THE ACTION FUNCTIONAL ON DUAL LEGENDRIAN SUBMANIFOLDS OF THE LOOP SPACE OF A CONTACT THREE DIMENSIONAL CLOSED MANIFOLD

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ABSTRACT OF THE DISSERTATION

The action functional on dual Legendrian submanifolds of the loop space of a contact three dimensional closed manifold

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The main object of my dissertation is the study of the action functional of a contact form on a three dimensional manifold. This is part of a long program started by Professor A. Bahri [3], [4] in constructing a contact homology giving information about the number of periodic orbits of the Reeb vector field, that is an attempt to approach the Weinstein conjecture for 3-manifolds. Even though it appears to be proved by C. Taubes in a series of paper (see for instance [39] for more details). Given a closed 3-dimensional manifold, we prove an S^1-homotopy equivalence between a subspace $\mathcal{C}_\beta$ of Legendrian curves and the free loop space. This space appears to be convenient from a variational point of view, in contact form geometry and used in the approach developed by A. Bahri. Indeed, it is the right space of variations on which we study the action functional. In a second part we study the Fredholm assumption for a modified version of the action functional on the variational space $\mathcal{C}_\beta$. That is, whether the functional is Fredholm or not. We take here the text-book case-study of a sequence of overtwisted contact forms on the 3-sphere introduced by Gonzalo and Varela [26]. We show that the Fredholm assumption does not hold. This is done by studying the dynamics of the contact form along a vector field on its kernel. We also prove the existence of a

foliation stuck between the contact form and its Legendre dual in the part where they have opposite orientation. In the last part we present an explicit computation of the Bahri contact homology for a sequence of tight contact structures in the torus. We also extend this result to the case of torus bundles over S^1. The homology that we find, allows us in particular to confirm the fact that the contact structures are not isotopic since it has different values for each structure.

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I wanted to finish this acknowledgement with few words about my adviser Professor Abbas Bahri. I can't find the words to describe his generosity and kindness. A person so dedicated to his work and responsibilities that even during his darkest days where his health and life were threatened, he would meet me and discuss diverse perspective of this dissertation. I remember the first time that I talked to him about being my adviser and his shocking answer: "stay away from me if you want to get a job". I was amazed

about his sincerity and his direct way of communication. Even with that answer we ended up working together. That gave me new perspectives to my old knowledge in the domain and I learned valuable tools too. But the best thing that I learned, was not in math, it was in life. The fact that never to give up and stay focused in your dream and vision, away from the insignificant polemics. In one sentence, Abbas Bahri is a person with a heart! He cares.

Dedication

This is dedicated to my beloved parents Dalila and Abdallah and my brothers Seyf and Oussama. I am thankful for their support during my whole academic career and I hope that one day I can repay them my debt. Thank you for being part of my life!

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Chapter 1

Introduction

Contact geometry as most of the mathematical subjects appears as a way of studying physical phenomena at its origin. It appeared first in the work of Sophus Lie in [30], [31] when he introduced the notion of contact transformation to study systems of differential equations. The first example of contact manifolds appeared also in the same work under the terminology of space of contact elements. Given a manifold M, a contact element on M is a hyperplane in a tangent space to M. Hence the space of contact elements on a n-dimensional manifold M is a $2n-1$ dimensional manifold. A definition for contact structures in general will be given later. The study of contact structures can also be considered part of dynamical systems since in contrast to integrable systems represented by foliations, contact structures are the extreme opposite, the maximum non-integrability.

From a topological point of view, the field is relatively new. Indeed the main results in the field were obtained around the 1970's and since then it experienced a huge activity. For instance the existence of contact structure on every 3-manifold was a break through in [34] by Martinet and then the topological manipulation of contact structure as plane distribution by Lutz in [32], [33].

Other than the intrinsic study from a topological point of view of contact manifolds, the field also relates to the long studied subject of Hamiltonian dynamics. Indeed, studying closed characteristics of a Hamiltonian system is a classical problem and it dates to Euler and Fermat and one can relate this to the Arnold conjecture about finding an estimate on the number of periodic orbits of a Hamiltonian system. (see [2]). Powerful tools were introduced in the study of this question, mainly in the work of Floer, [20], [21].

Now if we want to study periodic orbits in a given energy hypersurface, contact geometry enters the picture. The first pioneering results were obtained by P. Rabinowitz [37] and Viterbo [40]. A similar conjecture of Weinstein (see preliminaries for details) was formulated about the existence of closed orbit of the Reeb vector field [41].

In order to solve this problem there was an extensive work starting by splitting the set of contact structures to two categories tight and overtwisted, see [14]. Further classifications were done. The most classical one is the classification of overtwisted contact structures in the work of Eliashberg [18]. Since overtwisted contact structures are geometrically flexible, a proof of the Weinstein conjecture was given in that case by Hofer [27].

Several approaches where developed to solve the conjecture, variational approaches and Floer homological approaches. So far the Floer approach, based on pseudo-holomorphic curves in the symplectization appears to be unsuccessful indeed one can construct some topological objects but one either cannot compute it or the object gives no informations (see M-L. Yau [42]). Recently, in 2007, C. Taubes proved the conjecture in dimension 3, using an involved study of the Seiberg-Witten equations and it appears that it can be related to the Embedded contact homology [39].

Another approach was developed by A. Bahri [3], [5] and [6] that is purely variational and does not involve the study of pseudo-holomorphic curves. This approach consists of the study of the action functional in a particular space making the problem approachable in a certain way. In what follows, we will study different aspects of this approach, starting from the space of variations to the Fredholm assumption for certain exotic structures on the sphere. We will state here the general plan of the dissertation.

In chapter II, we give a basic introduction to contact manifolds and the different properties that contact structures enjoy. Basically, we state the classical results and we give a small introduction to the Weinstein conjecture with the different techniques used in there. We also present a concise introduction to Morse theory and the techniques used therein to extract informations about critical points.

In chapter III, we will focus on the study of the space $\mathcal{C}_\beta$, the space of variations on which we study the action functional, where β here is the Legendrian dual of the contact form by a vector field in its kernel. Indeed in order to develop a Morse theoretical approach one needs to study the topology of the underlying space to get an estimate on the number of critical points. In our case, we show that under mild assumptions, the space $\mathcal{C}_\beta$ has the same topology as the free loop space. Hence, in the best scenario (no bubbling and compactness) the problem of finding periodic orbits of the Reeb vector field is equivalent to the closed geodesic problem. This space can also be seen of a different importance. In fact, the object that we study is a subspace of the Legendrian loops and it can be seen as the set of zero Maslov index Legendrian loops, therefore the study of this space can be linked to the study of Legendrian knots.

In chapter IV, we consider the specific case of the third exotic structure of Gonzalo and Varela [26]. We study the violation of the Fredholm assumption for the action functional, which does not place us in the best case scenario, in fact this phenomena happens with conjunction with a formation of critical points at infinity. A similar study was done in [10] for the first exotic contact structure on the 3-sphere. Though the main difference is that in that case there exist a vector field in the kernel, inducing a Legendrian duality. In our case, with the vector field that we consider, the Legendre duality fails. We also prove the existence of a foliation in the part where the contact form and its "Legendre dual" have opposite orientations.

In the last chapter, we present a concrete computation of the contact homology in the case of a sequence of the tight contact structures of the torus and also for torus bundles over S^1. We show that the assumptions in the approach developed by A.Bahri hold and hence by explicit computations we give the homology and show its local stability.

Chapter 2

Preliminaries on contact topology

2.1 Contact Forms

Let M be a three dimensional smooth manifold. A contact form α on M is a 1-form such that $\alpha \wedge d\alpha \neq 0$ on M. We associate to such contact form a unique vector ξ field satisfying :

$$\begin{cases} \alpha(\xi) = 1 \\ d\alpha(\xi, \cdot) = 0 \end{cases}.$$

A contact structure is a completely non-integrable plane distribution. Locally we can represent the plane distribution by a contact form. That is the plane distribution is given by $\ker \alpha$. By Frobenius theorem, saying that the plane distribution is a contact structure is equivalent to the fact that α is a contact form. Hence, if we are given a global 1-form α that is contact, $\ker \alpha$ will defines a contact structure on M. As one can notice, two contact forms α_1 and α_2 define the same contact structure if there exists a non-vanishing function f on M such that $\alpha_1 = f\alpha_2$.
In the literature a contact manifold is defined to be a pair (M, H) where H is a maximal non-integrable plane distribution. If H is co-oriented (that is TM/H is an oriented line bundle) then there exists a global 1-form α such that $\ker \alpha = H$ (see for instance [22]). We will consider in what follows the co-oriented case and we will use the terminology of D. Blair [15] to call a contact manifold the pair (M, α).

The way to think about contact structures is to visualise them as a distribution of planes that turns in a monotonic fashion. In fact we have :

Proposition 2.1.1. *Consider a contact 3-manifold* (M, α) *and* $v \in \ker \alpha$ *then* $\ker \alpha$ *turns in a monotonic way in a transported frame along* v*.*

In the previous proposition, we mean that, if φ_t denote the flow of v, and starting from a point x_0, we define $e_1 = D\varphi_t(\xi)$ and $e_2 = D\varphi_t([\xi, v])$, then the trace of $\ker\alpha$ in the plane spanned by e_1 and e_2 turns in a monotonic way. Notice that we can take any other vector in $\ker\alpha$ at x_0 instead of $[\xi, v]$.

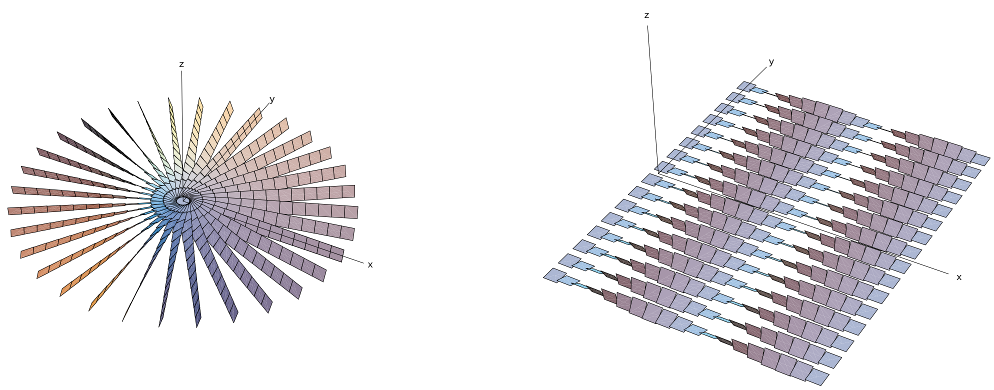


Figure 2.1: Contact structures in $\mathbb{R}^3$

Example 2.1.2. *a)The one 1-form defined on $\mathbb{R}^3$ by $\alpha_{st} = dz + xdy - ydx$ is a contact form. It defines the standard contact structure on $\mathbb{R}^3$. Notice that in polar coordinates this can be written as $dz + r^2 d\theta$.*
b)On S^3 considered as a sub-manifold of $\mathbb{R}^4$, with the coordinate system (x_1, x_2, x_3, x_4), the one form $\alpha = x_2dx_1 - x_1dx_2 + x_2dx_1 - x_1dx_2$ defines the standard contact structure on the sphere.
c)The family of 1-forms $\alpha_n = \cos(nz)dx - \sin(nz)dy$ defines a sequence of different contact structures on the torus T^3. See [28].

Now as for every structure and its morphisms, here we introduce the morphisms preserving the contact structure.

Definition 2.1.3. *A contactomorphism f from (M_1, α_1) to (M_2, α_2) is a diffeomorphism such that $f^*\alpha_2 = \lambda\alpha_1$ for a non-zero function λ.*

Example 2.1.4. *We consider on $\mathbb{R}^{\nvDash}$ the two contact forms, α_{st} and $\alpha_1 = dz - xdy$. Then the diffeomorphism $\Phi(x, y, z) = (2y, -x, xy + z)$ is a contactomorphism from $(\mathbb{R}^3, \alpha_{st})$ to $(\mathbb{R}^3, \alpha_1)$.*

Theorem 2.1.5 (Darboux)**.** *Let (M, α) be a contact manifold. For every $x \in M$ there exist U_x a neighborhood of x such that (U_x, α) is contactomorphic to the standard contact structure $(\mathbb{R}^3, xdy - ydx + dz)$. In fact one has that the previous contactomorphism comes from an isotopy.*

This tells us that there is no local geometry to be studied instead the global topology is more important.
Another important fact about the contact structures is the contact rigidity theorem also known as Gray's theorem :

Theorem 2.1.6 (Gray)**.** *If α_t, $t \in [0, 1]$ is a family of contact forms on a smooth closed manifold M, then there exists a diffeomorphism $\varphi : M \longrightarrow M$ isotopic to the identity such that $\varphi^*(\alpha_1) = f\alpha_0$ for some function $f \neq 0$.*

In other words this theorem says that any two contact forms linked by a path of contact forms are contactomorphic.

Now that we know that some manifolds carry contact structures, the natural question is : does any closed 3-manifold carry a contact structure?. The answer to this question is yes.

Theorem 2.1.7 (Martinet)**.** *Every smooth closed 3-manifold carries at least one contact structure.*

2.2 The link between contact and Symplectic geometry

We consider an even dimensional manifold M^{2n}. A symplectic form on M^{2n} is a 2-form ω such that :

a)$d\omega = 0$

b)w^n is a volume form on M.

Notice that the existence of a symplectic form induces a topological constraint on the manifold if it is compact. For instance, if M^{2n} is a closed symplectic manifold then $H^{2k}(M)$ (the De Rahm cohomology) is different from zero for all $0 \leq k \leq n$. For example, the 4-sphere cannot carry a symplectic structure.

Now we consider a closed 3-dimensional manifold, we can go from the contact context to the symplectic context in the following way :

Proposition 2.2.1. *α is a contact form on M if and only if $d(e^t\alpha)$ is a symplectic form on $\mathbb{R} \times M$.*

This process is called symplectification.

One can also go the other way around. That is, we consider a $2n$-dimensional symplectic manifold (Y, ω). A vector field X is called a Liouville vector field if $L_X(\omega) = \omega$, where L_X denote the Lie derivative with respect to X that is in the language of differential forms, $L_X = d \circ i_x + i_X \circ d$. Now the following holds

Proposition 2.2.2. *Let M be a sub-manifold of Y transverse to X. Then the 1-form $\alpha = \omega(X, \cdot)$ is a contact form on M.*

The manifold M is said to be of contact type.

Example 2.2.3. *a)Consider the symplectic manifold $(\mathbb{R} \times M)$ (where (M, α) is a contact manifold), then the vector field $\frac{\partial}{\partial t}$ is a Liouville vector field and $\{t\} \times M$ is a contact manifold for every $t \in \mathbb{R}$.*

b)We consider now a more explicit example which is $\mathbb{R}^4$ with its standard symplectic structure $w = dx_1 \wedge dx_2 - dx_2 \wedge dx_1 + dx_3 \wedge dx_4 - dx_4 \wedge dx_3$. The vector field $\frac{\partial}{\partial r} = x_1\partial x_1 + x_2\partial x_2 + x_3\partial x_3 + x_4\partial x_4$ is of contact type, and the 1-form $\alpha = i_{\frac{\partial}{\partial r}} w$ defines the standard contact structure on the 3-sphere.

2.3 Tight vs Overtwisted

There is a dichotomy in the space of contact structures. In fact we can split them into two categories : Tight and overtwisted.

Definition 2.3.1. *The contact structure* (M, α) *is said to be overtwisted if there exists an embedded disk* $D \subset M$ *such that* ∂D *is tangent to* $\ker \alpha$. *Such a disk is called an overtwisted disk (figure (2.2).*

A contact structure is said to be tight if it is not overtwisted.

Example 2.3.2. *a)The standard contact structures of the sphere and* $\mathbb{R}^3$ *are tight.*

b)The contact forms defined on the torus in the previous section are tight.

c)Using the polar coordinates again in $\mathbb{R}^3$ *we have that the contact form* $\alpha_3 = \cos(r)dz + r\sin(r)d\theta$ *is overtwisted. More generally, the one form* $\alpha_n = \cos(f_n(r))dz + sin(f_n(r))d\theta$, *where* f_n *is a strictly monotone function equal to* r^2 *near* $r = 0$ *and asymptotic to* $n\pi + \frac{\pi}{2}$ *at infinity, provides an overtwisted contact form for all* $n \geq 1$.

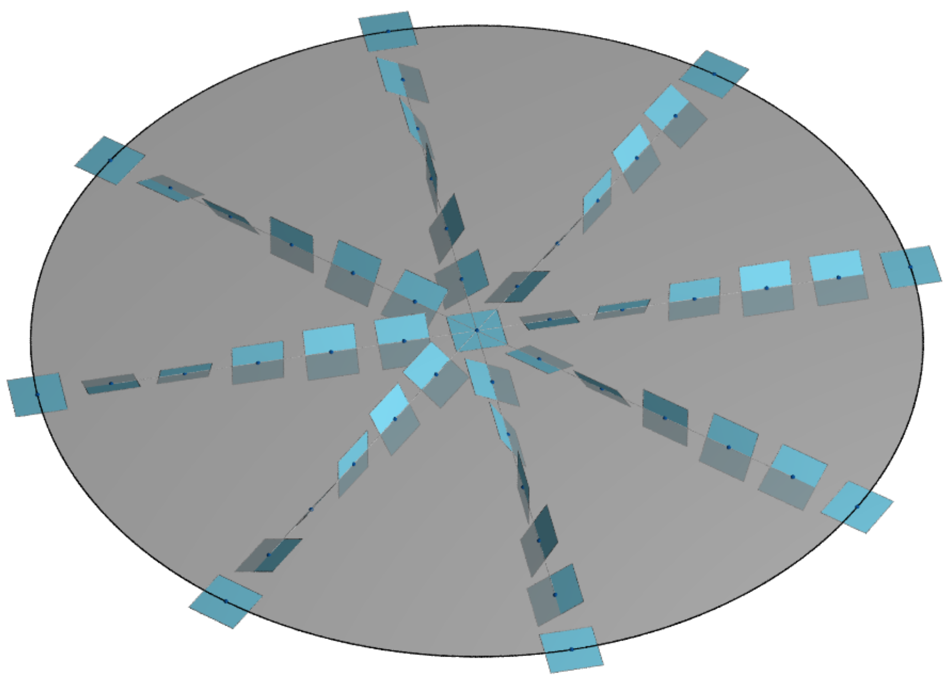


Figure 2.2: Overtwisted Disk

In fact one has :

Theorem 2.3.3 (Eliashberg [19])**.** *a) The contact structure induced by* α_{st} *on* S^3 *is the only tight one.*

b)Any tight contact structure on the torus is contactomorphic to one of the structures induced by α_n.

Another important example is the one of the exotic contact structures on S^3 introduced by Gonzalo-Varela in [26].

Theorem 2.3.4 (Gonzalo-Varela [26]). *For $n \geq 1$, the sequence of 1-forms α_n defined on the 3-sphere by*

$$\alpha_n = -(\cos(\frac{\pi}{4} + n\pi(x_1^2 + x_2^2))(x_2 dx_1 - x_1 dx_2) + \sin(\frac{\pi}{4} + n\pi(x_1^2 + x_2^2))(x_4 dx_3 - x_3 dx_4))$$

is a sequence of overtwisted contact form defining non contactomorphic contact structures.

In [35], V. Martino exhibited an explicit equation for an overtwisted disk for the Gonzalo-Varela forms. In fact, for $n \geq 1$, the disk

$$D = \{(x_1, x_2, x_3, x_4); x_1^2 + x_2^2 \leq \frac{3}{4n}, x_4 \geq 0, x_3 = \varepsilon\},$$

is an overtwisted disk for α_n, for ε small enough. The difference between the number of tight contact structures on the sphere and the overtwisted one is not shocking. In fact, the overtwisted contact structures are more abundant than the tight ones as we will see in the following two theorems.

Theorem 2.3.5 (Lutz [32]). *Let (M, α) be a contact manifold, then there exist an overtwisted contact structure homotopic to $\ker \alpha$ as a plane distribution.*

In fact the overtwisted contact form in the previous theorem is obtained from the first one by a Lutz twist along a transverse knot. In the other hand finding a tight contact structure is sometimes hard. For instance we have the following :

Theorem 2.3.6. *There exists a closed smooth manifold M with no tight contact structures.*

It is natural now to try to understand each one of the previous category. In fact the overtwisted contact structures are classified by there homotopy type as plane field distribution. This was the work of Eliashberg [18]:

Theorem 2.3.7. *Let M be a closed, compact 3-manifold. Let H be the set of homotopy classes of oriented plane fields on M and C_0 be the set of isotopy classes of oriented overtwisted contact structures on M. The inclusion map C_0 into H gives a homotopy equivalence.*

2.4 Legendrian curves

Let (M, α) be a contact manifold. A loop $x : S^1 \longmapsto M$ is said to be Legendrian if $\alpha_{x(t)}(\dot{x}(t)) = 0$ for all $t \in S^1$. That is the tangent vector to the curve belongs to the plane distribution $\ker \alpha$. In the other hand, a loop is called positively (resp. negatively) transverse if $\alpha_{x(t)}(\dot{x}(t)) > 0$ (resp. < 0) for all $t \in S^1$.

Theorem 2.4.1. *Let $x : S^1 \longrightarrow M$ be a knot, i.e. an embedding of S^1, in a contact 3manifold. Then x can be C^0approximated by a Legendrian knot isotopic to x. Alternatively, it can be C^0approximated by a positively as well as a negatively transverse knot.*

It is important to notice here that the approximation is in the C^0 sense. In the study of Legendrian knots, there are two classical invariants that are used. The Thurston-Bennequin number and the rotation number (also known as the Maslov index). Let us consider a homologically trivial knot x on a contact manifold (M, α) and let Σ a spanning surface for x, that is a surface such that $\partial\Sigma = x(S^1)$, (this exists since x is homologically trivial). The surface Σ is called a Seifert surface for x and $\partial\Sigma = x$.
The Thurston-Bennequin number denoted by $tb(x, [\Sigma])$ measures the twisting of the contact structure along the knot x. In fact, if x' is the curve obtained by pushing x out along a vector transverse to $\ker \alpha$, then

$$tb(x, [\Sigma]) = lk(x, x')$$

, where lk is the linking number of two loops. One can show that this number is independent of the transverse vector that we use to push x.
The rotation number $r(x, [\Sigma])$, measures the twisting of the tangent direction to x inside $\ker \alpha$. More precisely, we trivialize $\ker \alpha$ over Σ then $r(x, [\Sigma])$ is the winding number of

the tangent direction to x in that trivialization of $\ker \alpha$. For tight contact structures those invariants have a relation known as the Bennequin Inequality, here we will state it's most general form proved by Eliashberg :

Theorem 2.4.2. *Let x be a null-homologous Legendrian knot in a tight contact structure, and Σ a Seifert surface for x, then*

$$tb(x, [\Sigma]) + |r(x, [\Sigma])| \leq -\chi(\Sigma).$$

2.5 Hamiltonian systems between symplectic and contact geometry

Let us start first by the standard example in $\mathbb{C}^2$ with the coordinate system $z = (z_1, z_2)$. We consider a function H that has at most quadratic growth at infinity and we set our objective as solving the problem :

$$\begin{cases} z_1' = H_{z_2} \\ z_2' = -H(z_1) \\ z(0) = z(1). \end{cases} \tag{2.1}$$

That is finding a function $z : [0, 1] \longrightarrow \mathbb{C}^2$ satisfying the previous equation.
A compact way of writing this problem is to use the complex structure J so that solutions of (2.1) satisfies $z' = J\nabla H$. This problem is variational, that is, solutions if this problem are critical points of the functional F defined by

$$\int_0^1 Jz' \cdot z - \int_0^1 H(z(t))dt.$$

The natural space of variations here is $H^{\frac{1}{2}}_{per}([0, 1], \mathbb{C}^2)$. This space can be described in terms of Fourier series as

$$H^{\frac{1}{2}}_{per}([0, 1], \mathbb{C}^2) = \{u = \sum_{k \in \mathbb{Z}} e^{2ik\pi} a_k ; \sum_{k \in \mathbb{Z}} |a_k|^2 (1 + |k|) < \infty\}.$$

It is important to notice that the functional is strongly indefinite. In fact if we consider the differential part of the functional

$$A(z) = \int_0^1 Jz' \cdot z dt,$$

we can decompose the space of variations $H^{\frac{1}{2}}$ as follows

$$H^{\frac{1}{2}} = E^+ \oplus E^0 \oplus E^-,$$

where E^+ (resp. E^-) is the space on which the quadratic form A is positive definite (resp. negative definite) and E^0 is the kernel of A and it is a one dimensional space formed by the constant curves. Using this we can write

$$A(z) = |z^+|^2 - |z^-|^2.$$

As we can see the functional is not bounded neither from below nor from above. If we compute the gradient of our functional F we get

$$\partial F(z) = Jz' + H'(z) = Jz' + [z] + H'(z) - [z],$$

here $[z]$ is the average of z, that is $[z] = \int_0^1 z(t)dt$. This decomposition of ∂F tells us that it can be written as a bi-continuous operator $T = Jz' + [z]$ and a compact operator $K = H'(z) - [z]$. This is the classical setting for a variational problem. In particular this says that ∂F is a Fredholm operator of index zero.

Let us try to write this problem in a formal way. That we consider the standard symplectic structure of $\mathbb{C}^2 = \mathbb{R}^4$ defined by $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Given a Hamiltonian H, one can define the Hamiltonian vector field X_H by

$$i_{X_H} w_0 = dH.$$

One now can easily check that the flow of X_H corresponds to solutions of (2.1). So we want to find the periodic orbits of X_H.

If we set the map φ_t^H as the 1-parameter group of X_H then we have that

$$H(\varphi_t^H(x)) = H(x)$$

for all x, hence the level sets of H are invariant under the flow of X_H. From this one can ask the following question :

Given a hypersurface $S = H^{-1}(1)$ does it contain a Hamiltonian periodic orbit?

The Answer to this question is positive if the surface S is convex (or star shaped). This was the famous work of P. Rabinowitz in [37]. In fact in this case the periodic orbits corresponds to periodic orbits of the Reeb vector field corresponding to the contact structure induced by ω_0 on S.

Symplectic Framework

The previous procedure can be stated in a general manner for any symplectic manifold (W, ω). In this frame work one can state the Weinstein conjecture (1978) as follows

Conjecture 2.5.1. *Let* (M, α) *be a hypersurface of contact type with* $H^1(M) = 0$*, then* ξ*, has at least one periodic orbit.*

We see that the conjecture is true if we are in the framework of P. Rabinowitz, that is in $\mathbb{C}^n$ with its standard symplectic structure and M a convex hypersurface. In fact the complete result in $\mathbb{C}^n$ is due to Viterbo [40] :

Theorem 2.5.2. *Every hypersurface of contact type in* $(\mathbb{C}^n, \omega_0)$ *carries at least one periodic orbit.*

Now the most general statement of the Weinstein conjecture is as follows :

Conjecture 2.5.3. *Let* (M, α) *be a contact manifold, then* ξ*, has at least one periodic orbit.*

This problem is much harder and there are several ways of engaging it. The first one is by symplectization. This allws the use of previous theories on the pseudo-holomorphic curves. We state here one famous result by Hofer in the case of overtwisted contact structures.

Theorem 2.5.4 (Hofer [27])**.** *If* (M, α) *is an overtwisted contact structure, then* ξ *has at least one contractible periodic orbit.*

The approach of Hofer used the construction of a family pseudo-holomorphic curves with boundaries on the overtwisted disc. The idea is to show some monotonicity of this boundary that allows to say that the family will blow-up which yields the existence of a periodic orbit of the Reeb vectorfield. An interesting corollary of this theorem is :

Theorem 2.5.5. *All the contact forms of the torus defined in Subsection 1.1 are tight.*

Several other constructions were done using the symplectization to build a topological invariant as it was the case for the Arnold conjecture with the Floer homology.

Unfortunately, the construction of a stable Floer type homology in this case is much harder and even after construction the cases on which we can do explicit computations are very narrow, we can see for instance the work of Bourgeois [16], [17] and the references therein.

The contact framework

Another way to treat the problem is a Morse theoretical setting but this time without using a symplectization. We consider the action functional $J : H^1(S^1, M) \longrightarrow \mathbb{R}$ defined by

$$J(x) = \int_0^1 \alpha_{x(t)}(\dot{x}(t))dt.$$

It is easy to see that the critical points of this functional are the periodic orbits of ξ. The first problem that one encounter with this functional is the fact that it is strongly indefinite also it is not compact (the Palais-Smale condition does not hold).
Under a convexity assumption one can reduce the difficulty of the problem by restricting the functional to a smaller space of variations.

Definition 2.5.6. *We say that the contact form α admits a Legendre transform induced by v if :*
a)there exist a C^1 never vanishing vector field v in the kernel of α.
b)the 1-form $\beta(\cdot) = i_v d\alpha = d\alpha(v, \cdot)$, is a contact form with the same orientation as α.

Example 2.5.7. *a) The standard sphere.*
b)The first Gonzalo and Varela form as proved by V. Martino [35].
c)The tight contact forms on the torus (see Chapter 6).

Now we consider the space

$$\mathcal{C}_\beta = \{x \in \mathcal{L}_\beta; \alpha_x(\dot{x}) = c > 0\},$$

where c is a non-prescribed constant. Then the following holds :

Theorem 2.5.8. *The space $\mathcal{C}_\beta - M$ is a Hilbert manifold and its tangent space at a loop x is given by the set of vectors $Z = \lambda\xi + \mu v + \eta[\xi, v]$ such that :*

$$\begin{cases} \dot{\lambda} = b\eta - \int_0^1 b\eta \\ \dot{\eta} = \overline{\mu} b\eta + a\mu - \lambda b \end{cases} \tag{2.2}$$

where $\overline{\mu} = d\alpha(v, [v, [\xi, v]])$.

Theorem 2.5.9 (Bahri [3])**.** *The critical points of J restricted to $\mathcal{C}_\beta$ are the periodic orbits of ξ and those critical points have finite Morse index. Moreover, the difference between the Morse indices of the periodic orbits are the same whether J is restricted to $\mathcal{C}_\beta$ or the free loop space.*

The difference here is that the critical points and the asymptote of the functional are well understood. So one can start a Morse theoretical approach if we can settle the compactness issue. Another problem appears in this method, which is the Fredholm assumption. We will show In Chapter IV that for certain contact structures this does not hold, making the problem more challenging. But as the ultimate goal is to develop a Morse complex, the first thing that needs to be studied is the space of variations $\mathcal{C}_\beta$ and that will be the main topic of Chapter III.

2.6 Overview on Morse theory

In this section we will give a fast overview on the tools and objectives of Morse theory. Indeed, Morse theory is a way of making a link between the topological properties of a given manifold and the critical points and the differential properties of certain functions defined on the manifold. We will always assume in what follows that the manifold M is orientable.

Definition 2.6.1. *Let M be a smooth closed manifold, and $f : M \longrightarrow \mathbb{R}$ a C^2 function on M. f is said to be Morse if its critical points are non-degenerate. That is the Hessian of f at any critical point is invertible. In that case the number of negative eigenvalues of the Hessian is called the Morse index of the critical point and we write $i(x)$.*

The assumption that a function is Morse is not too restrictive, in fact we have

Proposition 2.6.2. *The set of Morse functions is G_δ dense in the space of C^2 functions on M.*

So for a generic perturbation, any function can be Made Morse. Now let us state the main theorem of Morse theory.

Theorem 2.6.3. *Let $f : M \longrightarrow \mathbb{R}$ be a Morse function. Assume that $a < b$ are two regular values of f, then if we write $M^r = \{x \in M; f(x) \leq r\}$, the following hold : i)If there is no critical values in (a, b) then M^a is a deformation retract of M^b.*
ii) If $c \in (a, b)$ a critical value corresponding to a unique critical point x_c, then M^b is obtained from M^a by attaching a cell of dimension index(x_c) along its boundary i.e. $M^b \simeq M^a \cup_f e_{i(x_c)}$.

This gives us a way to reconstruct a manifold given a Morse function on it.

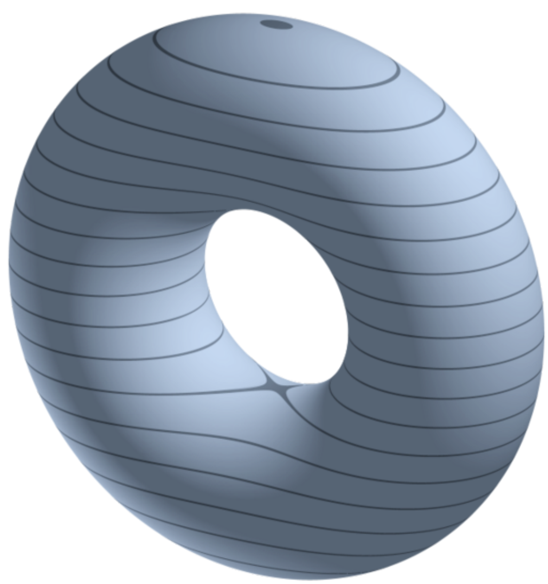


Figure 2.3: Critical points of the hight function

Example 2.6.4. *As an example, we consider the torus T^2 with f the height function (see figure (2.4)). We have that f has four critical points one minimum of index 0, two saddle points of index 1 and a maximum of index 2. Therefore, given those information* *we can reassemble T^2 this way :*

1- we start with a 2-disc.

2- we attach an interval to it, along the boundary and this gives us a cylinder.

3- we attach another interval along the boundary, and this gives us a torus minus a disc.

4- to finish we attach a disc by identifying the boundaries and this gives us back the torus.

Those steps are illustrated in figure (2.4).

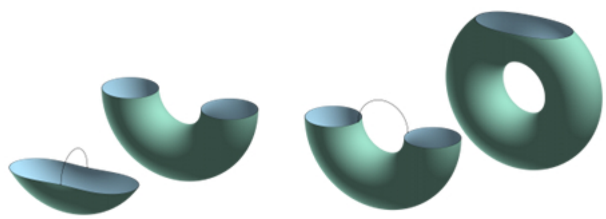


Figure 2.4: Attaching Cells

Now we will give a fast description of Morse homology and its construction. Given a Morse function f, we consider its descending gradient flow, defied by the ODE

$$\begin{cases} \frac{dx}{dt} = -\nabla f(x) \\ x(0) = x_0 \end{cases}$$

We will denote the flow by $\phi_t(x)$. It is worth mentioning that one can find different kind of normalization of the gradient flow in the literature, also one should know that the gradient depends on the Riemannian metric that one puts on the manifold.
Given a critical point $x_0 \in M$ of f, we define its unstable (Resp. stable) manifold and we write $W_u(x_0)$ (Resp. $W_s(x_0)$) by

$$W_u(x_0) = \{x \in M; \lim_{t \to -\infty} \phi_t(x) = x_0\}$$

Resp.

$$W_s(x_0) = \{x \in M; \lim_{t \to +\infty} \phi_t(x) = x_0\}.$$

One can prove that those two sets indeed have a submanifold structure. Now we give the following definition

Definition 2.6.5. *The gradient flow is said to be Morse-Smale if for any two critical points x_1 and x_0 we have that $W_u(x_1)$ and $W_s(x_0)$ intersect transversally.*

This Morse-Smale condition can be achieved by perturbing the Riemannian metric, indeed one can show that the set of metrics that induce this condition is generic.
So now one can state the following :

Theorem 2.6.6 (Banyanga [13])**.** *Consider a Morse-Smale function f defined on a compact manifold M, and let x_1 and x_2 be two critical points of f. Then* $\dim W_u(x_1) \cap W_s(x_2) = i(x_1) - i(x_2)$ *with the convention that the intersection is empty if the dimension is negative.*

The main ingredient for the proof of this theorem is the implicit function theorem. We will see indeed that in the infinite dimensional case, this might cease to happen if we lose the Fredholm condition.
Notice also that if $i(x_1) = i(x_2) + 1$ then $W_u(x_1) \cap W_s(x_2)$ is a 1 dimensional manifold. This with the fact that $\mathbb{R}$ acts on the flow line by time translation makes the set $\mathcal{M}(x_1, x_2) = \frac{W_u(x_1) \cap W_s(x_2)}{\mathbb{R}}$ a finite set. With this remark we can define now the Morse complex.
We set $Crit_k$ the set of critical points of f of index k, and we define the chain complex $C_*(f, M)$ as follow

$$C_k(f, M) = Crit_k \otimes \mathbb{Z},$$

with the boundary operator ∂ defined on the generator of the chain by

$$\partial x = \sum_{y; i(x) - i(y) = 1} \sharp \mathcal{M}(x, y) y,$$

where here $\sharp \mathcal{M}(x, y)$ is the number of elements counted with consideration of the orientation. In fact $\mathcal{M}(x, y)$ inherit an orientation from the one of M. But, even if M is non-orientable one in that case can consider the chain complex with coefficients in $\mathbb{Z}_2$.

Theorem 2.6.7. *The operator ∂ satisfies $\partial^2 = 0$ and the homology obtained from the chain complex is independent of the function f and the Riemannian metric. More precisely we have*

$$H_*(C_*(f, M)) = H_*(M).$$

Here $H_*(M)$ is the singular homology of M.

One can extract a lot of information from this theorem. For instance, if f is a smooth function on a manifold M then it has at list $\sum_k \dim H_k(M)$ critical points. So now since our objective is the study of contact structure, and ultimately solve the Weinstein conjecture, one can think about developing a Morse homology related to the action functional to exhibit periodic orbits of the Reeb vector field. Taking into account the remarks in the previous section about the space $\mathcal{C}_\beta$, in order to start a Morse theoretical approach one needs to understand the topology of the underlying space of variations ($\mathcal{C}_\beta$), and hope that if compactness holds that the number or periodic orbits is linked to the homology of the space of variations.

That is why, the first chapter of our study consists of understanding the topology of $\mathcal{C}_\beta$ and the second chapter is to see if the Fredholm assumption holds since it is crucial for the application of the implicit function theorem. Unfortunately, we show that this does not hold in certain cases. In the last chapter we give an application of this study on computing the contact homology for a sequence of contact forms defined on the torus T^3.

Chapter 3

The loop space In the presence of a contact form

3.1 Introduction

Let M be a 3-dimensional smooth compact orientable manifold, and α a one form on it. $\mathcal{L}_\alpha$ denote the space of Legendrian curves on M. This space is a subset of the free loop space of M denoted by $\Lambda(S^1, M)$. Now we recall a result of Smale [38] :

Theorem 3.1.1. *Let (M, α) be a contact manifold, then the injection*

$$j : \mathcal{L}_\alpha \hookrightarrow \Lambda(S^1, M)$$

is an S^1-equivariant homotopy equivalence.

Proof. The proof of this theorem follows from the following diagram :

$$\begin{tikzcd} \mathcal{L}_\alpha \arrow[r] \arrow[d, dotted, "i_1"] & \mathcal{P}\mathcal{L}_\alpha(M, x_0) \arrow[r, "\pi_1"] \arrow[d, dotted, "i_2"] & M \arrow[d, "\simeq"] \\ \Lambda(M) \arrow[r] & \mathcal{P}(M, x_0) \arrow[r, "\pi_2"] & M \end{tikzcd}$$

The map π_2 is the classical Serre fibration, and it suffices to show that indeed the second map satisfies the homotopy lifting property, which makes it a fibration. Now the space $\mathcal{P}(M, x_0)$ of paths starting at the point $x_0 \in M$ is contractible and the same holds for the space $\mathcal{P}\mathcal{L}_\alpha(M, x_0)$ of legendrian curves starting from x_0. The maps i_1 and i_2 are the natural injections.

The previous fibrations induces a long exact sequence on the homotopy groups that is

$$\begin{tikzcd} \cdots \arrow[r] & \pi_{n+1}(M) \arrow[r] \arrow[d, "\simeq"] & \pi_n(\mathcal{L}_\alpha) \arrow[r] \arrow[d, dotted, "i_*"] & \pi_n(\mathcal{P}\mathcal{L}_\alpha(M, x_0)) \arrow[r] \arrow[d, "\simeq"] & \pi_n(M) \arrow[r] \arrow[d, "\simeq"] & \cdots \\ \cdots \arrow[r] & \pi_{n+1}(M) \arrow[r] & \pi_n(\Lambda(M)) \arrow[r] & \pi_n(\mathcal{P}(M, x_0)) \arrow[r] & \pi_n(M) \arrow[r] & \cdots \end{tikzcd}$$

This give us the equality $\pi_n(\mathcal{L}_\alpha) = \pi_n(\Lambda(M))$ for all n. Using Whitehead's theorem, we have that the injection i induces a homotopy equivalence. $\square$

In this chapter we are going to prove a theorem that can be seen as related to the above theorem, only that the framework is different and the space $\mathcal{L}_\alpha$ is replaced by a smaller space $\mathcal{C}_\beta$, that appears to be convenient in contact form geometry as it is the space of variations of the functional J introduced in section 2.1. For more details we can see [3],[5] and [6]. Namely, we consider the following assumption :

In this chapter we are going to prove a theorem that can be seen as related to the above theorem only that the framework is different and the space $\mathcal{L}_\alpha$ is replaced by a smaller space $\mathcal{C}_\beta$, that appears to be convenient in contact form geometry as the space of variations of the functional J introduced in section 2.1. For more details we can see [3],[4] and [6]. Namely, we consider the following assumption :

(A) there exists a smooth vector field $v \in \ker(\alpha)$ such that the dual 1-form $\beta = d\alpha(v, \cdot)$ is a contact form with the same orientation than α.

In this paper we are going to prove a theorem that can be seen as related to the above theorem: the framework will be slightly different and the space $\mathcal{L}_\alpha$ will be replaced by a smaller space $\mathcal{C}_\beta$, that appears to be convenient in some variational problems in contact form geometry (see for instance [3],[4] and [6]). We will introduce the following assumption:

(A) there exists a smooth vector field $v \in \ker(\alpha)$ such that the dual 1-form $\beta = d\alpha(v, \cdot)$ is a contact form with the same orientation than α.

Under (A), we renormalize v onto λv so that $\alpha \wedge d\alpha = \beta \wedge d\beta$.

By Smale's theorem, we know that the injection $\mathcal{L}_\beta$ in $\Lambda(S^1, M)$ is an S^1-equivariant homotopy equivalence. We are interested in a space that is smaller than $\mathcal{L}_\beta$ and it is defined in the following way:

Let

$$\mathcal{C}_\beta = \{x \in \mathcal{L}_\beta; \alpha_x(\dot{x}) = c > 0\}$$

where c is a constant that varies with the curve x.

The space $\mathcal{C}_\beta$ is very useful in contact geometry and it is of independent interest in differential topology. For example, let us take the framework of (S^3, α_0), the standard contact form on S^3, and let

$$v = -x_4\partial_{x_1} - x_3\partial_{x_2} + x_2\partial_{x_3} + x_1\partial_{x_4}$$

be a Hopf fibration vector field in $\ker \alpha_0$. The space $\mathcal{C}_\beta$ can be identified as the lift to S^3 (according to some rules, see [3]) of the space $Imm_0(S^1, S^2)$ of immersed curves from S^1 into S^2 of Maslov index zero. Smale's theorem [38] asserts then that the injection $\mathcal{C}_\beta \hookrightarrow \Lambda(S^1, S^3)$ is an S^1-equivariant homotopy equivalence.

In this paper, we extend this result to a more general framework of (M, α) under (A) and an additional assumption that we introduce below. We need, in order to state this second assumption, to introduce the one-parameter group generated by v that we will denote by φ_s.

From [3] and [6] we know that the kernel of a contact form rotates monotonically in a frame transported by φ_s along v. Based on this fact we give the following definition.

Definition 3.1.2. *We say that* $\ker \alpha$ *turns well along* v*, if starting from any* x_0 *in* M*, the rotation of* $\ker \alpha$ *along the* v*-orbit in a transported frame exceeds* π*.*[1]

Our second assumption is therefore:

$$(B) \quad \ker \alpha \text{ turns well along } v$$

In this chapter, we will prove the following

Theorem 3.1.3. *Let* (M, α) *be a contact closed manifold. Then under the assumptions (A) and (B), the injection*

$$\mathcal{C}_\beta \hookrightarrow \Lambda(S^1, M)$$

is an S^1*-equivariant homotopy equivalence.*

[1] *It is in fact then infinite*

Let us recall first some properties that we will be using later. Given the contact form α, we will let ξ be its Reeb vector field. Namely, ξ is the unique vector satisfying

$$\alpha(\xi) = 1, \qquad d\alpha(\xi, \cdot) = 0$$

Therefore the following holds (see [3]):

Lemma 3.1.4 ([3]). *Under the assumption (A), let w be the Reeb vector field of the 1-form β, then there exist two functions τ and $\overline{\mu}$ such that:*

$$[\xi, [\xi, v]] = -\tau v, \qquad w = -[\xi, v] + \overline{\mu}\xi$$

where $\overline{\mu} = d\alpha(v, [v, [\xi, v]])$.

Notice also that with the previous notation, the following holds:

$$\dot{x} = a\xi + bv, \quad \forall x \in \mathcal{L}_\beta$$

Moreover if x is in $\mathcal{C}_\beta$ then a is a positive constant. One can show (see [3]) that $\mathcal{C}_\beta \setminus M$ has a Hilbert manifold structure. For $x \in \mathcal{C}_\beta$, the tangent space at the curve x is given by the set of vector fields

$$Z = \lambda\xi + \mu v + \eta w$$

with the coefficients λ, μ and η satisfying the following equations:

$$\begin{cases} \dot{\overline{\lambda + \overline{\mu}\eta}} = b\eta - \int_0^1 b\eta, \\ \\ \dot{\eta} = \mu a - \lambda b \end{cases} \tag{3.1}$$

where λ, μ and η are 1-periodic.

The proof of the main theorem requires several steps. We apply first Smale's theorem to conclude that the injection $\mathcal{L}_\beta \hookrightarrow \Lambda(S^1, M)$ is a homotopy equivalence. Next, we introduce an intermediate space $\mathcal{C}_\beta^+$ defined by

$$\mathcal{C}_\beta^+ = \{x \in \mathcal{L}_\beta; \alpha(\dot{x}) \geq 0\},$$

and we show that we can deform $\mathcal{L}_\beta$ to $\mathcal{C}_\beta^+$. This deformation is not continuous because "Dirac masses" (see below) along v are created through this transformation. We mean by a Dirac mass, a back and forth v-jump, inserted at a point of the curve. We will "solve the Dirac masses", showing how they are created along a smooth deformation in $\mathcal{L}_\beta$.

In a next and last step we "push" the curves of $\mathcal{C}_\beta^+$ into $\mathcal{C}_\beta$. This will be completed by constructing a flow that brings curves with $a \geq 0$ to curves with $a > 0$. Let us give a first a precise definition of the Dirac masses.

Definition 3.1.5. *Let x be a curve in $\mathcal{H}^1(S^1, M)$. We say that x has a Dirac mass at $x(t_0)$ if there exist t_1 and t_2 in S^1 such that x is tangent to v for $t \in [t_0, t_1]$, x tangent to $-v$ for $t \in [t_1, t_2]$ and $x(t_0) = x(t_2)$.*

Before going into more details, let us discuss the assumptions and let us give some examples of contact structures for which they hold.

Assumption (A) holds for a number of contact structures with suitable vector fields v in their kernel. For instance the standard contact form on S^3

$$\alpha_0 = x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4$$

and also the family of contact structures on T^3 given by

$$\alpha_n = cos(2n\pi z)dx + sin(2n\pi z)dy$$

All the contact forms in the previous examples are tight, but there are also overtwisted contact forms satisfying (A). This is the case of the first non-standard 1-form on S^3, given by Gonzalo-Varela in [26]:

$$\alpha_1 = -(\cos(\frac{\pi}{4} + \pi(x_3^2 + x_4^2))(x_2 dx_1 - x_1 dx_2) + \sin(\frac{\pi}{4} + \pi(x_3^2 + x_4^2))(x_4 dx_3 - x_3 dx_4))$$

where an (explicit) existence of a suitable v satisfying (A) is proved in [35].

The assumption (B) holds also for the previous mentioned examples; moreover this assumption has a deeper meaning. In fact, it was proved in the work of Gonzalo [25], that (B) holds if and only if α extends to a contact circle, namely there exists another contact form α_2 transverse to α with intersection the line spanned by v, such that

$$cos(s)\alpha + sin(s)\alpha_2$$

is a contact form for every $s \in \mathbb{R}$.

Let us observe that α_1 defined above represents the first example of an overtwisted contact circle on a compact manifold. In fact, see [23] for a question of Geiges and Gonzalo, where they give an example of an overtwisted contact circle on $\mathbb{R}^3$ and they observe that they don't know an explicit example of overtwisted contact circle on a closed manifold. α_1 with the v found in [35] is such an example.

Using this criteria one can give some conditions under which (B) holds:

Lemma 3.1.6. *Assume that (A) holds, then (B) holds if one of the following conditions is satisfied:*

(i) $\quad |\overline{\mu}| < 2$

(ii) $\quad$ *there exists a map u on M such that $\overline{\mu} = u_v$*

Moreover, if $\overline{\mu} = 0$ then α is tight.

Proof. We use the characterization stated above for contact circles.

Let s be a real number, and consider the 1-form

$$\alpha_s = \cos(s)\alpha + \sin(s)\beta;$$

then

$$\alpha_s \wedge d\alpha_s = \cos^2(s)\alpha \wedge d\alpha + \sin^2(s)\beta \wedge d\beta + \cos(s)\sin(s)(\alpha \wedge d\beta + \beta \wedge d\alpha)$$

Notice now, (see [3]), that $\alpha \wedge d\beta(\xi, v, w) = -\overline{\mu}$, thus we have

$$\alpha_s \wedge d\alpha_s(\xi, v, w) = 1 - \frac{\sin(2s)}{2}\overline{\mu}$$

and the conclusion follows for (i).

For (ii) we consider

$$\alpha_s = \cos(s)\alpha + \sin(s)e^u\beta$$

and the same computation yields

$$\alpha_s \wedge d\alpha_s = \cos^2(s)\alpha \wedge d\alpha + e^{2u}\sin^2(s)\beta \wedge d\beta + \sin(s)\cos(s)e^u(\alpha \wedge d\beta + \alpha \wedge du \wedge \beta)$$

Evaluating at $(\xi, v, [\xi, v])$ we get:

$$\alpha_s \wedge d\alpha_s = \cos^2(s) + e^{2u}\sin^2(s) + e^u \sin(s)\cos(s)(u_v - \overline{\mu})$$

therefore (ii) follows.

Now notice that if $\overline{\mu} = 0$ then we have what it is called a taut contact circle (in fact we have a Cartan structure), therefore based on the result of Geiges-Gonzalo [23], we have that α and β are tight. $\square$

3.2 Regularization

Since we are considering curves in $\mathcal{L}_\beta$ that we want to lift to C_β, the first difficulty that we will face are the degenerate curves, namely curves that are not generic in the sense that the components of the tangent vectors can have bad behaviour. Therefore, in this section we want to regularize the curves starting from a compact set of $\mathcal{L}_\beta$. This regularization will be done by the use of a flow on the curves that induces a heat flow on the components of the tangent vector making them smooth and having isolated zeros. The flow will be constructed on the tangent as a heat flow, but there is no guaranty that the flow is indeed a flow on curves. For that we will first approximate the deformation vector with a smooth one for which we know the local existence. Then we will show that when our approximation tends to the original flow, the maximal time of existence is bounded from below independently of the approximation. These statements will be made precise and clear in what follows.

Here we want to construct a flow on $\mathcal{L}_\beta$ that deforms compact sets of curves y of $\mathcal{L}_\beta$ into curves x with $\dot{x} = a\xi + bv$, where a and b are smooth and have zeros of finite order. This will be used in our proof below. In fact what is needed is just the fact that a has zeros of finite order but here we will prove the stronger result involving the preservation of the number of zeros of b along the deformation.

A first idea is to consider the flow defined by the vector field

$$Z = (\dot{a} + f)\xi + (\dot{b} + g)v + \eta[\xi, v]$$

where $\eta = \eta(a, b)$ satisfies the following differential equation:

$$\dot{\eta} = \overline{\mu} b\eta + \dot{b}a - \dot{a}b + ga - fb.$$

The vector field Z constructed in this way will generate a diffusion flow on a and b as follow:

$$\begin{cases} \dfrac{\partial a}{\partial s} = \ddot{a} + \dot{f} - b\eta \\ \\ \dfrac{\partial b}{\partial s} = \ddot{b} + \dot{g} + \eta(a\tau - \overline{\mu}_{\xi} b) \end{cases} \tag{3.2}$$

To ensure the periodicity of η, we set $f = -\kappa b$ and $g = \kappa a$ with $\kappa = \kappa(a, b)$ satisfying

$$\int_0^1 e^{-\int_0^r b(u,s)\overline{\mu}(u)du}(a\dot{b} - \dot{a}b)(r, s)dr + \kappa \int_0^1 e^{-\int_0^r b(u,s)\overline{\mu}(u)du}(a^2 + b^2)(r, s)dr = 0$$

The problem with this first attempt is that the previous system depends on the curve and we do not know so far how this vector Z acts on the curve and if it defines indeed a flow on $\mathcal{L}_\beta$. We will follow the same technique as in [6], to prove that indeed we have a flow on the curves that gives rise to the system defined above. Hence a first step consists of regularizing a and b by using a mollifier ϕ_ε and we use the classical Cauchy-Lipschitz theorem for the flow defined by the approximated vector field Z_ε. The second part consists of showing the convergence to the aimed system as ε converges to zero.

3.2.1 Approximated Flow

We consider the regularizing operator $\phi_\varepsilon : \mathcal{H}^1(S^1) \longrightarrow \mathcal{H}^2(S^1)$, for $\varepsilon > 0$, such that for $f \in \mathcal{H}^1(S^1)$, $\phi_\varepsilon(f)$ satisfies the following equation:

$$-\varepsilon\ddot{\phi}_\varepsilon(f) + \phi_\varepsilon(f) = f. \tag{3.3}$$

Notice that in terms of Fourier coefficients this corresponds to divide by $1 + \varepsilon k^2$.

Preliminary estimates

We have the following estimates for the operator ϕ_ε.

Lemma 3.2.1. *Let $f \in \mathcal{H}^1(S^1)$, then there exists $C > 0$ independent of ε such that* *(i)* $\|\phi_\varepsilon(f)\|_{\mathcal{H}^1} \leq C\|f\|_{\mathcal{H}^1}$

(ii) $\|\dot{\phi}_\varepsilon(f)\|^2_{\mathcal{H}^1} \leq \frac{C}{\varepsilon}\|f\|^2_{\mathcal{H}^1}$

(iii) $\|\phi_\varepsilon(f) - f\|^2_{L^2} \leq C\varepsilon\|f\|^2_{\mathcal{H}^1}$

Proof. (i) As it was defined $\phi_\varepsilon(f)$ satisfies

$$-\varepsilon\ddot{\phi}_\varepsilon(f) + \phi_\varepsilon(f) = f$$

So if we multiply the previous equation by $\phi_\varepsilon(f)$ we have

$$\|\phi_\varepsilon(f)\|^2_{L^2} \leq \|f\|_{L^2}\|\phi_\varepsilon(f)\|_{L^2} - \varepsilon\|\dot{\phi}_\varepsilon(f)\|^2_{L^2}$$

hence the inequality follows for the L^2 norm, and by linearity we have the same inequality for $\dot{f}$.

(ii) With the same idea, we find

$$\varepsilon\|\dot{\phi}_\varepsilon(f)\|^2_{L^2} \leq C\|f\|^2_{L^2}$$

Therefore the estimate follows by linearity.

(iii) From (3.3) we have that

$$\int_0^1 |f - \phi_\varepsilon(f)|^2 = \varepsilon\|\dot{\phi}_\varepsilon(f)\|^2_{L^2} - \varepsilon\|\dot{f}\|_{L^2}\|\dot{\phi}_\varepsilon(f)\|_{L^2}$$

Using (i) we have

$$\|f - \phi_\varepsilon(f)\|^2_{L^2} \leq 2\varepsilon\|\dot{f}\|^2_{L^2}.$$

□

We consider now the operator L_ε defined by

$$L_\varepsilon(f)(s,t) = \sum e^{-s\frac{k^2}{\varepsilon k^2+1}} e^{ikt} f_k,$$

where $f = \sum f_k e^{ikt}$. This operator satisfies for $g = L_\varepsilon(f)$,

$$\begin{cases} \dfrac{\partial g}{\partial s} - \ddot{\phi}_\varepsilon(g) = 0 \\ \\ g(0,t) = f(t) \end{cases}$$

Notice that L_0 corresponds to the inverse of the homogeneous heat operator.

Lemma 3.2.2. *The operator L_ε converges to L_0 in the operator norm from $\mathcal{H}^1(S^1)$ to $L^\infty(0,1,\mathcal{H}^1(S^1))$.*

Proof. Let $f \in \mathcal{H}^1(S^1)$, then

$$\|(L_\varepsilon - L_0)f\|^2_{\mathcal{H}^1} = \sum (1+k^2)|f_k|^2 |e^{-s\frac{k^2}{\varepsilon k^2+1}} - e^{-sk^2}|^2$$

but

$$|e^{-s\frac{k^2}{\varepsilon k^2+1}} - e^{-sk^2}| = e^{-sk^2}|1 - e^{\frac{-s\varepsilon k^2}{1+\varepsilon k^2}}| \leq e^{-sk^2}\varepsilon\frac{k^2 s}{1+\varepsilon k^2}$$

Therefore there exists a $C > 0$ independent of s and ε (an upper bound of $e^{-2u}u^2$) such that

$$\|(L_\varepsilon - L_0)f\|^2_{\mathcal{H}^1} \leq C\varepsilon^2 \|f\|^2_{\mathcal{H}^1}.$$

Hence $\|L_\varepsilon - L_0\|_{L^\infty(0,1,\mathcal{H}^1)}$ converges to zero as ε tends to zero with a rate of at least ε. $\square$

A similar lemma holds for the operator $\tilde{L}_\varepsilon$ corresponding to the general solution of:

$$\begin{cases} \dfrac{\partial g}{\partial s} - \ddot{\phi}_\varepsilon(g) = f \\ \\ g(0,t) = 0 \end{cases}$$

Lemma 3.2.3. *Let $f \in L^\infty(0,\epsilon,\mathcal{H}^l(S^1))$, where $\epsilon < 1$ fixed, then $g = \tilde{L}_\varepsilon(f) \in L^\infty(0,\epsilon,\mathcal{H}^l(S^1)$. Moreover, there exists C independent of ε, such that*

$$\|\tilde{L}_\varepsilon(f)\|^2_{\mathcal{H}^l}(s) \leq C \int_0^s \left(\|f\|^2_{\mathcal{H}^{l-1}}(r) + \varepsilon\|f\|^2_{\mathcal{H}^l}(r)\right) dr$$

for all $l \geq 1$.

Proof. We consider the Fourier expansion of $f = \sum_k f_k(s)e^{ikt}$, then for $u = \tilde{L}_\varepsilon(f)$ we have that

$$u_k(s) = \int_0^s e^{-\frac{k^2}{1+\varepsilon k^2}(s-r)} f_k(r)dr$$

Thus

$$|u_k^2| \leq \frac{1+\varepsilon k^2}{k^2}\int_0^s |f_k|^2(r)dr$$

Therefore

$$\|u\|_{\mathcal{H}^l}^2 \leq C \int_0^s \|f\|_{\mathcal{H}^{l-1}}^2(r) + \varepsilon \|f\|_{\mathcal{H}^l}^2(r) dr$$

$\square$

Estimates along the flow

We consider now the component

$$\eta(x,a,b) = e^{\int_0^t b(u,s)\overline{\mu}(u)du}\Big(\int_0^t e^{-\int_0^r b(u,s)\overline{\mu}(u)du}((a\dot{b} - \dot{a}b) + \kappa(a^2+b^2)(r,s)dr\Big)$$

where

$$\kappa(x,a,b) = -\frac{\int_0^1 e^{-\int_0^r b(u,s)\overline{\mu}(u)du}(a\dot{b} - \dot{a}b)(r,s)dr}{\int_0^1 e^{-\int_0^r b(u,s)\overline{\mu}(u)du}(a^2+b^2)(r,s)dr}.$$

The constant κ is computed so that η is 1-periodic. Notice that we use (x,a,b) instead of just x since we are interested more in the coefficients a and b.

Similarly, we take $\lambda(x,a,b) = \dot{a} - \kappa b$ and $\mu(x,a,b) = \dot{b} + \kappa a$.

We define now

$$\eta_\varepsilon(x,a,b) = \eta(x,\phi_\varepsilon(a),\phi_\varepsilon(b)), \qquad \lambda_\epsilon(x,a,b) = \lambda(x,\phi_\varepsilon(a),\phi_\varepsilon(b))$$

$$\mu_\epsilon(x,a,b) = \mu(x,\phi_\varepsilon(a),\phi_\varepsilon(b)), \qquad \kappa_\varepsilon(x,a,b) = \kappa(x,\phi_\varepsilon(a),\phi_\varepsilon(b))$$

The vector field

$$Z_\varepsilon = \lambda_\varepsilon \xi + \mu_\varepsilon v + \eta_\varepsilon[\xi, v]$$

is then locally Lipschitz and hence the flow

$$\begin{cases} \dfrac{\partial x}{\partial s} = Z_\varepsilon(x) = \lambda_\varepsilon \xi + \mu_\varepsilon v + \eta_\varepsilon[\xi,v] \\ \\ x(0) = x_0 \end{cases} \tag{3.4}$$

has a unique solution that exists in $[0, s_0(\varepsilon))$, by the standard Cauchy-Lipschitz theorem. It is important to notice that this flow will not stay in $\mathcal{L}_\beta$, in fact it will be defined in a neighborhood of x_0 in $\mathcal{H}^2(S^1, M)$. But hopefully, when ε converges to zero the limiting flow will be in $\mathcal{L}_\beta$.

We want to have good estimates on the coefficients a and b as ε converges to zero.

Using these notations we have that under the flow generated by Z_ε (see [6]):

$$\begin{cases} \dfrac{\partial a}{\partial s} = \dot{\lambda}_\varepsilon - b\eta_\varepsilon + c\mu_\varepsilon \\ \\ \dfrac{\partial b}{\partial s} = \dot{\mu}_\varepsilon + (\tau a - \overline{\mu}_\xi b)\eta_\varepsilon + c(\tau\lambda_\varepsilon - \overline{\mu}_\xi \mu_\varepsilon) \\ \\ \dfrac{\partial c}{\partial s} = \dot{\eta}_\varepsilon - \overline{\mu} b\eta_\varepsilon - \mu_\varepsilon a + \lambda_\varepsilon b - \overline{\mu}\mu_\varepsilon c \\ \\ a(0) = a_0,\ b(0) = b_0,\ c(0) = 0. \end{cases} \tag{3.5}$$

where c is the component of $\dot{x}$ along $[\xi, v]$. Existence is not found directly from the system itself, but instead it follows from the one in (3.4), since this is the evolution of the components of the tangent vector to the curve evolving under the flow generated by Z_ε. These functions $\overline{\mu}$, τ, are given as functions of s. That is $\overline{\mu}(x(s))$ and $\tau(x(s))$, where $x(s)$ is the solution of (3.4). We then reformulate the previous system as a fixed point problem. We will then derive appropriate estimates on (3.5) that will allow us to establish existence of the limiting flow as $\varepsilon \to 0$.

For this purpose, we consider the operator F_ε defined by :

$$F_\varepsilon(x, \dot{x}) = \begin{bmatrix} -\kappa_\varepsilon \dot{\phi}_\varepsilon(b) - b\eta_\epsilon + c\mu_\varepsilon \\ \kappa_\varepsilon \dot{\phi}_\varepsilon(a) + \eta_\varepsilon(a\tau - \overline{\mu}_\xi b) + c(\tau\lambda_\varepsilon - \overline{\mu}_\xi \mu_\varepsilon) \\ \dot{\eta}_\varepsilon - \overline{\mu} b\eta_\varepsilon - \mu_\varepsilon a + \lambda_\varepsilon b - \overline{\mu}\mu_\varepsilon c \end{bmatrix}$$

This evolution equation follows from the proposition in Appendix.

Now we define the space $\mathcal{B}_\varepsilon$, for $\varepsilon > 0$, as follows:

$$\mathcal{B}_\varepsilon = \{x(s,t) \in L^\infty(0, \varepsilon, \mathcal{H}^2(S^1)); \dot{x}(s,t) \in L^\infty(0, \varepsilon, \mathcal{H}^1(S^1))\}.$$

So F_ε sends $\mathcal{B}_\epsilon$ to itself. Also, we can define the operator T_ε by

$$T_\varepsilon(x, \dot{x}) = L_\varepsilon \begin{bmatrix} a_0 \\ b_0 \\ 0 \end{bmatrix} + \begin{bmatrix} \tilde{L}_\varepsilon(-\kappa_\varepsilon \dot{\phi}_\varepsilon(b) - b\eta_\epsilon + c_\varepsilon \mu_\varepsilon) \\ \tilde{L}_\varepsilon(\kappa_\varepsilon \dot{\phi}_\varepsilon(a) + \eta_\varepsilon(a\tau - \overline{\mu}_\xi b) + c_\varepsilon(\tau\lambda_\varepsilon - \overline{\mu}_\xi \mu_\varepsilon)) \\ \int_0^s e^{\int_s^r (\overline{\mu}\mu_\varepsilon)dr}(\dot{\eta}_\varepsilon - \overline{\mu} b\eta_\varepsilon - \mu_\varepsilon a + \lambda_\varepsilon b)ds \end{bmatrix}$$

So that the fixed point of T_ε corresponds to the solution of the system (3.5). In fact we have

$$\frac{\partial}{\partial t}T_\varepsilon(x,\dot{x}) = \ddot{\phi}_\varepsilon(L_\varepsilon \begin{bmatrix} a_0 \\ b_0 \\ 0 \end{bmatrix}) + \begin{bmatrix} \ddot{\phi}_\varepsilon(\tilde{L}_\varepsilon(-\kappa_\varepsilon\dot{\phi}_\varepsilon(b) - b\eta_\varepsilon + c_\varepsilon\mu_\varepsilon)) \\ \ddot{\phi}_\varepsilon(\tilde{L}_\varepsilon(\kappa_\varepsilon\dot{\phi}_\varepsilon(a) + \eta_\varepsilon(a\tau - \overline{\mu}_\xi b) + c_\varepsilon(\tau\lambda_\varepsilon\overline{\mu}_\xi\mu_\varepsilon))) \\ 0 \end{bmatrix} +$$

$$+ \begin{bmatrix} -\kappa_\varepsilon\dot{\phi}_\varepsilon(b) - b\eta_\epsilon + c_\varepsilon\mu_\varepsilon \\ \kappa_\varepsilon\dot{\phi}_\varepsilon(a) + \eta_\varepsilon(a\tau - \overline{\mu}_\xi b) + c_\varepsilon(\tau\lambda_\varepsilon - \overline{\mu}_\xi\mu_\varepsilon) \\ \dot{\eta}_\varepsilon - \overline{\mu}b\eta_\varepsilon - \mu_\varepsilon a + \lambda_\varepsilon b - \overline{\mu}\mu_\varepsilon c \end{bmatrix}.$$

In what follow we will use $\|f\|$ instead of $\|f(x(\cdot))\|$ for the functions depending on the curve (such as τ, $\overline{\mu}$, etc.).

Lemma 3.2.4. *Let $x \in \mathcal{B}_\epsilon$ such that $\|\dot{x}\|_{L^2} \geq \delta > 0$, then there exist three positive constants C_1, C_2 and C_3 independent of ε and x, but possibly depending on δ, such that*

$$\|T_\varepsilon(x,\dot{x})\|_{\mathcal{H}^1}(s) \leq \|\dot{x}_0\|_{\mathcal{H}^1} + e^{C_3\int_0^s \|\dot{x}\|_{\mathcal{H}^1}} \int_0^s (C_1 + C_2\sqrt{\varepsilon})e^{C_3\|\dot{x}\|_{\mathcal{H}^1}(r)}dr$$

Proof. We first need an estimate on λ_ε, μ_ε and η_ε. In order to do that, an estimate on the variable κ_ε is necessary.

By the very definition of κ we get

$$|\kappa_\varepsilon| \leq C\frac{(\|a\|_{\mathcal{H}^1} + \|b\|_{\mathcal{H}^1})^2 e^{2\|b\|_{\mathcal{H}^1}\|\overline{\mu}\|_{L^\infty}}}{\|a\|_{L^2}^2 + \|b\|_{L^2}^2}$$

Using the previous estimate and Lemma (2.1.), we have

$$\|\lambda_\varepsilon(x,\dot{x})\|_{\mathcal{H}^1} \leq \frac{1}{\sqrt{\varepsilon}}\|a\|_{\mathcal{H}^1} + \frac{(\|a\|_{\mathcal{H}^1} + \|b\|_{\mathcal{H}^1})^2 e^{2\|b\|_{\mathcal{H}^1}\|\overline{\mu}\|_{L^\infty}}}{\|a\|_{L^2}^2 + \|b\|_{L^2}^2}\|b\|_{\mathcal{H}^1}$$

A similar estimate holds for μ_ε, that is

$$\|\mu_\varepsilon(x,\dot{x})\|_{\mathcal{H}^1} \leq \frac{1}{\sqrt{\varepsilon}}\|b\|_{\mathcal{H}^1} + \frac{(\|a\|_{\mathcal{H}^1} + \|b\|_{\mathcal{H}^1})^2 e^{2\|b\|_{\mathcal{H}^1}\|\overline{\mu}\|_{L^\infty}}}{\|a\|_{L^2}^2 + \|b\|_{L^2}^2}\|a\|_{\mathcal{H}^1}$$

The estimate for η_ε is a little bit different

$$\|\eta_\varepsilon\|_{\mathcal{H}^2} \leq C(\|b\|_{\mathcal{H}^1}(\|b\|_{\mathcal{H}^1}+\|a\|_{\mathcal{H}^1}+\|c\|_{\mathcal{H}^1}+1)e^{2\|b\|_{\mathcal{H}^1}\|\overline{\mu}\|_{L^\infty}}\Big(\frac{1}{\varepsilon}(\|a\|_{\mathcal{H}_1}+\|b\|_{\mathcal{H}^1})^2+\frac{\|a\|_{\mathcal{H}^1} + \|b\|_{\mathcal{H}^1}}{\|a\|_{L^2} + \|b\|_{L^2}}\Big)$$

And by taking again another derivative we have the desired estimate. In fact we have

$$\|\dot{\eta}_\varepsilon\|_{L^2} \leq Ce^{\|b\|_{\mathcal{H}^1}\|\overline{\mu}\|_{L^\infty}}(\|\overline{\mu}\|_{L^\infty}\|b\|_{L^2} + e^{\|b\|_{\mathcal{H}^1}\|\overline{\mu}\|_{L^\infty}}(\|a\|_{\mathcal{H}^1}^2 + \|b\|_{\mathcal{H}^1}^2 + |\kappa_\varepsilon|(\|a\|_{L^2}^2 + \|b\|_{L^2}^2))$$

Now

$$\| - \kappa_\varepsilon \dot{\phi}_\varepsilon(b) - b\eta_\epsilon + c\mu_\varepsilon\|_{\mathcal{H}^1} \leq C\big(\frac{1}{\sqrt{\varepsilon}}|\kappa_\varepsilon|\|b\|_{\mathcal{H}^1} + \|b\|_{\mathcal{H}^1}\|\eta_\varepsilon\|_{\mathcal{H}^1} + \|c\|_{\mathcal{H}^1}\|\mu_\varepsilon\|_{\mathcal{H}^1}\big)$$

Also

$$\|\kappa_\varepsilon \dot{\phi}_\varepsilon(a) + \eta(a\tau - \overline{\mu}_\xi b) + c_\varepsilon(\tau\lambda_\varepsilon - \overline{\mu}_\xi \mu_\varepsilon)\|_{\mathcal{H}^1} \leq C\big(\frac{1}{\sqrt{\varepsilon}}|\kappa_\varepsilon|\|a\|_{\mathcal{H}^1} +$$

$$+ \|\dot{x}\|_{L^2}\|\eta_\varepsilon\|_{\mathcal{H}^1}(\|a\|_{\mathcal{H}^1} + \|b\|_{\mathcal{H}^1}) + \|\dot{x}\|_{L^2}\|c\|_{\mathcal{H}^1}(\|\mu_\varepsilon\|_{\mathcal{H}^1} + \|\lambda_\varepsilon\|_{\mathcal{H}^1})\big)$$

It is crucial to notice that η_ε satisfies

$$\dot{\eta}_\varepsilon = \overline{\mu}\eta_\varepsilon + \phi_\varepsilon(a)\mu_\varepsilon - \lambda_\varepsilon \phi_\varepsilon(b)$$

Therefore we have

$$\|\dot{\eta}_\varepsilon - \overline{\mu}b\eta_\varepsilon - \mu_\varepsilon a + \lambda_\varepsilon b\|_{\mathcal{H}^1} = \|\overline{\mu}\eta_\varepsilon(\phi_\varepsilon(b) - b) + \mu_\varepsilon(\phi_\varepsilon(a) - a) + \lambda_\varepsilon(b - \phi_\varepsilon(b))\|_{\mathcal{H}^1}$$

$$\leq \|\overline{\mu}\|_{\mathcal{H}^1}\|\eta_\varepsilon\|_{\mathcal{H}_1}\|\phi_\varepsilon(b) - b\|_{L^2} + \|\overline{\mu}\|_{L^2}\|\eta_\varepsilon\|_{L^2}\|b\|_{\mathcal{H}^1} +$$

$$+ \|\mu_\varepsilon\|_{L^2}\|a\|_{\mathcal{H}^1} + \|\mu_\varepsilon\|_{\mathcal{H}^1}\|a - \phi_\varepsilon(a)\|_{L^2} +$$

$$+ \|\lambda_\varepsilon\|_{\mathcal{H}^1}\|b - \phi_\varepsilon(b)\|_{L^2} + \|\lambda_\varepsilon\|_{L^2}\|b\|_{\mathcal{H}^1}$$

Using Lemma (2.1.) we have

$$\|\dot{\eta}_\varepsilon - \overline{\mu}b\eta_\varepsilon - \mu_\varepsilon a + \lambda_\varepsilon b\|_{\mathcal{H}^1} \leq C\big(\sqrt{\varepsilon}(\|\overline{\mu}\|_{\mathcal{H}^1}\|\eta_\varepsilon\|_{\mathcal{H}_1}\|b\|_{\mathcal{H}^1} + \|\mu_\varepsilon\|_{\mathcal{H}^1}\|a\|_{\mathcal{H}^1} + \|\lambda_\varepsilon\|_{\mathcal{H}^1}\|b\|_{\mathcal{H}^1}) +$$

$$+ (\|\overline{\mu}\|_{L^\infty}\|\eta_\varepsilon\|_{L^2}\|b\|_{\mathcal{H}^1} + \|\mu_\varepsilon\|_{L^2}\|a\|_{\mathcal{H}^1} + \|\lambda_\varepsilon\|_{L^2}\|b\|_{\mathcal{H}^1})\big)$$

Now for the L^2 norm we have

$$\|\dot{\eta}_\varepsilon - \overline{\mu}b\eta_\varepsilon - \mu_\varepsilon a + \lambda_\varepsilon b\|_{L^2} \leq C\big(\sqrt{\varepsilon}(\|\overline{\mu}\|_{\mathcal{H}^1}\|\eta_\varepsilon\|_{\mathcal{H}_1}\|b\|_{\mathcal{H}^1} + \|\mu_\varepsilon\|_{\mathcal{H}^1}\|a\|_{\mathcal{H}^1} + \|\lambda_\varepsilon\|_{\mathcal{H}^1}\|b\|_{\mathcal{H}^1})\big)$$

Let us set

$$Y = \dot{\eta}_\varepsilon - \overline{\mu}b\eta_\varepsilon - \mu_\varepsilon a + \lambda_\varepsilon b, \qquad A = -\overline{\mu}\mu_\varepsilon$$

So if we write a curve $x \in \mathcal{H}^2(S^1, M)$ as

$$\dot{x} = a\xi + bv + c[\xi, v]$$

then c satisfies

$$\frac{\partial}{\partial s}c = Ac + Y$$

From this equality we have

$$\frac{\partial}{\partial s}\|c\|_{L^2} \le C\|A\|_{L^2}\|c\|_{L^2} + \|Y\|_{L^2}$$

thus

$$\|c\|_{L^2} \le C\int_0^s e^{\int_r^s \|A\|_{L^2}dr}\|Y\|_{L^2}$$
$$\le C\sqrt{\varepsilon}e^{\int_0^s \|\dot{x}\|_{\mathcal{H}^1}}\int_0^s \|\dot{x}\|^3_{\mathcal{H}^1}ds$$

In a similar manner we have

$$\frac{\partial}{\partial s}(\|\dot{c}\|_{L^2}) \le C\|A\|_{L^2}\|\dot{c}\|_{L^2} + \|A\|_{\mathcal{H}^1}\|c\|_{L^2} + \|Y\|_{\mathcal{H}^1}$$

Thus

$$\|\dot{c}\|_{L^2} \le Ce^{C\int_0^s \|\dot{x}\|_{\mathcal{H}^1}}\int_0^s \|\dot{x}\|^4_{\mathcal{H}^1}ds$$

Hence

$$\|c\|_{\mathcal{H}^1} \le C_1 e^{C\int_0^s \|\dot{x}\|_{\mathcal{H}^1}}\int_0^s (1+\|\dot{x}\|^4_{\mathcal{H}^1})ds$$

And the conclusion of the lemma follows from the estimate that we got on the operator $\tilde{L}_\varepsilon$ in Lemma 2.3. □

We set $s_0(\varepsilon)$ the existence time of the solution of system (3.5). Then we have the following

Theorem 3.2.5. *There exists $\varepsilon_0 > 0$, and $\sigma > 0$ such that for every $\varepsilon < \varepsilon_0$, $s_0(\varepsilon) > \sigma$.*

The proof follows from a Gronwall type inequality. Let us first, state and prove the general inequality:

Lemma 3.2.6. *Let y_ε be a family of non-negative C^1 functions such that*

$$y_\varepsilon \le C_\varepsilon f\Big(\int_0^s y_\varepsilon(r)dr\Big)$$

for an increasing and positive function f. Then the blow-up time of y_ε depends only on C_ε. More precisely, if C_ε is bounded then the blow-up time is bounded away from zero.

By blow-up time in the previous lemma, we mean the time $T \in (0, +\infty]$ such that $\lim_{t\longrightarrow T}|f(t)| = +\infty$.

Proof. For the sake of simplicity we will remove the index ε for now. Let us set $U = \int_0^s y$, then one has

$$U' \leq Cf(U)$$

Hence if we set G to be the anti-derivative of $\frac{1}{f}$, then we get

$$G(U(s)) - G(U(0)) \leq Cs.$$

Therefore,

$$U(s) \leq G^{-1}(Cs + G(0)),$$

thus

$$y \leq Cf(G^{-1}(Cs + G(0))).$$

And result of the lemma follows. $\square$

Proof. (of Theorem) We have that the solution x_ε (or more precisely $\dot{x}_\varepsilon$) is a fixed point of T_ε. From the previous lemma we have

$$\|\dot{x}_\varepsilon\|_{\mathcal{H}^1}(s) \leq \|\dot{x}_0\|_{\mathcal{H}^1} + e^{C_3 \int_0^s \|\dot{x}\|_{\mathcal{H}^1}}(C_1 + \sqrt{\varepsilon}C_2) \int_0^s e^{C_3\|\dot{x}\|_{\mathcal{H}^1}}(r)dr$$

Hence if we set if we set $y = \|\dot{x}\|_{\mathcal{H}^1}$, then we have

$$y \leq \|\dot{x}_0\|_{\mathcal{H}^1} + e^{C_3 \int_0^s y}(C_1 + \sqrt{\varepsilon}C_2) \int_0^s e^{C_3 y}$$

Now by using the Jensen inequality for the convex function $u \longmapsto e^{C_3 u}$ and assuming that $s \leq 1$ we have

$$y \leq \|\dot{x}_0\|_{\mathcal{H}^1} + (C_1 + \sqrt{\varepsilon}C_2)\Big(\int_0^s e^{C_3 y}\Big)^2$$

Setting $H = e^{C_3 y}$ we get

$$H \leq e^{\|\dot{x}_0\|_{\mathcal{H}^1} + (C_1 + \sqrt{\varepsilon}C_2)(\int_0^s Hz)^2}$$

so we can apply the previous lemma for H and $f(t) = e^{||\dot{x}_0||_{\mathcal{H}^1} + (C_1 + \sqrt{\varepsilon}C_2)t^2}$. Since the constants never blow-up in our case, we have the result of the theorem, that is the blow-up time for H and hence for $\|\dot{x}\|_{\mathcal{H}^1}$ is bounded from below independently on ε. $\square$

3.2.2 Convergence

Now to see the convergence of the solution as ε tends to zero, we need the following

Lemma 3.2.7. *If x_ε is the solution of the flow defined above, then:*

(i) c_ε converges strongly to zero in the L^2-norm

(ii) a_ε and b_ε converge strongly in the L^2-norm to a solution of the flow (3.2).

Proof. (i) Notice that

$$c_\varepsilon(s) = \int_0^s e^{\int_s^r (\overline{\mu}\mu_\varepsilon)dr} (\overline{\mu}\eta_\varepsilon(\phi_\varepsilon(b_\varepsilon) - b_\varepsilon) + \mu_\varepsilon(\phi_\varepsilon(a_\varepsilon) - a_\varepsilon) + \lambda_\varepsilon(b_\varepsilon - \phi_\varepsilon(b_\varepsilon))(r)dr.$$

Since we have boundedness in the $\mathcal{H}^1$ sense for $s \in [0, s_0]$ we can extract a convergent subsequence in all the L^p; this, combined with the boundedness of η_ε, μ_ε and λ_ε, gives us the convergence to zero in L^2.

(ii) Let us go back to the fixed point formulation,

$$\dot{x}_\varepsilon = T_\varepsilon(\dot{x}_\varepsilon)$$

that is

$$a_\varepsilon = L_\varepsilon(a_0) + \tilde{L}_\varepsilon(-\kappa_\varepsilon\dot{\phi}_\varepsilon(b_\varepsilon) - b_\varepsilon\eta_\epsilon + c_\varepsilon\mu_\varepsilon)$$

The term $c_\varepsilon\mu_\varepsilon$ converges strongly to zero in L^2. Also we have convergence of the term $b_\varepsilon\eta_\epsilon$ to $b\eta$ and κ_ε to κ. This tells us that $\tilde{L}_\varepsilon(\dot{\phi}_\varepsilon(b_\varepsilon))$ converges in L^2. So one can check that the limit is $\tilde{L}_0(\dot{a})$. The same holds for b_ε, hence we can send ε to 0 to get a limiting system of the form:

$$\begin{cases} \dfrac{\partial a}{\partial s} = \ddot{a} - \kappa\dot{b} - b\eta \\ \\ \dfrac{\partial b}{\partial s} = \ddot{b} + \kappa\dot{a} + \eta(a\tau - \overline{\mu}_\xi b) \end{cases}$$

□

Therefore we proved that the system induces indeed a flow on the curves of $\mathcal{L}_\beta$ and it has a regularizing effect on a and b caused by the diffusion operator and this leads to the main result of this section.

3.3 From $\mathcal{L}_\beta$ to $\mathcal{C}_\beta^+$

In this section we will lift curves having isolated zeros of a from a compact set of curves in $\mathcal{L}_\beta$ to the space $\mathcal{C}_\beta^+$. This is where the assumption (B) is crucial. Since flowing along v will always transport vectors with negative ξ components to vectors with positive component. But first we want to give a rigorous definition on the transport along a curve in $\mathcal{L}_\beta$ by extending its tangent vector in a small neighborhood. The lifting process is not continuous, in fact, we will show that this procedure will lead to the formation of some "singularities" that will be removed later.

3.3.1 Extending the tangent vector $\dot{x}$ of a curve x in $\mathcal{L}_\beta$

First we describe here a way of extending the tangent vector $\dot{x}$ to a curve x in $\mathcal{L}_\beta$ which we take with the $\mathcal{H}^2$-topology, to a neighborhood of it, near a point $x(t)$ where $\dot{x}(t)$ is non zero. This becomes a vector field in that neighborhood allowing us to define a transport map along the curve. We consider a curve $x \in \mathcal{L}_\beta$. Let S be a disk at $x(0)$ (we are assuming that $\dot{x}(0)$ is non zero) transverse to the curve and tangent to $[\xi, v]$ at $x(0)$(see figure 3.1 below). These construction is local so it can be done by local diffeomorphism in $\mathbb{R}^3$. Now for any $y_0 \in S$, we consider the solution of the dynamical system

$$y'(t) = a(t)\xi(y(t)) + b(t)v(y(t))$$

with $y(0) = y_0$. Since a and b are in $\mathcal{H}^1$, by the continuous dependence on the initial conditions, there exists a small neighborhood of x such that every point of it lies on exactly one of those orbits for a suitable y_0 and hence we associate to it the tangent to the orbit at that point, and this gives an extension of $\dot{x}$.

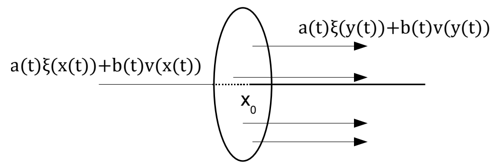


Figure 3.1: Extension of the tangent vector

It is important to notice that in fact this construction does not really give a vector field if the curve is not embedded. If the curve self-intersects then there could be a problem, but notice that the transport map is well defined by use of the dynamical system above.

3.3.2 First deformation and formation of Dirac masses

Let us consider a compact set of curves in $\mathcal{L}_\beta$, which we can view as given by a map

$$f : S^l \mapsto \mathcal{L}_\beta$$

where S^l is the l-sphere for $l \in \mathbb{N}$. We claim (see section 2) that after a small C^1-perturbation, we may assume that all curves in $K = f(S^l)$ have $\dot{x} = a\xi + bv$, with a and b having a finite number of zeros. We will use this result only for the case when a has a finite number of zeros.

Hence curves in the class K can be seen as pieces of curves with $a > 0$ and pieces with $a < 0$. In order to start our argument, we consider the case when the the curve has only two distinct zeros at t_0 and t_1. The argument will be extended later using a filtration adapted to the number of zeros of a. The filtration will include degenerate zeros of various orders.

Let us consider a curve $x(t)$ under the assumptions described above and assume that a is negative in (t_0, t_1). We will lift in the sequel the pieces with $a < 0$ by using the transport map φ_t. Let us now describe this procedure.

Let $t_2 = \frac{t_0+t_1}{2}$. Then, since $a(t_2) \neq 0$, we can define

$$s_0 = \inf\{s > 0; D\varphi_s(\dot{x}(t_2)) = \gamma\xi, \gamma > 0\}.$$

Therefore if we consider the function

$$g(s,t) = \varphi_s^* \beta_{x(t)}(\dot{x}(t))$$

then one has

$$g(s_0, t_2) = 0$$

and

$$\partial_s g(s,t) = \varphi_s^* \left(\mathcal{L}_v \beta\right)\left(\dot{x}(t)\right)$$

$$= d\beta_{\varphi_s(x(t))}(v, D\varphi_s(\dot{x}(t))).$$

Hence, at (s_0, t_2), we get

$$\partial_s g(s_0, t_2)) = -\alpha(v, [v, \gamma\xi]) = -\gamma < 0.$$

Thus, by mean of the implicit function theorem, we can define the piece of curve $y(t) = \varphi_s(t)(x(t))$, for $t \in (t_0, t_1)$. Notice that

$$\dot{y}(t) = D\varphi_{s(t)}(\dot{x}(t)) + \dot{s}(t)v = \gamma\xi + \dot{s}(t)v.$$

If we now close the curve by pieces of v, we transform our original curve x to a curve $\tilde{x}$ in $\mathcal{L}_\beta$ having pieces with $a > 0$, pieces with $a = 0$ and some isolated zeros of a (see figure 3.2).

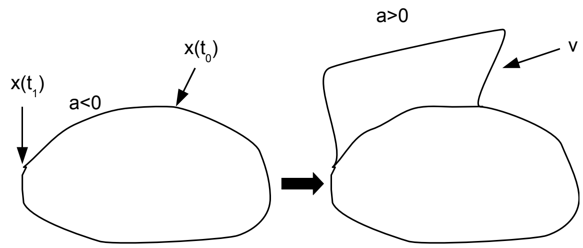


Figure 3.2: Lifting the negative parts

Notice that we can transform this process into a gradual process that will not take place in $\mathcal{L}_\beta$, that is, taking the map $f : S^l \longmapsto \mathcal{L}_\beta$ we construct a homotopy $U : [0,1]\times S^l \longmapsto \Lambda(M)$ such that $U(0,\cdot) = f(\cdot)$ and $U(1,\cdot)$ is valued in $\mathcal{C}^+_\beta$. Since the injection of $\mathcal{L}_\beta \hookrightarrow \Lambda(M)$ is a homotopy equivalence and since $\mathcal{C}^+_\beta$ injects into $\Lambda(M)$, this will lead us, after we resolve the issue of the "Dirac masses", to the fact that $\mathcal{C}^+_\beta \hookrightarrow \Lambda(M)$ is an S^1-homotopy equivalence. In fact, by using the previous construction we see that the new curve that we get in $\mathcal{C}^+_\beta$ reads as $y(t) = \varphi_{s(t)}(x(t))$, thus the homotopy that one might consider would be $H(t,l) := \varphi_{ls(t)}(x(t))$ (see figure 3.3).

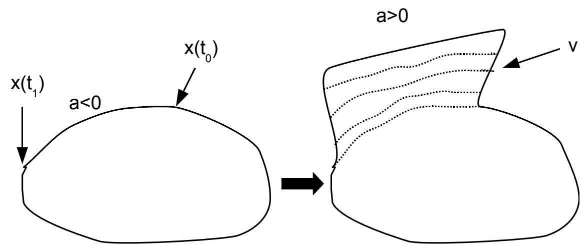


Figure 3.3: The lifting as a deformation

Now thinking of the base curve, the zeros t_0 and t_1 of a can come to each other, collide and cancel as x varies in S^l and $f(x)$ varies in $K = f(S^l)$. Tracking our construction over this deformation we see how a Dirac Mass, that is a back and forth run along v, can be created as two zeros of a come to each other and collapse (see figure 3.4).

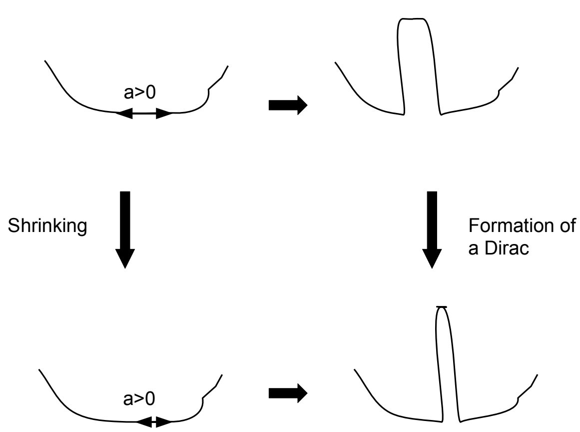


Figure 3.4: Formation of a “Dirac Mass”

More complicated phenomena take place as we resolve the case of collapse of zeros of a with higher multiplicity, also as x varies in S^l. Let us understand first the collapse of two zeros.

3.3.3 Simple Dirac mass

Here we consider a curve x, such that $\dot{x} = a\xi + bv$ where a is positive everywhere except at a point $x(t_0)$ where there is a back and forth v jump of length $l > 0$. The length here is to be understood as the amount of time spent between two points with the parametrisation of the curve in $[0, 1]$. This can be made more rigorous by considering the metric g on M such that $||\xi|| = ||v|| = ||[\xi, v]|| = 1$ and parametrise the curve proportionally to arclength. In fact any metric will do in our construction.
Consider the family $y(t, s)$ for $s \in [0, 1]$, that coincides with the curve x and at $x(t_0)$, $y(t, s)$ has a back and forth v jump of length sl. By using a lemma in [3], we recognize a process with which the Dirac mass can be gradually removed (see figure 3.5). What we mean here is the existence of a deformation supported in the neighborhood of the Dirac mass that decreases it gradually. We will construct in what follow a cancellation process that works for nearly Dirac masses and it converges to the one stated above as the size of the intermidiate ξ piece tends to zero.

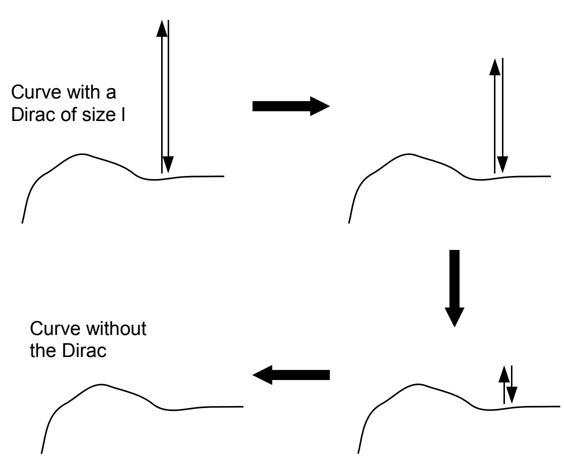


Figure 3.5: Cancellation of a Dirac mass

3.3.4 Cancellation Process

Here, let us consider a curve close to x, in the case it supports a "nearly" Dirac mass, namely the former back and forth run along v is now "opened" by a little bit: a small curve of length ε, where $\dot{x} = a\xi + bv$ and $a > 0$ is inserted in between the forth and back run along v.

To remove this nearly Dirac mass with a process that coincides with the one in the previous subsection as ε goes to zero, we construct a deformation vector $Z = \lambda\xi + \mu v + \eta[\xi, v]$ along the curve. By removing we mean constructing a deformation on the curves that decreases gradually the size of the back and forth v-run making the process continuous. This is done by first taking $-v$ at B, in fact, if we want it to be adapted to the length of the size of the Dirac mass, then we should take $-lv$ instead at B, where l is the length of the v jump. We will disregard this fact for now. So, after taking $-v$ at B, we transport it along $-\dot{x}$ ($\dot{x}$ is close to ξ on the "opening" so that the extension described above is possible) till we reach A. Eventually, we will have a non-zero $[\xi; v]$ component for the transported vector at A. We then transport, by using the transport map of v, our vector from A to C and we adjust the v-length of this v-jump, adding or subtracting a δsv, so that the v component of the transported vector at C is zero. At C, we know that $a = 0$ and this is inconvenient. Therefore, we transport it a bit further to a point p where $a, |b| > c > 0$,(see figure 3.6).

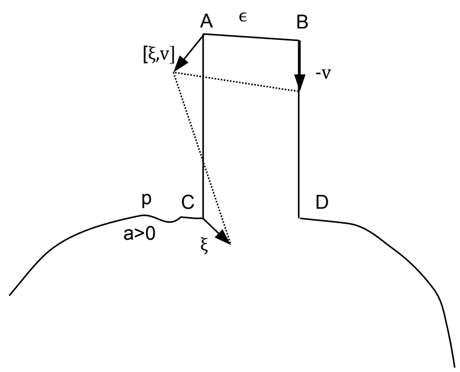


Figure 3.6: Nearly Dirac mass

The requirement that $|b| > c$ at p is not needed, but will be used to prove a stronger result of deformation, along which the number of zeros of b does not increase over the deformation. It can be dropped for the proof of Theorem 1. We need now to compensate the vector that we got at p and to derive precise estimates on a and b as they are deformed to adjust the variation of the curve induced by the tangent vector described above.

The main idea of the compensation is first to span the kernel of α at p. This will be done by using a combination of two process involving the introduction of a small perturbation to the curve.

The first process is meant to generate a v component at a given point on the curve. Namely, given a point x_1 on a curve x of $\mathcal{C}_\beta$ (in fact we only need that the point x_1 is located in a portion of the curve with $a > 0$) we can construct a small variation of the curve near x_1 so that we get a vector almost equal to v at that point. In fact, this can be described as changing b to λb, in a small interval before x_1, with $\lambda > 1$ to generate v and $\lambda < 1$ to generate $-v$ (see figure 3.7).

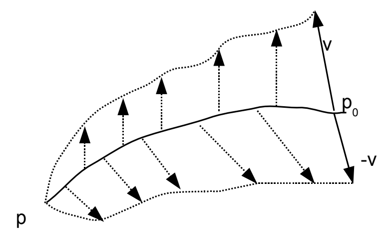


Figure 3.7: Perturbation of b to create v

The second step consists on generating a $[\xi, v]$ component at a given point of a curve x in $\mathcal{C}_\beta$. This will be done by transporting v (and this is where the combination comes in) along $\dot{x}$ to the point x_2 where we want to get the $[\xi, v]$ component (see figure 3.8); of course we will have other components at that point but we will show that they have a minor contribution.

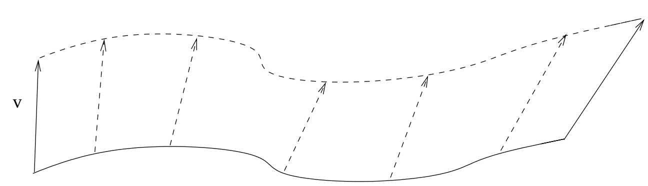


Figure 3.8: Creation of a $[\xi, v]$ component

We will show that a combination of these two steps, after a careful choice of the points in the curve for inserting them, indeed span the kernel of α (after a projection parallel to $\dot{x}$) at the desired point. The last step then consists of removing the undesired components along ξ. This will be done by using the transport map along the curve as described in paragraph 3.1, as we will transport the vector $a\xi + bv$ along itself, and by adjusting the length, we can cancel the additional component to find a resultant vector in kernel of α.

All the previous construction will be made precise as it depends tightly on the choice of the points and on the portions of the curve on which the deformation be built (see figure 3.9).

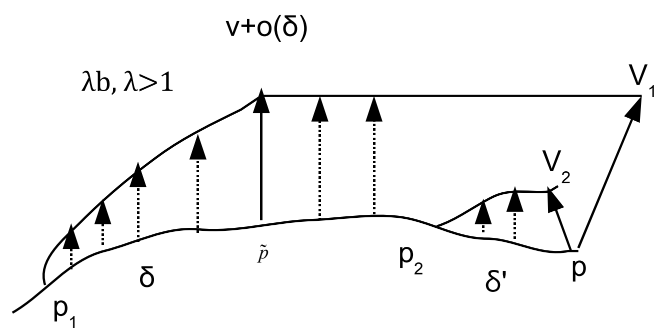


Figure 3.9: Combination of the processes

We will find a vector $Z = \lambda\xi + \mu v + \eta[\xi, v]$ which is locally Lipschitz in the $\mathcal{H}^2$ topology in a neighborhood of the curve x (that is the topology of $\mathcal{L}_\beta$). It is important to follow the full construction: we start from a compact set K in $\mathcal{L}_\beta$ endowed with the $\mathcal{H}^2$ topology and we deform it, by using the regularizing flow in section 2, to a compact set $\tilde{K}$ of smooth curves. This construction that follows is done on curves in $\tilde{K}$, and it is done curve by curve. It will be made continuous by using a partition of unity adapted to a covering of $\tilde{K}$ in the $\mathcal{H}^2$ topology: these details are in the Appendix.

Let us start now by detailing the construction that will be carried on four steps as mentioned above.

Let us consider the following system of differential equations:

$$\begin{cases} \dot{\lambda} = b\eta \\ \\ \dot{\mu} = (b\overline{\mu}_\xi - a\tau)\eta + h_\delta \\ \\ \dot{\eta} = b\overline{\mu}\eta + \mu a - \lambda b \end{cases} \tag{3.6}$$

The first equation of this system tells us that $a = \alpha(\dot{y})$ remains constant along the variation. The second equation provides us the variation of b. The third equation tells us that the curve stays in the space $\mathcal{L}_\beta$. In fact the second equation corresponds to the introduction of a small variation of b using the function h_δ that will be chosen depending on the situation, also the initial conditions will vary with the different cases of h_δ that we will consider. Notice here that a and b are continuous since they are in $\mathcal{H}^1$. Also the functions τ and $\overline{\mu}$ are C^1, so the differential system is then understood in the classical sense.

Case $h_\delta \neq 0$

We will give here the details on how to generate the vector v at a point as it was mentioned above.

In what follows for every $\delta > 0$, h_δ denote a positive function, with compact support in $(0, \delta)$ such that $h_\delta = 1$ in the interval $[\frac{\delta}{4}, \frac{3\delta}{4}]$. We will first describe the process that will be used starting from any point x_0 in the curve, and then insert them at specific points to get the construction described above.

Consider the following system:

$$\begin{cases} \dot{\lambda} = b\eta \\ \\ \dot{\mu} = (b\overline{\mu}_\xi - a\tau)\eta + h_\delta \\ \\ \dot{\eta} = b\overline{\mu}\eta + \mu a - \lambda b \\ \\ \lambda(0) = \eta(0) = \mu(0) = 0 \end{cases} \tag{3.7}$$

As mentioned before the second equation provides us the variation of b. So here this will induce a change of b in the positive direction. We solve this system in the interval $[0; 2\delta]$ so that there is no zero of b in this interval, i.e. $\frac{1}{c} < b < c$; this is always possible since b is real analytic and it is possible for all the curves in an $\mathcal{H}^2$ neighborhood of the curve (see section 2).

A study of the previous system leads us the following lemma that will be proved in the rest of this paragraph:

Lemma 3.3.1. *Let us assume that Z_1 satisfies (3.7), then:*

$$\begin{cases} \|\lambda\|_\infty \leq C(\|a\|_\infty + \|b\|_\infty)\delta^2 \\ \\ \|\mu - \int_0^t h_\delta(s)ds\|_\infty \leq C(\|a\|_\infty + \|b\|_\infty)\delta^2 \\ \\ \|\eta\|_\infty \leq C(\|a\|_\infty + \|b\|_\infty)\delta^2 \end{cases} \tag{3.8}$$

Proof. Let A be the matrix of the previous differential system, that is

$$A = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & (b\overline{\mu}_\xi - a\tau) \\ -b & a & b\overline{\mu} \end{pmatrix},$$

and let R denote the resolvent of the system, that is $\dot{R} = AR$ with $R(0) = id$. Then, if

$Z_1 = \begin{pmatrix} \lambda \\ \mu \\ \eta \end{pmatrix}$, we have

$$Z_1(t) = R(t) \int_0^t R^{-1}(s) \begin{pmatrix} 0 \\ h_\delta(s) \\ 0 \end{pmatrix} ds.$$

Now since $\dot{R} = AR$ we have that

$$|R(t) - id| \leq \int_0^t |A(s)| ds + \int_0^t |A(s)||R(s) - id| ds, \tag{3.9}$$

It follows from Gronwall's lemma that

$$|R(t) - id| \leq \int_0^t |A(s)| ds e^{\int_0^t |A(s)| ds}, \tag{3.10}$$

Hence for $t \in [0, \delta]$ we have

$$|R(t) - id| \leq C \int_0^t |A(s)| ds.$$

Then

$$|Z_1(t) - \begin{pmatrix} 0 \\ \int_0^t h_\delta(s) ds \\ 0 \end{pmatrix}| \leq C \int_0^t \int_0^s |A(u)| du ds.$$

Therefore we deduce the result of the lemma. □

According to the previous lemma we can estimate the change of b and a along the deformation introduced by the vector field Z_1 above. Knowing that, once extended, the evolution equations of a and b read as

$$\frac{\partial a}{\partial s} = \dot{\lambda} - b\eta$$

and

$$\frac{\partial b}{\partial s} = \dot{\mu} + (a\tau - \overline{\mu}_\xi b)\eta.$$

We get that a is unchanged and after a bootstrapping argument (see Appendix) we have

$$|b(s,t) - b(t)| \leq Cs|h_\delta| \leq Cs.$$

Case $h_\delta = 0$

We now consider the same system of equations, but with $h_\delta = 0$ and with initial conditions non-zero, that is:

$$\begin{cases} \dot{\lambda} = b\eta \\ \\ \dot{\mu} = (b\overline{\mu}_\xi - a\tau)\eta \\ \\ \dot{\eta} = b\overline{\mu}\eta + \mu a - \lambda b \\ \\ \lambda(0) = \eta(0) = 0, \ \mu(0) = 1 \end{cases} \tag{3.11}$$

This will allow us to generate a non-trivial $[\xi, v]$ component at the point p, with of course an extra term r that needs to be removed in a later stage.

If Z_2 is a solution of this equation then we have $Z_2(t) = R(t)Z_2(0)$. Notice now that

$$|\big(R(t) - id - \int_0^t A(s)ds\big)Z_2(0)| = |\int_0^t A(s)(R(s) - id)dsZ_2(0)|$$

Using the estimate (3.10) we have

$$|Z_2(t) - Z_2(0) - \int_0^t A(s)Z_2(0)| \leq C\delta^2(||a||_\infty + ||b||_\infty)^2$$

Therefore we have

$$\|\eta - \int_0^t a(s)ds\|_\infty \leq C\delta^2(\|a\|_\infty + \|b\|_\infty)^2$$

and

$$\|\mu - 1\|_\infty \leq C\delta^2(\|a\|_\infty + \|b\|_\infty)^2$$

We will set

$$\theta_\delta = \delta^2(\|a\|_\infty + \|b\|_\infty), \qquad \tilde{\theta}_\delta = \delta^2(\|a\|_\infty + \|b\|_\infty)^2$$

Combination

Now we will use a combination of these processes starting at specific points on the curve to span the kernel of α at p. So here, given a point p on the curve we will use 3 points

p_1, $\overline{p}$ and p_2, and we re-parametrize our curve so that zero corresponds to the point $p_1 = x(0)$ whereas the time δ will correspond to the point $\overline{p}$, and take $p_2 = x(2\delta - \delta')$, $p = x(2\delta)$, where here $0 < \delta' << \delta$. We will provide more details about the values of δ and δ' in the sequel. Also δ does not need to be small.

From p_1 we use the construction done with (3.7) up to time δ. Then again, use the process described by (3.11) starting from $\overline{p}$ with initial condition the resultant vector from the first construction, till we reach the point p. And to finish we run again the first process (that is using (3.7)) starting from p_2 till we reach p (see figure 3.9).

Let us see what are the vectors formed now at the point p. From the first and the second process we get a vector

$$V_1 = \int_0^\delta h_\delta(s)ds\Big[\big(1 + O(\tilde{\theta}_\delta) + \frac{b}{a}O(\theta_\delta)\big)v +$$

$$+\big(\int_0^\delta a(s)ds + O(\tilde{\theta}_\delta)\big)[\xi, v] + \frac{1}{a}O(\theta_\delta)(a\xi + bv) + O(\delta\theta_\delta)\Big]$$

and from the third process, we have

$$V_2 = O(\theta_{\delta'})(a\xi + bv) + \big(\int_0^{\delta'} h_{\delta'}(s)ds + O(\theta_{\delta'})\big)v + O(\theta_{\delta'})[\xi, v]$$

Now we compute the determinant $det(P(V_1), P(V_2))$, where P is the projection, on $\ker\alpha$, parallel to $a\xi + bv$, we find:

$$\begin{vmatrix} 1 + O(\tilde{\theta}_\delta) + \frac{b}{a}O(\theta_\delta) & O(\tilde{\theta}_\delta) \\ \\ \int_0^{\delta'} h_{\delta'}(s)ds + O(\theta_{\delta'}) & O(\theta_{\delta'}) \end{vmatrix}$$

The dominant term of this determinant is

$$O(\theta_{\delta'}) - \int_0^{\delta'} h_{\delta'}(s)ds \int_0^\delta a(s)ds = O(\theta_{\delta'}) - \delta\delta' a_0 + o(\delta\delta')$$

Since $\delta' << \delta$ this determinant is bounded away from zero.

Now the global estimate on b after extension the of the deformation vector Z, reads as follows:

$$|b(s,t) - b(t)| \leq Cs|h_\delta(t) + h_{\delta'}(t)| \leq Cs.$$

Compensation of ξ

Notice that now the only part that needs compensation is the ξ component. Since we extended the velocity vector of the curve to some small $\mathcal{H}^2$ neighborhood of the curve, by transporting $a\xi + bv$ from p_2, we get a non-zero ξ component at p. Notice that this corresponds to the use of the transport map $\phi_{t(x)}(x)$ where here $t(x)$ is the necessary time to be able to compensate the given ξ component. This can be made precise if we get the right estimates on the transported vector from B. Let S be the section at p of ker_α and S_2 a section of $\ker_\alpha$ at B. We consider also the section $\tilde{S}_2 = D\phi_t(S_2)$ the image of the section S_2 under the diffeomorphism ϕ_{t_0} where t_0 is the necessary time to reach p starting from B. Now we want to find a way of projecting the section $\tilde{S}_2$ on S using the diffeomorphism ϕ_t. In fact, we have

$$D(\phi_t(p))(\cdot) = D\phi_t(\cdot) + dt(\cdot)(a\xi + bv)$$

evaluating at $t = 0$, the previous equation reads as

$$D(\phi_t(p))(X) = X + dt(X)(a\xi + bv)$$

for every $X \in T_pM$. Therefore we can always project on S by taking $dt(X) = \frac{\alpha(X)}{a}$, noticing that $dt(X) = 0$ means we are already in S, and if $dt(X) \neq 0$ then by taking $\phi_{st}(p)$ and adjusting the s we can always compensate the ξ component. The same procedure can be done for the section spanned by the vectors V_1 and V_2 at p and projecting them on S to get components free from ξ.

Now one needs to estimate the size of the component that needs to be compensated since the previous procedure corresponds to an increase or decrease in time. Hence it will change the parametrization of our curve.

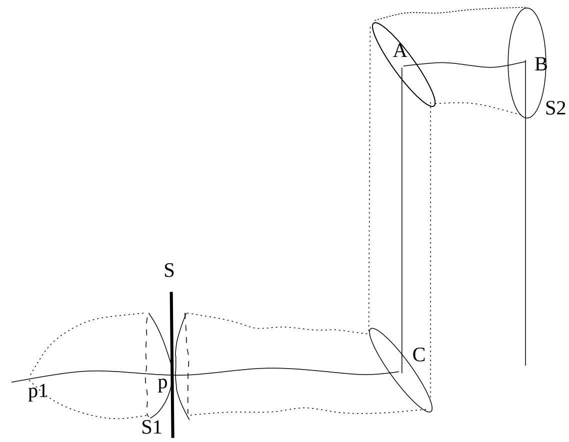


Figure 3.10: Compensation of ξ

Let ε be the opening of the nearly Dirac mass. Since we are transporting the vector $-v$ starting from B, the transport equation is equivalent to solving

$$\begin{cases} \dot{\lambda} = b\eta \\ \dot{\mu} = (b\overline{\mu}_{\xi} - a\tau)\eta \\ \dot{\eta} = b\overline{\mu}\eta + \mu a - \lambda b \\ \lambda(0) = \eta(0) = 0, \ \mu(0) = -1 \end{cases} \tag{3.12}$$

This last system behaves as (3.11), starting from the point $\tilde{p}$. Thus, it holds

$$\mu = -1 + O(\tilde{\theta}_{\varepsilon}), \qquad \eta = \int_0^{\varepsilon} a(s)ds + O(\tilde{\theta}_{\varepsilon})$$

Since the transport equation is linear, we have that at p all the components of the transported vector are $O(\varepsilon)$ and so, $|dt| = O(\varepsilon)$. Notice now that if the initial length is l then the new length will be $l' = l + O(\epsilon)$ and therefore the rescaled b is $\tilde{b} = \frac{b}{1+O(\epsilon)}$. Again this gives a final estimate on b along the variation as follows:

$$|b(s,t) - b(t)| \leq CsO(\epsilon). \tag{3.13}$$

Proposition 3.3.2. *There exists $\varepsilon_0 > 0$ such that if the opening of the nearly Dirac mass is $\varepsilon < \varepsilon_0$, then the nearly Dirac mass can be gradually removed.*

Proof. Recall that from the previous construction, b will only change in the portion $[0, 2\delta]$ between p_1 and p. In that region we have that $\frac{1}{c} < b < c$, hence from the estimate (3.13) we have

$$\frac{1}{c} - Cs\varepsilon \leq b(s,t) \leq c + Cs\varepsilon.$$

Therefore, given a nearly Dirac mass of length l, if we take $\varepsilon < min(cCl, \frac{1}{2cCl})$ we find that

$$\frac{1}{2c} \leq b(t,s) \leq 2c$$

for $s \in [0, l]$. Thus the process can be completed and the nearly Dirac mass can be removed with a control on the number of zeros of b. $\square$

After this compensation is done, we can see that this process will cancel the nearly Dirac mass, in fact if we let Z the deformation vector built in the previous construction, then if we start by $-v$ at B, it is enough to check the behaviour of $\int b$. We have

$$Z \cdot \int b = \int \dot{\mu} + (a\tau - \overline{\mu}_\xi)\eta$$

By splitting the integral into two pieces we see that:

from D to B we have $\eta = 0$ hence $Z \cdot \int_{[DB]} b = -1$;

from B to A we have $Z \cdot b = 0$ hence $Z \cdot \int_{[BA]} b = 0$.

Proposition 3.3.3. *Let l be the length of the nearly Dirac mass, then if l tends to zero, the deformation tends to the identity.*

Proof. One has to notice that, the previous construction was made regardless of the length of the nearly Dirac mass, and this can be adapted: instead of transporting $-v$ from the point B, we start by transporting $-lv$. Since the deformation was made using linear differential equations, one has that the new deformation vector is $\tilde{Z}(x) = l(x)Z(x)$, hence if l tends to zero, the deformation tends to identity. $\square$

3.3.5 Case of a double zero

In this case we will consider two nearly Dirac masses, that is 3 v-pieces, that might converge to a single jump. First thing to notice is that we can do our construction and build the deformation vector in two different ways, but we can convex combine them since they have independent supports assuming that the length of each intermediate piece is less that the ε_0 that we took in the case of a single nearly Dirac mass.

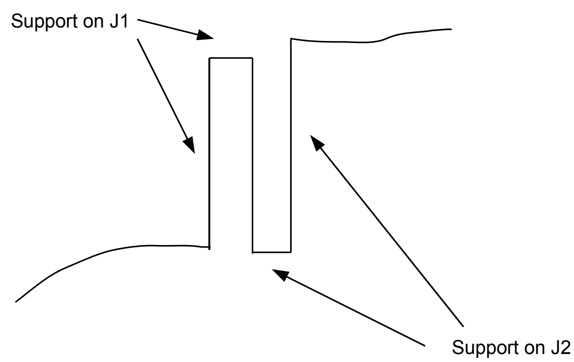


Figure 3.11: Case of a double zero

Hence, the two procedures can be run together without interfering, leading to a case where we have two positive (or negative jumps) linked by a piece of curve. Hence, we can convex combine them to end up with a step-like curve that moves along the convex combination between the two extremal parts that are curves with a single v jump.

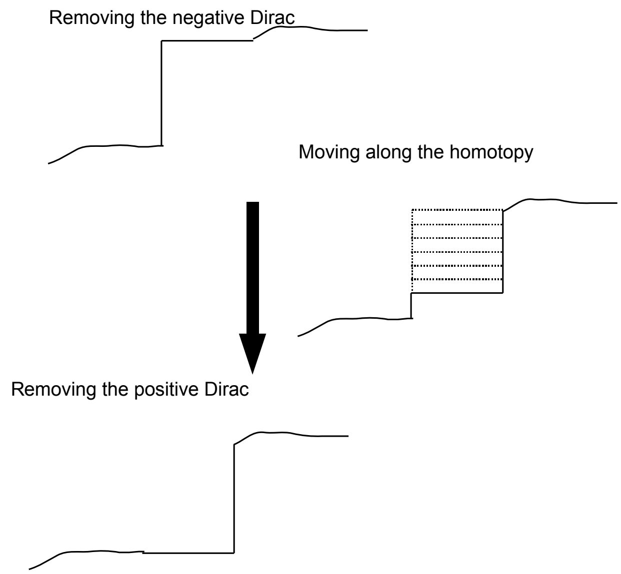


Figure 3.12: Convex combination of the two process

Another way to do this (which will be useful in the case of large multiplicity) is to build the deformation vector starting from one nearly Dirac mass and then crossing the other to finish the compensation from the other side (see figure 3.13).

In this case we need $\varepsilon_1+\varepsilon_2 < \varepsilon_0$ and since the construction can be made from both sides, they can be superposed since in the common support, it is just a transport equation that conserves all the quantities and hence they can be convex combined to get the same result as mentioned above.

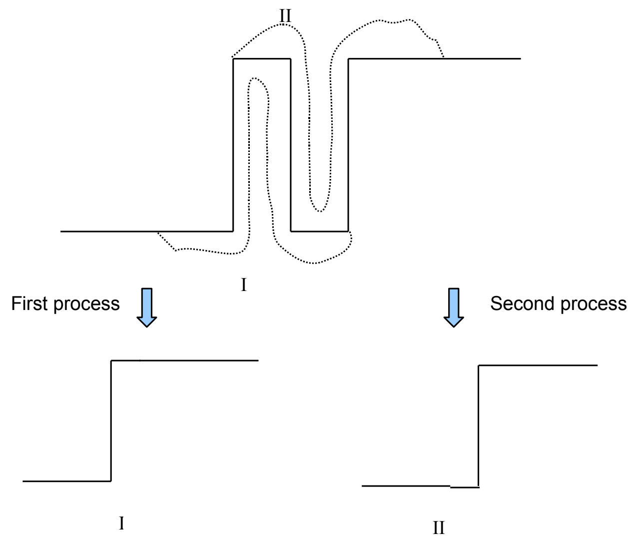


Figure 3.13: Second method

3.3.6 Case of large multiplicity

To clarify the construction let us first take a zero of order 3.

If we assume that, $\sum_{i=1}^{k} \varepsilon_i < \varepsilon_0$ then we can remove the nearly Dirac masses by building the decreasing vector on the sides (as in figure 3.14).

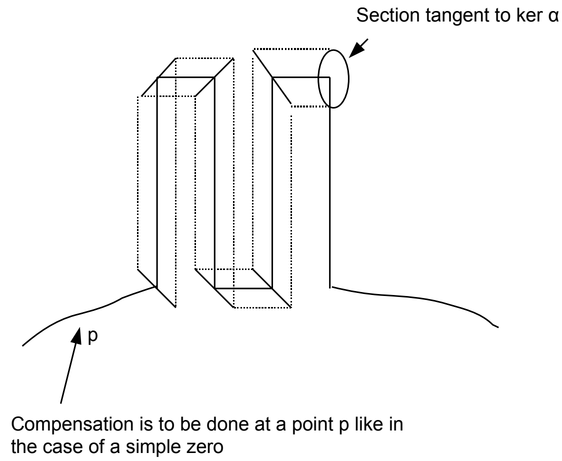


Figure 3.14: The cancellation of multiple Dirac masses

Hence they can be convex combined to lead to a situation of multiple positive v jumps linked by small pieces, as shown in the following figure 3.15.

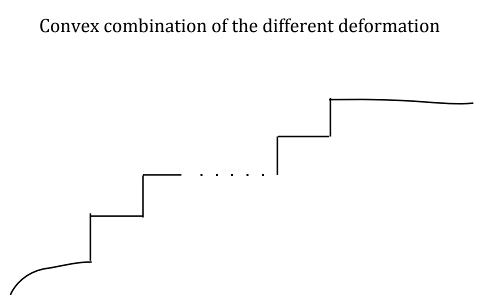


Figure 3.15: Multiple convex combinations.

3.4 "Pushing" in $\mathcal{C}_\beta$

After Cancelling the singularities that appears during the lifting process, we end-up with curves in $\mathcal{C}_\beta^+$ having consistent pieces with $a > 0$. In this section we will proceed with the final step which consists of pushing curves from $\mathcal{C}_\beta^+$ into $\mathcal{C}_\beta$ continuously. This again will be done by the use of a flow that is constructed in a similar way as in section 2. Indeed, we will construct a flow that induces a heat type flow on the component along ξ and by the use of a result of S. Angenent [1], we will see that after small time, the curves will be deformed into ones in $\mathcal{C}_\beta$.

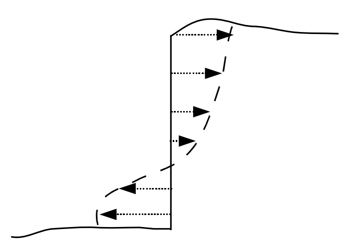


Figure 3.16: Removing the v pieces

First let us recall that we are deforming a curve x in $\mathcal{L}_\beta$, that is $\dot{x} = a\xi + bv$, along a vector field $Z = \lambda\xi + \mu v + \eta[\xi, v]$, and we have:

$$\begin{cases} \dfrac{\partial a}{\partial s} = \dot{\lambda} - b\eta \\ \\ \dfrac{\partial b}{\partial s} = \dot{\mu} + (a\tau - \overline{\mu}_\xi b)\eta \end{cases} \tag{3.14}$$

We will assume in what follows that a is not identically zero. That is we do not consider periodic orbits of v. In fact this can be always assumed after deforming our compact set of curves using the vector field Z constructed in Appendix B.

Now we will focus on the first equation of (3.14), that is the evolution of a. So in this part we take $\lambda = \dot{a} + f$, $\mu = \dot{b}$ and η satisfying the usual equation of $\mathcal{L}_\beta$, i.e $\dot{\eta} = \overline{\mu} b\eta + \mu a - \lambda b$. Hence:

$$\eta = e^{\int_0^t b(u,s)\overline{\mu}(u)du} \int_0^t e^{-\int_0^r b(u,s)\overline{\mu}(u)du}((a\dot{b} - \dot{a}b) - bf)(r,s)dr.$$

So if we look at the evolution of a, we get

$$\frac{\partial a}{\partial s} = \ddot{a} + \dot{f} - b\eta$$

Therefore, if we can find a function f such that $\dot{f} - b\eta = ah$, we can insure the positivity of a starting from a non-negative initial data (as it will be explained later on). But this is equivalent to solving the linear non-homogenous integro-differential equation

$$\dot{f} + be^{\int_0^t b(u,s)\overline{\mu}(u)du} \int_0^t e^{-\int_0^r b(u,s)\overline{\mu}(u)du} bf dr =$$

$$= be^{\int_0^t b(u,s)\overline{\mu}(u)du} \int_0^t e^{-\int_0^r b(u,s)\overline{\mu}(u)du}(a\dot{b} - \dot{a}b)dr + ah$$

Notice that we need to find a periodic solution to this equation, so we define the operator K on the space $C_{per}([0,1])$ in the following way:

$$K(f)(t) = \int_0^t b[-\int_0^l e^{\int_r^l b(u,s)\overline{\mu}(u)du} bf dr + \int_0^l e^{\int_r^l b(u,s)\overline{\mu}(u)du}(a\dot{b} - \dot{a}b)dr] + ahdl$$

Since we want periodicity, we will take $h = c(f)a$ where $c(f)$ satisfies

$$\int_0^1 b[-\int_0^l e^{\int_r^l b(u,s)\overline{\mu}(u)du} bf dr + \int_0^l e^{\int_r^l b(u,s)\overline{\mu}(u)du}(a\dot{b} - \dot{a}b)dr]dl = c(f)\int_0^1 a^2(l)dl$$

Notice that $c(f)$ is an affine function of f, thus $c(f) = c_1(f) + c_2$. Therefore the final form of the operator K is

$$K(f) = \int_0^t \int_0^l e^{\int_r^l b(u,s)\overline{\mu}(u)du}(a\dot{b} - \dot{a}b)dr + c_2 a^2 dl + T(f)(t),$$

where $T(f)$ is the bounded linear operator on $C([0,1])$ defined by

$$T(f)(t) = \int_0^t -b\int_0^l e^{\int_r^l b(u,s)\overline{\mu}(u)du} bf dr + c_1(u)a^2 dl$$

So the problem now is reduced to find a fixed point for the operator K. For that we will use the contraction mapping theorem for an iterate of K. The main estimate that is needed reads as

$$\|K^n(f_1) - K^n(f_2)\| \leq \frac{\|T\|^n}{n!}\|f_1 - f_2\|$$

where $\|\cdot\|$ stands for the L^∞ norm.

Thus we have the existence and the uniqueness of f and this leads to the diffusion equation

$$\frac{\partial a}{\partial s} = \ddot{a} + ca^2$$

To be more precise about the existence of this flow, one should follow the same procedure as in section 2. That is, we need to regularize the coefficients of the deformation vector to get classical existence, then we need to show that indeed we have convergence to a flow on the curves. Since the procedure is similar to that in section 2, we will omit it.

Now we refer to the work of Angenent [1], about the zeros of parabolic equations of the form

$$\frac{\partial a}{\partial s} = \ddot{a} + g_1\dot{a} + g_2 a$$

We know that the number of zeros of a is non-increasing and if we have $a(s, t_0) = \dot{a}(s, t_0) = 0$ then the flow will move toward the direction canceling the zero. In our case all the curves in $\mathcal{C}^+_\beta$ have $a \geq 0$, hence if a is not identically zero then along the flow a will become strictly positive: that is $a(s,t) > 0$ for $s > 0$.

3.5 Some remarks on the case when β in not a contact form

If we look back at our proof we see that to show this equivalence we needed first Smale's theorem. So in a first step, we will show that there is indeed an S^1-equivariant homotopy equivalence between $\mathcal{L}_\beta$ and $\Lambda(M)$, after a possible generic small perturbation of v. The next step is to try to build the same flows but in a different frame, since v and $[\xi, v]$ are not always linearly independent in this case. The main issue here is the structure of the space $\mathcal{L}_\beta$ in fact we can see that it generically has finitely many singularities preventing it from being a manifold.

3.5.1 The Fibration

In this subsection we will prove the following :

Theorem 3.5.1. *For a generic perturbation of v, the injection $i : \mathcal{L}_\beta \longrightarrow \Lambda(M)$ is an S^1-equivariant homotopy equivalence,*

The proof will be made through several Lemmata.

Lemma 3.5.2. *Every two points in M can be connected by a path tangent to* $\ker\beta$.

Proof. Let $p = *\beta \wedge d\beta$. If p_1 and p_2 are two points in $[p > 0]$ then since β is contact in that region then we can connect them with a path tangent to $\ker\beta$. The same result is true for the region $[p < 0]$. Now we want to show that given a point $p_1 \in [p = 0]$, we can connect it to a point $p_2 \in [p > 0]$. Locally (for a small perturbation) we can assume that $[p = 0] = [z = 0]$ and $p_1 = (0, 0, 0)$ then in a small neighborhood of p_1, $\alpha = xdy + df + dz$ and $\xi = \frac{\partial}{\partial z}$. We can write the vector $v = m\partial_x + l\partial_y + C\partial_z$. Hence $p = ml_z - lm_z$. Without loss of generality we assume that $l(p_1) \neq 0$. Let $a > 0$. We consider the functions $z(t) = ta$ and $x(t) = t$. For y we take it to be the solution of

$$\begin{cases} y' = \frac{m(t,y(t),ta)}{l(t,y(t),ta)} \\ y(0) = 0 \end{cases}$$

Then since $\beta(\cdot) = -mdx + ldy$ we have that $\gamma(t) = (x(t), y(t), z(t))$ is tangent to $\ker\beta$ and it connects p_1 with a point p_2 with $z > 0$ and p_2' with $z < 0$.
An easier way would be to take the curve $(0, 0, t)$ which is obviously tangent to $\ker\beta$.

□

Lemma 3.5.3. *The projection $\pi : \mathcal{P}_\beta \longrightarrow M$ satisfies the homotopy lifting property.*

Proof. Consider a homotopy $G : [0, 1] \times X \longrightarrow M$, and let $f_0 = G(0, \cdot)$. Consider a lifting $\tilde{f}_0$ of f_0. That is for $\gamma \in \mathcal{P}_\beta$, $\tilde{f}_0(1) = f_0(\gamma(1))$. We know so far that every two points can be linked by a path in $\mathcal{P}_\beta$. Consider the map $\tilde{G} : [0, 1] \times X \longrightarrow \mathcal{P}_\beta$ defined by $\tilde{G}(t, x)$ is the path tangent to $\ker\beta$ connecting $\tilde{f}_0(x)(0)$ to $G(t, x)$. Then, by construction $\tilde{G}(t, x)(1) = G(t, x)$. Thus the homotopy lifting property holds. □

Proof of Theorem 3.2.1. According to the second Lemma we have that $\mathcal{L}_\beta \longrightarrow \mathcal{P}_\beta \longrightarrow M$ is a Serre fibration. Hence using the homotopy groups long exact sequence for fibrations and the fact that $\mathcal{P}$ is contractible, we have that $\pi_k(\mathcal{L}_\beta) = \pi_{k+1}(M) = \pi_k(\Lambda(M))$. Hence by Whitehead's theorem, the injection induces a homotopy equivalence. $\square$

3.5.2 Extension of the Deformations Constructed Earlier

Let us state now the changes that needs to be made. In fact the global construction is the same one needs only to change some equations and adapt them to the problem. The main change comes from the fact that $(\xi, v, [\xi, v])$ is not a frame any more. So we we consider a vector field $w \in ker\alpha$ so that $\beta(w) = 1$. Now all our construction will be written in the frame (ξ, v, w). Let $Z = \lambda\xi + \mu v + \eta w$, then Z is a tangent vector to $\mathcal{L}_\beta$ at x if and only if

$$\dot{\eta} = d\beta(\dot{x}, w)\eta + (\mu a - \lambda b)p.$$

In the other hand, Z is tangent to $\mathcal{C}_\beta$ at x if and only if

$$\begin{cases} \dot{\lambda} = b\eta - \int_0^1 b\eta \\ \dot{\eta} = d\beta(\dot{x}, w)\eta + (\mu a - \lambda b)p \end{cases}$$

These changes will induce some changes on the regularization flow is section 2. In fact, this will allow us to regularize closed curves not every where tangent to $[p = 0]$. That is curves satisfying $\int_0^1 (a^2 + b^2)p^2 dt > 0$. Now it remains to treat the curves tangent $[p = 0]$. But notice that those closed curves are periodic orbits of the vector field $p_v\xi - p_\xi v$ hence they can assumed to be regular.

For the flow in section 4, this has no influence.

Appendix A. Extension of the deformation vector

In this appendix we will see how we can extend the vector field constructed in section 3 to a global deformation on $\mathcal{C}_\beta^+$.

Before we start our extension, let us recall how one can compute the evolution of the tangent to a curve along a deformation vector. We consider here a curve $x \in \mathcal{H}^2(S^1, M)$

such that

$$\dot{x} = a\xi + bv + c[\xi, v]$$

and we also consider a vector field

$$Z = \lambda\xi + \mu v + \eta[\xi, v]$$

Proposition 3.5.4. *Let us assume that x evolves under the flow of Z, that is*

$$\frac{\partial x}{\partial s} = Z(x),$$

then the following hold:

(i) $\dfrac{\partial a}{\partial s} = \dot{\lambda} - \eta b + \mu c,$

(ii) $\dfrac{\partial b}{\partial s} = \dot{\mu} + \eta(\tau a - \overline{\mu}_\xi b) + c(\overline{\mu}_\xi \mu - \lambda\tau)$

(iii) $\dfrac{\partial c}{\partial s} = c\mu\overline{\mu} + \dot{\eta} - \overline{\mu} b\eta + \mu a - \lambda b$

Proof. (i) Notice that $a = \alpha(\dot{x})$, hence

$$\frac{\partial}{\partial s} a = Z \cdot a = (Z \cdot \alpha)(\dot{x}) + \alpha(Z \cdot \dot{x}) =$$

$$= d\alpha(Z, \dot{x}) + \alpha(\dot{Z}) = \dot{\lambda} - \eta b + \mu c$$

(ii) We consider the 1-form $\gamma(\cdot) = -d\alpha(\cdot, [\xi, v])$ so we have $b = \gamma(\dot{x})$. Therefore

$$\frac{\partial}{\partial s} b = d\gamma(Z, \dot{x}) + \gamma(\dot{Z}).$$

Now

$$d\gamma(Z, \dot{x}) = (\lambda b - \mu a) d\gamma(\xi, v) + (\lambda c - \eta a) d\gamma(\xi, [\xi, v]) + (\mu c - b\eta) d\gamma(v, [\xi, v]),$$

but

$$d\gamma(\xi, v) = \xi\gamma(v) - v\gamma(\xi) - \gamma([\xi, v]) = 0,$$

$$d\gamma(\xi, [\xi, v]) = -\gamma([\xi, [\xi, v]]) = \tau$$

and

$$d\gamma(v, [\xi, v]) = -\gamma([v, [\xi, v]]) = d\alpha([v, [\xi, v]], [\xi, v]) = \overline{\mu}_\xi.$$

Thus

$$\frac{\partial}{\partial s} b = \dot{\mu} + \eta(-\overline{\mu}_\xi b + \tau a) + c(-\lambda\tau + \overline{\mu}_\xi \mu).$$

(iii) Here $c = -\beta(\dot{x})$, therefore

$$\frac{\partial}{\partial s} c = -d\beta(Z, \dot{x}) - \beta(\dot{Z}) =$$

$$= -(\lambda b - \mu a) d\beta(\xi, v) - (\lambda c - \eta a) d\beta(\xi, [\xi, v]) - (c\mu - \eta b) d\beta(v, [\xi, v]).$$

A similar computation to the one in (ii) shows that

$$d\beta(\xi, v) = 1, \qquad d\beta(\xi, [\xi, v]) = 0$$

and

$$-d\beta(v, [\xi, v]) = d\alpha(v, [v, [\xi, v]]) = \overline{\mu}$$

Hence

$$\frac{\partial}{\partial s} c = \dot{\eta} - (\lambda b - \mu a) - \overline{\mu}\eta b + c\mu\overline{\mu}$$

□

Given a curve $x \in \tilde{K} \cap \mathcal{C}_\beta^+$, where $\tilde{K}$ is the image of the compact set $K \subset \mathcal{L}_\beta$ under the regularizing flow constructed in section 2, such that $\dot{x} = a\xi + bv$. The construction of the vector in section 3 depends on the point p and on the zeros of a at the points B and C as shown in figure (3.9). So we write this vector $Z_{p,C,B}$, noticing that the same construction works for φ_1 and φ_2 close to a and b in $\mathcal{H}^2(S^1, M)$ with p, A and B the same. Therefore, there exist two neighborhood $U(a)$ and $U(b)$ for which the vector field $Z_{p,C,B}$ is well defined. This constitutes an open cover of the space $\mathcal{H}^2(S^1, M)$ for a and b.

Now, since $\mathcal{H}^2(S^1, M)$ is paracompact, we can extract a refined cover $(U_i)_{i\in I}$ that is locally finite and an adapted partition of unity $(\psi_i)_{i\in I}$. We then use the global deformation

$$Z = \sum_{i\in I} \psi_i Z_{p_i, C_i, B_i}.$$

Observe that each Z_{p_i,C_i,B_i} allows us to compensate our combination of deformations decreasing the "Dirac mass" from A to C. The deformation from A to C does not depend on p_i, C_i, B_i. Then Z will also "compensate" the deformation. To complete this part, we need to show that indeed Z is a vector field that defines a flow (at least locally). For instance if we can show that $Z_{p,C,B}$, is Lipschitz, then the proof is finished.

Lemma 3.5.5. *Consider a vector field $V \in T\Lambda(M)$, such that for $x \in \Lambda(M)$*

$$V(x) = \lambda(x)\xi + \mu(x)v + \eta(x)[\xi, v].$$

Then, if the functions λ, μ, and η are Lipschitz then so is V.

Proof. Let us fix $x \in \Lambda(M)$, then there exists a neighborhood U_x of x in $\Lambda(M)$ such that for every $\tilde{x} \in U_x$, there exists $h \in x^*TM$ such that $\tilde{x}(t) = exp_{x(t)}(h(t))$. Hence, this brings the study to curves in $\mathcal{H}^2_{loc}(S^1, \mathbb{R}^3)$.
The vector field V in this case can be seen as acting on h, since $V(\tilde{x})(t) = V(exp_{x(t)}(h(t))$. This yields the regularity of V, given the regularity of the coefficients. $\square$

We consider now the vector field $Z_{p,C,B}$ constructed on a given curve x. This vector contains mainly two parts. The first one is obtained by transporting $-v$, and it depends smoothly on the curve since it depends on the transport equation of the curve. The second part is the one obtained by solving a differential system of the form $\dot{Z} = AZ + H$. If we show that the component of the solution Z has Lipschitz dependence on the curve, then combined with the previous lemma, this proves the result.

Lemma 3.5.6. *The resolvent R of the system satisfying $\dot{R} = AR$, as function of the curve, is Lipschitz.*

Proof. The proof is a computational consequence of the formula

$$R(\tilde{x})(t) = R(x)(t) + R(x)(t) \int_0^t R(x)^{-1}(s)(A(x) - A(\tilde{x}))(s)R(\tilde{x})(s)ds.$$

$\square$

Let us consider now for $x_0 \in \mathcal{C}_\beta^+$ the solution to the flow generated by Z. That is

$$\begin{cases} \dfrac{\partial x}{\partial s} = Z(x) \\ \\ x(0) = x_0, \end{cases}$$

So for $0 < s < s_0$, $x(s)$ will be in a certain neighborhood U of x_0. Hence $Z(x(s)) = \sum_{i=1}^n \psi_i Z_{p_i,C_i,B_i}$. Thus we have

$$\frac{\partial b}{\partial s} = \sum_{i=1}^n \psi_i(x(s))(h_{\delta,i} + h_{\delta',i})$$

So we have

$$|b(0,t) - b(s,t)| \leq s(\delta + \delta')$$

Now adapted to the opening ε and the length l of the nearly Dirac mass we have

$$|b(0,t) - b(s,t)| \leq C\varepsilon l(\delta + \delta')$$

Appendix B. Perturbation of the periodic orbits of v

Let us consider now the periodic orbits of v, if there is any. We want to perturb them using the flow of a vector field Z so that they have a part with $a \neq 0$. Let us recall that the variation of a along a vector field $Z = \lambda\xi + \mu v + \eta[\xi, v]$ is given by

$$\frac{\partial a}{\partial s} = \dot{\lambda} - b\eta$$

Hence we want to solve the following system:

$$\begin{cases} \dot{\lambda} - b\eta = h \\ \dot{\eta} = \overline{\mu} b\eta - \lambda b \end{cases}$$

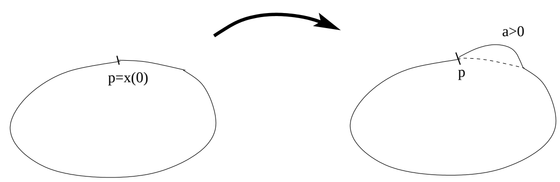
For a certain $h \geq 0$. This can be written as

$$\dot{X} = bAX + H$$

where

$$X = \begin{bmatrix} \lambda \\ \eta \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ \overline{\mu} & -1 \end{bmatrix}, \quad H = \begin{bmatrix} h \\ 0 \end{bmatrix}$$

We take a point $p = x(0)$ where $b \neq 0$ then it is easy to see that for t_0 small enough $R(t_0) - id$ is invertible, where $R(t)$ is the resolvent of the system. This follows from the fact that $R(t) - id = tA(0) + o(t)$, hence $det(R(t) - id) = -bt^2 + o(t^2)$.



So we take h to be supported in the interval $[0, t_0]$. It is important to notice that Z depends on p and t_0 hence we can write it as Z_{p,t_0}. Now we need to extend this deformation globally. In a similar way as before we can take an adapted partition of unity (U_i, ψ_i) to the periodic orbits of v, so that the vector field Z is globally defined by $Z = \sum_i \psi_i Z_{p_i,t_{0,i}}$.

Chapter 4

Violation of the Fredholm assumption

In this chapter we study the violation of the Fredholm assumption of the action functional related to the sequence of overtwisted contact structures on the sphere introduced by Gonzalo and Varela. Then, we exhibit a foliation stuck between the contact form and its Legendre dual in the region where they have opposite orientations. Let us start first by introducing the concept of a Fredholm operator and its basic properties.

4.1 First properties of Fredholm operators

Definition 4.1.1. *Consider two Banach spaces E and F. A bounded linear operator A is said to be Fredholm if $\ker A$ and $coker A = F/Range(A)$ are finite dimensional subspaces and $Range(A)$ is closed. In that case we define the index of A by*

$$ind(A) = \dim \ker(A) - \dim coker(A).$$

Fredholm operators are in some sense the closest to linear operators in finite dimensional setting.

Example 4.1.2. *If K is a compact operator on a Banach space E, then $Id + K$ is a Fredholm operator of index zero. This follows from the Fredholm alternative for compact operators.*

Now we can extend this to non-linear operator on Banach manifolds as follows

Definition 4.1.3. *Let J be a functional between two Banach manifolds M and N. Then J is said to be Fredholm if $DJ(x) : T_xM \longrightarrow T_{J(x)}N$ is a Linear Fredholm operator for all $x \in M$.*

In fact one can show that the index is constant on each connected component of M.

Theorem 4.1.4 (Sard-Smale). *Let $J : M \longrightarrow N$ be a C^k Fredholm map, then the set of regular values of J is G_δ dense in N provided that $k > index(J)$.*

Theorem 4.1.5 (Zeros of Fredholm sections). *Consider a Banach vector bundle $E \longrightarrow M \times S$, and F a smooth section from $M \times S$ to E. Assume that for every $(p, s) \in M \times S$ such that $F(p, s) = 0$ we have :*

i)$DF(p, s) : T_{(p,s)}M \times S :\longrightarrow T_{(p,s,0)}E \longrightarrow E_{(p,s)}$ is surjective

ii)$D_pF_s :\longrightarrow T_pM \longrightarrow E_{(p,s)}$ is Fredholm

Then for a generic $s \in S$,

a)$F_s^{-1}(0_E) \in M$ is a Banach sub-manifold of dimension $index(D_pF_s)$

b)$T_pF_s^{-1}(0_E) = \ker D_pF_s : T_pM \longrightarrow E_{(p,s)}$

4.2 The General setting for the Action Functional

Let M be a 3-dimensional closed compact manifold and α a contact form. That is $\alpha \wedge d\alpha$ is a volume form on M. We want to study the variations of the functional $J(x) = \int_0^1 \alpha_{x(t)}(\dot{x}(t))dt$. The choice of the space of variations is important in this case. For instance, if we consider the variations on the free loop space, the critical points are the periodic orbits of the Reeb vector field. But since the functional is strongly indefinite, the Morse index and co-index are both infinite for each critical point. A better space $\mathcal{C}_{beta}$ was introduced in the work of A. Bahri [3] makes the study of the variations easier. In fact, the restriction of the functional to C_β has only the periodic orbits of the Reeb vector field as critical points. Moreover, the Morse index is finite and the difference of Morse indices in $\mathcal{C}_\beta$ is the same as in the free loop space. And as we saw in the previous chapter, that under mild assumptions there is an S^1-equivariant homotopy equivalence between the free loop space and $\mathcal{C}_\beta$. Therefore from a Morse theoretical perspective, the study of the periodic orbits of the Reeb vector field, seems natural with the functional J restricted to the space $\mathcal{C}_\beta$. However, another unusual phenomena appears in this study that makes the application of the classical Morse theory approach much harder. This is the violation of the Fredholm assumption. Roughly speaking, the linearised operator of the action functional on the tangent space to $\mathcal{C}_\beta$ does not have the form $T+K$, where T is bicontinuous and K is compact. Hence, the classical variational theory does not apply. This fact from functional analysis has the following geometric consequence presented in the figure (4.2).

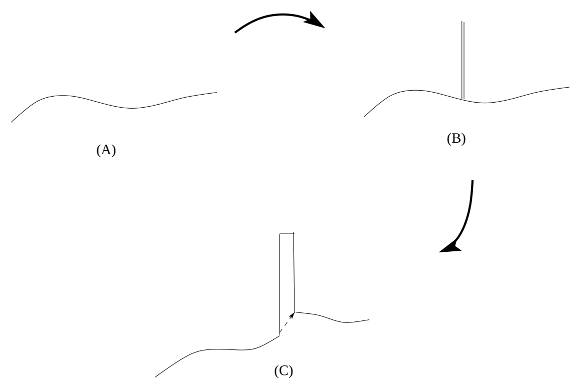


Figure 4.1: Violation of Fredholm assumption

For the sake of exposition, let us assume in a first case that the form α admits a Legendre transform, that is there exists a non-vanishing vector field v in the kernel of α such that the 1-form $\beta = d\alpha(v, \cdot)$ is a contact form with the same orientation. In this setting, (see [5]) we take our space of variations the set $\mathcal{C}_\beta$ defined by :

$$\mathcal{C}_\beta = \{x \in H^1(S^1, M); \alpha(\dot{x}) = c > 0\},$$

where c in the previous definition is a non prescribed constant.

A. Bahri proves in [5] that the Fredholm assumption is violated at a given point of the curve if there exists $s \in \mathbb{R}$ such that

$$\alpha_{\varphi(-s)}(D\varphi(s)(\xi) > 1.$$

In fact this follows from the expansion of the functional J near a critical point with a back and forth v-run inserted on it (see figure (4.1)). If we call x_ε the new perturbed curve, we have that

$$J(x_\varepsilon) = J(x) + \varepsilon(1 - \alpha_{\varphi(-s)}(D\varphi(s)(\xi)) + o(\varepsilon)$$

Hence, if $\alpha_{\varphi(-s)}(D\varphi(s)(\xi) > 1$ we have an extra decreasing direction. This can be thought of as presented in the following figure :

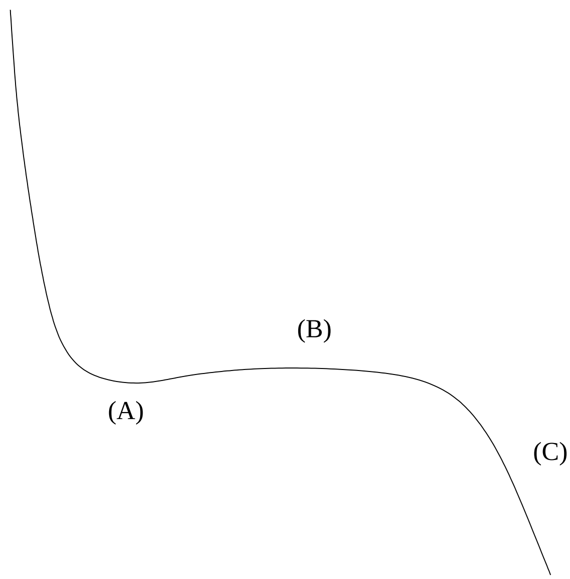


Figure 4.2: Geometric interpretation

The (A) part represent the variation without the "Dirac mass" inserted. The critical point is "genuine" for example a minimum.

The (B) part, corresponds to the insertion of the Dirac mass. The functional increases a tiny bit, as small as we wish, with the opening of the Dirac mass. It is zero at the Dirac mass.

In the last part, that is (C), the Dirac mass has reached a certain "height", that is the back and forth or forth and back run along v has an appropriate size. "Opening" the Dirac mass, that is increasing the length of the ξ-piece inserted between the vertical v-portions (see figure (4.1)), lower the functional between the critical level.

There are other cases where the Fredholm assumption is much harder to exhibit, especially for circle bundles. For example one can check that all the tight contact forms of the torus T^3 admit a Legendre transform and $\alpha_{\varphi(-s)}(D\varphi(s)(\xi) \leq 1$, but there exists $s \neq 0$ such that the equality holds.

The objective of this chapter is to show that the sequence of overtwisted contact structures on S^3 introduced by Gonzalo-Varela [26] defined by

$$\alpha_n = -(\cos(\frac{\pi}{4}+n\pi(x_3^2+x_4^2))(x_2dx_1-x_1dx_2)+\sin(\frac{\pi}{4}+n\pi(x_3^2+x_4^2))(x_4dx_3-x_3dx_4)),$$

fails to satisfy the Fredholm assumption. In fact we will study a toy model of those structures for the case $n = 3$. The general case can be deduced by a more involved but similar dynamics of the vector field v. The Fredholm violation was first studied by A. Bahri in [10] for the first exotic contact form α_1 of J. Gonzalo and F.Varela, which admits a Legendre transform, as it was proved in the work of V. Martino [35]. In our case, the vector field v that we consider does not induce a Legendre transform.

4.3 Definition and first properties

In all what follows we will consider S^3 as a sub-manifold of $\mathbb{R}^4$ with the coordinate system (x_1, x_2, x_3, x_4). We will use the notation $r_1 = x_1^2+x_2^2$ and $r_2 = x_3^2+x_4^2$. Therefore, S^3 is defined by $r_1 + r_2 = 1$.

We consider now the 3^{rd} Gonzalo-Varela form, defined by

$$\alpha = -(A_3(x_2dx_1 - x_1dx_2) + B_3(x_4dx_3 - x_3dx_4),$$

where $A_3 = \cos(\frac{\pi}{4} + 3\pi r_2)$ and $B_3 = \sin(\frac{\pi}{4} + 3\pi r_2)$. We will first investigate the dynamics of a vector field on its kernel, knowing that the hypothesis (H_2) does not hold. And then we will show that the Fredholm assumption does not hold as well, similarly to the case $n = 1$, [10].

Let $\tilde{A}_3 = A_3 + 3\pi r_1 B_3$, and $\tilde{B}_3 = B_3 + 3\pi r_2 A_3$. Hence, if we consider the vector field

$$\zeta = -(\tilde{B}_3 x_2\partial_{x_1} - x_1\partial_{x_2} + \tilde{A}_3 x_4\partial_{x_3} - x_3\partial_{x_4})$$

then the Reeb vector field $\xi = \frac{\zeta}{\alpha(\zeta)}$.

For simplicity we will use this notation for the following vector fields:

$$\begin{cases} x_2\partial_{x_1} - x_1\partial_{x_2} &= \partial_{\theta_1} \\ x_4\partial_{x_3} - x_3\partial_{x_4} &= \partial_{\theta_2} \\ x_1\partial_{x_1} + x_2\partial_{x_2} &= \partial_{r_1} \\ x_3\partial_{x_3} + x_4\partial_{x_4} &= \partial_{\theta_1} \end{cases}$$

With this notations, we define the following vector fields that will be studied later in this paper.

$X = \sqrt{2}((\frac{B_3}{r_1}\partial_{\theta_1}) - (\frac{A_3}{r_2}\partial_{\theta_2})$

$Y = \frac{1}{r_1}\partial_{r_1} + \frac{1}{r_2}\partial_{r_2}$

$X_0 = \partial_{\theta_1} + \partial_{\theta_2}$.

Also define the two functions $a = x_1x_3 + x_2x_4$ and $b = x_1x_4 - x_2x_3$. Notice that one can see that $a = sqrtr_1r_2\cos(\theta_1 - \theta_2)$ and $\bar{=}\sqrt{r_1r_2}sin(\theta_2 - \theta_1)$. This later remark is useful in proving the continuity of a certain vector field defined later. Then the we have the following

Lemma 4.3.1. *i)*$[\zeta, X_0] = [X, X_0] = [Y, X_0] = 0$
ii) $X_0 \cdot a = X_0 \cdot b = 0$

The vector fields X and Y generates the kernel of α whenever they are defined. But the vector field $v = aY + bX$ is in the kernel and globally defined and C^1. Another important thing to deduce from the previous Lemma is the fact that X_0 generates a symmetry in our setting.

A word of warning here: we will use here, in order to define v the notations of V.Martino [35] where he used $v = aX + bY$. The notation are not to be confound the a and b that are the component of the tangent vector $\dot{x} = a\xi + bv$ to a curve x of $\mathcal{C}_\beta$. We will in the sequel be studying v and $\ker\alpha$ along v, so no confusion is possible. In what follows, a and b therefore refer to the components of v along X and Y.

We will start by studying the dynamics of v. So let us first write down the different differential relations between the quantities defined previously.

Proposition 4.3.2. *i)* $Y \cdot r_1 = 2,\ Y \cdot r_2 = -2$

ii) $\zeta \cdot a = -(\tilde{A}_3 - \tilde{B}_3)b,\ \zeta \cdot b = (\tilde{A}_3 - \tilde{B}_3)a$

iii) $Y \cdot a = \frac{r_2 - r_2}{r_2 r_1} a,\ Y \cdot b = \frac{r_2 - r_2}{r_2 r_1} b$

iv) $X \cdot a = -\sqrt{2} b \frac{A_3 r_1 + B_3 r_2}{r_1 r_2},\ X \cdot b = \sqrt{2} a \frac{A_3 r_1 + B_3 r_2}{r_1 r_2}$

v) $Y \cdot \tilde{A}_3 = 6\pi(2B_3 - 3\pi r_1 A_3),\ Y \cdot \tilde{B}_3 = -6\pi(2A_3 - 3\pi r_2 B_3)$

vi) $v \cdot r_1 = 2a,\ v \cdot r_2 = -2a$

vii) $v \cdot a = \frac{(r_2 - r_2)a^2}{r_2 r_1} - \sqrt{2} b^2 \frac{A_3 r_1 + B_3 r_2}{r_1 r_2}$.

It is important to notice here that v does not induce a Legendre transform for α_3. More precisely we have

Proposition 4.3.3. *The 1-form* $\beta(\cdot) = d\alpha_3(v, \cdot)$ *is not a contact form. In fact,*

$$\beta \wedge d\beta(\xi, v, [\xi, v]) = \sqrt{2}\left(B\tilde{A}r_1 + A\tilde{B}r_2\right)(\tilde{A} - \tilde{B}) + 6\pi a^2 \left[\tilde{A}\,(3\pi r_1 B - 2A) - \tilde{B}\,(2B - 3\pi r_2 A)\right] \tag{4.1}$$

As presented down in figure (4.9), we can see where it is negative in the $(a, y)-$plane. We can even show that there is no v in the kernel of α_3 making β a contact form.

4.4 Dynamics of v

The detailed study that follows for the dynamics of v along α_3 follows the analysis completed by A.Bahri for α_1. However, there are several differences when it comes to the violation of the Fredholm assumption. The phenomena is different for α_3 than it is for α_1.

From now on, we will write A for A_3 and B for B_3. We then write down the dynamical system generated by v as follow

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & \frac{a}{r_1} x_1 + \sqrt{2} \frac{bBx_2}{r_1} \\ \dot{x}_2 & = & \frac{a}{r_1} x_2 + \sqrt{2} \frac{bBx_1}{r_1} \\ \dot{x}_3 & = & -\frac{a}{r_2} x_3 - \sqrt{2} \frac{bAx_4}{r_2} \\ \dot{x}_1 & = & \frac{a}{r_2} x_4 - \sqrt{2} \frac{bAx_3}{r_2} \end{array} \right. \tag{4.2}$$

4.4.1 Evolution on the (a,y)-variable

This system presents many symmetries that we can use to understand its evolution. Let $r_2 = y$, then we have the following

$$v \cdot b = \frac{((2y-1) + \sqrt{2}(A(1-y) + By))}{y(1-y)}$$

and

$$\begin{cases} v \cdot y = -2a \\ v \cdot a = \frac{((2y-1)+\sqrt{2}(A(1-y)+By))a^2}{y(1-y)} - \sqrt{2}(A(1-y) + By) \end{cases} \tag{4.3}$$

So we define

$f(y) = \frac{-1}{2}\frac{(2y-1)+\sqrt{2}(A(1-y)+By)}{y(1-y)}$, $g(y) = (A(1-y) + By)$ and $p = -2a$.

Hence the system (4.3) reads as follow :

$$\begin{cases} v \cdot y = p \\ v \cdot p = f(y)p^2 + 2\sqrt{2}g(y) \end{cases} \tag{4.4}$$

. Also if we start from $b = b_0$, we have that along v,

$$b = b_0 e^{\int_{y_0}^{y} f(x)dx}.$$

Proposition 4.4.1. *i)the system has 3 equilibrium points that can be identified by the value of y, two attractive s^+, s^- and one repulsive $s_0 = \frac{1}{2}$*
ii) the orbits of this system are closed, except for the two homo-clinic reaching s_0 in infinite time.

Proof. Let us consider a function h, such that $h'(t) = e^{-\int_0^t f(s)ds}$. And let $u = h(y)$, then it is easy to see that u satisfies

$$u'' = 2\sqrt{2}g(y)h'(y).$$

This reads also as

$$u'' = 2\sqrt{2}g(h^{-1}(u))h'(h^{-1}(u)) = F(u),$$

which is a regular Newtonian second order dynamical system. The main results that one can conclude from this :

i) All bounded orbits are closed except the ones travelled in infinite time between unstable equilibriums.

ii)The equilibrium points are zeros of g.

Hence, to get periodicity, it is enough to show that all the orbits in interest are bounded. Notice that we are only interested in the orbits located in the region corresponding to $0 < y < 1$. Indeed, since $p = \pm 2\sqrt{y(1-y)}$ satisfies (4.4.1), and that is an ellipse of equation $p^2 + 4(y - \frac{1}{2})^2 = 1$, in the phase plane that contains all the region that we are interested in, all the orbits are bounded and hence periodic except the ones mentioned before.

For the stability of the equilibrium values, it is enough to check the following determinant $\begin{vmatrix} 0 & 2\sqrt{2}g'(y) \\ 1 & 2f(y)p \end{vmatrix}$

And it is easy to check that $g'(y) = \tilde{B} - \tilde{A}$ and g has exactly 3 zeros in $[0,1]$, that we are going to order as follow $s^- < \frac{1}{2} < s^+$. The zeros are symmetric with respect to $\frac{1}{2}$ also s^- and s^+ are located in the region $\tilde{A} - \tilde{B} > 0$ in the other hand $\frac{1}{2}$ is located in the region $\tilde{A} - \tilde{B} < 0$, as shown in the figure (4.3) and this finishes the proof of the proposition. $\square$

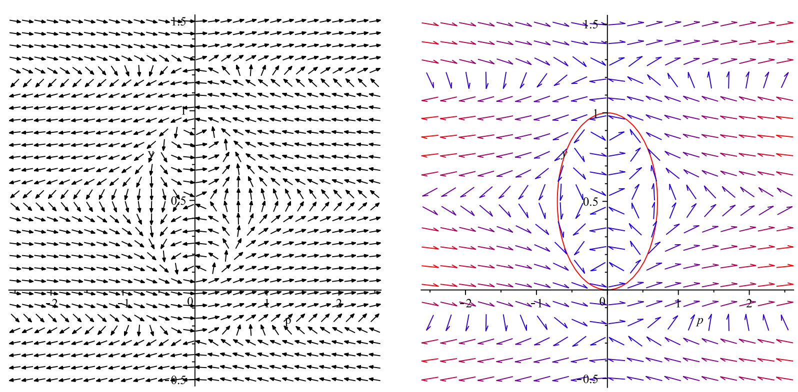


Figure 4.3: Flow of v

4.4.2 Total Rotation

Let us consider y as a variable of t along the v trajectories. We want to compute the total rotation along v. Let us use the following complex notation as in [10]. That is, we set

$$z = \frac{x_1 + ix_2}{\sqrt{r_1}}$$

and

$$z_1 = \frac{x_3 + ix_4}{\sqrt{r_2}}$$

Therefore, the system (4.4) becomes,

$$\begin{cases} \dot{z} &= -i\frac{\sqrt{2}bB}{r_1}z \\ \dot{z_1} &= i\frac{\sqrt{2}bA}{r_2}z_1 \end{cases} \tag{4.5}$$

Hence we deduce that $z = e^{i\phi}z_0$ and $z_1 = e^{i\psi}z_{0,1}$. Where $\phi(t) = -\int_0^t \frac{\sqrt{2}bB}{r_1}$, and $\psi(t) = \int_0^t \frac{\sqrt{2}bA}{r_2}$. The total Rotation R is then defined to be the difference between the values of ϕ within a period of time

$$R(y) = \int_{T^-}^{T^+} \frac{\sqrt{2}bA}{r_2}.$$

We will have two formulas for R depending on the type of orbits.

Definition 4.4.2. *We call orbits of type I, the orbits concentrating around s^+ or s^-, that is the inner orbits like in figure (4.3), and call orbits of type II, the one crossing $\frac{1}{2}$, that is the ones turning around the type I orbits.*

Type I orbits

Let us start by studying the type I orbits. Consider for instance, an orbit associated to s^+. Call T^- and T^+ the first times before and after the orbit crosses s^+, starting from a crossing of s^+. Then

$$R(y) = \int_{T^-}^{T^+} \frac{\sqrt{2}bA}{r_2}$$

and take $y_M = \max y$, knowing that $b = b_0 e^{\int_{y_0}^{y} f(x)dx}$, we get at $y = y_M$,

$$b = \sqrt{y_M(1 - y_M)}e^{\int_{y_M}^{y} f(x)dx}$$

also if we consider the function

$$k(y) = \sqrt{y(1-y)}e^{-2\int_{\frac{1}{2}}^{y} f(x)dx}$$

then we have

$$b(y) = \sqrt{k(y_M)}e^{\int_{\frac{1}{2}}^{y} f(x)dx}$$

and

$$a(y) = \sqrt{k(y) - k(y_M)}e^{\int_{\frac{1}{2}}^{y} f(x)dx}$$

so we can write

$$R(y_M) = \int_{T^-}^{T^+} \frac{\sqrt{2}bA}{-2ay}y'$$

$$= \int_{y_m}^{y_M} \frac{\sqrt{2}\sqrt{k(y_M)}e^{\int_{\frac{1}{2}}^{y} f(x)dx}\cos(\frac{\pi}{4}+3\pi y)}{-2\sqrt{k(y)-k(y_M)}e^{\int_{\frac{1}{2}}^{y} f(x)dx}y}$$

$$= -\sqrt{2k(y_M)}\int_{y_m}^{y_M} \frac{\cos(\frac{\pi}{4}+3\pi y)}{2y\sqrt{k(y)-k(y_M)}}dy$$

Where y_m is the minimal value of y in the orbit.

Notice that y_M and y_m are related through k. In fact looking at the graph of k we can see how they are related even though we do not have an explicit formula for it (see figure (4.4)).

Hence R depends only on y_M.

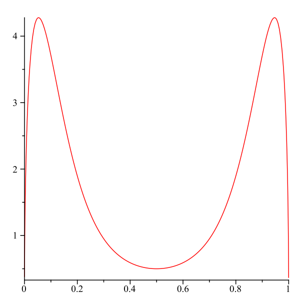


Figure 4.4: The graph of k

Type II orbits

For the type II orbits the formula is more explicit since there is more symmetries with respect to $y = \frac{1}{2}$, and hence using a similar computation we get :

$$R(y_M) = -\sqrt{2k(y_M)} \int_{1-y_M}^{y_M} \frac{\cos(\frac{\pi}{4} + 3\pi y)}{2y\sqrt{k(y) - k(y_M)}} dy$$

$$= -2\sqrt{2k(y_M)} \int_{\frac{1}{2}}^{y_M} \frac{\cos(\frac{\pi}{4} + 3\pi y)}{y\sqrt{k(y) - k(y_M)}} dy - 2\tan^{-1}\left(\sqrt{\frac{1}{4k(y_M) - 1}}\right)$$

4.5 Conjugate points

We will distinguish the two cases depending on the number zeros of a between the crossings with s^+, that is if it is even or odd.

4.5.1 The even case

Let us consider first the Type I orbits.

Notice here that for the type I orbits b is never zero. Let us assume for as a first step that $a \neq 0$ at the conjugate points.

Notice that since X_0 is transported along v then, at two conjugate points $\alpha(X_0)$ is the same, but $\alpha(X_0) = g(y)$. Hence two conjugate points must have the same image with g, that is if we take a look at the graph of g (figure (4.5)) we see that we have two cases, either they are on the same torus (when $y > s^+$), or they can be in two different tori (when $\frac{1}{2} < y < s^+$, in fact they coincide when y is a critical point of g, that is when $\tilde{A} = \tilde{B}$).

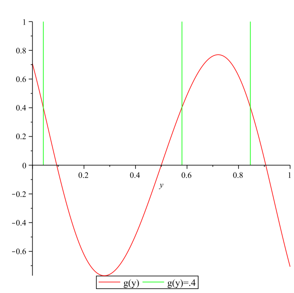


Figure 4.5: Graph of g

Let us call ψ the transport map from a torus to it self, that is it sends a point in the torus to the second intersection of the v orbit with it, as represented in the figure (4.6).

Lemma 4.5.1. *i) The torus $y = s^+$ is a characteristic surface.*
ii)If z and $\psi(z)$ are conjugate, then $d\psi_z = id$.

Proof. Clearly, since X_0 is v-transported, then

$$d\psi(X_0) = X_0 + \mu v,$$

but since $a \neq 0$ and ψ is a map from the torus to itself, then $\mu = 0$ and by density we get that

$$d\psi(X_0) = X_0,$$

even when $a = 0$. So if $z \in T_r = [y = r]$ then its conjugate should be $\psi(z)$. Now let us look at $d\psi(X)$: Since z and $\psi(z)$ are conjugate, the kernel is mapped to itself, that is, since $a \neq 0$,

$$d\psi(X) = \theta X + \mu v$$

but using the same argument as before we have

$$d\psi(X) = \theta X$$

. If we look at a now, we have that since it depends on r along v, $a(z) = a(\psi(z))$, and by taking the differential we get $da_z = da_{\psi z}(d\psi)$, therefore

$$da_z(X) = da_{\psi z}(d\psi(X)) = \theta da_{\psi z}(X)$$

and assuming that b is not maximal, we get that$\theta = 1$ since $da_z(X) = da_{\psi z}(X)$. So we have

$$d\psi(X) = X$$

Now we claim that

$$d\psi_z(\zeta) = \zeta).$$

For that we define Γ the differential of the transport along v from $r = s^+$ to $r = r_0$. And L the differential of the transport from $[z_0, z]$ to $[\psi(z_0), \psi(z)]$. And to finish take Γ_1 the differential of the transport map from $\psi(z_0)$ to $\psi(z)$. Thus we have

$$\Gamma_1 = L \circ \Gamma \circ L^{-1}$$

and

$$d\psi_{z_0} = \Gamma_1^{-1} \circ d\psi_z \circ \Gamma$$

$$= L \circ \Gamma^{-1} \circ L^{-1} \circ d\psi_z \circ \Gamma$$

since L is just a rotation and X and X_0 commutes, we have

$$d\psi_{z_0}(\Gamma^{-1}(X)) = L \circ \Gamma^{-1}(X).$$

Writing $\Gamma^{-1}(X) = \theta X_0 + \gamma\zeta$, where $\gamma \neq 0$, we have

$$d\psi_{z_0}(\theta X_0 + \gamma\zeta) = L(\theta X_0 + \gamma\zeta).$$

Using the fact that it is just a rotation again we get the claim.

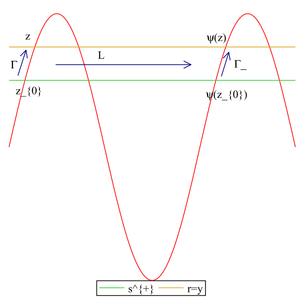


Figure 4.6: Conjugate points

Now it remains to show that s^+ is also a characteristic surface.

Let us again consider the map ψ from s^+ to itself. We have that ψ sends X_0 to itself and of course v to itself since it is the 1-parameter group of v.

So we need to check that now what is the image of ζ. In fact we have $d\psi(\zeta) = a_1 X_0 + b_1\zeta$, Since $a(z) = a(\psi(z))$ we have

$$da_z(\zeta) = da_{\psi(z)}(d\psi(\zeta))$$

Therefore

$$-b(\tilde{A} - \tilde{B}) = -bb_1(\tilde{A} - \tilde{B})$$

So if $b \neq 0$ we have that $b_1 = 1$ and $\alpha_z = \alpha_{\psi(z)}(d\psi)$. $\square$

Type II orbits

For the type II orbits we have the same results for the conjugate points in the same torus. But there is a difference from the previous case since if we take a look at the graph of g, and we take the set $g(y) = c$, we find two cases, the first one which correspond to one torus and that is similar to the type I orbits. The other case is when it has 3 intersections. Let us consider the case where we have 3. We have then one torus bellow s^- and two tori between $\frac{1}{2}$ and s^+ (by symmetry we have a similar behaviour from the other side). So we have the following result :

Lemma 4.5.2. *If we consider the map ψ as before, we have :*
i)if z and $\psi(z)$ are conjugate, then $d\psi_z = id$
ii) the torus $y = \frac{1}{2}$ is a characteristic surface.
iii)the tori $y = s^+$ and $y = s^-$ are characteristic and conjugate to each other.

The part i) is similar to the previous type of orbits, so let us just check ii) and iii)

The only thing that one needs to check is what happens to the transport of ζ but that depends only on how we write our map, since a is the same.

Let us take the case of figure (4.5), and take the tori 1,2 and 3. When 1 moves down toward $\tilde{A} = \tilde{B}$ the torus 2 moves upward toward it and 3 moves to $y = 0$.

4.5.2 The odd case.

In this part we will exhibit conjugate points separated by an odd number of zeros of a. We will do the proof for the case of type I orbits. The type II case is similar up to a small modification because of the rotation formula.

Define the map $l : S^3 \longrightarrow S^3$ by $l(x_1, x_2, x_3, x_4) = (x_3, x_4, -x_1, -x_2)$, It is in fact the action of the complex structure of $\mathbb{C}^2$. We consider also the map $\theta : s^+ \longrightarrow s^+$ that maps the point A (the intersection of the v-orbit with s^+) the point B which is the next intersection. The following lemma holds :

Lemma 4.5.3. *Assume that y_m is not a critical point of the total rotation R. Then : i)there exists $m > 0$ such that $d\theta(\xi) = -\xi - \frac{m^2}{2}X_0$ and $d\psi(\xi) = \xi - m^2 X$.*

Proof. First recall that $\psi(z_0, z) = (e^{iR(y_M)}z_0, e^{iR(y_M)}z)$, therefore

$$d\psi(\xi) = \xi + iR_\xi \psi(z_0, z)$$

Let us compute now what is R_ξ. Recall that

$$R = \int_{T^-}^{T^+} \frac{\sqrt(2)bA}{r_2}$$

thus

$$\begin{aligned} R_\zeta &= \int_{T^-}^{T^+} \frac{\sqrt(2)aA(\tilde{A} - \tilde{B})}{r_2} \\ &= -\frac{1}{2}\int_{T^-}^{T^+} \frac{\sqrt(2)A(\tilde{A} - \tilde{B})}{r_2}y' \\ &= -\frac{1}{2}\int_{y_m}^{y_M} \frac{\sqrt(2)A(y)(\tilde{A} - \tilde{B})}{y}dy \end{aligned}$$

And this can be easily seen to be negative since $y_M > s^+$. Noting that $i\psi(z_0, z) = -\frac{1}{\sqrt{r_1}}\partial_{\theta_1} - \frac{1}{\sqrt{r_2}}\partial_{\theta_2} = c_1\zeta + c_0 X_0$ Where here $c_1 = \frac{\sqrt(r_2) - \sqrt(r_1)}{r_2 r_1(\tilde{A} - \tilde{B})}$ and $c_0 = \frac{-1}{\sqrt{r_2}} + c_1\tilde{A}$. It is easy to check that at s^+, c_0 is a negative number and c_1 is a positive number. Therefore we have $d\psi(\xi) = (1 + \gamma)\xi - m^2 X$.

□

Let A, B, C, D and E as described before. Take two points A' between D and A and C' between C and E.

Lemma 4.5.4. *The rotation between A' and C' is more than 2π.*

Proof. Consider the vector field $T_0 = X_0 - \alpha_{A'}(X_0)\xi$ we transport this vector to A, and take the time of transport to be δ. Since X_0 is transported we need only to worry about the transport of ξ. So let us write down the transport equations of ξ along v in the (ξ, v, w), where w is a vector such that $\beta(w) \neq 0$.

$$\begin{cases} \dot{\lambda} = \eta \\ \dot{\eta} = d\beta(v,w)\eta - \lambda\mathcal{P}\lambda(0) = 1, \eta(0) = 0 \end{cases}$$

Here $\mathcal{P} = *\beta \wedge d\beta$. Therefore if we set

$$M(t) = \begin{pmatrix} 0 & 1 \\ -\mathcal{P} & e \end{pmatrix}$$

we have using Gronwall's inequality, for $Y(t) = \begin{pmatrix} \lambda \\ \eta \end{pmatrix}$,

$$||Y(\delta) - Y(0) - \int_0^\delta MY(0)dt|| \leq C\delta^2 e^{\delta C}.$$

Hence, the transported ξ at A reads as

$$\xi + C\delta\mathcal{P}w + O(\delta^2)$$

But $\mathcal{P}$ is positive in the neighborhood of s^+. Thus the transport of the vector T_0 that we call T_1 is

$$T_1 = X_0 - \alpha_{A'}(X_0)(\xi + C\delta w + O(\delta^2))$$

We can take $w = X_0$ in the neighborhood of s^+ since it is different from ξ, thus we can finally write T_1 as

$$T_1 = (1 + C\delta\alpha_{A'}(X_0))X_0 - \alpha_{A'}(X_0)(\xi + O(\delta^2),$$

$$= (1 + C\delta\alpha_{A'}(X_0))(X_0 - \frac{\alpha_{A'}(X_0)}{1 + C\delta\alpha_{A'}(X_0)}(\xi + O(\delta^2))$$

Again we transport T_1 to C. Using (i) we have that The transport of T_1 denoted by T_2 reads as

$$T_2 = (1 + C\delta\alpha_{A'}(X_0))(X_0 - \frac{\alpha_{A'}(X_0)}{1 + C\delta\alpha_{A'}(X_0)}[(1+\gamma)\xi - m^2X_0 + O(\delta^2)])$$

$$= (1{+}C\delta\alpha_{A'}(X_0))(1{+}\frac{\alpha_{A'}(X_0)}{1+C\delta\alpha_{A'}}m^2)[X_0{-}(\frac{\alpha_{A'}(X_0)(1+\gamma)}{(1+C\delta\alpha_{A'}(X_0))((1+\frac{\alpha_{A'}(X_0)}{1+C\delta\alpha_{A'}}m^2))}){(\xi{+}O(\delta^2)]}$$

Since $\gamma > 0$ and $\alpha_{A'}(X_0) < 0$, we have that the ξ component of T_2 is larger than the one is T_1 for δ small. Hence if we take the vector T at C' and transport it back to C, knowing that $\alpha_{C'}(X_0) = \alpha_{A'}(X_0)$ we will always have a smaller component along ξ compared to T_2 hence then in order to have the same component we should come from a point farther than C' that is bellow C'. $\square$

From this lemma we get that there exist two conjugate points separated by a zero of a between the points D and E To see this, we star moving A' back, thus the point C' will go backward and so does the point C''. If the point C' and C'' coincide along the way before crossing a zero of A, then this leads to a critical point of R which is rejected by assumption, same holds if this happens at $a = 0$. Thus, by continuity, there exist a position r_0 for which A' and C'' are in the same torus $y = r_0$ and separated by just one zero.

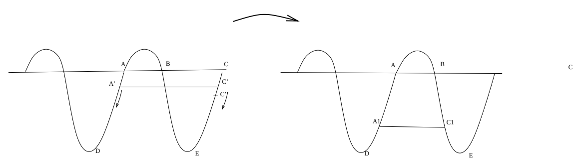


Figure 4.7: Conjugate points in the odd case

4.6 The Fredholm aspect

In this section we will use the results from the study of the dynamics of v to prove that indeed the Fredholm assumption is violated.

Definition 4.6.1. *If ϕ_s denote the one parameter group of v, we set*

$$A^+ = \{x_0 \in M; \alpha_{x_0}(D\phi_s(\xi(x_{-s})) = \alpha_{x_0}(D\phi_s(\xi(D\phi_{-s}(x_0))) > 1, \textit{for a certain } s > 0\},$$

and in a similar way we define

$$A^+ = \{x_0 \in M; \alpha_{x_0}(D\phi_s(\xi(x_{-s})) = \alpha_{x_0}(D\phi_s(\xi(D\phi_{-s}(x_0))) > 1, \textit{for a certain } s < 0\}\,.$$

So basically, A^+ is the set of points from which the Fredholm assumption is violated by a positive back and forth v-jump, and A^- is the set of points from which the Fredholm assumption is violated by a negative back and forth v-jump.
The main result of this section can be stated as follow (compare to Proposition 3 and 4 of [10]) :

Proposition 4.6.2. *Every point in $S^3 - (s^+ \cup s^- \cup T_0 \cup C)$ is in A^+ and in A^-, where here C is the set of critical points of the total rotation function R.*

Lemma 4.6.3. *All the points in Type I orbits, except the characteristic tori and the critical points of R, belong to A^+ or A^-.*

The key here is to use the result of the previous section about the existence of conjugate points separated by an odd number of zeros of a.

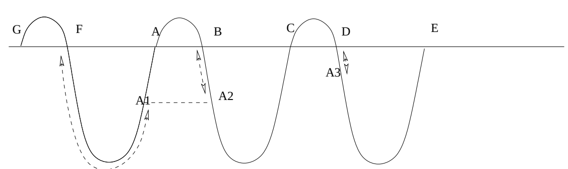


Figure 4.8: Transport along v and violation of Fredholm Assumption

Proof. Consider a type I orbit as in figure (4.8) such that y_m is not a critical point of R, for instance we will assume that the total rotation between two points below the characteristic torus is more than 2π. Then we know that we have two conjugate points A_1 and A_2. Then after 2π rotation, the transport map along v will map F to B and A_1 is mapped to A_2. Hence by monotonicity of the rotation we have that the portion $[F, A_1]$ is mapped to $[B, A_2]$. Now notice that starting from A_2, since B is mapped to D, we have that there exist a point in the interval $[D, E]$ where $\ker \alpha$ makes a 2π turn.

Let us call that point A_3. We claim that A_3 is above the torus containing A_2. In fact it cannot be exactly at the torus since y_m is not a critical point of R. So it is either below or above it. Assume it is below it. Then by continuity we have that the rotation is less that 2π which is impossible.

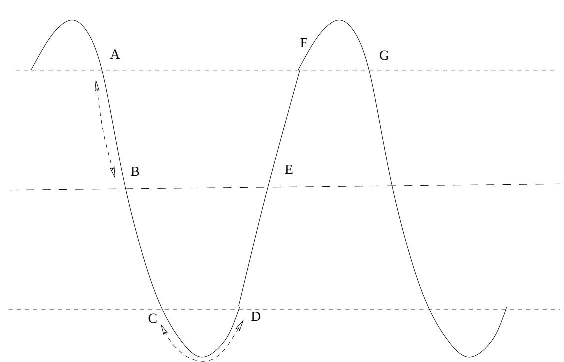
So now we have a point A_3 of coincidence for α. By iterating the process, we find a sequence of points A_k of coincidence of α such that they converge to the characteristic torus. Hence if s_k the time corresponding to the point A_k we have $\varphi^*_{s_k}\alpha = \lambda_k\alpha$. Knowing that X_0 is transported along v we have that λ_k converges to zero as k goes to infinity. Thus starting from a point in $[F, A_1]$ by transport, we can make $\alpha_{x_0}(D\phi_s(\xi(x_{-s}))$ arbitrarily large making it a point in A^+. To show that it is in A_- it is enough o iterate the process in the other direction.

Notice that the proof is the same if we assume that the rotation is less that 2π. $\square$

Lemma 4.6.4. *For the type II orbits the regions $y > s^+$ and $y < s^-$, are parts of A^+.*

One can do the same procedure to get the result but for parts of the v orbits above s^+ and below s^- and for rotation 4π instead of 2π. This is because the two tori s^+ and s^- are characteristic and conjugate. An important thing to notice here is the fact that if we try to do the procedure for points between s^+ and s^- then we get stopped by the the two preceding conjugate tori and we cannot keep turning. Still, even with that we have

Lemma 4.6.5. *Any point of a type II orbit is in A^+.*



Proof. What makes it work in this case is that the portion of orbit $[A, B]$ is mapped to $[C, D]$. In fact, since X_0 is transported along v and $\alpha(X_0) = 0$ exactly at the characteristic tori, then we have that the total rotation from A to C is 2π by monotonicity of the rotation of $\ker\alpha$. Now after mapping that portion to $[C, D]$ we are again in the region $y < s^-$ and using the 4π transport map as in Lemma 4.6.3, we have the result of the lemma.

□

There is another important property that one can notice by studying the variations of a and b along ξ. If we call $h = \alpha(\zeta)$ then

$$\xi = \frac{1}{h}\zeta$$

and we have the following identities :

Lemma 4.6.6. *If we set* $\tau = \frac{(\tilde{A}-\tilde{B})^2}{h^2}$ *then we have :*

i)$\xi\xi a = -\tau a$, $\xi\xi b = -\tau b$.

ii)$[\xi, [\xi, v]] = -\tau v$

Hence the characteristic length is determined exactly by τ which governs the same behaviour of a and b along ξ, and since ξ is always tangent to the tori $r = cte$ we have that τ is constant and thus a and b are linear combination of sine and cosine, hence in a piece of characteristic length we can reduce them to be as small as we can. And thus the part $|a| < c_0$ in the orbit will be in the type I v orbits and using the preceding procedure we get the Fredholm violation.

4.7 The function $\frac{a}{b}$

In the previous sections we saw that β changes its behaviour from a positively oriented contact form to a negatively oriented one (see figure (4.9)). We recall that $\mathcal{P} = -d\alpha(v, [\xi, v])$. The sign of $\mathcal{P}$ determines the behaviour of β. In particular, in the region $\mathcal{P} < 0$, we have two contact forms α and β that are transverse to each other and turns in opposite directions. So we expect the existence of a foliation stuck between them. In this section we will exhibit such foliation in the region $p \leq \delta$ for δ small and positive. This can be stated as follows.

Proposition 4.7.1. *There exist $\delta > 0$ and a function F defined on the set $[\mathcal{P} \leq \delta]$ such that $dF(v) = 0$ and $dF(\xi) > 0$ for every $x \in [\mathcal{P} < 0]$.*

This supports the more general conjecture by A.Bahri in [5] that there is a foliation γ transverse to α and β in the region $\beta \wedge d\beta < 0$.

Proof. This map will be constructed in several steps depending on the type of orbits and the range of y.

First we will construct the function in the region $\left\{y > \frac{1}{2}\right\}$. We take the map F to be constant on the trajectories of v and equal to the value of $\frac{a}{b}$ at $y = s^+$. That is

$$F(x) = \frac{a}{b}(\varphi_{t^+(x)}(x)) + \frac{a}{b}(\varphi_{t^-(x)}(x)),$$

where $t^+(x)$ (resp. $t^-(x)$) is the time needed for the v orbit to hit s^+ flowing with positive times (resp. flowing backward).

Now to show that this map is well defined we need the following : Assume that $b = 0$, that is $a^2 = y(1-y)$. Replacing it in (4.1) we get the existence of $c > 0$ such that $\mathcal{P} > c$ (see figure (4.10)). So if we pick $\delta = \frac{c}{2}$, then the map is well defined. In fact as stated before $b = 0$ corresponds to the v orbit representing the ellipse containing all the other orbits (see figure (4.9)).

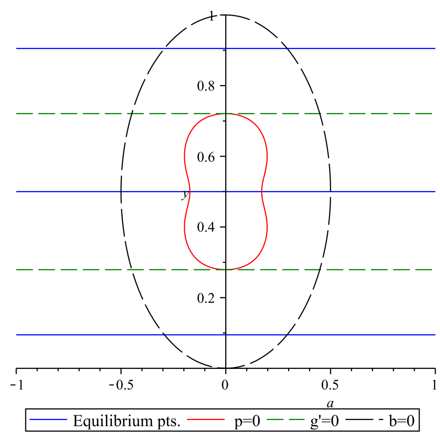


Figure 4.9: Different zones in the (a,y)-plane

In fact from Proposition 4.3.3, we have if $b = 0$, $a^2 = y(1-y)$ thus

$$\beta \wedge d\beta(\xi, v, [\xi, v]) = \sqrt{2}\left(B\tilde{A}r_1 + A\tilde{B}r_2\right)(\tilde{A}-\tilde{B}) + 6\pi r_2(1-r_2)\left[\tilde{A}\left(3\pi r_1 B - 2A\right) - \tilde{B}\left(2B - 3\pi r_2 A\right)\right]$$

And this is represented by figure below.

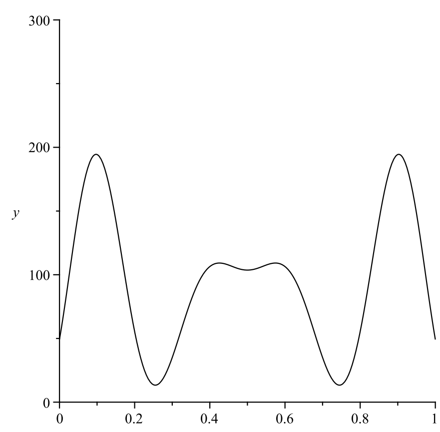


Figure 4.10: The value of $\mathcal{P}$ when $b = 0$.

And every point $x \in [\mathcal{P} < 0]$ is contained in an orbit crossing $s^{\pm}$, except of course the points in $r_2 = \frac{1}{2}$ with $a = 0$, for those by continuity, we assign to them the value of points in the homoclinic orbit as they are reached in infinite time.

Also, since F is constant along the orbits of v, $dF(v) = 0$. Now the only thing we need to check is $dF(\zeta) > 0$. Note that since F is defined using the transport along v, we have

$$dF(\zeta) = d(\frac{a}{b})(D\varphi_{(\varphi_{t^+(x)}}(x)(\zeta) + dt^+(x)v) + d(\frac{a}{b})(D\varphi_{(\varphi_{t^-(x)}}(x)(\zeta) + dt^-(x)v)$$

$$= d(\frac{a}{b})(D\varphi_{(\varphi_{t^+(x)}}(x)(\zeta)) + d(\frac{a}{b})(D\varphi_{(\varphi_{t^-(x)}}(x)(\zeta))$$

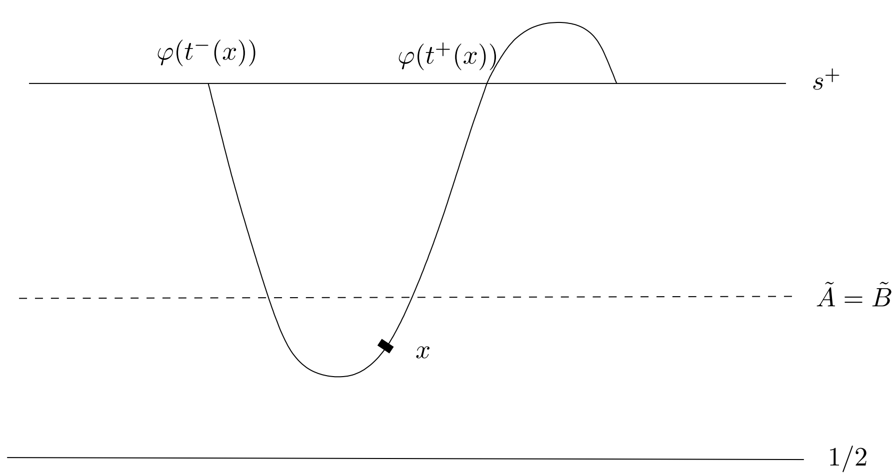
Writing $D\varphi_{t^+(x)}(\zeta) = \theta_1 X_0 + \mu_1\zeta$ and $D\varphi_{t^-(x)}(\zeta) = \theta_2 X_0 + \mu_2\zeta$ we get

$$dF(\zeta) = d(\frac{a}{b})(\theta_1 X_0 + \mu_1\zeta)_{|\varphi(t^+(x))} + d(\frac{a}{b})(\theta_2 X_0 + \mu_2\zeta)_{|\varphi(t^-(x))}.$$

Now it is easy to see that $d(\frac{a}{b})(X_0) = 0$. Thus, the only term that needs to be studied is $d(\frac{a}{b})(\zeta)$ and using Proposition 4.3.2 we have

$$d(\frac{a}{b})(\zeta) = \frac{(\tilde{A} - \tilde{B})y(1-y)}{b^2}.$$

Therefore, if we show that μ_1 and μ_2 have the same fixed sign, we are done. If $\mu_1 = 0$ then ζ is transported to X_0 and since X_0 is transported along v we have that $\zeta_x = \theta X_0$ which is impossible unless $\tilde{A} = \tilde{B}$. But in that set $\mathcal{P} > 0$ unless $a = 0$ and hence, for $\mathcal{P} < 0$, $\mu_1 \neq 0$ and the same hold for μ_2. Now to see that they do have the same sign, it is enough to notice that any type I orbit having a point in $[\mathcal{P} < 0]$ crosses the torus defined by $\tilde{A} = \tilde{B}$ twice in a single period. Also μ_2 (or the component of the transport of ζ on $zeta$) is zero exactly at the crossing. Hence μ_2 changes sign twice. Therefore by continuity, $\mu_1 + \mu_2$ has a fixed sign in the set $\mathcal{P} < 0$.



□

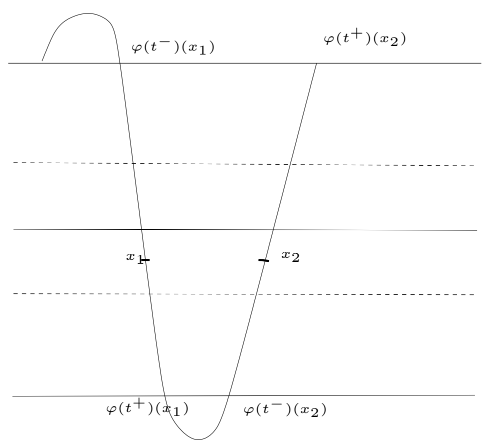
Remark : The previous construction works for $p \leq 0$ except for the circle defined by $p = 0$ and $\tilde{A} = \tilde{B}$.

By symmetry now, we construct the map F on the orbits in $r < \frac{1}{2}$, for type I orbits by taking the intersection now with s^- the symmetric of s^+.

The second step now is to define F for the points in a type II orbit. Here we can define F is a similar way, that is

$$F(x) = \frac{a}{b}(\varphi_{t^+(x)}(x)) + \frac{a}{b}(\varphi_{t^-(x)}(x)),$$

where $t^+(x)$ (resp. $t^-(x)$) is the first time of crossing with $s^\pm$ when flowing in the positive direction (resp. when flowing backward).



Following the same procedure we see that indeed F satisfies the properties in Proposition 4.7.1. Now it remains to define it for the two homoclinic orbits and to show that F is continuous shifting from one orbit to the other.

In deed the continuity will follow from the definition of F for the homoclinic ones since they present an intermediate configuration between the type I and type II orbits.

Let us consider one homoclinic orbit. For instance the one crossing s^+. Notice that it does cross it twice say for instance at two points x_1 and x_2. Then $F(x) = \frac{a}{b}(x_1) + \frac{a}{b}(x_2)$, and by symmetry the same for the second homoclinic orbit intersecting s^-. The continuity now follows by symmetry of the orbits.

4.8 Periodic orbits and Morse index

In this part we will compute the indices of the periodic orbits of ξ and also the corresponding critical values.

Recall that the index of a periodic orbit is related to the rotation of v, (see [4]). So we will first study the rotation of v along ξ. Notice first that if $r_2 \neq 0, 1$ ξ is tangent to the torus $r_2 = cte$. Hence the closed orbits corresponds to

$$\frac{\tilde{A}}{\tilde{B}} = \frac{p}{q}$$

for $p, q \in \mathbb{Z}$ with the convention that p has the same sign as $\tilde{A}$ and q has the same sign as $\tilde{B}$. The corresponding period then, is $\mathcal{T} = \frac{2\pi h|q|}{|\tilde{B}|} = \frac{2\pi hq}{\tilde{B}}$.

Since X and ξ commute, we can follow the rotation of v through its projection along X that is b.

Along the trajectory of ξ, we have that $b'' + (\frac{\tilde{A}-\tilde{B}}{h})^2 b = 0$, hence

$$b = b_0 \cos(|\frac{\tilde{A} - \tilde{B}}{h}|t + \theta_0),$$

we can take it to be $b = \sqrt{r_2(1-r_2)} \cos(|\frac{\tilde{A}-\tilde{B}}{h}|t)$. Hence, the number of zeros of b in a period of time is given by

$$\frac{|\tilde{A} - \tilde{B}|\tau}{h\pi} = 2|p - q|.$$

Now we need to determine the direction of rotation to find out if the rotation is positive or negative. For that we notice that $\dot{b} = \frac{(\tilde{A}-\tilde{B})}{h} a$ and from the previous computations we have

$$\dot{b} = -\sqrt{r_2(1-r_2)} \frac{|(\tilde{A} - \tilde{B})|}{h} \sin(|\frac{\tilde{A} - \tilde{B}}{h}|t).$$

Therefore

$$a = -\sqrt{r_2(1-r_2)} sig(\tilde{A} - \tilde{B}) \sin(|\frac{\tilde{A} - \tilde{B}}{h}|t).$$

Thus the direction of rotation is determined by the sign of $(\tilde{A} - \tilde{B})$ and the index is

$$i = 2sign(\tilde{A} - \tilde{B})|p - q|,$$

This can also read as

$$\begin{aligned} i &= 2sign(\tilde{A}-\tilde{B})|q||\frac{\tilde{A}}{\tilde{B}}-1| \\ &= 2sign(\tilde{A}-\tilde{B})\frac{|q|}{|\tilde{B}|}|\tilde{A}-\tilde{B}| \\ &= 2(\tilde{A}-\tilde{B})\frac{q}{\tilde{B}} = 2(p-q), \end{aligned}$$

the same hold for the iterated orbit

$$i_k = 2k(p-q).$$

It is important to notice that the rotation of β is not determined just by the term $\tilde{A}-\tilde{B}$ as shown in the formula of proposition 4.3.3. However, each periodic orbit of ξ that starts in the region $\tilde{A}-\tilde{B}<0$ crosses the part $\mathcal{P}<0$. That explains why the index is negative for those types of orbits. Indeed the rotation in the part $\mathcal{P}<0$ is greater then the rotation contained in the portion of the orbit in which $\mathcal{P}>0$. This is because the crossing happens when $a=0$ and if $a=0$ and $\tilde{A}-\tilde{B}<0$ then $\mathcal{P}<0$.

Now if c is the corresponding critical value, we have,

$$\begin{aligned} c &= k\int_0^\tau \alpha(\xi)dt = k\frac{2\pi hp}{\tilde{A}} \\ &= 2\pi k(A\tilde{B}r_1\frac{p}{\tilde{A}} + B\tilde{A}r_2\frac{q}{\tilde{B}}) \\ &= 2\pi k(Aqr_1 + Bpr_2). \end{aligned}$$

This can be written as

$$c = 2\pi kq(Ar_1 + Br_2) + 2\pi kBr_2(p-q).$$

Notice now that

$$i_k = 2k(p-q) = 2kq(\frac{\tilde{A}}{\tilde{B}}-1),$$

using the sign convention that we took, we have

$$i_k = 2k\frac{q}{\tilde{B}}(\tilde{A}-\tilde{B}).$$

Hence $2kq = \frac{\tilde{B}i_k}{\tilde{A}-\tilde{B}}$, thus c reads

$$c = \pi i_k\frac{h}{\tilde{A}-\tilde{B}}.$$

This formula is not valid for $\tilde{A}-\tilde{B}=0$ but it extends to the zero case since the index becomes zero and $c_0=\frac{2k\pi hp}{\tilde{A}}$ and this is for only two values of r_2. We can see in figure (4.11) the graph of $cv=\pi\frac{h}{\tilde{A}-\tilde{B}}$ as a function of r_2 of course there is two discontinuity corresponding to the zero case.

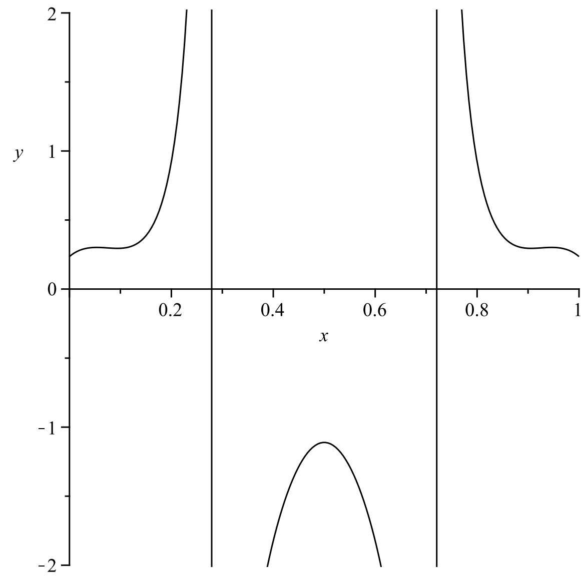


Figure 4.11: Critical values

The case $\tilde{A}-\tilde{B}=0$ corresponds to closed orbits of X_0 and there is also a full circle of them. since there is the action of $[\xi, v]$ that makes a full loop this time. This case is similar to the one in the torus T^3 that w Another important remark is the fact that all the indices are even. Hence the circle of orbits can be split into a minimum corresponding to the strict and odd index and a maximum of even index.

It remains to study now the case of the two circles $r_2=0,1$.

For that we will consider the first case, that is $r_2=0$. The orbit is then of period $T_1=\frac{2\pi h}{\tilde{B}}$ and notice that $\tilde{\tau}=(\frac{\tilde{A}-\tilde{B}}{h})^2$, hence if we consider the differential equation

$$\ddot{\eta}+\tilde{\tau}\eta=0$$

in the interval $[0,T_1]$, the number of zeros of η is at most

$$2\frac{\tilde{A}-\tilde{B}}{\tilde{B}}=6\pi$$

Therefore the H_0^1 index i_0 satisfies $18 \leq i_0 \leq 19$ and the Morse index of O_0 satisfies

$$19 \leq i_{O_0} \leq 20$$

And for the iterated orbit we get

$$19k \leq i_{O_0}^k \leq 20k.$$

Chapter 5

Contact homology of the Torus

In this chapter we will consider a family of contact structures on the torus T^3 and we will compute their relative Contact Homology. We will set the problem in a suitable variational framework and we will use the techniques developed by A.Bahri in his works [3], [7], [8] and with Y.Xu in [12].

Let us then define the torus $T^3 = S^1 \times S^1 \times S^1$, parameterized with coordinates

$$(x, y, z) \in [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi]$$

and by identifying 0 and 2π. On the torus we consider the family of infinitely many differential one-forms defined by

$$\alpha_n = \cos(nz)dx + \sin(nz)dy, \qquad n \in \mathbb{N}$$

A direct computation shows that

$$d\alpha_n = n\sin(nz)dx \wedge dz - n\cos(nz)dy \wedge dz$$

and consequently

$$\alpha_n \wedge d\alpha_n = -ndx \wedge dy \wedge dz$$

Therefore, for every $n \in \mathbb{N}$, (T^3, α_n) is a contact manifold, with contact structure given by $\sigma_n = \ker(\alpha_n)$. In particular by a classification result due to Y.Kanda [28], we have that every tight contact structure on T^3 is contactomorphic to one of the α_n; moreover for $n \neq m$, the contact structures σ_n and σ_m are not contactomorphic.

Our main result is the following:

Theorem 5.0.1. *Let g be an homotopy class of the two-dimensional torus T^2, then for* *every* $n \in \mathbb{N}$*, we have*

$$H_k(\alpha_n, g) = \begin{cases} \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \ n \ \textit{times}, & \textit{if } k = 0, 1 \\ 0, & \textit{if } k > 1 \end{cases} \tag{5.1}$$

We will prove that the homology is locally stable, namely we will consider small perturbations of the forms in the family $\{\alpha_n\}$ and we will show the our computations still hold.

We will also show some additional algebraic relations between the contact homologies of the family $\{\alpha_n\}$: in particular we will exhibit an equivariant homology reduction under the action of $\mathbb{Z}_k$, that is for every integer k, we will prove the existence of a morphism

$$f_* : H_*(\alpha_{kn}, g) \longrightarrow H_*(\alpha_n, g)$$

that corresponds to an equivariant homology reduction under the action of the group $\mathbb{Z}_k$, namely

$$H_*(\alpha_n, g) = H_*^{\mathbb{Z}_k}(\alpha_{kn}, g)$$

Finally, in the last section, we will consider the case of a more general 2-torus bundles over S^1

$$T^2 \times \mathbb{R}/(x, y, z) = (A(x, y), z + 2\pi)$$

where A is a given matrix in $SL_2(\mathbb{Z})$. We will consider the families of contact forms introduced by Giroux [24] of the following form

$$\alpha_h = \cos(h(z))dx + \sin(h(z))dy$$

with h a strictly increasing function. We will prove that for the related contact structures Theorem 5.0.1 still holds.

Other results on Homology computations are in the works of F.Bourgeois [16] and F.Bourgeois-V.Colin [17], where the authors compute the homology using the cylindrical contact homology which coincides with our result if we disregard the degeneracy. Also in his thesis dissertation E.Lebow [29] computed the embedded contact homology for 2-torus bundles which appears to be very different from the result that we find here.

5.1 General setting of the problem

We recall here some of the objects that were described in previous chapters and that will be needed in our investigation. Given a contact manifold (M, α) and v in its kernel inducing a Legendrian duality, we define the action functional

$$J(x) = \int_0^1 \alpha(\dot{x}) \tag{5.2}$$

on the subspace of the H^1-loops on M:

$$C_\beta = \{x \in H^1(S^1; M)\ s.t.\ \beta(\dot{x}) = 0;\ \alpha(\dot{x}) = \text{strictly positive constant}\},$$

where β here is the dual form of α that is $\beta = d\alpha(v, \cdot)$. Now if $\xi \in TM$ denotes the Reeb vector field of α, i.e.

$$\alpha(\xi) = 1, \qquad d\alpha(\xi, \cdot) = 0 \tag{5.3}$$

then the following result by A.Bahri-D.Bennequin holds [3]:

Theorem 5.1.1. *J is a C^2 functional on C_β whose critical points are of finite Morse index and are periodic orbits of ξ.*

Now, for the sake of computations, we rescale v such that

$$\alpha \wedge d\alpha = \beta \wedge d\beta$$

then in particular we have:

$$d\alpha(v, [\xi, v]) = -1$$

Moreover we introduce the functions τ and $\bar{\mu}$ defined by:

$$[\xi, [\xi, v]] = -\tau v$$

and

$$\bar{\mu} = d\alpha(v, [v, [\xi, v]])$$

so that the Reeb vector field of β is

$$w = \bar{\mu}\xi - [\xi, v]$$

We note that a general tangent vector z to M reads as

$$z = \lambda\xi + \mu v + \eta w$$

for some functions λ, μ, η. Also, a curve x belongs to C_β if

$$\dot{x} = a\xi + bv$$

for some function b and with a being a positive constant. Therefore, if z is tangent to C_β at x, it holds:

$$\begin{cases} \dot{\overline{\lambda + \overline{\mu}\eta}} = b\eta - \int_0^1 b\eta \\ \\ \dot{\eta} = a\mu - b\lambda \\ \\ \lambda, \mu, \eta \quad \text{1-periodic} \end{cases}$$

The second derivative of J at a critical point x $(b = 0)$ reads as:

$$J''(x) \cdot z \cdot z = \int_0^1 \dot{\eta}^2 - a^2\eta^2\tau \tag{5.4}$$

We will also need the transport maps ψ_s and ϕ_s of ξ and v respectively, namely the one parameter group of diffeomorphism generated by the flows

$$\begin{cases} \dfrac{d}{ds}\big(\psi_s(x)\big) = \xi_{\psi_s(x)} \\ \\ \psi_0(x) = x \end{cases} \tag{5.5}$$

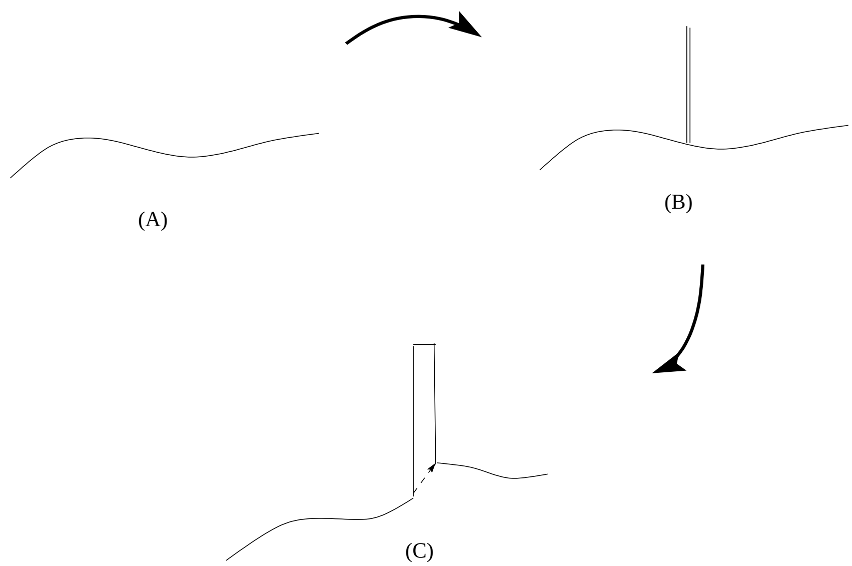
and

$$\begin{cases} \dfrac{d}{ds}\big(\phi_s(x)\big) = v_{\phi_s(x)} \\ \\ \phi_0(x) = x \end{cases} \tag{5.6}$$

The major difficulties that show up in the variational analysis of this functional are the lack of compactness (that is the Palais-Smale condition does not hold) and the loss of the Fredholm condition as it was described for instance in the case of the exotic form on the sphere. In fact the linearized operator is not Fredholm in general and this is a serious issue in the Morse theoretical methods since one cannot apply the implicit function theorem anymore and therefore the Morse lemma does not hold. We know that the Fredholm assumption is violated for the standard contact structure α_0 on S^3 and three first exotic structure of Gonzalo and Varela [26] and the third one as described in the previous chapter. Recall the following lemma

Lemma 5.1.2 (Bahri [6])**.** *If $\phi^*_{-s}(\alpha)(\xi) < 1$, for every $s \neq 0$, then J satisfies the Fredholm condition.*

We will show that in our framework Fredholm does not hold. In fact, we will see that we have situations for which there exists $s \neq 0$, such that $\phi^*_{-s}(\alpha)(\xi) = 1$.



In order to prove Theorem (5.0.1), we will first compute explicitly all the quantities defined in this variational framework for our family of contact forms $\{\alpha_n\}$.

Later, since for our model we will show that the second derivative of J has a direction of degeneracy corresponding to the action of $[\xi, v]$, the critical points will come in circles. This degeneracy will be removed by a small perturbation of the functional in a neighborhood of the critical points in order to "break the symmetry".

Then, in order to compute explicitly the homology in our framework, we need to worry about the non-compactness due to the presence of asymptotes. To deal with that we will show that the the critical points at infinity have always higher energy so that they cannot interact with our critical points, that is cancelations cannot occur. Hence the problem will come down in counting the number of periodic orbits. The idea is the same as in the theory of critical points at infinity, namely after compactifying the space, by adding the asymptotes, the classical Morse theory tells us that indeed $\partial^2 = 0$, but in this situation the boundary operator ∂ has two components $\partial = \partial_{per} + \partial_{\infty}$. The operator ∂_{per} counts the number of pseudo-gradient flow lines between periodic orbits (actual critical points) and ∂_{∞} counts the flow lines between critical points at infinity and periodic orbits. Therefore to show that we have compactness in the homology theory developed by A.Bahri in [3], [4] and [5], we need that $\partial^2_{per} = 0$. Now if we compute

$$\partial^2 = \partial^2_{per} + \partial^2_{\infty} + \partial_{per}\partial_{\infty} + \partial_{\infty}\partial_{per}$$

Hence if we show that $\partial_{per}\partial_{\infty} + \partial_{\infty}\partial_{per} = 0$ when applied to periodic orbits, then compactness holds.

Finally, since the Fredholm condition is violated, we will show however that the homology is locally stable along isotopies.

In the last two sections we will first show also some additional algebraic relations between the contact homologies of the family $\{\alpha_n\}$ and then we will consider the case of a more general 2-torus bundles over S^1.

5.2 Proof of Theorem (5.0.1)

Here we compute explicitly all the quantities defined in the previous section for our family of contact forms $\{\alpha_n\}$.

The Reeb vector field ξ_n is given by:

$$\xi_n = \cos(nz)\partial_x + \sin(nz)\partial_y$$

Now if we set:

$$v_n = \frac{1}{n}\partial_z$$

then we have $v_n \in \ker(\alpha_n)$ and

$$\beta_n(\cdot) := d\alpha_n(v_n, \cdot) = -\sin(nz)dx + \cos(nz)dy$$

Since

$$d\beta_n = n\cos(nz)dx \wedge dz + n\sin(nz)dy \wedge dz$$

and

$$\beta_n \wedge d\beta_n = -ndx \wedge dy \wedge dz$$

therefore, with this choice of the vector field v_n, we obtain that hypotheses (i) and (ii) are fulfilled; moreover

$$\alpha_n \wedge d\alpha_n = \beta_n \wedge d\beta_n$$

Furthermore we compute

$$[\xi_n, v_n] = \sin(nz)\partial_x - \cos(nz)\partial_y$$

thus $[\xi_n, [\xi_n, v_n]] = 0$ and so τ_n identically vanishes. Also, since

$$w_n = -[\xi_n, v_n]$$

is the Reeb vector field for β_n, then $\bar{\mu}$ must be zero.

Therefore, by using (5.4), the second derivative of J at a critical point x reduces to:

$$J''(x) \cdot z \cdot z = \int_0^1 \dot{\eta}^2 \tag{5.7}$$

Notice that since $\tau = 0$ we have a direction of degeneracy corresponding to η constant. So the critical points will come in circles generated by the action of $[\xi, v]$. The next Lemma shows how to perturb the functional near the critical sets, in order to "break the symmetry" and avoid degeneracy. First let us compute explicitly also the transport maps (5.5) and (5.6):

$$\psi_s(x, y, z) = \big(\cos(nz)s + x, \sin(nz)s + y, z\big) \tag{5.8}$$

and

$$\phi_s(x, y, z) = \big(x, y, z + \frac{s}{n}\big) \tag{5.9}$$

Lemma 5.2.1. *There exists a perturbed functional J_ε, for small $\varepsilon > 0$, in a neighborhood of the critical sets of J, such that J_ε is equal to J outside this neighborhood, and it has exactly 2 critical points inside it: a minimum and a maximum.*

Proof. From the equation (5.8) we see that we have periodicity for the orbits of ξ if there exists z such that $\tan(nz)$ is rational. With the same z also the orbits of $[\xi, v]$ are closed and since $[\xi, v]$ is transported along ξ we have that the set of critical points has two different S^1-actions: the first is the natural one due to the translation on time along the curve itself, and the second one due to the action of $[\xi, v]$ that gives rise to the degeneracy.

Now we want to describe the tangent space of C_β at a critical point. We know that if

$$Z = \lambda\xi + \mu v + \eta[\xi, v]$$

is tangent to C_β, we need only the function η to describe completely the tangent space; in particular at a critical point $x = a\xi$, it holds $\dot{\eta} = a\mu$.

In addition, the set of critical points is a submanifold of C_β, endowed with the S^1-action given by $[\xi, v]$: if Z is tangent to this submanifold, we have $\mu = 0$ and therefore η is constant. Moreover, the normal space to the submanifold is given by the functions $\eta \in H^1$ that are orthogonal to the constants, namely the normal space to the submanifold is generated by the vector fields Z (tangent to C_β) having

$$\eta \in H^1(S^1; \mathbb{R}), \qquad s.t. \qquad \int_0^1 \eta(t)dt = 0$$

Since the second derivative of J at a critical point x reads as $J''(x) \cdot Z \cdot Z = \int_0^1 \dot{\eta}^2$, we see that for a non vanishing normal variation, we have $J''(x) \cdot Z \cdot Z > 0$ and this shows indeed that the critical sets are isolated.

Hence we can split the tangent space to C_β at a critical point x in the following way:

$$T_x C_\beta = \{\theta\} \oplus \{\eta\}, \qquad \theta \in \mathbb{R}, \quad \eta \in H^1(S^1; \mathbb{R}), \quad \int_0^1 \eta = 0$$

Now we want to construct a tubular neighborhood around the orbit of $[\xi, v]$: so by means of the exponential map (C_β is an Hilbert manifold) we will consider the neighborhood around the critical set given by

$$\theta + s\eta, \quad s \in [0,1], \quad \theta \in \mathbb{R}, \quad \eta \in H^1, \quad \int_0^1 \eta = 0, \quad \|\eta\|_{H^1} \leq \delta$$

Therefore our functional reads in this neighborhood as $\tilde{J}(\theta, \eta)$, and we note that by construction $\frac{\partial \tilde{J}}{\partial \theta} \equiv 0$. Now we will perturb it in the following way

$$\tilde{J}_\varepsilon(\theta, \eta) = \tilde{J}(\theta, \eta) + \varepsilon w\big(\|\eta\|_{H^1}\big) f(\theta)$$

where f is a smooth function on S^1 having exactly 2 critical points, and $w(r)$ is a cut-off function that vanishes outside $|r| \geq \delta$ and it is equal to 1 for $|\delta| \leq \delta/2$. Now by choosing suitable small constants ε, δ and the bump function w, we get that the functional $\tilde{J}_\varepsilon$ is equal to the old functional outside this neighborhood, and it has exactly 2 critical points inside it: a minimum and a maximum. □

Now we recall that in this setting A.Bahri introduced different pseudo-gradient flows. For instance in [11], [6] it is was shown that the natural L^2-pseudo-gradient for J on C_β is not the right flow to consider since at the blow-up time there is the presence of an absolutely continuous part adding up to the Diracs therefore another flow was constructed that does the right decreasing. We will consider the second flow defined in [7].

It is shown for this flow the existence of critical points at infinity made by alternating v- and ξ-pieces. We define the set

$$\Gamma_{2k} = \{\gamma \in C_\beta, ab = 0\}$$

that is the set of curves in C_β made by k v_n-pieces and k ξ_n-pieces. Then we consider the set of variation at infinity, namely

$$\bigcup_{k \geq 0} \Gamma_{2k}$$

and on this set we define the functional at infinity

$$J_\infty(\gamma) = \sum_{k=0}^{k=\infty} a_k$$

The critical points of this functional are what we call critical points at infinity, and we have and the exact characterization for them. First we need the following two definitions:

Definition 5.2.2. *A v-jump between two points x_0 and $x_1 = x(s_1)$, $s_1 \neq 0$, is a v-jump between conjugate points if it holds:*

$$\left(\phi_{s_1}^* \alpha\right)_{x_1} = \alpha_{x_0}$$

In other words conjugate points are points on the same v-orbit such that the form α is transported onto itself by the transport map along v.

Definition 5.2.3. *A ξ-piece $[x_0; x_1]$ of orbit is characteristic if v completes exactly a number $k \in \mathbb{Z}$ of half revolutions from x_0 to x_1.*

It holds (see [7]):

Proposition 5.2.4. *A curve in $\bigcup_{k\geq 0} \Gamma_{2k}$ is a critical point at infinity if it satisfies one of the following assertions:*

(1) *the v-jumps are between conjugate points. These critical points are denoted in the sequel "true" critical points at infinity;*

(2) *the ξ-pieces have characteristic length, and in addition the v-jumps send $ker(\alpha)$ to itself.*

In our case we see from the transport equation (5.8) along ξ_n, that we cannot have ξ-pieces with characteristic length, thus all the critical points at infinity are "true". Also, we see from the transport equation (5.9) along v_n, that each point has n conjugate points corresponding to the translation along z by $\frac{2\pi}{n}$. Next we check the validity of the Fredholm condition. We have:

Lemma 5.2.5. *The Fredholm assumption is violated.*

Proof. By using Lemma (5.1.2) and by a straightforward computation, if we just compute the transport of ξ_n along v_n, we get

$$\left(\phi_s^* \alpha_n\right)(\xi_n) = \cos(s) \leq 1$$

Since we can have $s \neq 0$ such that equality occurs, then Fredholm does not hold. $\square$

Next, in order to compute explicitly the Homology it suffices to show that there is no interaction between periodic orbits and critical points at infinity, in the sense that there are no flow lines among them. This is what we prove in the following:

Lemma 5.2.6. *There is no interaction between the periodic orbits of ξ_n and the critical points at infinity.*

Proof. First from the classification result Lemma (5.2.4), the critical points at infinity for our model are just periodic orbits of ξ_n with some additional back and forth v-jumps of length multiple of $\frac{2\pi}{n}$. An interesting case happens when $n = 1$ since each point in the orbit of ξ_n can be conjugate only to itself along v_n, so we have periodic orbits linked by v-cycles.

Now by knowing the trivial splitting of the fundamental group of the torus, that is

$$\pi_1(T^3) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

we denote by

$$P_3 : \pi_1(T^3) \longrightarrow \mathbb{Z}$$

the natural projection on the third component, namely: if

$$[\gamma] \in \pi_1(T^3), \qquad [\gamma] = (m, n, k)$$

then $P_3([\gamma]) = k$. Next we explicitly note now that if x is a periodic orbit of the Reeb vector field ξ_n, then $P_3([x]) = 0$; moreover any v-cycle will add a pure third component, thus it will have projection non zero. In particular if x_∞ is a critical point at infinity, let us suppose with m v-cycles (with orientation) attached to a periodic orbit of ξ, then

$$P_3([x_\infty]) = \sum_{i=1}^{m} k_i$$

where k_i is the number of iterations of the k-th v-cycle counted with its orientation. Therefore we deduce that a periodic orbit and a critical point at infinity can interact if and only if $P_3([x_\infty]) = 0$. Notice that since the strict index of the periodic orbits is zero, and the index of the critical points at infinity is at least 1, then trivially we have that $\partial^2_{per} = 0$. But notice that we have a richer structure here built by the tower of critical points at infinity above each critical point. $\square$

Now we will prove the main Theorem.

Proof. (of Theorem 5.0.1)

Let g be an homotopy class in T^2, $(g \in \pi_1(T^2))$. then g reads as:

$$g = m\overline{x} + l\overline{y}$$

where $\overline{x}$ and $\overline{y}$ are the generators of $\pi_1(T^2)$. Hence since

$$\xi_n = \cos(nz)\partial_x + \sin(nz)\partial_y$$

we get that g contains exactly n periodic orbits of ξ_n (in fact we have n circles of critical points). By breaking the symmetry each circle can be seen as a min and a max with zero boundary operator between them. Therefore

$$H_k(\alpha_n, g) = \begin{cases} \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \text{ n times}, & \text{if } k = 0, 1 \\ 0, & \text{if } k > 1 \end{cases} \tag{5.10}$$

□

In the last part of this section we will show that our computations are "locally stable", that is we are interested in small perturbations of the contact forms in the family $\{\alpha_n\}$. Thus, let us suppose that α is a form in the previous family, and let us consider a perturbed form $\tilde{\alpha} := u\alpha$, where $u \in C^2(M, \mathbb{R})$ and $\|1 - u\|_{C^2}$ is small. Hence we will get a new functional $\tilde{J}$ whose critical points $\tilde{x}$ will be in a L^∞ neighborhood of the original ones x. We will show that in fact $J(\tilde{x}) \geq J(x)$. In order to do that we first use a result in [11]: every curve $x_0 \in C_\beta$ in a L^∞ neighborhood of a critical point x can be represented by a curve x_1 made only by pieces of ξ-orbit and finitely many $\pm v$-jumps, and in addition this is a minimizing process, i.e. $J(x_1) \leq J(x_0)$. In particular in our situation, in order to stay in a given homotopy class, the v-jumps need to be small, moreover since the ξ-pieces have the z component constant, we need the sum of the $\pm v$-jumps to be zero: hence we can think to have finitely many nearly "Dirac masses" placed on the original critical point x. Now we can obtain this broken curve x_1 from a critical point x by pushing along a deformation vector Z having η such that:

$$\ddot{\eta} = \sum_{i=1}^{k} \pm A_i (\delta_{t_i^-} - \delta_{t_i^+})$$

where k is the number of the "Dirac masses", $A_i > 0$ represent the jump in the v direction, and t_i^-, t_i^+ are the times where the jumps occur. If we compute the second variation along this Z, we get:

$$J''(x)ZZ = \int_0^1 \dot{\eta}^2 = -\int_0^1 \eta\ddot{\eta} = \sum_{i=1}^{k} \pm A_i \big(\eta(t_i^+) - \eta(t_i^-)\big)$$

Now let us consider disjoint intervals $[T_i^-, T_i^+]$, each of them containing $[t_i^-, t_i^+]$, with $T_{i-1}^+ = T_i^-$ and $T_i^+ = T_{i+1}^-$. By a direct computation we find

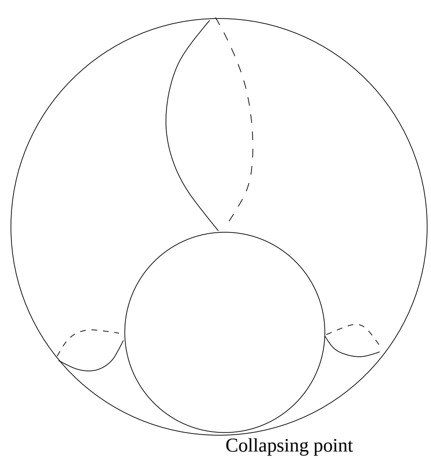
$$J''(x)ZZ = \sum_{i=1}^{k} A_i^2 \big(t_i^+ - t_i^-\big)\Big(1 - \frac{t_i^+ - t_i^-}{T_i^+ - T_i^-}\Big) > 0$$

Therefore Z is a strictly increasing direction for J and this proves the local stability.

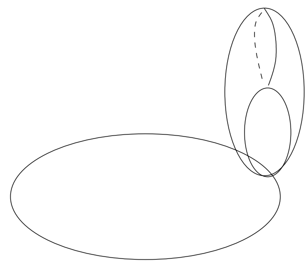
Finally, we want to show a strict relation between our structures and some spaces of configurations. So, given a periodic orbit, let us consider the set $\tilde{\Gamma}_{2k}$ made by the periodic orbit with attached $+k$ v-orbits and $-k$ v-orbits. Studying this space corresponds to understand the configuration space of signed particles on S^1. This was studied in a paper by D. McDuff [36] in which she gives a full description of the space of configuration of signed particles, denoted by $C^{\pm}$, as follows:

Theorem 5.2.7 (McDuff [36]). *If M is a manifold without boundary, there is a homotopy equivalence between $C^{\pm}(M)$ and $\Gamma^{\pm}$ the space of compactly supported sections from M to $E^{\pm}$ the bundle over M constructed by taking at each point of $x \in M$ the set $S_x \times S_x/D$. Here S_x is the unit sphere in the tangent space at x and D the diagonal.*

For instance, one sees that the space $\tilde{\Gamma}_4$ (made by the periodic orbit with two v periodic orbits attached to it with opposite orientations) has the topology of S^2 with two points identified. The identification comes from the fact that if the two v-orbits coincide at the same point, they cancel each other. In particular, in the case $n = 1$, the space $\tilde{\Gamma}_4$ coincides with the space Γ_4 with the v-pieces having opposite orientations.



Indeed because of the extra S^1-action that we have, the full structure can be seen as in the figure below.



Circle of periodic orbits with the extra structure of critical points at infinity

5.3 More Structures

In this section we will give some algebraic relations between the different contact homologies of the family $\{\alpha_n\}$.

Theorem 5.3.1. *Let p and k be a positive integers, then there exists a morphism*

$$f_* : H_*(\alpha_{kp}, g) \longrightarrow H_*(\alpha_p, g).$$

Moreover, this homomorphism corresponds to an equivariant homology reduction under the action of the group $\mathbb{Z}_k$, that is

$$H_*(\alpha_p, g) = H_*^{\mathbb{Z}_k}(\alpha_{kp}, g).$$

Proof. Let us consider the action of the group $\mathbb{Z}_k$ on the torus by translating the third component, namely the action generated by $f(x, y, z) = (x, y, z + \frac{2\pi}{k})$. We notice that the contact form is invariant under f, that is

$$f^*\alpha_{kp} = \cos(kp(z + \frac{2\pi}{k}))dx + \sin(kp(z + \frac{2\pi}{k}))dy = \alpha_{kp}$$

Therefore also the functional $J_{\alpha_{kp}}$ is invariant under this action. We recall that at the chain level the boundary operator ∂ counts the number of orbits of a decreasing pseudo-gradient for J. For two periodic orbits x_1 and x_2 of ξ we define $\langle x_1, x_2\rangle$ as the number of gradient flow lines from x_1 to x_2, if the index difference is one. With this notation we have that

$$\partial x_1 = \sum_{i_{x_k}=i_{x_1}-1} \langle x_1, x_k \rangle x_k$$

Next we define

$$C_n(\alpha_p, g) := Crit_n(J_{\alpha_p}, g) \otimes \mathbb{Z}$$

where $Crit_n(J_{\alpha_p}, g)$ is the set of critical points of J_{α_p} in the homotopy class $g \in \pi_1(T^2)$ with Morse index n. We notice that

$$Crit_n(J_{\alpha_{kp}}, g)/\mathbb{Z}_k = Crit_n(J_{\alpha_p}, g)$$

Therefore

$$C_n^{\mathbb{Z}_k}(\alpha_{kp}, g) := Crit_n(J_{\alpha_{kp}}, g)/\mathbb{Z}_k \otimes \mathbb{Z} = C_n(\alpha_p, g)$$

so we can define the surjective group homomorphism

$$f_* : C_*(\alpha_{kp}, g) \longrightarrow C_*(\alpha_p, g)$$

induced on the quotient by the group action of $\mathbb{Z}_k$ on the generators. We claim that this is indeed a chain map. In fact the boundary operator on the quotient chain is defined by

$$\partial_{\mathbb{Z}_k} \tilde{x}_1 = \sum_{\tilde{x}_i \in Crit_{n-1}(J_{\alpha_{kp}}, g)/\mathbb{Z}_k} \sum_{j=1}^{k} < x_1, x_i^j > \tilde{x}_i$$

where $\tilde{x}_1 = f_n(x_1)$ and $\{x_i^j\}_j = f_*^{-1}(\tilde{x}_i)$. It is easy to see now that $\partial^2_{\mathbb{Z}_k} = 0$ and f_* is indeed a chain map by construction. In fact, we have

$$f_* \partial x_1 = f_*(\sum_{x_i \in Crit_{n-1}(J_{\alpha_{kp}}, g)} < x_1, x_i > x_i)$$

$$= \sum_{x_i \in Crit_{n-1}(J_{\alpha_{kp}}, g)} < x_1, x_i > f_*(x_i),$$

by grouping the terms with the same image under f we get that

$$\partial_{\mathbb{Z}_k} \tilde{x}_1 = f_* \partial.$$

Now using this fact we have

$$\partial^2_{\mathbb{Z}_k} \tilde{x}_1 = \partial_{\mathbb{Z}_k} f_*(\partial x_1) = f_* \partial^2 x_1 = 0.$$

Thus it descends to a morphism in the homology level. $\square$

Then one has the following commuting diagram :

$$
\begin{tikzcd}
& H_*(\alpha_{pq},g) \arrow[rr, "\partial_{pq}"] \arrow[dl, "f^q_*"] \arrow[dd, dotted] & & H_{*-1}(\alpha_{pq},g) \arrow[dl, "f^q_{*-1}"] \arrow[dd, "f^p_{*-1}"] \\
H_*(\alpha_p,g) \arrow[rr, "\partial_p"] \arrow[dd, "f^p_*"] & & H_{*-1}(\alpha_p,g) \arrow[dd, "f^p_{*-1}"] & \\
& H_*(\alpha_q,g) \arrow[rr, dotted] \arrow[dl, dotted, "f_q^*"] & & H_{*-1}(\alpha_q,g) \arrow[dl, "f^q_{*-1}"] \\
H_*(\alpha_1,g) \arrow[rr, "\partial"] & & H_{*-1}(\alpha_1,g) &
\end{tikzcd}
$$

Moreover if we consider one of the faces of the previous diagram we have for $p_1, \cdots, p_k$, k positive integers:

$$
\begin{tikzcd}
\cdots \arrow[r] & H_*(\alpha_{p_k \cdots p_1},g) \arrow[r, "\partial_{p_k \cdots p_1}"] \arrow[d, "f^{p_k}_*"] & H_{*-1}(\alpha_{p_k \cdots p_1},g) \arrow[r] \arrow[d, "f^{(p_k}_{*-1}"] & \cdots \\
& \vdots \arrow[d, "f^{p_2}_*"] & \vdots \arrow[d, "f^{p_2}_{*-1}"] & \\
\cdots \arrow[r] & H_*(\alpha_{p_1},g) \arrow[r, "\partial_{p_1}"] \arrow[d, "f^{p_1}_*"] & H_{*-1}(\alpha_{p_1},g) \arrow[r] \arrow[d, "f^{p_1}_{*-1}"] & \cdots \\
\cdots \arrow[r] & H_*(\alpha_1,g) \arrow[r, "\partial"] & H_{*-1}(\alpha_1,g) \arrow[r] & \cdots
\end{tikzcd}
$$

5.4 Torus Bundles

We consider now the case of more general 2-torus bundles over S^1. Given a matrix $A \in SL_2(\mathbb{Z})$, we define the space

$$Y_A = T^2 \times \mathbb{R}/(x, y, z) = (A(x, y), z + 2\pi).$$

We recall that the fundamental group of Y_A, is $\pi_1(Y_A) = \mathbb{Z} \times \mathbb{Z} \rtimes_A \mathbb{Z}$. From the work of Giroux [24], we know that these spaces contains infinitely many contact structures, given by a fixed contact form α. The construction of such structures starts by taking a strictly increasing function h and considering the contact form α_h on $\mathbb{R}^3$ defined by

$$\alpha_h = \cos(h(z))dx + \sin(h(z))dy$$

We state then the result of Giroux as follow:

Theorem 5.4.1 ([24]). *Let A be a matrix in $SL_2(\mathbb{Z})$ then:*

a) For every $n \geq 0$ there exists a contact structure on $\mathbb{R}^3$ given by the 1-form

$$\alpha_{h_n} = \cos(h_n(z))dx + \sin(h_n(z))dy,$$

that is invariant under the action of the fundamental group of Y_A and the increasing function h satisfies :

$$2\pi n \leq h_n(z + 2\pi) - h_n(z) < 2\pi(n + 1)$$

b) The contact structure descends to a contact structure on Y_A, depending only on n up to isotopy

c) All these contact structures are homotopic as plane fields on Y_A.

Remark 5.4.2. *We explicitly note that the family of contact forms we considered in the first part of the paper correspond to the choice of $h_n(z) = nz$, with $A = I_2$.*

We are going to compute for these contact forms all the quantities needed in order to apply the variational method. First we have that the Reeb vector field of α_{h_n} is given by

$$\xi_{h_n} = \cos(h_n(z))\partial_x + \sin(h_n(z))\partial_y$$

Then, by straightforward computations we get the following

Lemma 5.4.3. *The 1-form $\beta_{h_n} = d\alpha_{h_n}(v_n, \cdot)$ is a contact form with the same orientation than α_{h_n} on Y_A with*

$$v_{h_n} = \frac{1}{h_n'(z)}\partial_z$$

Therefore hypotheses (i) and (ii) are fulfilled and

$$\alpha_{h_n} \wedge d\alpha_{h_n} = \beta_{h_n} \wedge d\beta_{h_n}$$

Moreover τ_{h_n} and $\bar{\mu}_{h_n}$ are zero.

Also

Lemma 5.4.4. *The transport maps ψ_s and ϕ_s of ξ_{h_n} and v_{h_n} respectively are given by:*

$$\psi_s(x, y, z) = \Big(\frac{\cos(h_n(z))s}{h_n'(z)} + x, \frac{\sin(h_n(z))s}{h_n'(z)} + y, z\Big) \tag{5.11}$$

and

$$\phi_s(x, y, z) = \Big(x, y, z + \frac{s}{h_n'(z)}\Big) \tag{5.12}$$

Now we can check the Fredholm condition. We have:

Lemma 5.4.5. *The Fredholm assumption is violated for all the contact forms α_{h_n}.*

Proof. By using again Lemma (5.1.2), if we compute the transport of ξ_{h_n} along v_{h_n}, we get

$$\big(\phi_s^*\alpha_{h_n}\big)(\xi_{h_n}) = \cos\Big(h_n\big(z + \frac{s}{h_n'(z)}\big) - h_n(z)\Big) \leq 1$$

Hence Fredholm does not hold. □

Moreover by the transport equations for ξ_{h_n} and v_{h_n} we see that there are no ξ-pieces with characteristic length. Regarding the conjugate points, different scenarios might happen. We will distinguish two cases.

Case 1: the conjugate points are in different fibers.

In fact for two points to be conjugate we need to have

$$h_n(z + \frac{s}{h_s'(z)}) - h_n(z) = 0 mod(2\pi)$$

By using the fact that

$$2n\pi < h_n(z + 2\pi) - h_n(z) \leq 2(n + 1)\pi$$

there exist n values s, with $0 < s \leq 2\pi h_n'(z)$, such that $h_n(z + \frac{s}{h_n'(z)}) - h_n(z)$ is a multiple of 2π, and this corresponds to conjugate points in different fibers (see figure (5.1)).

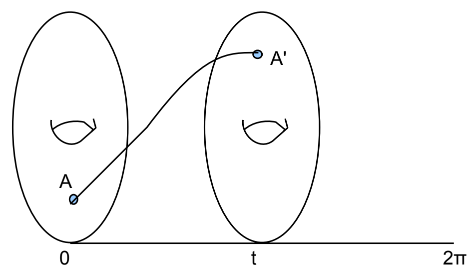


Figure 5.1: Conjugate points

Case 2 : the conjugate points are in the same fiber.

This case happens in the particular situation when

$$h_n(z + 2\pi) - h_n(z) = 2(n + 1)\pi$$

and so the conjugate point in the same fiber is achieved when $s = 2\pi h_n'(z)$ (see figure (5.2)).

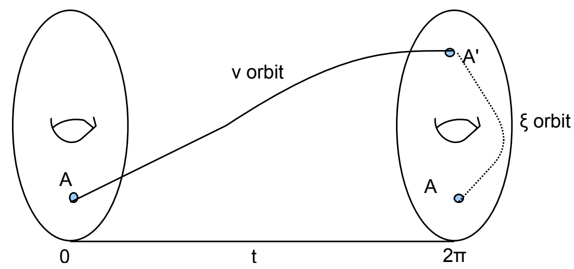


Figure 5.2: Conjugate points and Fibers

Another important thing to notice is that the orbits of the Reeb vector field are tangent to the fibers, thus if two conjugate points are in a different fibers we need more that one v-piece to be able to close the curve as critical point at infinity.
On the other hand, in the case where the conjugate points are in the same fiber, we can close the orbit by a ξ-piece, but in this situation we are in a different homotopy class (as in the case of T^3 considered in the previous sections). Hence in order to be able to close the curve staying in the homotopy class containing the periodic orbits, we need to have at least one v-piece in the opposite direction.
Then with the same reasoning as in the case of the torus T^3, we have that the index of the critical points at infinity is strictly greater than zero in a given homotopy class. In order to compute the homology, we need just to find the index of the periodic orbits, but since τ is zero, we can proceed as in the previous case of the torus T^3: therefore there is no interaction between the periodic orbits and the critical points at infinity.
Now let us fix an homotopy class $g \in \pi_1(T^2)$, with $g = (a, b) \in \mathbb{Z} \oplus \mathbb{Z}$. Since the periodic orbits are all tangent to the fibers then the periodicity condition is equivalent to $\tan(h_n(z)) = \frac{a}{b}$ and this corresponds to n periodic orbits. Finally we have proved the following

Theorem 5.4.6. *For any given contact structure of the form α_{h_n} on Y_A, if $g \in \pi_1(T^2)$,* *we have:*

$$H_k(\alpha_{h_n}, g) = \begin{cases} \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \ n \ times, & if\ k = 0, 1 \\ 0, & if\ k > 1 \end{cases} \tag{5.13}$$

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