UNIVERSAL LABELING ALGEBRAS AS INVARIANTS OF LAYERED GRAPHS

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ABSTRACT OF THE DISSERTATION

Universal Labeling Algebras as Invariants of Layered Graphs

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In this work we will study the universal labeling algebra $A(\Gamma)$, a related algebra $B(\Gamma)$, and their behavior as invariants of layered graphs. We will introduce the notion of an upper vertex-like basis, which allows us to recover structural information about the graph Γ from the algebra $B(\Gamma)$. We will use these bases to show that several classes of layered graphs are uniquely identified by their corresponding algebras $B(\Gamma)$. We will use the same techniques to construct large classes of nonisomorphic graphs with isomorphic $B(\Gamma)$. We will also explore the graded structure of the algebra $A(\Gamma)$, using techniques developed by C. Duffy, I. Gelfand, V. Retakh, S. Serconek and R. Wilson to find formulas for the Hilbert series and graded trace generating functions of $A(\Gamma)$ when Γ is the Hasse diagram of a direct product of partially ordered sets.

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Chapter 1

Introduction

A directed graph $\Gamma = (V, E)$ consists of a set V of vertices, together with a set E of ordered pairs of elements of V called edges. Given an edge $e = (v, w)$, we call v the "tail" of e and write $v = t(e)$, and we call w the "head" of v and write $w = h(e)$. A **path** is a sequence of edges $(e_1, e_2, \ldots, e_n)$ satisfying $h(e_i) = t(e_{i+1})$ for all $1 \leq i < n$. A **layered** graph is a directed graph $\Gamma = (V, E)$ whose vertex set V is divided into a sequence of layers $V_0, V_1, V_2, V_3, \ldots$ such that each directed edge in E travels exactly one layer down.

Let $\Gamma = (V, E)$ be a layered graph, let A be an algebra over some field F, and let f be a map from E to A. To each path $\pi = (e_1, \ldots, e_n)$ in Γ, we associate a polynomial $p_\pi = (t - f(e_1))(t - f(e_2)) \ldots (t - f(e_n))$ in $A[t]$. We say that the ordered pair (A, f) is a **labeling** of Γ if $p_{\pi_1} = p_{\pi_2}$ for any paths π_1 and π_2 with the same starting and ending vertices. Each graph Γ has a universal labeling, given by $(A(\Gamma), f_\Gamma)$. The algebra $A(\Gamma)$ is called the **universal labeling algebra** for the graph Γ.

These algebras arose from the constructions occurring in the proof of Gelfand and Retakh's Vieta theorem [2] for polynomials over a noncommutative division ring. There they show how to write such a polynomial in a central variable t with a specified set of roots $\{x_i \mid 1 \leq i \leq n\}$ in the form

$$f(t) = (t - y_1)(t - y_2) \ldots (t - y_n).$$

The expressions for the y_i depend on the ordering of the roots, and lead to labelings of the Boolean lattice.

Universal labeling algebras induce an equivalence relation $\sim_A$ on the collection of layered graphs, given by $\Gamma \sim_A \Gamma'$ if and only if $A(\Gamma) \cong A(\Gamma')$. One of the major goals of this work is to explore the equivalence classes of this relation. In order to do so, we will work with a related algebra $B(\Gamma)$. We will use $B(\Gamma)$ to demonstrate that several interesting collections of layered graphs have equivalence classes consisting of one isomorphism class of layered graphs. These include complete layered graphs, Boolean lattices, and lattices of subspaces of finite-dimensional vector spaces over finite fields. We will also give examples of nonisomorphic graphs which have isomorphic $B(\Gamma)$.

The algebras $A(\Gamma)$ have interesting structures as graded algebras. In [7], Retakh, Serconek, and Wilson give a linear basis for $A(\Gamma)$ which allows us to compute the Hilbert series of $A(\Gamma)$. In [1], Duffy generalizes this result to study the graded trace generating functions for automorphisms of $A(\Gamma)$ induced by graph automorphisms acting on Γ. Here we will use the same tools to calculate Hilbert series and graded trace generating functions for Hasse diagrams of direct products of posets and certain sublattices of the Young lattice.

Structure of the Paper

We will begin in Chapter 2 with the construction of the algebra $A(\Gamma)$. This algebra has a filtration which allows us to consider the associated graded algebra $grA(\Gamma)$. If Γ satisfies certain additional conditions, we can pass from the associated graded algebra to its quadratic dual $B(\Gamma)$. We will discuss how, under the correct set of hypotheses, each of these algebras can be presented as a quotient of $T(V_+)$, the free associative algebra generated by the vertices of Γ of nonzero rank.

We are interested in studying the equivalence relation on layered graphs which sets $\Gamma \sim_B \Gamma'$ whenever $B(\Gamma) \cong B(\Gamma')$. One reasonable question to consider is, given two layered graphs $\Gamma = (V, E)$ and $\Gamma' = (W, F)$, which isomorphisms $\phi : T(V_+) \to T(W_+)$ will induce isomorphisms from $B(\Gamma)$ to $B(\Gamma')$? In Chapter 3, we will answer this question by considering a certain collection of linear subspaces κ_a of $B(\Gamma)$. In Chapter 4, we will introduce a construction which allows us to find these subspaces.

In Chapter 5, we will use the tools developed in Chapters 3 and 4 to give examples of several important collections of layered graphs which have equivalence classes consisting of one layered graph isomorphism class. Specifically, we will deal with complete layered graphs, Boolean lattices, and the lattice of subspaces of a finite-dimensional vector space over a finite field.

There also exist equivalence classes under this relation which have size greater than one. Chapter 6 explores the equivalence relation $\sim_B$ on the class of layered graphs with only two nontrivial layers. In the process, we will demonstrate that large collections of two-layered graphs are equivalent with respect to this relation.

Chapters 7 and 8 focus on the algebra $A(\Gamma)$. Specifically, we calculate the Hilbert series and graded trace functions for selected automorphisms of $A(\Gamma)$. In Chapter 7, we study Hilbert series and graded trace functions when Γ is the Hasse diagram of a direct product of posets. Chapter 8 focuses on the Young lattice.

Chapter 2

Preliminaries

2.1 Ranked Posets and Layered Graphs

Recall that a **directed graph** $G = (V, E)$ consists of a set V of vertices, together with a set E of ordered pairs of vertices called **edges**. We define functions t and h from E to V called the **tail** and **head** functions, respectively, such that for $e = (v, w)$, we have $t(e) = v$ and $h(e) = w$.

In [4], Gelfand, Retakh, Serconek, and Wilson define a **layered graph** to be a directed graph $\Gamma = (V, E)$ such that $V = \bigcup_{i=0}^{n} V_i$, and such that whenever $e \in E$ and $t(e) \in V_i$, we have $h(e) \in V_{i-1}$. For any $v \in V$, define $S(v)$ to be the set $\{h(e) : t(e) = v\}$. Let V_+ be the collection of vertices in $V \setminus V_0$. The layered graphs that we will consider in this paper all have the property that for any $v \in V_+$, the set $S(v)$ is nonempty.

Recall that if $(P, \leq)$ is a partially ordered set, and $x, y \in P$, we write $x \gtrdot y$ and say that x **covers** y if the following holds:

i) $x > y$.

ii) For any $z \in P$, such that $x \geq z \geq y$, we have $x = z$ or $y = z$.

A **ranked poset** is a partially ordered set $(P, \leq)$ together with rank function $|\cdot| : P \to \mathbb{N}$ satisfying the following two properties:

i) $(x \gtrdot y) \Rightarrow (|x| = |y| + 1)$.

ii) $|x| = 0$ if and only if x is minimal in P.

Notice that if $\Gamma = (V, E)$ is a layered graph as defined above, then we can associate to

Γ a partially ordered set $(V, \leq)$, with the partial order given by $w \leq v$ if and only if there exists a directed path in Γ from v to w. If Γ satisfies the additional property that $S(v) = \emptyset$ if and only if $v \in V_0$, then we can define a rank function on $(V, \leq)$, given by $|v| = i$ if and only if $v \in V_i$. When it is convenient, we will treat the graph Γ as a ranked partially ordered set of vertices, writing $v \geq w$ to indicate that there is a directed path from v to w, $v \gtrdot w$ to indicate that there is a directed edge from v to w, and $|v| = i$ to indicate that $v \in V_i$.

Also notice that any ranked partially ordered set $(P, \gtrdot)$ has as its Hasse diagram a layered graph (P, E_P), with edges given by (p, q) for each covering relation $p \gtrdot q$. When it is convenient, we will equate P with its Hasse diagram so that we can talk about the algebra $A(P)$ for any ranked poset P. Clearly, there is a one-to-one correspondence between layered graphs of this type and ranked posets.

2.2 Universal Labeling Algebras

Let $\Gamma = (V, E)$ be a layered graph. We call an ordered n-tuple of edges $\pi = (e_1, e_2, \ldots, e_n)$ a **path** if $h(e_i) = t(e_{i+1})$ for all $1 \leq i < n$. We will occasionally find it useful to refer to the notion of a **vertex path**, a sequence $(v_1, \ldots, v_n)$ of vertices such that for every $1 \leq i < n$ there exists $e = (v_i, v_{i+1}) \in E$. Each path $(e_1, \ldots, e_n)$ has an associated vertex path given by $(t(e_1), t(e_2), \ldots, t(e_n), h(e_n))$, and each vertex path $(v_1, \ldots, v_n)$ has an associated path $((v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n))$.

Let $\Gamma = (V, E)$ be a layered graph, and fix a field F. Let A be an F-algebra, and let $f : E \to A$ be a set map. To each path $\pi = (e_1, \ldots, e_n)$ in Γ, we associate a polynomial

$$p_{f,\pi} = (t - f(e_1))(t - f(e_2)) \ldots (t - f(e_n)) \in A[t],$$

where t is a central indeterminate. We write $||\pi|| = n$, $t(\pi) = t(e_1)$, and $h(\pi) = h(e_n)$. Whenever we have two paths π_1 and π_2 with $t(\pi_1) = t(\pi_2)$ and $h(\pi_1) = h(\pi_2)$, we write $\pi_1 \approx \pi_2$.

Definition 1. The ordered pair (A, f) is called a **Γ-labeling** if it satisfies $p_{f,\pi_1} = p_{f,\pi_2}$ whenever $\pi_1 \approx \pi_2$. Given two Γ-labelings (A, f) and (A', f'), the algebra map $\phi : A \to$

A' is an Γ**-labeling map** if $\phi \circ f = f'$.

For each layered graph, we define an algebra $A(\Gamma)$ as follows: let $T(E)$ be the free associative algebra on E over F, and to each path $\pi = (e_1, \ldots, e_n)$, associate the polynomial

$$P_\pi = (t - e_1)(t - e_2) \ldots (t - e_n) = \sum_{i=0}^{\infty} e(\pi, i) t^i \in T(E)[t].$$

Define $A(\Gamma)$ to be $T(E)/R$, where R is the ideal generated by the set

$$\{e(\pi_1, i) - e(\pi_2, i) \mid \pi_1 \approx \pi_2 \text{ and } 0 \leq i \leq ||\pi_1|| - 1\}.$$

This is precisely the algebra introduced by Gelfand, Retakh, Serconek, and Wilson in [4].

Proposition 1. *Let $\Gamma = (V, E)$ be a layered graph, let $A(\Gamma) = T(E)/R$ as defined as above, and let $f_\Gamma : E \to A(\Gamma)$ be defined by $f_\Gamma(e) = e + R$. Then $(A(\Gamma), f_\Gamma)$ is the universal Γ-labeling.*

Proof. Let (A, f) be an arbitrary Γ-labeling. We need to show that there exists a unique Γ-labeling map $\phi : A(\Gamma) \to A$. Let $g : E \to T(E)$ be the canonical embedding of E into the free algebra $T(E)$. Then there exist unique algebra isomorphisms $\psi_{A(\Gamma)} : T(E) \to A(\Gamma)$ and $\psi_A : T(E) \to A$ satisfying $\psi_{A(\Gamma)} \circ g = f_\Gamma$ and $\psi_A \circ g = f$. We know that $\psi_{A(\Gamma)}$ is onto, with kernel R.

Notice that for any path π in Γ, we have

$$p_{f,\pi} = \sum_{i=0}^{\infty} f(e(\pi, i)) t^i \in A[t].$$

Since $p_{f,\pi_1} = p_{f,\pi_2}$ in $A[t]$ for any $\pi_1 \approx \pi_2$, it follows that $f(e(\pi_1, i)) = f(e(\pi_2, i))$ in A whenever $\pi_1 \approx \pi_2$ and $0 \leq i \leq ||\pi_1|| - 1$. It follows that $e(\pi_1, i) - e(\pi_2, i) \in \ker(\psi_A)$ for all $\pi_1 \approx \pi_2$ and $0 \leq i \leq ||\pi_1|| - 1$. Thus $R \subseteq \ker \psi_A$, and ψ_A factors uniquely through $A(\Gamma)$ as $\psi_A = \phi \circ \psi_{A(\Gamma)}$. Thus ϕ is the unique Γ-labeling map from $A(\Gamma)$ to A, and our proof is complete. $\square$

2.3 Presentation of $A(\Gamma)$ as a Quotient of $T(V_+)$

In the case where $\Gamma = (V, E)$ has a unique minimal vertex $*$, we can choose a distinguished edge $e_v = (v, w)$ for each vertex $v \in V_+$, where $V_+ = (V \setminus V_0)$. By following this sequence of distinguished edges down through the graph, we can associate to each vertex v a distinguished path $\pi_v = (e_1, e_2, \ldots, e_{|v|})$ from v to $*$, where $e_1 = e_v$, and where $e_i = e_{h(e_{i-1})}$ for each $1 < i \leq |v|$. This symbol π_v will also occasionally be used to designate the vertex path $(t(e_1), t(e_2), \ldots, t(e_{|v|}), h(e_{|v|}))$.

We can define a map $\phi : T(V_+) \to A(\Gamma)$, taking each vertex v to the sum of the edges in π_v. That is, if $\pi_v = \big(e_1, \ldots, e_{|v|}\big)$, then $\phi(v) = e_1 + \ldots + e_{|v|}$. Let $e = (v, w)$ be an arbitrary edge in E. Then the path obtained by adding the edge e onto the beginning of the path π_w will start at v and end at $*$, just like the path π_v. Thus we have $P_{\pi_v} = (t - e)P_{\pi_w}$.

If $\pi_v = \big(e_1, \ldots, e_{|v|}\big)$ and $\pi_w = \big(f_1, \ldots, f_{|w|}\big)$, this gives us

$$e_1 + e_2 + \ldots + e_{|v|} = e + f_1 + f_2 + \ldots + f_{|w|}.$$

Clearly, this implies that $e = \phi(v) - \phi(w)$, and so ϕ is a surjective map from $T(V_+)$ to $A(\Gamma)$.

It follows that there exists a presentation of $A(\Gamma)$ as a quotient of $T(V_+)$. To obtain a generating set of relations, we take the map $\psi : T(E) \to T(V_+)$ defined by

$$\psi((v, w)) = \begin{cases} v - w & \text{if } w \neq * \\ v & \text{if } w = * \end{cases}$$

If we define $\tilde{e}(\pi, i) = \psi(e(\pi, i))$, then we have $A(\Gamma) = T(V_+)/R_V$, where R_V is the ideal generated by the set

$$\{\tilde{e}(\pi_1, i) - \tilde{e}(\pi_2, i) \mid \pi_1 \approx \pi_2 \text{ and } 0 \leq i \leq ||\pi_1|| - 1\}.$$

2.4 A Basis For $A(\Gamma)$

Here we will recall the basis for $A(\Gamma)$ given by Gelfand, Retakh, Serconek, and Wilson in [4]. Let $\Gamma = (V, E)$ be a layered graph. For every ordered pair (v, k) with $v \in V$

and $0 \le k \le |v|$, we define $\tilde{e}(v,k) = \tilde{e}(\pi_v, k)$. We define $\mathbb{B}_1$ to be the collection of all sequences of ordered pairs

$$\mathbb{b} = ((b_1, k_1), \ldots, (b_n, k_n)),$$

and for each such sequence, we define

$$\tilde{e}(\mathbb{b}) = \tilde{e}(b_1, k_1) \ldots \tilde{e}(b_n, k_n).$$

Given (v,k) and $(v', k'$ with $v, v' \in V$, $0 \le k \le |v|$, and $0 \le k', \le |v'|$, we say that (v,k) "covers" (v', k'), and write $(v,k) \vDash (v', k')$ if $v > v'$ and $|v| - |v'| = k$.

Define $\mathbb{B}$ to be the collection of sequences

$$\mathbb{b} = ((b_1, k_1), \ldots, (b_n, k_n))$$

of ordered pairs such that for any $1 \le i < n$, $(b_i, k_i) \nvDash (b_{i+1}, k_{i+1})$. In [4], Gelfand, Retakh, Serconek, and Wilson show that $\{\tilde{e}(\mathbb{b}) : \mathbb{b} \in \mathbb{B})$ is a basis for $A(\Gamma)$.

2.5 Uniform Layered Graphs

Most of the examples we work with here are uniform layered graphs, defined as follows:

Definition 2. For each $v \in V_{>1}$, we define an equivalence relation $\sim_v$ on $S(v)$ to be the transitive closure of the relation $\approx$ on $S(v)$ given by $w \approx_v u$ whenever $S(w) \cap S(u) \neq \emptyset$. A graph Γ is said to be a **uniform layered graph** if for any $v \in V_{>1}$, all elements of $S(v)$ are equivalent under $\sim_v$.

It will sometimes be useful to be able to consider one of the following equivalent definitions:

Proposition 2. *Let Γ be a layered graph. Then Γ is uniform if and only if for any v, x, x' with $x \lessdot v$ and $x' \lessdot v$, there exist seqences of vertices $x_0, \ldots, x_s$ and $y_1, \ldots, y_s$ such that*

i) $x = x_1$ and $x' = x_s$.

ii) For all i such that $0 \le i \le s$ we have $x_i \lessdot v$.

iii) For all i such that $1 \leq i < s$, we have $y_i \lessdot x_{i-1}$ and $y_i \lessdot x_i$.

Proof. This follows directly from the definition of the relation $\sim_v$. □

Proposition 3. *Let Γ be a layered graph with unique minimal vertex $*$. Then Γ is uniform if and only if for any two vertex paths $(v_1, v_2, \dots, v_n)$ and $(w_1, w_2, \dots, w_n)$ with $v_1 = w_1$ and $v_n = w_n = *$, there exists a sequence of vertex paths $\pi_1, \pi_2, \dots, \pi_k$, each path beginning at v_1 and ending at $*$, such that for $1 \leq i < k$, the vertex paths π_i and π_{i+1} differ by at most one vertex.*

Proof. We will induct on n, the number of vertices in the paths. The result clearly holds for $n = 1, 2$.

Let Γ be a uniform layered graph, let $n > 2$, and assume that the result holds for all paths with fewer than n vertices. Let $(v_1, v_2, \dots, v_n)$ and $(w_1, w_2, \dots, w_n)$ be paths satisfying $v_1 = w_1$ and $v_n = w_n = *$. By Proposition 2, there exist sequences of vertices $x_0, \dots, x_s$ and $y_1, \dots, y_s$ such that

i) $x = x_1$ and $x' = x_s$.

ii) For all i such that $0 \leq i \leq s$ we have $x_i \lessdot v$.

iii) For all i such that $1 \leq i < s$, we have $y_i \lessdot x_{i-1}$ and $y_i \lessdot x_i$.

Let $y_0 = v_3$, and let $y_{s+1} = w_3$. Then for $0 \leq i \leq s$, our induction hypothesis tells us that there exists a sequence of vertex paths $\pi^i_1, \pi^i_2, \dots, \pi^i_{k_i}$ such that

a) $\pi^0_1 = (v_2, \dots, v_n)$ and $\pi_{k_s} = (w_2, \dots, w_n)$.

b) For $1 \leq i \leq s-1$, $\pi^i_1 = x_i \wedge \pi_{y_i}$ and $\pi^i_{k_i} = x_i \wedge \pi_{y_{i+1}}$.

c) For $1 \leq j < k_i$, the paths π^i_j and π^i_{j+1} differ by only one vertex.

The path-sequence that we wish to obtain is given by

$$v_1 \wedge \pi^1_1, (v_1, x_1) \wedge \pi^1_2, \dots, (v_1, x_1) \wedge \pi^1_{k_1}, (v_1, x_2) \wedge \pi^2_1, \dots, (v_1, x_2) \wedge \pi^2_{k_2}, \dots$$

$$\dots, (v_1, x_s) \wedge \pi^s_1, \dots, (v_1, x_s) \wedge \pi^s_{k_s} = (w_1, w_2, \dots, w_n).$$

Conversely, suppose that for any pair of vertex paths $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ satisfying $v_1 = w_1$ and $v_n = w_n = *$, we have a sequence of vertex paths $\pi_1, \ldots \pi_k$ beginning at v_1 and ending at $*$ such that for $1 \leq i < k$, the vertex paths π_i and π_{i+1} differ at at most one vertex.

Let v, x, and x' be vertices in Γ such that $x \lessdot v$ and $x' \lessdot v$. Then $v \wedge \pi_x$ and $v \wedge \pi_{x'}$ are paths which both start at v and end at $*$. Thus there exists a sequence of paths $\pi_1, \ldots, \pi_k$ beginning at v and ending at $*$, and differing in each step by at most one vertex, such that $\pi_1 = v \wedge \pi_x$ and $\pi_k = v \wedge \pi_{x'}$. For $1 \leq i \leq k$, let x_i be the second vertex on path π_i, and let y_i be the third vertex on path π_i. Then the sequences $x_1, \ldots, x_k$ and $y_2, \ldots, y_{k-1}$ satisfy conditions i-iii from Proposition 2. $\square$

We will also find the following corollary useful.

Corollary 4. *Let Γ be a uniform layered graph with unique minimal vertex $*$. Then for any two vertex paths $(v_1, v_2, \ldots, v_n)$ and $(w_1, w_2, \ldots, w_n)$ with $v_1 = w_1$, there exists a sequence of vertex paths $\pi_1, \pi_2, \ldots, \pi_k$, each path beginning at v_1, such that for $1 \leq i < k$, the vertex paths π_i and π_{i+1} differ by at most one vertex.*

In [6], Retakh, Serconek, and Wilson prove that for a uniform layered graph Γ with unique minimal vertex $*$, $A(\Gamma) \cong T(V_+)/R_V$, where R_V is the ideal generated by

$$\{v(w-u) - u^2 + w^2 + (u-w)x : v \in V_{>1}, u, w \in S(v), x \in S(v) \cap S(w)\}.$$

2.6 The Associated Graded Algebra $gr A(\Gamma)$

Let V be a vector space, with filtration

$$V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots \subseteq V_i \subseteq \ldots$$

such that $V = \sum_i V_i$. Then V is isomorphic as a vector space to the graded vector space $\bigoplus_i V_{[i]}$, where $V_{[0]} = V_0$, and $V_{[i+1]} = V_{i+1}/V_i$.

If W is a subspace of V, then we have

$$(W \cap V_0) \subseteq (W \cap V_1) \subseteq \ldots \subseteq (W \cap V_i) \subseteq \ldots,$$

with $W = \sum_i (W \cap V_i)$. It follows that W is isomorphic as a vector space to the graded vector space $\bigoplus_i W_{[i]}$, where $W_{[0]} \cong (W \cap V_0)$, and for each i,

$$W_{[i+1]} \cong (W \cap V_{i+1})/(W \cap V_i) \cong ((W \cap V_{i+1}) + V_i)/V_i.$$

Thus $W_{[i+1]}$ is isomorphic to the subspace of $V_{[i+1]}$ given by

$$\{w + V_i \mid w \in W \cap V_{i+1}\}.$$

We can think of this as the collection of "leading terms" of elements of $W \cap V_{i+1}$.

Now consider the algebra $T(V)$ for some graded vector space $V = \bigoplus_i V_{[i]}$. Every element of $T(V)$ can be expressed as a linear combination of elements of the form $v_1 v_2 \ldots v_n$, where each v_t is homogeneous—that is, $v_t \in V_{[k]}$ for some k. For each homogeneous $v \in V[k]$, we will write use the notation $|v| = k$. The grading of V induces a filtration

$$T(V)_0 \subseteq T(V)_1 \subseteq \ldots \subseteq T(V)_i \subseteq \ldots$$

on $T(V)$, with

$$T(V)_i = span\{v_1 \ldots v_n : |v_1| + \ldots + |v_n| \leq i\}.$$

Clearly this filtration induces a grading $T(V) = \bigoplus_i T(V)_{[i]}$, where

$$T(V)_{[i]} = span\{v_1 \ldots v_n : |v_1| + \ldots |v_n| = i\}.$$

Let I be an ideal of $T(V)$, and consider the algebra $A = T(V)/I$. The filtration on $T(V)$ will induce a filtration

$$A_0 \subseteq A_1 \subseteq \ldots A_i \subseteq \ldots$$

on A. If I is homogeneous with respect to the grading on $T(V)$, then A inherits this grading. Otherwise, we can consider a structure called the associated graded algebra, denoted grA and given by

$$grA = \bigoplus A_{[i]},$$

where $A_{[0]} = A_0$, $A_{[i+1]} = A_{i+1}/A_i$, and where multiplication is given by

$$(x + A_m)(y + A_n) = xy + A_{m+n}.$$

The associated graded algebra grA is isomorphic to A as a vector space, but not necessarily as an algebra.

For $A = T(V)/I$, we can understand the structure of grA by considering the graded structure of the ideal I. As a vector space, I is isomorphic to the vector space $grI = \bigoplus (grI)_{[i]}$, where $(grI)_{[0]} = I \cap T(V)_0$, and

$$(grI)_{[i+1]} = \{w + T(V)_i : w \in I \cap T(V)_{i+1}\} \subseteq T(V)_{[i]}.$$

If we think of $T(V)$ as the direct sum $\bigoplus T(V)_{[i]}$, then we can think of grI as the collection of sums of leading terms of elements of I. This is a graded ideal in the graded algebra $T(V)$. From [6], we have the following result:

Lemma 5. *Let V be a graded vector space and I an ideal in $T(V)$. Then*

$$gr(T(V)/(I)) \cong T(V)/(gr(I)).$$

In the case of the universal labeling algebra $A(\Gamma)$ for a layered graph Γ with unique minimal vertex, we have $A(\Gamma) = T(V_+)/R_V$, where $T(V_+)$ is the tensor algebra over the vector space generated by the vertices in V_+. This means that $A(\Gamma)$ inherits a filtration from $T(V_+)$, given by

$$T(V_+)_i = span\{v_1 \dots v_n : |v_1| + \dots + |v_n| \leq i\}.$$

We will refer to this as the **vertex filtration**. This means that $A(\Gamma)$ has an associated graded algebra $grA(\Gamma)$.

2.7 Basis for $grA(\Gamma)$

In [4], a basis for $grA(\Gamma)$ is constructed as follows: For each $\mathbb{b} \in \mathbb{B}_{\mathbb{1}}$, with $|\tilde{e}(\mathbb{b})| = i$ with respect to the vertex-filtration, define $\overline{e}(\mathbb{b}) = \tilde{e}(\mathbb{b}) + A(\Gamma)_{i-1}$ in $grA(\Gamma)$. Then for any distinct $\mathbb{b}$ and $\mathbb{b}'$ in $\mathbb{B}$, we have $\overline{e}(\mathbb{b}) \neq \overline{e}(\mathbb{b}')$, and the set $\{\overline{e}(\mathbb{b}) : \mathbb{b} \in \mathbb{B}\}$ is a basis for $grA(\Gamma)$.

With a little extra notation, we can describe this basis in another way.

Definition 3. For every ordered pair (v,k) with $v \in V$, $0 \le k \le |v|$ with $\pi_v = (v_1, v_2, \ldots, v_n)$ as a vertex path, we define a monomial

$$m(v,k) = v_1 v_2 \ldots v_k \in T(V_+).$$

For each $\mathbb{b} = ((b_1,k_1),\ldots,(b_n,k_n)) \in \mathbb{B}_1$, we define

$$m(\mathbb{b}) = m(b_1,k_1)\ldots m(b_n,k_n).$$

Proposition 6. *Let Γ be a layered graph, and let ϕ be the quotient map from $T(V_+)$ to $T(V_+)/grR \cong grA(\Gamma)$. Then for all $\mathbb{b} \in \mathbb{B}_1$, we have $\phi(m(\mathbb{b})) = \overline{e}(\mathbb{b})$.*

Proof. We know that for (v,k) with $v \in V$, $0 \le k \le |v|$, and with vertex path $\pi_v = (v_1,\ldots,v_n)$, we define $e(v,k)$ by

$$(t-(v_1,v_2))(t-(v_2,v_3))\ldots(t-(v_{n-1},v_n)) = \sum_{i=0}^{|v|} e(v,i)t^i.$$

Thus we have

$$\tilde{e}(v,k) = \sum_{1\le i_1<i_2<\ldots i_k<n} \left(v_{i_1} - v_{(i_1+1)}\right)\ldots\left(v_{i_k} - v_{(i_k+1)}\right),$$

and the highest-order term of $\tilde{e}(v,k)$ with respect to the vertex filtration is the monomial $v_1v_2\ldots v_k$, or $m(v,k)$. It follows that $\overline{e}(v,k) = \phi(m(v,k))$, and so by extension we have $\overline{e}(\mathbb{b}) = \phi(m(\mathbb{b}))$ for all $\mathbb{b} \in \mathbb{B}_1$. □

Corollary 7. *For any distinct $\mathbb{b}$ and $\mathbb{b}'$ in $\mathbb{B}$, we have $\phi(m(\mathbb{b})) \ne \phi(m(\mathbb{b}'))$, and the set $\{\phi(m(\mathbb{b})) : \mathbb{b} \in \mathbb{B}\}$ is a basis for $grA(\Gamma)$.*

Notice that any monomial $a \in T(V_+)$ is expressible as $m(\mathbb{b})$ for some $\mathbb{b} \in \mathbb{B}$.

Definition 4. Let $a = v_1,\ldots v_l$ be a monomial in $T(V_+)$. Borrowing terminology from [3], we define $s(a)$, the **skeleton** of a, to be the sequence of integers $(n_1,\ldots,n_t)$ satisfying

i) $n_1 = 1$.

ii) If $n_k < l+1$, then

$$n_{k+1} = \min\left(\{j > n_k : v_j < v_{n_k} \text{ or } |v_j| \ne |v_{n_k}| + n_k - j\} \cup \{l+1\}\right).$$

iii) $t = \min\{i : n_i = l + 1\}$.

We define $\mathbb{b}_a \in \mathbb{B}$ to be

$$((v_{n_1}, n_2 - n_1), (v_{n_2}, n_3 - n_2), \ldots, (v_{n_{t-1}}, n_t - n_{t-1}))$$

It is easy show from these definitions that $m(\mathbb{b}_a) = a$.

2.8 A Presentation of $grA(\Gamma)$ For Uniform Layered Graphs

If Γ is a uniform layered graph, then R_V is generated by

$$\{v(w-u) - u^2 + w^2 + (u-w)x : v, u, w \in V_+, u, w \in S(v), x \in S(v) \cap S(w)\}.$$

The discussion in section 2.6 tells us that grR_V contains the set of leading terms:

$$\{v(w-u) : v, u, w \in V_+, u, w \in S(v)\}.$$

Call the ideal generated by this set R_{gr}. The map $\phi : T(V_+) \to T(V_+)/grR$ factors uniquely through $T(V_+)/R_{gr}$, giving us

$$T(V_+) \xrightarrow{\phi'} T(V_+)/R_{gr} \xrightarrow{\phi''} T(V_+)/grR.$$

We will use this notation for these three maps in the discussion that follows.

Proposition 8. *Let Γ be a uniform layered graph. If $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ are vertex paths in Γ with $v_1 = w_1$, then*

$$\phi'(v_1 \ldots v_n) = \phi'(w_1 \ldots w_n)$$

Proof. Let $(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots v_n)$ and $(v_1, \ldots, v_{i-1}, v'_i, v_{i+1}, \ldots v_n)$ be two vertex paths, differing by only one vertex. Since $v_i, v'_i \lessdot v_{i-1}$, we have $v_{i-1}(v_i - v'_i) \in R_{gr}$. Thus if we consider

$$\phi'(v_1 \ldots v_{i-1} v_i v_{i+1} \ldots v_n) - \phi'(v_1 \ldots v_{i-1} v'_i v_{i+1} \ldots v_n),$$

we find that it is equal to

$$\phi'(v_1 \ldots v_{i-2}(v_{i-1}(v_i - v'_i))v_{i+1} \ldots v_n),$$

which is zero, since $v_{i-1}(v_i - v_i')$ is in the kernel of ϕ'. This simplification, in combination with Corollary 4 and the uniformity of Γ give us our result. $\square$

Proposition 9. *Let $\Gamma = (V, E)$ be a uniform layered graph, and let a be a monomial in $T(V_+)$. Then there exists a monomial $a' \in T(V_+)$ such that $\phi'(a) = \phi'(a')$, and $\mathbb{b}_{a'} \in \mathbb{B}$.*

Proof. Let $\Gamma = (V, E)$ be a uniform layered graph, and let $a = v_1, \dots, v_l$ be a monomial in $T(V_+)$ with skeleton $(n_1, \dots, n_t)$. If $\mathbb{b}_a \notin \mathbb{B}$, then there exists $1 < i < t$ such that

$$(v_{n_{i-1}}, n_i - n_{i-1}) \models (v_{n_i}, n_{i+1} - n_i).$$

We will induct on $l - r_a$, where

$$r_a = \min\{n_i : (v_{n_{i-1}}, n_i - n_{i-1}) \models (v_{n_i}, n_{i+1} - n_i)\} \cup \{t\}$$

If $l - r_a = 0$, then $\mathbb{b}_a \in \mathbb{B}$. Now assume that $l - r_a > 0$, and that the result holds for all monomials with larger r-values.

We have

$$a = v_1 \dots v_{(n_{(r_a-1)}-1)} m(v_{n_{(r_a-1)}}, n_{r_a} - n_{(r_a-1)}) v_{n_{r_a}} \dots v_l$$

Since we have

$$(v_{n_{(r_a-1)}}, n_{r_a} - n_{(r_a-1)}) \models (v_{n_{r_a}}, n_{r_a+1} - n_{r_a}),$$

we know that there exists a vertex path $(w_1, \dots, w_h)$ with $w_1 = v_{n_{(r_a-1)}}$ and $w_h = v_{n_{r_a}}$. From Proposition 8, we know that

$$\phi'\left(m\left(v_{n_{(r_a-1)}}, n_{r_a} - n_{(r_a-1)}\right)\right) = \phi'\left(w_1 \dots w_{h-1}\right),$$

and that

$$\phi'(w_1, \dots, w_h) = \phi'\left(m\left(v_{n_{(r_a-1)}}, n_{r_a} - n_{(r_a-1)} + 1\right)\right).$$

It follows that

$$\phi'\left(m\left(v_{n_{(r_a-1)}}, n_{r_a} - n_{(r_a-1)}\right) v_{n_{r_a}}\right) = \phi'\left(m\left(v_{n_{(r_a-1)}}, n_{r_a} - n_{(r_a-1)} + 1\right)\right).$$

Thus we have $\phi'(a) = \phi'(a')$, where

$$a' = v_1 \dots v_{(n_{r_a-1}-1)} m\left(v_{n_{(r_a-1)}}, n_{r_a} - n_{(r_a-1)} + 1\right) v_{(n_{r_a}+1)} \dots v_l$$

Say $a' = v'_1, \dots v'_l$, and $s(a') = (n'_1, \dots, n'_{t'})$. By construction, we have

(i) $v_i = v'_i$ for $1 \le i \le n_{r_a}$.

(ii) $n_j = n'_j$ for $j < n_{r_a}$.

(iii) $n'_{r_a} \ge n_{r_a} + 1$.

Together, (i) and (ii) tell us that

$$\left(v'_{n'_{i-1}}, n'_i - n'_{i-1}\right) \not\models \left(v'_{n'_i}, n'_{i+1} - n'_i\right)$$

for $i < n'_{r_a}$. Thus if

$$r_{a'} = \min\{n'_i : (v'_{n'_{i-1}}, n'_i - n'_{i-1}) \models (v'_{n'_i}, n'_{i+1} - n'_i)\} \cup \{t'\},$$

then $r_{a'} \ge n'_{r_a} > n_{r_a}$. Thus by the inductive hypothesis there exists a monomial $a'' \in T(V_+)$ with $\phi'(a') = \phi'(a'')$ and $\mathbb{b}_{a''} \in \mathbb{B}$. We have $\phi'(a) = \phi'(a'')$, and this completes our proof. □

Proposition 10. *For any distinct $\mathbb{b}$ and $\mathbb{b}'$ in $\mathbb{B}$, we have $\phi'(m(\mathbb{b})) \neq \phi'(m(\mathbb{b}'))$, and the set $\{\phi'(m(\mathbb{b})) : \mathbb{b} \in \mathbb{B}\}$ is a basis for $T(V_+)/R_{gr}$.*

Proof. For any $\mathbb{b} \in \mathbb{B}$, we have $\phi(m(\mathbb{b})) = \phi'' \circ \phi'(m(\mathbb{b}))$. Thus the elements of

$$\{\phi'(m(\mathbb{b})) : \mathbb{b} \in \mathbb{B}\}$$

are distinct and linearly independent by Corollary 7. Since the set

$$\{a \in T(V_+) : a \text{ a monomial}\}$$

spans $T(V_+)$, it follows that the set

$$\{\phi'(a) : a \text{ a monomial}\}$$

spans $T(V_+)/R_{gr}$. Thus the set

$$\{\phi'(m(\mathbb{b})) : \mathbb{b} \in \mathbb{B}\}$$

spans $T(V_+)/R_{gr}$ as a consequence of Proposition 9. This gives us our result. □

Corollary 11. *The associated graded algebra $grA(\Gamma)$ is isomorphic to $T(V_+)/R_{gr}$, where R_{gr} is the ideal generated by the set*

$$\{v(w-u) : v, u, w \in V_+, u, w \in S(v)\}$$

Proof. For any $\mathbb{b} \in \mathbb{B}$, we have

$$\phi''(\phi'(m(\mathbb{b})) = \phi(m(\mathbb{b})).$$

Since ϕ'' maps a basis of $T(V_+)/R_gr$ bijectively onto a basis of $grA(\Gamma)$, it follows that ϕ'' is a bijection, and thus an isomorphism. □

2.9 The Quadratic Dual $B(\Gamma)$

We begin this section by recalling the following definitions, as presented in [5]:

Definition 5. An algebra A is called **quadratic** if $A \cong T(W)/\langle R\rangle$, where W is a finite-dimensional vector space, and $\langle R\rangle$ is an ideal generated by some subspace R of $W \otimes W$.

Definition 6. If R is a subspace of $W \otimes W$, then $R^{\perp}$ is the subspace of $(W \otimes W)^*$ generated by the set of elements $x \in (W \otimes W)^*$ such that for all $y \in W \otimes W$, we have $\langle x, y\rangle = 0$.

Definition 7. Let $A \cong T(W)/\langle R\rangle$ be the quadratic algebra associated to some particular subspace $R \subseteq W \otimes W$. The quadratic dual $A^!$ of A is defined to be the algebra $T(W^*)/\langle R^{\perp}\rangle$.

When Γ is a uniform layered graph with unique minimal vertex, $grA(\Gamma) \cong T(V_+)/R_V$ is a quadratic algebra with quadratic dual $T((V_+)^*)/(R_V)^{\perp}$. Since R_V is generated by

$$\{v(w-u) : v \in V_{>1}, u, w \in S(v)\},$$

$(R_V)^{\perp}$ is generated by the collection of $x^*y^* \in (V_+)^* \otimes (V_+)^*$ such that

$$\langle x^*, v\rangle\langle y^*, w-u\rangle = 0$$

for all $v \in V_{>1}$ and $u, w \in S(v)$. In [1], Duffy shows that this is the ideal generated by

$$\{v^*w^* : v \not> w\} \cup \left\{v^* \sum_{v > w} w^*\right\}.$$

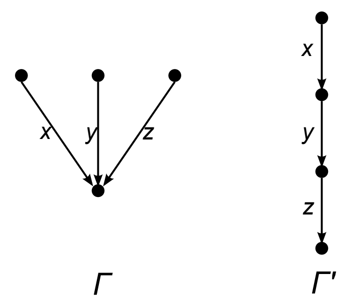
Changing the generators of our algebra, we define $B(\Gamma) = T(V_+)/R_B$, where R_B is the ideal generated by

$$\{vw : v \not> w\} \cup \left\{ v \sum_{v > w} w \right\}.$$

Notice that $B(\Gamma) \cong T((V_+)^*)/(R_V)^\perp$, the quadratic dual of $grA(\Gamma)$. Throughout much of this paper, we will focus our attention on the algebra $B(\Gamma)$.

2.10 Using $A(\Gamma)$ and $B(\Gamma)$ as Layered Graph Invariants

To each layered graph Γ, we have associated an algebra $A(\Gamma)$. It is natural to ask how this algebra behaves when considered as an invariant of layered graphs. Unfortunately, if we only consider its structure as an algebra, it is not a particularly strong invariant. Consider the two graphs below:



We have $A(\Gamma) \cong A(\Gamma') \cong F\langle x, y, z\rangle$. This is unfortunate, as one would hope that an invariant of layered graphs would be able to distinguish between graphs with different numbers of layers, or graphs whose layers have a different number of vertices. This clearly does neither. To capture these properties, we will need to consider some additional structure on $A(\Gamma)$.

Notice that $A(\Gamma)$ is a quotient of $T(V_+)$ by a homogeneous ideal. Thus there is a grading on $A(\Gamma)$ given by $A(\Gamma) = \bigoplus A(\Gamma)_{[i]}$, with

$$A(\Gamma)_{[i]} = span\{v_1 v_2 \ldots v_i : v_1, \ldots, v_i \in V_+\}.$$

$A(\Gamma)$ also has a filtration given by the level of the vertices in the graph. So $A(\Gamma) = \bigcup A(\Gamma)_i$, where

$$A(\Gamma)_i \leq span\left\{ v_1 v_2 \dots v_j : \sum_{k=1}^{j} |v_k| \leq i \right\}.$$

When we allow ourselves to consider these pieces of structure, we obtain an invariant that can distinguish between graphs that have layers of different sizes.

Definition 8. We say $\Gamma \sim_A \Gamma'$ if and only if there exists an isomorphism $\phi : A(\Gamma) \to A(\Gamma')$ which preserves both the grading and the filtration described above.

Proposition 12. *If* $\Gamma = (V, E)$, $\Gamma' = (V', E')$, *and* $\Gamma \sim_A \Gamma'$, *then* $|V_i| = |V_i'|$ *for all* $i \in \mathbb{N}$.

Proof. We have

$$|V_i| = \dim \left(\left(A(\Gamma)_1 \cap A(\Gamma)_{[i]} \right) / \left(A(\Gamma)_1 \cap A(\Gamma)_{[i-1]} \right) \right)$$

and

$$|V_i'| = \dim \left(\left(A(\Gamma')_{[1]} \cap A(\Gamma')_i \right) / \left(A(\Gamma')_{[1]} \cap A(\Gamma')_{i-1} \right) \right).$$

Any isomorphism $\phi : A(\Gamma) \to A(\Gamma')$ that preserves the grading and the filtration will map the subspace from the first expression onto the subspace in the second expression. The result follows. $\square$

Similarly, $B(\Gamma)$ has a double grading given by

$$B(\Gamma)_{m,n} = \text{span}\left\{ v_1 \dots v_m : \sum_{i=1}^{m} |v_i| = n \right\}.$$

Definition 9. We say $\Gamma \sim_B \Gamma'$ if and only if there exists an isomorphism $\phi : B(\Gamma) \to B(\Gamma')$ which preserves the double grading.

Notice that whenever $B(\Gamma)$ and $B(\Gamma')$ are defined, we have

$$(\Gamma \sim_A \Gamma') \Rightarrow (\Gamma \sim_B \Gamma').$$

Thus $\sim_B$ gives us a coarser partition of these layered graphs. In particular, if the $\sim_B$ equivalence class consists of a single isomorphism class, then so does the $\sim_A$ equivalence class. If the $\sim_B$ equivalence class of a certain graph is small and easy to describe, we know that the $\sim_A$ equivalence class is contained in this small, easily-describable set of layered graphs.

2.11 Notation and Formulas for Layered Graphs

Let Γ be a layered graph. Recall that whenever there is a directed edge from v to w, we write $v \gtrdot w$, and say that v "covers" w. $S(v)$ is the set $\{w : v \gtrdot w\}$.

Definition 10. Throughout this work, we will use the following notation: For any subset $T \subseteq V_n$,

(i) $S(T) = \bigcup_{t \in T} S(t)$, the set of all vertices covered by some vertex in T.

(ii) $I(T) = \bigcap_{t \in T} S(t)$, the set of vertices covered by every vertex in T.

(iii) $N(T) = I(T) \setminus S\left(\overline{T}\right)$, the set of vertices covered by exactly the vertices in T.

(iv) $\sim_T$ is the equivalence relation obtained by taking the transitive closure of the relation

$$R_T = \{(v, w) : \exists t \in T, \{v, w\} \subseteq S(t)\}.$$

(v) $\mathscr{C}_T$ is the collection of equivalence classes of V_{n-1} under $\sim_T$.

(vi) $k_T = |\mathscr{C}_T|$.

(vii) $k_T^T = k_T - |V_{n-1}| + |S(T)|$.

Notice that k_T^T is also the number of equivalence classes of $S(T)$ under $\sim_T$. Since $\emptyset \subseteq V_n$ for multiple values of n, we will use the notation $\emptyset_n$ to indicate that $\emptyset$ is being considered as a subset of V_n.

Proposition 13. *If Γ is a layered graph, and $A \subseteq V_n$, then we have the following:*

(i) $|V_{n-1}| = k_{\emptyset_n}$

(ii) $|V_{n-1} \setminus S(A)| = k_A - k_A^A$,

(iii) $|S(A)| = k_{\emptyset_n} - k_A + k_A^A$

Proof. Statement (i) follows immediately from the definition of k_A. Statement (ii) follows from the definition of k_A^A. Statement (iii) follows easily from statements (i) and (ii). $\square$

Proposition 14. *If $A \subseteq V_n$, then*

$$|I(A)| = \sum_{B \subseteq A} (-1)^{|B|}(k_B - k_B^B)$$

Proof. We know that $I(A) = \bigcap_{t \in A} S(t)$, so

$$\begin{aligned}
|I(A)| &= \left| \bigcap_{t \in A} S(t) \right| \\
&= \left| \overline{\left(\bigcup_{t \in A} \overline{S(t)} \right)} \right| \\
&= |V_{n-1}| - \left| \bigcup_{t \in A} \overline{S(t)} \right| \\
&= |V_{n-1}| - \sum_{\emptyset \neq B \subseteq A} (-1)^{|B|-1} \left| \bigcap_{t \in B} \overline{S(t)} \right| \\
&= |V_{n-1}| + \sum_{\emptyset \neq B \subseteq A} (-1)^{|B|} \left| \bigcap_{t \in B} \overline{S(t)} \right| \\
&= \sum_{B \subseteq A} (-1)^{|B|} \left| \overline{S(B)} \right| \\
&= \sum_{B \subseteq A} (-1)^{|B|}(k_B - k_B^B)
\end{aligned}$$

$\square$

Proposition 15.

$$|N(A)| = \sum_{B \supseteq \overline{A}} (-1)^{|B|-|\overline{A}|} \left(k_B - k_B^B\right)$$

Proof. $N(A)$ is the collection of vertices in V_{n-1} that are covered by all vertices in A, but not by any of the vertices in $\overline{A}$, so

$$\begin{aligned}
N(A) &= I(A) \cap \overline{S\left(\overline{A}\right)} \\
&= I(A) \cap \left(\overline{\bigcup_{t \in \overline{A}} S(t)}\right) \\
&= I(A) \setminus \left(\bigcup_{t \in \overline{A}} S(t)\right)
\end{aligned}$$

Therefore we have

$$\begin{aligned}
|N(A)| &= |I(A)| - \left| I(A) \cap \left(\bigcup_{t \in \overline{A}} S(t)\right) \right| \\
&= |I(A)| - \left| \left(\bigcup_{t \in \overline{A}} (I(A) \cap S(t))\right) \right| \\
&= |I(A)| - \left| \left(\bigcup_{t \in \overline{A}} I(A \cup \{t\})\right) \right| \\
&= |I(A)| - \sum_{\emptyset \neq C \subseteq \overline{A}} (-1)^{|C|-1} \left| \bigcap_{t \in C} I(A \cup \{t\}) \right| \\
&= |I(A)| - \sum_{\emptyset \neq C \subseteq \overline{A}} (-1)^{|C|-1} |I(A \cup C)|
\end{aligned}$$

By Proposition 14, this gives us

$$\begin{aligned}
&= \sum_{C \subseteq \overline{A}} (-1)^{|C|} \left(\sum_{B \subseteq A \cup C} (-1)^{|B|} \left| \overline{S(B)} \right| \right) \\
&= \sum_{\substack{B \subseteq A \cup C \\ C \subseteq \overline{A}}} (-1)^{|B|+|C|} \left| \overline{S(B)} \right| \\
&= \sum_{B \subseteq V_n} \sum_{i=0}^{\left|\overline{A} \setminus B\right|} (-1)^{|B|+\left|B \cap \overline{A}\right|+i} \binom{\left|\overline{A} \setminus B\right|}{i} \left| \overline{S(B)} \right| \\
&= \sum_{B \supseteq \overline{A}} (-1)^{|B|+\left|\overline{A}\right|} \left| \overline{S(B)} \right| \\
&= \sum_{B \supseteq \overline{A}} (-1)^{|B|-\left|\overline{A}\right|} \left(k_B - k_B^B \right)
\end{aligned}$$

$\square$

Chapter 3

Isomorphisms of $B(\Gamma)$

3.1 The Algebra $B(\Gamma)$

Recall that $B(\Gamma) = T(V_+)/R_B$, where R_B is the ideal generated by

$$\{vw : v \not> w\} \cup \left\{ v \sum_{v > w} w : v \in V \right\}.$$

This algebra has a double grading given by

$$B(\Gamma)_{m,n} = \text{span}\left\{ v_1 \dots v_m : \sum_{i=1}^{m} |v_i| = n \right\}$$

Definition 11. For notational convenience, we define $B_n = B(\Gamma)_{1,n} = \left\{\sum_{v \in V_n} \alpha_v v\right\}$, the linear span of the vertices in V_n.

Definition 12. Given an element $a = \sum_{v \in V_n} \alpha_v v \in B_n$, let $A_a = \{v \in V_n : \alpha_v \neq 0\}$. Then we define

i) $S(a) = S(A_a)$

ii) $I(a) = I(A_a)$

iii) $\sim_a = \sim_{A_a}$

iv) $k_a = k_{A_a}$

v) $k_a^a = k_{A_a}^{A_a}$

3.2 The Subspaces κ_a

Definition 13. For each element $a \in B_n$, we define a map

$$L_a : B_{n-1} \to B(\Gamma)$$

such that $L_a(b) = ab$. We define κ_a be the kernel of the map L_a.

Looking at the ideal R, it is clear that for any $v \in V_n$, we have

$$\kappa_v = \text{span}\left(\{w : w \not\lessdot v\} \cup \left\{\sum_{w \lessdot v} w\right\}\right)$$

It follows that if $|S(v)| > 1$, then for any vertex w, we have $w \in S(v)$ if and only if $w \notin \kappa_v$ when both v and w are considered as algebra elements. We can also obtain the following results about the structure of κ_a:

Lemma 16. *For any nonzero $a = \sum_{v \in V_n} \alpha_v v$ in B_n,*

$$\kappa_a = \bigcap_{\alpha_v \neq 0} \kappa_v$$

Proof. Clearly, if $b \in \bigcap_{\alpha_v \neq 0} \kappa_v$, then $b \in \kappa_a$, which implies that $\bigcap_{\alpha_v \neq 0} \kappa_v \subseteq \kappa_a$.

To obtain $\kappa_a \subseteq \bigcap_{\alpha_v \neq 0} \kappa_v$, we note that $B(\Gamma)$ can be considered as a direct sum of vector spaces

$$B(\Gamma) = F \bigoplus_{v \in V_+} vB(\Gamma),$$

where F is the base field. If $b \in \kappa_a$, then

$$\sum_{v \in V_n} \alpha_v vb = 0,$$

and so we must have $vb = 0$ for every v such that $\alpha_v \neq 0$. Thus we have $b \in \bigcap_{\alpha_v \neq 0} \kappa_v$, and hence $\kappa_a \subseteq \bigcap_{\alpha_v \neq 0} \kappa_v$. $\square$

Proposition 17. *For any $a = \sum \alpha_v v \in B_n$,*

$$\kappa_a = span\left\{\sum_{w \in C} w : C \in \mathscr{C}_a\right\}$$

Proof. In light of Lemma 16, this statement reduces to showing that for any subset $A \subseteq V_n$,

$$\bigcap_{v \in A} \kappa_v = \text{span}\left\{\sum_{w \in C} w : C \in \mathscr{C}_A\right\}$$

We have

$$\begin{aligned}
\bigcap_{v\in A}\kappa_v &= \bigcap_{v\in A}\left(\text{span}\left(\{w : w \not\lessdot v\}\cup\left\{\sum_{w\lessdot v} w\right\}\right)\right)\\
&= \bigcap_{v\in A}\left\{\left(\sum_{w\in V_{n-1}}\beta_w w\right) : \beta_w=\beta_{w'} \text{ if } \{w,w'\}\subseteq S(v)\right\}\\
&= \left\{\left(\sum_{w\in V_{n-1}}\beta_w w\right) : \beta_w=\beta_{w'}\text{if } \{w,w'\}\subseteq S(v) \text{ for some } v\in A\right\}\\
&= \left\{\left(\sum_{w\in V_{n-1}}\beta_w w\right) : \beta_w=\beta_{w'} \text{ if } \{w,w'\}\subseteq C\in\mathscr{C}_A\right\}\\
&= \text{span}\left\{\sum_{w\in C} w : C\in\mathscr{C}_A\right\}
\end{aligned}$$

□

Often it will be useful to be able to refer to κ_A for a subset $A \subseteq V_n$:

Definition 14.

$$\kappa_A = \bigcap_{v\in A}\kappa_v.$$

Notice that $\kappa_A = \kappa_a$ for $a = \sum_{v\in A} v \in B_n$.

From Proposition 17, it is easy to see that the following corollaries hold:

Corollary 18. *For any $a \in B_n$, $\dim(\kappa_a) = k_a$, and for any $A \subseteq V_n$, $\dim(\kappa_A) = k_A$.*

Corollary 19. *For any $a \in B_n$, $|S(a)| = |V_{n-1}| - k_a + k_a^a$, and $|S(A)| = |V_{n-1}| - k_A + k_A^A$*

In general, we cannot recover information about k_a^a from $B(\Gamma)$, but in the special case where v is a vertex in V_n, we have $k_v^v = 1$, and so $|S(v)| = |V_{n-1}| - k_v + 1$.

We make the following observations about κ_a:

Proposition 20. *Let $a = \sum \alpha_v v \in B_n$, and let w be a vertex in V_n such that $\alpha_w \neq 0$. Then the following statements are true:*

i) $\kappa_a \subseteq \kappa_w$

ii) $k_a \leq k_w$

iii) $S(w) \subseteq S(a)$

We have equality in i) if and only if we have equality in ii). If $|S(w)| > 1$, then we have equality in iii) if and only if we have equality in i) and ii). In the case where $|S(w)| = 1$, then equality in iii) implies equality in i) and ii), but the other direction of implication does not hold.

Proof. All this is obvious from Lemma 16 and Proposition 17, except that $\kappa_a = \kappa_w$ implies $S(a) = S(w)$ in the case where $|S(w)| > 1$. Suppose $\kappa_a = \kappa_w$. Notice that for $u \in V_{n-1}$ we have $u \in S(a)$ if and only if $u \notin \kappa_a$, and $u \in S(w)$ if and only if $u \notin \kappa_w$. Thus $u \in S(a)$ if and only if $u \in S(w)$, so $S(a) = S(w)$. $\square$

3.3 Isomorphisms from $B(\Gamma)$ to $B(\Gamma')$

Let $\Gamma = (V, E)$, and $\Gamma'(W, F)$ be uniform layered graphs. Here we consider $B(\Gamma) = T(V_+)/R$ and $B(\Gamma') = T(W_+)/R'$. A natural question to ask is which doubly graded algebra isomorphisms between $T(V_+)$ and $T(W_+)$ induce isomorphisms between the doubly graded algebras $B(\Gamma)$ and $B(\Gamma')$. The answer turns out to be fairly simple:

Theorem 21. *Let $\Gamma = (V, E)$ and $\Gamma' = (W, F)$ be uniform layered graphs with algebras $B(\Gamma) = T(V_+)/R$ and $B(\Gamma') = T(W_+)/R'$ resepectively, and let*

$$\phi : T(V_+) \to T(W_+)$$

be an isomorphism of doubly graded algebras. Then ϕ induces a doubly graded algebra isomorphism from $B(\Gamma)$ to $B(\Gamma')$ if and only if $\kappa_{\phi(v)} = \phi(\kappa_v)$ for all $v \in V$.

For this result, we will need the following lemma:

Lemma 22. *Let $\Gamma = (V, E)$ and $\Gamma'(W, F)$ be uniform layered graphs, and let*

$$\phi : T(V_+) \to T(W_+)$$

be an isomorphism of doubly graded algebras. Then the following are equivalent:

(i) For any $a \in B_n$, $\phi(\kappa_a) = \kappa_{\phi(a)}$.

(ii) For any $v \in V_n$, $\phi(\kappa_v) = \kappa_{\phi(v)}$.

Proof. Clearly, we have (i) $\Rightarrow$ (ii). To show that (ii) $\Rightarrow$ (i), let

$$a = \sum \alpha_v v.$$

We know from Lemma 16 that

$$\kappa_a = \bigcap_{\alpha_v \neq 0} \kappa_v.$$

Since ϕ is a bijection, this implies that

$$\phi(\kappa_a) = \bigcap_{\alpha_v \neq 0} \phi(\kappa_v) = \bigcap_{\alpha_v \neq 0} \kappa_{\phi(v)}$$

Let $\phi(v) = \sum \beta_{v,w} w$. Then

$$\kappa_{\phi(v)} = \bigcap_{\beta_{v,w} \neq 0} \kappa_w,$$

and

$$\phi(a) = \sum \left(\alpha_v \left(\sum \beta_{v,w} w \right) \right) = \sum \alpha_v \beta_{v,w} w,$$

so we have

$$\phi(\kappa_a) = \bigcap_{\substack{\alpha_v \neq 0 \\ \beta_{v,w} \neq 0}} \kappa_w = \kappa_{\phi(a)}$$

$\square$

Proof of Theorem 21. For the purposes of this proof, we will let $B_n = B(\Gamma)_{1,n}$, and $B'_n = B(\Gamma')_{1,n}$. We wish to show that $\phi(R) = R'$ if and only if $\kappa_{\phi(v)} = \phi(\kappa_v)$ for all $v \in V_n$. We have

$$R = \langle ab : a \in B_n, b \in \kappa_a \text{ or } b \notin B_{n-1} \rangle$$

and so

$$\phi(R) = \langle \phi(ab) : a \in B_n, b \in \kappa_a \text{ or } b \notin B_{n-1} \rangle.$$

We also have

$$R' = \langle ab : a \in B'_n, b \in \kappa_a \text{ or } b \notin B'_{n-1} \rangle.$$

Suppose that $\kappa_{\phi(v)} = \phi(\kappa_v)$ for all $v \in V_+$. Then by Lemma 22, we have $\kappa_{\phi(a)} = \phi(\kappa_a)$ for any $a \in B_n$. This means that for any a, we have $\kappa_a = \phi^{-1}(\kappa_{\phi(a)})$, and thus $\kappa_{\phi^{-1}(a)} = \phi^{-1}(\kappa_a)$.

Let ab be one of the generators of R. Then $a \in V_n$ and either $b \in \kappa_a$ or $b \notin B_{n-1}$. If $b \in \kappa_a$, we have

$$\phi(ab) = \phi(a)\phi(b)$$

We know $\phi(b) \in \phi(\kappa_a) = \kappa_{\phi(a)}$, so $\phi(ab) \in R'$. Otherwise, $b \notin B_{n-1}$, and so we have $\phi(a) \in B'_n$ and $\phi(b) \notin B'_{n-1}$, so $\phi(a)\phi(b) = \phi(ab)$ is in R'. It follows that $\phi(R) \subseteq R'$.

Any element of R' takes the form ab, for $a \in W_n$ and $b \in \kappa_a$ or $b \notin B'_{n-1}$. If $b \in \kappa_a$, we have

$$ab = \phi(\phi^{-1}(a)\phi^{-1}(b))$$

We know that $\phi^{-1}(a) \in V_n$, and $\phi^{-1}(b) \in \phi^{-1}(\kappa_a) = \kappa_{\phi^{-1}(a)}$. Thus $\phi^{-1}(a)\phi^{-1}(b) \in R$, and so $ab \in \phi(R)$. If $b \notin B'_{n-1}$, then we have $\phi^{-1}(a) \in B_n$ and $\phi^{-1}(b) \notin B_{n-1}$, and so $\phi^{-1}(ab) \in R$. It follows that $ab \in \phi(R)$, and so $R' \subseteq \phi(R)$. This means that $R' = \phi(R)$.

Conversely, suppose $\kappa_{\phi(v)} \neq \phi(\kappa_v)$ for some $v \in V_n$. Then there must be some a such that

$$a \in \left(\kappa_{\phi(v)} \setminus \phi(\kappa_v)\right) \cup \left(\phi(\kappa_v) \setminus \kappa_{\phi(v)}\right)$$

If $a \in (\kappa_{\phi(v)} \setminus \phi(\kappa_v))$, then we have $\phi^{-1}(a) \notin \kappa_v$, so $v\phi^{-1}(a) \notin R$. However, we also have $a \in \kappa_{\phi(v)}$, so $\phi(v\phi^{-1}(a)) = \phi(v)a \in R'$. Since ϕ is a bijection, this means $R' \neq \phi(R)$.

If $a \in (\phi(\kappa_v) \setminus \kappa_{\phi(v)})$, then $v\phi^{-1}(a) \in R$, but $\phi(v\phi^{-1}(a)) = \phi(v)a \notin R$. Since ϕ is a bijection, $R' \neq \phi(R)$.

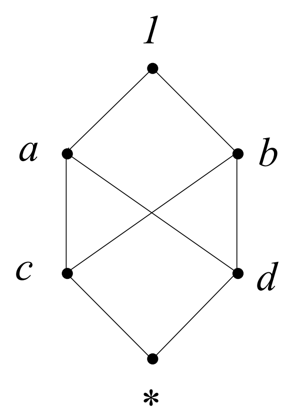
□

Chapter 4

Upper Vertex-Like Bases

We are interested in studying the equivalence classes of layered graphs under $\sim_A$ and $\sim_B$. As previously discussed, for uniform layered graphs Γ and Γ', whenever $\Gamma \sim_A \Gamma'$, we also have $\Gamma \sim_B \Gamma'$, so for now we will focus on the relation $\sim_B$. Suppose that for a particular graph Γ, we are given the doubly-graded algebra $B(\Gamma)$, but no additional information about the graph. What information about Γ can we recover?

If we could somehow identify the vertices in $B(\Gamma)$, we could recover quite a bit of information. In particular, we would know $S(v)$ for every vertex v of degree greater than 1. Unfortunately, it is not always possible to recover the vertices from the algebra. Consider the following graph Γ:



It is easy to verify from Proposition 21 that the algebra map ϕ given by

$$\phi(v) = \begin{cases} a+b & \text{if } v = a \\ v & \text{if } v \in V_+ \setminus \{a\} \end{cases}$$

is an automorphism of $B(\Gamma)$ which does not fix the vertices. Thus we cannot, in general, identify which elements of $B(\Gamma)$ are the vertices. However, it is possible to find a collection of algebra elements that in some sense “act like” the vertices. In this chapter, we will give a construction for these elements.

4.1 Upper Vertex-Like Bases

We will continue to use the notation B_n for $B(\Gamma)_{1,n}$, the linear span of V_n in $B(\Gamma)$.

Definition 15. If $\Gamma = (\bigcup_{i=0}^{\infty} V_i, \bigcup_{i=1}^{\infty} E_i)$ is a layered graph, then the **restriction of** Γ **to** n is

$$\Gamma|_n = \left(\bigcup_{i=0}^{n} V_i, \bigcup_{i=0}^{n} E_i\right)$$

Notice that if Γ is a uniform layered graph, then so is $\Gamma|_n$, and that $B(\Gamma|_n)$ is the subalgebra of $B(\Gamma)$ generated by $\bigcup_{i=1}^{n} V_i$.

Definition 16. A basis L for B_n is called an **upper vertex-like** basis if there exists a doubly-graded algebra isomorphism

$$\phi : B(\Gamma|_n) \to B(\Gamma|_n)$$

such that ϕ fixes $\bigcup_{i=0}^{n-1} V_i$, and $\phi(V_n) = L$.

Proposition 23. *A basis L for B_n is upper vertex-like if and only if there exists a bijection $\phi : V_n \to L$ such that $\kappa_v = \kappa_{\phi(v)}$ for all $v \in V_n$.*

Proof. This follows from Theorem 21. Define

$$\phi' : T\left(\bigcup_{i=0}^{n} V_i\right) \to T\left(\bigcup_{i=0}^{n} V_i\right)$$

to be the graded algebra isomorphism given by

$$\phi'(v) = \begin{cases} v & \text{if } v \in \bigcup_{i=0}^{n-1} V_i \\ \phi(v) & \text{if } v \in V_n \end{cases}$$

Notice that ϕ' fixes all of $B(\Gamma|_{n-1})$. Since $\kappa_v \subseteq B_{n-1}$, we have $\phi(\kappa_v) = \kappa_v$. Thus $\kappa_v = \kappa_{\phi(v)}$ is exactly the condition we need for ϕ' to induce an isomorphism on $B(\Gamma|_n)$.

Conversely, if L is upper vertex-like, then there exists $\phi : B(\Gamma|_n) \to B(\Gamma|_n)$ which fixes $\bigcup_{i=0}^{n-1} V_i$. The restriction of ϕ to V_n is a bijection from V_n to L. Theorem 21 tells us that $\phi(\kappa_v) = \kappa_{\phi(v)}$. Since ϕ fixes κ_v, we have our result. $\square$

Since an upper vertex-like basis L for B_n is indistinguishable from the actual set of vertices in $B(\Gamma|_n)$, the collection of κ-subspaces associated with L is identical to the collection associated to the actual vertices. For convenience, here we will use the notation $\kappa(a)$ rather than κ_a.

Proposition 24. *Let $\Gamma = (\bigcup_{i=0}^{\infty} V_i, E)$ be a uniform layered graph, and let L be an upper vertex-like basis for $B(\Gamma)_{1,n}$. Then there exists a bijection*

$$\psi : V_n \to L$$

such that for any $A \subseteq V_n$, we have

$$\kappa\left(\sum_{v\in A} v\right) = \kappa\left(\sum_{v\in A} \psi(v)\right)$$

Proof. Since L is upper vertex-like, there exists an isomorphism

$$\phi : B(\Gamma|_n) \to B(\Gamma|_n)$$

which fixes V_i for $1 \le i \le n-1$, and takes V_n to L. Let $\psi = \phi|_{V_n}$. Since $\phi(\kappa(a)) = \kappa(\phi(a))$ for any $a \in B_n$, we have

$$\phi\left(\kappa\left(\sum_{v\in A} v\right)\right) = \kappa\left(\phi\left(\sum_{v\in A} v\right)\right) = \kappa\left(\sum_{v\in A} \phi(v)\right) = \kappa\left(\sum_{v\in A} \psi(v)\right),$$

and since ϕ fixes all elements of $B(\Gamma_{n-1})$, we have

$$\phi\left(\kappa\left(\sum_{v\in A} v\right)\right) = \kappa\left(\sum_{v\in A} v\right)$$

The result follows. $\square$

This allows us to recover some information about the structure of the graph Γ from the algebra $B(\Gamma)$.

4.1.1 Example: Out-Degree of Vertices

If $L = \{b_1, \ldots, b_{\dim(B_n)}\}$ is an upper vertex-like basis for B_n and $\dim(\kappa_{b_i}) = d$, then there exists $v \in V_n$ with $\dim(\kappa_v) = d$. We have

$$|S(v)| = V_i - d + 1,$$

and so Γ must have a vertex in V_n with out-degree $\dim(B_n) - d + 1$. In fact, since we can calculate $\dim(\kappa_{b_i})$ for all $1 \leq i \leq \dim(B_n)$, these upper vertex-like bases allow us to calculate the degree sequence of the entire graph. More precisely, this argument gives us

Proposition 25. *Let Γ be a directed graph, and let L be an upper vertex-like basis for $B_n \subseteq B(\Gamma)$. Then multiset $\{|S(v)| : v \in V_n\}$ is equal to the multiset*

$$\{\dim(B_n) - \dim(\kappa_b) + 1 : b \in L\},$$

and thus can be calculated from $B(\Gamma)$.

4.1.2 Example: Size of Intersections

Let $L = \{b_1, \ldots, b_{\dim(B_n)})$ be an upper vertex-like basis for B_n, and suppose we have $\dim(\kappa_{b_i}) = d_i$, $\dim(\kappa_{b_j}) = d_j$, and $\dim(\kappa_{b_i} \cap \kappa_{b_j}) = d_{ij}$. Then we have v_i and v_j such that $\dim(\kappa_{v_i}) = d_i$, $\dim(\kappa_{v_j}) = d_j$, and $\dim(\kappa_{v_i} \cap \kappa_{v_j}) = d_{ij}$. By Proposition 14 and Corollary 18, we can conclude that

$$\begin{aligned} |I(\{v_i, v_j\})| &= \dim(B_n) - (d_i - 1) - (dj - 1) + d_{ij} - k_{\{v_i,v_j\}}^{\{v_i,v_j\}} \\ &= \dim(B_n) - d_i - d_j + d_{ij} + \left(2 - k_{\{v_i,v_j\}}^{\{v_i,v_j\}}\right) \end{aligned}$$

If the intersection is nontrivial, then $2 - k_{\{v_i,v_j\}}^{\{v_i,v_j\}} = 1$. Otherwise, $2 - k_{\{v_i,v_j\}}^{\{v_i,v_j\}} = 0$. We cannot calculate $k_{\{v_i,v_j\}}^{\{v_i,v_j\}}$ from $B(\Gamma)$, as we'll see in Chapter 6. However, we can conclude the following:

Proposition 26. *Let Γ be a layered graph, let L be an upper vertex-like basis for $B_n \subseteq B(\Gamma)$, with bijection $\phi : V_n \to L$ satisfying $\kappa_{\phi(v)} = \kappa_v$ for every $v \in V_n$. Then*

$$|I(\{v_i, v_j\}| = \dim(B_n) - \dim(\kappa_{\phi(v_i)}) - \dim(\kappa_{\phi(v_j)}) + \dim(\kappa_{\phi(v_i+v_j)}) + 1$$

if the value on the right is greater than or equal to 2. Otherwise $I(\{v_i, v_j\})$ is 0 or 1.

4.2 Constructing an Upper Vertex-Like Basis

Our goal in this section is to prove the following theorem, which shows that we can construct an upper vertex-like basis for B_n using only the information given to us by the doubly-graded algebra $B(\Gamma)$. Our main result is the following theorem:

Theorem 27. *Let $L = \{b_1, b_2, \dots, b_{|V_n|}\}$ be a basis for B_n, such that for any i and for any $a \in (V_n \setminus span\{b_1, \dots, b_{i-1}\})$, we have $k_{b_i} \geq k_a$. Then L is an upper vertex-like basis.*

This construction resulted from an attempt to prove Proposition 25, that the out-degree sequence of Γ can be calculated from the doubly-graded algebra $B(\Gamma)$. The argument relies heavily on the chain of subspaces

$$F_{|V_{n-1}|} \subseteq F_{|V_{n-1}|-1} \subseteq \dots F_2 \subseteq F_1 = B_n$$

given by

$$F_i = span\{v \in V_n : k_v \geq i\}$$

We will use this notation for the duration of this section. Recall that from Corollary 25, we know that for any $a \in B_n$, we have

$$|S(a)| = |V_{n-1}| - k_a + k_a^a.$$

For a vertex v, we know that $k_v^v = 1$, so we have

$$|S(v)| = |V_{n-1}| - k_v + 1,$$

and thus

$$F_i = span\{v \in V_n : |S(v)| \leq |V_{n-1}| - i + 1\}$$

Thus if it is possible to calculate the dimension of the F_i from the information given to us by the doubly-graded algebra $B(\Gamma)$, then we can also calculate the out-degree sequence. We have the following result:

Proposition 28. *For $i = 1, \dots, |V_{n-1}|$, let F_i be defined as above. Then we have*

$$F_i = span\{a \in B_n : k_a \geq i\}$$

Proof. Clearly, $F_i \subseteq span\{a \in B_n : k_a \geq i\}$. To obtain the opposite inclusion, suppose $a \in B_n$ satisfies $k_a \geq i$. We know from Proposition 20 that

$$a \in span\{v \in V_n : k_v \geq k_a\} \subseteq F_i,$$

and so $span\{a \in B_n : k_a \geq i\} \subseteq F_i$, and our proof is complete. $\square$

This gives us our result about out-degree, but it falls short of giving us a construction for an upper vertex-like basis. According to Proposition 23, an upper vertex-like basis for B_n will consist of a linearly independent collection of algebra elements whose κ-subspaces "match" the κ-subspaces of the elements of V_n. Proposition 28 gives us the tools that we need to construct a basis whose κ-subspaces have the correct dimensions.

Definition 17. We say that a basis $\{x_1, \ldots, x_{\dim(V)}\}$ for a vector space V is **compatible** with the chain of subspaces

$$V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots \subseteq V$$

if for each i, $span\{x_1, \ldots, x_{\dim(V_i)}\} = V_i$.

Lemma 29. *Let L be defined as in Theorem 27. Then L is compatible with*

$$F_{|V_{n-1}|} \subseteq F_{|V_{n-1}|-1} \subseteq \ldots F_2 \subseteq F_1 = B_n.$$

Proof. Recall that F_i is spanned by $\{v \in V_n : k_v \geq i\}$. Thus if $j < \dim(F_i)$, then there exists $v \in F_i \setminus span\{b_1, \ldots, b_j\}$, and this v will satisfy $k_v \geq i$. Thus by the definition of L, we have $k_{b_{j+1}} \geq k_v \geq i$. Thus

$$span\{b_1, \ldots, b_{\dim(F_i)}\} \subseteq span\{a \in B_n : k_a \geq i\} = F_i,$$

and so

$$span\{b_1, \ldots, b_{\dim(F_i)}\} = F_i$$

□

This shows that the κ-subspaces of L have the correct dimension. Our next goal is to show that the κ-subspaces of L all appear as κ-subspaces associated to vertices.

Lemma 30. *Let L be a basis for B_n satisfying the hypothesis of Theorem 27. Then for each $b \in L$, there exists $w \in V_n$ such that $\kappa_b = \kappa_w$.*

Proof. According to Proposition 20, any element $a \in B_n$ is in $span\{v \in V_n : \kappa_v \supseteq \kappa_a\}$. Thus either there exists $w \in V_n$ with $\alpha_w \neq 0$ and $\kappa_a = \kappa_w$, or we have

$$a \in span\{v \in V_n : \kappa_v \supsetneq \kappa_a\} \subseteq span\{v \in V_n : k_v \geq k_a + 1\} \subseteq F_{k_a+1}$$

By Lemma 28, this gives us

$$a \in span\{b \in L : k_b \geq k_a + 1\},$$

and so $a \notin L$.

It follows that for any $b = \sum_{v \in V_n} \beta_v v \in L$, there exists $w \in V_n$ such that $\beta_w \neq 0$ and $\kappa_b = \kappa_w$. $\square$

All that remains is to show that the multiplicity with which each κ-subspace appears in L matches the multiplicity with which it appears in V_n.

Lemma 31. *Let L be a basis for B_n, satisfying the hypothesis of Theorem 27. Then for each $w \in V_n$, we have*

$$|\{b \in L : \kappa_b = \kappa_w\}| = |\{v \in V_n : \kappa_v = \kappa_w\}|$$

Proof. Notice that

$$\{b + F_{k_w+1} : b \in L, k_b = k_w\}$$

and

$$\{v + V_{k_w+1} : v \in V_n, k_v = k_w\}$$

are both bases for the space F_{k_w+1}/F_{k_w}. We know that for each $b \in L$ with $\kappa_b = \kappa_w$, we have $b \in span\{v \in V_n : \kappa_v \supseteq \kappa_w\}$, and so

$$span\,\{b + F_{k_w+1} : b \in L, \kappa_b = \kappa_w\} \subseteq span\,\{v + F_{k_w+1} : v \in V_n, \kappa_v = \kappa_w\}$$

$\square$

A simple dimension argument now gives us

Corollary 32. *Let L be a basis for B_n satisfying the hypothesis of Theorem 27. Then for each $w \in V_n$, we have*

$$|\{b \in L : \kappa_b = \kappa_w\}| = |\{v \in V_n : \kappa_v = \kappa_w\}|$$

Now we are ready to prove that L is an upper vertex-like basis.

Proof of Theorem 27. By Corollary 32, we know that for every $w \in V_n$, we have

$$|\{b \in L : \kappa_b = \kappa_w\}| = |\{v \in V_n : \kappa_v = \kappa_w\}|$$

Thus there exists a bijective map $\phi : V_n \to L$ such that $\kappa_{\phi(v)} = \kappa_v$. $\square$

Chapter 5

Uniqueness Results

Upper vertex-like bases allow us to find the collection of κ-subspaces associated to the vertices, and to count the multiplicity of each of these κ-subspaces. However, there are cases in which we can do even better—identifying the linear span of a particular vertex in $B(\Gamma)$. Recall that for any $a \in B_n$,

$$a \in span\{v \in V_n : \kappa_v \supseteq \kappa_a\}$$

Thus if there exists a vertex $w \in V_n$ such that

$$\{v \in V_n : \kappa_v \supseteq \kappa_w\},$$

then for any element b of an upper vertex-like basis L for B_n satisfying $\kappa_b = \kappa_w$, we have $b \in span\{w\}$. In cases where this condition is fairly common, this allows us to almost entirely determine the structure of Γ from the algebra $B(\Gamma)$.

We will use both poset and layered-graph notation in the discussion that follows. We will identify the poset P with the layered graph associated to its Hasse diagram. For ease of notation, we will use the notation $P_{\geq j}$ for $\bigcup_{i \geq j} P_i$.

5.1 Non-Nesting Posets

Definition 18. Let P be a ranked poset with a unique minimal vertex $*$. We will say that P has the **non-nesting property** if for any two distinct elements p and q with out-degree greater than 1, we have $S(p) \not\subseteq S(q)$.

Theorem 33. *Let P be a finite poset with the non-nesting property, such that $|S(p)| > 1$ whenever $\rho(p) > 1$. If Q is a poset with $Q \sim_B P$, then $P_{\geq 2} \cong Q_{\geq 2}$*

Proof. Suppose there is a doubly graded algebra isomorphism from $B(P)$ to $B(Q)$. We will equate the vertices in $B(P)$ with their images in $B(Q)$, allowing us to work entirely inside the algebra $B(Q)$.

For each level i, we can find an upper vertex-like basis L_i for B_i. The construction of L_i depends on the algebra, not on the original poset, so L_i is upper vertex-like for both P and for Q. That is to say, there exist two bijections

$$\phi_i : L_i \to P_i \quad \text{and} \quad \psi_i : L_i \to Q_i$$

such that $\kappa_{\phi_i(\mathcal{A})} = \kappa_{\mathcal{A}}$ and $\kappa_{\psi_i(\mathcal{A})} = \kappa_{\mathcal{A}}$ for every $\mathcal{A} \subseteq L_i$.

Define $\xi_i : P_i \to Q_i$ by $\xi_i = \phi_i^{-1} \circ \psi_i$. This is a bijection, and for any $\mathcal{A} \subseteq P_i$, we have

$$\kappa_{\xi_i(\mathcal{A})} = \kappa_{\phi_i^{-1}(\psi_i(\mathcal{A}))} = \kappa_{\psi_i(\mathcal{A})} = \kappa_{\mathcal{A}}$$

For any $q \in Q_i$, $a \in B_i$, we know from Theorem 20 that $\kappa_a = \kappa_q$ if and only if

$$a \in \text{span}\{r \in Q_i : \kappa_r \subseteq \kappa_q\} = \text{span}\{r \in Q_i : S(q) \subseteq S(r)\}$$

For $i > 1$, the non-nesting property tells us that the rightmost set is equal to $span\{q\}$, and so a is a constant multiple of q. In particular, for each $p \in P_i$, $\xi_i(p)$ is a scalar multiple of p. We will write $\xi_i(p) = \alpha_p p$.

Now define a bijection

$$\xi : P_{\geq 2} \to Q_{\geq 2}$$

such that $\xi(p) = \xi_i(p)$ for every $p \in P_i$. We claim that this is an isomorphism of posets. To prove this, we must show that for any $r \in P_{\geq 2}$, we have $r \in S(p)$ if and only if $\xi(r) \in S(\xi(p))$.

Let $p \in P_i$ for $i > 2$. We know that

$$\kappa_p = \text{span}\left(\left\{\sum_{r \in S(p)} r\right\} \cup \{r : r \notin S(p)\}\right),$$

and that $|S(p)| > 1$. It follows that for $r \in P_{i-1}$ we have $r \in S(p)$ if and only if $r \notin \kappa_p$. A similar argument shows that $\xi(r) \in S(\xi(p))$ if and only if $\xi(r) \notin \kappa_{\xi(p)}$. This gives us

$$\big(r \in S(p)\big) \Leftrightarrow \big(r \notin \kappa_p\big) \Leftrightarrow \big(\alpha_r r \notin \kappa_p\big) \Leftrightarrow \big(\xi(r) \notin \kappa_{\xi(p)}\big) \Leftrightarrow \big(\xi(r) \in S(\xi(p))\big)$$

□

Corollary 34. *Given any finite atomic lattice P whose Hasse diagram is a uniform layered graph, the poset $P_{\geq 2}$ is determined up to isomorphism by $B(P)$.*

Proof. Since P is a finite lattice, it has a unique minimal element $\hat{0}$. Since P is atomic, any element p of rank two or greater satisfies $|S(p)| > 1$. All that remains is to show that P satisfies the non-nesting property.

Suppose p and q are elements of P with $S(p) \subseteq S(q)$. Then either p and q are both atoms and $S(p) = S(q) = \hat{0}$, or we have $\bigvee S(p) = p$ and $\bigvee S(q) = q$. This means that

$$p \vee q = \left(\bigvee S(p)\right) \vee \left(\bigvee S(q)\right) = \bigvee S(q) = q,$$

and so we have $q \geq p$. Since $S(p) = S(q)$, we know that $\rho(p) = \rho(q)$, and so it follows that $q = p$. □

5.2 The Boolean Algebra

The Boolean algebra satisfies the non-nesting property. This makes it fairly easy to show that it is uniquely identified by its algebra $B(\Gamma)$.

Proposition 35. *Let $2^{[n]}$ be the Boolean lattice, and let Γ be another uniform layered graph with $\Gamma \sim_B 2^{[n]}$. Then Γ and $2^{[n]}$ are isomorphic as layered graphs.*

Proof. Let $2^{[n]}$ be the Boolean lattice, and let $\Gamma = (V_0 \cup \ldots \cup V_n, E)$ be a uniform layered graph with unique minimal vertex such that there exists an isomorphism of doubly-graded algebras from $B\left(2^{[n]}\right)$ to $B(\Gamma)$. We equate the vertices in $B\left(2^{[n]}\right)$ with their images in $B(\Gamma)$. For each $i > 2$, define ξ_i as in the proof of Theorem 33. Then since $2^{[n]}$ is a finite atomic lattice, the map

$$\xi : 2^{[n]}{}_{\geq 2} \to \Gamma_{\geq 2}$$

given by $\xi(p) = \xi_i(p)$ for $p \in \binom{[n]}{i}$ is an isomorphism of posets. We would like to define an extension ξ' of ξ, such that ξ' is an isomorphism from $2^{[n]}$ to Γ.

We will use the following in our construction:

Claim 1: For every $w \in V_1$, we have $|\{v \in V_2 : v \not\geq w\}| = \binom{n-1}{2}$.

Claim 2: For every $A \subseteq V_2$ with $|A| = \binom{n-1}{2}$, we have $\dim(\kappa_A) > 1$ if and only if there exists $w \in V_1$ with $A = \{v \in V_2 : v \not\geq w\}$.

We will begin by proving Claim 1. We know that ξ satisfies $\kappa_{\xi(p)} = \kappa_p$ for any $p \in \binom{[n]}{2}$. Since $|S(p)| = 2$ for every $p \in \binom{[n]}{2}$, Corollary 25 gives us

$$\dim(\kappa_p) = n - 1$$

for all $p \in \binom{[n]}{2}$, and thus

$$\dim(\kappa_v) = n - 1$$

for all $v \in V_2$. This tells us that $|S(v)| = 2$ for all $v \in V_2$.

Furthermore, for any $p \neq q$ in $\binom{[n]}{2}$, we have $\kappa_p \neq \kappa_q$. Since ξ is a bijection, this tells us that $\kappa_v \neq \kappa_w$ for any $v \neq w$ in V_2. Thus $S(v) \neq S(w)$ whenever $v \neq w$ in V_2. We have

$$|V_1| = \dim(B_1) = n,$$

and

$$|V_2| = \dim(B_2) = \binom{n}{2},$$

so each possible pair of vertices in V_1 appears exactly once as $S(v)$ for some $v \in V_1$. It follows that for any $w \in V_1$, $|\{v \in V_2 : v \not\geq w\}|$ is exactly the number of pairs of vertices in V_1 that exclude w. There are exactly $\binom{n-1}{2}$ such pairs. This proves Claim 1.

To prove Claim 2, we begin by associating to each subset $A \subseteq V_2$ a graph G_A with vertex set V_1, and with edge set

$$\{S(v) : v \in A\}.$$

Corollary 18 implies that k_A counts the number of connected components of G_A. If the connected components have sizes $i_1, i_2, \ldots, i_{k_A}$, then we have

$$|A| \leq \binom{i_1}{2} + \binom{i_2}{2} + \ldots + \binom{i_{k_A}}{2}$$

Thus if $|A| = \binom{n-1}{2}$, then

$$\binom{n-1}{2} \le \binom{i_1}{2} + \binom{i_2}{2} + \ldots + \binom{i_{k_A}}{2}.$$

Notice that for $i \ge j \ge 1$, we have

$$\begin{aligned}
\binom{i}{2} + \binom{j}{2} &= i(i-1) + j(j-1) \\
&\le i(i-1) + i(j-1) \\
&\le i(i-1) + (j-1)(j-2) + i(j-1) \\
&\le \binom{i}{2} + \binom{j-1}{2} + i(j-1) \\
&= \binom{i+j-1}{2},
\end{aligned}$$

with equality if and only if $j = 1$. This gives us

$$|A| = \binom{n-1}{2} \le \binom{n+1-k_A}{2},$$

which implies that $k_A \le 2$. If $k_A = 2$, we have

$$\begin{aligned}
\binom{n-1}{2} &\le \binom{i}{2} + \binom{j}{2} \\
&\le \binom{i+j-1}{2} \\
&= \binom{n-1}{2},
\end{aligned}$$

where i and j are the sizes of the two connected components of G_A, with $i \ge j$. The inequality in the second line is an equality if and only if $j = 1$. Thus when $|A| = \binom{n-1}{2}$ and $k_A > 1$, G_A consists of one isolated vertex w, together with the complete graph on $V_1 \setminus \{w\}$. Thus A is exactly the collection of vertices $\{v \in V_2 : v \not\geq w\}$, which proves Claim 2.

Now we are prepared to prove the theorem. For each $i \in [n]$, let $\mathscr{A}_i = \binom{[n]\setminus\{i\}}{2}$, and let $A_i = \xi(\mathscr{A}_i)$. Since $\kappa_{A_i} = \kappa_{\mathscr{A}_i}$, we have $k_{A_i} = k_{\mathscr{A}_i} = 2$. Since $|A_i| = |\mathscr{A}_i| = \binom{n-1}{2}$, Claim 2 tells us that for each $i \in [n]$, there exists a unique $w_i \in V_1$ such that $A_i = \{v \in V_2 : v \not\geq w_i\}$. We define our function $\xi' : 2^{[n]} \to \Gamma$ so that

$$\xi'(p) = \begin{cases} * & \text{if } p = \emptyset \\ w_i & \text{if } p = \{i\} \\ \xi(p) & \text{else} \end{cases}$$

We wish to show that this function is an isomorphism of posets.

First, notice that for $i \neq j$, we have $\kappa_{\mathscr{A}_i} \neq \kappa_{\mathscr{A}_j}$. This implies that $A_i \neq A_j$, and so $w_i \neq w_j$. Thus ξ' is a bijection.

To show that ξ' preserves order, we must show that whenever $i \in [n]$ and $p \in \binom{[n]}{2}$, we have $w_i \lessdot \xi'(p)$ if and only if $i \in p$. If $i \in p$, then $\xi'(p) \notin A_i$. Since $A_i = \{v \in V_2 : v \not\gtrdot w_i\}$, it follows that $w_i \lessdot \xi'(p)$. Conversely, if $i \notin p$, then $\xi'(p) \in A_i$, and so $w_i \not\lessdot \xi'(p)$. It follows that ξ' is an isomorphism of partially ordered sets, and so $\Gamma \cong 2^{[n]}$. □

5.3 Subspaces of a Finite-Dimensional Vector Space over $\mathbb{F}_q$

Proposition 36. *Let X be an n-dimensional vector space over $\mathbb{F}_q$, the finite field of order q, and let P_X be the set of subspaces of X, partially ordered by inclusion. Let Γ be another uniform layered graph with $\Gamma) \sim_B P_X$. Then Γ and P_X are isomorphic as layered graphs.*

Proof. The case where $n \leq 2$ is trivial, so we will assume that $n \geq 3$. We will use P_i to denote the collection of i-dimensional subspaces of X. Let $\Gamma = (V_0 \cup V_1 \cup \ldots \cup V_n, E)$ be a uniform layered graph with unique minimal vertex such that there exists an isomorphism from $B(P_X)$ to $B(\Gamma)$. Once again we will equate the vertices in $B(P_X)$ with their images in $B(\Gamma)$, and we will define ξ_i as in the proof of Theorem 33. Again our poset P_X is a finite atomic lattice, so the map

$$\xi : (P_X)_{\geq 2} \to \Gamma_{\geq 2}$$

is an isomorphism of posets. Again we are looking for an extension ξ' of ξ that is an isomorphism from our whole poset P_X to Γ. Our proof will mirror that of Proposition 35, and we will make use of the following two facts:

Claim 1: For every $w \in V_1$, we have

$$|\{v \in V_2 : v \not\gtrdot w\}| = \frac{(q^n - q^2)(q^{n-1} - 1)}{(q-1)(q^2-1)}.$$

Claim 2: For every $A \in V_2$ with

$$|A| = \frac{(q^n - q^2)(q^{n-1} - 1)}{(q-1)(q^2-1)},$$

we have $k_A > 1$ if and only if there exists $w \in V_1$ with $A = \{v \in V_2 : v \not\geqslant w\}$.

We begin by proving Claim 1. We know that P_2 consists of $\frac{(q^n-1)(q^{n-1}-1)}{(q-1)(q^2-1)}$ planes. It follows that V_2 consists of $\frac{(q^n-1)(q^{n-1}-1)}{(q-1)(q^2-1)}$ vertices. If we wish to show that

$$|\{v \in V_2 : v \not\geqslant w\}| = \frac{(q^n - q^2)(q^{n-1} - 1)}{(q-1)(q^2-1)},$$

it will suffice to show that

$$|\{v \in V_2 : v \geqslant w\}| = \frac{(q^n - 1)(q^{n-1} - 1)}{(q-1)(q^2-1)} - \frac{(q^n - q^2)(q^{n-1} - 1)}{(q-1)(q^2-1)} = \frac{(q^{n-1} - 1)}{(q-1)}$$

Let $v, v' \in V_2$ with $v \neq v'$. Since ξ is a bijection, we have $\xi^{-1}(v) \neq \xi^{-1}(v')$. Since two planes intersect in a unique line, we have

$$|I(\{\xi^{-1}(v), \xi^{-1}(v')\})| = 1.$$

By Proposition 26, this means that

$$\dim(B_n) - k_{\{\xi^{-1}(v)\}} - k_{\{\xi^{-1}(v')\}} + k_{\{\xi^{-1}(v), \xi^{-1}(v')\}} + 1 = 1$$

Since $k_A = k_{\xi(A)}$ for any $A \subseteq P_2$, this gives us

$$\dim(B_n) - k_{\{v\}} - k_{\{v'\}} + k_{\{v,v'\}} + 1 = 1,$$

and another application of Proposition 26 gives us

$$I(\{v, v'\}) \leq 1$$

for any $v \neq v'$. Each plane in P_2 contains $q+1$ lines. It follows that each vertex $v \in V_2$ satisfies $|S(v)| = q + 1$. Thus if we have $w \in V_1$ such that

$$|\{v \in V_2 : v \geqslant w\}| > \frac{(q^{n-1} - 1)}{(q-1)},$$

then there must exist at least

$$q\left(\frac{q^{n-1} - 1}{q - 1} + 1\right) + 1 = \frac{q^n + q^2 - q - 1}{q - 1}$$

distinct vertices in V_1. However, we know that

$$|V_1| = \frac{q^n - 1}{q-1} < \frac{q^n + q^2 - q - 1}{q-1},$$

so it follows that

$$|\{v \in V_2 : v \gtrdot w\}| \leq \frac{(q^{n-1}-1)}{(q-1)}$$

for all $w \in V_1$. We have

$$\begin{aligned}
\sum_{w \in V_1} |\{v \in V_2 : v \gtrdot w\}| &= (q+1)|V_2| \\
&= (q+1)\left(\frac{(q^n-1)(q^{n-1}-1)}{(q-1)(q^2-1)}\right) \\
&= \left(\frac{(q^n-1)}{(q-1)}\right)\left(\frac{(q^{n-1}-1)}{(q-1)}\right) \\
&= |V_1|\left(\frac{(q^{n-1}-1)}{(q-1)}\right)
\end{aligned}$$

It follows that

$$|\{v \in V_2 : v \gtrdot w\}| = \frac{(q^{n-1}-1)}{(q-1)}$$

for each $w \in V_1$, and Claim 1 is proved.

We know that any two planes in P_2 intersect in exactly one line. Thus for any $\mathscr{A} \subseteq P_2$, we have

$$\kappa_{\mathscr{A}} = \operatorname{span}\left(\left\{\sum_{l \in S(\mathscr{A})} l\right\} \cup \{w \in V_1 : w \notin S(\mathscr{A})\}\right),$$

and thus $\dim(\kappa_{\mathscr{A}}) = |P_1| - |S(\mathscr{A})| + 1$ for all $\mathscr{A} \subseteq P_2$. In particular, when $|\mathscr{A}| = \frac{(q^n-q^2)(q^{n-1}-1)}{(q-1)(q^2-1)}$, we have $k_{\mathscr{A}} > 1$ if and only if $\mathscr{A}$ is the collection of planes which do not contain a particular line l. There are $\frac{(q^n-1)}{(q-1)}$ such subsets. This means that there are exactly $\frac{(q^n-1)}{(q-1)}$ subsets $A \subseteq V_2$ with $|A| = \frac{(q^n-q^2)(q^{n-1}-1)}{(q-1)(q^2-1)}$ and $k_A > 1$. Since there are $|V_1| = \frac{(q^n-1)}{(q-1)}$ subsets of the form $\{v \in V_1 : v \not\gtrdot w\}$ for some $w \in V_1$, this must be the complete list of subsets $A \subseteq V_2$ with $|A| = \frac{(q^n-q^2)(q^{n-1}-1)}{(q-1)(q^2-1)}$ and $k_A > 1$. This proves Claim 2.

Now we can construct our extension ξ'. For each point $p \in P_1$, define $\mathscr{A}_p$ to be the collection of lines in P_2 which do not contain p, and let $A_p = \xi(\mathscr{A}_p)$. Since $\kappa_{A_p} = \kappa_{\mathscr{A}_p}$, we have $k_{A_p} = 2$, and thus there exists a unique $w_p \in V_1$ such that $A_p = \{v \in V_2 : v \not> w_p\}$. We define our function $\xi' : P_X \to \Gamma$ as follows:

$$\xi'(x) = \begin{cases} * & \text{if } x = \emptyset \\ w_x & \text{if } x \in P_1 \\ \xi(x) & \text{else} \end{cases}$$

Since $\kappa_{\mathscr{A}_p} \neq \kappa_{\mathscr{A}_q}$ whenever $p \neq q$ in P_1, we know that $\xi'(p) \neq \xi'(q)$, and thus ξ' is a bijection.

All that remains is to show that for $p \in P_1$ and $l \in P_2$, we have $w_p \lessdot \xi'(l)$ if and only if $p \in l$. If $p \in l$, then $\xi'(l) \notin A_p$, and since $A_p = \{v \in V_2 : v \not> w_p\}$, it follows that $w_p \lessdot \xi'(l)$. Conversely, if $p \notin l$, then $\xi'(l) \in A_p$, and so $w_p \not\lessdot \xi'(p)$. It follows that ξ' is an isomorphism of partially ordered sets, and that $\Gamma \cong P_X$.

$\square$

Chapter 6

Two-Layered Graphs

We will call the layered graph $\Gamma = \{V, E\}$ a two-layered graph if $V = V_0 \cup V_1 \cup V_2$. It seems reasonable to focus some attention on the collection of two-layered graphs with unique minimal vertices, since their structure is especially simple. We will begin by showing that if Γ is a two-layered graph with a unique minimal vertex, then Γ can be determined up to isomorphism from the values of k_A and k_A^A for $A \subseteq V_2$.

6.1 Structure of Two-Layered Graphs

Proposition 37. *Let Γ and Γ' be two-layered graphs with unique minimal vertices $*$ and $*'$. If there exists a bijection $\phi : V_2 \to V_2'$ such that $|N(A)| = |N(\phi(A))|$ for all $A \subseteq V_2$, then $\Gamma \cong \Gamma'$.*

Proof. Notice that the sets $N(A)$ such that $A \subseteq V_2$ form a partition of V_1, and that the sets $N(\phi(A))$ such that $A \subseteq V_2$ form a partition of V_1'. Since $|N(A)| = |N(\phi(A))|$ for all $A \subseteq V_2$, there must exist a bijection

$$\psi : V \to V'$$

satisfying the following three conditions:

i) $\psi(*) = *'$

ii) $\psi(v) = \phi(v)$ for all $v \in V_2$

iii) $\psi(w) \in N(\phi(A)) = N(\psi(A))$ if and only if $w \in N(A)$.

We claim that ψ is a graph isomorphism from Γ to Γ'.

Let (x, y) be an edge in Γ. If $x \in V_1$ and $y = *$, then $(\psi(x), \psi(y))$ is clearly an edge in Γ'. If $x \in V_2$ and $y \in V_1$, then $y \in N(A)$ for some A containing x. Thus $\psi(y) \in (N(\psi(A)),$ and $\psi(x) \in \psi(A)$, so $(\psi(x), \psi(y))$ is an edge in Γ'.

Let $(\psi(x), \psi(y))$ be an edge in Γ'. If $\psi(x) \in V_1'$ and $\psi(y) = *'$, then $x \in V_1$ and $y = *$, so (x, y) is clearly an edge in Γ. If $\psi(x) \in V_2'$ and $\psi(y) \in V_1'$, then $\psi(y)$ in $N(\psi(A))$ for some A containing X. This means that $y \in N(A)$, so (x, y) is an edge in Γ.

$\square$

In fact, we have the following stronger statement:

Proposition 38. *Given a set V and a function $\boldsymbol{N} : \mathscr{P}(V) \to \mathbb{N}$, we can construct a two-layered graph Γ such that V is the set of vertices in the second layer, and $|N(A)| = \boldsymbol{N}(A)$ for any $A \subseteq V$. This graph is unique up to isomorphism.*

Proof. Let $V_2 = V$. For each $A \subseteq V_2$, define a set $n(A)$ such that $|n(A)| = \mathbf{N}(A)$. Let $V_1 = \bigcup n(A)$, and for $v \in V_2$, $w \in n(A)$, let $v \gtrdot w$ if and only if $v \in A$. Set $w \gtrdot *$ if and only if $w \in V_1$. Then $n(A) = N(A)$ for all $A \subseteq V_2$, so $|N(A)| = \mathbf{N}(A)$. Uniqueness follows from Proposition 37. $\square$

This means that a two-layered graph Γ can be given by the ordered pair (V, E), where V is the collection of vertices and E is the collection of edges, or by the ordered pair $(V_2, \mathbf{N})$, where V_2 is the collection of top-level vertices, and $\mathbf{N} : \mathscr{P}(V_2) \to \mathbb{N}$.

6.2 Blueprints For Two-Layered Graphs

Propositions 15 and 37 give us the following corollary:

Corollary 39. *Let Γ and Γ' be two-layered graphs with unique minimal vertices $*$ and $*'$. If there exists a bijection $\phi : V_2 \to V_2'$ such that $k_A = k_{\phi(A)}$ and $k_A^A = k_{\phi(A)}^{\phi(A)}$ for all $A \subseteq V_2$, then $\Gamma \cong \Gamma'$.*

Recall that k_A^A represents the number of equivalence classes of $S(A)$ under the relation $\sim_A$. Equivalently, we could consider the graph $G = (V_2, E_\sim)$, where

$$E_\sim = \{\{v, v'\} : S(v) \cap S(v') \neq \emptyset\}$$

Then k_A^A is the number of components of the induced subgraph on A. Clearly, the k_A^A will be entirely determined by the values of $k_{\{x,y\}}^{\{x,y\}}$ for $x \neq y \in V_n$.

Thus complete information about a two-layered graph Γ can be given by the ordered triple $(V_2, E_\sim, \mathbf{k})$, where $\mathbf{k} : \mathscr{P}(V_2) \to \mathbb{N}$ is the function given by $\mathbf{k}(A) = k_A$.

Definition 19. We call an arbitrary triple of the form $(V, , E_\sim, \mathbf{k})$ with V a set, $E_\sim$ a collection of two-element subsets of V, and $\mathbf{k} : \mathscr{P}(V_2) \to \mathbb{N}$ a **blueprint**.

Definition 20. Given such blueprint $(V, E_\sim, \Bbbk)$, and a subset $A \subset V$, we define the following:

i) $(k_\beta)_A = \mathbf{k}(A)$

ii) $(k_\beta)_A^A$ is the number of connected components of the subgraph of $(V, E_\sim)$ induced by A.

iii) $\mathbf{S}_\beta(A) = (k_\beta)_\emptyset - (k_\beta)_A + (k_\beta)_A^A$

iv) $\mathbf{I}_\beta(A) = \sum_{B \subseteq A} (-1)^{|B|} \left((k_\beta)_B - (k_\beta)_B^B \right)$

v) $\mathbf{N}_\beta(A) = \sum_{B \supseteq \overline{A}} (-1)^{|B| - |\overline{A}|} \left((k_\beta)_B - (k_\beta)_B^B \right)$

When there is no chance of ambiguity, we will drop the β in the subscript.

Definition 21. We say that a blueprint $\beta = (V, E_\sim, \mathbf{k})$ is **graph-inducing** if $\mathbf{N}(A) \geq 0$ for all $A \subseteq V$, since we can construct a two-layered graph by taking $(V, \mathbf{N})$. We call this the graph **induced by** β.

Definition 22. We call a blueprint $\beta = (V, E_\sim, \mathbf{k})$ **valid** if there exists a two-layered graph $\Gamma = (V_0 \cup V_1 \cup V_2, E)$ and a bijection $\phi : V_2 \to V$ such that $\mathbf{k}(\phi(A)) = k_A$ for all $A \subseteq V_2$, and $\{\phi(x), \phi(y)\} \in E_\sim$ if and only if $S(x) \cap S(y) \neq \emptyset$.

In this case, Γ is isomorphic to the graph induced by β, and we say that β validly induces Γ.

Proposition 40. *Let Γ be a layered graph, validly induced by a blueprint $\beta = (V, E_\sim, \mathbf{k})$. If we identify the the second layer of vertices in Γ with V, we have*

i) $(k_\beta)_A = k_A$

ii) $(k_\beta)_A^A = k_A^A$

iii) $\boldsymbol{S}_\beta(A) = |S(A)|$

iv) For nonempty A, $\boldsymbol{I}_\beta(A) = |I(A)|$

v) $\boldsymbol{N}(A) = |N(A)|$

Proof. All of this is clear from the definitions. □

Valid blueprints must satisfy certain properties. Clearly if β is valid, then it is also graph-inducing, and the values of the five functions described above must all be positive. In addition, we have the following:

Proposition 41. *If $\beta = (V, E_\sim, \boldsymbol{k})$ is valid, then $\boldsymbol{k}(A) \geq \boldsymbol{k}(B)$ whenever $A \subseteq B$.*

Proof. Let Γ be the graph induced by β. Then $\mathbf{k}(A) = k_A$, and $\mathbf{k}(B) = k_B$. It is clear from the definition of k_A and k_B that whenever $A \subseteq B$, $k_A \geq k_B$. □

Proposition 42. *Let β be a valid blueprint. If*

$$(k_\beta)_\emptyset - (k_\beta)_{\{x\}} - (k_\beta)_{\{y\}} + (k_\beta)_{\{x,y\}} > 0,$$

then $(k_\beta)_{\{x,y\}}^{\{x,y\}} = 1$.

Proof. Since β is valid, $(k_\beta)_{\{x,y\}}^{\{x,y\}} = 1$ if and only if $\mathbf{I}(\{x,y\}) \neq \emptyset$. We have

$$\begin{aligned}
\mathbf{I}(\{x,y\}) &= \Big((k_\beta)_\emptyset - (k_\beta)_\emptyset^\emptyset\Big) - \Big((k_\beta)_{\{x\}} - (k_\beta)_{\{x\}}^{\{x\}}\Big) \\
&\quad - \Big((k_\beta)_{\{y\}} - (k_\beta)_{\{y\}}^{\{y\}}\Big) + \Big((k_\beta)_{\{x,y\}} - (k_\beta)_{\{x,y\}}^{\{x,y\}}\Big) \\
&= (k_\beta)_\emptyset - \big((k_\beta)_{\{x\}} - 1\big) - \big((k_\beta)_{\{y\}} - 1\big) + \Big((k_\beta)_{\{x,y\}} - (k_\beta)_{\{x,y\}}^{\{x,y\}}\Big) \\
&= \big((k_\beta)_\emptyset - (k_\beta)_{\{x\}} - (k_\beta)_{\{y\}} + (k_\beta)_{\{x,y\}}\big) + 2 - (k_\beta)_{\{x,y\}}^{\{x,y\}}
\end{aligned}$$

If $(k_\beta)_\emptyset - (k_\beta)_{\{x\}} - (k_\beta)_{\{y\}} + (k_\beta)_{\{x,y\}} > 0$, then

$$\mathbf{I}(\{x,y\}) \geq 3 - (k_\beta)_{\{x,y\}}^{\{x,y\}},$$

and since $(k_\beta)_{\{x,y\}}^{\{x,y\}} \in \{1,2\}$, this means

$$\mathbf{I}(\{x,y\}) \geq 1,$$

and so $(k_\beta)_{\{x,y\}}^{\{x,y\}} = 1$. □

6.3 Removing Edges from Blueprints

If two graphs Γ and Γ' have the same blueprint, then they are isomorphic, and so clearly $B(\Gamma) = B(\Gamma')$. We also know that given just the algebra $B(\Gamma)$ of a particular graph Γ, we can find an upper vertex-like basis L for V_2, so we have a bijection $\phi : V_2 \to L$ with $\mathbf{k}(A) = k_{\phi(A)}$ for any $A \subseteq V_2$. This means that the only piece of the blueprint for Γ that is not necessarily recoverable from $B(\Gamma)$ is $E_\sim$.

Definition 23. If $\beta = (V, E_\sim, \mathbf{k})$ is a blueprint and $x, y \in V$, with $x \neq y$, we define

$$\beta \setminus \{x,y\} = (V, E_\sim \setminus \{x,y\}, \mathbf{k})$$

We ask the question: If $\beta = (V, E_\sim, \mathbf{k})$ is a valid blueprint and $\{x,y\} \in E_\sim$, under what circumstances is $\beta \setminus \{x,y\}$ a valid blueprint? And how are the algebras of the corresponding graphs related?

We can identify some situations in which $\beta \setminus \{x,y\}$ cannot be a valid blueprint. A fairly simple one is the following:

Proposition 43. *If $\beta = \{V, E_\sim, \boldsymbol{k}\}$ is a valid blueprint with $\boldsymbol{I}_\beta(\{x,y\}) \geq 2$ for some $x, y \in V$, then $\beta \setminus \{x,y\}$ is not a valid blueprint.*

Proof. We know that

$$\begin{aligned}
\mathbf{I}_{\beta\setminus\{x,y\}}(\{x,y\}) &= \left((k_\beta)_\emptyset - (k_\beta)_\emptyset^\emptyset\right) - \left((k_\beta)_{\{x\}} - (k_\beta)_{\{x\}}^{\{x\}}\right) \\
&\quad - \left((k_\beta)_{\{y\}} - (k_\beta)_{\{y\}}^{\{y\}}\right) + \left((k_\beta)_{\{x,y\}} - (k_\beta)_{\{x,y\}}^{\{x,y\}}\right) \\
&= (k_\beta)_\emptyset - \left((k_\beta)_{\{x\}} - 1\right) - \left((k_\beta)_{\{y\}} - 1\right) + \left((k_\beta)_{\{x,y\}} - 1\right),
\end{aligned}$$

and that

$$\begin{aligned}
\mathbf{I}_{\beta\setminus\{x,y\}}(\{x,y\}) &= \Big((k_{\beta\setminus\{x,y\}})_{\emptyset} - (k_{\beta\setminus\{x,y\}})^{\emptyset}_{\emptyset}\Big) - \Big((k_{\beta\setminus\{x,y\}})_{\{x\}} - (k_{\beta\setminus\{x,y\}})^{\{x\}}_{\{x\}}\Big) \\
&\quad - \Big((k_{\beta\setminus\{x,y\}})_{\{y\}} - (k_{\beta\setminus\{x,y\}})^{\{y\}}_{\{y\}}\Big) + \Big((k_{\beta\setminus\{x,y\}})_{\{x,y\}} - (k_{\beta\setminus\{x,y\}})^{\{x,y\}}_{\{x,y\}}\Big) \\
&= (k_\beta)_{\emptyset} - \big((k_\beta)_{\{x\}} - 1\big) - \big((k_\beta)_{\{y\}} - 1\big) + \big((k_\beta)_{\{x,y\}} - 2\big)
\end{aligned}$$

It follows that $I_{\beta\setminus\{x,y\}} > 0$. If $\beta \setminus \{x,y\}$ were valid, it would follow that $\{x,y\} \in E_\sim \setminus \{x,y\}$. Clearly this is not the case, and so we conclude that $\beta \setminus \{x,y\}$ is not a valid blueprint. □

Other situations in which $\beta \setminus \{x,y\}$ cannot be a valid blueprint involve cycles in the graph $(V, E_\sim)$. For instance, if this graph includes a triangle with vertices, x, y, and z, such that $I(x,y,z) = \emptyset$ in the graph induced by β, then $\beta \setminus \{x,y\}$ will not be valid.

Proposition 44. *Let $\beta = \{V, E_\sim, \boldsymbol{k}\}$ be a valid blueprint, and let $x,y,z \in V$. If $\binom{\{x,y,z\}}{2} \subseteq E_\sim$ and $\boldsymbol{I}(\{x,y,z\}) = \emptyset$, then $\beta \setminus \{x,y\}$ is not a valid blueprint.*

Proof. Let $\mathbf{I} = \mathbf{I}_\beta$, $k = k_\beta$, and $\mathbf{I}' = \mathbf{I}_{\beta\setminus\{x,y\}}$, $k' = k_{\beta\setminus\{x,y\}}$. Then

$$\begin{aligned}
\mathbf{I}'(\{x,y,z\}) - \mathbf{I}(\{x,y,z\}) &= \left(\sum_{B\subseteq\{x,y,z\}} (-1)^{|B|}\left((k')_B - (k')^B_B\right)\right) \\
&\quad - \left(\sum_{B\subseteq\{x,y,z\}} (-1)^{|B|}\left(k_B - k^B_B\right)\right)
\end{aligned}$$

We know that $\mathbf{I}(\{x,y,z\}) = 0$, and that $(k')_A = k_A$ for any $A \subseteq V$, so this gives us

$$\begin{aligned}
\mathbf{I}'(\{x,y,z\}) &= \sum_{B\subseteq\{x,y,z\}} (-1)^{|B|}\left(k^B_B - (k')^B_B\right) \\
&= \left(k^{\emptyset}_{\emptyset} - (k')^{\emptyset}_{\emptyset}\right) - \left(k^{\{x\}}_{\{x\}} - (k')^{\{x\}}_{\{x\}}\right) - \left(k^{\{y\}}_{\{y\}} - (k')^{\{y\}}_{\{y\}}\right) \\
&\quad - \left(k^{\{z\}}_{\{z\}} - (k')^{\{z\}}_{\{z\}}\right) + \left(k^{\{x,y\}}_{\{x,y\}} - (k')^{\{x,y\}}_{\{x,y\}}\right) \\
&\quad + \left(k^{\{x,z\}}_{\{x,z\}} - (k')^{\{x,z\}}_{\{x,z\}}\right) + \left(k^{\{y,z\}}_{\{y,z\}} - (k')^{\{y,z\}}_{\{y,z\}}\right) \\
&\quad - \left(k^{\{x,y,z\}}_{\{x,y,z\}} - (k')^{\{x,y,z\}}_{\{x,y,z\}}\right)
\end{aligned}$$

All of these values are easy to calculate, and we find that

$$\mathbf{I}(\{x, y, z\}) = -1$$

If $\beta \setminus \{x, y\}$ were a valid blueprint, then all values of $\mathbf{I}'$ would be non-negative, so we can conclude that $\beta \setminus \{x, y\}$ is not valid. $\square$

We can also show that if $(V, E_\sim)$ contains as an induced subraph a cycle of length greater than 3, the removal of any of the edges of the cycle will result in an invalid blueprint.

Proposition 45. *Let $\beta = \{V, E_\sim, \boldsymbol{k}\}$ be a valid blueprint. Suppose there exist x, y, and $x_1, \ldots x_n$ with $n \geq 2$ such that the collection of edges in the subgraph of $(V, E_\sim)$ induced by $\{x, x_1, \ldots, x_n, y\}$ is*

$$\{\{x, y\}, \{x, x_1\}, \{x_1, x_2\}, \ldots, \{x_{n-1}, x_n\}, \{x_n, y\}\}$$

Then $\beta \setminus \{x, y\}$ is not a valid blueprint.

To show this it will be useful to have the following:

Lemma 46. *If $\beta = (V, E_\sim, \boldsymbol{k})$ is a valid blueprint, and $A \subseteq V$, then*

$$\beta|_A = \left(A, E_\sim \cap \binom{A}{2}, \boldsymbol{k}|_{\mathscr{P}(A)}\right)$$

is also valid.

Proof. Let $\beta = (V_2, E_\sim, \mathbf{k})$, and let $\Gamma = (V_0 \cup V_1 \cup V_2, E)$ be a graph induced by β. For $A \subseteq V_2$, define Γ' to be the subgraph of Γ induced by the set of vertices $V_0 \cup V_1 \cup A$. We claim that $\beta|_A$ induces Γ'.

Let $\phi : A \to A$ be the identity map. Clearly for any $B \subseteq A$ we have

$$k_B = \mathbf{k}(B) = \mathbf{k}|_A(B),$$

and for any $x \neq y \in A$,

$$\{\phi(x), \phi(y)\} = \{x, y\} \in E_\sim \cap \binom{A}{2}$$

if and only if $\{x, y\} \in E_\sim$, which is true if and only if $S(x) \cap S(y) \neq \emptyset$. Thus $\beta|_A$ is a valid blueprint, inducing Γ'. $\square$

Now we can prove Proposition 45.

Proof of Proposition 45. Let $\beta = \{V, E_\sim, \mathbf{k}\}$ be a valid blueprint. By Lemma 46, we may assume that

$$V = \{x, x_1, \ldots, x_n, y\}$$

and that

$$E_\sim = \{\{x, y\}, \{x, x_1\}, \{x_1, x_2\}, \ldots, \{x_{n-1}, x_n\}, \{x_n, y\}\}$$

Let $\mathbf{N} = \mathbf{N}_\beta$, $k = k_\beta$, $\mathbf{N}' = \mathbf{N}_{\beta \setminus \{x,y\}}$, and $k' = k_{\beta \setminus \{x,y\}}$. We have

$$\mathbf{N}(V) - \mathbf{N}'(V) = \sum_{B \subseteq V} (-1)^{|B|} \left((k')^B_B - k^B_B \right)$$

We know that $\mathbf{N}(V) = 0$, and that

$$(k')^B_B - k^B_B = \begin{cases} 1 & \text{if } \{x, y\} \subseteq B \neq V \\ 0 & \text{else} \end{cases}$$

Thus we have

$$\begin{aligned} -\mathbf{N}'(V) &= \sum_{\{x,y\} \subseteq B \neq V} (-1)^{|B|} \\ &= \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} \\ &= (-1)^{n+1}, \end{aligned}$$

so

$$\mathbf{N}'(V) = (-1)^n$$

If n is odd, then $\beta \setminus \{x, y\}$, is not graph inducing, and so clearly cannot be valid. If n is even, then the graph induced by $\beta \setminus \{x, y\}$ must have a vertex in $N(V)$. This implies that $S(x) \cap S(x_2) \neq \emptyset$, but we have $\{x, x_2\} \notin (E_\sim \setminus \{x, y\})$. Thus $\beta \setminus \{x, y\}$ cannot be valid. $\square$

Suppose β and $\beta \setminus \{x, y\}$ are both valid blueprints. What can we say about the graphs that they induce? What about the algebras associated to those graphs?

Proposition 47. *Suppose $\beta = (V, E_\sim, \boldsymbol{k})$, and $\beta \backslash \{x,y\}$ are both valid blueprints. Then there exists exactly one set $A \subseteq (V \setminus \{x,y\})$ such that*

$$N_\beta(\{x,y\} \cup A) = 1$$

For all other $B \subseteq (V \setminus \{x,y\})$, we have

$$N_\beta(\{x,y\} \cup B) = 0$$

Furthermore, for any $B \in V$, we have

$$\boldsymbol{N}_{\beta\backslash\{x,y\}}(B) = \begin{cases} \boldsymbol{N}_\beta(B) - 1 & \text{if } B = (\{x,y\} \cup A) \text{ or } B = A \\ \boldsymbol{N}_\beta(B) + 1 & \text{if } B = (\{x\} \cup A) \text{ or } B = (\{y\} \cup A) \\ \boldsymbol{N}_\beta(B) & \text{else} \end{cases}$$

Proof. For ease of notation, we will let $k = k_\beta$, and $k' = k_{\beta\backslash\{x,y\}}$ for the duration of this proof.

We will begin by establishing the existence and uniqueness of the set A. Let $\Gamma = (\{*\} \cup V_1 \cup V, E)$ be the graph induced by β. Since $\{x,y\} \in E_\sim$, there exists some vertex $w \in V_1$ such that $x \gtrdot w$ and $y \gtrdot w$. If we take

$$A = \{v \in V : v \gtrdot w\},$$

then we have $\mathbf{N}_\beta(A) \geq 1$.

If there were another set $B \subseteq V \setminus \{x,y\}$

$$\mathbf{N}_\beta(\{x,y\} \cup B) \geq 1,$$

or if we had any set $A \subseteq (V \setminus \{x,y\})$ with

$$\mathbf{N}_\beta(\{x,y\} \cup A) > 1,$$

then there would exist at least two vertices w and w' in V_1 which were covered by both x and y. Thus we would get $\mathbf{I}_\beta(\{x,y\}) \geq 2$. However by Proposition 43, this would imply that $\beta \setminus \{x,y\}$ is not a valid blueprint. It follows that $\mathbf{N}_\beta(A) = 1$, and that for any other $B \subseteq (V \setminus \{x,y\})$, $\mathbf{N}_\beta(B) = 0$.

It remains to show that the values of $\mathbf{N}_{\beta\setminus\{x,y\}}$ listed above are correct. Clearly, $k_B^B = (k')_B^B$ unless $\{x,y\} \subseteq B$. Since $N(A \cup \{x,y\}) = 1$, we know that there exists some $w \in V_1$ such that $v \gtrdot w$ for every $v \in A$. It follows that the subgraph of $(V, E_\sim)$ induced by A is the complete graph on A. This means that if $\{x,y\} \subseteq B$ and there exists $z \in A \cap B$, then $k_B^B = (k')_B^B$.

We claim that if $\{x,y\} \subseteq B$ and $A \cap B = \emptyset$, then $(k')_B^B = k_B^B + 1$. To prove this, we need to show that x and y are in two different connected components of $\left(B, (E_\sim \setminus \{\{x,y\}\}) \cap \binom{B}{2}\right)$, the subgraph of $(V, E_\sim \setminus \{\{x,y\}\})$ induced by B. Clearly there is no edge from x to y. If there exists $z \in B$ such that $\{x,z\}$ and $\{z,y\}$ are both in $E_\sim$, then the graph induced by β has a triangle on the vertices x, y, and z. Since $z \notin A$, then the vertex shared by x and y is not shared by z, so $I(\{x,y,z\}) = 0$. But this would imply that $\beta \setminus \{x,y\}$ is not valid by Proposition 44.

Thus if there is a path from x to y in $\left(B, E_\sim \cap \binom{B}{2}\right)$, it must have length greater than two. Let $(x, x_1, \ldots, x_n, y)$ be a minimal such path. Then the subgraph of $(V, E_\sim)$ induced by these vertices is a cycle of length greater than three. This again would imply that $\beta \setminus \{x,y\}$ is not valid, by Proposition 45.

Thus it must be the case that if $\{x,y\} \subseteq B$ and $A \cap B = \emptyset$, then x and y are in two different connected components of $\left(B, E_\sim \cap \binom{B}{2}\right)$, and thus $(k')_B^B = k_B^B + 1$.

We have

$$\mathbf{N}_\beta(C) - \mathbf{N}_{\beta\setminus\{x,y\}}(C) = \sum_{B \supseteq \overline{C}} (-1)^{|B| - |\overline{C}|} \left((k')_B^B - k_B^B\right),$$

and we have established that

$$(k')_B^B - k_B^B = \begin{cases} 1 & \text{if } \{x,y\} \subseteq B \text{ and } A \cap B = \emptyset \\ 0 & \text{else} \end{cases}$$

It follows that

$$\mathbf{N}_\beta(C) - \mathbf{N}_{\beta\setminus\{x,y\}}(C) = \sum_{\substack{B \supseteq \left(\overline{C} \cup \{x,y\}\right) \\ B \cap A = \emptyset}} (-1)^{|B| - |\overline{C}|}$$

If $A \not\subseteq C$, then the set $\{B : \overline{C} \subseteq B \text{ and } B \cap A = \emptyset\} = \emptyset$, so

$$\mathbf{N}_\beta(C) - \mathbf{N}_{\beta\setminus\{x,y\}}(C) = 0$$

If $A \subseteq C$ and $C \cap \overline{(A \cup \{x,y\})} \neq \emptyset$, then

$$\mathbf{N}_\beta(C) - \mathbf{N}_{\beta\setminus\{x,y\}}(C) = \sum_{i=0}^{|\overline{A}|-|\overline{C}\setminus\{x,y\}|} (-1)^i \binom{|\overline{A}| - |\overline{C} \setminus \{x,y\}|}{i} = 0,$$

since $|\overline{A}| - |\overline{C} \setminus \{x,y\}| > 0$.

Thus we have proved our result for all but the four exceptional cases, all of which are easy:

$$\mathbf{N}_\beta(A \cup \{x,y\}) - \mathbf{N}_{\beta\setminus\{x,y\}}(A \cup \{x,y\}) = \sum_{\substack{B \supseteq \overline{A} \\ B \cap A = \emptyset}} (-1)^{|B|-|\overline{A}+2|} = 1$$

$$\mathbf{N}_\beta(A \cup \{x\}) - \mathbf{N}_{\beta\setminus\{x,y\}}(A \cup \{x\}) = \sum_{\substack{B \supseteq \overline{A} \\ B \cap A = \emptyset}} (-1)^{|B|-|\overline{A}+1|} = -1$$

$$\mathbf{N}_\beta(A \cup \{y\}) - \mathbf{N}_{\beta\setminus\{x,y\}}(A \cup \{y\}) = \sum_{\substack{B \supseteq \overline{A} \\ B \cap A = \emptyset}} (-1)^{|B|-|\overline{A}+1|} = -1$$

$$\mathbf{N}_\beta(A) - \mathbf{N}_{\beta\setminus\{x,y\}}(A) = \sum_{\substack{B \supseteq \overline{A} \\ B \cap A = \emptyset}} (-1)^{|B|-|\overline{A}|} = 1$$

□

Finally we can state exactly when, for a given valid blueprint β, $\beta \setminus \{x,y\}$ is also valid:

Theorem 48. *Let $\beta = \{V, E_\sim, \boldsymbol{k})$ be a valid blueprint, and let $\{x,y\} \in E_\sim$. Then $\beta \setminus \{x,y\}$ is a valid blueprint if and only if*

i. Whenever $\binom{\{x,y,z\}}{2} \subseteq E_\sim$, $\boldsymbol{I}(\{x,y,z\}) > 0$.

ii. For any $n > 1$ distinct $x_1, \ldots, x_n$, not equal to x or y, the subgraph of $(V, E_\sim)$ induced by $x, y, x_1, \ldots, x_n$ is not a cycle.

iii. There exists a unique $A \subseteq (V \setminus \{x,y\})$ such that $\boldsymbol{N}_\beta(A \cup \{x,y\}) = 1$. For all other sets $B \subseteq (V \setminus \{x,y\})$, we have $\boldsymbol{N}_\beta(A \cup \{x,y\}) = 0$.

iv. $\boldsymbol{N}_\beta(A) > 0$.

Proof. We have already shown that all of these conditions are necessary. We would like to show that they are sufficient. Let $\beta = (V_2, E_\sim, \mathbf{k})$ be a blueprint satisfying all of these conditions. Let $\Gamma = (V_0 \cup V_1 \cup V_2, E)$ be the graph induced by β. By condition iii, there exists a unique vertex $w \in V_1$ such that $w \lessdot x$ and $w \lessdot y$. Then

$$A = \{v \in V_2 \mid v \gtrdot w\} \setminus \{x, y\}$$

By condition iv, there exists at least one vertex $w' \in N(A)$. We delete the edge (x, w) and replace it with the edge (x, w') to obtain a new graph $\Gamma' = (V_0 \cup V_1 \cup V_2, E')$. We wish to show that $\beta \setminus \{x, y\}$ is valid, inducing Γ'.

We will need to distinguish the k, N and S functions associated to Γ from those associated to Γ', so we will use k', N', and S' for the latter.

Our bijection from V_2 to V_2 will be the identity. We need to show that $\mathbf{k}(B) = (k')_B$ for all $B \subseteq V_2$, and that $\{a, b\} \in (E_\sim \setminus \{x, y\})$ if and only if $S'(a) \cap S'(b) \neq \emptyset$.

We know that $\mathbf{k}(B) = k_B$. Thus we need to show that $(k')_B = k_B$ for all $B \in V_2$.

Case 1: If $x \notin B$, then clearly $(k')_B = k_B$.

Case 2: If $x \in B$ and there exists $z \in B \cap A$, then we have $z \gtrdot w$ and $z \gtrdot w'$, so $S'(B) = S(B)$. If $y \in B$, then there exists a path from y to x through z, so $(k')^B_B = k^B_B$. If y is not in B, then $(k')^B_B = k^B_B$ trivially. In either case, we have

$$|S(B)| - |S'(B)| = k_\emptyset - k_B + k^B_B - (k')_\emptyset + (k')_B - (k')^B_B$$

and so

$$0 = (k')_B - k_B$$

Case 3: If $x \in B$, $B \cap A = \emptyset$, and $y \notin B$, then no element of B other that x covers w or w'. Thus $S'(B) = (S(B) \setminus \{w\}) \cup \{w\}$, and $|S'(B)| = |S(B)|$. Again $(k')^B_B = k^B_B$ trivially, so we get $(k')_B = k_B$ as above.

Case 4: If $x \in B$, $B \cap A = \emptyset$, and $y \in B$, then in Γ, both x and y cover w, and no element of B covers w'. In Γ', only y covers w, and now x covers w', so $S'(B) = S(B) \cup \{w'\}$, and $|S'(B)| = |S(B)| + 1$. Since $B \cap A = \emptyset$, we know that x and y are in different connected components of the subgraph of $(V_2, E_\sim \setminus \{x, y\})$ induced by B. Thus $(k')^B_B = k^B_B + 1$, and we get $(k')_B = k_B$, as above.

So we have established that $(k')_B = k_B$ for all $B \subseteq V_2$. Now we must show that $\{a,b\} \in (E_\sim \setminus \{x,y\})$ if and only if $S'(a) \cap S'(b) \neq \emptyset$. This is clearly true unless one of the vertices is x, and we have a situation where $S(x) \cap S(t) = \{w\}$ for some vertex $t \neq y$. However, if this is the case, then $t \in A$, and so $t \gtrdot w'$, so $S'(x) \cap S'(t) = w'$. This gives us our result.

□

6.4 Nonisomorphic Graphs With Isomorphic Algebras

Theorem 49. *Let $\beta = (V_2, E_\sim, \boldsymbol{k})$ be a valid blueprint such that $\beta \setminus \{x,y\}$ is also valid. Let $\Gamma = (V_0 \cup V_1 \cup V_2, E)$ be the graph induced by β, and let Γ' be the graph induced by $\beta \setminus \{x,y\}$. Then $B(\Gamma) \cong B(\Gamma')$.*

Proof. According to Theorem 48, we may assume without loss of generality that $\Gamma' = (V_0 \cup V_1 \cup V_2, E')$, where $E' = E \setminus (x,w) \cup (x,w')$, where w and w' are defined as above.

We will need to distinguish between κ_v when v is being considered as a vertex of Γ, and when it is being used as a vertex of Γ', so we will use $\kappa(v)$ for the former and $\kappa'(v)$ for the latter. Similarly, we define S to be the S-function associated to Γ, and S' to be the S-function associated to Γ'. We will find it useful to use the following notation: $S_x = S(x) \setminus w$, and $S_y = S(y) \setminus w$.

We define a set

$$\mathscr{T} = \big(\{u \in V_1 : u \in S(v) \text{ for some } v \text{ with } S(v) \cap S(x) \neq \emptyset\} \setminus \{w, w'\}\big)$$

We define an isomorphism $\phi : T(V_+) \to T(V_+)$ such that ϕ fixes $V_1 \setminus \{w, w'\}$, and

$$\phi(w) = w - \sum_{u \in \mathscr{T}} u$$

$$\phi(w') = w + \sum_{u \in \mathscr{T}} u$$

We wish to show that ϕ induces an isomorphism $\psi : B(\Gamma) \to B(\Gamma')$. This will be the case if $\phi(\kappa(v)) = \kappa'(v)$ for all $v \in V_2$.

We have

$$\kappa(x) = \mathrm{span}\left(\overline{S(x)} \cup \left\{\sum_{u \in S(x)} u\right\}\right),$$

so

$$\phi(\kappa(x)) = \mathrm{span}\left(\left(\overline{S(x)} \setminus \{w'\}\right) \cup \left\{w' + \sum_{u \in \mathscr{T}} u\right\} \cup \left\{\sum_{u \in S(x)} u - \sum_{u \in \mathscr{T}} u\right\}\right)$$

Since $S_x \subseteq \mathscr{T}$, this gives us

$$\phi(\kappa(x)) = \mathrm{span}\left(\left(\overline{S(x)} \setminus \{w'\}\right) \cup \left\{w' + \sum_{u \in \mathscr{T}} u\right\} \cup \left\{w - \sum_{u \in (\mathscr{T} \setminus S_x} u\right\}\right),$$

and since $(\mathscr{T} \setminus S_x) \subseteq \left(\overline{S(x)} \setminus \{w'\}\right)$, this means that

$$\begin{aligned}
\phi(\kappa(x)) &= \mathrm{span}\left(\left(\overline{S(x)} \setminus \{w'\}\right) \cup \left\{w' + \sum_{u \in S_x} u\right\} \cup \{w\}\right) \\
&= \mathrm{span}\left(\overline{S'(x)} \cup \left\{\sum_{u \in S'(x)} u\right\}\right) \\
&= \kappa'(x)
\end{aligned}$$

Thus the condition holds for x. We also have

$$\kappa(y) = \mathrm{span}\left(\overline{S(y)} \cup \left\{\sum_{u \in S(y)} u\right\}\right)$$

Since $w' \in N(A)$, we know that $w' \notin S(y)$. Therefore,

$$\phi(\kappa(y)) = \mathrm{span}\left(\left(\overline{S(y)} \setminus \{w'\}\right) \cup \left\{w' + \sum_{u \in \mathscr{T}} u\right\} \cup \left\{\sum_{u \in S(y)} u - \sum_{u \in \mathscr{T}} u\right\}\right)$$

Notice that $S(y) \cap \mathscr{T} = \emptyset$. Thus $\mathscr{T} \subset \left(\overline{S(y)} \setminus \{w'\}\right)$, and so

$$\begin{aligned}
\phi(\kappa(y)) &= \mathrm{span}\left(\left(\overline{S(y)} \setminus \{w'\}\right) \cup \{w'\} \cup \left\{\sum_{u \in S(y)} u\right\}\right) \\
&= \mathrm{span}\left(\overline{S(y)} \cup \left\{\sum_{u \in S(y)} u\right\}\right) \\
&= \kappa'(y)
\end{aligned}$$

Thus the condition holds for y. Now take an arbitrary $v \in (V_2 \setminus \{x, y\})$.

Case 1: $S(v) \cap S(x) = \emptyset$. In this case $S(v) \cap \mathscr{T} = \emptyset$, and $\{w, w'\} \subseteq \overline{S(v)}$, so

$$\begin{aligned}
\phi(\kappa(v)) &= \operatorname{span}\left(\left(\overline{S(v)} \setminus \{w, w'\}\right) \cup \left\{w - \sum_{u \in \mathscr{T}} u\right\}\right. \\
&\qquad \left. \left\{w' + \sum_{u \in \mathscr{T}} u\right\} \cup \left\{\sum_{u \in S(v)} u\right\}\right) \\
&= \operatorname{span}\left(\left(\overline{S(v)} \setminus \{w, w'\}\right) \cup \{w, w'\} \cup \left\{\sum_{u \in S(v)} u\right\}\right) \\
&= \operatorname{span}\left(\overline{S(v)} \cup \left\{\sum_{u \in S(v)} u\right\}\right) \\
&= \kappa'(v)
\end{aligned}$$

Case 2: $v \in A$. In this case, both w and $w\prime$ are in $S(v)$, so the $\sum_{u \in \mathscr{T}} u$ terms cancel easily, and $\phi(\kappa(v)) = \kappa'(v)$ with no additional work.

Case 3: $v \notin A$, and $S(v) \cap S(x) \neq \emptyset$. In this case, we have $S(v) \subseteq \mathscr{T}$, so

$$\begin{aligned}
\phi(\kappa(v)) &= \operatorname{span}\left(\left(\overline{S(v)} \setminus \{w, w'\}\right) \cup \left\{w - \sum_{u \in \mathscr{T}} u\right\}\right. \\
&\qquad \left. \left\{w' + \sum_{u \in \mathscr{T}} u\right\} \cup \left\{\sum_{u \in S(v)} u\right\}\right) \\
&= \operatorname{span}\left(\left(\overline{S(v)} \setminus \{w, w'\}\right) \cup \left\{w - \sum_{u \in (\mathscr{T} \setminus S(v))} u\right\}\right. \\
&\qquad \left. \left\{w' + \sum_{u \in (\mathscr{T} \setminus S(v))} u\right\} \cup \left\{\sum_{u \in S(v)} u\right\}\right),
\end{aligned}$$

And since $(\mathscr{T} \setminus S(v)) \subseteq \left(\overline{S(v)} \setminus \{w, w'\}\right)$, this gives us

$$\begin{aligned}
\phi(\kappa(v)) &= \operatorname{span}\left(\left(\overline{S(v)} \setminus \{w, w'\}\right) \cup \{w, w'\} \cup \left\{\sum_{u \in S(v)} u\right\}\right) \\
&= \operatorname{span}\left(\overline{S(v)} \cup \left\{\sum_{u \in S(v)} u\right\}\right) \\
&= \kappa'(v)
\end{aligned}$$

□

Chapter 7

Hilbert Series and Graded Trace Generating Functions of Direct Products of Posets

As we stated before, the algebra $A(\Gamma)$ has a natural graded structure inherited from $T(E)$. In this chapter, we will explore the Hilbert series and graded trace generating functions of $A(\Gamma)$ for several important classes of layered graphs. Some background on incidence algebras and generalized layered graphs will be useful for this discussion.

7.1 Incidence Algebras and the Möbius Function

Let P be a locally-finite poset. That is to say, for any $p, q \in P$, the set $\{r : p \leq r \leq q\}$ is finite. Fix a field F. The **incidence algebra** $I(P)$ of P is the set of functions

$$f : P \times P \to F$$

satisfying $f(p, q) = 0$ whenever $p \not\leq q$ in P. Addition is given by

$$(f + g)(p, q) = f(p, q) + g(p, q),$$

and multiplication is given by

$$(fg)(p, q) = \sum_{p \leq r \leq q} f(p, r)g(r, q).$$

If we define V_P to be a vector space over F with basis P, then each $f \in I(P)$ extends uniquely to a linear map

$$f' : V_P \otimes_F V_P \to F$$

given by setting $f'(p \otimes q) = f(p, q)$ for any $p, q \in P$. Thus it is easy to see that $I(P)$ is isomorphic to the algebra of linear maps

$$f' : V_P \otimes_F V_P \to F$$

satisfying $f(p \otimes q) = 0$ whenever $p \not\leq q$, with addition given by

$$(f' + g')(p \otimes q) = f'(p \otimes q) + g'(p \otimes q),$$

and multiplication map defined by

$$(f'g')(p \otimes q) = \sum_{r \in P} f(p \otimes r)g(r \otimes q).$$

Notice that since $\{r : p \leq r \leq q\}$ is finite, this will always be a finite sum. For our purposes, we will identify $I(P)$ with this algebra of linear maps.

The multiplicative identity in this algebra is the function δ given by

$$\delta(p \otimes q) = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{else} \end{cases}$$

One combinatorially important function in $I(P)$ is the **zeta function** ζ, given by

$$\zeta(p \otimes q) = \begin{cases} 1 & \text{if } p \leq q \\ 0 & \text{else} \end{cases}$$

This is invertible in $I(P)$. Its inverse is called the **Möbius function**, and denoted μ. It can be constructed recursively by taking

$$\mu(p \otimes p) = 1$$

for all $p \in P$, and

$$\mu(p \otimes q) = -\sum_{p \leq r < q} \mu(r \otimes q).$$

In this way, we obtain

$$(\mu\zeta)(p \otimes p) = \zeta(p \otimes p)\mu(p \otimes p) = 1 = \delta(p \otimes p),$$

and

$$(\mu\zeta)(p \otimes q) = \sum_{p \leq r \leq q} \mu(p \otimes r)\zeta(r \otimes q) = \mu(p \otimes q) + \sum_{p \leq r < q} \mu(r \otimes q) = 0 = \delta(p \otimes r).$$

Notice that this implies that $\mu(p \otimes q)$ depends entirely on the interval $[p, q]$, and can be calculated without full information about the poset P.

Example: If $P = 2^{[n]}$, the Boolean algebra, we have

$$\mu(p \otimes q) = \begin{cases} (-1)^{|q|-|p|} & \text{if } p \leq p \\ 0 & \text{else} \end{cases}$$

Proof. Clearly if $p \not\leq q$, then $\mu(p \otimes q) = 0$. If $p \leq q$, then the interval $[p, q]$ is isomorphic to $2^{[|q|-|p|]}$. So it will suffice to show that for the lattice $2^{[n]}$, we have

$$\mu(\emptyset \otimes [n]) = (-1)^n.$$

This is clearly the case for $n = 0$, since $\mu(\emptyset \otimes \emptyset) = 1$. Using this as a base case, we can induct on n. If the result holds for all smaller numbers, then we have

$$\begin{aligned} \mu(\emptyset \otimes [n]) &= -\sum_{S \subsetneq [n]} \mu(\emptyset \otimes S) \\ &= -\sum_{i=0}^{n-1} (-1)^i \binom{n}{i} \\ &= (-1)^n \end{aligned}$$

This gives us our result, and our proof is complete. □

We wish to consider a generalization of the incidence algebra. For a locally-finite ranked poset P and a vector space V_P with basis P, we define $I_z(P)$ to be the set of functions

$$f : V_P \otimes_F V_P \to F[z]$$

satisfying $f(p \otimes q) \in Fz^{|q|-|p|}$ for $p, q \in P$ with $p \leq q$ in P, and $f(p \otimes q) = 0$ for $p, q \in P$ with $p \not\leq q$. $I_z(P)$ is an algebra with addition given by

$$(f + g)(p \otimes q) = f(p \otimes q) + g(p \otimes q)$$

and

$$(fg)(p \otimes q) = \sum_{r \in P} f(p \otimes r) g(r \otimes q).$$

This algebra has identity element δ_z, given by

$$\delta_z(p \otimes q) = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{else} \end{cases}$$

It also has a generalized zeta function given by

$$\zeta_z(p \otimes q) = \begin{cases} z^{|q|-|p|} & \text{if } p = q \\ 0 & \text{else} \end{cases}$$

This generalized matrix is invertible in $I_z(P)$, with inverse μ_z given by

$$\mu_z(p \otimes q) = \mu(p \otimes q) z^{|q|-|p|}$$

Notice that $I_1(P)$ is the usual incidence algebra, with $\delta_1 = \delta$, $\zeta_1 = \zeta$, and $\mu_1 = \mu$. We will use the notation δ_z^P, ζ_z^P, and μ_z^P when we wish to make the specific poset P explicit.

For our purposes, P will be finite. In this case, each element $f \in I_z(P)$ corresponds to a unique matrix $M(f) = [f_{pq}]$ with rows and columns indexed by elements of P, and with $f_{pq} = f(p \otimes q)$. Most of our concrete calculations will be done in this context, because it reduces the calculation of μ_z from ζ_z to matrix inversion. In this context, for $v, w \in V_P$, we have

$$f(v \otimes w) = v^T M(f) w.$$

In particular, $f(\mathbb{1} \otimes \mathbb{1})$ is the sum of the entries of $M(f)$.

7.2 Generalized Layered Graphs

In our discussion of graded trace generating functions, it will be useful to be able to refer to the notion of a generalized layered graph. A more thorough development of these ideas can be found in **??**.

Definition 24. A **generalized layered graph** is a directed graph $\Gamma = (V, E)$ such that $V = V_0 \cup V_2 \cup \ldots \cup V_n$, and such that for every $e \in E$, $|t(e)| > |h(e)|$. We define $l(e) = |t(e)| - |h(e)|$ to be the **length** of the edge e.

An algebra $A(\Gamma)$ can be constructed, generalizing the notion of the universal labeling algebra to generalized layered graphs. The construction is presented in detail in in **??**. To each edge e, we associate $l(e)$ generators $a_1(e), a_2(e), \ldots, a_{l(e)}(e)$. We take

$$E^{\#} = \{a_i(e) : e \in E, 1 \leq i \leq l(e)\},$$

and associate a polynomial $P_e \in T(E^{\#})[t]$ to each $e \in E$, given by

$$P_e(t) = 1 + \sum_{i=1}^{l(e)} (-1)^i a_i(e) t^i.$$

We form an ideal of relations R, generated by the relations obtained by taking any two paths $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_m)$ with $t(e_1) = t(f_1)$ and $h(e_n) = h(f_m)$ and setting

$$P_{e_1} P_{e_2} \ldots P_{e_n} = P_{f_1} P_{f_2} \ldots P_{f_m}$$

$A(\Gamma)$ is defined to be $T(E^{\#})/R$. Notice that if Γ is a layered graph, this is entirely equivalent to the construction described in Chapter **??**.

7.3 Hilbert Series and Graded Trace Functions of $A(\Gamma)$

Recall that given a graded algebra $A = \bigoplus A_i$, the Hilbert series is the polynomial

$$H(A, z) = \sum_i \dim(A_i) z^i,$$

the generating function for the dimension of the graded pieces. A linear basis for $A(P)$ is calculated in [4], and in [7], it is used to calculate the Hilbert series for $A(P)$. The result is given by

$$H(A(\Gamma), z) = \frac{1 - z}{1 - z\mu_z^P(\mathbb{1} \otimes \mathbb{1})},$$

where $\mathbb{1} = \sum_{p \in P} p \in V_P$. Note that since P is finite, this is a finite sum. In **??**, we see that this formula holds for generalized layered graphs as well.

In [1], Duffy shows that this generalization can be used to calculate the graded trace generating functions of certain automorphisms of $A(P)$. Any automorphism σ of a graded algebra $A = \bigoplus A_i$ will act on each graded piece of A as a vector space automorphism. The graded trace generating function of σ acting on A is

$$Tr_\sigma(A, z) = \sum_i Tr(\sigma|_{A_i}) z^i$$

If $\sigma' : P \to P$ is an automorphism of the poset P, then σ' will induce an automorphism σ of the algebra $A(\Gamma)$. Such automorphisms fix the linear basis defined by Retakh, Serconek, and Wilson in [4], and this means that the trace of σ on each of the graded pieces will be equal to the number of fixed basis elements in that particular piece. If we define P^σ to be the subposet of P containing only the elements of P that are fixed by σ', then

$$Tr_\sigma(A(P), z) = \frac{1 - z}{1 - z\mu_z^{P^\sigma}(\mathbb{1} \otimes \mathbb{1})},$$

the Hilbert series of the algebra $A(P^\sigma)$.

7.4 Hilbert Series of Direct Products of Posets

If P and Q are ranked posets, then their direct product $P \times Q$ is the ranked poset with order given by

$$(p, q) \geq_{P \times Q} (p', q')$$

if and only if $p \leq_P p'$ and $q \leq_Q q'$, and rank function given by

$$|(p, q)|_{P \times Q} = |p|_P + |q|_Q.$$

The Hilbert series of the universal labeling algebras of direct products of posets take on a particularly nice form:

Theorem 50. *Let P and Q be finite ranked posets. Then*

$$H(A(P \times Q), z) = \frac{1 - z}{1 - z\left(\mu_z^P\left(\mathbb{1}_P \otimes \mathbb{1}_P\right)\right)\left(\mu_z^Q\left(\mathbb{1}_Q \otimes \mathbb{1}_Q\right)\right)}$$

The vector space $V_{P\times Q}$ is naturally isomorphic to $V_P \otimes V_Q$, and under this isomorphism, $\mathbb{1}_{P\times Q}$ maps to $\mathbb{1}_P \otimes \mathbb{1}_Q$.

$I_z(P) \otimes I_z(Q)$ is the collection of module maps

$$\phi : V_P \otimes V_P \otimes V_Q \otimes V_Q \to F[z]$$

satisfying

$$\phi(p \otimes p' \otimes q \otimes q') \in F z^{|p'|-|p|+|q'|-|q|}$$

whenever $p \leq p'$ and $q \leq q'$, and

$$\phi(p \otimes p' \otimes q \otimes q') = 0$$

otherwise. This corresponds to the set of maps

$$\phi' : (V_P \otimes V_Q) \otimes (V_P \otimes V_Q) \to F[z]$$

satisfying

$$\phi'((p \otimes q) \otimes (p' \otimes q')) \in F z^{|(p',q')|-|(p,q)|}$$

whenever $p \leq p'$ and $q \leq q'$, and

$$\phi'((p \otimes q) \otimes (p' \otimes q')) = 0$$

otherwise. In this form, it is clear that $I_z(P) \otimes I_z(Q) \cong I_z(P \times Q)$. Under this isomorphism, $\zeta_z^{P\times Q}$ corresponds to $\zeta_z^P \otimes \zeta_z^Q$, and $\mu_z^{P\times Q}$ corresponds to $\mu_z^P \otimes \mu_z^Q$. Thus if we identify $(V_P \otimes V_Q) \otimes (V_P \otimes V_Q)$ with $(V_P \otimes V_P) \otimes (V_Q \otimes V_Q)$, we find that

$$\mu_z^{P\times Q}\left(\mathbb{1}_{P\times Q} \otimes \mathbb{1}_{P\times Q}\right) = \left(\mu_z^P\left(\mathbb{1}^P \otimes \mathbb{1}^P\right)\right)\left(\mu_z^Q\left(\mathbb{1}^Q \otimes \mathbb{1}^Q\right)\right)$$

Thus we can conclude that

$$H(A(P \times Q), z) = \frac{1-z}{1 - z\left(\mu_z^P\left(\mathbb{1}_P \otimes \mathbb{1}_P\right)\right)\left(\mu_z^Q\left(\mathbb{1}_Q \otimes \mathbb{1}_Q\right)\right)}$$

7.5 Graded Trace Functions and Direct Products

Let P be the direct product of n copies of Q, and let σ be the automorphism of P which cyclically permutes the copies of Q in P. Then $(q_1, q_2, \ldots, q_n) \in P$ is fixed by σ if and only if $q_1 = q_2 = \ldots = q_n$. Thus P^σ is isomorphic to Q as a poset, but with the ranking of each element multiplied by n. For ease of notation, we will call this poset ${}^{\times n}Q$. We have

$$\zeta_z^{\times n Q} = \zeta_{(z^n)}^{Q}$$

In general, if $P = Q^n$, there is a natural action of S_n on P, permuting the copies of Q^n. Let $\sigma \in S_n$ with cycle decomposition $\sigma_1 \ldots \sigma_r$, and let i_j be the length of the cycle σ_j. Then $P^\sigma \cong \left({}^{\times i_1}Q\right) \times \ldots \times \left({}^{\times i_r}Q\right)$. In this case,

$$Tr_\sigma(A(P), z) = \frac{1-z}{1 - z \prod_{j=1}^{r} \mu^{Q}_{\left(z^{i_j}\right)} \left(\mathbb{1}_Q \otimes \mathbb{1}_Q\right)}$$

More generally, suppose $P = \prod_{k=1}^{n} Q_k$. Let σ be any automorphism of P which permutes the copies of isomorphic Q_k. Again, we can break σ into cycles $\sigma_1 \ldots \sigma_r$, with i_j the length of the cycle σ_j. Then

$$Tr_\sigma(A(P), z) = \frac{1-z}{1 - z \prod_{j=1}^{r} \mu^{(Q_j)}_{\left(z^{i_j}\right)} \left(\mathbb{1}_{Q_j} \otimes \mathbb{1}_{Q_j}\right)}$$

7.6 Example 1: The Boolean Algebra

Let $P = \{x, y\}$, with $x > y$. The Boolean algebra 2^n is a product of n copies of P. We have

$$M(\zeta_z^P) = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}.$$

It follows that

$$M(\mu_z^P) = \begin{bmatrix} 1 & -z \\ 0 & 1 \end{bmatrix},$$

and thus

$$\mu_z^P(\mathbb{1} \otimes \mathbb{1}) = 2 - z,$$

and so the Hilbert series of 2^n is given by

$$H(A(P), z) = \frac{1 - z}{1 - z(2 - z)^n}$$

If σ is an automorphism of P permuting the n intervals, with cycle decomposition $\sigma_1 \dots \sigma_r$, with each cycle σ_j of length i_j, then

$$Tr_\sigma(A(P), z) = \frac{1 - z}{1 - z \prod_{j=1}^{r} \left(2 - z^{i_j}\right)}$$

7.7 Example 2: Factors of n

If $n = p_1^{s_1} p_2^{s_2} \dots p_k^{s_k}$ for k distinct primes $p_1, \dots, p_k$, then the poset of factors of n, ranked by number of prime factors, can be decomposed as a product of k chains of lengths $s_1, s_2, \dots, s_k$.

If $P = \{x_1, \dots, x_r\}$, with $x_i \leq x_j$ if and only if $i \leq j$, and $|x_i| = i$, then

$$M(\zeta_z^P) = \begin{bmatrix} 1 & t & t^2 & \cdots & t^r \\ 0 & 1 & t & \cdots & t^{r-1} \\ 0 & 0 & 1 & \cdots & t^{r-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

and so

$$M(\mu_z^P) = \begin{bmatrix} 1 & -t & 0 & \cdots & 0 \\ 0 & 1 & -t & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -t \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

It follows that

$$\mu_z^P(\mathbb{1}_P \otimes \mathbb{1}_P) = s + 1 - sz,$$

and so the Hilbert series of the poset of factors of n is given by

$$H(A(P),z) = \frac{1-z}{1-z\prod_{i=1}^{k}(s_i+1-s_iz)}$$

If σ is an automorphism of P permuting the chains of the same length, with cycle decomposition $\sigma_1\ldots\sigma_r$ with each cycle σ_j of length i_j permuting cycles with length r_j, then

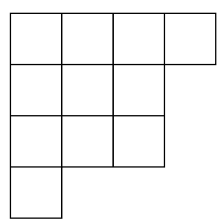
$$Tr_\sigma(A(P),z) = \frac{1-z}{\prod_{j=1}^{r}\left(s_j+1-s_jz^{i_j}\right)}$$

Chapter 8

Some Results for Young Lattices

Definition 25. The **Young lattice**, denoted Y, consists of all nonincreasing integer sequences $(\lambda_1, \lambda_2, \lambda_3, \ldots)$ with only finitely many nonzero entries. Given $\overline{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ and $\overline{\gamma} = (\gamma_1, \gamma_2, \gamma_3, \ldots)$ in Y, we define $\overline{\lambda} \geq \overline{\gamma}$ if and only if $\lambda_i \geq \gamma_i$ for all i.

It is often convenient to visualize such the elements of Y using Young diagrams. The Young diagram of an element $\overline{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ of Y consists of a collection of boxes arranged into rows, with λ_i boxes in the i'th row. For example, the Young diagram of the sequence $(4, 3, 3, 1, 0, \ldots)$ is



Given $\overline{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ and $\overline{\gamma} = (\gamma_1, \gamma_2, \gamma_3, \ldots)$, we have $\overline{\lambda} \geq \overline{\gamma}$ if and only if the Young diagram for $\overline{\gamma}$ sits inside the Young diagram for $\overline{\lambda}$. In this context it is easy to see that Y is a lattice. The join of two elements $\overline{\lambda} \vee \overline{\gamma}$ is the union of the two Young diagrams, and the meet $\overline{\lambda} \wedge \overline{\gamma}$ is the intersection.

It is also clear that Y is a ranked poset. Given $\overline{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \ldots)$, we have $|\overline{\lambda}| = \sum_{i=1}^{\infty} \lambda_i$. The zero sequence is the unique minimal element of rank zero.

We cannot calculate Hilbert series and graded trace functions for the whole poset Y. Since Y is an infinite poset, the number of generators of $A(\Gamma)$ is also infinite, which means that the graded pieces will not have finite dimension. This means that in order to calculate Hilbert series and graded trace functions, some reasonable finite sublattice of Y must be considered.

In this chapter, we will focus on the sublattice $Y_{m\times n} \subseteq Y$ consisting of all elements of Y whose Young diagrams have at most m rows, with at most n blocks in each row. That is, $Y_{m\times n}$ consists of sequences $\overline{\lambda} = (\lambda_i)_{i=1}^{\infty}$ such that $\lambda_i = 0$ for all $i > m$, and $\lambda_i \leq n$ for all i.

This sublattice arises naturally in algebraic geometry. It is isomorphic to the lattice of Schubert varieties in the Grassmannian $G(m,n)$, ordered by containment.

8.1 Calculating μ on the Young Lattice

In this section, we will use the interpretation of μ as a function

$$\mu : P \times P \to F,$$

defined recursively by

$$\mu\left(\overline{\lambda}, \overline{\lambda}\right) = 1,$$

and

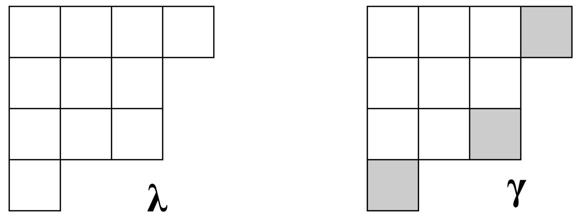
$$\mu\left(\overline{\lambda}, \overline{\gamma}\right) = -\sum_{\overline{\lambda} < \overline{\xi} \leq \overline{\gamma}} \mu\left(\overline{\xi}, \overline{\gamma}\right).$$

Definition 26. We define the relation $\preceq$ on Y such that for $\overline{\lambda} = (\lambda_1, \lambda_2, \ldots)$ and $\overline{\gamma} = (\gamma_1, \gamma_2, \ldots)$, we have $\overline{\gamma} \preceq \overline{\lambda}$ if

(i) $\overline{\lambda} \leq \overline{\gamma}$

(ii) Whenever $\lambda_i > \gamma_i$, we have $\lambda_i = \gamma_i + 1 > \lambda_{i+1}$

Notice that the above definition essentially states that $\overline{\gamma} \preceq \overline{\lambda}$ if and only if the Young diagram for $\overline{\gamma}$ can be obtained from the Young diagram of $\overline{\lambda}$ by cutting off some subset of its corners. So in the figure below, if the diagram on the left is the Young diagram of $\overline{\lambda}$, then the Young diagram of any $\overline{\gamma}$ satisfying these conditions is going to look like the diagram on the right, with some subset of the shaded squares missing.



Proposition 51. *Let $\overline{\lambda}, \overline{\gamma} \in Y$. If $\overline{\gamma} \preceq \overline{\lambda}$, then $\mu(\overline{\gamma}, \overline{\lambda}) = (-1)^{|\overline{\lambda}|-|\overline{\gamma}|}$. Otherwise, $\mu(\overline{\gamma}, \overline{\lambda}) = 0$.*

Proof. Suppose $\overline{\gamma} \preceq \overline{\lambda}$. Then $\overline{\gamma}$ can be obtained from $\overline{\lambda}$ by removing some subset of the free corners of its Young diagram. The interval $[\overline{\gamma}, \overline{\lambda}]$ consists of all possible intermediate steps. This will be isomorphic to the Boolean lattice $2^{[|\overline{\lambda}|-|\overline{\gamma}|]}$, and thus $\mu(\overline{\gamma}, \overline{\lambda}) = (-1)^{|\overline{\lambda}|-|\overline{\gamma}|}$.

Also notice that the collection of $\overline{\lambda}$ with $\overline{\lambda} \succeq \overline{\gamma}$ is precisely the interval $\left[\overline{\gamma}, \bigvee_{\overline{\delta} \gtrdot \overline{\gamma}} \overline{\delta}\right]$.

In order to obtain a contradiction, we will assume that $\overline{\lambda}$ is an element of Y with minimal rank such that

i) $\overline{\lambda} \geq \overline{\gamma}$

ii) $\overline{\lambda} \not\succeq \overline{\gamma}$

iii) $\mu(\overline{\gamma}, \overline{\lambda}) \neq 0$

Let $\overline{\zeta} = \bigvee_{\overline{\xi} \in D} \overline{\xi}$, where

$$D = \left\{ \overline{\xi} \in Y : \overline{\xi} \in \left[\overline{\gamma}, \bigvee_{\overline{\delta} \gtrdot \overline{\gamma}} \overline{\delta}\right] \text{ and } \overline{\xi} < \overline{\lambda} \right\}$$

It is not difficult to see that $\overline{\zeta} \in D$. Thus $D = [\overline{\gamma}, \overline{\zeta}]$. We know that

$$\mu(\overline{\gamma}, \overline{\lambda}) = -\sum_{\overline{\gamma} \leq \overline{\delta} < \overline{\lambda}} \mu(\overline{\gamma}, \overline{\delta})$$

Minimality of λ tells us that

$$\mu(\overline{\gamma}, \overline{\lambda}) = -\sum_{\overline{\xi} \in D} \mu(\overline{\gamma}, \overline{\xi}),$$

and since D is isomorphic to a Boolean lattice, it follows that $\mu(\overline{\gamma}, \overline{\lambda}) = 0$, giving us our contradiction. $\square$

8.2 Hilbert Series

Now consider the sublattice $Y_{m\times n} \subseteq Y$ consisting of all elements of Y whose Young diagrams have at most m rows, with at most n blocks in each row. That is, $Y_{m\times n}$ consists of sequences $\overline{\lambda} = (\lambda_i)_{i=1}^{\infty}$ such that $\lambda_i = 0$ for all $i > m$, and $\lambda_i \leq n$ for all i.

Fact 1. The Hilbert series $h(z)$ for $A(Y_{m\times n})$ is given by

$$\frac{1-z}{1-\sum_{k\geq 0}\binom{m}{k}\binom{n}{k}(1-z)^k}$$

Proof. Recall that for any poset P, the Hilbert series for $A(P)$ is given by

$$h(z) = \frac{1-z}{1-z\mu_z^P(\mathbb{1}\otimes\mathbb{1})},$$

with μ_z^P given by

$$\mu_z^P(p\otimes q) = \mu(p,q)z^{|q|-|p|}.$$

It follows that

$$\mu_z^P(\mathbb{1}\otimes\mathbb{1}), = \sum_{p,q}\mu(p,q)z^{|q|-|p|}$$

Given our analysis of the behavior of μ on the poset Y, the Hilbert series for $A(Y_{m\times n})$ will be given by

$$\frac{1-z}{1-\sum_i(-1)^iX_iz^i},$$

where X_i is the number of intervals $[\overline{\gamma},\overline{\lambda}]$ in Y with $|\overline{\lambda}|-|\overline{\gamma}| = i$ and $\mu(\overline{\gamma},\overline{\lambda}) \neq 0$. Thus we are interested in calculating these X_i.

First, we notice that any element of Y can be defined uniquely by the placement of its corners. We use the notation (x, y) to indicate the x'th row and the y'th column. If the corners of $\overline{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ are at $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$, for $a_1 < a_2 < \ldots < a_k$, then

$$\lambda_i = \begin{cases} a_1 & \text{if } i \leq a_1 \\ b_j & \text{if } a_{j-1} < i \leq a_j \\ 0 & \text{if } i > a_k \end{cases}$$

Notice that the ascending sequence of a's must be paired with a corresponding descending sequencd of b's, so in fact the Young diagram can be defined entirely by a set of

k rows and k columns in which the corners appear. Thus the number of diagrams in $Y_{m\times n}$ with k corners is $\binom{m}{k}\binom{n}{k}$.

For any $\lambda \in Y$, the collection of $\overline{\gamma}$ such that $\mu(\overline{\gamma}, \overline{\lambda})$ is nonzero is precisely the collection of $\overline{\gamma}$ that can be obtained by removing some subset of the corners of $\overline{\gamma}$. The collection of $\overline{\gamma}$ such that, in addition, $|\overline{\lambda}| - |\overline{\gamma}| = i$ is the collection of $\overline{\gamma}$ that can be obtained by removing a subset of the corners of size exactly i. Thus if $\overline{\lambda}$ has k corners, the number of $\overline{\gamma}$ such that $|\overline{\lambda}| - |\overline{\gamma}| = i$ and $\mu(\overline{\gamma}, \overline{\lambda}) \neq 0$ is precisely $\binom{k}{i}$.

Thus we have $X_i = \sum_{k\geq i}\binom{m}{k}\binom{n}{k}\binom{k}{i}$, and so

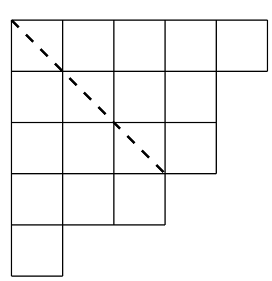
$$\begin{aligned}
\sum_i (-1)^i X_i z^i &= \sum_i (-1)^i \left(\sum_{k\geq i}\binom{m}{k}\binom{n}{k}\binom{k}{i}\right) z^i \\
&= \sum_k \binom{m}{k}\binom{n}{k}\sum_{i\leq k}(-1)^i\binom{k}{i} z^i \\
&= \sum_k \binom{m}{k}\binom{n}{k}(1-z)^k
\end{aligned}$$

This completes our proof. □

8.3 Graded Trace of Young Lattices

The Young lattice Y, and the lattices $Y_{n\times n}$ defined in the previous section have a single automorphism σ which takes each partition $\overline{\lambda} = (\lambda_1, \lambda_2, \ldots)$ to its conjugate partition $\overline{\lambda'} = (\lambda'_1, \lambda'_2, \ldots)$, where $\lambda'_i = |\{j : \lambda_j \geq i\}|$. The collection of partitions which are fixed under σ is exactly the collection of partitions whose young diagrams are symmetric across the diagonal

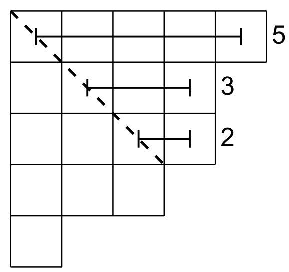
Example: The partition $(5, 4, 4, 3, 1)$ is symmetric across the diagonal, and thus will be fixed by the automorphism σ.



Proposition 52. *Let $Y_{n\times n}$ be as defined above, and let σ be the isomorphism taking each partition to its conjugate. Then*

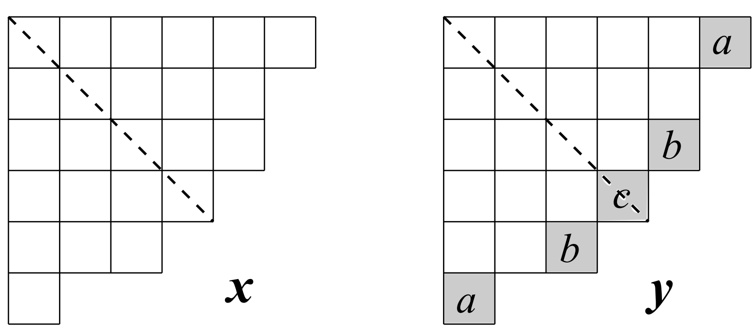
$$Tr_\sigma(A(Y_{n\times n}), z) = \frac{1-z}{1 - z\left(\sum_k (1-z^2)^k \left(\binom{n}{2k} + (1-z)\binom{n}{2k-1}\right)\right)}$$

In order to prove this, we will need to explore the Möbius function of $Y^\sigma_{n\times n}$, the lattice of symmetric Young diagrams. We can put these partitions in one-to-one correspondence with strictly decreasing sequences $\mathbf{x} = \langle x_1, x_2, x_3, \ldots\rangle$, where x_i is the number of blocks in the i'th column and beyond in the i'th row of the Young diagram. So for instance, the partition $(5, 4, 4, 3, 1)$ corresponds to the sequence $\langle 5, 3, 2\rangle$



If we consider the poset $Y^\sigma_{n\times n}$, the lattice of partitions corresponding to symmetric Young diagrams with at most n rows and columns, we find again that the intervals $(\mathbf{y}, \mathbf{x})$ such that $\mu(y, x) \neq 0$ correspond to a partition $\mathbf{y}$ whose Young diagram is obtained from the Young diagram of $\mathbf{x}$ by removing some collection of the corners of x, this time removing symmetric pairs together.

So if $\mathbf{x}$ is the diagram on the left, then $\mu(\mathbf{y}, \mathbf{x}) \neq 0$ if $\mathbf{y}$ is one of the partitions obtained by deleting some subset of the pairs labeled with a and b and the singleton labeled c.



More formally,

Proposition 53. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be partitions with symmetric young diagrams represented by the strictly decreasing sequences* $\langle x_1, x_2, \ldots \rangle$ *and* $\langle y_1, y_2, \ldots, \rangle$ *as described above. Suppose* $\mathbf{x}$ *and* $\mathbf{y}$ *satisfy*

i) $y_i \leq x_i$ *for all* i*.*

ii) If $y_i < x_i$*, then* $y_i = x_i - 1$*.*

iii) If $y_i < x_i$*, then either* $y_{i+1} = 0$ *or* $y_{i+1} = y_i - 1$*.*

Then $\mu(\mathbf{y}, \mathbf{x}) = |\{i : y_i < x_i\}|$*. Otherwise,* $\mu(\mathbf{y}, \mathbf{x}) = 0$*.*

Proof. Just as before, if $\mathbf{x}$ and $\mathbf{y}$ satisfy conditions i-iii, then the interval $[\mathbf{x}, \mathbf{y}]$ is isomorphic to a Boolean lattice, and it is easy to verify that $\mu(\mathbf{y}, \mathbf{x}) = |\{i : y_i < x_i\}|$.

Given a particular $\mathbf{y}$, the collection of $\mathbf{x}$ such that $\mathbf{x}$ and $\mathbf{y}$ satisfy i-iii is precisely the interval $\left[\mathbf{y}, \left(\bigvee_{\mathbf{v} \gtrdot \mathbf{y}} \mathbf{v}\right)\right]$. Suppose $\mathbf{x}$ is an element which is minimal with respect to the following three conditions:

a) $\mathbf{x} \geq \mathbf{y}$

b) $\mathbf{x}$ and $\mathbf{y}$ do not satisfy i-iii

c) $\mu(\mathbf{y}, \mathbf{x}) \neq 0$.

Let $\mathbf{z} = \bigvee_{\mathbf{w} \in D} \mathbf{w}$, where

$$D = \left\{ \mathbf{w} \in Y^{\sigma} : \mathbf{w} \in \left[\mathbf{y}, \left(\bigvee_{\mathbf{v} \gtrdot \mathbf{y}} \mathbf{v}\right)\right] \text{ and } \mathbf{w} < \mathbf{x} \right\}$$

We find that $D = [\mathbf{y}, \mathbf{z}]$. Minimality of $\mathbf{x}$ tells us that $\mu(\mathbf{y}, \mathbf{x}) = -\sum_{\mathbf{w} \in D} \mu(\mathbf{y}, \mathbf{w})$, and since D is isomorphic to a Boolean lattice, this gives us $\mu(\mathbf{y}, \mathbf{x}) = 0$. This contradiction completes our proof. □

With this fact in place, we can prove Proposition 52

Proof of Proposition 52. We are interested in finding

$$\sum_{\mathbf{x} \leq \mathbf{y}} \mu(\mathbf{x}, \mathbf{y}) z^{|\mathbf{y}| - |\mathbf{x}|},$$

where the ranking is inherited from the original poset $Y_{n\times n}$.

Each symmetric Young diagram in $Y_{n\times n}$ can be represented by a strictly decreasing sequence of positive integers, where the first is less than or equal to n. We can also think of this simply as a subset S of n. Each gap in the sequence $1, 2, 3, \ldots, \max S$ is going to create a pair of symmetric corners. If $1 \notin S$, there will be one more pair of symmetric corners. If $1 \in S$ there will be an additional unmatched corner on the diagonal.

If we wish to count the number of diagrams with a certain collection of corners, we can think of encoding S as follows: We define a subset T_S by

$$T_S = \{i : i \in S\} \cup \{i : i-1 \in S, i \notin S\} \cup \{i : i-1 \notin S, i \in S\}$$

Any S yields a $T_S \subseteq [n+1]$, and any even-sized subset of $[n+1]$ has an interpretation as a unique corresponding S.

If $|T_S| = 2k$ and $1 \notin T_S$, then the Young diagram corresponding to S has k symmetric pairs of corners, and no corner along the diagonal. Thus there are $\binom{n}{2k}$ such diagrams. Each of these is the top of $\binom{k}{i}$ intervals of length $2i$, and no intervals of odd length. The Möbius function of each of these intervals is $(-1)^i$.

If $|T_S| = 2(k+1)$ and $1 \in T_S$, then the Young diagram corresponding to S has k symmetric pairs of corners and a single corner on the diagonal. There are $\binom{n}{2k-1}$ such diagrams. Each of these is the top of $\binom{k}{i}$ intervals of length $2i$, and $\binom{k}{i}$ intervals of length $2i+1$. The Möbius function of the intervals of even length is $(-1)^i$. The Möbius function of the intervals of odd length is $(-1)^{i+1}$.

It follows that

$$\begin{aligned}
\sum_{\mathbf{x}\leq\mathbf{y}} \mu(\mathbf{x},\mathbf{y}) z^{|\mathbf{y}|-|\mathbf{x}|} &= \sum_k \binom{n}{2k} \sum_i \binom{k}{i} (-1)^i z^{2i} \\
&\quad + \sum_k \binom{n}{2k-1} \sum_i \binom{k}{i} (-1)^i z^{2i} \\
&\quad + \sum_k \binom{n}{2k-1} \sum_i \binom{k}{i} (-1)^{i+1} z^{2i+1} \\
&= \sum_k (1-z^2)^k \left(\binom{n}{2k} + (1-z) \binom{n}{2k-1} \right)
\end{aligned}$$

The result follows. $\square$

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