ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE CONFORMAL QUOTIENT EQUATION

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ABSTRACT OF THE DISSERTATION

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We classify all radial admissible solutions to the conformal quotient equation on the punctured Euclidean space and prove that an admissible solution to the conformal quotient equation with an isolated singular point is asymptotic to a radial solution. We also provide an alternative proof to obtain higher order expansion of solutions using analysis of the linearized operators. This dissertation is based on a preprint of the author [89].

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Dedication

To my parents Gangqiang Wang and Huanxi Peng

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Chapter 1

Introduction

One of the most important problems in conformal geometry is the Yamabe Problem, which is to determine whether there exists a conformal metric with constant scalar curvature on any closed Riemannian manifold. In what follows, let $n \geq 3$ and (M^n, g_0) be an n-dimensional compact Riemannian manifold with metric g_0, R_{g_0} be the scalar curvature of metric g_0. Denoting the conformal change of metric as $g = u^{\frac{4}{n-2}} g_0$ for some positive function u, R_g and R_{g_0} are related by the equation:

$$-\Delta_{g_0} u + \frac{n-2}{4(n-1)} R_{g_0} u = \frac{n-2}{4(n-1)} R_g u^{\frac{n+2}{n-2}}, \tag{1.0.1}$$

where Δ_{g_0} is the Laplace-Beltrami operator. The Yamabe problem is to solve (1.0.1) for a positive function u with $R_g \equiv c$ for some constant c.

The Yamabe problem has been solved through the works of Yamabe [90], Trudinger [83], Aubin [1], and Schoen [77]. A crucial ingredient in their solution is to verify a criteria for compactness of a minimizing sequence of the functional associated with (1.0.1) and in the solution by Aubin, the non-vanishing of the Weyl tensor (a local conformal invariant) plays the role in dimensions $n \geq 6$ for (M^n, g) which is not locally conformally flat. The remaining cases require a global invariant and Schoen used the Positive Mass Theorem to complete the solution. An important fact is that $SO(n+1, 1)$, the group of conformal transformations of the n-sphere $\mathbb{S}^n$ with the round metric, is non-compact. In [79], Schoen proved that if (M, g) is locally conformally flat and is not conformally diffeomorphic to standard spheres, then the space of all solutions to (1.0.1) is compact. When (M^n, g) is not locally conformally flat, the same conclusion has been proved to hold by Li-Zhang [63] and Marques [68] independently in dimensions $n \leq 7$. For $n = 3, 4, 5$, see works of Druet [22, 23], Li-Zhu [65] and Li-Zhang [62]. For $8 \leq n \leq 24$, it has been proved that this compactness result holds under the assumption that the Positive Mass Theorem holds in these dimensions; see Li-Zhang [63, 64] for $8 \leq n \leq 11$, and Khuri-Marques-Schoen [45] for $12 \leq n \leq 24$. On the other hand, there exist counterexamples in dimensions $n \geq 25$; see Brendle [6] for $n \geq 52$, and Brendle and Marques [7] for $25 \leq n \leq 50$. For the corresponding problem to prescribe scalar curvature, see, e.g., [3, 12, 17, 18, 19, 20, 41, 46, 55, 56, 57, 75, 80].

In recent years, there are many works on studying a fully nonlinear Yamabe Problem. We recall the Schouten tensor

$$A_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)}g\right),$$

where Ric_g denotes the Ricci tensor of g. This tensor arises naturally in the decomposition of the full Riemannian curvature tensor

$$Rm = W_g + A_g \odot g,$$

where W_g is the Weyl tensor and $\odot$ denotes the Kulkari-Nomizu product [4]. Since W_g is conformally invariant, the behavior of the full curvature tensor under a conformal change of metric is entirely determined by the Schouten tensor. Let F denote any symmetric function of the eigenvalues, which is homogeneous of degree one, and consider the equation

$$F(g) := F\left(g^{-1}A_g\right) = \text{constant}. \tag{1.0.2}$$

Under the conformal change $g = e^{2\omega}g_0$, the Schouten tensor transforms as

$$A_g = \left[\nabla^2\omega + d\omega \otimes d\omega - \frac{1}{2}|\nabla\omega|^2 g_0\right] + A_{g_0},$$

so equation (1.0.2) is equivalent to

$$F\left(g^{-1}\left(\nabla^2\omega + d\omega \otimes d\omega - \frac{|\nabla\omega|^2}{2}g_0 + A_{g_0}\right)\right) = \text{constant}.$$

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ be the set of eigenvalues of a symmetric $n \times n$ matrix A and for $1 \leq k \leq n$, σ_k denote the kth elementary symmetric function of the eigenvalues

$$\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k},$$

for the case of $F = \sigma_k^{\frac{1}{k}}$, the equation (1.0.2) has become known as the σ_k-Yamabe equation :

$$\sigma_k(g) = \text{constant}. \tag{1.0.3}$$

Let

$$\Gamma_k^+ = \{\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n \,|\, \sigma_j(\Lambda) > 0, \forall 1 \leq j \leq k\},$$

and we say $g \in \Gamma_k^+$ if $g^{-1} \cdot A_g \in \Gamma_k^+$ for every point $x \in M$. We call ω k-admissible or in the Γ_k^+ class in some region if $e^{-2\omega} g_0 \in \Gamma_k^+$ there. If the metric g is k-admissible, then the linearization of (1.0.3) at g is elliptic, and Guan-Viaclovsky-Wang[34] also proved the algebraic fact that $\lambda(A_g) \in \Gamma_k^+$ for $k \geq \frac{n}{2}$ implies the positivity of the Ricci tensor (see also [14, 38]). When $k = 1$, $\sigma_1(A_g) = \frac{1}{2(n-1)} R_g$ and (1.0.3) is the Yamabe equation. When $k = n$, $\sigma_n(A_g) = \text{determinant of } A_g$. Fully nonlinear elliptic equations involving $F\left(\lambda\left(\nabla^2 u\right)\right)$ have been investigated in the classic paper of Caffarelli-Nirenberg-Spruck [8]. Fully nonlinear elliptic equations involving the Schouten tensor and applications to geometry and topology have been studied extensively in and after the pioneering works of Viaclovsky [86, 87, 88] and Chang-Gursky-Yang [13, 14, 15, 16].

When $k \neq \frac{n}{2}$ and the manifold (M, g) is locally conformally flat, the equation (1.0.3) is the Euler-Lagrange equation of the variational functional $\int \sigma_k(A_g) dv_g$, and in the exceptional case $k = \frac{n}{2}$, the integral $\int \sigma_k(A_g) dv_g$ is a conformal invariant, see viaclovsky [86]. We remark that when $n = 4$ and $k = 2$, the invariance of the integral $\int \sigma_2(A_g) dv_g$ is a reflection of the Chern-Gauss-Bonnet formula [4]

$$8\pi^2 \chi(M) = \int_M (\frac{1}{4}|W|^2 + 4\sigma_2(A_g) dv_g.$$

In the case that (M^n, g) is locally conformally flat, there is much progress on the study of the σ_k equation when the conformal structure admits metrics whose Schouten tensor belongs to the cone Γ_k^+. This is largely due to the result of Schoen-Yau [81] which assures that the developing map of locally conformally flat manifolds having nonnegative Yamabe invariants realizes the holonomy cover as a domain in $\mathbb{S}^n$, and in addition, the complement of Ω has small Hausdorff dimension: $\dim(\mathbb{S}^n \backslash \Omega) \leq \frac{n-2}{2}$, so that the method of moving planes may be used to derive a priori estimates for such equations. In particular Li-Li [52] classified all of the solutions of the equation (1.0.3) on $\mathbb{R}^n$, thus providing a priori estimates for this equation in the locally conformally flat case. Subsequently, their results has been generalized to much more general classes of symmetric functions F, see Li-Li [51, 53, 54]. In addition, Guan-Wang [36] applied the heat flow associated to the σ_k $(k \neq \frac{n}{2})$ equation to derive the conformally invariant Sobolev inequality for locally conformally flat manifolds. When (M^n, g) is locally conformally flat, the existence of solution to (1.0.3) was obtained by Guan-Wang and Li-Li independently in the above mentioned papers. In the case of general manifold, the existence of solution to (1.0.3) was given by Chang-Gursky-Yang [14] first for $k = 2$, $n = 4$. As for the existence result in arbitrary dimension $n \geq 3$, when $k = 2$, see Ge-Wang [27] and Sheng-Trudinger-Wang [82]; When $k = \frac{n}{2}$, see Li-Nguyen [61] and Trudinger-Wang [84]; When $k > \frac{n}{2}$, see Gursky-viaclovsky [39, 40].

The proof of existence and compactness of solutions to (1.0.3) involves a bubbling analysis, and the applications of various estimate techniques to determine the growth rate of solutions at isolated singular points. Recall that a classical theorem of Bôcher [5] asserts that any positive harmonic function in the punctured ball $B_1 \backslash \{0\} \subset \mathbb{R}^n$ can be expressed as the sum of a multiple of the fundamental solution of the Laplace equation and a harmonic function in the whole unit ball B_1. This can be viewed as a statement on the asymptotic behavior of a positive harmonic function near its isolated singularities. And a remarkable work related to the Yamabe problem by Caffarelli-Gidas-Spruck [9] proved the asymptotic radial symmetry of positive singular solutions to the conformal scalar curvature equation

$$-\Delta u(x) = \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}}(x) \tag{1.0.4}$$

on a punctured ball, and further proved that such solutions are asymptotic to radial singular solutions to (1.0.4) on $\mathbb{R}^n \backslash \{0\}$. Specifically, for any singular solution $u(x)$ to (1.0.4) in $B_1 \backslash \{0\}$, there exists a radial singular solution $u^*(|x|)$ to (1.0.4) on $\mathbb{R}^n \backslash \{0\}$ and some $\alpha > 0$ such that

$$u(x) = u^*(|x|)\,(1 + O(|x|^{\alpha})) \quad \text{as} \quad |x| \to 0. \tag{1.0.5}$$

In the same paper, they also proved that any positive solution to

$$-\Delta u(x) = \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}}(x) \quad \text{ on } \quad \mathbb{R}^n$$

is of the form

$$u_a(x) = \left(\frac{2a}{1 + a^2|x - \bar{x}|^2}\right)^{\frac{n-2}{2}}$$

for some $\bar{x} \in \mathbb{R}^n$ and $a > 0$. We note $u_a(x)$ is a sequence of functions whose norm are constant in $W^{1,2}(\mathbb{R}^n)$, and the weak limit of the sequence is zero; hence it does not have a convergent subsequence in $L^{\frac{2n}{n-2}}$. This lack of compactness turns out to be at the heart of many problems. A key ingredient in the proof of [9] uses a "measure theoretic" variation of the moving plane technique, while Korevaar-Mazzeo-Pacard-Schoen [47] proved (1.0.5) through an analysis of linearized operators at those global singular solutions. For the study of singular solutions to (1.0.1) in the positive scalar curvature case, Schoen-Yau [81] proved that if a complete conformal metric g exists on a domain $\Omega \subset \mathbb{S}^n$ with $\sigma_1(A_g)$ having a positive lower bound, then the Hausdorff dimension of $\partial\Omega$ has to be $\leq (n-2)/2$. In [78] Schoen constructed complete conformal metrics on $\mathbb{S}^n \backslash \Lambda$ when Λ is either a finite discrete set on $\mathbb{S}^n$ containing at least two points or a set arising as the limit set of a Kleinian group action. Later Mazzeo-Pacard gave another proof of this result in [70]; they also proved in [69] that if $\Omega \subset \mathbb{S}^n$ is a domain such that $\mathbb{S}^n \backslash \Omega$ consists of a finite number of disjoint smooth submanifolds of dimension $1 \leq k \leq (n-2)/2$, then one can find a complete conformal metric g on Ω with its scalar curvature identical to $+1$. For the negative scalar curvature case, the results of Loewner-Nirenberg [66], Aviles [2], and Veron [85] imply that if $\Omega \subset \mathbb{S}^n$ admits a complete, conformal metric with negative constant scalar curvature, then the Hausdorff dimension of $\partial\Omega > (n-2)/2$. Loewer-Nirenberg [66] also proved that if $\Omega \subset \mathbb{S}^n$ is a domain with smooth boundary $\partial\Omega$ of dimension$> (n-2)/2$, then there exists a complete conformal metric g on Ω with $\sigma_1(A_g) = -1$. This result was later generalized by D. Finn [24] to the case of $\partial\Omega$ consisting of smooth submanifolds of dimension$> (n-2)/2$ and with boundary, see also [48, 49, 67, 74, 72] and the references therein.

Advancing beyond the singular Yamabe Problem case, there are some relevant research on the singular solution to (1.0.3). Chang-Han-Yang [10] classified all possible radial solutions to (1.0.3) in $\Gamma_k^{\pm}$ class on an annular domain including punctured ball and punctured Euclidean space. In [58], Li proved that an admissible solution with an isolated singularity at $0 \in \mathbb{R}^n$ to (1.0.2) is asymptotically radially symmetric. Later

Han-Li-Teixeira [43] studied the singular solution of (1.0.3) on a punctured ball when $2 \leq k \leq n$. Let (r, θ) be the polar coordinates such that $x = r\theta = |x|\theta$ with $\theta \in \mathbb{S}^{n-1}$, $t = -\ln r$ be the cylindrical variable, and

$$g = u^{\frac{4}{n-2}}(x)|dx|^2 = e^{-2\omega(t,\theta)}(dt^2 + d\theta^2).$$

Then with the flexibility of treating (1.0.3) either as an equation for $u(x)$ on $B_R \backslash \{0\}$ or as an equation for $\omega(t, \theta)$ on a cylinder $\{(t, \theta) : t > -\ln R, \theta \in \mathbb{S}^{n-1}\}$ with respect to the background metric $dt^2 + d\theta^2$, they were able to prove (1.0.5) and

$$|\omega(t, \theta) - \omega^*(t)| \leq Ce^{-\alpha t} \text{ for } t \to \infty.$$

Also Chang-Hang-Yang [11] proved that if $\Omega \subset \mathbb{S}^n$ $(n \geq 5)$ admits a complete, conformal metric g with

$$\sigma_1(A_g) \geq c_1 > 0, \ \sigma_2(A_g) \geq 0, \text{ and } |R_g| + |\nabla_g R_g| \leq c_0, \tag{1.0.6}$$

then $\dim(\mathbb{S}^n \backslash \Omega) < (n-4)/2$. This has been generalized by M. Gonzalez [29] and Guan-Lin-Wang [32] to the case of $2 < k < n/2$: if $\Omega \subset \mathbb{S}^n$ admits a complete, conformal metric g with

$$\sigma_1(A_g) \geq c_1 > 0, \ \sigma_2(A_g), \cdots \sigma_k(A_g) \geq 0, \text{ and } (1.0.6)$$

then $\dim(\mathbb{S}^n \backslash \Omega) < (n - 2k)/2$. Gonzalez also showed in [30] that isolated singularities of C^3 solutions of (1.0.3) with finite volume are bounded, among other statements. For the degenerate case of (1.0.2), Li [59] proved that a locally Lipschitz viscosity solution in $\mathbb{R}^n \backslash \{0\}$ must be radially symmetric about $\{0\}$, see Li-Nguyen [60] for recent development in this direction.

We studied a special case of (1.0.2) with $F = \left(\frac{\sigma_k}{\sigma_l}\right)^{\frac{1}{k-l}}$, $1 \leq l < k \leq n$, i.e., (1.0.2) is the conformal quotient equation:

$$\frac{\sigma_k(g)}{\sigma_l(g)} = c \tag{1.0.7}$$

or, equivalently,

$$\frac{\sigma_k}{\sigma_l}\left(\nabla^2\omega + d\omega \otimes d\omega - \frac{|\nabla\omega|^2}{2}g_0 + A_{g_0}\right) = ce^{2(l-k)\omega}, \tag{1.0.8}$$

where c is a positive constant. Similar to the work [10] of Chang, Han and Yang, we prove Theorem 2.2.1 in [89], which is the a classification result on radial solutions of (1.0.8) in the Γ_k^+ class on the entire $\mathbb{R} \times \mathbb{S}^{n-1}$. We state below a part of Theorem 2.2.1, which omits the results for the case $2k \geq n$ and $h > 0$.

Theorem 1.0.1. *Any radial solution $\xi(t) := \omega(t,\theta)$ of (1.0.8) in the Γ_k^+ class on the entire $\mathbb{R} \times \mathbb{S}^{n-1}$, when $1 \leq l < k \leq n$ and c is a positive constant, normalized to be $2^{l-k}\binom{n}{k}/\binom{n}{l}$, has the property that $1 - \xi_t^2 > 0$ for all t. Furthermore, $h := e^{(2k-n)\xi(t)}(1-\xi_t^2(t))^k - e^{(2l-n)\xi}(1-\xi_t^2(t))^l$ is a nonnegative constant. Moreover, if $h = 0$, these solutions give rise to the round spherical metric on $\mathbb{R}^n \cup \{\infty\} = \mathbb{S}^n$. If $h > 0$ and $2k < n$, then $h \leq h^* := (\frac{n-2k}{n-2l})^{\frac{n-2k}{2}} - (\frac{n-2k}{n-2l})^{\frac{n-2l}{2}}$ and $\xi(t)$ is periodic and gives rise to metric $g = \frac{e^{-2\xi(ln|x|)}}{|x|^2}|dx|^2$ on $\mathbb{R}^n \backslash \{0\}$ which is complete.*

When $l = 0$, $\sigma_l(g) = 1$, then (1.0.8) is (1.0.3), so our result is a generalization of [10] for the Γ_k^+ class part. We note that [10] just need the condition that $\xi(t)$ is well defined for all $t \in \mathbb{R}$ to exclude $h < 0$ case, however, the phase plane of radial solutions of (1.0.8) is different from (1.0.3) and we also use the "$\xi(t) \in \Gamma_k^+$ condition to prove h is nonnegative in Theorem 1.0.1. This theorem also exemplifies the result in Li [58] which says any solution $\omega(t,\theta)$ in the Γ_k^+ class to (1.0.2) must have a lower bound. Although the analysis involved here is of elementary nature, these results provide useful guidance in studying the behavior of singular solutions in punctured balls. Actually with some a priori estimates and asymptotically radially symmetric properties for general conformal equation established by Li [58] and the local first and second order gradient estimates by Guan-Lin-Wang [33] and Li [59], we find that any admissible solution with an isolated singularity at $0 \in \mathbb{R}^n$ is asymptotic to a radial solution of the above theorem, i.e., we prove the following theorem in [89], which is the generalization of the results in [43].

Theorem 1.0.2. *Let $\omega(t,\theta)$ be a smooth solution to (1.0.8) on $\{t > t_0\} \times \mathbb{S}^{n-1}$ in the Γ_k^+ class, where $n \geq 3, 1 \leq l < k \leq n$, and c is normalized to be $\frac{2^{l-k}\binom{n}{k}}{\binom{n}{l}}$. Then there exist a radial solution $\omega^*(t)$ to (1.0.8) on $\mathbb{R} \times \mathbb{S}^{n-1}$ in the Γ_k^+ class, and constants $\alpha > 0, C > 0$ such that*

$$|\omega(t,\theta)-\omega^*(t)| \le Ce^{-\alpha t} \quad \textit{for } t > t_0+1. \tag{1.0.9}$$

First a linearization analysis and integral estimate showed that the radial average of $\omega(t,\theta)$ (denoted as $\gamma(t)$) solves some perturbation forms of both (2.2.1) and

$$e^{(2k-n)\xi(t)}(1-\xi_t^2(t))^k - e^{(2l-n)\xi}(1-\xi_t^2(t))^l = h. \tag{1.0.10}$$

Note that (1.0.10) is the first integral of (2.2.1). By exploiting the perturbed ODE satisfied by $\gamma(t)$, we prove that the radial average $\gamma(t)$ is approximated by a (translated) radial solution to (2.2.1) as $t\to\infty$, moreover, Li [58] showed that the radial average of the solution is a good approximation to the solution as $t\to\infty$, thus we arrive at Theorem 1.0.2. We note that when the radial average is bounded, a key ingredient is a general asymptotic approximation result for solutions of certain ODEs (see Theorem B in Section 3.1) by Han-Li-Teixeira [43]; when the radial average is unbounded from above and $2k>n$, a new estimate technique was provided to dig out the information

$$\lim_{t\to\infty}\gamma_t(t)=1$$

which leads to the solution of Theorem 1.0.2; also a Pohozeav type identity established by Han [42] plays a important role in the proof especially for the $2k<n$ and $h=0$ case (see proof of Claim 3.2.1). The study of singular solutions of equations of the above type is related to the characterization of the size of the limit of the image domain in $\mathbb{S}^n$ of the developing map of a locally conformally flat n-manifold. More specifically, one is led to look for necessary/sufficient conditions on a domain $\Omega\subset\mathbb{S}^n$ so that it admits a metric g which is pointwisely conformal to the standard metric on $\mathbb{S}^n$, complete, and with its Weyl-Schouten tensor A_g in the $\Gamma_k^{\pm}$ class. Theorem 1.0.2 would also be helpful to study the blow-up phenomenon of solutions to more general equations to prescribe arbitrary function $f(x)$ for σ_k/σ_l. We note that for such prescribing problem, the case $n=4$, $k=2$ and $l=1$ was solved in [14]. When the underlying manifold (M,g_0) is locally conformally flat, thanks to the work of Schoen-Yau on the developing map, one can even solve a more general equation, see [31, 54, 37]. When $k>\frac{n}{2}$, the work of [40] and the local estimates of [33] implies the existence. See also [25, 26] for the existences on general compact Riemannian manifold.

Inspired by the work of Korevaar-Mazzeo-Pacard-Schoen [47] and Han-Li-Teixeira [43], we obtain higher order expansions for solutions to (1.0.8) in [89]:

Theorem 1.0.3. *Let $\omega(t,\theta)$ be a solution to* (1.0.8) *on $\{t > t_0\} \times \mathbb{S}^{n-1}$ in the Γ_k^+ class, where $n \geq 3$, $0 < l < k \leq \frac{n}{2}$, and the constant c is normalized to be $2^{l-k}\binom{n}{k}/\binom{n}{l}$, and let $\omega^*(t) = \xi_h(t+\tau)$ be the radial solution to* (1.0.8) *on $\mathbb{R} \times \mathbb{S}^{n-1}$ in the Γ_k^+class for which* (1.0.9) *holds. Let $\{Y_j(\theta) : j = 0, 1, \cdots\}$ denote the set of normalized spherical harmonics. In the case $k < \frac{n}{2}$ and $h > 0$ let ρ be the infimum of the positive characteristic exponents defined through Floquent theory to the linearized equation of* (1.0.8) *at $\omega^*(t)$ corresponding to higher order spherical harmonics $Y_j(\theta)$, $j > n$; in the case $k = \frac{n}{2}$ or $h = 0$, a similar notion of ρ can also be defined in a straightforward fashion. Then $\rho > 1$ and there is a*

$$\omega_1(t,\theta) = \sum_{j=1}^{n} a_j e^{-t-\tau}\left(1 + \xi_h'(t+\tau)\right) Y_j(\theta)$$

which is a solution to the linearized equation of (1.0.8) *at $\omega^*(t)$, such that*

$$|\omega(t,\theta) - \omega^*(t) - \omega_1(t,\theta)| \leq Ce^{-\min\{2,\rho\}t} \quad \textit{for } t > t_0 + 1 \tag{1.0.11}$$

This theorem requires some knowledge on the spectrum of the linearized operator of (1.0.8). We first establish the linearization equation and then a long algebraic computation helps us to prove $\rho_j > 1$ for all $\lambda_j \geq 2n$ as in Lemma 4.2.4 in Section 4.2. After this, for a non-homogeneous form of the linearization equation, we apply a decomposition of the solutions with Wronskian function and Maximum principle to get the higher order estimate, and then an elaborated iteration argument leads to Theorem 1.0.3. When we apply those analysis on the conformal quotient equation, the computation is highly more involved than the case in [43]. However, we introduce a crucial quantity $P(\xi,\xi_t) := \left(\frac{e^{-2\xi}}{1-\xi_t^2}\right)^{k-l}$ and $q(\xi,\xi_t) = \frac{P-1}{1-P^{\frac{l}{k}}}$ to simplify the algebraic expression to a quadratic function of $q(\xi,\xi_t)$ and then the computation is straightforward. Note that if $l = 0$ as in [43], the algebraic expression would be a linear function of $q(\xi,\xi_t)$, and it is easy to verify Lemma 4.2.4. Also due to some new observations, we obtain the anticipated estimate with a different approach (see Section 4.2 for more description).

The analysis of linearized operator should be useful in constructing solutions to (1.0.8) on $\mathbb{S}^n \backslash \Lambda$, and in analyzing the moduli space of solutions to (1.0.8) on $\mathbb{S}^n \backslash \Lambda$, when Λ is a finite set. Actually Mazzieri-Ndiaye claimed in [73] the existence of constant positive σ_k-curvature metrics which are complete and conformal to the standard metric on $\mathbb{S}^n \backslash \Lambda$, where $\Lambda \subset \mathbb{S}^n$ is a finite number of symmetrically balanced points of cardinality at least 2, and n, k are positive integers such that $2 \leq 2k < n$. So we expect to obtain the same result for σ_k/σ_l using this linearization analysis. Also our knowledge of the spectrum of the linearized operator to (1.0.8) immediately yields Fredholm mapping properties of these operators on appropriately defined spaces as in [71, 72].

This thesis is organized as follows. In Chapter 2, we establish a classification of radial solutions to the conformal quotient equation. In Chapter 3, we prove Theorem 1.0.2 by exploiting the ODE satisfied by the radial average. In Chapter 4, we give proofs for Theorem 1.0.2 and Theorem 1.0.3 by an analysis of the linearized operator. We provide a proof for the phase plane results in Appendix A.

Chapter 2

Classification of radial solutions to the conformal quotient equation

2.1 Phase plane

We begin with a phase plane result and will provide a proof in the Appendix A. We consider the level set given by

$$G(x,y) = e^{(2k-n)x}(1-y^2)^k - e^{(2l-n)x}(1-y^2)^l = h \tag{2.1.1}$$

in the region $\Omega \equiv \{(x,y) \in \mathbb{R}^2 |\, |y| < 1\}$. We also set $\Omega^- \equiv \{(x,y) \in \mathbb{R}^2 | -1 < y < 0\}$, then

$$G(x,y) = G(x,-y) \qquad \forall (x,y) \in \Omega^-.$$

Let S_h denote the level set given by (2.1.1), then we have the following phase plane result:

Case h=0 The level set S_h is

$$1 - y^2 = e^{-2x} \Longleftrightarrow x = -\frac{1}{2}\ln(1-y^2). \tag{2.1.2}$$

Set $W \equiv \{(x,y) \in \Omega \,|\, x < -\frac{1}{2}\ln(1-y^2)\}$ and $V \equiv \{(x,y) \in \Omega \,|\, x > -\frac{1}{2}\ln(1-y^2)\}$, then obviously $\forall (x,y) \in \Omega$ solving (2.1.1), we have

$$(x,y) \in W \Longleftrightarrow h < 0 \quad \text{while} \quad (x,y) \in V \Longleftrightarrow h > 0.$$

Case h>0 The level set S_h is classified in the following subcases:

(P1) If $1 \leq l < k < \frac{n}{2}$ & $0 < h < h^* := (\frac{n-2k}{n-2l})^{\frac{n-2k}{2}} - (\frac{n-2k}{n-2l})^{\frac{n-2l}{2}}$, then S_h is a closed curve in V.

(P2) If $1 \le l < k < \frac{n}{2}$ & $h = h^*$, then S_h is the point $\left(\frac{1}{2(k-l)} \ln\left(\frac{n-2l}{n-2k}\right), 0\right)$.

(P3) If $1 \le l < k < \frac{n}{2}$ & $h > h^*$, there is no solution for (2.1.1).

(P4) If $1 \le l < k = \frac{n}{2}$ & $h < 1$, then (2.1.1) defines a function $y := y(x)$ for large x in Ω^-, and $y(x)$ is strictly decreasing. Besides, $y \to -\sqrt{1 - \sqrt[k]{h}}$ as $x \to \infty$. Roughly S_h is a U-shaped curve that opens right in V.

(P5) If $1 \le l < k = \frac{n}{2}$ & $h \ge 1$, there is no solution for (2.1.1).

(P6) If $1 \le l < k$, $\frac{n}{2} < k$ & $h > 0$, then (2.1.1) defines a function $y := y(x)$ for large x in Ω^-, and $y(x)$ is strictly decreasing. Besides, $y(x) \to -1$ in Ω^- as $x \to \infty$. Roughly S_h is a U-shaped curve that opens right in V.

Case h<0 We introduce the notations below:

$$\underline{x}(y) = \frac{1}{2} \ln\left(\frac{(\frac{l}{k})^{\frac{1}{k-l}}}{1-y^2}\right) \qquad |y| < 1 \tag{2.1.3}$$

$$\tilde{x}(y) = \frac{1}{2} \ln \frac{\left(\frac{2l-n}{2k-n}\right)^{\frac{1}{k-l}}}{1-y^2} \qquad |y| < 1 \tag{2.1.4}$$

$$(\kappa_h^*, \zeta_h^*) \equiv \left(-\sqrt{1 - \left(\frac{l}{k}\right)^{\frac{1}{k-l}} \left(\frac{h}{\left(\frac{l}{k}\right)^{\frac{k}{k-l}} - \left(\frac{l}{k}\right)^{\frac{l}{k-l}}}\right)^{\frac{2}{n}}}, -\frac{1}{n} \ln\left(\frac{h}{\left(\frac{l}{k}\right)^{\frac{k}{k-l}} - \left(\frac{l}{k}\right)^{\frac{l}{k-l}}}\right)\right)$$

$$(\tilde{\zeta}_h, \pm\tilde{\kappa}_h) = \left(-\frac{1}{n} \ln\left(\frac{h\left(\frac{2l-n}{2k-n}\right)^{\frac{-l}{k-l}}}{\left(\frac{2l-2k}{2k-n}\right)}\right), \pm\sqrt{1 - \left(\frac{2l-n}{2k-n}\right)^{\frac{1}{k-l}} \left(\frac{h\left(\frac{2l-n}{2k-n}\right)^{\frac{-l}{k-l}}}{\left(\frac{2l-2k}{2k-n}\right)}\right)^{\frac{2}{n}}}\right)$$

$$\hbar = \left(\frac{l}{k}\right)^{\frac{2k-n}{2(k-l)}} - \left(\frac{l}{k}\right)^{\frac{2l-n}{2(k-l)}}, \quad \wp = \left(\frac{n-2l}{n-2k}\right)^{\frac{2k-n}{2(k-l)}} - \left(\frac{n-2l}{n-2k}\right)^{\frac{2l-n}{2(k-l)}}$$

and the level set S_h is classified in the following subcases:

(N1) If $1 \le l < k$, $l < \frac{n}{2}$ & $h < \hbar$, then S_h intersects with the x-axis only at point $(x_h^*(0), 0)$, and (2.1.1) defines a function $y := y(x)$ in $\{(x,y) \,|\, x < \frac{1}{2(k-l)} \ln(\frac{l}{k})\} \bigcap \Omega^-$, which is strictly increasing. Moreover, $y(x) \to -1$ in Ω^- as $x \to -\infty$. Roughly S_h is an U-shaped curve that opens left in W.

(N2) If $1 \leq l < k$, $l < \frac{n}{2}$ & $\hbar \leq h < 0$, then S_h intersects with the x-axis only at point $(x_h^*(0), 0)$ and it also intersects with the curve (2.1.3) at $(\zeta_h^*, \pm\kappa_h^*)$. Moreover, in Ω^-, (2.1.1) defines a function $x := x(y) \quad -1 < y < 0$ and $x(y)$ satisfies

$$\begin{cases} \frac{\partial x(y)}{\partial y} > 0 & \text{if } -1 < y < \kappa_h^* \\ \frac{\partial x(y)}{\partial y} < 0 & \text{if } k_h^* < y < 0 \end{cases}$$

(N3) If $1 < l = \frac{n}{2} < k$ & $h \leq -1$, (2.1.1) has no solution.

(N4) If $1 < l = \frac{n}{2} < k$ & $-1 < h \leq \hbar = \frac{l}{k} - 1$, then S_h intersects with the x-axis only at point $(x_h^*(0), 0)$. (2.1.1) defines a function $y = y(x)$ in $\{(x, y) \,|\, x < \frac{1}{2(k-l)} \ln(\frac{l}{k})\} \bigcap \Omega^-$, which is strictly increasing there. Moreover, $y(x) \to -\sqrt{1 - \sqrt[l]{-h}}$ in Ω^- as $x \to -\infty$. Roughly S_h is a U-shaped curve that opens left in W.

(N5) If $1 < l = \frac{n}{2} < k$ & $\hbar < h < 0$, then S_h intersects with the x-axis only at point $(x_h^*(0), 0)$, and in Ω^- (2.1.1) defines a function $x := x(y) \quad -\sqrt{1 - \sqrt[l]{-h}} < y < 0$ and $x(y)$ satisfies

$$\begin{cases} \frac{\partial x(y)}{\partial y} > 0 & \text{if } -\sqrt{1 - \sqrt[l]{-h}} < y < \kappa_h^* \\ \frac{\partial x(y)}{\partial y} < 0 & \text{if } k_h^* < y < 0 \end{cases}$$

(N6) If $\frac{n}{2} < l < k \,\&\, \hbar \leq h < 0$, then S_h is a closed curve, which intersects with (2.1.3) at $(\zeta_h^*, \pm\kappa_h^*)$.

(N7) If $\frac{n}{2} < l < k \,\&\, \wp < h < \hbar$, then S_h is a closed curve, which doesn't intersect with (2.1.3), and all $(x, y) \in \Omega$ solving (2.1.1) satisfies $x < \frac{1}{2} \ln(\frac{l}{k})^{\frac{1}{k-l}}$. Also it intersects with (2.1.4) at $(\tilde{\zeta}_h, \pm\tilde{\kappa}_h)$, and in Ω^- (2.1.1) defines a function $y = y(x)$, which satisfies

$$\begin{cases} \frac{\partial y}{\partial x} < 0 & \text{if } x < \tilde{\zeta}_h \\ \frac{\partial y}{\partial x} > 0 & \text{if } x > \tilde{\zeta}_h \end{cases} \qquad (x, y) \in \Omega^-.$$

(N8) If $\frac{n}{2} < l < k \,\&\, h = \wp$, S_h is a point $(\tilde{x}(0), 0)$.

(N9) If $\frac{n}{2} < l < k \,\&\, h < \wp$, (2.1.1) has no solution.

2.2 Classification of radial solutions

Chang, Han and Yang [10] showed that if $\omega(t,\theta) =: \xi(t)$ is a function of t, then

$$\sigma_k(g) = 2^{1-k}\binom{n}{k}(1-\xi_t^2)^{k-1}\left[\frac{k}{n}\xi_{tt} + \left(\frac{1}{2}-\frac{k}{n}\right)(1-\xi_t^2)\right]e^{2k\xi}.$$

See Chapter 1 for notations. As a computation in [43], the conformal quotient equation

$$\frac{\sigma_k(g)}{\sigma_l(g)} = c$$

in the radial case becomes

$$\frac{2^{1-k}\binom{n}{k}(1-\xi_t^2)^{k-1}\left[\frac{k}{n}\xi_{tt} + \left(\frac{1}{2}-\frac{k}{n}\right)(1-\xi_t^2)\right]e^{2k\xi}}{2^{1-l}\binom{n}{l}(1-\xi_t^2)^{l-1}\left[\frac{l}{n}\xi_{tt} + \left(\frac{1}{2}-\frac{l}{n}\right)(1-\xi_t^2)\right]e^{2l\xi}} = c. \tag{2.2.1}$$

When the positive constant c is normalized to $2^{l-k}\binom{n}{k}/\binom{n}{l}$, (2.2.1) is just

$$\begin{aligned}&(1-\xi_t^2)^{k-1}\left[\frac{k}{n}\xi_{tt} + \left(\frac{1}{2}-\frac{k}{n}\right)(1-\xi_t^2)\right]e^{2k\xi}\\ &= (1-\xi_t^2)^{l-1}\left[\frac{l}{n}\xi_{tt} + \left(\frac{1}{2}-\frac{l}{n}\right)(1-\xi_t^2)\right]e^{2l\xi},\end{aligned}$$

and multiplying both sides by $ne^{-n\xi}\xi_t$, we get (1.0.10), i.e.,

$$e^{(2k-n)\xi(t)}(1-\xi_t^2(t))^k - e^{(2l-n)\xi}(1-\xi_t^2(t))^l = h$$

where h is constant. Since $\frac{\sigma_k}{\sigma_l}$ has a fixed sign on $\mathbb{R}\times\mathbb{S}^{n-1}$, then either $1-\xi_t^2 > 0$ or $1-\xi_t^2 < 0$ for all $t \in \mathbb{R}$, and actually we can claim

$$1-\xi_t^2 > 0$$

by a contradiction argument. Indeed $\xi(t) \in \Gamma_t^+$ and $k \geq 2$ imply $\sigma_1, \sigma_2 > 0$, if $1-\xi_t^2 < 0$, then

$$\begin{aligned}\sigma_2 > 0 &\Longrightarrow \frac{2}{n}\xi_{tt} + (\frac{1}{2}-\frac{2}{n})(1-\xi_t^2) < 0,\\ \sigma_1 > 0 &\Longrightarrow \frac{1}{n}\xi_{tt} + (\frac{1}{2}-\frac{1}{n})(1-\xi_t^2) > 0.\end{aligned}$$

Combing these two inequalities,

$$(1-\frac{n}{4})(1-\xi_t^2) > \xi_{tt} > (1-\frac{n}{2})(1-\xi_t^2),$$

a contradiction. Furthermore, as a work [10] of Chang, Han and Yang, we proved the following Theorem , which is a classification result on radial solutions of (2.2.1) in the Γ_k^+ class on the entire $\mathbb{R}\times\mathbb{S}^{n-1}$.

Theorem 2.2.1. *Any radial solution $\xi(t) := \omega(t,\theta)$ of (2.2.1) in the Γ_k^+ class on the entire $\mathbb{R}\times\mathbb{S}^{n-1}$, when $1 \le l < k \le n$ and c is a positive constant, normalized to be $2^{l-k}\binom{n}{k}/\binom{n}{l}$, has the property that $1-\xi_t^2 > 0$ for all t. Furthermore, $h := e^{(2k-n)\xi(t)}(1-\xi_t^2(t))^k - e^{(2l-n)\xi}(1-\xi_t^2(t))^l$ is a nonnegative constant. Moreover,*

1. If $h = 0$, then $u^{\frac{4}{n-2}}(|x|) = (\frac{2\rho}{|x|^2+\rho^2})^2$ for some positive parameter ρ. So these solutions give rise to the round spherical metric on $\mathbb{R}^n \cup \{\infty\} = \mathbb{S}^n$.

2. If $h > 0$, then the behavior of u is classified according to the relation between $2k$ and n :

- If $2k < n$, then h has the further restriction $h \le h^* := (\frac{n-2k}{n-2l})^{\frac{n-2k}{2}} - (\frac{n-2k}{n-2l})^{\frac{n-2l}{2}}$ and $\xi(t)$ is a periodic function of t, giving rise to a metric $g = \frac{e^{-2\xi(ln|x|)}}{|x|^2}|dx|^2$ on $\mathbb{R}^n\backslash\{0\}$ which is complete. Note that the case $h = h^*$ gives rise to the cylindrical metric $\frac{|dx|^2}{|x|^2}$ on $\mathbb{R}^n\backslash\{0\}$.

- If $2k = n$, then h satisfies the further restriction $h < 1$ and as $|x| \to 0$, $g = u^{\frac{4}{n-2}}(|x|)|dx|^2$ has the asymptotic
$$g \sim |x|^{-2\left(1-\sqrt{1-\sqrt[k]{h}}\right)}|dx|^2 = e^{-\left(2\sqrt{1-\sqrt[k]{h}}\right)t}(dt^2+d\theta^2),$$
and as $|x| \to \infty$,$g = u^{\frac{4}{n-2}}(|x|)|dx|^2$ has the asymptotic
$$g \sim |x|^{-2\left(1+\sqrt{1-\sqrt[k]{h}}\right)}|dx|^2 = e^{\left(2\sqrt{1-\sqrt[k]{h}}\right)t}(dt^2+d\theta^2).$$
Thus g gives rise to a metric on $\mathbb{R}^n\backslash\{0\}$ singular at 0 and ∞ which behaves like the cone metric, is incomplete with finite volume.

- If $2k > n$, then $u^{\frac{4}{n-2}}(|x|)$ has an asymptotic expansion of the form
$$u^{\frac{4}{n-2}}(|x|) = \rho^{-2}\left\{1 - \sqrt[k]{h}\cdot\frac{k}{2k-n}\left(\frac{|x|}{\rho}\right)^{2-\frac{n}{k}} + \cdots\right\}$$
as $|x| \to 0$, where $\rho > 0$ is a positive parameter, thus $u(|x|)$ has a positive, finite limit, but $u_{rr}(|x|)$ blows up at $|x| \to 0$. The behavior of u as $|x| \to \infty$ can be described similarly. Putting them together, we conclude that $u^{\frac{4}{n-2}}(|x|)|dx|^2$ extends to a $C^{2-\frac{n}{k}}$ metric on $\mathbb{S}^n$.

Proof. We consider

$$e^{(2k-n)\xi}(1-\xi_t^2)^k - e^{(2l-n)\xi}(1-\xi_t^2)^l = h$$

with h as a constant, and first show $h \geq 0$ in our setting with a contradiction argument. When $1 \leq l < k$ & $l < \frac{n}{2}$, let ξ be a solution satisfying (2.2.1) and (1.0.10) with $\xi(0) = x_h^*(0)$ and $(-T_h, T_h)$ be the maximal open interval of ξ. If $h \leq (\frac{l}{k})^{\frac{2k-n}{2(k-l)}} - (\frac{l}{k})^{\frac{2l-n}{2(k-l)}} < 0$, then by the description for the level set in the above section, we have $|\xi_t| < 1$ and $\lim_{t\to T}\xi = -\infty$, thus $T_h = \infty$ and ξ_t is always decreasing as t increases, which implies that $\xi_{tt} \leq 0$ for all t. Now we can show that $\sigma_k\left(e^{-2\xi}(dt^2 + d\theta^2)\right) \leq 0$ everywhere and the solution is not in the Γ_k^+ class. Assume $\sigma_k\left(e^{-2\xi}(dt^2 + d\theta^2)\right) > 0$ for some point $(t_0, \theta_0) \in \mathbb{R} \times \mathbb{S}^{n-1}$, then at (t_0, θ_0),

$$\frac{k}{n}\xi_{tt} + (\frac{1}{2} - \frac{k}{n})(1-\xi_t^2) > 0.$$

However in the equality

$$\frac{2^{-k}\binom{n}{k}}{2^{-l}\binom{n}{l}} = \frac{2^{1-k}\binom{n}{k}(1-\xi_t^2)^{k-1}\left(\frac{k}{n}\xi_{tt} + (\frac{1}{2}-\frac{k}{n})(1-\xi_t^2)\right)e^{2k\xi}}{2^{1-l}\binom{n}{l}(1-\xi_t^2)^{l-1}\left(\frac{l}{n}\xi_{tt} + (\frac{1}{2}-\frac{l}{n})(1-\xi_t^2)\right)e^{2l\xi}},$$

we have $e^{2k\xi} < e^{2l\xi}$ and $(1-\xi_t^2)^{k-1} < (1-\xi_t^2)^{l-1}$, so

$$\frac{k}{n}\xi_{tt} + (\frac{1}{2} - \frac{k}{n})(1-\xi_t^2) > \frac{l}{n}\xi_{tt} + (\frac{1}{2} - \frac{l}{n})(1-\xi_t^2),$$

hence

$$\xi_{tt} > 1 - \xi_t^2 > 0,$$

a contradiction. If $(\frac{l}{k})^{\frac{2k-n}{2(k-l)}} - (\frac{l}{k})^{\frac{2l-n}{2(k-l)}} < h < 0$, then from the above section , the level set S_h intersects with (2.1.3) at $(\zeta_h^*, \pm\kappa_h^*)$. And as $\xi \to \zeta_h^*$, and $\xi_t \to \kappa_h^*$, the coefficient of ξ_{tt} will approach 0 while the other parts will approach $\frac{1}{2}(1-\frac{k}{l})(1-\xi_t^2)$ in (2.2.1), thus

$$\begin{aligned}\lim_{\xi\to\zeta_h^*}\xi_{tt} &= \lim_{\xi\to\zeta_h^*}\frac{\left[\left(\frac{1}{2}-\frac{k}{n}\right) - \left(\frac{1}{2}-\frac{l}{n}\right)e^{2(l-k)\xi}(1-\xi_t^2)^{l-k}\right]}{\frac{l}{n}e^{2(l-k)\xi}(1-\xi_t^2)^{l-k} - \frac{k}{n}} \\ &= \infty,\end{aligned}$$

which implies that $0 < T_h < \infty$ and $\lim_{t\to T}\xi(t) = \zeta_h^*$. This contradicts the fact that the solution is defined all over $\mathbb{R} \times \mathbb{S}^{n-1}$. If ξ is a solution defined on other pieces of the level set, the maximal interval of t is still finite, a contradiction. The exclusion of other cases with $h<0$ is similar. Thus the solutions $\xi(t)$ in our theorem must satisfy (1.0.10) with $h \geq 0$ and

1. when $h=0$ we have
$$1-\xi_t^2 = e^{-2\xi},$$
so the solution is $r^{-2}e^{-2\xi} = u^{\frac{4}{n-2}}(|x|) = (\frac{2\rho}{|x|^2+\rho^2})^2$ for some positive parameter ρ and these solutions give rise to the round spherical metric on $\mathbb{R}^n \cup \{\infty\} = \mathbb{S}^n$.

2. when $h>0$, we consider it in the following subcases:

   (a) If $2k<n$, then by the phase plane results `(P1)(P2)(P3)` in Section 2.1, $0 < h \leq h^* := (\frac{n-2k}{n-2l})^{\frac{n-2k}{2}} - (\frac{n-2k}{n-2l})^{\frac{n-2l}{2}}$. If $0<h<h^* := (\frac{n-2k}{n-2l})^{\frac{n-2k}{2}} - (\frac{n-2k}{n-2l})^{\frac{n-2l}{2}}$, then we can verify that the solution is in the Γ_k^+ class. By contradiction if $\sigma_k\left(e^{-2\xi}(dt^2+d\theta^2)\right) \leq 0$, then
$$\frac{k}{n}\xi_{tt} + (\frac{1}{2}-\frac{k}{n})(1-\xi_t^2) \leq 0.$$
However in the equality
$$\frac{2^{-k}\binom{n}{k}}{2^{-l}\binom{n}{l}} = \frac{2^{1-k}\binom{n}{k}(1-\xi_t^2)^{k-1}\left(\frac{k}{n}\xi_{tt} + (\frac{1}{2}-\frac{k}{n})(1-\xi_t^2)\right)e^{2k\xi}}{2^{1-l}\binom{n}{l}(1-\xi_t^2)^{l-1}\left(\frac{l}{n}\xi_{tt} + (\frac{1}{2}-\frac{l}{n})(1-\xi_t^2)\right)e^{2l\xi}},$$
we have $1-\xi_t^2 > e^{-2\xi}$ which implies that
$$\frac{k}{n}\xi_{tt} + (\frac{1}{2}-\frac{k}{n})(1-\xi_t^2) \geq \frac{l}{n}\xi_{tt} + (\frac{1}{2}-\frac{l}{n})(1-\xi_t^2),$$
hence
$$\xi_{tt} \geq 1-\xi_t^2 > 0,$$
which implies $\sigma_k\left(e^{-2\xi}(dt^2+d\theta^2)\right) > 0$, a contradiction. Meanwhile, for any $1 \leq i < k$, we have
$$\begin{aligned}&\frac{i}{n}\xi_{tt} + (\frac{1}{2}-\frac{i}{n})(1-\xi_t^2)\\ =\ &\frac{i}{k}\left(\frac{k}{n}\xi_{tt} + (\frac{1}{2}-\frac{k}{n})(1-\xi_t^2)\right) + \left(\frac{1}{2}-\frac{i}{2k}\right)(1-\xi_t^2)\\ >\ &0,\end{aligned}$$

so $\sigma_i\left(e^{-2\xi}(dt^2+d\theta^2)\right)>0$. Thus the solution ξ is in the Γ_k^+ class and $\xi(t)$ is a periodic function of t, giving rise to a metric $g=\frac{e^{-2\xi(ln|x|)}}{|x|^2}|dx|^2$ on $\mathbb{R}^n\backslash\{0\}$ which is complete. Note that the case $h=h^*$ gives rise to the cylindrical metric $\frac{|dx|^2}{|x|^2}$ on $\mathbb{R}^n\backslash\{0\}$.

(b) If $2k=n$, then by the phase plane results (P4)(P5) in Section 2.1, h satisfies the further restriction $h<1$. Furthermore $T_h=\infty$ and ξ is the Γ_k^+ class. Also as $t\to\infty$, $\xi\to\infty$ & $\xi_t\to\sqrt{1-\sqrt[k]{h}}$, thus

$$\xi\sim\left(\sqrt{1-\sqrt[k]{h}}\right)t$$

which implies when $|x|\sim 0$, we have

$$g\sim|x|^{-2(1-\sqrt{1-\sqrt[k]{h}})}|dx|^2.$$

Similarly when $|x|\sim\infty$, we have

$$g\sim|x|^{-2(1+\sqrt{1-\sqrt[k]{h}})}|dx|^2.$$

These are incomplete, finite volume metrics on $\mathbb{R}^n\backslash\{0\}$, corresponding to conical metrics on $\mathbb{S}^n\backslash\{0,\infty\}$.

(c) If $2k>n$, then by the phase plane result (P6) in Section 2.1, $T_h=\infty$ and ξ is in the Γ_k^+ class. Also as $t\to\infty$, $\xi\to\infty$ & $\xi_t\to 1$. Moreover, we can show $\zeta(t):=e^{(2l-n)\xi}(1-\xi_t^2)^l\to 0$ as $t\to\infty$, or else there exists sequences $\{t_j\}_{1\le j}$ and some $\epsilon>0$ such that as $t_j\to\infty$,

$$\zeta(t_j)=e^{(2l-n)\xi(t_j)}(1-\xi_t^2(t_j))^l>\epsilon.$$

However, from the definition of $\zeta(t)$, we get

$$e^{2\xi(t_j)}(1-\xi_t^2(t_j))=\zeta(t_j)^{\frac{1}{l}}e^{\frac{n}{l}\xi(t_j)},$$

plugging it into the curve (1.0.10), then as $t_j\to\infty$

$$\begin{aligned} h &= \zeta(t_j)^{\frac{k}{l}}e^{(\frac{nk}{l}-n)\xi(t_j)}-\zeta(t_j)\\ &> \epsilon^{\frac{k}{l}}e^{(\frac{nk}{l}-n)\xi(t_j)}-\epsilon\\ &\to \infty, \end{aligned}$$

a contradiction. Thus as $t \to \infty$

$$1 - \xi_t^2 \sim \sqrt[k]{h} e^{\frac{n-2k}{k}\xi},$$

$$1 - \xi_t \sim \frac{\sqrt[k]{h} e^{\frac{n-2k}{k}\xi}}{2},$$

and taking integral, we get as $t \to \infty$,

$$\xi - t = c + \frac{\sqrt[k]{h}}{2} \cdot \frac{k}{2k-n} e^{\frac{n-2k}{k}\xi} + h.o.t.$$

for some constant c. Then $u^{\frac{4}{n-2}}(|x|)$ has an asymptotic expansion of the form

$$u^{\frac{4}{n-2}}(|x|) = \rho^{-2} \left\{ 1 - \sqrt[k]{h} \cdot \frac{k}{2k-n} \left(\frac{|x|}{\rho} \right)^{2-\frac{n}{k}} + \cdots \right\}$$

as $|x| \to 0$, where $\rho = e^c > 0$ is a positive parameter. The analysis near x at ∞ can be carried out in a similar way. Thus we conclude that $v^{-2}|dx|^2$ extends to a $C^{2-\frac{n}{k}}$ metric on $\mathbb{S}^n$.

$\square$

Chapter 3

Exploiting the ODE satisfied by the radial average

3.1 Several preliminary properties

In this section we present some needed preliminary properties for solutions derived in [58]. Recall that under the Euclidean coordinates, we introduce $u(x)$ to write the conformal change of metric g as

$$g = u^{\frac{4}{n-2}}(x)|dx|^2 = e^{-2\omega(t,\theta)}(dt^2 + d\theta^2).$$

Then (1.0.7) is equivalent to

$$\begin{aligned} &\frac{\sigma_k\left(-(n-2)u(x)\nabla^2 u(x) + n\nabla u(x)\otimes\nabla u(x) - |\nabla u(x)|^2 Id\right)}{\sigma_l\left(-(n-2)u(x)\nabla^2 u(x) + n\nabla u(x)\otimes\nabla u(x) - |\nabla u(x)|^2 Id\right)} \\ &\quad = c\cdot\frac{2^k\cdot(n-2)^{-2k}\cdot u^{\frac{2kn}{n-2}}(x)}{2^l\cdot(n-2)^{-2l}\cdot u^{\frac{2ln}{n-2}}(x)}. \end{aligned} \tag{3.1.1}$$

The following theorem is drawn from Theorem $1.1'$ and 1.3 in [58].

Theorem A (Y. Y. Li). *Suppose that $u \in C^2(B_2\backslash\{0\})$ is a positive solution to* (3.1.1) *on $B_R\backslash\{0\}$, then*

$$\limsup_{x\to 0}|x|^{\frac{n-2}{2}}u(x) < \infty, \tag{3.1.2}$$

and there exists some constant $C>0$ such that

$$|u(x) - \bar{u}(|x|)| \leq C|x|\cdot\bar{u}(|x|) \tag{3.1.3}$$

for all $0<|x|\leq 1$, where

$$\bar{u}(|x|) = \frac{1}{|\partial B_{|x|}(0)|}\int_{\partial B_{|x|}(0)} u(y)d\sigma(y)$$

is the spherical average of $u(x)$ over $\partial B_{|x|}(0)$.

As in the previous section, in terms of $t = -\ln r = -\ln |x|$, let

$$U(t,\theta) = r^{\frac{n-2}{2}} u(r\theta) = e^{-\frac{n-2}{2}\omega(t,\theta)}$$

$$\beta(t) = |S^{n-1}|^{-1} \int_{S^{n-1}} U(t,\theta) d\theta$$
$$\gamma(t) = |S^{n-1}|^{-1} \int_{S^{n-1}} \omega(t,\theta) d\theta, \tag{3.1.4}$$

then the inequality (3.1.2) is reformulated as

$$U(t,\theta) \le C \ \text{ and } \ e^{-2\omega(t,\theta)} \le C, \tag{3.1.5}$$

plus

$$|U(t,\theta) - \beta(t)| \le C\beta(t)e^{-t},$$

and

$$|\hat{\omega}(t,\theta)| := |\omega(t,\theta) - \gamma(t)| \le Ce^{-t}. \tag{3.1.6}$$

We also have the gradient estimate for positive singular solutions $u(x)$ in the Γ_k^+ class to (3.1.1) on $B_R(0)\backslash\{0\}$.

Proposition 3.1.1. *Let $u(x)$ be a positive singular solution to* (3.1.1) *on $B_R\backslash\{0\}$ in the Γ_k^+ class, $U(t,\theta), \beta(t), \omega(t,\theta)$ and $\gamma(t)$ be defined above. Then for any $0 < \delta$ small, there exists a constant $C > 0$ depending on δ such that*

$$|\nabla^j_{t,\theta}(U(t,\theta) - \beta(t))| \le C\beta(t)e^{-(1-\delta)t}$$
$$|\nabla^j_{t,\theta}(\omega(t,\theta) - \gamma(t))| \le Ce^{-(1-\delta)t} \tag{3.1.7}$$

for all $t \ge 0$ and $j = 1, 2$.

We now provide an argument for (3.1.7). First, (3.1.5) and the gradient estimates for solutions to (1.0.8), see [33], give a bound $B > 0$ depending on $j > 0$ and C in (3.1.7), such that

$$|\nabla^j_{t,\theta}\omega(t,\theta)| \le B, \tag{3.1.8}$$

which leads to

$$|\nabla^j_{t,\theta}\gamma(t)| \leq B. \tag{3.1.9}$$

Combining them together, we get

$$|\nabla^j_{t,\theta}(\omega(t,\theta) - \gamma(t))| \leq 2B$$

This estimate, together with (3.1.6) and interpolation, proves (3.1.7).

We also quote Theorem 3 in [43] here, which is an elaborated inductive argument and plays a key role in proving Theorem 1.0.2.

Theorem B (Z.-C. Han-Y. Y. Li-E. V. Teixeira). *Consider a solution $\beta(t)$ to*

$$\beta''(t) = f(\beta'(t), \beta(t)) + e_1(t) \;\; t \geq 0$$

where f is locally Lipschitz, and $e_i(t)$ is considered as a perturbation term with $e_1(t) \to 0$ as $t \to \infty$ at a sufficiently fast rate to be specified later. Suppose that $|\beta(t)| + |\beta'(t)|$ is bounded over $t \in [0, \infty)$. Then by a compactness argument there exists a sequence of $t_i \to \infty$ and a solution $\psi(t)$ to

$$\psi''(t) = f(\psi'(t), \psi(t)) \tag{3.1.10}$$

which exists for all $t \in R$ such that

$$\beta(t_i + \cdot) \to \psi \;\; in \; C^1_{loc}(-\infty, \infty) \; as \; i \to \infty \tag{3.1.11}$$

Suppose $\psi(t)$ is a periodic solution with $T \geq 0$, thus for some finite $m \leq M$

$$\psi(R) = [m, M].$$

We may do a time translation for $\psi(t)$ so that $\psi(0) = m, \psi'(0) = 0$, then the approximation property (3.1.11) *can be reformulated as, for some s,*

$$\beta(t_i + \cdot) - \psi(-s + \cdot) \to 0 \; in \; C^1_{loc}(-\infty, \infty) \; as \; i \to \infty$$

Suppose that (3.1.10) *has a first integral in the form of*

$$H(\psi'(t), \psi(t)) = 0$$

for some continuous function $H(x, y)$, where H satisfies the following non-degeneracy condition, depending on

1. *($\psi(t) := m$ is a constant): there exist some $\epsilon_1 > 0, A > 0$ and $l > 0$*

$$H(x, y) \geq A(|x|^l + |y - m|^l) \tag{3.1.12}$$

for any (x, y) with $|x| + |y - m| \leq \epsilon_1$;

2. *($\psi(t)$ is non-constant): there exists some $\epsilon_1 > 0$, $A > 0$ and $l > 0$,*

$$|H(0, y)| = |H(0, y) - H(0, m)| \geq A|y - m|^l \tag{3.1.13}$$

for any y with $|y - m| \leq \epsilon_1$.

Suppose also that $\beta(t)$ has H as an approximate first integral

$$|H(\beta'(t), \beta(t))| \leq e_2(t) \text{ for } t \geq 0$$

where $e_2(t) \to 0$ as $t \to \infty$. Without loss of generality, we may suppose that $e_2(t)$ is monotone non-increasing in t. Finally suppose that

$$\int_0^\infty \left(((e_2(t))^{\frac{1}{l}} + sup_{\tau \geq t}|e_1(\tau)|\right) dt < \infty$$

Then there exists some s_∞ and $C > 0$ such that

$$\begin{aligned} & |\beta(t) - \psi(t - s_\infty)| + |\beta'(t) - \psi'(t - s_\infty)| \\ \leq \quad & C \int_{t-(T+2)}^\infty \left((e_2(t'))^{\frac{1}{l}} + sup_{\tau \geq t'}|e_1(\tau)|\right) dt' \to 0 \text{ as } t \to \infty. \end{aligned}$$

3.2 Perturbed ODEs

Let $\omega(t, \theta)$ be a positive solution of (1.0.8) on $B_R\backslash\{0\}$ in the Γ_k^+ class, where the constant c is normalized to be $2^{l-k}\binom{n}{k}/\binom{n}{l}$, and $\gamma(t)$ is defined in (3.1.4). We can make

mathn 3.2.1. If $1 \leq l < k \leq n$, there exists some constant h such that

$$\begin{aligned} & \left\{ 2(1 - \gamma_t^2)^{k-1} \left[\frac{k}{n}\gamma_{tt} + \frac{n - 2k}{2n}(1 - \gamma_t^2) \right] + \eta_1(t) \right\} e^{2k\gamma} \\ & = \left\{ 2(1 - \gamma_t^2)^{l-1} \left[\frac{l}{n}\gamma_{tt} + \frac{n - 2l}{2n}(1 - \gamma_t^2) \right] + \eta_2(t) \right\} e^{2l\gamma}, \end{aligned} \tag{3.2.1}$$

$$\left[(1 - \gamma_t^2)^k + \eta_3(t) \right] \cdot e^{(2k-n)\gamma} - \left[(1 - \gamma_t^2)^l + \eta_4(t) \right] \cdot e^{(2l-n)\gamma} = h \tag{3.2.2}$$

where $\eta_i(t)$, for $i = 1, \cdots 4$, have the decay rate $\eta_i(t) = O(e^{-2(1-\delta)t})$ as $t \to \infty$, for arbitrarily small $\delta > 0$ as in (3.1.7).

Proof. We first prove (3.2.1). By equation (1.0.7) we have

$$2^k \cdot \frac{1}{\binom{n}{k}} \sigma_k(g_0^{-1} \cdot A_g) \cdot e^{2k\omega(t,\theta)} = 2^l \cdot \frac{1}{\binom{n}{l}} \sigma_l(g_0^{-1} \cdot A_g) \cdot e^{2l\omega(t,\theta)}.$$

Let

$$\sigma_k(A_\omega) := \sigma_k(g_0^{-1} \cdot A_g)$$

be a functional of $\omega(t,\theta)$, then with $\hat{\omega}(t,\theta) = \omega(t,\theta) - \gamma(t)$, we have the following expansion

$$\sigma_k(A_\omega) = \sigma_k(A_{\gamma(t)}) + L_{\gamma(t)}[\hat{\omega}(t,\theta))] + \hat{\eta}_1(t,\theta)$$

where $L_{\gamma(t)}$ denotes the linearized operator for $\sigma_k(A_{\omega(t,\theta)})$ at $\gamma(t)$, and $\hat{\eta}_1(t,\theta)$ satisfies $|\hat{\eta}_1(t,\theta)| = O(e^{-2(1-\delta)t})$ as $t \to \infty$ by (3.1.7) and (3.1.9). Next,

$$e^{2k\omega(t,\theta)} = e^{2k\gamma(t)} \cdot e^{2k\hat{\omega}(t,\theta)}$$

and

$$e^{2k\hat{\omega}(t,\theta)} = 1 + 2k\hat{\omega}(t,\theta) + \hat{\eta}_2(t,\theta)$$

where $|\hat{\eta}_2(t,\theta)| = O(e^{-2t})$ as $t \to \infty$ since (3.1.6). Putting them together, we have

$$\begin{aligned}
& \sigma_k(g_0^{-1} \cdot A_g) \cdot e^{2k\omega(t,\theta)} \\
= \; & \left(\sigma_k(A_{\gamma(t)}) + L_{\gamma(t)}[\hat{\omega}(t,\theta))] + \hat{\eta}_1(t,\theta)\right)(1 + 2k\hat{\omega}(t,\theta) + \hat{\eta}_2(t,\theta)) \cdot e^{2k\gamma(t)} \\
= \; & e^{2k\gamma(t)} \Big\{ \sigma_k(A_{\gamma(t)}) + L_{\gamma(t)}[\hat{\omega}(t,\theta)] + \hat{\eta}_1(t,\theta) + \sigma_k(A_{\gamma(t)}) \cdot 2k\hat{\omega}(t,\theta) \\
& \quad + L_{\gamma(t)}[\hat{\omega}(t,\theta)] \cdot 2k\hat{\omega}(t,\theta) + \hat{\eta}_1(t,\theta) \cdot 2k\hat{\omega}(t,\theta) + \sigma_k(A_{\gamma(t)})\hat{\eta}_2(t,\theta) \\
& \quad + L_{\gamma(t)}[\hat{\omega}(t,\theta)]\hat{\eta}_2(t,\theta) + \hat{\eta}_1(t,\theta)\hat{\eta}_2(t,\theta) \Big\}.
\end{aligned}$$

Integrating over $\theta \in \mathbb{S}^{n-1}$, we have

$$\frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \sigma_k(A_{\gamma(t)}) d\theta = 2^{1-k} \tbinom{n}{k} (1 - \gamma_t^2)^{k-1} \left[\frac{k}{n}\gamma_{tt} + \frac{n-2k}{2n}(1 - \gamma_t^2) \right]$$

$$\frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} L_{\gamma(t)}[\hat{\omega}(t,\theta))] d\theta = 0$$

$$\frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \sigma_k(A_{r(t)}) \cdot 2k\hat{\omega}(t,\theta) d\theta = 0$$

$$\frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} L_{\gamma(t)}[\hat{\omega}(t,\theta)] \cdot 2k\hat{\omega}(t,\theta) d\theta = O(e^{-(2-\delta)t})$$

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\hat{\eta}_1(t,\theta)\cdot 2k\hat{\omega}(t,\theta)d\theta = O(e^{-(3-2\delta)t})$$
$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\sigma_k(g_0^{-1}\cdot A_{\gamma(t)})\hat{\eta}_2(t,\theta)d\theta = O(e^{-2(1-\delta)t})$$
$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}L_{\gamma(t)}[\hat{\omega}(t,\theta)]\hat{\eta}_2(t,\theta)d\theta = O(e^{-3(1-\delta)t})$$
$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\hat{\eta}_1(t,\theta)\hat{\eta}_2(t,\theta)d\theta = O(e^{-4(1-\delta)t})$$

as $t\to\infty$. Similarly we have the same estimate for the right side, and thus we have (3.2.1).

Next we are ready to prove (3.2.2). First we show that, when $2k\geq n$ or $h\neq 0$, (3.2.2) follows from (3.2.1). We use C to denote a positive constant, which may vary according to the context. Multiplying both sides of (3.2.1) by $ne^{-n\gamma(t)}\gamma_t(t)$, one has

$$\left[e^{(2k-n)\gamma(t)}(1-\gamma_t^2)^k - e^{(2l-n)\gamma(t)}(1-\gamma_t^2)^l\right]_t = ne^{-n\gamma(t)}\gamma_t(t)\left[e^{2k\gamma(t)}\eta_1(t)-e^{2l\gamma(t)}\eta_2(t)\right]. \tag{3.2.3}$$

By the estimates (3.1.5) and (3.1.6), $\gamma(t)\geq -C\ \forall t\in[0,\infty)$. For its upper bound, there are two possibilities: $\gamma(t)\leq C$, $\forall t\in[0,\infty)$, or $\sup_{t\in[0,\infty)}\gamma(t)=\infty$. We consider them separately:

- If $\gamma(t)\leq C$, $\forall t\in[0,\infty)$, then the right hand of (3.2.3) is of the order $O(e^{-2(1-\delta)t})$ from the gradient estimates (3.1.9), and the decay rates of $\eta(t)$. It then follows that for some constant h, we have

  $$e^{(2k-n)\gamma(t)}(1-\gamma_t^2)^k - e^{(2l-n)\gamma(t)}(1-\gamma_t^2)^l = h + O(e^{-2(1-\delta)t}),$$

  thus (3.2.2) holds.

- If $\sup_{t\in[0,\infty)}\gamma(t)=\infty$ and $2k>n$, we claim that $\gamma(t)$ is strictly increasing when t is large enough, or else there must exist an increasing sequence $\{t_j\}_{j\in\mathbb{N}}$ such that

  $$t_j\to\infty\,\&\,\gamma(t_j)\to\infty \quad \text{as } j\to\infty,$$

  $$\gamma(t_j)=\max_{t\in[0,t_j]}\gamma(t),\ \ \&\ \ \gamma(t_j)>2\gamma(t_{j-1})\quad \forall j\geq 2,$$

  and

  $$\gamma_t(t_j)=0\ \ \forall j\in\mathbb{N}.$$

By taking the integral on $[t_{j-1}, t_j]$ for both sides of (3.2.3), we have

$$\begin{aligned} & e^{(2k-n)\gamma(t)}\left(1-\gamma_t^2(t)\right)^k - e^{(2l-n)\gamma(t)}\left(1-\gamma_t^2(t)\right)^l \Big|_{t_{j-1}}^{t_j} \\ = & \int_{t_{j-1}}^{t_j} n e^{-n\gamma(s)}\gamma_s(s)\left[e^{2k\gamma(s)}\eta_1(s) - e^{2l\gamma(s)}\eta_2(s)\right] ds. \end{aligned} \tag{3.2.4}$$

However, as $j \to \infty$, the left side of equation (3.2.4) satisfies

$$\begin{aligned} & \left[e^{(2k-n)\gamma(t_j)}\left(1-\gamma_t^2(t_j)\right)^k - e^{(2l-n)\gamma(t_j)}\left(1-\gamma_t^2(t_j)\right)^l\right] \\ & \quad - \left[e^{(2k-n)\gamma(t_{j-1})}\left(1-\gamma_t^2(t_{j-1})\right)^k - e^{(2l-n)\gamma(t_{j-1})}\left(1-\gamma_t^2(t_{j-1})\right)^l\right] \\ = & \left[e^{(2k-n)\gamma(t_j)} - e^{(2l-n)\gamma(t_j)}\right] - \left[e^{(2k-n)\gamma(t_{j-1})} - e^{(2l-n)\gamma(t_{j-1})}\right] \\ > & \frac{1}{2}e^{(2k-n)\gamma(t_j)}, \end{aligned}$$

while the right hand side of (3.2.4) satisfies

$$\begin{aligned} & \int_{t_{j-1}}^{t_j} n e^{-n\gamma(s)}\gamma_s(s)\left[e^{2k\gamma(s)}\eta_1(s) - e^{2l\gamma(s)}\eta_2(s)\right] ds \\ \leq & C\int_{t_{j-1}}^{t_j}\left(e^{(2k-n)\gamma(s)}|\eta_1(s)| + e^{(2l-n)\gamma(s)}|\eta_2(s)|\right) ds \\ \leq & Ce^{(2k-n)\gamma(t_j)}\int_{t_{j-1}}^{t_j} e^{-2(1-\delta)s} ds \\ \leq & Ce^{(2k-n)\gamma(t_j)}e^{-2(1-\delta)t_{j-1}} \\ < & \frac{1}{2}e^{(2k-n)\gamma(t_j)}, \end{aligned}$$

a contradiction. Hence when t is large enough, $\gamma(t)$ is strictly increasing and goes to ∞, and with a similar argument, we can show that $1-\gamma_t^2 \to 0$ as $t \to \infty$. Indeed, if there exists $\epsilon > 0$ and an increasing sequence $\{\bar{t}_j\}_{j\in\mathbb{N}}$ such that

$$\begin{aligned} \bar{t}_j & \to \infty \\ 1-\gamma_t^2(\bar{t}_j) & > \epsilon \ \forall j \in \mathbb{N} \\ \gamma(\bar{t}_j) & \to \infty, \end{aligned}$$

then by taking integral on $[\bar{t}_{j-1}, \bar{t}_j]$, we get

$$\begin{aligned} & e^{(2k-n)\gamma(t)}\left(1-\gamma_t^2(t)\right)^k - e^{(2l-n)\gamma(t)}\left(1-\gamma_t^2(t)\right)^l \Big|_{\bar{t}_{j-1}}^{\bar{t}_j} \\ = & \int_{\bar{t}_{j-1}}^{\bar{t}_j} n e^{-n\gamma(s)}\gamma_s(s)\left[e^{2k\gamma(s)}\eta_1(s) - e^{2l\gamma(s)}\eta_2(s)\right] ds, \end{aligned}$$

which leads to

$$\frac{\epsilon^k}{2} e^{(2k-n)\gamma(\bar{t}_j)} \quad < \quad C e^{(2k-n)\gamma(\bar{t}_j)} e^{-2(1-\delta)\bar{t}_j}$$

as $j \to \infty$, a contradiction. Thus $1 - \gamma_t^2 \to 0$ as $t \to \infty$, consequently,

$$\lim_{t\to\infty} \gamma_t(t) = 1.$$

Now let

$$\eta_3(t) \quad = \quad e^{(n-2k)\gamma(t)} \int_0^t n e^{-n\gamma(s)} \gamma_s(s) \left[e^{2k\gamma(s)} \eta_1(s) - e^{2l\gamma(s)} \eta_2(s) \right] ds,$$

then clearly

$$\frac{d}{dt} \left[e^{(2k-n)\gamma(t)} \left(\left(1 - \gamma_t^2(t)\right)^k + \eta_3(t) \right) - e^{(2l-n)\gamma(t)} \left(1 - \gamma_t^2(t)\right)^l \right] \quad = \quad 0,$$

so there exists a constant h such that

$$e^{(2k-n)\gamma(t)} \left(\left(1 - \gamma_t^2(t)\right)^k + \eta_3(t) \right) - e^{(2l-n)\gamma(t)} \left(1 - \gamma_t^2(t)\right)^l = h.$$

Moreover, as $t \to \infty$,

$$\begin{aligned} |\eta_3(t)| &\leq C e^{(n-2k)\gamma(t)} \int_0^t e^{(2k-n)\gamma(s)} \cdot e^{-2(1-\delta)s} ds \\ &\leq C e^{(n-2k)(1-\epsilon)t} \int_0^t e^{(2k-n)(1+\epsilon)s} \cdot e^{-2(1-\delta)s} ds \\ &\leq C e^{-2[1-(\delta+(2k-n)\epsilon)]t}, \end{aligned}$$

where δ and ϵ could be arbitrarily small, so we have (3.2.2) with $\eta_4 = 0$.

- If $\sup_{t\in[0,\infty)} \gamma(t) = \infty$, and $2k < n$, it then follows that for some constant h, we have

$$e^{(2k-n)\gamma(t)}(1 - \gamma_t^2)^k - e^{(2l-n)\gamma(t)}(1 - \gamma_t^2)^l = h + O(e^{-2(1-\delta)t}). \tag{3.2.5}$$

If $h \neq 0$, then $\gamma(t)$ is strictly increasing and $\lim_{t\to\infty} \gamma(t) = \infty$, or else there exists an increasing sequence $\{t_j\}_{j\in\mathbb{N}}$ such that

$$\begin{aligned} t_j &\to \infty \,\&\, \gamma(t_j) \to \infty \\ \gamma_t(t_j) &= 0 \quad \forall j \in \mathbb{N}. \end{aligned}$$

Then as $j \to \infty$, the left side of (3.2.5)

$$e^{(2k-n)\gamma(t)}(1-\gamma_t^2)^k - e^{(2l-n)\gamma(t)}(1-\gamma_t^2)^l \to 0$$

while the right side of (3.2.5) is h, a contradiction. Consequently by a similar argument as above we can get (3.2.2) with

$$\eta_3 = -e^{(n-2k)\gamma(t)} \int_t^{\infty} ne^{-n\gamma(s)}\gamma_s(s)\left[e^{2k\gamma(s)}\eta_1(s) - e^{2l\gamma(s)}\eta_2(s)\right] ds$$

and $\eta_4 = 0$.

- If $2k = n$, it is a similar argument as the case $\gamma(t) \le C$ on $[0, \infty)$.

However we are unable to get (3.2.2) from (3.2.1) directly when $2k < n$ and $h = 0$ and have to utilize an approach with Pohozaev identity from [42]. Let us assume $n \neq 2k$ & $n \neq 2l$ and during this proof we use $\eta(t)$ to denote a quantity which has the decay rate $\eta(t) = O(e^{-2(1-\delta)t})$ as $t \to \infty$, and it may vary according to the context. Let T_a^b and $\bar{T}_a^b$ denote the components of Newton Tensor T_{k-1} and T_{l-1} respectively, then by [42], we have the following identity:

$$\begin{aligned}
< X, \nabla\sigma_k >&= \frac{1}{2k-n}\nabla_a\left(T_b^a\nabla^b(\mathtt{div}X) + 2k\sigma_k X^a\right)\\
< X, \nabla\sigma_l >&= \frac{1}{2l-n}\nabla_a\left(\bar{T}_b^a\nabla^b(\mathtt{div}X) + 2l\sigma_l X^a\right),
\end{aligned}$$

so let ϕ_t denote the local one-parameter family of conformal diffeomorphism of (M, g)generated by X, then from the equation (1.0.7)

$$\begin{aligned}
0 &= \frac{d}{dt}|_{t=0}(\sigma_k - c\sigma_l)\\
&\quad < X, \nabla(\sigma_k - c\sigma_l) >\\
&= < X, \nabla\sigma_k > -c < X, \nabla\sigma_l >\\
&= \frac{1}{2k-n}\nabla_a\left(T_b^a\nabla^b(\mathtt{div}X) + 2k\sigma_k X^a\right) - \frac{c}{2l-n}\nabla_a\left(\bar{T}_b^a\nabla^b(\mathtt{div}X) + 2l\sigma_l X^a\right).
\end{aligned}$$

Now taking $M = \mathbb{R} \times \mathbb{S}^{n-1}$, $g = e^{-2\omega(t,\theta)}(dt^2 + d\theta^2)$ and $X = \partial_t$, we get

$$\int_{\mathbb{S}^{n-1}}\left[\frac{1}{2k-n}\left(2k\sigma_k - nT_{1b}\omega_{tb}e^{2k\omega}\right) - \frac{c}{2l-n}\left(2l\sigma_l - n\bar{T}_{1b}\omega_{tb}e^{2l\omega}\right)\right]e^{-n\omega}d\theta = h \tag{3.2.6}$$

where h is a constant. Let $\tilde{L}_{\gamma(t)}$ and $\bar{L}_{\gamma(t)}$ denote the linearized operators for $\sum_{b=1}^{n} T_{b1}\omega_{bt}$ and $\sum_{b=1}^{n} \bar{T}_{b1}\omega_{bt}$ at $\gamma(t)$, then a computation in [43] shows that

$$\begin{aligned}\sum_{b=1}^{n} T_{b1}\omega_{bt} &= \frac{2k}{n}\cdot 2^{-k}\tbinom{n}{k}(1-\gamma_t^2)^{k-1}\gamma_{tt} + \tilde{L}_\gamma[\hat{\omega}(t,\theta)] + O(e^{-2(1-\delta)t}),\\ \sum_{b=1}^{n} \bar{T}_{b1}\omega_{bt} &= \frac{2l}{n}\cdot 2^{-l}\tbinom{n}{l}(1-\gamma_t^2)^{l-1}\gamma_{tt} + \bar{L}_\gamma[\hat{\omega}(t,\theta)] + O(e^{-2(1-\delta)t}).\end{aligned}$$

So we can write (3.2.6) as

$$h = E + F + G$$

where

$$\begin{aligned}E &= \int_{\mathbb{S}^{n-1}} -\frac{1}{2k-n}\left((1-\gamma_t^2)^{k-1}\gamma_{tt}e^{2k\omega}\right)e^{-n\omega}\cdot 2k\cdot 2^{-k}\tbinom{n}{k}d\theta\\ &\quad+\int_{\mathbb{S}^{n-1}} \frac{c}{2l-n}\left((1-\gamma_t^2)^{l-1}\gamma_{tt}e^{2l\omega}\right)\cdot e^{-n\omega}\cdot 2l\cdot 2^{-l}\tbinom{n}{l}d\theta,\end{aligned}$$

$$F = \int_{\mathbb{S}^{n-1}} \left(\frac{2k}{2k-n} - \frac{2l}{2l-n}\right)\cdot\sigma_k e^{-n\omega}d\theta,$$

$$\begin{aligned}G &= \int_{\mathbb{S}^{n-1}} \frac{-n}{2k-n}\left(\tilde{L}_\gamma[\hat{\omega}(t,\theta)] + \eta(t)\right)e^{(2k-n)\omega}d\theta\\ &\quad+\int_{\mathbb{S}^{n-1}} \frac{cn}{2l-n}\left(\bar{L}_\gamma[\hat{\omega}(t,\theta)] + \eta(t)\right)e^{(2l-n)\omega}d\theta.\end{aligned}$$

Clearly

$$G = \eta(t)\cdot e^{(2k-n)\gamma(t)} + \eta(t)\cdot e^{(2l-n)\gamma(t)},$$

and

$$\begin{aligned}F &= \int_{\mathbb{S}^{n-1}} \frac{n(2l-2k)}{(2k-n)(2l-n)}\sigma_k e^{-n\omega}d\theta\\ &= \int_{\mathbb{S}^{n-1}} \frac{2n(l-k)2^{-k}\binom{n}{k}}{(2k-n)(2l-n)}\\ &\quad\times\left\{2(1-\gamma_t^2)^{k-1}\left[\frac{k}{n}\gamma_{tt} + \frac{n-2k}{2n}(1-\gamma_t^2)\right] + \eta(t)\right\}e^{(2k-n)\gamma}d\theta,\end{aligned}$$

and by (3.2.1),

$$\begin{aligned}E &= \int_{\mathbb{S}^{n-1}} e^{-n\gamma}\cdot 2^{-k}\tbinom{n}{k}\left(\frac{-2k}{2k-n}\right)(1-\gamma_t^2)^{k-1}\gamma_{tt}e^{2k\gamma}d\theta\\ &\quad+\int_{\mathbb{S}^{n-1}} e^{-n\gamma}\cdot 2^{-k}\tbinom{n}{k}\left(\frac{2l}{2l-n}\right)(1-\gamma_t^2)^{l-1}\gamma_{tt}e^{2l\gamma}d\theta\\ &\quad+\int_{\mathbb{S}^{n-1}} \eta(t)\cdot e^{(2k-n)\gamma(t)} + \eta(t)\cdot e^{(2l-n)\gamma(t)}d\theta\end{aligned}$$

$$
\begin{aligned}
= \;& \int_{\mathbb{S}^{n-1}} e^{-n\gamma} \cdot 2^{-k} \tbinom{n}{k} \left(\frac{-2n}{2k-n} \right) \\
& \quad \times \left[\frac{k}{n} \left(1-\gamma_t^2\right)^{k-1} \gamma_{tt} + \frac{n-2k}{2n} \left(1-\gamma_t^2\right)^k \right] e^{2k\gamma} d\theta \\
& - \int_{\mathbb{S}^{n-1}} e^{-n\gamma} \cdot 2^{-k} \tbinom{n}{k} \cdot \left(1-\gamma_t^2\right)^k e^{2k\gamma} d\theta \\
& + \int_{\mathbb{S}^{n-1}} e^{-n\gamma} \cdot 2^{-k} \tbinom{n}{k} \left(\frac{2n}{2l-n} \right) \\
& \quad \times \left[\frac{l}{n} \left(1-\gamma_t^2\right)^{l-1} \gamma_{tt} + \frac{n-2l}{2n} \left(1-\gamma_t^2\right)^l \right] e^{2l\gamma} d\theta \\
& + \int_{\mathbb{S}^{n-1}} e^{-n\gamma} \cdot 2^{-k} \tbinom{n}{k} \cdot \left(1-\gamma_t^2\right)^l e^{2l\gamma} d\theta \\
& + \eta(t) \cdot e^{(2k-n)\gamma(t)} + \eta(t) \cdot e^{(2l-n)\gamma(t)} \\
= \;& \int_{\mathbb{S}^{n-1}} e^{(2k-n)\gamma} \cdot 2^{-k} \tbinom{n}{k} \cdot \left(\frac{2n}{2l-n} - \frac{2n}{2k-n} \right) \\
& \quad \times \left[\left(1-\gamma_t^2\right)^{(k-1)} \left(\frac{k}{n} \gamma_{tt} + \frac{n-2k}{2n} (1-\gamma_t^2) \right) \right] d\theta \\
& + \int_{\mathbb{S}^{n-1}} 2^{-k} \tbinom{n}{k} \left[- \left(1-\gamma_t^2\right)^k e^{(2k-n)\gamma} + \left(1-\gamma_t^2\right) e^{(2l-n)\gamma} \right] d\theta \\
& + \eta(t) \cdot e^{(2k-n)\gamma(t)} + \eta(t) \cdot e^{(2l-n)\gamma(t)} \\
= \;& -F + \int_{\mathbb{S}^{n-1}} 2^{-k} \tbinom{n}{k} \\
& \quad \times \left[\left((1-\gamma_t^2)^k + \eta(t) \right) e^{(2k-n)\gamma} - \left((1-\gamma_t^2)^l + \eta(t) \right) e^{(2l-n)\gamma} \right] d\theta.
\end{aligned}
$$

Combining them together, we have

$$
\left[(1-\gamma_t^2)^k + \eta_3(t) \right] \cdot e^{(2k-n)\gamma} - \left[(1-\gamma_t^2)^l + \eta_4(t) \right] \cdot e^{(2l-n)\gamma} = h
$$

for some constant h, where $\eta_3(t)$ and $\eta_4(t)$ have the decay rate $O\left(e^{-2(1-\delta)t}\right)$ as $t \to \infty$. Combing them together, we have (3.2.2) in all cases. □

3.3 Asymptotic behavior of singular solutions

Now we are ready to prove Theorem 1.0.2. We first establish the following claim, which help us to exclude the $2k < n$ & $h < 0$ case, then we apply Theorem B to (3.2.1) and (3.2.2) to describe asymptotic behavior of singular solutions.

mathn 3.3.1. Let $u(x)$ be a positive solution to (3.1.1) in the Γ_k^+ class in a punctured ball $B_R\backslash\{0\}$, then

$$\liminf_{x\to 0}|x|^{\frac{n-2}{2}}u(x)>0\Longrightarrow 2k<n \text{ and } h>0.$$

Especially in the case of $2k<n$, we always have $h\geq 0$, and

- If $h>0$, then $\liminf_{x\to 0}|x|^{\frac{n-2}{2}}u(x)>0$ and for some $\epsilon>0$, we have

  $$1-\gamma_t^2\geq\epsilon$$

  for all sufficiently large t.

- If $h=0$, then $\gamma_t(t)>0$ for t large, and $\lim_{t\to\infty}\gamma(t)=\infty$.

Proof. If $\liminf_{x\to 0}|x|^{\frac{n-2}{2}}u(x)>0$, then by (3.1.5) and (3.1.6), we have

$$-C\leq\gamma(t)\leq C.$$

So by a rescaling and compactness argument on the translations to $\gamma(t)$, with the help of (3.1.8) and (3.2.2), One produces a limiting $\hat{\gamma}(t)$ which exists and is bounded for all $t\in R$ and satisfies the equation (2.2.1) with

$$e^{(2k-n)\hat{\gamma}(t)}(1-\hat{\gamma}_t^2(t))^k-e^{(2l-n)\hat{\gamma}(t)}(1-\hat{\gamma}_t^2)^l=h$$

for some h. But according to Theorem 2.2.1, the classification result says that no bounded admissible solution of (2.2.1) exists for all $t\in\mathbb{R}$ with $h\leq 0$ or $2k\geq n$, which implies $h>0$ and $2k<n$. And from (3.2.2), there is some $\epsilon>0$ depending on h such that $1-\gamma_t^2>\epsilon$ for all sufficiently large t.

On the other hand, if $\liminf_{x\to 0}|x|^{\frac{n-2}{2}}u(x)=0$ then

$$\lim_{x\to 0}\sup\gamma(t)=\infty. \tag{3.3.1}$$

So when $2k<n$, it follows from (3.1.8) and (3.2.2) that $h=0$. Conversely if $h=0$, it follows from above that $\liminf_{x\to 0}|x|^{\frac{n-2}{2}}u(x)=0$. In addition, it follows (3.2.2) that, for sufficiently large t, $\gamma_t(t)=0$ can occur only near $\gamma(t)=0$. Together with (3.3.1), we see that $\gamma_t(t)>0$ for sufficiently large t and $\lim_{t\to\infty}\gamma(t)=\infty$. $\square$

The above claim leads to:

Proof of Theorem 1.0.2. By Claim 3.3.1, we just need to handle the four cases slightly differently: Case (a), $2k < n$ and $h > 0$; Case (b), $h = 0$; Case (c), $2k > n$ and $h \neq 0$; Case (d), $2k = n$, and $h \neq 0$. The proof of case (a) needs the help of Theorem B.

Case (a) $2k < n$ and $h > 0$: For each h, $\gamma(t)$ is bounded, thus equation (3.2.2) is transformed into

$$e^{(2k-n)\gamma}(1-\gamma_t^2)^k - e^{(2l-n)\gamma}(1-\gamma_t^2)^l = h + \eta(t)$$

where $\eta(t) = O(e^{-2(1-\delta)t})$ as $t \to \infty$, so h here is subject to the same upper bound as in Theorem 1.0.1:

$$h \le h^* = \left(\frac{n-2k}{n-2l}\right)^{\frac{n-2k}{2}} - \left(\frac{n-2k}{n-2l}\right)^{\frac{n-2l}{2}}.$$

Let

$$H(x,y) = h - \left[e^{(2k-n)y}(1-x^2)^k - e^{(2l-n)y}(1-x^2)^l\right].$$

For each $h \in (0, h^*]$, we have proved that $\gamma(t)$ satisfies that $|\gamma(t)|$ and $|\gamma'(t)|$ are bounded over $t \in [0, \infty)$, and by (3.2.1) and (3.2.2), it is a solution to the following condition

$$\gamma''(t) = f(\gamma'(t), \gamma(t)) + e_1(t) \qquad t \ge 0$$

$$|H(\gamma'(t), \gamma(t))| \le e_2(t) \qquad t \ge 0$$

where f is a locally Lipschitz function and $e_1(t), e_2(t) = O(e^{-2(1-\delta)t})$. Furthermore,

- When the solution of $H(x,y) = 0$ is a point , we take $l = 2$ to satisfy (3.1.12) .
- When the solution of $H(x,y) = 0$ is not a point, we take $l = 1$ to satisfy (3.1.13).

So we can apply Theorem B and (3.1.3) to get Theorem 1.0.2.

Case (b) $h = 0$: Using $\lim_{t\to\infty} \gamma(t) = \infty$ back into (3.2.2), which now takes the form

$$e^{2k\gamma}\left\{(1-\gamma_t^2)^k + \eta_3(t)\right\} - e^{2l\gamma}\left\{(1-\gamma_t^2)^l + \eta_4(t)\right\} = 0,$$

we see that $1-\gamma_t^2(t) := \eta(t) \to 0$ as $t \to \infty$. Since $\gamma_t(t) > 0$ for sufficiently large t, we conclude that $1-\gamma_t(t) \to 0$ as $t \to \infty$. As a consequence, $\gamma(t) \geq (1-\epsilon)t + \gamma_0$ for large t and some $\epsilon > 0$ small and γ_0. This would imply through (3.2.2) that

$$|\eta(t)| \leq Ce^{-\frac{2(1-\delta)}{k}t}$$

for some constant $C > 0$ and for large t, or as $t \to \infty$,

$$\eta^l \gg \eta^k \gg O(e^{-2(1-\delta)t}).$$

However (3.2.2) shows

$$\begin{aligned} \eta^{k-l} &\approx e^{2(l-k)\gamma(t)} \\ \eta &\approx e^{-2\gamma(t)}, \end{aligned}$$

a contradiction. Finally, we have

$$|\gamma_t(t) - 1| = |\sqrt{1-\eta(t)} - 1| \leq Ce^{-\frac{2(1-\delta)}{k}t},$$

from which we conclude that

$$\gamma(t) - t = \tau + O(e^{-\frac{2(1-\delta)t}{k}})$$

for some τ as $t \to \infty$. Note that $\xi_0(t) = \ln\cosh(t)$, the solution to (2.2.1) with $h = 0$, to which (3.2.1) is a perturbation, satisfies $\xi_0(t) = t - \ln 2 + O(e^{-2t})$. Therefore, using also (3.1.6),

$$\omega(t,\theta) = \gamma(t) + \hat{\omega}(t,\theta) = \xi_0(t + \tau + \ln 2) + O(e^{-\frac{2(1-\delta)t}{k}})$$

as $t \to \infty$, which is (1.0.5). Furthermore,

$$\begin{aligned} u(x) &= e^{-\frac{n-2}{2}(\omega(t,\theta)-t)} = e^{-\frac{n-2}{2}\left(\xi_0(t+\tau+\ln 2)-t+O\left(e^{-\frac{2(1-\delta)t}{k}}\right)\right)} \\ &= u^*(|x|)e^{O\left(e^{-\frac{2(1-\delta)t}{k}}\right)} \\ &= u^*(|x|)\left(1 + O\left(e^{-\frac{2(1-\delta)t}{k}}\right)\right) \end{aligned}$$

where

$$\begin{aligned} u^*(|x|) &= e^{-\frac{n-2}{2}(\xi_0(t+\tau+\ln 2)-t)} \\ &= \left(\frac{4e^\tau}{4e^{2\tau} + |x|^2}\right)^{\frac{n-2}{2}} \end{aligned}$$

is a positive radial solution to (3.1.1) on $\mathbb{R}^n\backslash\{0\}$. We also find in this case that

$$\lim_{x\to 0} u(x) = e^{-\frac{n-2}{2}\tau} := u(0) > 0$$

exists, with

$$\begin{aligned} |u(x)-u(0)| &\leq |u(0)||e^{-\frac{n-2}{2}\left(\hat{\omega}(t,\theta)+O(e^{-\frac{2(1-\delta)t}{k}})\right)} - 1| \\ &\leq Ce^{-\frac{2(1-\delta)t}{k}} \\ &\leq C|x|^{\frac{2(1-\delta)}{k}}. \end{aligned}$$

<u>**Case (c)** $2k > n$ **and** $h \neq 0$**:**</u> By Claim 3.3.1, $\liminf_{x\to 0} |x|^{\frac{n-2}{2}} u(x) = 0$ which implies

$$\lim_{t\to\infty} \gamma(t) = \infty \tag{3.3.2}$$

Equation (3.2.2) implies that, for large t, $\gamma'(t) = 0$ can occur only when $\gamma(t)$ is near certain finite value. Together with (3.3.2), this implies

$$\gamma_t > 0 \qquad \text{for } t \text{ large}$$

which, together with (3.2.2), implies that $(1-\gamma_t^2)^k \to 0$ as $t \to \infty$. Then the conclusion of (1.0.5) is proved in an almost identical way as was done above for the $h = 0$ case.

<u>**Case (d)** $2k = n$ **and** $h \neq 0$**:**</u> First from Claim 3.3.1,

$$\lim_{t\to\infty} \gamma(t) = \infty.$$

In [7], they proved: If $e^{-2\omega(t,\theta)}(dt^2 + d\theta^2) \in \Gamma_2^+$ for all $\theta \in \mathbb{S}^{n-1}$ at some t, then

$$1 - \gamma_t^2 + \frac{1}{|S^{n-1}|}\int_{S^{n-1}} |\nabla\hat{\omega}|^2 d\theta \geq 0,$$

and in (3.2.2), we have $\forall\epsilon{>}0$,

$$\begin{cases} -\epsilon < (1-\gamma_t^2)^k + \eta_3 < 1+\epsilon \\ 0 < e^{-(2l-n)\gamma} < \epsilon \end{cases} \quad \text{when } t \text{ is large enough} \quad .$$

Thus

$$0 < h \leq 1.$$

Moreover, if $h = 1$, then when t is large enough, $(1-\gamma_t^2)^k + \eta_3 > \frac{1}{2}$ by (3.2.2), which implies $(1-\gamma_t^2)^l > \frac{1}{4}$, thus

$$(1-\gamma_t^2)^l + \eta_4 > 0.$$

Still using (3.2.2),

$$(1-\gamma_t^2)^k + \eta_3 > 1,$$

from which we obtain

$$\gamma_t = O(e^{-(1-\delta)t})$$

and

$$\lim_{t\to\infty} \gamma(t) = \gamma(1) + \int_1^\infty \gamma_t dt < \infty,$$

a contradiction. So h must satisfy

$$0 < h < 1.$$

Arguing as before, we can also show $\gamma_t > 0$ when t is large enough. Then (3.2.2) implies that $(1-\gamma_t^2)^k \to h$ as $t \to \infty$; and $e^{-n\gamma(t)} = O(e^{-\alpha t})$ as $t \to \infty$ for some $\alpha > 0$ depending on $0 < h < 1$. Now with $\eta(t) := 1 - \gamma_t^2(t)$, we find

$$\begin{aligned} \eta^k(t) &= h + e^{(2l-n)\gamma(t)} \left((1-\gamma_t^2)^l + \eta_4(t) \right) - \eta_3(t) \\ &= h + O(e^{-\alpha t}) \end{aligned}$$

as $t \to \infty$, and

$$\gamma_t(t) = \sqrt{1-\eta(t)} = \sqrt{1-\sqrt[k]{h}} + O(e^{-\alpha t})$$

which implies that $\gamma(t) = \sqrt{1-\sqrt[k]{h}}t + \gamma_0 + O(e^{-\alpha t})$, for some γ_0. Similarly, $\xi_h(t)$ satisfies $\xi_h(t) = \sqrt{1-\sqrt[k]{h}}t + \xi_0 + O(e^{-\alpha t})$ for some ξ_0. Thus for some τ, we have

$$\gamma(t) = \xi_h(t+\tau) + O(e^{-\alpha t})$$

and

$$u(x) = e^{-\frac{n-2}{2}(\omega(t,\theta)-t)} = e^{-\frac{n-2}{2}(\gamma(t)-t+\hat{\omega}(t,\theta))},$$

from which we find that

$$|x|^{\frac{n-2}{2}\left(1-\sqrt{1-\sqrt[k]{h}}\right)} u(x) = e^{-\frac{n-2}{2}\left(\gamma(t)-\sqrt{1-\sqrt[k]{h}}t+\hat{\omega}(t,\theta)\right)}$$

extends to a $C^\alpha(B_R)$ positive function for some $\alpha > 0$.

□

Chapter 4

Analysis of the linearized operator

4.1 Linearization of the conformal quotient equation

In this section we provide a proof of Theorem 1.0.3 by an analysis of the linearized operator of (1.0.8) at a radial solution $\xi(t)$ to (1.0.8), where c is normalized to be $2^{l-k}\binom{n}{k}/\binom{n}{l}$. Clearly $\xi(t)$ satisfies

$$(1-\xi_t^2)^{k-1}\left[\frac{k}{n}\xi_{tt}+(\frac{1}{2}-\frac{k}{n})(1-\xi_t^2)\right]e^{2k\xi}=(1-\xi_t^2)^{l-1}\left[\frac{l}{n}\xi_{tt}+(\frac{1}{2}-\frac{l}{n})(1-\xi_t^2)\right]e^{2l\xi}.$$

Let $P(\xi,\xi_t):=\left(\frac{e^{-2\xi}}{1-\xi_t^2}\right)^{k-l}$, and we use P to denote the function $P(\xi,\xi_t)$ all through the paper and we obtain

$$\begin{aligned}
\xi_{tt} &= \left\{\frac{\left(\frac{e^{-2\xi}}{1-\xi_t^2}\right)^{k-l}\cdot(\frac{n}{2k}-\frac{l}{k})-(\frac{n}{2k}-1)}{1-\frac{l}{k}\cdot\left(\frac{e^{-2\xi}}{1-\xi_t^2}\right)^{k-l}}\right\}(1-\xi_t^2) \\
&= \left(\frac{P(\frac{n}{2k}-\frac{l}{k})-(\frac{n}{2k}-1)}{1-P\frac{l}{k}}\right)(1-\xi_t^2) \\
&= \left(1+\frac{n}{2k}\cdot\frac{P-1}{1-P\frac{l}{k}}\right)(1-\xi_t^2). \qquad (4.1.1)
\end{aligned}$$

Since $\xi(t)$ satisfies $1-\xi_t^2>0$ and the first integral

$$e^{(2k-n)\xi}(1-\xi_t^2)^k-e^{(2l-n)\xi}(1-\xi_t^2)^l=h,$$

P must satisfy

$$\begin{cases}0<P<1 & \text{if } h>0\\ P=1 & \text{if } h=0.\end{cases}$$

Next we have the linearization of $\sigma_k(A_g)$ at $g = e^{-2\xi(t)}(dt^2 + d\theta^2)$ (for reference see [43]),

$$\begin{aligned} L_\xi(\phi) &= \frac{(1-|\xi_t^2|^2)^{k-2}}{2^{k-2}}\binom{n-1}{k-1} \\ &\quad \times \left[A(t)\phi_{tt}(t,\theta) + B(t)\phi_t(t,\theta) + C(t)\Delta_\theta\phi(t,\theta)\right], \end{aligned}$$

where

$$\begin{aligned} A(t) &= \frac{1-|\xi_t|^2}{2} \\ B(t) &= -\xi_t(t)\left[(k-1)\xi_{tt}(t) + \frac{n-2k}{2}(1-|\xi_t|^2)\right] \\ C(t) &= \frac{k-1}{n-1}\xi_{tt}(t) + \frac{n-2k+1}{n-1}\cdot\frac{1-|\xi_t|^2}{2}. \end{aligned}$$

We use $\dot{L}(\phi)$ to denote the linearization of $\sigma_l(A_g)$ at $g = e^{-2\xi(t)}(dt^2 + d\theta^2)$ while $\dot{B}(t)$ and $\dot{C}(t)$ is defined correspondingly as well.

When c is normalized to be $2^{l-k}\binom{n}{k}/\binom{n}{l}$, the linearization of PDE (1.0.8) at $\xi(t)$ is

$$L_\xi(\phi) + 2k\sigma_k\phi - (ce^{-2(k-l)\xi}\dot{L}_\xi(\phi) + 2l\sigma_k\phi) = 0.$$

A direct computation shows

$$\begin{aligned} ce^{-2(k-l)\xi}\dot{L}_\xi(\phi) &= \frac{2^{-k}\binom{n}{k}}{2^{-l}\binom{n}{l}}\cdot e^{-2(k-l)\xi}\cdot\frac{(1-|\xi_t|^2)^{l-1}}{2^{l-1}}\binom{n-1}{l-1} \\ &\quad \times\left[\phi_{tt} + \frac{\dot{B}(t)}{A(t)}\phi_t + \frac{\dot{C}(t)}{A(t)}\Delta_\theta\phi(t,\theta)\right] \\ &= \frac{(1-|\xi_t|^2)^{k-1}}{2^{k-1}}\binom{n-1}{k-1}\cdot\frac{l}{k}\cdot\left(\frac{e^{-2\xi}}{1-|\xi_t|^2}\right)^{k-l} \\ &\quad \times\left[\phi_{tt} + \frac{\dot{B}(t)}{A(t)}\phi_t + \frac{\dot{C}(t)}{A(t)}\Delta_\theta\phi(t,\theta)\right], \end{aligned}$$

and also

$$\begin{aligned} \sigma_k\phi &= 2^{1-k}\binom{n}{k}(1-\xi_t^2)^{k-1}\left[\frac{k}{n}\xi_{tt} + (\frac{1}{2} - \frac{k}{n})(1-\xi_t^2)\right]\phi \\ &= \frac{(1-|\xi_t|^2)^{k-1}}{2^{k-1}}\binom{n-1}{k-1}\left[\xi_{tt} + (\frac{n}{2k} - 1)(1-|\xi_t^2|)\right]\phi \\ &= \frac{(1-|\xi_t|^2)^{k-1}}{2^{k-1}}\binom{n-1}{k-1}\left[\frac{n}{2k}\left(\frac{P-1}{1-P\frac{l}{k}} + 1\right)(1-\xi_t^2)\right]\phi. \end{aligned}$$

Let $Y_j(\theta)$ be the normalized eigenfunctions of Δ_θ on $L^2(\mathbb{S}^{n-1})$ and λ_j be the eigenvalues of Δ_θ on $L^2(\mathbb{S}^{n-1})$ associated with $Y_j(\theta)$. Thus

$$\lambda_0 = 0, \;\; \lambda_1 = \cdots = \lambda_n = n-1, \;\; \lambda_j \geq 2n, \; \text{for } j > n.$$

If we take the projection of $\phi(t,\theta)$ into spherical harmonics:

$$\phi(t,\theta) = \sum_j \phi_j(t) Y_j(\theta),$$

then $\phi_j(t)$ satisfies the ODE

$$\begin{aligned} F_{\lambda_j}(\phi_j(t)) : &= \left\{ \phi_j'' + \frac{B(t)}{A(t)}\phi_j' + \left[-\lambda_j \frac{C(t)}{A(t)} + n\left(\frac{P-1}{1-P\frac{l}{k}} + 1 \right)(1-\xi_t^2) \right] \phi_j \right\} \\ &\quad - \frac{l}{k} \cdot P \left\{ \phi_j'' + \frac{\dot{B}(t)}{A(t)}\phi_j' + \left[-\lambda_j \frac{\dot{C}(t)}{A(t)} + \frac{n}{P}\left(\frac{P-1}{1-P\frac{l}{k}} + 1 \right)(1-\xi_t^2) \right] \phi_j \right\} \\ &= 0 \end{aligned} \tag{4.1.2}$$

4.2 Decay rate

Now we try to figure out the decay rate of the solutions of (4.1.2). Note that in $F_{\lambda_j}(\phi_j(t))$, the term $\frac{B(t)}{A(t)}\phi_j' - \frac{l}{k} \cdot \frac{\dot{B}(t)}{A(t)}\phi_j'$ is

$$\xi_t \left[\left(1 - (n-1)\frac{C(t)}{A(t)} \right) - \frac{l}{k} \cdot P \left(1 - (n-1)\frac{\dot{C}(t)}{A(t)} \right) \right] \phi_j'$$

and when $2k < n$, it can always alter its sign as $t \to \infty$, and hard to be controlled in computation. In [43], $l = 0$ and they introduced an auxiliary function $V(t)$ to remove this term involving ϕ_j', more specifically, they obtained

$$V(t) F_{\lambda_j}\left[V^{-1}\phi_j \right] = \phi_{tt} + E(t)\phi(t)$$

to go through the computation. But when $l \neq 0$, it is hard to recover the anticipated results by their method, however, we observe (4.2.1) and with the estimate $|\xi_t| < 1$ we have

$$\begin{aligned} &\xi_t \left[\left(1 - (n-1)\frac{C(t)}{A(t)} \right) - \frac{l}{k} \cdot P \left(1 - (n-1)\frac{\dot{C}(t)}{A(t)} \right) \right] \\ < \;& \left(1 - (n-1)\frac{C(t)}{A(t)} \right) - \frac{l}{k} \cdot P \left(1 - (n-1)\frac{\dot{C}(t)}{A(t)} \right), \end{aligned}$$

and this upper bound help us to have the following properties for the decay rate of solutions:

Fact 4.2.1. *When $2k \leq n$ and $h \geq 0$*

$$F_{\lambda_j}[e^{\pm t}] < 0$$

for all $\lambda_j \geq 2n$ and $t \in \mathbb{R}^+$.

Proof. First we state the following claim and provide a proof in the end,

$$\left[1-(n-1)\frac{C(t)}{A(t)}\right] - \frac{l}{k}\cdot P\left[1-(n-1)\frac{\dot{C}(t)}{A(t)}\right] < 0. \tag{4.2.1}$$

By this we have

$$\begin{aligned}
F_{\lambda_j}[e^{-t}] &= e^{-t}\Bigg\{1-\frac{B(t)}{A(t)} \\
&\quad + \left[-\lambda_j\frac{C(t)}{A(t)} + n\left(\frac{P-1}{1-P\frac{l}{k}}+1\right)(1-\xi_t^2)\right]\Bigg\} \\
&\quad - e^{-t}\cdot\frac{l}{k}\cdot P\Bigg\{1-\frac{\dot{B}(t)}{A(t)} \\
&\quad + \left[-\lambda_j\frac{\dot{C}(t)}{A(t)} + \frac{n}{P}\left(\frac{P-1}{1-P\frac{l}{k}}+1\right)(1-\xi_t^2)\right]\Bigg\} \\
&= e^{-t}\Bigg\{1-\xi_t\left(1-(n-1)\frac{C(t)}{A(t)}\right) \\
&\quad + \left[-\lambda_j\frac{C(t)}{A(t)} + n\left(\frac{P-1}{1-P\frac{l}{k}}+1\right)(1-\xi_t^2)\right]\Bigg\} \\
&\quad - e^{-t}\cdot\frac{l}{k}\cdot P\Bigg\{1-\xi_t\left(1-(n-1)\frac{\dot{C}(t)}{A(t)}\right) \\
&\quad + \left[-\lambda_j\frac{\dot{C}(t)}{A(t)} + \frac{n}{P}\left(\frac{P-1}{1-P\frac{l}{k}}+1\right)(1-\xi_t^2)\right]\Bigg\} \\
&\leq e^{-t}\Bigg\{1-\left(1-(n-1)\frac{C(t)}{A(t)}\right) \\
&\quad + \left[-\lambda_j\frac{C(t)}{A(t)} + n\left(\frac{P-1}{1-P\frac{l}{k}}+1\right)\right]\Bigg\} \\
&\quad - e^{-t}\cdot\frac{l}{k}\cdot P\Bigg\{1-\left(1-(n-1)\frac{\dot{C}(t)}{A(t)}\right) \\
&\quad + \left[-\lambda_j\frac{\dot{C}(t)}{A(t)} + \frac{n}{P}\left(\frac{P-1}{1-P\frac{l}{k}}+1\right)\right]\Bigg\}
\end{aligned}$$

If we replace λ_j by $2n$ and use $\mathcal{F}_{2n}(t)$ to denote the right side of the last inequality, then the above expression becomes

$$\begin{aligned}
F_{2n}[e^{-t}] \leq \mathcal{F}_{2n}(t) \quad := \quad & -e^{-t}\left\{\left[(n+1)\frac{C(t)}{A(t)} - n\left(\frac{P-1}{1-P\frac{l}{k}}+1\right)\right]\right. \\
& \left. -\frac{l}{k}\cdot P\left[(n+1)\frac{\dot{C}(t)}{A(t)} - \frac{n}{P}\left(\frac{P-1}{1-P\frac{l}{k}}+1\right)\right]\right\} \\
= \quad & -e^{-t}(n+1)\left\{\left[\frac{C(t)}{A(t)} - \frac{n}{n+1}\left(\frac{P-1}{1-P\frac{l}{k}}+1\right)\right]\right. \\
& \left. -\frac{l}{k}\cdot P\left[\frac{\dot{C}(t)}{A(t)} - \frac{n}{(n+1)P}\left(\frac{P-1}{1-P\frac{l}{k}}+1\right)\right]\right\} \qquad (4.2.2)
\end{aligned}$$

Let $q(\xi,\xi_t) = \frac{P-1}{1-P\frac{l}{k}}$ and use q to denote $q(\xi,\xi_t)$ all through the paper, then $P = \frac{q+1}{1+q\frac{l}{k}}$ and obviously $-1 < q \leq 0$ since $0 < P \leq 1$. By (4.1.1) we have

$$\begin{aligned}
\frac{C(t)}{A(t)} &= \frac{1}{A(t)}\cdot\left(\frac{k-1}{n-1}\xi_{tt} + \frac{n-2k+1}{n-1}\cdot\frac{1-\xi_t^2}{2}\right) \\
&= \frac{1}{A(t)}\cdot\left[\frac{k-1}{n-1}\left(1+\frac{n}{2k}\cdot\frac{P-1}{1-P\frac{l}{k}}\right)(1-\xi_t^2) + \frac{n-2k+1}{n-1}\cdot\frac{1-\xi_t^2}{2}\right] \\
&= 1 + \frac{k-1}{n-1}\cdot\frac{n}{k}\cdot\frac{P-1}{1-P\frac{l}{k}} \\
&= 1 + \frac{k-1}{n-1}\cdot\frac{n}{k}\cdot q.
\end{aligned}$$

Now we plug it into (4.2.2) to get

$$\begin{aligned}
F_{2n}\left[e^{-t}\right] \leq \mathcal{F}_{2n}(t) \quad = \quad & -e^{-t}(n+1)\left\{\left[\left(1+\frac{k-1}{n-1}\cdot\frac{n}{k}\cdot q\right) - \frac{n}{n+1}(q+1)\right]\right. \\
& \left. -\frac{l}{k}\left[\left(1+\frac{l-1}{n-1}\cdot\frac{n}{k}\cdot q\right)P - \frac{n}{(n+1)}(1+q)\right]\right\} \\
= \quad & -e^{-t}(n+1)\frac{1}{1+\frac{l}{k}q}\left\{1-\frac{l}{k}+\frac{(k-1)n}{(n-1)k}q(1+\frac{l}{k}q)\right. \\
& \left. -\frac{n}{n+1}(q+1)(1-\frac{l}{k})(1+\frac{l}{k}q) - \frac{nl(l-1)}{k^2(n-1)}q(q+1)\right\} \\
= \quad & -e^{-t}\frac{n+1}{1+\frac{l}{k}q}\left\{\left(\frac{nl(k-l)}{k^2(n-1)} - \frac{n(k-l)l}{(n+1)k^2}\right)q^2\right. \\
& \left. +\left(\frac{n(k-l)(k+l-1)}{k^2(n-1)} - \frac{n(k-l)(k+l)}{(n+1)k^2}\right)q + \frac{1}{n+1}\left(1-\frac{l}{k}\right)\right\} \\
= \quad & -e^{-t}\frac{k-l}{k(1+\frac{l}{k}q)}\left\{\frac{2nl}{k(n-1)}q^2 + \frac{n\left(2(k+l)-n-1\right)}{k(n-1)}q + 1\right\}.
\end{aligned}$$

Next we consider two cases:

- If $2(k+l)-n-1 \leq 0$, then the parabola

$$\frac{2nl}{k(n-1)}q^2 + \frac{n\left(2(k+l)-n-1\right)}{k(n-1)}q + 1 \geq 1$$

for all $q \in (-1,0]$, which implies that $F_{2n}\left[e^{-t}\right] \leq \mathcal{F}_{2n}(t) < 0$.

- If $2(k+l)-n-1 > 0$, we let $a = \frac{n}{2} - k$, and $a \geq 0$ by assumption, then the discriminant of the parabola is

$$\begin{aligned}
\Delta &= \left(\frac{n(2(k+l)-n-1)}{k(n-1)}\right)^2 - 4 \cdot \frac{2nl}{k(n-1)} \\
&= \frac{n}{k^2(n-1)^2}\Big(n\left(2(k+l)-n-1\right)^2 - 8lk(n-1)\Big) \\
&= \frac{n}{k^2(n-1)^2}\big(n(2l-2a-1)^2 - 4l(n-2a)(n-1)\big) \\
&\leq \frac{n(2l-2a-1)}{k^2(n-1)^2}\left(n(2l-2a-1) - 2(n-2a)(n-1)\right) \\
&\leq \frac{n(2l-2a-1)}{k^2(n-1)^2}\left(n(n-4a-1) - 2(n-2a)(n-1)\right) \\
&< \frac{n(2l-2a-1)}{k^2(n-1)^2}(-2an-2a) \\
&< 0.
\end{aligned}$$

We still have

$$\frac{2nl}{k(n-1)}q^2 + \frac{n\left(2(k+l)-n-1\right)}{k(n-1)}q + 1 > 0,$$

and thus

$$F_{2n}[e^{-t}] \leq \mathcal{F}_{2n}(t) < 0.$$

Also we find the coefficient of λ_j in $F_{\lambda_j}\left[e^{-t}\right]$ is

$$-e^{-t}\left(\frac{C(t)}{A(t)} - \frac{l}{k}P\frac{\dot{C}(t)}{A(t)}\right) = \frac{\mathcal{F}_{2n}(t) - n(1-\frac{l}{k})(q+1)e^{-t}}{n+1} < 0,$$

which implies

$$F_{\lambda_j}\left[e^{-t}\right] < F_{2n}\left[e^{-t}\right] < 0,$$

similarly we have $F_{\lambda_j}\left[e^{t}\right] < 0$.

Finally it remains to verify (4.2.1), indeed,

$$
\begin{aligned}
& \left[1-(n-1)\frac{C(t)}{A(t)}\right]-\frac{l}{k}\cdot P\left[1-(n-1)\frac{\dot{C}(t)}{A(t)}\right] \\
= & \left[1-(n-1)\left(1+\frac{k-1}{n-1}\cdot\frac{n}{k}q\right)\right]-\frac{l}{k}\cdot\frac{q+1}{1+q\frac{l}{k}}\left[1-(n-1)\left(1+\frac{l-1}{n-1}\cdot\frac{n}{k}q\right)\right] \\
= & \frac{1}{k(k+ql)}\{[(2-n)k-(k-1)nq](k+ql)-l(q+1)[(2-n)k-(l-1)nq\} \\
= & \frac{1}{k(k+ql)}\left\{(2-n)k^2+((2-n)kl-(k-1)nk)q-(k-1)nlq^2\right. \\
& \left.-(2-n)kl+((l-1)nl-(2-n)kl)q+(l-1)nlq^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{1}{k(k+ql)}\left\{(2-n)k(k-l)+(k-l)(1-k-l)nq+nl(l-k)q^2\right\} \\
= & \frac{l-k}{k(k+ql)}\left\{nlq^2+n(k+l-1)q+(n-2)k\right\},
\end{aligned}
$$

Since the axis of symmetry of parabola $\frac{l-k}{k}\left\{nlq^2+n(k+l-1)q+(n-2)k\right\}$ is

$$x=-\frac{n(k+l-1)}{2nl}$$

and obviously

$$-1>-\frac{n(k+l-1)}{2nl},$$

we have $\frac{l-k}{k}\left\{nlq^2+n(k+l-1)q+(n-2)k\right\}$ is decreasing when $q>-1$, thus

$$
\begin{aligned}
& \left[1-(n-1)\frac{C(t)}{A(t)}\right]-\frac{l}{k}\cdot P\left[1-(n-1)\frac{\dot{C}(t)}{A(t)}\right] \\
< & \frac{l-k}{k}\left\{nl(-1)^2+n(k+l-1)(-1)+(n-2)k\right\} \\
= & \frac{l-k}{k}\{n-2k\} \\
\leq & 0
\end{aligned}
$$

for all $q\in(-1,0]$. □

Fact 4.2.2. *When $\frac{n}{2}<k<n$ and $h>0$ we have*

$$F_{\lambda_j}\left[e^{-t}\right]<0$$

for all $\lambda_j\geq 2n$ and t large enough. There also exists some small $\lambda>0$ such that

$$F_{\lambda_j}\left[e^{\lambda t}\right]<0$$

for all $\lambda_j \geq 2n$ and t large enough.

Proof. First as $t \to \infty$, $\xi \to \infty$ and $1 - \xi_t^2 \to h^{\frac{1}{k}} e^{(\frac{n}{k}-2)\xi} \to 0$, and also

$$(1-\xi_t^2)^k = he^{(n-2k)\xi} + (1-\xi_t^2)^l e^{(2l-2k)\xi},$$

we get

$$P \;=\; \left(\frac{e^{-2\xi}}{1-\xi_t^2}\right)^{k-l} \to h^{-\frac{k-l}{k}} e^{-\frac{n(k-l)}{k}\xi} \to 0,$$

thus $q \to -1$ and

$$\begin{aligned} F_{2n}\left[e^{-t}\right] &\to e^{-t}\left\{1-\left[1-(n-1)\left(1+\frac{k-1}{n-1}\cdot\frac{n}{k}(-1)\right)\right]-2n\left(1+\frac{k-1}{n-1}\cdot\frac{n}{k}(-1)\right)\right\} \\ &= -e^{-t}(n+1)\cdot\frac{n-k}{k(n-1)} \\ &< 0, \end{aligned}$$

Moreover, as $t \to \infty$, $-\frac{C(t)}{A(t)} - P\frac{l}{k}\frac{\dot{C}(t)}{A(t)} \to \frac{k-n}{k(n-1)} < 0$, which implies

$$\begin{aligned} F_{\lambda_j}\left[e^{-t}\right] &< F_{2n}\left[e^{-t}\right] \\ &< 0 \end{aligned}$$

as $t \to \infty$. Similarly there exist some $\lambda > 0$ such that

$$F_{\lambda_j}\left[e^{\lambda t}\right] < 0$$

as $t \to \infty$. □

Using the same argument as in the proof of Proposition 2 in [43], we find out the the solution basis $\{\phi_0^+(t), \phi_0^-(t)\}$ of (4.1.2) for $\lambda_0 = 0$ and $\lambda_j = n-1$:

- $\lambda_0 = 0$ $\phi_0^+(t) = \partial_t \xi(t)$ $\phi_0^-(t) = \partial_h \xi_h(t)$
- $\lambda_j = n-1$ $\begin{cases} \phi_0^+(t) = [1-\partial_t\xi(t)]\, e^t, \quad \phi_0^-(t) = [1+\partial_t\xi(t)]\, e^{-t} & \text{if } h>0 \\ \phi_0^+(t) = \cosh^{-1}(t)\int_0^1 (\cosh(s))^n ds, \quad \phi_0^-(t) = \cosh^{-1}(t) & \text{if } h=0 \end{cases}$
  and when $h > 0$, $\phi_0^+(t)$ grows unbounded and $\phi_0^-(t)$ decays exponentially as $t \to \infty$ if $k < n$.

For $\lambda_j \geq 2n$, it is not easy to write down the solution basis ϕ_0^+ and ϕ_0^-. However by Fact 4.2.1 and 4.2.2 we can still use the comparison principle to prove that ϕ_0^+ grows unbounded and $\phi_0^-(t)$ decays faster than e^{-t} as $t \to \infty$ if $k < n$. Consequently we establish the following proposition. For details of the comparison principle proof see proposition 2 of [43].

Proposition 4.2.3. *For all solutions $\xi_h(t)$ to* (4.1.2) *with $h \geq 0$, $k < n$ and $j \geq 1$, the following holds:*

1. The basis of solution space to $F_{\lambda_j}[\phi] = 0$ contains a pair of linearly independent solutions on $\mathbb{R}$, of which one grows unbounded and the other decays exponentially as $t \to \infty$;

2. Any solution of $F_{\lambda_j}[\phi] = 0$ which is bounded for $\mathbb{R}^+$ must decay exponentially;

3. Any solution of $F_{\lambda_j}[\phi] = 0$ which is bounded for all of $\mathbb{R}$ must be identically 0;

4. Any non-zero solution of $F_{\lambda_j}[\phi] = 0$ which is bounded for all of $\mathbb{R}^+$ must be unbounded on $\mathbb{R}^-$.

While the asymptotic behavior of the solution basis is sufficient for providing a proof for Theorem 1.0.2, Theorem 1.0.3 requires some more detailed knowledge about the linearized operator to (1.0.8). More specifically, the decay rates of bounded solutions to $F_j[\phi] = 0$ on $\mathbb{R}^+$ need to be faster than e^{-t} when $\lambda_j \geq 2n$. In the case $2k < n$ and $h > 0$, L_j is an ordinary differential operator with periodic coefficient, so by Floquent Theory, it has a set of well defined characteristic roots which give the exponential decay/grow rates to solution ϕ of $F_j[\phi] = 0$ on $\mathbb{R}$, Indeed Theorem 5.1 in Chap. 3 of [21] which applied to F_j implies that $F_j[\phi] = 0$ has a set of fundamental solutions in the form of $e^{\rho_j t} p_1(t)$ and $e^{-\rho_j t} p_2(t)$ for some periodic functions $p_1(t)$ and $p_2(t)$, where $\rho_j \neq 0$. When $h = 0$, $\xi(t) = \ln\cosh(t)$, and (4.1.2) take the form of

$$F_j[\phi_j] = \phi_j'' + (2-n)\tanh(t)\phi_j' + \left[-\lambda_j + n\cosh^{-2}(t)\right]\phi_j, \tag{4.2.3}$$

so a similar notion for characteristic roots can be defined, which is also the case when $2k = n$. Fact 4.2.1 implies the following

Lemma 4.2.4. *When $2k \le n$ and $h \ge 0$, there is a $\beta_* > 1$ such that for all $\lambda_j \ge 2n$, the associated ρ_j satisfies $\rho_j \ge \beta_*$.*

4.3 Expansion in terms of Wronskian

With the knowledge from the previous section, we can now establish

Proposition 4.3.1. *When $2k \le n$, and suppose that $\phi(t,\theta) \to 0$ as $t \to \infty$ uniformally in $\theta \in \mathbb{S}^{n-1}$ and also satisfies*

$$L_\xi(\phi) + 2k\sigma_k\phi - \left(ce^{-2(k-l)\xi}\dot{L}_\xi(\phi) + 2l\sigma_k\phi\right) = r(t,\theta) \tag{4.3.1}$$

for all $t > t_0$ and $\theta \in \mathbb{S}^{n-1}$. Suppose that for some $0 < \beta < \beta_*$ and $\beta \ne 1$, $|r(t,\theta)| \lesssim e^{-\beta t}$. Then there exist constants a_j for $j = 1, \cdots, n$ such that

$$|\phi(t,\theta) - \sum_{j=1}^{n} a_j e^{-t}(1+\xi_t(t))Y_j(\theta)| \lesssim e^{-\beta t}. \tag{4.3.2}$$

In fact, when $\beta_* \le \beta < \rho_{n+1}$, it continues to hold and when $\beta > \rho_{n+1}$,we will have

$$|\phi(t,\theta) - \sum_{j=1}^{n} a_j e^{-t}(1+\xi_t(t))Y_j(\theta)| \lesssim e^{-\rho_{n+1}t}; \tag{4.3.3}$$

when $\beta = 1$,

$$|\phi(t,\theta) - \sum_{j=1}^{n} a_j e^{-t}(1+\xi_t(t))Y_j(\theta)| \lesssim te^{-t}; \tag{4.3.4}$$

and when $\beta = \rho_{n+1}$,

$$|\phi(t,\theta) - \sum_{j=1}^{n} a_j e^{-t}(1+\xi_t(t))Y_j(\theta)| \lesssim te^{-\rho_{n+1}t}.$$

Proof. Define

$$\hat{\phi}(t,\theta) = \phi(t,\theta) - \sum_{j=0}^{n} \pi_j\left[\phi(t,\theta\right] Y_j(\theta)$$

where $\phi_j(t) := \pi_j\left[\phi(t,\theta)\right]$ is the L^2 orthogonal projection of $\phi(t,\theta)$ onto $\operatorname{span}\{Y_j(\theta)\}$. Then

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} \hat{\phi}(t,\theta)Y_j(\theta)d\theta &= \int_{\mathbb{S}^{n-1}} \nabla\hat{\phi}(t,\theta)\cdot\nabla Y_j(\theta)d\theta \qquad (4.3.5) \\ &= \int_{\mathbb{S}^{n-1}} \Delta_\theta Y_j(\theta)\hat{\phi}(t,\theta)d\theta = 0 \end{aligned}$$

for $j = 0, \cdots, n$. As a consequence,

$$\begin{cases} \int_{\mathbb{S}^{n-1}} \Delta_\theta \phi(t,\theta)\hat{\phi}(t,\theta)d\theta & = -\int_{\mathbb{S}^{n-1}} |\nabla_\theta \hat{\phi}(t,\theta)d\theta \\ \int_{\mathbb{S}^{n-1}} \phi_t(t,\theta)\hat{\phi}(t,\theta)d\theta & = \int_{\mathbb{S}^{n-1}} \hat{\phi}_t(t,\theta)\hat{\phi}(t,\theta)d\theta = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}^{n-1}} |\hat{\phi}(t,\theta)|^2 d\theta. \end{cases} \tag{4.3.6}$$

In the following we will prove separately the expected decays for $\hat{\phi}(t,\theta)$ and $\phi_j(t) := \pi_j[\phi(t,\theta)]$, for $j = 0, 1, \cdots, n$. We first estimate $\phi_j(t) = \pi_j[\phi(t,\theta)]$ for $j = 0, \cdots, n$. Multiplying both sides of (4.3.1) by

$$\left\{ \frac{(1-|\xi_t|^2)^{k-1}}{2^{k-1}} \binom{n-1}{k-1} \left(1 - \frac{l}{k}P\right) \right\}^{-1} Y_j(\theta),$$

and integrating over $\theta \in \mathbb{S}^{n-1}$, we obtain

$$\phi_j^{''}(t) + \Phi\phi_j^{'}(t) + \Psi\phi_j(t) = \hat{r}_j(t) \tag{4.3.7}$$

where

$$\begin{cases} \Phi = -\xi_t\left((n-2) + n\cdot\frac{P-1}{k-Pl}\left(\frac{k^2-Pl^2}{k-Pl} - 1\right)\right) \\ \Psi = 2n\left\{\frac{(n+1)l}{2k(n-1)}q^2 + \frac{(n+1)(k+l)-2n}{2(n-1)k}q + \frac{1}{2}\right\} - \frac{n}{1-P\frac{l}{k}}\left(\frac{P-1}{1-P\frac{l}{k}} + 1\right)\left(1-\frac{l}{k}\right)\xi_t^2 \\ \hat{r}_j(t) = \int_{\mathbb{S}^{n-1}} \hat{r}(t,\theta)Y_j(\theta)d\theta. \end{cases}$$

We will write out the details for the $h > 0$ case; the $h = 0$ has the same linearization of (4.2.3) as $\sigma_k = c$ case, so it was solved by [43]. For $j = 1, \cdots, n$, $\lambda_j = n-1$, and $\phi_1^-(t) := e^{-t}(1+\xi_t'(t))$, $\phi_1^+ := e^t(1-\xi_t'(t))$ form a basis of solutions to the homogeneous equation

$$\phi_j^{''}(t) + \Phi\phi_j^{'}(t) + \Psi\phi_j(t) = 0.$$

Since $\phi_j(t)$ is a solution to (4.3.4) and $\phi_j(t) \to 0$ as $t \to \infty$, by the variation of constant formula,

$$\phi_j(t) = c\phi_1^-(t) + \phi_1^-(t)\int_0^t \frac{\phi_1^+(s)\hat{r}_j(s)}{W_1(s)}ds + \phi_1^+(t)\int_0^t \frac{\phi_1^-(s)\hat{r}_j(s)}{W_1(s)}ds \tag{4.3.8}$$

for some constant c, where

$$W_1(s) = \phi_1^+(s)\phi_1^{-'}(s) - \phi_1^-(s)\phi_1^{+'}(s)$$

is the Wronskian of $\left\{\phi_1^-(t), \phi_1^+(t)\right\}$, and satisfies

$$W_1^{'}(s) = -\Phi(t)W_1(s).$$

Integrating this equation out, using

$$\Phi = -\xi_t\left((n-2) + n\cdot\frac{P-1}{k-Pl}\left(\frac{k^2-Pl^2}{k-Pl}-1\right)\right),$$

we find

$$\begin{aligned} W_1(t) &= (\text{cosst.})e^{\int_0^t \xi_s\left((n-2)+n\cdot\frac{P-1}{k-Pl}\left(\frac{k^2-Pl^2}{k-Pl}-1\right)\right)ds} \\ &= (\text{cosst.})e^{\int_0^t \xi_s\left((n-2)+n\left(\frac{P(k-l)}{k(k-Pl)}-\frac{1}{k}\right)\left(\frac{Pl(k-l)}{k-Pl}+(k-1)\right)\right)ds} \\ &= (\text{cosst.})e^{\int_0^t \xi_s\left(\left(\frac{n-2k}{k}\right)+n\frac{P^2(k-l)}{k(k-Pl)^2}+\frac{P(k-l)(k-l-1)}{k(k-Pl)}\right)ds} \\ &= (\text{cosst.})e^{\frac{n-2k}{k}\xi(t)}\cdot e^{\int_{\xi(0)}^{\xi(t)}\left(n\frac{P^2(k-l)}{k(k-Pl)^2}+\frac{P(k-l)(k-l-1)}{k(k-Pl)}\right)d\xi(s)}. \end{aligned}$$

If $2k < n$, then $\xi(t)$ is bounded and $\xi_t(t)$ is a function of $\xi(t)$, thus $W_1(t)$ has a positive upper and lower bound. If $2k = n$, then $P \sim h^{-\frac{k-l}{k}}e^{-2(k-l)\xi} \to 0$ as $t \to \infty$, thus

$$1 \le e^{\int_{\xi(0)}^{\xi(t)}\left(n\frac{P^2(k-l)}{k(k-Pl)^2}+\frac{P(k-l)(k-l-1)}{k(k-Pl)}\right)d\xi(s)} \le C.$$

Consequently $W_1(t)$ also has a positive upper and lower bound. According to our assumption on the decay rate of $r(t,\theta)$, we have

$$|r_j(s)| \le Ce^{-\beta s}.$$

Thus

$$\left|\int_t^\infty \frac{\phi_1^-(s)\hat{r}_j(s)}{W_1(s)}ds\right| \lesssim \int_t^\infty e^{-(1+\beta)s}ds \lesssim e^{-(1+\beta)t}$$

from which we deduce that

$$\left|\phi_1^+(t)\int_t^\infty \frac{\phi_1^-(s)\hat{r}_j(s)}{W_1(s)}ds\right| \lesssim e^{-\beta t}.$$

When $\beta \ne 1$, we also have

$$\left|\int_0^t \frac{\phi_1^+(s)\hat{r}_j(s)}{W_1(s)}ds\right| \lesssim \int_0^t e^{(1-\beta)s}ds \lesssim e^{(1-\beta)t}$$

from which we deduce that

$$\left|\phi_1^-(t)\int_0^t \frac{\phi_1^+(s)\hat{r}_j(s)}{W_1(s)}ds\right| \lesssim e^{-\beta t}.$$

Putting these estimates into (4.3.8), we have

$$\left|\phi_j(t) - ce^{-t}(1+\xi'(t))\right| \lesssim e^{-\beta t}.$$

When $\beta = 1$, (4.3.8) gives the modified estimate.

For $j = 0$, $\phi_0^+(t) = \xi_h'(t)$ and $\phi_0'(t) = \partial_h \xi_h(t)$ also form a solution basis to the homogeneous equation

$$\phi_j''(t) + \Phi\phi_j'(t) + \Psi\phi_j(t) = 0.$$

Since $\phi_0(t)$ is a solution to (4.3.7) and $\phi_0(t) \to 0$ as $t \to \infty$, a variant of (4.3.8) gives:

$$\phi_0(t) = -\phi_0^-(t)\int_t^\infty \frac{\phi_0^+(s)\hat{r}_0(s)}{W_0(s)}ds + \phi_0^+(t)\int_t^\infty \frac{\phi_0^-(s)\hat{r}_0(s)}{W_0(s)}ds, \tag{4.3.9}$$

where

$$W_0(s) = \phi_0^+(t)\phi_0^{-'}(t) - \phi_0^-(t)\phi_0^{+'}(t) \tag{4.3.10}$$

is the Wronskian of $\{\phi_0^-(t), \phi_0^+(t)\}$, and also satisfies

$$W_0'(s) = -\frac{B(s)}{A(s)}W_0(s).$$

Thus, as for $W_1(s)$, $W_0(s)$ is a periodic function, having a positive upper and lower bound. Let $T(h)$ denote the minimal period of the solution $\xi_h(t)$. Then $\xi_h(t+T(h)) = \xi_h(t)$. Differentiating in h, we obtain

$$\phi_0^-(t+T(h)) + T'(h)\xi_h'(t+T(h)) = \phi_0^-(t), \tag{4.3.11}$$

which implies that $\phi_0^-(t)$ grows in t at most linearly. Then (4.3.10) would imply that $|\phi_0(t)| \lesssim te^{-\beta t}$. This is not quite as claimed, but is good enough to be used in our iterative argument in proving (1.0.9). To obtain the more precise estimate (4.3.2), note that (4.3.11) implies that

$$p(t) := \phi_0^+(t) + \frac{T'(h)}{T(h)}t\xi_h'(t) = \phi_0^-(t) + \frac{T'(h)}{T(h)}t\phi_0^+(t)$$

is $T(h)$ periodic. Thus we can express $\phi_0'(t)$ as $p(t) - \frac{T'(h)}{T(h)}t\phi_0^+(t)$ in (4.3.9) to obtain

$$\begin{aligned}
\phi_0(t) &= -\left(p(t) - \frac{T'(h)}{T(h)}t\phi_0^+(t)\right)\int_t^\infty \frac{\phi_0^+(s)\hat{r}_0(s)}{W_0(s)}ds \\
&\quad +\phi_0^+\int_t^\infty \frac{\left(p(s) - \frac{T'(h)}{T(h)}s\phi_0^+(s)\hat{r}_0(s)\right)}{W_0(s)}ds \\
&= -p(t)\int_t^\infty \frac{\phi_0^+(s)\hat{r}_0(s)}{W_0(s)}ds + \phi_0^+(s)\int_t^\infty \frac{p(s)\hat{r}_0(s)}{W_0(s)}ds \\
&\quad -\frac{T'(h)}{T(h)}\phi_0^+(t)\int_t^\infty\int_s^\infty \frac{\phi_0^+(\tau)\hat{r}_0(s)}{W_0(s)}d\tau ds,
\end{aligned}$$

from which $|\phi_0(t)| \lesssim e^{-\beta t}$ follows.

Finally, we estimate the decay rate of $\hat{\phi}(t,\theta)$. Multiplying both sides of (4.3.1) by

$$\left\{ \frac{(1-|\xi_t|^2)^{k-1}}{2^{k-1}} \binom{n-1}{k-1} \left(1 - \frac{l}{k}P\right) \right\}^{-1} \hat{\phi}(t,\theta),$$

integrating over $\theta \in \mathbb{S}^{n-1}$ and using (4.3.5) and (4.3.6), we find

$$\begin{aligned} &\int_{\mathbb{S}^{n-1}} \left\{ \hat{\phi}_{tt}(t,\theta)\hat{\phi}(t,\theta) + \Phi\hat{\phi}_t(t,\theta)\hat{\phi}(t,\theta) \right. && (4.3.12) \\ &\quad \left. + \frac{k-l}{1-\frac{l}{k}P} \cdot \frac{n}{k}(1+q)(1-|\xi_t^2|)| \left|\hat{\phi}(t,\theta)\right|^2 \right\} d\theta \\ &- \int_{\mathbb{S}^{n-1}} \left\{ \frac{\lambda_j}{1-\frac{l}{k}P} \left[\frac{C(t)}{A(t)} - \frac{l}{k}P\frac{\hat{C}(t)}{A(t)} \right] \left|\hat{\phi}_\theta(t,\theta)\right|^2 \right\} d\theta \\ = &\int_{\mathbb{S}^{n-1}} \hat{r}(t,\theta)\hat{\phi}(t,\theta)d\theta, && (4.3.13) \end{aligned}$$

where

$$\hat{r}(t,\theta) = \left\{ \frac{(1-|\xi_t|^2)^{k-1}}{2^{k-1}} \binom{n-1}{k-1} \left(1 - \frac{l}{k}P\right) \right\}^{-1} r(t,\theta) \approx r(t,\theta)$$

Defining

$$y(t) = \sqrt{\int_{\mathbb{S}^{n-1}} \left|\hat{\phi}(t,\theta)\right|^2 d\theta},$$

then

$$y'(t) = \int_{\mathbb{S}^{n-1}} \hat{\phi}_t(t,\theta)\hat{\phi}(t,\theta)d\theta/y(t) \ \text{ whenever}\, y(t) > 0,$$

and

$$y(t)y^{''}(t) = \int_{\mathbb{S}^{n-1}} \left\{ \hat{\phi_{tt}}(t,\theta)\hat{\phi}(t,\theta) + \left|\hat{\phi_t}(t,\theta)\right|^2 \right\} d\theta - \left|y^{'}(t)\right|^2.$$

The Cauchy-Schwarz inequality implies that

$$\left|y'(t)\right|^2 \leq \int_{\mathbb{S}^{n-1}} \left|\hat{\phi}_t(t,\theta)\right|^2 d\theta.$$

Using these relations and plugging

$$\int_{\mathbb{S}^{n-1}} \left|\nabla_\theta\hat{\phi}(t,\theta)\right|^2 d\theta \geq 2n \int_{\mathbb{S}^{n-1}} \left|\hat{\phi}(t,\theta)\right|^2 d\theta$$

into (4.3.12) , we obtain

$$y(t)y^{''}(t) + \Phi y(t)y^{'}(t) + \Psi y^2(t) \geq -\left|\left|\hat{r}(t,\cdot)\right|\right|_{L^2(\mathbb{S}^{n-1})} y(t),$$

whenever $y(t) > 0$, from which we deduce

$$y^{''}(t) + \Phi y(t) + \Psi y(t) \geq - ||\hat{r}(t, \cdot)||_{L^2(\mathbb{S}^{n-1})}$$

whenever $y(t) > 0$. According to our assumption on $r(t, \theta)$, we have

$$||\hat{r}(t, \cdot)||_{L^2(\mathbb{S}^{n-1})} \leq Ce^{-\beta t}$$

for some constant $C > 0$. By Lemma 4.2.4,

$$(\partial_{tt} + \Phi\partial_t + \Psi)\left(e^{-\beta t}\right) \leq -\epsilon e^{-\beta t}$$

for some $\epsilon > 0$ when $\beta < \beta_*$. So $z(t) := \frac{C}{\epsilon}e^{-\beta t}$ satisfies

$$(\partial_{tt} + \Phi\partial_t + \Psi)(z(t) - y(t)) \leq 0 \tag{4.3.14}$$

whenever $y(t) > 0$. We also know that $y(t) \to 0$ as $t \to \infty$. We may choose $C > 0$ large so that $z(0) \geq y(0)$. Then we claim that $z(t) - y(t) \geq 0$ for all $t \geq 0$, for if not, $\min(z(t) - y(t)) < 0$ is finite, and is attained at some t_*, so $y(t_*) > z(t_*) > 0$, (4.3.14) holds at $t = t_*$, and $\partial_t(z(t) - y(t))|_{t=t_*} = 0$, as well as $\partial_{tt}(z(t) - y(t))|_{t=t_*} \geq 0$. This contradicts (4.3.14). Thus we conclude

$$\sqrt{\int_{\mathbb{S}^{n-1}} \left|\hat{\phi}(t,\theta)\right|^2 d\theta} = y(t) \leq \frac{C}{\epsilon}e^{-\beta t}.$$

We can now bootstrap this integral estimate to obtain a pointwise decay estimate

$$\left|\hat{\phi}(t,\theta)\right| \lesssim e^{-\beta t}.$$

When $\beta \geq \beta_*$, we can simply split those components ϕ_j of ϕ with $\lambda_j = 2n$ from $\hat{\phi}(t,\theta)$, and estimate them as we did for ϕ_j, $j = 0, \cdots, n$, and estimate $\hat{\phi}(t,\theta)$ with an exponential decay rate. □

With Proposition 4.3.1 and a iteration argument from[43], we proved Theorem 1.0.3.

Proof of Theorem 1.0.3. Our starting point is still

$$L_{\xi_h(t+\tau)}(\phi) + 2k\sigma_k\phi - \left(ce^{-2(k-l)\xi_h(t+\tau)}\dot{L}_{\xi_h(t+\tau)}(\phi) + 2l\sigma_k\phi\right) + Q(\phi) = 0,$$

and our premise is

$$|Q(\phi)| \lesssim e^{-2\alpha t} \quad \text{whenever we have } |\phi, \partial\phi, \partial^2\phi| \lesssim e^{-\alpha t}.$$

In Theorem 1.0.2 we have established

Step 1: For some $\alpha_0 > 0$, $|\phi, \partial\phi, \partial^2\phi| \lesssim e^{-\alpha_0 t}$.If $\alpha_0 \geq \rho_{n+1}$, we stop and have now proved $|\omega(t,\theta) - \xi_h(t+\tau)| = |\phi(t,\theta)| \lesssim e^{-\rho_{n+1}t}$, where $\rho_{n+1} > \sqrt{2}$; if $1 \leq \alpha_0 < \rho_{n+1}$, we jump to Step 3; if $\alpha_0 \leq 1$, we move to

Step 2: Recall that we now have $|Q(\phi)| \lesssim e^{-2\alpha_0 t}$. If $2\alpha_0 > \rho_{n+1}$, then we can apply Proposition 4.3.1 directly to conclude our proof; If $1 < 2\alpha_0 \leq \rho_{n+1}$, then we certainly still have $|Q(\phi)| \lesssim e^{-2\alpha t}$ for some $1 < 2\alpha < \rho_{n+1}$ and can apply Proposition 4.3.1 to imply that

$$|\omega(t,\theta) - \xi_h(t+\tau) - \sum_{j=1}^{n} a_j e^{-(t+\tau)}(1 + \xi_h^{'}(t+\tau))Y_j(\theta)| \lesssim e^{-2\alpha t},$$

for some constants a_j for $j = 1, \cdots, n$, and jump to Step 3; if $2\alpha_0 \leq 1$, we may take α_0 to satisfy $2\alpha_0 < 1$ and apply Proposition 4.3.1 to imply that

$$|\phi(t,\theta) - \sum_{j=1}^{n} a_j e^{-(t+\tau)}(1 + \xi_h^{'}(t+\tau))Y_j(\theta)| \lesssim e^{-2\alpha_0 t},$$

for some constant a_j for $j = 1, \cdots, n$. This certainly implies that

$$|\phi(t,\theta)| \lesssim e^{-2\alpha_0 t}. \tag{4.3.15}$$

Next we use higher derivative estimates for $\omega(t,\theta)$ and $\xi_h(t+\tau)$ and interpolation with (4.3.15) to obtain

$$|\phi, \partial\phi, \partial^2\phi| \lesssim e^{-2\alpha' t}$$

for any $\alpha' < \alpha_0$. Now we go back to the beginning of Step 2 and repeat the process with a new $\alpha_1 > \alpha_0$ to replace the α_0 there, say, $\alpha_1 = 1.8\alpha_0$. After a finite number of steps, we will reach a stage where $2\alpha > 1$ and ready to move onto

Step 3: At this stage, we have $|\phi(t,\theta)| \lesssim e^{-t}$. Repeating the last part of Step 2 involving the derivative estimates for $\omega(t,\theta)$ and $\xi_h(t+\tau)$ to bootstrap the estimate for $Q(\phi)$ to $|Q(\phi)| \lesssim e^{-\alpha t}$, with α can be as close to 2 as one needs. Then, depending on whether $\rho_{n+1} \geq 2$ or otherwise, one can apply Proposition 4.3.1 to obtain (4.3.2) and

(4.3.3). In the first case, we can continue the iteration until $2\alpha > 2$. But due to the presence of $e^{-(t+\tau)}(1+\xi_h^{'}(t+\tau))Y_j(\theta)$ in the estimate for ϕ, the estimate for $Q(\phi)$ cannot be better than e^{-2t}. This explains the appearance of $\min\{2, \rho_{n+1}\}$ in (1.0.11). $\square$

Appendix A

Proof for the phase plane

Now we prove the result in Section 2.1. Remember that we consider the level set given by

$$G(x,y) = e^{(2k-n)x}(1-y^2)^k - e^{(2l-n)x}(1-y^2)^l = h \tag{2.1.1}$$

where $(x,y) \in \Omega \equiv \left\{(x,y) \in \mathbb{R}^2 | \, |y| < 1\right\}$ and h is a constant. We also recall that $\Omega^- \equiv \left\{(x,y) \in \mathbb{R}^2 \,|\, -1 < y < 0\right\}$ and the level set is symmetric with respect to x-axis, i.e.,

$$G(x,y) = G(x,-y) \tag{A.0.1}$$

for all $(x,y) \in \Omega$. Moreover, we set $W := \left\{(x,y) \in \Omega \,|\, x < -\frac{1}{2}\ln\left(1-y^2\right)\right\}$ and $V := \left\{(x,y) \in \Omega \,|\, x > -\frac{1}{2}\ln\left(1-y^2\right)\right\}$, and $\forall (x,y) \in \Omega$ solving (2.1.1),

$$(x,y) \in W \Longleftrightarrow h < 0 \quad \text{while} \quad (x,y) \in V \Longleftrightarrow h > 0. \tag{A.0.2}$$

Proof of (P1) (P2) (P3) in Case $h > 0$

Lemma A.0.2. *If $1 < l < k < \frac{n}{2}$ & $h > 0$, then $\frac{\partial G}{\partial y} > 0$ in $V \cap \Omega^-$. Besides, the set $\Re \equiv \left\{(x,y) \in \Omega \,|\, \frac{\partial G}{\partial x} = 0\right\}$ is the curve*

$$\bar{x}(y) = \frac{1}{2}\ln\frac{\left(\frac{n-2l}{n-2k}\right)^{\frac{1}{k-l}}}{(1-y^2)} \qquad |y| < 1 \tag{A.0.3}$$

and it stays in V. Moreover, for all $(x,y) \in \Omega$ we have

$$\begin{cases} \frac{\partial G}{\partial x} > 0 & \text{if } x < \bar{x}(y) \\ \frac{\partial G}{\partial x} < 0 & \text{if } x > \bar{x}(y) \end{cases} \tag{A.0.4}$$

Proof. First if $1 < l < k < \frac{n}{2}$, we have that

$$\begin{aligned}\frac{\partial G}{\partial y} &= (-2y)\left(e^{(2k-n)x}k(1-y^2)^{k-1} - e^{(2l-n)x}l(1-y^2)^{l-1}\right) \\ &= (-2y)(1-y^2)^{-1}\left(ke^{(2k-n)x}(1-y^2)^{k} - le^{(2l-n)x}(1-y^2)^{l}\right) \\ &> 0\end{aligned}$$

in $V \cap \Omega^-$. Secondly we solve

$$\frac{\partial G}{\partial x} = (2k-n)e^{(2k-n)x}(1-y^2)^k - (2l-n)e^{(2l-n)x}(1-y^2)^l = 0$$

in Ω to obtain the curve

$$\bar{x}(y) = \frac{1}{2}\ln\frac{\left(\frac{n-2l}{n-2k}\right)^{\frac{1}{k-l}}}{(1-y^2)} \qquad |y| < 1,$$

and obviously

$$\bar{x}(y) > -\frac{1}{2}\ln\left(1-y^2\right) \qquad \text{if } |y| < 1,$$

so $\Re$ must stay in V. Consequently for all $(x, y) \in \Omega$, we have $\frac{\partial G}{\partial x} > 0$ if $x < \bar{x}(y)$ and $\frac{\partial G}{\partial x} < 0$ if $x > \bar{x}(y)$. □

Lemma A.0.3. *If* $1 < l < k < \frac{n}{2}$ *, then* $G(x, y)$ *achieves its maximum* $h^* := (\frac{n-2k}{n-2l})^{\frac{n-2k}{2}} - (\frac{n-2k}{n-2l})^{\frac{n-2l}{2}}$ *only at* $\left(\frac{1}{2(k-l)}\ln\frac{n-2l}{n-2k}, 0\right)$ *in* V*. Moreover, if* $0 < h < h^*$*, the level set given by* (2.1.1) *is a closed curve in* V*; and if* $h = h^*$*, it is the point* $(\bar{x}(0), 0)$*. All the sets* $\{(x, y)\,|\,G(x, y) = h\}$, $0 < h < h^*$ *foliate* V *and there is no solution to* (2.1.1) *in* Ω *if* $h > h^*$.

Proof. By Lemma A.0.2 $\frac{\partial G}{\partial y} > 0$ in $V \cap \Omega^-$ and (A.0.1), we have

$$G(x, y) < G(x, 0) \qquad \forall (x, y) \in V \text{ with } y \neq 0. \tag{A.0.5}$$

Also

$$\begin{cases} \frac{\partial G}{\partial x}(x, 0) > 0 & \text{if } x < \bar{x}(0) \\ \frac{\partial G}{\partial x}(x, 0) < 0 & \text{if } \bar{x}(0) < x, \end{cases} \tag{A.0.6}$$

which implies

$$G(x, 0) < G\left(\bar{x}(0), 0\right) \equiv h^* \qquad \forall x > 0 \text{ and } x \neq \bar{x}(0). \tag{A.0.7}$$

Combining $(A.0.5)$ and $(A.0.7)$,

$$G(\bar{x}(0),0)=\max_{(x,y)\in V}G(x,y)>G(x,y)\qquad\forall(x,y)\in V\,\&\,(x,y)\neq(\bar{x}(0),0).$$

Besides,

$$G(0,0)=\lim_{x\to\infty}G(x,0)=0,$$

so when $0<h<h^*$, by $(A.0.6)$, there are just $a_1(h)$ & $a_2(h)$ such that $0<a_1(h)<\bar{x}(0)<a_2(h)<\infty$ and

$$G(a_1(h),0)=G(a_2(h),0)=h.$$

If $x\in(a_1(h),a_2(h))$, we have $G(x,0)>h$ & $G(x,-\sqrt{1-e^{-2x}})=0$, so there must exist one unique $y(x)$ such that $-\sqrt{1-e^{-2x}}<y(x)<0$ solving

$$G(x,y(x))=h.$$

However, if $(x,y)\in V$ with $0<x<a_1(h)$ or $x>a_2(h)$, then $G(x,y)>h$, so the solution of (2.1.1) must be the set $\{(x,\pm y(x))\,|\,a_1(h)<x<a_2(h)\}$ by symmetry $(A.0.1)$, whose graph is a closed curve in V. When $h=h^*$, the graph of (2.1.1) is the point $(\bar{x}(0),0)$. There is no solution to (2.1.1) if $h>h^*$. □

Proof of (P4) (P5) in Case $h>0$

Lemma A.0.4. *If $1\le l<k=\frac{n}{2}$ & $h>0$, then $\frac{\partial G}{\partial y}>0$ in $V\cap\Omega^-$ and $\frac{\partial G}{\partial x}>0$ in Ω. Consequently in $V\cap\Omega^-$ the function $y(x)$ solving* (2.1.1) *is well defined and strictly decreasing.*

Proof. Similarly as Lemma $A.0.2$, we have

$$\begin{aligned}\frac{\partial G}{\partial y} &= (-2y)(1-y^2)^{-1}\left(k(1-y^2)^k-le^{(2l-2k)x}(1-y^2)^l\right)\\ &= (-2y)(1-y^2)^{l-1}e^{(2l-2k)x}(ke^{(2k-2l)x}(1-y^2)^{k-l}-l)\\ &> 0\end{aligned}$$

in $V\cap\Omega^-$, and also in Ω,

$$\frac{\partial G}{\partial x}=(n-2l)e^{(2l-n)x}(1-y^2)>0. \tag{A.0.8}$$

Thus

$$\frac{\partial y}{\partial x} = -\frac{\frac{\partial G}{\partial x}}{\frac{\partial G}{\partial y}} < 0$$

in $V \cap \Omega^-$, which implies that $y(x)$ is well defined and strictly decreasing in $V \cap \Omega^-$. $\square$

Lemma A.0.5. *If $1 \leq l < k = \frac{n}{2}$ & $0 < h < 1$, the level set given by (2.1.1) is a U-shaped curve which opens right. Moreover the function $y(x)$ solving (2.1.1) is strictly decreasing and $y \to -\sqrt{1 - \sqrt[k]{h}}$ as $x \to \infty$ in Ω^-. If $1 \leq l < k = \frac{n}{2}$ & $h \geq 1$, there is no solution for (2.1.1) in Ω.*

Proof. By Lemma A.0.4, if $1 \leq l < k = \frac{n}{2}$, then

$$G(x, y) \leq G(x, 0) < \lim_{x \to \infty} G(x, 0) = 1$$

for all $(x, y) \in V$. Thus when $h \geq 1$, there is no solution for (2.1.1) in Ω. If $0 < h < 1$, the function $y(x)$ solving (2.1.1) is well defined and strictly decreasing in $V \cap \Omega^-$. Besides, $y \to -\sqrt{1 - \sqrt[k]{h}}$ as $x \to \infty$ in Ω^-, and by symmetry the level set is a U-shaped curve that opens right in V. $\square$

Proof of (P6) in Case $h > 0$

Lemma A.0.6. *If $1 \leq l < k$, $\frac{n}{2} < k$ & $h > 0$, then $\frac{\partial G}{\partial y} > 0$ in $V \cap \Omega^-$ and $\frac{\partial G}{\partial x} > 0$ in V, thus the function $y(x)$ solving (2.1.1) is well defined and strictly decreasing in Ω^-. Moreover, $y(x) \to -1$ as $x \to \infty$ in Ω^- and the level set given by (2.1.1) is a U-shaped curve that opens right.*

Proof. If $1 \leq l < k$, $\frac{n}{2} < k$ & $h > 0$, then

$$\begin{aligned}
\frac{\partial G}{\partial x} &= (2k-n)e^{(2k-n)x}(1-y^2)^k - (2l-n)e^{(2l-n)x}(1-y^2)^l \\
&= e^{(2l-n)x}(1-y^2)^l((2k-n)e^{(2k-2l)x}(1-y^2)^{k-l} - (2l-n)) \\
&> 0
\end{aligned}$$

in V and same as above $\frac{\partial G}{\partial y} > 0$ in $\Omega^- \cap V$. Thus in $\Omega^- \cap V$,

$$\frac{\partial y}{\partial x} = -\frac{\frac{\partial G}{\partial x}}{\frac{\partial G}{\partial y}} < 0,$$

so the function $y(x)$ solving (2.1.1) is well defined and strictly decreasing in $V \cap \Omega^-$. We claim $y(x) \to -1$ as $x \to \infty$ in Ω^-, or else there exists one $0 > c > -1$ such that $y(x) > c$, then

$$h \equiv \lim_{x \to \infty} G(x, y(x)) > \frac{1}{2} \lim_{x \to \infty} e^{(2k-n)x}(1 - c^2)^k = \infty$$

a contradiction. Hence $y(x) \to -1$ as $x \to \infty$ in Ω^- and by symmetry the level set given by (2.1.1) is a U-shaped curve that opens right in V. □

Proof of (N1) (N2) in Case $h < 0$

Lemma A.0.7. *If $1 \le l < k$, $l < \frac{n}{2}$ & $h < 0$, then*

$$\frac{\partial G}{\partial x} > 0 \qquad (x, y) \in W,$$

and $\forall\ y \in (-1, 1)$, there exists a unique $x_h^*(y)$ such that

$$G(x_h^*(y), y) = h.$$

Moreover, $x_h^*(0) < 0$ and

$$\lim_{y \to -1} x_h^*(y) = -\infty. \tag{A.0.9}$$

Proof. If $1 \le l < \frac{n}{2} \le k$, then in Ω,

$$\begin{aligned} \frac{\partial G}{\partial x} &= e^{(2l-n)x}(1 - y^2)^l((2k - n)e^{(2k-2l)x}(1 - y^2)^{k-l} - (2l - n)) \\ &> 0. \end{aligned}$$

With $(A.0.6)$, $(A.0.8)$, we obtain that if $1 \le l < k$, $l < \frac{n}{2}$, then

$$\frac{\partial G}{\partial x} > 0 \quad (x, y) \in W.$$

Besides, $\forall y \in (-1, 1)$

$$\begin{cases} G(x, y) = 0 & \text{if } (x, y) \in \partial V \\ \lim_{x \to \infty} G(x, y) = -\infty & . \end{cases}$$

Thus there is one unique $x_h^*(y)$ such that $(x_h^*(y), y) \in W$ by $(A.0.2)$ and

$$G(x_h^*(y), y) = h.$$

Specifically $x_h^*(0) < 0$. Also we claim

$$\lim_{y \to -1} x_h^*(y) = -\infty,$$

or else there exist b and a sequence $\{y_k\}_{k \in \mathbb{Z}}$ such that $y_k \xrightarrow{k \to \infty} -1$ & $x_h^*(y_k) > b$, it implies

$$\begin{aligned} h &= \lim_{k \to \infty} G(x_h^*(y_k), y_k) \\ &> \lim_{k \to \infty} -e^{(2l-n)b}(1 - y_k^2)^l \\ &= 0, \end{aligned}$$

contradiction. □

Lemma A.0.8. *If* $1 \le l < k$, $l < \frac{n}{2}$ *&* $h < 0$, *set* $\aleph \equiv \{(\underline{x}(y), y) \in \Omega\}$ *where*

$$\underline{x}(y) = \frac{1}{2} \ln \left(\frac{(\frac{l}{k})^{\frac{1}{k-l}}}{1 - y^2} \right) \qquad |y| < 1, \tag{A.0.10}$$

then $\aleph$ *stays in* W, *and also* $\{(x, y) \in \Omega \,|\, \frac{\partial G}{\partial y} = 0\} = \aleph \cup \{(x, 0) \,|\, x \in R\}$. *Moreover, for all* $(x, y) \in \Omega^-$,

$$\begin{cases} \frac{\partial G}{\partial y} < 0 & \textit{if } x < \underline{x}(y) \\ \frac{\partial G}{\partial y} > 0 & \textit{if } x > \underline{x}(y). \end{cases} \tag{A.0.11}$$

Proof. If $1 \le l < k$, $l < \frac{n}{2}$ & $h < 0$,

$$\begin{aligned} \frac{\partial G}{\partial y} &= e^{(2k-n)x} k (1 - y^2)^{k-1}(-2y) - e^{(2l-n)x} l (1 - y^2)^{l-1}(-2y) \\ &= (-2y) e^{(2l-n)x} (1 - y^2)^{l-1} \left(k e^{(2k-2l)x} (1 - y^2)^{k-l} - l \right), \end{aligned}$$

by solving $\frac{\partial G}{\partial y} = 0$, we obtain $y = 0$ or the curve

$$\underline{\mathrm{x}}(y) = \frac{1}{2} \ln \left(\frac{(\frac{l}{k})^{\frac{1}{k-l}}}{1 - y^2} \right) \qquad |y| < 1.$$

Thus $\{(x, y) \in \Omega \,|\, \frac{\partial G}{\partial y} = 0\} = \aleph \cup \{(x, 0) \,|\, x \in R\}$ and obviously

$$\underline{\mathrm{x}}(y) < -\frac{1}{2} \ln(1 - y^2) \qquad |y| < 1,$$

so it stays in W. Moreover for all $(x, y) \in \Omega^-$ with $y \neq 0$,

$$\begin{cases} \frac{\partial G}{\partial y} < 0 & \text{if } x < \underline{\mathrm{x}}(y) \\ \frac{\partial G}{\partial y} > 0 & \text{if } x > \underline{\mathrm{x}}(y). \end{cases}$$

□

Lemma A.0.9. *Let* $\hbar := (\frac{l}{k})^{\frac{2k-n}{2(k-l)}} - (\frac{l}{k})^{\frac{2l-n}{2(k-l)}}$ *and if* $1 \leq l < k$, $l < \frac{n}{2}$ & $h < \hbar$, *then* $\frac{\partial G}{\partial y}(x_h^*(y), y) < 0$ *and consequently* $x_h^*(y)$ *satisfies*

$$\frac{\partial x_h^*(y)}{\partial y} = -\frac{\frac{\partial G}{\partial y}\left(x_h^*(y), y\right)}{\frac{\partial G}{\partial x}\left(x_h^*(y), y\right)} > 0.$$

Thus the level set given by (2.1.1) *is an U-shaped curve that opens left in* W. *If* $1 \leq l < k$, $l < \frac{n}{2}$ & $\hbar \leq h < 0$, *let* $\kappa_h^* \equiv -\sqrt{1 - (\frac{l}{k})^{\frac{1}{k-l}} \left(\frac{h}{(\frac{l}{k})^{\frac{k}{k-l}} - (\frac{l}{k})^{\frac{l}{k-l}}} \right)^{\frac{2}{n}}}$, *then* $x_h^*(y)$ *satisfies*

$$\begin{cases} \frac{\partial x_h^*(y)}{\partial y} > 0 & \text{if } -1 < y < \kappa_h^* \\ \frac{\partial x_h^*(y)}{\partial y} < 0 & \text{if } k_h^* < y < 0. \end{cases}$$

First we consider the equation system in Ω,

$$\begin{cases} G(x, y) = h \\ x = \underline{\mathrm{x}}(y) \end{cases} . \tag{A.0.12}$$

If $1 \leq l < k$, $l < \frac{n}{2}$ & $h < \hbar$, there is no solution to $(A.0.12)$, and $x_h^*(0) < \underline{\mathrm{x}}(0)$, thus

$$x_h^*(y) < \underline{\mathrm{x}}(y).$$

By $(A.0.11)$, $\frac{\partial G}{\partial y}(x_h^*(y), y) < 0$ and by $(A.0.4)$ & $(A.0.8)$, $\frac{\partial G}{\partial x}(x_h^*(y), y) > 0$, so

$$\frac{\partial x_h^*(y)}{\partial y} = -\frac{\frac{\partial G}{\partial y}\left(x_h^*(y), y\right)}{\frac{\partial G}{\partial x}\left(x_h^*(y), y\right)} > 0.$$

If $1 \leq l < k$, $l < \frac{n}{2}$ & $\hbar \leq h < 0$, let $\zeta_h^* \equiv \underline{\mathrm{x}}(\kappa_h^*) = -\frac{1}{n} \ln \left(\frac{h}{(\frac{l}{k})^{\frac{k}{k-l}} - (\frac{l}{k})^{\frac{l}{k-l}}} \right)$, then $(A.0.12)$ has the solutions

$$(\zeta_h^*, \pm\kappa_h^*). \tag{A.0.13}$$

When $-1 < y < k_h^*$, since $(A.0.9)$, we have $x_h^*(y) < \underline{\mathrm{x}}(y)$, thus $\frac{\partial G}{\partial y}(x_h^*(y), y) < 0$ and

$$\frac{\partial x_h^*(y)}{\partial y} = -\frac{\frac{\partial G}{\partial y}\left(x_h^*(y), y\right)}{\frac{\partial G}{\partial x}\left(x_h^*(y), y\right)} > 0.$$

When $\kappa_h^* < y < 0$, we have $x_h^*(y) > \underline{\mathrm{x}}(y)$, thus $\frac{\partial G}{\partial y}(x_h^*(y), y) > 0$ and

$$\frac{\partial x_h^*(y)}{\partial y} = -\frac{\frac{\partial G}{\partial y}\left(x_h^*(y), y\right)}{\frac{\partial G}{\partial x}\left(x_h^*(y), y\right)} < 0.$$

Proof of (N3) (N4) (N5) in Case $h < 0$

Lemma A.0.10. *If* $1 < l = \frac{n}{2} < k$*, then* $\frac{\partial G}{\partial x} > 0$ *in* Ω *and also in* Ω^-

$$\begin{cases} \frac{\partial G}{\partial y} < 0 & \text{if } x < \underline{x}(y) \\ \frac{\partial G}{\partial y} > 0 & \text{if } x > \underline{x}(y) \end{cases}$$

and then

$$G(x,y) > \inf_{(x,y)\in\Omega} G(x,y) = -1 \qquad (x,y) \in \Omega. \tag{A.0.14}$$

So if $1 \le l < k \le \frac{n}{2}$ & $-1 < h < \hbar$*, then* $\frac{\partial G}{\partial y}(x_h^*(y), y) < 0$ *and consequently* $x_h^*(y)$ *satisfies*

$$\frac{\partial x_h^*(y)}{\partial y} = -\frac{\frac{\partial G}{\partial y}\left(x_h^*(y), y\right)}{\frac{\partial G}{\partial x}\left(x_h^*(y), y\right)} > 0.$$

Thus the level set given by (2.1.1) *is an* U*-shaped curve that opens left in* Ω*. If* $1 \le l < k \le \frac{n}{2}$ & $\hbar \le h < 0$*, let* $\kappa_h^* \equiv -\sqrt{1 - \left(\frac{l}{k}\right)^{\frac{1}{k-l}} \left(\frac{h}{\left(\frac{l}{k}\right)^{\frac{k}{k-l}} - \left(\frac{l}{k}\right)^{\frac{l}{k-l}}}\right)^{\frac{2}{n}}}$*, then* $x_h^*(y)$ *satisfies*

$$\begin{cases} \frac{\partial x_h^*(y)}{\partial y} > 0 & \text{if } -1 < y < \kappa_h^* \\ \frac{\partial x_h^*(y)}{\partial y} < 0 & \text{if } k_h^* < y < 0. \end{cases}$$

If $1 < l = \frac{n}{2} < k$ & $h \le -1$*, there is no solution for* (2.1.1).

Proof. If $1 < l = \frac{n}{2} < k$, then for $(x,y) \in \Omega$,

$$\frac{\partial G}{\partial x} = (2k-n)e^{(2k-n)x}(1-y^2)^k > 0 \tag{A.0.15}$$

and similar to Lemma A.0.8, we have that in Ω^-

$$\begin{cases} \frac{\partial G}{\partial y} < 0 & \text{if } x < \underline{\mathrm{x}}(y) \\ \frac{\partial G}{\partial y} > 0 & \text{if } x > \underline{\mathrm{x}}(y), \end{cases} \tag{A.0.16}$$

thus for any $(x,y) \in \Omega$, $\ln\left(\frac{l}{k}\right)^{\frac{1}{k-l}} < \min_{y\in(-1,1)} \underline{\mathrm{x}}(y) < \underline{\mathrm{x}}(y)$, consequently by $(A.0.1)$, $(A.0.15)$ and $(A.0.16)$, we obtain

$$\begin{aligned} G(x,y) &\ge G\left(\min\left\{x, \ln\left(\frac{l}{k}\right)^{\frac{1}{k-l}}\right\}, y\right) \\ &\ge G\left(\min\left\{x, \ln\left(\frac{l}{k}\right)^{\frac{1}{k-l}}\right\}, 0\right) \\ &> \lim_{x\to-\infty} G(x,0) \\ &= -1, \end{aligned}$$

so we have $(A.0.14)$. If $1 < l = \frac{n}{2} < k$ & $h \leq -1$, there is no solution to (2.1.1). The proof for $1 \leq l < k \leq \frac{n}{2}$ & $-1 < h < (\frac{l}{k})^{\frac{2k-n}{2(k-l)}} - (\frac{l}{k})^{\frac{2l-n}{2(k-l)}}$ and $1 \leq l < k \leq \frac{n}{2}$ & $(\frac{l}{k})^{\frac{2k-n}{2(k-l)}} - (\frac{l}{k})^{\frac{2l-n}{2(k-l)}} \leq h < 0$ is same as Lemma A.0.9. □

Proof of (N6) (N7) (N8) (N9) in Case $h < 0$

Lemma A.0.11. *If $\frac{n}{2} < l < k$, the set $\mathcal{O} = \{(x, y) \in \Omega \mid \frac{\partial G}{\partial x} = 0\}$ is the curve*

$$\tilde{x}(y) = \frac{1}{2} \ln \frac{\left(\frac{2l-n}{2k-n}\right)^{\frac{1}{k-l}}}{1 - y^2} \qquad |y| < 1, \tag{A.0.17}$$

and $\tilde{x}(y) < \underline{x}(y) < -\frac{1}{2}\ln(1-y^2) \quad \forall |y| < 1$, which is defined in $(A.0.10)$. Consequently we have

$$\begin{cases} \frac{\partial G}{\partial x} < 0 \,\&\, \frac{\partial G}{\partial y} < 0 & \textit{if}\, x < \tilde{x}(y) \\ \frac{\partial G}{\partial x} > 0 \,\&\, \frac{\partial G}{\partial y} < 0 & \textit{if}\, \tilde{x}(y) < x < \underline{x}(y) \\ \frac{\partial G}{\partial x} > 0 \,\&\, \frac{\partial G}{\partial y} > 0 & \textit{if}\, \underline{x}(y) < x. \end{cases}$$

Proof. If $\frac{n}{2} < l < k$, then we solve

$$\begin{aligned} 0 &= \frac{\partial G}{\partial x} \\ &= (2k-n)e^{(2k-n)x}(1-y^2)^k - (2l-n)e^{(2l-n)x}(1-y^2)^l \\ &= e^{(2l-n)x}(1-y^2)^l\left((2k-n)e^{(2k-2l)x}(1-y^2)^{k-l} - (2l-n)\right) \end{aligned}$$

in Ω to obtain the solution

$$\tilde{x}(y) = \frac{1}{2} \ln \frac{\left(\frac{2l-n}{2k-n}\right)^{\frac{1}{k-l}}}{1 - y^2} \qquad |y| < 1.$$

Obviously

$$\tilde{x}(y) < \underline{\mathrm{x}}(y) < -\frac{1}{2}\ln(1-y^2) \quad \forall |y| < 1$$

and consequently

$$\begin{cases} \frac{\partial G}{\partial x} < 0 \,\&\, \frac{\partial G}{\partial y} < 0 & \text{if}\, x < \tilde{x}(y) \\ \frac{\partial G}{\partial x} > 0 \,\&\, \frac{\partial G}{\partial y} < 0 & \text{if}\, \tilde{x}(y) < x < \underline{\mathrm{x}}(y) \\ \frac{\partial G}{\partial x} > 0 \,\&\, \frac{\partial G}{\partial y} > 0 & \text{if}\, \underline{\mathrm{x}}(y) < x. \end{cases} \tag{A.0.18}$$

□

Lemma A.0.12. *If $\frac{n}{2} < l < k$, then $G(x,y)$ achieves it is minimum $\wp := \left(\frac{n-2l}{n-2k}\right)^{\frac{2k-n}{2(k-l)}} - \left(\frac{n-2l}{n-2k}\right)^{\frac{2l-n}{2(k-l)}}$ at point $(\tilde{x}(0),0)$. Also*

1. *If $\hbar \le h < 0$, the level set given by (2.1.1) is a closed curve, and it has intersection points $(A.0.13)$ with $(A.0.10)$.*

2. *If $\wp < h < \hbar$, the level set given by (2.1.1) is a closed curve. This curve and $(A.0.17)$ have two intersection points*

$$(\tilde{\zeta}_h, \pm\tilde{\kappa}_h) = \left(-\frac{1}{n}\ln\left(\frac{h\left(\frac{2l-n}{2k-n}\right)^{\frac{-l}{k-l}}}{\left(\frac{2l-2k}{2k-n}\right)}\right), \pm\sqrt{1-\left(\frac{2l-n}{2k-n}\right)^{\frac{1}{k-l}}\left(\frac{h\left(\frac{2l-n}{2k-n}\right)^{\frac{-l}{k-l}}}{\left(\frac{2l-2k}{2k-n}\right)}\right)^{\frac{2}{n}}}\right),$$

   Besides, $\forall (x,y) \in \Omega$ solving (2.1.1) satisfies $x < \frac{1}{2}\ln(\frac{l}{k})^{\frac{1}{k-l}}$, and in Ω^- $y(x)$ is well defined,

$$\begin{cases} \frac{\partial y}{\partial x} < 0 & \text{if } x < \tilde{\zeta}_h \\ \frac{\partial G}{\partial y} > 0 & \text{if } x > \tilde{\zeta}_h \end{cases} \qquad (x,y) \in \Omega^-.$$

3. *If $h = \wp$, (2.1.1) is a point $(\tilde{x}(0),0)$.*

4. *If $h < \wp$, (2.1.1) has no solution.*

Proof. If $\frac{n}{2} < l < k$, then

$$\begin{cases} G(x,y) = 0 & \text{if } (x,y) \in \partial V \\ G(x,y) = 0 & \text{if } (x,y) \in \partial\Omega \\ \lim_{x\to-\infty} G(x,y) = 0 & \text{uniformally for } y \in (-1,1). \end{cases}$$

The critical point of $G(x,y)$ in W is only

$$(\tilde{x}(0),0),$$

and $G(\tilde{x}(0),0) = \wp < 0$, thus $G(x,y)$ achieves it is minimum $\wp := \left(\frac{n-2l}{n-2k}\right)^{\frac{2k-n}{2(k-l)}} - \left(\frac{n-2l}{n-2k}\right)^{\frac{2l-n}{2(k-l)}}$ at point $(\tilde{x}(0),0)$. If $\hbar \le h < 0$, consider the equation system $(A.0.12)$ in

Ω, and we obtain the solutions $(A.0.13)$. Since $(A.0.18)$ and $h > G(\underline{\mathrm{x}}(0), 0) > G(\tilde{x}(0), 0)$, there exist only two points $a_3(h)$ & $a_4(h)$ such that

$$G(a_3(h), 0) = G(a_4(h), 0) = h,$$

and also $0 > a_4(h) > \underline{\mathrm{x}}(0) > \tilde{x}(0) > a_3(h)$. Repeating using $(A.0.18)$ and $(A.0.1)$, we see the level set given by (2.1.1) is a closed curve with x ranging from $a_3(h)$ to ζ_h^*. The proof of the remaining cases is similar. $\square$

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