QUANTIFYING ALGEBRAIC PROPERTIES OF SURFACE GROUPS AND 3-MANIFOLD GROUPS

BY PRIYAM PATEL

A dissertation submitted to the

Graduate School—New Brunswick

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Feng Luo

and approved by

New Brunswick, New Jersey

May, 2013

ABSTRACT OF THE DISSERTATION

Quantifying Algebraic Properties of Surface Groups and 3-Manifold Groups

by Priyam Patel

Dissertation Director: Feng Luo

A group G is *residually finite* (RF) if for every nontrivial element $g \in G$, there exists a finite index subgroup G' of G such that $g \notin G'$. A group G is called *locally extended residually finite* (LERF) if for any finitely generated subgroup S of G and any $g \in G - S$, there exists a finite index subgroup G' of G which contains S but not g. Quantifying these algebraic finiteness properties refers to bounding the indexes of the finite index subgroups G' in each of the definitions above. In this dissertation we quantify Peter Scott's theorem that surface groups are LERF in terms of geometric data. In the process, we will quantify the fact that surface groups are residually finite and quantify another result by Scott that any closed geodesic in a surface lifts to an embedded loop in a finite cover. We also extend the methods used in the 2-dimensional case to quantify the residual finiteness of particular 3-manifold groups.

Acknowledgements

First and foremost, I thank my advisor, Feng Luo, for his guidance in the writing of this thesis and for suggesting these problems to me. I am deeply indebted to him for his unwavering support and for all that he has taught me throughout my time at Rutgers.

I would like to sincerely thank Xiaochun Rong, Steve Ferry and Ben McReynolds for agreeing to serve on my thesis committee. Additionally, I would like to thank Zheng-Chao Han, Amy Cohen and Michael Weingart for their help and generosity over the last few years.

I thank Ian Agol for his valuable suggestions and for sharing his mathematical ideas with me, especially those that helped prove the results of Chapter 4.

I also thank Peter Scott for conversations that led to improvements of Theorems 3.5.1 and 4.3.3.

I must acknowledge Tian Yang for his help with the proof of Lemma 4.3.1, and thank him for all of the fun and enlightening mathematical discussions we have had.

I am immensely grateful for the love and encouragement I have always received from my parents. I was able to accomplish this goal because of their sacrifices. I also thank my sisters, my brother-in-law, and the rest of my family, for their support and love.

Finally, I must thank my friends: Liz Vocaturo, Justin Bush, David Duncan, Brian Garnett, Ramya Kanukollu, Ashwin Kamath, Krina Patel, Glen Wilson and Philomena Chu, for making the last five years exciting and unforgettable.

This work was partially supported by the GAANN fellowship and by a research fellowship from NSF DMS 1222663 and NSF DMS 1105808.

Parts of this thesis, including all of Chapter 3, have been accepted for publication by the American Mathematical Society to appear in the Proceedings of the AMS.

Dedication

To my sisters Chandani and Ekta:

Thank you for always believing I could accomplish this goal and for helping me every step of the way. You mean the world to me.

To my nephew Rahul:

Thank you for always reminding me that I can overcome any obstacle.

Table of Contents

Chapter 1

Introduction

A group G is *residually finite* (RF) if for every nontrivial element $g \in G$, there exists a finite index subgroup G' of G such that $g \notin G'$. A group G is called *locally extended residually finite* (LERF) if for any finitely generated subgroup S of G and any $g \in G - S$, there exists a finite index subgroup G' of G which contains S but not g. The LERF property for groups is also often referred to as *subgroup separability.* A group G with a subgroup S is *S-residually finite* (or *S-subgroup separable*) if for any element g of $G - S$, there is a subgroup G' of finite index in G which contains S but not g. Hence, a group is LERF if it is S-subgroup separable for all finitely generated $S < G$.

Residual finiteness and LERF-ness are considered two of the most important finiteness properties for groups and have been studied by algebraists since the 1940s. In the 1960s, geometers and topologists began studying RF-ness and LERF-ness in the context of the fundamental groups of surfaces and 3-manifolds, which I will refer to as *surface groups* and *3-manifold groups* respectively. In [4], Baumslag proves that all surface groups are RF (Hempel gives a shorter proof in [11]), and in [20], Peter Scott shows that surface groups are LERF. In [12], John Hempel proves that the fundamental groups of compact 3-manifolds are RF, and more recently Ian Agol proves that closed hyperbolic 3-manifold groups are LERF in [2].

A natural question that arises when studying these algebraic finiteness properties is to ask how big we must take the finite index subgroups G' in order to satisfy the definitions. Bounding the indexes of the G' in each of the definitions above gives a *quantification* of the residual finiteness or LERF-ness of a group. In recent years, there has been significant work done in an effort to quantify residual finiteness. In [6], Bou-Rabee introduced quantifying residual finiteness for important classes of finitely generated groups, including free groups, the first Grigorchuck group, finitely generated nilpotent groups and certain arithmetic groups. In [8], Buskin gives an improvement on Bou-Rabee's bound in the case of free groups. Rivin addresses similar questions [19]. Additionally, in the case of nonabelian free groups Bou-Rabee and McReynolds [7] and Kassabov and Matucci [15] give lower bounds for the indexes of the subgroups G', again in terms of word length.

The goal of this dissertation is to quantify Peter Scott's result that surface groups are LERF by giving an estimate on the indexes of the subgroups G' in terms of geometric data. In the process, we also quantify the residual finiteness of surface groups. The flavor of our results is rather different from the work on quantifying residual finiteness cited above, namely in the significant use of hyperbolic geometry to obtain the quantification.

Peter Scott also shows in [20] that any closed geodesic in a surface Σ lifts to an embedded loop in a finite cover of Σ. The first result I will present in this dissertation is the following theorem, which quantifies the above result.

Theorem 1.0.1. *Let Σ be a compact surface of negative Euler characteristic. Then there exists a hyperbolic metric on Σ so that any closed geodesic of length ℓ lifts to an embedded loop in a finite cover whose index is bounded by* $16.2\,\ell$.

The idea for the proof of this theorem came from [20]. We tessellate the hyperbolic plane by regular, right-angled pentagons as Scott does in his paper, which will induce a tessellation on Σ and on any cover of Σ. For any compact subsurface S of a surface tessellated by these pentagons, we are able to estimate the area of the smallest, closed, convex union of pentagons Y containing S, which we call the *convexification* of S. Using the upper bound on the area of Y and some other geometric results, we obtain the bound of Theorem 1.0.1 in Section 3.4.

Our main result, Theorem 3.6.1, quantifies Peter Scott's LERF theorem. The statement is fairly complicated, but a special case of it is the RF case stated below.

Theorem 1.0.2. *Let Σ be a compact surface of negative Euler characteristic. There exists a hyperbolic metric on Σ so that for any $\alpha \in \pi_1(\Sigma) - \{id\}$, there exists a subgroup H' of $\pi_1(\Sigma)$, such that $\alpha \notin H'$. The index of H' is bounded by $32.3\,\ell$, where ℓ is the length of the unique geodesic representative of α.*

We prove Theorem 1.0.2 in Section 3.5 and in Section 3.6 we will see that the quantification of LERF-ness will be a natural extension of the proof of Theorem 1.0.2. Both proofs rely on the Poincaré Polygon Theorem (see §2.4.1), which will explain the significance of obtaining the convex space Y described above.

We should note that in [20], there is a gap in Peter Scott's argument that surface groups are LERF. He addresses and fills in this gap in his paper [21]. In our proof of Theorem 3.6.1, we will make use of the Neilson convex region of a surface, also called the convex core, as Scott does in [21] to avoid the gap in his original paper.

In Chapter 4, we study the 3-dimensional analog of Theorem 1.0.2. The first step in tackling this problem is to understand what the analog of the tessellation by regular, right-angled pentagons is in hyperbolic 3-space, $\mathbb{H}^3$. Let P be any compact polyhedron in $\mathbb{H}^3$, all of whose dihedral angles are $\pi/2$. We will refer to P as an *all right polyhedron* and denote by Γ the group of isometries of $\mathbb{H}^3$ generated by reflections in the faces of P. The images of P under the action of Γ tessellate $\mathbb{H}^3$. These all right polyhedra are the analog of the regular, right-angled pentagons in $\mathbb{H}^2$.

Let M be a compact hyperbolic 3-manifold that is tiled by copies of P, one of these all right polyhedra. Following the work of Agol, Long and Reid in [3] and making use of a crucial fact introduced by Agol in [1], we quantify the residual finiteness of $\pi_1(M)$ with the following theorem:

Theorem 1.0.3. *Let M be any compact hyperbolic 3-manifold tessellated by copies of P, an all right polyhedron. Then for any $\alpha \in \pi_1(M) - \{id\}$, there exists a subgroup H' of $\pi_1(M)$, such that $\alpha \notin H'$. The index of H' is bounded by*

$$\frac{2\pi \sinh^2(\ln(\sqrt{3}+\sqrt{2}) + d_P)}{V_P}\,\ell,$$

where ℓ is the length of the unique geodesic representative of α, d_P is the diameter of P and V_P is the volume of P.

We prove the theorem in Section 4.3 by obtaining an upper bound on the volume of the smallest, closed, convex union of polyhedra Y_C containing a compact set C in any 3-manifold tessellated by copies of an all right polyhedron. Again, Y_C is referred to as the *convexification* of C. The proof relies on the fact that polyhedra sufficiently far away from the compact set C cannot lie in Y_C. In [1], Ian Agol gives an explicit bound on what "sufficiently far" means. We prove a special case of the bound from [1] in Section 4.3, and then apply the analog of the technique used for hyperbolic surfaces to obtain the desired bound on the volume of Y_C in terms of geodesic length in M.

Throughout this dissertation, we will make use of several standard results of hyperbolic geometry. We discuss the foundations of hyperbolic geometry and these standard results in the next chapter.

Chapter 2

Hyperbolic Geometry and Topology

2.1 Models of Hyperbolic Space

We let $\mathbb{H}^2$ denote the Poincaré half-plane model of hyperbolic 2-space. That is, $\mathbb{H}^2 = \{z = x + iy \in \mathbb{C} : y > 0\}$ endowed with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

of constant negative curvature. $\mathbb{H}^2$ is isometric to $\mathbb{D}^2 = \{z = x + iy \in \mathbb{C} : |z| < 1\}$ endowed with the metric

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2} \; .$$

$\mathbb{D}^2$ is called the Poincaré Disc Model of hyperbolic 2-space.

Similarly, we can define the Poincaré half-space model of hyperbolic 3-space as $\mathbb{H}^3 = \{(x, y, u) \in \mathbb{R}^3 : u > 0\}$ with the metric

$$ds^2 = \frac{dx^2 + dy^2 + du^2}{u^2},$$

and the Poincaré ball model $\mathbb{D}^3 = \{(x, y, u) \in \mathbb{R}^3 : x^2 + y^2 + u^2 < 1\}$, with the metric

$$ds^2 = \frac{4(dx^2 + dy^2 + du^2)}{(1 - x^2 - y^2 - u^2)^2}.$$

We will use the two models interchangeably, and unless otherwise stated, the properties discussed for one model will carry over to analogous properties in the other model.

The geodesics, or shortest curves, in $\mathbb{H}^2$ and $\mathbb{H}^3$ are lines and semi-circles orthogonal to $\partial\mathbb{H}^2$ and $\partial\mathbb{H}^3$, respectively, as seen in Figure 2.1 below.

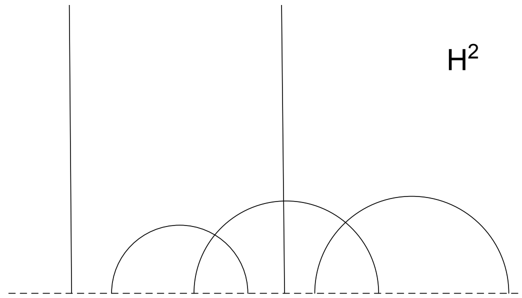


Figure 2.1

The contrast between the geometry of hyperbolic and Euclidean space is well demonstrated through the study of basic shapes, such as triangles. Figure 2.2 below shows a hyperbolic triangle in $\mathbb{H}^2$. One should note that the angles in the triangle look relatively small. Unlike Euclidean triangles, the angles of a hyperbolic triangle need not sum to π. An application of the Gauss-Bonnet Theorem [18] gives us the following lemma, which tells us that the angles of a triangle in hyperbolic space actually always sum to less than π.

Lemma 2.1.1. *Let T be a triangle in $\mathbb{H}^2$ with angles α, β, and γ. Then* $\text{Area}(T) = \pi - \alpha - \beta - \gamma$.

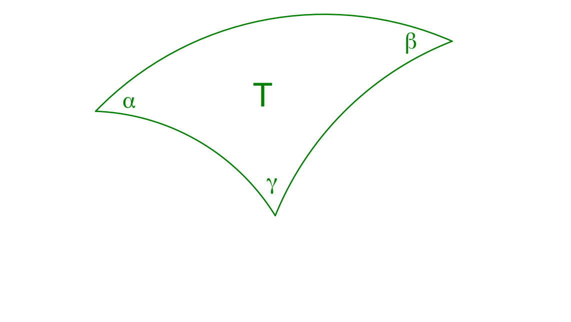


Figure 2.2

Note that Lemma 2.1.1 tells us that there is no notion of similar triangles in hyperbolic space. If two triangles have the same angles they are congruent. We also note that Lemma 2.1.1 does not rule out the possibility of a triangle in $\mathbb{H}^2$ having vertices in $\partial\mathbb{H}^2$. We call a triangle T in $\mathbb{H}^2$ with all vertices at $\partial\mathbb{H}^2$ an *ideal triangle*. In this case, Area$(T) = \pi$ since $\alpha = \beta = \gamma = 0$. Figure 2.3 below shows an ideal triangle in $\mathbb{D}^2$, (2.3 (a)), and two different examples of ideal triangles in $\mathbb{H}^2$ (2.3 (b) and (c)).

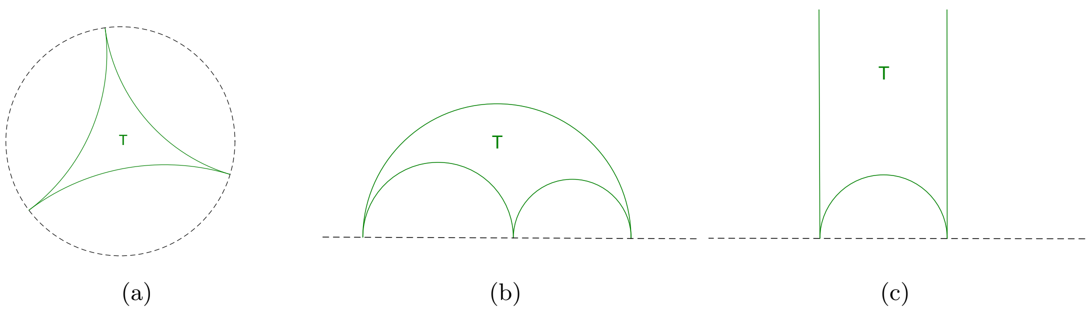


Figure 2.3

We now state the hyperbolic law of sines and cosines [16, §1.5] for a hyperbolic triangle T with angles α, β and γ, and with edges of lengths a, b and c opposite the angles α, β and γ, respectively.

Sine Law: $\dfrac{\sinh a}{\alpha} = \dfrac{\sinh b}{\beta} = \dfrac{\sinh c}{\gamma}$

Cosine Law I: $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma$

Cosine Law II: $\cosh c = \dfrac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta}$

Now consider the very special case of the triangle T of Figure 2.4, with one vertex at i, two vertices at infinity, an edge of length a and the indicated angle $\pi(a)$.

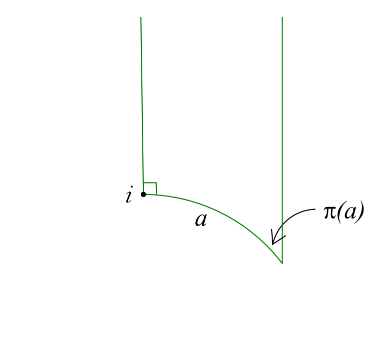


Figure 2.4

We have the following relations for such a triangle, which are called the angle of parallelism laws [16] and which we make use of in the proof of Lemma 3.3.2.

Lemma 2.1.2 (Angle of Parallelism Laws)**.** *With T a triangle of the form in Figure 2.4, the following 3 relations hold:*

i. $\tan(\pi(a)) = \dfrac{1}{\sinh(a)}$

ii. $\sin(\pi(a)) = \dfrac{1}{\cosh(a)}$

iii. $\cos(\pi(a)) = \dfrac{1}{\tanh(a)}$

2.2 Classification of Isometries in $\mathbb{H}^2$ and $\mathbb{H}^3$

In this section, we give a classification of all isometries of $\mathbb{H}^2$ and $\mathbb{H}^3$, beginning with a classification of the orientation preserving isometries of $\mathbb{H}^2$.

The orientation preserving isometries of $\mathbb{H}^2$, $\mathrm{Iso}^+(\mathbb{H}^2)$, are the Möbius transformations

$$\left\{ z \longmapsto \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}.$$

Since each of these transformations can be represented by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of determinant 1 and accounting for the fact that $\dfrac{az+b}{cz+d} = \dfrac{-az-b}{-cz-d}$, we get that $\mathrm{Iso}^+(\mathbb{H}^2) \cong \mathrm{PSL}(2, \mathbb{R})$. Each element $\gamma \in \mathrm{Iso}^+(\mathbb{H}^2)$ can be classified as one of the following:

- *loxodromic:* γ is conjugate to an isometry of $\mathbb{H}^2$ of the form $z \longmapsto \lambda z$, where $\lambda \in \mathbb{R} - \{0\}$, and γ has exactly two fixed points which both lie on $\partial\mathbb{H}^2$. In this case, there exists a geodesic axis L (between the fixed points of γ) that is invariant under the action of γ on $\mathbb{H}^2$.

- *parabolic:* γ is conjugate to an isometry of $\mathbb{H}^2$ of the form $z \longmapsto z + r$, where $r \in \mathbb{R} - \{0\}$, and γ has exactly one fixed point in $\partial\mathbb{H}^2$.

- *elliptic:* γ is conjugate to a rotation of $\mathbb{H}^2$, and γ has exactly one fixed point in $\mathbb{H}^2$. Considering γ as an element of $\mathrm{Iso}^+(\mathbb{D}^2)$, an elliptic isometry is conjugate to a rotation about the origin.

The orientation reversing isometries of $\mathbb{H}^2$ are of the form $z \longmapsto \dfrac{a(-\overline{z})+b}{c(-\overline{z})+d}$ such that $ad-bc=1$. Thus, each orientation reversing isometry of $\mathbb{H}^2$ is either a loxodromic isometry composed with $-\overline{z}$, a parabolic isometry composed with $-\overline{z}$ or an elliptic isometry composed with $-\overline{z}$.

Next, we present the classification of orientation preserving isometries, $\mathrm{Iso}^+(\mathbb{H}^3)$, of $\mathbb{H}^3$. Every element of $\mathrm{Iso}^+(\mathbb{H}^3)$ arises as an extension of a linear fractional transformation, $\Phi(z)$, of $\partial\mathbb{H}^3 = \widehat{\mathbb{C}}$, where $\Phi(z) = \dfrac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$ and $ad-bc \neq 0$. We can classify the isometries by the number of fixed points of $\Phi(z)$:

- If $\Phi(z)$ has exactly one fixed point in $\widehat{\mathbb{C}}$, the extension, $\widetilde{\Phi}(z)$, of Φ to $\mathbb{H}^3$ is conjugate to an isometry of the form $z \longmapsto z + w_0$, where $w_0 \in \mathbb{C} - \{0\}$. Such an isometry is called *parabolic*.

- If $\Phi(z)$ has exactly two fixed points in $\widehat{\mathbb{C}}$, the extension, $\widetilde{\Phi}(z)$, of Φ to $\mathbb{H}^3$ is conjugate to an isometry of the form $z \longmapsto \lambda z$, where $\lambda = re^{i\theta} \in \mathbb{C} - \{0\}$. If $|\lambda| \neq 1$, such an isometry is called a *homothety rotation.* $\widetilde{\Phi}$ fixes the geodesic axis, L, between to the two fixed points of Φ in $\widehat{\mathbb{C}}$.

  In the case where $|\lambda| = 1$, $\lambda = e^{i\theta}$ and $\widetilde{\Phi}$ is conjugate to a pure rotation. Such an isometry is called *elliptic* and $\widetilde{\Phi}$ fixes every point of L.

Moving on to the orientation reversing isometries of $\mathbb{H}^3$, we first note that each such isometry is an extension, $\widetilde{\Phi}$, of an anti-linear fractional map, Φ on $\widehat{\mathbb{C}}$, where $\Phi(z) = \dfrac{a\overline{z}+b}{c\overline{z}+d}$, $a, b, c, d \in \mathbb{C}$ and $ad-bc \neq 0$. Thus each orientation reversing isometry is one of the following:

- *orientation reversing parabolic:* Φ is conjugate to an isometry of the form $z \longmapsto -\overline{z} + ri$, where $r \in \mathbb{R}$, and the extension $\widetilde{\Phi}$ is conjugate to the composition of a Euclidean reflection across a vertical plane with a horizontal translation along a nonzero vector parallel to that plane.

- *orientation reversing homothety rotation:* Φ is conjugate to an isometry of the form $z \longmapsto -\lambda\overline{z}$, where $|\lambda| \neq 1$, and the extension $\widetilde{\Phi}$ is conjugate to the composition of a homothety rotation with a reflection across a vertical Euclidean plane passing through the origin of $\mathbb{H}^3$.

- *orientation reversing elliptic:* Φ is conjugate to an isometry of the form $z \longmapsto \dfrac{\lambda}{\overline{z}}$, where $|\lambda| = 1, \lambda \neq 1, -1$, and the extension $\widetilde{\Phi}$ is conjugate to the composition of an inversion about a hemisphere centered at the origin of $\mathbb{H}^3$ with a pure rotation about the vertical u-axis of $\mathbb{H}^3$ by an angle θ that is not an integer multiple of π.

- *pure inversion:* Φ is conjugate to an isometry of the form $z \longmapsto \dfrac{1}{\overline{z}}$ and the extension $\widetilde{\Phi}$ is conjugate to a pure inversion about hemisphere centered at the origin $\mathbb{H}^3$, or Φ is conjugate to $z \longmapsto \dfrac{-1}{\overline{z}}$ and the extension $\widetilde{\Phi}$ is conjugate to the composition of an inversion about a hemisphere with a rotation by π.

We now make an observation for which we need the following definition:

Definition. Let Γ be a subgroup of $\mathrm{Iso}(\mathbb{H}^3)$ and let $C(\mathbb{H}^3/\Gamma)$ denote the convex core of $\mathbb{H}^3/\Gamma$. Γ is *geometrically finite* if for every $r > 0$, the r neighborhood $N_r(C(\mathbb{H}^3/\Gamma))$ of the convex core has finite volume.

If γ is an orientation preserving or reversing homothety rotation of $\mathbb{H}^3$, the cyclic subgroup of $\mathrm{Iso}(\mathbb{H}^3)$ generated by γ is geometrically finite. We will make use of this observation while proving the results of Chapter 4.

2.3 Hyperbolic Surfaces and 3-Manifolds

Definition. A *hyperbolic n-manifold* is an n-manifold that admits a complete Riemannian metric of constant sectional curvature equal to -1. We can equivalently define a hyperbolic n-manifold as the quotient of hyperbolic n-space $\mathbb{H}^n$ by a subgroup of $\mathrm{Iso}(\mathbb{H}^n)$ acting freely and properly discontinuously.

If Σ is a compact hyperbolic surface, the classification of isometries (§2.2) and the definition above imply that the elements $\gamma \in \pi_1(\Sigma)$ must correspond to orientation preserving or reversing loxodromic isometries. This fact will be useful in Chapter 3 once we interpret the residual finiteness and LERF-ness properties of surface groups geometrically.

Similarly, if M is a compact hyperbolic 3-manifold, then the elements of $\pi_1(M)$ must correspond to orientation preserving or orientation reversing homothety rotations.

2.4 Basic Lemmas of Hyperbolic Geometry

In this section we will mention two theorems of hyperbolic geometry that will be useful to us throughout this dissertation. We begin with a discussion about Fuchsian groups and fundamental domains.

Definition. A *Fuchsian group* is a discrete subgroup of $\mathrm{Iso}(\mathbb{H}^2)$.

Definition. A *fundamental domain* for the action of a group of isometries $G < \mathrm{Iso}(\mathbb{H}^n)$ on $\mathbb{H}^n$, is a closed subset $F \subset \mathbb{H}^n$ such that $\bigcup_{g \in G} g\, F = \mathbb{H}^n$ and $g\mathring{F} \cap \mathring{F} = \varnothing$.

2.4.1 Poincaré's Theorem

We first present the Poincaré Polygon Theorem, which allows us to construct many examples of Fuchsian groups from convex polygons in $\mathbb{H}^2$. We will follow the setup and statement of the theorem in [13, §3.9].

Let $P \subset \mathbb{H}^2$ be a closed, convex polygon. If x is a point of ∂P, we let $\theta(x)$ denote the interior angle of P at x and note that $0 < \theta(x) \leq \pi$. We say that P has a *side pairing* if we can label the sides of P in pairs $(s_i, s_i')_{i \in I}$ where $s_i \neq s_i'$ and there exists $\gamma_i \in \mathrm{Iso}(\mathbb{H}^2)$ such that $\gamma_i(s_i) = s_i'$ for each i. We also require that under γ_i, the side of s_i in P maps to the side of s_i' that is not in P.

Let X be the quotient space $P/\sim$, where $x \sim \gamma_i(x)$ when $x \in s_i$. Let Z be the image of ∂P in X and let $V_z = \{x \in P : x \mapsto z \text{ under the quotient map } P \to X\}$.

Theorem 2.4.1 (Poincaré Polygon Theorem)**.** *Let P be a closed, convex polygon with a side pairing and with X and Z as above. If X with the quotient metric is complete and for every $z \in Z$ there is an integer $n_z \geq 1$ such that*

$$\sum_{x \in V_z} \theta(x) = \frac{2\pi}{n_z},$$

then the subgroup $G := \langle \gamma_1, \gamma_2, \ldots, \gamma_m \rangle < \mathrm{Iso}(\mathbb{H}^2)$*, is discrete and* $\mathring{P}$ *is a fundamental domain for* G*.*

In [5, §9.8], Beardon gives a more general version of this theorem that is also independent of dimension. The following example from [5] is one application of Beardon's version of Poincaré's Theorem, which will play a key role in the proofs of our results.

Example 1. Let $P \subset \mathbb{H}^2$ be a closed convex polygon with r sides and with angles $\dfrac{\pi}{n_j}$ at the vertices v_j $(j = 1, \ldots, r)$. For each side s_i we let γ_i be a reflection in the side s_i. Then $G := \langle \gamma_1, \ldots, \gamma_r \rangle$ is discrete and $\mathring{P}$ is a fundamental domain for G.

The 3-dimensional analog of the Poincaré Polygon Theorem is often called the Poincaré Polyhedra Theorem. An application of the theorem is the following example:

Example 2. Let $P \subset \mathbb{H}^3$ be an all right polyhedron as defined in Chapter 1. Then the group G of isometries generated by reflections in the faces of P is a discrete subgroup of $\mathrm{Iso}(\mathbb{H}^3)$ and $\mathring{P}$ is a fundamental domain for G.

2.4.2 The Collaring Theorem

The next theorem we will discuss is the Collaring Theorem [17], which describes the structure of a hyperbolic Riemann surface around a simple closed geodesic. We will follow the setup and statement of the theorem in [13, §3.8].

Definition. Let γ be a simple closed geodesic of length ℓ in a hyperbolic Riemann surface X. If the δ-neighborhood

$$A_\delta(\gamma) := \{x \in X : d(x, \gamma) < \delta\}$$

is isometric to the δ-neighborhood of the unique simple closed geodesic on the cylinder of modulus π/ℓ, we say that γ admits a δ-*collar*.

Next we define the collaring function $\eta(\ell)$ by

$$\eta(\ell) = \frac{1}{2} \ln \frac{\cosh(\ell/2) + 1}{\cosh(\ell/2) - 1}.$$

Theorem 2.4.2 (The Collaring Theorem)**.** *Let X be a complete hyperbolic Riemann surface, and let $\Gamma := \{\gamma_1, \gamma_2, \ldots\}$ be a collection of disjoint simple closed geodesics, where each γ_i has length ℓ_i. Then $A_{\eta(\ell_i)}(\gamma_i)$ are collars around the γ_i and they are disjoint.*

We make use of this theorem in Section 3.6 when extending our quantification of the residual finiteness of hyperbolic surface groups to a quantification of LERF-ness.

Chapter 3

Quantifying Residual Finiteness and LERF-ness of Hyperbolic Surface Groups

3.1 Preliminaries

Following [20], we let $P \subset \mathbb{D}^2$ be a regular, right-angled pentagon. Let Γ be the group of isometries of $\mathbb{D}^2$ generated by reflections in the five sides of P. By the Poincaré Polygon Theorem, P is a fundamental domain for the action of Γ on $\mathbb{D}^2$, and the images of P under Γ tessellate $\mathbb{D}^2$. Let $T = \{gP : g \in \Gamma\}$ be the tessellation.

Let $F = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$. In his paper [20], Scott shows that there exists a fundamental domain for the action of $\pi_1(F)$ on $\mathbb{D}^2$ consisting of four regular, right-angled pentagons whose sides have been identified in such a way that $\pi_1(F) < \Gamma$. Therefore, F can be tiled by these regular, right-angled pentagons. He then shows that every closed surface Σ of negative Euler characteristic covers F. That is, there exists a covering map $r : \Sigma \longrightarrow F$ and an induced map $r_* : \pi_1(\Sigma) \longrightarrow \pi_1(F)$ on their fundamental groups. r_* is injective as an induced map on π_1 of a covering map, and therefore, $\pi_1(\Sigma) < \pi_1(F) < \Gamma$. This tells us that there exists a fundamental domain for the action of $\pi_1(\Sigma)$ on $\mathbb{D}^2$ preserving the tiling, and thus, Σ can also be tiled by these pentagons. In fact, the argument above shows that any cover of F can be tiled by such pentagons.

Pulling back the metric induced by the tiling on F via the covering map r gives us a hyperbolic metric on Σ, which we will call the *standard metric* throughout the dissertation. All of our results will be for surfaces endowed with this standard metric. The fact that the standard metric is a hyperbolic metric is key. In his paper [21], Peter Scott demonstrates the gap in his original paper [20] with a counterexample for his argument in the Euclidean case. The special properties of hyperbolic space are precisely what allows his revised argument and all of our arguments to work.

One should note that for a hyperbolic surface Σ, $\pi_1(\Sigma)$ acts on $\mathbb{D}^2$ as the deck transformation group for the universal covering space. Therefore, the elements of $\pi_1(\Sigma)$ are isometries of $\mathbb{D}^2$, and as discussed in §2.3, they must be of loxodromic type. If $\alpha \in \pi_1(\Sigma)$, we make a slight abuse of notation and, as a convention, will call the unique geodesic representative in this homotopy class α as well.

Let Σ be a closed hyperbolic surface, tiled by regular, right-angled pentagons, and let $\alpha \in \pi_1(\Sigma) - \{id\}$. Let X be the cover of Σ corresponding to $\langle\alpha\rangle$, the cyclic subgroup of $\pi_1(\Sigma)$ generated by α. Since $\pi_1(X) \cong \langle\alpha\rangle \cong \mathbb{Z}$, X must be an open annulus or an open Möbius band depending on if α is an orientation preserving or orientation reversing loxodromic isometry. In both cases, there exists a lift of α that is the unique simple closed geodesic in X, which we will call $\overline{\alpha}$.

We then have the following sequence of covering maps:

$$\mathbb{D}^2 \xrightarrow{q} X \xrightarrow{p} \Sigma \xrightarrow{r} F$$

Since X is a cover of F, the argument above shows that X can be tiled by regular, right-angled pentagons.

3.2 Convexification

Definition. Let N be a subsurface of a hyperbolic surface M. N is *convex* if for every path $\gamma \subset N$, the geodesic γ^* homotopic rel endpoints to γ is also contained in N. N is *locally convex* if each point $x \in N$ has a neighborhood in N isometric to a convex subset of $\mathbb{H}^2$.

Definition. Let N be a subsurface of a hyperbolic surface M tiled by regular, right-angled pentagons. The *convexification* of N is the smallest, closed, convex union of pentagons in M that contains N.

Recall that q is the covering map $q : \mathbb{D}^2 \longrightarrow X$. Let $S = q(T)$, so that S consists of the pentagons that tile X. Let $S_0 \subset S$ be the union of all pentagons $P_i \in S$ such that

$P_i \cap \overline{\alpha} \neq \varnothing$. We choose the basepoint, a, of $\overline{\alpha}$ to be on a geodesic edge of a pentagon in S_0 for reasons that will become obvious later. Our first goal will be to convexify, i.e. obtain the convexification of, S_0, which we do by adding pentagons along ∂S_0 in an effort to cure any non-convex portions. We obtain a locally convex subsurface Y consisting of a union of pentagons in our tiling of X. In section 8.3 of his notes [22], Thurston shows that for a complete hyperbolic manifold, local convexity implies convexity and so Y will be the desired convexification of S_0. In this section, we aim to prove the following theorem:

Theorem 3.2.1. *We can convexify S_0 by adding pentagons in our tiling of X, so that any pentagon added has non-empty intersection with S_0.*

Recall that the pentagons of S have all angles equal to $\frac{\pi}{2}$. Therefore, S_0 can fail to be convex if three pentagons of S_0 form an angle of $\frac{3\pi}{2}$ at a vertex on ∂S_0. Points of ∂S_0 with interior angles equal to π will not be referred to as vertices of S_0. Thus, all of the vertices of S_0 either have interior angle equal to $\frac{3\pi}{2}$ or $\frac{\pi}{2}$.

Definition. If a vertex, v, of S_0 has interior angle $\frac{3\pi}{2}$, we will call v a *bad corner.* If a vertex, v, of S_0 has interior angle $\frac{\pi}{2}$, we will call v a *good corner.*

Definition. Choosing an orientation for each boundary component of ∂S_0, we will say that corners of ∂S_0 are *consecutive* if they occur consecutively with respect to the chosen orientation.

Our results will be independent of the choice of orientation in this definition.

In order to obtain the convexification we must first understand how bad the boundary components of S_0 can be, which we quantify by how many consecutive bad corners occur along it.

Lemma 3.2.2. *Two bad corners never occur consecutively on a boundary component of S_0.*

Proof. Let $\widetilde{\alpha}$ be one lift of $\overline{\alpha}$ to $\mathbb{D}^2$. Lift every pentagon of S_0 to its lift that intersects $\mathring{\widetilde{\alpha}}$. Doing so, we have lifted all of S_0 and ∂S_0. We call their lifts $\widetilde{S_0}$ and $\widetilde{\partial S_0}$. Now suppose we have a bad corner in S_0 formed by three pentagons P_1, P_2, $P_3 \in S_0$. We have lifted these three pentagons to $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$ in $\mathbb{D}^2$. By construction, $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$ intersect $\widetilde{\alpha}$ and form a bad corner, B_1, on $\widetilde{\partial S_0}$. Translating by an element of Γ, we may assume that $\widetilde{P}_1$, $\widetilde{P}_2$ and $\widetilde{P}_3$ form the region in Figure 3.1 (a) below.

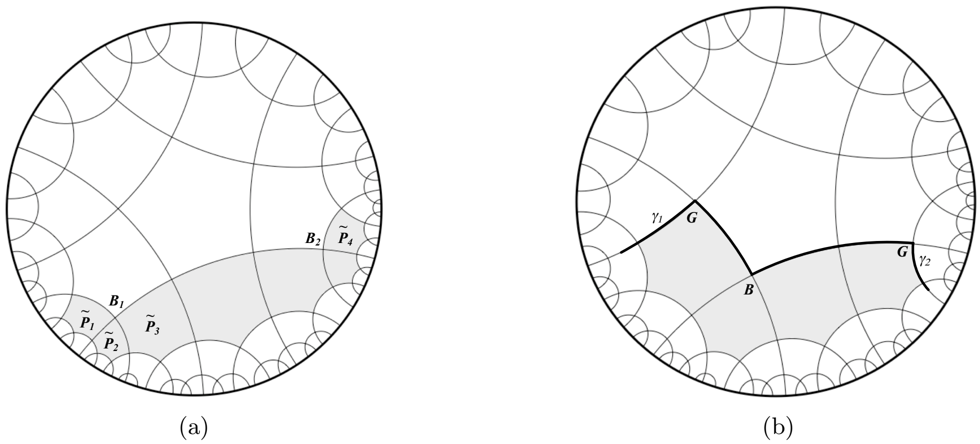


Figure 3.1

If we hit a second consecutive bad corner, B_2, travelling along $\widetilde{\partial S_0}$, then there exists a pentagon $\widetilde{P}_4 \in \widetilde{S_0}$, as in Figure 3.1 (a) above, that is one of the pentagons that forms B_2. Since $\widetilde{P}_1$, $\widetilde{P}_4 \in \widetilde{S_0}$, $\widetilde{\alpha}$ must intersect both of these pentagons, but $\widetilde{\alpha}$ cannot intersect any of the unshaded pentagons between them. There is no such geodesic in $\mathbb{D}^2$. Thus, there can never be two consecutive bad corners along a boundary component of S_0.

□

The result above is independent of the choice of orientation for the boundary component containing B_1, so that two consecutive bad corners can never occur along a boundary component of S_0 with respect to either of the two choices of orientation for the boundary component.

To each of the boundary components of S_0 we can associate a word, w, in the letters G and B by reading off whether the consecutive corners are good or bad along the boundary component. For example, if a boundary component contains two bad corners and four good corners the word w could be $w = GBGBGG$. For long words we will write $w = \cdots GBGGBGG \cdots$, by which we mean that we are reading off only a portion of the corners along the boundary component. An immediate consequence of Lemma 3.2.2 is the following.

Corollary 3.2.3. *Let B_0 be a bad corner of a boundary component of S_0. Then the word w associated to that boundary component of S_0 must be of the form $w = B_0, w = B_0G, w = GB_0$ or $w = \cdots GB_0G \cdots$.*

Proof. If B_0 is the only corner of that boundary component, then $w = B_0$. If the boundary component consists of exactly two corners, Lemma 3.2.2 tells us that $w \neq B_0B_1$ where B_1 is another bad corner since bad corners never occur consecutively along ∂S_0. Thus, $w = B_0G$ or $w = GB_0$. If the boundary component has three or more corners, then again by Lemma 3.2.2 we know that two bad corners never occur consecutively along ∂S_0 regardless of the orientation we choose for the boundary component. Thus, $w \neq \cdots GB_0B_1 \cdots$ and $w \neq \cdots B_1B_0G \cdots$, and w must therefore be of the form $\cdots GB_0G \cdots$.

□

Now we will attempt to convexify S_0 by adding pentagons near the bad corners and show that the number of pentagons needed can be bounded.

Proof of Theorem 3.2.1. So long as we are not in the case where $w = B$, $w = BG$ or $w = GB$ (see the note below), the picture at each bad corner looks like Figure 3.2 below. If we extend the far geodesic edges of the good corners and add the pentagons that intersect S_0 lying between these two extended geodesic segments, we will have convexified this portion of S_0.

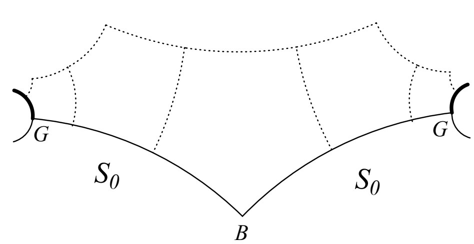


Figure 3.2

Note: If $w = B$, the lift of this component of ∂S_0 to $\mathbb{D}^2$ will look exactly like the figure above. In this case, the geodesics with the bold extensions will have been identified in S_0 and we will have convexified this boundary component. In fact, this one step is enough to obtain the convexification for the case where $w = GB$ or BG as well. So in these cases the convexification procedure is complete at this point.

We follow this procedure for every bad corner of S_0. As shown earlier, the bad corners of S_0 are separated by one or more good corners. We will see below that it is slightly easier to convexify portions of boundary components where bad corners are separated by two or more good corners so we will handle this case first.

Case 1: Suppose B_1 is a bad corner of S_0 and that travelling along ∂S_0, the next bad corner we hit, B_2, is separated from B_1 by two or more good corners. Then Figure 3.3 (a) below shows that we have convexified the region around B_1 and B_2 and we need not add any more pentagons between the two bad corners.

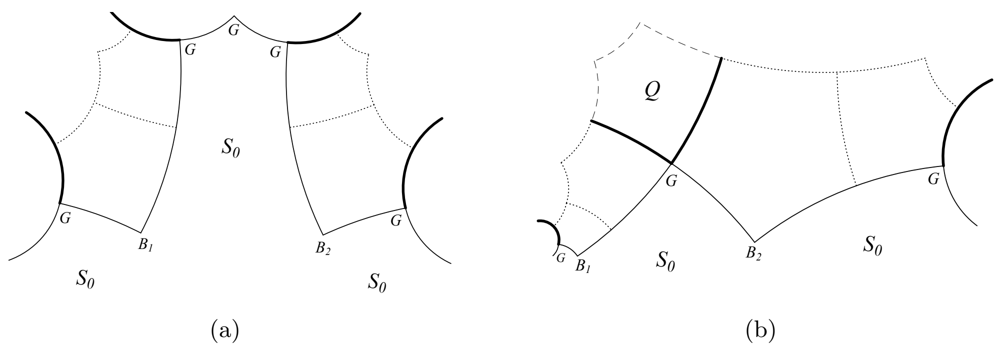


Figure 3.3

Case 2: Suppose that B_2 is separated from B_1 by only one good corner, that is $w = \cdots GB_1GB_2G\cdots$. Then we will need to add one more pentagon of S to S_0, which will still have non-empty intersection with S_0.

As seen in Figure 3.3 (b) above, we have created a new bad corner during our attempt to convexify S_0. However, we can simply add in the one missing pentagon, Q, which still intersects ∂S_0 at one point.

After adding all such pentagons Q, we have a set Y containing S_0 and obtained by adding pentagons that all intersect S_0. Y is locally convex, and by the comments above

Y is therefore the convexification of S_0 we were looking for.

$\square$

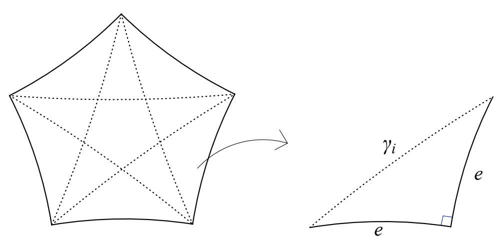
3.3 Bounding the number of pentagons in Y

Our next goal will be to obtain an upper bound on the number of pentagons in Y, the convex set obtained in the proof of Theorem 3.2.1. This bound will play a crucial role in proving our main result, Theorem 3.6.1.

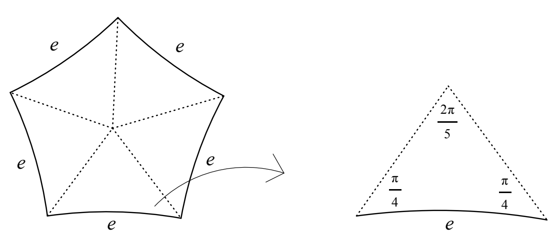
Lemma 3.3.1. *The diameter of each pentagon in our tiling of* $\mathbb{D}^2$*, and therefore in our tiling of* X*, is* $\cosh^{-1}\left(\left(1+2\cos\dfrac{2\pi}{5}\right)^2\right)$.

Proof. Each pentagon in our tiling has angles $\dfrac{\pi}{2}$. For simplicity we will work with a regular, right-angled pentagon P centered at the origin of $\mathbb{D}^2$.

The longest geodesic segments between any two points of P are represented by the dotted lines in Figure 3.4 (a). Call these five segments $\gamma_1, \ldots, \gamma_5$ of lengths $\ell_1, \ldots, \ell_5$ respectively. It turns out that ℓ_i are all equal.



(a)



(b)

Figure 3.4

First we will find the lengths of the sides of P, which are also all equal. Break P into the five triangles in Figure 3.4 (b) above. Each triangle has angles $\frac{2\pi}{5}$, $\frac{\pi}{4}$ and $\frac{\pi}{4}$ with the side of P, whose length we will call e_P, opposite the $\frac{2\pi}{5}$ angle. Using a hyperbolic Law of Cosines (§2.1), we have that $e_P = \cosh^{-1}\left(1 + 2\cos\left(\frac{2\pi}{5}\right)\right)$.

Now going back to Figure 3.4 (a), we see that each of the γ_i forms a side of a hyperbolic triangle opposite a right angle where the other two sides are of length e_P. Using another hyperbolic Law of Cosines (§2.1) we have that $\ell_i = \cosh^{-1}\left((\cosh e_P)^2\right) = \cosh^{-1}\left(\left(1 + 2\cos\frac{2\pi}{5}\right)^2\right) \approx 1.167$.

□

The length ℓ_i is the diameter of P and we now call this diameter d_0. We will use d_0 to bound the number of pentagons in Y, but first we need the following lemma.

Lemma 3.3.2. *Let $\Omega \subset \mathbb{H}^2$ be the region in Figure 3.5. Then $Area(\Omega) = \ell_0 \sinh b$, where ℓ_0 is the length of the geodesic segment between $r_0 i$ and $R_0 i$.*

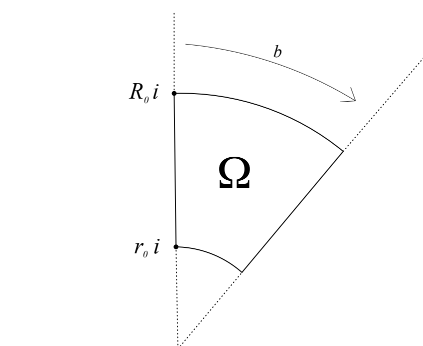


Figure 3.5

Proof. Let $\pi(b)$ be the angle of parallelism (see §2.1).

$$Area(\Omega) = \iint\limits_{\Omega} \frac{dx \wedge dy}{y^2} = \int_{\frac{\pi}{2}-\pi(b)}^{\frac{\pi}{2}} \int_{r_0}^{R_0} \frac{dr \wedge d\theta}{r \sin^2\theta} = \ln\left(\frac{R_0}{r_0}\right) \int_{\frac{\pi}{2}-\pi(b)}^{\frac{\pi}{2}} \frac{1}{\sin^2\theta}\, d\theta =$$

$$\ell_0 \left(-\cot\theta \Big|_{\frac{\pi}{2}-\pi(b)}^{\frac{\pi}{2}} \right) = \ell_0 \cot\left(\frac{\pi}{2} - \pi(b)\right).$$

By the angle of parallelism laws, $\cot\left(\frac{\pi}{2} - \pi(b)\right) = \sinh b$, so that $Area(\Omega) = \ell_0 \sinh b$.

□

Theorem 3.3.3. *Let ℓ be the length of $\overline{\alpha}$ in X, and hence, the length of α in Σ. Then $Area(Y) \leq 2\,\ell \sinh(2d_0)$.*

Proof. Let $Z = \{x \in X : d(x, \overline{\alpha}) \leq 2d_0\}$. We know that every pentagon of Y either intersects $\overline{\alpha}$ (and is an element of S_0) or intersects S_0. Thus, $\sup_{y \in Y} \{d(y, \overline{\alpha})\} \leq 2\,d_0$, so that $Y \subseteq Z$.

Recall that a is the basepoint of $\overline{\alpha}$. Choose points z_1 and z_2 on the two different boundary components of Z, such that $d(z_i, a) = 2\,d_0$. Let β_i be the geodesic segment between a and z_i, for $i = 1, 2$. Then a lift of Z to $\mathbb{H}^2$ looks like the region in Figure 3.6, where $\widetilde{\beta}_i$ and $\widetilde{\beta}'_i$ are two lifts of β_i, for $i = 1, 2$, and $\widetilde{\alpha}$ is a lift of $\overline{\alpha}$. We say that Z "opens" along the β_i.

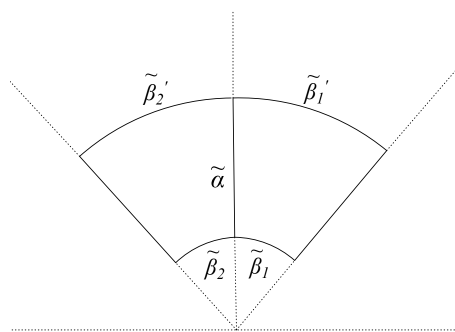


Figure 3.6

Since the lengths of $\widetilde{\beta}_1$, $\widetilde{\beta}'_1, \widetilde{\beta}_2$, $\widetilde{\beta}'_2$ are all equal to $2\,d_0$, we can apply Lemma 3.3.2, which tells us that $Area(Z) = 2\ell \sinh(2\,d_0)$. Thus, we have $Area(Y) \leq Area(Z) = 2\ell \sinh(2\,d_0)$.

□

Note that when Σ in unorientable, there may not be two boundary components of Z. We can still open Z in the same way as described above, and the lift to $\mathbb{D}^2$ will have the same area as the region in Figure 3.6, the only difference being that the labeling of $\widetilde{\beta}'_1$ and $\widetilde{\beta}'_2$ should be exchanged.

Corollary 3.3.4. *If Y consists of k pentagons, then $Area(Y) = k\,\frac{\pi}{2} \leq 2\,\ell \sinh(2\,d_0)$. Solving for k we have, $k \leq \dfrac{4\,\sinh(2\,d_0)}{\pi}\,\ell \approx 16.131\,\ell$. Thus, $k < 16.2\,\ell$.*

□

Corollary 3.3.5. *If $\overline{\alpha}$ is the image of a geodesic line in our tessellation of $\mathbb{D}^2$, then $k < 3.1\,\ell$.*

Proof. Since $Y = S_0$, $Y \subseteq \{x \in X : d(x, \overline{\alpha}) \leq d_0\}$. Therefore, $k \leq \dfrac{4\sinh(d_0)}{\pi}\ell \approx 3.081\,\ell < 3.1\,\ell$.

□

We should note that the bounds given in Corollaries 3.3.4 and 3.3.5 are not sharp. By going to the $2d_0$ neighborhood of the geodesic $\overline{\alpha}$ we are overestimating the area of the convex union of pentagons Y, and therefore overestimating the number of pentagons in the set. Also, in the case where $\overline{\alpha}$ is the image of a geodesic line in our tessellation, we can give the exact number of pentagons in $Y = S_0$ to be $k = \dfrac{2\ell}{e_P}$ where e_P is the length of the edges in the regular right-angled pentagons computed above. We use the bound in Corollary 3.3.5 because it is analogous to the bound in Corollary 3.3.4 and using this estimate eliminates the need for a separate argument in the proof of Theorem 3.6.1.

3.4 Lift to Finite Cover

In this section we will prove Theorem 1.0.1 in the closed case and then show that the argument easily extends to the case of a compact surface with boundary.

Let Σ be a closed surface endowed with the standard metric defined in Section 3.1. Recall that X is the cover of Σ corresponding to $\langle\alpha\rangle$, and that $\mathbb{D}^2 \xrightarrow{q} X$ is the universal covering map. Therefore, the deck transformation group $\langle\alpha\rangle$ acts by isometries on $\mathbb{D}^2$. The axis of the isometries of $\langle\alpha\rangle$ is the geodesic line in $\mathbb{D}^2$ consisting of all of the lifts of $\overline{\alpha}$. Call this axis L.

Let $\widetilde{Y}$ be the set of all lifts of the pentagons in Y to $\mathbb{D}^2$, and let the set of isometries of $\mathbb{D}^2$ consisting of reflections in the sides y_i of $\widetilde{Y}$ be denoted by $R = \left\{R_{y_i} : y_i \text{ is a side of } \widetilde{Y}\right\}$. Then $\widetilde{Y}$ is a fundamental domain for the action of $\langle R\rangle$ on $\mathbb{D}^2$ by the Poincaré Polygon Theorem (§2.4.1). Note that $\langle R\rangle < \Gamma$ since the sides of $\widetilde{Y}$

are lines in our tessellation of $\mathbb{D}^2$, and a reflection in any of these lines is an element of Γ.

Let $K = \langle R, \alpha \rangle$, and let $\widetilde{\alpha}$ be one lift of $\overline{\alpha}$ to $\mathbb{D}^2$. Next, lift each pentagon of Y to one of its lifts so that the result is a connected union of k pentagons in $\mathbb{D}^2$ containing $\widetilde{\alpha}$. Call this union of k pentagons $\overline{Y}$.

Lemma 3.4.1. *$\overline{Y}$ is a fundamental domain for the action of $K = \langle R, \alpha \rangle$ on $\mathbb{D}^2$.*

Proof. We know that $\bigcup\limits_{\alpha^n \in \langle \alpha \rangle} \alpha^n \overline{Y} = \widetilde{Y}$, where $\widetilde{Y}$ is the set of all lifts of the pentagons of Y. Since $\widetilde{Y}$ is a fundamental domain for $\langle R \rangle$, we also know that $\bigcup\limits_{r \in \langle R \rangle} r\,\widetilde{Y} = \mathbb{D}^2$. Therefore, $\bigcup\limits_{k \in K} k\,\overline{Y} = \mathbb{D}^2$. Now must show that $k\,\mathring{\overline{Y}} \cap \mathring{\overline{Y}} = \varnothing$, for all $k \in K - \{id\}$.

Since $\langle \alpha \rangle$ is the deck transformation group for $\mathbb{D}^2 \xrightarrow{q} X$ and $\overline{Y}$ contains only one lift of each pentagon in Y, we know that $\alpha^n \mathring{\overline{Y}} \cap \mathring{\overline{Y}} = \varnothing$ for all $\alpha^n \in \langle \alpha \rangle - \{id\}$.

Next, recall that R_{y_i} denotes a reflection in the side y_i of $\widetilde{Y}$. If $\alpha^n y_1 = y_2$ where y_1 and y_2 are sides of $\widetilde{Y}$, then we have the relation $\alpha^n R_{y_1} = R_{y_2} \alpha^n$ in K, in other words $\alpha^n R_{y_1} \alpha^{-n} = R_{y_2}$. In fact, there exists a group homomorphism $\phi : \langle \alpha \rangle \longrightarrow Aut(\langle R \rangle)$, defined by $\phi(\alpha^n) \to \phi_{\alpha^n}$, where $\phi_{\alpha^n}(r) = \alpha^n r \alpha^{-n}$ for all $r \in \langle R \rangle$ and $\alpha^n \in \langle \alpha \rangle$. Therefore, $K = \langle R \rangle \rtimes_\phi \langle \alpha \rangle$, and it follows that every element of K can be written as $r\alpha^n$ where $r \in \langle R \rangle$.

Now, $\widetilde{Y}$ is a fundamental domain for $\langle R \rangle$, so $r\,\mathring{\widetilde{Y}} \cap \mathring{\widetilde{Y}} = \varnothing$ for every $r \in \langle R \rangle - \{id\}$. Let $k = r\alpha^n \in K - \{id\}$. If $r \neq id$ then $\alpha^n \mathring{\overline{Y}} \subset \mathring{\widetilde{Y}}$ and $r\alpha^n \mathring{\overline{Y}} \cap \mathring{\widetilde{Y}} = \varnothing$, and hence $k\mathring{\overline{Y}} \cap \mathring{\overline{Y}} = \varnothing$. If $r = id$ then $\alpha^n \neq id$ and we have shown above that in this case $k\mathring{\overline{Y}} \cap \mathring{\overline{Y}} = \alpha^n \mathring{\overline{Y}} \cap \mathring{\overline{Y}} = \varnothing$.

$\square$

Lemma 3.4.2. *Let $K' = K \cap \pi_1(\Sigma)$. Then, $[\pi_1(\Sigma) : K'] \leq [\Gamma : K] = k$.*

Proof. Since $\overline{Y}$ is a fundamental domain for K and consists of k pentagons, we know that $[\Gamma : K] = k$. Poincaré's Theorem [14] states that if H_1 and H_2 are subgroups of a group G, then $[H_1 : H_1 \cap H_2] \leq [G : H_2]$. The lemma follows by letting $G = \Gamma$, $H_1 = \pi_1(\Sigma)$ and $H_2 = K$ in Poincaré's Theorem.

$\square$

Next, we let N be the cover of Σ corresponding to the subgroup $K' = K \cap \pi_1(\Sigma)$ of $\pi_1(\Sigma)$. That is $N = \mathbb{D}^2/K'$. Let $s : \mathbb{D}^2 \longrightarrow N$ be the covering map.

Lemma 3.4.3. *The image of $\widetilde{\alpha}$ under s is an embedded loop in N.*

Proof. Since K' is a subgroup of K, the covering map $f : \mathbb{D}^2 \longrightarrow \mathbb{D}^2/K$ factors as $f = u \circ s$ where s and u are the covering maps in the sequence $\mathbb{D}^2 \xrightarrow{s} \mathbb{D}^2/K' \xrightarrow{u} \mathbb{D}^2/K$. Let $\alpha' = s(\widetilde{\alpha})$. We will first show that $f(\widetilde{\alpha})$ is an embedded loop in $\mathbb{D}^2/K$, and then use this fact to show that α' is an embedded loop in $N = \mathbb{D}^2/K'$.

Since $\alpha \in K$ we know that $f(\widetilde{\alpha})$ is certainly a loop in $\mathbb{D}^2/K$. By Lemma 3.4.1 the set $\overline{Y}$, defined above, is a fundamental domain for the action of K on $\mathbb{D}^2$. Recall that $\overline{\alpha}$ is the lift of α that is the unique simple closed geodesic in the cover X. Also recall that $\widetilde{\alpha}$ is the only lift of $\overline{\alpha}$ in $\overline{Y}$, and that $\widetilde{\alpha}$ is a simple geodesic arc in $\overline{Y}$ with endpoints in $\partial\overline{Y}$. Since $\overline{Y}$ is a fundamental domain for the action of K, the restriction of f to $\mathring{\overline{Y}}$ is a homeomorphism into $\mathbb{D}^2/K$. Therefore, $\mathring{\widetilde{\alpha}}$ also projects by a homeomorphism into $\mathbb{D}^2/K$ since $\mathring{\widetilde{\alpha}} \subset \mathring{\overline{Y}}$. Thus, $f(\widetilde{\alpha})$ is an embedded loop in $\mathbb{D}^2/K$.

Now, since $\alpha \in K'$ we know that $\alpha' = s(\widetilde{\alpha})$ is also a loop in $N = \mathbb{D}^2/K'$. But, $f(\widetilde{\alpha}) = u(s(\widetilde{\alpha}))$ so that if $s(\widetilde{\alpha})$ is not an embedded loop in N, $f(\widetilde{\alpha})$ cannot be an embedded loop in $\mathbb{D}^2/K$. That is, if x_1 and x_2 are two points of $\widetilde{\alpha}$ that are identified under the map s, then $f(x_1) = u(s(x_1)) = u(s(x_2)) = f(x_2)$ is a self intersection point of $f(\widetilde{\alpha})$. Thus, α' must be an embedded loop in $N = \mathbb{D}^2/K'$.

□

We have now proved Theorem 1.0.1, and actually have proved the following stronger result.

Theorem 3.4.4. *Let Σ be a closed surface of negative Euler characteristic, endowed with the standard metric. For every closed geodesic α in Σ, there exists a finite cover X_α of Σ in which α lifts to an embedded loop. The index of the cover is bounded by 16.2ℓ, where ℓ is the length of the geodesic α. If α is the image in Σ of a geodesic line in our tessellation of $\mathbb{D}^2$, the index of the cover is bounded by $3.1\,\ell$.*

Proof. Our finite cover X_α is the cover $N = \mathbb{D}^2/K'$ of Lemma 3.4.3, and the lift of α that is an embedded geodesic loop is α'. $\pi_1(N) = K'$ and by Lemma 3.4.2 and Corollary 3.3.4, $[\pi_1(\Sigma) : K'] < k < 16.2\ell$. Thus, the index of X_α as a cover of Σ is bounded by 16.2ℓ. If α is the image of a line in our tiling of $\mathbb{D}^2$, $k < 3.1\,\ell$ and the result follows.

□

We have proved Theorem 3.4.4 for any closed surface of negative Euler characteristic with the standard metric. However, the theorem also holds for compact surfaces with boundary of negative Euler characteristic. In [20], Peter Scott showed that for the closed case, Σ can be tiled by regular, right-angled pentagons, and thus proved that $\pi_1(\Sigma) < \pi_1(F) < \Gamma$. It turns out that compact surfaces with boundary, of negative Euler characteristic, can also be tiled by regular, right-angled pentagons. We endow such a surface Σ with the metric obtained through this tiling by pentagons, and call this the standard metric as well. With the standard metric, the universal cover of Σ is a convex, non-compact polygon in $\mathbb{D}^2$, which we call $\widetilde{\Sigma}$. Therefore, $\pi_1(\Sigma)$ acts on $\widetilde{\Sigma}$ by isometries, but these isometries extend to isometries of $\mathbb{D}^2$. Then considering $\pi_1(\Sigma)$ as a group of isometries of $\mathbb{D}^2$, we have that $\pi_1(\Sigma) < \Gamma$. Once we have this result, the rest of the proof follows exactly as in the closed case.

3.5 Hyperbolic Surface Groups are Residually Finite

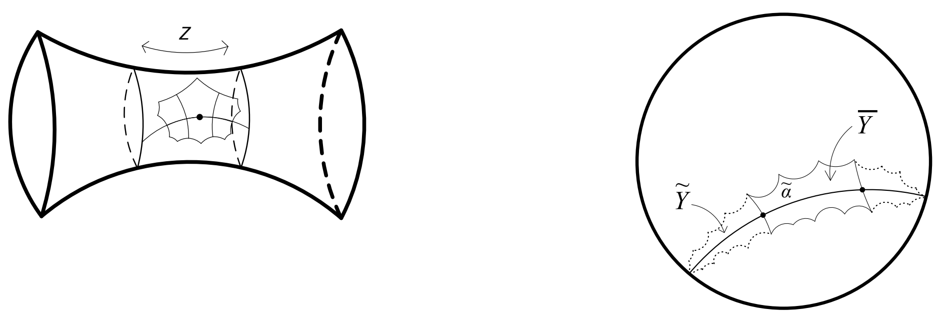
Definition. A group G is said to be *residually finite (RF)* if for every non-trivial element $g \in G$, there is a subgroup G' of finite index in G that does not contain g.

Let Σ be a closed surface or a compact surface with boundary. From [20], we know that $\pi_1(\Sigma)$ is RF. We will quantify this result by proving the following theorem, which is a slightly stronger result than Theorem 1.0.2.

Theorem 3.5.1. *Let Σ be a compact surface of negative Euler characteristic endowed with the standard metric. For any $\alpha \in \pi_1(\Sigma) - \{id\}$, there exists a subgroup H' of $\pi_1(\Sigma)$, such that $\alpha \notin H'$. Additionally, $[\pi_1(\Sigma) : H'] < 32.3\,\ell$, where ℓ is the length of the geodesic α. If α is the image of a geodesic line in our tessellation of $\mathbb{D}^2$, $[\pi_1(\Sigma) : H'] < 6.2\,\ell$.*

Proof. Let $\alpha \in \pi_1(\Sigma) - id$. Using the same notation as in Section 3.4, we let $\widetilde{\alpha}$ be one lift of $\overline{\alpha}$ to $\mathbb{D}^2$. We claim that we can lift Y to $\mathbb{D}^2$ so that the result is a convex union of k pentagons. The convexity of the lift is crucial since we will want to apply the Poincaré Polygon Theorem to prove the result above.

Let Z be the set defined in the proof of Theorem 3.3.3. We lift Z to a lift in $\mathbb{D}^2$ that contains $\widetilde{\alpha}$. Recall that in the proof of Theorem 3.3.3 we lifted Z so that it opened along the geodesics β_i. Instead we lift Z so that it opens along the geodesic line of our tessellation of X containing the basepoint of $\overline{\alpha}$ and shown on the left in Figure 3.7 (a) below. Call this set $\overline{Z} \subset \mathbb{D}^2$ and lift every pentagon of Y to its lift that lies in $\overline{Z}$. The result is a convex, connected union of k pentagons which we will call $\overline{Y}$ (see Figure 3.7 (b)).



(a) (b)

Figure 3.7

Let $\widetilde{\alpha}_1$ be one of the two lifts of $\overline{\alpha}$ that share endpoints with $\widetilde{\alpha}$. Let $\overline{Y}_1$ be the lift of Y containing $\widetilde{\alpha}_1$. Then $Y' = \overline{Y} \cup \overline{Y}_1$ is a convex union of $2k$ pentagons in $\mathbb{D}^2$, such that one endpoint of $\widetilde{\alpha}$ is contained in the interior of Y'.

Let H be the group of isometries of $\mathbb{D}^2$ generated by reflections in the sides of Y'. Then $H < \Gamma$, and Y' is a fundamental domain for the action of H on $\mathbb{D}^2$ by the Poincaré Polygon Theorem (§2.4.1). Since Y' contains $2k$ pentagons, $[\Gamma : H] = 2k$. Letting $b : \mathbb{D}^2 \longrightarrow \mathbb{D}^2/H$ be the covering map, we then have that the restriction of b to $\mathring{Y}'$ is a homeomorphism onto its image. Since one endpoint of $\widetilde{\alpha}$ is contained in $\mathring{Y}$, $b(\widetilde{\alpha})$ is cannot close up to become a loop in $\mathbb{D}^2/H$, and $\alpha \notin H$.

Now, let $H' = H \cap \pi_1(\Sigma)$. Then, $\alpha \notin H'$ and $[\pi_1(\Sigma) : H'] \leq [\Gamma : H] = 2k$. The result follows from Corollary 3.3.4 and Corollary 3.3.5.

□

One should note that the bound in Theorem 3.5.1 can certainly be improved. Instead of adding one more set of k pentagons to $\overline{Y}$, we could have simply added a small number of pentagons in order to encapsulate one of the endpoints of $\widetilde{\alpha}$ and retain convexity. We used the method above to simplify the proof.

3.6 Hyperbolic Surface Groups are LERF

Definition. A group G with a subgroup S is *S-residually finite* if for any element g of $G - S$, there is a subgroup G' of finite index in G which contains S but not g. A group G is called *locally extended residually finite (LERF)* if G is S-residually finite for every **finitely generated** subgroup S of G.

Let Σ be a closed surface or a compact surface with boundary. From [20], we know that $\pi_1(\Sigma)$ is LERF, and we will attempt to quantify this result. Just as for Theorem 3.4.4, we will prove the result in the closed case, and then will see that the compact with boundary case follows immediately.

Let Σ be a closed surface of negative Euler characteristic with the standard metric, and let S be a finitely generated subgroup of $\pi_1(\Sigma)$ with $g \in \pi_1(\Sigma) - S$. If S is a finite index subgroup, then S itself is the required subgroup for the LERF condition. Thus, we will be interested in the case where S is a finitely generated, infinite index subgroup of $\pi_1(\Sigma)$.

Let X be the cover of Σ corresponding to such a subgroup S. If we pull back the standard metric on Σ to X, then X is a noncompact hyperbolic surface of finite type. Thus, $\pi_1(X) \cong S$ is a free group of rank n.

Let $\gamma \in \pi_1(\Sigma) - S$. As per our convention, we also let γ be the unique geodesic representative in this homotopy class and let ℓ_γ be the length of γ. Let $\widetilde{\gamma}$ be a lift of γ to X. Since $\gamma \notin S$, $\widetilde{\gamma}$ is a (non-closed) geodesic path in X.

Let $C(X)$ be the convex core of X, that is, $C(X)$ is the smallest, closed, convex subsurface of X with geodesic boundary, such that $i : C(X) \longrightarrow X$ is a homotopy equivalence. Choose the basepoint, x_0, of X to be in $C(X)$, and let $\alpha_1, \ldots, \alpha_m$ be the geodesic boundary components of $C(X)$ of lengths $\ell_1, \ldots, \ell_m$, respectively.

We will quantify Peter Scott's LERF theorem by proving the following result.

Theorem 3.6.1. *Let Σ be a compact surface of negative Euler characteristic with the standard metric. If S is an infinite index, finitely generated subgroup of $\pi_1(\Sigma)$ and $\gamma \in \pi_1(\Sigma) - S$ as described above, then there exists a finite index subgroup K' of $\pi_1(\Sigma)$, such that $S \subseteq K'$ and $\gamma \notin K'$. When the rank of S is $n \geq 2$, the index of K' in $\pi_1(\Sigma)$ can be bounded as follows. If $\widetilde{\gamma} \subset C(X)$,*

$$[\pi_1(\Sigma) : K'] < 4n - 4 + 8.1(\ell_1 + \cdots + \ell_m) \tag{3.1}$$

and if $\widetilde{\gamma} \not\subset C(X)$,

$$[\pi_1(\Sigma) : K'] < 4n - 4 + \frac{2 \sinh\left[(\ell_\gamma / e_P + 2)\, d_0\right]}{\pi} (\ell_1 + \ell_2 + \cdots + \ell_m), \tag{3.2}$$

where e_P is the length of the edges and d_0 is the diameter in a regular, right-angled, hyperbolic pentagon calculated in Section 3.3. If α_j is the image of a line in our tessellation of $\mathbb{D}^2$ for some j, the coefficient of ℓ_j can be improved to 1.6 instead of 8.1 in equation (1).

In the case where the rank of S is $n = 1$, we must double the coefficients of the ℓ_1 in equations (1) and (2) and we arrive at the following bounds:
If $\widetilde{\gamma} \subset C(X)$,

$$[\pi_1(\Sigma) : K'] < 16.2\, \ell_1 \tag{3.3}$$

and if $\widetilde{\gamma} \not\subset C(X)$,

$$[\pi_1(\Sigma) : K'] < \frac{4 \sinh\left[(\ell_\gamma / e_P + 2)\, d_0\right]}{\pi} \ell_1. \tag{3.4}$$

Proof. We handle the case where $n \geq 2$ and will elaborate on the case where $n = 1$ at the end of the proof.

We will be interested in extending $C(X)$ at the boundary components α_i in order to obtain a convex union of pentagons containing $\widetilde{\gamma}$ in its interior. Then we will apply the same methods we used to prove the RF case.

Let S_i be the set of pentagons in the tiling of X whose intersection with α_i is non-empty. Let Y_i be the one sided convexification, i.e. the convexification of the side of α_i in $X - C(X)$, of each set S_i, obtained by the procedure in Section 3.2 and shown in Figure 3.8 below.

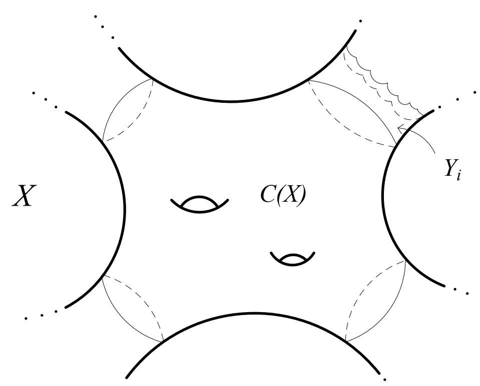


Figure 3.8

Note that though X may be unorientable, it makes sense to talk about the side of α_i in $C(X)$ and the side of α_i in $X - C(X)$ because α_i admits a bi-collar neighborhood in X that is orientable.

<u>Case 1:</u> Suppose $\widetilde{\gamma}$ is completely contained in $C(X)$. Then $Y_i \subset C(X) \cup Z_i$, where $Z_i = \left\{ x \in \overline{X - C(X)} : d(x, \alpha_i) \leq 2d_0 \right\}$. From the results of Section 3.3, we know $Area(Z_i) = \sinh(2\, d_0)\, \ell_i$.

Since $\pi_1(C(X)) \cong \pi_1(X) \cong S$, we know $\chi(C(X)) = 1 - n$. Then by the Gauss-Bonnet Theorem [18], we also have that $Area(C(X)) = 2\pi(n-1)$.

Let $Y = C(X) \cup Y_1 \cup \cdots \cup Y_m$. Then $Y \subset C(X) \cup Z_1 \cup \cdots \cup Z_m$, and Y is a convex union of k' pentagons, where $k' \dfrac{\pi}{2} = Area(Y) \leq 2\pi(n-1) + \sinh(2\, d_0)\, \ell_1 + \cdots + \sinh(2\, d_0)\, \ell_m$. Therefore,

$$k' < 4n - 4 + 8.1(\ell_1 + \cdots + \ell_m).$$

Note: If any α_i is the image of a line in our tessellation of $\mathbb{D}^2$, then S_i is automatically convex, so that $Y_i = S_i$. Therefore, $Y_i \subset C(X) \cup Z_i^*$ where $Z_i^* = \{x \in \overline{X - C(X)} : d(x, \alpha_i) \leq d_0\}$, and $Area(Z_i^*) = \sinh(d_0)\,\ell_i \approx 1.55\,\ell_i$. This gives us the improvement on the bound stated in Theorem 3.6.1.

Let $\widetilde{Y} \subset \mathbb{D}^2$ be the set of all lifts of the pentagons in Y. As before, let $R = \left\{R_{y_i} : y_i \text{ is a side of } \widetilde{Y}\right\}$ be the set of isometries of $\mathbb{D}^2$ consisting of reflections in the sides, y_i of $\widetilde{Y}$. Then $\widetilde{Y}$ is a fundamental domain for the action of $\langle R \rangle$ on $\mathbb{D}^2$, and $\langle R \rangle < \Gamma$.

Let W be a fundamental domain for the action of S on $\mathbb{D}^2$ so that $W/S = X$ and W is a union of pentagons in our tessellation of $\mathbb{D}^2$. Lift each pentagon of Y to one of its lifts so that the result is a connected union of k' pentagons contained in W. We call this union of k' pentagons $\overline{Y}$.

Let $K = \langle R, S \rangle$, and let $X' = \mathbb{D}^2/K$. The remainder of the proof follows the arguments of Section 5 very closely. By an extension of the reasoning in Section 5, we have that $K = \langle R, S \rangle = \langle R \rangle \rtimes_\phi S$, and $\overline{Y}$ is a fundamental domain for the action of K on $\mathbb{D}^2$. Thus, $[\Gamma : K] = k'$. Since $\widetilde{\gamma}$ is contained in $C(X) \subset \mathring{Y}$, all lifts of $\widetilde{\gamma}$ are contained in the interior of $\overline{Y}$. The image of $\widetilde{\gamma}$ is, therefore, not a loop in $X' = \mathbb{D}^2/K$ and $\gamma \notin K$.

Letting $K' = K \cap \pi_1(\Sigma)$, we have that $S \subset K'$, $\gamma \notin K'$ and

$$[\pi_1(\Sigma) : K'] \leq [\Gamma : K] = k' < 4n - 4 + 8.1(\ell_1 + \cdots + \ell_m).$$

Explanation of equation (3): If the rank of S is $n = 1$, the cover X corresponding to S is an open annulus or an open Möbius band as described in Section 3.1. In this case, $C(X) = \alpha_1$, and $\gamma \subseteq C(X)$. The analog of the sets Z_i defined above is the set $Z = \left\{x \in \overline{X - C(X)} : d(x, \alpha_1) \leq 2d_0\right\} = \{x \in X : d(x, \alpha_1) \leq 2d_0\}$. We know from the calculation in Theorem 3.3.3 that $Area(Z) = 2\sinh(2\,d_0)\,\ell_1$, explaining the doubling of the coefficient of ℓ_1.

<u>Case 2:</u> Suppose that $\widetilde{\gamma}$ is not completely contained in $C(X)$. Recall that we chose the basepoint of X to be in $C(X)$ so that one endpoint of $\widetilde{\gamma}$ is $x_0 \in C(X)$. Then $\widetilde{\gamma}$ crosses a boundary component of $C(X)$, say α_1, at some point and enters $X - C(X)$. Of course $\widetilde{\gamma}$ may extend past Y_1, and thus may not be contained in the convex space

Y. Thus, we will add pentagons to Y_1 until we have encapsulated the portion of the curve $\widetilde{\gamma}$ in the non-compact region bounded by α_1. We repeat this procedure for each Y_i so that we have encapsulated all of $\widetilde{\gamma}$ in a larger convex union of pentagons and then apply the same method as in Case 1.

Let ∂Y_1 be the portion of the boundary of Y_1 contained in $X - C(X)$. Let U_1 be the set obtained from Y_1 by adding all pentagons in the tessellation of X that intersect ∂Y_1. Let ∂U_1 be the portion of the boundary of U_1 contained in $X - C(X)$. Since Y_1 is convex along ∂Y_1, U_1 will be convex along ∂U_1. When we extend Y_1 in this fashion we say that we have added *one layer of pentagons* to Y_1. If we repeat this procedure for U_1, we say that we have added *two layers of pentagons* to Y_1, and so on.

It is not too hard to see that $d(\partial Y_1, \partial U_1) \geq e_P$, where $e_P \approx 1.062$ is the length of an edge of a pentagon in our tiling. Now, recall that ℓ_γ is the length of γ, and hence, the length of $\widetilde{\gamma}$. Let Y_i' be the set obtained by adding ℓ_γ / e_P layers of pentagons to Y_i, for $i = 1, \ldots, m$. Then $Y' = C(X) \cup Y_1' \cup \cdots \cup Y_m'$ is a convex extension of Y, and we can ensure that $\widetilde{\gamma}$ is contained in the interior of Y'.

Let $Z_i' = \left\{ x \in \overline{X - C(X)} : d(x, \alpha_i) \leq (\ell_\gamma / e_P + 2)\, d_0 \right\}$. Then, $Y_i' \subset C(X) \cup Z_i'$, and we have that $Area(Z_i') = \sinh[(\ell_\gamma/e_P\, +\, 2)\, d_0]\, \ell_i$. It then follows that if Y' consists of k'' pentagons,

$$k'' < 4n - 4 + \frac{2\,\sinh[(\ell_\gamma/e_P\, +\, 2)\, d_0]}{\pi}\, (\ell_1 + \ell_2 + \cdots \ell_m).$$

We follow the proof of Case 1 replacing the set Y with Y' to obtain a subgroup K', such that $S \subset K'$ and $\gamma \notin K'$. In this case,

$$[\pi_1(\Sigma) \!:\! K'] \leq [\Gamma \!:\! K] = k'' < 4n - 4 + \frac{2\,\sinh\left[(\ell_\gamma/e_P\, +\, 2)\, d_0\right]}{\pi}\, (\ell_1 + \ell_2 + \cdots + \ell_m).$$

Explanation of equation (4): Again, in the case where the rank of S is $n = 1$, the analog of the sets Z_i' is $Z' = \left\{ x \in \overline{X - C(X)} : d(x, \alpha_1) \leq (\ell_\gamma / e_P + 2)\, d_0 \right\} = \{x \in X : d(x, \alpha_1) \leq (\ell_\gamma/e_P + 2)\, d_0\}$ so that $Area(Z') = 2\sinh[(\ell_\gamma/e_P\, +\, 2)\, d_0]\, \ell_1$, explaining the doubling of the coefficient.

$\square$

We can now prove the case where Σ is a compact hyperbolic surface with geodesic boundary. We know that the universal cover of Σ is a convex, non-compact polygon in $\mathbb{D}^2$, which we call $\widetilde{\Sigma}$. By our comments after Theorem 3.4.4, we also know that $\pi_1(\Sigma) < \Gamma$ when we consider $\pi_1(\Sigma)$ as a group of isometries of $\mathbb{D}^2$.

At each boundary component of Σ we will glue in a non-compact region, as in Figure 3.9 below. We call this new surface Σ'.

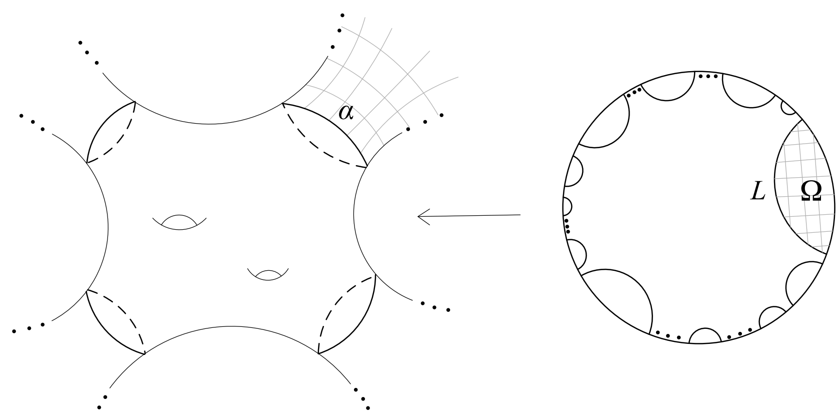


Figure 3.9

In $\mathbb{D}^2$, one such gluing along a boundary component, α, corresponds to the gluing of the region Ω to $\widetilde{\Sigma}$, as in the figure above. The gluing occurs along the geodesic line L, whose image in Σ is α. Most importantly, $\pi_1(\Sigma) \cong \pi_1(\Sigma')$ and $\pi_1(\Sigma') < \Gamma$.

Now we can follow the proof of the closed case for $\pi_1(\Sigma')$. Thus, $\pi_1(\Sigma')$ is LERF, and $\pi_1(\Sigma)$ is, therefore, LERF since the groups are isomorphic. The same bounds stated in Theorem 3.6.1 will hold.

Chapter 4

Quantifying Residual Finiteness of 3-manifold Groups

4.1 Preliminaries

Let P be an all right-angled finite volume polyhedron in hyperbolic 3-space, and let Γ be the Kleinian group generated by reflections in the faces of P. Let M be a compact hyperbolic 3-manifold tiled by copies of P, that is $\pi_1(M) < \Gamma$.

In this chapter, we aim to quantify the residual finiteness of $\pi_1(M)$ where M is as above. In order to obtain a quantification for the residual finiteness of $\pi_1(M)$, we need an upper bound on the volume of the smallest, closed, convex union of polyhedra G containing a compact set C in a cover of M. We refer to G as the *convexification* of C.

In [3], Agol, Long and Reid prove that Γ is H- subgroup separable for all geometrically finite subgroups $H < \Gamma$ by showing that the convexification of a compact set in $\mathbb{H}^3/H$ always involves a finite number of polyhedra. The proof relies on the fact that polyhedra sufficiently far away from the compact set C cannot lie in its convexification. In [1], Ian Agol gives an explicit bound on what "sufficiently far" means. Once we have the bound in Agol's note, we can apply the analog of our technique in Chapter 3 to obtain the desired bound on the volume of G in terms of geodesic length in M. The rest of the argument follows as is in the 2-dimensional case.

4.2 Agol's Lemma

In his note, Agol proves a lemma, which will be the basis for our proof of the quantification that $\pi_1(M)$ is residually finite. We will use his notation throughout this chapter.

Begin with a geometrically finite subgroup Φ of Γ, and let C be a compact subset of $\mathbb{H}^3/\Phi$. Next, let Y be the convex hull of C in $\mathbb{H}^3/\Phi$, and let $N_R(Y)$ be the R-neighborhood of Y, where R is a constant that Agol determines later.

As in Scott's paper [20] we consider the half spaces bounded by the geodesic hyperplanes in our tessellation of $\mathbb{H}^3$ by copies of P. We let $\widetilde{Y}$ be the set of all lifts of Y under the covering map $\mathbb{H}^3 \longrightarrow \mathbb{H}^3/\Phi$, and we let $\widetilde{G}$ be the intersection of all half spaces containing $\widetilde{Y}$. $\widetilde{G}$ is the union of polyhedra in the tessellation, and maps down to a convex suborbifold G of $\mathbb{H}^3/\Phi$. Additionally, G is a union of polyhedra in the tessellation of $\mathbb{H}^3/\Phi$ and is the convexification of the compact set C that we mentioned above.

Lemma 4.2.1 (Agol)**.** *If a polyhedron in our tessellation of $\mathbb{H}^3/\Phi$ is in G, then is must intersect $N_R(Y)$. We can take $R = \ln(\sqrt{3}+\sqrt{2})$.*

We give a proof of this lemma for a special case in the next section.

4.3 Using Agol's Lemma for RF

Let α be a nontrivial element of $\pi_1(M)$, and let ℓ be the length of the geodesic representative of α. As in the previous chapter, we will also call the geodesic representative α. Now, let $\Phi = \langle\alpha\rangle$, the cyclic subgroup of $\pi_1(M)$ generated by α.

We first note that Φ is a geometrically finite subgroup of Γ. Let $X = \mathbb{H}^3/\Phi$ be the cover of M corresponding to the subgroup $\Phi = \langle\alpha\rangle$, and let $\overline{\alpha}$ be the unique simple closed geodesic lift of α to $\mathbb{H}^3/\Phi$. The convex core of X is simply $\overline{\alpha}$, so every r-neighborhood of the core of X certainly has finite volume.

A natural approach to quantifying the residual finiteness of $\pi_1(M)$ would be to use the analog of the convexification argument of Chapter 3. Letting S_0 be the union of all polyedra in X that intersect $\overline{\alpha}$, we can still talk about "good" (convex) portions and "bad" (non-convex) portions of the boundary of S_0. More specifically, a bad portion of the boundary would consist of 3 copies of P, our all right polyhedron, forming an interior dihedral angle of $\dfrac{3\pi}{2}$ in S_0. A similar argument to that of Lemma 3.2.2 shows that two bad portions cannot occur consecutively along ∂S_0. However, we believe that the analog of the filling argument used to prove Theorem 3.2.1 would be much more difficult to prove in the 3-dimensional case. By appealing to Agol's Lemma, we eliminate the need for an algorithmic procedure that tells us the number of polyhedra in convexification of S_0.

Following the setup of Agol's Lemma, we take $\overline{\alpha}$ to be our compact subset C of $\mathbb{H}^3/\Phi$, where $\Phi = \langle\alpha\rangle$. Y is the convex hull of $C = \overline{\alpha}$, which is equal to C itself. We give a proof of Agol's Lemma in this case.

Lemma 4.3.1. *Let G is the union of polyhedra that is the convexification of $\overline{\alpha}$, formed by the procedure mentioned above. Then any polyhedron $P_i \in G$ must intersect $N = N_R(\overline{\alpha})$ where $R = \ln(\sqrt{3}+\sqrt{2})$.*

Proof. Let $\widetilde{Y}$ be the set of all lifts of $\overline{\alpha}$ to $\mathbb{D}^3$. Then $\widetilde{Y}$ forms a geodesic axis that is invariant under the action of $\langle\alpha\rangle$ on $\mathbb{D}^3$. Let $\widetilde{N}$ be the lift of N to $\mathbb{D}^3$, that is $\widetilde{N}$ forms an R-neighborhood around $\widetilde{Y}$. As defined above, we let $\widetilde{G}$ be the intersection of all half spaces in our tessellation containing $\widetilde{Y}$, which maps down to our convexification G. Suppose P is a polyhedron in our tessellation of $\mathbb{D}^3$ which does not intersect $\widetilde{N}$. We aim to show that the extension of one of the faces of P to a hyperplane in $\mathbb{D}^3$ must separate P from $\widetilde{Y}$, showing that $P \notin \widetilde{G}$ and proving the lemma.

Using Agol's notation, we let e be the cell of P which is closest to $\widetilde{Y}$ and let k =codim(e). Note that k is also the number of faces of P that intersect e. Take a shortest geodesic $\overline{py}$ from e to $\widetilde{Y}$, which intersects $\widetilde{Y}$ at a point y and e at a point p. Then $\ell(\overline{py}) \geq R$ since P does not intersect the R-neighborhood of $\widetilde{Y}$, and $\overline{py}$ is necessarily an orthogeodesic. We let j be a hyperplane through y that is perpendicular to $\overline{py}$, and therefore contains $\widetilde{Y}$, which separates e from $\widetilde{Y}$.

Case k= 1: We first handle the case where e is a face of P, that is the case where $k = 1$. Then $\overline{py}$ is an orthogeodesic between the hyperplane e and the hyperplane j. Taking any $R > 0$ tells us that e itself is a face of P whose hyperplane extension separates P from $\widetilde{Y}$ (see Figure 4.1 below).

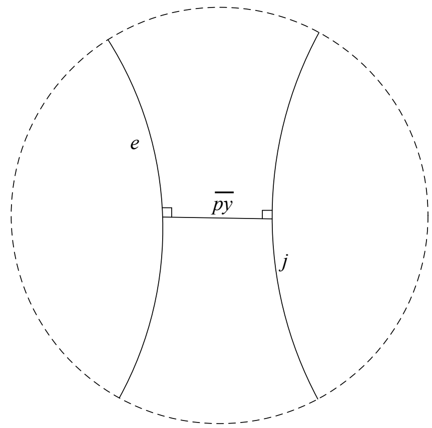
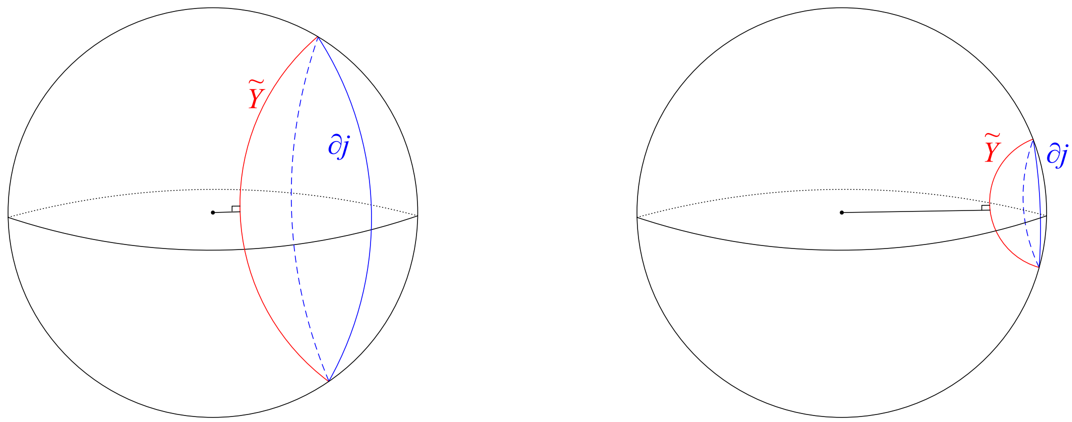


Figure 4.1

Case k = 3: Next we will handle the hardest case where e is a vertex of P and $k = 3$. First, we move $e = p$ to the origin of $\mathbb{D}^3$ by isometries. We define the three hyperplanes $L_x = \{(x, y, u) \in \mathbb{D}^3 : x = 0\}$, $L_y = \{(x, y, u) \in \mathbb{D}^3 : y = 0\}$ and $L_u = \{(x, y, u) \in \mathbb{D}^3 : u = 0\}$. The 3 faces of P that intersect e necessarily lie in three hyperplanes L_1, L_2 and L_3 that are the isometric image of L_x, L_y and L_u. This follows from the fact that P is a polyhedron whose dihedral angles are all right angles. Letting $\partial L_i = \partial\mathbb{D}^3 \cap L_i$, we see that ∂L_1, ∂L_2 and ∂L_3 form an all right angled spherical triangle in $\partial\mathbb{D}^3$.

We can assume that $\widetilde{Y}$ passes through the north pole of the hyperplane j, which will be important for our calculation of R. We apply isometries to $\mathbb{D}^3$ until y, and therefore $\overline{py} = \overline{0y}$, lies on the line formed by $L_u \cap L_x$ and $\widetilde{Y}$ lies in L_x. If the distance between P and $\widetilde{Y}$, i.e. the length of $\overline{0y}$, is large, then $j \cap \partial\mathbb{D}^3 := \partial j$ is a small circle on $\partial\mathbb{D}^3$, and if the distance is short, then ∂j forms a large circle on $\partial\mathbb{D}^3$, as can be seen in Figure 4.2 below. That is, the distance $d(P, \widetilde{Y})$ and the radius, r, of ∂j have an inverse proportional relationship.

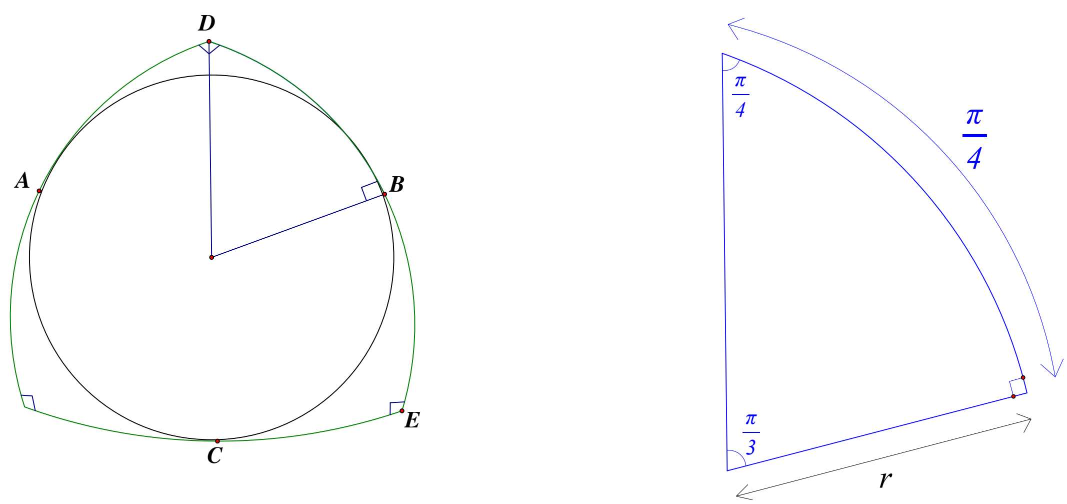


(a) (b)

Figure 4.2

What we are then looking for is the threshold, R, of the distance between P and $\widetilde{Y}$ so that the radius r of ∂j is small enough that ∂j can be inscribed in an all right angled spherical triangle formed by the boundaries of three pairwise orthogonal planes. Then if $d(P, \widetilde{Y}) > R$, at least one of the three circles ∂L_i cannot intersect ∂j, and thus, one of the three hyperplanes L_i must separate P from $\widetilde{Y}$.

We begin by calculating the radius of a circle inscribed in such a spherical triangle.



(a) (b)

Figure 4.3

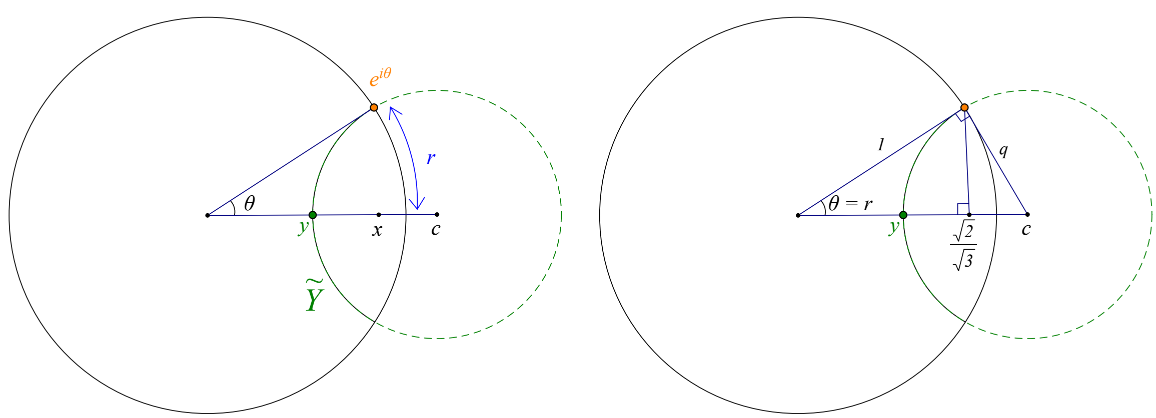
In Figure 4.3 (a) above, A, B and C are the midpoints of the three edges in our spherical triangle, which are also the points of tangency for the inscribed circle. Recall that the triangle is formed by the intersection of three pairwise orthogonal hyperplanes with $\partial\mathbb{D}^3$, the unit sphere in $\mathbb{R}^3$. Thus, the length of the edge $\overline{DE}$ is equal to $\dfrac{\pi}{2}$, and the length of $\overline{DB}$ is $\dfrac{\pi}{4}$.

To calculate the radius r of the inscribed circle we apply the following spherical Cosine Law. Let T be a spherical triangle with angles α, β and γ, and with edges of lengths a, b and c opposite the angles α, β and γ, respectively. Then,

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a.$$

For the triangle in Figure 4.3 (b) above, we have $\cos\dfrac{\pi}{4} = \sin\dfrac{\pi}{2}\sin\dfrac{\pi}{3}\cos r$, and therefore, $r = \cos^{-1}\left(\dfrac{\sqrt{2}}{\sqrt{3}}\right)$.

Now we calculate the distance $R =$ length of $\overline{0y} = d(P, \widetilde{Y})$ so that the radius r of ∂j is $\cos^{-1}\left(\dfrac{\sqrt{2}}{\sqrt{3}}\right)$. Consider the cross sectional view formed by the intersection of the hyperplane L_x with our setup in Figure 4.2 above. This view is represented in Figure 4.4 (a) below.



(a) (b)

Figure 4.4

Since $\partial\mathbb{D}^3 \cap \partial L_x$ is a unit circle we know that $\theta = r = \cos^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right)$. Therefore, the point x in the figure is $\cos\theta = \cos r = \frac{\sqrt{2}}{\sqrt{3}}$. The point c is the center of the circular completion of our geodesic $\widetilde{Y}$ and with q the radius of this circle, we see that $y = c - q$.

Using the similar triangles in Figure 4.4 (b) we calculate that $c = \frac{\sqrt{3}}{\sqrt{2}}$ and an application of the Pythagorean theorem shows that $q = \frac{1}{\sqrt{2}}$. Thus, $y = \frac{\sqrt{3}-1}{\sqrt{2}}$ and $R = d(0, \widetilde{Y}) = \ln\left(\frac{1+y}{1-y}\right)$, which by a simple calculation gives us $R = \ln(\sqrt{2}+\sqrt{3})$.

Case k = 2: The case where e is an edge of P can be handled in a similar but much simpler way. In this case, we show that $R = \ln(\sqrt{2}+1)$.

We assume the same setup as in the previous case where the point p on the edge e that is closest to $\widetilde{Y}$ is at the origin of $\mathbb{D}^3$ and y lying on $L_u \cap L_x$. The extensions of the two faces of P that intersect e form a pair of orthogonal hyperplanes, L_1 and L_2, in $\mathbb{D}^3$. Their boundaries, ∂L_1 and ∂L_2, form a spherical bi-disk with angles $\frac{\pi}{2}$. We are looking for the threshold R such that ∂j, and thus $\widetilde{Y}$, is tangent to such a spherical bi-disk at the endpoints of $\widetilde{Y}$. Then, if $d(0, \widetilde{Y}) > R$, one of the ∂L_i cannot intersect ∂j, and one of the hyperplanes L_i must therefore separate P from $\widetilde{Y}$.

A cross sectional view of this situation is shown in Figure 4.5 below. Given the triangle in the figure, we know that $c = \sqrt{2}$ so that $y = \sqrt{2} - 1$. Thus, $R = \ln\left(\frac{1+\sqrt{2}-1}{1-\sqrt{2}+1}\right) = \ln(\sqrt{2}+1)$.

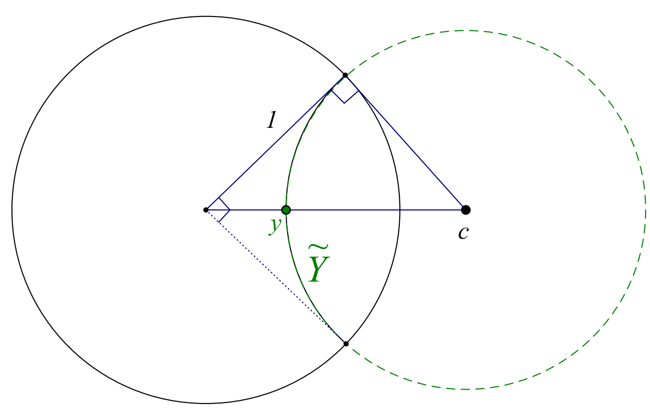


Figure 4.5

We take R to be the largest of the values in these three cases, i.e. $R = \ln(\sqrt{2}+\sqrt{3})$. If P does not intersect $\widetilde{N} = N_R(\widetilde{Y})$, then there is a face of P whose hyperplane extension separates P from $\widetilde{Y}$ so that $P \notin \widetilde{G}$. Therefore, if a polyhedron P_i in $\mathbb{H}^3/\langle\alpha\rangle$ is in the convexification G of $\overline{\alpha}$, P_i must intersect the R-neighborhood N of $\overline{\alpha}$.

□

Given the above lemma, we have that $G \subset N_{R+d_P}(\overline{\alpha})$, where d_P is the diameter of P. The following lemma allows us to calculate the volume of $N_{R+d_P}(\overline{\alpha})$. It is the 3-dimensional analog of the calculation of Lemma 3.3.2.

Lemma 4.3.2. *We let Ω be the region in $\mathbb{H}^3$ shown in Figure 4.6 below.*

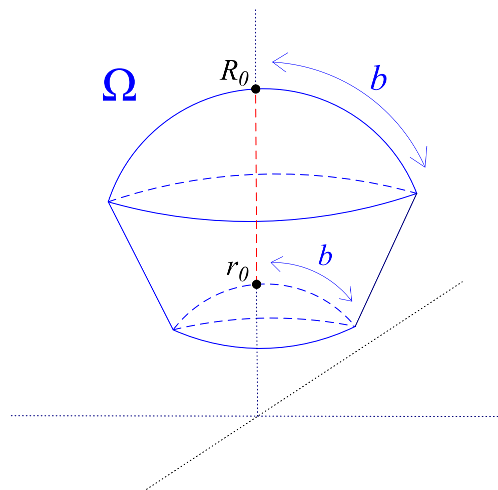


Figure 4.6

Then $\mathrm{Vol}(\Omega) = \pi \sinh^2(b)\,\ell$, *where* $\ell = \ln(R_0/r_0)$ *is the length of the geodesic between the points* R_0 *and* r_0 *in* $\mathbb{H}^3$.

Proof. For this volume calculation we find it convenient to use spherical coordinates. The volume form on $\mathbb{H}^3$, $\dfrac{dx \wedge dy \wedge du}{u^3}$, becomes $\dfrac{1}{r}\tan\phi\sec^2\phi\, dr \wedge d\phi \wedge d\theta$. Note that the intersection of Ω with the plane L_y forms the angle of parallelism picture from Figures 3.5 and 3.6. Let $\pi(b)$ be the angle of parallelism. A simple calculation tells us that the range of values for ϕ in Ω is then $[\,0,\, \pi/2 - \pi(b)\,]$. Therefore,

$$Vol(\Omega) = \iiint\limits_{\Omega} \frac{1}{r}\ \tan\phi\ \sec^2\phi\ dr \wedge d\phi \wedge d\theta\ \ =$$

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}-\pi(b)} \int_{r_0}^{R_0} \frac{1}{r} \tan\phi\ \sec^2\phi\ dr \wedge d\phi \wedge d\theta\ \ =$$

$$\ln\left(\frac{R_0}{r_0}\right) \int_0^{2\pi} \int_0^{\frac{\pi}{2}-\pi(b)} \tan\phi\ \sec^2\phi\ d\phi \wedge d\theta\ \ =\ \ \ell \left[\frac{\tan^2\phi}{2} \Big|_0^{\frac{\pi}{2}-\pi(b)} \right] \int_0^{2\pi} d\theta =$$

$$\frac{\pi}{\tan^2(\pi(b)}\,\ell\ \ =\ \ \pi\sinh^2(b)\,\ell,$$

with the last equality coming from the angle of parallelism laws.

□

We can lift $N_{R+d_P}(\overline{\alpha})$ isometrically to $\mathbb{H}^3$ so that it forms a region like Ω from the previous lemma where $b = R + d_P$. Lemma 4.3.2 then implies that $\mathrm{Vol}(G) < \mathrm{Vol}(N_{R+d_P}(\overline{\alpha})) = \pi\sinh^2(R + d_P)\,\ell$, where ℓ is the length of α. Thus, we have the following theorem:

Theorem 4.3.3. *Let M be as above. Then for any $\alpha \in \pi_1(M) - \{id\}$, there exists a subgroup H' of $\pi_1(M)$, such that $\alpha \notin H'$. The index of H' is bounded by*

$$\frac{2\pi\sinh^2(\ln(\sqrt{3}+\sqrt{2}) + d_P)}{V_P}\,\ell,$$

where ℓ is the length of the unique geodesic representative of α, d_P is the diameter of P and V_P is the volume of P.

Proof. We know that $Vol(G) < \pi\sinh^2(R + d_P)\,\ell$ so if G consists of k polyhedra,

$$k < \frac{\pi\sinh^2(ln(\sqrt{3}+\sqrt{2}) + d_P)}{V_P}\,\ell.$$

Let $\widetilde{\alpha}$ be one lift of $\overline{\alpha}$ to $\mathbb{H}^3$. Using the analog of the argument in Section 3.5, we can lift G to $\mathbb{H}^3$ so that the result is a connected, convex union of k polyhedra denoted by $\overline{G}$. The convexity of the lift is crucial since we will want to apply the Poincaré Polyhedra Theorem to prove the result above.

Let $\widetilde{\alpha}_1$ be one of the two lifts of $\overline{\alpha}$ that share endpoints with $\widetilde{\alpha}$. Let $\overline{G}_1$ be the convex lift of G containing $\widetilde{\alpha}_1$. Then $G' = \overline{G} \cup \overline{G}_1$ is a convex union of $2k$ polyhedra in $\mathbb{H}^3$, such that one endpoint of $\widetilde{\alpha}$ is contained in the interior of G'.

Let H be the group of isometries of $\mathbb{H}^3$ generated by reflections in the sides of G'. Then $H < \Gamma$, and G' is a fundamental domain for the action of H on $\mathbb{H}^3$ by the Poincaré Polyhedra Theorem. Since G' contains $2k$ polyhedra, $[\Gamma : H] = 2k$. Letting $b : \mathbb{H}^3 \longrightarrow \mathbb{H}^3/H$ be the covering map, we then have that the restriction of b to $\mathring{G}'$ is a homeomorphism onto its image in $\mathbb{H}^3/H$. Thus, by the same reasoning as in the 2-dimensional case, $b(\widetilde{\alpha})$ is not a loop in $\mathbb{H}^3/H$, and $\alpha \notin H$.

Now, let $H' = H \cap \pi_1(M)$. Then, $\alpha \notin H'$ and $[\pi_1(M) : H'] \leq [\Gamma : H] = 2k$. The result follows.

$\square$

Bibliography

[1] I. Agol, *Geometrically Finite Subgroup Separability for the Figure-8 Knot Group*, Preliminary Report/ Private Communication.

[2] I. Agol, D. Groves, and J. Manning, *The virtual Haken conjecture*, Preprint. arXiv:1204.2810 [math.GT].

[3] I. Agol, D. D. Long, and A. W. Reid, *The Bianchi groups are separable on geometrically finite subgroups*, Ann. of Math. (2) **152** (2001), no. 3, 599-621.

[4] G. Baumslag, *On generalised free products*, Mathematische Zeitschrift **78** (1962), 423-438.

[5] A. F. Beardon, *The Geometry of Discrete Groups*, Springer-Verlag, Berlin and New York, 1983.

[6] K. Bou-Rabee, *Quantifying Residual Finiteness*, Journal of Algebra **323** (2010), 729-737.

[7] K. Bou-Rabee and D. B. McReynolds, *Asymptotic growth and least common multiples in groups*, Bull. London Math. Soc. **43(6)** (2011), 1059-1068.

[8] N. V. Buskin, *Efficient separability in free groups*, Siberian Mathematical Journal **50(4)** (2009), 603-608.

[9] A. J. Casson and S. A. Bleiler, *Automorphisms of Surfaces after Nielsen and Thurston*, Cambridge University Press, Cambridge, 1988.

[10] M. Hall, *Coset representation in free groups*, Trans. Amer. Math. Soc. **67** (1949), 421-432.

[11] J. Hempel, *Residual Finiteness of Surface Groups*, Proc. Amer. Math. Soc. **32** (1972), 323.

[12] J. Hempel, *Residual Finiteness for 3-manifolds*, Combinatorial group theory and topology (Alta Utah 1984), 379-396, Ann. of Math. Stud., 111, Princeton Univ. Press, Princeton, NJ, 1987.

[13] J. H. Hubbard, *Teichmüller Theory and Applications to Geometry, Topology, and Dynamics*: *Volume I*, Matrix Editions, Ithaca, NY, 2006.

[14] N. Jacobson, *Basic Algebra I*: *Second Edition*, W. H. Freeman and Company, San Francisco, 1985.

[15] M. Kassabov and F. Matucci, *Bounding the Residual Finiteness of Free Groups*, Proc. Amer. Math. Soc. **139** (July 2011), no. 7, 2281-2286.

[16] S. Katok, *Fuchsian Groups*, University of Chicago Press, Chicago, 1992.

[17] L. Keen, *Collars on Riemann Surfaces*, Discontinuous Groups and Riemann Surfaces, 263-268, Ann. of Math. Stud., 79, Princeton Univ. Press, Princeton, NJ, 1974.

[18] J. M. Lee, *Riemannian Manifolds: An Introduction to Curvature*, Springer-Verlag, New York, NY, 1997.

[19] I. Rivin, *Geodesics with one self-intersection, and other stories*, Adv. Math **231 (5)** (2012), 2391-2412.

[20] P. Scott, *Subgroups of Surface Groups Are Almost Geometric*, London Math. Soc. **2 (17)** (1978), 555-565.

[21] P. Scott, *Correction to 'Subgroups of Surface Groups Are Almost Geometric'*, London Math. Soc. **2 (32)** (1985), 217-220.

[22] W. P. Thurston, *The Geometry and Topology of Three-Manifolds.* Princeton University Notes.

Vita

Priyam Patel

2013	Ph. D. in Mathematics, Rutgers University
2004-08	B.A. in Mathematics from New York University
2004	Graduated from Bayonne High School

2013	Graduate assistant, Department of Mathematics, Rutgers University
2012	Teaching assistant, Department of Mathematics, Rutgers University
2011-2012	GAANN Fellow, Department of Mathematics, Rutgers University
2009-2011	Teaching assistant, Department of Mathematics, Rutgers University
2008-2009	GAANN Fellow, Department of Mathematics, Rutgers University