THE SKEIN ALGEBRA OF ARCS AND LINKS AND THE DECORATED TEICHMÜLLER SPACE

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ABSTRACT OF THE DISSERTATION

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Dissertation Director: Feng Luo

This dissertation is based on a joint work with Dr. Julien Roger. We define an associative $\mathbb{C}[[h]]$–algebra $\mathcal{AS}_h(\Sigma)$ generated by framed arcs and links over a punctured surface Σ which is a quantization of the Poisson algebra $\mathcal{C}(\Sigma)$ of arcs and curves on Σ. We also construct a Poisson algebra homomorphism from $\mathcal{C}(\Sigma)$ to the space of smooth functions on the decorated Teichmüller space endowed with the Weil-Petersson Poisson structure. The construction relies on a collection of geodesic lengths identities in hyperbolic geometry which generalizes Penner's Ptolemy relation, the trace identity and Wolpert's cosine formula.

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Dedication

To those who do or do not understand me.

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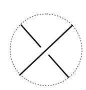
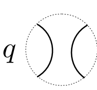
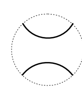
Chapter 1

Introduction

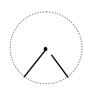
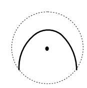
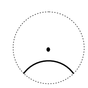
Let Σ be a surface with punctures obtained from a closed oriented surface $\overline{\Sigma}$. The first goal of this dissertation is to extend the notion of skein algebra generated by links over Σ by adding arcs connecting the punctures. The second goal is to relate this algebra to the decorated Teichmüller space of Σ in the context of quantization of Poisson algebras. As a motivation for this work, let us recall some of the key steps of a similar construction in the case of the skein algebra of links.

Let q be the formal power series $e^{\frac{h}{4}} \in \mathbb{C}[[h]]$. The skein algebra $\mathcal{S}_h(\Sigma)$, introduced by Przytycki [17] and Turaev [19], is the $\mathbb{C}[[h]]$–algebra generated by isotopy classes of framed links in $\Sigma \times [0,1]$ subject to the Kauffman bracket skein relation

 $= q$  $+ \ q^{-1}$ 

as well as the framing relation  $= -q^2 - q^{-2}$. In [20], Turaev studied the relationship between the skein algebra and the Lie algebra of curves on Σ introduced by Goldman [9]. In turn, in the work of Goldman, the Lie bracket on curves is related to the Weil-Petersson Poisson structure on the $SL_2(\mathbb{C})$–character variety $X(\Sigma)$ of Σ. A direct relationship between the skein algebra and the character variety is given by Bullock [6], with the construction of a surjective homomorphism between the commutative algebra $\mathcal{S}_0(\Sigma)$ and the coordinate ring of $X(\Sigma)$. This map turns out to be an isomorphism by the work of Przytycki and Sikora [18]. Up to a sign, it assigns to each free homotopy class of curves γ in Σ its trace function tr_γ on $X(\Sigma)$, given by evaluating the trace of representations along γ. One of the key ingredients is then given by the trace identities which relate the product of traces of two intersecting curves with the traces of their resolutions at one point. This identity, in turn, comes from the classical formula $trA \cdot trB = trAB + trAB^{-1}$ relating traces in $SL_2(\mathbb{C})$. Using this isomorphism, Bullock, Frohman and Kania-Bartoszyńska [7] showed that the skein algebra is in fact a quantization of the character variety for the Goldman-Weil-Petersson bracket, in the sense of deformation of Poisson algebras. The key fact is that $\mathcal{S}_0(\Sigma)$, endowed with the Poisson bracket inherited from the commutator on $\mathcal{S}_h(\Sigma)$, is isomorphic *as a Poisson algebra* to the coordinate ring of $X(\Sigma)$.

Our goal is to extend some of the steps described above by introducing framed arcs in the definition of the skein algebra for a surface with punctures. We define a generalized framed link to be the embedding of a collection of annuli and strips in $\Sigma \times [0, 1]$, so that the ends of the strips are above the punctures (see Section 2 for a more precise definition). A component given by a strip will be called a framed arc. Then the *skein algebra of arcs and links* $\mathcal{AS}_h(\Sigma)$ of Σ will be generated by isotopy classes of generalized framed links. For the relations, the usual skein relation still applies for crossings occurring above Σ, where some of the strands can be arcs. When two arcs meet at a puncture we have the so-called *puncture-skein relation*

 $= \frac{1}{v}\Big(q^{\frac{1}{2}}$  $+ q^{-\frac{1}{2}}$  $\Big).$

Here v is a central element associated to the puncture whose meaning will be made clearer later on. The framing relation still applies and is supplemented with the *puncture relation*  $= q + q^{-1}$.

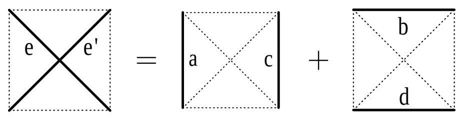
In the non-quantum case, we consider the algebra $\mathcal{C}(\Sigma)$ generated by arcs and curves on Σ itself subject to (classical) skein relations. It has a Poisson bracket given by resolutions of intersections inside the surface and at the punctures which generalizes Goldman's Lie bracket on loops. Using arguments similar to the ones in [7], we show that this bracket comes from the commutator in $\mathcal{AS}_h(\Sigma)$ that is

Theorem 1.0.1. *$\mathcal{AS}_h(\Sigma)$ is a quantization of $\mathcal{C}(\Sigma)$*

The next step of our construction is to relate the algebra $\mathcal{C}(\Sigma)$ to the $SL_2(\mathbb{R})$–character variety, or in our case to the Teichmüller space $\mathcal{T}(\Sigma)$ of Σ. In fact, since Σ has punctures we are led to work with a variant of this space, the so-called decorated Teichmüller space $\mathcal{T}^d(\Sigma)$ introduced by Penner [15]. It is a bundle over $\mathcal{T}(\Sigma)$ with fiber $\mathbb{R}^s_{>0}$, where s is the number of punctures. Given a hyperbolic metric $m \in \mathcal{T}(\Sigma)$, the choice of a point in the fiber corresponds to fixing the length of a horocycle at each of the punctures of Σ. This, in turn, permits to assign a well-defined length $l(\alpha)$ to each arc α between punctures. A more convenient quantity is the so-called lambda-length of α given by $\lambda(\alpha) = e^{\frac{l(\alpha)}{2}}$. They satisfy the Ptolemy relation

$$\lambda(e)\lambda(e') = \lambda(a)\lambda(c) + \lambda(b)\lambda(d)$$

where a, b, c, d are the consecutive edges of a square and e and e' are its diagonals. Graphically, the Ptolemy relation can the be rewritten:



which we want to interpret as a (non-quantum) skein relation.

Using these notions, we obtain the following theorem.

Theorem 1.0.2. *There is a well-defined homomorphism of Poisson algebras*

$$\Phi \colon \mathcal{C}(\Sigma) \to C^\infty(\mathcal{T}^d(\Sigma))$$

Up to signs, the map sends links to their traces, arcs to their lambda-lengths and punctures to horocycle lengths around them. The Poisson structure on $\mathcal{T}^d(\Sigma)$ is an extension of the usual Weil-Petersson Poisson bracket on $\mathcal{T}(\Sigma)$ and was described by Mondello [13]. The proof of the theorem relies on a collection of identities which generalizes Penner's Ptolemy relation, the trace identity and Wolpert's cosine formula [21] for the Poisson bracket of two trace functions. These identities are derived in turn from a set of cosine laws which can be found in the appendix of [10] by Guo and Luo.

Combining Theorems 1.0.1 and 1.0.2, it is tempting to interpret $\mathcal{AS}_h(\Sigma)$ as a quantization of the decorated Teichmüller space. This would require restricting the range of the homomorphism in Theorem 1.0.2 to a carefully chosen subalgebra so that the map becomes surjective and possibly an isomorphism. The image of this homomorphism is essentially generated by traces and lambda-lengths. It is known that, for a surface without punctures, traces alone generate the coordinate ring over Teichmüller space in the algebro-geometric sense. The question is then if this fact generalizes in some sense to the decorated Teichmüller space. Another approach would be to use the fact that, given a maximal collection of edges on Σ, that is, given an ideal triangulation of Σ, lambda-lengths along them form a system of coordinates over $\mathcal{T}^d(\Sigma)$. In addition, trace functions have a simple expression as Laurent polynomials in these coordinates (see Lemma 3.4.1), so the two points of view might be related by restricting to the algebra of Laurent polynomials in these coordinates. However, it is not clear at this point how well this fact translates at the level of the skein algebra $\mathcal{AS}_h(\Sigma)$.

The fact that the trace identity and the Ptolemy relation can be combined into generalized skein relations involving both arcs and curves has been used recently in works of Dupont and Palesi [8] and Musiker and Williams [14], in the context of cluster algebras associated to triangulated surfaces. It would be interesting to see if our work applies to the context of quantum cluster algebras as defined by Berenstein and Zelevinsky [2]. Closely related to these considerations is the construction of so-called quantum trace functions in the context of the quantization of Teichmüller space. This problem was solved recently by Bonahon and Wong in [4, 5] using the skein relation in a crucial way. In turn, their construction is based on the use of shear coordinates [3] which are closely related to lambda-length. We hope that our work could shed new light on the relationship between the skein algebra and the quantum Teichmüller space.

Chapter 2

The algebraic aspect

2.1 The skein algebra of arcs and links

Let $\overline{\Sigma}$ be a closed oriented surface and let V be a (possibly empty) finite subset of $\overline{\Sigma}$. We consider the surface with punctures $\Sigma = \overline{\Sigma} - V$ and call V the set of *punctures* of Σ. As a first step, we need to generalize the notion of a framed link in the 3-manifold $\Sigma \times [0,1]$ to allow for components joining the punctures.

Definition 2.1.1. *A continuous map* $\alpha = \coprod_i \alpha_i \sqcup \coprod_j l_j$ *from a domain* D *consisting of a finite collection of strips* $\coprod_i [0,1] \times (-\epsilon, \epsilon)$ *and annuli* $\coprod_j S^1 \times (-\epsilon, \epsilon)$ *into* $\overline{\Sigma} \times [0,1]$ *is called a* generalized framed link *in* $\Sigma \times [0,1]$ *if*

(1) α *is an injection into* $\overline{\Sigma} \times (0,1)$*;*

(2) each l_j *is an embedding into* $\Sigma \times [0,1]$*;*

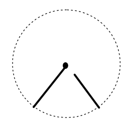
(3) the restriction of each α_i *to* $(0,1) \times (-\epsilon, \epsilon)$ *is an embedding into* $\Sigma \times [0,1]$*;*

(4) the restriction of each α_i *to* $\{0,1\} \times (-\epsilon, \epsilon)$ *is an orientation preserving embedding into* $V \times [0,1]$.

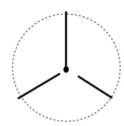
Two generalized framed links α *and* β *are* isotopic *if there exists a continuous map* $H\colon D \times [0,1] \to \overline{\Sigma} \times [0,1]$ *such that* $H_0 = \alpha$ *and* $H_1 = \beta$*, and* H_t *is a generalized framed link for each* $t \in (0,1)$.

Each strip α_i in a generalized link α will be called a *framed arc*, with the understanding that such a component can be "knotted". Condition (1) implies that each arc ends at a different height above the punctures of Σ. Condition (4) prevents a framed arc from doing a "half-twist" between two punctures.

Some conventions are needed when considering a diagram of a generalized link projected onto Σ. Firstly, modulo an isotopy, we can assume that the framing of a generalized link always points in the vertical direction. On a link diagram this will correspond to the direction pointing toward the reader. Secondly, we use the usual convention to encode which strand of a generalized link passes over another in $\Sigma \times [0, 1]$ and we assume that the diagram only possesses ordinary double points in Σ. We generalize this convention as follows when two strands of arcs meet at a puncture:



Here the left strand ends above the right one at the puncture. However, there might be more than two strands meeting at a puncture. In this case, such a picture



will be supplemented with an explanation of the respective positions of the strands lying under the top one.

As is well known, two diagrams correspond to isotopic framed links if and only if one can be obtained from the other by a sequence of Reidemeister Moves II and III. This is also true in the case of a generalized link if we add the move described in Figure 2.1, which we will call *Reidemeister Move II′*. Indeed, this move is obtained by replacing one of the crossings in Rei-

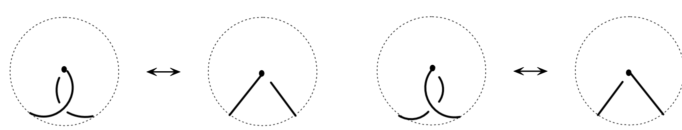
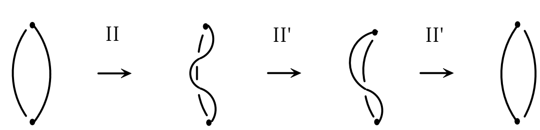


Figure 2.1: Reidemeister Move II′.

demeister Move II by a crossings at a puncture. The only possible move obtained by replacing both of the crossings in Reidemeister Move II by crossings at punctures is a composition of Reidemeister Moves II and II′ as follows:



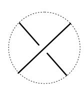
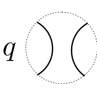
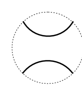
There are no analogous moves to Reidemeister Move III, since a strand of a generalized link cannot be isotoped through a puncture.

Given two generalized links α and β in $\Sigma \times [0,1]$ we define the *stacking of* α *over* β to be the link in $\Sigma \times [0,1]$ obtained by rescaling α to $\Sigma \times [\frac{1}{2}, 1]$ and β to $\Sigma \times [0, \frac{1}{2}]$ and taking there union. This operation is compatible with isotopies.

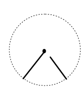
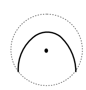
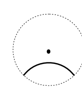
With these definitions and conventions we define the skein algebra of arcs and links over Σ as follows:

Definition 2.1.2. *Let* $\mathbb{C}[[h]][\mathcal{L}, V^{\pm 1}]$ *be the free* $\mathbb{C}[[h]]$*-module generated by the set of isotopy classes of generalized framed links* $\mathcal{L}$ *in* $\Sigma \times [0,1]$ *and the set* $V^{\pm 1}$ *of punctures* v *and there formal inverses* v^{-1}*. Let* q *be the formal power series* $e^{\frac{h}{4}} \in \mathbb{C}[[h]]$*. The* skein algebra of arcs and links $\mathcal{AS}_h(\Sigma)$ *is the quotient of* $\mathbb{C}[[h]][\mathcal{L}, V^{\pm 1}]$ *by the sub-module generated by the following relations:*

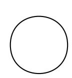
(1) **Kauffman Bracket Skein Relation:** *For a crossing in the surface, we have*

 $= q$  $+ \; q^{-1}$  ;

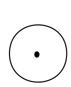
(2) **Puncture-Skein Relation:** *For a crossing at a puncture* v*, we have*

 $= \frac{1}{v}\Big(q^{\frac{1}{2}}$  $+ \; q^{-\frac{1}{2}}$  $\Big)$;

(3) **Framing Relation:** *For the isotopy class of a trivial circle, we have*

 $= -q^2 - q^{-2}$;

(4) **Puncture Relation:** *For the isotopy class of a circle around a puncture, we have*

 $= q + q^{-1}$.

The multiplication in $\mathcal{AS}_h(\Sigma)$ *is defined by:*

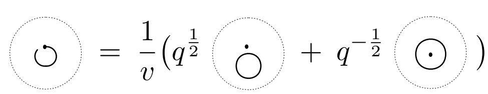
- *the product* $\alpha \cdot \beta$ *of* α *and* β *in* $\mathcal{L}$ *is obtained by stacking* α *over* β*;*
- *the elements of* $V^{\pm 1}$ *are central and* $v \cdot v^{-1} = 1$ *for each* $v \in V$*.*

Some comments are in order to justify this definition. Firstly, note that if V is empty, then $\mathcal{AS}_h(\Sigma)$ corresponds to the usual Kauffman bracket skein algebra as defined in [17] and [19]. Secondly, The choice of the coefficients $q^{\pm 1/2}$ in the puncture-skein relation turns out to be essential for proving that this algebra is well-defined and that the product is associative. It will also have a geometrical justification which will be explained in Section 3. Finally, the central elements v associated to the punctures will be related to geometric data given by the choice of horocycle lengths around the punctures. At the algebraic level however, one could work with the quotient of $\mathcal{AS}_h(\Sigma)$ by the module generated by $V^{\pm 1}$, and carry the computations that will follow in the same way.

We recall that a $\mathbb{C}[[h]]$-module M is called *topologically free* if there exists a $\mathbb{C}$-vector space $\mathcal{V}$ so that M is isomorphic to $\mathcal{V}[[h]]$ (see for example [11]). We have the following

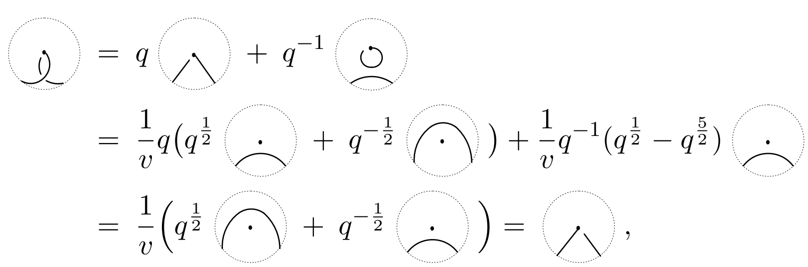
Theorem 2.1.3. *The skein algebra* $\left(\mathcal{AS}_h(\Sigma), \cdot\right)$ *is a well defined topologically free associative* $\mathbb{C}[[h]]$*–algebra.*

Proof. In order to verify the well definition of the multiplication, it suffices to show that it is invariant under Reidemeister Moves II, II$'$ and III. The invariance under Reidemeister Moves II and III is the same as in [17]. For Reidemeister Move II$'$, we calculate that



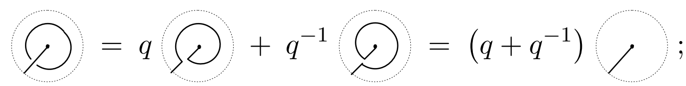
$$= \frac{1}{v}(q^{\frac{1}{2}}(-q^2 - q^{-2}) + q^{-\frac{1}{2}}(q + q^{-1})) = \frac{1}{v}(q^{\frac{1}{2}} - q^{\frac{5}{2}}),$$

where v is the puncture and the second equality is from the framing and the puncture relations. With this, we have



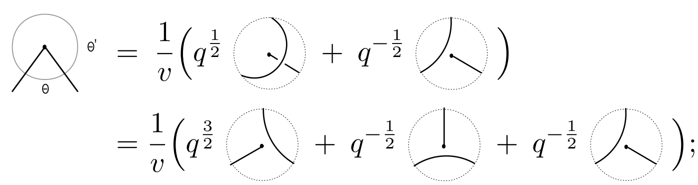
where the first equality is from the Kauffman bracket skein relation and the second equality is from the puncture-skein relation and the previous calculation. The well definition under the other Reidemeister Move II$'$ is verified similarly.

To show that $a \cdot \odot = \odot \cdot a = (q + q^{-1})a$, the only case we need to consider is when $\odot$ is a circle around a puncture v and a is a framed arc with v one of its end points. For $a \cdot \odot$, we have

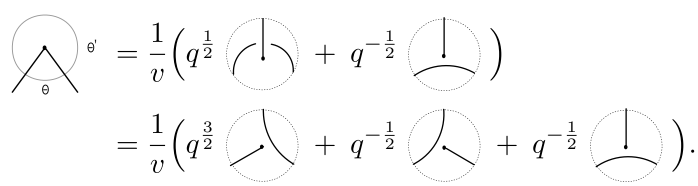


and similarly for $\odot \cdot a = (q + q^{-1})a$.

When three links cross inside the surface, the associativity follows from the same arguments as in [17], and similarly if some intersections happen at a puncture *as long as there are no triple points*. If three arcs a, b and c meet at a puncture v, say in counterclockwise order, we have for $(a \cdot b) \cdot c$ that



and for $a \cdot (b \cdot c)$ that



The case when a, b and c are ordered clockwise is similar.

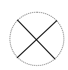
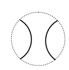
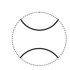
The proof that $\mathcal{AS}_h(\Sigma)$ is topologically free is in the spirit of [17]. A diagram of a generalized framed link in $\Sigma \times (0,1)$ is a graph in $\overline{\Sigma}$ which is four-valent in Σ and many-valent at V with crossings and vertical framing. Two diagrams represent the same generalized framed link if and only if they differ by a sequence of isotopies of Σ and Reidemeister Moves II, II$'$ and III. A diagram $\mathcal{S}$ is called a state if $\mathcal{S}$ does not contain trivial circles, circles around a puncture and crossings neither in the surface nor at punctures. If $\mathcal{W}$ is the $\mathbb{C}$-vector space generated by the set of states on Σ, then we claim that $\mathcal{AS}_h(\Sigma) \cong \mathcal{V}[[h]]$ where $\mathcal{V} = \mathcal{W} \otimes \mathbb{C}[V^{\pm 1}]$, i.e., $\mathcal{AS}_h(\Sigma)$ is topologically free. Indeed, we first resolve the crossings at punctures once at a time to get diagrams with crossings only in the surface. Then, we resolve the crossings in the surface once a time to obtain a diagram without crossings. Finally, we send trivial circles to $-q^2 - q^{-2}$ and circles around a puncture to $q + q^{-1}$. $\square$

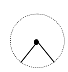
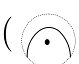
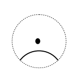
Remark 2.1.4. In the rest of this dissertation, we also call an element of $\mathcal{V} = \mathcal{W} \otimes \mathbb{C}[V^{\pm}]$ a *state*.

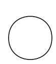
2.2 The Poisson algebra of curves on a surface

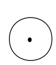
The classical counterpart of the skein algebra can be defined in terms of curves on the surface Σ itself. We define a *generalized curve* to be a union of immersed closed curves and arcs in Σ with ends at the punctures. Any two generalized curves α and β will be considered to be *equivalent* if one can be obtained from the other by a sequence of isotopies of Σ and Reidemeister Moves II, II$'$ and III. We do not however identify curves differing by a Reidemeister Move I.

Definition 2.2.1. *The* algebra of curves $\mathcal{C}(\Sigma)$ *on* Σ *is the quotient of the* $\mathbb{C}$*-vector space generated by the equivalence classes of generalized curves on* Σ*, the punctures of* Σ *and their formal inverses, modulo the subspace generated by the following relations:*

(1$'$) **Kauffman Bracket Skein Relation:**  $=$  $+$  *for an intersection in* Σ*;*

(2$'$) **Puncture-Skein Relation:**  $= v^{-1}($  $+$  $)$ *for an intersection at* v*;*

(3$'$) **Framing Relation:**  $= -2$*;*

(4$'$) **Puncture Relation:**  $= 2$.

The product $\alpha \cdot \beta$ *of two generalized curves* α *and* β *is given by taking their union.*

The fact that $\mathcal{C}(\Sigma)$ is a well defined commutative algebra follows from the same arguments as for $\mathcal{AS}_h(\Sigma)$. These two algebras are related naturally as follows: let $p\colon \mathcal{AS}_h(\Sigma) \to \mathcal{C}(\Sigma)$ be the map which to a generalized link in $\Sigma \times [0,1]$ with vertical framing associates its projection on Σ. We also let $p(h) = 0$ and $p(v) = v$ for every puncture. In this way, p maps relations (1) – (4) to the corresponding relations (1$'$) – (4$'$), and maps the stacking of generalized framed links in $\Sigma \times [0,1]$ to the union of generalized curves on Σ.

Proposition 2.2.2. *The quotient map* $\overline{p}\colon \mathcal{AS}_h(\Sigma)/h\mathcal{AS}_h(\Sigma) \to \mathcal{C}(\Sigma)$ *is an isomorphism of* $\mathbb{C}$*–algebras.*

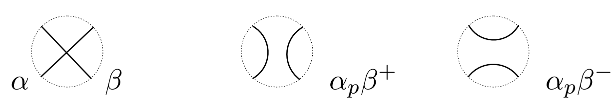
Proof. Since $\mathcal{AS}_h(\Sigma) \cong \mathcal{V}[[h]]$ is topologically free, each element $a \in \mathcal{AS}_h(\Sigma)$ can be uniquely written as a power series $\sum a_k h^k$ with coefficients $a_i \in \mathcal{V}$. By the definition of p, we have $p(a) = p(a_0)$. Remember that the elements of $\mathcal{V}$ are diagrams without crossings neither in Σ nor at V, hence p is injective on $\mathcal{V}$. Since $a_0 \in \mathcal{V}$, we have $p(a_0) = 0$ if and only if $a_0 = 0$. As a consequence, $\ker p = h\mathcal{AS}_h(\Sigma)$ and p induces a $\mathbb{C}$–algebra isomorphism $\overline{p}\colon \mathcal{AS}_h(\Sigma)/h\mathcal{AS}_h(\Sigma) \to \mathcal{C}(\Sigma)$. $\square$

In [9], Goldman defines a Lie bracket on the free algebra generated by free homotopy classes of curves on Σ. It can be described in terms of resolutions of intersections and is of a purely topological nature. Following this construction, we consider the *Goldman bracket* on $\mathcal{C}(\Sigma)$ to be the bilinear map $\{\ ,\ \}\colon \mathcal{C}(\Sigma) \times \mathcal{C}(\Sigma) \to \mathcal{C}(\Sigma)$ defined as follows:

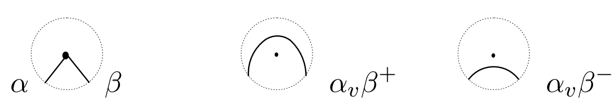
- for a puncture v and a generalized curve α, we let $\{v, \alpha\} = 0$;
- for two generalized curves α and β, we let

$$\{\alpha, \beta\} = \frac{1}{2} \sum_{p \in \alpha \cap \beta \cap \Sigma} (\alpha_p \beta^+ - \alpha_p \beta^-) + \frac{1}{4} \sum_{v \in \alpha \cap \beta \cap V} \frac{1}{v} (\alpha_v \beta^+ - \alpha_v \beta^-).$$

Here, the positive resolution $\alpha_p\beta^+$ of α and β at p is obtained by going along β toward p then turning right at p, and the negative resolution $\alpha_p\beta^-$ by going along β then turning left (see the figure below);



the positive resolution $\alpha_v\beta^+$ of α and β at the puncture v is obtained by going along β toward v then turning right around v, and the negative resolution $\alpha_p\beta^-$ by going along β then turning left (see the figure below).

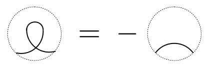


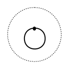
With these notions, we have the following theorem.

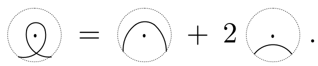
Theorem 2.2.3. *The algebra* $\big(\mathcal{C}(\Sigma), \cdot, \{\ ,\ \}\big)$ *is a well defined Poisson algebra.*

The following Lemma is needed in the proof of Theorem 2.2.3.

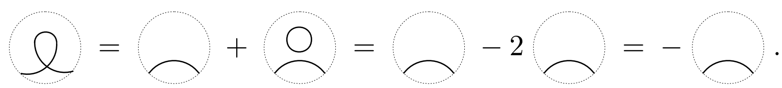
Lemma 2.2.4. *We have the following identities:*

(1) 

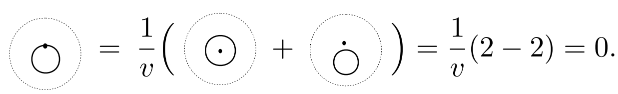
(2)  $= 0$*, and*

(3) 

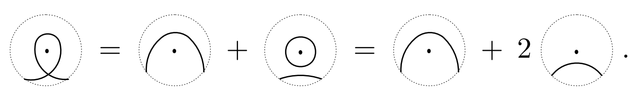
Proof. For (1), we have



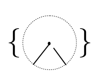
For (2), we have

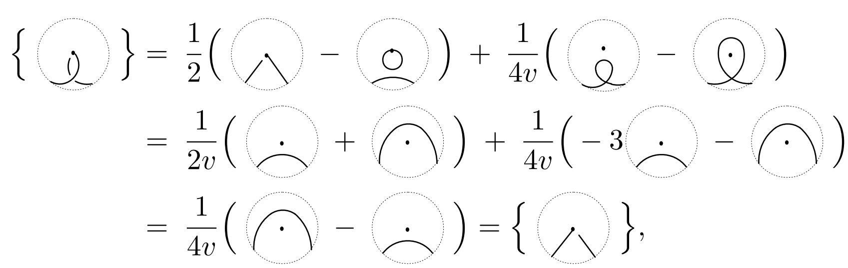


For (3), we have



□

Proof of Theorem 2.2.3. In order to verify the well definition, it suffices to show that $\{\ ,\ \}$ is invariant under Reidemeister Moves II, II$'$ and III. The invariance under Reidemeister Moves II and III follows from arguments in [9]. For the invariance under Reidemeister Move II$'$, we denote by  the sum of the terms in $\{\alpha, \beta\}$ coming from the crossings in the dotted circle. We have



where the second equality follows from Lemma 2.2.4. The anti-symmetry of $\{\ ,\ \}$ follows from the fact that $\alpha_x\beta^{\pm} = \beta_x\alpha^{\mp}$ for each $x \in \alpha \cap \beta$ either in the surface or at the punctures. The verification of the Jacobi identity is in the spirit of Goldman [9] separating the following two cases:

(1) $\alpha \cap \beta \cap \gamma \cap V = \emptyset$, and

(2) $\alpha \cap \beta \cap \gamma \cap V \neq \emptyset$.

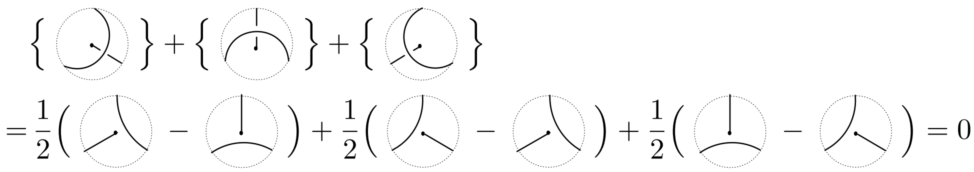
In case (1), we let α, β and γ be three generalized curves on Σ. We let $c(x,y) = \frac{1}{4}$ if $x, y \in \Sigma$ and $c(x,y) = \frac{1}{16}x^{-1}y^{-1}$ if $x, y \in V$; and if only one of x and y, say x, is a puncture of Σ, we let $c(x,y) = \frac{1}{8}x^{-1}$. Then we have

$$\begin{aligned}
&\{\{\alpha,\beta\},\gamma\} \\
&= \sum_{\substack{x \in \alpha\cap\beta \\ y \in \beta\cap\gamma}} c(x,y)\big((\alpha_x\beta^+)_y\gamma^+ - (\alpha_x\beta^+)_y\gamma^- - (\alpha_x\beta^-)_y\gamma^+ + (\alpha_x\beta^-)_y\gamma^-\big) \\
&+ \sum_{\substack{x \in \alpha\cap\beta \\ z \in \gamma\cap\alpha}} c(x,z)\big((\alpha_x\beta^+)_z\gamma^+ - (\alpha_x\beta^+)_z\gamma^- - (\alpha_x\beta^-)_z\gamma^+ + (\alpha_x\beta^-)_z\gamma^-\big),
\end{aligned}$$

and

$$\begin{aligned}
&\{\{\beta,\gamma\},\alpha\} \\
&= \sum_{\substack{y \in \beta\cap\gamma \\ z \in \gamma\cap\alpha}} c(y,z)\big((\beta_y\gamma^+)_z\alpha^+ - (\beta_y\gamma^+)_z\alpha^- - (\beta_y\gamma^-)_z\alpha^+ + (\beta_y\gamma^-)_z\alpha^-\big) \\
&+ \sum_{\substack{y \in \beta\cap\gamma \\ x \in \alpha\cap\beta}} c(y,x)\big((\beta_y\gamma^+)_x\alpha^+ - (\beta_y\gamma^+)_x\alpha^- - (\beta_y\gamma^-)_x\alpha^+ + (\beta_y\gamma^-)_x\alpha^-\big).
\end{aligned}$$

By definition, we have that $(\alpha_x\beta^+)_y\gamma^+ = (\beta_y\gamma^+)_x\alpha^-$, $(\alpha_x\beta^+)_y\gamma^- = (\beta_y\gamma^-)_x\alpha^-$, $(\alpha_x\beta^-)_y\gamma^+ = (\beta_y\gamma^+)_x\alpha^+$ and $(\alpha_x\beta^-)_y\gamma^- = (\beta_y\gamma^-)_x\alpha^+$ for each $x \in \alpha\cap\beta$ and $y \in \beta\cap\gamma$, so the summands in the first row of the expansion of $\{\{\alpha,\beta\},\gamma\}$ cancels out the summands in the second row of the expansion of $\{\{\beta,\gamma\},\alpha\}$. By the similar reason, the summands in the second row of the expansion of $\{\{\alpha,\beta\},\gamma\}$ and the first row of the expansion of $\{\{\beta,\gamma\},\alpha\}$ cancel out the summands in the expansion of $\{\{\gamma,\alpha\},\beta\}$. Hence $\{\{\alpha,\beta\},\gamma\} + \{\{\beta,\gamma\},\alpha\} + \{\{\gamma,\alpha\},\beta\} = 0$. In case (2), let $v \in \alpha\cap\beta\cap\gamma$, and we may without loss of generality assume that α, β and γ are counter-clockwise ordered at v. Then all the summands in $\{\{\alpha,\beta\},\gamma\} + \{\{\beta,\gamma\},\alpha\} + \{\{\gamma,\alpha\},\beta\}$ cancel out in pairs as in case (1) except three summands around v which are from $\frac{1}{4}v^{-1}\{\alpha_v\beta^+,\gamma\}$, $\frac{1}{4}v^{-1}\{\beta_v\gamma^+,\alpha\}$ and $\frac{1}{4}v^{-1}\{\gamma_v\alpha^+,\beta\}$ respectively; and for the sum of them, we have



The Leibniz rule is directly from the definition and that $(\alpha \cdot \beta) \cap \gamma = (\alpha \cup \beta) \cap \gamma = (\alpha \cap \gamma) \cup (\beta \cap \gamma)$. $\square$

A topologically free $\mathbb{C}[[h]]$–algebra A_h is called a *quantization* of a Poisson algebra A if there is a $\mathbb{C}$–algebra isomorphism $\Theta\colon A_h/hA_h \to A$ such that

$$\Theta\Big(\frac{\bar{\alpha} \cdot \bar{\beta} - \bar{\beta} \cdot \bar{\alpha}}{h}\Big) = \{\alpha, \beta\}$$

for all $\bar{\alpha} \in \Theta^{-1}(\alpha)$ and $\bar{\beta} \in \Theta^{-1}(\beta)$ (see for example [12]). Using the isomorphism from Proposition 2.2.2, we obtain the following theorem.

Theorem 2.2.5. *The $\mathbb{C}[[h]]$–algebra $\mathcal{AS}_h(\Sigma)$ is a quantization of $\mathcal{C}(\Sigma)$ via the $\mathbb{C}$–algebra isomorphism $\overline{\mathrm{p}}$.*

Proof. This is an analogue of the arguments in [7]. Given a diagram on the surface, we let $p^{\pm}(\mathcal{S})$ respectively be the number of positive and negative resolutions in the surface used to obtain the state $\mathcal{S}$, and let $v^{\pm}(\mathcal{S})$ respectively be the number of positive and negative resolutions at the punctures used to obtain $\mathcal{S}$. Keeping track of the crossings, we have

$$\{\alpha, \beta\} = \sum_{\mathcal{S}} \Big(\frac{1}{2}\big(p^+(\mathcal{S}) - p^-(\mathcal{S})\big) + \frac{1}{4}\big(v^+(\mathcal{S}) - v^-(\mathcal{S})\big)\Big)\mathcal{S},$$

where the summation is taken over all states $\mathcal{S}$ obtained from resolving $\alpha \cup \beta$, and

$$\begin{aligned}\bar{\alpha} \cdot \bar{\beta} - \bar{\beta} \cdot \bar{\alpha} = \sum_{\mathcal{S}} \Big(& q^{(p^+(\mathcal{S}) - p^-(\mathcal{S})) + \frac{1}{2}(v^+(\mathcal{S}) - v^-(\mathcal{S}))} \\ & - q^{-(p^+(\mathcal{S}) - p^-(\mathcal{S})) - \frac{1}{2}(v^+(\mathcal{S}) - v^-(\mathcal{S}))}\Big)\mathcal{S},\end{aligned}$$

in which the coefficient of h is exactly $\{\alpha, \beta\}$. $\square$

In particular, the proof of Theorem 2.2.5 explains the relationship between the coefficients $q^{\pm 1/2}$ in the puncture-skein relation used in the definition of $\mathcal{AS}_h(\Sigma)$ and the coefficient $\frac{1}{4}$ in front of the puncture terms in the Goldman bracket on $\mathcal{C}(\Sigma)$. Both of these choices were essential at some point in the well-definition of $\mathcal{AS}_h(\Sigma)$ and $\{\ ,\ \}$ and turn out to be related to the geometric aspects of the theory described in the next chapter.

Chapter 3

Relationship with hyperbolic geometry

An essential aspect of the skein algebra is its relationship with the character variety $X(\Sigma)$. This connection can be found in the work of Turaev [20] and was unraveled by the work of Bullock, Frohman and Kania-Bartoszyńska [7] and Przytycki and Sikora [18]. In our context the right framework is that of the decorated Teichmüller space and the notion of λ-length, which as we will see can be understood as generalized trace functions.

3.1 The decorated Teichmüller space and its Poisson structure

As before we let Σ be a surface with a set of punctures $V = \{v_1, \ldots, v_s\}$. We consider the *cusped Teichmüller space* $\mathcal{T}_c(\Sigma)$ defined as the set of isotopy classes of complete hyperbolic metrics on Σ with finite area. A *decoration* r of a hyperbolic metric $m \in \mathcal{T}_c(\Sigma)$ is given by a choice of a positive real number $r_i = r(v_i)$ associated to each of the punctures. Geometrically, a decoration should be interpreted as a choice of a horocycle of length r_i centered at each of the punctures v_i of Σ for the metric m. The *decorated Teichmüller space* $\mathcal{T}^d(\Sigma)$ is then the set of decorated hyperbolic metrics (m, r), $m \in \mathcal{T}_c(\Sigma)$. It was introduced by Penner in [15]. Topologically, it is the fiber bundle $\mathcal{T}^d(\Sigma) = \mathcal{T}_c(\Sigma) \times \mathbb{R}^V_{>0}$ over the cusped Teichmüller space.

One of the reasons for introducing the notion of decorations is to permit the measure of the length of arcs between punctures. More precisely, given a decorated hyperbolic metric (m, r) and an arc α in Σ between two punctures, consider a geodesic lift $\widetilde{\alpha}$ of α to the universal cover $\mathbb{H}^2$ of (Σ, m). The *length* $l(\alpha)$ of α for (m, r) is then defined to be the signed length of the segment of $\widetilde{\alpha}$ between the horocycles given by the decoration, where the sign is chosen to be positive if the horocycles do not intersect and negative if they do. A number of properties concerning length of arcs can in fact be expressed in terms of the associated *λ-length* $\lambda(\alpha) = e^{\frac{l(\alpha)}{2}}$. In particular, if a, b, c, d are the consecutive sides of a square in Σ and e, e' are its diagonals, they satisfy the Ptolemy relation [16]

$$\lambda(e)\lambda(e') = \lambda(a)\lambda(c) + \lambda(b)\lambda(d).$$

Let T be an ideal triangulation of Σ, that is, a maximal collection of isotopy classes of arcs between punctures in Σ which decompose the surface into ideal triangles. We let E be the set of edges of T. Then the associated lengths $l(e)$, $e \in E$, form a coordinate system on $\mathcal{T}^d(\Sigma)$. In these coordinates, Mondello [13] introduces a Poisson bi-vector field on $\mathcal{T}^d(\Sigma)$ defined as follows: For two geodesic arcs α and β on a decorated hyperbolic surface Σ meeting at a puncture v, let θ_v be the generalized angle from α to β, that is, the length of the horocycle segment measured counterclockwise from α to β, and let θ'_v be the generalized angle from β to α at v. Then the Poisson bi-vector field

$$\Omega_{WP} = \frac{1}{4} \sum_{v \in V} \sum_{\substack{\alpha,\beta \in E \\ \alpha \cap \beta = v}} \frac{\theta'_v - \theta_v}{r(v)} \frac{\partial}{\partial l(\alpha)} \wedge \frac{\partial}{\partial l(\beta)}$$

on the decorated Teichmüller space descends to the usual Weil-Petersson Poisson bi-vector field on the Teichmüller space $\mathcal{T}_c(\Sigma)$. It can be shown by a direct calculation that this expression is invariant under a diagonal switch. As a consequence, Ω_{WP} is independent of the choice of the ideal triangulation T.

If α is a closed curve on Σ and $m \in \mathcal{T}_c(\Sigma)$, we consider the quantity $\lambda(\alpha) = 2\cosh\frac{l(\alpha)}{2}$ where $l(\alpha)$ is the length of the geodesic representative of α for m. Up to a sign, it is equal to the trace $tr(\rho(\alpha))$ of the monodromy representation $\rho\colon \pi_1(\Sigma) \to PSL_2(\mathbb{R})$ associated to m. We purposefully used the same notations as for λ-lengths and call $\lambda(\alpha)$ the *generalized trace* of α, where α can be an arc or a closed curve on Σ.

The goal of this chapter is to construct a map from the algebra of curves $\mathcal{C}(\Sigma)$ to the algebra of functions over $\mathcal{T}^d(\Sigma)$ by associating to a generalized curve the product of the generalized traces of its components. One issue however is the fact that elements of $\mathcal{C}(\Sigma)$ are not identified up to Reidemeister Move I, and hence we need to introduce the following definition.

Definition 3.1.1. *In Figure 3.1, we call c a* curl *and p its* vertex. *If α is an arc or a non-null-homotopic curve on Σ, then the* curling number $c(\alpha)$ *of α is the maximal number of the* one way Reidemeister Move I *(Figure 3.1) α carries; and if α is a null-homotopic curve, then $c(\alpha)$ is the corresponding number plus one.*

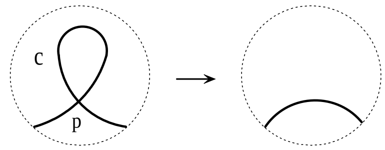


Figure 3.1: One way Reidemeister Move I.

Example 3.1.2. $c(\odot) = 0$ and $c(\bigcirc) = 1$.

Example 3.1.3. Since a geodesic minimizes self-intersections, the curling number of a geodesic is always 0.

Using this definition, if $\alpha = \alpha_1 \cup \cdots \cup \alpha_n$ is a generalized curve, that is, a union of equivalence classes of arcs and closed curves on Σ, then we let $c(\alpha) = \sum_i c(\alpha_i)$ and $\lambda(\alpha) = \prod_i \lambda(\alpha_i)$. We recall that, if (m, r) is a decorated hyperbolic metric, then $r(v)$ denotes the length of the horocycle at the puncture v of Σ.

Theorem 3.1.4. *The map $\Phi\colon \mathcal{C}(\Sigma) \to C^\infty(\mathcal{T}^d(\Sigma))$ defined on the generators by $\Phi(v) = r(v)$ if v is a puncture and $\Phi(\alpha) = (-1)^{c(\alpha)}\lambda(\alpha)$ if α is a generalized curve is a well defined Poisson algebra homomorphism with respect to $\{\ ,\ \}$ on $\mathcal{C}(\Sigma)$ and the Weil-Petersson Poisson bracket on $C^\infty(\mathcal{T}^d(\Sigma))$ associated to the bi-vector field Ω_{WP}.*

The remainder of this article will be dedicated to the proof of this theorem. Firstly, we are going to derive a series of lengths identities in hyperbolic geometry which generalize the Ptolemy relation, the trace identity and Wolpert's cosine formula for the Weil-Petersson Poisson bracket of length functions. Together with an analysis of the behavior of the curling number under resolutions, half of these identities can be combined into generalized trace identities which imply that the map defined above is an algebra homomorphism. Finally, combined with a lemma about the expression of generalized trace functions in terms of the λ-lengths associated to the edges of a fixed ideal triangulation, the other half will give the Poisson algebra homomorphism.

3.2 The lengths identities

In this chapter we are going to derive a series of identities involving geodesic lengths of curves and arcs between horocycles which are the heart of the proof of Theorem 3.1.4. They rely on a set of "cosine laws" for various types of generalized hyperbolic triangles which can be found in the appendix of [10]. The twisted version of them are also needed and included in the appendix to this dissertation.

Throughout this section, we will fix a punctured surface Σ and a decorated hyperbolic metric $(m, r) \in \mathcal{T}^d(\Sigma)$. We recall that if α and β are two geodesics on Σ for (m, r), then the angle from α to β at $p \in \alpha \cap \beta$ in Σ is the angle measured counterclockwise from α to β, and the generalized angle from α to β at $v \in \alpha \cap \beta$ at V is the length of the horocycle segment measured counterclockwise from α to β.

Lemma 3.2.1. *Let α and β be two closed geodesics of lengths a and b, and let θ be the angle from α to β at $p \in \alpha \cap \beta$. If x and y respectively are the lengths of the geodesic representatives of $\alpha_p\beta^+$ and $\alpha_p\beta^-$, then we have*

(1) $\cosh\frac{x}{2} + \cosh\frac{y}{2} = 2\cosh\frac{a}{2}\cosh\frac{b}{2}$,

(2) $\cosh\frac{x}{2} - \cosh\frac{y}{2} = 2\sinh\frac{a}{2}\sinh\frac{b}{2}\cos\theta$.

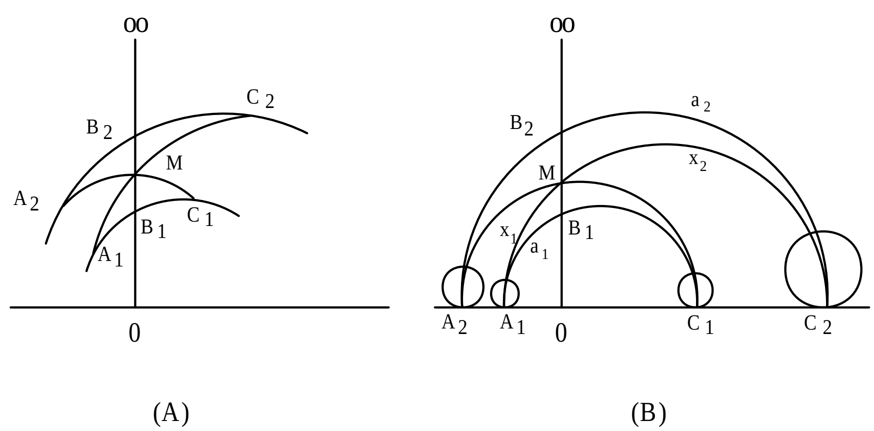


Figure 3.2:

Proof. Formula (1) is from the trace identity $trA \cdot trB = trAB + trAB^{-1}$. For formula (2), let us look at (A) of Figure 3.2. Let $\overline{0\infty}$ be a lift of the geodesic β in the universal cover $\mathbb{H}^2$ of Σ. Let $\{B_i\}_{i\in\mathbb{Z}}$ be the lifts of p on $\overline{0\infty}$ so that $|B_iB_{i+1}| = b$. Let A_i and C_i for $i = 1, 2$ be the points on the lift of the geodesic α passing through B_i so that $|A_iB_i| = |B_iC_i| = \frac{a}{2}$, hence $|A_iC_i| = a$ and $\overline{A_iC_i}$ is a lift of α for $i = 1, 2$. Now take the mid-point

M of $\overline{B_1B_2}$, and connect A_1 and C_2 to M by geodesics. Since $|A_1B_1| = |B_2C_2| = \frac{a}{2}$ and $|B_1M| = |MB_2| = \frac{b}{2}$, and $\angle A_1B_1M = \angle MB_2C_2 = \pi - \theta$, the triangles A_1B_1M and MB_2C_2 are isometric, hence the anlges $\angle A_1MB_1 = \angle B_2MC_2$ and $\angle B_1A_1M = \angle MC_2B_2$. Therefore, the points A_1, M and C_2 are on the geodesic representing a lift of $\alpha_p\beta^+$, and $|A_1M| = |MC_2| = \frac{x}{2}$. By the same argument, we have that A_2, M and C_1 are on a lift of $\alpha_p\beta^-$ and $|A_2M| = |MC_1| = \frac{y}{2}$. Applying the cosine law to the triangles A_1B_1M and A_2B_2M respectively, we have $\cos(\pi - \theta) = \frac{-\cosh\frac{x}{2}+\cosh\frac{a}{2}\cosh\frac{b}{2}}{\sinh\frac{a}{2}\sinh\frac{b}{2}}$ and $\cos\theta = \frac{-\cosh\frac{y}{2}+\cosh\frac{a}{2}\cosh\frac{b}{2}}{\sinh\frac{a}{2}\sinh\frac{b}{2}}$. Since $\cos(\pi-\theta) = -\cos\theta$, the difference of the two equalities implies formula (2). Note that from the sum of these two equalities we also get formula (1). $\square$

Lemma 3.2.2. *Let α be a geodesic arc of length a and let β be a closed geodesic of length b. Let θ be the angle from α to β at $p \in \alpha \cap \beta$. If x and y respectively are the lengths of geodesic representatives of the arcs $\alpha_p\beta^+$ and $\alpha_p\beta^-$, then we have*

(1) $e^{\frac{x}{2}} + e^{\frac{y}{2}} = 2e^{\frac{a}{2}}\cosh\frac{b}{2}$,

(2) $e^{\frac{x}{2}} - e^{\frac{y}{2}} = 2e^{\frac{a}{2}}\sinh\frac{b}{2}\cos\theta$.

Proof. Let us look at (B) of Figure 3.2. Let $\overline{0\infty}$ be a lift of α in the universal cover $\mathbb{H}^2$. Let $\{B_i\}_{i\in\mathbb{Z}}$ be the lifts of p on $\overline{0\infty}$ so that $|B_i, B_{i+1}| = a$, and let A_i and C_i for $i = 1, 2$ be the end points of the lifts of β passing through B_i. Let M be the intersection of $\overline{0\infty}$ and the geodesic connecting A_1 and C_2. Let a_1 be the distance from B_i to the horocycle centered at A_i and let a_2 be the distance from B_i to C_i for $i = 1, 2$ so that $a_1 + a_2 = a$, and let x_1 be the distance from M to the horocycle centered at A_1 and let x_2 be the distance from M to the horocycle centered at C_2 so that $x_1 + x_2 = x$. Since $\angle A_1B_1M = \angle C_2B_2M$ and $\angle A_1MB_1 = \angle C_2MB_2$, we have that the ideal triangles A_1B_1M and C_2B_2M of type $(0, 1, 1)$ are isometric which implies that $|B_1M| = |MB_2| = \frac{b}{2}$. Applying the cosine law to the triangle A_1B_1M, we have $\cos(\pi-\theta) = \frac{-e^{x_1}+e^{a_1}\cosh\frac{b}{2}}{e^{a_1}\sinh\frac{b}{2}}$. Applying the sine law to the triangles A_1B_1M and C_2B_2M, we have $\frac{e^{a_1}}{e^{x_1}} = \frac{\sin\angle A_1MB_1}{\sin\angle A_1B_1M} = \frac{\sin\angle C_2MB_2}{\sin\angle C_2B_2M} = \frac{e^{a_2}}{e^{x_2}}$, hence $\frac{a_2-a_1}{2} = \frac{x_2-x_1}{2}$. With this the cosine law above becomes $\cos(\pi-\theta) = \frac{-e^{\frac{x}{2}}+e^{\frac{a}{2}}\cosh\frac{b}{2}}{e^{\frac{a}{2}}\sinh\frac{b}{2}}$. By the same argument to the generalized triangles A_2B_2M and B_1C_1M, we have $\cos\theta = \frac{-e^{\frac{y}{2}}+e^{\frac{a}{2}}\cosh\frac{b}{2}}{e^{\frac{a}{2}}\sinh\frac{b}{2}}$. Formula (1) is from the sum of the two equalities above and formula (2) follows from the difference of them. $\square$

Lemma 3.2.3. *(**Penner [16]**) Let α and β be two geodesic arcs of lengths a and b, and let θ be the angle from α to β at $p \in \alpha \cap \beta$. If x and x' respectively are the lengths of the geodesic representatives of the arc components of $\alpha_p\beta^+$, and y and y' respectively are the lengths of the geodesic representatives of the arc components of $\alpha_p\beta^-$, then we have*

(1) $e^{\frac{x}{2}}e^{\frac{x'}{2}} + e^{\frac{y}{2}}e^{\frac{y'}{2}} = e^{\frac{a}{2}}e^{\frac{b}{2}}$,

(2) $e^{\frac{x}{2}}e^{\frac{x'}{2}} - e^{\frac{y}{2}}e^{\frac{y'}{2}} = e^{\frac{a}{2}}e^{\frac{b}{2}}\cos\theta$. □

Lemma 3.2.4. *If α and β are two geodesic arcs of lengths a and b meeting at a puncture v, and θ and θ' respectively are the generalized angle from α to β and the generalized angle from β to α. Let r be the length of the horocycle centered at v, and let x and y respectively be the lengths of the geodesic representatives of $\alpha_v\beta^+$ and $\alpha_v\beta^-$. Then we have*

(1) $e^{\frac{x}{2}} + e^{\frac{y}{2}} = re^{\frac{a}{2}}e^{\frac{b}{2}}$,

(2) $e^{\frac{x}{2}} - e^{\frac{y}{2}} = (\theta' - \theta)e^{\frac{a}{2}}e^{\frac{b}{2}}$.

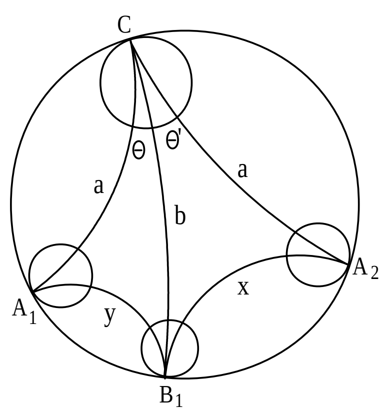


Figure 3.3:

Proof. Let us look at Figure 3.3. Let C be a lift of v in the universal cover $\mathbb{H}^2$, and let $\overline{A_1C}$ and $\overline{B_1C}$ be the lifts of α and β passing through C respectively. Then $\overline{A_2B_1}$ and $\overline{A_1B_1}$ are respectively lifts of $\alpha_v\beta^+$ and $\alpha_v\beta^-$. Applying the cosine law to the ideal triangles CA_2B_1 and CA_1B_1, we have $\theta' = e^{\frac{x-a-b}{2}}$ and $\theta = e^{\frac{y-a-b}{2}}$. Since $r = \theta + \theta'$, the sum of the two equalities above implies formula (1), and the difference implies formula (2). □

For each point of self-intersection p of an arc or a closed curve α, one of its two resolutions at p is connected and the other one is not. We call the former one the *non-separating resolution* of α and the later one the *separating resolution* of α. Note that if α is an arc, then the separating resolution of α consists of an arc and a closed curve, which we call the *arc component* and the *closed component* respectively. We have

Lemma 3.2.5. *Let α be a closed geodesic or a geodesic arc. Then the curling number $c(\beta)$ of the non-separating resolution β of α at each of its points of self-intersection is at most 1; and the only possibility that $c(\beta) = 1$ is as shown in Figure 3.4.*

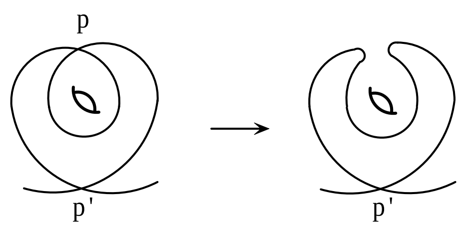


Figure 3.4: The possibility that $c(\beta) = 1$.

Proof. If $c(\beta) > 0$, then let c be one of its curl and let p' be the vertex of c. Let α_1 and α_2 be the components of $\alpha - p$, there are the following three cases:

(a) $p' \in \alpha_i$ and $c \subset \alpha_i$ for $i = 1$ or 2,

(b) $p' \in \alpha_1 \cap \alpha_2$, or

(c) $p' \in \alpha_i$ but $c \not\subseteq \alpha_i$ for $i = 1$ or 2.

If (a) occurred, then the geodesic α would contain a curl, which is excluded; and if (b) occurred, then α would contain a bi-gon, which is also excluded. Therefore, the only possibility is (c). In this case, if $p' \in \alpha_1$, then $\alpha_2 \subset c$ and c is the unique curl in β. Since the curl c is contractible, the arc α_2 must be simple, which is the case as in Figure 3.4. $\square$

Lemma 3.2.6. *If α is a closed geodesic of length a and p is one of its self-intersection points. Let x and y respectively be the lengths of the geodesic representatives of the two components of the separating resolution of α, and let z be the length of the geodesic representative of the non-separating resolution β of α.*

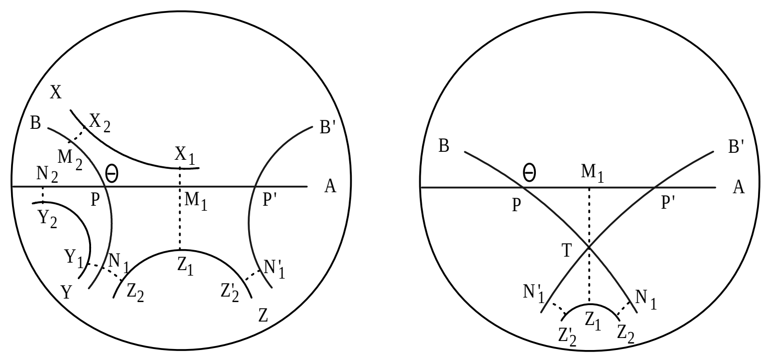
(1) If $c(\beta)=0$*, then*

$$\cosh\frac{a}{2}=2\cosh\frac{x}{2}\cosh\frac{y}{2}+\cosh\frac{z}{2}.$$

(2) If $c(\beta)=1$*, then*

$$\cosh\frac{a}{2}=2\cosh\frac{x}{2}\cosh\frac{y}{2}-\cosh\frac{z}{2}.$$

Moreover, the formulae still hold when some components of the resolutions of α *are circles around a puncture.*



(A) (B)

Figure 3.5:

Proof. For (1), let us look at (A) of Figure 3.5. Let P be a lift of p in the universal cover $\mathbb{H}^2$, and let θ the angle between the two lifts A and B of α passing through P. Let X and Y be the corresponding lifts of the components of the separating resolution of α, and let Z be the corresponding lift of the non-separating resolution of α. Let $M_1\in A$, $N_1\in B$ and $Z_1, Z_2\in Z$ such that $|M_1Z_1|$ realizes the distance $d(A,Z)$ and $|N_1Z_2|$ realizes the distance $d(B,Z)$. Let P' be the lift of p on A next to P and let B' be the lift of α passing through P'. Let $N_1'\in B'$ and $Z_2'\in Z$ such that $|N_1'Z_2'|$ realizes $d(B',Z)$. Then $\overline{N_1'Z_2'}$ and $\overline{N_1Z_2}$ are the lifts of the same geodesic segment, which implies that $|N_1'Z_2'|=|N_1Z_2|$. Since $\angle N_1'P'M_1=\angle N_1PM_1$, the generalized triangles $M_1Z_1Z_2'N_1'P'$ and $M_1Z_1Z_2N_1P$ of type $(1,-1,-1)$ are isometric. Therefore, the lengths $|Z_2'Z_1|=|Z_1Z_2|=\frac{z}{2}$ and $|P'M_1|=|PM_1|\doteq a_1$. Let $M_1''\in A$ and $X_1\in X$ such that $|M_1''X_1|$ realizes $d(A,X)$, and let $N_1''\in B$ and $Y_1\in Y$ such that $|N_1''Y_1|$ realizes $d(B,Y)$. By a similar argument as above, we see that $|PM_1''|=\frac{1}{2}|P'P|=a_1$ and $PN_1''=PN_1\doteq a_2$. Let $M_2\in B$, $N_2\in A$, $X_2\in X$ and $Y_2\in Y$ such that $|M_2X_2|$

realizes $d(B, X)$ and $|N_2Y_2|$ realizes $d(A, Y)$. Then as above, we have $|PM_2| = a_1$ and $|PN_2| = a_2$. Since $|PM_1| = |PM_2| = \frac{1}{2}|P'P|$, the points M_1 and M_2 cover the same point on the surface Σ, hence X_1 and X_2 cover the same point on Σ and $|X_1X_2| = x$. The same argument implies $|Y_1Y_2| = y$. Applying the cosine law to the generalized triangles $PM_1X_1X_2M_2$ and $PN_1Y_1Y_2N_2$ of type $(1, -1, -1)$, we have $\cos\theta = \frac{-\cosh x+\sinh^2\frac{a_1}{2}}{\cosh^2\frac{a_1}{2}} = \frac{-\cosh y+\sinh^2\frac{a_2}{2}}{\cosh^2\frac{a_2}{2}}$, which implies $\sin^2\frac{\theta}{2} = \frac{\cosh\frac{x}{2}\cosh\frac{y}{2}}{\cosh\frac{a_1}{2}\cosh\frac{a_2}{2}}$. Applying the cosine law to the generalized triangle $PM_1Z_1Z_2N_1$ of the same type, we have $\cos(\pi-\theta) = \frac{-\cosh\frac{z}{2}+\sinh\frac{a_1}{2}\sinh\frac{a_2}{2}}{\cosh\frac{a_1}{2}\cosh\frac{a_2}{2}}$. From the last two equations and the identity $\cos(\pi-\theta) = 2\sin^2\frac{\theta}{2} - 1$, we get the result. Note that when some components of the resolutions of α are curves around a puncture, then the corresponding lengths x, y or z tend to 0, and the corresponding generalized triangles become union of generalized ideal triangles of type $(0, 1, 1)$. Applying the cosine law for such triangles we get formula (1) in these degenerated cases.

For (2), let us look at (B) of Figure 3.5. Similarly Applying the cosine law to the generalized triangles $PM_1X_1X_2M_2$ and $PN_1Y_1Y_2N_2$ of type $(1, -1, -1)$, we have $\cos\theta = \frac{-\cosh x+\sinh^2\frac{a_1}{2}}{\cosh^2\frac{a_1}{2}} = \frac{-\cosh y+\sinh^2\frac{a_2}{2}}{\cosh^2\frac{a_2}{2}}$, which implies $\sin^2\frac{\theta}{2} = \frac{\cosh\frac{x}{2}\cosh\frac{y}{2}}{\cosh\frac{a_1}{2}\cosh\frac{a_2}{2}}$. Since $c(\beta) = 1$, there is an intersection T of B and B' and the generalized triangle $PM_1Z_1Z_2N_1$ of type $(1, -1, -1)$ is twisted. Applying the cosine law for such generalized triangle (see Appendix), we have $\cos(\pi-\theta) = \frac{\cosh\frac{z}{2}+\sinh\frac{a_1}{2}\sinh\frac{a_2}{2}}{\cosh\frac{a_1}{2}\cosh\frac{a_2}{2}}$, and the identity $\cos(\pi-\theta) = 2\sin^2\frac{\theta}{2} - 1$ implies the result. $\square$

Lemma 3.2.7. *If α is a geodesic arc of length a and p is one of its points of self-intersection. Let x be the length of the geodesic representative of closed component of the separating resolution of α and let y be the length of the geodesic representative of the arc component in the separating resolution of α. Let z be the length of the geodesic representative of the non-separating resolution β of α.*

(1) If $c(\beta) = 0$, then

$$e^{\frac{a}{2}} = 2\cosh\frac{x}{2}e^{\frac{y}{2}} + e^{\frac{z}{2}}.$$

(2) If $c(\beta) = 1$, then

$$e^{\frac{a}{2}} = 2\cosh\frac{x}{2}e^{\frac{y}{2}} - e^{\frac{z}{2}}.$$

Moreover, the formulae still hold when the closed component in the separating resolution of α *is a circle around a puncture.*

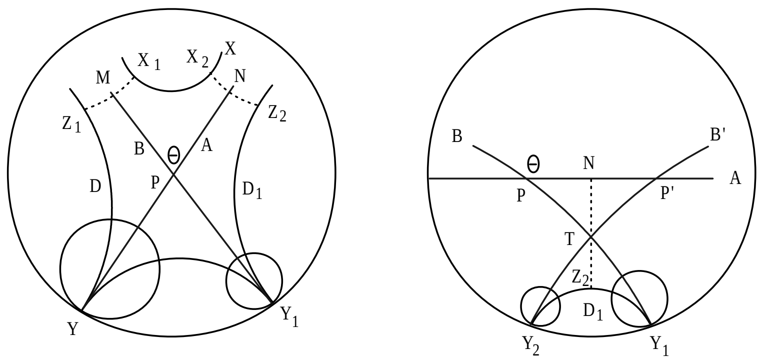


Figure 3.6:

Proof. For (1), let P in (A) of Figure 3.6 be a lift of p in the universal cover $\mathbb{H}^2$, and let A and B be the two lifts of α passing through P with θ the angle between them at p. Let the end point Y of A and the end point Y_1 of B respectively be the lifts of the two end points of α so that $\overline{YY_1}$ is a lift of the geodesic representative of the arc component of the separating resolution of α. Let X be the corresponding lift of the geodesic representative of the closed curve component of the separating resolution of α, and let D and D_1 be the lifts of the geodesic representative of the non-separating resolution of α. We take points X_1 and $X_2 \in X$, $M \in B$ and $N \in A$ such that $|MX_1|$ realizes $d(B, X)$ and $|NX_2|$ realizes $d(A, X)$. Since $\overline{MX_1}$ and $\overline{NX_2}$ cover the same curve on Σ, we have $|MX_1| = |NX_2|$. Applying the sine law to the generalized triangle PMX_1X_2N of type $(-1,-1,1)$, we have $|PM| = |PN| \doteq \frac{a_3}{2}$. Suppose $M' \in A'$, $N' \in A$, $Z_1 \in D$ and $Z_2 \in D_1$ are the points such that $|M'Z_1|$ realizes $d(B, D)$ and $|N'Z_2|$ realizes $d(A, D_1)$. Then $\overline{M'Z_1}$ and $\overline{N'Z_2}$ cover the same curve on Σ, hence $|M'Z_1| = |N'Z_2|$. Applying the cosine law the the generalized ideal triangles PYZ_1M' and PY_1Z_2N' of type $(-1,0,1)$, we see that $\sinh|PM'| = \frac{1+\cos(\pi-\theta)\cosh|M'Z_1|}{\sin(\pi-\theta)\sinh|M'Z_1|} = \frac{1+\cos(\pi-\theta)\cosh|N'Z_2|}{\sin(\pi-\theta)\sinh|N'Z_2|} = \sinh|PN'|$. Therefore, we have $|PM'| = |PN'|$. Hence $M' = M$, $N' = N$ and $|PM'| = |PN'| = \frac{a_3}{2}$. Let H_Y and H_{Y_1} respectively be the horocycles centered at Y and Y_1, and $a_1 = d(P, H_Y)$, $a_2 = d(P, H_{Y_1})$, $z_1 = d(Z_1, H_Y)$ and $z_2 = d(Z_2, H_{Y_1})$. Then $a = a_1 + a_2 + a_3$ and $z = z_1 + z_2$. Applying the cosine law to the generalized ideal triangles PYZ_1M, we have

$\cos(\pi-\theta) = \frac{-e^{z_1}+e^{a_1}\sinh\frac{a_3}{2}}{e^{a_1}\cosh\frac{a_3}{2}}$. From the sine law to the generalized ideal triangles PYZ_1M and PY_1Z_2N, we have $\frac{e^{z_1}}{e^{a_1}} = \frac{\sin(\pi-\theta)}{\sinh|MZ_1|} = \frac{\sin(\pi-\theta)}{\sinh|NZ_2|} = \frac{e^{z_2}}{e^{a_2}}$. Hence $\frac{a_2-a_1}{2} = \frac{z_2-z_1}{2}$. With this the cosine law above becomes $\cos(\pi-\theta) = \frac{-e^{\frac{z}{2}}+e^{\frac{a_1+a_2}{2}}\sinh\frac{a_3}{2}}{e^{\frac{a_1+a_2}{2}}\cosh\frac{a_3}{2}}$, hence $e^{\frac{z}{2}} = e^{\frac{a_1+a_2}{2}}(\sinh\frac{a_3}{2} + \cosh\frac{a_3}{2}\cos\theta)$. Applying the cosine law to the generalized triangle PMX_1X_2N, we have $\cos\theta = \frac{-\cosh x+\sinh^2\frac{a_3}{2}}{\cosh^2\frac{a_3}{2}}$, which implies $2\cosh\frac{x}{2} = 2\cosh\frac{a_3}{2}\sin\frac{\theta}{2}$; and the cosine law to the generalized ideal triangle PYY_1 of type $(0,0,1)$ gives $e^{\frac{y}{2}} = e^{\frac{a_1+a_2}{2}}\sin\frac{\theta}{2}$. Therefore, we have $2\cosh\frac{x}{2}e^{\frac{y}{2}} + e^{\frac{z}{2}} = e^{\frac{a_1+a_2}{2}}(\sinh\frac{a_3}{2} + \cosh\frac{a_3}{2}\cos\theta + 2\cosh\frac{a_3}{2}\sin^2\frac{\theta}{2}) = e^{\frac{a_1+a_2}{2}}e^{\frac{a_3}{2}} = e^{\frac{a}{2}}$.

For (2), let us look at (B) of Figure 3.6. Similarly applying the cosine law to the generalized triangle PMX_1X_2N, we have $\cos\theta = \frac{-\cosh x+\sinh^2\frac{a_3}{2}}{\cosh^2\frac{a_3}{2}}$, which implies $2\cosh\frac{x}{2} = 2\cosh\frac{a_3}{2}\sin\frac{\theta}{2}$; and the cosine law to the generalized ideal triangle PYY' of type $(0,0,1)$ gives $e^{\frac{y}{2}} = e^{\frac{a_1+a_2}{2}}\sin\frac{\theta}{2}$. When $c(\beta) = 1$, there is an intersection T of A' and A'' and the generalized triangles PNZ_2Y' and $P'NZ_2Y''$ of type $(0,1,-1)$ are twisted. Applying the cosine law to PNZ_2Y', we have $\cos(\pi-\theta) = \frac{e^{z_1}+e^{a_1}\sinh\frac{a_3}{2}}{e^{a_1}\cosh\frac{a_3}{2}}$. From the sine law to the generalized ideal triangles PNZ_2Y' and PNZ_2Y'', we have $\frac{e^{z_1}}{e^{a_1}} = \frac{\sin(\pi-\theta)}{\sinh|NZ_2|} = \frac{e^{z_2}}{e^{a_2}}$, which implies $\frac{a_2-a_1}{2} = \frac{z_2-z_1}{2}$. With this the cosine law above becomes $\cos(\pi-\theta) = \frac{e^{\frac{z}{2}}+e^{\frac{a_1+a_2}{2}}\sinh\frac{a_3}{2}}{e^{\frac{a_1+a_2}{2}}\cosh\frac{a_3}{2}}$, hence $-e^{\frac{z}{2}} = e^{\frac{a_1+a_2}{2}}(\sinh\frac{a_3}{2} + \cosh\frac{a_3}{2}\cos\theta)$. Therefore, we have $2\cosh\frac{x}{2}e^{\frac{y}{2}} - e^{\frac{z}{2}} = e^{\frac{a}{2}}$. $\square$

Lemma 3.2.8. *If α is a geodesic arc of length a both of whose end points meet at a puncture v, and r is the length of the horocycle centered at v. Let x and y respectively be the length of the geodesic representative of $\alpha_v^{\pm}$. Then we have*

$$e^{\frac{a}{2}} = \frac{2}{r}(\cosh\frac{x}{2} + \cosh\frac{y}{2}).$$

Moreover, the formula still holds when some of the components of the resolutions of α are circles around a puncture.

Proof. In Figure 3.7, let V be the lift of v and let H_V be the lift of the horocylce centered at V, let $\overline{AV}$ and $\overline{A_1V}$ be the lifts of α passing through V in the universal cover $\mathbb{H}^2$. Let θ_1 be the generalized angle between $\overline{AV}$ and $\overline{A_1V}$ and let $\overline{BB'}$ be the corresponding lift of the geodesic representative of the homotopy class of α_v^+. We take the point A, A_1, B and B' such that $|AB|$ realizes the distance from $\overline{AV}$ to $\overline{BB'}$ and $|A_1B'|$ realizes the distance from

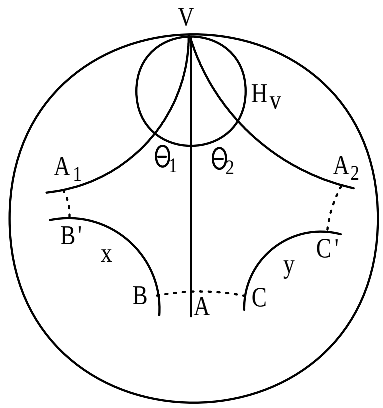


Figure 3.7:

$\overline{A_1V}$ to $\overline{BB'}$. Since $\overline{AB}$ and $\overline{A_1B'}$ cover the same line in Σ, we have $|AB| = |A_1B'|$ and $|BB'| = x$. By the sine law to the generalized triangle $CABB'A'$ of type $(0,-1,-1)$, we have $\frac{e^{d(A,H_V)}}{\sinh|A_1B'|} = \frac{e^{d(A_1,H_V)}}{\sinh|AB|} = 1$, which implies that $d(A,H_V) = d(A_1,H_V) = \frac{a}{2}$. Applying the cosine law to the generalized triangle $CABB'A'$, we have $\theta_1^2 = \frac{\cosh x+1}{\frac{e^a}{2}}$, which implies that $\theta_1 = \frac{2\cosh\frac{x}{2}}{e^{\frac{a}{2}}}$. Similarly, let $\overline{A_2V}$ be the other lift of α adjacent to $\overline{AV}$ and let θ_2 be the generalized angle between $\overline{AV}$ and $\overline{A_2V}$, and have $\theta_2 = \frac{2\cosh\frac{y}{2}}{e^{\frac{a}{2}}}$, from which and the previous identity the formula follows. □

3.3 Generalized trace identities and the algebra homomorphism

Combining the results from the previous section, we obtain the following generalized trace identities.

Proposition 3.3.1. *(a) For a generalized curve α with p one of its self-intersection points in Σ, let α_1 and α_2 be the components of the separating resolution of α at p and let β be the non-separating resolution of α at p. Then we have*

$$(-1)^{c(\alpha)}\lambda(\alpha) = (-1)^{c(\alpha_1)+c(\alpha_2)}\lambda(\alpha_1)\lambda(\alpha_2) + (-1)^{c(\beta)}\lambda(\beta).$$

(b) Let α and β be two generalized curves with $p \in \Sigma$ one of their intersections. If γ_1 and γ_2 are the resolutions of α and β at p, then we have

$$(-1)^{c(\alpha)+c(\beta)}\lambda(\alpha)\lambda(\beta) = (-1)^{c(\gamma_1)}\lambda(\gamma_1) + (-1)^{c(\gamma_2)}\lambda(\gamma_2).$$

Proof. By (1) of Lemma 3.2.1 - 3.2.3, Lemma 3.2.6 and 3.2.7, the formulae (a) and (b) are true when α and β are geodesics. In the general case that the generalized curves are not all geodesics, we use induction. If a generalized curve α has only one self-intersection p which is a vertex of a curl C, then we let α' be the generalized curve obtained from α by removing C via a Reidemeister Move I. We let α_1 and α_2 be the components of the separating resolution of α at p, and let β be the non-separating resolution of α at p. Note that one of α_1 and α_2, say α_1, is a trivial loop since p is the vertex of the curl C; and α_2 and β are equivalent to α'. We have $(-1)^{c(\alpha)}\lambda(\alpha) = -(-1)^{c(\alpha')}\lambda(\alpha')$ and $(-1)^{c(\alpha_1)+c(\alpha_2)}\lambda(\alpha_1)\lambda(\alpha_2) + (-1)^{c(\beta)}\lambda(\beta) = -(-1)^{c(\alpha_2)}2\lambda(\alpha_2) + (-1)^{c(\beta)}\lambda(\beta) = -(-1)^{c(\alpha')}\lambda(\alpha')$. Hence (a) is true in this case. If α has only one self-intersection and no curl, then α is equivalent to a geodesic, and (a) is true by Lemma 3.2.6 and 3.2.7. If two simple generalized curves α and β have only one intersection, then α and β are equivalent to geodesics, and (b) is true by (1) of Lemma 3.2.1 - 3.2.3. Now we assume that formula (a) holds when the number of self-intersections of α is less then n, and formula (b) holds when the number of crossings $\alpha \cup \beta$ is less than n.

For (a), if the number of self-intersections of α is equal to n, there are the following two cases to be considered:

(1) p is not a vertex of a bigon bounded by α; and

(2) p is a vertex of a bigon B bounded by α.

We denote by $\overline{\gamma}$ the unique geodesic in the homotopy class of a generalized curve γ. In case (1), we have that $c(\alpha) = c(\alpha_1) + c(\alpha_2)$ and $0 \leqslant c(\beta) - c(\alpha) \leqslant 1$. Moreover, $c(\beta) - c(\alpha) = 1$ if and only if the non-separating resolution β' of $\overline{\alpha}$ contains a curl, i.e., $c(\beta) - c(\alpha) = c(\beta')$. The way to see this is that we first push all the curls together by isotopy so that away from the curls the curve is equivalent to a geodesic and then apply Proposition 3.2.5. In this case, we have

$$\begin{aligned}(-1)^{c(\alpha)}\lambda(\alpha) &= (-1)^{c(\alpha)}\lambda(\overline{\alpha}) \\ &= (-1)^{c(\alpha)}\big(\lambda(\overline{\alpha_1})\lambda(\overline{\alpha_2}) + (-1)^{c(\beta')}\lambda(\beta')\big) \\ &= (-1)^{c(\alpha_1)+c(\alpha_2)}\lambda(\alpha_1)\lambda(\alpha_2) + (-1)^{c(\beta)}\lambda(\beta).\end{aligned}$$

In case (2), there is a curl C generated from the bigon B whose vertex is the other vertex p' of B . If C is in one of the component of the separating resolution of α, say α_1, then let α_1'

be the curve obtained from α_1 by removing C via a Reidemeister Move I. Let α' be the curve obtained from α by removing B via a Reidemeister Move II. Then we have $c(\alpha') = c(\alpha)$ and $c(\alpha_1') = c(\alpha_1) - 1$, and formula (a) is equivalent to

$$(-1)^{c(\beta)}\lambda(\beta) = (-1)^{c(\alpha')}\lambda(\alpha') + (-1)^{c(\alpha_1')+c(\alpha_2)}\lambda(\alpha_1')\lambda(\alpha_2),$$

which holds by the inductive assumption to formula (a). Indeed, the generalized curves α_1' and α_2 are the components of the separating resolution of β at p' and α' is the non-separating resolution of β at p', and the number of self-intersections of β is less than n. If $C \subset \beta$, then let β' be the curve obtained from β by removing C via a Reidemeister Move I. We have $c(\beta') = c(\beta) - 1$, and that formula (a) is equivalent to

$$(-1)^{c(\alpha_1)+c(\alpha_2)}\lambda(\alpha_1)\lambda(\alpha_2) = (-1)^{c(\alpha')}\lambda(\alpha') + (-1)^{c(\beta')}\lambda(\beta'),$$

which holds by the induction assumption to formula (b). Indeed, the generalized curves α' and β' are resolutions of $\alpha_1 \cup \alpha_2$ at p'.

For (b), we have to consider the following two cases:

(1) p is not a vertex of a bigon bounded by α and β; and

(2) p is a vertex of a bigon B bounded by α and β.

We denote by $\overline{\gamma}$ the unique geodesic in the homotopy class of a generalized curve γ. In case (1), the resolutions p do not change the number of curls. We have $c(\alpha) + c(\beta) = c(\gamma_1) = c(\gamma_2)$, and

$$\begin{aligned}
(-1)^{c(\alpha)+c(\beta)}\lambda(\alpha)\lambda(\beta) &= (-1)^{c(\alpha)+c(\beta)}\lambda(\overline{\alpha})\lambda(\overline{\beta}) \\
&= (-1)^{c(\alpha)+c(\beta)}\big(\lambda(\overline{\gamma_1}) + \lambda(\overline{\gamma_2})\big) \\
&= (-1)^{c(\gamma_1)}\lambda(\gamma_1) + (-1)^{c(\gamma_2)}\lambda(\gamma_2).
\end{aligned}$$

In case (2), there is a curl C generated from the bigon B whose vertex is the other vertex p' of B. If $C \subset \gamma_1$, say, then let γ_1' be the curve obtained from α_1 be removing C via a Reidemeister Move I. Let α' and β' be the curve obtained from α and β by removing B via a Reidemeister

Move II. We have $c(\alpha') = c(\alpha)$, $c(\beta') = c(\beta)$ and $c(\gamma_1') = c(\gamma_1) - 1$, and the result is equivalent to

$$(-1)^{c(\gamma_2)}\lambda(\gamma_2) = (-1)^{c(\alpha')+c(\beta')}\lambda(\alpha')\lambda(\beta') + (-1)^{c(\gamma_1')}\lambda(\gamma_1'),$$

which holds by formula (a), since α' and β' are the components of the separating resolution of γ_2 at p' and γ_1' is the non-separating resolution of γ_2 at p'. $\square$

Proposition 3.3.2. *(a) For an arc α both of whose end points are at the same puncture v, let β and γ be the resolutions of α at v, and let $r(v)$ be the length of the horocycle centered at v. Then we have*

$$(-1)^{c(\alpha)}\lambda(\alpha) = \frac{1}{r(v)}\big((-1)^{c(\beta)}\lambda(\beta) + (-1)^{c(\gamma)}\lambda(\gamma)\big).$$

(b) Let α and β be two arcs intersecting at a puncture v. If γ_1 and γ_2 are the resolutions of α and β at v, then we have

$$(-1)^{c(\alpha)+c(\beta)}\lambda(\alpha)\lambda(\beta) = \frac{1}{r(v)}\big((-1)^{c(\gamma_1)}\lambda(\gamma_1) + (-1)^{c(\gamma_2)}\lambda(\gamma_2)\big).$$

Proof. By Lemma 3.2.8, part (a) is true when α is a geodesic. In the general case when α is not a geodesic, there are the following two cases:

(1) v is not a vertex of a generalized bigon bounded by α, that is, a bigon with one of its vertices a punctue; and

(2) v is a vertex of a generalized bigon B bounded by α.

In case (1), we have $c(\alpha) = c(\beta) = c(\gamma)$, and the formula follows from the case that α is geodesic. In case (2), we see one of the resolutions of α at v, say γ, contains a curl from the generalized bigon B, then the other resolution β is the one enclosing the puncture v. We let β_1 be the non-separating resolution of β and let β_2 be the component of the separating resolution of β which is not a circle around v. Then we have $c(\gamma) = c(\beta_2) + 1$ and $\lambda(\gamma) = \lambda(\beta_2)$, hence $(-1)^{c(\gamma)}\lambda(\gamma) = -(-1)^{c(\beta_2)}\lambda(\beta_2)$. By Lemma 3.3.1, we have $(-1)^{c(\beta)}\lambda(\beta) = (-1)^{c(\beta_1)}\lambda(\beta_1)+(-1)^{c(\beta_2)}2\lambda(\beta_2)$, which implies that $(-1)^{c(\beta)}\lambda(\beta)+(-1)^{c(\gamma)}\lambda(\gamma) = (-1)^{c(\beta_1)}\lambda(\beta_1)+(-1)^{c(\beta_2)}\lambda(\beta_2)$. Let α' be the curve obtained from α by removing the generalized bigon B via a Reidemeister Move II$'$, then β_1 and β_2 are the resolutions of α' at v. By case (1) and the last equation above, we have

$$\begin{aligned}(-1)^{c(\alpha)}\lambda(\alpha) &= (-1)^{c(\alpha')}\lambda(\alpha') \\ &= \frac{1}{r(v)}\big((-1)^{c(\beta_1)}\lambda(\beta_1) + (-1)^{c(\beta_2)}\lambda(\beta_2)\big) \\ &= \frac{1}{r(v)}\big((-1)^{c(\beta)}\lambda(\beta) + (-1)^{c(\gamma)}\lambda(\gamma)\big)\end{aligned}$$

Formula (b) is a consequence of Lemma 3.2.4 (1); and the proof is similar to that of (a). □

Combining Propositions 3.3.1 and 3.3.2, we obtain the following intermediate theorem.

Theorem 3.3.3. *The map $\Phi\colon \mathcal{C}(\Sigma) \to C^\infty(\mathcal{T}^d(\Sigma))$ defined in Theorem 3.1.4 is a well defined commutative algebra homomorphism.*

3.4 The homomorphism of Poisson algebras

To complete the proof of Theorem 3.1.4, we need the following lemma.

Lemma 3.4.1. *Let T be an ideal triangulation of a punctured surface Σ with a set of edges E. Suppose α is a generalized curve on Σ and $i(\alpha, e)$ is the number of intersection points of α and $e \in E$. Then the product $\alpha \cdot \prod_{e\in E} e^{i(\alpha,e)}$ in $\mathcal{C}(\Sigma)$ can be expressed as a polynomial P_α with variables in E.*

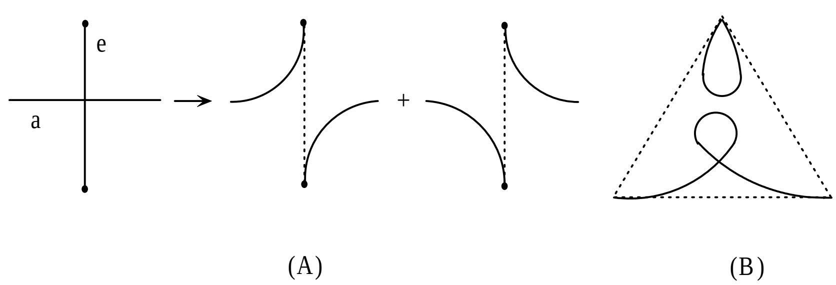


Figure 3.8:

Proof. Let $e \in E$ such that $\alpha \cap e \neq \emptyset$ and $p \in \alpha \cap e$. As in (A) of Figure 3.8, each resolution of $\alpha \cdot e$ at p has less intersection number with e than α does. Resolving the product $\alpha \prod_{e\in E} e^{i(\alpha,e)}$ at each point of intersection $p \in \alpha \cap (\bigcup_{e\in E} e) \cap \Sigma$, we see that each component of the final resolution has no intersection with each edge $e \in E$ in the surface, hence must lie in a triangle in T. Since a triangle is contractible, each component of the final resolution is either 0 or up to sign an edge $e \in E$ (as in (B) of Figure 3.8). $\square$

Proof of Theorem 3.1.4. For the Poisson structures, we let T be a triangulation of Σ with a set of edges E. If e and e' are two edges in E meeting at a puncture v, and x and y are the resolutions of e and e' at v, then $\{e, e'\} = \frac{1}{4}v^{-1}(x - y)$. By Theorem 3.3.3 and Lemma 3.2.4 (2), we have $\Phi(\{e, e'\}) = \frac{1}{4r(v)}(e^{\frac{l(x)}{2}} - e^{\frac{l(y)}{2}}) = \frac{1}{4}\frac{\theta'-\theta}{r(v)}e^{\frac{l(e)}{2}}e^{\frac{l(e')}{2}}$. We also have that $\Omega_{WP}(e^{\frac{l(e)}{2}}, e^{\frac{l(e')}{2}}) = \frac{1}{16}\frac{\theta'-\theta}{r(v)}e^{\frac{l(e)}{2}}e^{\frac{l(e')}{2}}$. If e and e' are two disjoint edges in E, then $\Phi(\{e, e'\}) = 4\Omega_{WP}(e^{\frac{l(e)}{2}}, e^{\frac{l(e')}{2}}) = 0$. Therefore, we have that $\Phi(\{e, e'\}) = 4\Omega_{WP}(\lambda(e), \lambda(e'))$ for all pairs of edges in E. Now for each generalized curve α, by lemma 3.4.1, we have $\alpha\prod_{e\in E} e^{i(\alpha,e)} = P_\alpha$ for some polynomial P_α with variables in E. Since $\{,\}$ is a Poisson bracket, we have $\{\alpha\prod_{e\in E} e^{i(\alpha,e)}, e_0\} = \prod_{e\in E} e^{i(\alpha,e)}\{\alpha, e_0\} + \sum_{e'\in E}\alpha\prod_{e\neq e'} e^{i(\alpha,e)}\{e', e_0\}$ for each edge e_0 in E, from which we see that $\prod_{e\in E} e^{i(\alpha,e)}\{\alpha, e_0\} = \{P_\alpha, e_0\} - Q$ for some polynomial Q in α, e and $\{e, e_0\}$ in which the degrees of α and $\{e, e_0\}$ are equal to 1. Since Φ is a $\mathbb{C}$–algebra homomorphism and Ω_{WP} is a bi-vector field, we have

$$P_\alpha(\lambda(e)) = (-1)^{c(\alpha)}\lambda(\alpha)\prod_{e\in E}\lambda(e)^{i(\alpha,e)},$$

and

$$\begin{aligned}&\Omega_{WP}\big((-1)^{c(\alpha)}\lambda(\alpha)\prod_{e\in E}\lambda(e)^{i(\alpha,e)}, \lambda(e_0)\big)\\ =&\prod_{e\in E}\lambda(e)^{i(\alpha,e)}\Omega_{WP}\big((-1)^{c(\alpha)}\lambda(\alpha), \lambda(e_0)\big) + (-1)^{c(\alpha)}Q_\lambda,\end{aligned}$$

where Q_λ is the value of Q at $\lambda(\alpha)$, $\lambda(e)$ and $\Omega_{WP}(\lambda(e), \lambda(e_0))$. As a consequence, since $\lambda(e) \neq 0$ for each $e \in E$, we have

$$\begin{aligned}\Phi(\{\alpha, e_0\}) &= \frac{\Phi(\{P_\alpha, e_0\}) - \Phi(Q)}{\prod_{e\in E}\Phi(e)^{i(\alpha,e)}}\\ &= \frac{4\Omega_{WP}\big(P_\alpha(\lambda(e)), \lambda(e_0)\big) - (-1)^{c(\alpha)}4Q_\lambda}{\prod_{e\in E}\lambda(e)^{i(\alpha,e)}}\\ &= 4\Omega_{WP}\big((-1)^{c(\alpha)}\lambda(\alpha), \lambda(e_0)\big).\end{aligned}$$

For two generalized curves α an β, we let $\alpha\prod_{e\in E} e^{i(\alpha,e)} = P_\alpha$ and $\beta\prod_{e\in E} e^{i(\beta,e)} = P_\beta$ as in Lemma 3.4.1. Then we have $\prod_{e\in E} e^{i(\alpha,e)+i(\beta,e)}\{\alpha, \beta\} = \{P_\alpha, P_\beta\} - R$, where R is a polynomial in α, β, e, $\{\alpha, e\}$, $\{e, \beta\}$ and $\{e, e'\}$ such that the degrees of α, β, $\{\alpha, e\}$, $\{e, \beta\}$ and $\{e, e'\}$ are all equal to 1. Therefore, we have $\Phi(R) = 4R_\lambda$, where R_λ is the value of R at $\lambda(\alpha)$, $\lambda(\beta)$, $\lambda(e)$, $\Omega_{WP}(\lambda(\alpha), \lambda(e))$, $\Omega_{WP}(\lambda(e), \lambda(\beta))$ and $\Omega_{WP}(\lambda(e), \lambda(e'))$, and

$$\begin{aligned}\Phi(\{\alpha, \beta\}) &= \frac{\Phi(\{P_\alpha, P_\beta\}) - \Phi(R)}{\prod_{e\in E} \Phi(e)^{i(\alpha,e)+i(\beta,e)}} \\ &= \frac{4\Omega_{WP}\big(P_\alpha(\lambda(e)), P_\beta(\lambda(e))\big) - (-1)^{c(\alpha)+c(\beta)} 4R_\lambda}{\prod_{e\in E} \lambda(e)^{i(\alpha,e)+i(\beta,e)}} \\ &= 4\Omega_{WP}\big((-1)^{c(\alpha)}\lambda(\alpha), (-1)^{c(\beta)}\lambda(\beta)\big).\end{aligned}$$

Let $\pi\colon \mathcal{T}^d(\Sigma) \to \mathbb{R}^V_{>0}$ be the projection onto the fiber. By Mondello [13], the kernel of Ω_{WP} is the pull-back $\pi^*(T^*\mathbb{R}^V_{>0})$ of the cotangent space of $\mathbb{R}^V_{>0}$. Since $d(r(v)) = \pi^*(dv) \in \pi^*(T^*\mathbb{R}^V_{>0})$, we have

$$\Phi(\{v, \alpha\}) = 4\Omega_{WP}\big(r(v), (-1)^{c(\alpha)}\lambda(\alpha)\big) = 0$$

for each puncture v and each generalized curve α. $\square$

As a consequence of Theorem 3.1.4, Wolpert's cosine formula generalizes to the bi-vector field Ω_{WP} as follows:

Corollary 3.4.2. *Let θ_p be the angle from α to β at $p \in \alpha\cap\beta$ in Σ. If α and β are two geodesic arcs, then let θ_v be the generalized angle from α to β and let θ'_v be the generalized angle from β to α at a puncture $v \in \alpha \cap \beta$. We have*

$$\Omega_{WP}(l(\alpha), l(\beta)) = \frac{1}{2} \sum_{p\in\alpha\cap\beta\cap\Sigma} \cos\theta_p + \frac{1}{4} \sum_{v\in\alpha\cap\beta\cap V} \frac{\theta'_v - \theta_v}{r(v)}.$$

Proof. We let $\lambda'(\alpha) = \sinh\frac{l(\alpha)}{2}$ if α is a closed curve on Σ, and let $\lambda'(\alpha) = \frac{1}{2}e^{\frac{l(\alpha)}{2}}$ if α is an arc on Σ. By Theorem 3.1.4 and (2) of Lemma 3.2.1 - 3.2.4, we have

$$
\begin{aligned}
&\Omega_{WP}(l(\alpha), l(\beta)) \\
&= \frac{1}{\lambda'(\alpha)\lambda'(\beta)}\Omega_{WP}(\lambda(\alpha), \lambda(\beta)) \\
&= \frac{1}{4\lambda'(\alpha)\lambda'(\beta)}\Phi(\{\alpha, \beta\}) \\
&= \frac{1}{4\lambda'(\alpha)\lambda'(\beta)}\Phi\big(\frac{1}{2}\sum_{p\in\alpha\cap\beta\cap\Sigma}(\alpha_p\beta^+ - \alpha_p\beta^-) + \frac{1}{4}\sum_{v\in\alpha\cap\beta\cap V}\frac{1}{v}(\alpha_v\beta^+ - \alpha_v\beta^-)\big) \\
&= \frac{1}{8}\sum_{p\in\alpha\cap\beta\cap\Sigma}\frac{\lambda(\alpha_p\beta^+) - \lambda(\alpha_p\beta^-)}{\lambda'(\alpha)\lambda'(\beta)} + \frac{1}{16}\sum_{v\in\alpha\cap\beta\cap V}\frac{1}{r(v)}\frac{\lambda(\alpha_v\beta^+) - \lambda(\alpha_v\beta^-)}{\lambda'(\alpha)\lambda'(\beta)} \\
&= \frac{1}{2}\sum_{p\in\alpha\cap\beta\cap\Sigma}\cos\theta_p + \frac{1}{4}\sum_{v\in\alpha\cap\beta\cap V}\frac{\theta'_v - \theta_v}{r(v)}.
\end{aligned}
$$

$\square$

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