PERCENTS ARE NOT NATURAL NUMBERS

by

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ABSTRACT OF THE DISSERTATION

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Adults are prone to treating percents, one representational format of rational numbers, as novel cases of natural number. This suggests that percent values are not differentiated from natural numbers; a conceptual shift from the natural numbers to the rational numbers has not yet occurred. This is most surprising, considering people are inundated with rational numbers all around them, from the "% Daily Values" on nutrition labels to sales and discounts in stores to the constant ups and downs of gas prices. From an early age, humans have a rather robust concept of natural number, but this earlier knowledge seems to act as a barrier to future learning of rational number. While participants performed better on one-statement problems with a single percent, they ignored the principles of rational number when presented with two-statement percent problems. For example, using rational numbers involves dropping the successor principle, a principle that applies to the natural numbers. Problems presented algebraically gave participants great difficulty, though performance improved when subjects were able to replace the variable y with a natural number (100). When using a numerical hint (substituting in a value during a training phase), participants still did not improve their performance on algebra problems in posttest. Error analyses revealed a strong tendency for students to interpret the problems as novel examples of natural number problems. This research discusses the failure to understand percents in light of overuse or failure to apply the natural number successor principle, division, false cognates, and the absence of knowledge that relates multiple rational number representations to a common underlying knowledge structure. The amount of previous math experience is also correlated with the ease of moving between natural number and rational number concepts.

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I. Introduction

"Mathematics is written for mathematicians" (Copernicus, 1543).

It is commonplace to hear adults say, "Oh I'm just not good at math" with a dismissive wave of the hand and no sign of shame or embarrassment (Willingham, 2009). Our society has accepted the idea that math is not for everyone (Stevenson & Stigler, 1992). This perception contrasts with almost all research findings with *young* children, be it done by Mathematics educators, Cognitive Scientists, Psychologists, Linguists, or Neuroscientists. All agree that when children start school, they know quite a bit about the mathematical role of counting, ordering, addition and subtraction with the positive natural numbers. However, this facility does not guarantee the subsequent learning about rational numbers, whether these are in fraction, decimal or percent formats.

The research that bears on my dissertation shows that people consistently make errors in equations with rational numbers, be these about the mechanics, the rules, the language, the symbols, and/or the algorithms of mathematics. More often than not, people treat rational number formats as if they were novel cases of natural number problems and do so without checking if the answer actually makes sense in the context of the problem (e.g., a 20% increase in x followed by a 20% decrease in x results in x). This research takes a closer look at this tendency by uncovering the source of the errors. Doing so signals a major concern of cognitive scientists, the nature of conceptual change. It also contributes to the observation of mathematical illiteracy on the part of many adults in this country. While most people living in numerate environments typically develop *some* sense of natural number, this does not guarantee their ability to "do math" in novel situations (Carraher, Carraher, & Schliemann, 1985). As is the case of language, at an early age we possess some knowledge of the core domain of natural number, which lays the groundwork for attending to learning about the verbal instantiation of verbal counting and its relation to natural number arithmetic and one's ability to use verbal counting (Gelman & Gallistel, 1978). This is the case even though the inputs for early mathematical learning are more variable than are those for language learning (Gleitman & Papafragou, 2004). Although many children arrive at school with an ability to count as well as add and subtract, the relation of mathematical knowledge to its symbols and format typically requires teaching in a classroom or work environment. The continued development of mathematical knowledge depends on the child's dedicated effort as well as well-prepared inputs.

In this chapter, I discuss the differences between natural number and rational number, with a focus on percent formats. While humans have an innate concept of natural number, rational numbers are novel concepts that involve a new mental representation (Carey, 2009; Gelman, 1991). As a result, the odds are that the input will be assimilated to existing knowledge structures. This sets up the need for a conceptual change.

The focus of the dissertation is on the rational number format, one that is very much present in the everyday environment, including the discussion of increase and decrease of budgets, newspaper ads of sales, up and down costs of gas, and so on. I emphasize the importance of understanding and differentiating natural and rational numbers, as well as illustrate the underlying difficulties with rational number concepts.

In Chapter 2, I ask which types of rational number representations are more easily understood than others and look at performance differences in problems dealing with a single percent statement (e.g. an increase of 20%) versus a two-statement problem where the second statement is dependent upon the outcome of the first (e.g. an increase of 20% later followed by a decrease of 30%). In Chapter 3, I present a study aimed at improving performance on two-statement percent problems by providing hints to students. Perhaps, students need to be reminded that they can do these problems. All of the studies in this dissertation are concerned with the conditions that might bring out an understanding of rational numbers. The ultimate goal is to determine whether college students can do the problems once they are given practice trials. If they are not successful, their errors are carefully analyzed to gain insight regarding the barriers to conceptual change. Thus, in Chapter 4, I explore the patterns of errors that occurred in Chapter 3. In Chapter 5, I sum up the conclusions as well as consider the implications of these results for various research topics. A possible framework for the learning to use percents with understanding is also suggested.

The next section of the introduction discusses conceptual development in the number domain as it is viewed in Developmental Psychology and Cognitive Science. The research leaves little room to doubt that humans do have a number sense, which develops “on the fly,” without formal teaching. In providing a brief overview of the differences between the domains of the natural numbers and the rational numbers, I highlight the underlying difficulties involved in moving on from natural number understanding. I discuss difficult areas of mathematics where even educated adults continue to make many errors with rational numbers, across several formats. Examples are provided to illustrate the extensive infiltration of these errors in our everyday life. Finally, research that helps get beyond these errors will be explored and considered.

Conceptual development: the participant as the active learner

A recurring theme throughout my work is the question of conceptual change and development – can students successfully shift from learning about one concept (natural numbers) to another one (rational number representations of percents)? Conceptual development is most likely to be achieved when existing conceptual structures overlap with new ones. Restructuring of existing knowledge structures is needed when new information is incompatible with the existing knowledge (Carey, 2009; Carey & Spelke, 1994; Gelman & Williams, 1998; Vosniadou, 1994). When the new framework or structure does not readily map onto available cognitive structures, it becomes more likely that the problem's information will be misconstrued; inevitably, errors will occur (Hartnett & Gelman, 1998).

Our cognitive development relies on our ability to gain knowledge by using and/or constructing mental representations; we are active participants in our own development. This constructivist theory of learning allows us to create meaning from different experiences as we build upon our own knowledge structures. But there is a problem with the student as the learner. They can interpret data in ways that do not overlap with what is intended by those who offer inputs. If the learner already has a mental structure than can assimilate the data, then uptake is very likely. But if what is offered does not fit, then the risk of misinterpretation is very high.

Numbers

From a formal perspective, there are several kinds of numbers. These form a hierarchy with complex numbers at the top and their subdivision into real and imaginary numbers. Within the real number system, there are rational and irrational numbers. Rational numbers are numbers that can be expressed as the ratio of two cardinal integers or the division of one cardinal number by another. Integers include both positive and negative values (…–3, –2, –1, 0, 1, 2, 3,…). Thus, the subsets that fall under the umbrella of rational numbers include the whole numbers (0, 1, 2, 3…), natural numbers or counting numbers (1, 2, 3…), zero, negative numbers (-1, -2, -3…), and fractions (½, ¾, etc). Irrational numbers are those numbers that cannot be expressed as the ratio of two integers and which include, but are not limited to, examples such as non-repeating decimals (.349681884…) and radicals (√2).

The difference relevant to this paper is the discrete nature of natural numbers versus the continuous nature of rational numbers. The concept of number is greatly expanded when moving from natural numbers to rational numbers. The conceptual framework that must be constructed is unlike the one with the natural numbers; rather, it must be embedded into the existing knowledge structure (Gelman, 1991). Natural numbers are discrete in that no two numbers can have the same successor. In contrast, rational numbers are defined by the relationship between two integers; thus, there are an infinite and continuous number of divisions possible between any two numbers (Gelman, 1991).

It is important to note that zero is a non-positive, non-negative, integer, whole number, and rational number. It is not categorized as a natural number until it becomes part of the group of positive and negative integers under addition and subtraction. The acceptance of zero as a number is a late development; the related acceptance of negative integers occurred even later. For a long time, the Greeks refused to accept negatives, zero, irrational numbers and complex numbers (e.g., imaginary numbers) as part of their number system since they could not construct a physical representation of these numbers. The number zero was finally accepted in the sixteenth century (Boyer, 1944). Near the end of the nineteenth century, a formal approach was introduced to include and accept these entities in the definition of number – even if they do not refer to counted items. A formal definition of cardinal numbers requires that there be a number that serves as the identity element such that when it is added to an integer "*a*", the value of *a* is unchanged. Zero is also essential in order for natural number arithmetic to be closed (i.e., so that $a - a = -a + a =$ Additive Identity Principle).

The meaning of zero changes when division is considered; it raises the question of what it means (or rather, does not mean) to divide by zero. The concept of zero and the properties of zero are extremely relevant to the discussion of moving from natural number concepts to rational number ones (in particular, see the section entitled "Importance of understanding division and its relationship to rational numbers").

Research is mixed on the acquisition of natural numbers and rational numbers (for a review, see Carey, 2009). There are many arguments for when the counting numbers are learned and what other mental structures need to be in place to learn about rational numbers; the mental structures for rational numbers are inconsistent with those for natural numbers (e.g., the successor principle must be dropped, division is closed, etc.).

While there is a great deal of evidence supporting innate constructs of natural number (e.g., Gallistel & Gelman, 1992), a division occurs in addressing these other arguments.

However, the purpose of this section is not to argue when nor how we develop concepts of natural number. In fact, as will be seen in my research (Chapters 2 and 3), children and adults are so proficient with natural numbers that this presents them with a possible stumbling block to their learning about other classes of number. An overwhelming number of adults default to natural numbers when thinking about rational number data.

The difficulty to shift from natural numbers to rational numbers

Conceptual development occurs when existing conceptual structures are consistent with new ones. Thus, learning the counting numbers and their governing principles of addition and subtraction can be a gateway into learning about more advanced and related concepts, for example, that there is no end to the natural numbers. Yet, as will soon be discussed, rational numbers differ from natural numbers in many ways that conflict with the existing cognitive framework for natural numbers. The reason for this can be attributed, in part, to the different principles underlying the different types of number. I focus on one principle, in particular, to account for children's and adults' great difficulty with rational numbers – the successor principle.

The role of the successor principle in number concepts – in children and in adults

The successor principle states that for every natural number there is a unique number that comes "next." Children acquire knowledge about natural numbers and their successors much more easily than with numbers containing rational number notation. With the natural numbers, the most primitive way to gauge emerging counting skills in children is to see if they use as many counting tags as there are items to be counted. Touching objects is a very important part of learning to count – it helps children keep separate the counted from the to-be-counted items (Gelman & Gallistel, 1978). This is not the case with rational numbers. When looking at fractions, for example, children between 5 to 7 years old are not successful on the task of rank ordering fractions from the smallest to the largest (Hartnett & Gelman, 1998). The problem: rational numbers do not have unique successors. There is no longer a unique "next" after every number. Children are extremely reliant on count-based strategies, ones that only work to render a fraction in special part-whole contexts. Yet, there is an infinite number of numbers that fall between any two rational numbers. Further, the numbers in the numerator and denominator no longer indicate individual succession; now the numbers on the "top" and on the "bottom" must be viewed together in order to arrive at a new type of rational number, one that is generated by dividing the numerator by the denominator. For instance, 3 is greater than 2, but 1/3 is not greater than 1/2. Thus, it is not surprising that children are not very good at explaining the relationship between the numerator and the denominator in fractions, much less explain *why* there are two numbers in a fraction (Hartnett & Gelman, 1998; Smith, Solomon, & Carey, 2005). This failure also perpetuates the difficulty in ranking fractions from smallest to largest. In ordering tasks with rational numbers, unlike integers, it is extremely difficult to think about what comes next.

In trying to rank order numbers, first-, second-, and third-grade children sorted a set of cards labeled with representations of whole numbers, zero, or fraction representations equivalent to one called “mathematical wholes” (e.g., 2/2, 15/15, etc.). But they did not put the mathematical wholes where they placed “1”, emphasizing that these fractions are treated as novel examples of natural number even though they are equivalent (Gelman, 2000). To be successful on such tasks, children must understand that there are an infinite number of answers that might come after a fraction, such as ¼. There is no unique “next” number. What is perhaps even harder to understand, is that there are an infinite number of answers for what comes next after a decimal, such as .25. Immediately, .26 might come to mind, but what about .255… what about .259… what about .25000001?

Smith et al. (2005) further investigated this question. Children were shown two cards with quantities on them and were asked to choose the card with the larger value. They received values such as .65 versus .8, 2.09 versus 2.9 and 1/75 versus 1/56. The importance of the relationship between the numerator and denominator was ignored such that children judged 1/75 as larger than 1/56, justifying their response by ignoring the numerator (i.e., 75 is larger than 56). They falsely stated that a value of 2.09 is larger than 2.9 because the value has more digits. Judging .65 as greater than .8 occurred as well, except when students recognized place value and added a 0 to represent .8 as .80. Those who interpreted these decimals as percents or as money seemed to perform better. Overall, the results were analogous to those from similar studies (e.g., Gelman, 1991) and show that children are making whole number errors.

Comparable misinterpretations are found in adults. Drawing from three populations of older students (upper secondary level, student teachers, and math majors at the university level), Merenluoto and Lehtinen (2002) asked participants how many values were between two given natural numbers. Many students said that the answer could not be defined. When probed with "what value comes next?" some responded with "the one with the most decimals;" meanwhile, most student teachers and a small number of math majors consistently responded with "add one," a clear indication of the successor principle in operation. With these responses, a conflict exists between the discrete nature of natural numbers and the continuous nature of rational numbers.

The treatment of rational number values as ones about natural number are very real and very present in everyday life. One such example is in the medical field. When getting treated with chemotherapy, a family member of mine was looking through her blood charts. She was questioning how many values are between 9.9 and 10 – when does it becomes 10, when does it become 10.2, etc.? The answer is not a simple one as illustrated by earlier examples and the continuous nature of rational number. In the context of my relative, she was looking at her hemoglobin scale and comparing her current level to past levels. She knew that her doctor wanted her number to be above 10; falling below this number might have warranted a transfusion. Is being 0.1 below the designated level enough of a difference to be concerned? For that reason, she wanted to know how much distance there was between 9.9 and 10 in order to gauge if this discrepancy was dangerous. Ultimately, for this example, the unit of measurement is the decision criteria and the resolution of the numerical scale can suggest what the decision threshold should be. Indeed, understanding measurement and what counts as a meaningful unit of difference is critically related to the interpretation of a rational number.

In sum, not being able to count objects tangibly by "adding 1" might account for the failure to shift to learning about rational numbers. Rational numbers are not about discrete entities. There is not a comparable system for these concepts of quantities as there is for the system of natural numbers since they do not always represent collections of countable objects. Indeed, the strong tendency of young children to count "things" makes it very difficult. This is true of all numbers that are not the natural numbers (e.g., negative numbers). Thus, new strategies to order numbers must be derived. These new strategies are constructed by overlapping new knowledge structures with existing knowledge structures.

Importance of understanding division and its relationship to rational numbers

Understanding division and the infinite divisibility of numbers between 0 and 1 is directly relevant to moving from a natural number concept to a rational number concept (Smith et al., 2005). When a division symbol is found between two integers, the final value is not defined by any one of those integers, but instead it is defined by the relationship between the two. Division in itself helps define rational numbers, as can be seen from examples such as "2.1 children in the average American household" or "2/17 cut to the budget." But division, a closed operation, is itself a difficult concept to grasp. In order for division to be closed, which entails that any natural number divided by any other number produces a unique number of the same set, it is necessary to add rational numbers to the concept of number. This notion is clearly problematic for many people who seem to hold the false belief that you can throw away the denominator.

Smith et al. (2005) asked children if they could "get" to zero by repeatedly dividing by 2. Responses showed that children treated repeated division of a number akin to natural numbers that are repeatedly subtracted. The same pattern occurs in adults (in Chapters 2, 3, and 4). Some students even believed that the answer would eventually result in a negative value, though the initial dividend was positive. This response is indicative of believing that the number line has gaps and is not in fact "dense" – that is, there is an infinite number of values found between any two numbers on a number line, which represents a continuous quantity. This is a stark contrast to a number line with only natural numbers.

Carey (2009, p. 352-359) cites the two Quinean conceptual systems (Quine, 1960) to explain the importance of acquiring the idea of infinite divisibility to rational numbers. In the first, students are able to make comparisons, add, subtract and multiply. It is not until a developmental shift occurs to the second that they are able to fully understand division as the inverse of multiplication; consequently, the student's foundational concept of number is threatened and must change. By separating the concept of division from subtraction, Carey points out the very nature of number and its principles must be reconceptualized. One cannot equate successive subtractions with the operation of division any more than one can equate successive additions with multiplication. A new knowledge structure, or what Gelman (2009) dubs a noncore domain, must be built. It is well known that this is a very difficult task, one that takes time as well as effort on the part of both the learner and the teacher.

False cognates interfere with number concepts

Rational numbers provide a new framework for looking at the natural numbers. As previously mentioned, natural numbers are a subset of rational number. The number 2, as a natural number, represents two objects or entities. However, when this same number 2 is treated as a rational number, it represents a division relationship between two values, 2 and 1, 4 and 2, 6 and 3, etc. (Merenluoto & Lehtinen, 2004).

The previous example illustrates how multiple meanings can be paired with one and the same word, e.g., “bat” (made of wood, or flies at night). Understanding mathematics can be compared to learning a language. In fact, while the notation systems are not the same, there is still much that is similar as regards the underlying concepts in language and mathematics. Language itself is a domain that is made up of principles at various levels and, accordingly, it has rules about its entities and how they combine (Gelman, 2002). Principles are organized to dictate what are acceptable combinations of syntax, semantics, and morphology. You find this in math, just as you do in language.

Fully mastering the verbal counting system does not guarantee understanding of all numerical and mathematical concepts. In fact, often the language of math introduces false cognates for rational numbers: multiplying no longer means “gets bigger” (e.g., 3*(1/3) = 1, a value smaller than the original value of 3). A fraction of a number does not refer to a countable entity (as in, “each teacher has 22 1/3 students in his/her class”). A value of 9.10 (nine point ten) does not come after 9.9 (nine point nine). The latter error is very similar to one that children make in counting: “thirty eight, thirty nine, thirty ten…” The language “point ten” interferes with the rational number representation (9.1).

Children have a difficult time understanding the formal concept of inverse relations, that as one quantity increases, another decreases (Piaget, 1952). This is yet another example of false cognates seen in the environment surrounding them. The faster a parent drives, the shorter amount of time it will take to get to the destination. Given a set amount of food, the amount of food available per person decreases with an increasing amount of guests at a party. Understanding such concepts might contribute to beginning to understand division (Bryant, 1997; Squire & Bryant, 2003). Misconceptions might be attributed to falsely applying a positive relationship or an opposite relationship (switching the relation) between the divisor and the quotient. Making an opposite error could mean the child is incorrectly assuming that a larger divisor results in a larger quotient, along with not understanding what the denominator means in a fraction (Smith et al., 2005); in comparing two groups, children might believe the bigger group will get 'more' because of the perceptual salience of a larger versus smaller group (Squire & Bryant, 2003).

Not only is the language of number riddled with false cognates, but so are the symbols. While children do understand the value of cardinality is increased or decreased as a result of addition and subtraction (Sarnecka & S. Gelman, 2004), they have trouble understanding the new values of number once a symbol is introduced. The expression 2/4 can occur in both mathematics and music. In mathematics, this symbol is equivalent to 4/8, 200/800, 4/16, etc. In music, 2/4 is approximately equal to 6/8, where two triplets are to be played within two beats (Gelman & Brenneman, 1994). The meaning of these symbols is tied up with the domain. In mathematics, when a number is preceded by a negative sign, the larger the integer, the smaller the numerical value actually is.

To be specific, the understanding of a rational number requires knowing that there are domain-specific notations with their own meaning. Recall the earlier example of division and the fact that the separate numerals of 4 and 9 take on new meaning when joined by a division sign. This entails recognizing that a fraction is really a tri-partite symbol. Similarly, there are symbolic formats for decimals and percents. To further complicate matters, it turns out that percent statements do not refer to the exact same value as fractions. Thus, it may come as no surprise that some confusion about the notation ensues.

The importance of multiple representations of number in concept learning

The domains of mathematics have principles, symbols, and combination rules that are in place for a reason – they dictate the syntax of the language of math. When the relationship between equivalent representations is not recognized, errors inevitably follow. Earlier, I spoke of the student having to build a new conceptual construct when novel information does not fit the existing framework. Structural mapping supports learning about data that fit what is already known. The mind finds it easiest to learn more about that which it already knows; thus, a relationship must be made between the novel concept and an existing one. So, it is no surprise that young learners actively engage their environment with any available structures, even if they are but of skeletal form (e.g., as in Gelman, 2000). In rational numbers, the relationship between counting and the arithmetic representation might be very difficult to attain. Here, what is already known of the natural numbers and their structure conflicts with the principles of a rational number and almost serve as a "barrier" instead of a "bridge" to later learning, as demonstrated by Hartnett and Gelman (1998). This evidence conveys the difficulties that exist in understanding equivalent forms of mathematical representations.

It is critical to recognize the range of problems to which a concept may apply. Often, children are taught a very specific set of problem types that fall under the umbrella of one number concept. It limits their ability to apply a concept to a novel problem when that concept seems restricted. Perhaps, children simply need more practice with a more expansive breadth of examples. Learning from at least two mathematical examples that are similar to a test problem has been shown to yield better performance than when learning from just one (Craig & Ryan, 2010; Gick & Holyoak, 1983; Zur, 2003). Gick and Holyoak found that, in addition to solving two comparable problems, putting them side-by-side to compare allowed for easy transfer and thus, greater improvements on subsequent tests. Similarly, Zur showed that children were able to successfully solve a second problem in a set more rapidly than the first problem; she explained that the children were able to recognize an arithmetic relationship between the two problems and transfer a mathematical principle from one to the next. Rittle-Johnson and Star (2007) further showed that comparing alternative solution methods has great value conceptually and procedurally for seventh-grade students. This articulation helps contribute to higher-order thinking (Zaslavsky & Shir, 2005), which entails critical thinking and problem solving skills.

Alternatively, Wu (2009) discussed what can happen when multiple examples of the same number concept are not taught alongside one another. He pointed out that while finite decimals are merely a special class of fractions, fractions and decimals are taught separately in schools rather together. This results in either: a) leading students to believe that these numbers cannot be related to each other, or b) avoiding an opportunity for students to realize that these numbers are related to each other. In any case, it cripples the student's ability to easily move from one representation to another. Further research (Kellman, Massey, Roth, Burke, Zucker, Saw, Aguero, & Wise, 2008) suggested that instead of introducing the simpler concept first, multiple concepts could be presented from the beginning so that the students might learn to perceive the relation between the alternative representations from the outset. This contrasts with common classroom approaches where simpler concepts are first introduced and then built upon to create more complex concepts.

In my research (in Chapters 2 and 3), training exercises and hints were designed to promote recall and demonstrate transfer between problems with different rational number representations. The idea was that students had not thought about the problems in a long time and needed a probe or hint. The assumption was that students at a good state university might not have used their knowledge for awhile.

Consistent mathematical errors in our environment

People are not always aware of their current knowledge and how new knowledge may contradict it. If they do not recognize a need for changing their existing knowledge structure, learning is jeopardized; they will be more likely to ignore the demands of a novel problem and default to existing knowledge (Gallistel, 2007; Vosniadou, 1999). One such example is Leron's (2010) bat and ball problem: "A bat and a ball cost $1.10 in total. The bat costs a dollar more than the ball. How much does the ball cost?" This problem was originally introduced in Daniel Kahneman's 2002 Nobel Prize speech, as well as being further explored by Kahneman and Frederick (2005), the latter a former MIT Sloan School of Management professor now at Yale University. They were astonished to find that less than half of 3,000 subjects from eight different universities were able to correctly answer 5 cents. Instead, students went with their first instinct of 10 cents. Just to get an idea of some of the numbers and the subjects' intellectual abilities, 50% of Princeton students (47/93) and 56% of University of Michigan students (164/293) gave the wrong answer. So why do so many people who meet requirements for admissions to prestigious universities err on such a simple problem? One reason might be that students defaulted to natural number arithmetic by failing to consider the relationship between the bat and ball. In their mind, the ball was represented as x and the bat was represented as \$1 such that $x + \$1 = \$1.10$. The final cost ($x + x + 1 = \$1.10$) was dependent upon this relationship; one quantity was mapped onto another. This error might also be associated with heuristics and biases, namely, attribute substitution. Often, humans find it easier to rely on the automaticity of an intuitive judgment system, rather than reflect on more complicated problem (Kahneman, 2011).

Being able to understand the mathematical equivalences of different rational number representations becomes extremely relevant to consumers, especially around the holiday shopping season. During Black Friday (2011), known as the busiest shopping day of the year in the United States, one store offered consumers 40% off their entire purchase for the day. Come the weekend, a new deal was offered where shoppers could get \$30 off a purchase of \$75 or more. Many shoppers might have seen this value of \$30, a numerical number instead of a rational number (i.e., percent) and smacked their foreheads, thinking they should have waited for this sale. But in actuality, no more money was being saved. In fact, the customer was at a disadvantage if his/her purchase cost even one cent more than the $75 needed to get the discount. As 30/75 = 40%, a shopper would receive the same (if not a worse) discount than if they had shopped on Black Friday. And, of course, it follows that the store strategically marks their merchandise as $69.50, $74.50 or $79.50, just to name a few, so that the consumer *has* to spend more than the $75 in order to get the discount. The question becomes: how do shoppers decide what has more value – the percent of discount or the actual dollar amount discount? The answer is directly related to their knowledge of equivalent rational number representations.

"Percent" as a rational number

The previous example focuses attention on percent as a rational number format. The concept of "percent" as a number is interesting in and of itself. A percent, such as 40%, is visibly a number, but it only makes sense in terms of some referent or base value. The context defines "how much." For instance, the possibility of a university incurring a 1% budget cut might send some (most likely, the university's employees) into an uproar while others might not blink an eye. The 1% is contingent upon the base value. So, if the budget is 1.2 billion, a 1% cut means the university loses 120 million in funding! While 1% seems like a "small" number, 120 million certainly does not seem as insignificant. Yet, the two are equal. How can that be? Davis (1988) defines the representation of a percent as a function, much like a fraction where the relation between two variables or numbers is compared. People often think of 50% as ½ and sometimes fail to acknowledge that it is ½ *of* another value, ultimately yielding an entirely new number.

Thus, percent is a variable number whose value is dependent upon a base value, which can be ever–changing. Further, the meaning of the base value is tied to the domain. To illustrate, consider the time signature of 2 over 4 (2/4) in music. In music, 2/4 is equal to 6/8, but not 600/1200 (Gelman & Brenneman, 1994). Intervals are controlled by the theory of timing in music and not mathematical equivalences. This additional example further stresses that the domain matters, as does the representation (Gelman & Brenneman, 1994).

Jacobs and Gelman (2010) show that adults, for the most part, fail to calculate two-statement percent problems dealing with increases and decreases. Other research also has shown that people treat percents as a function of addition or subtraction of natural numbers, without taking into account that there is a multiplicative factor. This overextension of a mathematical rule is similar to another one mentioned earlier: when children apply the simple addition rule for whole numbers to fractions, instead of finding the least common denominator. Colby and Chapman (2011) designed their study around two promotional statements on a sign in the Bronx Zoo eco-restroom: 1) "A standard flush toilet uses 99% more water than a toilet that uses foam," and 2) "Uses half as much water." Did the Zoo mean to be redundant (as 99% more is equal to 1.99 times as much as the eco-toilet which is almost equal to "twice as much") or did they instead mean that the eco-toilet uses "1% of the amount used by the regular toilet?" The two are not equivalent; thus, the point of reference is pivotal. The direction from the referent, or base value, is just as important. Consider costs: moving from a less pricey item "*x*" to one that is 99% more will result in a more expensive item that is 199% of *x* (100% represents the original *x* plus "99% more *of x*"), a ratio of about 2:1. In contrast, moving from a more expensive item to one that is 99% less results in an item that is 1% of the original cost *x*, a ratio of about 1:100. Results from the study demonstrated that people tend to see 99% as almost as big as possible; it is close to 100%, which is erroneously believed to be a cap. Yet, it is simply a little less than twice as much as the original price.

Adults make this same error of using the wrong mathematical rule when looking a percent scale. As another example, in order to regain a 33% loss, as occurred in the 2000-2001 U.S. recession, a 50% gain would be needed to at least recover that loss. Kruger and Vargas (2008) attribute these errors to biases in judgment that increase especially as motivation decreases. The authors set out to show that while identical percents do not correspond to identical magnitudes, it is incorrect to also assume that different percents correspond to different magnitudes. In other words, if $150 is the referent, 33% less yields $100, whereas if $100 is the referent, 50% more yields $150. Yet, if an identical starting value of $100 is the referent for a 33% loss *and* a 50% gain, the resulting magnitudes are quite different ($67 and $150, correspondingly). These errors are analogous to treating the values as natural numbers rather than rational numbers. The result: undesirable consequences for the investor. Kruger and Vargas showed that the perceived differences can downplay a riskier investment or cause a mediocre investment to look more attractive than it really was; ultimately, the framing manipulation greatly influenced the consumer's purchasing decisions. They found that subjects were more likely to perceive a greater discount when given a less expensive referent as compared to when the more expensive item is the referent. The authors offer the loss aversion component of prospect theory as another interpretation of their experiments (Kahneman & Tversky, 1979). In mathematics and in logic, we learn that if

A is X greater than B, then B must be X less than A. Adults who overextend this rule when X equals a percent (as in Kruger & Vargas, 2008) are incorrectly applying natural number arithmetic (addition and subtraction) where it is not warranted.

Overview of dissertation research

This review served as an update on certain assumptions illustrating that the psychology of number has been studied in a number of ways. First, humans are equipped with a sense of positive natural number under addition and subtraction, much as they are endowed with an ability to learn language. Results from previous studies, along with the ones in the subsequent chapters, reveal a strong tendency for individuals to resist instruction about rational numbers. The error analyses for problems couched in percent terms help demonstrate that there needs to be a way to accomplish a conceptual change. The errors reveal a robust trend of children and adults alike to incorrectly reinterpret the problems as if they were confronted with a format for natural number. In doing so, they also illustrate a failure of understanding regarding equivalence classes of rational number.

It remains an open question as to whether schooling, in the interim, impacts this tendency to default to a natural number solution. The focus here is on the fact that many well-educated adults do not understand rational numbers. This is true with all representations, whether they be fractions, decimals, or percents. Much of my research centers specifically on college students' interpretation of percents and the errors made in their calculations with them. A main result is that a significant number of these students have little understanding of the different formats of these numbers.

In two-statement problems with successive percent increases and decreases, I expect adults to default to treating these values as natural numbers if they have not made the conceptual move from natural number concepts to rational number concepts. This is to say that their earliest competence will have interfered with their ability to learn more complex math concepts like percents and fractions. Since we do not have the innate ability to represent advanced concepts in math, I further expect a student's previous math experience to influence the errors made. By exploring different learning and training activities, students are able to see more than one example of a given problem; thus, (as in Chapter 3) they should be able to not only represent a problem at the algebraic level, but should also be able to move fluently between levels and representations. Finally, by taking a closer look at the patterns of conceptual errors (Chapter 4), I propose a framework for training studies about percents that encompasses the lessons learned from the previous and current research as well as the theories in Cognitive Science (Chapter 5). As the concept of number is greatly expanded when moving from natural to rational numbers, the conceptual framework that must be constructed is different from the one with natural numbers so as to embed it into the existing framework.

II. Exploring errors in one- and two-statement percent problems

The focus of this chapter is on the errors discussed in Chapter 1. To follow up, I ran a pilot study with a relatively small number of college students. They had an unexpected amount of difficulty working with percents. This finding provided the motivation for the two follow-up studies with large samples. The small pilot study and the first of the two subsequent studies are reported in this chapter. The second one is presented in Chapter 3. All studies focus on exploring the extent of difficulty undergraduates have with rational numbers, when they are presented in problems containing percents.

I hypothesized that students would make errors with the various representations of rational number, especially when they are in algebraic formats. In the Two-Statement Percent Study below, I expected participants to perform better on problems dealing with numerical values as this more closely resembles examples in the world around us.

Study 1: Pilot Study

A small training study was run with Psychology undergraduate students. Students (N = 27 undergraduates (25 females)[1]) were presented one-statement percent problems with fraction, decimal, and percent and bar graph formats. Examples of these are shown in Figure 2.1.1.

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[1] The bias for females was not expected.
[2] Natural number is a subset of rational numbers.
[3] SAS and all other SAS Institute Inc. product or service names are registered trademarks or trademarks of SAS Institute Inc. in the USA and other countries. ® indicates USA registration. The data analysis and output for the chapter was generated using SAS software, Version 9.3 of the SAS System for Unix. Copyright, SAS Institute Inc. SAS and all other SAS Institute Inc. product or service names are registered

Of particular interest was whether errors could reflect the use of incorrect rules and principles that underlie the participant's choice of algorithms for achieving answers. If so, these errors would be informative about the kinds of mathematical syntactic rules that were at risk.

Decimal example

- The stock you invested in in 2002 has now decreased by **.80** of its original value *x*. What is the new value of the stock? Please represent your final answer in *decimal* form.

Fraction example

- The stock you invested in in 2002 has now increased by **2/5** of its original value *x*. What is the new value of the stock? Please represent your final answer in *fractional* form.

Percents and graphs example

- The stock you invested in in 2002 has now increased by **10%** of its original value *x*. Please draw a new *bar on the graph* below so that its height shows the stock's value now (in "2009").

*Example of graph given with "percents and graphs" format **

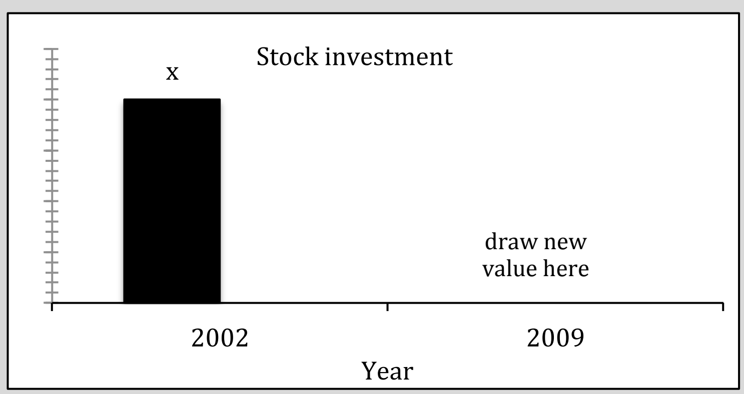

Figure 2.1.1. Examples of representational formats used in the pretest and posttest of the pilot study. The same word problems are given in this particular figure to highlight the different formats of rational number representations. The actual word problems seen each varied in content (i.e., problems pertaining to cost, health, ownership, etc.), representational format (e.g., fraction, decimal, percent), and rational number values (e.g., 0.80, 2/5, 10%).
*Note: Bar graphs were used in this pilot study.

Most often, we see percents represented in terms of decimals. Less frequently, but still somewhat often, we see them represented as fractions. The decision to use bar graphs, instead of another type of visual format such as a pie chart or a line graph, etc., followed from a desire to make the problems across representational formats as similar in structure as possible. Further, bar graphs are frequently seen in public media (e.g., the business section of newspapers).

The study was run in three phases as a pretest-training-posttest design. The test items were presented on paper and individuals could take as much time as they wanted. Questions on the pretest and posttest presented percent word problems containing one of three types of numerical formats: each participant saw two decimal problems, two fraction problems, and two bar graph / percent problems in *each* the pretest and posttest (see Figure 2.1.1 for examples). Only one format appeared per question. Training varied between participants so that they were exposed to practice with decimal formats ("decimal training") or fraction formats ("fraction training") between pretest and posttest. The training consisted of a worksheet that guided participants through analogous numerical and algebraic problems in the format of the training condition to which they were randomly assigned.

Were some formats easier to use than others?

Undergraduates performed significantly better on questions paired with bar graphs (M =80.60% of students answered these questions correctly) than on those with decimal (M = 53.30%) or fraction (M = 60.10%) formats for their pretest and posttest items, $\chi^2(2, N = 27) = 11.41$, $p = .003$. By definition, a bar graph is a visual display of data; the length of its bars is equal to the frequency of each element in a set of data. Thus, a plausible explanation for this finding might very well be that converting problems a value that resembles frequency, in the form of bar graphs, causes the problem to be easier to comprehend (Brase, Cosmides, & Tooby, 1998; Hoffrage, Gigerenzer, Krauss, & Martignon, 2002). The other two formats dealing with rational number (i.e., decimals and fractions) presented considerable difficulty to students.

Effects of training on performance

While decimal training (Figure 2.1.2, left panel) had little to no effect on performance across representations, fraction training (Figure 2.1.2, right panel) yielded significant improvements from pretest to posttest in every category: decimal (*M* = 25.00% correct to *M* = 53.60%), fraction (*M* = 35.70% to *M* = 57.10%), and bar graph with percent (*M* = 67.90% to *M* = 89.30%).

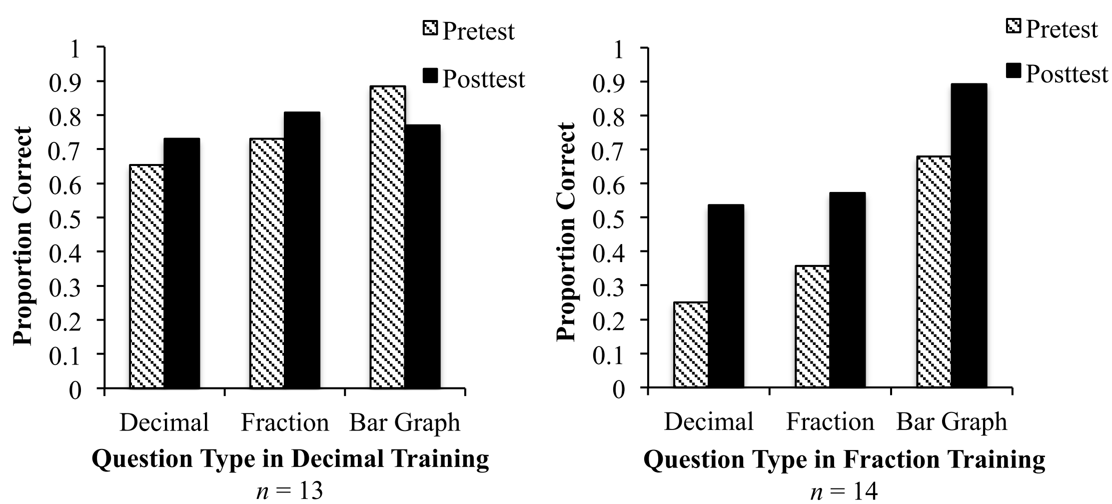

Figure 2.1.2. Pilot study: training and test interaction. Though there were few participants, a mixed logistic regression was run with accuracy (correct/incorrect) as the dependent variable (as in previous work by Jacobs & Gelman, 2010). Not only was there an effect of training (p = .049), but there was also an interaction of test (pretest versus posttest) and training on accuracy (p = .031). Fraction training (right panel) seemed to improve performance across representations from pretest to posttest moreso than decimal training (left panel).

An explanation for this might be in part, due to the exposure of multiple representations. As decimals are often the preferred representational format for percents, perhaps exposing participants to another representation of which they are less familiar, helped aid their conceptual understanding of using percents.

Two-statement percent problems

The study also included additional algebra problems in the pretest and in the posttest that were comprised of two-statement percent problems (e.g., a 50% increase of *y* and then later, a 50% decrease off the "new" amount) which were different from the one-statement problems (e.g., a 30% decrease of *y* or a 10% increase of *y*). One of these two-statement problems appeared in the pretest and another of these problems appeared in the posttest. In the two-statement problems, the effect of the second percent was dependent upon the outcome of the first percent statement. When two percent values appeared in a problem, participants did not perform nearly as well as they had when a single percent statement was presented in a problem. Less than 25% of the participants were able to successfully answer the two-statement percent problem.

The next study is dedicated to a focused follow-up of this class of errors.

Study 2: Two-Statement Percent Study

The results of the pilot study led to the hypothesis that students would perform poorly on percent problems containing two statements (i.e., where the second percent statement is dependent upon the result of the first). The results in the current study replicate and extend the findings on individual problems with both percent increases and decreases. Two-statement percent problems also varied on whether they contained an algebraic or numerical representation of a value. This study was not a pretest – posttest study; rather, it consisted of three questions.

In addition, there were two different numerical values across problems (either 50% or 30% appeared in each question) to determine whether values other than 50% increased error rates, possibly which people make without realizing it. Also of interest was whether the individuals changed the problems from a percent representational format to another format. Finally, individuals with higher math SAT scores and more previous math background (math level) were expected to do better overall.

Method

Participants

Data with a large, heterogeneous population was collected on two-statement percent problems with successive increases and/or decreases (e.g., "The cost of an item increases by 50%. It later decreases by 50%. What is the final cost of the item in terms of x?"). Data was collected from a secure online computer system (in a prescreen) from 1648 participants who had signed up for experiments in the Psychology department at a large university with a diverse population. Only data from adults (defined as being at least 18 years old) were looked at in this study as the IRB considers a participant under the age of 18 to be a child and thereby requires different approval. Thus, 19 participants

(<18 years old) were excluded from the final data set, leaving N = 1629 adults (807 females, 822 males).

Participants' previous mathematical knowledge (math level*)* was obtained for subsequent data analyses. Students were grouped in one of two categories: 1) below Calculus 1 (663 participants), or 2) Calculus 1 and above (942 participants). 24 of the participants did not type in a response for previous math course. Participants were also grouped by their Math SAT score (see Figure 2.2.1). 378 of the 1629 participants did not report their Math SAT scores.

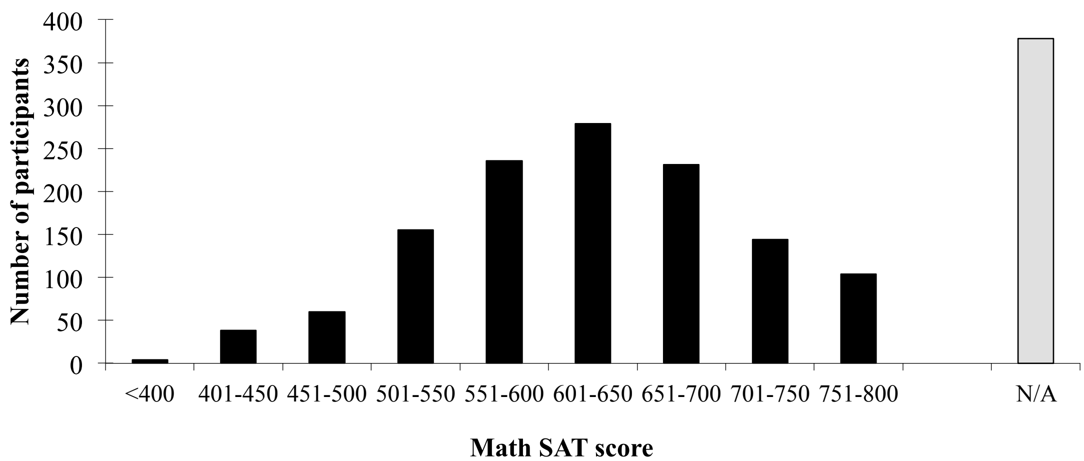

Figure 2.2.1. Participants' (N = 1629) math SAT scores revealed a normal distribution, representative of the diverse population at this large university. A subset of the group consisted of students who did not report their scores (n = 378); they were labeled as "N/A."

Materials and Design

Participants answered three test questions and additional demographic questions in an online prescreen necessary to partake in Psychology studies at Rutgers University – New Brunswick. The items within this particular block of questions were randomized.

Question. I manipulated the type of question with the aim to further explore the preliminary results found in pilot data, by looking at successive percent increases and/or decreases (Figure 2.2.2). In earlier pilot work, there was no order effect of direction (i.e., “increase by __% then decrease by __%” vs. “decrease __% then increase __%”) within a problem with multiple percent processes. Thus, all questions in this study were phrased as first a percent decrease and then a percent increase. The 50% – Algebraic question, was the same 50% increase/decrease algebraic question that also appeared in the pilot study. The 50% – Numerical question embedded numerical values within the 50% – Algebraic question. And the 30% – Algebraic question was similar to the 50% – Algebraic, only it presented 30% values rather than 50% values. The order in which these questions were seen was randomized.

50% – Algebraic question

- **The original cost of an object is "y" dollars. At first, that cost is reduced by 50%. Later that new cost is then increased by 50%. Please represent the final cost of the object using the variable "y".**

50% – Numerical question

- **The original cost of an object is $100 dollars. At first, that cost is reduced by 50%. Later that new cost is then increased by 50%. What is the final cost of the object?**

30% – Algebraic question

- **The original cost of an object is "y" dollars. At first, that cost is reduced by 30%. Later that new cost is then increased by 30%. Please represent the final cost of the object using the variable "y".**

Figure 2.2.2. The three questions seen by all participants ($N = 1629$) in the Two-Statement Percent Study. These questions were randomized in a prescreen task on the computer. Participants typed in their responses, which were not limited in length. Red font is used in this paper to illustrate the differences between questions; in the actual task, the font was black.

Coding. To start, 4887 responses to the test questions (1629 participants x 3 questions) were coded separately by two coders; inter-rater reliability was greater than 99.88%. Disagreements were coded again by a third coder and this information was used to resolve the differences. All three coders had extensive mathematical backgrounds. The tasks facing the coders were separated into the following categories:

i. *Accuracy.* Responses were grouped into one of two categories, correct or incorrect, for performance on each of the 3 questions.

ii. *Response representation*. Since the participants were given open-ended questions, their responses were sorted according to type of representation: decimal, fraction, percent, integer (which includes an "invisible" coefficient of "1" in front of a variable), multiple representations, verbal expression or statement, and "none of the above."

Results

There were several major findings in this study. As accuracy responses were correct or incorrect, chi-square tests within a logistic regression were used to evaluate performance.

How did accuracy differ between questions?

First, there was an effect for type of question ($\chi^2(1, N = 1629) = 152.39$, $p <$ .0001); undergraduates' accuracy was higher when dealing with numerical values rather than algebraic representations (50% – Algebraic: $M = 54.57\%$ correct, 50% – Numerical :

M = 75.14%, 30% – Algebraic: M = 37.57%; see Figure 2.2.3). Here, as compared to the pilot study, when more than sixty times as many adults were asked to give an algebraic solution to a 50% – Algebraic problem, only 892 out of 1629 participants were accurate. Performance was even worse when the same question was asked with other percent values (e.g., 30% in 30% – Algebraic), dropping to only 612 out of the 1629 participants arriving at a correct answer. Interestingly, when students were able to use a numerical value of $100 (i.e., in "Calculate with original price of $100"), their performance increased and 1224 out of the 1629 participants were accurate.

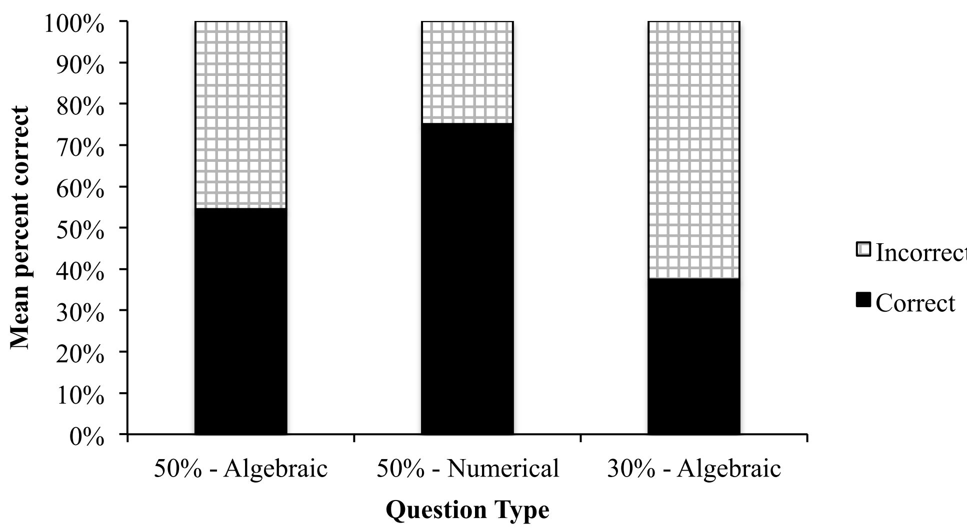

Figure 2.2.3. Mean accuracy on three questions (N = 1629). While participants performed better on the 50% – Numerical question than the other two, notice that no question results in scores at ceiling.

Was there an effect of math SAT or math level on accuracy?

A logistic regression was run with accuracy as the dependent variable, question type as the within-subject independent variable and math SAT as a continuous covariate.

With the exclusion of participants who did not provide their math SAT score, an otherwise normally-distributed population yielded an effect for math SAT score, $\chi^2(1, N = 1629) = 142.41$, $p < .0001$. Those with lower math SAT scores showed poorer performance across questions. An interaction was also found for math SAT and question, $\chi^2(1, N = 1629) = 45.57$, $p < .0001$, where the effect of question type was smaller for those participants with higher math SAT scores. (Figure 2.2.4)

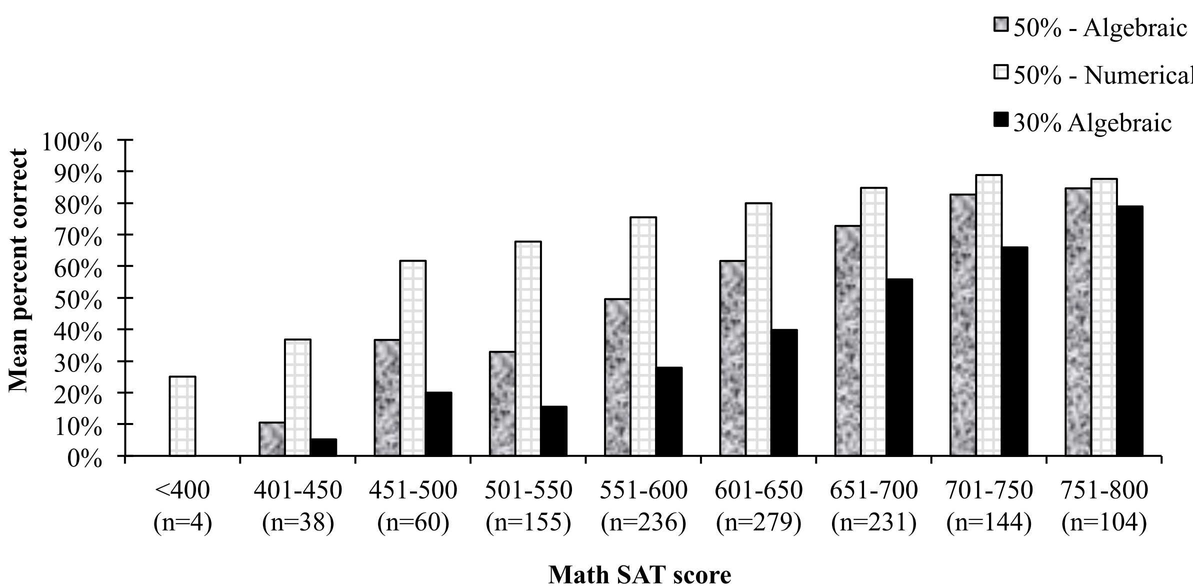

Figure 2.2.4. Effect of math SAT score and question on accuracy. Bars represent the three questions and are organized by math SAT score. Students with higher scores performed better on the questions.

A logistic regression was then run with accuracy as the dependent variable, question type as the within-subject independent variable and math level as a continuous covariate. Not only was there an effect for math level for those students who reported their scores ($\chi^2(1, N = 1605) = 22.34$, $p < .0001$), but there was also an interaction of math level and question ($\chi^2(2, N = 1629) = 6.57$, $p = .0375$). Of the 942 participants who have taken at least Calculus I, 69.32% answered the 50% – Algebraic question correctly, 82.91% answered the 50% – Numerical question correctly, and 49.89% answered the 30% – Algebraic question correctly. Out of the 663 participants who have taken a math course lower than Calculus I, 34.99% answered the 50% – Algebraic question correctly, 64.71% answered the 50% – Numerical question correctly, and 21.12% answered the 30% – Algebraic question correctly. The effect of question type was smaller for participants who had taken Calculus I than for those participants who had taken below Calculus I.

Did response representation influence accuracy?

Next, I looked at the types of representations that participants used when expressing their answers. Figure 2.2.5 shows the number of participants using each of the six representation types. It is evident that most answers were represented in a decimal format (e.g., 0.75*y*). Indeed, the six representation types differed in their frequency, ($\chi^2(6, N = 1629) = 551.66, p < .0001$). In addition, the distribution across the representation types differed between the two algebraic problems ($\chi^2(6, N = 1629) = 144.65, p < .0001$, Figure 2.2.5) with the 30% – Algebraic problem showing more decimal and fewer fraction representations than the 50% – Algebraic problem.

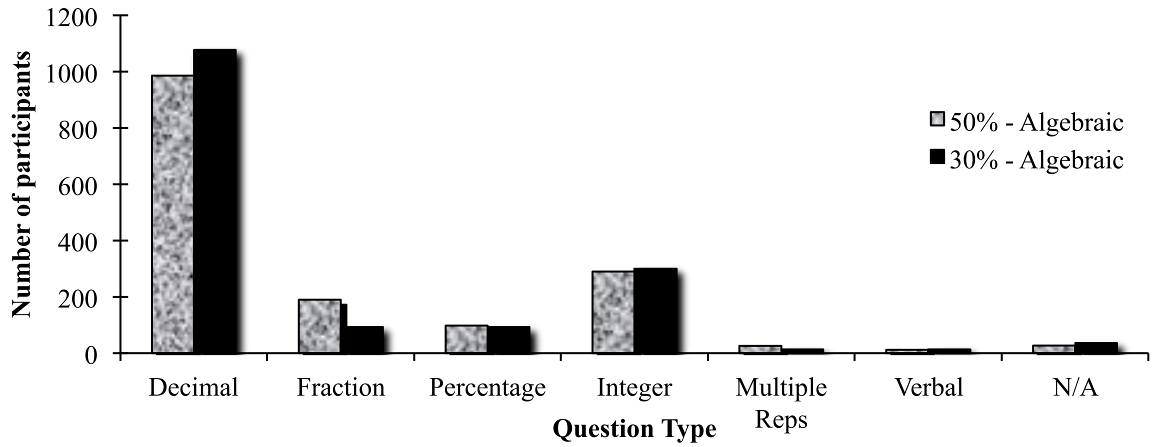

Figure 2.2.5. Patterns of response representation: with which representation did participants (N = 1629) respond? The majority of responses were rewritten as decimal representations. Those who did not fit in any of the six categories were coded as “N/A.”

Was there a gender difference on accuracy?

Chi-square tests revealed gender differences on accuracy across all three of the test questions. Men outperformed women on the 50% – Algebraic question (64.2% correct compared to 44.7% correct; $\chi^2(1, N = 1629) = 62.46$, $p < .0001$), the 50% – Numerical question (81.6% correct compared to 68.5% correct; $\chi^2(1, N = 1629) = 37.44$, $p < .0001$) and the 30% – Algebraic question (48.5% correct compared to 26.4% correct; $\chi^2(1, N = 1629) = 85.15$, $p < .0001$).

What was the effect of showing “scratch work” on accuracy?

As the responses were typed into a computer, a surprising number of participants showed an extensive amount of scratch work, working out the problem in full, rather than computing the final answer with mental math (e.g., as in Anderson, Reder, Simon, 1996; Lave, 1988). Some responses were simplified (e.g., (¾)y, .75y) and others were left unsimplified (e.g., (75/100)y, .50y + .25y, (y – .5y) + .5(y – .5y)). Thus, a category called "work shown" was created. Participants were grouped into two groups: those who illustrated the processes by which they arrived at the correct or incorrect answer (whether it be addition, subtraction, multiplication, division, FOIL-ing {a mnemonic for a series of ordered distributive processes}, a combination of these, and/or a verbal description) were coded as "showed work" and those who simply gave a final answer were coded as "did not show work."

Whether or not participants showed their scratch work yielded the opposite effect of what would be expected. Those who did not show work actually performed better, resulting in an effect of work shown on accuracy ($\chi^2(1, N = 1629) = 12.54, p < .0001$). Participants who did not show their work in the 50% – Algebraic question were more likely to arrive at the correct answer than those who did show their work. In the 30% – Algebraic question, participants were also more likely to arrive at the correct answer if they did not show their work than those who did. (Figure 2.2.6)

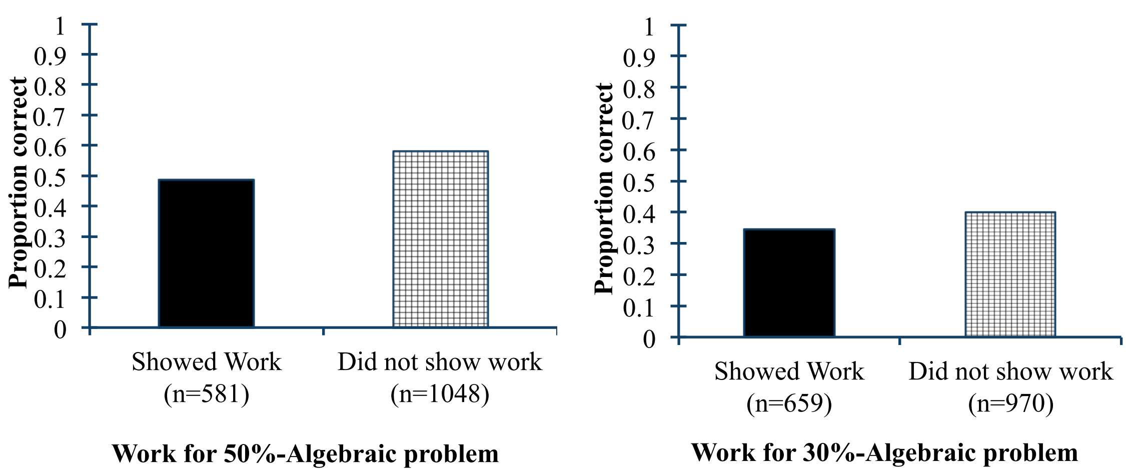

Figure 2.2.6. Work shown on accuracy in two-statement percent problems (computer task).

DISCUSSION

When increase and decrease problems were presented individually in one-statement percent problems, there was no significant difference between participants' performance on increase problems versus participants' performance on decrease problems. *However*, the results from the 50% and 30% successive increase and decrease problems in the Two-Statement Percent Study hint at more difficulty with problems involving multiple processes. What are the reasons for this discrepancy?

For the 50% increase/decrease problem (as well as the 30% – Algebraic question), the most frequent incorrect responses given were "x," "it remains 'x' dollars, the original cost at the start of [season]," or "back to the original price." An interpretation of this result stems from participants incorrectly treating these problems as natural number arithmetic problems where $x + y - y = x$, or more specifically, $x + 50 - 50 = x$. This error was illustrated in students' incorrect raw answers (for some examples, see Figure 2.2.7). It demonstrates a failure to move from natural number concepts to rational number concepts. These results provide a very strong clue that these adults would not be very good at dealing with any two-statement percent problem with the following format: *(cost)* decreased by *x*% <u>and</u> (*cost*) increased by *x*%.

The answers given to these increase and decrease problems are quite astounding. This kind of example occurs in everyday life and yet, the results here strongly suggest a lack of understanding, quite possibly due to the influence of simple rules of arithmetic. The errors committed reflect a real problem with comprehending figures about the stock market, a budget and almost anything that is numerically-related to a person's daily life.

$y - .50 + .50 = y$

$y + (1/5)y - (1/5)y = y$

$y - (1/2)y = (1/2)y$ " then " $(1/2)y + (1/2)y = 1y$

$y - (y \times .50) + (y \times .50)$

$[y - (y*0.50)] + [y + (y*0.50)]$

$(y*.5) - 50 + ((y*.5) + 50)$

$y - (y / 50) + (y / 50)$

Figure 2.2.7. Examples of students' incorrect responses on the 50% – Algebraic question. Each line represents a different student's response. All students' work shows their default to natural number arithmetic.

We know there is a strong tendency of elementary school children to interpret fraction problems as natural number problems (see Chapter 1). It has remained an open question as to whether school, in the interim, impacts this tendency to default to a natural number solution. Unlike elementary school children who have been in a curriculum trajectory that has involved various kinds of instruction about rational numbers (Common Core State Standards Initiative, 2010), adults do get a great deal of experience with percent increase and decrease problems in their everyday life, along with any higher math courses they have taken. Thus, the seeming robustness of the tendency to default to a natural number solution is most surprising.

Consequently, the results in the current study revealed that math knowledge level, along with math SAT scores, might be important predictors of success. This may be the result of more experience with numbers and algebraic formats and, hence, more fluency and familiarity with mathematics, especially as it appears in our daily lives. From the findings in this study, it follows that either more math background has a positive influence on participants' ability to solve novel problems or that those with the ability to solve these problems are more likely to take more advanced math courses. Similar results have been found in previous research.

In general, participants struggled across questions, especially those dealing with algebraic values rather than numerical ones. The ability to "plug in" a specific numerical value (e.g., $100) might explain higher performance on the 50%–Numerical question as compared to the other two non-numerical questions. Plugging in a value is most applicable to what is done in everyday life. Substituting a numerical value in for a variable was employed even in solving the 50% – Algebraic and 30% –Algebraic questions. In fact, giving an integer (or natural number) response was the second most common response representation (Figure 2.2.5).

The responses for the two algebra problems were overwhelmingly written in decimal even though each question was presented in a percent format. In Chapter 3, I take advantage of this result by presenting some participants with both algebraic and numerical problems to see if exposure to more than one format increases their performance.

Researchers assume that simple arithmetic is relatively easy for children to learn about because it is based on a core domain of positive integers that combine with addition (Gelman, 1998). More complex concepts, like fractions and percentages, etc., do not map onto existing concepts (e.g., Carey, 2009; Gallistel & Gelman, 1992; Gelman, 1991). But understanding alternative concepts of number and their formats (e.g., fractions) is critical for effectively working with algebraic techniques. It is widely known among educators that rational numbers are a watershed. Children who cannot master them are unlikely to go on to learn mathematics. They no longer can resort to their fingers as a counting prop and diagrams of pizza pies (which appears frequently in textbooks across the United States) are really part-whole counting props. In comparing students of both high and low math abilities across Key Developmental Understandings (KDU) constructs, Simon (2006) stresses that it is crucial to understand the idea of a rational number (here, a fraction) as a quantity, rather than an arrangement. For example, whether a student is splitting a square into two triangular halves or a square into two rectangles, the half that is a rectangle and the half that is a triangle are each half of the same whole so they are equivalent. While, students can easily learn that the parts must be equal, they need to understand correspondence between numerator and denominator. The difficulty lies in understanding fractions as a *quantity*. The same difficulty was seen with percents in this study.

Concerning the rules governing math principles and their operations, there was a significant demonstration of a failure to understand the logic of multiplying by a percent. Similar to Kintsch and Greeno's (1985) findings, false cognates may be hindering student performance. A problem that involves increasing by a percent might convey processes of addition rather than of multiplication. This might be a plausible error since multiplication can be converted to the repeated addition of integers. Division differs, however. Errors in the Two-Statement Percent Study were sometimes the result of not understanding division. One example, where a 50% decrease was represented as "*y*/50" (Figure 2.2.7), illustrates the student's difficulty with concepts of division, decrease, and percent. What is happening here? Apparently, the rational number was erroneously converted to a natural number. Yet, that is not possible since nonzero rational numbers are closed under division such that another nonzero rational number always results. So, while students seem to understand that the base value y must be more if increased and less if decreased, they err in retrieving the knowledge structure consistent with the rational number input (Hartnett & Gelman, 1998).

In response to the algebraic questions, many participants entered work on the computer. This provided the coders some insight as to how they arrived at the final answer. Therefore, "work shown" was coded for insight about the nature of the students' errors. It turned out that those who showed work (compared with those who did not show any work at all) did not know what they were doing. Individuals who showed their work often made mistakes, contrary to what is usually taught and assumed in Mathematics Education. However, in this study, participants had to type their responses. It is quite possible that participants might have put more scratch work down on paper before keying their responses into the computer; thus, their final answer did not reveal how much work was needed. In the following study (in Chapter 3), a switch was made to paper and pencil tests in order to make it easier for people to show their work. As will be seen, coders were better able to arrive at an error-coding scheme by looking at the scratch work that was made alongside a question. Of interest is whether the solution paths detected will align with the expert – novice distinction. That is, do successful individuals recognize the relevant information from the problem and employ efficient solution strategies, and do poor performing individuals incorporate irrelevant information into their lengthier solution strategies? Retrieving relevant information can greatly affect student performance. Examples of efficient versus inefficient strategies from the Two-Statement Percent Study can be seen in Figure 2.2.8 (Patel, Jacobs & Gelman, 2011).

Here, lengthier solutions were characterized by employing the addition rule and shorter solutions were characterized by the multiplication rule (with addition performed mentally). Either way, the results on what is seen in work shown will have implications for theories of conceptual change.

In sum, the findings relating to the default to natural number in the current study were surprising given the participants are at a good college. Perhaps, they were surprised to encounter the test items in the same preliminary test pool with a large number of social psychology items. If so, simply providing a hint in between a pretest and posttest should help them catch on and perform better. This is done in the study presented in Chapter 3. The hint also serves a second purpose: it offers people a second notational format. Overall accuracy, problem-solving representational formats, and types of errors will be specific areas of analysis. The focus will be on specific error patterns, the effect of problem-solving representational formats, and types of problems that everyone finds easy. The data found in my studies highlight the fact that even undergraduates can have persistent problems with rational number concepts and notations.

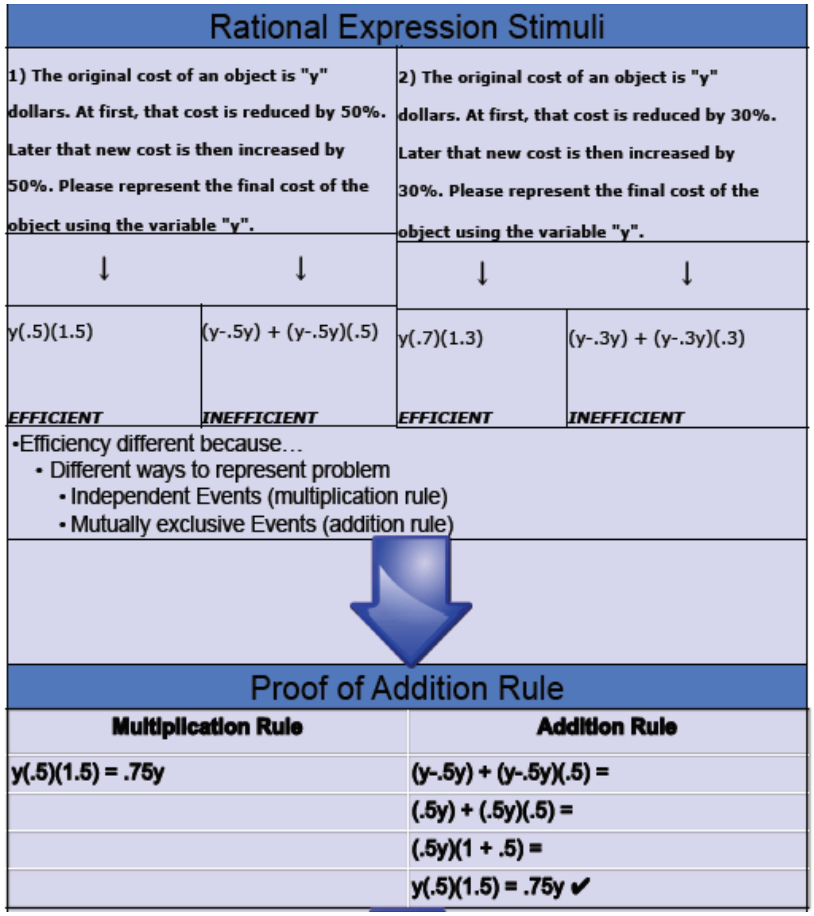

Figure 2.2.8 Efficient versus inefficient solution strategies that appeared in Patel, Jacobs, and Gelman (2011). Given a problem where there is a cost decrease of 50% followed by a cost increase of 50%, the answer, using an algebraic term of "*y*", would be .75*y*. However, instead of using an efficient multiplicative process (e.g., *y* * (.5) * (1.5)), beginners choose a complicated, less efficient additive process (e.g., (*y* – .5*y*) + (*y* – .5*y*)*(.5)). Though both responses are correct, the latter yields more errors when it comes to simplifying an equation.

III. Two-Statement Hint Study
Part I: Hint strategies

Introduction

Teaching students to recognize different methods for solving a problem helps them develop a greater understanding of concepts (Gelman, 1986). Although the processes that lead up to a solution might be different, they must all share common principles and honor the underlying constraints on any algorithm. Often, children memorize techniques for arriving at these solutions. Accordingly, they tend to have more success on one solution technique than on another. This was illustrated in Chapter 2 – participants were more successful on numerical problems than on algebraic problems.

The algebraic problems require students to represent their responses in a rational number format of their choosing, as there is often more than one correct solution representation. As an example, a 20% decrease could be re-represented in decimal notation (e.g., $y - .20y = .80y$), in fraction notation (e.g., $y - (1/5)y = (4/5)y$), etc. However, the nature of the problem changes when a student is asked to "plug in" a value, as in the 50% – Numerical problem. Substituting in a value (e.g., 100) for the variable y causes the rational number (e.g., $.80y$) to take on the format of a natural number[2]: 100 – 20 = 80. Students who do not fully understand rational numbers might not realize that they are still working with a rational number problem. Those who have undergone a conceptual change from natural numbers to rational numbers can more easily recognize the similarities of the solution structures than those who have not; thus, they are more fluent in moving between the representations.

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[2] Natural number is a subset of rational numbers.

As the participants in my studies were all adults, it must be assumed that they have learned about rational number. This aligns with the Common Core State Standards (2010) for math; students should be able to solve real-life and mathematical problems using numerical and algebraic expressions and equations. Relevant to these dissertation studies, one standard "CCSS.Math.Content.7.EE.B.3" provides an example that demonstrates the conversion between formats: "If a woman making $25 an hour gets a 10% raise, she will make an additional 1/10 of her salary an hour, or $2.50, for a new salary of $27.50." This example illustrates what a student should know by the time they leave the seventh grade.

Participants have been taught rational number concepts in school and they have also had experience with these values in every day life. Therefore, perhaps only a reminder is needed to provide a hint for solving these problems. In the current study, I utilized a concept on which students were more successful (e.g., natural number concepts in Chapter 2) and employed it as a hint between a pretest and a posttest of rational number problems (e.g., substituting in a value for the variable). I hypothesized that providing a numerical hint might cue participants to recognize the similar knowledge structures between the numerical and rational number problems; thus, their performance would increase on posttest. As in the previous chapter, it was also of interest to look at the variables affecting participants' accuracy, such as showing one's work, math level, and solution strategies.

Method

Participants

The participants were 134 undergraduates from a large university with a diverse population. Participants were recruited from undergraduate Psychology courses and were compensated through either extra credit in their respective courses or through fulfillment of a required number of credits all Psychology students at this university must take. Six participants voluntarily withdrew and four participants were excluded due to incomplete data, leaving $N = 124$ (81 females, 43 males). 57 participants (44f, 13m) were at a Math level below Calculus I, 46 (27f, 19m) took Calculus I, and 21 (10f, 11m) took Calculus II or higher. All participants were at least 18 years old, fulfilling IRB requirements in place for this study. Human subjects' approval and all informed consents were obtained before the study began.

Design and Procedure

The study involved four phases in which participants were asked to answer questions with paper and pencil, and without the use of a calculator. Phase I served as a pre-test, Phase II as a hint phase, Phase III as a post-test, and Phase IV as a collection of demographic information, previous math background information, and confidence levels. There were 8 test questions in each test phase (Phase I and Phase III), 4 questions in the hint phase (Phase II), and 5 questions in Phase IV. Similar to the Chapter 2 studies, other variables of interest included previous math level, math SAT score, gender, time (to complete the tasks), and confidence level in performance.

Test was a within-subjects variable in which all participants saw both a pre-test in Phase I and a post-test in Phase III. The 16 test questions consisted of algebraic word problems following the general format of "There are *y* units. *y* is increased by (__)%. *y* is then decreased by (__)%. What is the final value, in terms of *y*?" This follows the same question structure that appeared in my earlier studies. All questions were approximately the same number of words and same length so length of question would not be a factor. Additionally, the language used for mathematical procedures was identical in each problem (i.e., the use of "decrease" instead of "discount" and "increase" instead of "improved" or "rose"). Questions were counterbalanced for the variable order so that participants in order 1 saw questions A-H (see Table A.1) in their pre-test and questions I-P in their post-test. Participants in order 2 saw questions I-P as their pre-test and A-H as their post-test. Direction (a decrease by a percent followed by an increase by a percent, or vice versa) and the size of the percent values seen were also counterbalanced in the questions seen in each test (pre and post). For percent, two-digit values that end in 0 were used (10%, 20%, …80%) and participants saw different combinations of these values; each percent pair was seen only once. These combinations were also counterbalanced for "large-small," a variable defined by whether a large percent or a small percent is seen first in the problem (see Table 3.1 below).

The problems and the task

Before beginning, participants were walked through an overall description of what would be seen and what was expected of them. They were reminded that this was not a test and that there was more than one way to solve the problems. Participants were encouraged to use the ample space provided on sheets of paper to show their work if needed. The experimenters told participants that the researcher's goal was to look at how undergraduates solve these problems. They were also given the option to share their email addresses if they were interested in learning about the results of the study in the future. Students were encouraged to take as much or as little time as needed and were timed for how long they took to do the problems.

Participants were told not to go back to a previous section (e.g., Phase I) once they moved onto a new one (e.g., Phase II). As most of the participants were run in either groups of two or larger, they were all asked to sign an honor code; certain aspects of the environment, such as honor codes, make morality more salient and can encourage honest actions. Simply exposing participants to a moral code can drastically reduce cheating; signing a moral code can eliminate it entirely (Shu, Gino, & Bazerman, 2011).

Table 3.1
Test design with phase layout

Phase for Order 1	***Question***	***Percent***	***Direction***	***Phase for Order 2***
Pre-test	A	50%, 10% *	Increase, Decrease	Post-test
Pre-test	B	40%, 60%	Decrease, Increase	Post-test
Pre-test	C	20%, 30%	Increase, Decrease	Post-test
Pre-test	D	70%, 20% *	Decrease, Increase	Post-test
Pre-test	E	80%, 40% *	Increase, Decrease	Post-test
Pre-test	F	30%, 70%	Increase, Decrease	Post-test
Pre-test	G	60%, 50% *	Decrease, Increase	Post-test
Pre-test	H	10%, 80%	Increase, Decrease	Post-test
Post-test	I	40%, 70%	Increase, Decrease	Pre-test
Post-test	J	50%, 80%	Decrease, Increase	Pre-test
Post-test	K	30%, 10% *	Increase, Decrease	Pre-test
Post-test	L	10%, 60%	Decrease, Increase	Pre-test
Post-test	M	80%, 20% *	Increase, Decrease	Pre-test
Post-test	N	20%, 40%	Increase, Decrease	Pre-test
Post-test	O	60%, 30% *	Decrease, Increase	Pre-test
Post-test	P	70%, 50% *	Increase, Decrease	Pre-test

Note: The problems were counterbalanced so that some questions had a larger percent appearing before a smaller percent* (e.g., 80% increase followed by 40% decrease) and other questions had a smaller percent appearing before a larger one (e.g., 30% increase followed by 70% decrease). The table also illustrates that Direction and Test were counterbalanced.

Hint Phase

There were two groups in the hint phase (Phase II) and participants were randomly assigned to one of the two.

i. *Control*. The control condition involved repetition of what was seen in the pre-test, a continuation of problems presented in an algebraic format (i.e., “Solve in terms of y”). There were 4 questions in this condition. No feedback was given. Repetition was chosen as the control because that has been what is customarily taught and encouraged in many math classes. Two versions of this group condition, order 1 and order 2, counterbalanced test.

ii. *Numerical hint*. There were also 4 questions in this condition, only instead of seeing “*y*,” participants saw a numerical value (e.g., 100 tickets). No feedback was given here, either. Two versions of this group condition, order 1 and order 2, counterbalanced test.

Coding responses

The codes were established by studying the participants’ data sheets. A small sample of responses was compared by at least two coders to define levels for each variable; thereafter, responses were coded with set guidelines by one experimenter. Student work on each problem was coded for:

i. *Accuracy*. Responses were coded as correct or incorrect on each of the 16 questions appearing in Phase I and Phase III.

ii. *Work shown*. All participants’ responses for the 16 questions in both phases were coded into one of the three levels:

a. Showed no work

b. Showed a little work (i.e., simple arithmetic or one to two non-repeating lines of work)

c. Showed a lot of work (i.e., more than two lines of non-repeating work)

Results

Data were analyzed using general linear models that were run using the *proc genmod* command in SAS/STAT® software.[3] This procedure is used to fit a generalized linear model (GLM) to the data by maximum likelihood estimation of the parameter vector β. As there were multiple dichotomous dependent variables (the 16 questions), a binary distribution (*dist = bin* command) and a logistic (*link = logit*) were embedded into the general linear models. Therefore, this procedure fit a model to repeated categorical responses that were correlated and clustered; the output yielded generalized estimating equations (GEE), an extension of the GLM. Accordingly, quasi-likelihood estimation was used rather than maximum likelihood estimation. A quasi-likelihood estimate of β arises from the maximization log likelihood estimation without assuming that the response is normally distributed. As the GEE is an estimating procedure, empirical estimates of the standard errors and covariances were looked at and compared with the model-based estimates (the latter was done using chi-square tests and likelihood ratio tests).

[3] SAS and all other SAS Institute Inc. product or service names are registered trademarks or trademarks of SAS Institute Inc. in the USA and other countries. ® indicates USA registration. The data analysis and output for the chapter was generated using SAS software, Version 9.3 of the SAS System for Unix. Copyright, SAS Institute Inc. SAS and all other SAS Institute Inc. product or service names are registered trademarks or trademarks of SAS Institute Inc., Cary, NC, USA.

The strategy for variable selection entailed fitting a sequence of models, beginning with the simplest. The importance of each additional explanatory variable was assessed and then kept (if important) or removed from the model (if it made it weaker). Additionally, nonparametric tests such as chi-square tests were used to look at differences between variables and Spearman's rank order correlation coefficient (i.e., Spearman's rho) was used to look at the relationship strength between variables. (See Table B.1 for correlations.)

While math SAT scores and math levels were collected from participants, both were not included in the GEE model as they were quantifying similar measures ($r_s[91] = .52, p < .0001$). Thus, each variable was added to the model separately to measure its importance; math SAT score resulted in a weaker model whereas the inclusion of math level generated a stronger model. Math level was also more highly correlated with accuracy ($r_s[124] = .48, p < .0001$). Therefore, math level was used in the analyses rather than math SAT score.

The GEE model

Preliminary analyses were run in an effort to exclude design variables such as order, direction, the number fifty, and the large-small comparison from the GEE model. These variables were looked at across all questions rather than across subjects. Hence, the large N is attributed to 124 participants times the number of questions answered (out of 16). There was no effect of order on accuracy ($\chi^2(1, N = 1979) = 1.14$, n.s.). In other words, there was no difference for whether participants received questions A through H first or I through P first in this study. There were also no effects on accuracy of having 50% in the problem ($\chi^2(1, N = 1979) = 0.17$, n.s.), nor for direction on accuracy ($\chi^2(1, N$

= 1979) = 0.78, n.s.), nor for large-small on accuracy (χ^2(1, N = 1979) = 0.12, n.s.). Thus, the variables of order, direction, fifty, and large-small were not included in the GEE full model.

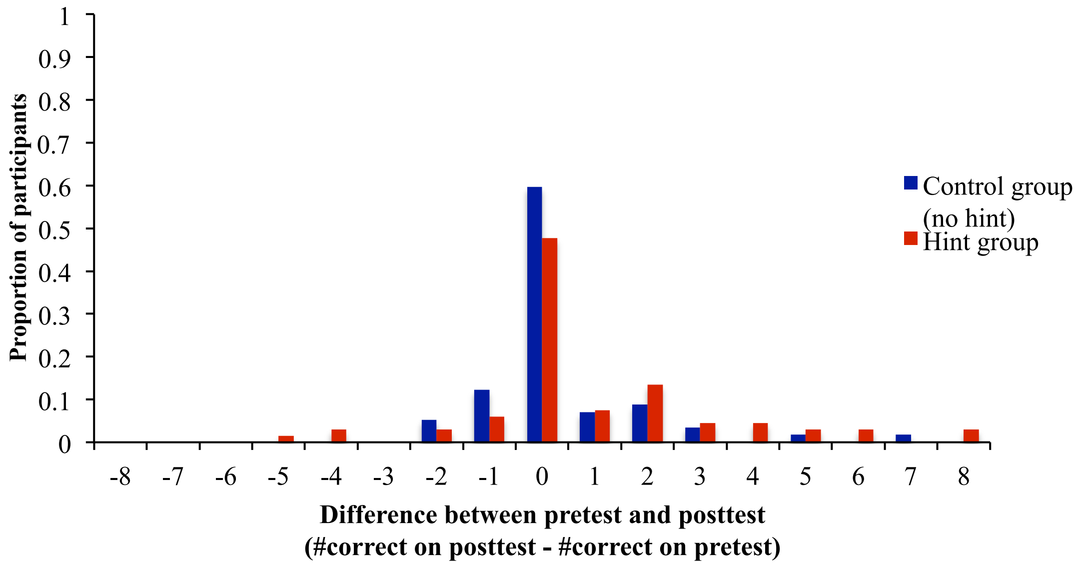
In the full model below (Table 3.2), I looked at the empirical estimates of hint phase, hint phase score, test, work shown, math level, and gender on accuracy. Gender was not significant (β = -0.20, Z = -0.61, n.s.). Contrary what was expected, there was no interaction of hint phase and test such that performance was no different in the pretest and the posttest, regardless of membership in the hint phase groups (control group or hint group). No reliable interactions were included in the table below as they lessened the strength of the models.

Table 3.2
GEE parameter estimates: Full model (N = 124)
Output of hint phase, hint phase score, test, work shown, math level, and gender on accuracy

Parameter		β	SE (β)	95% Confidence Limits		Z
(Intercept)		-1.35	0.45	-2.23	-0.46	-2.98***
Hint phase		-1.12	0.35	-1.81	-0.43	-3.19**
Hint phase score		1.12	0.14	0.85	1.38	8.15***
Test	post	0.16	0.16	-0.15	0.47	0.98
Test	pre	(ref.)				
Work shown		1.09	0.23	0.65	1.54	4.80***
Math level		0.92	0.21	0.51	1.33	4.37***
Gender		-0.20	0.32	-0.83	0.43	-0.61

Note. '*' = p < .05, '**' = p<.01, and '***' = p < .001

Hint phase, hint phase score, work shown, and math level were all significant predictors of accuracy. Students' accuracy was consistent across questions such that there was no difference between their pretest and posttest (test) accuracy (χ^2(1, N = 1979) = 1.07, n.s.) and thus, there was no effect of test, (β = 0.16, Z = 0.98, n.s.). Figure 3.1 shows that the majority of participants scored the same number of questions correct on the posttest as on the pretest ($\chi^2(1, N = 832$ correct responses$) = 0.10$, n.s.); this was the case for the control group (176 correct responses on pretest, 191 correct responses on posttest) and for the hint group (228 correct responses on pretest, 237 correct responses on posttest).



(a)

(b)

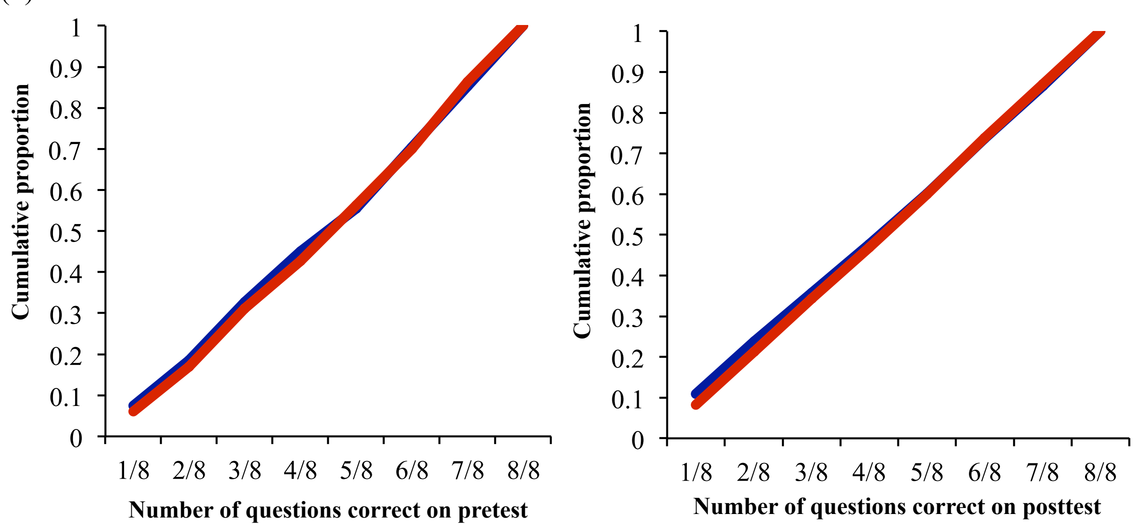


Figure 3.1. Test consistency in hint phase groups. (a) The top panel reveals that the majority of participants received posttest scores similar to their pretest scores. There is a slight skew to the left for the hint group. (b) In the bottom panel, cumulative relative frequency graphs also show little to no difference between the control group (in blue) and the hint group (in red) across the 8 questions in pretest (left) and 8 questions in posttest (right).

Hint phase and hint phase score

Unexpectedly, the control condition (i.e., repetition) for hint phase seemed to be a greater cause of accuracy than the hint condition where participants were given a numerical hint, according to the full GEE model in Table 3.2 (β = -1.12, Z = -3.19, p = 0.001). Yet, a chi-square test follow-up revealed no differences between hint phase groups on overall percent accuracy across all sixteen questions ($\chi^2(1, N = 1979) = 2.25$, n.s.) (see top panel of Figure 3.2).

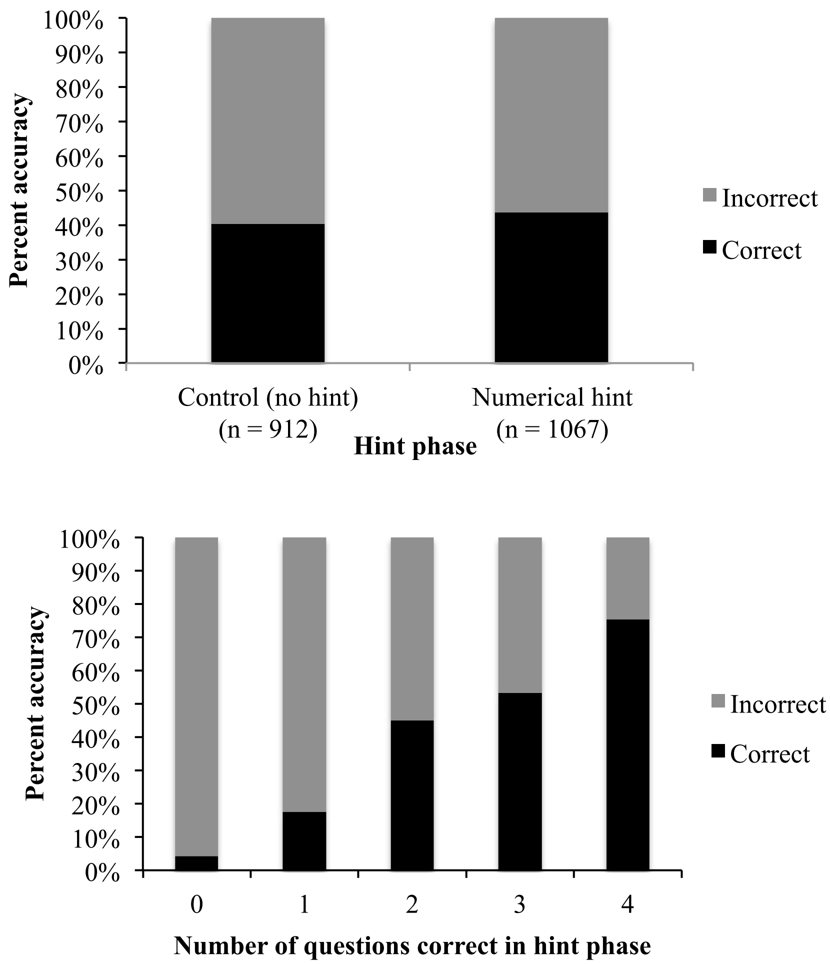


Figure 3.2. The effects of type of hint and hint score on accuracy. The top panel reveals no difference in performance between the control and numerical hint phases. The bottom panel demonstrates how correctly answering more questions in the hint phase was positively related to total accuracy across pretest and posttest.

In taking a closer look, participants who were more accurate on the hint phase problems also tended to do well on the pretest and posttest problems ($\beta = 1.12$, $Z = 8.15$, $p < 0.001$). There was also a significant difference between the hint phase scores on accuracy across all questions, lending further support to the findings in the GEE model ($\chi^2(4, N = 1979) = 677.06$, $p < .0001$) (see bottom panel of Figure 3.2).

Math level

Math levels contributed to accuracy scores such that a student with a higher math level was more likely to be accurate ($\beta = 0.93$, $Z = 4.50$, $p < .0001$) (Figure 3.3).

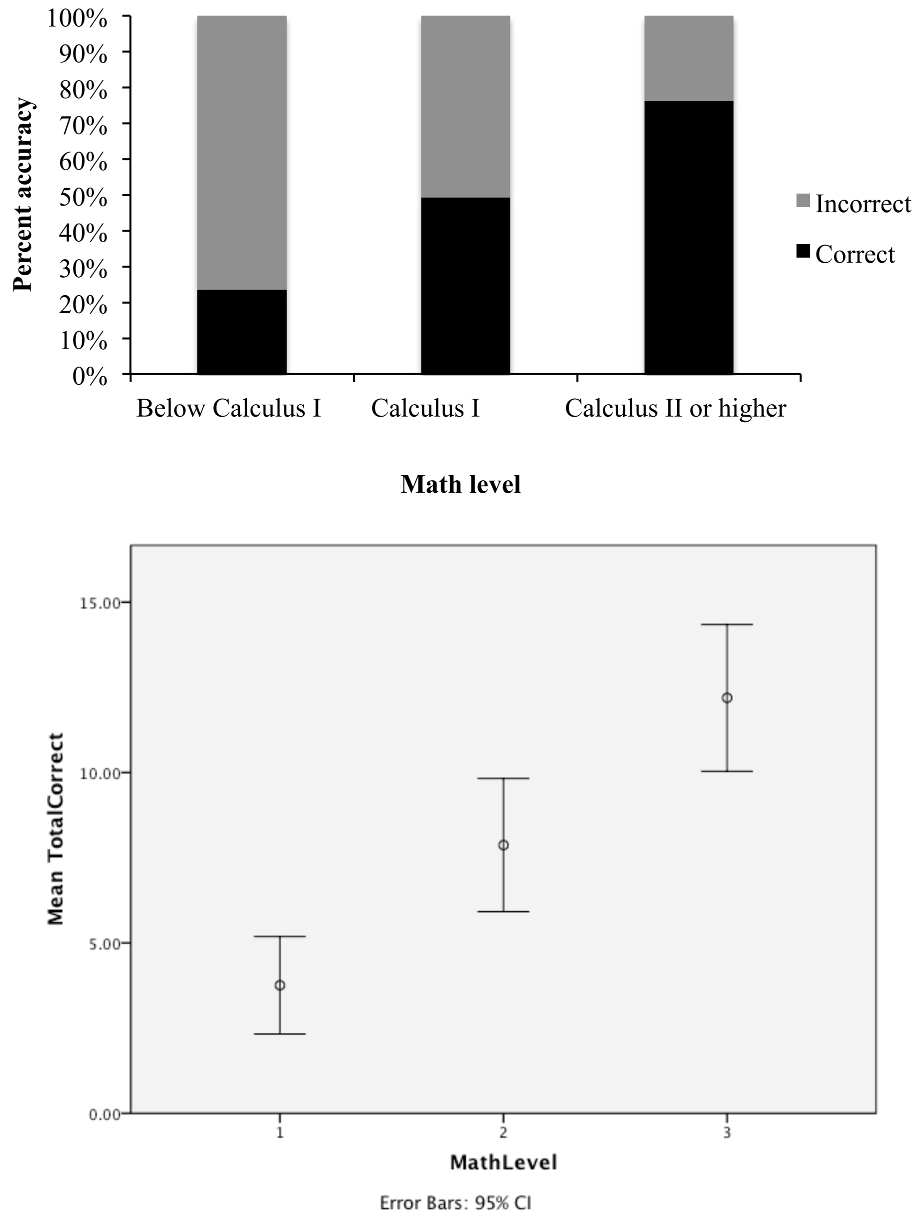


Figure 3.3. The top panel shows the percent accuracy by math level across all questions. The bottom panel shows the mean total correct out of the 16 questions given (where math level 1 refers to below calculus I, 2 refers to calculus I, and 3 refers to calculus II or higher). Participants with more math experience were more likely to be accurate on questions.

Work shown across responses

Similar to the results in Chapter 2, showing work was a significant predictor of participant's accuracy ($\beta = 1.09$, $Z = 4.80$, $p < .0001$). Overall, most participants showed at least a little work, or 1 to 2 lines of non-repeating arithmetic, when answering the questions. However, unlike the results in Chapter 2 where there were only two levels (showed work or did not show work), three levels were scored in the current experiment. Apparently, providing paper and pencil facilitated this tendency to show work. The results in the current study differed from those in the previous study such that answers here were more accurate if participants showed more scratch work than none at all or too little ($\chi^2(2, N = 1978) = 230.51, p < .0001$) (see Figure 3.4). Only 5.4% of the answers that occurred without showing work were correct ($n = 15$ responses). By contrast, 44.3% of answers showing a little work were correct ($n = 633$) and 67.9% showing a lot of work were correct ($n = 184$).

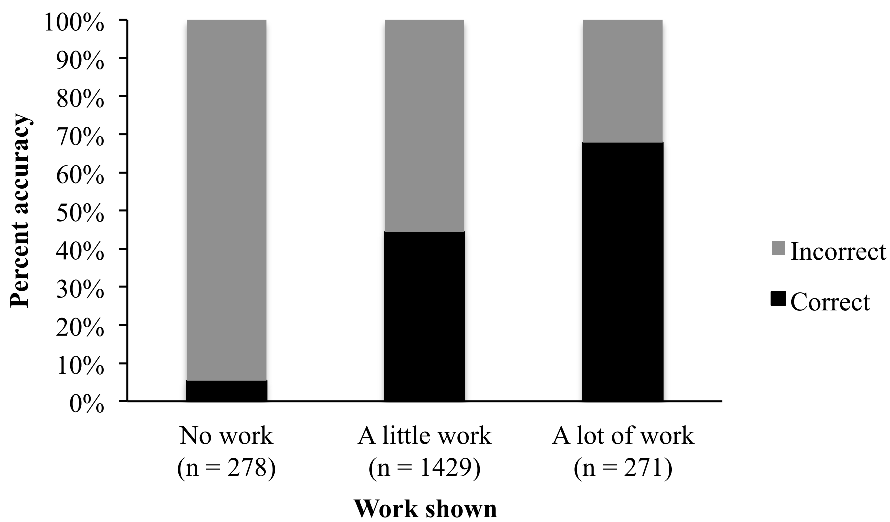


Figure 3.4. Accuracy by work shown.

Gender

Gender was not a significant predictor for accuracy across the sixteen test questions as seen in the GEE model. However, a chi-square test did reveal an effect of gender on accuracy, such that males were more accurate ($M = 8.98$, $SE = 1.03$) than were females ($M = 5.51$, $SE = 0.68$), ($\chi^2(2, N = 124) = 14.12, p < .0001$). It is beyond the nature of this work to attempt to account for the different outcomes for the different measures.

Confidence ratings and completion time on accuracy

Responses on the sixteen questions were collapsed within participants to yield total number of questions correct. Using Spearman's rho (r_s), correlations were run to measure the strength of the relationships. Confidence ratings were highly correlated with the total number of questions correct across pretest and posttest ($r_s[124] = .59$, $p < .0001$). Participants who were more accurate were also more likely to be confident in their accuracy. The relationship of completion time to accuracy was just barely significant ($r_s[124] = .18$, $p = .041$), such that participants who took longer to complete the tasks had a greater number of questions correct across pretest and posttest.

Ceiling effects

Removing the 19 participants who scored at ceiling during pretest (all eight questions correct) revealed that the hint phase condition no longer significantly affected accuracy ($\beta = -0.52$, $Z = -1.32$, n.s.) (see Table 3.3). Of these 19 participants, 12 were in the control group and 7 were in the hint group. With this smaller population ($N = 105$), I looked at output models for the two hint phase groups separately; a model was run for the control (no hint) group (Table B.2) and another model was run for the hint group (Table B.3). In the control group, now there was no effect of work shown on accuracy ($\beta = 0.39$, $Z = 1.00$, n.s.) nor was there an effect of math level on accuracy ($\beta = 0.15$, $Z = 0.55$, n.s.).

In the hint group (Table B.3), there were still effects of these variables on accuracy, similar to the results found in Table 3.2 and Table 3.3.

Table 3.3
GEE parameter estimates: Model without pretest ceiling group (N = 105)
Output of hint phase, hint phase score, test, work shown, math level, and gender on accuracy

Parameter		β	SE (β)	95% Confidence Limits		Z
(Intercept)		-4.51	0.73	-5.94	-3.08	-6.18***
Hint phase		-0.52	0.39	-1.29	0.25	-1.32
Hint phase score		0.87	0.14	0.59	1.15	6.03***
Test	post	0.20	0.19	-0.18	0.57	1.02
Test	pre	(ref.)				
Work shown		1.17	0.23	0.72	1.63	5.08***
Math level		0.78	0.21	0.36	1.20	3.65***
Gender		-0.05	0.35	-0.73	0.63	-0.14

Note. '*' = p < .05, '**' = p<.01, and '***' = p < .001

Discussion

Overall, numerical hints did not help participants significantly improve their performance any more than pure repetition (the control condition in the hint phase). In fact, students who were performing well on the pretest continued to perform well on the posttest and students who were performing poorly on the pretest continued to perform poorly on the posttest. Given the depth of natural number errors, it is unlikely that students simply forgot what they once knew. This suggests that a more intensive training, with multiple kinds of data will be needed to accomplish learning.

Initially, hint phase (control {no hint} group or hint group) was shown to be a significant predictor of accuracy because it included those participants who scored at ceiling on pretest (and accuracy measured performance throughout the task, not only on posttest). In removing those participants, the hint phase was no longer significant, supporting the results showing no difference between the control and hint groups, and little to no difference between the pretest and posttest scores. There happened to be more participants scoring at ceiling in the control group (where there also appeared to be more success), so removing these students resulted in no difference between the hint phases.

The failure to remind participants during the hint phase also relates directly to the misunderstanding of rational number and the principles underlying this type of number. Specifically, the inability to see natural number as a type of rational number concerns the successor principle. The successor principle states that there is always a unique next but students must recognize that this is impossible for rational numbers – there are an infinite number of values between any two rationals. Understanding rational numbers does not call for replacing the current notion of a successor for natural numbers but it does require giving it up for rational numbers.

> "The process of extending the number concept to rational numbers requires meta-conceptual awareness of the differences in the two kinds of numbers, conscious thinking in constructing a parallel mental model for numbers, and metacognitive control in the selection to appropriate rules of operation depending on the task at hand" (Merenluoto & Lehtinen, 2004).

In order to connect a novel concept to an existing knowledge structure, different mental formats must be in place (Carey, 2009). From the results in this study, it is clear that many students have not managed to construct the new format required for rational numbers.

The default to use natural number is a potent case of students' difficulty of getting beyond what they know well. Rational numbers are interpreted as novel cases of natural number, given the lack of knowledge about non-core knowledge regarding rational numbers. Mastery of complex numbers, such as rational number, requires considerable work on the part of the learner as well as a teacher who can move individuals forward with relevant lessons. The trick is to think of relevant lessons – ones that overlap at least to some extent with natural number. Otherwise, a conceptual move from natural numbers (core knowledge) to rational numbers (non-core knowledge) is not likely to take place (Gelman, 2009; Gelman & Gallistel, 1978). What is a bit surprising is how many students might have so little experience working with rational numbers that they default to using natural numbers.

It is also possible, as Merenluoto and Lehtinen (2004) suggest, that earlier failures experienced in mathematics also involved undetected defaults to discrete numbers. If so, this would mean that such students kept failing in their math classes, even if they really tried. The outcome could be an aversion to taking any more mathematics in school.

Showing one's work in this study may have helped individuals monitor their errors and correct them. It could be that paper and pencil tests are preferable to computerized ones for this very reason. For students who made errors, their work allowed us to pinpoint a pattern of errors. When one works on a computer, it is hard to check, erase, and start over again. Therefore, computer versions of the task favor those who know the correct answer and do not have to work through a problem. This conjecture deserves attention in future research. For now, we move onto analyzing students' work and the relation of some variables of interest.

Chapter 4 takes a closer look at these patterns and at the nature of the errors and finds a ubiquitous tendency for those who err to default, one way or another, to natural number arithmetic.

IV. Two-Statement Hint Study
Part II: Error patterns

The data discussed in this chapter are the errors made by participants whose success and failure tendencies were reported in Chapter 3. In addition to scoring error rates, I went on to do an analysis of the nature of errors produced during the two-statement percent problems. The code captures different levels of errors. Analyses of these allowed us to determine that subjects with limited math backgrounds had a robust tendency to interpret the problems as if they were novel cases of natural number.

Introduction

Percent and percentage are sometimes used interchangeably, but this is not always appropriate. Recall from Chapter 1 that percent can be defined in terms of a function (Davis, 1988) that requires a referent, or a base value, in order to take on meaning. Percent can be expressed by the symbol "%" and is used most frequently when preceded by a number (e.g., 20%). In contrast, "percentage" often refers to the product when taking a percent of a base value. In other words, (percent)*(base value) = percentage (e.g., 20% x 50 = 10). The percentage refers to a relationship rather than symbol or a function. In the above example, 50 can be called the base value, the referent, or the "whole," just to name a few terms. Depending on the circumstances, there are also many different names for percentage (or in the previous example, "10") such as the percent change, the "discount," and the "part."

In my studies, I frequently use algebraic values for the base amount so that 20% is the percent and (20%)y, or .20y (represented as a decimal), is the percentage. The new value results from the addition of the percentage to the base value if there is an increase occurring ($y + .20y$) or from the subtraction of the percentage from the base value if a decrease takes place ($y – .20y$).

Conceptually, what should happen when there is more than a single percent statement in the problem (as in both Chapter 2 and 3)? The student must recognize that this is a novel problem dealing with rational numbers. The format of the problem needs to be redefined so that in the second step of the process the “new value 1” takes place of the (original) base value. Table 4.1 broadly outlines three solution strategies seen in students’ work (from Chapter 3). In the second step of these problems, percent B is based on the new value rather than the original base value. An advanced problem-solver may opt to find the solution in fewer steps as illustrated in the table below. The use of a beginner’s solution is very piece-wise without an obvious goal, where the student painstakingly and vigilantly breaks the problem into parts. The intermediate solution strategy eliminates two of the steps in the beginner’s solution and demonstrates understanding that the base value is increased or decreased as a function of itself (rather than add or subtract the percentage back to the base value). Yet, the participant still fails to master the shortest solution path. To move from the intermediate level to the more advanced solution strategy, the mechanics of the proof require one substitution of nv1 (the new value after the first process) in place of the (original) base value. The application of multiplying three values demonstrates knowledge of the multiplicative, additive, and distributive properties.

Table 4.1
Problem solving strategies for two-statement percent problems

Solution strategies	Example: 40% increase of y followed by 10% decrease of new value
Beginner strategy	
Step 1: (percent A)(base value) = percentage A	Step 1: $(.40)(y)$
Step 2: base value ± percentage A = new value 1 ("nv1")	Step 2: $y + .40y$
Step 3: (percent B)(nv1) = percentage B	Step 3: $(.10)(y)(y + .40y)$
Step 4: nv1 ± percentage B = new value 2 ("nv2")	Step 4: $(y + .40y) – (.10)(y)(y + .40y)$
Intermediate strategy	
Step 1: (base value)(1 ± (percent A)) = nv1	Step 1: $(y)(1 + .40)$
Step 2: (nv1)(1 ± (percent B)) = nv2	Step 2: $(1.40y)(1 – .10)$
Advanced strategy	
Step 1: (base value)(1 ± (percent A))(1 ± (percent B)) = nv2	Step 1: $(y)(1 + .40)(1 – .10)$

Here, it is important to note briefly a topic mentioned earlier that is of interest to math educators and cognitive scientists, alike – the novice-expert distinction. As it turns out, the distinction of the strategies in Table 4.1 are reminiscent of the novice-expert distinction. However, too many mistakes were made for students to be considered novices as discussed in the literature (e.g., Simon & Simon, 1978). Thus, the beginner strategy is only consistent with the definition of a novice in that becoming more skillful on a task (e.g., using an advanced strategy) depends upon gaining more experience. While there is almost always more than one way to solve a problem, it is quite obvious that some strategies are quicker and more efficient than others. As illustrated in Chapter 2, beginners were identified by their unnecessarily long solution strategies in a computer task; participants were more prone to making errors. However, Chapter 3 revealed that showing work was an important predictor in accuracy in a paper and pencil task. That does not necessarily mean that experts are the students who show more work; it simply means that until students are comfortable taking shortcuts (e.g., the advanced strategy), longer solutions (e.g., the beginner or intermediate strategies) can help them keep track and even check their work. These longer solution strategies can contain extraneous and/or redundant information, underscoring our guess that they are not akin to expert solutions. While all students have important prior knowledge about these domains, more advanced students are able to quickly identify similar relations in the domain of mathematics.

Additionally, the open-ended design of the response section allowed for the identification of certain errors, especially those categorized as conceptual errors. Students who make conceptual errors demonstrate the inability to move from one concept to another; they are unable to move the concept and the mathematics forward together. This is evidence for the person to have achieved a conceptual change.

The factors of interest to the current study are embedded in the solution strategies seen earlier (Table 4.1). In looking at work shown, I identified the type of error made and the location of the error in considering that these problems essentially are comprised of two parts. It was also of interest to follow up on the natural number arithmetic error. Whereas in Chapters 2 and 3, I referred to a 50% increase followed by a 50% decrease as a 50% – Algebraic or a 50% – Numerical problem, the problems in this study differ. Here, the percents in each problem are always different numerical values from one another. Moving forward, I refer to each problem by abbreviating the direction of each percent. For example, the shorthand notation for a 60% decrease followed by a 10% increase shall be called "60dec~10inc." "30inc~40dec" references a problem with a 30% increase followed by a 40% decrease.

Method

Participants

The participants were the same sample as those in the Two-Statement Hint Study (Chapter 3).

Design and Procedure

Each participant was given 8 questions in Phase I and 8 questions in Phase III. Phase II, the hint phase, and Phase IV, the collection of demographic information, was not coded any further than was previously described in Chapter 3. As the 16 responses in Phase I and Phase III were previously coded for accuracy (correct or incorrect), only incorrect answers were looked at in the current chapter to find patterns of errors.

Coding responses. Student work was grouped by:

i. *Error location by problem part*. For problems where the participants showed work (see Chapter 3), the solution to each problem involved two steps. Therefore, incorrect responses were coded as:

   a. Error on <u>only</u> the first part of the problem
   b. Error on <u>only</u> the second part of the problem
   c. Error on final simplification of the problem (correct setup but wrong final answer due to simplification/ addition/ subtraction error in last step)
   d. Entire problem incorrect (both parts were wrong)

ii. *Type of error*. Participants' incorrect responses were labeled according to the type of error committed. In the cases where a single error could not be isolated, multiple codes were given. Codes "a" through "d" were considered conceptual errors and codes e through g were considered calculation errors.

The codes were:

a. Natural number arithmetic error (NNA error)

   i. Sublevels are exemplified below in Table 4.2 using 40inc~70dec as a problem example

Table 4.2
Levels of natural number arithmetic error

Level of NNA error	Example of NNA error	Explanation of error
Level 1 (Beginner level error)	$+ .40y - .70y = -.30y$ $+ .40y - .70y = .30y$	Addition/subtraction of percentages (sometimes ignoring the direction)
Level 2 (Intermediate level error)	$y + .40y - .70y = .70y$	First part correct; for second part, participant incorrectly subtracted percent of initial base value rather than calculating the percent of the new base value before addition/subtraction
Level 3 (Advanced level error)	$(y + .4y) + (y - .7y))$	Percentages calculated correctly; incorrect last step of addition/ subtraction operation rather than multiplication

b. Failure for brackets or order of operation error

c. Multiplication / division of given percents (e.g., “.20*.30*y for a 20dec~30inc problem)

d. Missing decimal

e. Incomplete answer (e.g., participant takes percent of variable but fails to add or subtract it from original quantity)

f. Missing variable

g. Final simplification error (usually with addition, multiplication, etc.)

h. Incorrect values used or ambiguous answer given

Results

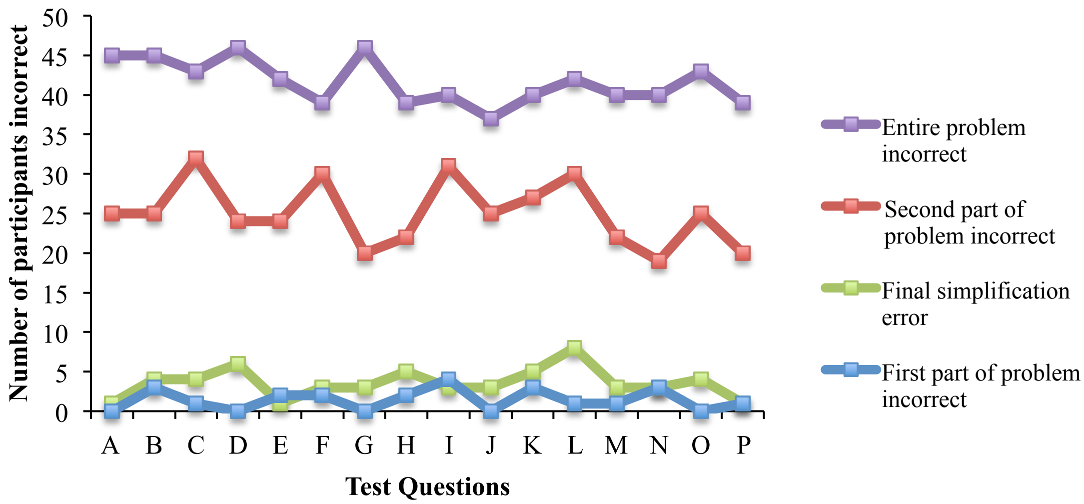
Error analyses tell us what type of issues participants encounter and how they interpret the problem. A rather large percentage of the students defaulted one way or another – some made less conspicuous natural number errors while others made more obvious mistakes. Below, I look at the location of these errors and the types of errors.

Were errors more common in one part of the problem than in another (i.e., error location)?

The percent problems required two separate operations on the two percents presented. Step 1 involved calculating the percentage of the first base value and then adding or subtracting the original base value. The result was a new value. Step 2 involved calculating the percentage of this new value and adding or subtracting the percentage to or from that new value. The result was a second new value. Above (Table 4.1), I discussed expected solution strategies that would be used to look at participants' work. If students showed work, it allowed experimenters to code a student's solution as correct in the first and second parts of the problems (and likewise, as incorrect).

How was each part of the problem coded? While the intermediate solution clearly shows the problem separated into two parts, the other strategies may not be as clear. In the beginner solution, the first two steps focus on the first percent and thus comprise the first part of the problem; the second two steps focus on the second percent for the second part of the problem. If a participant used the advanced solution strategy, getting each part of the problem correct would require having the correct multipliers (or multiplicands). If both parts of the problem were incorrect, *or* if the participant yielded an incorrect final answer with no work shown, the code "entire problem incorrect" was assigned. An additional code, "final simplification error," was used when the participant's incorrect final answer resulted from an error in the final simplification step, most often from an addition or multiplication error; otherwise, the student gave the correct problem structure and values. Differences can be seen between questions and between and within participants in Figure 4.1.

A Friedman's test was run to test for differences between the locations of errors. There was a significant difference in the number of errors on each question depending on the location of the error, $\chi^2(16) = 45.65$, $p < .001$ (Figure 4.1). Post-hoc analysis with Wilcoxon signed-rank tests was conducted with a Bonferroni correction applied, resulting in a significant level of $p < .008$ (.05/(6 tests)). Median (IQR) error location levels for the first part incorrect, second part incorrect, final simplification step incorrect, and all incorrect were 1.00 (0 to 2.75), 25.00 (22 to 29.25), 3.00 (3 to 4.75), and 41.00 (39.25 to 44.50), respectively. The first part incorrect was statistically different from the second part incorrect ($Z = -3.52$, $p < .001$), the final simplification step incorrect ($Z = -2.94$, $p < .001$), and the entire problem incorrect ($Z = -3.52$, $p < .001$). The second part incorrect was statistically different from the final simplification step incorrect ($Z = -3.52$, $p < .001$) and the entire problem incorrect ($Z = -3.52$, $p < .001$). Lastly, there was a difference between the final simplification step incorrect and the entire problem incorrect ($Z = -3.52$, $p < .001$).



(a)

(b)

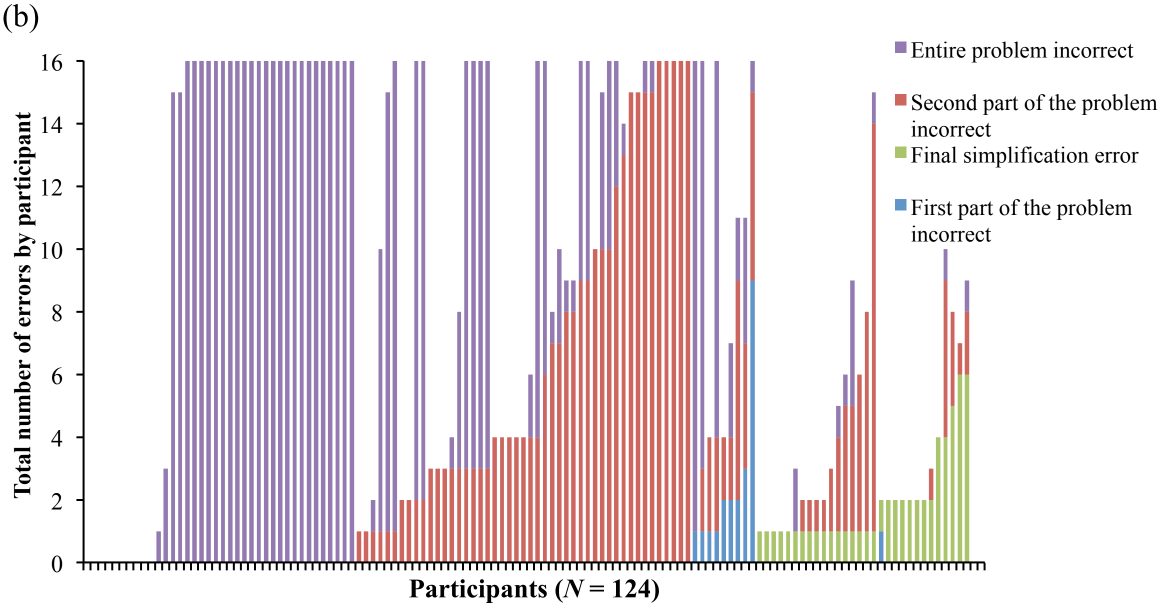


Figure 4.1. Error location by (a) question in the top panel and by (b) individual participants (N = 124) in the bottom panel. Each question is labeled A through P and corresponds to how they appear in Table A.1. 124 participants answered questions A, B, C, D, I, J, K, L, M, N, O, P. 123 participants answered questions E, F, G, H.

Was there a specific type of error that was more common than another?

Coding for type of error on each question allowed the experimenters to assign a name to every type of error that was made. Most participants made one key error that resulted in their incorrect final response. However, there were a small number of students within each question that were not assigned *only* one error code. Because of the small *n*, all responses with more than one error were consolidated into its own category.

The isolated errors were separated into two groups – those that conveyed conceptual misunderstanding and those that were computational/ calculation errors (see Figure 4.2). Conceptual errors consisted of natural number arithmetic (NNA) errors, failure for brackets and order of operation errors, multiplying or dividing the given percents, missing decimal(s), and incomplete answers. Computational errors included missing variable(s) and final simplification errors. Items that did not fit in these groups were incorrect and ambiguous values / responses.

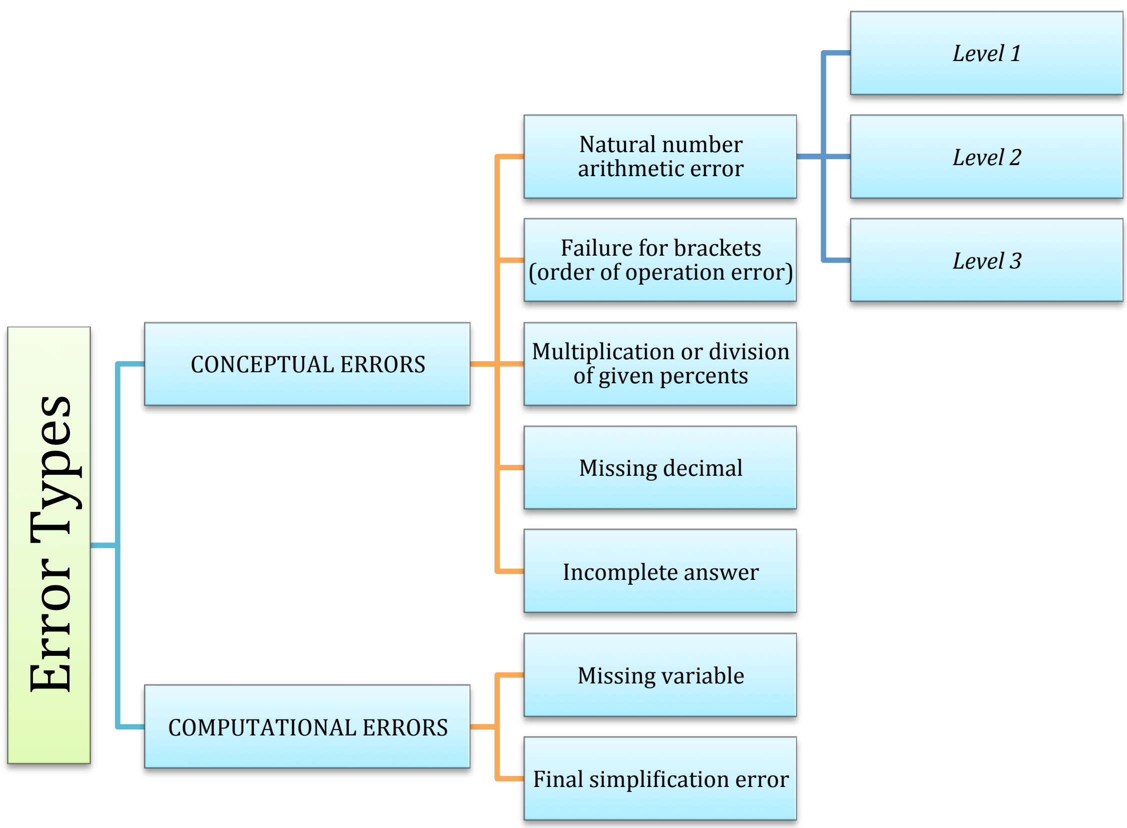


Figure 4.2. Error types. Errors showed both conceptual and computational errors. The most significant conceptual error, natural number arithmetic, was further split into 3 levels.

In looking at the total number of conceptual and computational errors made within each participant, a Wilcoxon signed-ranks test showed a significant change between the two groups of errors ($Z = -6.56$, $p < .001$). Seventy-three participants made more conceptual errors than computational ones, 26 made more computational than conceptual errors, and with 25 students, there were no differences in the type of error made. Within each of these groups, patterns of errors emerged as can be seen in Figures 4.3 and 4.4.

A Friedman's test was run to test for differences between the conceptual errors. There was a significant difference in the number of errors on each question depending on the type of conceptual error, $\chi^2(16) = 55.10$, $p < .001$ (Figure 4.3). The mean number of incorrect responses on each of the sixteen questions revealed the large gap between NNA errors ($M = 40.69$, $SD = 3.61$) and the others: bracket error ($M = 0.88$, $SD = 0.72$), multiplication/division of given percents ($M = 4.19$, $SD = 1.80$), missing decimal ($M = 0.44$, $SD = 0.51$) and incomplete responses ($M = 1.94$, $SD = 1.34$). A post-hoc analysis was not run given the small *n* for all errors other than the NNA error; there was no overlap between error types.

A Friedman's test revealed a significant difference between the two computational errors, $\chi^2(16) = 16.00$, $p < .001$ (Figure 4.4). More errors were found due to simplification errors in the final step ($M = 5.37$, $SD = 3.32$) than due to missing a variable ($M = .06$, $SD = .25$). Due to the small *n*, a post-hoc analysis was not run.

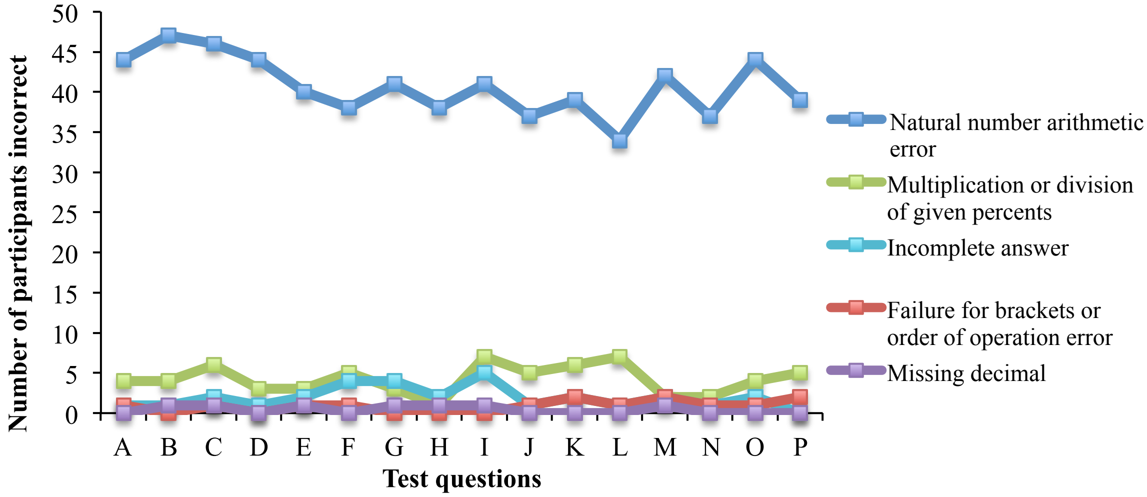


Figure 4.3. Conceptual errors by question. Across all questions, the natural number arithmetic error was the most common. Each question is labeled A through P and corresponds to how they appear in Table A.1. 124 participants answered questions A, B, C, D, I, J, K, L, M, N, O, P. 123 participants answered questions E, F, G, H.

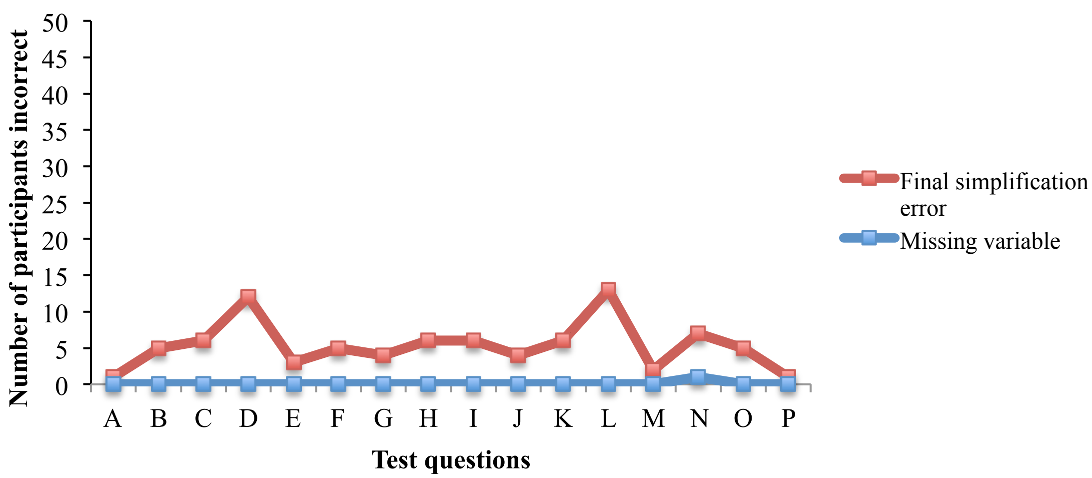


Figure 4.4. Computational errors by question. Each question is labeled A through P and corresponds to how they appear in Table A.1. 124 participants answered questions A, B, C, D, I, J, K, L, M, N, O, P. 123 participants answered questions E, F, G, H.

Did natural number arithmetic errors differ from one another?

A large subset of students made a recurrent and widespread type of error across all questions. In Chapters 2 and 3, I called this error the "natural number arithmetic error" and I continue to do so here, sometimes with the shorthand "NNA error." In the previous section, each incorrect response was coded with only one error code (e.g., decimal error *or* NNA error *or* bracket error). There were a very small number of responses that warranted multiple codes but, because of the small *n*, these responses were not included in earlier analyses (see Figures 4.3 and 4.4). Now, in looking specifically at NNA errors, I chose to include all NNA errors whether they appeared in isolation or alongside another error. An example of this might be the student who responded with "+ 40 – 70 = 30," for a 40inc~70dec problem. This particular response had three error codes (NNA error, missing variable, and missing decimal) and was included in the subset of NNA errors that is now presented.

There were three types of errors that were broken down within this code. Within almost every question, a Level 2 error (e.g., "$y + .40y - .70y$" for a 40inc~70dec) was the most common error, followed closely by a Level 1 type of error. There were very few Level 3 errors. As described in *coding responses* in Methods, a Level 3 error entailed incorrectly applying a final arithmetic step (using addition or subtraction rather than multiplication). Participants making a Level 3 error did get the individual parts of the problem correct.

There was a significant difference between the levels of natural number arithmetic errors, such that some error levels were more frequent than others, $\chi^2(16) = 30.13$, $p < .001$ (Figure 4.5). Post-hoc analysis with Wilcoxon signed-rank tests was conducted with a Bonferroni correction applied, resulting in a significant level set a $p < .016$ (.05/(3 tests)). Medians (IQR) for the Level 1, Level 2, and Level 3 errors were 17.00 (15.25 to 19.75), 24.00 (23.00 to 26.00), and 0.00 (0 to 1.00), respectively. Level 1 was statistically different from both Level 2 ($Z = -3.29$, $p < .001$) and from Level 3 ($Z = -3.52$, $p < .001$). Level 2 was also significantly different from Level 3 ($Z = -3.53$, $p < .001$).

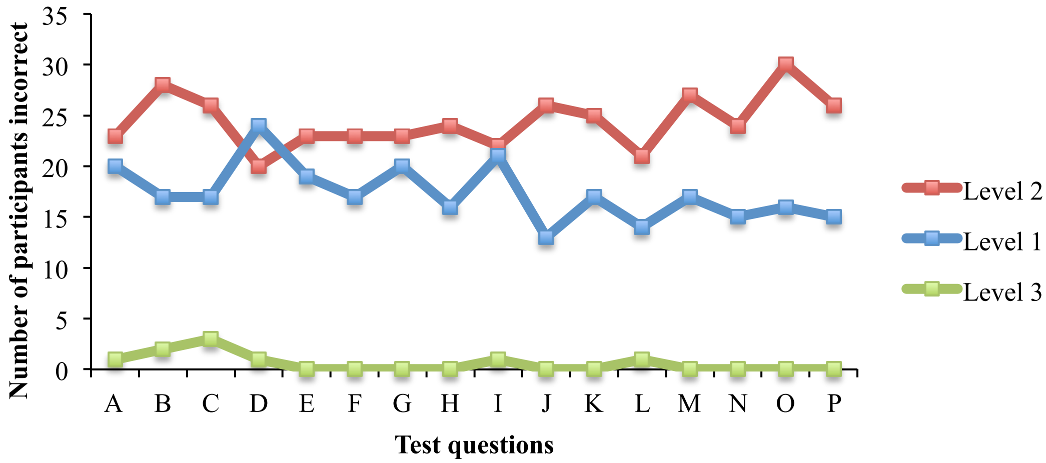


Figure 4.5. Levels of natural number arithmetic errors by question. Each question is labeled A through P and corresponds to how they appear in Table A.1. 124 participants answered questions A, B, C, D, I, J, K, L, M, N, O, P. 123 participants answered questions E, F, G, H.

Discussion

There is evidence that participants are good at solving percent problems with one-statement – or at least, they do not perform as poorly as when solving a problem with two statements. A very interesting pattern existed in the location of the errors in these two-statement percent problems. While many participants were able to get the first part of the problem correct, the second part yielded a great deal of errors. Students recognize that the first part of the problem involves a new knowledge structure, but fail in recognizing that same need for the second part. Instead, they revert to their core knowledge of natural number.

In the second stage of the problem, students might have had the correct numerical values in place, but they failed at using the new value 1 (nv1) as the new base value in the second part of the problem. Rather, the participants added or subtracted the percent of the original base value, as illustrated in this 80inc~40dec problem: $(y + (y*.80)) - (.40)*y$. Within error location, a third level was created in order to describe students who made simplification errors such as subtracting instead of adding: $(y - .10y) + .60(y - .10y) = .9y + .60(.90y) = .36y$. In this 10dec~60inc problem, the work is correct but the final distribution should have yielded $1.44y$, not $.36y$. This level of error was also accounted for when coding for type of error. This finding further supports the results in Chapter 2 and demonstrates that participants are much better at dealing with a one-statement percent problem than a two-statement problem. It becomes even clearer that there is a problem when participants try to combine the percents with addition and subtraction rather than treat them as values that should be multiplied.

The default to natural numbers is undeniably evident in my research. As predicted, the natural number arithmetic errors were, by far, the most common type of conceptual error made. The three levels of this type of error were created to narrow down and categorize any differences (see Table 4.2 in Methods). Making a Level 1 error corresponded to having the least amount of conceptual knowledge about how to set up and solve the problem. In this level of error, participants combined two percentages with addition or subtraction. Many students held false cognates and directly translated the words increase and decrease as key words for the positive and negative signs: a 50% decrease was $-.50y$ and an 80% increase was $+.80y$. This violation occurred in each part of the problem, and so it followed that a 50dec~80inc problem translated to $-.50y +.80y$.

Depending on the participant's adherence to the laws of addition and subtraction, this might have simplified to .30y or 1.3y. In problems where the decrease was a larger percent than the increase, some participants took liberties in treating the expression as an absolute value (such that for a 30dec~20inc, responses might have been +.10y, –.10y, or .50y). In the previous examples, it is clear that students do not recognize percent as a type of number different from a natural number.

The Level 2 NNA error coincided with making an error on the second part of the problem. For a 50dec~80inc, one participant wrote "30% more than y originally was," or algebraically, $y + .30y$. That is akin to saying $y - .50y + .80y = 1.3y$, which was consistent with a Level 2 NNA error. Most participants making this error responded with the latter expression of $y - .50y + .80y$. In contrast to Level 1, participants in Level 2 had some idea that a percent of the base value must be added to or subtracted from that original base value in order get the new value. They knew the base value played a role other than just calculating the percentage. They recognized a novel concept and treated the first part of the problem as a rational number problem. Thus, students correctly calculated the percent change (the percentage added to/subtracted from the base value) and found a new value. However, the default to a natural number strategy soon occurred. After correctly calculating the second percent of the new value, participants failed to add that percentage back to that new value. Conceptually, the participant understood what the final value should look like in the first step (e.g., the result of a percent decrease or increase). The failure lay in the second step where the student's final answer was the new percentage rather than the second new value.

Lastly, with a Level 3 natural number arithmetic error, participants understood the direction of each percent change, but they incorrectly used addition rather than multiplication to find the final answer. False cognates may be responsible. As multiplication with rational numbers does not always yield a larger product, students may default to addition which always results in a larger number when dealing with positive numbers (rational or natural). Using the same 50dec~80inc problem as above, a Level 3 response resulted in ($.5y + 1.8y$) when the answer should be ($.5y$)(1.8).

I now turn to discuss the language of symbols and their principles in mathematics. The order of operations in mathematics is analogous in many ways to the comma usage in language. A problem such as 20/10/5 yields an entirely different interpretation when you insert parentheses to form 20/(10/5). Similarly, we can change the interpretation of "eats shoots and leaves" [like a panda] by inserting commas to form "eats, shoots, and leaves" [like a mobster] (Truss, 2006). In the current research, a participant who made a bracket placement error unknowingly violated the orders of operation. In a 20inc~30dec problem, (y +.20y) – (.30 x .20y + y) was an incorrect response due to missing parentheses around (.20y + y) in the latter part of the problem. This translates to an order of operation error since the participant did not uphold the rule of distribution – the percent, 30%, cannot be multiplied by only the percentage of the original base value, but rather must be multiplied by the entire new value, .20y + y. Thus, we see that, with bracket errors, there is a failure of order rules. These order rules are akin to tree branching rules that organize which values are constrained under a certain set of procedures, and in what order those procedures should be calculated. This becomes a serious problem with algebra.

In looking at the raw data, there were many addition and subtraction errors. For a decrease in percentage, a common error was seen in switching the order of subtraction (e.g. a 60% decrease erroneously became $.60y - y$). This demonstrates a poor understanding of the commutative property. While A + B = B + A exemplifies the commutative property of addition, there is no such commutative property for subtraction since A – B ≠ B – A. Two entities can be switched and still equal each other when adding or multiplying, but the same cannot be said for subtraction or division. Not understanding subtraction also led some participants to express a decrease (of 60%) as $-.60y$.

When reasoning about mathematical statements is absent, errors can show up in the failure to honor the structure of the processes. For example, errors can result from incorrectly lining up decimals in addition and subtraction (Rittle-Johnson, Siegler, & Alibali, 2001), or, in multiplication, as a failure to shift sequential rows one number at a time to the left (Wu, 2009). Both types of error demonstrate misinterpretations of rational number; specifically, these errors suggest that the basic concept of place values might not be understood. Other multiplication errors might be caused by using an unnecessarily lengthy strategy, when a more efficient strategy can, by definition, be quicker and also can demonstrate an understanding of the mathematical axioms at task. While students often incorrectly apply a previously learned strategy to a new problem (Hiebert & Carpenter, 1992; Hartnett & Gelman, 1998), sometimes they simply choose a drawn-out method to fit the problem into a familiar algorithm and fail to re-represent the problem more efficiently.

In terms of efficiency, moving easily between representational formats is not just recommended, but rather it is integral to developing the concept of rational number. It provides more opportunities for students to recognize that rational numbers are different from natural numbers. By comparing equivalent values in various formats, students are able to connect similar relations. In participants' scratch work, the majority re-represented percents (e.g., 40%**y*) as decimals (e.g., .40*y*). Whether or not students do this because of over-exposure or over-familiarity with decimals (e.g., due to school-based math), or because of competence limitations (e.g., because fractions are difficult), remains a question. However, moving from percents to decimals as well as to other formats sometimes resulted in the incorrect re-representation of a value. This incorrect translation seemed to occur more when there was a decrease. A few participants translated a 20% decrease to (1/2)*y*, demonstrating difficulty with the concept of a fraction. Others represented a decrease by placing the percent's numerical value in the denominator of a fraction (e.g., (1/40)*y* for a 40% decrease) and represented an increase of 40% as .40*y* (without adding this percentage to *y*). A 40% decrease should be *y* – .40*y* = .60*y* or (60/100)*y* or (3/5)*y*. Somehow, decrease translated to "put value in denominator" and increase translated to "put value in numerator as a ratio to 100." A 40% increase should be *y* + .40*y* = 1.40*y* or (140/100)*y* or (7/5)*y*. Only 1.40*y* in the increase problem bears some resemblance to the actual solution. The increase would be correct if asked to give the percentage rather than the new value.

As illustrated in the advanced strategy (Table 4.1), the most efficient solution strategy for a two-statement percent problem is to multiply two values such as (.60*y*) and (1.60) for a 40dec~60inc problem. Participants seemed to look for this shortcut method, but failed to recognize the principles underlying percents; thus, errors resulted (e.g., (.40y)*(.60y) for a 40dec~60inc problem). Some participants got the first part of the problem correct, but then multiplied (if the problem called for an increase) or divided (if the problem called for a decrease). In this scenario, a 40dec~60inc problem resulted in (.6y)(.6) where the first part was correct and a 50inc~10dec problem resulted in (1.5y)/(.1), where only the first part is correct.

Ignoring the problem's question ("what is the new/ final value") also resulted in what experimenters called incomplete final answers. The students understood they needed to calculate a percent change, but then failed to add or subtract it back to the original value. An example is (y + .80y)(.40) for an 80inc~40dec problem. Notice that the student found the percentage (or "discount") of the new amount but failed to subtract it from the new 1.8y value. Thus the problem was considered incomplete.

An aside. While the focus of this dissertation was not on computational errors, it should come as no surprise that they resulted from hasty calculations or from neglecting to check one's own work. A short list of examples from students' work follows in Table 4.3. The most common computational error occurred in the final simplification step of the problems when participants showed their work.

Table 4.3
Computational mistakes in students' work

Problem	**Student's raw work**	**Error explanation**
10inc~20dec	1.1 – .8(1.1y) = 1.1y – .3y = .8y	Final simplification error: Correct structure but incorrect multiplication when distributing
80inc~40dec	(y + .80) – .40	Missing variable
30inc~10dec	(y + .3y) * (y – .1y) = 1.3y + .9y	Final simplification error; did not follow through with multiplication

Conclusions

Overall, the errors made on two-statement percent problems were very informative. It is evident that students have at least a rudimentary understanding of what a decrease or increase means. Decrease means less than (the original) and increase means more than (the original). Furthermore, as seen within the natural number arithmetic errors, it is possible that students have an internal concept that taking a percent of a base value yields a value smaller than the original base amount. Yet, instead of multiplying by a fraction (<|1|), students subtract to convey the idea that the final amount must be smaller. Thus, a major problem lies in false cognates – the translation of decrease and increase to corresponding mathematical symbols and the conceptual framework surrounding these processes.

When students learn a new mathematical representation (e.g., percents), their expectations about what is true may get in the way of what actually is true and relevant to the task at hand (Walkington, Sherman, & Petrosino, 2012). Before learning about rational numbers, the rules governing basic mathematical operations must be in place. Otherwise, errors will no doubt occur. As seen here, participants might hold the expectation that calculating the percent yields not only the percentage but also the new value. With a decrease or discount problem, when you take a percent of a base value, you do wind up with a value less than the original. That is the expectation. But expectations about what the result should be and intuition can get in the way. In this study, the intuition of increasing and decreasing percents translated (more often than not) to adding and subtracting percents. There was a default to natural numbers and intuition was wrong.

Failure in expectations or intuitions will frustrate students to make necessary connections between a familiar solution model and the demands of a novel problem. This increases the chance that they will treat the problems as disconnected rather than ones with related representations. In failing to recognize the problems as novel, it follows that students will default to natural numbers. Thus, learning would be most effective if it supported the development of strategies that connected concepts of natural number to those of rational number. I discuss teaching and learning strategies to improve students' understanding in Chapter 5.

V. Contributions to a framework for understanding percents

The kinds of errors that students make are informative about what to teach, correct or remind students about the nature of percents. Errors can show us the components of what it means to arrive at the correct answer and to do something right. I have shown that the difficulty with percent problems is due to what is known in Cognitive Science as a failure in conceptual change – in this case, a failure to realize that rational numbers are a different kind of number than are natural numbers (or the counting numbers). The formal properties of natural and rational numbers are different and even contradictory. Whereas natural numbers are expressed as unique whole numbers, each of which has a unique successor, rational numbers are written as tripartite symbols and lack unique successors. Further, different operations involve different conditions regarding closure. Any natural number added to any other natural number always yields another natural number. However, the same is not true for one natural number divided by another. To accomplish closure under division, it is necessary to acknowledge the existence of decimals, another kind of representation of a fraction. Put differently, various arithmetic facts do not stand alone; they are embedded in structures that are defined by different structures. Further, the evidence supports the idea that natural numbers and their learning constitute a core domain, that is, one that leads to early learning without formal instruction (Gelman, 2000). The more that is known about a domain, the easier it is to understand and recognize novel problems.

Rational numbers are not part of the natural number core domain. Instead, they are considered as an example of non-core domains, the kind that requires formal instruction and dedicated work on the part of the learner. Therefore, it should be expected that students will make mistakes. They might even fail to recognize their mistakes and, instead, treat problems as novel cases to which to apply their knowledge of natural numbers.

In mathematics, a problem is facilitated when we can draw similar relations to other familiar problems. Simply stated, new knowledge is easier to gain if we can make connections to facts and methodology already in place (Hartnett & Gelman, 1998). Learning is jeopardized when those connections are not made. Success in making the shift from natural numbers to percents depends on students' ability to attend to only the relevant features of the problem and block out the noise of irrelevant features (Chi, Feltovich, & Glaser, 1981). In Chapter 4, the advanced solution strategy (see Table 4.1) demonstrates the student's aptitude for consolidating mathematical processes and arriving at the most efficient solution strategy, both in length and solution time. This is one objective in learning – the student should eventually be able to recognize comparable methods and choose the more efficient method to solve the problem. In contrast, the beginner might show an excessive amount of work or sometimes no work at all. Often, the beginner defaults to natural number arithmetic because a) (s)he does not realize that the demands of the novel concept require a new knowledge structure for learning to take place, or possibly because b) reverting to what one knows is simply easier.

Learning comes in progressive stages that are dependent upon conceptual change in an individual. Since division is a process that gives people great difficulty, let me illustrate this with a basic example learned in elementary school – how do we move from adding to multiplying and then back to more complex processes of adding values that have been multiplied? Once the basic concepts (e.g., the algorithms and the place rules) involved in the processes of addition and multiplication of natural numbers are understood, it is then important for children to learn (and, to some degree, memorize) the single-digit multiplication table. Knowing these products can facilitate multiplying other natural numbers, no matter how large or how many. This same idea can be applied generally to a complex problem. If a student is able to automatically retrieve the answers to a simple subtraction process, (s)he can then focus more of his/her working memory on a more complicated problem of long division (Lee, Ng, & Fong Ng, 2009; Clarke, Ayres, & Sweller, 2005; Mayer, 2001; also see Sweller, van Merriënboer, & Paas, 1998, for a review). However, as mentioned earlier, calling upon the numerical combinations that yield the correct division answer is not an easy task in itself. Overall, the lack of automaticity with basic arithmetic facts and procedures interferes with success on various advanced math problems.

Fluency and accuracy should be considered together – as students are introduced to more advanced concepts, the basic building blocks are assumed to already be in place (Kellman et al., 2008). Not understanding the material, or taking too long to understand it, may impede a student's progress. Throughout these dissertation studies, students misinterpreted the data in a way that highlighted their difficulty to overlap percent formats with what is already known about natural number formats. The default to natural number arithmetic in the two-statement percent problems (Chapters 2 and 3) illustrates that they have not yet made the conceptual move from natural number to rational number concepts. The failure to remind participants by providing a hint (Chapter

3) also begs the question as to whether they ever fully acquired these concepts in the first place.

In Chapter 2, students were more proficient with using natural numbers than they were with using rational numbers (percents) in an algebraic context. The relative ease that comes along with using natural numbers seems to serve as a major obstacle once presented with other number formats. Among students who struggled to separate natural number concepts from rational number ones, a widespread default to natural number followed (Chapter 4). These errors are much like false autobiographical memories – they are not intentional but a result of an active mind's engagement of a given context.

Perhaps, this coincides with the problems that students encounter when trying to apply school-based mathematics procedures to real world problems (Masingila, Davidenko, & Prus-Wisniowska, 1996; Walkington et al., 2012). In our every day lives, we are presented problems with multiple representational formats (e.g., "Take $10 off your purchase after taking 10% off the price") and there are often multiple methods that can be employed to arrive at a solution. The framework below suggests the use of different solution methods to strengthen a greater understanding of concepts (e.g., Gelman, 1986); not including different methods and/or formats fosters a disconnect between mathematical concepts rather than encourages linking similar relations. While the solution processes might be different, they must share the common mathematical principles that underlie understanding. It therefore would seem to follow that students should be taught to share and compare multiple representations. This recommendation is a part of the curriculum recommended in the United States by the Common Core State Standards (2010).

Finally, false cognates may play a large role in the default to natural numbers. In the previous chapter, many of the conceptual errors were due to a default to natural number arithmetic error, where participants added a percent to another percent or to another value when they should have multiplied it by a base value. One suspicion is that this error may be attributed to a misunderstanding of the language, “increase” and “decrease.” When students read a word problem, they are really decoding the meanings and inferring missing information from the text. In Kintsch and Greeno (1985), the authors assumed that children knew the meaning of abstract mathematical terms such as “have, give, all, more, less, etc.” Understanding the meanings of these words were essential in order to solve the problem. Similarly, in percent problems, words such as “increase” and “decrease” need to be understood contextually. Natural number arithmetic errors revealed that participants correctly translated these words to mean “more” or “less” than the base, but then defaulted to adding and subtracting. Multiplication and division can also cause an increase and a decrease to occur, though there are more restrictions when dealing with non-natural numbers (e.g., the closed nature of division). These false cognates are addressed in the proposed framework.

Proposals for teaching, development and learning

In Table 5.1, I summarize the findings in my studies by offering a framework, in the form of a related set of proposals, for introducing two-statement algebraic percent problems. This framework might allow students to proceed from what is known to what is not known. A problem involving multiple processes presumably requires that the problem solver invoke specific existing knowledge about the use of one-statement percent problems and their representations. With minor modifications, the same framework can be used for one-statement percent problems.

The suggested learning framework compares processes with percents to those with natural numbers in similar word problems. Comparisons are used as they promote deeper understanding of the links between abstract concepts and solution models (Kintsch & Kintsch, 1995; Hattikudur & Alibali, 2011). Beginners are encouraged to show their work and talk through the problems with their peers; being an active learner is essential to learning and development (e.g., National Science Foundation, 1996; Springer, Stanne, & Donovan, 1999). Also highlighted is the importance of allowing time for students to produce and discuss various ways to problem solve amongst each other (Stigler & Hiebert, 1999; Rittle-Johnson & Star, 2007). The comparison structure addresses difficulty with natural number as it relates to the successor principle and false cognates, and encourages learning with multiple examples. It is designed as a bridge between learning about single percent processes and processes involving more than one percent. These models are not recommended as substitutions for introductory models; rather, one-statement percent problems should be addressed before or alongside the models below.

An outline of the learning framework follows (see Table 5.1). The rationale for each step is explained as they relate to Cognitive Science, previous research and the findings in this dissertation:

1) Compare and contrast models have been known to increase students' understanding of concepts (e.g., Gelman, 1969; Gentner, Loewenstein, & Thompson, 2003; Kellman et al., 2008), making them more flexible problem-

solvers (Star & Rittle-Johnson, 2009). Put side-by-side, the student is able to see almost identical wording of a problem with natural naturals compared to one with percents. The structure is the same, yet the values (here, 20 and 20% and 10 and 10%) hold different meanings.

2) The student is asked to think about what the final answer should be – greater than, less than, or equal to the original value. This step provides a way to predict and later check the final answer (as in Zur & Gelman, 2004). Providing reference points allows students to set an expected estimate and later see if their answer violates their expectation.
3) A major difference between the problems can be seen in getting students to recognize that the percent acquires meaning once "attached," or applied, to the base value (e.g., y).
4) The student is then asked to define the operational processes that correspond to the increase and decrease.
5) A solution expression is created. Students can note the difference between natural number solutions and rational number solutions.
6) Students are not required to stay within the confines of the problem – they can be encouraged to think about different scenarios in which the outcome might be the same or different. It is important to make these models applicable to the real world – a world where the outputs are always changing based on varying inputs. Giving children personal standards, also known as a reference point strategy, helps the values becomes meaningful (Joram, Subrahmanyam, &

Gelman, 1998). However, these points of reference should be of a continuous nature rather than that of a part-whole relationship.

Table 5.1
Proposed learning framework: Differentiating between two-statement natural number problems and two-statement percent problems

	Natural number process	**Percent process**
1) Present word problem	John buys <u>y apples</u> from one market. After going to another market, the number of apples he has <u>increases by 20</u>. Later that day, his friends come over and the number of apples <u>decreases by 10</u>. How many apples does John now have?	Jane buys <u>y apples</u> from one market. After going to another market, the number of apples she has <u>increases by 20%.</u> Later that day, her friends come over and the number of apples <u>decreases by 10%</u>. How many apples does Jane have now?
2) Compare y to final amount	First new amount should yield a result greater than y	Final amount should be greater than y
3a) Establish independent or dependent meaning of 20	Independent	Dependent (on base value y)
3b) Establish independent or dependent meaning of 10	Independent	Dependent (on new resulting value from first process)
4) Define operational meaning of increase and decrease	Add then subtract	Multiply then add, multiply then subtract
5) Generate solution expression	$y + 20 - 10$	$(y(.20) + y) - (.10)\ (y(.20) + y)$ or $(1.20)(.90)y$ *
6) Encourage creative thinking and group discussion: Is it possible for the final amount to be less than y? How?	Ex: "When his friends eat more than y + 20 apples," "If John started out with less than y apples," etc.	Ex: "When Jane starts with 0 apples"

* *Note:* Here is an opportunity to introduce different representations: "How did we get from 20% to .20? Are there other ways to represent 20%? Would these be proper substitutions? Why or why not?"

It is important to note that, in some cases, natural number arithmetic *can* be applicable to percent problems with multiple processes. An example of an acceptable situation might be two coupons that are both applied to the base value (or, the original price). If 20% is taken off the base, x, and 10% is taken off the base, x, the results yields $(x - .20x) - .10x$, which displays natural number arithmetic characteristics. This exception emphasizes how critical it is to stress the difference between the two usages of number and to teach them alongside one another. The goal is to get students to recognize key components of each type of problem so they do not default to what is most familiar.

Closing statements

Learning rational numbers is quite difficult. The default to natural number arithmetic in adults was seen throughout my work and thus, it emphasizes the great need for creating new learning models for students. The learning model suggested in this final chapter took into consideration the results of previous research, current research, and the lessons from both Cognitive Science and Mathematics.

With or without schooling, everyone has a sense of numbers, but only of the natural numbers as they are part of a knowledge structure in our core domain. The work presented in this dissertation allows us to see the continued pervasive failure for adults to make the shift from natural number concepts to those dealing with rational numbers (e.g., percents in my studies). As conceptual change must take place to move between these different number concepts, considerable effort is needed from the learner. This turns out to be quite difficult as it is the principles that are changing. The learner must constantly overcome demands of a novel problem by recognizing that it does not fit in a current knowledge structure. And even then, the learner may continue to default to natural number and false cognates. By learning about the novel concept alongside a comparable one (or two), it is possible that (s)he may develop into a flexible problem solver and thus, have an easier time transferring between similar items. Accordingly, learning with multiple equivalent formats may help students recognize that rational numbers are in a class of their own, different from the natural numbers. In sum, developing a more successful framework for teaching rational numbers is critical in helping people understand that rational numbers, especially percents, are not natural numbers.

APPENDIX A
Stimuli

Table A.1
Test questions from Chapters 3 and 4

Question	Question content
A	In 2009, a large university accepted *y* student applicants. In 2010, due to a large number of applicants, the university increased the number accepted by 50% of *y*. In 2011, the university then decreased that number by 10%. How many applicants did the university accept in 2011, in terms of *y*?
B	A private company normally hires *y* new employees every year. Due to budget cuts, the number of hires had been decreased by 40%. At the last minute, the company received a large donation, allowing them to then increase the number of hires by 60%. How many employees can this company now hire, in terms of *y*?
C	A cereal box normally contains *y* cheerios. The cereal company increases the number of cheerios by 20% of *y*. After packaging costs rise, the company then decreases the number of cheerios by 30%. How many cheerios can now be found in the cereal box, in terms of *y*?
D	The original price of an item in a department store is listed as *y* dollars. The price of the item is decreased by 70% of *y* and sells very well. Therefore, the store decides to increase the discounted price by 20%. What is the final cost of the item, in terms of *y*?
E	Sally owns *y* acres of land. She inherits more land, increasing her new plot by 80% of *y*. Several years later, she decides to sell some of her land, decreasing it by 40%. How much land does she now own, in terms of *y*?
F	While warming up for a race, John's heart-rate was *y* beats per minute (bpm). During the race, his heart-rate increased by 30% of *y*. His heart-rate then decreased by 70% while he was stretching. What is John's heart-rate while he was stretching, in terms of *y*?
G	A patient has an illness that causes her temperature to be *y* degrees Fahrenheit. The doctors prescribe a medication and her temperature decreases by 60%. Since it is too low, another medication is prescribed, which increases her temperature by 50% from what it had dropped to. What is the patient's new temperature, in terms of *y*?
H	In the middle of the semester, a student had accumulated *y* points towards his final grade. After factoring in extra credit assignments, his points increased by 10% of *y*. Due to many absences and missing assignments, his points then decreased by 80%. What is the final number of points he has, in terms of *y*?
I	In 2007, a football stadium sold *y* seats. After a winning season, the number of seats sold increased in 2008 by 40% of *y*. In 2009, the stadium was undergoing construction so the number of seats sold decreased by 70% of what it had been in 2008. How many seats were sold in 2009, in terms of *y*?
J	A company has *y* employees in their database. When the company switches ownership, it decreases the number of employees by 50%. A few years later, there is room for expansion and the company increases its size by 80%. How many employees does the company now have, in terms of *y*?

K	A vegetable bag normally contains *y* lentil beans. The company increases the number of lentils by 30% of *y*. After packaging costs rise, the company then decreases the number of lentils by 10%. How many lentil beans can now be found in the vegetable bag, in terms of *y*?
L	The original price of an item in a department store is listed as *y* dollars. The price of the item is decreased by 10% of *y* and sells very well. Therefore, the store decides to increase the discounted price by 60%. What is the final cost of the item, in terms of *y*?
M	Sheryl owns *y* paintings. She moves to a large house and buys more paintings, increasing the number by 80% of *y*. Several years later, she moves to a smaller apartment and sells the paintings, decreasing the number by 20% of what she had in her house. How many paintings does she now own, in terms of *y*?
N	Joe weighs *y* pounds. To get ready for wrestling season, he needs to increase his weight by 20% of *y*. Once wrestling is over, he decides to lose weight for the summer and decreases his weight by 40%. What should Joe's weight be for wrestling season, in terms of *y*?
O	A patient's white blood cell count is *y*. He takes a medication that causes his cell count to decrease by 60%. He follows up with additional medication, which increases his white blood cell count by 30% from what it had dropped to. What is the patient's new white blood cell count, in terms of *y*?
P	Several years ago, the price of gas was *y* dollars per gallon. During the recession, the price of gas increased by 70% of *y*. When the economy starts booming again, the price of gas is expected to decrease by 50% of the cost during the recession. When the economy booms, what will be the expected price of gas per gallon, in terms of *y*?

Note. Questions A-H appeared in *Order 1* as the pre-test and in *Order 2* as the post-test. Questions I-P appeared in *Order 1* as the post-test and in *Order 2* as the pre-test.

APPENDIX B
Analyses

Before analyses were run in Chapter 3, Spearman's correlation coefficients were computed to look at the relationship between the variables of interest. Table B.1 displays all of these correlations.

Table B.1
Correlations among Chapter 3 study variables

	Accuracy	Math level	Math SAT	Hint phase group	Hint phase score	Showed work	Gender
Accuracy	______	0.39***	-0.05*	0.03	0.58***	0.33***	-0.21
Math level		______	0.11***	0.01	0.28***	0.20***	-0.24***
Math SAT			______	0.01	-0.06**	-0.05*	-0.15***
Hint phase group				______	0.29***	-0.05*	-0.16***
Hint phase sco					______	0.27***	-0.31***
Showed work						______	0.02
Gender							______

Note. '*' = p < .05, '**' = p<.01, and '***' = p < .001

After removing participants who were scoring at ceiling on the pretest, the two hint phase groups were looked at separately. Output for the control group, in Table B.2, revealed that showing work and math level were no longer significant predictors of accuracy. In the hint group, in Table B.3, there still exist effects of these variables on accuracy.

Table B.2
GEE parameter estimates: Control (no hint) group without pretest ceiling group (N = 45)
Output of hint phase score, test, work shown, math level, and gender on accuracy

Parameter		β	SE (β)	95% Confidence Limits		Z
(Intercept)		-3.49	0.78	-5.02	-1.97	-4.50***
Hint phase score		1.14	0.14	0.87	1.40	8.39***
Test	post	0.46	0.34	-0.20	1.12	1.38
Test	pre	(ref.)				
Work shown		0.39	0.39	-0.37	1.15	1.00
Math level		0.15	0.28	-0.40	0.71	0.55
Gender		-0.46	0.44	-1.33	0.41	-1.04

Note. '*' = p < .05, '**' = p<.01, and '***' = p < .001

Table B.3

GEE parameter estimates: Hint group without pretest ceiling group (N = 60)

Output of hint phase score, test, work shown, math level, and gender on accuracy

Parameter		β	SE (β)	95% Confidence Limits		Z
(Intercept)		-5.27	0.96	-7.16	-3.39	-5.48***
Hint phase score		0.53	0.23	0.08	0.97	2.32*
Test	post	0.06	0.25	-0.43	0.55	0.25
Test	pre	(ref.)				
Work shown		1.81	0.28	1.25	2.36	6.37***
Math level		0.95	0.25	0.46	1.44	3.81***
Gender		-0.30	0.45	-1.19	0.59	-0.66

Note. ‘*’ = $p < .05$, ‘**’ = $p<.01$, and ‘***’ = $p < .001$

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Curriculum Vitae

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NJ Department of Education, Alternate Route (Provisional Teacher Program), 2006-2007

Rutgers College – Rutgers University, Psychology, B.A., 2001 – 2005

ADDITIONAL RESEARCH EXPERIENCE

Rutgers University

Cognitive Development and Learning Lab (Advisor: Dr. Rochel Gelman) 2008-2010

Graduate Assistant

MetroMath (Advisors: Dr. Gerald Goldin and Dr. Yakov Epstein) 2007-2008

Graduate Fellow

Visual Cognition Lab (Advisor: Dr. Jacob Feldman) Summer 2005

Lab Assistant

TEACHING EXPERIENCE

Rutgers University

Infant and Child Development Lab Summer 2010, Fall 2010, Fall 2011, Spring 2012

Course Instructor

Quantitative Methods in Psychology (lecture course) Summer 2011

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Teaching Assistant and Recitation Instructor

New Jersey Public Schools 2005-2007

Mathematics Teacher. Ramsey High School, Ramsey, NJ

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