OPTIMAL PROCUREMENT POLICY FOR

COST CONSCIOUS RETAILER

By SICONG HOU

A thesis submitted to the

Graduate School-New Brunswick

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree of

Master of Science

Graduate Program in

Operations Research

Written under the direction of

Professor Melike Baykal-Gürsoy

and approved by

________________________

________________________

________________________

New Brunswick, New Jersey

May, 2013
ABSTRACT OF THE THESIS

OPTIMAL PROCUREMENT POLICY FOR

COST CONSCIOUS RETAILER

By SICONG HOU

Thesis Director:

Professor Melike Baykal-Gürsoy

Manufacturers whose products primarily rely on expensive raw materials need to make well informed procurement decisions in order to reduce their production costs. Decision making is especially difficult when there is uncertainty about the raw material cost as well as the product demand. This research considers an optimum procurement and production planning problem under uncertainty. Basic raw material is a rare metal with a dynamically and stochastically changing price. The manufacturer has the dual sourcing option: buy from the spot market; or to sign long term contracts. Long term contracts can be signed only at the beginning of every year with a yearly duration. In addition, the contract price depends on the average spot price during some interval. We first consider a simplified version of the problem without long-term contract, and assume a base-stock procurement and inventory control policy. We model the problem as a Markov chain and present some numerical examples demonstrating the effect of various parameters on the optimum base-stock value. Then, we investigate a case with dual procurement option, and present our results.
ACKNOWLEDGEMENT

It is with immense gratitude that I acknowledge the support and help of my thesis advisor Dr. Melike Baykal-Gürsoy, who continually and convincingly conveyed a spirit of adventure in regard to research and scholarship, without her continuous guidance and persistent help this thesis would not have been possible.

I would like to express my appreciation to my committee member and my program director Dr. Endre Boros for his support and encouragement during the research.

I would also like to thank Dr. András Prékopa for serving on my thesis committee and giving me valuable suggestions.

In addition, I take the opportunity to thank Mr. Bora and Mr. Figueroa as well as all graduate students of Rutgers Center for Operations Research for their support and help.
DEDICATION

This thesis is dedicated to my parents who have given me the opportunity of an education from the best institutions and support throughout my life. They have always stood by me and gave me the strength to carry out my studies.
# Table of Contents

Abstract .......................................................................................................................................................... ii

Acknowledgements .................................................................................................................................. iii

Dedication .................................................................................................................................................... iv

List of Tables .............................................................................................................................................. vii

List of Figures ........................................................................................................................................... viii

1. Introduction ............................................................................................................................................ 1

2. Literature Review ........................................................................................................................................ 3

3. A simplified Markov Chain Model of the Inventory/Production Control of a Cost Conscious Retailer ......................................................................................................................... 7

   3.1 Assumptions and Notations ...................................................................................................... 7

   3.2 Model Description ......................................................................................................................... 9

   3.3 Optimization .................................................................................................................................. 17

   3.4 Numerical Results ........................................................................................................................ 19

4. A Case Study ......................................................................................................................................... 30

   4.1 Introduction ................................................................................................................................... 30

   4.2 Data Preparation and Forecasting ........................................................................................... 32
List of Tables

3.1. Price Inventory transition probability matrix ................................................................. 8

3.2. Price Inventory transition probability matrix for K=2 .................................................. 12

3.3. Price Inventory transition probability matrix for K=3 .................................................. 12

3.4. Price Inventory transition probability matrix for K=4 .................................................. 13

3.5. Price Inventory transition probability matrix for K=5 and K=6 ......................... 14

3.6. Price Inventory transition probability matrix for $K \geq 6$ ........................................ 15

3.7. Effect of $\pi_H$ and $Y_H$ on $K^*$ .................................................................................. 20

3.8. Effect of $\pi_H$ and 1-p on $K^*$ ................................................................................... 22

3.9. Effect of $\pi_H$ and h on $K^*$ ...................................................................................... 23

3.10: Effect of 1-p and $Y_H$ on $K^*$ ................................................................................ 25

3.11. Effect of h and $Y_H$ on $K^*$ .................................................................................. 26

3.12. Effect of h and 1-p on $K^*$ ...................................................................................... 28

3.13. Effect of $\pi_H, Y_H, h$ and p on $K^*$ ....................................................................... 29

4.1. Predicted price from SPSS “autoarima” ........................................................................ 45

4.2: Predicted demand from SPSS “autoarima” .................................................................. 45
## List of Figures

3.1. Visualization of the dependence of $K^*$ on $\pi_H$ and $Y_H$ .......................................................... 21

3.2. Visualization of the dependence of $K^*$ on $\pi_H$ and $1-p$ ......................................................... 22

3.3. Visualization of the dependence of $K^*$ on $\pi_H$ and $h$ ............................................................. 24

3.4. Visualization of the dependence of $K^*$ on $1-p$ and $Y_H$ ........................................................... 25

3.5. Visualization of the dependence of $K^*$ on $h$ and $Y_H$ .............................................................. 27

3.6. Visualization of the dependence of $K^*$ on $h$ and $1-p$ ............................................................... 28

4.1. Weekly historical data of spot prices ............................................................................................... 33

4.2. AC and PAC plots of original data ................................................................................................. 34

4.3. 1-differenced and mean subtracted data of spot prices ............................................................... 36

4.4. AC and PAC plots of 1-differenced and mean subtracted data ..................................................... 37

4.5. SPSS “autoarima” fit and forecast ................................................................................................. 40

4.6. Eviews ARMA(1,1) fit and forecast ............................................................................................... 41

4.7. Excel ARMA(3,1) fit and forecast ................................................................................................. 42

4.8. R GARCH fit and forecast .............................................................................................................. 43
Chapter 1

Introduction

Procurement problems, particularly direct procurement problems concerning mainly raw material and production goods, are becoming more and more prominent in supply chain management these days because they greatly affect the total expenditure of many companies. This is especially true when the raw material is an expensive rare metal. This research considers an optimum procurement and production planning problem under raw material price and demand uncertainty. The raw material used to produce an intermediate product is a rare metal with dynamically and stochastically changing prices. The demand for the intermediate product is also random but reasonably stationary.

The manufacturer has dual sourcing options; they can either buy from the spot market, or sign a long term forward contract. That is, dual sourcing is comprised of forward contract buying and spot market buying. Forward contract buying provides a long term stable supply at a fixed price for a fixed delivery time. While spot buying, although is more flexible with the delivery times and the order quantity, is more risky and uncertain. For instance, you might not able to purchase a large volume when the price is good because other companies already took the advantage first. According to Zhang et al. (2011), in recent years spot markets have emerged for a wide variety of commodities, and companies are starting to use it to
incorporate with the traditional forward contracts especially in dynamic random access memory procurements. The market has been a mix of private, bilateral contracts and spot market trading for years, where 80% of the intermediate products are sold through a fixed price long term contract.

For large manufacturers, especially in chemical and pharmaceutical industries whose products primarily rely on expensive raw materials, their production costs can be largely reduced if good strategies be adopted when making procurement decisions. However, these complex strategies of decision making are very difficult considering the demand uncertainty and randomness of the raw material costs. Most of the mathematical models fail to predict a jump or slump on the prices because sudden events that cause such a jump or slump to happen are impossible to predict sometimes. Relatively, the demand of products is also uncertain but may be easier to predict since we have a more stable requirement for the materials globally. Thus the strategy that is applied in order to compensate the possible failure of forecasting raw material price becomes very important for the dual sourcing scenario.

In Chapter 2, we present the literature survey. Chapter 3 considers a simplified model of the procedure for a problem without long term contract. The effect of various parameters on the optimum order quality is investigated. A case in which both long term contract and spot buying are available is studied in Chapter 4 with real world data and situations. Finally, in Chapter 5, conclusions and future research are discussed.
Chapter 2

Literature Review

Many optimization approaches based on periodic updates have been used in commodity procurement. Secomandi and Kekre (2009) consider the problem of reselling the commodity back to the market in the forward process. They suggest not only to update the forward price, but also to update the demand forecast. Thus the whole dynamic optimization can provide a better solution overall. Optimal cost value function is provided through a Markov decision process formulation.

For a typical periodic-review inventory control model with stochastic demand, Goel and Gutierrez (2012) characterized the optimal procurement policy and developed a computational algorithm to obtain the optimal policy thresholds. The model they developed has a total cost consists of procurement cost and a market-determined economic cost of holding inventory, and this economic cost can be significantly reduced according to the numerical results from this paper. Also they considered a dynamic program with a risk-neutral valuation approach for minimizing the total procurement cost.

In comparing the forward contract with the total order quantity commitment, the latter is more flexible according to the paper of Zhang, Chen, Hua and Xue (2011). Hence, it is easier to use it and take advantage of the dual sourcing. In this
paper they suppose a stochastic demand and the spot price can be stationary or non-stationary, but it is independent of the demand process.

Two of the papers focused on a dual sourcing scenario of periodic-review inventory model. First one by Inderfurth, Kelle and Kleber (2013) assume random stationary demand and a capacity reservation contract for the sourcing. In this modeling part, they used dynamic programming recursive equations. However, when formulating the models, this paper focuses more on the structure analysis, because of the special reservation capacity decision needed to be considered. The model can be extended to be optimal for infinite horizon, too. A heuristic approach is used for determining iterative parameters step by step. The second one by Seifert, Thonemann and Hausman (2004) mentioned that in these problems the demand and the spot price are usually positively correlated, and thus assume a bivariate distribution rather than a normal distribution. Their paper then deals with dynamic pricing and determines the contract-to-spot ratio for purchases, where in spot market we need to consider a hedging situation since it allows salvage. Based on the assumed distribution, we can adjust the multiplier of the deviation to set different policies for different level of risks you wish to take. Also, a spot price premium case was studied, as well as a pure contract sourcing case.

Other two papers focused on a multi-period model with a single product. Nagarajan and Rajagopalan (2009) first consider a comprehensive environment and the equilibrium order quantities have the special property such that each player ignores the strategy of his opponent under reasonable conditions on the cost
parameters. Secondly, they show that under certain conditions, the equilibrium quantities in the finite horizon game can be reduced to a multi-period single-product inventory model for each player. Then they approach to the equilibrium by proving the deterministic demand identifies a unique Nash equilibrium while stochastic demand derived from a decoupling result. Thus, the equilibrium strategies of two players is simply to solve a single-product in n-period stochastic inventory model with partial backlogging instead of lost sales, where the demand of non-substituting customers from the current period is back logged to the next period.

Hwang and Hahn (2000) studied a periodic review inventory model for a single perishable product and demand rate of the item is dependent on the current inventory level. The proof first shows some properties about the ordering quantity. Since the item have a fixed life time, two different kind of order up to values are introduced depending on the outdated items. They approach to solve the two different situations by using linear programming and then apply enumeration method which has a similar approach to dynamic programming.

The last optimal procurement policy paper we reviewed here by Polatoglu and Sahin (2000), studied a (s,S) type inventory control model under price-dependent demand relationship. The main difference in the model of this paper is that the price is a decision variable here and brings two opposing cost-related effects. The solution approach is also dynamic programming. However, an n-period pseudo-profit function is introduced instead. And finally, the problem is solved by
investigating the critical inventory level of \((s,S)\) under certain sufficient condition. Some special situations such as No-fixed-ordering-cost, Non-stationary extensions and deterministic demand are also considered in the end of this paper.
Chapter 3

A simplified Markov Chain Model of the Inventory/Production Control of a Cost Conscious Retailer

In this section our goal is to build a Markov chain model for base-stock type procurement policies, and then analyze the effects of various parameters on the optimum order quantity. Dual sourcing will not be considered in this chapter but will be discussed in detail in Chapter 4.

3.1 Assumptions and Notations

We begin modeling by making some necessary assumptions for the inventory/production control policy. The procurement policy is adopted depending on the price level and the inventory on hand. We assume the order of each month’s operations are as follows: Production facility consumes items from the inventory to meet demand, notice demand is always met here since we have the minimum inventory level that is equal to the maximum daily demand. Then we observe the current price level and the number of items in stock, in order to decide on how many items to purchase. Lastly, after we finish the purchasing stage, we check our new inventory level and set it as the next inventory level.
To simplify the problem, we assume only two price levels: $H$, denoting the state of high price level, and $L$, denoting the state of low price level. The unit cost for low price level is denoted as $Y_L$ and for high price level is denoted as $Y_H$.

Let $C(n)$ denote the raw material price at time $n$. Assume that the price transition probability matrix $P$ is given as follows:

\[
P = \begin{pmatrix}
    H & L \\
    H & 1-a & a \\
    L & b & 1-b \\
\end{pmatrix}
\]

Table 3.1: Price Inventory transition probability matrix

Here, $a$ denote the conditional probability that the price will be low at the next time period given that it is currently high, i.e., $a = P(C(n+1) = L / C(n) = H)$. $b$ is similarly defined as $b = P(C(n+1) = H / C(n) = L)$.

Also, we assume that there are historical data such that we could use to obtain information about the price distribution. Assume that the historical data indicates that in the long-run 20% of the time the price is at the high state and 80% of the time it is at the low state. Next, we set the expected cycle length to be 25 days which means in every 25 days we will have the above probability distribution for the high and low prices.
We assume the demand for raw material is distributed in a very simplified fashion; demand on day \( n \) is equal to 1 item with probability \( p \) and 2 items with probability \( 1-p \). That is, let \( D_n \) denote the demand quantity at time \( n \), we have:

\[
P\{D_n = 1\} = p, \quad P\{D_n = 2\} = 1 - p
\]

When the price is low, we purchase enough items to stock up to the maximum inventory of \( K \) items. When the price is high, we do not purchase unless the number of items we have on hand are less than the maximum daily demand. More specifically, if the price level is high we only purchase when the inventory level is below 2 and if so, we purchase to bring the inventory level back to the minimum of 2 items.

Lastly, we assume that the total cost of inventory/production management consists of the inventory holding cost and the procurement cost, where the holding cost is \( $h/\text{day} \) per item and the procurement cost is the total cost of all the purchases made.

### 3.2 Model Description

Now we are ready for the modeling of our inventory management problems. The objective of our model is to optimize the inventory/production policy we utilized, and therefore to minimize the expected total cost under such a policy. Later on we will see that the expected total cost is decided by the base-stock value,
which is $K$. However, we would like to obtain the actual values of $a$ and $b$ before we calculate the expected total cost and start the optimization.

Let $\pi_H$ and $\pi_L$ denote the stationary probability of price states H and L respectively. In Table 3.1 matrix we can obtain all the transition probabilities between the states by having the following relations:

$$\pi_H = 0.2 \text{ and } \pi_L = 0.8$$

$$a \geq 0, b \geq 0$$

$$\begin{bmatrix} \pi_H, \pi_L \end{bmatrix} = \begin{bmatrix} \pi_H, \pi_L \end{bmatrix} \ast \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

$$\Rightarrow a = 4b \ldots \ldots \ldots \ldots \ldots (1)$$

and $a \geq 0, b \geq 0$

Next, we use the fact that the cycle length is on the average of 25 days, where 20% of a cycle is at the high state, and 80% of a cycle is at the low state, in order to calculate the exact $a$ and $b$ values. Let the first day be day 0 and since a cycle is 25 days, next cycle starts at day 25. Consider the probability that the first time a high state goes to low state; it is exactly the probability for the number of consecutive days that a high state stay at high, before it goes to low. The corresponding expected values for the number of consecutive days at the high state before it goes to low is exactly equal to the total number of days at high state.

$$P \{X=n\} = P \{\text{the first time } H \text{ goes to } L \text{ at day } n\} = (1-a)^n \ast a$$
Let $X$ denotes the number of consecutive days at H before it goes to L in one cycle. Since we know that for a geometrically distributed random variable $X$, the expected value is:

$$ E(X) = \frac{1 - a}{a} $$

It means that the expected number of days at H is equal to $\frac{1-a}{a}$ as well. Similarly, we can obtain the expected number of days at L is equal to $\frac{1-b}{b}$. Now, we also know that the expected cycle length is 25 days, so:

$$ \frac{1-a}{a} + \frac{1-b}{b} = 25 \ldots \ldots \ldots \ldots \ldots (2) $$

We can then obtain $a$, $b$ values by combining equations (1) and (2) above, getting results for the price transition probability matrix:

$$ a = \frac{20}{108}, \, b = \frac{5}{108} $$

Now we can start to build the detailed model of the inventory control problem. We want to have this matrix because we can then use it to compute the expected total cost which is ultimately what we intend to optimize. Let $\{C(n)I(n)\} n \geq 0$ denote the two-dimensional state, where $C(n)$ is the price state and $I(n)$ is the inventory on hand at the beginning of nth time period. For example, $Hm$ denotes the state of high price level with inventory of $m$ items, and similarly $Lm$ denotes the state of low price level with inventory of $m$ items.
The matrix size of the probability transition matrix of \( \{C(n)I(n)\} n \geq 0 \) depends on the base-stock value \( K \). Thus the matrix will be defined as \( P_k \). This also implies that our main optimization of the procurement policy is based on value \( K \) as well.

First, let \( K = 2 \) which is the minimum case, we have the following matrix that is similar to Table 3.1:

\[
P_2 = \begin{pmatrix}
    & H2 & L2 \\
H2 & 1-a & a \\
L2 & b & 1-b
\end{pmatrix}
\]

Table 3.2: Price Inventory transition probability matrix for \( K = 2 \)

Next, we let \( K = 3 \) and obtain the Table 3.3 below. However, we notice that the states \( H3 \) and \( L2 \) are not accessible from any states, not even themselves. This means their values are zeros so we do not have to consider them at all. Thus we should simplify the matrix in a more compact fashion on the following:

\[
P_3 = \begin{pmatrix}
    & H2 & H3 & L2 & L3 \\
H2 & 1-a & 0 & 0 & a \\
H3 & 1-a & 0 & 0 & a \\
L2 & 0 & 0 & 0 & 0 \\
L3 & b & 0 & 0 & 1-b
\end{pmatrix}
\]
Table 3.3: Price Inventory transition probability matrix for K=3

Notice the above matrix is very similar to the Table 3.2 matrix the inventory levels. Then we look at Table 3.4 for K=4 below, in this case we have the probability to jump to either H2 or H3 from L4, because the demand consumptions are \( P\{D_n = 1\} = p \) and \( P\{D_n = 2\} = 1 - p \). We show only the simplified matrix after we deleted all the columns and rows of inaccessible states:

Table 3.4: Price Inventory transition probability matrix for K=4

Next, let us look at K=5 and K=6 on Table 3.5, where a pattern on the price inventory transition probability matrix can be found. Because when we reach a low state, we always purchase up to stock level K, therefore there will always be only
one low state that is accessible, the LK. In contrast, we will never be able to reach HK because no states could ever access HK.

\[
P_3 = \begin{array}{cccc}
& H2 & H3 & H4 & L5 \\
H2 & 1-a & 0 & 0 & a \\
H3 & 1-a & 0 & 0 & a \\
H4 & (1-a)*(1-p) & (1-a)*p & 0 & a \\
L5 & 0 & b*(1-p) & b*p & 1-b \\
\end{array}
\]

\[
P_6 = \begin{array}{cccccc}
& H2 & H3 & H4 & H5 & L6 \\
H2 & 1-a & 0 & 0 & 0 & a \\
H3 & 1-a & 0 & 0 & 0 & a \\
H4 & (1-a)*(1-p) & (1-a)*p & 0 & 0 & a \\
H5 & 0 & (1-a)*(1-p) & (1-a)*p & 0 & a \\
L6 & 0 & 0 & b*(1-p) & b*p & 1-b \\
\end{array}
\]

Table 3.5: Price Inventory transition probability matrix for K=5 and K=6

Lastly we conclude the detailed price inventory transition probability matrix for K ≥ 6 in the following Table 3.6:
Table 3.6: Price Inventory transition probability matrix for $K \geq 6$

Finally, we will define the expected total cost in one cycle with respect to $K$. Also, we let the probability of states with high price level and $m$ items in inventory denote as $\pi_{Hm}$, while the probability of states with low price level and $m$ items denote as $\pi_{Lm}$. We have:

$$E(\text{total cost}) = E(\text{inventory cost}) + E(\text{procurement cost}) \quad \cdots \cdots \cdots \cdots (3)$$

And:

$$E(\text{inventory cost}) = h \cdot \left[ (k \cdot \pi_{LK}) + \sum_{i=2}^{K-1} (i \cdot \pi_{Hi}) \right] \quad \cdots \cdots \cdots \cdots (4)$$

Also:

$$E(\text{procurement cost}) = E(\text{procurement cost on } Y_L) + E(\text{procurement cost on } Y_H)$$
With:

\[ E (\text{procurement cost on } Y_L) \]
\[ = Y_L \left[ \pi_{LK} \cdot (1 - b) \cdot [1 \cdot p + 2 \cdot (1 - p)] \right] + Y_L \sum_{i=2}^{K-1} \{\pi_{Hi} \cdot a \}
\]
\[ \cdot [(k - i + 1) \cdot p + (k - i + 2) \cdot (1 - p)] \quad \cdots \cdots \cdots \cdots \cdots (5) \]

\[ E (\text{procurement cost on } Y_H) \]
\[ = Y_H \{\pi_{H2} \cdot (1 - a) \cdot [1 \cdot p + 2 \cdot (1 - p)] + \pi_{H3} \cdot (1 - a) \}
\]
\[ \cdot [0 \cdot p + 1 \cdot (1 - p)] \} \quad \cdots \cdots \cdots \cdots \cdots (6) \]

Therefore, we plug in the equations (4), (5), (6) into equation (3) to get the expected total cost function below:

\[ E (\text{total cost}) = h \cdot [(k \cdot \pi_{LK}) + \sum_{i=2}^{K-1} (i \cdot \pi_{Hi})] + Y_L \cdot [\pi_{LK} \cdot (1 - b) \cdot [1 \cdot p + 2 \cdot (1 - p)] + Y_L \sum_{i=2}^{K-1} \{\pi_{Hi} \cdot a \cdot [(k - i + 1) \cdot p + (k - i + 2) \cdot (1 - p)]}
\]
\[ + Y_H \{\pi_{H2} \cdot (1 - a) \cdot [1 \cdot p + 2 \cdot (1 - p)] + \pi_{H3} \cdot (1 - a) \cdot [0 \cdot p + 1 \cdot (1 - p)] \} \quad \cdots \cdots \cdots \cdots \cdots (7) \]

In equation (7), parameters a, b, p, h, Y_L and Y_H are all actual values. Also, \( \pi_{Hi} \) and \( \pi_{LK} \) are variables that depend on the value of K. Thus the expected total cost depends solely on the base-stock value K, which means to optimize the expected total cost we simply need to optimize the value of K.
3.3 Optimization

After obtaining the expected total cost function, we will begin this section with the minimization of the expected total cost and solve the optimal inventory/production management problem with respect to $K$. That is, we will be able to optimize our procurement policy if we can find the optimal value of the base-stock value $K^*$. 

In order to minimize equation (7) with respect to $K$, we need to first obtain all the stationary probability of state $\pi_{Hi}$ and $\pi_{Li}$, $i = 2 \ldots k - 1$. We can do so by analyzing the price inventory transition probability matrices. For $K=2$, the stationary probability of the matrix is obtained by solving the balance equations and the normalization equations.

\[
\begin{align*}
(1 - a) \cdot \pi_{H2} + b \cdot (\pi_{L2}) &= \pi_{H2} \\
a \cdot \pi_{H2} + (1 - b) \cdot \pi_{L2} &= \pi_{L2} \\
\pi_{H2} + \pi_{L2} &= 1
\end{align*}
\]

Thus, we apply a linear transformation to the above system of linear equations, so that they are represented in the matrix form equations:

\[
\begin{bmatrix}
1 & 1 \\
a & -b
\end{bmatrix}
\begin{bmatrix}
\pi_{H2} \\
\pi_{L2}
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

Notice one of the equations above is redundant when we are performing the transformation.

Similarly, we approach the other equations respect of $K$ in the same way and present their matrix equations in the following:
When $K=3$,

\[
\begin{bmatrix}
1 & 1 \\
-a & b
\end{bmatrix} \ast \begin{bmatrix}
\pi_{H2} \\
\pi_{L3}
\end{bmatrix} = \begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

Notice in this case the solutions for the states are essentially the same as $K=2$ case.

When $K=4$,

\[
\begin{bmatrix}
1 & 1 & 1 \\
-a & 1-a & b \ast (1-p) \\
0 & -1 & b \ast p \\
a & a & -b
\end{bmatrix} \ast \begin{bmatrix}
\pi_{H2} \\
\pi_{H3} \\
\pi_{L4}
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}
\]

However, we notice that in the above matrix,

\[
[-a \quad 1-a \quad b \ast (1-p)] + [0 \quad -1 \quad b \ast p] = [a \quad a \quad -b]
\]

This means there exist linear dependency between the rows, and thus we can reformulate the equations for $K=4$ again after omitting the linearly dependent row:

\[
\begin{bmatrix}
1 & 1 & 1 \\
-a & 1-a & b \ast (1-p) \\
0 & -1 & b \ast p \\
\end{bmatrix} \ast \begin{bmatrix}
\pi_{H2} \\
\pi_{H3} \\
\pi_{L4}
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

We will do this linear dependency check for all the other cases and reformulate them in a cleaner fashion when necessary. So when $K \geq 5$, in general we will have:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & \cdots & 1 \\
-a & 1-a & (1-a) \ast (1-p) & 0 & \cdots & 0 \\
0 & -1 & (1-a) \ast p & (1-a) \ast (1-p) & 0 \\
0 & 0 & -1 & (1-a) \ast p & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & b \ast (1-p) \\
\end{bmatrix} \ast \begin{bmatrix}
\pi_{H2} \\
\pi_{H3} \\
\pi_{H4} \\
\vdots \\
\pi_{L(K-1)} \\
\pi_{LK}
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix}
\]
Since the solutions of $\pi_{hi}$ and $\pi_{lki}$, $i = 2 \ldots k - 1$ can be solved from above nonhomogeneous matrix equations with $p = 0.5$ and the values of $a = \frac{20}{108}$, $b = \frac{5}{108}$ we obtained before. We can finally compute equation (7) with respect to $K$ by substituting the values of all the $\pi$s. Also, recall we assumed the following parameters for calculations before: $Y_L = 15$, $Y_H = 25$, $h = 0.5$.

In order to find the optimum $K^*$, we use complete enumeration. We utilize Matlab to numerically evaluate the expected total cost for $K=1, 2, 3, 4 \ldots$ until the minimum cost value is detected, since the cost is monotonically changing with respect to $K$. In our case, the optimum base-stock value $K^*=9$ with $E$ (total cost) = 23.89.

The details of the coding from Matlab program for a general case that $K \geq 5$ can be found at Appendix A.

3.4 Numerical Results

Next, we present numerical results showing the effect of various parameters on optimum base-stock value $K^*$. We choose parameters $\pi_H$, $Y_H$, $p$ and $h$, then we present their influences in pairs on the sensitivity of $K^*$.

Firstly, we begin by changing the stationary probability of high states, $\pi_H$ and high price value, $Y_H$. Let $\pi_H = 10\%, 20\%, 40\%, 80\%$, and $Y_H=20$, 25, 50, 100. Since increasing the probability of high states, and increasing high price value would force
the procurement policy to be more efficient at the low price state, both suggests an increase in the optimal base-stock value, $K^*$. Table 3.7 presents $K^*$ values in response to the changes in $\pi_H$ and $Y_H$ while keeping $p$ and $h$ constant. Figure 3.1 presents the visualization of such response for $K^*$. We can see that $K^*$ is more sensitive to $\pi_H$ than $Y_H$.

<table>
<thead>
<tr>
<th>$\pi_H$</th>
<th>20</th>
<th>25</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>4</td>
<td>5</td>
<td>7</td>
<td>10</td>
</tr>
<tr>
<td>20%</td>
<td>7</td>
<td>9</td>
<td>14</td>
<td>19</td>
</tr>
<tr>
<td>40%</td>
<td>16</td>
<td>20</td>
<td>31</td>
<td>41</td>
</tr>
<tr>
<td>80%</td>
<td>63</td>
<td>70</td>
<td>92</td>
<td>114</td>
</tr>
</tbody>
</table>

Table 3.7: Effect of $\pi_H$ and $Y_H$ on $K^*$
Next, we change the stationary probability of high states $\pi_H$ and the probability $(1-p)$ that demand is equal to 2 items and then evaluate $K^*$. We let $\pi_H = 10\%, 20\%, 40\%, 80\%$ and $1-p=0.2, 0.5, 0.7, 0.9$. Table 3.8 provides $K^*$ values to the changes in $\pi_H$ and $(1-p)$ while keeping $Y_H$ and $h$ constant. Figure 3.2 presents the visualization of such response for $K^*$. We see that $(1-p)$ only have a small impact on $K^*$ compared to $\pi_H$. Thus $K^*$ is more sensitive to $\pi_H$ than $(1-p)$.
Table 3.8: Effect of $\pi_H$ and $1-p$ on $K^*$

<table>
<thead>
<tr>
<th>$\pi_H$</th>
<th>10%</th>
<th>20%</th>
<th>40%</th>
<th>80%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>4</td>
<td>7</td>
<td>16</td>
<td>56</td>
</tr>
<tr>
<td>0.5</td>
<td>5</td>
<td>9</td>
<td>20</td>
<td>70</td>
</tr>
<tr>
<td>0.7</td>
<td>6</td>
<td>10</td>
<td>22</td>
<td>81</td>
</tr>
<tr>
<td>0.9</td>
<td>6</td>
<td>10/11</td>
<td>25</td>
<td>91</td>
</tr>
</tbody>
</table>

Figure 3.2: Visualization of the dependence of $K^*$ on $\pi_H$ and $1-p$
Next, we change the stationary probability of high states $\pi_H$ and the holding cost of $h$ and then evaluate $K^*$. We let $\pi_H = 10\%, 20\%, 40\%, 80\%$ and $h=0.1, 0.5, 2, 5$. Table 3.9 provides $K^*$ values to the changes in $\pi_H$ and $h$ while keeping $Y_H$ and $(1-p)$ constant. Figure 3.3 presents the visualization of such response for $K^*$. We see that both $\pi_H$ and $h$ have considerable influence on $K^*$. Thus $K^*$ is sensitive to both $\pi_H$ than $h$.

<table>
<thead>
<tr>
<th>$\pi_H$</th>
<th>0.1</th>
<th>0.5</th>
<th>2</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>10</td>
<td>5</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>20%</td>
<td>21</td>
<td>9</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>40%</td>
<td>45</td>
<td>20</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>80%</td>
<td>121/122</td>
<td>70</td>
<td>26</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 3.9: Effect of $\pi_H$ and $h$ on $K^*$
Next, we change high price value $Y_H$ and the probability $(1-p)$ that the demand is equal to 2 items and then evaluate $K^*$. We let $Y_H=20, 25, 50, 100$ and $1-p=0.2, 0.5, 0.7, 0.9$. Table 3.10 provides $K^*$ values to the changes in $Y_H$ and $(1-p)$ while keeping $\pi_H$ and $h$ constant. Figure 3.4 presents the visualization of such response for $K^*$. We see that $(1-p)$ only have a small impact on $K^*$ compared to $Y_H$. Thus $K^*$ is more sensitive to $Y_H$ than $(1-p)$.
### Table 3.10: Effect of 1-p and $Y_H$ on $K^*$

<table>
<thead>
<tr>
<th>1-p</th>
<th>0.2</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_H$</td>
<td>20</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>25</td>
<td>7</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>11</td>
<td>14</td>
<td>16</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>16</td>
<td>19</td>
<td>21</td>
</tr>
</tbody>
</table>

### Figure 3.4: Visualization of the dependence of $K^*$ on 1-p and $Y_H$
Next, we change high price value $Y_H$ and the holding cost of $h$ and then evaluate $K^*$. We let $Y_H = 20, 25, 50, 100$ and $h = 0.1, 0.5, 2, 5$. Table 3.11 provides $K^*$ values to the changes in $Y_H$ and $h$ while keeping $\pi_H$ and $(1-p)$ constant. Figure 3.5 presents the visualization of such response for $K^*$. We see that $Y_H$ have some impact on $K^*$, and $h$ have even greater influences on $K^*$. Thus $K^*$ is more sensitive to $h$ than $Y_H$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>0.1</th>
<th>0.5</th>
<th>2</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_H$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>19</td>
<td>7</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>25</td>
<td>21</td>
<td>9</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>50</td>
<td>26</td>
<td>14</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>100</td>
<td>31</td>
<td>23</td>
<td>9</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 3.11: Effect of $h$ and $Y_H$ on $K^*$
Lastly, we change the holding cost of $h$ and the probability $(1-p)$ that the demand is equal to 2 items and then evaluate $K^*$. We let $h=0.1, 0.5, 2, 5$ and $1-p=0.2, 0.5, 0.7, 0.9$. Table 3.12 provides $K^*$ values to the changes in $h$ and $(1-p)$ while keeping $\pi_H$ and $Y_H$ constant. Figure 3.6 presents the visualization of such response for $K^*$. We see that $(1-p)$ only have a small impact on $K^*$ compared to $h$. Thus $K^*$ is more sensitive to $h$ than $(1-p)$. 
Table 3.12: Effect of h and 1-p on K*

<table>
<thead>
<tr>
<th>h</th>
<th>1-p</th>
<th>0.1</th>
<th>0.5</th>
<th>2</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>17</td>
<td>7</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>21</td>
<td>9</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>23</td>
<td>10</td>
<td>4</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>26</td>
<td>10/11</td>
<td>4</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Figure 3.6: Visualization of the dependence of K* on h and 1-p
In summary, we can conclude that the sensitivity of $K^*$ is contributed mostly by $\pi_H$ and $h$, and moderately by $Y_H$, while the impact of $p$ on $K^*$ is rather small compared to the others. Table 3.13 shows the changes in $K^*$ by keeping one parameter constant and raising the value of the others. In conclusion, for solving the optimal procurement policy regarding optimal $K^*$ value, we find the percentages of different price states and the holding cost are the most important factors to be considered. The price values of states are next to consider, while the probability of different demand consumptions has relatively less impact on the system.

<table>
<thead>
<tr>
<th>$K^*$</th>
<th>$\pi_H \uparrow$</th>
<th>$Y_H \uparrow$</th>
<th>$p \uparrow$</th>
<th>$h \uparrow$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_H$</td>
<td></td>
<td>↑</td>
<td>↓</td>
<td>↓</td>
</tr>
<tr>
<td>$Y_H$</td>
<td>↑</td>
<td></td>
<td>↓</td>
<td>↓</td>
</tr>
<tr>
<td>$p$</td>
<td>↑</td>
<td>↑</td>
<td></td>
<td>↓</td>
</tr>
<tr>
<td>$h$</td>
<td>↑</td>
<td>↑</td>
<td>↓</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.13: Effect of $\pi_H$, $Y_H$, $h$ and $p$ on $K^*$
Chapter 4

A Case Study

4.1 Introduction

We consider a company, producing an intermediate material that is used in manufacturing military as well as consumer products. An expensive rare metal is required in the production as the raw material. Since the intermediate material is used in manufacturing military products, we do not allow any back ordering or lost sales. The procurement problem of rare metal, Y, is to minimize the total cost of the company including inventory holding, spot buying and contract buying costs during a finite time horizon.

We have made our case study based on the following assumptions: The procurement frequency is monthly. Spot buying is a purchase made in the current month and contract buying is a purchase made at the beginning of every year that lasts 12 months. That means if we make a long term contract at the beginning of the year, we pay the total cost at the beginning and receive a delivery every month throughout the year. Let \( C(m) \) denote the spot price at month \( m \), \( C'(m) \) denote the contract price at month \( m \), \( Q(m) \) denote the total contract amount for the year of month \( m \) and \( S(m) \) denote the amount of spot buying at month \( m \). Assume that the contract price has a discount of \( d \) dollars per kg on the spot price. All purchases are made at the beginning of the month while demands, \( D(m) \), come after the purchases.
are in the beginning of the month as well. At the end of the month we count the new inventory level \(I\), also notice all costs at the present have an interest rate of \(r\%\) as discount factor and the holding cost are \(h\) dollars per kg. It is required that we keep the inventory level at least twice as large as the last month's demand. Also, we assume the inventory level starts with initially 1000kg of rare metal \(Y\), and there is a capacity limit for spot buying so that the maximum amount that could be bought every month is 3000kg.

We use a monthly updated linear/dynamic programming model to optimize this procurement problem. However, some of the data we need are the predicated values that are coming from a time series autoregressive integrated moving average (ARIMA) model. The ARIMA model can be used to predict a series of future values when previous values over a certain time period is available. It fits the distribution of past data and builds a model to predict future values. More details of the times series model and ARIMA model can be found in Brockwell & Davis (2002). We use ARIMA to predict the demand and spot price for a 24month time period in 2010-2011. We utilize the actual data for those 2 years to validate our forecasting model. Lastly, we present the optimum procurement policy and compare it to the actual procurement policy of company X.
4.2 Data Preparation and Forecasting

The raw data sets we obtained from company X are weekly reports of the spot prices offered from Asian Metal, monthly consumption reports of the final product, frequency reports of company X’s purchasing history and their monthly inventory report.

As the first step of the data preparation, we focus on building a forecasting model to predict spot prices. So, it is necessary to clean the weekly spot price data and sort them by dates from 1/1/2001-7/30/2012. We could pivot the report to monthly now but since we want to use a time series model, more historical data is always better. Thus we keep it as weekly until we finish the prediction part and then we will turn the data into a monthly report. Figure 4.1 presents this weekly historical data for spot prices with a trend line. It shows rare metal Y spot price is very volatile since its usage is relatively narrow and is affected heavily by global politics. Its price is also affected by disasters, for example 2011 Japan’s earthquake and tsunami at Tōhoku made the price jump greatly from March to May as shown on the figure because Japan is a large consumer of rare metal Y.
Also, we want to make sure that the time series model is indeed a good choice here for our regression analysis and forecasting. So we attempt to find if there exists a linear/nonlinear relation between the data variables we have so far, which are the spot prices, consumptions, purchases and inventory. However, a correlations matrix and regression results would suggest there are no significant relations exist between the variables and maybe we should look into the relations between past week spot prices and current week spot prices. Then we see a strong linear relation between last week and current week spot prices with an adjusted R square > 0.95 here. So this confirms that we should consider a times series model for forecasting because of the strong relations between past and current week spot prices.

Figure 4.1: Weekly historical data of spot prices
To start the time series analysis, initially we would like to examine the autocorrelation (AC) plot and partial autocorrelation (PAC) plot with a default 95% confidence level on the weekly historical data (Figure 4.2). Also, notice that ARMA is used to build a stationary linear time series model while ARIMA is used to build a non-stationary linear model with trend in the mean. Combing Figure 4.1 and Figure 4.2, we can see that our data is probably non-stationary. Also from Figure 4.2 we can see a large set of correlated lags that suggests an ARIMA with 1-difference. Thus we might prefer an ARIMA model than an ARMA model for our data but that requires further investigation and comparison of both models, which we will perform on later stages.
So next we perform 1 differencing and mean subtraction on the original spot prices data and obtain the residuals as new data on Figure 4.3. Let $\bar{C}(m)$ denote the mean subtracted price at month $m$, $\bar{C}$ denote mean value of the spot prices and $W(m)$ denote the new residual price at month $m$.

$$\bar{C}(m) = C(m) - \bar{C}$$

$$W(m) = \bar{C}(m) - C(m-1)$$
Figure 4.3: 1-differenced and mean subtracted data of spot prices

Again we check its AC and PAC plots (Figure 4.4) which shows a good result with value of 1 as the correlation ratio for the first lag. Similarly if we do 2-difference and mean subtract on the data we get a worse result on the lags because of over-differencing. This indicates us to use the ARIMA model with exactly one time differenced. Next we can proceed to build our ARMA/ARIMA model with all the above information.
Figure 4.4: AC and PAC plots of 1-differenced and mean subtracted data
Let \( C(m) \) denote the predicted spot price on month \( m \), \( C'(m) \) denote the predicted contract price on month \( m \), \( W(m) \) denote the 1-differenced data on month \( m \), and \( d \) is a discount of contract price. Following are the formulas that we are going to use for our times series ARMA/ARIMA modeling:

\[
W(m) = C(m) - C(m-1)
\]

\[
C'(m) = C(m) - d
\]

For a ARMA(p,q) model with autoregressive of order \( p \) and moving average of order \( q \) on our data we will have:

\[
C(m) = \sum_{i=1}^{p} (\Phi_i \cdot C(m-i)) + \sum_{i=1}^{q} (\Theta_i \cdot \varepsilon(m-i)) + \varepsilon(m)
\]

Where, \( \varepsilon(m) \) is white noise, \( \Phi_i + \Theta_i \neq 0 \) for all \( i \), where \( \Phi_i \) and \( \Theta_i \) are parameters to be determined by the ARMA model we build to fit for data.

Also for a generalized ARIMA(p,1,q) model with autoregressive of order \( p \), 1-differenced and moving average of order \( q \) on our data we will have:

\[
W(m) = \sum_{i=1}^{p} (\Phi_i \cdot W(m-i)) + \sum_{i=1}^{q} (\Theta_i \cdot \varepsilon(m-i)) + \varepsilon(m)
\]

\[
\Rightarrow C(m) - C(m-1)
\]

\[
= \Phi_1 \cdot (C(m-1) - C(m-2)) + \Phi_2 \cdot (C(m-2) - C(m-3)) + \ldots
\]

\[
+ \Phi_p \cdot (C(m-p) - C(m-p-1)) + \sum_{i=1}^{q} (\Theta_i \cdot \varepsilon(m-i)) + \varepsilon(m)
\]
\[ C(m) = C(m - 1) + \Phi_1 \cdot C(m - 1) - \Phi_1 \cdot C(m - 2) + \Phi_2 \cdot C(m - 2) - \Phi_2 \cdot C(m - 3) + \cdots + \Phi_p \cdot C(m - p) - \Phi_p \cdot C(m - p - 1) + \sum_{i=1}^{q} (\Theta_i \cdot \epsilon(m - i)) + \epsilon(m) \]

\[ C(m) = C(m - 1) \cdot (1 + \Phi_1) + \sum_{i=2}^{p} C(m - i) \cdot (\Phi_i - \Phi_{i-1}) - \Phi_p \cdot C(m - p - 1) + \sum_{i=1}^{q} (\Theta_i \cdot \epsilon(m - i)) + \epsilon(m) \]

Here, \( \epsilon(m) \) is white noise, \( \Phi_i + \Theta_i \neq 0 \) for all \( i \), and \( \Phi_i \) and \( \Theta_i \) are parameters to be determined by the ARIMA model we build to fit for data.

Now we will use some statistical software to test our ARMA/ARIMA models and see which one will fit best for our data. Here we are using SPSS, Eviews, R and a time series plugin for Excel to compare their results, and therefore choose the relatively better fit among all.

Firstly we show the “autoarima” function from SPSS’s ARIMA fit and predictions on Figure 4.5, the X axis stands for the week number as time period and the Y axis stands for the rare metal spot price. Also in the later ARMA/ARIMA model figures from different software fits and predicts, unless specified, the X axis is always the time line for week number and the Y is always the spot price.
Figure 4.5: SPSS "autoarima" fit and forecast
Next let us look at Eviews’ ARMA(1,1) model on Figure 4.6, where for the first graph, the left Y axis is the residual thresholds and the right Y axis is the spot price. And the second graph is the forecasting plot for the spot price of 12 periods with a two standard deviation line.

Figure 4.6: Eviews ARMA(1,1) fit and forecast
Next we will exam the ARMA model from the time series add-in of Excel, that is ARMA(3,1) on Figure 4.7. Also we adjust the time period range a little here with 2 months less comparing to the previous model to see how it reflects on the new one.

Figure 4.7: Excel ARMA(3,1) fit and forecast
Lastly, a GARCH model (Figure 4.8) using R to build is also considered here but the fits are significantly worse than all the ARMA/ARIMA models and thus should not be our choice. Again we adjust the time period range with 2 months less to see how it reflects on the new one.

![Figure 4.8: R GARCH fit and forecast](image-url)
In summary, the time series model we choose to use for our forecasting should not only be a best fit of past data, but also a most accurate and correct model for forecasting data. Notice that from all of the above models we see a huge misfit around week 500-540. That is around year 2011 when Tōhoku earthquake and tsunami happened in Japan and its aftermath affected the market price of rare metal Y. We understand that such an unpredictable event happened all the time but it should not affect our model in a general sense. Now considering the fitting part, we conclude that ARMA(3,1) from Excel and “autoarima” from SPSS have a better performance. In order to see which forecast data performs better, since year 2010-2011 are the most unpredictable two years, we take the historical data from 2001-2009 and let all models to forecast 2010-2011 data to compare them with the actual data we have. Therefore although both ARMA(3,1) and “autoarima” were providing models with good fit for past data, “autoarima” from SPSS showed us a better forecasting ability. Thus we choose “autoarima” function from SPSS to do our forecasting with the results below on Table 4.1. As you can see, in a long turn, the price forecasting could make a large error comparing to the actual values. However, in our dynamic model for optimization we would only require for the next time period predicted value which usually is accurate enough and acceptable.

We can also do similar analysis on the demand forecasting part which contains a much smaller historical data set. Since we assume the company is always going to fulfill the demand, demand values are not as sensitive as the price. Thus we omit the procedures of the demand forecasting and provide the result directly on Table 4.2 below.
### Table 4.1: Predicted price from SPSS “autoarima”

<table>
<thead>
<tr>
<th>Date</th>
<th>Spot</th>
<th>Buy Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2010)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 P1</td>
<td>560</td>
<td>560</td>
</tr>
<tr>
<td>2 P2</td>
<td>560</td>
<td>560 560</td>
</tr>
<tr>
<td>3 P3</td>
<td>560</td>
<td>561 560</td>
</tr>
<tr>
<td>4 P4</td>
<td>560</td>
<td>561 560</td>
</tr>
<tr>
<td>5 P5</td>
<td>549</td>
<td>562 561</td>
</tr>
<tr>
<td>6 P6</td>
<td>530</td>
<td>562 561</td>
</tr>
<tr>
<td>7 P7</td>
<td>517</td>
<td>563 561</td>
</tr>
<tr>
<td>8 P8</td>
<td>529</td>
<td>563 562</td>
</tr>
<tr>
<td>9 P9</td>
<td>554</td>
<td>564 563</td>
</tr>
<tr>
<td>10 P10</td>
<td>566</td>
<td>565 564</td>
</tr>
<tr>
<td>11 P11</td>
<td>623</td>
<td>565 566</td>
</tr>
<tr>
<td>12 P12</td>
<td>754</td>
<td>566 567</td>
</tr>
<tr>
<td>(2011)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 P13</td>
<td>950</td>
<td>567 566</td>
</tr>
<tr>
<td>2 P14</td>
<td>1039</td>
<td>567 566</td>
</tr>
<tr>
<td>3 P15</td>
<td>1070</td>
<td>568 567</td>
</tr>
<tr>
<td>4 P16</td>
<td>1362</td>
<td>569 568</td>
</tr>
<tr>
<td>5 P17</td>
<td>1362</td>
<td>569 568</td>
</tr>
<tr>
<td>6 P18</td>
<td>1390</td>
<td>570 569</td>
</tr>
<tr>
<td>7 P19</td>
<td>1390</td>
<td>571 570</td>
</tr>
<tr>
<td>8 P20</td>
<td>1390</td>
<td>572 571</td>
</tr>
<tr>
<td>9 P21</td>
<td>1390</td>
<td>572 571</td>
</tr>
<tr>
<td>10 P22</td>
<td>1390</td>
<td>573 572</td>
</tr>
<tr>
<td>11 P23</td>
<td>1390</td>
<td>574 573</td>
</tr>
<tr>
<td>12 P24</td>
<td>1390</td>
<td>575 574</td>
</tr>
</tbody>
</table>

### Table 4.2: Predicted demand from SPSS “autoarima”

<table>
<thead>
<tr>
<th>Date</th>
<th>Demand</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2010)</td>
<td></td>
</tr>
<tr>
<td>1 D1</td>
<td>654</td>
</tr>
<tr>
<td>2 D2</td>
<td>388</td>
</tr>
<tr>
<td>3 D3</td>
<td>523</td>
</tr>
<tr>
<td>4 D4</td>
<td>621</td>
</tr>
<tr>
<td>5 D5</td>
<td>794</td>
</tr>
<tr>
<td>6 D6</td>
<td>863</td>
</tr>
<tr>
<td>7 D7</td>
<td>1084</td>
</tr>
<tr>
<td>8 D8</td>
<td>1085</td>
</tr>
<tr>
<td>9 D9</td>
<td>689</td>
</tr>
<tr>
<td>10 D10</td>
<td>1150</td>
</tr>
<tr>
<td>11 D11</td>
<td>917</td>
</tr>
<tr>
<td>12 D12</td>
<td>778</td>
</tr>
<tr>
<td>(2011)</td>
<td></td>
</tr>
<tr>
<td>1 D13</td>
<td>902</td>
</tr>
<tr>
<td>2 D14</td>
<td>786</td>
</tr>
<tr>
<td>3 D15</td>
<td>764</td>
</tr>
<tr>
<td>4 D16</td>
<td>924</td>
</tr>
<tr>
<td>5 D17</td>
<td>850</td>
</tr>
<tr>
<td>6 D18</td>
<td>856</td>
</tr>
<tr>
<td>7 D19</td>
<td>929</td>
</tr>
<tr>
<td>8 D20</td>
<td>911</td>
</tr>
<tr>
<td>9 D21</td>
<td>891</td>
</tr>
<tr>
<td>10 D22</td>
<td>1037</td>
</tr>
<tr>
<td>11 D23</td>
<td>1115</td>
</tr>
<tr>
<td>12 D24</td>
<td>1023</td>
</tr>
</tbody>
</table>
4.3 Modeling

After we did our data preparation including the data forecasting, we had all the data we need for the model. Now we have all the data including actual demand, predicted demand, actual spot price, predicted spot price, actual contract price, predicted contract price, actual procurement purchases and actual inventory levels. Notice for the spot price and demand in our dynamic programming, they will not all be the actual values but as a hybrid of both actual values and predicted values. And with the updates move on, our dynamic system will substitute the old predicted values with the recently obtained actual values and require new predicted values from ARMA/ARIMA forecasting model.

Next, we shall start to build our linear programming model to minimize the expected total cost of each month.

Total cost of month m:

\[
T(m) = \begin{cases}
  C(m) \cdot S(m) + C'(m) \cdot Q(m) + h \cdot I(m), & \text{for } m \text{ is a beginning month of the year} \\
  C(m) \cdot S(m) + h \cdot I(m), & \text{for } m \text{ otherwise}
\end{cases}
\]

\(C'(m)\) and \(Q(m)\) are determined by the beginning of the year since it is a long term contract price and will be decide at the beginning of month 1, 13, 25...etc.

The following constraints will ensure the validity of the assumptions.
Inventory level of month $m$:

$$I(m) = I(m - 1) + S(m) + Q(m) - D(m)$$

$$I(m) \geq \frac{1}{2} D(m - 1)$$

Procurement policy of month $m$:

$$0 \leq S(m) \leq 2$$

$$Q(m) > 0$$

Contract buy and spot buy relations at month $m$:

$$C'(m) = C(m) - d$$

With this linear program of every month $m$, we can then extend it to a multi-period mathematical programming model with the objective of minimizing the total cost of all time periods in a finite horizon. Let $M$ be the total number of months. Thus, we have the mathematical programming model as the following.

Minimize the total cost of all $M$ months considering interest rate $r$:

$$T(M) = \sum_{m=1}^{M} \left( \frac{1}{1 + r} \right)^m [C(m) \cdot S(m) \cdot + h \cdot I(m)] + \sum_{f=1,13,25...} \left( \frac{1}{1 + r} \right)^f C'(f) \cdot Q(f)$$

Such that:

$$I(m) = I(m - 1) + S(m) + Q(m) - D(m)$$

$$I(m) \geq \frac{1}{2} D(m - 1)$$
\[ C'(m) = C(m) - d \]

\[ 0 \leq S(m) \leq 2, \quad Q(m) > 0 \]

Notice that all the values can be imported from our data preparation stage, and then plugged into the above system starting for \( m=1 \)...

This means we have to run the linear programming repeatedly while updating the predicted data and plug in new actual data to solve the problem \( M \) times until the end. We approach to do so by using Xpress Mosel while incorporating it with updated ARIMA models imported from SPSS. Now let us try our algorithm based on all the assumptions we have made before and test its efficiency for the year of 2010-2011, since they are the most unpredictable years, and would sort of serve us as the worst case scenario. In this case \( M=24 \) and January of 2010 will be \( m=1 \).

The codes we use to build the above single time period model in Xpress Mosel can be found in Appendix B, where the spot price \( C \) and demand \( D \) are in a hybrid fashion containing both actual values from historical data and predicted values from ARIMA model.

### 4.4 Optimization Result

Eventually, we want to compare our optimum procurement policy with the actual procurement company X made during 2010-2011 and by inserting different
interest rate or holding cost we can also simulate the policy under different scenarios. Here we would like to show one of the most basic scenarios with \( h = 10, d = 50 \) and \( r = 0.0006 \). Also, since it is a simulation for year 2010-2011, \( M = 1...24 \) and \( F = 1...2 \). Since we are doing this linear programming optimization with updates, only every month's spot buying strategy will be adopted and the inventory will rely on the actually demand in the future rather than the predicted values from ARIMA model. That is, using the new spot price we observe, demand predictions from the ARIMA model and the policy we have adopted last month via the linear programming model, we run the linear program to compute a new inventory level, thus decide on the order quantity. The linear programming model is applied repeatedly in order to provide the next month's procurement policy as well as calculate the new inventory level. Eventually, we will reach our last time period and then we can consolidate the entire spot buy, contract buy and inventory level control record for all months.

In summary, by running the above Xpress Mosel model over 24 months with each time importing a new updated data from ARMA/ARIMA, we have the result for every month’s spot buying and contract buying amount, as well as the inventory levels. Finally, the optimum expected total cost is: $15,922,901, with a total purchase of 21937 kg of rare metal Y and average of $726/kg, the detailed optimal policy is at the following.

Spot buying quantity of month 1-24: all zeros kg.

Contract buying quantity of year 2010 and 2011: 11167kg and 10770kg.
Inventory level of the month 1-24: 1276kg, 1819kg, 2227kg, 2536kg, 2673kg, 2740kg, 2587kg, 2432kg, 2674kg, 2455kg, 2468kg, 2621kg, 2616kg, 2728kg, 2861kg, 2835kg, 2882kg, 2924kg, 2892kg, 2899kg, 2919kg, 2780kg, 2562kg, 2437kg.

When we compare the optimum procurement policy to company X’s actual purchases, we see that they have made a total purchase of 19983kg of rare metal Y and total cost of $17,165,381 with average purchase of $859/kg, we have spent 7.8% less on the total cost and 18.3% less per kg on rare metal of Y while doing more purchasing. Thus, it strongly suggests that our approach would provide considerable savings for the total cost.
Chapter 5

Conclusions

In this thesis, we study various methods for optimal procurement policy under different circumstances of our own. Firstly, in chapter 3, we use a policy that its form is given as a base-stock policy and try to find the best base-stock value $K^*$. Also, we investigate the impact of various parameters on the optimal base-stock value. Secondly, in chapter 4, we approach a real life scenario where we have to provide an optimal procurement policy from historical data. We applied time series modeling to provide raw material price and demand forecasts that will be used in a mathematical program. This mathematical program is solved recursively by updating the price and demand forecasts at each time step. In this case study, we show that inventory/procurement costs can be reduced by implementing our algorithm. In future, we plan to investigate properties and structure of the optimum policy for procurement and production planning. We are especially interested in threshold type policies that are easy to implement.
The coding for the expected total cost respect to different values of $K$ in Matlab when $K \geq 5$ is below.

```matlab
%K >= 5
function total = totalcost(K)

% Parameters
a = 20/108;
b = 5/108;
p = 0.5;
k = K-1;
h = 0.5;
YL = 15;
YH = 25;

% Build the matrix
A = [ones(1,k); zeros(1,k); zeros(k-2,1) (diag(-ones(1,k-2))+diag((1-a)*p*ones(1,k-3),1)+diag((1-a)*(1-p)*ones(1,k-4),2)) zeros(k-2,1)];

A(2,1) = -a;
A(2,2) = 1-a;
```
\[
A(2,3) = (1-a)*(1-p);
\]
\[
A(k-1,k) = b*(1-p);
\]
\[
A(k,k) = b*p;
\]
\[
B = \text{zeros}(k,1);
\]
\[
B(1) = 1;
\]

% Solve all the pi
mypiT = A\B;

mypi = (mypiT)';

display(mypi);

% Calculate total cost with respect K
total1 = 0;
total2 = 0;
for i = 2:k
    total1 = h*(mypi(k)*K+\text{sum}(i*mypi(i-1)));
    total2 = YL*(mypi(k)*(1-b)*(p+2*(1-p)))+YL*\text{sum}(mypi(i-1)*a*((K-i+1)*p+(K-i+2)*(1-p)));
end
\[ \text{total3} = YH \times (\text{mypi}(1) \times (1-a) \times (p+2\times (1-p)) + \text{mypi}(2) \times (1-a) \times (1-p)) \; ; \]

\[ \text{total} = \text{total1} + \text{total2} + \text{total3} ; \]

\[ \text{display}(K) ; \]
Appendix B

The coding in Xpress Mosel of a single time period mathematical model with data imported from ARIMA is below.

model ModelName

uses "mmxprs";

!parameters
declarations

M = 1..24 !insert all time periods

F = 1..2 !insert the number of years

h = 10 !insert holding cost here

r = 0.0006 !insert interest rate here

d = 50 !insert contract buy discount here

C: array(M) of real

dC: array(F) of real

D: array(M) of real

end-declarations
!import data from ARIMA model

initializations from 'OptimalPolicy.dat'

C D

dend-initializations

!setup values

declarations

S: array(M) of mpvar

Q: array(F) of mpvar

I: array(M) of mpvar

dend-declarations

!constraints

dC(1):= C(1) - d

dC(2):= C(13) - d

!insert more if there are more time period
\( I(1) = 1000 + S(1) + Q(1)/12 - D(1) \)

\[
\text{forall}(m \in 2..12) \ I(m) = I(m-1) + S(m) + Q(1)/12 - D(m)
\]

\[
\text{forall}(m \in 13..24) \ I(m) = I(m-1) + S(m) + Q(2)/12 - D(m)
\]

!insert more if there are more time period

\[
\text{forall}(m \in 2..24) \ I(m) \geq 2 * D(m-1)
\]

!insert more if there are more time period

\[
\text{forall}(m \in M) \ S(m) \leq 3000
\]

\[
\text{forall}(f \in F) \ Q(f) \geq 0
\]

!objective

\[
\text{cost} := dC(1) \cdot (1/1+r)^1 \cdot Q(1) + dC(2) \cdot (1/1+r)^{13} \cdot Q(2) +
\]

!insert more if there are more time period

\[
\sum(m \in M) C(m) \cdot (1/1+r)^m \cdot S(m) +
\]

\[
\sum(m \in M) h \cdot (1/1+r)^m \cdot I(m)
\]
!solve

minimize(cost)

writeln("Begin running model")

writeln("The minimum of the total cost is: ", getobjval)

forall(m in M) writeln("Spot buy quantity of month(", m, "): ", getsol(S(m)))

forall(f in F) writeln("Contract buy quantity of year("f", "): ", getsol(Q(f)))

forall(m in M) writeln("Inventory level of the month("m", "): ", getsol(I(m)))

writeln("End running model")

default

default
References


