$\mathcal{F}_\tau = \left\{S \subset \widehat{\mathbb{R}^3} : S \textit{ is a sphere orthogonal to } L_\tau \textit{ through } v\right\}.$$

If $v = \infty$, $\mathcal{F}_\tau$ consists of parallel Euclidean planes orthogonal to L_τ. For each $s \in \mathcal{F}_\tau$, $s \cap L_\tau$ has two points; one of which is v. Since τ is parabolic, $\tau(s) \neq s$ and $\tau(s) \cap s = \{v\}$. Similar to that of a pure hyperbolic isometry, the permuted pencils of τ serves as a reflection bank for factoring τ.

Theorem 3.3.1. *Let τ be a pure parabolic isometry of $\mathbb{H}^4$. Then for each $h \in \mathcal{F}_\tau$ there exist $h_1, h_2 \in \mathcal{F}_\tau$ such that*

$$\tau = R_{h_1}R_h = R_hR_{h_2}.$$

Proof. Suppose ∞ is the fixed point of τ. Then τ has a simple formula $x \mapsto x + b$ by computing two reflections across parallel Euclidean planes. The point $b \in \mathbb{R}^3$ is in the direction L_τ of τ by definition. There is a $\lambda \in \mathbb{R}$ such that λb is the intersection of h and L_τ. Then there are also $h_1, h_2 \in \mathcal{F}_\tau$ passing through $(\lambda + \frac{1}{2})b$ and $(\lambda - \frac{1}{2})b$ respectively. Computing the compositions $R_{h_1}R_h$ and $R_hR_{h_2}$ yield to a map that sends x to $x + b$.

If τ fixes $v \in \mathbb{R}^3$, it does not have a nice formula but still leaves $L_\tau \cup \{\infty\}$ invariant. Let b_τ be the point in L_τ with $\|b_\tau - v\| = 1$ in the side of $\tau(\infty)$. Specifically, b_τ is the same point defined previously. Then there is an affine homeomorphism $\mathbb{R} \leftrightarrow L_\tau$ so that each point in L_τ can be expressed as $\lambda b_\tau + v$ for some $\lambda \in \mathbb{R}$. Since τ is pure parabolic, the composition $\lambda \mapsto (\lambda b_\tau + v) \mapsto \tau(\lambda b_\tau + v)$ has a nice formula.

$$\lambda \mapsto \frac{\|\tau(\infty) - v\|\lambda}{\lambda + \|\tau(\infty) - v\|} b_\tau + v$$

The coefficient of b_τ can be extended to a Möbius map of $\widehat{\mathbb{R}}$.

If $h \in \mathcal{F}_\tau$, it intersects L_τ through v and another point that can be expressed as $2\lambda_h b_\tau + v$, unless h is a Euclidean plane. Suppose first that h is a sphere. Then the reflection R_h across h sends a point $\lambda b_\tau + v \in L_\tau$ to

$$\frac{\lambda_h \lambda}{\lambda - \lambda_h} b_\tau + v.$$

Let h_1 be the sphere centered at $\frac{\|\tau(\infty)-v\|\lambda_h}{\|\tau(\infty)-v\|+\lambda_h} b_\tau + v$ with radius equal to its center's distance to v. If this center is undefined, let h_1 be the Euclidean orthogonal complement of L_τ in $\mathbb{R}^3$. Let h_2 be the sphere centered at $\frac{\|\tau(\infty)-v\|\lambda_h}{\|\tau(\infty)-v\|-\lambda_h} b_\tau + v$ with radius radius equal to its distance to v. If $\lambda_h = \|\tau(\infty) - v\|$, let h_2 be the Euclidean plane in $\mathbb{R}^3$ orthogonal to L_τ at v. Then both $R_{h_1} R_h$ and $R_h R_{h_2}$ have the same action on $L_\tau \cup \{\infty\}$ as τ. It follows that $\tau = R_{h_1} R_h = R_h R_{h_2}$ since their formulas are equal.

If h is a Euclidean plane, let h_1 be the sphere centered at $\|\tau(\infty) - v\|$ with the same radius and let h_2 be the sphere centered at $-\|\tau(\infty) - v\|$ with the same radius as that of h_1. Then both $R_{h_1} R_h$ and $R_h R_{h_2}$ are equal to τ since their formulas are also equal. $\square$

Remark 3.3.1. *The coefficient of b_τ is a parabolic element of* $\mathrm{PSL}_2\mathbb{R}$ *that fixes* 0*. It is well known that elements of* $\mathrm{PSL}_2\mathbb{R}$ *can be factored into reflections across intervals. Hence the details of the computations above are omitted.*

The next theorem shows that the elements of $\mathcal{F}_\tau$ fill up $\widehat{\mathbb{R}^3}$. Thus the hyperplanes they bound fill up $\mathbb{H}^4$.

Theorem 3.3.2. *Let τ be a pure parabolic isometry of $\mathbb{H}^4$ with fixed point $v \in \widehat{\mathbb{R}^3}$. Then for each $x \in \widehat{\mathbb{R}^3} \setminus \{v\}$ there is a unique $h_x \in \mathcal{F}_\tau$ containing x.*

Proof. Let L_τ be the direction of τ. If $v = \infty$, pick a unit vector b_τ in L_τ. The Euclidean plane $P_x = \{y \in \mathbb{R}^3 : y \cdot b_\tau = b_\tau \cdot x\} \subset \mathbb{R}^3$ is orthogonal to $L\tau$. Then $P_x \cup \{\infty\} \in \mathcal{F}_\tau$. It is unique as P_x is the only plane that passes through $(b_\tau \cdot x) b_\tau \in L_\tau$.

If $v \neq \infty$, the point in $c_x \in L_\tau$ Euclidean-equidistant to x and v can be solved uniquely using Euclidean geometry.

$$c_x = \frac{\|v-x\|^2}{4b_\tau \cdot (x-v)} b_\tau + v$$

Then the sphere in $\mathbb{R}^3$ centered at c_x with radius $\|x - c_x\|$ is the unique element of $\mathcal{F}_\tau$ containing x. □

Corollary 3.3.1. *Let τ be a pure parabolic isometry of $\mathbb{H}^4$. Then for each $x \in \mathbb{H}^4$, there is a unique hyperplane h_x containing x such that $\partial h_x \in \mathcal{F}_\tau$.*

Proof. Using Poincaré extension, the arguments above work for $\mathbb{H}^4$. Let v be the fixed point and L_τ be the direction of τ. Since $x \in \mathbb{H}^4$, $x \neq v$. Let

$$P_x = \left\{y \in \mathbb{R}^4 : y \cdot b_\tau = x \cdot b_\tau\right\} \cap \mathbb{H}^4$$

where b_τ is a unit vector in the direction of τ. If $v = \infty$, then ∂P_x is orthogonal to L_τ so $\partial P_x \in \mathcal{F}_\tau$. If $v \neq \infty$ then either x is in the orthogonal complement P_v of L_τ through v in $\mathbb{R}^4$ or one can find a $c_x \in L_\tau$ so that $\|c_x - v\| = \|c_x - x\|$. Since $L_\tau \subset \mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$, $P_v \cap \mathbb{H}^4$ is a hyperplane and its boundary in $\widehat{\mathbb{R}^3}$ is an element of $\mathcal{F}_\tau$. If $x \notin P_v$, the sphere $s_x \subset \mathbb{R}^4$ centered at c_x is orthogonal to L_τ. Then $s_x \cap \mathbb{H}^4$ is a hyperplane containing x and its boundary in $\widehat{\mathbb{R}^3}$ is an element of $\mathcal{F}_\tau$. □

Similar to a hyperbolic isometry, τ has a natural dual pencil. However the lack of a second fixed point makes it a bit an obstacle to define its dual pencil. Intuitively, if τ fixes ∞, the dual pencil consists of Euclidean lines in the visual boundary parallel to its direction. If τ fixes a finite point, the dual pencil of τ are circles tangent to its direction at v. Moreover the concepts of tangency and parallel lines must be defined.

Definition 22. *Let τ be a pure parabolic isometry of $\mathbb{H}^4$ with fixed point $v \in \widehat{\mathbb{R}^3}$ and direction L_τ. The* farm *of τ is the Euclidean plane $G_\tau \subset \mathbb{R}^3 \times \{0\}$ orthogonal to L_τ through the origin if $v = \infty$ or through v if $v \neq \infty$.*

If G_τ is the farm of τ, then $G_\tau \cup \{\infty\} \in \mathcal{F}_\tau$. It is used here only as an index for defining the dual pencil.

Definition 23. *Let τ be a pure parabolic isometry of $\mathbb{H}^4$ with fixed point v and farm G_τ. The* dual pencil of τ *is the set*

$$\mathcal{D}_\tau = \begin{cases} \Big\{ \big\{ \lambda\tau(\infty) + x : \lambda \in \mathbb{R} \big\} \cup \{\infty\} : x \in G_\tau \Big\} & \text{if } v = \infty \\ \{c_x : x \in G_\tau\} & \text{if } v \neq \infty \end{cases}$$

where c_x is

$$\left\{ \frac{\cos\theta}{2}(x - v) + \frac{\sin\theta}{2} \frac{\|v - x\|}{\|\tau(\infty) - v\|}\big(\tau(\infty) - v\big) + \frac{1}{2}(v + x) \right\}_{\theta\in[0,2\pi)}.$$

The elements of $\mathcal{D}_\tau$ are orthogonal to G_τ. If $v = \infty$, they are in the same direction as $\tau(0)$ in a vector space manner. If $v \neq \infty$, then c_x has a formula that can be differentiated with respect to θ and one can see that the derivative at v, i.e. $\theta = -\pi$, is along L_τ. For each $d_x \in \mathcal{D}_\tau$ indexed by $x \in G_\tau$, $d_x \cap G_\tau = \{v, x\}$.

Any two distinct elements of $\mathcal{D}_\tau$ intersect only at v and form a unique sphere or Euclidean plane. If $v = \infty$, distinct elements $d_x, d_y \in \mathcal{D}_\tau$ intersecting G_τ at x and y respectively form a Euclidean plane orthogonal to G_τ at the Euclidean line connecting x and y. If $v \neq \infty$, $d_x \in \mathcal{D}_\tau$ paired with $L_\tau \cup \{\infty\} \in \mathcal{D}_\tau$ where $x \neq v$ form a unique Euclidean plane orthogonal to G_τ through the Euclidean line connecting v and x. If $d_x, d_y \in \mathcal{D}_\tau$ have $x \neq y \in G_\tau \setminus \{v\}$, the points v, x and y form a unique circle that extends to a unique sphere in $\mathbb{R}^3$ that is orthogonal to G_τ.

Similar to those of $\mathcal{F}_\tau$, the elements of $\mathcal{D}_\tau$ fill up $\widehat{\mathbb{R}^3}$ and the planes they bound fill up $\mathbb{H}^4$.

Theorem 3.3.3. *Let τ be a pure parabolic isometry of $\mathbb{H}^4$ with fixed point v. Then, for each $x \in \widehat{\mathbb{R}^3} \setminus \{v\}$ there is a unique $d_x \in \mathcal{D}_\tau$ that contain x.*

Proof. Let G_τ be the farm of τ. If $v = \infty$, there is a unique Euclidean line L_x orthogonal to G_τ through x. Then $L_x \cup \{\infty\} \in \mathcal{D}_\tau$. If $v \neq \infty$, either x is in the direction L_τ of τ which extends to a unique element $L_\tau \cup \{\infty\} \in \mathcal{D}_\tau$ or x is in a unique plane P_x containing $\{x\} \cup L_\tau$. Using Euclidean geometry, one can find the unique point $z_x \in G_\tau \cap P_x$ equidistant to x and v. A circle c_x can be formed in P_x with center z_x and passing through v and x. Then $c_x \in \mathcal{D}_\tau$. It is unique since z_x and P_x are unique. $\square$

Corollary 3.3.2. *Let τ be a pure parabolic isometry of $\mathbb{H}^4$. Then for each $x \in \mathbb{H}^4$, there is a unique plane $P_x \subset \mathbb{H}^4$ such that $\partial P_x \in \mathcal{D}_\tau$.*

Proof. Let v, L_τ and G_τ be the fixed point, direction and farm of τ respectively. If $x \in \mathbb{H}^4$ then it projects to a unique $x_\infty \in \mathbb{R}^3 \subset \partial\mathbb{H}^4$ by ignoring the last coordinate. If $x_\infty \in L_\tau$, then $L_\tau \cup \{\infty\}$ bounds a plane containing x. If $v = \infty$, there is a unique $d_x \in \mathcal{D}_\tau$ containing x_∞. Then the plane bounded by d_x contains x. If $v \neq \infty$, either $x_\infty \in L_\tau$ or not.

If $x_\infty \notin L_\tau$, then x_∞ can be further projected to G_τ by subtracting $\|\tau(\infty) - v\|^{-1}\big(\tau(\infty) - v\big) \cdot (x_\infty - v)$ from x_∞. Let x_{G_τ} be this projection of x_τ into G_τ. Then there is a unique $r > 0$ such that $r(x_{G_\tau} - v) + v$ is Euclidean-equidistant to v and x. A circle $c_x \subset \widehat{\mathbb{R}^3}$ can be constructed so that c_x is centered at $r(x_{G_\tau} - v) + v$ passing through v and tangent to L_τ. Then $c_x \in \mathcal{D}_\tau$ and the plane it bounds in $\mathbb{H}^4$ contains x since x has Euclidean distance to the center of c_x equal to the radius of c_x. The uniqueness of c_x comes from the uniqueness of the projection of x to G_τ and r. $\square$

It seems that the definitions of pencils, farm and direction of τ depend on whether τ fixes ∞ or not. However a pure parabolic isometry τ fixing ∞ can be conjugated to fix $v \in \mathbb{R}^3$ by a reflection $R_{(v,1)}$ so that $R_{(v,1)}\tau R_{(v,1)}$ has pencils equal to the images of the pencil elements of τ under $R_{(v,1)}$. In particular, $R_{(v,1)}$ is the reflection across the sphere with radius 1 centered at v.

$$R_{(v,1)}(x) = v + \frac{1}{\|x - v\|^2}(x - v)$$

By using $R_{(v,1)}$ we may now assume a pure parabolic isometry τ fixes ∞ and the claims about τ involving its pencils are true for any parabolic isometry with fixed point other than ∞. This is useful particularly on the properties of the invariant pencil defined as follows.

Definition 24. *Let τ be a pure parabolic isometry of τ fixing $v \in \widehat{\mathbb{R}^3}$ and with farm G_τ. The* invariant pencil of τ *is the set*

$$\mathcal{T}_\tau = \left\{ s \subset \widehat{\mathbb{R}^3} : s \textit{ is a sphere orthogonal to } G_\tau \textit{ and } v \in s \right\}.$$

If τ fixes ∞, every $t \in \mathcal{T}_\tau$ consists of extended Euclidean lines that belong to $\mathcal{D}_\tau$. Since τ leaves elements of $\mathcal{D}_\tau$ invariant, it also leaves elements of $\mathcal{T}_\tau$ invariant. Any other pure parabolic isometry is conjugate to one that fixes ∞ so it is true for any pure parabolic isometry.

3.3.1 Properties of the invariant pencil

Let τ be a pure parabolic isometry fixing v in $\widehat{\mathbb{R}^3}$.

1. For each $t \in \mathcal{T}_\tau$, $\tau(t) = t$ and t is orthogonal to each $s \in \mathcal{F}_\tau$.
2. If $t, u \in \mathcal{T}_\tau$ such that $u \cap t$ has more than one point but $u \neq t$, then $u \cap t \in \mathcal{D}_\tau$.
3. If a sphere $s \subset \widehat{\mathbb{R}^3}$ contains some $c \in \mathcal{D}_\tau$, then $s \in \mathcal{T}_\tau$.
4. For each pair $p, q \in \mathcal{D}_\tau$ with $p \neq q$, there is a unique $t \in \mathcal{T}_\tau$ containing $p \cup q$.
5. For each $d \in \mathcal{D}_\tau$, there are $t, u \in \mathcal{T}_\tau$ such that $t \cap u = d$.

The next objects to collect are the intersections of elements of $\mathcal{F}_\tau$ with those of $\mathcal{T}_\tau$.

Definition 25. *Let τ be a pure parabolic isometry of $\mathbb{H}^4$. The* half-turn bank of τ *is the set*

$$\mathcal{K}_\tau = \left\{ s \cap t \subset \widehat{\mathbb{R}^3} : s \in \mathcal{F}_\tau \text{ and } t \in \mathcal{T}_\tau \right\}.$$

Theorem 3.3.4. *Let τ be a pure parabolic isometry of $\mathbb{H}^4$ with fixed point $v \in \widehat{\mathbb{R}^3}$. Then c is a circle in $\widehat{\mathbb{R}^3}$ such that $v \in c \subset h$ for some $h \in \mathcal{F}_\tau$ if and only if $c \in \mathcal{K}_\tau$.*

Proof. Let c be a circle in $\widehat{\mathbb{R}^3}$ such that $v \in c \subset h$ for some $h \in \mathcal{F}_\tau$. It suffices to show that $c \in \mathcal{K}_\tau$ in the case where $v = \infty$. Then h is a Euclidean plane and c is a Euclidean line. Let t be the set $\{x + \lambda\tau(0) : x \in c \setminus \{\infty\}, \lambda \in \mathbb{R}\} \cup \{\infty\}$. Then t is orthogonal to h through c so $t \in \mathcal{T}_\tau$ and $c = h \cap t$. Hence, $c \in \mathcal{K}_\tau$.

Conversely if $c \in \mathcal{K}_\tau$, there are $h \in \mathcal{F}_\tau$ and $t \in \mathcal{T}_\tau$ such that $c = h \cap t$. Then $c \subset h$. Both h and t contain v so $v \in c$. □

Theorem 3.3.5. *Let τ be a pure parabolic isometry of $\mathbb{H}^4$. Then for each $k \in \mathcal{K}_\tau$, there exist $k_1, k_2 \in \mathcal{K}_\tau$ such that $\tau = H_{k_1} H_k = H_k H_{k_2}$.*

Proof. The argument for the pure hyperbolic case works exactly for the pure parabolic case. There are $s \in \mathcal{F}_\tau$ and $t \in \mathcal{T}_\tau$ such that $k = s \cap t$. Then there are $s_1, s_2 \in \mathcal{F}_\tau$ such that $\tau = R_{s_1} R_s = R_s R_{s_2}$. Let $k_1 = s_1 \cap t$ and $k_2 = s_2 \cap t$. Since s, s_1 and s_2 are orthogonal to t, $H_{k_1} = R_{s_1} R_t = R_t R_{s_1}$ and $H_{k_2} = R_{s_2} R_t = R_t R_{s_2}$. Then,

$$\begin{aligned} \tau &= R_{s_1} R_s & \qquad \tau &= R_s R_{s_2} \\ &= R_{s_1} R_t R_t R_s & &= R_s R_t R_t R_{s_2} \\ &= H_{k_1} H_k & &= H_k H_{k_2}. \end{aligned}$$

□

3.4 Pure loxodromic

Let δ be a pure hyperbolic isometry of $\mathbb{H}^4$. One can pick a pair a distinct hyperplanes t_1 and t_2 bounded by some elements of $\mathcal{T}_\delta$ so that $R_{t_1} R_{t_2}$ is a type I elliptic isometry. Let $\rho = R_{t_1} R_{t_2}$ and $P = t_1 \cap t_2$. Then P is the twisting plane of ρ and $\partial P \in \mathcal{D}_\delta$. It follows that $\mathcal{F}_\rho \subset \mathcal{T}_\delta$ and $\mathcal{F}_\delta \subset \mathcal{T}_\rho$ by definition. Since P contains the axis of δ, then ρ leaves it invariant. The composition $\delta\rho$ is therefore a hyperbolic isometry that leaves the axis of δ invariant. Moreover ρ and δ commute. To see this, express δ as $R_{h_1} R_{h_2}$ for some $h_1, h_2 \in \mathcal{F}_\delta$. Then every element of $\{h_1, h_2\}$ is orthogonal to every element of $\{\partial t_1, \partial t_2\}$ so each reflection in $\{R_{h_1}, R_{h_2}\}$ commutes with each reflection in $\{R_{t_1}, R_{t_2}\}$. These imply the following computations.

$$\begin{aligned} \delta\rho &= R_{h_1} R_{h_2} R_{t_1} R_{t_2} \\ &= R_{h_1} R_{t_1} R_{h_2} R_{t_2} \\ &= R_{t_1} R_{h_1} R_{t_2} R_{h_2} \\ &= R_{t_1} R_{t_2} R_{h_1} R_{h_2} \\ &= \rho\delta \end{aligned}$$

Two things can be observed from the computation. First, $\delta\rho$ is a composition of four reflections. Second, the expression $R_{h_1} R_{t_1} R_{h_2} R_{t_2}$ is a composition of half-turns about $h_1 \cap t_1$ and $h_2 \cap t_2$ so $\delta\rho$ can be expressed as such. If a hyperbolic isometry can be expressed as a composition of exactly four reflections, it is interesting to show that it can be expressed as a composition of two half-turns. In this section, we construct a set of possible planes in which $\delta\rho$ can be factored into half-turns about planes within the same set.

Definition 26. *A* pure loxodromic isometry γ *is a hyperbolic isometry of* $\mathbb{H}^4$ *that can be expressed as* $\delta\rho$ *where* δ *is a pure hyperbolic isometry leaving the axis of* γ *invariant and* ρ *is a type I elliptic isometry whose twisting plane contains the axis of* γ*.*

Using the same arguments above, it follows that if $\gamma = \delta\rho$ from the definition, then $\delta\rho = \rho\delta$. Furthermore such decompositions of γ into $\delta\rho$ is unique.

Theorem 3.4.1. *Let* γ *be a pure loxodromic isometry of* $\mathbb{H}^4$*. If* $\gamma = \delta_1\rho_1 = \delta_2\rho_2$ *where* δ_1, δ_2 *are pure hyperbolic isometries leaving the axis of* γ *invariant and* ρ_1, ρ_2 *are type I elliptic isometries whose twisting planes contain the axis of* γ*, then* $\delta_1 = \delta_2$ *and* $\rho_1 = \rho_2$*.*

Proof. Since the axes of δ_1 and δ_2 are equal, they must have the same pencils. In particular, $\mathcal{F}_{\delta_1} = \mathcal{F}_{\delta_2}$ and $\mathcal{T}_{\delta_1} = \mathcal{T}_{\delta_2}$. Express $\delta = R_{h_1}R_{h_2}$ for some $h_1, h_2 \in \mathcal{F}_{\delta_1}$. Then there is $h_3 \in \mathcal{F}_{\delta_2}$ such that $\delta_2 = R_{h_1}R_{h_3}$. The twisting planes of ρ_1 and ρ_2 intersect in at least a line so there is a hyperplane t_1 that contain both twisting planes. Then $\partial t_1 \in \mathcal{F}_{\rho_1} \cap \mathcal{F}_{\rho_2}$ so there are $t_2 \in \mathcal{F}_{\rho_1}$ and $t_3 \in \mathcal{F}_{\rho_2}$ such that $\rho_1 = R_{t_2}R_{t_1}$ and $\rho_2 = R_{t_3}R_{t_1}$. Then γ can be expressed as $R_{h_1}R_{h_2}R_{t_2}R_{t_1}$ or $R_{h_1}R_{h_3}R_{t_3}R_{t_1}$. It implies that $R_{h_2}R_{t_2} = R_{h_3}R_{t_3}$. Since t_2 and t_3 contain the fixed points of γ, then $t_2, t_3 \in \mathcal{T}_{\delta_1} = \mathcal{T}_{\delta_2}$. Each element of $\{h_2, h_3\}$ is orthogonal to every element of $\{t_2, t_3\}$ so $R_{h_2}R_{t_2}$ and $R_{h_3}R_{t_3}$ are half-turns about the same plane. Then $h_2 \cap t_2 = h_3 \cap t_3$ but since $\mathcal{F}_{\delta_1}$ has pairwise disjoint elements, $h_1 = h_3$. The equation $R_{h_2}R_{t_2} = R_{h_3}R_{t_3}$ implies that $R_{t_2} = R_{t_3}$. Thus $\delta_1 = \delta_2$ and $\rho_1 = \rho_2$. $\square$

Let $\delta\rho$ be the unique decomposition of a pure loxodromic isometry γ. Since $\mathcal{F}_\rho \subset \mathcal{T}_\delta$, the boundary of the twisting plane of ρ must be in $\mathcal{D}_\delta$. Hence δ leaves the twisting plane of ρ invariant together with the axis of γ. It follows that γ also leave the twisting plane of ρ invariant. The pure loxodromic isometry γ share both an axis with δ and an invariant plane with ρ. The uniqueness of ρ allows γ to have its own unique invariant plane.

Definition 27. *Let γ be a pure loxodromic isometry of $\mathbb{H}^4$. Let δ be the unique pure hyperbolic isometry and ρ be the unique type I elliptic isometry so that $\gamma = \delta\rho = \rho\delta$. Then δ is called* the dilation part of γ *and ρ is called the* rotational part of γ. *The* twisting plane of γ *is the set of fixed points of ρ.*

The pencils of γ are ready to be defined. They are derived from the pencils of its dilation and rotational parts.

Definition 28. *Let γ be a pure loxodromic isometry of $\mathbb{H}^4$ with dilation part δ and rotational part ρ. The* permuted pencil of γ *is defined to be the permuted pencil of δ. The* twisting pencil of γ *is defined to be the permuted pencil of ρ. The* dual pencil of γ *is defined as the dual pencil of δ. The twisting pencil of γ is denoted $\mathcal{R}_\gamma$.*

In the pure hyperbolic section, $\mathcal{F}_\gamma$ and $\mathcal{D}_\gamma$ are already defined even if γ is pure loxodromic. The new pencil here is $\mathcal{R}_\gamma$ which depends of the rotational part. Nonetheless the definitions above are consistent with those of pure hyperbolic.

3.4.1 Properties of the pencils of a pure loxodromic isometry

Let γ be a pure loxodromic isometry of $\mathbb{H}^4$ with twisting plane P.

1. For each $t \in \mathcal{R}_\gamma$, then $\gamma(t) \in \mathcal{R}_\gamma$, $\gamma(t) \cap t = \partial P$ and t is orthogonal to each $s \in \mathcal{F}_\gamma$.

2. If $t, u \in \mathcal{R}_\gamma$ with $u \neq t$, then $u \cap t = \partial P$.

3. If a sphere $s \subset \widehat{\mathbb{R}^3}$ contains some ∂P, then $s \in \mathcal{R}_\gamma$.

4. For each $d \in \mathcal{D}_\gamma \setminus \{\partial P\}$, there is a unique $t \in \mathcal{R}_\gamma$ containing $d \cup (\partial P)$.

Similar to $\mathcal{F}_\gamma$ and $\mathcal{D}_\gamma$, the elements of $\mathcal{R}_\gamma$ fill up $\widehat{\mathbb{R}^3}$ and the hyperplanes they bound fill up $\mathbb{H}^4$.

Theorem 3.4.2. *Let γ be a pure loxodromic isometry of $\mathbb{H}^4$ with twisting plane P. Then for each $x \in \widehat{\mathbb{R}^3} \setminus (\partial P)$, there is a unique $h_x \in \mathcal{R}_\gamma$ containing x.*

Proof. Let ρ be the rotational part of γ. Then there is a unique $h_x \in \mathcal{F}_\rho$ containing x. Since $\mathcal{F}_\rho = \mathcal{R}_\gamma$, then $h_x \in \mathcal{R}_\gamma$. □

Corollary 3.4.1. *Let γ be a pure loxodromic isometry of $\mathbb{H}^4$ with twisting plane P. Then for each $x \in \mathbb{H}^4 \setminus P$, there is a unique hyperplane h_x containing x such that $\partial h_x \in \mathcal{R}_\gamma$.*

Proof. Let ρ be the rotational part of γ. Then there is a unique $h_x \in \mathcal{F}_\rho$ so that the hyperplane it bounds contains x. Since $\mathcal{F}_\rho = \mathcal{R}_\gamma$, $h_x \in \mathcal{R}_\gamma$. □

The importance of $\mathcal{R}_\gamma$ is highlighted by the need to collect the planes in which γ can be factored into half-turns.

Definition 29. *Let γ be a pure loxodromic isometry of $\mathbb{H}^4$. The* half-turn bank of γ *is the set*

$$\mathcal{K}_\gamma = \left\{ s \cap t \subset \widehat{\mathbb{R}^3} : s \in \mathcal{F}_\gamma \text{ and } t \in \mathcal{R}_\gamma \right\}.$$

Corollary 3.4.2. *Let γ be a pure loxodromic isometry of $\mathbb{H}^4$ with twisting plane P. For each $x \in \mathbb{H}^4 \setminus P$, there is a unique plane k_x containing x such that $\partial k_x \in \mathcal{K}_\gamma$. For each $y \in \widehat{\mathbb{R}^3} \setminus (\partial P)$, there is a unique $k_y \in \mathcal{K}_\gamma$ containing y.*

Proof. Let L be the axis of γ. Let ρ be the rotational part and δ the dilation part of γ. Since $x \notin P$ implying $x \notin L$, there are unique $h_x \in \mathcal{F}_\delta$ and $t_x \in \mathcal{F}_\rho$ such that the hyperplanes they bound contain x. Let $k_x = h_x \cap t_x$; then $\partial k_x \in \mathcal{K}_\gamma$.

Since $y \notin \partial P$, there are unique $h_y \in \mathcal{F}_\delta$ and $t_y \in \mathcal{F}_\rho$ containing y. Let $k_y = h_y \cap t_y$; then $k_y \in \mathcal{K}_\gamma$. □

Theorem 3.4.3. *Let γ be a pure loxodromic isometry of $\mathbb{H}^4$ with axis L and twisting plane P. Then $Q \subset \mathbb{H}^4$ is a plane orthogonal to both P and L through a line in P if and only if $\partial Q \in \mathcal{K}_\gamma$.*

Proof. Suppose Q is a plane orthogonal to both P and L such that $Q \cap P$ is a line. Let $\ell = Q \cap P$. Since Q is orthogonal to L, then ℓ and L are perpendicular lines in P. Let x be the intersection point Q and L. Let ℓ' be the unique line in Q perpendicular to ℓ through x. Likewise ℓ' is perpendicular to L. There is a unique line ℓ_4 perpendicular to all L, ℓ and ℓ'. In particular, ℓ_4 is the unique line orthogonal to the hyperplane spanned by P and Q passing through x. Let h be the hyperplane spanned by ℓ, ℓ' and ℓ_4. Since L is perpendicular to all ℓ, ℓ' and ℓ_4, h is the orthogonal complement of L through x. Then $\partial h \in \mathcal{F}_\gamma$. Let t be the hyperplane spanned by L, ℓ and ℓ'. Therefore $P \subset t$ and $t \cap h = Q$. Hence $\partial t \in \mathcal{R}_\gamma$ and $\partial Q \in \mathcal{K}_\gamma$.

Conversely if $\partial Q \in \mathcal{K}_\gamma$, there are $h \in \mathcal{F}_\gamma$ and $t \in \mathcal{R}_\gamma$ such that $\partial Q = h \cap t$. Let $\hat{h}$ and $\hat{t}$ be the hyperplanes bounded by h and t respectively. Then $\hat{h}$ is orthogonal to to P through a line since ∂P is orthogonal to every element of $\mathcal{F}_\gamma$. Whereas $\hat{t}$ contains P so $Q = \hat{h} \cap \hat{t}$ must also intersect P in the line $P \cap \hat{h}$ since $Q \cap P = \left(\hat{h} \cap \hat{t}\right) \cap P = \hat{h} \cap \left(\hat{t} \cap P\right) = \hat{h} \cap P$. Then P is orthogonal to $\hat{h}$ through $P \cap Q$, so $Q \subset \hat{h}$ implies that P is also orthogonal to Q through the line $P \cap Q$.

Since L is a subset of $\hat{t}$ and L is orthogonal to $\hat{h}$ through x, then Q intersect L at x. But $Q \subset \hat{h}$ so Q is orthogonal to L. □

Theorem 3.4.4. *Let γ be a pure loxodromic isometry of $\mathbb{H}^4$. Then for each $k \in \mathcal{K}_\gamma$ there exist $k_1, k_2 \in \mathcal{K}_\gamma$ such that $\gamma = H_{k_1} H_k = H_k H_{k_2}$.*

Proof. Let δ be the dilation part of γ and ρ be the rotational part of γ. If $k \in \mathcal{K}_\gamma$, there are $s \in \mathcal{F}_\gamma$ and $t \in \mathcal{R}_\gamma$ such that $k = s \cap t$. Then there are $s_1, s_2 \in \mathcal{F}_\delta = \mathcal{F}_\gamma$ such that $\delta = R_{s_1} R_s = R_s R_{s_2}$ and there are $t_1, t_2 \in \mathcal{F}_\rho = \mathcal{R}_\gamma$ such that $\rho = R_{t_1} R_t = R_t R_{t_2}$. Each sphere of $\{s, s_1, s_2\}$ is orthogonal to each sphere in $\{t, t_1, t_2\}$. The following relations hold.

$$R_s R_{t_1} = R_{t_1} R_s \qquad\qquad H_k = R_s R_t$$

$$R_{t_2} R_s = R_s R_{t_2} \qquad\qquad H_k = R_t R_s$$

Let $k_1 = s_1 \cap t_1$ and $k_2 = s_2 \cap t_2$. Then $H_{k_1} = R_{s_1} R_{t_1} = R_{t_1} R_{s_1}$ and $H_{k_2} = R_{s_2} R_{t_2} =$

$R_{t_2}R_{s_2}$. Since $\gamma = \delta\rho = \rho\delta$,

$$\begin{aligned} \gamma &= \delta\rho & \gamma &= \rho\delta \\ &= R_{s_1}R_sR_{t_1}R_t & &= R_tR_{t_2}R_sR_{s_2} \\ &= R_{s_1}R_{t_1}R_sR_t & &= R_tR_sR_{t_2}R_{s_2} \\ &= H_{k_1}H_k & &= H_kH_{k_2} \end{aligned}$$

□

3.4.2 Twisting hyperplane

If γ is a pure loxodromic isometry of $\mathbb{H}^4$ with axis L and twisting plane P, then there is a unique hyperplane h that is orthogonal to P through L. Since γ leave both P and L invariant, the image of h is also a hyperplane orthogonal to P through L. The uniqueness of h makes it an invariant hyperplane of γ. Hence, h is called the *twisting hyperplane of* γ. In the boundary, the twisting hyperplane can be found by locating the sphere orthogonal to the boundary of the twisting plane through the fixed points of γ. Let δ and ρ be the respective dilation and rotational parts of γ. For each point $x \in L$, there is a unique plane d_x orthogonal to P through x. Then $\partial d_x \in \mathcal{D}_\rho$, and the union of all these planes for all $x \in L$ is h. Since h contains L and some d_x, then $\partial h \in \mathcal{T}_\delta \cap \mathcal{T}_\rho$. This is another way to show that $\gamma = \delta\rho$ leave h invariant.

3.5 Screw parabolic

Let τ be a pure parabolic isometry of $\mathbb{H}^4$. Let P be a plane bounded by an element in $\mathcal{D}_\tau$. If t_1 and t_2 are distinct hyperplanes intersecting at P, then $\partial t_1, \partial t_2 \in \mathcal{T}_\tau$. The composition $\rho = R_{t_1}R_{t_2}$ is a type I elliptic isometry fixing P pointwise. Both τ and ρ leave P invariant and so does $\tau\rho$. Since every $t \in \mathcal{F}_\rho$ contain ∂P which is in $\mathcal{D}_\tau$, then $\mathcal{F}_\rho \subset \mathcal{T}_\tau$. Likewise, every $h \in \mathcal{F}_\tau$ is orthogonal to ∂P so $\mathcal{F}_\tau \subset \mathcal{T}_\rho$. The composition $\tau\rho$ can be shown to be parabolic with the same fixed point as that of τ.

Let $x \in \left(\mathbb{H}^4 \setminus P\right) \cup \left(\widehat{\mathbb{R}^3} \setminus (\partial P)\right)$. Then there is a unique hyperplane t such that $x \in t \cup (\partial t)$ and $\partial t \in \mathcal{F}_\rho$. Since $x \notin P$ and $\rho(t) \cap t = P$, then $\rho(x) \neq x$. But

$\rho(\partial t) \in \mathcal{F}_\rho \subset \mathcal{T}_\tau$ so $\tau\rho(x) \in \rho(t) \cup \partial(\rho(t))$. Also $\rho(\partial t) \neq \partial t$ so $\tau\rho(x) \notin t \cup (\partial t)$. Hence, $\tau\rho(x) \neq x$.

If $x \in P \cup (\partial P)$, then $\tau\rho(x) = \tau(x)$ which is equal to x if and only if x is the fixed point of τ. Hence $\tau\rho$ is a parabolic isometry sharing the fixed point with τ.

Furthermore, τ and ρ commute. Let $h_1, h_2 \in \mathcal{F}_\tau$ so that $\tau = R_{h_1} R_{h_2}$. Since $\mathcal{F}_\tau \subset \mathcal{T}_\rho$ and $\mathcal{F}_\rho \subset \mathcal{T}_\tau$, every element of $\{h_1, h_2\}$ is orthogonal to every element of $\{\partial t_1, \partial t_2\}$. The argument is similar to the decomposition of a pure loxodromic isometry.

$$\begin{aligned}
\tau\rho &= R_{h_1} R_{h_2} R_{t_1} R_{t_2} \\
&= R_{h_1} R_{t_1} R_{h_2} R_{t_2} \\
&= R_{t_1} R_{h_1} R_{t_2} R_{h_2} \\
&= R_{t_1} R_{t_2} R_{h_1} R_{h_2} \\
&= \rho\tau
\end{aligned}$$

Definition 30. *A* screw parabolic isometry γ *is a parabolic isometry of* $\mathbb{H}^4$ *that can be expressed as* $\tau\rho$ *where* τ *is a pure parabolic isometry fixing the fixed point of* γ*, and* ρ *is a type I elliptic isometry whose twisting plane is bounded by an element of* $\mathcal{D}_\tau$*.*

If such a γ can be decomposed into $\tau\rho$, then $\tau\rho = \rho\tau$. Moreover, such a decomposition is unique.

Theorem 3.5.1. *Let* γ *be a screw parabolic isometry of* $\mathbb{H}^4$ *with fixed point* v*. If* $\gamma = \tau_1\rho_1 = \tau_2\rho_2$ *where* τ_1, τ_2 *are pure parabolic isometries fixing* v *and* ρ_1, ρ_2 *are type I elliptic isometries whose twisting plane are bounded by elements of* $\mathcal{D}_{\tau_1}$ *and* $\mathcal{D}_{\tau_2}$ *respectively, then* $\tau_1 = \tau_2$ *and* $\rho_1 = \rho_2$*.*

Proof. It is enough to show when $v = \infty$. Let E_1 and E_2 be the respective farms of τ_1 and τ_2; let $P_1 \in \mathcal{D}_{\tau_1}$ and $P_2 \in \mathcal{D}_{\tau_2}$ be the boundaries of the twisting planes of ρ_1 and ρ_2 respectively. Then each $E_i \cap P_i$ has two points. Let $c_1 \in E_1 \cap P_1 \setminus \{v\}$ and $c_2 \in E_2 \cap P_2 \setminus \{v\}$. Since $v = \infty$, there are $A, B \in \mathrm{SO}(3)$ and $a, b \in \mathbb{R}^3 \setminus \{0\}$ such that

$$\begin{aligned}
\tau_1(x) &= x + a & \qquad \rho_1(x) &= A(x - c_1) + c_1 \\
\tau_2(x) &= x + b & \qquad \rho_2(x) &= B(x - c_2) + c_2
\end{aligned}$$

for all $x \in \mathbb{R}^3$ with $c_1 \cdot a = 0 = c_2 \cdot b$, $\rho_1(a + c_1) = a + c_1$ and $\rho_2(b + c_2) = b + c_2$. Thus $Aa = a$; $Bb = b$. If $\gamma = \tau_1\rho_1 = \tau_2\rho_2$, then

$$A(x - c_1) + c_1 + a = B(x - c_2) + c_2 + b \tag{3.1}$$

for all $x \in \mathbb{R}^3$. For $x = 0$,

$$-Ac_1 + c_1 + a = -Bc_2 + c_2 + b. \tag{3.2}$$

Subtracting equation (3.2) from (3.1) yields $A(x) = B(x)$. Hence $A = B$.

Since $A \in \mathrm{SO}(3)$ is nontrivial, its fixed points are 1-dimensional so $b = \lambda a$ for some $\lambda \in \mathbb{R}$. Then $c_1 \cdot b = 0 = c_2 \cdot a$. The equation (3.2) implies

$$Ac_2 - Ac_1 = c_2 - c_1 + b - a.$$

Taking the Euclidean norms of both sides follows that $\|b - a\| = 0$ so $b = a$. But that also means $E_1 = E_2$ and is left invariant by A. Since A fixes only the origin in E_1, then $A(c_2 - c_1) = c_2 - c_1$ implies that $c_2 = c_1$. $\square$

Let $\tau\rho$ be the unique decomposition of a screw parabolic isometry γ. Since $\mathcal{F}_\rho \subset \mathcal{T}_\tau$, the boundary of the twisting plane of ρ must be in $\mathcal{D}_\tau$. Hence τ leaves the twisting plane of ρ invariant. It follows that γ also leave the twisting plane of ρ invariant. The screw parabolic isometry γ share both a fixed point with τ and an invariant plane with ρ. The uniqueness of ρ allows γ to have its own unique invariant plane.

Definition 31. *Let γ be a screw parabolic isometry of $\mathbb{H}^4$. Let τ be the unique pure parabolic isometry and ρ be the unique type I elliptic isometry so that $\gamma = \tau\rho = \rho\tau$. Then τ is called* the translation part of γ *and ρ is called the* rotational part of γ. *The* twisting plane of γ *is the set of fixed points of ρ.*

Definition 32. *Let γ be a screw parabolic isometry of $\mathbb{H}^4$ with translation part τ and rotational part ρ. The* permuted pencil of γ *is defined to be the permuted pencil of τ. The* twisting pencil of γ *is defined to be the permuted pencil of ρ. The* dual pencil of γ *is defined as the dual pencil of τ. The twisting pencil of γ is denoted $\mathcal{R}_\gamma$.*

3.5.1 Properties of the pencils of a screw parabolic isometry

Let γ be a screw parabolic isometry of $\mathbb{H}^4$ with twisting plane P.

1. For each $t \in \mathcal{R}_\gamma$, then $\gamma(t) \in \mathcal{R}_\gamma$, $\gamma(t) \cap t = \partial P$ and t is orthogonal to each $s \in \mathcal{F}_\gamma$.

2. If $t, u \in \mathcal{R}_\gamma$ with $u \neq t$, then $u \cap t = \partial P$.

3. If a sphere $s \subset \widehat{\mathbb{R}^3}$ contains some ∂P, then $s \in \mathcal{R}_\gamma$.

4. For each $d \in \mathcal{D}_\gamma \setminus \{\partial P\}$, there is a unique $t \in \mathcal{R}_\gamma$ containing $d \cup (\partial P)$.

The elements of $\mathcal{R}_\gamma$ fill up $\widehat{\mathbb{R}^3}$ and the hyperplanes they bound fill up $\mathbb{H}^4$.

Theorem 3.5.2. *Let γ be a screw parabolic isometry of $\mathbb{H}^4$ with twisting plane P. Then for each $x \in \widehat{\mathbb{R}^3} \setminus (\partial P)$, there is a unique $h_x \in \mathcal{R}_\gamma$ containing x.*

Proof. Let ρ be the rotational part of γ. Then there is a unique $h_x \in \mathcal{F}_\rho$ containing x. Since $\mathcal{F}_\rho = \mathcal{R}_\gamma$, then $h_x \in \mathcal{R}_\gamma$. □

Corollary 3.5.1. *Let γ be a screw parabolic isometry of $\mathbb{H}^4$ with twisting plane P. Then for each $x \in \mathbb{H}^4 \setminus P$, there is a unique hyperplane h_x containing x such that $\partial h_x \in \mathcal{R}_\gamma$.*

Proof. Let ρ be the rotational part of γ. Then there is a unique $h_x \in \mathcal{F}_\rho$ so that the hyperplane it bounds contains x. Since $\mathcal{F}_\rho = \mathcal{R}_\gamma$, $h_x \in \mathcal{R}_\gamma$. □

Definition 33. *Let γ be a screw parabolic isometry of $\mathbb{H}^4$. The* half-turn bank of γ *is the set*

$$\mathcal{K}_\gamma = \left\{ s \cap t \subset \widehat{\mathbb{R}^3} : s \in \mathcal{F}_\gamma \text{ and } t \in \mathcal{R}_\gamma \right\}.$$

Corollary 3.5.2. *Let γ be a screw parabolic isometry of $\mathbb{H}^4$ with twisting plane P. For each $x \in \mathbb{H}^4 \setminus P$, there is a unique plane k_x containing x such that $\partial k_x \in \mathcal{K}_\gamma$. For each $y \in \widehat{\mathbb{R}^3} \setminus (\partial P)$, there is a unique $k_y \in \mathcal{K}_\gamma$ containing y.*

Proof. Let L be the direction of γ. Let ρ be the rotational part and τ the translation part of γ. Since $x \notin P$, there are unique $h_x \in \mathcal{F}_\tau$ and $t_x \in \mathcal{F}_\rho$ such that the hyperplanes they bound contain x. Let $k_x = h_x \cap t_x$; then $\partial k_x \in \mathcal{K}_\gamma$.

Since $y \notin \partial P$, there are unique $h_y \in \mathcal{F}_\tau$ and $t_y \in \mathcal{F}_\rho$ containing y. Let $k_y = h_y \cap t_y$; then $k_y \in \mathcal{K}_\gamma$. $\square$

Theorem 3.5.3. *Let γ be a screw parabolic isometry of $\mathbb{H}^4$ with fixed point v and twisting plane P. Then $Q \subset \mathbb{H}^4$ is a plane orthogonal to P through a line bounded by v if and only if $\partial Q \in \mathcal{K}_\gamma$.*

Proof. Suppose Q is a plane orthogonal to P through a line bounded by v. There is a hyperplane t contaning $Q \cup P$ since they intersect in a line. Let $\ell = Q \cap P$. There is a point $x \in \partial\ell \setminus \{v\}$. Then there is a unique $h_x \in \mathcal{F}_\gamma$ that contain x. Since Q is orthogonal to P, then Q is contained in the hyperplane bounded by h_x. But $Q, P \subset t$ so $\partial t \in \mathcal{R}_\gamma$ and $\partial Q = (\partial t) \cap h_x$. Hence $\partial Q \in \mathcal{K}_\gamma$.

Conversely if $\partial Q \in \mathcal{K}_\gamma$, there are $h \in \mathcal{F}_\gamma$ and $t \in \mathcal{R}_\gamma$ such that $\partial Q = h \cap t$. Let $\hat{h}$ and $\hat{t}$ be the hyperplane bounded by h and t respectively. Then $\hat{h}$ is orthogonal to P through a line bounded by v. Also $\hat{t}$ contains P, so $Q = \hat{h} \cap \hat{t}$ implies $Q \cap P = \big(\hat{h} \cap t\big) \cap P = \hat{h} \cap \big(\hat{t} \cap P\big) = \hat{h} \cap P$. Then P is orthogonal to $\hat{h}$ through $P \cap Q$ and hence to Q. Since $v \in h \cap t$, then $v \in \partial Q$; $v \in \partial P$ implies that v bounds the line $P \cap Q$. $\square$

Theorem 3.5.4. *Let γ be a screw parabolic isometry of $\mathbb{H}^4$. Then for each $k \in \mathcal{K}_\gamma$ there exist $k_1, k_2 \in \mathcal{K}_\gamma$ such that $\gamma = H_{k_1} H_k = H_k H_{k_2}$.*

Proof. Let τ be the translation part of γ and ρ be the rotational part of γ. If $k \in \mathcal{K}_\gamma$, there are $s \in \mathcal{F}_\gamma$ and $t \in \mathcal{R}_\gamma$ such that $k = s \cap t$. Then there are $s_1, s_2 \in \mathcal{F}_\tau = \mathcal{F}_\gamma$ such that $\tau = R_{s_1} R_s = R_s R_{s_2}$ and there are $t_1, t_2 \in \mathcal{F}_\rho = \mathcal{R}_\gamma$ such that $\rho = R_{t_1} R_t = R_t R_{t_2}$. Each sphere of $\{s, s_1, s_2\}$ is orthogonal to each sphere in $\{t, t_1, t_2\}$. The following relations hold.

$$R_s R_{t_1} = R_{t_1} R_s \qquad\qquad H_k = R_s R_t$$
$$R_{t_2} R_s = R_s R_{t_2} \qquad\qquad H_k = R_t R_s$$

Let $k_1 = s_1 \cap t_1$ and $k_2 = s_2 \cap t_2$. Then $H_{k_1} = R_{s_1} R_{t_1} = R_{t_1} R_{s_1}$ and $H_{k_2} = R_{s_2} R_{t_2} =$

$R_{t_2}R_{s_2}$. Since $\gamma = \tau\rho = \rho\tau$,

$$\begin{aligned} \gamma &= \tau\rho \\ &= R_{s_1}R_sR_{t_1}R_t \\ &= R_{s_1}R_{t_1}R_sR_t \\ &= H_{k_1}H_k \end{aligned} \qquad\qquad \begin{aligned} \gamma &= \rho\gamma \\ &= R_tR_{t_2}R_sR_{s_2} \\ &= R_tR_sR_{t_2}R_{s_2} \\ &= H_kH_{k_2} \end{aligned}$$

□

3.6 More on Half-Turn Banks

The half-turn bank of an isometry of hyperbolic space serves as a collection of planes in which the isometry can be factored into a product of two half-turns. In this section, we show that any half-turn factorization of an orientation preserving isometry comes from the defined set called half-turn bank.

The half-turn bank $\mathcal{K}_\gamma$ of an isometry γ of $\mathbb{H}^4$ is a collection of circles with a property that for each $P \in \mathcal{K}_\gamma$, there are $Q, R \in \mathcal{K}_\gamma$ such that $\gamma = H_QH_P = H_PH_R$ where H_P, H_Q and H_R are half-turns about their respective planes. It is defined as intersections of hyperplanes in the permuted pencil and invariant/twisting pencils of γ.

In section 3.6.1, the common perpendicular line across a pair of ultra-parallel $(n-2)$-dimensional planes is constructed. In section 3.6.2 all possible combinations of pairs of planes in $\mathbb{H}^4$ are enumerated to show the result of composing half-turns about the planes. Together with classification of isometries, the combinations tell how the involved planes intersect whenever an isometry is factored into a product of two half-turns. Section 3.6.3 has the details of proving that all half-turn factorizations come from half-turn banks. The theorem implies that a pair of isometries is linked by a half-turn if and only if they have a common circle in their half-turn banks.

3.6.1 Codimension-2 version of common perpendicular

If P and Q are ultra-parallel hyperplanes in $\mathbb{H}^4$, there is a unique line perpendicular to them. If P and Q are ultra-parallel lines in $\mathbb{H}^4$, there is also a unique line perpendicular to them. In this section, the same result is shown for a pair of ultra-parallel $(n-2)$-dimensional subplanes of $\mathbb{H}^n$.

Theorem 3.6.1. *For each pair of ultra-parallel planes $\alpha, \beta \in \mathbb{H}^4$, there is a unique line orthogonal to both α and β.*

More generally, the definitions of half-turn banks can be extended to orientation preserving isometries of $\mathbb{H}^n$, but to justify its name, a more general statement can be used.

Theorem 3.6.2. *Let $n \geq 2$ be an integer. For each pair of ultra-parallel $(n-2)$-dimensional planes $\alpha, \beta \in \mathbb{H}^n$, there is a unique line orthogonal to both α and β.*

Proof. Theorem 3.6.2 implies Theorem 3.6.1, so constructing the common orthogonal line in a general setting is enough for a case in Theorem 3.6.3. Let P and Q be ultra-parallel $(n-2)$-dimensional planes of $\mathbb{H}^n$. If $n = 2$, then P and Q are distinct points, and the unique line connecting them are vacuously orthogonal to P and Q. If $n = 3$, then P and Q are ultra-parallel lines so there is a unique line perpendicular to both of them. If $n \geq 4$, we use the hyperboloid model of $\mathbb{H}^n$ embedded in the Lorentzian space $\mathbb{R}^{n,1}$. Then P and Q extend to $(n-1)$-dimensional vector subspaces P' and Q' respectively. Since P and Q are ultra-parallel, the span of $P' \cup Q'$ is the whole $\mathbb{R}^{n,1}$. However, its dimension can not add up to $2n-2$ but only a maximum of $n+1$.

Therefore, the dimension of the space-like subspace $P' \cap Q'$ is either $n-2$ or $n-3$. If it is $n-2$, then $\operatorname{span}(P' \cup Q')$ is n-dimensional and has P' and Q' as its hyperplanes with a unique common perpendicular line. If $\dim(P' \cap Q')$ is $n-3$, then $N = \left(P' \cap Q'\right)^L$ is a 4-dimensional time-like vector subspace nontrivially intersecting both P' and Q' due to their large dimensions. Since N contains P'^L and Q'^L, which respectively intersect P' and Q' trivially, $N \cap P'$ and $N \cap Q'$ are 2-dimensional subspaces. If they are time-like, then there is a unique line in $\mathbb{H}^n$ perpendicular to both $\mathbb{H}^n \cap N \cap P'$ and $\mathbb{H}^n \cap N \cap Q'$. We must show that $N \cap P'$ and $N \cap Q'$ are time-like vector subspaces.

Pick a time-like vector vector $x \in N$. Then $x \notin P'^L \cup Q'^L$ since P'^L and Q'^L are space-like. Let $w \in P'^L$ and $v \in Q'^L$ be nontrivial elements so they are linearly independent with x. Since ${P'}^L$ and ${Q'}^L$ are subsets of N, the spans of $\{x, w\}$ and $\{x, v\}$ are subspaces of N. Then the vector $-\frac{\langle x,w\rangle_L}{\langle w,w\rangle_L} w + x$ is an element of both N and P' since it is Lorentz orthogonal to $w \in {P'}^L$, and is a linear combination of w and x. Then its Lorentz norm can be completed as follows.

$$\begin{aligned}\left\| -\frac{\langle x, w\rangle_L}{\langle w, w\rangle_L} w + x \right\|_L^2 &= \frac{\langle x, w\rangle_L^2}{\|w\|_L^2} - 2\frac{\langle x, w\rangle_L^2}{\|w\|_L^2} + \|x\|_L^2 \\ &= -\frac{\langle x, w\rangle_L^2}{\|w\|_L^2} + \|x\|_L^2\end{aligned}$$

Since x is time-like and w is space-like, the quantity above is negative, making the vector time-like. Hence, $N \cap P'$ is a 2-dimensional time-like subspace of P'. By replacing the vector w with v, the same arguments show that $N \cap Q'$ is also a 2-dimensional time-like subspace of Q'.

$\square$

3.6.2 Combinations of a pair of planes

There are a few ways for two planes in $\mathbb{H}^4$ to intersect. They can intersect in a line, a point, or in a point at infinity. The last case is not technically an intersection but points in the two planes can be arbitrarily near.

Let P, Q be distinct planes in $\mathbb{H}^4$. There are four ways they can intersect or not intersect.

1. P and Q are ultra-parallel.

2. P and Q are tangent to a unique point x at infinity.

3. $P \cap Q$ is a line.

4. $P \cap Q$ consists of a single point p.

Let h be a half-turn of $\mathbb{H}^4$ about some plane α. Then h is a type-I elliptic isometry with a well-defined permuted pencil $\mathcal{F}_h$ and invariant pencil $\mathcal{T}_h$. Recall that $\mathcal{F}_h$ consists of spheres that contain the boundary of α at infinity while $\mathcal{T}_h$ consists of spheres orthogonal to the boundary of α through two points. For each $c \in \mathcal{T}_h$, the image of c under h is also c. For each $s \in \mathcal{F}_h$, then $h(s)$ must also be in $\mathcal{F}_h$; but h is an involution so $h(s) = s$. Equivalently h leaves all spheres in both $\mathcal{F}_h$ and $\mathcal{T}_h$ invariant.

In this section, the model of $\mathbb{H}^4$ being used is the upper half space of $\mathbb{R}^4$ so that the boundary at infinity is $\widehat{\mathbb{R}^3}$. Going back to P and Q, let P' be the boundary of P in $\widehat{\mathbb{R}^3}$ and Q' be that of Q. By conjugation, assume P' is a straight Euclidean line in $\widehat{\mathbb{R}^3}$ and Q' is a Euclidean circle, except in the second case where we may assume that $x = \infty$. Then the elements of $\mathcal{K}_{H_P}$ are precisely the circles in planes orthogonal to P' and centered in points of P'. We investigate the composition $H_P H_Q$ and its invariant subplanes based on the combination of P and Q.

P and Q are ultra-parallel

If P and Q are ultra-parallel, then there is a unique line N in $\mathbb{H}^4$ that is orthogonal to both P and Q. Both H_P and H_Q leave N invariant so $H_P H_Q$ must also leave N invariant. That leaves two options for $H_P H_Q$: either hyperbolic or elliptic. Suppose $H_P H_Q$ has a fixed point y inside $\mathbb{H}^4$. Then $H_P(y) = H_Q(y)$, so the midpoint between y and $H_P(y)$ is an element of both P and Q. This contradicts the hypothesis that P and Q are ultra-parallel.

Lemma 3.6.1. *Let P and Q be ultra-parallel planes in $\mathbb{H}^4$. Then $H_P H_Q$ is a hyperbolic isometry.*

P and Q are tangent to a unique point x at infinity

Let P and Q be distinct planes in $\mathbb{H}^4$ so that their boundaries meet at one point x at infinity. By conjugation, we may assume $x = \infty$ of $\widehat{\mathbb{R}^3}$. Then P' and Q' are Euclidean lines. If P' and Q' form a Euclidean plane, there are several Euclidean lines perpendicular to both P' and Q' but all of them identify a unique direction or vector $v \in \mathbb{R}^3$. If they do not form a plane, there is a unique Euclidean line N orthogonal to P' and Q', which identifies a direction $v \in \mathbb{R}^3$. In either case, all Euclidean lines with same direction as v is left invariant by $H_P H_Q$. Futhermore, $H_P H_Q$ is a parabolic translation

$(x \mapsto x + 2v, \quad x \in \mathbb{R}^3)$ if P' and Q' are coplanar or a screw parabolic isometry with twisting plane bounded by N if not. So if P and Q are tangent, $H_P H_Q$ is definitely parabolic.

Lemma 3.6.2. *Let P and Q be tangent planes in $\mathbb{H}^4$. Then $H_P H_Q$ is a parabolic isometry.*

$P \cap Q$ is a line

Let P and Q be planes of $\mathbb{H}^4$ intersecting in a line L. Then they form a unique hyperplane h and it is an element of both permuted pencils of H_P and H_Q. It follows that h is left invariant, and L is fixed pointwise by $H_P H_Q$. The plane τ orthogonal to h through L is also left by $H_P H_Q$, but since L is fixed pointwise, $H_P H_Q$ also fixes τ pointwise. To illustrate it with the model $\widehat{\mathbb{R}^3}$ as the boundary at infinity, assume P' is a Euclidean line and Q' is a circle intersecting P' in two points y_1 and y_2 . Then $Q' \cup P'$ forms a unique Euclidean plane $\hat{h}$. There is a unique circle $\hat{\tau}$ passing through y_1 and y_2 centered at their midpoint, and inside the Euclidean plane orthogonal to $\hat{h}$ through P'. Then $\hat{\tau}$ is orthogonal to both P' and Q' through $P' \cap Q'$. Both half-turns H_P and H_Q flip $\hat{\tau}$ across $P' \cap Q'$ so the composition $H_P H_Q$ fixes $\hat{\tau}$ pointwise. Since $\hat{\tau}$ is the unique plane orthogonal to h through $P \cap Q$, then $\hat{\tau}$ is the only fixed point set of $H_P H_Q$. It follows that $H_P H_Q$ is a type-I elliptic isometry. Moreover, its twisting plane is orthogonal to $\hat{h}$.

Lemma 3.6.3. *Let P and Q be distinct planes in $\mathbb{H}^4$ intersecting in a line. Then $H_P H_Q$ is a type-I elliptic isometry whose twisting plane is orthogonal to the hyperplane containing $P \cup Q$ through $P \cap Q$.*

$P \cap Q$ consists of a single point p

Suppose P and Q are planes $\mathbb{H}^4$ intersecting in a unique point p. Then for each $x \in \mathbb{H}^4 \setminus \{p\}$, we show that $H_P H_Q(x) \neq x$. Suppose first that $x \in Q$. Then $H_P H_Q(x) = H_P(x)$ which is not equal to x since $x \notin P$. For the rest of this combination of P and Q, assume that $x \in \mathbb{H}^4 \setminus Q$. If it happens that $H_Q(x) \in P$, then $H_P H_Q(x) = H_Q(x) \neq x$.

If $H_Q(x) \notin P$ but $x \in P$, then $H_P H_Q(x) \notin P$ so $H_P H_Q(x) \neq x$.

The last possibility is when both $H_Q(x)$ and x are outside $Q \cup P$. We prove that H_P can not map $H_Q(x)$ back to x. Suppose $H_P H_Q(x) = x$. Let m be the midpoint between $H_P(x)$ and x, and let L be the line connecting x to $H_Q(x)$. Then m is in Q since it is the midpoint between x and $H_Q(x)$. Likewise m must also be in P as $H_P(x) = H_Q(x)$. There is only one point in $P \cap Q$ so this midpoint must be p. Furthermore, L is orthogonal to both P and Q through p. It follows that the hyperplane orthogonal to L through p contains both P and Q. Since intersecting planes within a hyperplane must meet in at least a line, $P \cap Q$ must have at least a line which has more points other than p. This contradicts that P and Q intersect only in a point.

Lemma 3.6.4. *Let P and Q be distinct planes in $\mathbb{H}^4$ intersecting in a single point. Then $H_P H_Q$ is a type-II orientation preserving elliptic isometry.*

Since $H_P H_Q$ is type II elliptic, one might wonder where the invariant planes or lines are located. If P and Q are orthogonal complements of each other, then H_Q and H_P respectively rotate them half-way around. Otherwise, $H_P H_Q$ has a canonical pair of invariant planes. To locate these invariant planes, we can use the ball model of $\mathbb{H}^4$ embedded in $\mathbb{R}^4$ which is conformal to both the Euclidean geometry and the spherical geometry of S^3. The problem simplifies further if P and Q are conjugated to intersect at the origin. The following lemma locates the canonical invariant planes relative to P and Q.

Lemma 3.6.5. *Let P and Q be 2-dimensional vector subspaces of $\mathbb{R}^4$ intersecting trivially. Then there are 2-dimensional vector subspaces τ_1 and τ_2 orthogonal complements of each other such that τ_1 and τ_2 are orthogonal to P and Q through two separate lines.*

Proof. Let $U_P = S^3 \cap P$ and $U_Q = S^3 \cap Q$ be unit (Euclidean) circles. The Euclidean inner product restricted to $U_P \times U_Q$ has a maximum value realized by a pair (v_P, v_Q) since $U_P \times U_Q$ is compact and the inner product is continuous. The vectors v_P, v_Q can be augmented by $w_P \in P$ and $w_Q \in Q$ so that the sets $\{v_P, w_P\}$ and $\{v_Q, w_Q\}$ are orthonormal bases. Let $\tau_1 = \text{span}\{v_P, v_Q\}$ and $\tau_2 = \text{span}\{w_P, w_Q\}$. The dimension of τ_i is 2 since $v_P \neq v_Q$. It follows that $\tau_1 \cap \tau_2 = \{0\}$ and they intersect P and Q through two separate lines. What is left to show is that τ_1 and τ_2 are orthogonal to P, Q and each other. It is sufficient to show that $\langle v_P, w_Q\rangle = \langle v_Q, w_P\rangle = 0$.

The function $\theta \mapsto \langle(\cos\theta)v_P + (\sin\theta)w_P, v_Q\rangle$ is continuous and smooth with a maximum at $\theta = 0$. Its derivative $\theta \mapsto \langle-(\sin\theta)v_P + (\cos\theta)w_P, v_Q\rangle$ therefore has a zero value at $\theta = 0$. Hence $\langle v_Q, w_P\rangle = 0$. Similarly the function $\theta \mapsto \langle(\cos\theta)v_Q + (\sin\theta)w_Q, v_P\rangle$ has derivative $\theta \mapsto \langle-(\sin\theta)v_Q + (\cos\theta)w_Q, v_P\rangle$ with a zero value at $\theta = 0$, implying that $\langle w_Q, v_P\rangle = 0$. □

Corollary 3.6.1. *Let P and Q be orthogonal planes in $\mathbb{H}^4$ intersecting in a unique point. Then $H_P H_Q$ is a type II elliptic involution that leaves every line through $P \cap Q$ invariant.*

Proof. Since P and Q are orthogonal, they are left invariant by both H_P and H_Q. The half-turns are involutions themselves so applying $H_P H_Q$ twice to $P \cup Q$ is the identity map on $P \cup Q$. The conformal ball model can be used to conjugate $H_P H_Q$ into an element of SO(4), with p corresponding to the origin. Then P and Q form vector spaces that are orthogonal complements of each other. Any Euclidean orthonormal bases of them can be combined into an orthonormal basis $\mathcal{B}$ of $\mathbb{R}^4$. If $x \in \mathbb{H}^4 = B_4$ is outside $P \cup Q$, it can be expressed as a linear combination of vectors in $\mathcal{B} \subset \partial(P \cup Q)$. The composition $(H_P H_Q)^2$ as an element of SO(4) therefore maps x back to itself. Hence $H_P H_Q$ is an involution.

A matrix of SO(n) has its inverse and transpose equal, but if it is also an involution, then $H_P H_Q$ as a matrix is also symmetric and thus diagonalizable (Spectral Theorem). There are four linearly independent eigenvectors that correspond to lines in $\mathbb{H}^4$ that are pairwise perpendicular through p. The diagonal entries are all -1 since a 1 value would make $H_P H_Q$ have more than one fixed point and other values would make the matrix not in SO(4). Each vector is hence mapped into its opposite. In the conformal ball model of hyperbolic space, the action of $H_P H_Q$ on lines passing through p is reflection across p. □

Corollary 3.6.2. *Let P and Q be non-orthogonal planes in $\mathbb{H}^4$ intersecting in a unique* *point. Then $H_P H_Q$ is a type II elliptic isometry with a unique unordered pair of invariant planes orthogonal to each other through the fixed point of $H_P H_Q$.*

Proof. If P and Q intersect only in one point, then $H_P H_Q$ is type II elliptic isometry. There are planes τ_1 and τ_2 that are orthogonal to each other through the fixed point of $H_P H_Q$ and also orthogonal to both P and Q through separate lines. Both H_P and H_Q leave τ_1 and τ_2 invariant since they are orthogonal to the half-turns' fixed point sets. Then τ_1 and τ_2 are also left invariant by $H_P H_Q$.

It must be shown that τ_1 and τ_2 are the unique invariant planes. In order to show their uniqueness, $H_P H_Q$ can be conjugated to a 4×4 matrix in SO(4) so that the upper-left and lower-right 2×2 blocks are elements of SO(2) (Theorem 3.1.7). Since $H_P H_Q$ has only one fixed point, either both these blocks are diagonal matrices with -1 in its entries or one of these blocks is non-diagonalizable.

Recall that the construction of τ_1 allows it to have $v_P \in P$ and $v_Q \in Q$ so that the angle between v_P and v_Q is at minimum. If P and Q are not orthogonal, this angle is less than $\pi/2$. The action of $H_P H_Q$ on τ_1 is a composition of reflections across span$\{v_P\}$ and span$\{v_Q\}$. Thus, one of the non-diagonalizable block correspond to the rotation of τ_1 in an angle other than 0 and π. Then τ_1 is the unique invariant plane of $H_P H_Q$ with minimum angle or rotation. The orthogonal complement of τ_1 is τ_2 which is also unique. □

3.6.3 The half-turn bank is exhaustive

In this section, we show that every half-turn factorization of an orientation preserving isometry comes from its half-turn bank. The proof still uses cases but the lemmas from section 3.6.2 restrict the possibilities for how a pair of planes intersect.

Theorem 3.6.3. *Let γ be an orientation preserving isometry of $\mathbb{H}^4$. Then for every half-turn factorization $H_P H_Q$ of γ, the circles ∂P and ∂Q are elements of $\mathcal{K}_\gamma$.*

Proof. The proofs depend on the class of isometry of γ.

- γ is hyperbolic.

If γ is hyperbolic, let L be its axis. The only combination for P and Q is that they are ultra-parallel. The common perpendicular line of P and Q is left invariant by $H_P H_Q$ so it must be L. Then both P and Q are orthogonal to the axis of γ. Let h_P be the hyperplane spanned by L and P. Similarly, let h_Q be the hyperplane spanned by L and Q. Then $\partial h_P \in \mathcal{F}_{H_P}$ and $\partial h_Q \in \mathcal{F}_{H_Q}$ with $P \subset h_P$; $Q \subset h_Q$.

If $h_P = h_Q$, there are $f_P \in \mathcal{F}_{H_P}$ and $f_Q \in \mathcal{F}_{H_Q}$ such that $H_P = R_{f_P} R_{h_P}$ and $H_Q = R_{h_Q} R_{f_Q}$ are reflection factorizations. If $h_P = h_Q$, then $\gamma = H_P H_Q = R_{f_P} R_{f_Q}$ is a pure hyperbolic isometry with $\partial f_P, \partial f_Q \in \mathcal{F}_\gamma$ and $\partial h_P = \partial h_Q \in \mathcal{T}_\gamma$. So $\partial P = \partial(h_P \cap f_P) \in \mathcal{K}_\gamma$ and $\partial Q = \partial(h_Q \cap h_Q) \in \mathcal{K}_\gamma$.

If $h_P \cap h_Q$ is a plane τ, then τ intersects P and Q in two different lines. The half-turns H_P and H_Q reflect τ across these lines and thus leave τ invariant. If γ is pure loxodromic, then $h_P \neq h_Q$, so the twisting plane of γ matches τ. Still, the hyperplanes $h_Q^\perp$ orthogonal to h_Q through Q and $h_P^\perp$ orthogonal to h_P through P are also orthogonal to L so $\partial h_P^\perp, \partial h_Q^\perp \in \mathcal{F}_\gamma$. Both h_P and h_Q contain τ so $\partial h_P, \partial h_Q \in \mathcal{R}_\gamma$. Since $P = h_P \cap h_P^\perp$ and $Q = h_Q \cap h_Q^\perp$, then $\partial P, \partial Q \in \mathcal{K}_\gamma$.

- γ is parabolic.

If γ is parabolic, then P and Q are tangent at infinity. For simpler illustration, assume that the fixed point of γ is ∞. Then the boundaries $(\partial P, \partial Q)$ of P and Q are straight non-crossing Euclidean lines in $\mathbb{R}^3$. Any Euclidean line commonly perpendicular to the boundaries of P and Q forms the same direction exactly equal to that of γ. Recall that the direction of the map $x \mapsto Ax + b$ is the Euclidean line spanned by b in $\mathbb{R}^3$. Let B the direction of γ. Then B is either Euclidean-parallel or equal to any common perpendicular between ∂P and ∂Q. Let h_P and h_Q be the Euclidean planes orthogonal to B through ∂P and ∂Q respectively. Let f_P be the Euclidean plane orthogonal to h_P through ∂P and let f_Q be Euclidean plane orthogonal to h_Q through ∂Q. Then $h_P, f_P \in \mathcal{F}_{H_P}$ and $h_Q, f_Q \in \mathcal{F}_{H_Q}$. Since (h_P, f_P) and (h_Q, f_Q) are pairwise orthogonal, then $H_P = R_{h_P} R_{f_P}$ and $H_Q = R_{f_Q} R_{h_Q}$ are reflection factorizations.

If $f_P = f_Q$, then $\gamma = H_P H_Q = R_{h_P} R_{h_Q}$ is pure parabolic, so $f_P = f_Q \in \mathcal{T}_\gamma$ and

$h_P, h_Q \in \mathcal{F}_\gamma$. If $f_P \cap f_Q$ is a Euclidean line τ, it is left invariant by H_P, H_Q and hence by γ. This implies that τ bounds the twisting plane of γ and therefore $f_P, f_Q \in \mathcal{R}_\gamma$. Still, $\partial P, \partial Q \in \mathcal{K}_\gamma$.

- γ is type-I elliptic.

  If γ is type-I elliptic, then P and Q intersect in a line. Let τ be the twisting plane of γ. There is also a unique hyperplane h spanned by P and Q. Then τ is orthogonal to h through $P \cap Q$. The planes P, Q and τ pairwise intersect at $P \cap Q$ while τ is orthogonal to both P and Q through $P \cap Q$. Let h_P be the hyperplane spanned by τ and P; let h_Q be the hyperplane spanned by Q and τ. It follows that $\partial h \in \mathcal{T}_\gamma$ and $\partial h_P, \partial h_Q \in \mathcal{F}_\gamma$. Since $P = h_P \cap h$ and $Q = h_Q \cap h$, then $\partial P, \partial Q \in \mathcal{K}_\gamma$.

- γ is type-II elliptic.

  If γ is type-II elliptic, then P and Q intersect in a unique point p. Using the conformal ball model of $\mathbb{H}^4$ inside $\mathbb{R}^4$, we may assume that p is the origin. Then P and Q extend to Euclidean planes that intersect only at the origin. There are planes τ_1 and τ_2 that has properties in Lemma 3.6.5. Let h_P, h_Q, h_1 and h_2 be hyperplanes defined as follows.

  $$h_P = \text{span}(\tau_1 \cup P) \qquad h_Q = \text{span}(\tau_2 \cup Q)$$
  $$h_1 = \text{span}(Q \cup \tau_1) \qquad h_2 = \text{span}(P \cup \tau_2)$$

  The configuration of planes and hyperplanes yield to $\tau_1 \subset h_P \cap h_1$, $\tau_2 \subset h_Q \cap h_2$, $P = h_P \cap h_2$ and $Q = h_Q \cap h_1$.

  There are two options for γ and the planes τ_1, τ_2. Either γ has an associated angle that is not an integer multiple of π (Corollary 3.1.3) or γ is an involution (Corollary 3.6.1). In the former case, γ has a unique pair of invariant planes that must match τ_1 and τ_2. By Corollary 3.1.3, $\gamma = \rho_1\rho_2$ where ρ_1 and ρ_2 are type I elliptic isometries whose respective twisting planes are τ_1 and τ_2. In the latter case, P and Q are already in $\mathcal{K}_\gamma$.

  In any option for γ, the planes P, Q have their boundaries in as elements of $\mathcal{K}_\gamma$.

□

Corollary 3.6.3. *Let A and B be orientation preserving isometries of $\mathbb{H}^4$. Then the pair A, B is linked by half-turns if and only if $\mathcal{K}_A \cap \mathcal{K}_B$ is nonempty.*

Proof. If A and B are linked by half-turns, then there are planes α, β and δ such that $A = H_\alpha H_\beta$ and $B = H_\beta H_\delta$. But β is an element of both $\mathcal{K}_A$ and $\mathcal{K}_B$.

If $\mathcal{K}_A \cap \mathcal{K}_B$ is nonempty, let β be one of its elements. Then there are $\alpha \in \mathcal{K}_A$ and $\beta \in \mathcal{K}_B$ such that $A = H_\alpha H_\beta$ and $B = H_\beta H_\delta$. □

Chapter 4

Conditions for Linking in Dimension 4

In this chapter, some conditions for a pair of isometries in $\mathbb{H}^4$ to be linked are stated and proved. The main idea is to find a geometric or computational requirements for a given pair to be linked. The conditions are divided into cases depending on which type of isometry is given. Since the elliptic isometries are not included, there are total of ten cases. Note that in the definition of linked pairs, the order of the isometries matters. Whereas in this chapter, the order of the isometries do not matter once they are linked. This helps in forming an algorithm in the discreteness test of the group generated by the pair.

The conditions for linking pairs with twisting planes are quite demanding, justifying the theorem of Basmajian and Maskit [1] that linked pairs are of measure zero. Still, pencils can be a useful tool for finding a common perpendicular plane to factorize orientation preserving isometries. In particular, planes in which half-turn factoring are constructed all fit in the elements of the pencil of an isometry.

4.1 The Ten Cases

The following list is the ten cases for combinations of types of two isometries.

1. Both are pure hyperbolic.

2. Both are pure parabolic.

3. Pure hyperbolic and pure parabolic

4. Pure parabolic and pure loxodromic

5. Screw parabolic and pure hyperbolic

6. Screw parabolic and pure parabolic

7. Pure loxodromic and pure hyperbolic

8. Both are screw parabolic.

9. Both are pure loxodromic.

10. Screw parabolic and pure loxodromic

4.1.1 Both are pure hyperbolic

Theorem 4.1.1. *If A and B are pure hyperbolic isometries of $\mathbb{H}^4$ with ultra-parallel axes, then there is a plane P orthogonal to both axes of A and B. Hence, A and B are linked.*

Proof. Suppose both A and B are pure hyperbolic with ultra-parallel axes. Let L be the common perpendicular line between the axes of A and B. If the axes of A and B lie in a plane C, then $L \subset C$ and there are plenty of planes that are orthogonal to C containing L. Pick P to be any of those planes. Then $\partial P \in \mathcal{K}_A \cap \mathcal{K}_B$. If the axes of A and B do not lie in the same plane, the axis of A and L still lie in a plane P_1. As the axis of B and L intersect in a single point, so does P_1 and axis of B and they are contained is a unique hyperplane C. There is a unique plane P orthogonal to C through L. Since both axes of A and B lie in the hyperplane C, then P must be orthogonal to both of them. Then $\partial P \in \mathcal{K}_A \cap \mathcal{K}_B$. It follows that there are $P_1, P_2 \in \mathcal{K}_A$ and $P_3, P_4 \in \mathcal{K}_B$ such that $A = H_{P_1}H_P = H_P H_{P_2}$ and $B = H_{P_3}H_P = H_P H_{P_4}$. □

4.1.2 Both are pure parabolic

Theorem 4.1.2. *Let A and B be pure parabolic isometries of $\mathbb{H}^4$. Then A and B are linked.*

Proof. Suppose their fixed points x_A and x_B are not equal. Then there is a unique line $L =$ connecting x_A to x_B. Let $h_A \in \mathcal{F}_A$ be the element containing x_B and $h_B \in \mathcal{F}_B$ the element containing x_A. Then $h_B \cap h_A$ contains the line L. Either $h_A \cap h_B$ is a circle or

$h_A = h_B$. Let P be a circle in $h_A \cap h_B$ through x_A and x_B. Thus P is a circle in different or same spheres in $\mathcal{F}_A$ and $\mathcal{F}_B$. Then $P \in \mathcal{K}_A \cap \mathcal{K}_B$. Hence, there are $P_1, P_2 \in \mathcal{K}_A$ and $P_3, P_4 \in \mathcal{K}_B$ such that $A = H_{P_1} H_P = H_P H_{P_2}$ and $B = H_{P_3} H_P = H_P H_{P_4}$.

If $x_A = x_B$, there are still $h_A \in \mathcal{F}_A$ and $h_B \in \mathcal{F}_B$ that has three possibilities: $h_A = h_B$; $P = h_A \cap h_B$ is a circle; or $h_A \cap h_B = \{x_A\}$. In the first case, let $t \in \mathcal{T}_A$. Then $h_B \cap t \in \mathcal{K}_A \cap \mathcal{K}_B$ and so A and B are linked. In the second case, $P \in \mathcal{K}_A \cap \mathcal{K}_B$ and links A and B. In the last case, let $\mathcal{F}_A = \mathcal{F}_B$ which implies that $\mathcal{K}_B = \mathcal{K}_A$ whose elements make A and B linked. $\square$

4.1.3 Pure hyperbolic and pure parabolic

Theorem 4.1.3. *Let A and B be isometries of $\mathbb{H}^4$. Suppose A is pure hyperbolic, and B is pure parabolic and* $\operatorname{fix}(A) \cap \operatorname{fix}(B) = \emptyset$. *Then A and B are linked.*

Proof. Let v be the fixed point of B and L be the axis of A. Then there is $h_x \in \mathcal{F}_A$ containing v. If $\hat{h}$ is the hyperplane bounded by h_x, there is a unique point $a \in \hat{h} \cap L$. Then there is a unique $h_a \in \mathcal{F}_B$ which bounds a hyperplane that contains a. Since the hyperplanes bounded by h_x and h_a intersect in a, their intersection $h_x \cap h_a$ is at least a plane orthogonal to L and bounded by v. Otherwise $h_x = h_a$ and one can pick a plane $P \subseteq h_x \cap h_a$ bounded by v and passes through a. Then $P \in \mathcal{K}_A \cap \mathcal{K}_B$ so A and B are linked. $\square$

4.1.4 Pure parabolic and pure loxodromic

Let A be a pure parabolic isometry of $\mathbb{H}^4$ and B a pure loxodromic isometry of $\mathbb{H}^4$. Suppose the fixed points of A and B in $\widehat{\mathbb{R}^3}$ are disjoint. The following are conditions for A and B to be linked.

Theorem 4.1.4 (Condition 1)**.** *Suppose there is an $h \in \mathcal{F}_A \cap \mathcal{F}_B$. Then A and B are linked.*

Proof. Let L be the boundary of the twisting plane of B, and let x be the fixed point A. Since $L \in \mathcal{D}_B$, h intersects L in two points y and z. If x is equal to either y or

z, then L is the unique element of $\mathcal{D}_B$ containing x. It allows any plane of sphere t in $\widehat{\mathbb{R}^3}$ containing L to be an element of both $\mathcal{T}_A$ and $\mathcal{R}_B$. Hence, $h \cap t$ is an element of both $\mathcal{K}_A$ and $\mathcal{K}_B$ that links A and B. If x, y, and z are three distinct points, they form a unique circle L_2 that must be a subset of h. Then L_2 is perpendicular to L so it is in $\mathcal{K}_B$. It also passes through x so $L_2 \in \mathcal{K}_A$. Thus $L_2 \in \mathcal{K}_A \cap \mathcal{K}_B$ so A and B are linked. □

Theorem 4.1.5 (Condition 2). *If the fixed point of A is in the boundary of the twisting plane of B, then A and B are linked.*

Proof. Let x be the fixed point of A. Since x is not the fixed point of B, there is a unique $h_x \in \mathcal{F}_B$ containing x. Let L be the boundary of the twisting plane of B. Then $L \in \mathcal{D}_B$ intersects h in two points, one of which is x. Let m be the other intersection point. There is also a unique $h_m \in \mathcal{F}_A$ containing m. Either $h_m = h_x$ which is done in the first condition, or $h_m \cap h_x$ is a circle c since both h_m and h_x contain x and m. We must show $c \in \mathcal{K}_A \cap \mathcal{K}_B$. The intersection points of c and L are x and m, but $c \subset h_x \in \mathcal{F}_B$ so $c \in \mathcal{K}_B$. As A is pure parabolic, any circle inside any element of $\mathcal{F}_A$ and passing through x is an element of $\mathcal{K}_A$. Hence, $c \in \mathcal{K}_A$ as $x \in c \subset h_m \in \mathcal{F}_A$ and $x \in c$. Then other planes can be found so that A and B are linked. □

Theorem 4.1.6 (Condition 3). *Let $L \subset \widehat{\mathbb{R}^3}$ be the boundary of the twisting plane of B. If three points in L form a subset of h for some $h \in \mathcal{F}_A$, then A and B are linked.*

Proof. The three points determine L and are sufficient for L to fit inside h. Let $x \in \widehat{\mathbb{R}^3}$ be the fixed point of A. The case where $x \in L$ is is handled in the previous condition. We may assume $x \notin L$. Then there is a unique $c_x \in \mathcal{K}_B$ containing x. We must show $c_x \in \mathcal{K}_A$. As $c_x \in \mathcal{K}_B$, it must intersect L in two points which are in h. But h contains x since A is parabolic, so it has three points in common with c_x. They are sufficient for c_x to be a subset of h. A circle or line in h containing x is an intersection of h with some element of $\mathcal{T}_A$. Hence c_x having this property, must be in $\mathcal{K}_A$. This implies that A and B are linked. □

4.1.5 Screw parabolic and pure hyperbolic

Let A be screw parabolic and B be pure hyperbolic isometries of $\mathbb{H}^4$. Assume they have disjoint fixed points in the boundary. Throughout this case, let x be the fixed point of A and let L be the boundary of its twisting plane. Likewise, let d_x be the unique element of $\mathcal{D}_B$ containing x.

Theorem 4.1.7 (Condition 1)**.** *If there is an $h \in \mathcal{F}_A \cap \mathcal{F}_B$, then A and B are linked.*

Proof. Let a and b be points in $\widehat{\mathbb{R}^3}$ so that $d_x \cap h = \{x, b\}$ and $L \cap h = \{x, a\}$. If $a = b$, then $d_x = L$ so any sphere c containing L is an element of both $\mathcal{T}_B$ and $\mathcal{R}_A$. It follows that $c \cap h$ is an element of both $\mathcal{K}_A$ and $\mathcal{K}_B$. If $a \neq b$, the points a, b, and x form a unique circle L_2 in $\widehat{\mathbb{R}^3}$. Because $a, b, x \in h$, $L_2 \subset h$. We claim that $L_2 \in \mathcal{K}_A \cap \mathcal{K}_B$. First, L_2 intersects L in two points, a and x, while sitting in $h \in \mathcal{F}_A$. Hence, $L_2 \in \mathcal{K}_A$. Next, L_2 intersects d_x in two points, b and x while sitting in $h \in \mathcal{F}_B$. Thus, $L_2 \in \mathcal{K}_B$. The half-turn around L_2 is the desired common factor of A and B linking them. □

Theorem 4.1.8 (Condition 2)**.** *If $L \subset h$ for some $h \in \mathcal{F}_B$, then A and B are linked.*

Proof. The hypothesis implies that h is the unique element of $\mathcal{F}_B$ containing x. Let b be the intersection of d_x and h in $\widehat{\mathbb{R}^3}$ other than x. There is a unique $h_b \in \mathcal{F}_A$ containing b. Let a be the intersection of h_b and L other than x. Connect a, b and x with the unique circle L_2. Then L_2 is a subset of both h_b and h. It also connects L and d_x in two distinct points. Hence, $L_2 \in \mathcal{K}_A \cap \mathcal{K}_B$ so A and B are linked. □

Theorem 4.1.9 (Condition 3)**.** *If either $m \in L$ or d_x is orthogonal to k_m, then A and B are linked.*

The definition of m and k_m are as follows. There is a unique element $h_x \in \mathcal{F}_B$ containing x, and it is orthogonal to d_x through two intersection points, one of which is x. Let m be the other intersection point. If $m \notin L$ there are unique elements $h_m \in \mathcal{F}_A$ and $k_m \in \mathcal{K}_A$ containing m.

Proof. Suppose $m \in L$. It is possible that $d_x = L$ which implies that $h_m = h_x$ reducing the case to the previous condition. If $d_x \neq L$, $h_m \cap h_x$ is a circle c containing m and

x. It follows that $c \subset \mathcal{K}_A \cap \mathcal{K}_B$ since it intersects d_x and L orthogonally through two points. Hence, A and B are linked.

If $m \notin L$, the hypothesis requires that d_x is orthogonal to k_m. It follows that $k_m \subset h_x$ since all circles orthogonal to d_x at x and m are contained in h_x. Therefore, $k_m \in \mathcal{K}_B$ so A and B are linked. $\square$

Theorem 4.1.10 (Computational Condition)**.** *Let A be a screw parabolic isometry of $\mathbb{H}^4$ leaving the z-axis of $\widehat{\mathbb{R}^3}$ invariant and fixing ∞. Let B be a pure hyperbolic isometry of $\mathbb{H}^4$ fixing v and w in $\mathbb{R}^3$. If v and w are equidistant to the z-axis then A and B are linked.*

If $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3) \in \mathbb{R}^3$, we say that v and w are equidistant to the z-axis if $v_1^2 + v_2^2 = w_1^2 + w_2^2$.

Proof. The unique element $d_\infty \in \mathcal{D}_B$ containing ∞ is the Euclidean line connecting v and w. On the other hand, the unique element $h_\infty \in \mathcal{F}_B$ containing ∞ is the orthogonal complement of d_∞ through the midpoint m of v and w. Then

$$m = \left(\frac{v_1 + w_1}{2}, \frac{v_2 + w_2}{2}, \frac{v_3 + w_3}{2} \right).$$

Let $m = (m_1, m_2, m_3)$; then $m_1 = \frac{v_1+w_1}{2}$ and $m_2 = \frac{v_2+w_2}{2}$. If $m_1 = 0 = m_2$, then the hypothesis implies that $v_1 = v_2 = w_1 = w_2 = 0$ so d_∞ is the boundary of the twisting plane of A. It follows that h_∞ is a horizontal Euclidean plane and therefore, $h_\infty \in \mathcal{F}_A$. Hence, $m_1 = 0 = m_2$ implies the first condition which is proven to link A and B.

If one of m_1 and m_2 is nonzero, m does not lie in z-axis so the unique element k_m of $\mathcal{K}_A$ passing through m is the Euclidean line connecting m and $(0, 0, m_3)$. Computing the angle between d_∞ and k_m may come from the inner product $(-m_1, -m_2, 0) \cdot (v - m)$, but the hypothesis imply that $v - m = \left(0, 0, \frac{v_3 - w_3}{2}\right)$. It follows that k_m is orthogonal to d_∞ and it satisfies the third condition so A and B are linked. $\square$

Proposition 4.1.1. *If A is a screw parabolic isometry of $\mathbb{H}^4$ fixing ∞, leaving z-axis invariant, and B a pure hyperbolic isometry of $\mathbb{H}^4$ fixing v and w in $\mathbb{R}^3$, then conditions* 1, 2 *and* 3 *imply the computational condition.*

4.1.6 Screw parabolic and pure parabolic

Let A be screw parabolic and B be a pure parabolic isometries of $\mathbb{H}^4$ with different fixed points. Throughout this case, let x be the fixed point of A and y be the fixed point of B. Suppose L is the boundary of the twisting plane of A. There are unique elements $d_x \in \mathcal{D}_B$ containing x and $d_y \in \mathcal{D}_A$ containing y. If $y \notin L$, denote the unique element of $\mathcal{K}_A$ containing y by k_y.

Theorem 4.1.11 (Condition 1)**.** *If there is an $h \in \mathcal{F}_A \cap \mathcal{F}_B$, then A and B are linked.*

Proof. As A and B are both parabolic, h contains both x and y. Since $L \in \mathcal{D}_A$, it intersects h orthogonally in a point a other than x. If $a = y$, $L = d_x$ so any circle in h connecting x and a is an element of $\mathcal{K}_A \cap \mathcal{K}_B$ making A and B linked. If $a \neq y$ the points a, x and y form a circle c in h. The circle c intersects L in two points and also d_x in x and y. Hence, c is in both $\mathcal{K}_A$ and $\mathcal{K}_B$ so A and B are linked. □

Theorem 4.1.12 (Condition 2)**.** *If $y \in L$, then A and B are linked.*

Proof. There are spheres $h_x \in \mathcal{F}_B$ and $h_y \in \mathcal{F}_A$ containing both x and y. If $h_x = h_y$, A and B are linked as per previous condition. Otherwise, $h_x \cap h_y$ is a circle k passing through x and y. If $y \in L$, $k \in \mathcal{K}_A$ but k is also in $\mathcal{K}_B$ so A and B are linked. □

Theorem 4.1.13 (Condition 3)**.** *If $L \subset h$ for some $h \in \mathcal{F}_B$, then A and B are linked.*

Proof. We may assume $y \notin L$. Otherwise, the linking of A and B is implied by the previous condition. Then there is a unique $k_y \in \mathcal{K}_A$ containing y. It intersects L in a point p other than x. The circle k_y is uniquely determined by the points $x, p, y \in h$. Hence, $k_y \subset h$ so $k_y \in \mathcal{K}_B$ since B is pure parabolic. The half-turn about k_y links A and B. □

Theorem 4.1.14 (Condition 4)**.** *If $k_y \perp d_x$, then A and B are linked.*

If $y \notin L$, there is a unique $k_y \in \mathcal{K}_A$ containing y. If $y \in L$, there are many choices for $k_y \in \mathcal{K}_A$ but choosing $k_y \perp d_x$ is not necessary.

Proof. There is a unique $h_x \in \mathcal{F}_B$ containing x. It intersects d_x in x and y which are also in k_y. Every circle orthogonal to d_x through x and y must be a subset of h_x so $k_y \subset h_x$. The span of $k_y \cup d_x$ is an element of $\mathcal{T}_B$ so $k_y \in \mathcal{K}_B$. Therefore, A and B are linked. $\square$

4.1.7 Pure loxodromic and pure hyperbolic

Linking pairs of hyperbolic isometries are better expressed inside $\mathbb{H}^4$. The hyperplanes and planes in $\mathbb{H}^4$ bounded by the pencils have as much interesting properties.

Let A be a pure loxodromic isometry of $\mathbb{H}^4$ with axis Ax_A and twisting plane τ_A. Let B be a pure hyperbolic isometry of $\mathbb{H}^4$ with axis Ax_B. Assume Ax_A and Ax_B are disjoint. Then there is a unique line N perpendicular to both Ax_A and Ax_B. It intersects A in a point a and B in a point b. There is a unique hyperplane h_a orthogonal to Ax_A through a. Likewise, there is a unique hyperplane h_b orthogonal to Ax_B through b. Since $\mathrm{Ax}_A \subset \tau_A$, there is a unique line $L_a \subset \tau_A$ perpendicular to Ax_A through a. Throughout this case, A, B, Ax_A, Ax_B, τ_A, N, a, b and L_a are used.

Theorem 4.1.15 (Condition 1)**.** *If $L_a = N$, then A and B are linked.*

Proof. The hypothesis implies that τ_A intersects Ax_B at least at b. If $\mathrm{Ax}_B \subset \tau_A$, there are plenty of planes orthogonal to τ_A through N. Any of them is orthogonal to Ax_B. If Ax_B intersect τ_A only at b, they span a hyperplane h. There is a plane k orthogonal to h through N. Since h contains τ_A and Ax_B, k is also orthogonal to τ_a and Ax_B. (By dimension count, $k \subset h_b$.) Hence, k is an element of both $\mathcal{K}_A$ and $\mathcal{K}_B$, so A and B are linked. $\square$

Theorem 4.1.16 (Condition 2)**.** *If $L_a \subset h_b$, Then A and B are linked.*

Proof. If $L_a \subset h_b$, it is possible that $L_a = N$, which is the previous case. We may assume that $L_a \neq N$. Then L_a and N span a plane k which sits in h_a. It follows that $k \subset h_b$ since both L_a and N lie in h_b. Thus k is orthogonal to Ax_B and τ_A so $k \in \mathcal{K}_B$ linking A and B. $\square$

Computational procedure for testing the link between A and B.

1. Compute the common perpendicular N and its endpoints $n_1, n_2 \in \widehat{\mathbb{R}^3}$.

2. Locate the intersection points a and b with Ax_A and Ax_B respectively.

3. Compute the distance between a and b. Build the pure hyperbolic isometry P with fixed points n_1 and n_2 (i.e. along N) with translation length equal to that distance.

4. Compute L_a which is the perpendicular line to Ax_A through a and inside the twisting plane.

5. Compute/Locate M_a which is the image of L_a under P. Automatically, $P(a) = b$.

6. Compute the angle between Ax_B and M_a or check if M_a is perpendicular to Ax_B.

7. If they are perpendicular, A and B are linked.

4.1.8 Both are screw parabolic

Let A and B be screw parabolic isometries of $\mathbb{H}^4$, with disjoint fixed points x and y respectively. Let τ_A and τ_B be their respective twisting planes. Define $L_A = \partial\tau_A$ and $L_B = \partial\tau_B$. As $x \neq y$, there are unique elements $h_x \in \mathcal{F}_B$ and $h_y \in \mathcal{F}_A$ such that $x \in h_x$ and $y \in h_y$. The conditions in which a common orthogonal plane exists are quite restrictive, so A and B are highly unlikely linked.

Theorem 4.1.17 (Condition 1)**.** *If there is $h \in \mathcal{F}_A \cap \mathcal{F}_B$ and the points x, y, x_h and y_h form a circle, Then A and B are linked.*

Any $h \in \mathcal{F}_A$ intersect L_A in two points. Let x_h be the intersection point other than x. If $h \in \mathcal{F}_B$, let y_h be the element of $h \cap L_B$ other than y.

Proof. Let c be the circle formed by x, y, x_h and y_h as allowed by the hypothesis. We must show $c \in \mathcal{K}_A \cap \mathcal{K}_B$. Since $\{x, y, x_h, y_h\} \subset h$, $c \subset h$. Also c is orthogonal to L_A through $\{x, x_h\}$ so $c \in \mathcal{K}_A$. Similarly, c is orthogonal to L_B through $\{y, y_h\}$ so $c \in \mathcal{K}_B$. Hence A and B are linked. $\square$

Theorem 4.1.18 (Condition 2)**.** *If $y \in L_A$ and $x \in L_B$, then A and B are linked.*

Proof. We may assume $h_x \neq h_y$; otherwise $L_A = L_B$ making any circle in h_x that connects x to y an element of $\mathcal{K}_A \cap \mathcal{K}_B$. Let c be $h_x \cap h_y$ which is a circle containing $\{x, y\}$. The hypothesis imply that $h_y \cap L_A = \{x, y\} = h_x \cap L_B$, so c is orthogonal to both L_A and L_B. Therefore, $c \in \mathcal{K}_A \cap \mathcal{K}_B$. The existence of c makes A and B linked. □

Theorem 4.1.19 (Condition 3)**.** *If $h_y \cap L_A \subset h_x$, $h_x \cap L_B \subset h_y$ and $h_x \neq h_y$, then A and B are linked.*

Proof. Let c be $h_x \cap h_y$. Then c is a circle containing $(h_y \cap L_A) \cup (h_x \cap L_B)$. It is orthogonal to L_B and L_A through two distinct points each. Hence, $c \in \mathcal{K}_A \cap \mathcal{K}_B$. It follows that A and B are linked. □

Theorem 4.1.20 (Condition 4)**.** *If $L_B \subset h_y$ and $L_A \subset h_x$, then A and B are linked.*

Proof. Since $L_B \subset h_y$, $h_y \in \mathcal{R}_B$. Let $c = h_y \cap h_x$. Then $c \in \mathcal{K}_B$. Similarly, $L_A \subset h_x$ implies that $h_x \in \mathcal{R}_A$ so $c \in \mathcal{K}_A$. Thus A and B are linked. □

4.1.9 Both are pure loxodromic

Let A and B be pure loxodromic isometries of $\mathbb{H}^4$ with axes Ax_A, Ax_B and twisting planes τ_A, τ_B respectively. Assume Ax_A and Ax_B are disjoint. There is a unique line N perpendicular to both Ax_A and Ax_B. (Similar to case 7,) There are hyperplanes h_a orthogonal to Ax_A through a, and h_b orthogonal to Ax_B through b. There are also lines $L_a \subset \tau_A$ perpendicular to Ax_A through a, and $L_b \subset \tau_B$ perpendicular to Ax_B through b. Like the previous case, A and B are rarely linked.

Theorem 4.1.21. *If L_a and L_b are coplanar, then A and B are linked.*

Proof. Suppose first that N, L_a and L_b are distinct lines. The plane P containing L_a and L_b is unique. Since N connects a and b, N lie in P. Each of τ_A and τ_B intersect P in L_a and L_b respectively. So unless N coincide with either L_a or L_b, P is orthogonal to both τ_A and τ_B.

If only one of L_a and L_b, say L_a, is equal to N, τ_B is orthogonal to P. But $N, L_b \subset h_a$ so $P \subset h_a$ which makes P orthogonal to τ_A through L_a. Then P is the common orthogonal plane τ_A and τ_B.

If both L_a and L_b are equal to N, there are plenty of planes containing L_a and L_b. However, either $h_a \cap h_b$ is a plane Q of $h_a = h_b$ which contains planes Q through a and b. In both cases, Q is orthogonal to τ_A, τ_B and the axes $\mathrm{Ax}_A, \mathrm{Ax}_A$ so $Q \in \mathcal{K}_A \cap \mathcal{K}_B$. Hence A and B are linked. □

4.1.10 Screw parabolic and pure loxodromic

Let A be screw parabolic isometry of $\mathbb{H}^4$ fixing x and with twisting plane τ_A. Let B be a pure loxodromic isometry of $\mathbb{H}^4$ with axis Ax_B and twisting plane τ_B. Suppose x does not bound Ax_B. There is a unique $h_x \in \mathcal{F}_B$ that contains x. There is also $d_x \in \mathcal{D}_B$ containing x. Let x_2 be the intersection of d_x with h_x other than x. Since $x \neq x_2$, there are unique elements $d_2 \in \mathcal{D}_A$ and $h_2 \in \mathcal{F}_A$ containing x_2.

Let L_A be the boundary of τ_A and L_B be that of τ_B. Since $L_A \in \mathcal{D}_A$, $L_A \cap h_2$ has exactly two points. Let a be the element of $L_A \cap h_2$ other than x. Similarly, let b_1 and b_2 be the elements of $L_B \cap h_x$. Throughout this case, A, B, τ_A, τ_B, Ax_B, x, h_x, x_2, d_x, d_2, h_2, L_A, L_B, b_1 and b_2 are used consistently.

Theorem 4.1.22 (Condition 1)**.** *If there is an $h \in \mathcal{F}_A \cap \mathcal{F}_B$ and the points of $h \cap (L_A \cup L_B)$ form a circle, then A and B are linked.*

Proof. Since $x \in h$, then h is the unique element of $\mathcal{F}_B$ containing x. That is $h = h_x$. It follows that $x_2 \in h$ and h is the unique element of $\mathcal{F}_A$ containing x_2. So $h = h_x = h_2$. Hence,

$$\begin{aligned} h \cap (L_A \cup L_B) &= (h \cap L_A) \cup (h \cap L_B) \\ &= (h_2 \cap L_A) \cup (h_x \cap L_B) \\ &= \{x, a, b_1, b_2\}. \end{aligned}$$

Let c be the circle containing $\{x, a, b_1, b_2\}$ according to the hypothesis. Then $c \subset h$ since $\{x, a, b_1, b_2\} \subset h$. It is orthogonal to L_A through $\{x, a\}$ and to L_B through $\{b_1, b_2\}$.

Therefore, $c \in \mathcal{K}_A \cap \mathcal{K}_B$. It implies that A and B are linked. $\square$

Theorem 4.1.23 (Condition 2)**.** *If $a \in h_x$, $b_1, b_2 \in h_2$ and $h_x \neq h_2$, then A and B are linked.*

Proof. Let $c = h_x \cap h_2$. Then c is a circle since $h_x \neq h_2$. It is orthogonal to L_A through $\{a, x\}$ and to L_B through $\{b_1, b_2\}$. Thus, $c \in \mathcal{K}_A \cap \mathcal{K}_B$ so A and B are linked. $\square$

Theorem 4.1.24 (Condition 3)**.** *If there are $p_A \in \mathcal{F}_A$ and $p_B \in \mathcal{F}_B$ such that $L_B \subset p_A$ and $L_A \subset p_B$, then A and B are linked.*

Proof. Since $L_A \subset p_B$, then $p_B \in \mathcal{R}_A$ so $p_b \neq p_A$. Let $c = p_B \cap p_A$. Then $c \in \mathcal{K}_A$ as it is an intersection of a pair in $\mathcal{F}_A \times \mathcal{R}_A$. Likewise, $L_B \subset p_A$ implies that $p_A \in \mathcal{R}_B$, so c is also in $\mathcal{K}_B$. Hence A and B are linked. $\square$

Chapter 5

Discreteness Conditions

The linking of a pair has a few applications to discreteness of 2-generator subgroups of $\mathrm{Isom}(\mathbb{H}^n)$. If a group leaves a lower dimensional subplane C invariant, its discreteness can be determined by its restriction to C as long as its action on C is faithful. Section 5.1 shows a known result that extends to dimension 4 once a pair is linked. In section 5.2, some types of pairs in Chapter 4 are guaranteed to have lower dimensional invariant hyperplanes. Other pairs can have invariant planes if their rotational parts are half-turns. In section 5.3, the isometries are restricted in order to reduce the discreteness problem to lower dimensions.

5.1 Results from lower dimension

Determining and classifying discrete groups in hyperbolic geometry has been a hard problem in any dimension. In dimension 2, there is an algorithm by Gilman-Maskit [4] that completely determines whether a 2-generator group is discrete or not. This method does not extend to dimension 3 since it relies in comparing traces in $\mathrm{PSL}_2\mathbb{R}$ either greater than or less than, which does not work in complex numbers. In dimension 3 however, the linking by factoring allows a 2-generator group to be tested for discreteness in certain examples.

Definition 34. *Let $F = \{S_1, S_2, S_3\}$ be a set of extended Euclidean spheres in $\widehat{\mathbb{R}^n}$ as the visual boundary of $\mathbb{H}^{n+1}$. Then F is called a set of* non-separating disjoint spheres *if for each $\sigma \in F$, the elements of $F \setminus \{\sigma\}$ are in a connected component of $\widehat{\mathbb{R}^3} \setminus \sigma$.*

Definition 35. *Let A, B be orientation preserving isometries of $\mathbb{H}^n$ that are linked by half-turns H_α, H_β, H_δ for some $(n-2)$-dimensional planes α, β and δ. Then (A, B)*

has the non-separating disjoint sphere property *if there are extended Euclidean spheres C_1, C_2 and C_3 in the boundary at infinity such that $C_1 \supset \alpha$, $C_2 \supset \beta$, $C_3 \subset \delta$ and $\{C_1, C_2, C_3\}$ is a set of non-separating disjoint spheres.*

Theorem 5.1.1 (Gilman, [6])**.** *If a non-elementary marked group $G = \langle A, B\rangle \subset \mathrm{PSL}_2\mathbb{C}$ has the non-separating disjoint circle property, then G is discrete.*

Gilman and Keen [8] has another discreteness condition in dimension 3 on 2-generator elementary subgroups given by defining a core geodesic which is essentially the common perpendicular line between the axes of the generators. The core geodesic serves as the axis of a half-turn that links a pair. Every pair of isometries of $\mathbb{H}^3$ is linked by half-turns so requiring them to be linked is superfluous.

Theorem 5.1.2 (Gilman-Keen, [8])**.** *If all palindromes in a non-elementary group $G = \langle A, B\rangle \subset \mathrm{PSL}_2\mathbb{C}$ have axes that intersect the core geodesic L in a compact interval then G is discrete.*

In dimension 4 and above, Basmajian and Maskit [1] show that in a measure theoretic sense, almost all pairs of isometries are not linked. The linking must be imposed in order to use the arguments of Theorem 5.1.2 since it depends on the simultaneous factoring of A and B which is automatic in dimension 3. Fortunately, some conditions in Chapter 4 imply that certain linked pairs have invariant subplanes of lower dimension. The discreteness question becomes a lower dimensional problem and linking may not be necessary.

Meanwhile, linking by half-turns must be imposed in dimensions 4 or higher to extend Theorem 5.1.1. The conditions in Chapter 4 allow geometric criteria for a pair to be linked by half-turns.

Corollary 5.1.1 (nsds condition)**.** *Let A and B be linked pairs of orientation preserving isometries of $\mathbb{H}^4$ such that $\langle A, B\rangle$ is a non-elementary group, $A = H_{P_A}H_P$ and $B = H_PH_{P_B}$ for some planes $P_A, P, P_B \subset \mathbb{H}^4$. If there are non-separating and disjoint spheres $S_A, S, S_B \subset \widehat{\mathbb{R}^3}$ containing $\partial P_A, \partial P, \partial P_B$ respectively, then $\langle A, B\rangle$ is discrete.*

Proof. Let $G_3 = \langle H_{P_A}, H_P, H_{P_B}\rangle$. If a sphere σ contains the boundary of the twisting plane of a half-turn H, then $\sigma \in \mathcal{F}_H$ and there is $\sigma' \in \mathcal{F}_H$ such that $H = R_\sigma R_{\sigma'} = R_{\sigma'} R_\sigma$. The spheres σ and σ' are orthogonal so $R_{\sigma'}$ leaves σ invariant.

Going back to G_3, there is a side-pairing between the half-turns and F. In particular, H_{P_A} maps S_A to itself, H_P maps S to itself, and H_{P_B} maps S_B to itself. Since the elements of F are non-separating, they bound a polyhedron in $\mathbb{H}^4$ that satisfy the hypothesis of Poincaré Polyhedron Theorem [17]. The conclusion of this theorem shows that G_3 is discrete, but $\langle A, B\rangle < G_3$ so $\langle A, B\rangle$ is discrete. □

5.2 Pairs with Invariant Subplanes

A group of isometries of $\mathbb{H}^4$ with an invariant hyperplane or plane C can be tested for discreteness by restricting its action on C as long as no nontrivial element fixes C pointwise. If the group is discrete as isometries of C, then the group is discrete as isometries of $\mathbb{H}^4$. The discreteness problem reduces to lower dimension.

5.2.1 Classical Pairs

If A, B are isometries of $\mathbb{H}^4$ do not have rotational parts, they have at least one common invariant hyperplane. They are automatically linked as shown in Chapter 4. Therefore, the discreteness of $\langle A, B\rangle$ can be analyzed by treating its elements as isometries of $\mathbb{H}^3$.

Corollary 5.2.1. *For any pair of pure parabolic isometries A, B of $\mathbb{H}^4$, there is a hyperplane $C \subset \mathbb{H}^4$ that is left invariant by the group generated by A and B.*

Proof. Suppose first that A and B have a common fixed point x in the boundary. Pick any y in $\widehat{\mathbb{R}^3}$ other than x. Then there are planes and hyperplanes bounded by $h_1 \in \mathcal{F}_A$, $p_1 \in \mathcal{D}_A$, $h_2 \in \mathcal{F}_B$, $p_2 \in \mathcal{D}_B$ containing y. As p_1 and p_2 intersect at y, they are either equal or they span a sphere $\hat{C}$ that contain x and y. If $p_1 = p_2$, let $\hat{C}$ be any sphere containing p_1. The sphere $\hat{C}$ is an element of both $\mathcal{T}_A$ and $\mathcal{T}_B$ since it contains both p_1 and p_2. Hence, the hyperplane C bounded by $\hat{C}$ is left invariant by both A and B. The rest of the group $\langle A, B\rangle$ leave C invariant as well.

Suppose next that A fixes x and B fixes y in the boundary where $x \neq y$. Then there are planes and hyperplanes bounded by $p_1 \in \mathcal{D}_A$, $h_1 \in \mathcal{F}_A$ containing y and $p_2 \in \mathcal{D}_B$,

$h_2 \in \mathcal{F}_B$ containing x. If $p_1 = p_2$, any sphere $\hat{C}$ containing p_1 belong to $\mathcal{T}_A \cap \mathcal{T}_B$. If $p_1 \neq p_2$, they still intersect at x and y so they span a sphere $\hat{C}$ that consequently belongs to both $\mathcal{T}_A$ and $\mathcal{T}_B$. The hyperplane C bounded by $\hat{C}$ is therefore left invariant by A and B. This suffices to show C is left invariant by $\langle A, B\rangle$. □

Corollary 5.2.2. *Let A be pure hyperbolic and B be pure parabolic isometries of $\mathbb{H}^4$. Then there is a hyperplane in $\mathbb{H}^4$ that is left invariant by the group generated by A and B.*

Proof. Suppose first that the fixed points of A and B are three different points. Let x the fixed point of B and let y and z be those of A. Pick $h_x \in \mathcal{F}_A$ and $p_x \in \mathcal{D}_A$ containing x. Since p_x is a circle, $p_x \cap h_x$ have two points, one of which is x. Let x_2 be the other point. Then there are $h_2 \in \mathcal{F}_B$ and $p_2 \in \mathcal{D}_B$ containing x_2. The circles p_x and p_2 span a sphere $\hat{C}$ which bounds a hyperplane C. Since $\hat{C} \in \mathcal{T}_A \cap \mathcal{T}_B$, then $\langle A, B\rangle$ must leave C invariant.

Suppose the fixed point of B is common with one of the points of A. There is a $t \in \mathcal{T}_B$ containing the fixed point of A that is not common with B. Then t has the fixed points of A so $t \in \mathcal{T}_A$. Thus t is left invariant by A and B. □

Corollary 5.2.3. *If A and B are pure hyperbolic isometries of $\mathbb{H}^4$ with ultra-parallel axes, then there is a hyperplane C that is left invariant by the group $\langle A, B\rangle$.*

Proof. If the axes of A and B are subsets of the same plane P, let C be any hyperplane containing P. Otherwise, let N be the common perpendicular between their axes. The axis of A and N span a unique plane that intersect the axis of B in one point. Together $\mathrm{Ax}_A \cup \mathrm{Ax}_B \cup N$ span a unique hyperplane C. In both cases, there is a hyperplane C containing the axes of A and B. Since both A and B are pure hyperbolic, they leave C invariant. If W is a composition of A, B or their inverses, W also leaves C invariant. Hence the group $\langle A, B\rangle$ leaves C invariant. □

5.2.2 Pairs with Rotational Parts

Let A be an orientation preserving isometry of $\mathbb{H}^4$ with a twisting plane P and B be either a pure parabolic or pure hyperbolic isometry of $\mathbb{H}^4$. If $\partial P \in \mathcal{D}_B$, then B and therefore $\langle A, B\rangle$ leave P invariant. The discreteness of $\langle A, B\rangle$ can possibly be answered by treating A and B as isometries of $\mathbb{H}^2$. If A has a twisting hyperplane, it is possible for its boundary to be an element of $\mathcal{T}_B$. In this case, the twisting hyperplane is left invariant by the group $\langle A, B\rangle$ as acting on $\mathbb{H}^3$. If both A and B have rotational parts that are not half-turns, their twisting planes must match to have an invariant plane. If they both have twisting hyperplanes, those have to match to have an invariant hyperplane.

The question about how a pair is linked is a different problem. Indeed, one can impose conditions on a pair for them to be both linked and have an invariant subplane. Imposing the condition of having an invariant subplane together with linking reduces the question of discreteness to the known results in lower dimensions. However the conditions in Chapter 4 may imply that the invariant subplanes do not match. Hence the idea of finding an invariant subplane for both generators is too restrictive to use.

If the rotational part of an isometry A in a pair is a half-turn, then any hyperplane containing its twisting plane has its boundary in $\mathcal{R}_A$. If B leaves a hyperplane h invariant in $\mathcal{R}_A$, then $\langle A, B\rangle$ leaves h invariant but reverses the orientation of h even if A is orientation preserving in $\mathbb{H}^4$. The theorems mentioned here from [8] and [6] assume that the isometries are orientation preserving although the statements may still hold.

5.3 Restricting the Isometries

If A and B leave a common plane or hyperplane C invariant, the group $\langle A, B\rangle$ must also leave C invariant. Let $n = \dim(C)$ so n is either 2 or 3. The action of $\langle A, B\rangle$ on $\mathbb{H}^4$ restricts to an action on C which is isometric to $\mathbb{H}^n$. The discreteness of $\langle A, B\rangle <$ $\mathrm{Isom}^+(\mathbb{H}^4)$ reduces to testing the discreteness on its action on $\mathbb{H}^n$. There are non-separating disjoint circles [2, 4, 6, 16] and core geodesic [8] tests that can be used to check discreteness.

Theorem 5.3.1. *Let A and B be pure hyperbolic isometries of $\mathbb{H}^4$ with ultra-parallel axes. Let L be the common perpendicular line through their axes. If all palindromes have axes that intersect L in a compact subset, then $\langle A, B \rangle$ is discrete.*

Proof. If A and B are both pure hyperbolic with ultra-parallel axis, then by Theorem 4.1.1, there is a hyperplane C that is left invariant by $\langle A, B \rangle$. The action of $\langle A, B \rangle$ on $\mathbb{H}^4$ restricts to C which is isometric to $\mathbb{H}^3$. Let L be the common perpendicular line between the axes of A and B. By Lemma 5.1 of [8] the palindromes in A and B have axes intersecting L orthogonally. If these intersections form a compact subset, by Theorem 6.3 of [8], the group $\langle A, B \rangle$ is discrete. $\square$

Theorem 5.3.2. *Let A and B be pure parabolic isometries of $\mathbb{H}^4$ with different fixed points x and y. If all non-parabolic palindromes have axes that intersect the line $[x, y]$ connecting x to y in a compact subset, then $\langle A, B \rangle$ is discrete.*

Proof. By Corollary 5.2.1, there is a hyperplane C that is left invariant by the group $\langle A, B \rangle$. The action of this group can be restricted to $C \supset [x, y]$ which is isometric to $\mathbb{H}^3$. By Lemma 5.1 of [8] the non-parabolic palindromes in A and B have axes intersecting $[x, y]$ orthogonally. If these intersections form a compact subset, by Theorem 6.3 of [8], the group $\langle A, B \rangle$ is discrete. $\square$

Theorem 5.3.3. *Let A be pure hyperbolic and B be pure parabolic isometries of $\mathbb{H}^4$ with disjoint fixed points at infinity. If the axes of palindromes in A and B intersect the core geodesic in a compact subset, then $\langle A, B \rangle$ is discrete.*

Proof. By Corollary 5.2.2, there is a hyperplane C that is left invariant by the group $\langle A, B \rangle$. The action of this group can be restricted to a hyperplane C which contains the line L perpendicular to Ax_A and has $\mathrm{fix}(B)$ in its horizon. The restriction of the action of $\langle A, B \rangle$ on C has L as its core geodesic. Axes of non-parabolic palindromes in A and B intersect L orthogonally. If these intersections from a compact subset, by Theorem 6.3 of [8], the group $\langle A, B \rangle$ is discrete. $\square$

Theorem 5.3.4. *Let A and B be linked pairs with an invariant plane or hyperplane C and core geodesic L so that $\langle A, B\rangle$ is non-elementary. If all non-parabolic palindromes have axes that intersect L in a compact interval, then $\langle A, B\rangle$ is discrete.*

Chapter 6

Detailed Investigation of Enumeration Schemes

In this chapter we turn to a different topic, the enumeration of primitive words in a rank two free group. Every primitive word in such a group is conjugate to a unique palindrome or to a product of two palindromes. One enumeration scheme for primitives uses what are known as E-words. An E-word is either the unique palindrome in the conjugacy class of the word or a unique product of two palindromes that have previously occurred in the enumeration scheme. Here we present two alternative ways of listing or studying the E-words defined in [9] first by defining rational numbers called "orphans" and second by defining a new string called an E-sequence that comes from modifying the Gilman-Maskit algorithm in [4]. Farey words are the words that arise in the Keen-Series enumeration scheme [11]. Applications of the latter alternative mainly give comparisons of the Keen-Series Farey words [11] with E-words. Using the former alternative, the definition of E-words, which is a recursive definition, can also be modified so as to have an alternative terminating conditions. This gives quicker computations. The main result here is Theorem 6.1.2. An implementation of the enumeration scheme for E-words using Theorem 6.1.2 can be found in the webpage `https://pegasus.rutgers.edu/~benjsilv/turboenumerate.html`.

By studying the mapping classes of a punctured torus, one concludes that the primitive elements of a rank-2 free group can be indexed, up to conjugacy, by rational numbers and infinity. Gilman and Keen [9] derived an iteration scheme that takes a rational number and gives back a primitive element of the rank-2 free group F_2 with the options of providing its primitive associate. This iteration terminates on two conditions, when the argument or input is either 0 or ∞. In section 6.1.4, an alternative but equivalent conditions for terminating the recursive iteration is shown. These conditions enable a faster computation of the enumeration scheme whether manually or by a machine.

The Gilman-Maskit algorithm for determining the discreteness or non-discreteness of a two-generator subgroup of $\mathrm{PSL}_2\mathbb{R}$ stops with a pair of generators that are Farey words. The Farey words are primitive words that are indexed by rational numbers and infinity. The E-words, which are primitive words with either a palindromic form or a palindromic product form, are also indexed by rational numbers and infinity. In section 6.2, the Gilman-Maskit algorithm is modified so that the stopping generators are E-words.

We can view the enumeration scheme and the Gilman-Maskit algorithm as being reverses of each other in the following sense. The enumeration scheme begins by splitting a given rational number into the Farey sum of two other rational numbers. It keeps on splitting the other rational numbers until it ends with the E-word that correspond to either 0 or ∞. Technical details are given below, but roughly speaking both the modification of enumeration in section 6.1.4 and as well as the original definition of the enumeration scheme take a continued fraction expansion $[a_0; a_1, a_2, \ldots, a_k]$ and starts with a_k and runs down to a_0. The difference between these two is that the alternative procedure that comes out of the first modification terminates the recursion at $[a_0; 1]$ whereas the original one keeps on with the recursion. On the other hand, the modified Gilman-Maskit algorithm takes the same continued fraction expansion $[a_0; a_1, a_2, \ldots, a_k]$ but starts with a_0 and ends with a_k.

Finally, we apply our investigations of the different iteration schemes to obtain a theorem, Theorem 6.2.8 about the number of E-words of a given length within an interval.

6.1 Equivalent Conditions for Terminating Palindromic Primitives

As indicated above, it is well known that the conjugacy classes of the primitive elements of a rank-2 free group F_2 can be indexed by rational numbers and infinity up to taking inverses. Moreover, it is also known that for each conjugacy class of primitive elements, there is a representative that is either a palindrome or product of two palindromes. Gilman and Keen [9] prove these results by defining a function $E : \mathbb{Q} \cup \{\infty\} \to F_2 = \langle A, B \rangle$. This function is recursive and terminates on conditions $0 \mapsto A^{-1}$ and $\infty \mapsto B$. In this section, we give non-recursive formulas for this function E in cases where the rational number is an integer or reciprocal of an integer. These formulas serve as an alternative terminating conditions for the original enumerating scheme derived by Gilman and Keen [9].

The original definition of the enumeration scheme can be implemented and run in a machine without any modification. However every time a recursion "calls itself," the state of the previous "caller" is stored until the recursion stops calling itself. It is often efficient for a recursion to minimize calling itself in order to avoid wasted resources such as time and storage space. The non-recursive formulas for special cases reduce the self-calling of the recursion. An implementation of the enumeration scheme of E-words using Theorem 6.1.2 can be found in the webpage https://pegasus.rutgers.edu/~ benjsilv/turboenumerate.html. If the original definition is implemented instead, the machine can run out of allocated space for saving the state of an iteration and halt without the desired output. Hence applying Theorem 6.1.2 allows a running implementation to work on much more range of inputs.

6.1.1 Summary of the Gilman-Keen Enumeration Scheme

The notation used here for elements of $\mathbb{Q} \cup \{\infty\}$ are of the form p/q where $p \in \mathbb{Z}$, $q \in \mathbb{Z} \cap [0, \infty)$ and $\gcd(p, q) = 1$. The element ∞ is denoted by $1/0$. By definition, $\frac{1}{0}$ and $\frac{0}{1}$ are in lowest terms.

Definition 36. *Let $p/q, r/s \in \mathbb{Q} \cup \{\infty\}$. The pair p/q and r/s are called* Farey neighbors *if $|ps - rq| = 1$.*

If p/q and r/s are Farey neighbors, the Farey sum *of p/q and r/s is*

$$\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}.$$

Both p/q and r/s are Farey neighbors of their Farey sum. The Farey neighbors do not have transitive property. A rational number may have infinitely many Farey neighbors but the set rational numbers that are its Farey neighbors is certainly bounded. We give a name for its minimum and maximum.

Definition 37. *The smallest and largest Farey neighbors of a nonzero rational number p/q are called* parents *of p/q.*

The details of the Gilman-Keen enumeration scheme can be found in [9]. The following is a brief overview. Set $E_{0/1} = A^{-1}$ and $E_{1/0} = B$. For the rest of $\mathbb{Q}$, take the parents m/n and r/s of p/q such that $\frac{m}{n} < \frac{p}{q} < \frac{r}{s}$. Define $E_{p/q}$ recursively by

$$E_{p/q} = \begin{cases} E_{r/s}E_{m/n} & \text{if } pq \text{ is odd,} \\ E_{m/n}E_{r/s} & \text{if } pq \text{ is even.} \end{cases}$$

Definition 38. *The function $E : \mathbb{Q} \cup \{\infty\} \to F_2$ given by $p/q \mapsto E_{p/q}$ is called the* the Gilman-Keen enumeration scheme *or simply* enumeration scheme*.*

The definitions of $E_{0/1}$ and $E_{1/0}$ are called terminal conditions *since they do not require breaking a fraction into a Farey sum of their parents. Hence, we call the elements 0 and ∞ of $\mathbb{Q} \cup \{\infty\}$* orphans*.*

6.1.2 Non-recursive Formulas for Special Cases

In this section, formulas are given for $E_{n/1}$, $E_{1/n}$, $E_{p/(p+1)}$, and $E_{(p+1)/p}$ for all $n \in \mathbb{Z}$ and nonnegative integers p. Since the enumeration scheme is a recursive definition, the corresponding words of non-orphans are cumbersome to compute. However, formulas can be derived on some cases.

Lemma 6.1.1. *For $n > 1$, the parents of $\frac{1}{n}$ are $\frac{0}{1}$ and $\frac{1}{n-1}$. For $n > 0$, the parents of $\frac{n}{1}$ are $\frac{1}{0}$ and $\frac{n-1}{1}$.*

Proof. Since $\frac{1}{0} = \infty$, any other Farey neighbor of $\frac{n}{1}$ must be finite. Suppose p/q is a finite Farey neighbor of n. We may assume $q \geq 1$; otherwise, pass the negative sign to p. Then, $\frac{p}{q} < n \Rightarrow p < qn$. Since p/q is a Farey neighbor of n, $|p - qn| = 1$. Hence,

$qn - p = 1$, and

$$\begin{aligned} q \geq qn - p &\implies p \geq qn - q \\ &\implies p \geq q(n-1) \\ &\implies \frac{p}{q} \geq n - 1. \end{aligned}$$

Since $n-1$ is a Farey neighbor of n, $n-1$ must be the lower parent of n.

Next, we show the parents of $\frac{1}{n}$. Suppose $n > 1$ and $\frac{p}{q}$ is a Farey neighbor of $\frac{1}{n}$ with $\frac{p}{q} > \frac{1}{n}$. Since $n > 1$, $\frac{1}{n} > 0$ so we assume $p \geq 1$ and $q \geq 1$. Then $|pn - q| = 1$ and $pn > q \Rightarrow pn - q = 1$. Hence,

$$\begin{aligned} p \geq 1 &\implies p \geq pn - q \\ &\implies q \geq pn - p \\ &\implies q \geq p(n-1) \\ &\implies \frac{1}{n-1} \geq \frac{p}{q}. \end{aligned}$$

Since $\frac{1}{n-1}$ is a Farey neighbor of $\frac{1}{n}$, it is the greater parent of $\frac{1}{n}$. Lastly, if a Farey neighbor $\frac{p}{q} \leq \frac{1}{n}$, then $q - pn = 1 \Rightarrow q - 1 = pn \Rightarrow p = \frac{q-1}{n}$. Since $n > 1$ and we may assume that $q \geq 1$, it implies $p \geq 0$. Hence $\frac{p}{q} \geq 0$. Since 0 is a Farey neighbor of $\frac{1}{n}$, $\frac{0}{1}$ must be lower parent of $\frac{1}{n}$. □

Corollary 6.1.1. *If n is a negative integer, the parents of n are ∞ and $n+1$; the parents of $\frac{1}{n}$ are 0 and $\frac{1}{n+1}$ for $n < -1$.*

Proof. If $n < 0$, then $n+1$ is the greatest Farey neighbor of n other than ∞. Using similar methods, ∞ is the lowest possible parent of a negative rational number. On the other hand, if $n < -1$, then $-n > 1$, so the parents of $\frac{1}{-n}$ are 0 and $\frac{1}{-n-1} = -\frac{1}{n+1}$. Hence, the minimum and maximum Farey neighbors of $\frac{1}{n}$ are $\frac{-1}{-(n+1)}$ and 0 respectively. □

Lemma 6.1.2. *Let $p \in \mathbb{N}$. The parents of $\frac{p}{p+1}$ are $\frac{p-1}{p}$ and $\frac{1}{1}$. The parents of $\frac{p+1}{p}$ are $\frac{p}{p-1}$ and $\frac{1}{1}$.*

Proof. Suppose $\frac{m}{n} < \frac{p}{p+1}$ is a Farey neighbor with $n \geq 1$. Then $|mp+m-pn| = 1$ and $pn > mp + m \Rightarrow pn - mp - m = 1$.

$$\begin{aligned} n \geq 1 &\implies n \geq pn - mp - m \\ &\implies m + mp \geq pn - n \\ &\implies m(1+p) \geq n(p-1) \\ &\implies \frac{m}{n} \geq \frac{p-1}{p+1}. \end{aligned}$$

However,

$$\begin{aligned} -1 > 1 &\implies p - 1 > p + 1 \\ &\implies \frac{p-1}{p+1} > 1 > 1 - \frac{1}{p} = \frac{p-1}{p}. \end{aligned}$$

Since $\frac{p-1}{p}$ is a Farey neighbor of $\frac{p}{p+1}$, $\frac{p-1}{p}$ is a parent of $\frac{p}{p+1}$. In the case where $\frac{m}{n} > \frac{p}{p+1}$, we have $mp+m > np \implies mp+m-pn = 1$. Assuming $n \geq 1$, $n \geq mp+m-pn$. Then,

$$\begin{aligned} n + pn \geq mp + m &\implies n(1+p) \geq m(p+1) \\ &\implies n \geq m \\ &\implies \frac{m}{n} \leq 1. \end{aligned}$$

Since 1 is Farey neighbor of of $\frac{p}{p+1}$, 1 must be a parent of of $\frac{p}{p+1}$. $\square$

The following is initially an observation from computations by a machine.

Theorem 6.1.1. *For each $p \geq 0$, $E_{p/(p+1)} = \left(A^{-1}B\right)^p A^{-1}$ and $E_{(p+1)/p} = \left(BA^{-1}\right)^p B$.*

Proof. For $p = 0$, $E_{0/(0+1)} = E_{0/1} = A^{-1} = \left(A^{-1}B\right)^0 A^{-1}$ and $E_{(0+1)/0} = E_{1/0} = B = \left(BA^{-1}\right)^0 B$. Suppose the assertion is true for high enough p. Then,

$$E_{\frac{p+1}{p+2}} = E_{\frac{p}{p+1}} E_{\frac{1}{1}} = \left(A^{-1}B\right)^p A^{-1} \cdot BA^{-1} = \left(A^{-1}B\right)^{(p+1)} A^{-1}$$

$$E_{\frac{p+2}{p+1}} = E_{\frac{1}{1}} E_{\frac{p+1}{p}} = BA^{-1} \left(BA^{-1}\right)^p B = \left(BA^{-1}\right)^{(p+1)} B.$$

Note that $p(p+1)$ is always even, $\frac{p}{p+1} < 1$ and $\frac{p+1}{p} > 1$. $\square$

In computing a primitive word in the image of the enumerating scheme, the recursion eventually runs through the decreasing entries of a continued fraction $[a_0; a_1, \ldots, a_k]$.

The fraction $\frac{n}{1}$ has the form $[n;]$ and $\frac{1}{n}$ has the form $[0;n]$. Thus, a formula for these cases saves the iteration several steps. To construct more unified formulas, a function $s:\mathbb{R}\to\{-1,1\}$ is defined by

$$s(x)=\begin{cases}1 & \text{for } x\in(-\infty,0),\\ -1 & \text{for } x\in[0,\infty).\end{cases}$$

Theorem 6.1.2. *Let $n\in\mathbb{Z}$. Then,*

$$E_{\frac{n}{1}}=B^{\left\lceil\frac{|n|}{2}\right\rceil}A^{s(n)}B^{\left\lfloor\frac{|n|}{2}\right\rfloor}$$

and

$$E_{\frac{1}{n}}=A^{s(n)\left\lfloor\frac{|n|}{2}\right\rfloor}BA^{s(n)\left\lceil\frac{|n|}{2}\right\rceil}.$$

Proof. The cases for $n=-2,-1,0,1,2$ are given below. The inductive steps are shown for $n+2$ and $n-2$.

Case $n=0$:

$$B^{\left\lceil\frac{|0|}{2}\right\rceil}A^{s(0)}B^{\left\lfloor\frac{|0|}{2}\right\rfloor}=B^0A^{-1}B^0=A^{-1}=E_{0/1}$$

and

$$A^{s(0)\left\lfloor\frac{|0|}{2}\right\rfloor}BA^{s(0)\left\lceil\frac{|0|}{2}\right\rceil}=A^0BA^0=B=E_{1/0}.$$

Case $n=1$:

$$B^{\left\lceil\frac{|1|}{2}\right\rceil}A^{s(1)}B^{\left\lfloor\frac{|1|}{2}\right\rfloor}=B^1A^{-1}B^0=BA^{-1}=E_{1/1}$$

and

$$A^{s(1)\left\lfloor\frac{|1|}{2}\right\rfloor}BA^{s(1)\left\lceil\frac{|1|}{2}\right\rceil}=A^{-1\cdot 0}BA^{-1\cdot 1}=BA^{-1}=E_{1/1}.$$

Case $n=-1$:

$$B^{\left\lceil\frac{|-1|}{2}\right\rceil}A^{s(-1)}B^{\left\lfloor\frac{|-1|}{2}\right\rfloor}=B^1A^1B^0=BA=E_{-1/1}$$

and

$$A^{s(-1)\left\lfloor\frac{|-1|}{2}\right\rfloor}BA^{s(-1)\left\lceil\frac{|n|}{2}\right\rceil}=A^{1\cdot 0}BA^{1\cdot 1}=BA=E_{1/-1}.$$

Case $n = 2$**:**

$$B^{\left\lceil \frac{|2|}{2} \right\rceil} A^{s(2)} B^{\left\lfloor \frac{|2|}{2} \right\rfloor} = B^1 A^{-1} B^1 = BA^{-1}B$$

$$E_{2/1} = E_{1/1}E_{1/0} = BA^{-1} \cdot B$$

and

$$A^{s(2)\left\lfloor \frac{|2|}{2} \right\rfloor} B A^{s(2)\left\lceil \frac{|2|}{2} \right\rceil} = A^{-1}BA^{-1}$$

$$E_{1/2} = E_{0/1}E_{1/1} = A^{-1} \cdot BA^{-1}.$$

Case $n = -2$**:**

$$B^{\left\lceil \frac{|-2|}{2} \right\rceil} A^{s(-2)} B^{\left\lfloor \frac{|-2|}{2} \right\rfloor} = B^1 A^1 B^1 = BAB$$

$$E_{-2/1} = E_{-1/1}E_{1/0} = BA \cdot B$$

and

$$A^{s(-2)\left\lfloor \frac{|-2|}{2} \right\rfloor} B A^{s(-2)\left\lceil \frac{|-2|}{2} \right\rceil} = A^{1\cdot 1}BA^{1\cdot 1} = ABA$$

$$E_{1/-2} = E_{0/1}E_{-1/1} = A \cdot BA.$$

The following are the inductive steps. Assume that for n high or low enough, the assertions are true.

Case $n + 2$**:**

Assume $n \geq 0$. Then,

$$B^{\left\lceil \frac{|n+2|}{2} \right\rceil} A^{s(n+2)} B^{\left\lfloor \frac{|n+2|}{2} \right\rfloor} = B^{\left\lceil \frac{|n|}{2} \right\rceil + 1} A^{s(n)} B^{\left\lfloor \frac{|n|}{2} \right\rfloor + 1}$$

since $\frac{|n+2|}{2} = \frac{n}{2} + 1$ and $s(n) = s(n+2)$.

$$\begin{aligned}
E_{\frac{n+2}{1}} &= \begin{cases} E_{(n+1)/1}E_{1/0} = \left(E_{1/0}E_{n/1}\right) E_{1/0} & \text{if } n \text{ is even} \\ E_{1/0}E_{(n+1)/1} = E_{1/0}\left(E_{n/1}E_{1/0}\right) & \text{if } n \text{ is odd} \end{cases} \\
&= BE_{\frac{n}{1}}B \\
&= B\left(B^{\left\lceil \frac{|n|}{2} \right\rceil} A^{s(n)} B^{\left\lfloor \frac{|n|}{2} \right\rfloor}\right) B \\
&= B^{\left\lceil \frac{|n|}{2} \right\rceil + 1} A^{s(n+2)} B^{\left\lfloor \frac{|n|}{2} \right\rfloor + 1}.
\end{aligned}$$

$$\begin{aligned}
A^{s(n+2)\left\lfloor\frac{|n+2|}{2}\right\rfloor}BA^{s(n+2)\left\lceil\frac{|n+2|}{2}\right\rceil} &= A^{s(n)\left(\left\lfloor\frac{|n|}{2}\right\rfloor+1\right)}BA^{s(n)\left(\left\lceil\frac{|n|}{2}\right\rceil+1\right)} \\
&= A^{s(n)}A^{s(n)\left\lfloor\frac{|n|}{2}\right\rfloor}BA^{s(n)\left\lceil\frac{|n|}{2}\right\rceil}A^{s(n)} \\
&= A^{s(n)}E_{\frac{1}{n}}A^{s(n)}.
\end{aligned}$$

$$\begin{aligned}
E_{1/(n+2)} &= \begin{cases} E_{0/1}E_{1/(n+1)} = E_{0/1}\left(E_{1/n}E_{0/1}\right) & \text{if } n \text{ is even} \\ E_{1/(n+1)}E_{0/1} = \left(E_{0/1}E_{1/n}\right)E_{0/1} & \text{if } n \text{ is odd} \end{cases} \\
&= E_{0/1}E_{1/n}E_{0/1} \\
&= A^{s(n)}E_{1/n}A^{s(n)}
\end{aligned}$$

Case $n-2$**:**

Assume $n<0$. Then,

$$\frac{|n-2|}{2} = \frac{|(-1)(-n+2)|}{2} = \frac{-n+2}{2} = \frac{-n}{2}+1$$

and

$$s(n-2) = s(n).$$

Therefore,

$$\begin{aligned}
B^{\left\lceil\frac{|n-2|}{2}\right\rceil}A^{s(n-2)}B^{\left\lfloor\frac{|n-2|}{2}\right\rfloor} &= B^{\left\lceil\frac{|n|}{2}\right\rceil+1}A^{s(n)}B^{\left\lfloor\frac{|n|}{2}\right\rfloor+1} \\
&= BE_{n/1}B \\
&= E_{1/0}E_{n/1}E_{1/0} \\
&= E_{(n-1)/1}E_{1/0} \\
&= E_{(n-2)/1}.
\end{aligned}$$

And also,

$$\begin{aligned}
A^{s(n-2)\left\lfloor\frac{|n-2|}{2}\right\rfloor}BA^{s(n-2)\left\lceil\frac{|n-2|}{2}\right\rceil} &= A^{s(n)\left(\left\lfloor\frac{|n|}{2}\right\rfloor+1\right)}BA^{s(n)\left(\left\lceil\frac{|n|}{2}\right\rceil+1\right)} \\
&= A^{s(n)}E_{1/n}A^{s(n)} \\
&= E_{0/1}E_{1/n}E_{0/1} \\
&= E_{1/(n-1)}E_{0/1} \\
&= E_{1/(n-2)}
\end{aligned}$$

Note that the enumeration scheme terminates the recursion on the condition $E_{0/1} = A$ for negative numbers even if $\frac{0}{1} \mapsto A^{-1}$. $\square$

6.1.3 Examples

Some examples are shown in this section. The details below are not exhaustive in computing the parents of the given fraction, but Theorem 6.1.2 is directly applied.

$$\begin{aligned}
E_{17/3} &= E_{6/1}E_{11/2} = E_{6/1}\left(E_{5/1}E_{6/1}\right) \\
&= B^3A^{-1}B^3 \cdot \left(B^2A^{-1}B^2 \cdot B^3A^{-1}B^3\right) \\
&= B^3A^{-1}B^3 \cdot B^3A^{-1}B^5A^{-1}B^3 \\
&= B^3A^{-1}B^6A^{-1}B^5A^{-1}B^3 \\
E_{2/21} &= E_{1/11}E_{1/10} \\
&= A^{-5}BA^{-6} \cdot A^{-5}BA^{-5} \\
&= A^{-5}BA^{-11}BA^{-5} \\
E_{3/31} &= E_{1/10}E_{2/21} = E_{1/10}\left(E_{1/11}E_{1/10}\right) \\
&= A^{-5}BA^{-5} \cdot A^{-5}BA^{-6} \cdot A^{-5}BA^{-5} \\
&= A^{-5}BA^{-10}BA^{-11}BA^{-5}
\end{aligned}$$

$$\begin{aligned}
E_{31/9} &= E_{7/2}E_{24/7} \\
&= E_{7/2}\left(E_{17/5}E_{7/2}\right) \\
&= E_{7/2}\left(E_{7/2}E_{10/3}\right)E_{7/2} \\
&= E_{7/2}E_{7/2}\left(E_{3/1}E_{7/2}\right)E_{7/2} \\
&= \left(E_{7/2}\right)^2 E_{3/1}\left(E_{7/2}\right)^2 \\
&= \left(E_{3/1}E_{4/1}\right)^2 E_{3/1}\left(E_{3/1}E_{4/1}\right)^2 \\
&= \left(B^2A^{-1}B^1 \cdot B^2A^{-1}B^2\right)^2\left(B^2A^{-1}B^1\right)\left(B^2A^{-1}B^1 \cdot B^2A^{-1}B^2\right)^2 \\
&= B^2A^{-1}B^3A^{-1}B^4A^{-1}B^3A^{-1}B^4A^{-1}B^3A^{-1}B^3A^{-1}B^4A^{-1}B^3A^{-1}B^2
\end{aligned}$$

6.1.4 Alternative Termination Conditions

Since using Theorem 6.1.2 allows the enumeration scheme to terminate the recursion earlier, we conclude this section with the alternative but equivalent terminating conditions.

Theorem 6.1.3. *The Gilman-Keen enumeration scheme can have its recursion terminated using the conditions*

$$E_{\frac{n}{1}} = B^{\left\lceil \frac{|n|}{2} \right\rceil} A^{-1} B^{\left\lfloor \frac{|n|}{2} \right\rfloor} \qquad E_{\frac{1}{n}} = A^{-\left\lfloor \frac{|n|}{2} \right\rfloor} B A^{-\left\lceil \frac{|n|}{2} \right\rceil}$$

for $n \in \mathbb{Z} \cap [0, \infty)$ *; and*

$$E_{\frac{n}{1}} = B^{\left\lceil \frac{|n|}{2} \right\rceil} A B^{\left\lfloor \frac{|n|}{2} \right\rfloor} \qquad E_{\frac{1}{n}} = A^{\left\lfloor \frac{|n|}{2} \right\rfloor} B A^{\left\lceil \frac{|n|}{2} \right\rceil}$$

for $n \in \mathbb{Z} \cap (-\infty, 0)$*.*

Proof. First we must show that these terminating conditions give the same words as those of the original enumeration scheme. By Theorem 6.1.2, these conditions give the same words, but only allows the recursion to terminate early. Next, we show that these conditions eventually stops the iteration of the enumerations scheme. From [9], the iteration stops under only two conditions: $E_{0/1} = A^{-1}, A$ and $E_{1/0}$. If we have these two conditions, the iteration must stop. These two conditions are exactly the case $n = 0$, as shown in the proof of Theorem 6.1.2. Thus, the conditions above terminate the iteration. □

Remark:

It has been observed initially from machine computations that once the Theorem 6.1.2 is used as a terminating condition, the original ones are not needed anymore.

6.2 The Modified Gilman-Maskit Algorithm

A linear step in the Gilman-Maskit algorithm sends the ordered pair (g, h) to (g, gh). A Fibonacci step sends the pair (g, h) to (gh, g) [4, 13, 14]. Which step is used or picked depends on the traces of the new generators; the one with lower trace should occupy the left spot. It comes from the assumption that the old pair has $\mathrm{tr}^2(g) < \mathrm{tr}^2(h)$.

The main idea of a 'step' is to replace one of the two generators with their product. By keeping one of the generators, this procedure ensures that the groups generated by the old and new pairs are the same. The algorithm retains the generator with lower trace. The following is the proposed new 'step' in picking new generators from a given ordered pair (a, b).

conditions for a and b	preserve a	preserve b
both a and b are palindromes	(a, ba)	(ba, b)
a is not a palindrome	(a, ab)	(ab, b)
b is not a palindrome	(a, ab)	(ab, b)

Note that there are no assumptions about the traces of a and b, but it assumes a takes the left spot and both generators are either a palindrome or a product of palindromes. Since a generator is not usually expressed as product of other generators, there should be a new definition of a palindrome.

6.2.1 Summary of Gilman-Maskit Algorithm

The Gilman-Maskit algorithm takes two elements A and B of $\mathrm{PSL}_2\mathbb{R}$ and gives a definite output: either $\langle A, B\rangle$ is discrete; or not. The algorithm uses conditions, e.g. Poincaré polygon theorem or Jorgensen's inequality, to decide whether the group is discrete or not using the generators A and B. If it can not decide using A and B, the generators are combined to construct new generators to use for testing discreteness.

One such combination is the pair (A, AB) and the traces of their matrices are reduced after the iteration. Eventually the process of changing the generators stop and the algorithm makes a decision [4].

Other combinations and conditions are also used but the step that changes (A, B) into (A, AB) called here a Nielsen step is the main modification of this section.

6.2.2 Redefining Palindromes

Let L be a hyperbolic line or geodesic in $\mathbb{H}^3$, the 3-dimensional hyperbolic space. Although a subgroup of $\mathrm{PSL}_2\mathbb{R}$ acts on the upper-half space $\mathbb{H}^2$, the action always extend to $\mathbb{H}^3 \cup \widehat{\mathbb{C}}$. Typically, L is the common perpendicular line of the axes of a fixed pair of elements a_0 and b_0. Let $H_L \in \mathrm{PSL}_2, \mathbb{C}$ be the half-turn around L.

Definition 39. *Let $g \in \mathrm{PSL}_2\mathbb{C}$. We say g is a* palindrome with respect to L *if $g = H_L g^{-1} H_L$.*

If $g \in \langle A, B \rangle$, the definition does not depend on the factorization of g in terms of A and B. If L is the common perpendicular non-degenerate line through the axes of A and B, and g is a palindrome with respect to L, then g as a word in A and B reads the same forward and backward. The following is a reworded lemma in [8].

Theorem 6.2.1. *Let W be a word and in the generators A and B. Let L be the common perpendicular through the axes of A and B. Then W reads the same forward and backward if and only if W is a palindrome with respect to L.*

This definition allows the modification to be well-defined if a line L is fixed. To do this, fix the initial pair of generators (a_0, b_0) and construct their common perpendicular L as in [4] or [8]. Alternatively, we can keep track of the 'steps' and tell which of the generators or if none is the product of palindromes.

6.2.3 A Second Look at the Original Algorithm

The original Gilman-Maskit algorithm has a few assumptions about the given pair of generator (a, b). First it assumes that the fixed points are oriented so that the attracting points are on the opposite sides of the common perpendicular. Second it assumes $\mathrm{tr}^2(a) < \mathrm{tr}^2(b)$. Hence, the generator that needs to be retained is certainly a.

The order of the new generators is the difference between the linear and Fibonacci steps. The Fibonacci step seems to 'switch' the order done by the linear step. The F-sequence $(n_1, n_2, \ldots, n_k)$ in [7] is a notation for recording the iteration in the Gilman-Maskit algorithm that stores the number of consecutive linear steps before the next

Fibonacci step. Thus n_1 is the number of first consecutive linear steps before the first Fibonacci step, n_2 is the number of second consecutive linear steps, and $k-1$ (or k) is the number of all Fibonacci steps. The Fibonacci step is the only time the change-of-order is tracked in terms of recording the F-sequence.

The first assumption can be assured by replacing the generators by inverses whenever appropriate. The second assumption is assured by switching the order. These assurances are actually elements of $\mathrm{Aut}(F_2)$ that are usually ignored (on purpose) when recording the F-sequence and even when defining the algorithm itself.

In ignoring the action of certain elements of $\mathrm{Aut}(F_2)$, the palindromic and product-of-palindrome forms of the primitive elements are destroyed. Thus, the elements of $\mathrm{Aut}(F_2)$ used in every 'step' of the algorithm must be picked carefully to preserve the palindromic form. In particular, the 'switching' automorphism is not used at all.

Switching the generators is not essential to the original algorithm. The important things are that 1) one of the generators is preserved; and 2) the new generator is a product of the two previous ones. The proposed modification respects these two properties.

Remark:

It just turns out that the classification of steps into linear and Fibonacci is compatible with the Farey words. The winding steps defined in [13,14] are not elements of $\mathrm{Aut}(F_2)$ at all but it seems not to be destructive of the palindromic forms [8]. However, it does not produce the E-words in the p/q form.

6.2.4 New Linear and Fibonacci Steps

The original linear step seems to preserve the left generator and change the other. The original Fibonacci step turns the left generator into the right generator, and hence seems to change both generators. The F-sequence in [7] records the consecutive linear steps before a Fibonacci step. The proposed new steps here always preserve one generator including its position whether left or right. Instead of classifying the steps, we define a new sequence $[n_0; n_1, n_2, \ldots, n_k]$ called *E-sequence*. Let n_0 be the number of steps in preserving the initial right generator before changing it. Let n_1 be the number of steps in preserving the initial left generator before changing it. Let n_2 be the number of steps the next right generator is preserved. The rest of the n_is alternates between left and right generators. So for even i, n_i steps preserve the right generator; for odd i, n_i steps preserve the left generator. If the algorithm preserves the left generator first, we let $n_0 = 0$. All n_is take positive integers except n_0 can take a zero value. Hence, an E-sequence can take continued fraction expansion values of any positive rational number p/q.

6.2.5 From E-sequence to E-words

The main purpose of the modification is to end the algorithm with E-words. Since $\mathrm{tr}^2(ab) = \mathrm{tr}^2(ba)$ and the modification uses only Nielsen automorphisms, the proposed method stops the algorithm with the same number of steps and complexity as the original one. In section 6.2.7, we prove that this modification produces E-words in the end. More precisely and more strongly,

Theorem 6.2.2. *Let $[n_0; n_1, n_2, \dots, n_k]$ be the continued fraction expansion of the non-negative rational number p/q. Then the last changed generator of the modified Gilman-Maskit algorithm using the E-sequence $[n_0; n_1, n_2, \dots, n_k]$ is the E-word corresponding the rational number $-p/q$.*

Examples of the Modified Algorithm

The following shows the modified algorithm using the E-sequence $[5; 4, 3]$.

$$\begin{aligned}
(a,b) &\to (ba,b) \to (bab,b) \to \left(b^2ab,b\right) \to \left(b^2ab^2,b\right) \\
&\to \left(b^3ab^2,b\right) \to \left(b^3ab^2, b^3ab^3\right) \\
&\to \left(b^3ab^2, b^3ab^5ab^3\right) \to \left(b^3ab^2, b^3ab^5ab^5ab^3\right) \\
&\to \left(b^3ab^2, b^3ab^5ab^5ab^5ab^3\right) \to \left(b^3ab^5ab^5ab^5ab^3, b^3ab^5ab^5ab^5ab^3\right) \\
&\to \left(b^3ab^5ab^5ab^5ab^6ab^5ab^5ab^5ab^3, b^3ab^5ab^5ab^5ab^3\right) \\
&\to \left(b^3ab^5ab^5ab^5ab^6ab^5ab^5ab^5ab^6ab^5ab^5ab^5ab^3, b^3ab^5ab^5ab^5ab^3\right)
\end{aligned}$$

One can observe that the sum of the exponents of a is 13, and that of b is 68. Moreover, the continued fraction expansion of $\frac{68}{13}$ is $[5; 4, 3]$.

The following shows the algorithm using the E-sequence $[4; 3, 2]$.

$$\begin{aligned}(a, b) &\to (ba, b) \to \cdots \to \left(b^2ab^2, b\right) \to \left(b^2ab^2, b^3ab^2\right)\\ &\to \left(b^2ab^2, b^2ab^5ab^2\right) \to \left(b^2ab^2, b^2ab^5ab^4ab^2\right)\\ &\to \left(b^2ab^4ab^5ab^4ab^2, b^2ab^5ab^4ab^2\right)\\ &\to \left(b^2ab^4ab^5ab^4ab^4ab^5ab^4ab^2, b^2ab^5ab^4ab^2\right)\end{aligned}$$

In this example, the sum of the exponents of a is 7, and that of b is 30. Likewise, the continued fraction expansion of $\frac{30}{7}$ is $[4; 3, 2]$.

The following shows the algorithm using the E-sequence $[0; 3, 4]$.

$$\begin{aligned}(a, b) &\to (a, ba) \to (a, aba) \to \left(a, aba^2\right) \to \left(a^2ba^2, aba^2\right)\\ &\to \left(a^2ba^3ba^2, aba^2\right) \to \left(a^2ba^3ba^3ba^2, aba^2\right)\\ &\to \left(a^2ba^3ba^3ba^3ba^2, aba^2\right)\end{aligned}$$

The fraction for above is $\frac{4}{13}$ which has a continued fraction expansion of $[0; 3, 4]$.

Lastly, the example found in [9] is copied to the E-sequence $[3; 2, 4]$.

$$\begin{aligned}(a, b) &\to (ba, b) \to (bab, b) \to \left(b^2ab, b\right) \to \left(b^2ab, b^2ab^2\right)\\ &\to \left(b^2ab, b^2ab^3ab^2\right) \to \left(b^2ab^3ab^3ab^2, b^2ab^3ab^2\right)\\ &\to \left(b^2ab^3ab^4ab^3ab^3ab^2, b^2ab^3ab^2\right)\\ &\to \left(b^2ab^3ab^4ab^3ab^3ab^4ab^3ab^2, b^2ab^3ab^2\right)\\ &\to \left(b^2ab^3ab^4ab^3ab^4ab^3ab^3ab^4ab^3ab^2, b^2ab^3ab^2\right)\end{aligned}$$

The last modified generator looks like the E-word corresponding to $\frac{31}{9}$.

6.2.6 Reversing the Enumeration Scheme

In this section reversing the process of the Gilman-Keen enumeration scheme is shown. The definition of the scheme requires taking the parents of a given rational number.

While the parents exist and are well-defined for most rational numbers, their computations and ordering are cumbersome. In addition, the parents are broken further into "grandparents" until orphans are encountered. Every time a parent is not an orphan, another splitting into two parents must occur; the manual computations get worse.

In theory, one can start with the greatest grandparents of all elements which exactly are the orphans 0 and ∞. This section explains in detail how this process can be done. A typical pair of parents are noticeably Farey neighbors. Furthermore, their Farey sum is equal to their only 'child'. We prove these observations and show that the properties of Farey neighbors are also properties of parents. In the process of reversing the enumeration scheme, the modification of the Gilman-Maskit algorithm is also proven to stop with E-words.

The following are claims in [9]. The proofs are provided here.

Lemma 6.2.1. *Let $\frac{p}{q}$ be Farey neighbors with $\frac{p}{q} < \frac{r}{s}$. Assume $\frac{p}{q}$ and $\frac{r}{s}$ are in lowest terms. Then $\frac{p}{q} < \frac{p+r}{q+s} < \frac{r}{s}$; and the pairs $\frac{p}{q}, \frac{p+r}{q+s}$ and $\frac{p+r}{q+s}, \frac{r}{s}$ are Farey neighbors.*

Proof. We may assume $q \geq 1$ and $s \geq 1$ so that $q + s \geq 2$. Then,

$$\begin{aligned}
ps < qr &\implies ps + pq < rq + pq \\
&\implies p(s+q) < (r+p)q \\
&\implies \frac{p}{q} < \frac{r+p}{s+q}; \\
ps < qr &\implies ps + rs < qr + rs \\
&\implies (p+r)s < r(q+s) \\
&\implies \frac{p+r}{q+s} < \frac{r}{s}.
\end{aligned}$$

Since $|ps - qr| = 1$, we have

$$\begin{aligned}
|p(q+s) - q(p+r)| &= |pq + ps - qp - qr| \\
&= |ps - qr| = 1 \\
|(p+r)s - (q+s)r| &= |ps + rs - qr - qs| \\
&= |ps - qr| = 1
\end{aligned}$$

$\square$

Corollary 6.2.1. *If* $\dfrac{p}{q}$ *and* $\dfrac{r}{s}$ *are Farey neighbors, then the fractions* $\dfrac{p}{q}$, $\dfrac{r}{s}$, *and* $\dfrac{p+r}{q+s}$ *are in lowest terms.*

Lemma 6.2.2. *Let* $\frac{p}{q}$ *and* $\frac{r}{s}$ *be Farey neighbors expressed in lowest terms. Then* p *and* q *cannot be both even;* r *and* s *cannot be both even. Moreover,* p, q, r *and* s *cannot be all odd.*

Proof. If p and q are both even, then $\frac{p}{q}$ are not in lowest term since $\gcd(p, q) \geq 2$. Same is true with r and s. Suppose all integers p, q, r and s are odd. Then ps and rq are also odd, but $ps - qr$ is even. In particular $|ps - rq|$ is not 1. □

From this point onward, $\frac{p}{q}$ and $\frac{r}{s}$ are Farey neighbors reduced to lowest terms.

Lemma 6.2.3. *For each pair of Farey neighbors* $\frac{p}{q}$ *and* $\frac{r}{s}$*, only one of the following combinations hold.*

1. pq *is odd;* rs *is even.*

2. pq *and* rs *are even.*

3. pq *is even* rs *is odd.*

Proof. If pq is odd, both p and q are odd. By the lemma above, r and s cannot be both odd so one of them must be even. Hence rs is even. □

When it comes to listing possibilities, whether even or odd, of the integers p, q, r and s, two more combinations can be eliminated.

Lemma 6.2.4. *For each pair of Farey neighbors* $\frac{p}{q}$ *and* $\frac{r}{s}$*, the following combinations do not hold.*

1. p *and* r *are even;* q *and* s *are odd.*

2. p *and* r *are odd;* q *and* s *are even.*

Proof. If the fractions are Farey neighbors, $|ps - rq| = 1$. If any combinations above hold, both ps and rq are even. Hence, the difference of even numbers is even. In particular $|ps - rq|$ cannot equal to 1. □

Initially, odd-even combinations of four integers p, q, r and s add up to 16. However the preceding lemmas imply that there can be only 6 possibilities.

Theorem 6.2.3. *For each pair of Farey neighbors $\frac{p}{q}$ and $\frac{r}{s}$, only one of the following combinations hold.*

p	q	r	s	$(p+r)(q+s)$
even	*odd*	*odd*	*even*	*odd*
odd	*odd*	*even*	*odd*	*even*
odd	*odd*	*odd*	*even*	*even*
odd	*even*	*even*	*odd*	*odd*
even	*odd*	*odd*	*odd*	*even*
odd	*even*	*odd*	*odd*	*even*

Proof. The conditions where p and q are both even eliminate four conditions. The conditions where r and s are both even reduces 3 more. The case where all p, q, r and s are odd and lemma above make 3 more impossible. The remaining possibilities are 6 out of 16. $\square$

The Gilman-Keen enumeration scheme maps each element of $\mathbb{Q}\cup\{\infty\}$ to a primitive word in F_2 by defining a recursion. Recall that the word $E_{p/q}$ in F_2 corresponding to a positive rational number p/q.

$$E_{p/q} = \begin{cases} E_{r/s}E_{m/n} & \text{if } pq \text{ is odd,} \\ E_{m/n}E_{r/s} & \text{if } pq \text{ is even.} \end{cases}$$

The elements $\frac{m}{n}$ and $\frac{r}{s}$ of $\mathbb{Q}\cup\{\infty\}$ are the parents of $\frac{p}{q}$. For the orphans 0 and ∞, $E_0 = A^{-1}$ and $E_\infty = B$. It is also assumed that $\frac{m}{n} < \frac{r}{s} \leq \infty$.

It is probably well-known that the Farey sum of the parents of p/q is equal to p/q. It is also known that for any Farey neighbors $\frac{m}{n}$ and $\frac{r}{s}$ whose Farey sum is $\frac{p}{q}$, then $\frac{m}{n}$ and $\frac{r}{s}$ must be the parents of $\frac{p}{q}$. The proofs of well-known facts are provided here.

Fact 6.2.1. *If $\frac{p}{q}$ and $\frac{r}{s}$ are positive and Farey neighbors, then the parents of their Farey sum are exactly $\frac{p}{q}$ and $\frac{r}{s}$.*

Proof. Assume $\frac{p}{q} < \frac{r}{s}$. The case where $\frac{r}{s} = \infty$ imply that $\frac{p}{q}$ is an integer. This makes their Farey sum equal to the integer $p+1$. By Lemma 6.1.1, the parents of $p+1$ are p and ∞, which are p/q and r/s. So we can assume that $\frac{r}{s} < \infty$.

Suppose $\frac{m}{n}$ is a Farey neighbor with $\frac{m}{n} < \frac{p+r}{q+s}$. Assume $n \geq 1$. Then,

$$\frac{p+r}{q+s} < \frac{r}{s} \Rightarrow \frac{m}{n} < \frac{r}{s} \Rightarrow sm < nr \Rightarrow nr - sm > 0.$$

Since $nr - sm$ is an integer, $nr - sm \geq 1$. By Farey neighbor property,

$$\begin{aligned} mq + ms < np + nr &\implies np + nr - mq - sm = 1 \\ &\implies 1 - np + qm = nr - sm \geq 1 \\ &\implies qm - np \geq 0 \\ &\implies qm \geq np \\ &\implies \frac{m}{n} \geq \frac{p}{q}. \end{aligned}$$

Thus, p/q is a parent of $\frac{p+r}{q+s}$.

Suppose $\frac{m}{n}$ is a Farey neighbor of $\frac{p+r}{q+s}$ with $\frac{p+r}{q+s} < \frac{m}{n}$. Assume $n, q, s \geq 1$. Similarly,

$$\frac{p}{q} < \frac{p+r}{q+r} \Rightarrow \frac{p}{q} < \frac{m}{n} \Rightarrow qm - pn > 0.$$

The difference $qm - pn$ must be at least 1. Therefore,

$$\begin{aligned} pn + rn < qm + sm &\implies qm + sm - pn - rn = 1 \\ &\implies qm - pn = 1 + rn - sm \\ &\implies 1 + rn - sm \geq 1 \\ &\implies rn - sm \geq 0 \\ &\implies \frac{r}{s} \geq \frac{m}{n}. \end{aligned}$$

Thus, r/s is the other parent of $\frac{p+r}{q+s}$. $\square$

The theorem above and Lemma 6.2.1 allow one to guess the parents of a given rational number p/q. This is done by breaking p and q into sums $p = m+r$ and $q = n+s$, so that $|ms - rn| = 1$. On large numerators or denominators, the combinations can be cumbersome, but the goal is to reverse the process of the recursion defined in the enumeration scheme. More precisely, the goal is to determine $E_{p/q}$ starting from A and B instead of starting from computing the parents of p/q.

From this point onward, we assume Farey neighbors $\frac{p}{q}$ and $\frac{r}{s}$ have $\frac{p}{q} < \frac{r}{s}$. The following theorem derives the E-word corresponding to $\frac{p+r}{q+s}$.

Theorem 6.2.4. *Let $\frac{p}{q}$ and $\frac{r}{s}$ be nonnegative and Farey neighbors with $\frac{p}{q} < \frac{r}{s}$. Then, $E_{(p+r)/(q+s)}$ is a product of $E_{p/q}$ and $E_{r/s}$ determined by the following table.*

p	q	r	s	$(p+r)(q+s)$	$E_{(p+r)/(q+s)}$
even	odd	odd	even	odd	$E_{r/s}E_{p/q}$
odd	odd	even	odd	even	$E_{p/q}E_{r/s}$
odd	odd	odd	even	even	$E_{p/q}E_{r/s}$
odd	even	even	odd	odd	$E_{r/s}E_{p/q}$
even	odd	odd	odd	even	$E_{p/q}E_{r/s}$
odd	even	odd	odd	even	$E_{p/q}E_{r/s}$

Proof. Since the parents of $\frac{p+r}{q+s}$ are exactly $\frac{p}{q}$ and $\frac{r}{s}$, the residue class mod 2 of $(p+r)(q+s)$ can be determined by the residue class mod 2 of p, q, r, and s. The possible combinations are fully listed. Since $\frac{p}{q} < \frac{r}{s}$, the E-word corresponding to $\frac{p+r}{q+s}$ is determined in terms of the words corresponding to $E_{p/q}$ and $E_{r/s}$. $\square$

The image of the enumeration scheme is a set of palindromes or product of palindromes. Gilman and Keen [9] proved that $E_{p/q}$ is a palindrome if and only if pq is even. Hence, $E_{p/q}$ is not a palindrome if and only if pq is odd. Using the table in the theorem above, $E_{(p+r)/(q+s)}$ is a palindrome if either pq or rs is not a palindrome; and $E_{(p+r)/(q+s)}$ is not a palindrome if both pq and rs are palindromes.

Let $a = A^{-1}$ and $b = B$, where A and B are generators of rank-2 free group. Then $\langle a, b\rangle = \langle A, B\rangle$ and $(a, b) = \big(E_{0/1}, E_{1/0}\big)$. This initial condition has the rational number corresponding to the left generator less than that of the right generator.

Theorem 6.2.5. *Let $\frac{p}{q} < \frac{r}{s}$ be positive Farey neighbors. Let (a_1, b_1) be the new pair of generators after applying the modified algorithm step to the generators $\big(E_{p/q}, E_{r/s}\big)$. Then both a_1 and b_1 are E-words; $a_1 = E_{j/k}$ and $b_1 = E_{m/n}$ such that $\frac{j}{k} < \frac{m}{n}$. Either $\frac{j}{k}$ or $\frac{m}{n}$ is equal to $\frac{p+r}{q+s}$ so $\frac{j}{k}$ and $\frac{m}{n}$ are Farey neighbors.*

Proof. Let $a_0 = E_{p/q}$ and $b_0 = E_{r/s}$. Then (a_1, b_1) is one of the following.

conditions for a_0 and b_0	preserve a_0	preserve b_0
both a_0 and b_0 are palindromes	$(a_0, b_0 a_0)$	$(b_0 a_0, b_0)$
a_0 is not a palindrome	$(a_0, a_0 b_0)$	$(a_0 b_0, b_0)$
b_0 is not a palindrome	$(a_0, a_0 b_0)$	$(a_0 b_0, b_0)$

Since either a_0 or b_0 is preserved, we show that $a_0 b_0$ or $b_0 a_0$ is the E-word corresponding to $\frac{p+r}{q+s}$. If a_0 and b_0 are both palindromes, both pq and rs are even. Using the table in Theorem 6.2.3, $(p+r)(q+s)$ is odd in any possible combinations of parities (residue classes mod 2) of p, q, r and s. Also $\frac{p}{q}$ and $\frac{r}{s}$ are the parents of $\frac{p+r}{q+s}$. Hence, $E_{(p+r)/(q+s)} = E_{r/s} E_{p/q} = b_0 a_0$.

If either a_0 or b_0 is not a palindrome, then either pq or rs is odd, respectively. The same table shows $(p+r)(q+s)$ is even so $E_{(p+r)/(q+s)} = E_{r/s} E_{p/q} = a_0 b_0$.

The only thing left to show is that $\frac{j}{k} < \frac{m}{n}$. This is a simple application of Lemma 6.2.1 that says $\frac{p}{q} < \frac{p+r}{q+s} < \frac{r}{s}$. If $\frac{j}{k} = \frac{p}{q}$, $\frac{m}{n} = \frac{p+r}{q+s}$ so $\frac{j}{k}$ and $\frac{m}{n}$ are Farey neighbors. If $\frac{j}{k} = \frac{p+r}{q+s}$, $\frac{m}{n} = \frac{r}{s}$. In any case $\frac{j}{k} < \frac{m}{n}$. □

Corollary 6.2.2. *The modified algorithm step, applied finitely many times to a pair of primitive associates* $\left(E_{p/q}, E_{r/s}\right)$ *where* $\frac{p}{q} < \frac{r}{s}$, *provides a pair of E-words that generate the same group* $\left\langle E_{p/q}, E_{r/s} \right\rangle = \langle A, B \rangle$.

6.2.7 Proof of Theorem 6.2.2

Theorem 6.2.6. *Let* $[n_0; n_1, n_2, \ldots, n_k]$ *be the continued fraction expansion of the non-negative rational number* p/q. *Then the last changed generator of the modified Gilman-Maskit algorithm using the E-sequence* $[n_0; n_1, n_2, \ldots, n_k]$ *is the E-word corresponding the rational number* $-p/q$.

Proof. Let $[n_0; n_1, n_2, \ldots, n_k]$ be an E-sequence. Then the modified algorithm has outputs of E-words corresponding to the rational numbers p_i/q_i and r_i/s_i for $i = 0, 1, 2, \ldots, k$ in the following recursive formulas.

$$p_0 = 0 \qquad q_0 = 1 \qquad r_0 = 1 \qquad s_0 = 0$$

$$p_i = n_{2i-2} r_{i-1} + p_{i-1} \qquad r_i = n_{2i-1} p_i + r_{i-1}$$

$$q_i = n_{2i-2} s_{i-1} + q_{i-1} \qquad s_i = n_{2i-1} q_i + s_{i-1}$$

These recursions seem to work, both for E-sequence and fairy addition. Next thing to show is that

$$\frac{p_i}{q_i} = [n_0; n_1, n_2, \ldots, n_{2i-2}]$$

and

$$\frac{r_i}{s_i} = [n_0; n_1, n_2, \ldots, n_{2i-1}] .$$

There are formulas in [9] where *approximants* are defined as follows.

$$g_0 = n_0 \qquad h_0 = 1 \qquad g_1 = n_1 n_0 + 1 \qquad h_1 = n_1$$

$$g_i = n_i g_{i-1} + g_{i-2}$$

$$h_i = n_i h_{i-1} + h_{i-2}$$

It was claimed in [9] that $\dfrac{g_i}{h_i} = [n_0; n_1, n_2, \ldots, n_i]$. There is a way to relate $\frac{p_i}{q_i}$ and $\frac{r_i}{s_i}$ to $\frac{g_i}{h_i}$. In particular,

$$\frac{p_i}{q_i} = \frac{g_{2i-2}}{h_{2i-2}} \qquad \frac{r_i}{s_i} = \frac{g_{2i-1}}{h_{2i-1}}$$

for all $i \geq 1$. We show it as follows.

$$\begin{aligned} p_1 &= n_0 r_0 + p_0 & r_1 &= n_1 p_1 + r_0 \\ &= n_0 & &= n_1 n_0 + 1 \\ &= g_0 & &= g_1 \\ q_1 &= n_0 s_0 + q_0 & s_1 &= n_1 q_1 + s_0 \\ &= n_0 \cdot 0 + 1 & &= n_1 \cdot 1 + 0 \\ &= 1 & &= n_1 \\ &= h_0 & &= h_1 \end{aligned}$$

The assertions work for $i = 1$. To show that the formulas work for all other i, we show that they work for $i + 1$. That is,

$$p_{i+1} = g_{2(i+1)-2} \qquad q_{i+1} = h_{2i}$$
$$= g_{2i+2-2} \qquad r_{i+1} = g_{2i+1}$$
$$= g_{2i} \qquad h_{i+1} = h_{2i+1}.$$

The following are the computations.

$$\begin{aligned} p_{i+1} &= n_{2i}r_i + p_i & r_{i+1} &= n_{2i+1}p_{i+1} + r_i \\ &= n_{2i}g_{2i-1} + g_{2i-2} & &= n_{2i+1}g_{2i} + g_{2i-1} \\ &= g_{2i} & &= g_{2i+1} \\ q_{i+1} &= n_{2i}s_i + q_i & s_{i+1} &= n_{2i+1}q_{i+1} + s_i \\ &= n_{2i}h_{2i-1} + h_{2i-2} & &= n_{2i+1}h_{2i} + h_{2i-1} \\ &= h_{2i} & &= h_{2i+1} \end{aligned}$$

Now that p_i, q_i, r_i and s_i are consolidated to two formulas, g_i and h_i, except p_0, q_0, r_0 and s_0, it follows that

$$\frac{g_i}{h_i} = [n_0; n_1, n_2, \ldots, n_i].$$

□

Corollary 6.2.3. *Let $k \in \mathbb{N}$ be an even integer. For any continued fraction expansion $[a_0; a_1, a_2, \ldots, a_{k+1}]$,*

$$[a_0; a_1, a_2, \ldots, a_k] < [a_0; a_1, a_2, \ldots, a_{k+1}].$$

For any continued fraction expansion $[a_0; a_1, a_2, \ldots, a_k]$,

$$[a_0; a_1, a_2, \ldots, a_k] < [a_0; a_1, a_2, \ldots, a_{k-1}].$$

Proof. If $k = 0$, $[a_0; a_1, a_2, \ldots, a_{k-1}]$ is defined to be ∞ so the statement is equivalent to the fact that natural numbers are finite. The continued fraction expansion $[a_0; a_1, a_2, \ldots, a_k]$ can be regarded as an E-sequence that can be applied to the pair (a, b). The stopping generators correspond to the E-words $E([a_0; a_1, a_2, \ldots, a_k])$

and $E\left([a_0; a_1, a_2, \ldots, a_{k+1}]\right)$. Since k is even, $E\left([a_0; a_1, a_2, \ldots, a_k]\right)$ is the left generator. Also since the modification is designed to preserve the order of fractions, then $[a_0; a_1, a_2, \ldots, a_k] < [a_0; a_1, a_2, \ldots, a_{k-1}]$. Suppose that the E-sequence is extended further to $[a_0; a_1, a_2, \ldots, a_{k+1}]$. Then the set of stopping generators still contains E-word corresponding to $[a_0; a_1, a_2, \ldots, a_k]$ which is still in the left spot. Hence, $[a_0; a_1, a_2, \ldots, a_k] < [a_0; a_1, a_2, \ldots, a_{k+1}]$. $\square$

6.2.8 Consecutive Steps

In the theory of F-sequence, n consecutive linear steps have an simple formula $(a, b) \mapsto (a, a^n b)$. Equivalently, a is preserved in n consecutive steps. In the modified algorithm, there are more than one formula and not all of them are simple. The formulas depend on palindromic conditions of the current generators and which generator is preserved. There are six formulas shown in the following.

Case 1

Suppose a is preserved in n steps; both a and b are palindromes with respect to a fixed line. The next pair is (a, ba). Using the table in Theorem 6.2.5, ba is not a palindrome so the step that preserves a should be $(g, h) \mapsto (g, gh)$. Hence, the next pair of generators is (a, aba). Both a and aba are palindromes again so the next new generator is aba^2. Note that aba^2 is equal to $E_{1/3}$. The steps that change the right generator alternates between palindrome and non-palindrome. If $j = \lfloor n/2 \rfloor$, n steps preserving a ends in $\left(a, a^j b a^j\right)$ if n is even of $\left(a, a^j b a^j a\right)$ if n is odd. The right generator is exactly the E-word corresponding to $\frac{1}{n}$ in the pair (a, b). The formula for this case is

$$(a, b) \mapsto \left(a, E_{1/n}(a, b)\right).$$

Case 2

Suppose a is preserved in n steps; a is not a palindrome. The next pair of generators must be (a, ab). Since a is not a palindrome, ab is still a palindrome. The next pair after a step is $\left(a, a^2 b\right)$. Because the right generator is always a palindrome, the formula for this case is simple:

$$(a,b) \mapsto (a, a^n b)\,.$$

Case 3

Suppose a is preserved in n steps; b is not a palindrome. Like the second case, the next pair is (a, ab), but the new generators are both palindromes. The next new generators are aba, a^2ba, a^2ba^2, a^3ba^2, etc. For n steps, the right generator is a^jba^j if n is even or aa^jb^j if n is odd ($j = \lfloor n/2 \rfloor$). This is similar to E_n but switched a with b. Hence the formula for the third case is

$$(a,b) \mapsto (a, E_n(b,a))\,.$$

Case 4

Suppose b is preserved in n steps; both a and b are palindromes. The left generator becomes ba which is not a palindrome. The next generators are bab, b^2ab, b^2ab^2, b^3ab^2, etc. For n steps and $j = \lfloor n/2 \rfloor$, the last left generator is b^jab^j if n is even or bb^jab^j if n is odd. Thus, the formula for this case is

$$(a,b) \mapsto (E_n(a,b), b)\,.$$

Case 5

Suppose b is preserved in n steps; a not a palindrome. The next left generators are ab, bab, bab^2, b^2ab^2, b^2ab^3, etc. Likewise, it is either b^jab^j or b^jab^jb which is the E-word $E_{1/n}$ with switched a and b. So the formula for this case is

$$(a,b) \mapsto \big(E_{1/n}(b,a), b\big)\,.$$

Case 6

The last case is as simple as the second. Suppose b is preserved in n steps but is not a palindrome. The left generator for each step is a palindrome so the last left generator is ab^n. The formula is

$$(a,b) \mapsto (ab^n, b)\,.$$

The following table summarizes all the formulas.

	preserve a	preserve b
both a and b are palindromes	$\left(a, E_{\frac{1}{n}}(a,b)\right)$	$(E_n(a,b), b)$
a is not a palindrome	$(a, a^n b)$	$\left(E_{\frac{1}{n}}(b,a), b\right)$
b is not a palindrome	$(a, E_n(b,a))$	(ab^n, b)

6.2.9 Length of E-words

Theorem 6.2.7. *The length of $E_{m/n}$ in the generators $\{a,b\}$ is $|m|+|n|$. Moreover $E_{m/n}$ has $|m|$ b-factors and $|n|$ a-factors.*

Proof. Suppose $g, h \in F_2$; g has p b-factors and q a-factors. Suppose h has r b-factors and s a-factors. The modified algorithm applied to (g,h) replaces one of the generators with either gh or hg. Both gh and hg have $p+r$ b-factors and $q+s$ a-factors. More seriously, the algorithm starts with (a,b) and ends with $\left(E_{p/q}, E_{r/s}\right)$. The fraction 1/1 correspond to ba which is the very first new E-word of the algorithm. For $E_{1/1} = ba$, the assertion is true. Suppose this assertion is still true after the algorithm stops at the pair $\left(E_{p/q}, E_{r/s}\right)$. Then $E_{p/q}$ has p b-factors and q a-factors; $E_{r/s}$ has r b-factors and s a-factors. Continuing the algorithm just one step further yields a new E-word $E_{(p+r)/(q+s)}$. It is either $E_{p/q}E_{r/s}$ or $E_{r/s}E_{p/q}$. In any case, $E_{(p+r)/(q+s)}$ has $p+r$ b-factors and $q+s$ a-factors. □

6.2.10 E-words when $k \leq 2$

It is shown in section 6.1.4 what the form of E-words are if $k=0$ or precisely $p/q \in \mathbb{Z}$. In this section, we look at E-words up to $k=2$ and show what happens to "exponents" of a and b for any E-word. To alleviate complicated notations, we define $m_i = \lfloor \frac{n_i}{2} \rfloor$ and $M_i = \lceil \frac{n_i}{2} \rceil$. Then $m_i + M_i = n_i$, $m_i = M_i$ if n_i is even and $M_i = m_i + 1$ if n_i is odd. This notation is used as superscripts. For example, Theorem 6.1.2 shows $E_{n_0} = b^{M_i} a b^{m_i}$ and $E_{1/n_1} = a^{m_i} b a^{M_i}$.

For $p/q > 1$ and $k \geq 2$, four possible cases of an E-sequence can be fed to the modified Gilman-Maskit algorithm. The E-word corresponding to $[n_0; n_1, \ldots, n_k]$ is also a word in $E([n_0; n_1, \ldots, n_{k-2}])$ and $E([n_0; n_1, \ldots, n_{k-1}])$ if $k \geq 2$. By induction, $E([n_0; n_1, \ldots, n_k])$ is a word in $E([n_0; n_1, n_2])$ and $E([n_0; n_1])$.

Lemma 6.2.5. *Let $a, W, b \in F_n$. Then*

$$(aWb)^k = a(Wba)^k a^{-1}$$

for and $k \in \mathbb{N}$.

Proof. For $k = 0$, the statement is true. Suppose for k high enough, the statement is true. Then,

$$\begin{aligned}(aWb)^{k+1} &= (aWb)^k aWb = a(Wba)^k a^{-1}aWb \\ &= a(Wba)^k Wbaa^{-1} = a(Wba)^{k+1} a^{-1}.\end{aligned}$$

The following are the possible cases for an E-sequence with nonzero first entry and lengths or k equal to 1 and 2. ∎

Case $[\mathbf{odd}; n_1]$

Assume n_0 is odd and $n_1 \geq 1$. The modified algorithm turns the pair (a, b) into (E_{n_0}, b). Since n_0 is odd, E_{n_0} is not a palindrome. The stopping pair is $(E_{n_0}, (E_{n_0})^{n_1} b)$.

$$\begin{aligned}E_{n_0} &= b^{M_0} a b^{m_0} \\ (E_{n_0})^{n_1} b &= \left(b^{M_0} a b^{m_0}\right)^{n_1} b \\ &= b^{M_0} \left(a b^{m_0} b^{M_0}\right)^{n_1} b^{-M_0} b \\ &= b^{M_0} (ab^{n_0})^{n_1} b^{1-M_0} \\ &= b^{M_0} (ab^{n_0})^{n_1 - 1} a b^{M_0}\end{aligned}$$

Note that since $(E_{n_0})^{n_1} b = E([n_0; n_1])$ and $n_1 > 1$ the highest exponent is n_0.

Case $[\mathbf{even}; n_1]$

Assume $n_0 \geq 2$ is even and $n_1 \geq 1$. The pair (a, b) becomes the following pairs.

$$\begin{aligned}
(a,b) &\to (E_{n_0}, b) \to \left(E_{n_0}, E_{n_0}^{m_1} b E_{n_0}^{M_1}\right) \\
E_{n_0} &= b^{m_0} a b^{m_0} \\
E_{n_0}^{m_1} b E_{n_0}^{M_1} &= \left(b^{m_0} a b^{m_0}\right)^{m_1} b \left(b^{m_0} a b^{m_0}\right)^{M_1} \\
&= b^{m_0} \left(a b^{m_0} b^{m_0}\right)^{m_1} b^{-m_0} b b^{m_0} \left(a b^{m_0} b^{m_0}\right)^{M_1} b^{-m_0} \\
&= b^{m_0} \left(a b^{n_0}\right)^{m_1} b \left(a b^{n_0}\right)^{M_1} b^{-m_0} \\
&= b^{m_0} \left(a b^{n_0}\right)^{m_1 - 1} a b^{n_0} b \left(a b^{n_0}\right)^{M_1 - 1} a b^{n_0} b^{-m_0} \\
&= b^{m_0} \left(a b^{n_0}\right)^{m_1 - 1} a b^{n_0 + 1} \left(a b^{n_0}\right)^{M_1 - 1} a b^{m_0}
\end{aligned}$$

Case $[\mathbf{odd}; 1, n_2]$

Assume n_0 is odd and $n_2 \geq 1$. The modified algorithm turns (a, b) into the following pairs.

$$\begin{aligned}
(a,b) &\to (E_{n_0}, b) \to (E_{n_0}, E_{n_0} b) \\
&\to \left((E_{n_0} b)^{m_2} E_{n_0} (E_{n_0} b)^{M_2}, E_{n_0} b \right) \\
E_{n_0} b &= b^{M_0} a b^{M_0} \\
(E_{n_0} b)^{m_2} E_{n_0} (E_{n_0} b)^{M_2} &= \left(b^{M_0} a b^{M_0}\right)^{m_2} b^{M_0} a b^{m_0} \left(b^{M_0} a b^{M_0}\right)^{M_2} \\
&= b^{M_0} \left(a b^{M_0} b^{M_0}\right)^{m_2} b^{-M_0} b^{M_0} a b^{m_0} b^{M_0} \left(a b^{M_0} b^{M_0}\right)^{M_2} b^{-M_0} \\
&= b^{M_0} \left(a b^{n_0 + 1}\right)^{m_2} a b^{n_0} \left(a b^{n_0 + 1}\right)^{M_2} b^{-M_0} \\
&= b^{M_0} \left(a b^{n_0 + 1}\right)^{m_2} a b^{n_0} \left(a b^{n_0 + 1}\right)^{M_2 - 1} a b^{M_0}
\end{aligned}$$

Case $[\mathbf{even}; 1, n_2]$

Assume n_0 is even and $n_2 \geq 1$. The following pairs are encountered by the modified algorithm.

$$\begin{aligned}
(a, b) &\to (E_{n_0}, b) \to (E_{n_0}, bE_{n_0}) \\
&\to \left(E_{n_0} (bE_{n_0})^{n_2}, bE_{n_0}\right) \\
bE_{n_0} &= bb^{m_0}ab^{m_0} = b^{m_0+1}ab^{m_0} \\
E_{n_0} (bE_{n_0})^{n_2} &= b^{m_0}ab^{m_0} \left(b^{m_0+1}ab^{m_0}\right)^{n_2} \\
&= b^{m_0}ab^{m_0}b^{m_0+1} \left(ab^{m_0}b^{m_0+1}\right)^{n_2} b^{-m_0-1} \\
&= b^{m_0}ab^{n_0+1} \left(ab^{n_0+1}\right)^{n_2} b^{-m_0-1} \\
&= b^{m_0}ab^{n_0+1} \left(ab^{n_0+1}\right)^{n_2-1} ab^{m_0} \\
&= b^{m_0} \left(ab^{n_0+1}\right)^{n_2} ab^{m_0}
\end{aligned}$$

Note that the highest exponent is $n_0 + 1$.

Summary of cases $k \leq 2$:

E-sequence	stopping pair
$[\text{odd}; n_1]$	$\left(b^{M_0}ab^{m_0}, b^{M_0} (ab^{n_0})^{n_1-1} ab^{M_0}\right)$
$[\text{even}; n_1]$	$\left(b^{m_0}ab^{m_0}, b^{m_0} (ab^{n_0})^{m_1-1} ab^{n_0+1} (ab^{n_0})^{M_1-1} ab^{m_0}\right)$
$[\text{odd}; 1, n_2]$	$\left(b^{M_0} \left(ab^{n_0+1}\right)^{m_2} ab^{n_0} \left(ab^{n_0+1}\right)^{M_2-1} ab^{M_0}, b^{M_0}ab^{M_0}\right)$
$[\text{even}; 1, n_2]$	$\left(b^{m_0} \left(ab^{n_0+1}\right)^{n_2} ab^{m_0}, b^{m_0+1}ab^{m_0}\right)$

Note that the formulas above work even if $n_k = 1$.

Corollary 6.2.4. *For $n_0 > 0$, $k \geq 3$ and $p/q = [n_0; n_1, \ldots, n_k]$ $E_{p/q}$ is a word in a and b of the form*

$$b^{k_1}ab^{k_2}ab^{k_3} \cdots ab^{k_q}ab^{k_{q+1}}$$

where $k_1, k_{q+1} \in \{m_0, M_0\}$ and $\{k_2, k_3, \ldots, k_q\} = \{n_0, n_0 + 1\}$.

Proof. By Theorem 6.2.2, $E_{p/q}$ is the last modified generator in running the E-sequence $[n_0; n_1, \ldots, n_k]$. The modified algorithm has to output the words $E([n_0; n_1])$ and $E([n_0; n_1, n_2])$ in the middle of the process. Hence $E_{p/q}$ is a word in $E([n_0; n_1])$ and

$E([n_0; n_1, n_2])$. The possibilities of these words are listed in the preceding table. Any product or powers of them has $b^{m_0}b^{m_0}$, $b^{m_0}b^{m_0+1}$, $b^{M_0}b^{M_0}$ or $b^{m_0}b^{M_0}$ in its substring which simplifies to either b^{n_0} or b^{n_0+1}. The table also shows that $E_{p/q}$ is of the claimed form where $k_i \in \{n_0, n_0+1\}$ except k_1 and k_{q+1} which are in $\{M_0, m_0\}$.

In the second case in the table, m_1 and M_1 can possibly equal to 1 so there might be no b-exponent equal to n_0. We show that n_0 still appears as an exponent in $E([n_0; n_1, \ldots, n_k])$ if $k \geq 3$.

Let $n_1 > 1$. Suppose after running the E-sequence $[n_0; n_1]$ on (a, b), the new pair is (g, h). If n_0 is even, g is a palindrome. Continuing the E-sequence further to $[n_0; n_1, 1]$ turns g into either gh or hg.

$$
\begin{aligned}
gh &= b^{m_0}ab^{n_0}\left(ab^{n_0}\right)^{m_1-1}ab^{n_0+1}\left(ab^{n_0}\right)^{M_1-1}ab^{m_0} \\
&= b^{m_0}\left(ab^{n_0}\right)^{m_1}ab^{n_0+1}\left(ab^{n_0}\right)^{M_1-1}ab^{m_0} \\
hg &= b^{m_0}\left(ab^{n_0}\right)^{m_1-1}ab^{n_0+1}\left(ab^{n_0}\right)^{M_1-1}ab^{n_0}ab^{m_0} \\
&= b^{m_0}\left(ab^{n_0}\right)^{m_1-1}ab^{n_0+1}\left(ab^{n_0}\right)^{M_1}ab^{m_0}
\end{aligned}
$$

Hence $E([n_0; n_1, n_2])$, for $n_2 > 1$, is a word in $E([n_0; n_1])$ and $E([n_0; n_1, 1])$, but $E([n_0; n_1, n_2, n_3])$ is a word in $E([n_0; n_1])$ and $E([n_0; n_1, n_2])$. Hence, there is i with $2 \leq i \leq q$ so that $k_i = n_0$.

The same problem occurs in the last case where both primitive generators do not have n_0 as a b-exponent. We must also show that n_0 still appear as an exponent in $E([n_0; n_1, \ldots, n_k])$ for $k \geq 3$.

Let $n_2 \geq 1$. Suppose after running the E-sequence $[n_0; 1, n_2]$ on (a, b), the new pair is (g, h). If n_0 is even, g is a palindrome and h is not. Then the E-sequence $[n_0; 1, n_2, 1]$ turns (g, h) into (g, gh). Continuing further, $[n_0; 1, n_2, 2]$ stops with the pair (g, ghg) while $[n_0; 1, n_2, 1, 1]$ stops with the pair (ghg, gh). Thus $E([n_0; 1, n_2, 1, n_4])$ is a word in gh and ghg for $n_4 > 1$, and $E([n_0; 1, n_2, n_3])$ is a word in g and ghg for $n_3 > 2$.

$$\begin{aligned}
ghg &= b^{m_0}\left(ab^{n_0+1}\right)^{n_2} ab^{m_0} b^{m_0+1} ab^{m_0} b^{m_0}\left(ab^{n_0+1}\right)^{n_2} ab^{m_0} \\
&= b^{m_0}\left(ab^{n_0+1}\right)^{n_2} ab^{n_0+1} ab^{n_0}\left(ab^{n_0+1}\right)^{n_2} ab^{m_0} \\
gh &= b^{m_0}\left(ab^{n_0+1}\right)^{n_2} ab^{m_0} b^{m_0+1} ab^{m_0} \\
&= b^{m_0}\left(ab^{n_0+1}\right)^{n_2} ab^{n_0+1} ab^{m_0}
\end{aligned}$$

Hence both $E([n_0; 1, n_2, 1, n_4])$ and $E([n_0; 1, n_2, n_3])$ have b-exponents equal to n_0.

Remark: The E-word corresponding to $[n_0; 1, n_2, 1]$ still has no b-exponent equal to n_0. However, for a continued fraction expansion $[n_0; n_1, \ldots, n_k]$, n_k is required to be at least 2. $\square$

The case when $p/q < 1$ is similar but $n_0 = 0$ and the index shifts by 1. For example, the E-sequence $[0; n_1, n_2, n_3]$ is similar to $[n_1; n_2, n_3]$ which is greater than 1 as a rational number. In order to see the precise forms of E-words of $p/q < 1$, we go through the cases $n_0 = 0$ and $k \leq 3$.

Case $[0; \mathbf{odd}, n_2]$

Assume n_1 is odd and $n_2 \geq 1$. The modified algorithm runs this E-sequence as follows.

$$\begin{aligned}
(a, b) &\to \left(a, E_{1/n_1}\right) \to \left(a\left(E_{1/n_1}\right)^{n_2}, E_{1/n_1}\right) \\
E_{1/n_1} &= a^{m_1} ba^{M_1} \\
a\left(E_{1/n_1}\right)^{n_2} &= a\left(a^{m_1} ba^{M_1}\right)^{n_2} \\
&= aa^{m_1}\left(ba^{M_1} a^{m_1}\right)^{n_2} a^{-m_1} \\
&= a^{M_1}\left(ba^{n_1}\right)^{n_2} a^{-m_1} \\
&= a^{M_1}\left(ba^{n_1}\right)^{n_2-1} ba^{n_1-m_1} \\
&= a^{M_1}\left(ba^{n_1}\right)^{n_2-1} ba^{M_1}
\end{aligned}$$

Case $[0; \mathbf{even}, n_2]$

Assume $n_1 \geq 2$ is even and $n_1 \geq 1$. The pair (a, b) becomes the following pairs.

$$(a,b) \to \left(a, E_{1/n_1}\right) \to \left(\left(E_{1/n_1}\right)^{M_2} a \left(E_{1/n_1}\right)^{m_2}, E_{1/n_1}\right)$$

$$\begin{aligned}
E_{1/n_1} &= a^{m_1} b a^{m_1} \\
\left(E_{1/n_1}\right)^{M_2} a \left(E_{1/n_1}\right)^{m_2} &= \left(a^{m_1} b a^{m_1}\right)^{M_2} a \left(a^{m_1} b a^{m_1}\right)^{m_2} \\
&= a^{m_1} \left(b a^{m_1} a^{m_1}\right)^{M_2} a^{-m_1} a a^{m_1} \left(b a^{m_1} a^{m_1}\right)^{m_2} a^{-m_1} \\
&= a^{m_1} \left(b a^{n_1}\right)^{M_2 - 1} b a^{n_1+1} \left(b a^{n_1}\right)^{m_2 - 1} b a^{m_1}
\end{aligned}$$

Case $[0; \mathbf{odd}, 1, n_3]$

Assume n_1 is odd and $n_3 \geq 1$. The pair (a, b) turns into the following pairs.

$$\begin{aligned}
(a,b) &\to \left(a, E_{1/n_1}\right) \to \left(aE_{1/n_1}, E_{1,n_1}\right) \\
&\to \left(aE_{1/n_1}, \left(aE_{1/n_1}\right)^{M_3} E_{1/n_1} \left(aE_{1/n_1}\right)^{m_3}\right)
\end{aligned}$$

$$\begin{aligned}
aE_{1/n_1} &= a a^{m_1} b a^{M_1} \\
&= a^{M_1} b^{M_1} \\
\left(aE_{1/n_1}\right)^{M_3} E_{1/n_1} \left(aE_{1/n_1}\right)^{m_3} &= \left(a^{M_1} b a^{M_1}\right)^{M_3} a^{m_1} b a^{M_1} \left(a^{M_1} b a^{M_1}\right)^{m_3} \\
&= a^{M_1} \left(b a^{M_1} a^{M_1}\right)^{M_3} a^{-M_1} a^{m_1} b a^{M_1} a^{M_1} \left(b a^{M_1} a^{M_1}\right)^{m_3} a^{-M_1} \\
&= a^{M_1} \left(b a^{n_1+1}\right)^{M_3} a^{-1} b a^{n_1+1} \left(b a^{n_1+1}\right)^{m_3} a^{-M_1} \\
&= a^{M_1} \left(b a^{n_1+1}\right)^{M_3 - 1} b a^{n_1} b a^{n_1+1} \left(b a^{n_1+1}\right)^{m_3 - 1} b a^{M_1} \\
&= a^{M_1} \left(b a^{n_1+1}\right)^{M_3 - 1} b a^{n_1} \left(b a^{n_1+1}\right)^{m_3} b a^{M_1}
\end{aligned}$$

Case $[0; \mathbf{even}, 1, n_3]$

Assume n_1 is even and $n_3 \geq 1$. The pairs encountered by the modified algorithm are as follows.

$$\begin{aligned}
(a,b) &\to \left(a, E_{1/n_1}\right) \to \left(E_{1/n_1}a, E_{1/n_1}\right) \\
&\to \left(E_{1/n_1}a, \left(E_{1/n_1}a\right)^{n_3} E_{1/n_1}\right) \\
E_{1/n_1}a &= a^{m_1}ba^{m_1}a \\
&= a^{m_1}ba^{m_1+1} \\
\left(E_{1/n_1}a\right)^{n_3} E_{1/n_1} &= \left(a^{m_1}ba^{m_1+1}\right)^{n_3} a^{m_1}ba^{m_1} \\
&= a^{m_1}\left(ba^{n_1+1}\right)^{n_3} a^{-m_1}a^{m_1}ba^{m_1} \\
&= a^{m_1}\left(ba^{n_1+1}\right)^{n_3} ba^{m_1}
\end{aligned}$$

Summary of cases for $n_0 = 0$ and $k \leq 3$:

E-sequence	stopping pair
$[0;\text{odd};n_2]$	$\left(a^{M_1}\left(ba^{n_1}\right)^{n_2-1}ba^{M_1}, a^{m_1}ba^{M_1}\right)$
$[0;\text{even};n_2]$	$\left(a^{m_1}\left(ba^{n_1}\right)^{M_2-1}ba^{n_1+1}\left(ba^{n_1}\right)^{m_2-1}ba^{m_1}, a^{m_1}ba^{m_1}\right)$
$[0;\text{odd};1,n_3]$	$\left(a^{M_1}ba^{M_1}, a^{M_1}\left(ba^{n_1+1}\right)^{M_3-1}ba^{n_1}\left(ba^{n_1+1}\right)^{m_3}ba^{M_1}\right)$
$[0;\text{even};1,n_3]$	$\left(a^{m_1}ba^{m_1+1}, a^{m_1}\left(ba^{n_1+1}\right)^{n_3}ba^{m_1}\right)$

Likewise, the table works for n_2 of n_3 possibly equal to 1.

Corollary 6.2.5. *For $n_0 = 0$, $n_1 \geq 1$, $k \geq 4$ and $p/q = [n_0; n_1, \ldots, n_k]$, $E_{p/q}$ is a word in a and b of the form*

$$a^{k_1}ba^{k_2}ba^{k_3}\cdots ba^{k_p}ba^{k_{p+1}}$$

where $k_1, k_{p+1} \in \{m_1, M_1\}$ and $\{k_2, k_3, \ldots, k_p\} = \{n_1, n_1+1\}$.

Proof. The E-word corresponding to $[0; n_1, n_2, \ldots, n_k]$ is a word in the stopping pair of the E-sequence $[0; n_1, n_2, n_3]$. By induction, $E_{[0;n_1,\ldots,n_k]}$ is a word in the stopping pair of $[0; n_1, n_2]$. Looking at the table above, any word in a given stopping pair has either $a^{M_1+M_1}$, $a^{m_1+m_1}$, $a^{m_1+1+m_1}$ or $a^{m_1+M_1}$ in its substring which is equal to either

a^{n_1} or a^{n_1+1}. Hence, $E_{[0;n_1,\cdots,n_k]}$ is of the form $a^{k_1}ba^{k_2}b\cdots a^{k_p}ba^{k_{p+1}}$ where $k_1, k_{p+1} \in \{m_1, M_1\}$ and $k_2, k_3, \ldots, k_p \in \{n_1, n_1+1\}$. Using similar arguments in 6.2.4, there is $k_i = n_i$ if $k \geq 4$. $\square$

Theorem 6.2.8. *Let $n \in \mathbb{N}$. There exists a bijection between the set of nonzero integers relatively prime with n in the interval $[-n, n] \subset \mathbb{R}$ and the set of E-words of length n.*

Proof. Let Φ and Ψ be the sets defined as follows.

$$\Phi = \{p \in \mathbb{Z} : 0 < |p| < n \text{ and } \gcd(p, n) = 1\}$$

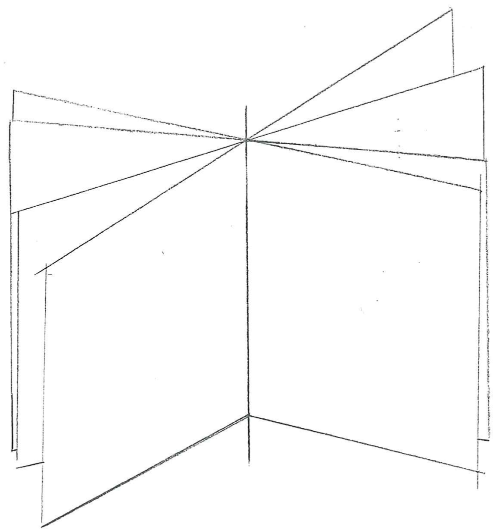
$$\Psi = \{(p, q) \in \mathbb{Z} \times \mathbb{N} : |p| + q = n,\ p \neq 0 \text{ and } \gcd(p, q) = 1\}$$

Define a function $f : \Phi \to \Psi$ given by $f(x) = (x, n - |x|)$. Let $x \in \Phi$. Then $|x| + (n - |x|) = n$, $\gcd(x, n) = 1$ and $n - |x| > 0$. Also $\gcd(x, n - |x|) = 1$ since a divisor both of x and $n - |x|$ is also a divisor of n. Hence $f(x) \in \Psi$. Let $(p, q) \in \Psi$. Then $q = n - |p|$ and $\gcd(p, n) = 1$ by a similar argument. Thus, $p \in \Phi$ and $f(p) = (p, q)$. If $x \neq y \in \Phi$, $f(x) = (x, n - |x|) \neq (y, n - |y|) = f(y)$. The set Ψ is exactly the index set of all E-words of length n. $\square$

Appendix A

Accompanying Figures

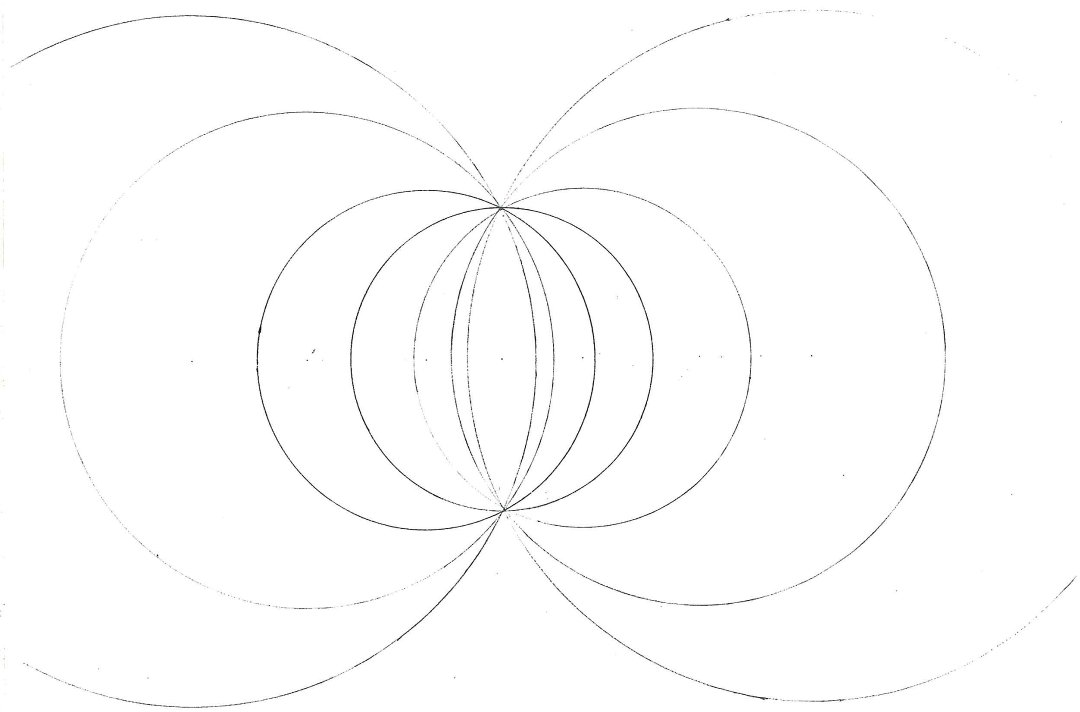
The following figures are aid for visualizing pencils of an isometry or a pir of isometries. Figures A.1 through A.5 approximates the pencils of various types. The rest, figures A.6 through A.28, are pictures involving pairs in Chapter 4.



Permuted pencil of a type I elliptic isometry that fixes ∞.

Same picture for the twisting pencil of either pure loxodromic or screw parabolic isometry fixing ∞.

Figure A.1: Elliptic Permuted Pencil Fixing ∞



Approximate figure of permuted pencil of a type I elliptic isometry fixing a circle.

Figure A.2: Elliptic Permuted Pencil Not Fixing ∞

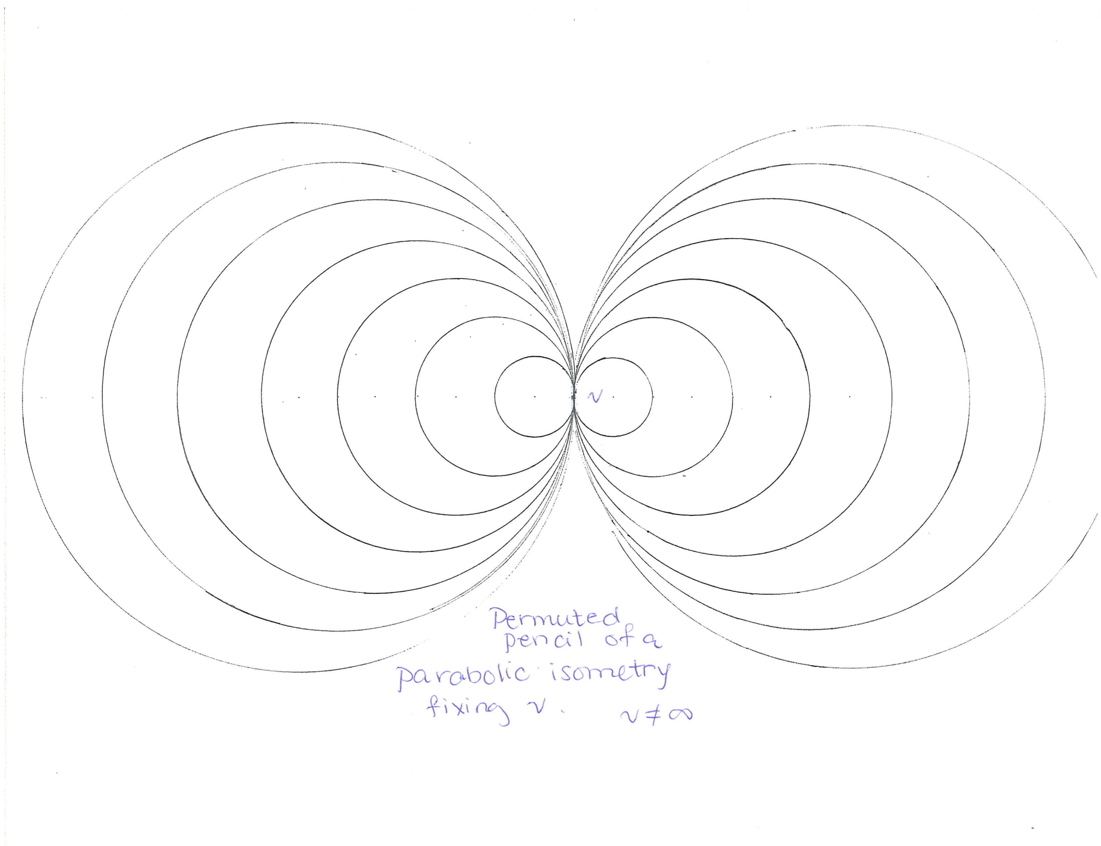


Figure A.3: Parabolic Permuted Pencil Not Fixing ∞

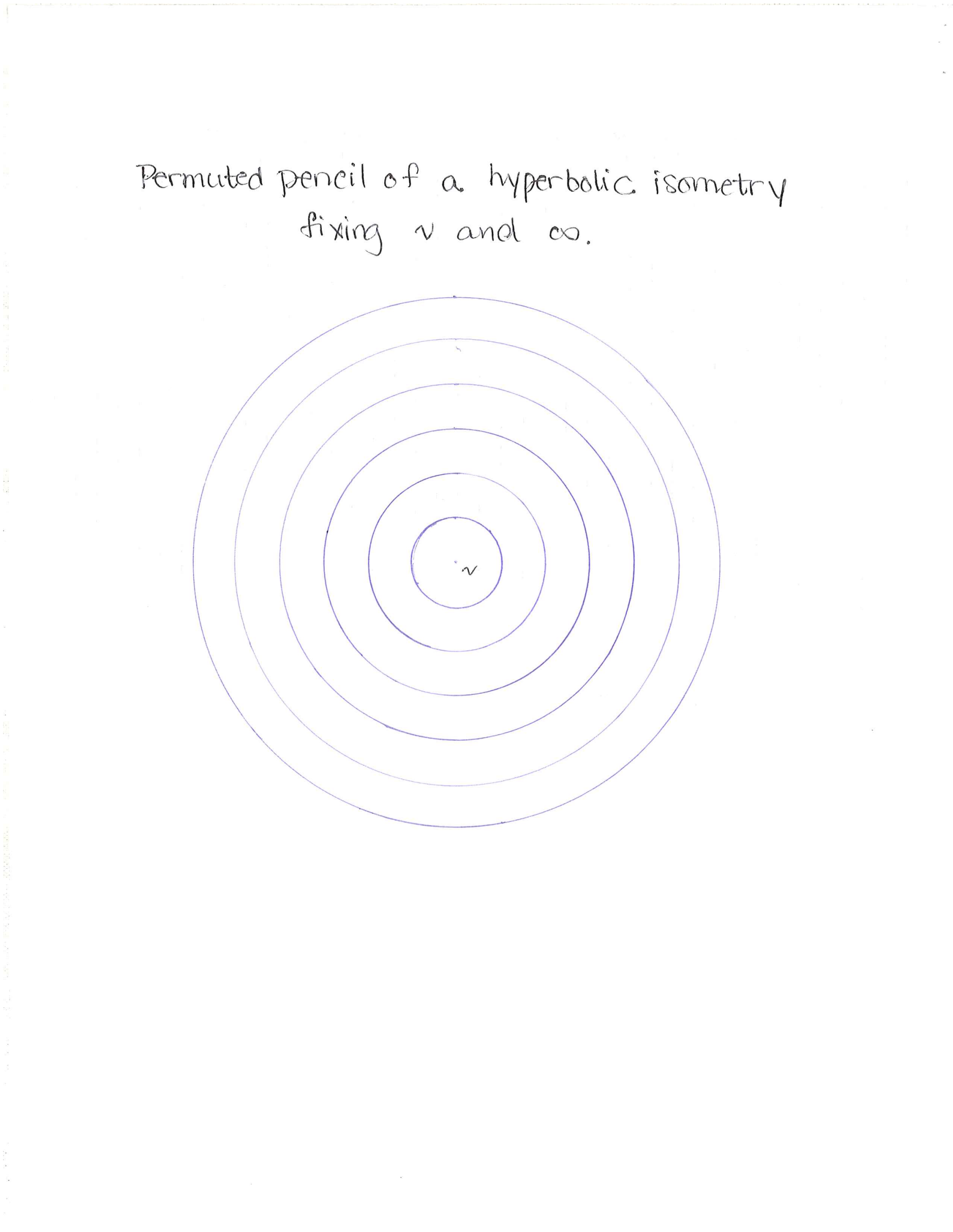


Figure A.4: Hyperbolic Permuted Pencil Fixing ∞

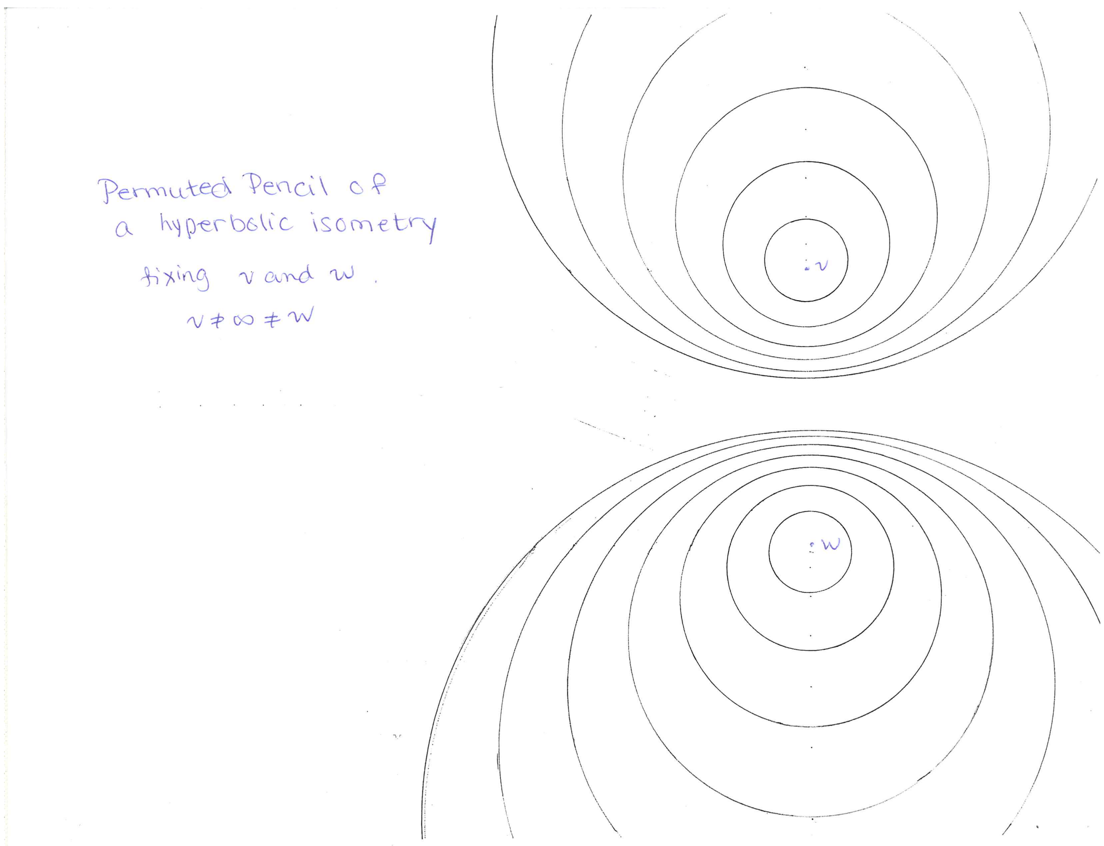


Figure A.5: Hyperbolic Permuted Pencil Not Fixing ∞

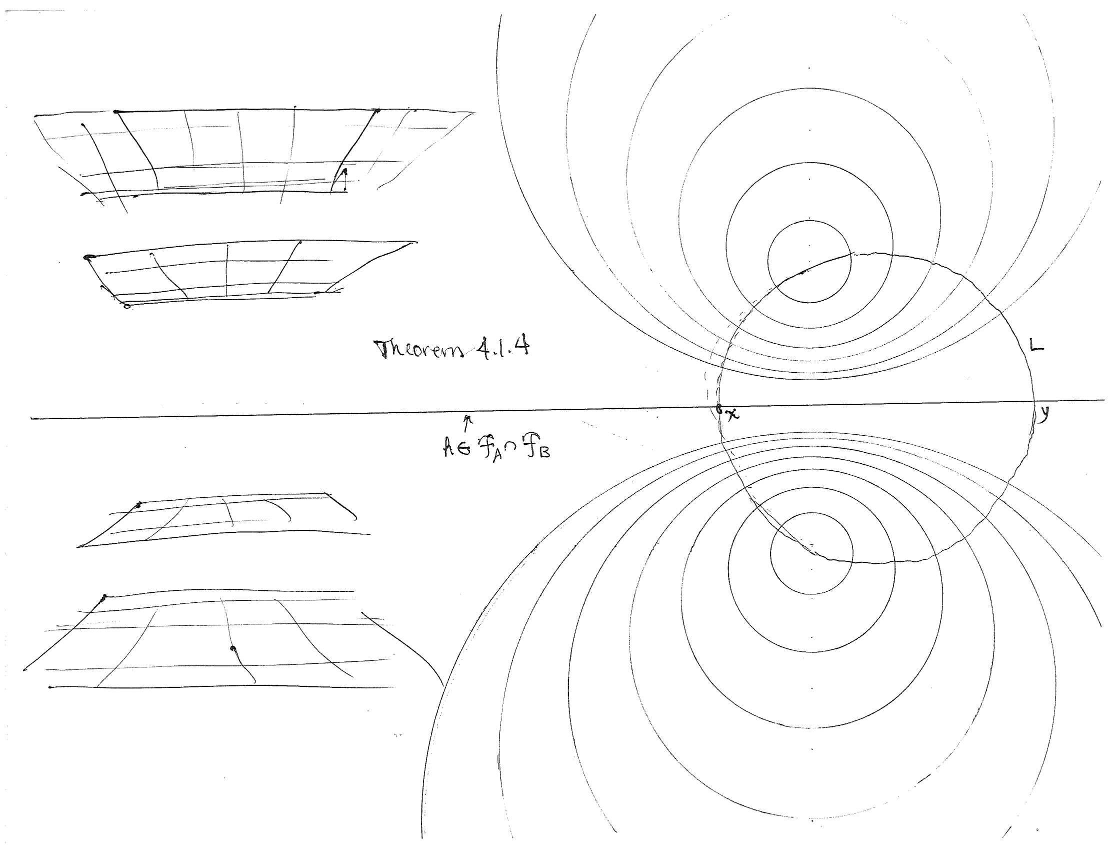


Figure A.6: Theorem 4.1.4

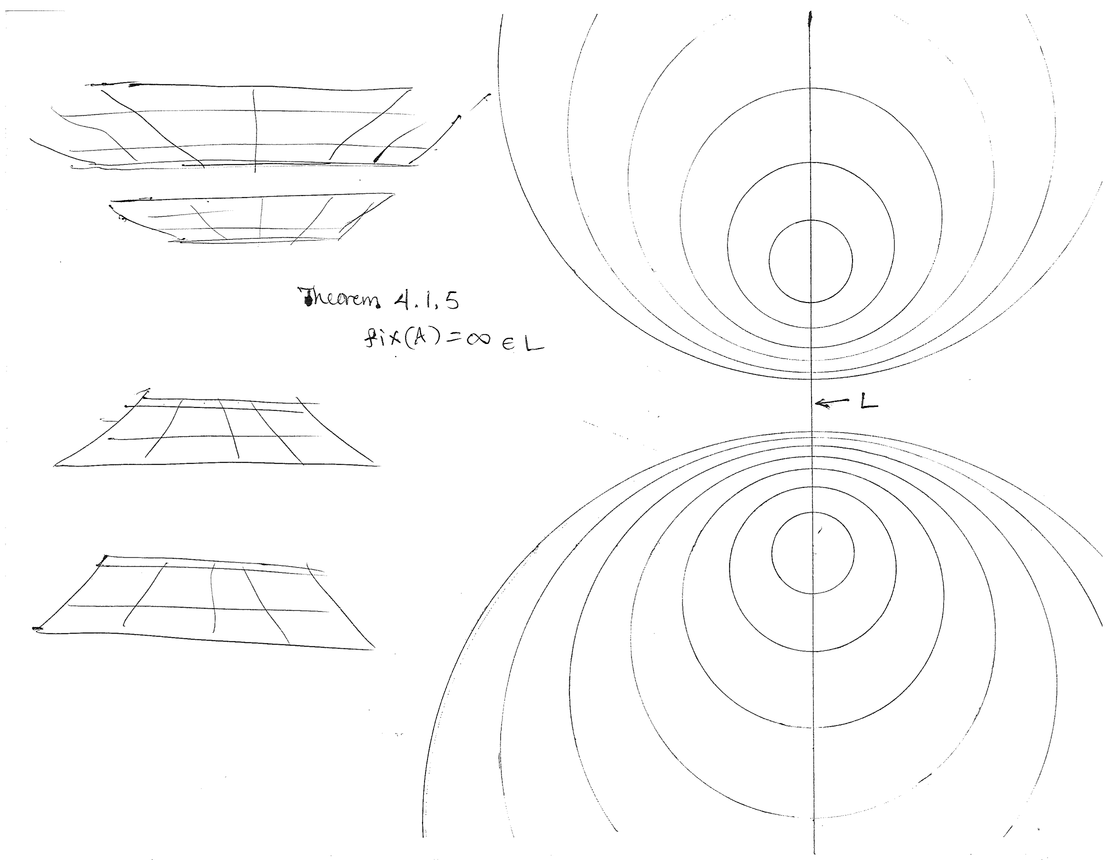


Figure A.7: Theorem 4.1.5

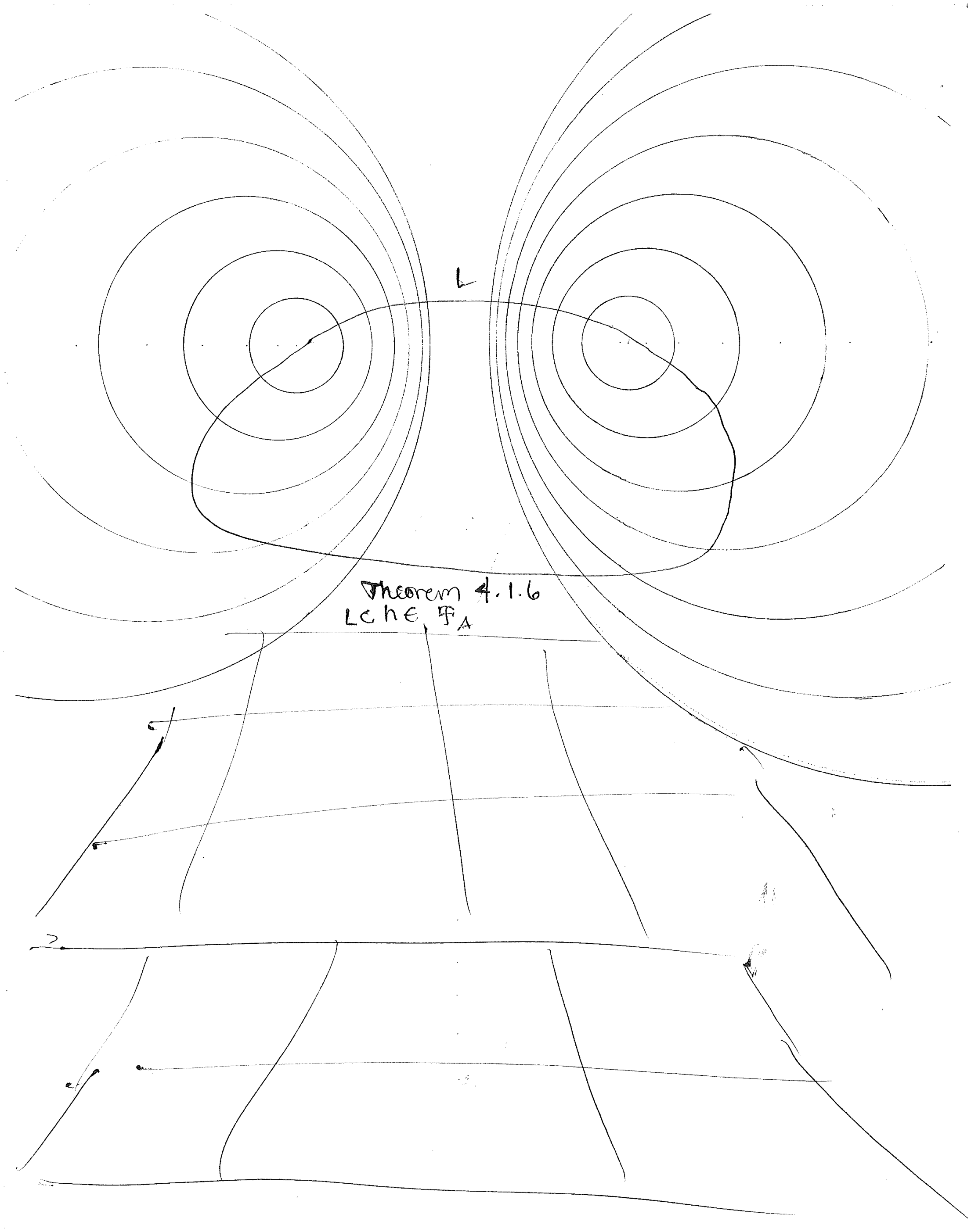


Figure A.8: Theorem 4.1.6

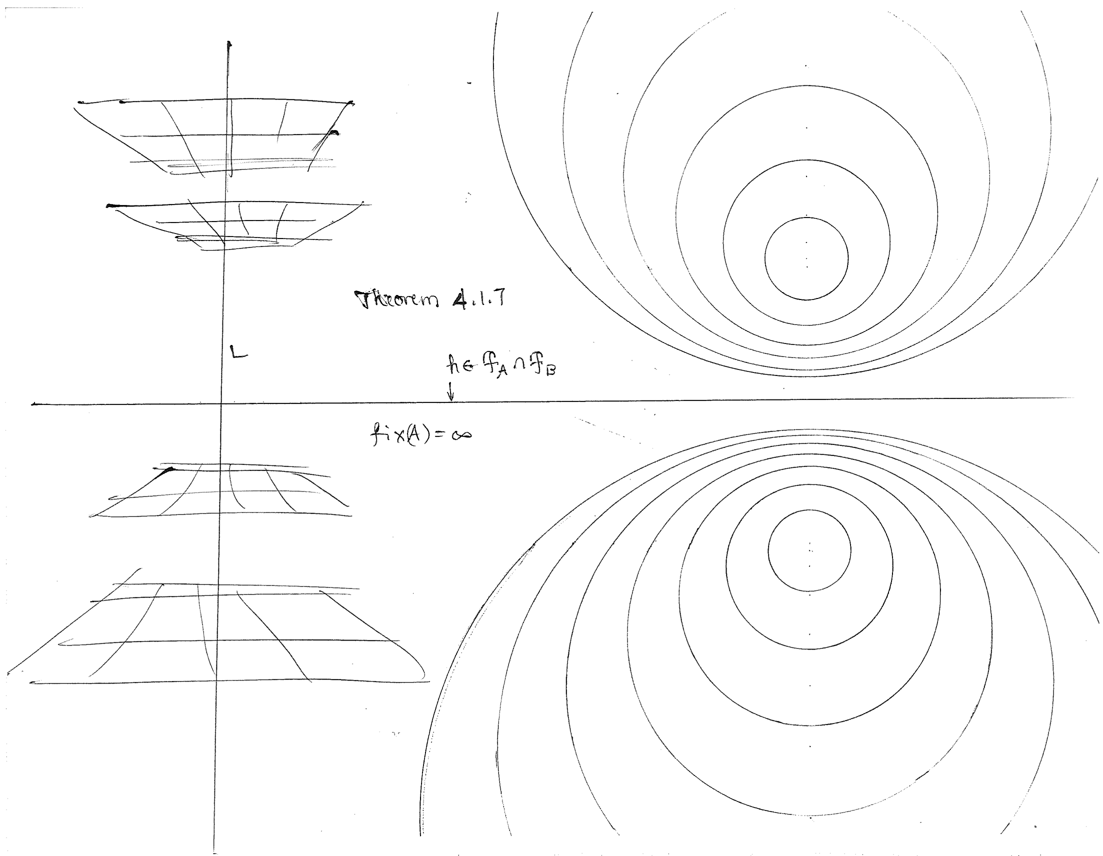


Figure A.9: Theorem 4.1.7

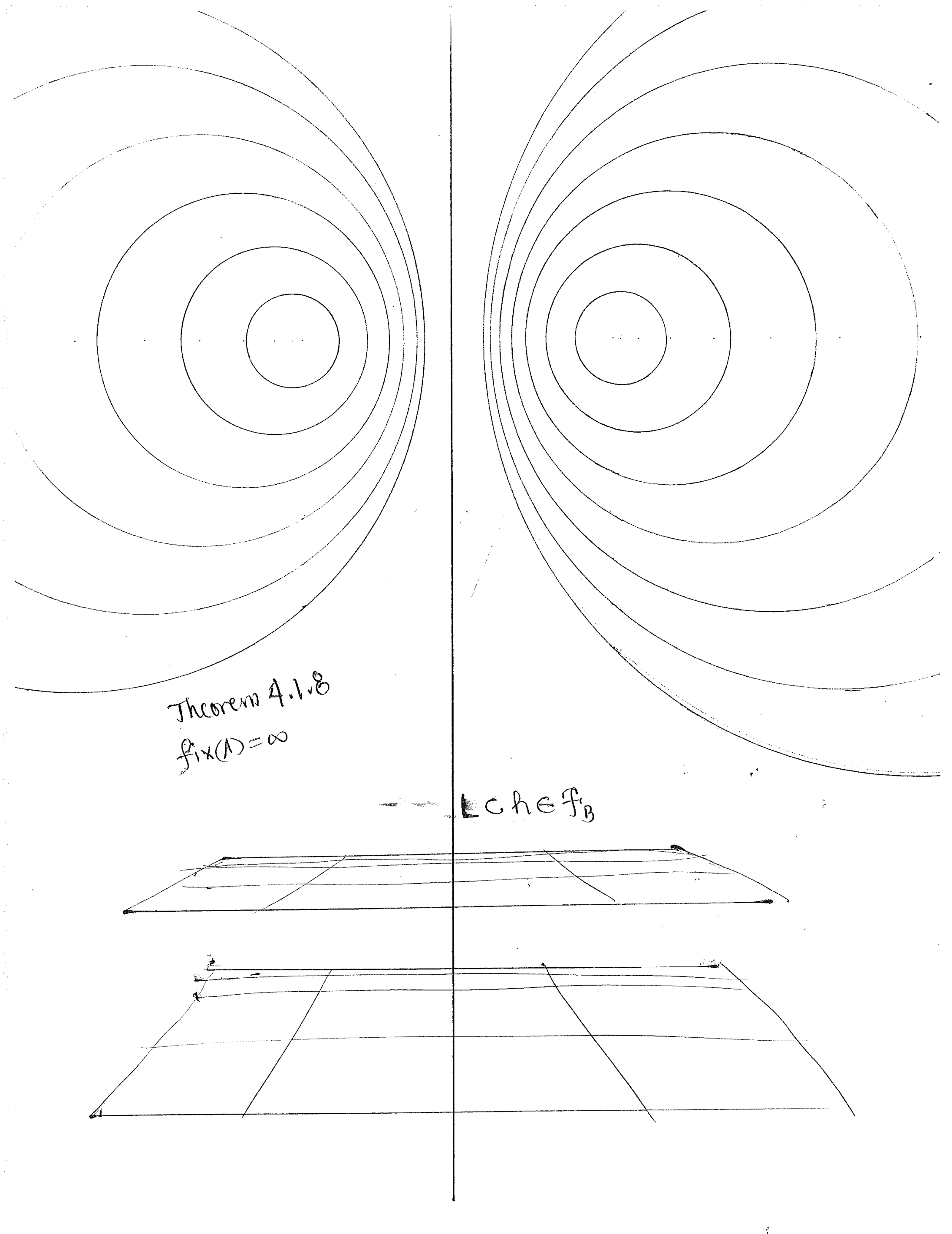


Figure A.10: Theorem 4.1.8

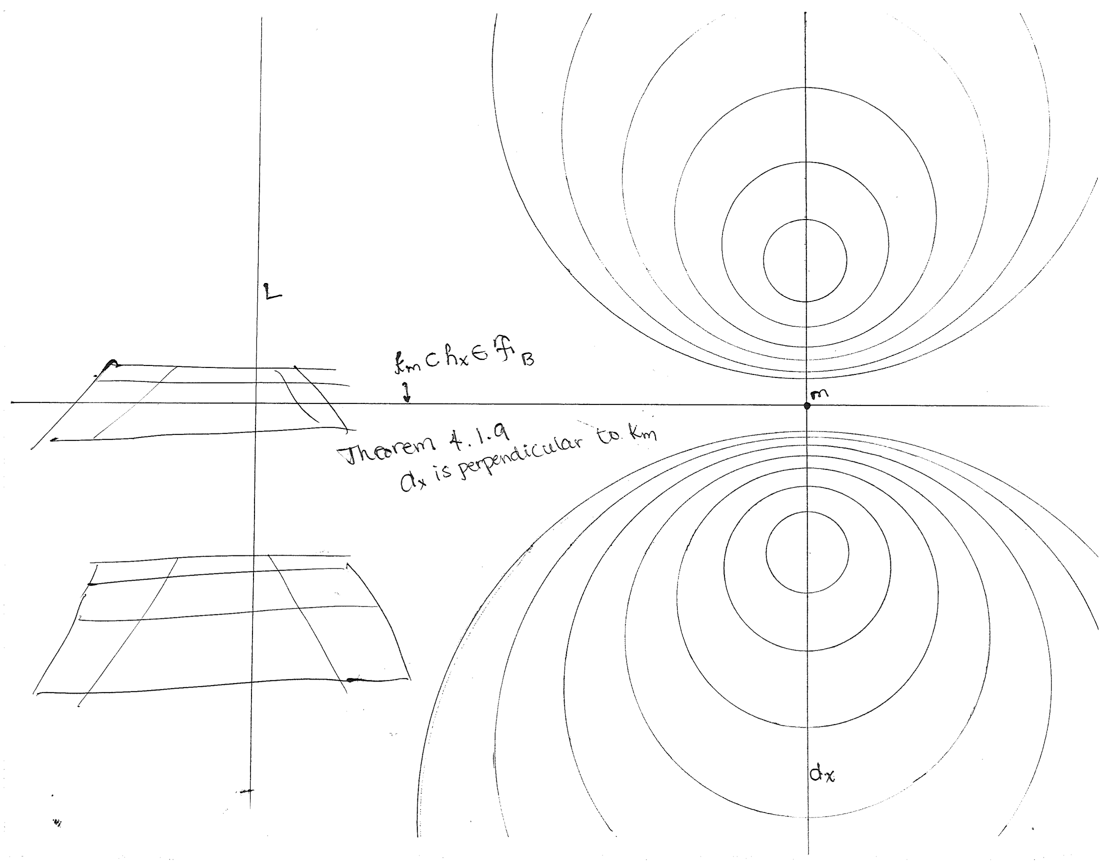


Figure A.11: Theorem 4.1.9: d_x is perpendicular to k_m

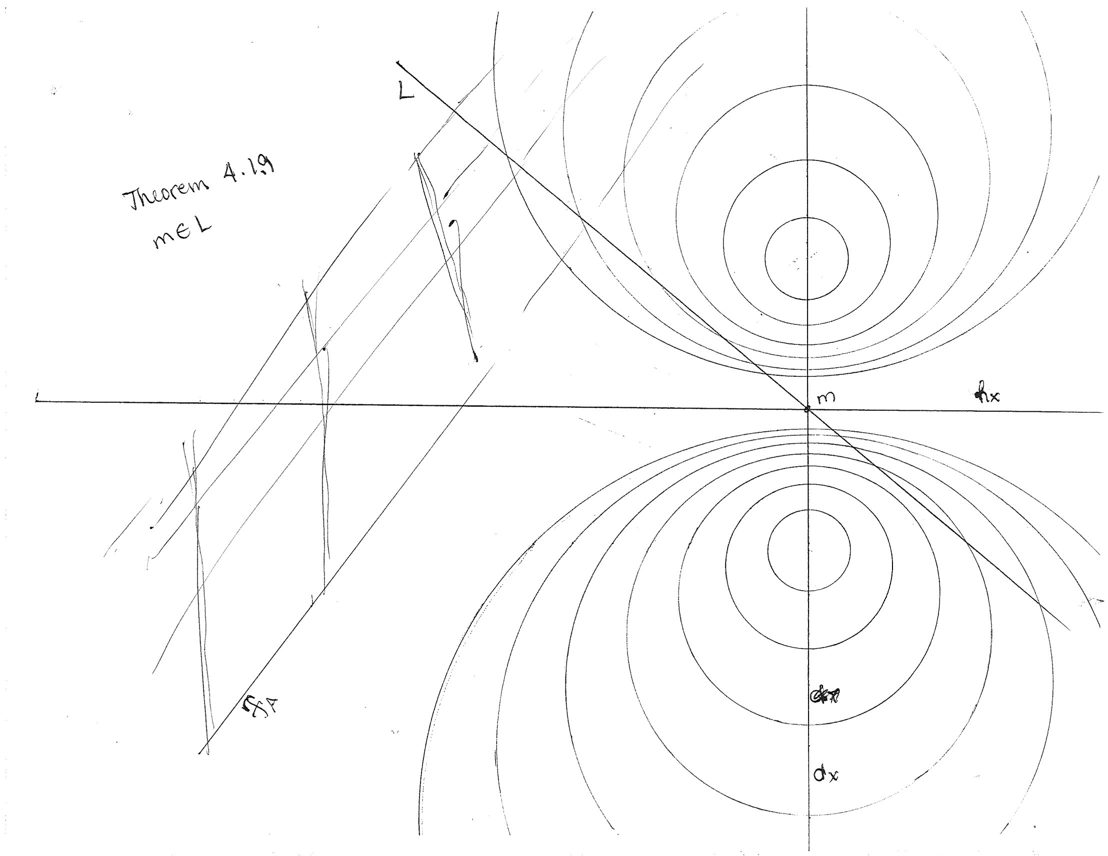


Figure A.12: Theorem 4.1.9: $m \in L$

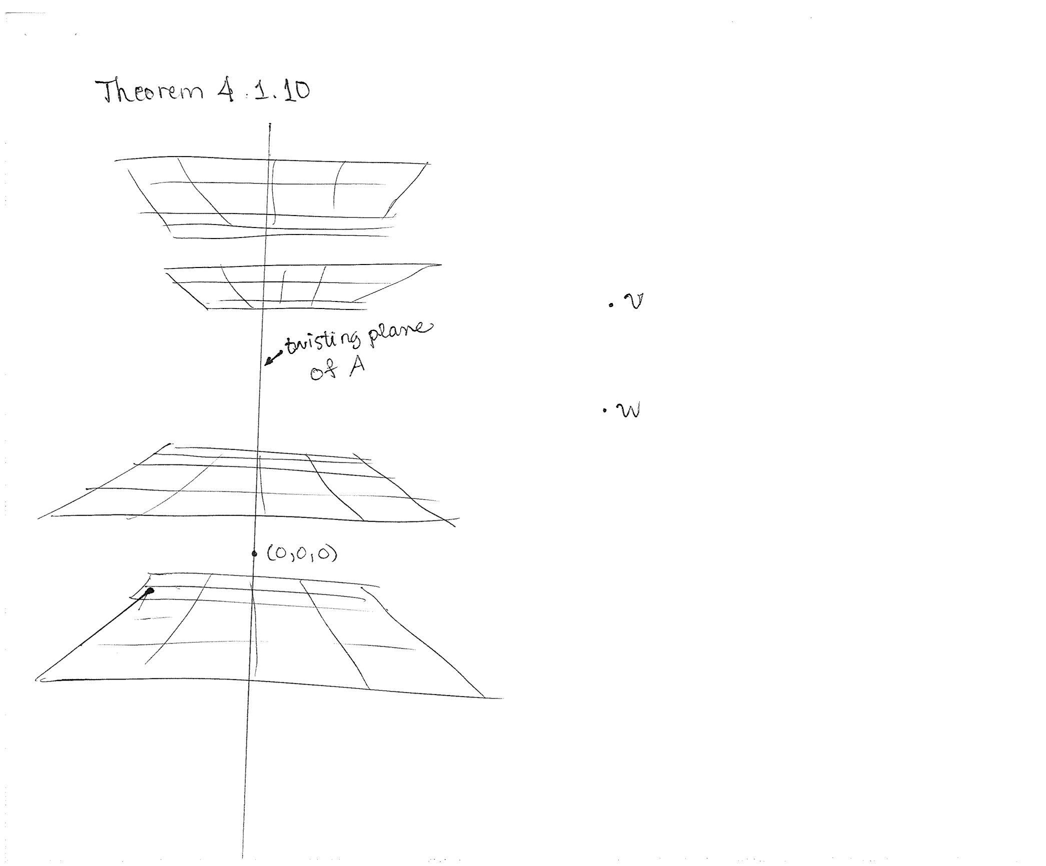


Figure A.13: Theorem 4.1.10

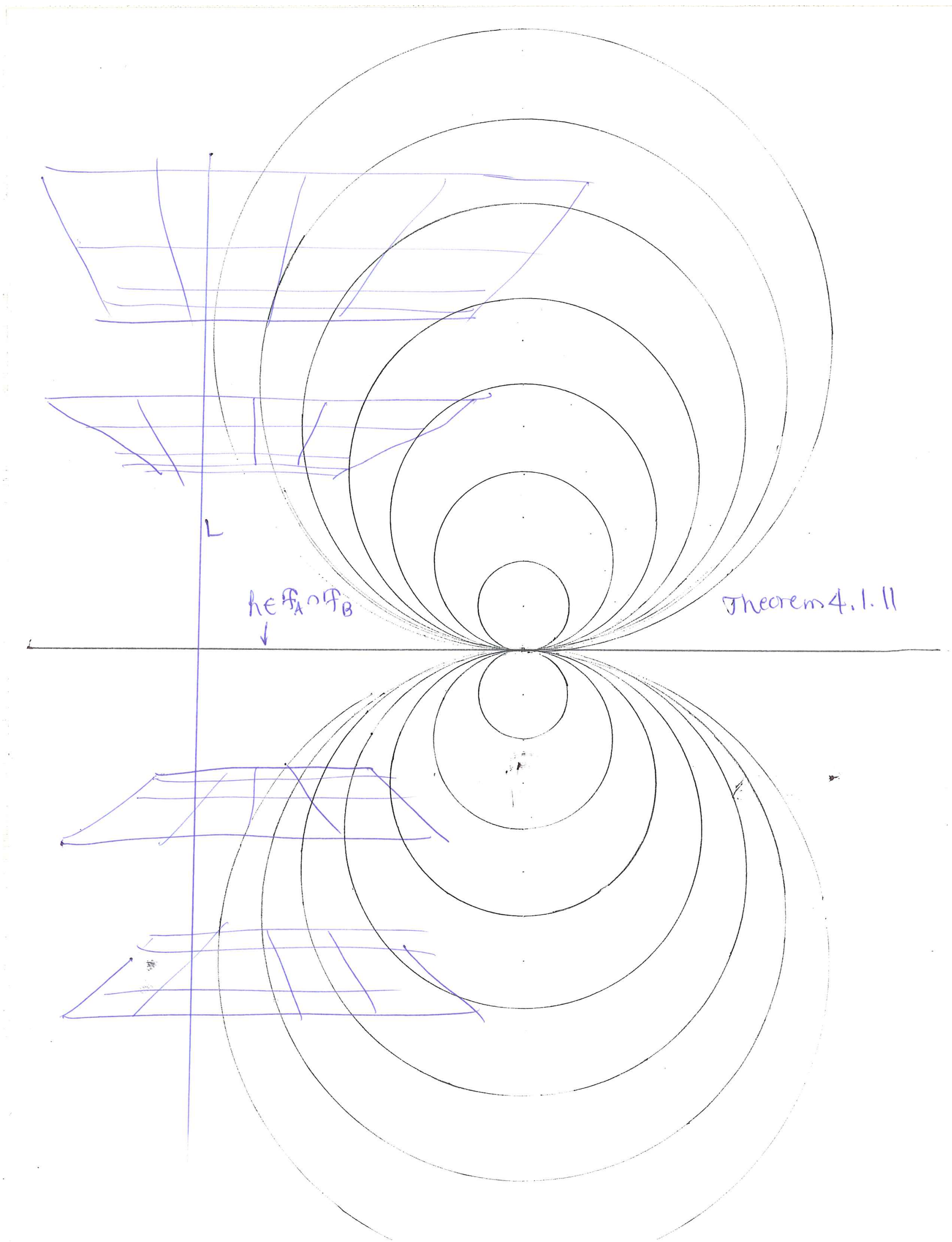


Figure A.14: Theorem 4.1.11

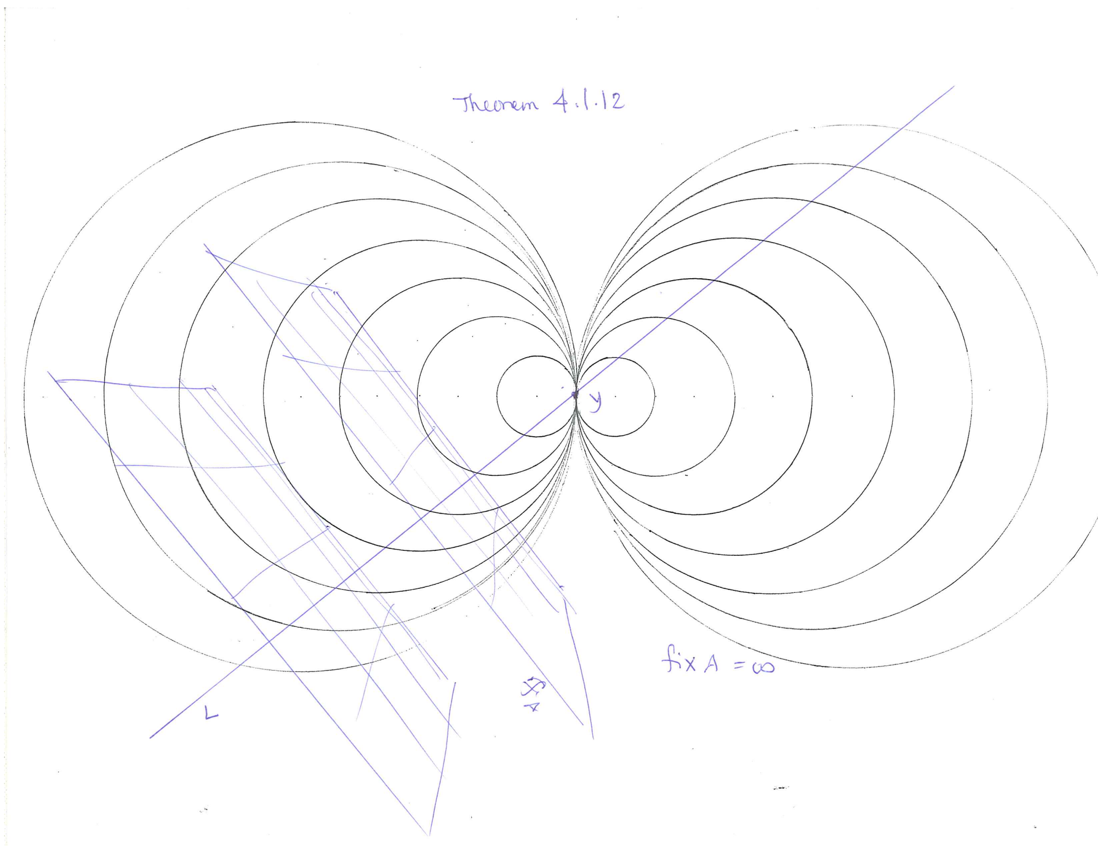


Figure A.15: Theorem 4.1.12

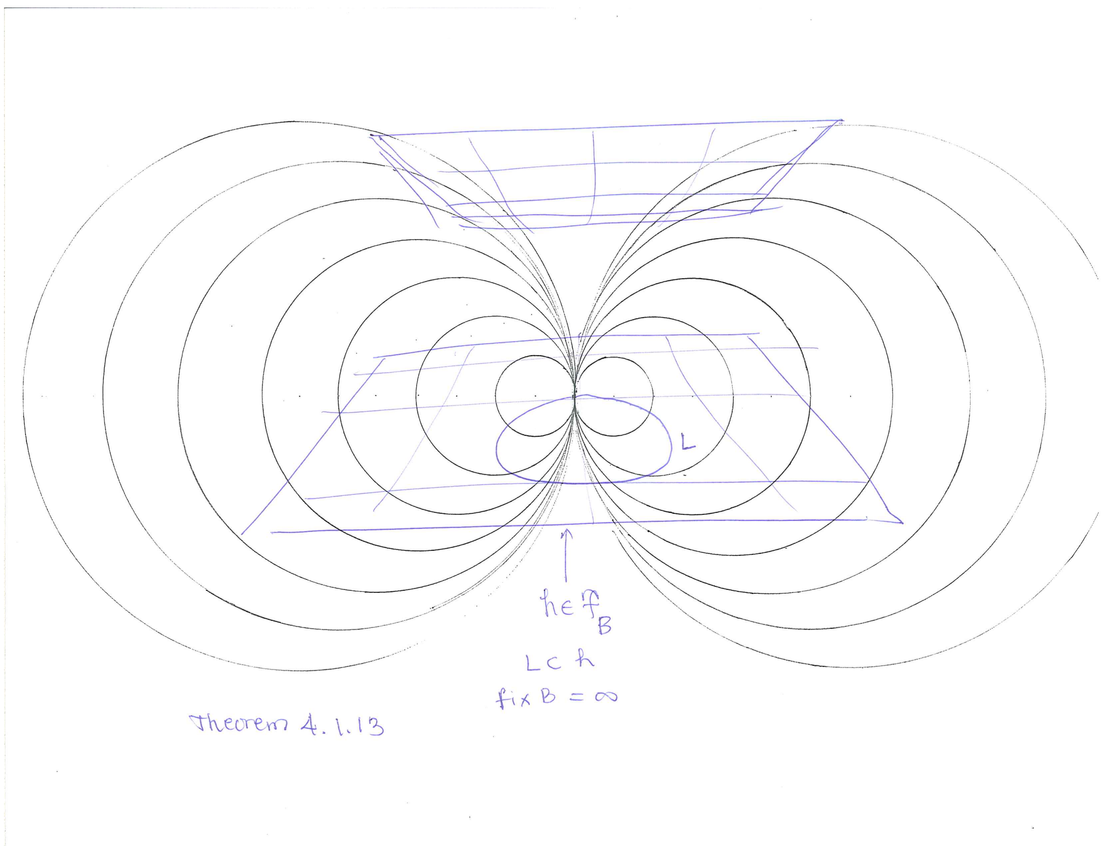


Figure A.16: Theorem 4.1.13

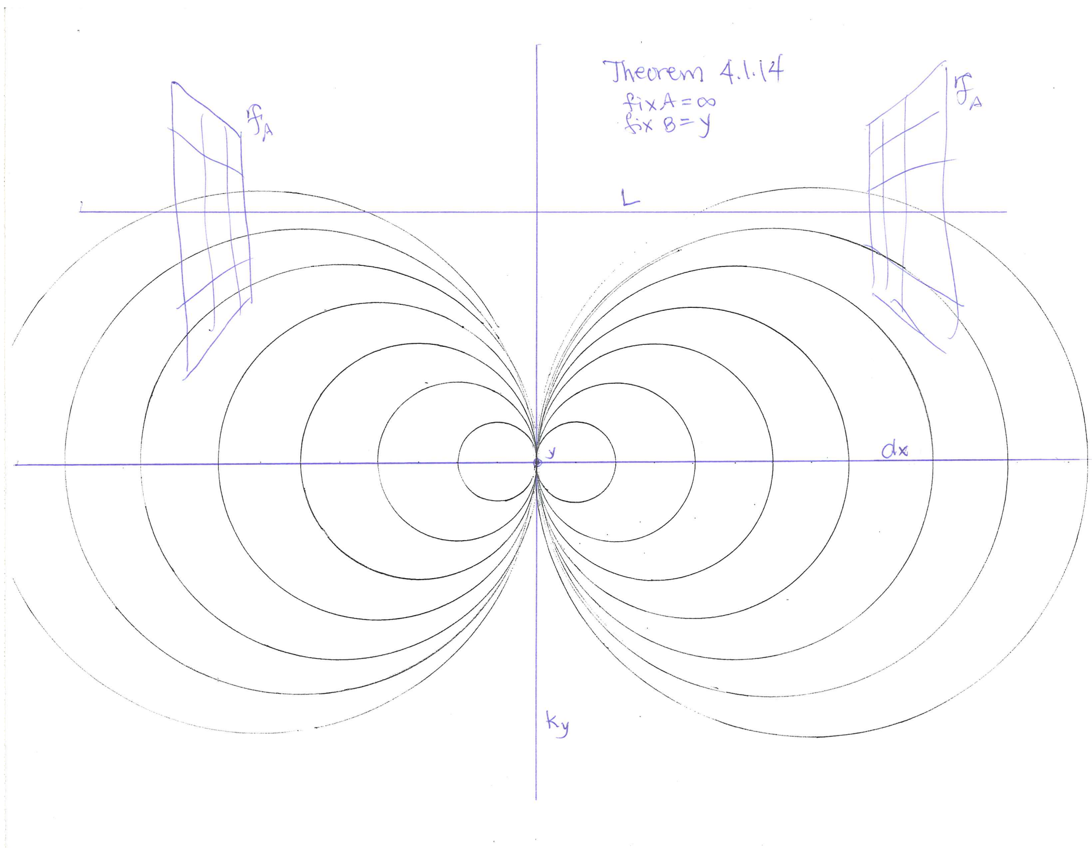


Figure A.17: Theorem 4.1.14

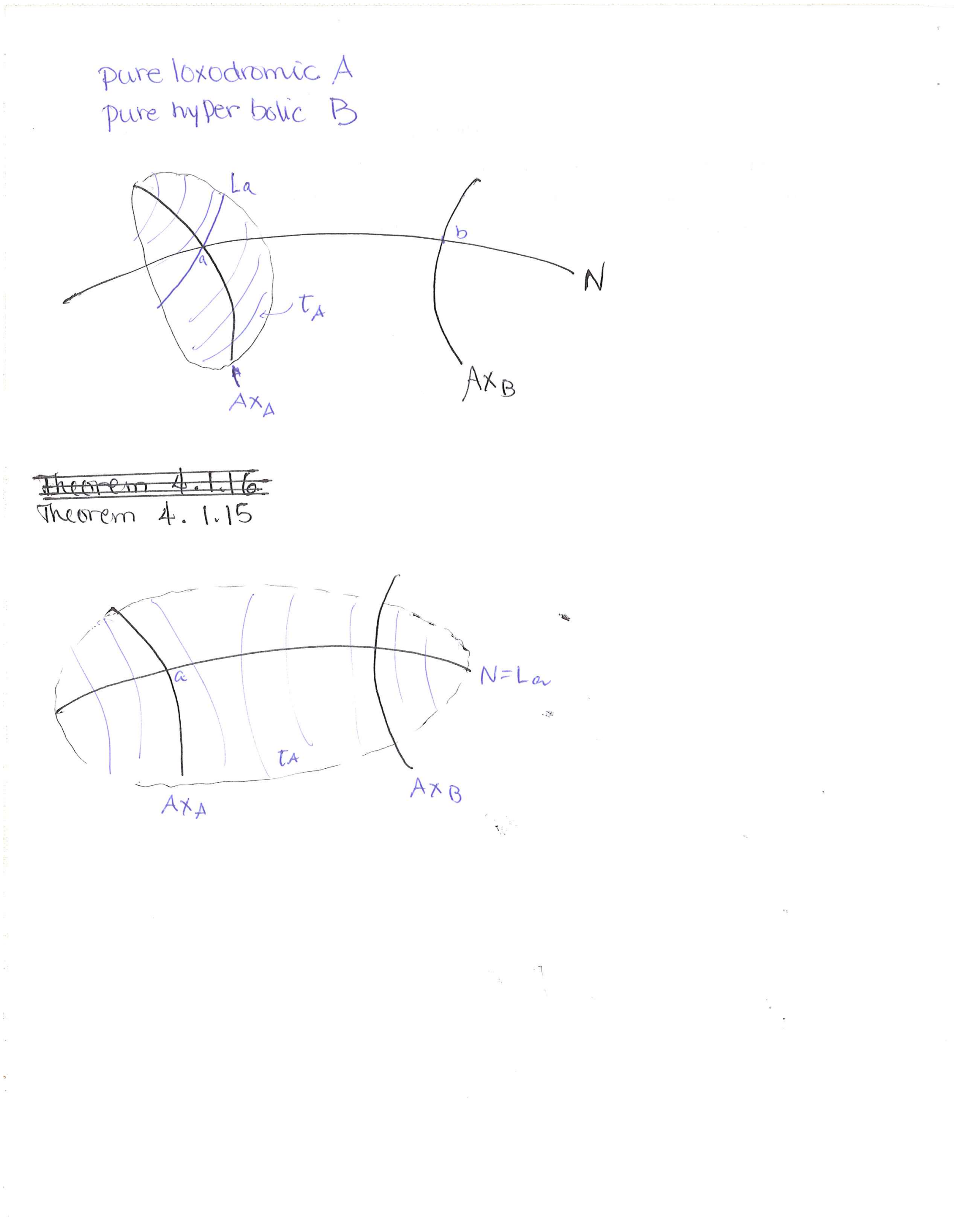


Figure A.18: Theorem 4.1.15

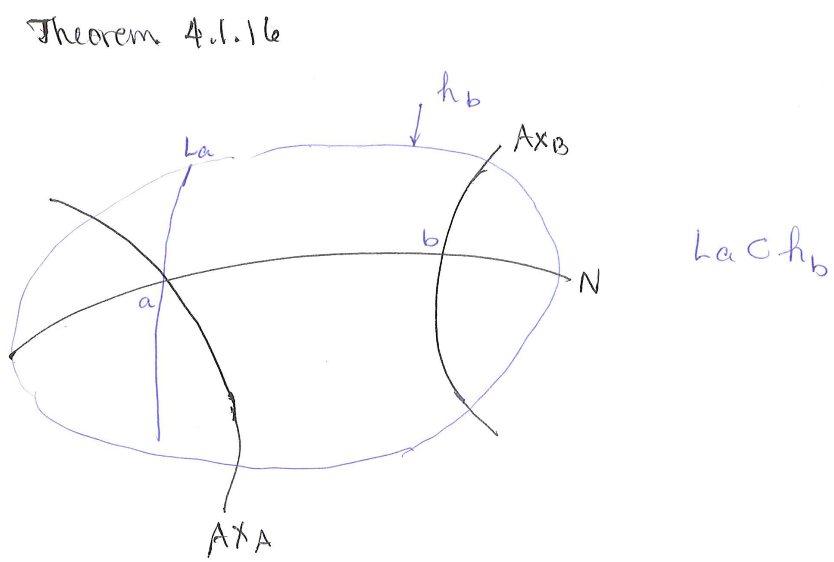


Figure A.19: Theorem 4.1.16

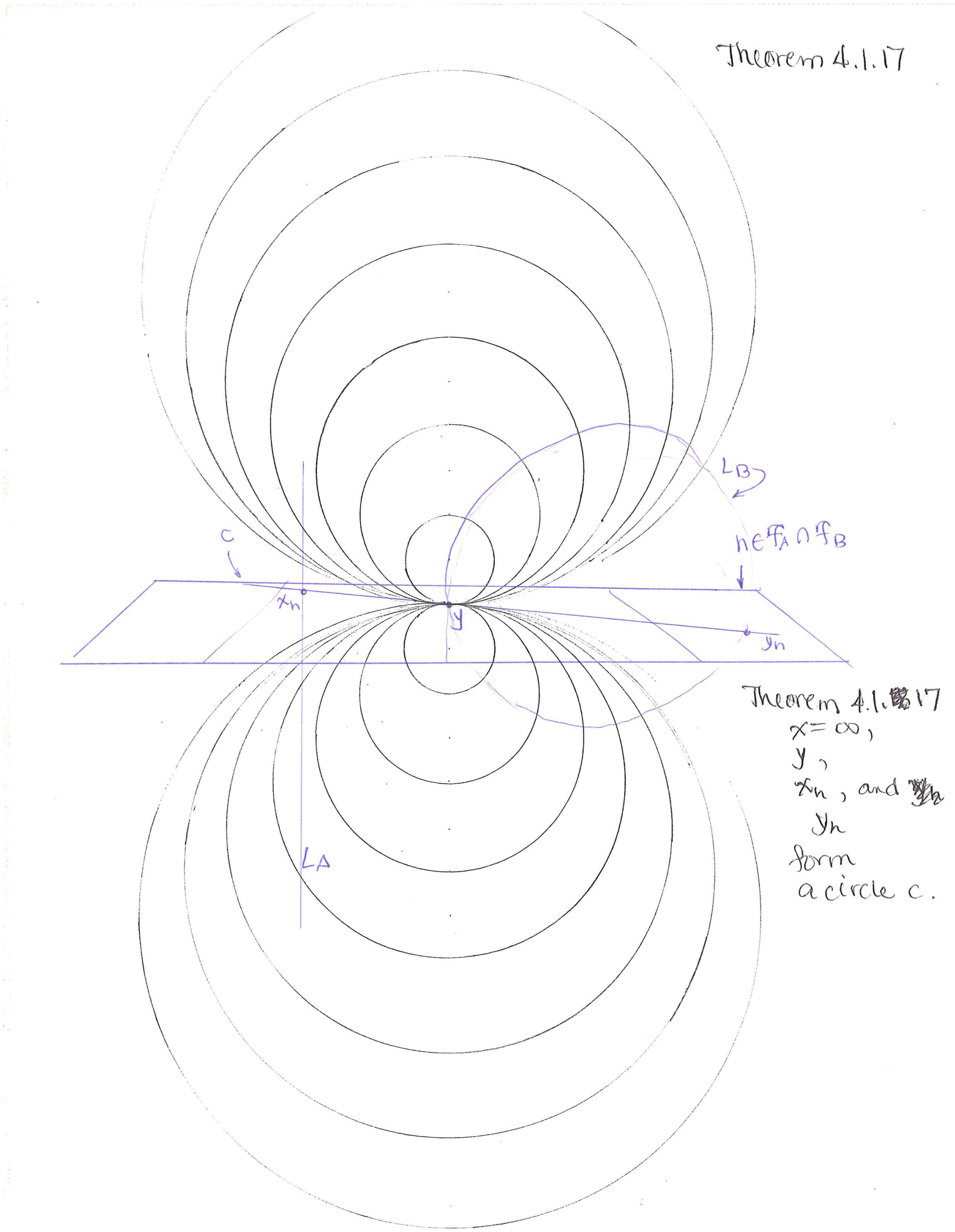


Figure A.20: Theorem 4.1.17

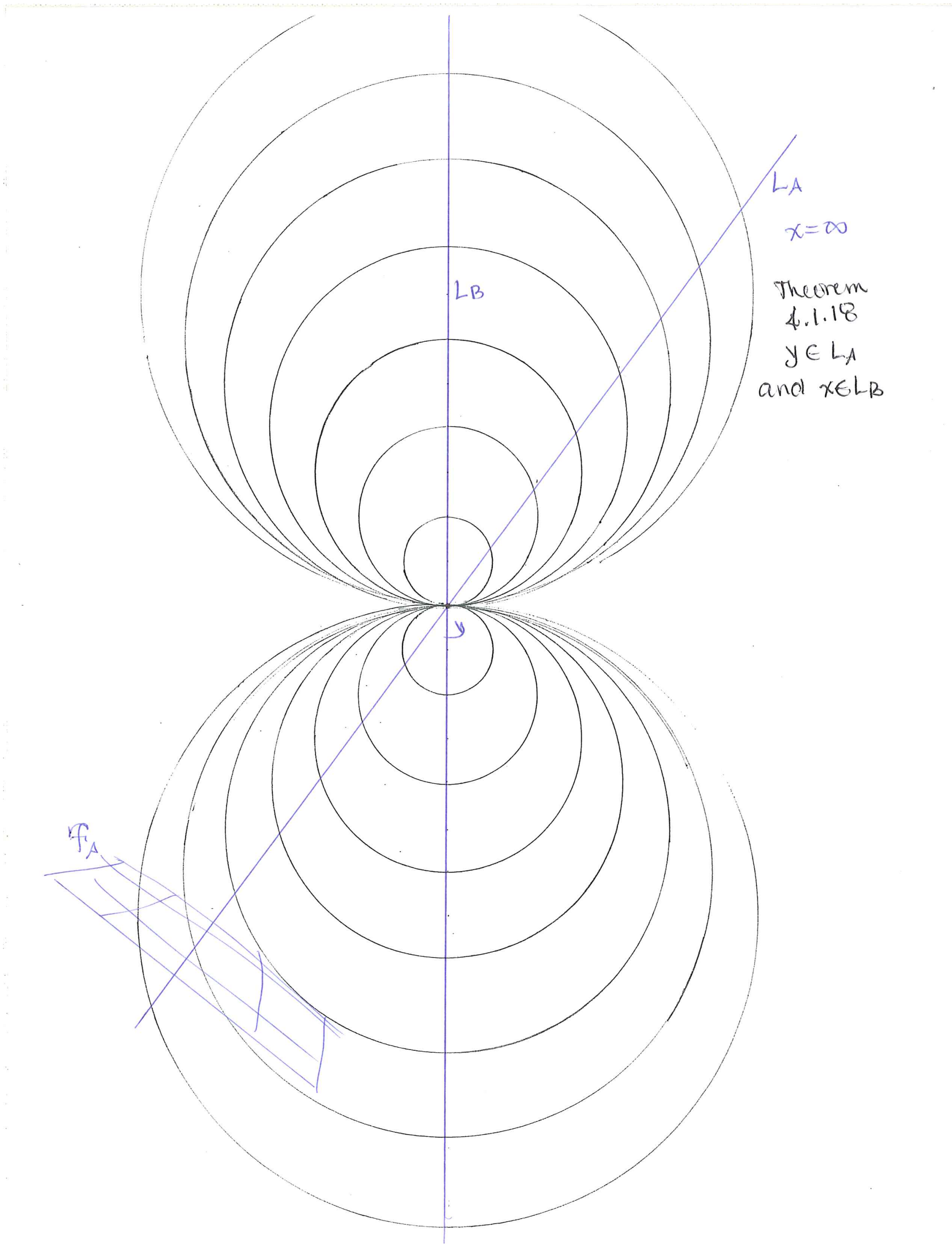


Figure A.21: Theorem 4.1.18

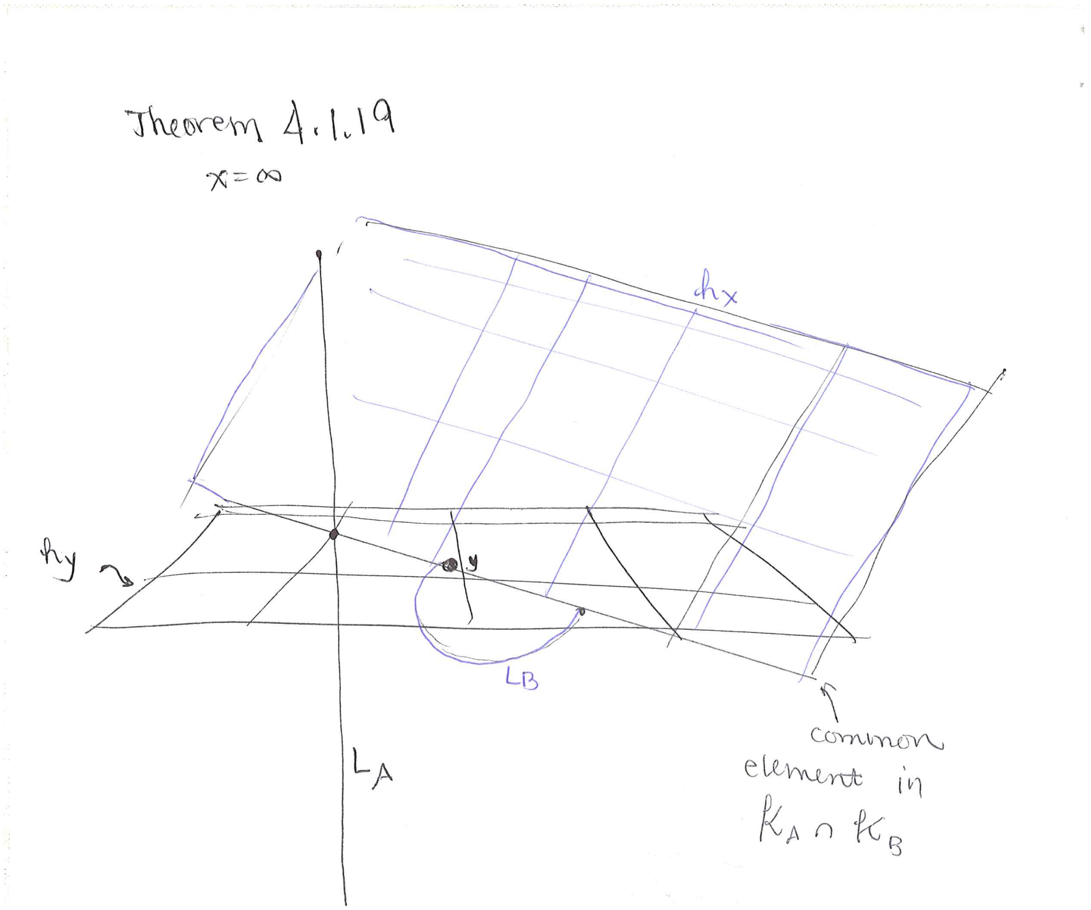


Figure A.22: Theorem 4.1.19

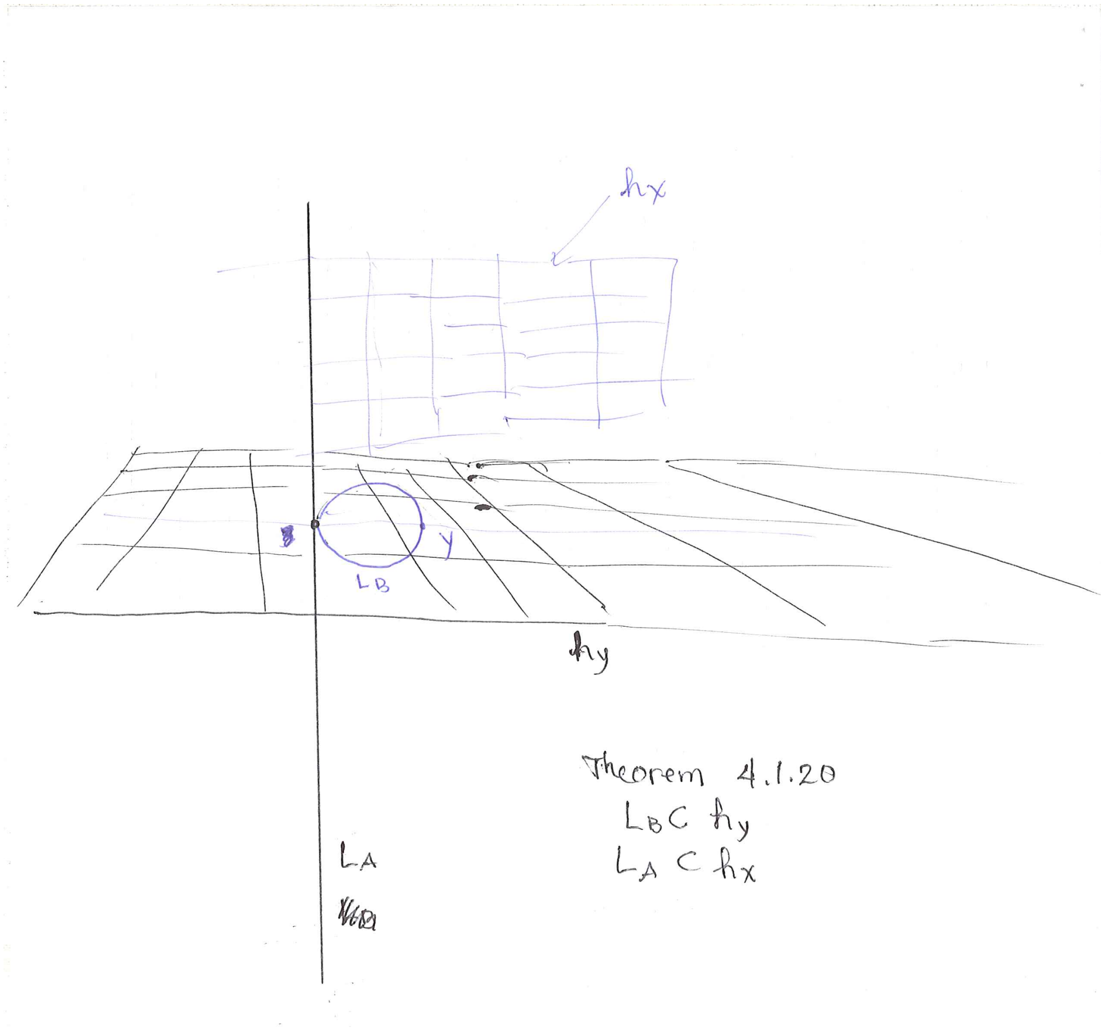


Figure A.23: Theorem 4.1.20

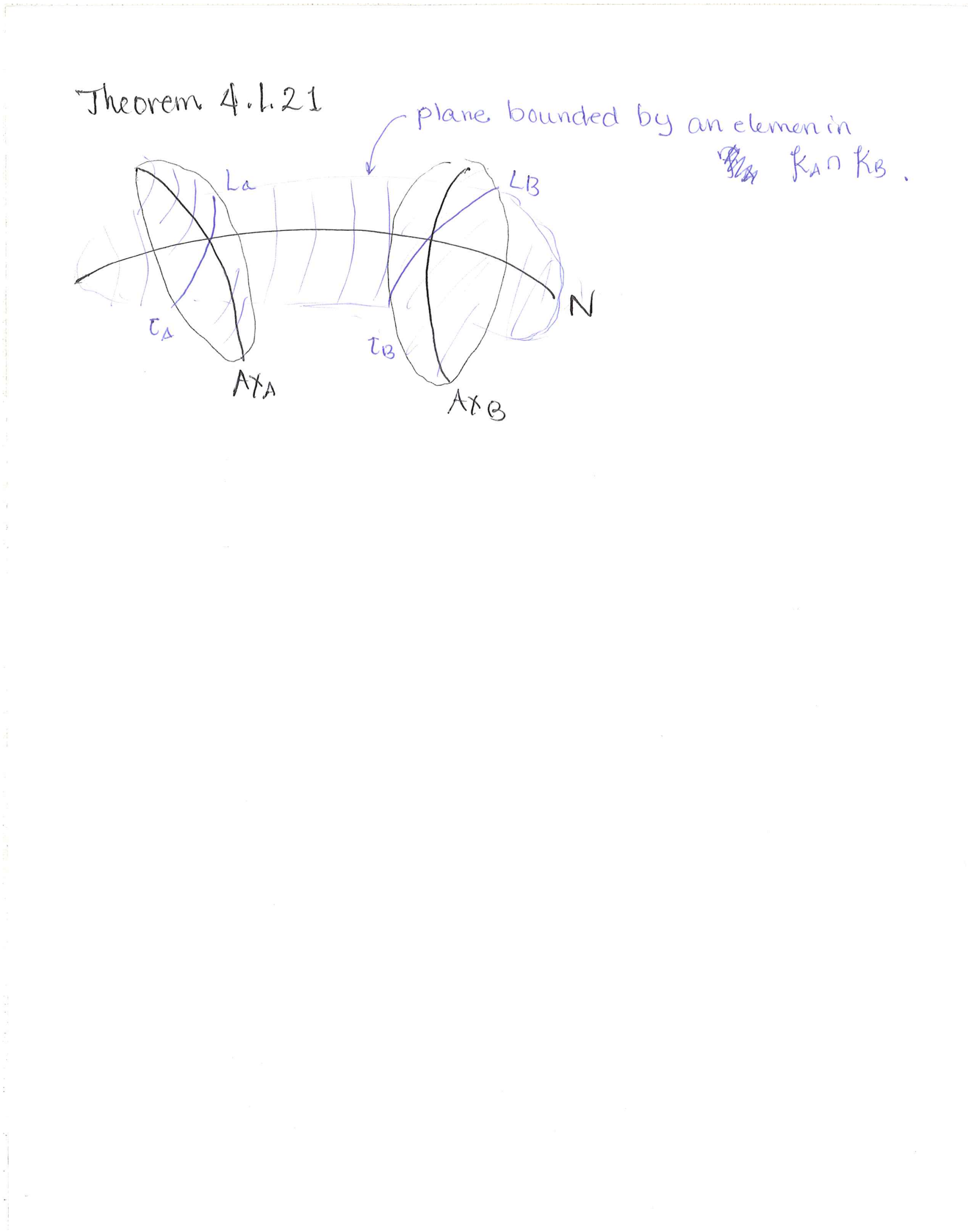


Figure A.24: Theorem 4.1.21

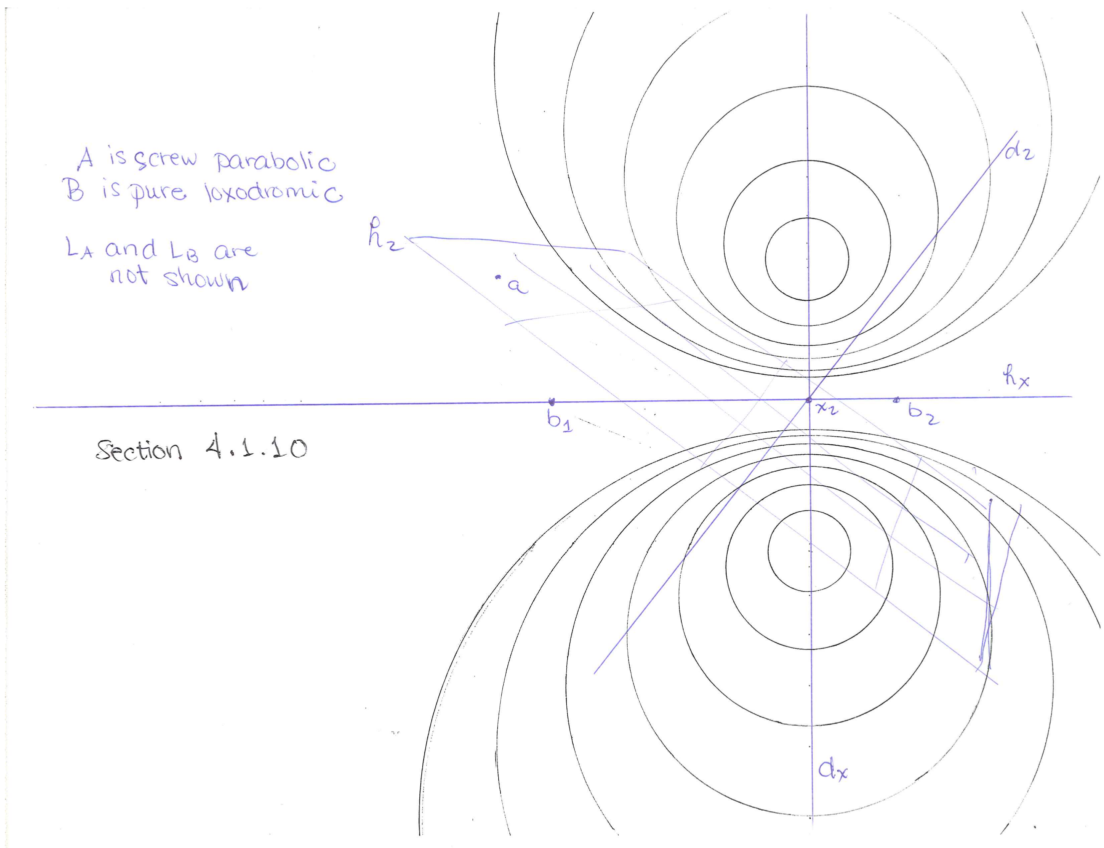


Figure A.25: General Figure for Section 4.1.10

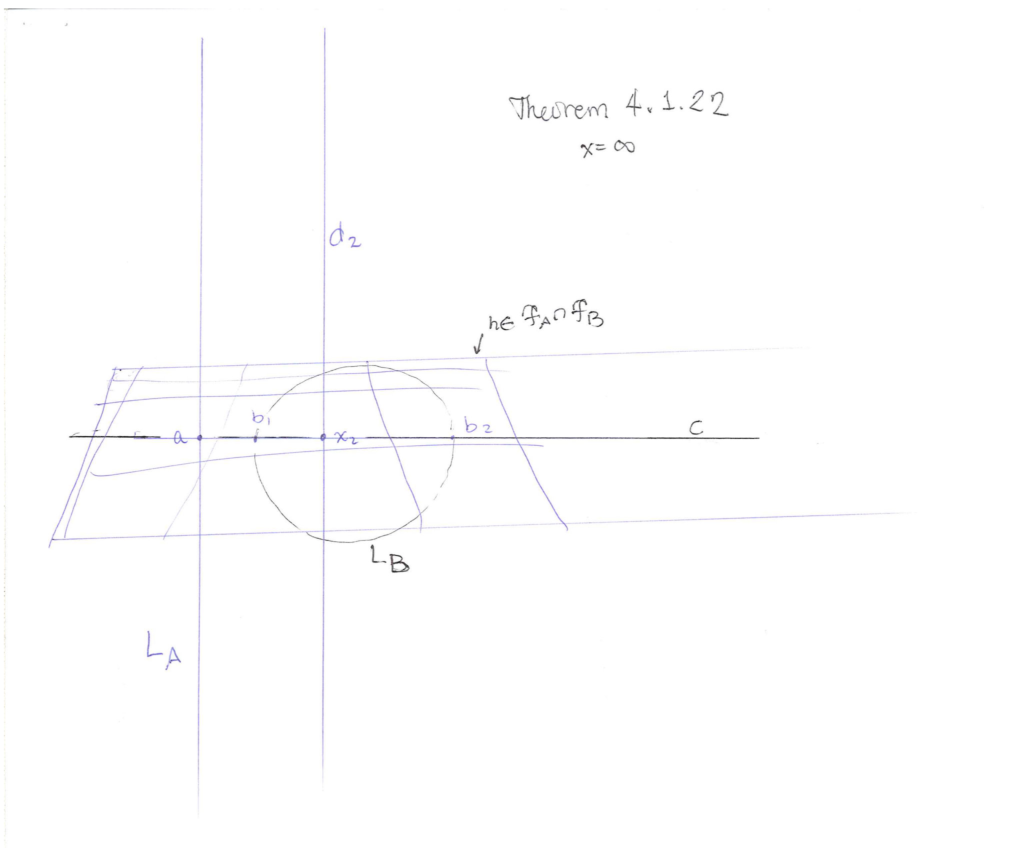


Figure A.26: Theorem 4.1.22

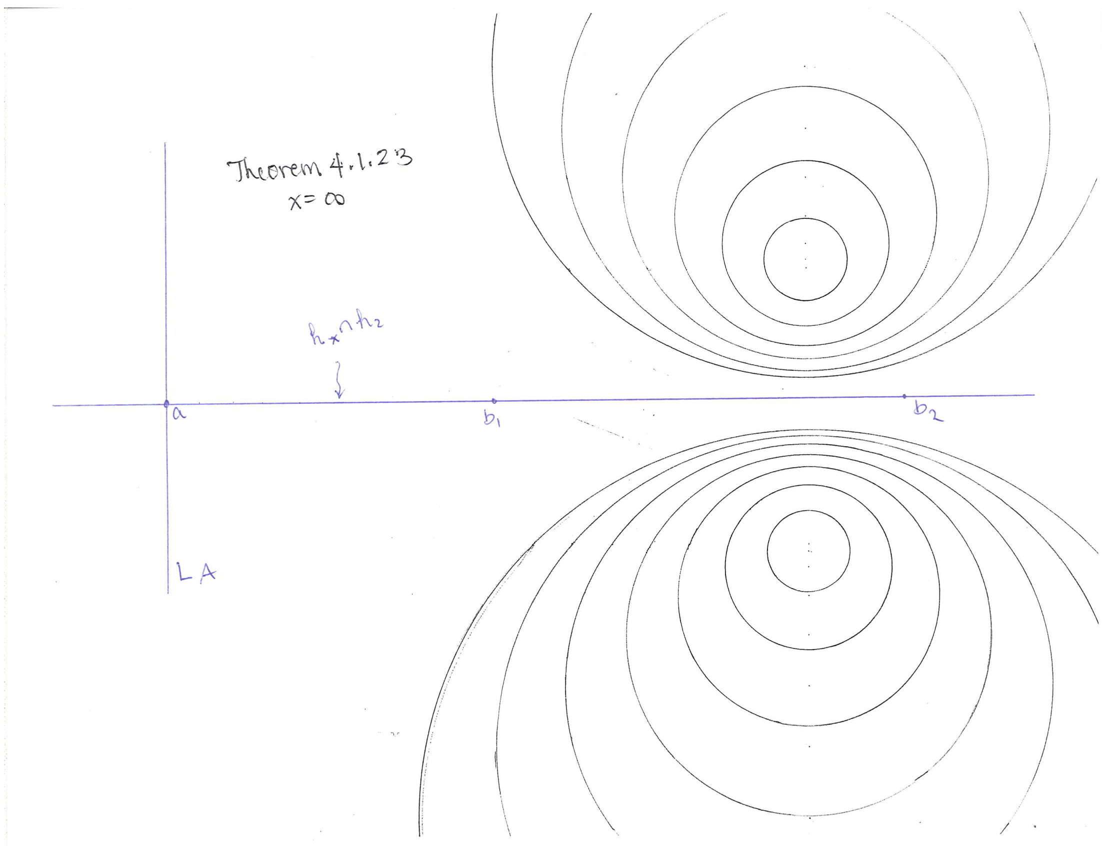


Figure A.27: Theorem 4.1.23

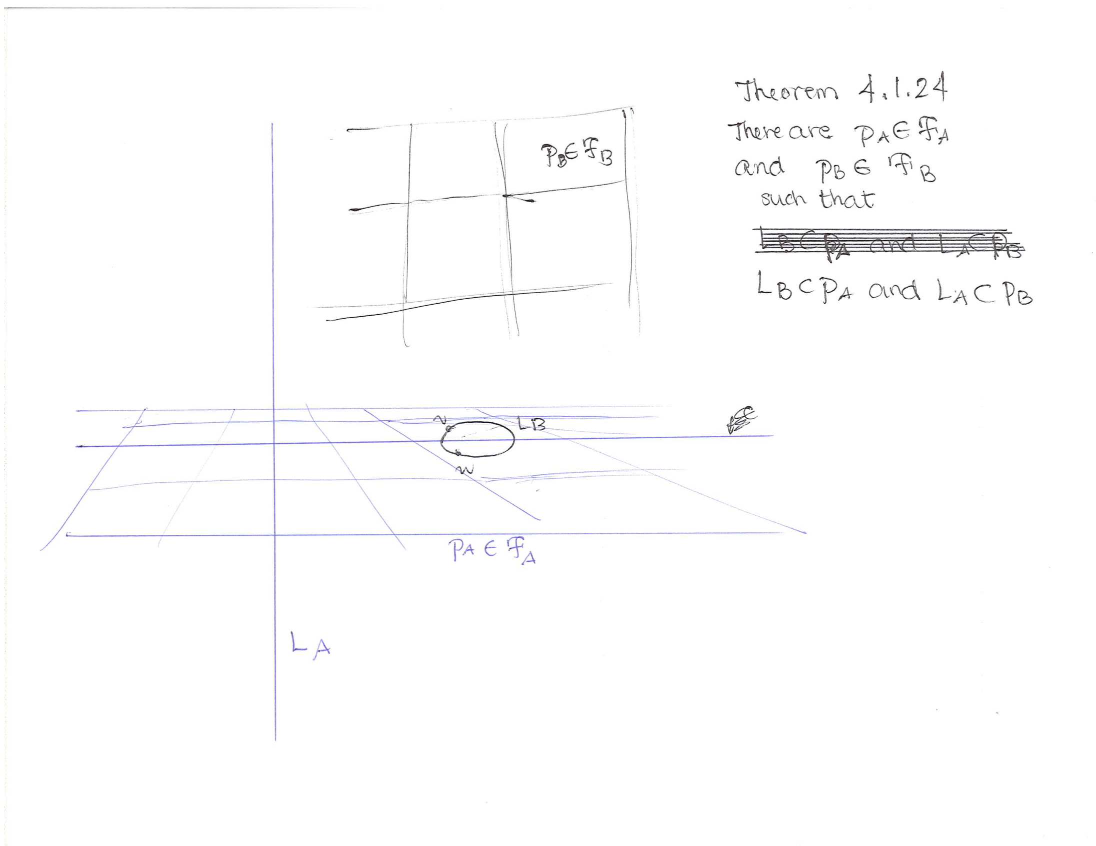


Figure A.28: Theorem 4.1.24

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2011	*Deformation Theory and Planar Families*, CUNY Hyperbolic Geometry Student Seminar (invited)
2011	*Primitive Words in Rank 2 Free Group*, CUNY Hyperbolic Geometry Student Seminar
2011	*The Dilatation Workshop*, Rutgers Newark Graduate Student Seminar
2012	*Pencils in Hyperbolic 4-Space*, CUNY Hyperbolic Geometry Student Seminar
2013	*Geometry Hodgepodge*, Rutgers Newark Graduate Student Seminar