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# EXPLORING AND JUSTIFYING IDEAS IN AN UNDERGRADUATE MATHEMATICS COURSE: A CASE STUDY 

BY<br>ANNA BROPHY<br>A dissertation submitted to<br>The Graduate School of Education<br>Rutgers, The State University of New Jersey<br>in partial fulfillment of the requirements<br>for the degree<br>Doctor of Education<br>Graduate Program in Mathematics Education

## Approved by

Carolyn A. Maher, Chair

Cindy E. Hmelo-Silver, Committee

Elizabeth B. Uptegrove, Committee

New Brunswick, New Jersey
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## ABSTRACT OF THE DISSERTATION

## Exploring and Justifying Ideas in an Undergraduate Mathematics Course:

A Case Study

## ANNA BROPHY

Dissertation Chairperson: Carolyn A. Maher, Ed. D.

The improvement of mathematics education relies very heavily on the improvement of undergraduate mathematics education for future teachers (National Research Council, 1989). It is important that undergraduate mathematics instruction for prospective teachers demonstrates techniques to be used in their future classrooms (Blair, 2006; Senk, Keller, \& Ferrini-Mundy, 2004). Specifically, pre-service teachers should develop an understanding of the mathematical processes of exploration and proof (Senk, Keller, \& Ferrini-Mundy, 2004).

If problems that encourage mathematical exploration and justification are to be brought into the undergraduate classroom, understanding how students build and justify their solutions will be of importance. The purpose of this research was to (1) investigate how undergraduate students enrolled in a mathematics course solve and justify their solution to a series of combinatorics tasks, (2) analyze the moves employed by the
instructor and (3) investigate how their solutions compare to the solutions of other students involved in the same problem-solving tasks.

This case study was conducted in a mathematics class at a liberal arts college. The six students in this class were all mathematics majors studying to be teachers. Using videotaped data and students' written work, a careful analysis of how the students built their solutions and justified their answers to three combinatoric problems was conducted.

It was found that the strategies and justifications used by the students in this study were similar to those used by participants in earlier studies. Furthermore, in investigating how the college math students built their solutions to the problems, it was found that the instructor played a critical role in the learning process.

Findings from this study verify that mathematical learning can take place in a college mathematics class that fosters mathematical exploration and justification with wellchosen tasks, collaboration with peers, and student-centered instruction. This study also has implications for implementation in other settings by providing examples of students' solutions to specific tasks as well as examples of how instructors can effectively interact with students in a mathematical classroom that nurtures the mathematical processes of conjecturing, generalizing, and justifying solutions to problems.

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## TABLE OF CONTENTS

ABSTRACT ..... I
ACKNOWLEDGEMENTS ..... III
LIST OF TABLES ..... VII
LIST OF FIGURES ..... VIII
I STATEMENT OF THE PROBLEM .....  1
II LITERATURE REVIEW AND THEORETICAL FRAMEWORK ..... 5
2.1 Introduction ..... 5
2.2 THEORETICAL FRAMEWORK ..... 5
2.3 Literature Review ..... 14
2.3.1 Reasoning and Proof ..... 14
2.3.2 Undergraduate Mathematics ..... 17
2.3.3 Discrete Mathematis: Combinatorics. ..... 19
2.3.4 Rutgers University - The Longitudinal Study ..... 22
2.3.5 The Combinatorics Problems ..... 24
2.3.6 Glossary of Terms ..... 27
2.3.7 The Towers Problem ..... 35
2.3.8 The Pizza Problem. ..... 49
2.3.9 Making Connections between Towers and Pizzas ..... 69
2.3.10 Ankur's Challenge ..... 78
III DESIGN OF THE STUDY ..... 101
3.1 BACKGROUND ..... 101
3.2 SUBJECTS ..... 101
3.3 Data ..... 101
3.4 SETTING ..... 102
3.5 TASKS ..... 102
3.6 DATA COLLECTION ..... 102
3.6.1 Video Recordings ..... 103
3.6.2 Students Written Work ..... 103
3.7 METHOD OF ANALYSIS OF THE VIDEO DATA ..... 103
3.7.1 Viewing ..... 103
3.7.2 Describing ..... 104
3.7.3 Identifying Critical Events ..... 104
3.7.4 Transcribing ..... 105
3.7.5 Coding - A Categorization Approach ..... 105
3.7.6 Constructing a Storyline ..... 107
3.7.7 Composing a Narrative ..... 107
3.7.8 Document Analysis. ..... 108
IV RESULTS ..... 109
4.1 The Towers Problem ..... 109
4.1.1 February 11, 2011 ..... 110
4.1.2 February 18, 2011 ..... 130
4.2 The Pizza Problem ..... 141
4.3 ANKUR's Challenge. ..... 174
V FINDINGS ..... 184
5.1 Introduction ..... 184
5.2 THE TOWERS PROBLEM ..... 184
5.2.1 Results ..... 184
5.2.2 Analysis of Strategies Compared to the Existing Research ..... 187
5.3 THE PIZZA PROBLEM ..... 192
5.3.1 Results ..... 193
5.3.2 Analysis of Strategies Compared to the Existing Research ..... 197
5.4 Connections Between The Towers Problem and The Pizza Problem ..... 202
5.4.1 Results ..... 203
5.4.2 Analysis of Strategies Compared to the Existing Research ..... 206
5.5 ANKUR's Challenge ..... 208
5.5.1 Results ..... 208
5.5.2 Analysis of Strategies Compared to the Existing Research ..... 211
5.6 The Role of the Teacher ..... 215
5.7 CONCLUSIONS ..... 238
5.7.1 Implications ..... 241
5.7.2 Limitations. ..... 243
5.7.3 Suggestions for Further Study ..... 243
VI APPENDICES ..... 244
A Outline of Course Schedule ..... 244
B Transcript 1.1 - February 11, 2011, Camera View One ..... 248
C Transcript 2.1 - February 18, 2011, Camera View One ..... 300
D Transcript 2.2 - February 18, 2011, Camera View Two ..... 358
VII REFERENCES ..... 410

## LIST OF TABLES

## Table

2.1 The first seven rows of Pascal's triangle ..... 35
5.1 Towers problem - initial building/organizing towers ..... 189
5.2 Towers problem - pattern recognition/formulas ..... 192
5.3 Towers problem - justifications ..... 192
5.4 Pizza problem - building/organizing pizzas ..... 202
5.5 Towes and pizza problem - connections ..... 208
5.6 Ankur' Challenge - initial building/organizing towers ..... 215
5.7 Teacher questioning - initial question after student exploration ..... 219
5.8 Teacher questioning that led to clarification ..... 222
5.9 Teacher suggestions to investigate smaller examples ..... 234
5.10 Teacher suggestions to test conjecture ..... 234

## LIST OF FIGURES

## Figure

2.1 Example of a four-tall tower. ..... 26
2.2 Example of a pair of opposites. ..... 27
2.3 Example of a pair of cousins. ..... 28
2.4 Example of towers forming the elevator pattern. ..... 29
2.5 Example of towers forming the staircase pattern ..... 30
2.6 Replication of Romina's original solution to Ankur's Challenge ..... 79
2.7 Replication of Romina's solution to Ankur's Challenge ..... 80
2.8 Replication of Mike and Ankur's first case of towers. ..... 82
2.9 Replication of Mike and Ankur's third case of towers. ..... 82
2.10 Diagram of Mary's solution to Ankur's Challenge. ..... 85
2.11 Diagram of Errol's solution to Ankur's Challenge. ..... 86
2.12 Diagram of Rob1 first six towers in his solution to Ankur's Challenge. ..... 87
2.13 Diagram of Rob1 second set of six towers. ..... 87
2.14 Diagram of Bernadette's solution to Ankur's Challenge. ..... 93
4.1 Camera view of the front board. ..... 109
4.2 Diagram of Jessica and Jamie's first organizational strategy of their towers. ..... 111
4.3 Diagram of Jessica and Jamie's eight towers organized in the staircase pattern.. ..... 111
4.4 Diagram of Jessica and Jamie's twelve towers ..... 112
4.5 Diagram of Jessica and Jamie's 20 towers. ..... 113
4.6 Diagram of Jessica and Jamie's 18 reorganized towers ..... 113
4.7 Camera view of Jessica and Jamie's 18 four-tall towers. ..... 115
4.8 Camera view of Jessica and Jamie's 16 four-tall towers. ..... 117
4.9 Camera view of Jessica and Jamie's reorganized 16 four-tall towers. ..... 118
4.10 Camera view of Jessica and Jamie's eight three-tall towers. ..... 119
4.11 Camera view of Jessica's notebook. ..... 122
4.12 Camera view of Jessica and Jamie's 16 towers during their presentation. ..... 123
4.13 Camera view of Kim and Francesca S.'s first organizational structure of their 16 towers ..... 124
4.14 Camera view of Kim and Francesca S.'s second organizational structure for their 16 towers. ..... 126
4.15 Camera view of Kim and Francesca S.'s third organizational structure for their 16 towers ..... 127
4.16 Camera view of Rebecca and Francesca C.'s presentation of towers. ..... 128
4.17 Camera view of front board. ..... 131
4.18 Camera view of Kim and Francesca S.'s 16 towers. ..... 132
4.19 Camera view of Kim's notebook. ..... 133
4.20 Diagram of Jessica and Jamie's 16 towers ..... 136
4.21 Camera view of Jessica and Jamie's four-tall towers. ..... 137
4.22 Diagram of Jessica and Jamie's cubes for their explanation of two-tall towers choosing from three colors. ..... 140
4.23 Replication of Kim's first drawing of her solution to the pizza problem. ..... 144
4.24 Replication of Francesca S.'s notebook of her solution to the pizza problem. ..... 145
4.25 Camera view of Francesca S.'s notebook of her solution to the pizza problem. ..... 145
4.26 Replication of Francesca S.'s notebook of her solution to the pizza problem ..... 147
4.27 Camera view of Francesca S.'s notebook of her solution to the pizza problem. ..... 147
4.28 Replication of Kim's second drawing of her solution to the pizza problem. ..... 148
4.29 Replication of Francesca S.'s notebook of her solution to the pizza problem. ..... 149
4.30 Replication of Francesca S.'s notebook for the pizza problem when choosing from three toppings. ..... 151
4.31 Camera view of Jessica's notebook. ..... 155
4.32 Camera view of Jessica and Jamie's towers representing the three-topping pizzas (before removing duplicates) ..... 157
4.33 Camera view of Jessica and Jamie's towers representing the three-topping pizzas (after removing duplicates). ..... 158
4.34 Camera view of Jessica and Jamie's towers representing the two-topping pizzas (before removing duplicates) ..... 159
4.35 Camera view of Jessica and Jamie's towers representing the two-topping pizzas (after removing duplicates). ..... 160
4.36 Camera view of white board. ..... 162
4.37 Replication of Jessica and Jamie's first six towers in their solution for Ankur's Challenge. ..... 175
4.38 Replication of Jessica and Jamie's first six towers in their solution for Ankur's Challenge. ..... 176
4.39 Camera view of Jessica's notebook of 12 towers found while working on Ankur's Challenge. ..... 176
4.40 Replication of Jessica and Jamie's list of 36 towers in their solution for Ankur's Challenge. ..... 178
4.41 Camera view of Kim and Francesca S.'s solution to Ankur's Challenge. ..... 181
4.42 Camera view of part of Rebecca and Francesca C.'s solution to Ankur's Challenge. ..... 182
4.43 Camera view of remaining part of Rebecca and Francesca C.'s
solution to Ankur's Challenge. ..... 183
4.44 Replication of Rebecca and Francesca C.'s solution to Ankur's Challenge ..... 183

## CHAPTER 1: STATEMENT OF THE PROBLEM

There is a shared view among mathematics educators that undergraduate mathematics education should provide alternative teaching techniques to the traditional style of lecturing (Blair, 2006; Ganter \& Barker, 2004). In particular, the use of active learning is emphasized. Active learning speaks to one of the seven transitions needed for the future of mathematics education in the United States described by the National Research Council (NRC, 1989). The NRC (1989) explains that this transition emphasizes that the learning and teaching of mathematics should shift from a body of laws to be memorized to an exploratory field where the mathematical processes of exploring and formulating conjectures is highlighted.

At Rutgers University, there is an extensive body of research involving classroom practices that embody this transformation suggested by the National Research Council. ${ }^{1}$ In the book, Combinatorics and Reasoning, Maher, Powell, and Uptegrove (2010) discuss the strand of research that focuses on the area of combinatorics. Maher et al. found that "in a program of carefully selected tasks, with minimal intervention by educators who pay careful attention to students' arguments and justifications, students can perform mathematically at high levels" (p. xvi). The combinatoric tasks that were chosen for the study give rise to the mathematical processes of exploration and justification. Maher et al. found that the students "began their investigations by searching for patterns, organizing solutions, searching for completeness, deriving strategies for

[^0]keeping track and checking, and then reorganizing justifications into arguments that were proof-like in structure" (p. 6).

The research team at Rutgers University has focused on many aspects of the mathematical process including using heuristics and applying personal representations to developing mathematical ideas and forms of reasoning. The research done at Rutgers University has mainly focused on students in grade two through high school. Minimal research on these specific tasks has been conducted at the college level. Glass (2001, 2010) studied college freshman working on these tasks. The current research project will add to the understanding of how undergraduate college students build and justify their solutions on specific combinatoric tasks.

There are four purposes of this research which focuses on developing mathematical ideas and forms of reasoning with undergraduate students enrolled in a mathematics course. These six undergraduate students are mathematics majors in their junior year of college studying to be teachers. The first purpose of this research is to understand how these students build their solutions to the tasks used for elementary and secondary students in the earlier studies. Second, what forms of reasoning do the college students use in justifying the solutions of these tasks?

In understanding how students build their solutions, it is important to consider the interventions of the instructor. A third purpose of the study is to analyze the instructor interventions in the problem-solving explorations of the six participating students. Fourth,
in addition to analyzing how the college students built their solutions, their approaches will be compared with the approaches of students from earlier research.

The questions that guide this study are:

1. How do pre-service teachers in an undergraduate math class build their solutions to the problems they investigate?
2. How, if at all, do they justify their solutions?
3. What role does the instructor play in the students' building and justifying of ideas? What types of interventions, if any, does she employ?
4. How do the solutions and justifications of these college students compare with the solutions of other students at various ages doing the same problems?

There is a need to understand how undergraduate students solve problems in an environment that encourages exploration and justification if an active learning style is to be incorporated in undergraduate education. Understanding the types of interventions the instructor used in the building of these ideas will also benefit future classrooms that will participate in an exploratory learning experience. Furthermore, understanding how learners develop mathematical ideas can benefit the teaching of mathematics. Careful and detailed analysis of the process in which learners build their mathematical ideas on specific problems can bring us closer to understanding the process. Different mathematical problems provoke different ways of thinking. If we can analyze learners in
different environments while keeping the task constant, perhaps we can better understand how students do mathematics on specific tasks. The more evidence we have on students working on the same tasks, the better we can understand the mathematical processes.

## CHAPTER 2: LITERATURE REVIEW AND THEORITICAL FRAMEWORK

### 2.1 Introduction

This chapter is organized into two sections. The first section explains the theoretical framework that guides this study. The second section contains the literature review. The literature review begins with an explanation of the importance of reasoning and justification in the school curriculum, how an active learning style suits the undergraduate level, and how problems in combinatorics fit into this scheme. The literature review continues with the three combinatoric tasks explored in this study and reviews the research on mathematical problem solving relevant to these three tasks.

### 2.2 Theoretical Framework

## Introduction

Under certain conditions the learning of mathematics can take place. These conditions are based on a setting where students are given an opportunity to explore and justify mathematical ideas in an environment where the communication of ideas is encouraged. These conditions also require appropriate mathematical tasks and an instructor who can guide the exploration and justification processes.

## Framework

Mathematicians solve problems through a process that involves exploration and justification (Fendel \& Resek, 1990). The exploration process, which might involve
pattern finding, making guesses, or looking at examples, is about the discovery of new ideas. Once a conjecture is made, the mathematician seeks to justify the solution.

Mathematics instruction should mimic the way mathematics is achieved and mathematical thinking occurs (Freudenthal, 1991; Pólya, 1945, 1954; Schoenfeld, 1992). "If the learning of mathematics has anything to do with the discovery of mathematics, the student must be given some opportunity to do problems in which he first guesses and then proves some mathematical fact on an appropriate level" (Pólya, 1954, p. 160). Schoenfeld (1992) argues that when students learn mathematics as a series of algorithms using drill-and-practice techniques, "they are not developing the broad set of skills Pólya and other mathematicians who cherish mathematical thinking have in mind" (pp. 56-57).

Students should first be given opportunities to explore mathematics and create ideas. Freudenthal (1991) calls this process "reinvention" and explains that "knowledge and ability, when acquired by one's own activity, stick better and are more readily available than when imposed by others" (p. 47). During this process of mathematical discovery, students build their own representations and understanding of the problem. Davis and Maher (1990) describe a series of steps that occurs in one's mind when encountered with a mathematical problem.

1. Build a representation for the input data.
2. From this data representation, carry out memory searches to retrieve or construct a representation of (hopefully) relevant knowledge that can be used in solving the problem or otherwise going further with the task.
3. Construct a mapping between the data representation and the knowledge representation.
4. Check this mapping (and these constructions) to see if they seem to be correct.
5. When the constructions and the mapping appear to be satisfactory, use technical devices (or other information) associated with the knowledge representation in order to solve the problem. (p. 65)

This cycle is based on the idea that new mathematical representations are created based on revising and extending previously built mathematical representations. This type of learning is grounded on a constructivist perspective of learning. As explained by O'Donnell and Hmelo-Silver (2013), "a constructivist perspective suggests that individuals create meaning using their prior understandings to make sense of new experience and construct new understandings" (p. 6). The idea that new knowledge is built from previous knowledge might seem simplistic; however, it becomes profound when we permit this type of learning in the classroom. As Davis and Maher (1997) explain, "it is the student who is doing the work of building or revising these personal representations" (p. 94).
"The term representation refers both to process and to product - in other words, to the act of capturing a mathematical concept or relationship in some form and to the form itself" (NCTM, 2000, p. 67). Representations are a vital part of mathematics because mathematics is about abstraction and generalization and it is these representations that symbolize mathematical concepts. Understanding the meaning of an abstract representation is much more valuable than focusing on the actual representation. Davis (1992) explains that by allowing students to invent representations and to create a personal representation of the task (as opposed to telling students what to do) the focus shifts to the meaning of these representations.

This process of exploration, also referred to as discovery-based learning or active learning, has many benefits. First, it allows the student to take ownership of their ideas. Francisco and Maher (2005) found that the ownership of mathematical ideas was central in students' success at problem solving. This act of construction builds the students' "mathematical power."

Students construct meaning as they learn mathematics. They use what they are taught to modify their prior beliefs and behavior, not simply to record and store what they are told. It is students' acts of construction and invention that build their mathematical power and enable them to solve problems they have never seen before. (NRC, 1989, p. 59)

This statement implies that greater transfer occurs when students construct their own mathematical understanding of the problem because they retain the mathematics best when they learn by internal construction (NRC, 1989, p. 59). Freudenthal (1991) also expressed this idea of greater transfer because mathematics, when constructed, "sticks better in one's head" (p. 47). However, it may not only be the product of the act of construction that enables greater transfer but the act itself that aids in transfer.

When students seek understanding, they must recognize if they grasp a concept and when they need more information (Bransford, Brown, \& Cocking, 2000). That is, they must engage in the metacognitive processes of "monitoring and control" and "selfregulation" (Schoenfeld, 1992). "Transfer can be improved by helping students become more aware of themselves as learners who actively monitor their learning strategies and
resources and assess their readiness for particular test and performances." (Bransford et al., 2000, p. 67)

However, reinvention of mathematical ideas is not enough; students must also learn to justify their solutions. Mathematicians justify their solutions not only to demonstrate the answer is true, but also to understand why it is true. The process of justifying a mathematical conjecture allows the mathematician to "make sense" of their find. The process of justifying is truly about understanding. Reasoning is central in this understanding (Ball \& Bass, 2003). Maher (2005) found that, when the responsibility of making sense of the solution was placed on the learner, this "led to careful reasoning and building of arguments" (p. 12).

By justifying their solutions, students monitor their learning and are forced to reexamine their solutions. Teaching practices that help students monitor their learning focus on sense-making, reflection and self-assessment (Bransford et al., 2000, p, 12). As shown, learning to monitor one's learning leads to greater transfer. This view is shared by Pólya. In order to justify the solution, students will have to reconsider their solution. Pólya (1945) explains that by reexamining and reconsidering the solutions, students can "consolidate their knowledge and develop their ability to solve problems" (pp. 14-15). He refers to this process as "looking back."

The two processes of mathematical exploration and justification are important in the learning and understanding of mathematics for many reasons. However, in the learning and teaching of mathematics, in order for students to engage in and learn from these
processes, problem solving should also involve 1) collaboration with peers, 2) proper teacher intervention, and 3) appropriate mathematical tasks.

Schoenfeld (1992) explains that mathematicians often discuss their ideas with their peers and "doing mathematics is increasingly coming to be seen as a social and collaborative act" (p. 29). Maher et al. (2010) explain that "a central component of the learning process is encouraging students to communicate their ideas" (p.3). By discussing their solutions with peers, it is possible that "cognitive conflicts will arise, inadequate reasoning will be exposed, and enriched understanding will emerge" (Springer, Stanne, \& Donovan, 1999, p. 25).

Furthermore, when learners are working with peers, the process of "looking back" might be enhanced. If students are encouraged to justify their solutions to the teacher and one another, they will have to go through a process of formulating and presenting an explanation of their solutions. Webb (2013) explains how by formulating an explanation, students will have to reorganize, transform, and clarify their explanation so that others can understand. The process of presenting these ideas may elicit many of the same processes as formulating the explanation, "especially when the presentation exposes contradictions or incompleteness of ideas that are recognized by the explainer or are pointed out by others" (p. 20). Webb further explains that listening is also an important part of this process.

Listeners may engage in processes analogous to those carried out by presenters. When comparing their own knowledge with what is being presented, listeners may recognize and fill in gaps in their own knowledge, recognize and correct
misconceptions, see contradictions that cause them to seek new information (e.g. by asking question), and generate new connections between their own ideas, or between their own and others' ideas. (p. 20)

Therefore, collaboration with peers can enhance mathematical understanding because, when learners share ideas, the need for clarification and reconsideration of the solution becomes important. However, the processes of exploration and justification cannot occur unless the students are given appropriate mathematical tasks. The tasks should be openended and allow for abstraction and generalization (Francisco \& Maher, 2005; Martino \& Maher, 1999; Maher et al., 2010). Francisco and Maher (2005) describe the value of supplying complex tasks as opposed introducing simple problems that make up a more complex problem. Francisco and Maher explain that "the opportunity to attend to the intricacies of a complex task provides the students with the opportunity to work on unveiling complex mathematical relationships, which enhances deep mathematical understanding" (p.371). However, the tasks must be appropriate for the students' knowledge base and the teacher will need to understand what constitutes challenging for the teacher's own students (Martino and Maher, 1999).

Careful consideration of the level of the mathematical task is important. As explained by Martino and Maher (1999), the teacher will have to determine the appropriate level that is considered challenging for their specific students. Vygotsky (1978) named this level the zone of proximal development. The zone of proximal development is "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving
under adult guidance or in collaboration with more capable peers" (Vygotsky, 1978, p. 86). "According to Vygotsky, the zone of proximal development is a level of competence on a task in which the student cannot yet master the task on his or her own but can perform the task with appropriate guidance and support from a more capable partner" (O'Donnell \& Hmelo-Silver, 2013, p. 8). According to this theory, the level of the task is important but only if there is appropriate guidance. Therefore, the role of the teacher is also very important.

The role of the teacher is critical in a classroom environment that fosters exploration, reinvention, and justification. In this environment, the teacher's role should be one of a listener and guide as opposed to a lecturer. The teacher must be able to provide timely, open-ended questions that promote conceptual understanding and problem-solving skills (Martino \& Maher, 1999). The knowledge of what and when to ask these questions will rely on acute listening, strong content knowledge, and knowledge of students' prior understanding.

Both Pólya (1945) and Freudenthal (1991) express how there is a fine line between helping too much and not at all. As Freudenthal explains, "guiding means striking a delicate balance between the force of teaching and the freedom of learning" (p. 55). Martino and Maher (1999) stress that the students must have time to explore the problem without any teacher intervention. The teacher should intervene only "after students have built a solution, consulted with each other and posed a solution that they believe is valid"
(p. 56). It is at this point that the students are ready to be challenged to explain their reasoning and justify their solutions.

Martino and Maher (1999) proposed four types of teacher questioning that will aid in fostering understanding. They suggest questioning that 1) facilitates justification, 2) offers opportunities for generalization, 3) invites opportunities to make connections, and 4) facilitates awareness of solutions presented by other students. Examples of the types of questions include: "Can you explain your solution to me?" "Can you convince the rest of us that your method works?" "Have you ever worked on a problem like this before?" "Is there anything about your solution that's the same as your classmate's?"

Martino and Maher further explain that the teacher must not only be knowledgeable about the content domain but must also have knowledge about the students. "Many times the teacher will have to make instructional decisions based upon students' ideas and actions that presented themselves during a prior lesson" (p. 54). As they explain, understanding the knowledge structure that might be available to the students is imperative to be able to encourage further thinking. This view is shared by Pólya for knowing how and when to guide. "The best is, however, to help the student naturally. The teacher should put himself in the student's place, he should see the student's case, he should try to understand what is going on in the student's mind, and ask a question or indicate a step that could have occurred to the student himself." (Pólya, 1954, p. 1)

To summarize, this study is situated in a theoretical perspective consistent with the one presented by Maher et al. (2010). That is, "students learn mathematics by engaging in
the process of building their own personal representations, communicating them as ideas, and then providing support for those ideas by reorganizing and restructuring representations" (p. 4). It is the combination of appropriate tasks, the process of explaining and justifying the solutions, and teacher questioning that promotes meaningful mathematical learning.

### 2.3 Literature Review

### 2.3.1 Reasoning and Proof

Reasoning and proof is one of the five process standards for all grade levels, prekindergarten through grade 12, set forth by the National Council of Teachers of Mathematics [NCTM]. As it is explained by NCTM, reasoning is an essential part of mathematics and should be a regular part of a student's mathematics education throughout all grade levels. "Being able to reason is essential to understanding mathematics. By developing ideas, exploring phenomena, justifying results, and using mathematical conjectures in all content areas and-with different expectations of sophistication-at all grade levels, students should see and expect that mathematics makes sense" (NCTM, 2000, p. 56).

The process described by NCTM of exploring and justification is akin to the description by Fendel and Resek (1990) on how mathematicians work. In the textbook titled Foundations of Higher Mathematics: Exploration and Proof, Fendel and Resek explain that mathematics entails exploration and proof. "In brief, exploration involves
examining a situation, with or without a particular question in mind, and discovering whatever you can about it. It involves 'messing around' with mathematical ideas - trying one thing and then another, looking at examples, making guesses, asking questions" (Fendel \& Resek, 1990, p. 3).

The process of exploration is about constructing new ideas. Ball and Bass (2003) explain that reasoning is a central instrument in this process. Instead of the phrase "exploration," they call this step the "reasoning of inquiry." Out of this process, conjectures are made and the mathematician will want to prove these conjectures. "Mathematical reasoning also functions centrally in justifying and proving mathematical claims, a process that we call the reasoning of justification" (Ball \& Bass, 2003, p. 30). As Fendel and Resek (1990) explain, exploration and proof are not separate identities, they are mutually supportive. The process of proof grows out of the process of exploration. According to Ball and Bass, reasoning is the central aspect of these processes.

The ultimate result of argumentation for a mathematician is a formal mathematical proof (Yackel \& Hanna, 2003, p. 228). "All stages of doing mathematics are concerned with acquiring understanding, and the separations between the stages are not always sharply defined. But the hallmark of the proof stage is that it is primarily concerned with acquiring certainty" (Fendel \& Resek, 1990, p. 19). A formal mathematical proof is series of steps and logic demonstrating certainty about the mathematical discovery using
specialized communication agreed upon by the mathematical community (Fendel \& Resek, 1990, p. 4).

Yackel and Hanna (2003) explain that "a good proof is one that also helps one understand the meaning of what is being proved: to see not only that it is true but also why it is true" (p.228). Mathematicians are concerned with understanding and to be able to understand a mathematical statement, one must understand why it is true. Ball and Bass (2003) state that "mathematical understanding is meaningless without a serious emphasis on reasoning" (p.28). As they explain, memorizing a series of steps in a procedure without understanding the reasons for these steps is analogous to reading a text without comprehension.

Therefore, mathematical instruction should emphasize reasoning. That is, it should highlight a need for understanding why a mathematical statement is true.

From children's earliest experiences with mathematics, it is important to help them understand that assertions should always have reasons. Questions such as "Why do you think it is true?" and "Does anyone think the answer is different, and why do you think so?" help students see that statements need to be supported or refuted by evidence. (NCTM, 2000, p. 56)

In summary, the central goal of mathematics is understanding. Students need to understand that we just don't do mathematics, we are concerned with why the mathematics we are doing is true. In order for students to learn to mathematically reason, teachers will need to "develop and learn practices to support such learning" (Ball \& Bass, 2003, p. 43). They will need to be equipped with mathematical tasks that promote mathematical reasoning (Ball \& Bass, 2003, p. 43). They will need to learn what it means
to reason mathematically and to be able to recognize mathematical reasoning when it occurs. "A challenge for mathematics educators is to design means to support teachers in developing forms of classroom mathematics practice that foster mathematics as reasoning and that can be carried out successfully on a large scale" (Yackel \& Hanna, 2003, p. 234).

### 2.3.2 Undergraduate Mathematics

In the publication, Everybody Counts: A report to the nation on the future of mathematics education (National Research Council [NRC], 1989), the National Research Council explains that improvement of mathematics education is dependent on the renewal of undergraduate mathematics education because most future teachers of mathematics are educated in our colleges and universities.

Undergraduate mathematics is the linchpin for revitalization of mathematics education. Not only do all the sciences depend on strong undergraduate mathematics, but also all students who prepare to teach mathematics acquire attitudes about mathematics, styles of teaching, and knowledge of content from their undergraduate experience. No reform of mathematics education is possible unless it begins with revitalization of undergraduate mathematics in both curriculum and teaching style. (p. 39)

There is a shared view among mathematics educators that undergraduate mathematics education should provide alternative teaching techniques to the traditional style of lecturing (Blair, 2006; Ganter \& Barker, 2004). In particular, the use of active learning is emphasized. As Blair (2006) explains, "active learning occurs in many formats such as collaborative learning, discovery-based learning, interactive lecturing and question posing, and writing. Whichever format is chosen, the goal of the activity should be to enhance conceptual understanding" (p. 54).

One of the formats of active learning, discovery-based learning, speaks to one of the seven transitions needed for the future of mathematics education in the United States described by the National Research Council (NRC). This transition emphasizes a need for exploration in mathematics. Discovery-based learning, described by Blair (2006), involves exploration by engaging students in the process of discovering concepts and patterns. The seventh transition, recommended by the NRC, emphasizes that the learning and teaching of mathematics should shift from a body of laws to be memorized to an exploratory field. The NRC states that teaching and learning of mathematics should focus on:

- Seeking solutions, not just memorizing procedures;
- Exploring patterns, not just learning formulas;
- Formulating conjectures, not just doing exercises.

As teaching begins to reflect these emphases, students will have opportunities to study mathematics as an exploratory, dynamic, evolving discipline rather than as a rigid, absolute, closed body of laws to be memorized. They will be encouraged to see mathematics as a science, not as a canon, and to recognize that mathematics is really about patterns and not merely about numbers. (NRC, 1989, p. 84)

This process of exploring patterns and formulating conjectures was described earlier by Ball and Bass (2003). They explained that this process requires mathematical reasoning and that future educators will need to be equipped with the knowledge and the mathematical tasks that will support and promote mathematical reasoning. If future mathematics classrooms are to support this type of learning, the future educators will need to be exposed to this type of learning. Since teachers are inclined to teach the way they were taught, it is important that undergraduate mathematics instruction for
prospective teachers demonstrates techniques to be used in their future classrooms (Blair, 2006; Senk, Keller, \& Ferrini-Mundy, 2004).

Mathematics classrooms, at all grade levels, should incorporate styles of instruction that emphasize the exploratory and justification aspect of the mathematical process. Most important is the education of our pre-service teachers at the undergraduate level because they are our future educators at the K-12 level. If our future educators are to emulate this type of instruction in the classroom, they will need to be exposed to it in their own education. It is essential that the mathematics courses taken by pre-service teachers develop "understanding of both mathematical content and mathematical processes such as defining, conjecturing and proving" (Senk, Keller, \& Ferrini-Mundy, 2004, p. 148). The improvement of mathematics education relies very heavily on the improvement in undergraduate education for future teachers. "Undergraduate mathematics is the bridge between research and schools and holds the power of reform in mathematics education" (NRC, 1989, p. 41).

### 2.3.3 Discrete Mathematics: Combinatorics

Principles and Standards for School Mathematics (National Council of Teachers of Mathematics [NCTM], 2000) recommends that discrete mathematics should be an important part of school mathematics and should be incorporated at all grade levels. "As an active branch of contemporary mathematics that is widely used in business and industry, discrete mathematics should be an integral part of the school mathematics curriculum, and these topics naturally occur throughout the other strands of mathematics"
(NCTM, 2000, p. 31). The three main areas of discrete mathematics recommended by NCTM are combinatorics, iteration and recursion, and vertex-edge graphs.

Discrete mathematics basically involves working with objects that are countable. That is, the objects in the set can be enumerated by the set of natural numbers. Discrete mathematics contrasts with continuous mathematics which involves continuous quantities. One of the branches of discrete mathematics is combinatorics. Simply put, combinatorics is concerned with counting objects of a set. "Combinatorics is the mathematics of systematic listing and counting. It facilitates solving problems such as determining the number of different orders for picking up three friends or counting the number of different computer passwords that are possible with five letters and two numbers" (Hart, Kenney, DeBellis, \& Rosenstein, 2008, p. 2).

Discrete mathematics is the basis of many other branches of mathematics including probability, statistics, and computer science. These topics were listed under the fifth transition for the future of mathematics education suggested by the NRC. This transition explains the need for a greater emphasis on "topics that are relevant to students' present and future needs" (NRC, 1989, p. 83).

Hart et al. (2008) explain the importance of discrete mathematics for the future of our children and the future of our nation. As they explain, since discrete mathematics is closely tied to technology, it is "particularly relevant in today's digital information age" (p. 4). Furthermore, Hart et al. explain how problems in discrete mathematics are "pedagogically powerful" because they not only include important mathematical content
but they also can be used to teach mathematical processes. They argue: "In working with discrete mathematics, students strengthen their skills in reasoning, proof, problem solving, communication, connections, and representation in many ways" (p. 5).

Freudenthal (1991) also explains how problems in discrete mathematics, specifically combinatorics, give rise to the need for conjecturing and creating convincing proofs, especially proofs in mathematical induction. Furthermore, Freudenthal explains the value of combinatorics in the process of discovery, which he calls "reinvention." Freudenthal explains:

Starting with numerical paradigms, guessing general relations, experiencing and satisfying needs for good definitions and convincing proofs, encountering mathematical induction thanks to these efforts, and using mathematical induction, first instinctively, then intentionally, and eventually in a more or less formally verbalised manner - all this together appears to be a most efficient course in reinvention. (p. 53)

According to Freduenthal (1991), reinvention involves discovery and organization and, in the context of the learning environment, stresses "guided reinvention." As explained, the student will discover something new to him but known to the guide in the process of guided reinvention. "Guiding means striking a delicate balance between the force of teaching and the freedom of learning" (p.55). Freduenthal explains the benefits of guided reinvention as an educational practice.

Learners should be allowed to find their own levels and explore the paths leading there with as much and as little guidance as each particular case requires. There are sound pedagogical arguments in favour of this policy. First knowledge and ability, when acquired by one's own activity, stick better and are more readily available than when imposed by others. Second discovery can be enjoyable and so learning by
reinvention may be motivating. Third it fosters the attitude of experiencing mathematics as a human activity. (p.47)

It has been shown that the educational value of solving problems in discrete mathematics, specifically combinatorics, is two-fold. Firstly, combinatorics is pertinent to mathematics that affect our professional and everyday lives especially in the technological society in which we live. Secondly and maybe most importantly, is the "pedagogically powerful" aspect of problems in combinatorics. If the future of mathematics education classrooms are ones in which the mathematical process of exploration and justification are to be nurtured, tasks in discrete mathematics will be very beneficial.

### 2.3.4 Rutgers University - The Longitudinal Study

An extensive body of research conducted at Rutgers University demonstrates how the use of well-chosen combinatoric tasks can engage students in the mathematical processes of exploration and justification. The problems were presented in a classroom community where students were encouraged to share ideas, there was minimal teacher intervention, and they were expected to justify their solutions. The researchers found that the students "created models, invented notation, and justified, reorganized, and extended previous ideas and understandings to address new challenges. That is, they performed mathematics: created mathematical ideas and reasoned mathematically" (Maher, Powell, \& Uptegrove, 2010, p. 203). Part of this body of research will be discussed in detail in sections to follow.

The research at Rutgers, informally called "the longitudinal study", began in 1987 in the blue-collar community of Kenilworth, New Jersey. The researchers were interested in "what mathematical concepts students could learn with minimal intervention from teachers" (Maher et al., 2010, p. 6). The interventions were videotaped and along with students' written work and researchers notes, the sessions were analyzed. The students in the study were exposed to different topics in mathematics but the major strand of tasks was grounded in the discipline of combinatorics. The researchers choose problems in combinatorics because "in working on these problems, students can find the need to organize their work systematically, look for patterns, and generalize their findings; also counting problems were at the time outside the regular elementary school curriculum and therefore unfamiliar to the students" (Maher et al., 2010, p. 11).

The longitudinal study contains research about students in grades one through high school. Some of the same students are followed from grade one though high school and beyond. Even at an early age, the students "began their investigations by searching for patterns, organizing solutions, searching for completeness, deriving strategies for keeping track and checking, and then reorganizing justifications into arguments that were prooflike in structure" (Maher et al., 2010, p. 6). In middle school, the researchers found that the students more clearly defined these forms of reasoning. By middle and high school, they could explain the underlying mathematical structures and make connections to mathematical concepts including the binomial expansion and Pascal's triangle.

The longitudinal study involved students from the Harding School in Kenilworth, New Jersey. However, the interventions were also conducted at two elementary schools in the suburban community of Colts Neck, New Jersey and the urban community of New Brunswick, New Jersey. Glass (2001) replicated some of the combinatorics tasks with a group of community college students. These students were enrolled in a liberal arts mathematics course. The data was collected, using videotapes and students' written work, over a two and a half year period starting in the spring semester of 1998 and concluding with the spring semester of 2000 .

The research at Rutgers University, along with the study by Glass (2001), has implications for teaching because it not only provides a detailed analysis on how mathematical ideas are developed and justified but it provides research of effective tasks that offer opportunities for students - young children through young adults - to explore patterns, formulate conjectures and justify their solutions.

### 2.3.5 The Combinatoric Problems

Three of the combinatorics problems encountered in the longitudinal study are found below. The research on these three problems will be discussed in detail in the next section. For each of these three tasks, common patterns, justifications, and organizational strategies were identified from the solutions of the students. A glossary of these schemes is included here as a reference. The three combinatorics problems are: (1) the four-tall towers problem, (2) the four-topping pizza problem, and (3) Ankur's challenge. The problems and their solutions are listed below as written in Combinatorics and Reasoning:

Representing, Justifying and Building Isomorphisms (Maher, Powell, \& Uptegrove, 2010).

## Four-Tall Towers

Your group has two colors of Unifix cubes. Work together and make as many different towers four cubes tall as is possible when selecting from two colors. See if you and your partner can plan a good way to find all the towers four cubes tall.

At each position in the tower, there are two color choices. Therefore, there are $2 \times 2 \times 2 \times 2=16$ possible towers that are four cubes tall. This can be generalized to an $n$-tall tower with two colors to choose from; there are $2 \times 2 \times 2 \ldots \times 2=2^{n}$ possible towers that are $n$ cubes tall, when there are two colors to choose from. This can also be generalized to an $n$-tall tower with $m$ colors to choose from; there are $m \times m \times m$. . $\times m=m^{n}$ possible towers that are $n$ cubes tall with $m$ colors to choose from. (Maher, Powell, \& Uptegrove, 2010, p. 207)

## The Four-Topping Pizza Problem

A local pizza shop has asked us to help design a form to keep track of certain pizza choices. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many different choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possible choices.

There are $2 \times 2 \times 2 \times 2=16$ possible pizzas. (Maher, Powell, \& Uptegrove, 2010, pp. 210-211)

## Ankur's Challenge

Find all possible towers that are four cubes tall, selecting from cubes available in three different colors, so that the resulting towers contain at least one of each color. Convince us that you have found them all.

Suppose the colors are red, blue, and green. We are counting the towers in three cases: (1) those with two red cubes, one blue cube and one green cube, (2) those with one red cube, two blue cubes, and one green cube, and (3) those with one red cube, one blue cube, and one green cube. The following equation gives the number of ways of selecting $m$ groups of objects of size $r_{1}$ through $r_{m}$ :

$$
\binom{n}{r_{1}, r_{2}, \ldots, r_{m}}=\frac{n!}{r_{1}!r_{2}!\ldots r_{m}!}, \text { where } \sum r_{i}=n
$$

So the number of four-tall towers containing exactly two red cubes, one blue cube, and two green cubes is:

$$
\binom{4}{211}=\frac{4!}{2!1!1!}=12
$$

Similarly, for the other two cases:

$$
\binom{4}{121}=\binom{4}{112}=12
$$

Hence, the number of towers with the required condition is $12+12+12=36$. (Maher, Powell, \& Uptegrove, 2010, pp. 212-213)

## Unifix Cubes

Unifix cubes are plastic cubes that come in a variety of colors. They have a top and a bottom and "lock" into each other to form towers. A four-tall tower consists of four cubes "locked" together. Since a cube has a vertical orientation, so does a tower. Therefore, the towers problem requires the student to produce all of the combinations of towers that can be made when selecting from cubes of different colors. This problem can be modified to make any height of a tower and selecting from any number of colors.


Figure 2.1. Example of a four-tall tower.

### 2.3.6 Glossary of Terms

When solving the towers problem, students often used patterns. Four major patterns are found throughout the literature and have been referenced as: opposites, cousins, staircase, and the elevator pattern.

## Opposites

This method is occasionally referred to as "pair-wise opposites." If you have a tower, then its opposite tower would have the opposite color cube from the original tower in each position. For example, if one tower is blue, red, blue, blue. Then the opposite tower would be red, blue, red, red (Maher \& Martino, 1996a; Maher \& Martino, 1996b; Martino, 1992).


[^1]
## Cousins

Cousins are towers that, when one is flipped, they are identical. For example, a pair of cousins would be red, red, red, blue and blue, red, red, red (Maher \& Martino, 1996a; Maher \& Martino, 1996b; Martino, 1992).


Figure 2.3. Example of a pair of cousins.

## Elevator

This technique is used for the towers that contain one cube that is a different color from all of the other cubes in that tower. To create different towers, the cube is systematically moved from position one to position $n$. The resulting towers, when placed next to each other, resemble an elevator. For example, in the third grade, a student in the Kenilworth study named Stephanie used this technique when solving the four-tall tower problem. She created four towers systematically. The towers contained three red cubes and one blue cube. She created the towers by moving the blue cube from position one to position two, to three, and four (Maher \& Martino, 1996a, 1996b; Martino, 1992).


Figure 2.4. Example of towers forming the elevator pattern.

## Staircase

This pattern describes a group of four towers. When using red and blue as the cube colors, the first tower would have a red on the bottom followed by three blues, the second tower would have two red cubes on the bottom followed by two blues, the third tower would have three reds on the bottom followed by one blue, and the last tower would contain all red cubes. When placed next to each other, the red cubes would form a staircase. For example, in the fourth grade, Stephanie used this technique when solving the five-tall tower problem (Maher, Sran \& Yankelewitz, 2010). See Figure 2.5.


Figure 2.5. Example of towers forming the staircase pattern.

## Controlling for variables

This phrase means to hold one variable constant while adjusting another variable.
Stephanie in grade four used this strategy when building towers that contain two cubes of the same color. She held the color of one of the tower positions constant while adjusting a second cube of the same color in the remaining tower positions (Maher \& Martino, 1996a).

## Tree diagram

A tree diagram is a systematic way to list all elements of a set. According to Tarlow (2010), an eleventh grade student named Shelly used this technique to solve the pizza problem. Shelly labeled the first node on the tree as plain. She then labeled the first four branches that stem from this node as one of the four pizza toppings. From each of these toppings, another branch extends listing another topping. However, she was careful not to repeat toppings. That is, from her first branch, that is labeled peppers, she had three
branches each for mushroom, sausage, and pepperoni. But, in the next branch which is labeled sausage, she only had mushroom and pepperoni because the combination of sausage and peppers was contained in the previous branch. She continued in this fashion to create 16 pizzas.

## Case Argument

A proof by cases is used in mathematics when it is easier to prove the statement by proving all of the smaller cases that make up the whole. For example, a justification could be made when showing that there are a total of eight towers three-tall when choosing from two colors by grouping the towers into cases based on a certain attribute. Then, in a complete argument by cases, each case would be proven to be true. Much of the research has shown that most of the students organized their cases by number of cubes of a certain color. For example, if the towers were three-tall choosing from two colors of cubes, the students would have four cases. They would organize their four cases as (1) towers with no cubes of that color, (2) towers with one cube of the particular color, (3) towers with two cubes of that color, and (4) towers with three cubes of the particular color. This particular organization strategy was the most abundant among the students but it was not the only choice. Stephanie, in grade four, provided a different approach to cases for the three-tall towers problem as "five individual cases (towers with no blue cubes, one blue cube, two blue cubes stuck together. Three blue cubes, and finally, two blue cubes separated by a red cube)" (Maher \& Martino, 1996b, p. 437).

## Inductive Argument

A mathematical proof by induction involves showing that a statement $p(n)$ is true for all whole number values of $n$, or for all values of $n$ greater than a given number. It involves the following steps: (1) show the statement is true for the first case (this is usually, $n=0, n=1$, or some other small value of $n$ ) and then (2) assume the statement is true for $n$ and prove that it is true for $n+1$. A proof of the formula $2^{n}$ as a solution of the $n$-tall tower problem selecting from two colors using mathematical induction is as follows:

Step One - show the formula is true for $n=1$. That is, show that there are two towers when the height is one cube. Since, there are only two colors to choose from, say red or blue, there are only two towers of one cube high. Therefore, the formula is true for $n=1$

Step Two - Assume true for $n$, prove true for $n+1$.

Proof: Assume true for $n$. That is, when you are choosing from two colors, there are $2^{n}$ different towers that are $n$-tall. To create all of the towers that are $n+1$ tall, you can take all of the existing $2^{n}$ different towers and add a cube to each one. Now, you have two choices for this cube. So each of the existing $2^{n}$ different towers can make two more towers. So the number of towers is two times $2^{n}$.

$$
\begin{aligned}
& 2^{n} * 2 \\
& 2^{n} * 2^{1} \\
& 2^{n+1}
\end{aligned}
$$

By induction, it has been proven that for towers $n$-tall, there are $2^{n}$ towers.

The inductive arguments that the students made are not as sophisticated as a mathematical proof by induction. However, there are some informal and basic ideas based on this type of proof imbedded in their arguments.

In grade four, Milin (a classmate of Stephanie) did his proof by induction with the actual cubes. He started with towers one cube high and built four towers two-high by adding a blue cube on one and a black on the other. He continued to do this for three-tall towers. When he is questioned about how many four-tall towers, he replied it would be 16 - two for each of the eight he had already made. And when asked about five-tall towers, he replied 32 (Alston \& Maher, 1993).

Milan used inductive reasoning to deduce that the number of towers doubles each time the height of the tower increases by one. He first demonstrated that his conjecture was true for when the towers are one cube tall. (This is the first step in a mathematical proof by induction.) He then said that the number of towers doubles when you go from 1tall to 2-tall because each 1-tall tower can be used to generate two 2-tall towers - because you can place either a blue cube or a black cube on the top of each tower. He also explains that this is also true when you go from 2-tall to 3-tall, etc. (This is the second step in a proof by induction.) Although he does not use a generalized formula, he demonstrated that the $(n+1)$-tall towers can be built from the towers $n$-tall.

## Isomorphism

Mathematicians use the term isomorphism, which translates to "same form," for mathematical systems that are "essentially the same" (Fendel \& Resek, 1990). The underlying structures of the solution to the four-topping pizza problem and the four-tall towers problem are isomorphic. The solution to both problems is two to the fourth power. In the pizza problem, the four represents the number of toppings. In the towers problem, the four is the height of the tower. The base of two in the solution represents the choice of colors in the towers problem. In the pizza problem, the two represents the inclusion or exclusion of the topping. Underneath, these two problems have the same mathematical structure. That is, they are isomorphic.

## Pascal's Triangle

The following triangular array of numbers is known as Pascal's triangle. The first and the last number in each row are ones. Starting with row two, the remaining numbers in the row can be found by adding the pair of numbers directly above. Also, the sum of each row equals a power of two and each number represents a combination. In general, if $n$ is equal to the row number and $r$ is the numbered entry in the $n$th row, then this entry is equal to ${ }_{n} C_{r}$ (the number of combinations of $n$ things taken $r$ at a time).

Table 2.1
First Seven Rows of Pascal's Triangle

|  | Pascal's Triangle | Row Sum | Row Sum expressed as a power of two |
| :---: | :---: | :---: | :---: |
| Row 0 | 1 | 1 | $2^{0}$ |
| Row 1 | 11 | 2 | $2^{1}$ |
| Row 2 | 121 | 4 | $2^{2}$ |
| Row 3 | $\begin{array}{llll}1 & 3 & 3 & 1\end{array}$ | 8 | $2^{3}$ |
| Row 4 | $\begin{array}{llllll}14 & 6 & 4 & 1\end{array}$ | 16 | $2^{4}$ |
| Row 5 | $\begin{array}{lllllll}1 & 5 & 10 & 10 & 5 & 1\end{array}$ | 32 | $2^{5}$ |
| Row 6 | $\begin{array}{lllllll}1 & 6 & 15 & 20 & 15 & 6 & 1\end{array}$ | 64 | $2^{6}$ |
| Row 7 | $\begin{array}{llllllll}1 & 7 & 21 & 35 & 35 & 21 & 7 & 1\end{array}$ | 128 | $2^{7}$ |

## Pascal's Identity

Pascal's identity, also known as the addition rule in Pascal's triangle, states that the $r$ th element in the $n+1$ row can be found by adding the two elements above it. That is, the $r$ th element in the $n+1$ row can be found by adding the $r$ th and the $(r-1)$ th element in the $n$th row. Mathematically, this is written as:

$$
\binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r} \quad \text { for } 1 \leq r \leq n+1
$$

For example, the third entry in the seventh row (the number 21) can be found by adding the second and third entries in the sixth row (6 and 15). [In this example, using the formula, $n$ is six and $r$ is 3.]

### 2.3.7 The Towers Problem - Building and Justifying a Solution

There is evidence of students solving this problem in elementary school (third, fourth, and fifth grades), high school (eleventh grade), and college. In the section that follows,
twenty-four solutions will be discussed in detail organized by grade level and concluding with an overall summary of solutions and justifications.

## Grade Three

Evidence of third graders working on the towers problem can be found in Martino (1992) and Martino and Maher (1999). Martino (1992) describes the work of two pairs of students that occurred on October 11, 1990. These pairs are: (1) Stephanie and Dana and (2) Michael and Jamie. Martino and Maher (1999) describe two students, Meredith and Jackie, working on this problem on December 10, 1992.

Stephanie and Dana began by building opposites. For most of the session, this was the only organization strategy they had for building towers. At the end of the session, to check to see that she had found all towers, Stephanie organized her towers with one blue cube in an elevator pattern. They also checked using the strategy of cousins. To justify that they had found all of the towers, Dana explained that they couldn't think of anymore and that every time they created a new tower, it was a duplicate.

On the same day, Michael and Jaime worked on this problem. Jaime also built towers using the opposite technique. They organized their 16 towers into 8 groups of opposite pairs. Michael and Jamie were not convinced that they had all of the towers because many of the other students in the classroom had found more than 16 towers. Once the other groups of students found duplicates and concluded that the answer was 16 , they were convinced that they had the correct answer.

Martino and Maher (1999) explain how Meredith and Jackie initially built towers in a random fashion. They then moved to organizing the towers into pairs of opposites. When questioned by the teacher as to how they knew that they had found all of the towers, Meredith explained that she could pick up an individual tower and check with each other tower and find that it wasn't a duplicate. As the teacher asked her how she knew that there weren't anymore, she rearranged her towers and organized them by cases based on the number of cubes of a particular color. Furthermore, she organized the towers with one cube of a particular color into an elevator pattern.

Meredith had difficulty explaining the case that contained towers with two cubes of a specific color. However, in an interview three days later, Meredith organized her towers so that she could justify that there were only six towers in the case of two of a specific color.

Meredith organized her six towers of height four into three pairs: a tower with exactly two yellow cubes separated by no red cubes and its "opposite," a tower with exactly two yellow cubes separated by one red cube and its "opposite" and a tower with exactly two yellow cubes separated by two red cubes and its "opposite." She then explained that there could not be a tower of height four with exactly two yellow cubes separated by three red cubes unless the tower violated the initial condition that it be four cubes tall. (Martino and Maher, 1999, p. 64)

Meredith worked on the pizza problem on March 15, 1993. She was asked if the pizza problem reminded her of any other problem. She replied that it reminded her of the towers problem. The connections she made can be found in the section "Making Connections between Towers and Pizzas."

## Grade Three Summary

All of the third graders used the pattern of opposites to create and organize their towers. Two of the three groups used the elevator pattern to show that they had found all of the towers with one cube of a specific color. Stephanie's group used the strategy of cousins as another way to organize their pairs. Stephanie's group justified that they had found all of the towers because anytime they created a new tower it was a duplicate. Michael's group wasn't convinced that they had the correct number of towers until the other students in the class had also found 16 towers. Only Meredith organized her towers by cases based on a specific color and was able to justify that she had found all of the towers within each case. She may have been able to have such a strong justification compared to the other third graders because of the types of questions that were asked by her teacher. As Martino and Maher (1999) report on page 75, "results from this research suggest that teacher questioning that is directed to probe for student justification of solutions has the effect of stimulating students to re-examine their original solution in an attempt to offer a more adequate explanation, justification and/or generalizations."

## Grade Four

There are many articles written about students solving the three-, four-, and five-tall towers problem in the fourth grade (Alston \& Maher, 1993; Maher, 1998; Maher \& Martino, 1996a; Maher \& Martino, 1996b; Maher \& Martino, 1997; Maher \& Martino, 1998; Martino, 1992).

Evidence of two fourth graders, Brandon and Justin, working on the four-tall towers problem on November 17, 1992 can be found in the article by Maher and Martino (1998). They created towers using cousins and opposites. When they realized that using both techniques created duplicates, they relied only on the opposite strategy and they organized their answer in eight pairs of opposite towers. After working on the pizza problem, Brandon was interviewed and asked if the pizza problem reminded him of any other problem. He replied that it reminded him of the towers problem. In this interview, conducted on April 5, 1993, he recreated the towers using opposites and when trying to make the connection, he organized his towers in cases based on the number of a certain colored cube. His three cases contained (1) the solids, (2) the towers containing one of a certain color, and (3) the towers containing two of a certain color. Furthermore, he organized his two groups with one cube of a certain color in an elevator pattern. The connections he made between the towers and the pizza problem are described in the section "Making Connections between Towers and Pizzas."

On February 6, 1992, Stephanie and Dana worked on solving the problem of finding all five-tall towers when selecting from two colors. These two students had worked together on the four-tall towers problem in the third grade. Similar to the third grade, they used the strategy of opposites to build their towers. As they checked the towers they had, they also used the cousins strategy. To check that they had found all of the towers, they organized their towers in groups that consisted of a tower, its opposite, its cousin, and the cousin's opposite (Maher \& Martino, 1996a; Maher \& Martino, 1996b; Martino, 1992).

In the same class as Stephanie and Dana, Michael and Milin worked together and they also began by creating opposites. They eventually built 30 towers and decided that they had found them all based on how much time passed before they found a duplicate. Since over 10 minutes had passed without finding a duplicate, they proclaimed that they were finished (Alston \& Maher, 1993; Martino, 1992).

On March 10, 1992, the task moved from building towers three-tall to justifying and providing a convincing argument. Four students, Milin, Stephanie, Jeff and Michelle, were interviewed and this session has come to be known as "The Gang of Four." During this session, Milin provided a proof by induction and Stephanie provided a proof by cases. Stephanie created five cases - towers with no blue cubes, towers with one blue cube, towers with exactly two adjacent blue cubes, towers with three blue cubes, and towers with two blue cubes apart. Milin did his proof by induction by drawing the four two-tall towers and showing how eight three-tall towers can be built from the four twotall towers (Maher, 1998; Maher \& Martino, 1996a; Maher \& Martino, 1996b; Maher \& Martino, 1997).

In an earlier interview, Milin did his proof by induction with the actual cubes. He started with the two towers one cube high. He built four towers two high by putting a blue cube on one of the one-tall towers and a black cube on the other one-tall tower. He continued to do this for three-tall towers. When he was questioned about how many fourtall towers, he replied it would be 16 - two for each of the eight he had already made. And when asked about five-tall towers, he replied 32 (Alston \& Maher, 1993).

## Grade Four Summary

All of the fourth graders created their towers using the strategy of opposites.
Brandon's group also used cousins to build their towers but they abandoned this strategy when it started to create duplicates. Stephanie's group used cousins as an organizational strategy. She and Dana created groups to justify they had found all of the towers. Their groups contained a tower, the opposite tower, the tower's cousin and the cousin's opposite. Brandon's group organized their 16 towers in eight groups of opposites and Milin and Michael justified that they could not find anymore because too much time had elapsed before they could think of another tower.

Brandon, Stephanie, and Milin all created sophisticated ways to justify that they had found all towers. Brandon and Stephanie did a proof by cases and Milin did a proof by induction. These students were given considerable time to think about their solutions and to revisit the problem. Also, the role of the teacher was very central in these interventions.

Importantly, these data show the advantage to revisiting tasks, group discussions about ideas, and sharing strategies. All of these components play a key role in the formulation and refinement of justifications. Stephanie and Milin, after having had multiple opportunities to think about and justify their ideas, presented a compelling argument to classmates during the group evaluation setting. (Maher, Sran, \& Yankelwitz, 2010, p. 43)

## Grade Five

In an article by Maher and Martino (1997), Stephanie created a proof by induction by recognizing a "doubling pattern." This understanding occurred after a series of episodes
of working with the towers over a year's time beginning in March of 1992. On February 26, 1993, Stephanie presented her explanation of the "doubling method" with her class. She used cubes and built the towers starting with towers one high. She explained that for each cube, there are two choices of colors to go on top, producing four towers two cubes high. She proceeded with this explanation until she created all 16 four-tall towers.

## Eleventh Grade

Evidence of the eleventh graders in the Kenilworth study solving the towers problem can be found in Tarlow (2004). On November 13, 1998, six of the students in the Kenilworth study worked on the towers problem in an after-school session. The six students worked in pairs: (1) Angela and Magda, (2) Michelle and Robert, and (3) Ali and Sherly. Of these six students, only Robert and Michelle had worked on the towers problem previously in the fourth grade.

## Angela and Magda

They first created their towers by building the towers that created an elevator pattern. They organized their towers in cases based on the number of blue cubes. They could not explain how they found all of the towers with two blues except that there were "no other possibilities." They decided to look at the towers that were three cubes tall and they found the answer to be eight. Angela came up with the formula $x^{n}$ where $x$ is equal to the number of colors and $n$ is equal to the height. However, they did not explain the reasoning for their (correct) formula.

## Michelle and Robert

Michelle and Robert initially worked separately. Robert built his towers using cases based on a specific color while Michelle randomly built her towers and organized them as opposites. Michelle justified that she had found them all because she could not think of anymore. Robert systematically demonstrated how he accounted for the towers with exactly two blues. He explained that he kept the top cube blue as he moved the other blue cube into all of the positions. Then he moved the top blue cube to the second position and moved the other blue cube into the remaining positions. Furthermore, Robert found a formula for the group of towers containing two of a color. His correct formula is $h^{*}(h / 2-$ $1 / 2$ ) where $h$ is the height of the tower. However, he was not able to explain why his formula worked.

The instructor asked Robert and Michelle to find the number of three-tall towers when choosing from two colors. After some thought, Robert replied that there would be eight because you could eliminate all of the four-tall towers that have a blue on top. The remaining eight four-tall towers with the yellow on top would create the eight three-tall towers once the top yellow cube is removed. Robert and Michelle realized that the number of towers doubled as the height of the towers increased and they explain that the formula is two raised to $n$ (where $n$ is equal to the height of the tower). They were not able to explain why the formula is two to the $n$. However, when asked how one could go from a one-tall tower to a two-tall tower, Robert explained that he could add a blue or a yellow cube to the top of the one-tall tower. Furthermore, Michelle and Robert
discovered that the formula for any number of colors for any height towers is $x^{n}$ where $x$ equals the number of colors and $n$ is equal to the height of the tower. However, although Robert was able to show inductively how the towers can be built, he was unable to explain why the formula is $x^{n}$.

## Sherly and Ali

Sherly and Ali initially organized their towers by opposites. To explain that they found all of the towers, they organized their towers by cases based on the number of a certain colored cube. They were able to explain the cases containing one of a certain color and three of another color by demonstrating the elevator pattern. However, they were unable to explain how they found all of the towers in the case containing two of one color and two of another color. They believed the answer to be 16 because, as they explained, four times four is sixteen. Based on this logic, they predicted that for towers that are three-tall, the answer would be nine because three times three is nine.

## Summary of the Eleventh Graders

The students used patterns to build their towers. These patterns included opposites and the elevator pattern. All three groups organized their towers by cases based on the number of a certain colored cube. All, except for Robert, had difficulty explaining that they had found all of the towers in the case containing towers with two blue cubes. Two of the groups discovered the formula to be $x^{n}$ where $x$ is equal to the number of colors and $n$ is equal to the height of the tower. However, they were not able to explain the
reasoning for this formula. Robert did explain how the towers could be built inductively by adding a cube to the top of the towers of the previous height. However, he did not make a connection from his inductive argument to the formula.

## College Students

Glass (2001) reported on 19 college students who solved the four-tall towers problem. Some of these students also solved extensions of this problem including the five-tall towers problem and towers four-tall choosing from three colors. Of these 19 students, 11 were highlighted for a case study. The remaining eight students' solutions were described when appropriate. Because the analysis of the remaining eight students' solutions are not as detailed, only the 11 students that were thoroughly described will be discussed.

In building the towers, most of these students used patterns to create the towers. Eight of the 11 students created towers by using the strategy of opposites. One student, Melinda, was reported to also use the strategy of cousins. Many students also used a staircase or elevator pattern to build their towers.

Once the towers were built, they had to justify that they had found all of the towers. Six of the students rearranged their towers using cases where each case was based on the number of a certain selected cube within the tower. For example, if the colors were yellow and red, the cases would be defined as (1) zero red, (2) one red, (3) two red, (4) three red, and (5) four red. One student, Errol, organized his 16 towers into two groups -
one group contained all of the towers with a red cube on top and the other group contained all of the towers with a yellow on top.

Three of the remaining four students, Melinda, Wesley, and Elizabeth, created their first two cases based on the elevator and staircase pattern and their opposite towers. Melinda organized the remaining eight towers using a mixture of cousins and opposites. Elizabeth organized her remaining eight towers using opposites. And Wesley organized his remaining eight towers into two groups - towers with a red cube on top and towers with a yellow cube on top. It is not clear how the remaining student, Donna, organized her towers.

Initially, when the students were asked how they knew that they had found all of the towers, many of them replied that they knew they were correct because 1) they couldn't find any more towers or 2 ) the other students had gotten the same answer. A few students doubted their answer because the instructor questioned if they were convinced that they had found them all. Melinda believed the answer to be a multiple of the height but could not justify this prediction. Four of the students (Wesley, Elizabeth, Stephanie, and Errol) initially believed that the reason the answer was 16 was because four times four is 16 and they predicted the answer 25 towers for five-tall towers. However, they were not able to justify this logic. Only Stephanie abandoned this prediction because she realized that the answer must be even because each tower had an opposite.

The instructor urged the students to continue to think about the reason for their solution. They investigated the five-tall towers. Two students, Melinda and Lisa, were not
able to justify their solutions further than, "we couldn't find anymore." Four students, Mike, Elizabeth, Rob1, and Donna, noticed that the number of towers doubled each time the height of the tower increased but were not able to explain why the number of towers doubled. After a class discussion on the fundamental counting principle, Elizabeth was able to explain that the total number of towers can be calculated by two to the height of the tower but she could not justify why this was true.

The remaining five students not only found that the number of towers doubled when the height of the towers increased by one but they also were able to explain the doubling pattern. They each explained separately that the towers doubled because there are two choices of colored cubes to add to the top of the tower. Furthermore, Errol explained in his homework how to build the towers inductively starting at the two-tall towers and building up to the four-tall towers. Rob2 also demonstrated in class how to build the fourtall towers inductively starting at the one-tall towers. Jeff and Rob2 (separately) predicted that for three-tall towers choosing from three colors, the answer would be 27 .

## College Students Summary

All of these students used patterns and organizational strategies to build their solutions. These strategies included opposites, cousins, staircase, and elevator patterns. Almost all of the students used the strategy of opposites to build or justify their solution. Many students organized their solutions into cases but not many of them could justify why the solution was 16 . Many of the students indicated that they knew they were finished because (1) they couldn't find any more towers or (2) they got the same answer
as other people in the class. Only five students were able to justify their solution mathematically.

The five students who justified their solution explained the reason for the doubling pattern. Although nine students recognized the doubling pattern, only these five could explain that, as the tower height increased, the number of towers doubled because you are able to add a choice of two cubes to the top of the tower. Furthermore, two of the students demonstrated how to build the towers inductively starting at the one-tall or two-tall towers.

## Overall Summary

At every grade, the students used the strategies of opposites, elevator, and staircase patterns to build and organize their towers. In the earlier years (grades three through four), the students often organized their towers in sets of opposites and often offered their initial justification as to (1) every time they found a new tower it was a duplicate or (2) it was the same answer as everyone else in the class. These were some of the same justifications that the college students gave as well.

Milin and Stephanie were able to provide mathematical justifications to their solutions but only after they had revisited the problem several times in the third and fourth grade. Milin and Stephanie were able to explain the doubling pattern and Milin provided a proof by induction. Stephanie, in the fourth grade, provided a proof by cases.

Brandon (fourth grade) and Meredith (third grade) were also able to provide a proof by cases in a separate interview with the teacher.

The eleventh graders and some of the college students discovered that the number of towers doubled as the height of the tower increased. The eleventh graders found the formula to be $2^{n}$ and the general formula to be $x^{n}$. However, they were not able to justify these formulas. The eleventh graders as well as some of the college students were able to explain, inductively, the reason for the doubling pattern.

### 2.3.8 The Pizza Problem - Building and Justifying a Solution

There is evidence of students solving this problem in elementary school (third, fourth, and fifth grades), high school (tenth and eleventh grades), and college. A total of 33 solutions to the four-topping pizza problem will be discussed in detail in this section.

## Third Grade

Evidence of two third graders working on the pizza problem can be found in an article by Martino and Maher (1999). Meredith and Sarah worked on this problem on March 15, 1993.

Meredith created a chart to build her pizzas. Across the top of the chart she wrote the pizza topping names and she used checks to construct her pizzas. She systematically constructed her pizzas using checks starting with the one-topping pizzas and continuing with the two-, three-, and four-topping pizzas. When creating the two-topping pizzas, she
used a systematic approach by first combining sausage with each of the other three toppings before moving onto the next topping. Sarah, Meredith's partner, made a list of pizzas using cases. However, her cases were based on a specific topping. That is, she listed all of the pizza combinations that included peppers. Then she listed all of the pizza combinations that included sausage (and not peppers), and so on.

## Summary of the Third Graders

Each girl organized their solution by cases. Meredith created her cases based on the number of toppings and Sarah created her cases based on a specific topping. Sarah listed her pizzas whereas Meredith created a chart and used checks to symbolize that the topping was included on the particular pizza. There is evidence of Meredith using a controlling for variables strategy when creating her two-topping pizza by creating all pizzas with sausage as a topping before moving on to the next topping.

## Fourth Grade

Evidence of six fourth graders working on the four-topping pizza problem can be found in Bellisio (1999). These students are in the Colts Neck school system and they were presented this problem on March 11, 1993. They worked in three groups of two. The three groups are Kevin and Steve, Alana and Jamie, and Colin and Brandon.

Kevin and Steve ultimately solved the problem by cases focusing on a specific topping. They first created all of the pizzas that had peppers, then all of the pizzas that had mushroom (without peppers), then all of the pizzas with sausage (without peppers or
mushroom), then all of the pizzas with pepperoni (without peppers, mushroom, or sausage), and, finally, the plain pizza. They found 16 pizzas. However, they did not have a strong explanation for the reason that they had found them all (Bellisio, 1999, p. 57).

They initially used the word pepper and the variables $\mathrm{p}, \mathrm{s}$, and m to indicated pepperoni, sausage, and mushroom respectively. However, Steve suggested they use a coding system instead. They decided to use the numbers $1,2,3$, and 4 to stand for the toppings. They used a 0 to represent the plain pizza. They used a circle and a combination of the numbers written inside the circle to represent the different pizzas. For example, a circle with a 123 in the middle of the circle represented a pizza with peppers, mushroom, and sausage. They provided a key to demonstrate which topping each number represented.

Alana and Jamie each created their own list of pizzas and checked with each other occasionally. Alana used triangles to represent her pizzas. In the triangles, she used symbols for each of the toppings and included a key in her final solution. She used a dot for peppers, a plus sign for sausage, a zero for pepperoni and a line (or a one) for mushroom. Jamie wrote her combinations using the whole word for each topping. They both created their pizzas based on the number of toppings. That is, they created all of the pizzas with one topping, two toppings, three toppings and then four toppings. Jamie controlled for variables when she created the two-topping pizzas. That is, she created all of the pizzas with pepperoni as one of the toppings. When she exhausted all possibilities for pepperoni, she created all of the pizzas with mushroom as a topping, careful not to
include pepperoni. Then she created all of the pizzas with sausage and pepperoni. They each found the total number of pizzas to be 15 . However, between the two of them, they had found all 16 pizzas. They were unaware that they had different pizzas listed for the three-topping pizzas.

Colin and Brandon worked separately but checked with each other periodically. They both created charts to organize their pizzas with the pizza toppings as headings for the columns in their charts. Colin used check marks to indicate inclusion of the particular topping while Brandon used a 1 to represent inclusion and a zero to represent that the topping was not on the particular pizza. Colin abbreviated the topping name while Brandon used P, S, M, and pepperoni to label his columns. Brandon redid his chart four times. In the last iteration of his chart, he used $\mathrm{P}, \mathrm{S}, \mathrm{M}$, and P to represent the toppings.

Colin started by creating all of the two-topping pizzas with peppers as one of the toppings. After placing a check in the peppers column, he systematically moved his check mark down in a staircase fashion to each of the three remaining toppings. He then created all of the two-topping pizzas with sausage as one of the toppings and systematically moved through the other toppings. He then created the four-topping pizza. Brandon, after several re-writes, created 16 pizzas first focusing on the pizzas with zero toppings, then the pizzas with one topping, then two toppings, three toppings, and finally, the pizzas with four toppings. He was systematic in creating his pizzas. For his onetopping pizzas, he created four pizzas by placing ones in a stair-like pattern to make sure that he included every topping. In his two-topping pizzas, he placed a one in the first
column and placed one in each of the remaining columns in a stair-like pattern until he exhausted all possibilities. This process is similar to the method Colin used to create his two-topping pizzas. He repeated this process for the remaining two-topping pizzas. He was not as systematic in creating the three-topping pizzas. After comparing, the boys agreed that the answer was 16 .

When asked by the researcher how they knew that they had found all of the possible pizzas, Colin could only explain that he had checked his chart with Brandon's. However, Brandon was able to explain his reasoning more clearly.
[He] was able to explain that he had started with pizzas with one topping, followed by two toppings, three toppings and then all four. He then explained how with multiple toppings he had begun with peppers in the left-hand column and combined that with each of the other toppings, going from left to right. He explained that when he began with sausages, there were fewer possibilities because sausages had already been paired with peppers, and so forth. He pointed at the entrees from left to right showing how he had combined toppings. He seemed very confident that he had found all of the possibilities but also gave the explanation that he had compared Colin's chart line for line with his to make sure they had found the possibilities. (Bellisio, 1999, p. 72)

## Summary of Fourth Graders

All of the students created and organized their pizzas by cases. Kevin and Steve organized their cases by a specific topping, starting with the pizzas that included peppers. The remaining students organized their pizzas by the number of toppings. Jamie listed her pizzas using the full name. Alana's and Steve's groups created pictures to represent their pizzas. Alana used symbols for toppings while Steve used numbers to represent toppings. Colin and Brandon both used charts to create their pizzas. Colin used a system of checks
and blanks while Brandon used 1's and 0's to represent inclusion and exclusion of a topping respectively.

Jamie, Brandon, and Colin all controlled for variables when creating their twotopping pizzas. That is, they kept one topping constant while they combined it with all of the other toppings. When they had exhausted all possibilities with that particular topping, they repeated the process with the next topping, careful not to include the topping that they had just held constant. The partners often checked with each other to verify their pizza combinations. To justify that they had found all of the possible pizzas, all but Brandon explained that they knew they were done because they had compared with each other. Brandon was the only student who could clearly explain how he had accounted for all of the possible pizza combinations.

## Fifth Grade

Two groups of fifth grade students worked on this problem. Nineteen students in a New Brunswick school system worked on this problem on March 30, 1993. Twelve students in the Kenilworth school system worked on this problem on April 2, 1993.

## New Brunswick - Grade 5

Evidence of the 19 children working on this problem can be found in Bellisio (1999). The children worked in groups: eight groups of two and one group of three. There was only one video camera in the classroom and it roamed from group to group. The Unifix cubes were available and four of the groups used them to solve the problem.

All four groups that used Unifix cubes organized their pizzas by cases based on the number of toppings. All four of the groups found the answer to be 16 . The groups were (1) Latima, Shauntee, and Ebonie, (2) Patrick and Benny, (3) Stephanie and LaToya, and (4) Desiree and Artesia. Each color cube represented a different topping and the height of the tower represented the number of toppings on a pizza. That is, a two-tall tower represented a pizza with two toppings. Latima and Desiree's group also included a yellow cube on the bottom of all of their towers to represent the actual pizza to which they "applied" toppings. So the height of each tower is one more than the number of toppings that are included. However, they all organized their towers by height. In doing so, they organized their pizzas by number of toppings.

All but Latima's group wrote a key to explain which color was connected to which topping. Only Patrick's group explained how they knew that they had found all 16 pizzas. Patrick explained that they knew they were finished because every time they found another pizza, it was already on their list. When building their towers, Desiree and Artesia organized them by number of toppings. However, when Artesia explained her solution to the researcher, she organized the towers differently. She organized her towers (pizzas) based on the bottom two cubes. That is, she grouped all of the towers that had a yellow and red cube on the bottom. These towers ranged from height two to height five. [Note: for her, the bottom cube represents the actual pizza base, not a topping.] She called this grouping a "family." However, she had difficulty organizing her pizzas in this manner and the instructor suggested that she go back to her original organizational
structure by number of toppings. Furthermore, Artesia explained to the class that when she built her two-topping pizzas, she first created all of the towers with a yellow cube (which represented a pizza) and a red cube (which represented peppers) and applied each of the other toppings. When she exhausted all of the combinations with yellow and red, she created all of the pizzas with yellow and another colored cube, careful not to duplicate any tower she had created. This is an example of controlling for variables. She kept one topping constant while she varied the second topping.

The remaining five groups did not use the Unifix cubes. These groups were: (1) Marcel and Frederick, (2) Kersa and Ebonie, (3) Ronald and Ivan, (4) Bhapur and Victor, and (5) Hector and Andre. They all found the answer to be 16. Three of the groups organized their pizzas by cases based on the number of toppings. The remaining two groups organized by cases but the cases that they used were not as identifiable.

Marcel and Frederick drew a giant circle and within the circle, they listed their pizzas organized by number of toppings. They supplied a key at the bottom of the circle to explain their representations. They used a C, S, M, P, and B to represent cheese, sausage, mushroom, peppers, and pepperoni respectively. They included a C (cheese) on each of the 16 pizzas. The instructor asked the group if they believed that there were any more pizzas and they replied that they did not believe there were anymore. She asked them to convince her. Marcel replied, "Because I considered all the things I could have done and it's just mixed up and each one is different and there is only one that I could find for each mixed up one" (Bellisio, 1999, p. 94).

Kersa and Ebonie organized their pizzas by cases based on the number of toppings. They listed their pizzas using the full names for the toppings (occasionally abbreviated saus. for sausage and mush. for mushroom). When they were asked how they knew that they had found all of the pizzas, they replied that whenever they created a new pizza, it was already on their list.

Hector and Andrew drew circles with letters inside the circles (representing the toppings) to represent their pizzas. They used S, M, Ps, and Pi to symbolize sausage, mushroom, peppers, and pepperoni respectively. They organized their pizzas by cases based on the number of toppings. They explained that found them all because they kept looking for more pizzas and they could not find any more that were different from the ones on their list.

Bhapur and Victor listed their pizzas using the entire name for the topping. Their list is organized as follows: 1) plain pizza, 2) one- and two-topping pizzas with peppers, 3 ) one- and two-topping pizzas with sausage (without peppers), 4) one- and two-topping pizzas with mushroom (without peppers and sausage), 5) one-topping mushroom pizza, 6) four-topping pizzas, and 7) three-topping pizzas. They grouped their pizzas together by number of toppings for the zero-, three-, and four-topping pizzas. However, when it came to the one- and two-topping pizzas, they organized them based on a specific topping. In this organizational strategy, it is evident that they controlled for variables when creating the two-topping pizzas. They justified that they had found all pizzas because they could
not find anymore. As they explained, anytime they created a new pizza, it was already on their list.

Ronald and Ivan also listed their pizzas and used the full topping name. They first listed all of the two-topping pizzas, then two of the three-topping pizzas, four-topping pizza, the one-topping pizzas, the plain, and the remaining three-topping pizzas. They organized by cases based on the number of toppings but only partially because the threetopping pizzas are not grouped together. It is not explained how they built their pizzas. However, it does appear that they controlled for variables when building their twotopping pizzas. They first listed all of the two-topping pizzas that contained sausage, then mushroom, then peppers.

## Summary of Fifth Grade New Brunswick Students

All nine groups organized their answers using cases based on the number of toppings. Two of these groups did not do a complete organization by cases. The four groups that used Unifix cubes to create their pizzas used similar strategies in that a certain color cube represented a topping and the height of the tower demonstrated the number of toppings on the pizza. The only difference was that two groups used a base cube to represent the pizza to which the toppings were applied.

The remaining five groups listed their pizzas using the full topping name, an abbreviation of the topping name, or a letter to represent the topping. Two groups used circles in their answer. Marcel and Frederick put their entire list within a giant circle.

Hector and Andre created 16 circles with combinations of letters inside the circles to represent each pizza.

There was evidence of three of the groups using a controlling for variables strategy when creating the two-topping pizzas. However, because there was only one video camera in the classroom, the construction of ideas of all of the students was not captured. With that in mind, only five of the groups explained how they knew that they had found all of the pizzas. All of them claimed that there were no more pizzas because they were not able to find anymore. Three of the groups further explained that anytime they created a new pizza, it was already on their list.

## Kenilworth - Grade 5

Twelve students in the Kenilworth school system worked on this problem on April 2, 1993 after working on the pizza problem with halves. [The pizza problem with halves involves finding all possible pizza combinations choosing from two toppings where the toppings can be placed on either the whole pizza or half of the pizza.] The twelve students in the classroom were: Romina, Brian, Ankur, Jeff, Michelle I., Michelle R., Matt, Stephanie, Amy-Lynne, Michael, Bobby, and Milan. Description of this session can be found in Bellisio (1999), Maher et al. (2010), Muter (1999), and Tarlow (2004). However, in each of these descriptions, only the work of six of the students is reported. Romina, Ankur, Jeff and Brian worked together. Stephanie and Matt worked together.

Romina's group decided to use P, S, M, and Pe to symbolize the peppers, sausage, mushroom, and pepperoni toppings. They also use Pl to represent a plain pizza. They systematically listed the 16 possible pizza combinations by first listing the plain and onetopping under the heading of "whole." They then listed the two-topping, three-topping, and four-topping pizzas under the heading of "mixed." As they listed the two-topping, they started with " P " for peppers and paired up P with each possible other topping. They moved on to " $S$ " for sausage and paired up $S$ with the remaining possible toppings. That is, they used a controlling for variables strategy when creating the two-topping pizzas. They organized their pizzas by cases by number of toppings. When asked how they know they had found all of the pizzas, Brian explains that they knew because they had a systematic way to create the pizzas (Maher et al., 2010).

Stephanie and Matt listed their pizzas using the letters c, pr, s, pp, and m for cheese, peppers, sausage, pepperoni and mushroom. They organized their pizzas by cases by number of topping. Matt explained how he created the two-topping pizzas by keeping the sausage constant as he added each topping to the pizza until he exhausted all of the possible combinations. Then, he repeated the process with the remaining toppings careful not to create a duplicate pizza. He used a controlling for variables strategy to systematically create his two-topping pizzas. Matt was able to thoroughly explain how he create his list of pizzas and how any other combination would be a duplicate of an existing pizza.

## Summary of Fifth Grade Kenilworth Students

Both groups organized their solution by cases based on the number of toppings and one member of each group demonstrated the same controlling for variables strategy when creating the two-topping pizzas. Each group used letters to symbolize their pizza toppings. The only differences in their solutions were in the choice of symbols for the cheese (plain pizza), peppers topping, and pepperoni topping. Both groups were able to explain that they found all of the pizzas by explaining how they built their pizzas and that there could not be anymore combinations.

## Tenth Grade

Evidence of five students working on the pizza problem can be found in Muter (1999) and Muter \& Uptegrove (2010). These five students, Ankur, Jeff, Romina, Brian and Mike, had all previously worked on this problem in the fifth grade.

On December 12, 1997 in an after-school session, these five students worked on the pizza problem. Initially, they discussed using factorials to solve the problem but soon realized that factorials did not work. At first, Romina and Jeff worked together and used letters to represent their pizza toppings. Ankur and Brian worked together and used numbers to represent the different pizza toppings. And Mike worked alone. Ankur, Romina, Jeff, and Brian decided that they should use one coding scheme and decided on the number coding scheme suggested by Ankur and Brian. These four students worked together and found 8 pizzas when choosing from three toppings, 16 pizzas when choosing
from four toppings and 31 pizzas when choosing from five toppings. They hypothesized that a doubling rule could work for this problem and they thought they should rethink the answer of 31 because it did not fit into the doubling pattern. At this point in the discussion, Michael, who had been working alone, presented his solution to his classmates.

Michael decided to use a binary coding system to create his pizzas. A one represented that the topping was on the pizza and a zero represented a topping not on the pizza. Each position represented a different topping. For example, for a five-topping pizza, the series of numbers 01000 could represent a pizza with one topping (i.e., mushroom). He explained that, using this system, he believed the answer to the five-topping pizza problem to be 32 . Furthermore, he explained that he believed the formula to be $2^{n}$ where $n$ is equal to the number of toppings.

Brian mentions that this problem reminded him of the towers problem. The students believed that the problems were similar but not the same. Ankur explained that in the towers problem the order of the cubes mattered but the order of the toppings on a pizza did not matter. However, because the class session was almost over, they did not discuss this idea much further until later sessions.

## Summary of the Tenth Graders

Unfortunately, the evidence does not show how, exactly, all of the students built their solutions to the pizza problem. However, the evidence shows how the students used
binary numbers to solve the pizza problem and how they understood the solution to be two to the $n$ where $n$ was equal to the number of toppings.

## Eleventh Grade

The study by Tarlow (2004) describes the solutions of eight students in the eleventh grade working on this problem on March 1, 1999. These students were a part of the Kenilworth study and five of them had already solved this problem in the fifth grade. Four of the students that had already worked on the problem were placed in a group together. They were Robert (Bobby), Stephanie, Shelly (Michelle), and Amy-Lynne. In the other group, this problem was novel for each student except for Michelle. This group was composed of Angela, Magda, Michelle, and Sherly. The students expressed that they only remembered the problem "a little."

Shelly and Stephanie initially wanted to solve the problem using factorials but were unsuccessful. Robert, Stephanie, Shelly, and Amy-Lynne each drew tree diagrams to solve the problem. When listing the toppings, the students used the full name of the topping, shorten names and symbols. When students used symbols, they used m and s for mushroom and sausage respectively. For peppers, "pp" and "pe" were used. And for pepperoni, "pep," "pr," and "p" were used. Robert is the only student who used subscripts. For peppers he used $p_{1}$ and for pepperoni he used $p_{2}$.

Stephanie, Shelly, and Amy-Lynne included plain as a topping when creating their tree diagrams. They each created every possible pizza and crossed out any duplicates.

Robert was more systematic when creating his tree diagram. Robert controlled for variables by finding all of the pizzas with peppers. Once he had exhausted all of the combinations with peppers, he created the next branch in his tree with sausage and was careful not to include peppers to avoid duplicates. He repeated this process until he exhausted all of the toppings.

All four students listed their pizzas based on cases determined by the number of toppings. The three girls made a connection from the number of pizzas to a row in Pascal's triangle. They believed that they had found all of the pizzas because the number of zero-topping, one-topping, two-topping, three-topping, and four-topping pizzas matched up to the fourth row in the triangle. However, they were unable to explain why. They were then instructed to find all the pizzas with five toppings and to explain how the addition rule in Pascal's triangle worked in terms of pizzas.

Angela, Magda, Michelle, and Sherly each started the problem by creating tree diagrams. However, they changed their approach because they thought the tree diagrams were confusing. Instead, they listed the pizzas based on the number of toppings. They found 16 pizzas and worked on the next problem which was to find the total number of pizzas with five toppings. After finding the answer to be 32, they investigated the number of three-topping pizzas and found the answer to be eight. They realized that the solution doubled each time you add a topping choice but they were unable to explain the doubling pattern. They also connected their solution to a row in Pascal's triangle. They were instructed to explain how the addition rule in Pascal's triangle worked in terms of pizzas.

## Summary - Eleventh Grade

A few of these students wanted to use combinatoric formulas to solve the problem but were unsuccessful. All of them initially used a tree diagram and all of them organized their final solution by cases. Angela's group started to use inductive reasoning. That is, Angela investigated the solution for towers three-tall, two-tall, etc. and found a pattern. However, her group could not justify why the number of towers doubled. Robert controlled for variables when creating his pizza using a tree diagram. Stephanie's group justified that their answer was correct because it matched up with Pascal's triangle although they could not explain why it matched up. They were instructed to explore this problem further by looking at the addition rule in Pascal's triangle in terms of pizzas. They made connections between the towers and the pizzas which is further explained in the next section, "Making Connections between Towers and Pizzas."

## College Summary

Glass (2001) reported on 19 college students who solved the four-topping pizza problem. They did not have Unifix cubes available and they worked on this problem about five weeks after working on the towers problem. The pizza problem these students encountered was slightly different that the one in this study. The difference is in the choice of toppings. These students were given the choice of pepperoni, green peppers, mushroom, and sausage. With these choices, these students did not have any topping names that started with the same letter. Some of the students were videotaped. The analysis is based on videotape (if available), instructor's field notes, and the students'
written work. The details to the solutions have been omitted because the solutions to the problem were very similar. Instead, a summary is provided.

All of the students organized by cases and all but two organized their cases by number of toppings. The remaining two students organized their cases based on a specific topping. Lisa (partnered with Yolanda) made a chart and used checks to keep track of her pizzas. The rest of the students listed their pizzas. When representing the topping, the students used the first initial of the topping, abbreviated the topping, or wrote the whole name of the topping.

Five students discussed using permutation and combination formulas to solve the problem. Only one student, who was currently taking a statistics course, was successful in solving the problem using combinations (Glass, 2010). All of the students were systematic when creating the two-topping pizzas. As Glass (2001) explains, "they held one topping fixed and paired with the each of the other toppings. They then moved to the next topping on the list" (p. 287). Some of her students only paired toppings that were not previously paired while the other students listed all two-topping pizzas and then eliminated the duplicates. One student, Jeff, found his two-topping pizzas by labeling the vertices of a rectangle with the names of each of the toppings and connecting the vertices with lines.

Four students recognized that the answer was a row in Pascal's triangle but they were unable to explain the connection. Two of these students justified that they knew they were finished because the numbers matched up to Pascal's triangle. No other
justifications of their solutions were provided. However, they further explored this problem by investigating the combinations for five-topping pizzas and investigating the relationship between the towers and the pizza problem. See the section titled, "Making Connections between Towers and Pizzas."

## Overall Summary

All of the students organized their solutions by cases and most organized their cases based on the number of toppings. However, five students organized their cases based on a specific topping. This was done at the third, fourth, fifth, and college levels.

At all of the ages, there is evidence of students being systematic when creating the two-topping pizzas. Most of the students listed their pizzas using the first initial of the topping, an abbreviated form of the topping name, or the whole name of the topping. Only two groups of students chose not to use any part of the topping name as a representation. Kevin and Steve (fourth grade) used the numbers 1, 2, 3, and 4 to symbolize the toppings and Alana (fourth grade) used a dot for peppers, a plus sign for sausage, a zero for pepperoni and a line (or a one) for mushroom.

At the younger ages, many students used pictures and symbols to represent the pizzas. The use of circles to represent pizzas was often used. At the high school and college levels, the students did not use pictures. However, they often wanted to use formulas that they had previously learned and most were unsuccessful in applying the formulas.

Only one class (fifth graders) had the Unifix cubes available when solving this problem. Almost half of the class used the cubes to solve the problem and they all used the cubes in a similar fashion. They all created the pizza combination using a specific colored cube to represent a topping and the height of the tower representing the number of toppings on a particular pizza.

Only the eleventh graders used tree diagrams to solve this problem. Four students organized their pizzas using charts and these students were either in elementary school or college. Meredith (third grader), Colin (fourth grader), and college students, Lisa and Yolanda, created very similar charts. The columns of their charts contained the topping names and each of them used check marks to indicate if the pizza contained the topping or not. Brandon (fourth grader) created a similar chart but used zeros and ones to indicate the absence or presence of the topping.

Michael (eleventh grade), like Brandon, used ones and zeros to indicate that a topping was included on the pizza or not. However, Michael did not list all of the possible pizzas using this method. Instead, he used this method to count the total number of threetopping, four-topping, and five-topping pizzas. Furthermore, based on this binary coding system, he was able to deduce that the formula for the total number of $n$-topping pizzas was $2^{n}$.

When asked how they knew that they had found all of the possible pizzas, the students in the elementary classrooms gave reasons such as "we checked with each other
and got the same answer," "we couldn't find anymore," and "anytime we created another pizzas, it was already on our list." These students were not introduced to further investigations. Only Brandon (fourth grade) could thoroughly and systematically explain how he accounted for all possible pizza combinations using his chart. Like Brandon, Matt (eleventh grade) was able to thoroughly explain how he knew there were no more pizzas by thoroughly explaining how he built and organized his pizzas. Many of the high school and college students connected their solution to Pascal's triangle or recognized that the solution doubled each time another topping choice was presented. However, not until they investigated deeper by looking at extensions of the four-topping pizza problem, Pascal's triangle, and the connection between the towers and the pizzas problem were they able to justify the solution of sixteen pizzas.

### 2.3.9 Making Connections between Towers and Pizzas

There is evidence of students understanding of the isomorphism between the towers and the pizza problem in elementary school (third and fourth grades), high school (tenth and eleventh grades), and college. Furthermore, the older students made connections with other mathematical concepts including Pascal's triangle and Pascal's identity.

## Grades Three and Four

Maher and Martino (1998) describe the connections Brandon (fourth grade) made between the pizza problem and the towers problem. In an interview on April 5, 1993, Brandon was asked if the pizza problem reminded him of any other problem that he had
worked on. He replied that it reminded him of the towers problem. In this interview, Brandon recreated his chart using zeros and ones for the pizza problem and organized the pizzas by cases based on the number of toppings. He rebuilt the towers using Unifix cubes using his technique of opposites. After he created all 16 towers and studied his pizza chart, he reorganized his towers into three groups. The three groups were: (1) the solid colored towers, (2) the eight towers that contain one of one color and three of the other color and (3) the towers that contain two of each color. He connected the towers containing one cube of a certain color with the pizzas with one topping. He then connected the solid towers with what he calls "the all group" (that is, he connected both of the solid towers with the pizza that contained all of the toppings). Finally, he connected the group of towers with two of each color to the pizzas with two toppings.

Brandon considered the solid yellow tower and the solid red tower as the pizza with everything and he categorized the towers with one yellow and three reds as well as the towers with one red and three yellows as the one-topping pizzas. Because he didn't quite fully make the connection, the interviewer asked Brandon to focus just on one color. He focused on the yellow cubes and rearranged his towers by cases based on the number of yellow cubes.

It was a this point in the interview that Brandon, enthusiastically, expressed that the group of four towers with exactly one yellow cube were like the four pizzas with the one topping in his chart, and placed each tower on top of its corresponding pizza on the chart. He explained how the red cubes in each tower corresponded to the "zero's" on his pizza chart and how the yellow cubes in each towers corresponded to the "one's" on his chart. He then confidently proceeded to match each of the sixteen
towers to each of the sixteen pizzas represented on his chart. (Maher and Martino, 1998, p. 88)

He further explained that it doesn't matter if you focus on the yellow cube to represent the toppings. He explained that the same connections could be made if he focused on the red cube to represent the inclusion of a topping.

Martino and Maher (1999) describe the connections made to the towers and the pizza problem by two third graders, Meredith and Sarah. Meredith and Sarah are asked if the pizza problem looked similar to any other problem they had worked on. They replied that it reminded them of the towers problem. Meredith used the cubes to create pizzas where a different colored cube represented a different topping and the height of the tower indicated how many toppings were contained on the pizza. Although this is a way to represent the pizzas using towers, it does not represent the isomorphism between the two problems.

The teacher decided to show the girls Brandon's binary chart for the pizza problem. She asked them if they understood the chart. She then explained to them that Brandon thought that this problem reminded him of the towers problem. After looking at the chart for some time, the girls indicated that they understood the chart and Sarah suggested that the zero code would be the red cube and the one code would be the yellow cube. The teacher asked them if they could build the towers to represent the pizzas. Even though they had made the connection between the red cube representing the topping being on the pizza, they wanted to create the towers using four different colors to represent the four
different toppings. The teacher asks them if it was possible to make the pizzas (towers) using two colors. The decided that they could and they created all of the pizzas using red and yellow cubes referring to Brandon's binary chart.

## Third and Fourth Grade Summary

For the third graders, it took some prodding by the teacher for the students to understand the isomorphism between the two problems. Without the teacher intervention, the girls consistently wanted to make the towers using different colored cubes. The teacher was able to use Brandon's solution as a tool to help the girls recognize the isomorphism between the problems.

Brandon was able to make the connection between the two problems after the teacher had encouraged him to focus on one specific color in the towers problem. After he rearranged his towers into groups based on the number of a specific color cube, he was able to make the connection between the pizzas and the towers. Furthermore, he was able to explain that it didn't matter if you focused on the yellow cube or the red cube to make the connection.

## Tenth Grade

In an after-school session, on December 19, 1997, five students came to understand the isomorphism between the towers problem and the pizza problem as described by Muter (1999) and Muter \& Uptegrove (2010). These five students (Romina, Michael, Jeff, Brian, and Ankur) had worked on the pizza problem a week earlier and at the end of
the session, thought there was a connection between the pizza problem and the towers problem but due to time constraints, were not able to investigate the connection further.

On December 19, they began the class session by recalling Micheal's binary coding scheme to solve the pizza problem and remembered the formula he had come up with for the pizza problem. The formula was $2^{n}$ where $n$ is equal to the number of toppings. They understood that the exponent was equal to the number of toppings but they had not, at this point, understood why the base of the formula was 2 .

Jeff and Michael discussed that if you keep the number of choices for colored cubes in the towers problem to two, the towers and the pizza problem are the same. The three other students were not convinced. Jeff explained that if, for example, you changed the height of the tower from two to three, that would be similar to changing the pizza problem from a two-topping pizza problem to a three-topping pizza problem.

Through further discussion, they were able to understand that the base of two in the pizza formula indicates that the topping would either be on the pizza or not on the pizza. They understood, from earlier investigations, that the base of two in the towers problem represented the two colors from which to choose and that the exponent, $n$, is equal to the height of the tower.

At the end of another session on January 9, 1998, the instructor asked the students if they could explain Pascal's triangle in terms of towers. Ankur explained how row four in Pascal's triangle represented the towers in the four-tall towers problem (when selecting
from two colors). Before they left for the evening, the teacher asked them to think about how the addition rule in Pascal's triangle worked (Pascal's identity). Specifically, she asked them to understand how the six in row four is produced from the three and three in row three.

On February 6, 1998, when they meet again, they were able to connect the towers and pizza problems to specific rows in Pascal's triangle. Furthermore, they were able to explain Pascal's identity using towers.

## Tenth Grade Summary

Over time, these tenth graders were able to understand the isomorphism between the towers and the pizza problem. They understood the formula to be $2^{n}$ for both problems and were able to explain what the base 2 and the exponent, $n$, represented in both problems. Furthermore, they made the connection to Pascal's triangle with both pizzas and towers and they were able to explain Pascal's identity using towers. These connections were made over a period of five after- school sessions.

## Eleventh Grade

As described in earlier sections, Angela, Magda, Michelle, Robert (Bobby), and Sherly worked on the towers problem on November 13, 1998. These five students, along with Stephanie, Shelly, and Amy-Lynn, worked on the pizza problem on March 1, 1999. After exploring the pizza problem, the groups further explored connections to the pizza
problem, Pascal's triangle, and the towers problem. Evidence of these investigations can be found in Tarlow (2004).

On March 1, 1999, the eight students were placed in two groups. Table A consisted of Robert (Bobby), Stephanie, Shelly (Michelle), and Amy-Lynn. Table B was composed of Angela, Magda, Michelle, and Sherly. Each of these groups quickly realized a connection to Pascal's triangle after completing the pizza problem. The instructor asked each of the groups to explain the addition rule in Pascal's triangle in terms of the pizzas.

## Table A

With some work, Stephanie was able to explain how the addition rule in Pascal's triangle works with pizzas. For example, the teacher asked Stephanie to explain how the 4 in the fourth row of Pascal's triangle is created, in terms of pizzas, from the 3 and the 1 in the third row of Pascal's triangle. She explained that "the one pizza [with three toppings] drops down and the three pizzas [with two toppings] get the new topping added to them. Together there are four pizzas with three toppings" (Tarlow, 2004, p. 139).

Robert, after looking at Pascal's triangle, figured the formula for the pizza problem to be $2^{n}$ where $n$ is equal to the number of toppings. However, he was not able to explain why the base of the formula is two. After hearing Stephanie's explanation of the addition rule in Pascal's triangle, Amy-Lynn believed that the two in Robert's formula is based on the fact that to create a new pizza, you either add the new topping or you do not.

The teacher asked them if this problem reminded them of any other problem that they worked on and they replied that it reminded them of the towers problem. Stephanie and Shelly explained the addition rule in Pascal's triangle using towers. Then, Robert explained the isomorphism between the towers and the pizza problem. He explained that the answer to both questions are determined by the formula $2^{n}$ where $n$ is equal to the height of the tower or the number of toppings. The base, 2 , which represents two colors in the towers problem, also indicates, in terms of pizzas, the two choices: with or without toppings. Furthermore, Stephanie explained how a particular position in the tower represents a particular topping.

Table B-Angela, Michelle, Sherly, and Magda

These four students determined that the number of pizzas doubled each time a new topping was introduced to the problem. They remembered that the number of towers doubled when the height increased but they were unable to explain the reason why the number of towers doubled. They mentioned that the two problems were not the same because the order of the colored cubes mattered in the towers problem but the order of the toppings did not.

They saw that the fourth row in Pascal's triangle was the same as their solution to the four-topping pizza problem and the instructor asked them to explain how the addition rule in Pascal's triangle worked in terms of pizzas. After some discussion, they were able
to explain the addition rule in terms of pizzas but they never described the connections between the towers and the pizza problem.

## Summary of the Eleventh Graders

All of the students connected the pizza solution to a row in Pascal's triangle and all of them were able to explain the addition rule in terms of pizza. Only Table A was able to completely explain the isomorphism between the towers and pizza problem. These four students at Table A had worked on the towers and the pizza problem in grades three, four, and five. Michelle at Table B was the only student that had work on these problems in earlier years.

## College Summary

Of the eleven students profiled in the study by Glass (2001), seven of them were able to describe the complete isomorphism between the pizza problem and the towers problem. That is, they were able to explain the base and exponent of the formulas for both problems. They were able to explain how the color of the cube in the tower represents whether the topping is on the pizza or not. And they were able to explain that a specific position in the tower represents a specific topping.

Of the remaining four students, Stephanie almost made the connection but was unable to explain completely that the position of the cube in the tower represented a specific topping. Melinda could explain what each tower represented in terms of pizzas except for the towers with two cubes of a specific topping. Rob1 believed they were related but
could not explain how. He tried to relate the color of the cube to a topping but did not investigate this further. Donna did not explain any connection at all.

Of the eleven students, five of the students connected the solution to the pizza problem to a row in Pascal's triangle. The remaining six students did not mention Pascal's triangle. Of the five students, two of these students explained the addition rule in Pascal's triangle using pizzas.

## Overall Summary

From as young as third grade, these students were able to recognize the isomorphism between the towers and the pizza problem. With some teacher intervention, the third and fourth graders were able to explain which pizza a specific tower represented. At the high school and college levels, the majority of the students were able to explain the formulas for both of the problems and make connections with mathematical concepts involving Pascal's triangle.

### 2.3.10 Ankur's Challenge

## Tenth Grade - Kenilworth Students

This problem was proposed to the Kenilworth students by one of the students, Ankur, at an after-school session on January 9, 1998. At the time, they were in the tenth grade and from the David Brearly High School in Kenilworth, New Jersey. They had been participants in the Rutgers University-Kenilworth Longitudinal study since the first
grade. There were five students present: Ankur, Michael, Jeff, Romina, and Brian. Ankur and Michael worked together in one group. Jeff, Romina, and Brian worked together in another. Evidence of their work can be found in Maher (2005) and Muter (1999).

Romina worked with Jeff and Brian on her solution. However, she began to work alone and the studies focus on her solution. She first decided that the towers must have two of a specific color. She used 1's, X's, and O's to represent three different colors. She first wrote 24 towers, horizontally as shown in Figure 2.6.

| 1 | 1 | X | O | X | O | 1 | X |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 1 | O | X | X | 1 | O | X |
| O | O | X | 1 | 1 | X | O | 1 |
| O | O | 1 | X | 1 | O | X | 1 |
| X | X | 1 | O | O | X | 1 | O |
| X | X | O | 1 | O | 1 | X | O |


| O | 1 | X | X | 1 | X | X | O |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | O | X | X | O | X | X | 1 |
| X | 1 | O | O | X | O | O | 1 |
| 1 | X | O | O | 1 | O | O | X |
| O | X | 1 | 1 | X | 1 | 1 | O |
| X | O | 1 | 1 | O | 1 | 1 | X |

Figure 2.6. Replication of Romina's original solution to Ankur's Challenge.

She explained that there are two additional groupings of six that could be listed and decided to create a more general way to write all 36 towers instead of listing the remaining two groups. After a few rewrites, she decided that the 1 would represent the duplicate color. She found all of the possible positions for the duplicate color and found six possible different position combinations.

She used a controlling for variables strategy by keeping the one in the first column and moving the other one to the other three positions. After she had exhausted all of the possibilities, she moved the one to the second position and moved the remaining one to the third and fourth position. And finally she moved the one to the third position and placed the remaining one in the fourth position.

She explained that in the other two spots, there must be one of each of the other two colors which she represented using an $X$ and an $O$. She explained that for each of the six towers that she had drawn, there are two possibilities each (to account for each $X$ and $O$ ). For example, the first tower drawn in the figure below represents the $1,1, O, X$ tower and the $1,1, X, O$ tower. Since there are two towers per drawing, there are a total of 12 possibilities. See Figure 2.7.

| 1 | 1 | O | X |
| :--- | :--- | :--- | :--- |
|  |  | X | O |


| 1 | O | 1 | X |
| :--- | :--- | :--- | :--- |
|  | X |  | O |


| 1 | O | X | 1 |
| :--- | :--- | :--- | :--- |
|  | X | O |  |


| O | 1 | 1 | X |
| :--- | :--- | :--- | :--- |
| X |  |  | O |


| O | 1 | X | 1 |
| :--- | :--- | :--- | :--- |
| X |  | O |  |


| O | X | 1 | 1 |
| :--- | :--- | :--- | :--- |
| X | O |  |  |

Figure 2.7. Replication of Romina's solution to Ankur's Challenge.

For each of these 12 towers, the same would be true for each of the other two colors, creating a total of 36 towers. That is, the one in the diagram above could represent each of the three different colors.

As described by Muter (1999), Michael and Ankur began the problem by explaining that there would be a total of 81 four-tall towers when choosing from three colors. They decided to create all of the towers, using paper and pencil. They used the numbers 1,2 , and 3 to represent the red, yellow, and blue cubes respectively. They also used the number zero to represent the variable cube. However, in using this method, they created many duplicates. They decided to focus on the complement of the problem instead. That is, they decided to find all of the four-tall towers that did not have at least one of each color.

While still working on the problem, Michael and Ankur listened to Romina's solution of 36 . They agreed that the answer was 36 . However, as they explained, they needed to understand the remaining 45 towers to be convinced. (That is, they needed to prove that the complement contained 45 towers.) They eventually created, on paper, the 45 towers using a series of numbers to represent the colors. They use 1,2 , and 3 to represent red, blue, and yellow. They used a zero to represent "any one of 3 except the one that's present" (Muter, 1999, p. 110).

In organizing the 45 towers in the complement, they created three cases. The first case contained all four-tall towers that have three cubes of one specific color and one cube of another color. See Figure 2.8.

| 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 | 0 | 0 | 0 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 2 | 3 | 1 | 2 | 3 | 0 | 0 | 0 | 1 | 2 | 3 |
| 1 | 2 | 3 | 0 | 0 | 0 | 1 | 2 | 3 | 1 | 2 | 3 |
| 0 | 0 | 0 | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |

Figure 2.8. Replication of Mike and Ankur's first case of towers.

Each column above represents two towers because the zero represents any of the other two colors not listed. Since there are 12 towers drawn, there are a total of 24 towers in this case. To build these towers, Ankur and Michael used a controlling for variables strategy. That is, they kept the variable cube (the zero cube) constant (in the same position) while they created each group of three towers. They also systematically moved the zero cube up a position each time they created a new set of three towers. The "zero row" moves up in a stair-like fashion.

The second case contained the solid towers. That is, this case contained three towers that each only contained one color - the all red tower, the all blue tower, and the all yellow tower.

The third case contained 18 towers with two cubes of one color and two cubes of another color. They created their towers as follows:

| 1 | 1 | 2 | 2 | 3 | 3 | 1 | 1 | 2 | 2 | 3 | 3 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 1 | 2 | 2 | 3 | 3 | 2 | 3 | 1 | 3 | 1 | 2 |
| 2 | 3 | 1 | 3 | 1 | 2 | 2 | 3 | 1 | 3 | 1 | 2 |
| 2 | 3 | 1 | 3 | 1 | 2 | 1 | 1 | 2 | 2 | 3 | 3 |
| 1 | 1 | 2 | 2 | 3 | 3 |  |  |  |  |  |  |
| 2 | 3 | 1 | 3 | 1 | 2 |  |  |  |  |  |  |
| 1 | 1 | 2 | 2 | 3 | 3 |  |  |  |  |  |  |
| 2 | 3 | 1 | 3 | 1 | 2 |  |  |  |  |  |  |

Figure 2.9. Replication of Mike and Ankur's third case of towers.

In creating this case, they also used a controlling for variables strategy. In the first group of six towers, they kept the top two cubes the same as they changed the bottom two cubes. The second group of six contains towers where the top and the bottom cubes are the same color and the middle cubes are the same color. They kept the top and the bottom cube constant as they changed the middle cubes. Then for the next six, they alternated the cubes that were the same.

## Summary of Tenth Graders

Romina solved this problem directly finding the 36 four-tall towers that contain at least one of each color. She solved this problem using cases by focusing on one specific color and justifying that there were 12 towers in this case. She used a controlling for variables strategy to create the towers within the case. She did not create all three cases. Instead, she justified that the remaining two cases would be created in the same way by replacing the duplicate color.

Ankur and Michael ultimately solved this problem by looking at the complement. They found the 45 towers that were contained within the complement by using cases. They created three cases based on the number of cubes of a certain color. The three cases were: (1) the towers containing all four cubes of a particular color, (2) the towers containing three cubes of a particular color and one different colored cube, and (3) the towers containing two cubes of a particular color and two cubes of one other color. They also used a controlling for variables strategy when creating their cases.

To represent the colors in the towers, Ankur and Mike used numbers while Romina used a $1, X$ and $O$. This group of high school students started to use binary numbers in their solutions to the pizza and towers problem prior to working on Ankur's Challenge after it was introduced to them by another student in the class (Maher, 2005; Muter, 1999).

## College Students

Glass (2001) has evidence of five college students at a community college solving this problem. The five students are Errol, Penny, Mary, Rob1, and Rob 2.

Errol, Penny, and Mary were in the same class in the spring of 1999. This session was not videotaped. The analysis is based on the instructor's field notes and the students' written work. The three students worked separately and this problem was introduced to them four weeks after they had worked on the four-tall towers problem when selecting from two colors.

Mary explained that there is one color that must appear twice while each of the other colors must appear once. She fixed the color that appears twice in the following cube positions: first and second, first and third, first and fourth, second and third, second and fourth, and the third and fourth. This strategy created six towers. For each of these six towers, there are two possible towers because the remaining colors can be switched. This method produced a total of 12 towers when one of the three colors appears twice. This would be true for the remaining two colors, creating a total of 36 towers. See Figure 2.10
for a diagram of her explanation using R to represent the color that appears twice. [The towers are represented horizontally.]


Figure 2.10. Diagram of Mary's solution to Ankur's Challenge.

Mary solved this problem directly using a cases approach. She focused on the 12 towers that had a certain duplicate color. Within this case, she used a controlling for variables strategy to create each of the six towers. That is, she kept the first cube constant while she moved the other cube to the second, third, and fourth position. After she had exhausted all of those possibilities, she started in the second position and repeated this process.

Penny created a tree diagram of all possible 81 towers four-tall choosing from three colors and then crossed out any towers that did not have at least one of each color. That is, she used a method of elimination to find her towers. As Glass (2001) explains, Errol uses an inductive method to find his towers:

He said that you could fix the first level as red. The second level could then be red, yellow, or blue. If the second level were red than the third and fourth level would have the other two colors yellow blue or blue yellow. If the second level were blue then the third and fourth level would contain at least one yellow. It could be yellow yellow, yellow red or red yellow, yellow blue or blue yellow. Similarly if the second level were yellow the third and fourth level could be blue blue, blue red or red blue,
blue yellow or yellow blue. This gives twelve combinations which you multiply by three since the first cube could be any of the three colors. (Glass, 2001, p. 236)

See Figure 2.11 for a diagram of Errol's solution.

| Fourth Level | B | Y | Y | R | Y | B | Y | B | R | B | Y | B |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Third Level | Y | B | Y | Y | R | Y | B | B | B | R | B | Y |
| Second Level | Red |  | Blue |  |  |  |  | Yellow |  |  |  |  |
| First Level | Red |  |  |  |  |  |  |  |  |  |  |  |

Figure 2.11. Diagram of Errol's solution to Ankur's Challenge.

Although this can be categorized as an inductive approach, Errol's strategy is also considered a cases approach using a controlling for variables strategy. His three cases are based on the color of the first cube. Then, he kept the second cube a constant color until he exhausted all of the remaining possibilities.

Rob1 worked on this problem in the spring of 1999, four weeks after working on the four-tall towers problem. He was not videotaped nor did he hand in his written work so only the instructor's field notes are available. He used the actual Unifix cubes to solve the problem. He focused on the towers that would have two yellow cubes. He created three towers with a blue cube on top, and systematically moved the red cube into the second, third, and fourth position, filling in the remaining cubes with yellow. He then fixed the top cubes as red, and systematically moved a blue cube into the second, third and fourth positions, filling in the remaining cubes with yellow. See Figure 2.12

| $B$ | $B$ | $B$ | $R$ | $R$ | $R$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $R$ | $Y$ | $Y$ | $B$ | $Y$ | $Y$ |
| $Y$ | $R$ | $Y$ | $Y$ | $B$ | $Y$ |
| $Y$ | $Y$ | $R$ | $Y$ | $Y$ | $B$ |

Figure 2.12. Diagram of Rob1 first six towers in his solution to Ankur's Challenge.

The next three towers are produced in the same manner, keeping a yellow cube fixed on the top and systematically moving a second yellow cube into the second, third and fourth position. Each of these positions would create two towers because the red and the blue cube can be in alternate positions. See Figure 2.13.

| Y | Y | Y | Y | Y | Y |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Y | R | R | Y | B | B |
| R | Y | B | B | Y | R |
| B | B | Y | R | R | Y |

Figure 2.13. Diagram of Rob1 second set of six towers.

There are a total of 12 towers with two yellow cubes. This would be the same for two red cubes and for two blue cubes, producing a total of 36 towers. He used a cases approach focusing on the towers with two yellows. He then broke his case up into three sub-cases based on the color of the top cube. Within these sub-cases, he used a controlling for variables strategy by systematically moving the other cube in a staircase fashion.

Rob2 worked on this problem in the spring of 2000, the same day as the four-tall towers problem choosing from two colors. He was videotaped. He initially approached this problem by creating all six three-tall towers that have one of each of the three colors.

He added one of the three colored cubes to the bottom of each of these six towers and then he added one of the three colored cubes to the top of each of the six three-tall towers. He removed any duplicates this approach created. However, he realized that he missed some towers and changed to solving the problem using a cases approach. (He missed the towers that would have the duplicate colored cubes in the middle of the tower.)

Rob2's second approach was based on focusing on one color, blue. He created his first three towers by keeping the two blues together and moving them to all possible positions. He explained that for each of these three towers, there would be two towers because the yellow and red cube can be alternated. He created the next two towers by separating the blue cubes by one cube and moving them into all possible positions. Finally, he created the last tower by separating the two blue cubes by two cubes. Each of these six towers can be multiplied by two. Then, he explained, this process can be repeated for each of the other two colors producing an answer of 36. Rob2 used a cases approach in this problem by focusing on one dominant color. Furthermore, he controlled for variables by initially keeping the blue cubes together and moving them into all possible positions. Then, he created the towers where the blue cubes would be separated by one cube and finally he created the towers where the two blue cubes would be separated by two cubes.

## Summary of College Students

All five students approached this problem directly. That is, they found the 36 towers four-tall towers that contain one of each color when selecting from three colors. All but two solutions involved a cases approach. Penny used a tree diagram to create all 81 towers that are four tall when selecting from three colors. She then eliminated the towers that did not contain one of each color. Rob2's first approach was inductive because he created new towers based on previously built towers. He created all three-tall towers that contained one of each color and then added cubes to either the top or the bottom of this tower. However, he was not successful in finding the answer using this approach and abandoned it for a cases approach.

The remaining four solutions involved a justification by cases. Although Glass (2001) categorized Errol's approach as inductive, it can also be viewed as cases. His three cases are based on the color of the first cube. He explains how 12 towers would be created with a particular color as the bottom cube. He does not create the remaining two cases but explains that the logic would be the same for the remaining two colors as the bottom cube.

Mary, Rob1, and Rob2 solved the problem by cases and created their cases based on a dominant color (that is, a color that would appear twice in the tower). Mary and Rob2 had similar approaches, creating six towers and explaining each possible position for the dominant color. They explained that for each of these six towers, the other two colored cubes would be alternated creating two towers each for a total of 12 towers. They did not
create the remaining two cases but explained that the same result would occur for the other two colors creating a total of 36 towers.

Rob1 also solved by cases based on a dominant color. However, he created three subcases to create the 12 towers by keeping the top cube constant with each of the three colors and systematically moving the same cube down in a stair-like fashion.

Mary, Errol, Rob1, and Rob2 all used a controlling for variables approach when creating their towers within their sub-case. Mary and Rob1 used similar approaches by keeping the top cube constant while systematically moving another cube into the second, third, and fourth positions. Once they exhausted all of the possibilities, they used a similar approach to create the next group of towers.

Errol's controlling for variables strategy was slightly different. He also kept the first cube constant. However, he then kept the second cube constant and changed the third and fourth cubes until he exhausted all possibilities. He then changed the second cube and repeated the process until he created all 12 towers.

Rob2 controlled for variables by keeping the number of cubes that separated the blue cubes constant until he exhausted all possibilities. That is, he first created all towers with blues together (separated by zero cubes). Then he created towers with blues separated by one cube then by two cubes.

## Other Undergraduate and Graduate Students

In a study by Glass and Maher (2004), the solutions of 22 students to Ankur's Challenge were analyzed and categorized. These students were in high school, undergraduate, or graduate school. Included in the 22 solutions are the solutions of Romina and the six solutions of the five students in the 2001 study by Glass. Glass and Maher organized the 22 solutions into four categories: (1) Justification by Cases, (2) Inductive Arguments, (3) Elimination Arguments, and (4) Analytic Method.

## Justification by Cases

Of the 22 solutions, nine solutions are categorized as using a justification by cases approach. These nine include the solutions of Romina, Rob1, Mary, and Rob2 ${ }^{2}$ which were previously discussed. The remaining five students included three undergraduate and two graduate students.

Two of the undergraduates in this study, April and Bernadette, and one graduate student, Traci, had similar solutions. April kept the blue cube on top of the towers to create her 12 towers while Bernadette and Traci, using the same logic, kept a constant colored cube on the bottom of the towers to create their 12 towers. That is, they created their cases based on a specific color cube in a specific position for all 12 towers (top or bottom).

[^2]April created her towers with a blue on top and then broke up this case into three subcases. The three sub-cases were blue as the second cube, purple as the second cube, and white as the second cube. Keeping the first and second cube constant, she listed all of the possible towers that could be made in the remaining two positions. She explained that this same logic would be used for a white on top and a purple on top.

Traci created her towers in a manner very similar to the way in which April created hers but used A, B, and C to represent her colors. She kept the bottom cube as color A and created three sub-cases based on the second cube. The first sub-case contains B as the second cube, the second sub-case contains C as the second cube and the third sub-case contains A as the second cube. Keeping the first and the second cube constant, she created all of the possible towers by listing the possibilities for the third and fourth cubes. She explained that 12 towers could also be created, in the same way, with B on the bottom and 12 towers could be created with C on the bottom.

Bernadette created her cases based on a specific color cube (blue) on the bottom of the tower. She then created three sub-cases. The first sub-case contained all towers with blue on the bottom and a second blue cube. She moved the second blue cube to all possible positions in a staircase fashion from second, to third, to the fourth position. She reasoned that the other two cubes could be either purple or white creating six towers in this sub-case. The next sub-case, still containing blue as the bottom cube, contains two purple cubes. There are three possible towers that can be created in this case as the white cube will fill the other positions. And the last sub-case, still containing blue on the
bottom contains two white cubes with purple as the last cube creating three towers. See
Figure 2.14.


| Sub-Case 1 | Sub-Case 2 | Sub-Case 3 |
| :---: | :---: | :---: |
| 6 Towers | 3 Towers | 3 Towers |

Figure 2.14. Diagram of Bernadette's solution to Ankur's Challenge.

As she explains there are also 12 towers for the purple on the bottom and 12 towers for the white on the bottom, for a total of 36 towers.

Joanne and her partner Donna (both undergraduate students) described each of the six possible positions for two cubes of the same color in a four-tall tower. They first described three towers that can be created with a blue as the top cube and systematically moved the second blue cube to the fourth, third, and second position. Then they created two towers where the blues are together in the second and third position and the third and fourth position. Finally, the last tower is created by keeping the blue cubes in the second and fourth position. Joanne's group explained that for each of the six possible arrangements for two cubes of the same color, two towers can be created by alternating the other two colors in the unfilled positions. As they explained, there are three different colors that could be the dominant color so there are six color combinations (3 dominant colors times 2 options per tower) for each for each of the six towers. Therefore, the answer is 36 (six times six) possible towers.

Tim, a graduate student, explained that there are six ways to create four-tall towers containing two each of two colors - say red and green. If you exchange a yellow cube for one of the red cubes, there are two possible ways to do this for each of the six towers. As he explained, there are 2 times $6=12$ ways to create a tower that contain a one yellow, one red, and two green cubes.

All of these solutions are done by justification by cases. Furthermore, four of the five students used a controlling for strategies approach when creating their towers within the sub-case. April and Traci kept the first and the second cube constant and exhausted all possibilities for the third and the fourth cube. After they had exhausted all possibilities, they kept the first cube constant and changed the color of the second cube. Keeping these two new cubes constant, they created the towers for all possible colors for the third and fourth tower. They repeated this process until they created all 12 towers within their subcase.

Bernadette's case was based on the color of the bottom cube (blue). Her three subcases involved the towers that had two blues, two purples and two whites. Keeping the blue on the bottom, she systematically moved the blue cube in a staircase fashion to create the blue case. To create the two purple and the two white cases, she kept the two cubes together to create two towers and the two cubes separated by one cube to create the third tower.

Joanne and Donna also created three towers similar to the way Bernadette did. They kept the two blues together and then apart. They also kept a blue on the top, and
systematically moved the blue to the second, third, and fourth position in a staircase fashion.

## Inductive Arguments

The solutions of four students are categorized under Inductive Arguments. This includes Errol's and Rob2's solutions as described previously and Frances and Christina. Frances (graduate student) created her towers the same way as Errol did by keeping the first cube as red and then breaking this case into three sub-cases where the second cube could be red, yellow, or blue. Keeping the first and second cube constant, she created all towers based on the possible colors for the third and fourth cube. As mentioned previously, this method could be categorized as an argument by cases. The cases are based on the first color cube. Furthermore, this case is broken up into sub-cases based on the second color cube. By keeping the two cubes constant, Frances was controlling her variables.

Christina started with the first cube being either A, B, or C. Starting with the cube A, she created all towers with $\mathrm{A}, \mathrm{B}$, and C as the second cube. Then, she added $\mathrm{A}, \mathrm{B}$, and C to each of these towers to create three-tall towers. She eliminated any towers that had three of one color. Finally, she added A, B, and C to the existing towers and eliminated any towers that did not have at least one of each color. She repeated this process with the towers that had B and C on the bottom. This is categorized as an inductive argument because she created new towers based on existing towers. That is, she created two-tall
towers based on one-tall towers. She created three-tall towers based on two-tall towers. And finally, she created four-tall towers based on the three-tall towers.

## Elimination Arguments

Four students, including Penny, are categorized under the Elimination Arguments. Robert (undergraduate), Liz (graduate), and Mary (graduate) started the problem with the number of four-tall towers when choosing from three colors (81 towers) and subtracted the towers that did not have at least one of each color. These three students used formulas, as opposed to creating subsets of the towers. All three subtracted towers when selecting from two colors. All three accounted for the possibility of counting the solid towers twice.

## Analytic Method

The last category, Analytic Method, contains only one solution, by graduate student Leana. Using factorials, Leana found all of the possible ways to arrange AABC. She then divided by two factorial to account for repetition. Using this mathematical method, she found 12 towers when A is repeated. She explained that she would do the same for B repeated and C repeated to produce a total of 36 towers.

## Summary

The dominant approach to this problem was justification by cases. However, the cases are done in different ways. Most students broke up their cases based on a dominant color
or by keeping a certain cube (the top or the bottom cube) constant. In creating the towers within the case, many students created sub-cases and used a controlling for variables technique to keep the towers organized.

Only the graduate students and one senior undergraduate student correctly used mathematical formulas to solve the problem (Glass \& Maher, 2004). Most of the students that used mathematical formulas were the students that used elimination arguments.

It could be argued that two of the examples that are listed under induction method are examples of justification by cases. Although they created their towers based on the choice for the first cube and then the second cube, because they are keeping this first cube a constant color, they are focusing on a specific sub-case of the total solution.

## Overall Summary

Nineteen solutions to Ankur's Challenge have been discussed. These students were in high school, undergraduate, or graduate school when solving these problems. The most popular type of justification was by cases. Eleven students directly found the 36 towers by breaking the solution up into cases. Ankur and Mike also justified their solution by cases but they found the solution indirectly by looking at the complement. A cases approach was used at each grade level.

In the solutions that used cases and solved the problem directly, there were two methods in determining their cases. They either based the cases on a dominant color cube
or by keeping the top (or bottom) cube a constant color. All but one student, Tim, controlled for variables when creating their towers within the sub-case.

There were seven solutions that did not involve cases. Four students used an elimination method, two students used an inductive method, and one student used an analytic method. Three of the four students that used an elimination method used formulas to solve the problem. Only one student, Penny, used a tree diagram. Penny created all 81 four-tall towers when selecting from three colors using a tree diagram and eliminated the towers that did not have one of each color. Christina and Rob2 approached the problem inductively. (Rob2 eventually abandoned this approach for a cases approach.) Leana, a graduate student, used combinatoric formulas to solve the problem analytically.

Five students solved the problem indirectly. That is, they did not approach the problem by finding the 36 towers immediately. Four of these students used an elimination method that involved finding the 81 towers and subtracting the towers that did not have at least one of each color. Mike and Ankur, two high school students, were the only ones that solved the problem by creating all of the towers in the complement (the towers that did not have one of each color).

To solve this problem, all of the students discussed either built the towers with cubes or wrote their solution on paper. For those that wrote their solution on paper, they either explained using words or representations. Of the representations used, most students used the first letter of the color of the cube they were representing. Three students used the
letters A, B, and C to represent the three different colors. Only the two high school students used numbers and x's and o's to represent the different colors.

The only distinctions that can be made between levels of academic study were in the use of representations, the use of formulas, and using the complement to solve the problem. The high school students were the only ones who used numbers and x's and o's to represent the colors. However, this group of high school students started to use binary numbers in their solutions to the pizza and towers problem prior to working on Ankur's Challenge after it was introduced to them by another student in the class (Maher, 2005; Muter, 1999). Only the graduate students and one senior undergraduate student correctly used mathematical formulas to solve the problem (Glass \& Maher, 2004). There was only one group of students that focused on the complement of the solution set; these students were in high school.

Glass and Maher (2004) described four major categories for solving this problem. These categories are 1) justification by cases, 2) inductive method, 3) elimination method, and 4) analytic method. In their article, all of the solutions discussed, except for the solution by Ankur and Mike, were classified into one of those four categories. Ankur and Mike solved the problem indirectly by looking at the complement. Although we only have one example of this method, it is uniquely different than the others. It could be argued that this method could be classified under justification by cases because the students built the towers in the complement by cases. However, a distinction should be made between an indirect proof and a direct proof by cases.

Furthermore, it has been shown that there is some overlap between the categories and it could be argued that some solutions fall into more than one category depending on the viewpoint. For example, it was argued that two explanations that were classified as inductive methods by Glass and Maher (2004) might better be classified as justification by cases. Nonetheless, there is definitely a pattern to which these students solve this problem regardless of age. It has been shown that this problem naturally gives rise to certain mathematical problem solving and justification strategies.

## CHAPTER 3: DESIGN OF THE STUDY

### 3.1 Background

This study took place in a mathematics course, Math Reasoning and Assessment, during the spring semester of 2011. The course is required for pre-service middle school math teachers at Felician College. The class met twice a week for one hour and 15 minutes. The data from videotaped problem-solving sessions focusing on combinatorics was analyzed for this study. These sessions occurred on February 11 and February 18. See Appendix A for an outline of the entire course schedule.

### 3.2 Subjects

Six undergraduate students in their junior year were enrolled in the course Math Reasoning and Assessment at Felician College in Rutherford, New Jersey during the spring semester of 2011. The students in the class were all mathematics majors studying to be teachers. All of the subjects were women. All six students agreed to be videotaped and all of them agreed that their work could be used for this study. There was one classroom instructor, Professor Elizabeth Uptegrove.

### 3.3 Data

To answer the research questions, data came from videos and student's written work.

### 3.4 Setting

This study is a component of a design study in the third year of a grant funded by the National Science Foundation (NSF) at Rutgers University and University of Wisconsin, Madison [award DRL-0822204] directed by Carolyn A. Maher. A component of the project is to build a repository to store a collection of video data and related metadata from earlier NSF funded projects. The videos and related metadata are being prepared for pre/in-service teacher interventions. This study extends the work of the grant by collecting and analyzing video data of students engaged in doing the mathematics before studying videos of childrens' reasoning.

### 3.5 Tasks

The students in the study worked on a counting/combinatorics strand of tasks used in earlier longitudinal and cross sectional research at Rutgers. The three tasks analyzed in this study are the towers problem, the pizza problem, and Ankur's Challenge. These tasks and the types of reasoning that are provoked from these tasks are explained in the literature review section.

### 3.6 Data Collection

The data collected included video recordings of the pre-service teachers working on the combinatorics tasks. The students' written work was also captured on camera.

### 3.6.1 Video Recordings

In this study, up to two video cameras were used to videotape the sessions. On February 11, 2011, there was one video camera in the classroom. On February 18, 2011, there were two video cameras in the room.

### 3.6.2 Students' Written Work

The students were encouraged to write their findings and justifications down on paper. Some of this written work was captured on videotape. This written work was useful in the analysis because it allowed the researcher to better understand the storyline.

### 3.7 Method of Analysis of the Video Data

This study used the analytical model for analyzing video data outlined by Powell, Francisco, and Maher (2003). Powell et al. (2003) describe seven non-linear phases of studying video data beginning with "viewing attentively" and ending with "composing narrative" (p. 413).

### 3.7.1 Viewing

The first step of the analytical model provided by Powell et al. (2003) is to watch the video several times to get a general idea of the content. This step allows the researcher to get familiar with the session(s). At this phase, the researcher viewed the data without any specific analytical viewpoint.

### 3.7.2 Describing

After the video is watched several times, the researcher records a description of the video. The analytical model suggests describing the video in even 2 to 5 minute intervals. Again, the descriptions should be descriptive only, devoid of any inferential remarks. These intervals should be time coded to allow the researcher to quickly find a particular event in future viewings. Not only do these descriptions enable the researcher to become more familiar with the data, they also allow other individuals to get an idea of the content of the videos.

### 3.7.3 Identifying Critical Events

At this stage of the study, the researcher identifies critical events. Critical events were first defined by Maher and Martino (1996a) as episodes that provide mathematical insights (p. 196). Powell et al. (2003) describe these events as events that may "either confirm or disaffirm research hypotheses; they may be instances of cognitive victories, conflicting schemes, or naïve generalizations; they may represent correct leaps in logic or erroneous application of logic; they may be any event that is somehow significant to a study's research agenda" ( p . 417).

As mentioned, critical events are significant to the research agenda. Identifying critical events was important because it enabled the researcher to chart
the development of ideas and to understand how these events influenced later thinking (Maher, 2002).

### 3.7.4 Transcribing

All of the data was transcribed to allow a more detailed analysis of the video. The transcripts are as close to exact as possible including not only verbal expressions but also gestures and descriptions of written work. Appendix A of the report Guidelines for Conducting Video Research in Education (Derry, 2007) provides a list of choices on how to transcribe common occurrences in speech and gestures along with providing strengths and weaknesses of each choice. This guideline was followed to provide consistency throughout the transcripts. Transcripts were verified by a graduate student for greater accuracy.

### 3.7.5 Coding - A Categorization Approach

The purpose for this step is to identify themes to understand the building of mathematical ideas, the justification of the solutions, and the teacher interventions. Research has shown that certain tasks tend to evoke certain types of justification and reasoning (see literature section for specifics). A "categorization approach" can be developed after the data was carefully studied (Barron, 2007, p. 160). These categories were based on the patterns and forms of reasoning that were found in the existing research and listed in Section 2.3.6. That is, in the towers problem, the researcher looked for evidence of using opposites, cousins,
staircase, elevator, or any other patterns that might have emerged. In all three of the problems, the researcher looked for evidence of controlling for variables, the use of tree diagrams, the use of pictures, or any other technique for building the solution. And in all three problems, the researcher looked for evidence of justifying by a cases argument, an inductive argument, or any other method for justification.

In analyzing the teacher's interventions, a categorization approach was also used. The initial categories used were based on the categories suggested by Martino and Maher (1999). The four types of teacher questioning they proposed was questions that 1) facilitate justification, 2) offer opportunities for generalization, 3) invite opportunities to make connections, and 4) facilitate awareness of solutions presented by other students. These categories were used and while analyzing the data, other categories emerged.

Furthermore, since the data set is small $(n=0)$, a detailed analysis producing a descriptive storyline was possible. Derry et al. (2010) refer to this step as a "play-by-play." "Play-by-play analyses are particularly effective at showing how the sequentially developing context relates to what happens next." (Derry et al., 2010, p. 22) Having a descriptive storyline enabled the researcher to identify the categories.

### 3.7.6 Constructing a Storyline

After transcribing and identifying critical events, Powell et al. (2003) suggest constructing a storyline. This phase of the analysis requires interpretation and inferences by the researcher based on the data provided. "Constructing a storyline requires the researcher to come up with insightful and coherent organizations of the critical events, often involving complex flowcharting" (Powell et al., 2003, p. 430). This flowcharting, also referred to as a trace, provides insight into a student's developing mathematical understanding (Maher, 2002).

### 3.7.7 Composing a Narrative

During this phase, the researcher would re-examine the whole data set and completed analysis of critical events and storylines. According to Powell, Francisco and Maher (2003), this phase actually occurs from the beginning of the research. "Researchers' questions as well as data-gathering procedures and media all imply explicit or implicit choices informed by open or hidden, conscious or unexamined theoretical perspectives. It is in this sense that the construction of a narrative begins at the initiation of research and accounts for why somewhere within a research report, researchers outline their theoretical biases." (Powell et al., 2003, p. 431)

### 3.7.8 Document Analysis

This step is not one of the seven steps outlined by Powell et al. (2003). The students' written work that was captured on videotape was examined to aid in the data analysis and construction of the storyline.

## CHAPTER 4: RESULTS

### 4.1 The Towers Problem

The towers problem was introduced on February 11, 2011 and discussed again on February 18, 2011. Both of these sessions were videotaped. The problem was presented on the board (see figure 4.1). The board read, "You have two colors of Unifix cubes to choose from. How many towers that are 4 cubes tall is it possible to build? Part 1: What's the answer? Part 2: Convince me that your answer is correct."

```
        The Towers Problem
    - You have two colors of Unifix cubes to
        choose from. How many towers that
        are 4 cubes tall is it possible to build?
        Part I:What's the answer?
        Part 2: Convince me that your answer is
        correct.
```

Figure 4.1. Camera view of the front board.

When discussing towers, the first cube described is the top cube and the fourth cube is the bottom cube. For example, RRWW will symbolize a tower with two reds cubes on top, followed by two white cubes.

### 4.1.1 February 11, 2011

On Friday, February 11, all six of the students were present. The class began with the instructor introducing the towers problem. There was one camera and three groups of two. One of the groups of two had already worked on the problem in another class. These students, Francesca C. and Rebecca, were not filmed building their solution. However, they are filmed explaining their solution to the problem. The other two groups were: (1) Jessica and Jamie and (2) Kim and Francesca S. Each of the three groups used the Unifix cubes to solve the problem and each found the answer to be 16 .

## Jessica and Jamie (Red and White cubes)

The video began by showing Jessica and Jamie building their towers. Jessica immediately builds an all white tower and an all red tower. They then start to build towers that have two reds and two whites. Jamie creates a tower that is WRWR. Jessica then quickly builds its opposite RWRW. At this point, Jessica says, "Because that's the opposite one?" [Line 1.1.5] Jamie agrees. They continue to use this strategy of opposites. Jamie builds RRWW and then Jessica builds WWRR. [Lines 1.1.1-1.1.10]

The camera focuses on the other group at this point. When it returns to Jessica and Jamie, they have built RRRW and RWWW. Their strategy has changed. This set of towers is not an opposite. If flipped, it would be an opposite. Jamie creates the WWWR tower and they spend some time trying to find with which tower it is to be paired. They rearrange the three towers so that WWWR is paired with RWWW. This pairing falls
under the strategy of "cousins." Jessica builds WWWR (which they already have) to create the opposite of RRRW. At this point, they have built towers by trial and error and then created the opposite for the tower. They have 10 towers (one of these towers, WWWR, is a duplicate). [Lines 1.1.11-1.1.18] Their organizational structure is as shown in Figure 4.2 (the duplicate is emphasized in bold).

| R | W | W | R | R | W | R | $\mathbf{W}$ | R | $\mathbf{W}$ |
| :--- | :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| R | W | R | W | R | W | W | $\mathbf{W}$ | R | $\mathbf{W}$ |
| R | W | W | R | W | R | W | $\mathbf{W}$ | R | $\mathbf{W}$ |
| R | W | R | W | W | R | W | $\mathbf{R}$ | W | $\mathbf{R}$ |

Figure 4.2. Diagram of Jessica and Jamie's first organizational strategy of their towers.

Jessica decides to reorganize the towers they have built. She takes four of the towers and organizes them in a staircase pattern. That is, she organizes them so that she has no white, one white on the bottom, two white on the bottom, and then three white on the bottom. [Lines 1.1.18-1.1.20] She then begins to organize another four so that they are the opposite of these four with no red, one red on the bottom, two red on the bottom, and then three red on the bottom. However, when she gets to the WRRR tower, she realizes that she does not have that tower and takes apart the duplicate WWWR and creates the WRRR tower. They now have a total of ten towers. [Lines 1.1.21-1.1.28] They have RWRW and WRWR plus the eight towers as shown in Figure 4.3.

| R | R | R | R | W | W | W | W |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| R | R | R | W | R | W | W | W |
| R | R | W | W | R | R | W | W |
| R | W | W | W | R | R | R | W |

Figure 4.3. Diagram of Jessica and Jamie's eight towers organized in the staircase pattern.

They stare at these 10 towers for a few seconds and Jessica says, "Oh! In the middle, remember?" and she creates a RWWR tower. [Line 1.1.29] Jamie makes the opposite of this tower WRRW. They now have 12 towers. They have them organized as shown in Figure 4.4. [Line 1.1.34]:

| R | R | R | R | R | R | W | W | W | W | W | W |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| R | R | R | W | W | W | R | R | R | W | W | W |
| R | R | W | W | W | R | W | R | R | R | W | W |
| R | W | W | W | R | W | R | W | R | R | R | W |

Figure 4.4. Diagram of Jessica and Jamie's 12 towers.

The first and the last group of four are the staircase pattern. In the middle they have the towers with two reds and two whites, along with their opposites.

They look at their towers for a few seconds and count the number of whites in each tower. Jessica then says, "Remember, move it down the line?" [Line 1.1.41] They create four towers in an elevator pattern. Each of these towers has three whites and one red. They then create the opposites of these towers using three reds and one white. In addition to the 12 towers, they now have eight more towers. See Figure 4.5 (duplicates emphasized in bold).

| $\mathbf{R}$ | R | R | $\mathbf{W}$ | $\mathbf{W}$ | W | W | $\mathbf{R}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{R}$ | R | W | $\mathbf{R}$ | $\mathbf{W}$ | W | R | $\mathbf{W}$ |
| $\mathbf{R}$ | W | R | $\mathbf{R}$ | $\mathbf{W}$ | R | W | $\mathbf{W}$ |
| $\mathbf{W}$ | R | R | $\mathbf{R}$ | $\mathbf{R}$ | W | W | $\mathbf{W}$ |


| R | $\mathbf{R}$ | R | $\mathbf{R}$ | R | R | W | W | $\mathbf{W}$ | W | $\mathbf{W}$ | W |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| R | $\mathbf{R}$ | R | $\mathbf{W}$ | W | W | R | R | $\mathbf{R}$ | W | $\mathbf{W}$ | W |
| R | $\mathbf{R}$ | W | $\mathbf{W}$ | W | R | W | R | $\mathbf{R}$ | R | $\mathbf{W}$ | W |
| R | $\mathbf{W}$ | W | $\mathbf{W}$ | R | W | R | W | $\mathbf{R}$ | R | $\mathbf{R}$ | W |

Figure 4.5. Diagram of Jessica and Jamie's 20 towers.

The top eight towers are the towers with one cube of a certain tower, organized as in the elevator pattern. The bottom two groups of four on each side contain the towers using the staircase pattern. The middle four towers are the towers with whites together and whites apart (along with their opposites).

Jessica decides to reorganize the towers based on the number of reds. However, during this process, they realize they have a duplicate and they remove one RWWW and one WRRR. [Line 1.1.78] They now have 18 towers as follows as shown in Figure 4.6 (duplicates emphasized in bold).

| $\mathbf{R}$ | R | R | W | $\mathbf{W}$ | W | W | R |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{R}$ | R | W | R | $\mathbf{W}$ | W | R | W |
| $\mathbf{R}$ | W | R | R | $\mathbf{W}$ | R | W | W |
| $\mathbf{W}$ | R | R | R | $\mathbf{R}$ | W | W | W |


| R | $\mathbf{R}$ | R | R | R | W | W | W | $\mathbf{W}$ | W |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| R | $\mathbf{R}$ | R | W | W | R | R | W | $\mathbf{W}$ | W |
| R | $\mathbf{R}$ | W | W | R | W | R | R | $\mathbf{W}$ | W |
| R | $\mathbf{W}$ | W | R | W | R | W | R | $\mathbf{R}$ | W |

Figure 4.6. Diagram of Jessica and Jamie's 18 reorganized towers.

The instructor asks them how they built their towers. Jessica explains that they built the towers that have two of a color and they have four of them as shown. [Lines 1.1.881.1.93] She then explains that they did the towers that have one red and the towers that have one white. They have a total of eight towers with these two groups as shown. [Lines 1.1.94-1.1.96] They explain that they have two towers that are all of one color. [Line 1.1.98] Next, they move the RRRW and RRWW with the solid RRRR. And they move WWRR and WWWR with the solid WWWW to form another grouping. [Line 1.1.98]

They conclude that they have found 18 towers. The instructor asks them if they are finished and they say that they think there are more towers. They tell her that they would like to keep working on it. The instructor leaves them to think. They focus on the four towers that have two of each color and they conclude that "together or separate" are the only way they can do two of a color. [Line 1.1.109] They try to build more towers but they indicate that they believe that they have found all of them. The camera focuses on the other group.

The camera returns with the instructor asking them to explain what they have found. They still have 18 towers and they show the instructor how they have grouped them together. Their organizational strategy has not changed since the last time the camera was focused on them. The top two groups of four are the elevator pattern. The middle four on the bottom are the two whites "together and separate" and their opposites. The two groups of three are towers created by the staircase pattern, minus the last tower which is
included in the elevator pattern. The organizational structure still contains duplicates. The duplicates are within the staircase and elevator patterns. See Figure 4.7.


Figure 4.7. Camera view of Jessica and Jamie's 18 four-tall towers.

The instructor asks them to explain their organizational strategy again to her. Jessica explains the group that has three reds and one white and the group that has three whites and one red. [Lines 1.1.160-1.1.164] There are a total of eight towers. As Jessica explains the group with RRRR, RRRW, RRWW, they realize they have a duplicate when Jessica pulls the RRRW from the group with one white.

| 1.1 .159 | $12: 48$ | Instructor | Ok, so, alright, explain your groupings one more <br> time. |
| :--- | :--- | :--- | :--- |
| 1.1 .160 | $12: 51$ | Jessica | Alright. So this one is, we have three reds and <br> one white in all of these. [Indicating Set 1, <br> $R R R W, R R W R, R W R R, W R R R$ ] |
| 1.1 .161 | $12: 55$ | Instructor | Okay. |


| 1.1 .162 | $12: 56$ | Jessica | So we just went up the line [pointing to each <br> individual white cube in Set ] to where each one <br> of them could be/so/look different. |
| :--- | :--- | :--- | :--- |
| 1.1 .163 | $12: 58$ | Instructor | Ok, three reds and one white. I believe that's the <br> only way to do three reds and one white. |
| 1.1 .164 | $13: 02$ | Jessica | [Pointing to Set 2 - WWWR, WWRW, WRWW, <br> RWWW] And then we did the opposite with three <br> whites and one red. |
| 1.1 .165 | $13: 05$ | Instructor | Okay. <br> 1.1 .166 <br> $13: 06$ |
| Jessica | With this one [Indicating Set 3 RRRR, RRRW, <br> RRWW]. Let me just pull this down so you can <br> see [moving tower RRRW from Set l to Set 3]. Oh <br> maybe not cause... [Putting RRRW back in Set 1] |  |  |
| 1.1 .167 | $13: 12$ | Instructor | I see a problem now that you pulled that one <br> down. Pull that one back down again. |
| 1.1 .168 | $13: 16$ | Jessica | [Putting RRRW back into Set 3] We've have two <br> of the same. |
| 1.1 .169 | $13: 15$ | Instructor | Yes you do. |

After realizing they have a duplicate RRRW, they realize they also have a duplicate WWWR. They come to the conclusion that the answer is 16 towers. [Lines 1.1.170 -

### 1.1.186]

They organize the towers into 6 groups. These groups are (1) two groups of two towers each with two cubes "together or separate" (and their opposites), (2) two groups of three towers with the elevator pattern of one cube starting at the second cube, and (3)
two groups of three towers in the staircase pattern starting with a solid, then one cube on bottom, then two cubes on bottom. See Figure 4.8.


Figure 4.8. Camera view of Jessica and Jamie's 16 four-tall towers.

They reorganize their towers into four groups and show the instructor what they have found (see Figure 4.9). Two of the groups are the staircase pattern with solid, one cube on bottom, two cubes on bottom, and three cubes on bottom. Jessica explains the group of four towers, each containing two colors as "keeping the two apart, keeping the two together." [Line 1.1.209] The fourth group is a group of four with one different color in the second position and then one different color in third position as shown below. The girls do not explain why these four are grouped together.


Figure 4.9. Camera view of Jessica and Jamie's reorganized 16 four-tall towers.

After they show the instructor their towers they explain to her that they believe the answer should be something mathematical. [Line 1.1.211] Jessica says, "Maybe it has to do with squares." [Line 1.1.212] They guess that the number of towers that are three-tall should be nine. Jessica remarks, "If we had 16 for four, maybe three would be nine." [Line 1.1.216] The instructor replies, "Well why don't you try three and see how that works out." [Line 1.1.217]

They work on building the three-tall towers using black and white cubes. Jessica immediately builds three towers in the staircase pattern of BBB, WBB, WWB, and Jamie adds WWW to her collection. They move the WWW and make the two opposite towers of the ones already created - they are BWW and BBW. Next, they make BWB and WBW. They have a total of eight towers (see Figure 4.10).


Figure 4.10. Camera view of Jessica and Jamie's eight three-tall towers.

Jessica says, "Notice this is eight and that was 16. Maybe we will have four for two and it can be like colors squared, not colors, um... two, yeah, two raised to a certain power." [Lines 1.1.244-1.1.245] They build the towers that are two-tall and find four towers. Jessica says, "That's what is it. The powers of 2." [Line 1.1.249] They say that five-tall will be 32 because that is two to the fifth. They tell the instructor that they believe they have figured it out. She says that she will be with them in a moment as she is listening to Kim and Francesca S.'s explanation to the towers problem. The camera focuses on this group for a short period of time.

The instructor and the camera return to Jamie and Jessica's group. They explain to the instructor that after they finished building the eight three-tall towers, they decided build the two-tall towers and found four towers. They realized that it was powers of two. As they are explaining what they have found to the instructor, they discover that the power is equal to the height of the tower.

| 1.1 .260 | $21: 15$ | Instructor | Now I am ready to hear what you guys have to <br> say. [Camera turns and focuses on Jamie and <br> Jessica's group.] |
| :--- | :--- | :--- | :--- |
| 1.1 .261 | $21: 22$ | Jamie | We think we figured it out. We think it's the <br> powers of two. |
| 1.1 .262 | $21: 27$ | Jessica | Yeah, because we, we... when you told us to do <br> [inaudible, pointing to 8 towers, 3-tall each] we <br> only got eight. So we were like let's go down to <br> two and see what we get there and we got 4. <br> Indicating group of 4 towers - each two-tall] So <br> you have two raised to the second power. |
| 1.1 .263 | $21: 38$ | Jamie | Do you know what it is? It's whatever number of <br> towers - |
| 1.1 .264 | $21: 40$ | Jessica | That's the power. |
| 1.1 .265 | $21: 41$ | Jamie | That's the power. |
| 1.1 .266 | $21: 43$ | Instructor | Oh.... |
| 1.1 .267 | $21: 44$ | Jamie | Two squared, two to the third, two to the fourth. |
| 1.1 .268 | $21: 46$ | Instructor | So you could tell me how many there's gonna be <br> five-tall - without doing it? |
| 1.1 .269 | $21: 50$ | Jessica | That's 32. |

The instructor tells them that what they have discovered is very nice. She asks them to explain their organizational strategy for the three tall towers. Jessica explains that, with the group of three towers that form the staircase pattern (BBB, WBB, WWB) that they could not put another tower there because it would be WWW and that is already in the other group. [Line 1.1.272] She explains that because of this, there are no more towers in
this group. The instructor says, "Okay, so that is like proof by contradiction - if you can't go any further down because there's no place else to go, right?" [Line 1.1.275] They agree.

The instructor asks them to repeat their formula again. Jamie explains that it is two to the power where the power is the height. [Line 1.1.278] And Jessica hypothesizes that the two is equal to the number of colors. The instructor asks what they think the answer is if there are three colors.

| 1.1 .282 | $22: 58$ | Jessica | And two is the, the amount of um, the colors. I'm <br> thinking. |
| :--- | :--- | :--- | :--- |
| 1.1 .283 | $23: 04$ | Instructor | Ok, so maybe you want, might need a piece of <br> paper for this. Suppose there was three colors - <br> what's it gonna be? |
| 1.1 .284 | $23: 11$ | Jamie | Three raised to the.... |
| 1.1 .285 | $23: 12$ | Jessica | To however tall it is. |
| 1.1 .286 | $23: 15$ | Instructor | So, why don't you get a third color? |

She suggests that they start with three colors two-tall. [Line 1.1.289] Jessica and Jamie work on this problem while the camera focuses on the other group.

When the camera returns, it is shown that they have built nine towers that are two-tall when choosing from three colors. They explain to the instructor that they found nine towers. She asks them for the general formula for any color, any height.

| 1.1 .316 | $26: 33$ | Instructor | I already asked you... the extensions was if there $n$ <br> cubes tall and you got two colors to choose from. <br> You know the answers for that. Now you got $m$ <br> colors to choose from, I want that equation. |
| :--- | :--- | :--- | :--- |
| 1.1 .317 | $26: 46$ | Jessica | Oh, 'cause we figured out when you have 2 colors <br> to the $n$. So now it's going to be $m \ldots$ so however <br> many total.... Oh, $m$ to the $n$. |
| 1.1 .318 | $26: 59$ | Jamie | Right. |

Jessica writes in her notebook both formulas $2^{n}$ and $m^{n}$ where $m$ is equal to the colors and $n$ is equal to the height of the tower (see Figure 4.11). [Line 1.1.319]


Figure 4.11. Camera view of Jessica's notebook.

After each group is finished working on the towers problem, each group presents their findings to the class. This group was the last group to present. When they present their four-tall towers, their organizational strategy has changed. They now have four groups of four. However, they do not explain this different organizational strategy. The top two groups contain the towers that create a staircase pattern. The bottom left group contain the towers were two whites are "together or separate" as well as their opposites. And the
last group contains the solids and two more towers with two cubes of two colors together and their opposite. See Figure 4.12.


Figure 4.12. Camera view of Jessica and Jamie's 16 towers during their presentation.

Jamie and Jessica do not explain their organizational strategy for their 16 towers to the class. Instead, they explain the general rule for the towers problem. They explain that the formula is the number of colors raised to the height of the tower. [Lines 1.1.3601.1.373] They demonstrate the formula using three colors. They have built nine two-tall towers when choosing from three colors. When the height is two, they explain to the class that they have nine towers. [Lines 1.1.374-1.1.378] The instructor begins a class discussion about the reasons why the formula is $3^{n}$. This class discussion will be described after the explanation of Rebecca and Francesca C.'s results.

Kim and Francesca S. (Blue and Yellow cubes)

The camera has focused on Jamie and Jessica for the first ten minutes of this session. Therefore, the camera did not capture Kim and Francesca S. building their solution to the towers problem. When the camera first focuses on this group, the instructor asks them to explain to her their strategy for building their towers. They explain that they built them by doing opposites. Kim explains, "Like, I would do one thing and then she would do the opposite." [Line 1.1.124] The instructor asks her to explain opposite. She says, "Meaning, like for this one there's blue, yellow, blue, blue. So then the opposite is yellow, blue, yellow, yellow." [Lines 1.1.126-1.1.128] They have built 16 towers and they have organized them into six groups as shown in Figure 4.13.


Figure 4.13. Camera view of Kim and Francesca S.'s first organizational structure for their 16 towers.

They explain that there is a group of two towers that are all of one color. The group of three contains a tower with one blue, two blue, three blue in a staircase pattern. The group of five contains an alternating blue and yellow tower with its opposite. This group
of five also contains the three towers that create the elevator pattern: one yellow, two yellow and three yellow.

The instructor explains to them that she does not see a pattern to their organizational structure. Kim and Francesca S. explain to her that they are organized based on the order of the cubes.

| 1.1 .143 | $11: 27$ | Kim | It depends on the order. |
| :--- | :--- | :--- | :--- |
| 1.1 .144 | $11: 28$ | Francesca S. | Yeah. |
| 1.1 .145 | $11: 29$ | Instructor | Okay. |
| 1.1 .146 | $11: 29$ | Kim | There's a specific order. |
| 1.1 .147 | $11: 30$ | Instructor | Okay.... So explain the order. Explain why this <br> goes here [pointing to BYYB, YBBY] and not with <br> those over there [pointing to YBYB, BYBY] |
| 1.1 .148 | $11: 38$ | Francesca S. | [Points to YBYB, BYBY] Because these are <br> alternating. These are like [inaudible, pointing to <br> BYYB, YBBY]. |
| 1.1 .149 | $11: 41$ | Instructor | Okay, so... so these are blue. You mean, so this is <br> two... you know, this is sort of like what she said <br> in the video - these are two took apart [indicating <br> YBYB, BYBY] and these are two stuck together, <br> kind of? [Indicating BYYB, YBBY]. |
| 1.1 .150 | $11: 52$ | Kim | Yeah, yeah. |

The instructor suggests to them to organize the towers such that all the towers with one color are together, all the towers with two colors are together, and all the towers with
three are together. They continue to work on this and the camera focuses back on Jessica and Jamie.

At 20:54, the camera returns to Kim and Francesca S. They have 16 towers and they have organized them into five groups. The groups are composed of the towers that contain zero blue cubes, one blue cube, two blue cubes, three blue cubes, and four blue cubes. See Figure 4.14.


Figure 4.14. Camera view of Kim and Francesca S.'s second organizational structure for their 16 towers.

The instructor explains that she can see the pattern for all of the groups except for the group of six in the middle. She explains that she does not see how these towers form a group. Kim asks, "Like a pattern?" [Line 1.1.258] She agrees and leaves them to work on it. The camera leaves them as well.

At 23:46 and then again at 28:14, Kim and Francesca S. present their solution. They explain that they have five groups. They are no blue, one blue, two blue (these six towers
are broken into two sub-groups - "two blues are stuck together" and "separated blues"), three blue, and all blue. [Lines 1.1.328-1.1.343]


Figure 4.15. Camera view of Kim and Francesca S.'s third organizational structure for their 16 towers.

## Rebecca and Francesca C. (Blue and Orange cubes)

These girls are not filmed building their towers but the class and the videographer focus on their towers at 29:33. They have towers that are one-tall, two-tall, three-tall, and four-tall. Each of these groups of towers of different heights is organized. Francesca C. explains how they organized the four-tall towers. They have all blue, three blues, a middle group of six with two blues that were broken into pairs (opposites), three orange, and all orange. [Lines 1.1.345-1.1.350] Below these towers of four, they have towers that are three-tall, two-tall and one-tall. See Figure 4.16 (the single blue cube is in Rebecca's hand).


Figure 4.16. Camera view of Rebecca and Francesca C.'s presentation of towers.
Rebecca explains that if you start with towers that are one cube tall, there are only two towers. She describes how to build towers one cube taller based on the towers of the previous height. She explains that for the towers that are one cube taller, you can add either a blue or an orange cube to the top of each tower.

| 1.1 .352 | $30: 47$ | Rebecca | So, if you have just one-tall tower, you only have <br> two [indicating one blue cube and one orange <br> cube] And then in order to get the second one with <br> yellow, you can add a blue and you'll get this one <br> $[B O]$. Or you can add another orange and you'll get <br> this one [OO]. So, for each tower, you add one or <br> the other to get the next group. To double it. |
| :--- | :--- | :--- | :--- |
| 1.1 .353 | $31: 06$ | Rebecca | See with this one, you can either add a blue to get <br> that one, or an orange to get the next one. |

She reiterates and explains, "So then for each tower.... to make it one cube higher, you can add either an orange or a blue, so it would essentially double what you have." [Line 1.1.357]

The instructor says, "Many of you discovered the rule - it doubles - and this is the explanation. Right? This is the reason why it doubles. So there's more than just, yeah, we see a pattern its time two. Here's the reason why its time two. Right? Inductive reasoning, right? See, that wasn't too bad." [Line 1.1.358]

Next, Jessica and Jamie explain the general rule for the towers problem and demonstrate the solution to the number of towers two-tall when choosing from three colors as described earlier. After their explanation, the instructor asks them to show the group how many towers there would be for one-tall towers choosing from three colors. They say there are three towers. The instructor asks the class to explain why the base is equal to the number of colors. Rebecca explains that to build a new tower from the previous one, you have three colors to choose from so the towers triple each time.

| 1.1 .390 | $35: 07$ | Rebecca | For each one-tall tower you can, for this one <br> [indicating B] you can add either a brown, a green, <br> or a maroon. For this one, [indicating G] you can <br> add either a brown, a green, or a maroon. So, for <br> each one, there's three possible towers you can <br> make to create it two tall. So, you add a green, you <br> know, you can add a green, you can add a maroon, <br> or you can add a brown. So you end up with, you <br> know, three more from what you already have. |
| :--- | :--- | :--- | :--- |
| 1.1 .391 | $35: 33$ | Instructor | Does that make sense to everybody? |
| 1.1 .392 | $35: 35$ | Francesca C. | So the answers triple. |
| 1.1 .393 | $35: 36$ | Instructor | That's right - the other one was doubled and this <br> one now is tripled. |


| 1.1 .394 | $35: 40$ | Francesca C. | Because it's three colors. |
| :--- | :--- | :--- | :--- |

The instructor introduces Ankur's Challenge and for the remainder of the class session, these students work on solving Ankur's Challenge.

### 4.1.2 February 18, 2011

During the first 20 minutes of the class on Friday, February 18, the class revisits the towers problem. Only four students were present and there were two videographers. They worked in groups of two. The groups were the same as February 11, 2011. They were (1) Jessica and Jamie and (2) Kim and Francesca S. The instructor began the session with a PowerPoint slide on the board that reads as follows, "The Towers Problems. Summarize our previous results: Two colors, four cubes tall: 16. You organized your towers by number of blue cubes. How many towers for 0 blue, 1 blue, 2 blue, 3 blues, and 4 blues? Two colors, $n$ cubes tall: $2^{n}$ Why is it $2^{n} ? m$ colors, $n$ cubes tall: $m^{n}$ Why is it $m^{n}$ ?"(See Figure 4.17)

## Summarize our previous results: <br> Two colors, four cubes tall: 16 <br> - You organized the towers by \# of blue cubes <br> - How many towers for 0 blue, 1 blue, 2 blue, 3 blues, and 4 blues? <br> Two colors, $n$ cubes tall: $\mathbf{2}^{\text {n }}$ <br> - Why is it $2^{m}$ ? <br> $m$ colors, $n$ cubes tall: $m^{n}$ <br> - Why is it m?

Figure 4.17. Camera view of front board.

Kim and Francesca S. (Blue and Orange cubes)

Kim and Francesca S. begin by building their towers in the staircase pattern.
Francesca S. builds the towers that contain one blue on top, two blues on top, three blues on top, and the all blue tower. Kim builds the opposites of these four towers. She builds the towers that contain one orange on top, two oranges on top, three oranges on top, and the all orange tower. They have a total of eight towers. [Lines 2.1.1-2.1.12] They build four more towers by building a tower and the opposite. These towers contain two blues (BOBO, OBOB, OBBO, BOOB). [Lines 2.1.12-2.1.23]

They recognize that they are missing four because they understand the answer to be 16. However, they are not sure which four they are missing. They sit silently. Using trial and error, Kim makes the BOBB tower and asks Francesca S. if they have created that one yet. [Line 2.1.27] Francesca S. replies that they have not. Kim, again using trial and
error, makes another tower - BBOB. [Line 2.1.29] (At this point, they do not build the opposites of these two newly created towers.) They now have 14 towers. The instructor joins their group and tells them that they are missing some. They agree. She instructs them to reorganize their towers so they have no blue, one blue, two blue, three blue and four blue. [Line 2.1.33]

As they organize the towers in this manner, they realize they are missing two towers with one blue cube. They build these two towers, OBOO and OOBO. These towers are the opposite of the towers they had just created. They continue to organize their towers. In the end, they have five groups organized by number of blues (see Figure 4.18). [Lines 2.1.34-2.1.68]


Figure 4.18. Camera view of Kim and Francesca S.'s 16 towers.

They are instructed to write the number of no blue towers, one blue, two blues, etc. on their paper. Kim writes the number of towers for the zero blue case, one blue, two blue, three blue and four blue case (see Figure 4.19).


Figure 4.19. Camera view of Kim’s notebook.

The instructor asks them to explain, on their paper, why the formula is $2^{n}$. The instructor joins Jessica and Jamie's group and Jessica says, "We did "why is it two to the $n$ ?" It's two because the two represents the number of colors. Two is the base." [Line 2.1.93] Kim and Francesca S. proceed to engage in a conversation about the formula after hearing Jessica.

| 2.1 .94 | $10: 24$ | Kim | That's true. There are two colors. [laughing] |
| :--- | :--- | :--- | :--- |
| 2.1 .95 | $10: 29$ | Francesca S. | Yeah, it's the two colors and the four cubes. |
| 2.1 .96 | $10: 32$ | Kim | So then two to the fourth equals sixteen. |

The camera focuses on Kim's paper. She writes that the "two means blue/orange" and " $n$ cubes tall." Francesca has, on her paper, that two is the number of colors and $n$ is the height of the tower. The instructor asks them if they remember the inductive explanation that Rebecca had given a week earlier.

| 2.1 .120 | $13: 30$ | Instructor | Do you remember when she was doing her little <br> proof? She started out with something like... I'm <br> going to take this away and put it back later. [She <br> takes apart one of the towers previously built.] She <br> started out with there's one tall towers, right? [She <br> puts a single blue cube down and a single orange <br> cube down.] You actually missed this discussion in <br> class on Wednesday so it's good to go over. <br> [Talking to Francesca S.] That's it, right? [pointing <br> to the single orange and single blue cube] This is it. |
| :--- | :--- | :--- | :--- |
| 2.1 .121 | $13: 46$ | Kim | Oh, then don't you add one and then it would be like <br> one.... |
| 2.1 .122 | $13: 50$ | Instructor | You add... Well, sort of, yes. |
| 2.1 .123 | $13: 53$ | Instructor | Now this is... How does this relate to what we are <br> doing induction? Here's step one - $n$ tall towers. <br> One tall towers, right? Step one, you pick some low <br> number. |
| 2.1 .124 | $14: 03$ | Instructor | Step two: you say "I'm at some height." We don't <br> have to think about that too much, but. What do you <br> do for each one of these? You started to say it... <br> You can either do what or what? |
| 2.1 .125 | $14: 13$ | Kim | Onh, you can put the blue on it or you can put the <br> orange on it. |
| Instructor | Right, so each one, you can put either a blue or an <br> orange and that gives you two choices. There's the <br> induction part - no matter where you start the next <br> one is going to be twice as many because you can do <br> either the blue or the yellow. |  |  |
|  |  |  |  |

After this discussion, Kim, Francesca S., and the instructor join Jessica and Jamie to listen to Jessica and Jamie explain towers when choosing from three colors. (This
conversation will be described after the description of Jessica and Jamie building their towers.)

## Jessica and Jamie (Blue and White cubes) - Second Camera View

Jessica builds the all blue tower and the blue towers with one white cube. At the same time, Jamie is building their opposites - the all white tower and the white towers with one blue cube. They both build the towers with one cube using the elevator pattern. They separate the towers they have created into 4 groups: (1) all blue, (2) towers containing one white cube, (3) towers containing one blue cube, and (4) all white. [Lines 2.2.12.2.3]

Jessica starts to build the towers with two blue by doing opposites. She creates BBWW and then WWBB. They decide that they will each build the towers with 2 blues and then compare and take out any extras. Jessica builds BWBW and WBWB. She builds BWWB and WBBW. Jamie has built four towers - BBWW, WWBB, BWBW, and WBWB. They realize that all four of Jamie's towers are contained in Jessica's group of six. They disregard Jamie's four towers. They have organized their towers by groups based on the number of blue cubes in a tower as shown in Figure 4.20. [Lines 2.2.62.2.16]
B
W
B
W
B W
B W

| W | B | B | B | B | W | W | W |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| B | W | B | B | W | B | W | W |
| B | B | W | B | W | W | B | W |
| B | B | B | W | W | W | W | B |


| W | B | B | W | W | B |
| :---: | :---: | :---: | :---: | :---: | :---: |
| B | W | B | W | B | W |
| B | W | W | B | W | B |
| W | B | W | B | B | W |

Figure 4.20. Diagram of Jessica and Jamie's 16 towers.

The instructor tells them to write, in their notebook, the number of towers in each group. As Jessica starts to write based on the number of blues, she says, "Alright, so four blue can only have one tower, three blue we have four towers, and then two blue is six towers. And I have to do one blue with the whites. [She takes the cubes and starts to build towers.]" [Line 2.1.25] Jessica explains to Jamie that she has to create the towers with one blue and the tower will all white cubes while Jamie must create the towers that have one white and the all blue tower. (They do not realize that Jamie's group of towers containing three white is equivalent to a group containing one blue.)

Jessica creates the towers with one blue and no blues. Jamie creates the towers with one white and no white. As they are creating these towers, the instructor questions them as to whether they must build those towers. At this point, they realize that these towers
would be the same as towers they have already created. Regardless, they create them. They have nine groups of towers as shown in Figure 4.21. [Lines 2.2.26-2.2.40]


Figure 4.21. Camera view of Jessica and Jamie's four-tall towers.

Jessica and Jamie write their results in their notebooks. They explain that the answer is the same whether they base the solution on the number of blue cubes or the number of white cubes. As Jessica writes in her notebook, she says, "One, four, six, four, one. And the same thing for white." [Line 2.2.41] Jamie replies, "Is the same thing." [Line 2.2.42]

The next assignment is to explain why the formula is $2^{n}$. Jamie immediately says, "Oh, because of the two color thing?" [Line 2.2.45] Jessica agrees, "Because it's two colors." [Line 2.2.46] They do not verbally explain why the exponent $n$. Instead, they answer the question, why is it $m^{n}$ ? Jessica explains that $m$ is equal to the number of colors and Jamie says that $n$ represents the height of the tower.

| 2.2 .50 | $07: 43$ | Jessica | Why is it $m$ to the $n$ ? Because $m$ represents the <br> number of colors. |
| :--- | :--- | :--- | :--- |
| 2.2 .51 | $08: 02$ | Jamie | Amount of colors. |
| 2.2 .52 | $08: 03$ | Jessica | Different colors, maybe? |
| 2.2 .53 | $08: 03$ | Jamie | Uh-huh. And $n$ represents the height of the tower? |
| 2.2 .54 | $08: 19$ | Jessica | Yeah. [Jamie and Jessica write in their <br> notebooks.] |

The instructor asks them to explain, to her, their answers to why the formula is $2^{n}$. They tell her that the two represents the two colors and $n$ is equal to the height of the tower. They also explain that, in the formula $m^{n}, m$ is the number of colors and $n$ is the height of the towers. She agrees but explains to them that they did not explain why it is $2^{n}$. She says, "I understand two different colors but you didn't exactly give me $100 \%$ reason why it's two to the $n$ as opposed to, say, two times $n$. "Why is it two to the $n$ power?" is the question." [Line 2.2.65]

The instructor asks them if they remember Rebecca's explanation from the previous week. Jamie asks if it has to do with choices. Jamie says, "Yeah, the choices that you are allowed and you can't have the same thing so that eliminates like the extra choices?" [Line 2.2.71] She tries to explain further, but cannot. Jessica says, "... the only thing I can remember truthfully is that you have two choices. You can either add a white one on or you can add a blue one on." [Line 2.2.73] They proceed to talk about induction and the formula $m^{n}$.
\(\left.$$
\begin{array}{|l|l|l|l|}\hline 2.2 .74 & 10: 27 & \text { Instructor } & \begin{array}{l}\text { Okay, we said this last time. That's sort of like } \\
\text { induction. Right? So no matter where you're } \\
\text { starting from, like if you're starting from these } \\
\text { [points to the four towers with three blue cubes and } \\
\text { one white cube], each one of these, say it again, } \\
\text { you can.... }\end{array}
$$ <br>

\hline 2.2 .75 \& 10: 38 \& Jessica \& You can either have a white one or a blue one.\end{array}\right\}\)| 2.2 .76 | $10: 40$ | Instructor | And the fact that you have two choices means <br> you're multiplying by two. So that's what I'm <br> getting at - multiplying by two, multiplying by two, <br> means two to the $n$. |
| :--- | :--- | :--- | :--- |
| 2.2 .77 | $10: 48$ | Jessica | Oh, that's right. <br> And so similarly for $m$ to the $n$, the $m$ choices. <br> Which means every single time you know you <br> have [inaudible]. |
| 2.2 .78 | $10: 51$ | Instructor | Yeah, you have that many choices; you have to <br> keep multiplying by that. |
| 2.2 .79 | $10: 58$ | Jessica |  |

The instructor instructs them to write their findings in their notebook and to prepare to explain $3^{n}$ to Kim and Francesca S.

They discuss towers choosing from three colors and decide to use blue, white and yellow cubes. They position a yellow cube on its side and one cube of each color above it to represent the three towers that are two-tall when choosing from three colors where yellow is the bottom cube. They repeat this process for blue as the bottom cube and for white as the bottom cube. They have the cubes, laid on their sides, as shown in Figure

### 4.22. [Lines 2.2.86-2.2.97]

$\begin{array}{lllllllll}Y & B & W & Y & B & W & Y & B & W \\ & Y & & & B & & & W & \end{array}$

Figure 4.22. Diagram of Jessica and Jamie's cubes for their explanation of two-tall towers choosing from three colors.

They discuss inductively how to build the two-tall towers when choosing from three colors.

| 2.2 .98 | $13: 29$ | Jessica | So one tall you have three choices which is three <br> to the first power. |
| :--- | :--- | :--- | :--- |
| 2.2 .99 | $13: 35$ | Jessica | Two tall, you have three more choices per cube. |
| 2.2 .100 | $13: 39$ | Jamie | Which equals nine. |
| 2.2 .101 | $13: 39$ | Jessica | Which equals nine. |

They are asked to explain the formula for towers (any height) choosing from three colors to the class. At this point, the instructor, Kim and Francesca S. joins them to listen to the explanation. Jamie and Jessica take turns explaining.

| 2.2 .109 | $14: 11$ | Jamie | One tall, okay, one-tall would be three to the one <br> which is three. |
| :--- | :--- | :--- | :--- |
| 2.2 .110 | $14: 16$ | Jessica | Because you only have three choices. |
| 2.2 .111 | $14: 17$ | Jamie | 'Cause you can only have three choices. |
| 2.2 .112 | $14: 20$ | Jessica | Then when you get to two tall, you have three <br> choices per the one that you already have. So <br> you have the yellow can either be yellow- <br> yellow, yellow-blue, or yellow-white. Blue can <br> be blue-yellow, blue-blue, or blue-white. And |


|  |  |  | white can be white-yellow, white-blue, or white- <br> white. So, since you have three choices each <br> time, it's three squared this time... um.. I'm <br> trying to think... |
| :--- | :--- | :--- | :--- |
| 2.2 .113 | $14: 47$ | Instructor | That's it! Three squared is... |
| 2.2 .114 | $14: 49$ | Jessica | Three squared is nine. So you have nine total <br> choices, nine total ways that you can do it. |

The instructor asks the class to tell her how many towers there are if the towers are three-tall. Francesca S. says, "Nine squared." [Line 2.2.116] Kim says, "Nine cubed." [Line 2.2.118] The instructor indicates that they are incorrect and Kim says, "I mean three cubed." [Line 2.2.120] The instructor replies, "Yes." Jessica and Jamie start to create a third row that would demonstrate towers that are three-tall when choosing from three colors (three to the third power). Jessica remarks that it is like the "tree-thing." They do not create all 27 towers. [Lines 2.2.122-2.2.124]

The instructor says, "Okay, so, that's great! There's your induction, right? No matter where you are at, you can always go to the next step with times three." [Line 2.2.125] She asks the class if they are satisfied with that explanation. They reply that they are and they begin to work on the pizza problem.

### 4.2 The Pizza Problem

On February $18^{\text {th }}$, after revisiting the towers problem, the students work on the pizza problem. There are four students present and two videographers. The students are paired
as previous problems. Kim and Francesca S. are one group while Jamie and Jessica are the other group. They start the pizza problem around minute 17:20. After they have finished solving the pizza problem, they work on explaining the isomorphism between the pizza problem and the towers problem.

The instructor presents the pizza problem with four toppings on a PowerPoint slide. The slide reads:

The Pizza Problem<br>There are four possible pizza toppings:<br>Sausage<br>Peppers<br>Pepperoni<br>Mushrooms<br>You can have a plain pizza (no toppings), or a pizza with any combination of the above toppings. How many pizzas is it possible to make?<br>Part 1: What's the answer?<br>Part 2: Convince me that your answer is correct.

## Kim and Francesca S.

Kim and Francesca S. write in their notebooks to solve this problem. They work separately but occasionally talk to each other. Kim suggests that they use a "tree." Francesca S. says, "It's probably easier for me to just list it." [Line 2.1.165] They are writing in their notebooks.

The camera focuses on Kim's paper and she is drawing a modified tree diagram. On the top of her paper, she has "Plain Pizza." Underneath this heading, she has a big circle
with four branches. Each of the branches is labeled Sausage, Peppers, Pepperoni, and Mushroom. [Line 2.1.166]

Then, off the Sausage branch, Kim creates three more branches and labels each of them - Peppers, Pepperoni, and Mushroom. Off of the Peppers branch, she creates two branches and labels them Pepperoni and Mushroom. Off of the Pepperoni branch she creates one branch and labels it Mushroom. She says, "And by the time you get to the mushroom, there's like nothing." [Line 2.1.176]

On the side of the paper, she writes pepperoni, sausage, and peppers. As she writes them, she says, "Then you have plain pepperoni, then you have plain sausage, plain peppers". [Line 2.1.177] She numbers the pizzas she has created starting with the plain pizza as \#1. She counts that she has 11 pizzas. See Figure 4.23.


Figure 4.23. Replication of Kim's first drawing of her solution to the pizza problem.
The instructor looks at Kim's work and asks her to explain what she has done. She explains that she has a plain pizza, pepperoni pizza, sausage pizza, and peppers pizza. The instructor asks, "Another question - how come you left out mushroom?" [Line 2.1.191] Kim replies, "Oh! Ok. Mushroom." [Line 2.1.192] She adds mushroom to the list and labels this as \#5. (She does not realize that she has the mushroom pizza as \#11.)

Kim continues to explain to the instructor her pizzas. She explains that the other pizzas are the pizzas with two toppings (she does not go through them). She then realizes that she has not done the pizzas with three toppings and renumbers her pizzas so that she now has 12 pizzas (the mushroom pizza is counted twice). [Lines 2.1.193-2.1.200]

The camera focuses on Francesca S.'s paper. She has the pizzas listed. The videographer asks her what she is doing. She replies, "I'm just doing with the one topping, then two topping, then three topping, then four topping. Like each different that's like the easiest way to do it." [Line 2.1.201] Her pizzas are listed as shown in Figures 4.24 and 4.25.

1 Plain
2 S
3 Pep 1 Topping
4 Pepperoni
5 Mushroom
6 S, P
7 S, Peperoni
2 Toppings
8 S, Mush
9 Pep, Pepperoni
10 Pep, Mushrooms
Figure 4.24. Replication of Francesca S.'s notebook of her solution to the pizza problem.


Figure 4.25. Camera view of Francesca S.'s notebook of her solution to the pizza problem.

Francesca S. pauses in the middle of trying to write the eleventh pizza. With her pencil, she silently goes over her two topping pizzas. She checks these pizzas by starting with sausage and running through her one topping pizzas. That is, she points to "S" (sausage) and then points to "Pep" (peppers). She points to "S" (sausage) and then points to "pepperoni." She points to "S" (sausage) and then points to "mushroom." She then moves to "Pep" (peppers) and points to "pepperoni." She points to "Pep" (peppers) and points to "mushroom." [Line 2.1.202] She erases the pepperoni and writes mushroom. But then she quickly erases the mushroom. Kim asks her how many she has so far and she replies, "Ten." [Lines 2.1.203-2.1.205]

Francesca S. begins to write the three-topping pizzas. The first three-topping pizza she writes down is S, Pep, Pepperoni. Kim asks her, "So when you get to three topping, there would only be one, right?" [Line 2.1.208] Francesca S. replies, "Yeah, but you have to put like mushroom. Cause there's sausage, peppers, and pepperoni, but where do you stick mushrooms? You know what I mean? Like, you have to make like a new one." [Line 2.1.209] She continues writing her list and compares with Kim. They both found a total of four with three toppings. [Lines 2.1.224-2.1.238] See Figure 4.26 and Figure 4.27.

1 Plain
2 S
3 Pep
4 Pepperoni
5 Mushroom
6 S, Pep
7 S, Pepperoni 2 Topping
8 S, Mush
9 Pep, Pepperoni
10 Pep, Mushrooms
Figure 4.26. Replication of Francesca S.'s notebook of her solution to the pizza problem.


Figure 4.27. Camera view of Francesca S.'s notebook of her solution to the pizza problem.

Francesca S. asks Kim how many two-topping pizzas she has. Kim replies that she has seven. Francesca S. explains that she only has five two-topping pizzas. The instructor suggests that they compare answers. They begin to compare their two-topping pizzas and, immediately, Kim realizes that she has a mushroom pizza listed under her two-topping pizzas. She crosses this pizza out and replies that she has six, two-topping pizzas. See Figure 4.28. [Lines 2.1.239-2.1.246]


3 Toppings<br>Pepperoni, Peppers, Mushroom<br>Sausage, Pepperoni, Peppers<br>Mushroom, Sausage, Peppers<br>Mushroom, Pepperoni, Sausage

Figure 4.28. Replication of Kim's second drawing of her solution to the pizza problem.

Kim and Francesca S. continue to compare their two- topping pizzas and Francesca S. discovers that she has missed pepperoni and mushroom. She adds this two-topping pizza to her list. While the camera was focused on Kim's paper, Francesca S. had added the four topping pizza. Her list is now as appears in Figure 4.29.

1 Plain
2 S
3 Pep
4 Pepperoni
5 Mushroom
6 S, Pep
7 S, Pepperoni
8 S, Mush
9 Pep, Pepperoni
10 Pep, Mushrooms
Pepe, Mushroom
Figure 4.29. Replication of Francesca S.'s notebook of her solution to the pizza problem.

Both of the girls have found 16 pizzas. They tell the instructor that they believe they are finished. The instructor asks them to organize their results by writing down the number of plain pizzas, the number of one-topping pizzas, the number of two-topping pizzas, etc. She also asks them to come up with a convincing argument that they have found them all. [Lines 2.1.261-2.1.272]

They both write in their notebooks that there is one pizza for the plain, four for the one-topping pizzas, six for the two-topping pizzas, four for the three-topping pizzas, and one for the pizza with all of the toppings. Francesca S. suggests that it might have something to do with $m^{n}$.

| 2.1 .293 | $31: 07$ | Francesca S. | No, I'm trying to think of something that has to do <br> with like $m$ to the $n$. |
| :--- | :--- | :--- | :--- |
| 2.1 .294 | $31: 10$ | Instructor | Ah, okay. |
| 2.1 .295 | $31: 11$ | Francesca S. | Like, if that's the reason for... [inaudible] |


| 2.1 .296 | $31: 14$ | Instructor | Well, then. My next question was... So what do you <br> think the answer would be if you had five toppings or <br> if you had three toppings? |
| :--- | :--- | :--- | :--- |
| 2.1 .297 | $31: 22$ | Francesca S. | Yeah, that's what I'm not sure 'cause I don't know <br> like the only way to get 16 would be four to the <br> second, four to the two. |
| 2.1 .298 | $31: 29$ | Instructor | So, if that's true, what do you think you would get if <br> there was only three toppings to choose from? |
| 2.1 .299 | $31: 36$ | Francesca S. | Would it be nine? I don't know if it would be nine. |
| 2.1 .300 | $31: 38$ | Instructor | Well, why don't you do a three topping case and see <br> what you get? |

Kim says, "Maybe it's the toppings like you are raising it to the toppings. It's like, 'cause you're having one pizza but like you can raise it to however many toppings there are." [Line 2.1.308] Francesca S. replies, "Yeah, something like that." [Line 2.1.309] But then Kim retracts her statement, "But you wouldn't raise one to the sixteenth - that wouldn't make sense." [Line 2.1.312] Francesca S. replies, "I know. I want to see what I get for three." [Line 2.1.313] She is working on listing all pizzas when choosing from three toppings. See Figure 4.30.

S, P, M
Plain
1 Topping
S
P
M

2 Topping
S, P
S, M
P, M
3 Topping
S, P, M
Figure 4.30. Replication of Francesca S.'s notebook for the pizza problem when choosing from three toppings.

After she completes her list, she says, "The three topping is nine." [Line 2.1.322] As Francesca S. shows the instructor her solution, she realizes that she only has eight. The instructor asks her to explain her eight to Kim. Together, they discuss that for four toppings, they got 16. For three toppings, they have eight. They ask each other if they would get four pizzas if there were only two toppings. Francesca S. decides to write out the pizzas if there were two toppings from which to choose. She finds four pizzas. [Lines 2.1.323-2.1.345]

Kim and Francesca S. find that if there was only one topping to choose, there would be two pizzas and if there were no toppings, there would be one plain pizza. Kim questions if the answer is 32 pizzas if there were five toppings from which to choose. Kim calls the instructor over by saying, "We think we found a pattern." [Line 2.1.350] They explain to the instructor the pattern that they have found.

| 2.1 .353 | $36: 42$ | Instructor | Ok, so, what have you got here? |
| :--- | :--- | :--- | :--- |
| 2.1 .354 | $36: 45$ | Kim | Well the fifth one we kinda guessed. |
| 2.1 .355 | $36: 47$ | Francesca S. | Yeah, we guessed it goes down by - you divide it by <br> two [Indicating pattern for toppings: 32, 16, 8, 4, 2, <br> l] |
| 2.1 .356 | $36: 51$ | Instructor | You divide it by two when you're going down - <br> what do you do when you're going up? |
| 2.1 .357 | $36: 54$ | Francesca S. | Multiply by two. |

Francesca S. explains her lists of pizzas that she had created when choosing from three toppings, two toppings, and one topping. After this explanation, Kim suggests that the formula is $2^{n}$.

| 2.1 .371 | $37: 31$ | Instructor | [Pointing to Plain - 1; 1 topping S; P; - 2; 2 topping <br> $-S, P-1]$ Oh, yeah. Plain, both, and one of each, very <br> nice. Okay. So, you have the inductive rule here, <br> right? It looks like to me. To add another topping, <br> you multiply by two. Right? |
| :--- | :--- | :--- | :--- |
| 2.1 .372 | $37: 46$ | Kim | So that's $n$ squared? I mean not $n$ squared. Two to the <br> $n$. |
| 2.1 .373 | $37: 51$ | Instructor | Two to the $n$. Okay. You have an explicit formula. <br> So, now you're telling me that when you have $n$ <br> toppings, the number of possible pizzas is...Ok, ok, <br> now the next question is why? |

They explain that it is $2^{n}$ because the $n$ represents the number of toppings. The instructor replies, " $N$ represents how many toppings there are. Yeah, but how comes it's two to the $n$ ? How come it's not 2 times $n$ ? How come it's not $n$ squared or some other thing? How come it's two to the $n$ ? What is it that makes that..." [Line 2.1.380]

With the instructor, they discuss that the pizza problem and the towers problem are isomorphic. They discuss that they found the same formula for both problems. The instructor points to the pattern of numbers they found for the solution to the towers problem $(1,4,6,4,1)$ and points to the pattern of numbers they found for the solution to the pizza problem $(1,4,6,4,1)$. Kim asks, "Is that supposed to happen?" [Line 2.1.390] The instructor replies, "You're gonna tell me why it happens. What are we talking about here? We're gonna get... We're talking isomorphism here - these are the same problems." [Line 2.1.392]

Francesca S. asks, in regards to the base of 2 in the pizza formula, "Is it because you can have a pizza that is plain and a pizza that has toppings? Is that what the two stands for?" [Line 2.1.394] The instructor tells them to work on that idea and she leaves them to investigate the isomorphism between the two problems.

They discuss that, in the towers problem, the two represents the number of colors and the $n$ is equal to the height of the towers. However, they do not get further than this. They get distracted and talk about eating pizza. Their focus returns when the whole class
discusses the similarities between both problems. [This discussion is described under the heading "Class Discussion".]

Jessica and Jamie (Black, Yellow, White, and Blue cubes) - Second Camera View

Jessica and Jamie begin to solve the pizza problem using paper and pencil. They decide on different variables for the toppings. They decide on S for sausage, Pe for peppers, Pi for pepperoni, and $M$ for mushroom. Jamie suggests using PL for plain pizza but Jessica explains that a plain pizza would be a pizza with a lack of toppings. [Lines 2.2.136-2.2.142]

They begin to write a list that contains the number of pizzas based on the number of toppings. They write that they would have one pizza for a zero toppings (a plain pizza) and for one topping, they would have four pizzas. For two toppings, they decide that they would have 12 pizzas. Jamie explains, "Two toppings, that would be - it would be like sausage-peppers, sausage-pepperoni, sausage-mushroom. So that's three for each one so it's twelve. Isn't it twelve?" [Line 2.2.146] Jessica agrees that for each of the four toppings, there would three pizzas. They agree that there are 12 two-topping pizzas and Jessica writes in her notebook " $3 \times 4=12$ ". [Lines 2.2.147-2.2.149]

For pizzas with three toppings, they start to write each combination. Jessica writes S-$\mathrm{Pe}-\mathrm{Pi}, \mathrm{S}-\mathrm{Pi}-\mathrm{M}$, and S-Pe-M. They say that is should be the same as the two topping pizzas and should be $3 \times 4$. However, they are not convinced and decide to come back to this case. They move on to the four topping pizza and they decide that there is only one pizza.

| 2.2 .158 | $19: 56$ | Jamie | Maybe it's the same -that's hard to believe. |
| :--- | :--- | :--- | :--- |
| 2.2 .159 | $19: 59$ | Jessica | We'll go back to this one. Right now we have <br> three times four. |
| 2.2 .160 | $20: 03$ | Jamie | And then the four is just one. |



Figure 4.31. Camera view of Jessica's notebook.

Jessica says that it might be similar to the answer to the towers problem. Jessica says, "I don't know I have a feeling that it should be like the four again because when we did the cubes problem, remember? It was one, four, [looks through her notebook]. It was one, four, six, four, one. Or that's just maybe something else." [Line 2.2.163] Jessica suggests that they use the Unifix cubes to work on the three-topping pizzas. They assign a color cube to a topping. They use yellow for sausage, blue for peppers, white for pepperoni, and black for mushroom. They discuss that this problem might be similar to the towers
problem in the number of choices of toppings. However, they do not discuss this idea very much. They move on to creating pizzas, using towers.

| 2.2 .182 | $21: 08$ | Jamie | You know? You know what I mean like how you <br> have three choices? |
| :--- | :--- | :--- | :--- |
| 2.2 .183 | $21: 12$ | Jessica | Yeah, it's like the towers. |
| 2.2 .184 | $21: 12$ | Jamie | Maybe it's something to the - maybe you can <br> express it as $m$ to the.... or something. See what I <br> mean? Instead of colors, make it choices. |
| 2.2 .185 | $21: 21$ | Jessica | Yeah. |
| 2.2 .186 | $21: 22$ | Jamie | So like if you have four choices it would be like <br> your colors. Right? |
| 2.2 .187 | $21: 30$ | Jessica | Yeah, I think we should just try to work it out first <br> to figure out if we can see the pattern. |

They focus on the three- topping pizzas. They first create all three-tall towers with yellow on top (sausage on the pizza). They create yellow-blue-white (sausage, peppers, pepperoni), yellow- black-white (sausage, mushroom, pepperoni), and yellow-blue-black (sausage, peppers, mushroom). They check to see if there are any other combinations. Jamie suggests putting black in the middle and creating yellow-black-white. But Jessica explains that, in this problem, the order of the cube doesn't matter. That is, yellow-blackwhite would be the same pizza as yellow-white-black; both of these towers would be sausage, pepperoni, and mushroom. Jamie agrees. [Lines 2.2.188-2.2.231]

They decide to make all of the pizzas, keeping the top cube constant. Jamie focuses on the group of towers with a white cube on top (pizzas containing pepperoni). Jessica focuses on the group of towers with a blue cube on top (pizzas containing peppers). They discuss that, in the end, they will have to take some towers out (there will be duplicates). They discuss that, in this problem, order doesn't matter. [Lines 2.2.232-2.2.251]

Before they build each group of towers, they write the combinations in their notebooks. Then, using these lists as a guide, they build the towers. They come up with 12 three-tall towers. Each group of towers has one of the toppings as the top cube. See Figure 4.32. [Lines 2.2.252-2.2.269]


Figure 4.32. Camera view of Jessica and Jamie's towers representing the three-topping pizzas (before removing duplicates).

They remove the duplicates and they end up with four pizzas. See Figure 4.33. [Lines 2.2.271-2.2.284]

Figure 4.33. Camera view of Jessica and Jamie's towers representing the three-topping pizzas (after removing duplicates).

Jessica says that she now doubts that the number of two-topping pizzas is 12 . They write that the number of no topping, one-topping, two-topping, three-topping, and fourtopping pizzas is $1,4,12,4,1$. They both indicate that they believe that the number of two-topping pizzas should be six because that would be consistent with the results that they had for the towers problem. They question why the numbers would be the same and they do not have an answer.

| 2.2 .189 | $29: 52$ | Jessica | This is going to be six. Because notice it was six <br> with the towers - it's the same. |
| :--- | :--- | :--- | :--- |
| 2.2 .190 | $30: 00$ | Jamie | But why is it the same? |
| 2.2 .191 | $30: 02$ | Jessica | I don't know. |
| 2.2 .192 | $30: 03$ | Jamie | So let's do this one if this one's six then we know <br> that... |
| 2.2 .193 | $30: 07$ | Jessica | Then we know it's the same. Alright. So it's <br> going to be two colors. |

They decide to build the two-topping pizzas. They use the same strategy for the twotopping pizzas as they did for the three-topping pizzas. They build the two-topping pizzas keeping one of the toppings constant, on the top of the tower. Jamie builds the set that has sausage (yellow) on the top. Jessica builds three sets of towers with pepperoni (white) on the top, mushroom (black) on top, and peppers (blue) on top. Jessica mistakenly creates a WBl tower when she is creating the towers with blue on top. They come up with 12 towers. See Figure 4.34. [Lines 2.2.293-2.2.328]


Figure 4.34. Camera view of Jessica and Jamie's towers representing the two-topping pizzas (before removing duplicates).

Jessica removes the duplicates. She looks for duplicates by gathering all of the towers that have a black cube (she takes out 3 towers). Then she groups together all towers that have one white cube and takes out two towers. She groups the towers with one yellow cube and takes out one. They are left with six towers. See Figure 4.35. [Lines 2.2.329 -


Figure 4.35. Camera view of Jessica and Jamie's towers representing the two-topping pizzas (after removing duplicates).

They realize that they have the same answer as the towers problem: $1,4,6,4,1$ but they do not understand why the two problems are the same. They discuss that in the towers problem, the position of the cube mattered but in the pizza problem, the position of the topping doesn't matter.

| 2.2 .338 | $33: 28$ | Jamie | It's the same. |
| :--- | :--- | :--- | :--- |
| 2.2 .339 | $33: 29$ | Jessica | Yeah, but why is it the same? |
| 2.2 .340 | $33: 31$ | Jamie | Alright, let's think about this. See now, why I <br> don't understand why it's the same is because in <br> the towers problem... |
| 2.2 .341 | $33: 39$ | Jessica | It was positional. And this one isn't positional |
| 2.2 .342 | $33: 43$ | Jamie | It's not. |
| 2.2 .343 | $33: 45$ | Jessica | That's why it doesn't make sense of why it's the <br> same numbers, right? |

They explain to the instructor how they solved the problem by creating the pizzas using Unifix cubes. After hearing their explanation, the instructor asks them to write what
they have found in their notebook and to explore the isomorphism between the pizza problem and the towers problem. [Lines 2.2.339-2.2.354]

Jessica and Jamie discuss how for pizzas with two toppings, the towers are two-tall and for pizzas with all four toppings, the tower is four-tall. They say that they believe there is a connection. They start with the pizza with no toppings. Jamie says it is like the one color, one-tall tower. But Jessica explains that they didn't have a color for that; they only had colors for the toppings. They decide that the plain pizza is just the odd pizza and that it is known that there is one. Jessica writes in her notebook that there is one plain pizza because of "common knowledge." [Lines 2.2.355-2.2.363]

For the one-topping pizzas, they say that there are four choices. They decide the formula is four to the one because, they explain, the number of colors is the base. When they get to pizzas with two toppings, they indicate that they are confused because they cannot come up with a formula, using exponents, which will give them the answer of six. They decide that this answer doesn't fit any mathematical formula. They move on to the four-topping pizza, skipping the three-topping pizzas and explain that there is one pizza with four toppings because of "common knowledge." They decide to look at Pascal's triangle but they are unable to discover a connection. [Lines 2.2.364-2.2.413]

## Class Discussion

The camera focuses on the board where the instructor has written some of the solutions to the towers and the pizza problem. The instructor has been working with Kim
and Francesca S. She has written the following results for the towers and pizza problem including the formulas for both problems. See Figure 4.36.


Figure 4.36. Camera view of white board.

As a class, they discuss why the base of the pizza formula is two. Francesca S. asks, "Is it because it is one plain pizza and one pizza with toppings?" [Line 2.1.430] They discuss this idea and asks Francesca S. what she should write on the board. Francescan S., replies "Like, one plain or either one that has toppings on it." [Line 2.1.441] The instructor writes on the board " 2 equals 2 types - plain and toppings." Jamie and Jessica remark that they haven't even gotten that far with the problem. The instructor explains how Jamie and Jessica did not get this relationship. The instructor explains that, in their problem, the number of colors is equal to the number of toppings and the heights of the towers vary. The instructor remarks that she believes that Kim and Francesca's
relationship is more straightforward and explains how they found the answer is eight when only three toppings are available and the answer is four when only two toppings are available. [Lines 2.1.442-2.1.472]

The instructor tells the class that she wants them to explain the complete isomorphism. They have discussed much of the isomorphism already but they need to explain what the two in the formula represents in terms of pizzas. She also indicates that she wants them to be able to connect a random tower to a specific pizza. Jessica remarks that she is having a hard time understanding how this is possible because, as she says, "If two is our base, there's nothing in the $n$ that, that's a whole number, that can give us six." [Line 2.1.275] (She doesn't realize that this formula is used for the total number of pizzas. She is trying to equate this formula to a subset of the total of pizzas.) The instructor explains that $2^{n}$ is the formula for the total number of pizzas. [Lines 2.1.4732.1.476]

Jamie asks if this has anything to do with Pascal's triangle. Jessica writes Pascal's triangle on the board and they discuss. They see that in row four, there is the sequence of numbers similar to the pizza problem and the towers problem: $1,4,6,4,1$. They discuss this row in terms of towers and pizzas. That is, they discuss that the first one represent the pizza with zero toppings and the towers with zero blue cubes. They discuss how the first four represents the four pizzas with one topping and the four towers with one blue cube. They proceed to discuss all of the numbers in this manner. Kim also remarks that the sum
of the rows in Pascal's triangle forms the pattern: 1, 2, 4, 8, 16, 32. [Lines 2.1.4772.1.545]

After this conversation, the instructor again indicates that she wants them to explain the complete isomorphism between the two problems and she suggests that they focus on row three of Pascal's triangle because she believes it would be easier considering there would only be eight towers and eight pizzas.

| 2.1 .546 | $54: 22$ | Instructor | Now, suggestion. Row four is kind of a pain because <br> you have sixteen. Row three, you only got eight. I <br> would suggest to work with row three because eight <br> is easier. Now, not only do you have the numbers <br> that go there, you can actually write each of the eight <br> pizzas and each of the eight towers. |
| :--- | :--- | :--- | :--- |
| 2.1 .547 | $54: 44$ | Instructor | The towers that goes with that one. The pizza that <br> goes with that one. The three towers, list them. The <br> three pizzas, list them. Write them all eight things <br> down in their groups and see if you can just look at <br> them and see exactly how they're related to each <br> other. Do you know what I am saying? You're going <br> to have three towers in this group, you're going to <br> have three pizzas in this group. How can you match <br> them up? Okay? |

The students, with their videographers, return to their groups to work on the isomorphism.

## Kim and Francesca S. (Blue and Orange cubes)

Francesca S. asks the instructor if each color cube would be a different topping. The instructor explains that Jessica and Jamie took that approach and they had towers that were different heights. The instructor suggests that they first build the eight three-tall towers and determine how the pizzas, when choosing from three toppings, are the same.
[Lines 2.1.550-2.1.552]

Kim and Francesca S. build the towers using blue and orange cubes and group them according to the number of blue cubes. The instructor indicates that this grouping produces the same pattern as the third row in Pascal's triangle: $1,3,3,1$ and the same pattern as the towers. After looking at this, Francesca S. connects the tower will all orange cubes with the plain pizza and the towers with all blue cubes as the pizza will all of the toppings. As she explains the all blue tower, she realizes the blue cube indicates a topping

| 2.1 .595 | $58: 26$ | Francesca S. | All blue would be the three toppings. |
| :--- | :--- | :--- | :--- |
| 2.1 .596 | $58: 27$ | Kim | Oh, gotcha. Oh. |
| 2.1 .597 | $58: 29$ | Francesca S. | And then this one would be one topping [Points to <br> the three towers with one blue cube]. Oh, my - <br> Okay. Look - this is one topping because it has <br> one blue. [Points to the three towers with one blue <br> cube.] Two toppings because it has two blue. <br> [Points to the three towers with two blue cubes.] |
| 2.1 .598 | $58: 39$ | Kim | And that's three... |


| 2.1 .599 | $58: 42$ | Francesca S. | Yeah. So we're like using blue. So like let's use <br> blue. Do you know what I mean? So one topping <br> is one blue. [She writes in her notebook: 0 blue; 1 <br> topp-> 1 blue; 2 topping-> 2 blue; 3 blue] |
| :--- | :--- | :--- | :--- |

Kim calls the instructor over so that they can explain what they have found to her. The instructor indicates that she likes what they have found and explains that she understands that the all orange tower is a plain pizza and an all blue tower is a pizza with all of the toppings. However, she tells them to explain to her, specifically, what each of the remaining six towers are in terms of pizzas. She asks them to explain which pizza the tower with a blue on top (BOO) would be. They reply that it is sausage. She indicates that she wants them to connect each of the remaining five towers with a specific pizza. She leaves them to discuss. [Lines 2.1.604-2.1.617]

They decide that blue on top is sausage. Francesca S. says, "Yeah, so blue on top is sausage. Blue in the second, I put, is peppers and then the third one's mushroom." [Line 2.1.681] They focus on the towers with two blue cubes. The instructor joins them again and they focus on the OBB tower. [This tower is peppers and mushroom.] Kim says it is sausage-peppers and Francesca S. says it sausage-mushroom or peppers-mushroom because mushroom is on the bottom. The instructor reiterates what they have said about each of the towers with one blue cube is in terms of pizzas explaining also which toppings are not on the pizza.

| 2.1.639 | 1:02:12 | Instructor | And the top one is sausage. Alright. So you're saying this one has mushrooms on it [points to OOB tower]. But what about sausage and peppers? It doesn't have sausage and peppers, you're telling me. |
| :---: | :---: | :---: | :---: |
| 2.1.640 | 1:02:20 | Francesca S. | Yeah. |
| 2.1.641 | 1:02:21 | Instructor | Okay, and this one has [points to the OBO tower]... which one - this one has peppers but it doesn't have mushroom or sausage. And this one [points to the BOO tower] doesn't have mushroom, doesn't have peppers but it does have sausage. |
| 2.1.642 | 1:02:30 | Francesca S. | Yeah. |
| 2.1.643 | 1:02:31 | Instructor | Now you said this one [points to the OBB tower] |
| 2.1.644 | 1:02:32 | Francesca S. | Oh, so this one has sausage. No, it doesn't. |

Francesca S. concludes that the OBB tower is pepper and mushroom. [Line 2.1.650] The instructor asks Francesca S. to explain to Kim why she thinks OBB is a pepper and mushroom pizza. The instructor leaves them to discuss.

Francesca S. explains to Kim that if the first cube is sausage and the first cube is orange, then there is no sausage on that pizza but then she says she is not sure. They discuss this idea for a bit and discuss how they are both confused. Francesca S. now states that if there are two blues, then it would be a mushroom pizza. She says, "But I think this whole thing is mushrooms. [Points to both blue cubes in OBB.]" [Line 2.1.668] Kim calls the instructor over and asks her, "Okay, so these two together, they would be one topping or are they different toppings? [Points to the two blues in the OBB tower.]

Because they're the same color." [Line 2.1.677] The instructor replies that they need to figure that out. Francesca S. now disagrees and explains that OBB is pepper and mushroom because there is no sausage. The instructor replies, "So you're saying that the position makes a difference?" [Line 2.1.680] Francesca S. agrees. Francesca S. points to the BOB tower and correctly identifies this as the sausage and mushroom pizza. Kim declares that she is completely confused. [Lines 2.1.681-2.1.689]

The instructor explains that they have already told her that the position of each cube represents a specific topping. She asks them to explain the difference between a blue cube and an orange cube. They cannot articulate their answer. The instructor points to the BOO tower and says that blue on the first cube means that sausage is there. They agree. She then points to the OBB tower and asks what the orange cube on top means. Francesca S. replies that it means that sausage is not there. [Lines 2.1.690-2.1.704]

The instructor continues comparing towers with an orange cube in a specific position and a blue cube in the same position. The students can identify, within a specific tower, that the pizza has the specific topping when the blue cube is there but does not have the specific topping when the orange cube is there. However, when asked to explain, in general, the difference between an orange cube and a blue cube, they cannot.

| 2.1.728 | 1:06:49 | Francesca S. | Oh, two blue and one orange -this one's sausage <br> and peppers. [Points to the BBO tower.] |
| :--- | :---: | :--- | :--- |
| 2.1.729 | $1: 06: 51$ | Instructor | Okay, so specifically blue means and yellow means, <br> orange means what? Blue means? You told me, I |


|  |  |  | think. |
| :---: | :---: | :---: | :---: |
| 2.1.730 | 1:07:00 | Francesca S. | Blue is sausage. |
| 2.1.731 | 1:07:02 | Instructor | No it isn't. [Kim laughs.] |
| 2.1.732 | 1:07:05 | Instructor | You told me the top is sausage. |
| 2.1.733 | 1:07:06 | Francesca S. | Yeah, top is sausage. The order matters not the colors, I don't think. |
| 2.1.734 | 1:07:10 | Instructor | Well, the color... Now, how do you know that there is sausage on this pizza [holds up the OBB tower] and that there isn't sausage on this pizza? [Holds up the BOO tower.] |
| 2.1.735 | 1:07:18 | Kim | Because of the top. |
| 2.1.736 | 1:07:18 | Francesca S. | Yeah, because the tops are switched. |
| 2.1.737 | 1:07:19 | Instructor | Yeah, so what specifically does blue mean and what specifically does orange mean? |
| 2.1.738 | 1:07:23 | Francesca S. | Okay, blue means there's sausage on the pizza; orange means there's no sausage. |
| 2.1.739 | 1:07:29 | Instructor | Yeah, but down here, but down here what does this blue and this orange mean? [She holds up the $B O B$ tower and the BBO tower.] |
| 2.1.740 | 1:07:33 | Kim | Well, they both have sausage on them. |
| 2.1.741 | 1:07:34 | Instructor | No, I mean the ones at the bottom. |
| 2.1.742 | 1:07:35 | Kim | Oh, um... |
| 2.1.743 | 1:07:37 | Francesca S. | Mushroom. |
| 2.1.744 | 1:07:37 | Kim | One has mushroom and one doesn't. |
| 2.1.745 | 1:07:39 | Instructor | Yeah, which one doesn't? |
| 2.1.746 | 1:07:40 | Kim | That one. [Points to the BBO tower.] |


| 2.1 .747 | $1: 07: 41$ | Instructor | So what does yellow, orange mean in isolation <br> without referring to a specific topping? What does <br> orange tell you both about sausage and about <br> mushroom? |
| :--- | :--- | :--- | :--- |
| 2.1 .748 | $1: 07: 51$ | Francesca S. | That there's both on it? |
| 2.1 .749 | $1: 07: 55$ | Instructor | Well, maybe you want to think about it. |

Francesca S. says that she thinks she understands it. The instructor compares two towers again and eventually, Kim is able to articulate that the blue cube means that the topping is there and the orange means that the topping is not there.

| 2.1 .759 | $1: 08: 20$ | Instructor | Right? Now, what does this tell you about peppers? <br> [Holding the $O B O$ and $O O B$ towers.] |
| :--- | :--- | :--- | :--- |
| 2.1 .760 | $1: 08: 25$ | Kim | That one does and one doesn't. |
| 2.1 .761 | $1: 08: 26$ | Instructor | Yeah. And how do you know that one does and one <br> doesn't? |
| 2.1 .762 | $1: 08: 29$ | Kim | Because the blue means... |
| 2.1 .763 | $1: 08: 31$ | Francesca S. | 'Cause peppers is the second. |
| 2.1 .764 | $1: 08: 31$ | Instructor | The blue means? |
| 2.1 .765 | $1: 08: 32$ | Francesca S. | Peppers. |
| 2.1 .766 | $1: 08: 33$ | Kim | That it's there. |
| 2.1 .767 | $1: 08: 35$ | Instructor | Say it again. Repeat yourself. |
| 2.1 .768 | $1: 08: 37$ | Kim | That it's there. |
| 2.1 .769 | $1: 08: 38$ | Instructor | The blue means that it's there. And what does the <br> orange mean? |


| 2.1 .770 | $1: 08: 40$ | Kim | That it's not there. |
| :--- | :---: | :--- | :--- |
| 2.1 .771 | $1: 08: 41$ | Instructor | Bingo! |

They are instructed to write the complete isomorphism for homework.

## Jessica and Jamie (Black and White cubes) - Second Camera View

The camera focuses on Jamie and Jessica working on pizzas with three toppings. They write down that the number of toppings is equal to the height of the towers. Jessica suggests that they build the towers first. They use black and white cubes and build eight towers. They organize them as no white (one tower), one white (three towers), two white (three towers), and three white (one tower). The instructor tells them to list the pizzas based on no toppings, one topping, two topping, and the three toppings. She instructs them to write the pizzas in their notebooks instead of creating them with cubes. [Lines 2.2.541-2.2.582]

They decide to use sausage, pepperoni, and mushroom. They write for no toppings, there would be one. They move on to the one-topping pizzas but they start to use four toppings. They start writing the pizzas with three-toppings choosing from four toppings. After they write these pizzas, they realize they have been doing it incorrectly. They erase what they have written and write for one topping there would be three pizzas: S (sausage), P (peppers - they changed from pepperoni), and M (mushroom). For two-
topping pizzas, they get three pizzas: SP, SM, PM and for three toppings, they get one pizza: SPM. [Lines 2.2.583-2.2.604]

Jessica picks up the tower with all black cubes and that it is the plain pizza. She says, "Well look at this. [She holds up the BBB tower.] Right? Our plain is without toppings, this is without white. So without white so..." [Line 2.2.608] Jamie agrees. Jessica picks up the towers with one white (they are organized in an elevator pattern) and she points to the white cubes, as she goes down the elevator, "sausage, pepper, mushroom". Jamie agrees. She picks up the three towers with two white cubes and names each of the towers as one of the two-topping pizzas. She names the first tower, BWW, as sausage and peppers, the second tower, WBW, as sausage and mushroom, and the third tower, WWB, as peppers and mushrooms. And the last tower, WWW, is sausage, peppers, and mushroom. They agree that white is the topping. Jamie says, "The white are the topping." [Line 2.2.612] Jessica replies, "For us, white is toppings." [Line 2.2.614] Jessica remarks that she is bothered. She says, "It just bothers me because the towers are positional and these aren't." [Line 2.2.618] (Note: they have not connected the topping with a specific position in the tower.)

The instructor requests that they explain, to her, the relationship that they have found between the towers and the pizza problems. They explain that the white cube equals the toppings. Jessica explains that the tower without white represents the plain pizza, the towers with one white cube represent the one-topping pizzas, the towers with two white cubes represent the two-topping pizzas, and the tower that is all white represents the
pizza with everything. The three towers with one white cube represent the one topping pizzas. The instructor asks them to explain exactly which pizzas are represented by the towers. During this explanation, Jessica matches a specific position to a topping.

| 2.2.628 | 1:02:36 | Instructor | Okay, that's perfect. But you know what I'm gonna ask next which is - which pizza is this and this and this? [She points to the following towers: $W B B, B W B$, and $B B W$.] Exactly which pizza? |
| :---: | :---: | :---: | :---: |
| 2.2.629 | 1:02:46 | Jamie | Okay, so, this is... it's that one right? |
| 2.2.630 | 1:02:49 | Jessica | This one, we like decided it would be sausage [points to the white cube in WBB], pepperoni [points to the white cube in $B W B$ ], mushroom [points to the white cube in $B B W$ ]. |
| 2.2.631 | 1:02:53 | Instructor | Sausage, pepperoni, mushroom. Okay. So then what's this one? [Points to the BWW tower.] |
| 2.2.632 | 1:02:57 | Jessica | Um, let me put them in the correct order. [She rearranges the 3-tall towers with two white cubes as - WWB, WBW, and BWW] So it's sausage and pepperoni [she points to the two white cubes in $W W B]$. Sausage and mushrooms [she points to the two white cubes in WBW]. Pepperoni and mushrooms [she points to the two white cubes in $B W W]$. Notice that it's sausage, pepperoni's in the second place, and mushroom is always in the third place. |
| 2.2.633 | 1:03:11 | Instructor | Okay, alright, so there's your homework. Complete description of the isomorphism. |

Class ends and they are instructed to write the complete isomorphism for homework.

### 4.3 Ankur's Challenge

The instructor introduces Ankur's Challenge, around minute 36, after the students finished solving the four-tall towers problem when selecting from two colors on February 11, 2011. The instructor introduces the problem verbally. She says, "Okay... and this builds on something that Rebecca noticed which is when you only got two-tall and three colors, some of them don't have all three colors in them. So here you got four-tall and three colors to choose from. Each cube [sic] must have at least one of each color. So how many can you make?" [Line 1.1.397] She writes on the board " 4 tall 3 colors must have at least 1 of each color".

All six students were present. There was one camera and three groups each composed of two students. The three groups were: (1) Jessica and Jamie, (2) Kim and Francesca S., and (3) Rebecca and Francesca C. The camera focuses on Jessica and Jamie building their solution to the problem. At the end of the tape, each group explains their solutions. That is, Kim, Francesca S., Rebecca, and Francesca C. were not filmed building their solutions. [When discussing towers, the first cube described is the top cube and the fourth is the bottom cube. For example, RRWW will mean two reds on top, followed by two whites.]

## Jessica and Jamie (Brown, Maroon, and Green cubes)

After the instructor introduces the problem, Jessica says, "It's three to the fourth minus three." [Line 1.1.398] She explains that she is subtracting the three solid colored
towers. The instructor says to her, "Think about that some more. Okay, that was a good start though." [Line 1.1.401]

Jessica turns to Jamie and says "there's probably more." [Line 1.1.402] They decide to work on the problem using paper and pen as opposed to building the towers with the actual cubes. They use the letter B for brown (although they occasionally call B blue), the letter M for maroon, and the letter G for green. They decide to write all of the towers that have two browns. They write the first two towers by keeping the two brown cubes together and systematically moving them up the tower. For each of these towers, they create the opposite tower by switching the green and maroon cubes (keeping the brown cubes in the same positions). They have six towers, written on their paper. See Figure 4.37. [Lines 1.1.415-1.1.433]

| G | M | M | G | B | B |
| :--- | :--- | :--- | :--- | :--- | :--- |
| M | G | B | B | B | B |
| B | B | B | B | G | M |
| B | B | G | M | M | G |

Figure 4.37. Replication of Jessica and Jamie's first six towers in their solution for Ankur's Challenge

They discuss whether they have found them all and remark that the number seems low. Jamie explains that they didn't do the towers with just one brown on the bottom and Jessica agrees. They create another six towers with the browns separated. Again, they write down the tower's opposite after creating a new tower. See Figure 4.38 and Figure 4.39. [Lines 1.1.434-1.1.450]

| B | B | G | M | B | B |
| :--- | :--- | :--- | :--- | :--- | :--- |
| M | G | B | B | M | G |
| G | M | M | G | B | B |
| B | B | B | B | G | M |

Figure 4.38. Replication of Jessica and Jamie's first six towers in their solution for Ankur's Challenge.


Figure 4.39. Camera view of Jessica's notebook of 12 towers found while working on Ankur's Challenge.

They have a total of 12 towers. They discuss how if there are 12 towers for two browns, there would be a total of 36 towers with two of each color. That is, 12 times 3 is equal to 36. [Line 1.1.453] Jessica remarks that she thinks this number still seems low and questions if there is any other way to build the towers. Jessica explains that she thinks that makes sense because the towers they are taking out of the group would be the towers that have three of one color and one of another color. She remarks that this would contain a lot of towers. They discuss that, if the total number of towers is 81 (three to the fourth power) and the answer they got is 36 , they would be taking 45 towers out of the group. [Lines 1.1.459-1.1.463] Jamie remarks that this means that they are taking 15 out for each color but she questions why. Jamie explains that since there are three colors, you would be taking 5 out for each section. She says "You know what I mean like five times
three - so five from the blue [sic], five from the green, five from the maroon or brown, whatever." [Line 1.1.476] Jessica explains that one of those five would be the solid tower.

The instructor asks them to explain what they have gotten so far. They explain that they listed all of the towers that have two browns and came up with 12 towers. They then explain that you would have to multiply this number by three because you would get the same amount of towers for two maroons and for two greens. So the total number of towers is 36 . Jamie explains that they subtracted 36 from the 81 total towers and got 45 . She tells the instructor that since they are taking out 45 towers, there would be 15 towers for each case (45 divided by three). And then within each case, you are taking out five for each color (because 15 divided by three is five). The instructor says that she doesn't understand the last part. The girls say that they are still working on that part. The instructor leaves them to work some more. [Lines 1.1.485-1.1.494]

The girls decide to write out all of the towers that have two greens and all of the towers that have two maroons. They create these towers using the two browns as a model. They came up with a total of 36 towers. See Figure 4.40. [Lines 1.1.503-1.1.511]

| G | M | M | G | B | B | B | B |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| M | G | B | B | B | B | M | G |
| B | B | B | B | G | M | G | M |
| B | B | G | M | M | G | B | B |


| G | M | B | B |
| :---: | :---: | :---: | :---: |
| B | B | M | G |
| M | G | B | B |
| B | B | G | M |


| B | M | M | B | G | G | G | G |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| M | B | G | G | G | G | M | B |
| G | G | G | G | B | M | B | M |
| G | G | B | M | M | B | G | G |


| M | B | G | G |
| :--- | :--- | :--- | :--- |
| G | G | M | B |
| B | M | G | G |
| G | G | B | M |


| B | G | B | G | M | M | M | M |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| G | B | M | M | M | M | B | G |
| M | M | M | M | B | G | G | B |
| M | M | G | B | G | B | M | M |


| M | M | G | G |
| :--- | :--- | :--- | :--- |
| B | G | M | M |
| M | M | B | G |
| G | B | M | M |

Figure 4.40. Replication of Jessica and Jamie's list of 36 towers in their solution for Ankur's Challenge.

After they have written all 36 towers in Jessica's notebook, Jessica remarks that they have to come up with a reason why it is 36 . They focus on the 45 towers. Jessica writes that for each "dominant" color, there would be 15 towers taken out. (She uses the term dominant color to indicate the towers with two cubes of a certain color.) One of each of
these sets of 15 towers would be the solid tower. They discuss the 14 towers that would have three of the "dominant" color. They focus on the brown dominant case. Jamie says that there would be two towers that have three browns and Jessica says that this is not true. [Lines 1.1.513-1.1.534]

Jessica decides to build the towers with three browns using the Unifix cubes. She builds four towers with three browns and one maroon. She builds four towers with three browns and one green. They found a total of eight towers with three browns and one of the other colors. They conclude that for each dominant color, there would be eight towers with three of the dominant color and one of the other. Of the 45 towers, they have accounted for 27 of them. That is, three solid towers plus 24 (eight times three) towers with three of the dominant color. The instructor stops them at this point because she wants everyone to explain their solution. (They have not accounted for the 18 towers that contain two of each color.) [Lines 1.1.535-1.1.559]

The instructor asks Jamie and Jessica to explain their solution to the class. Jessica explains that, since there has to be one of each color, the towers would have two of one of the colors. They call this the dominant color. She focuses on the brown dominant case and explains that they created the first six towers by keeping the brown together and moving them up the tower. The green and maroon would fill in the other cubes. Each time they created a tower, they created the opposite (switching only the green and the maroon). Next, they created their towers by separating the brown cubes. She explains that there was no other way to put the browns so they were convinced that there were 12
towers. She explains that you would also get 12 towers for the maroon dominant case and 12 towers for the green dominant case. They conclude that 12 times three is 36 . [Lines 1.1.567-1.1.586]

The instructor asks Kim and Francesca S. to go next because she says that they also did a proof by cases but different cases.

Kim and Francesca S. (Blue, Yellow, and White cubes)

Kim and Francesca S. tell the class that they got an answer of 36 as well. They have built 12 towers using Unifix cubes. The camera did not show them building these towers. However, they now explain how they came to their solution. The instructor introduces their solution by saying that their first case is the case where white is on top. They have twelve towers with white on top. These 12 towers are separated into 3 sub-cases where each of these sub-cases is based on the color of the second cube. The first sub-case contains 2 towers that have white on top and white as the second cube. The second subcase contains 5 towers that have white on top and blue as the second cube. The third subcase contains 5 towers that have white on top and yellow as the second cube. See Figure 4.41. [Lines 1.1.587-1.1.588]


Figure 4.41. Camera view of Kim and Francesca S.'s solution to Ankur's Challenge.

They explain that there are a total of 12 towers in the white on top case and that there would also be 12 towers for the blue on top case as well as the yellow on top case for a total of 36. [Lines 1.1.589-1.1.590]

## Rebecca and Francesca C. (Blue, Yellow, and Red cubes)

Rebecca and Francesca C. have written their solution on the board. They explain that they focused on the towers that do not have one of each color. The first group they focused on was the towers with two colors each - they call these towers the "doubles." Francesca C. explains that these would be the six towers with just blue. She says "And it turns out that for just the blue, like cause we did blue and yellow and then blue and red. For just the blue, there was six uh.... six different ones that had to be excluded." [Line 1.1.593] (However, this statement is incorrect. There would be twelve towers that contain two blues. Six of these twelve towers would contain two blues and two reds. And six towers would contain two blues and two yellows. The six on the board should have
contained only two colors.) She further explains that you would have to multiply these six by three to get 18. See Figure 4.42 .


Figure 4.42. Camera view of part of Rebecca and Francesca C.'s solution to Ankur's Challenge.

Francesca C. then explains that they did the same for solids. For solids, there would be one tower for each color for a total of three. [Line 1.1.594] Rebecca describes the "triples." She explains, "Yeah, for triples, we just wrote up there all the yellows. So, if you have three yellows and one of the other colors, there is eight different ways that you can do it. And then if you have the three colors, you have yellow, blue and red, so you can multiply it by three. And then you added them all up, which equals 45 and 81 minus 45 is $36 . "$ [Line 1.1.595] See Figure 4.44 for a complete image of Rebecca and Francesca C.'s solution.


Figure 4.43. Camera view of remaining part of Rebecca and Francesca C.'s solution to Ankur's Challenge.

Doubles x $3 \quad$ Singles x $3 \quad$ Triples x 3


Figure 4.44. Replication of Rebecca and Francesca C.'s solution to Ankur's Challenge.

Jessica indicates that she now understands the towers that her and Jamie were missing when they were trying to figure out the 45 towers. She says, "No but I like theirs. Now I understand what we were missing - it's their doubles." [Line 1.1.597] The instructor concludes the class by showing the students how she incorrectly solved Ankur's Challenge. She informs them that their homework assignment is to figure out where the error occurs in her solution.

## CHAPTER 5: FINDINGS

### 5.1 Introduction

The objective of this research is to examine how a group of math majors in their junior year of college built and justified their solutions to a series of combinatorics tasks. Furthermore, understanding how these students' strategies and justifications compared to the solutions of other students at various ages will also be investigated. In the sections that follow, a brief summary of how each of the group of students in this study solved a particular task will be followed by an analysis of how these results compare to the current body of research. These sections are organized by task starting with the towers problem, followed by the pizza problem and the connections made between the two problems, and concluding with Ankur's Challenge.

Additionally, in understanding how the students built their solutions to these tasks, it is important to consider the role of the instructor. Following the examination of the solutions to these tasks, the moves made by the instructor, which were an integral part of the building of these solutions, will be discussed.

### 5.2 The Towers Problem

### 5.2.1 Results

Jessica and Jamie - Summary

Jessica and Jamie started building their towers using opposites and then also employed the strategy of cousins. They initially organized their towers in groups of opposites. However, one of their pairs was also a cousin and mixing these strategies created a duplicate. They continued making pairs of opposite towers and found 18 towers. They rearranged their organizational structure into five groups which included two incomplete staircase patterns (these groups contained three towers each), two elevator patterns (four towers each), and the remaining four towers grouped as opposites. After the instructor asked them to explain their organizational strategy, they realized they had duplicates and they rearrange their towers into the pattern of staircase and pair-wise opposites (abandoning the elevator pattern).

At first, Jessica and Jamie suggested that the reason the answer was 16 was because four squared is 16 and hypothesized that for the three-tall towers, the answer should be nine. After the suggestion of the instructor to build the towers three-tall, they found the answer to be eight and realized the total number of towers doubled each time the height of the tower increased. After looking at the doubling pattern, they were able to find the formula to be $2^{n}$. They explained that the exponent $n$ represented the height of the towers and hypothesized that the base of two represented the number of colors. After the suggestion of the instructor to test their hypothesis by building towers two-tall when choosing from three colors, they found the general formula for the towers problem. That is, they said that the formula for any number of colors and any height of towers is $m^{n}$. They did not explain the reason for either of the formulas ( $2^{n}$ or $m^{n}$ ). A week later, after
the class revisited the towers problem, Jessica and Jamie were able to explain to the class the inductive rule for towers when choosing from three colors.

## Kim and Francesca S. - Summary

Kim and Francesca S. were filmed less often then Jamie and Jessica. Kim and Francesca S. built their towers using the strategy of opposites. They organized their 16 towers into groups containing the staircase patterns and the remaining towers as opposites. The instructor suggested that they reorganize their towers based on the number of a specific color cube. When asked to explain their solution to the class, they organized their towers into cases based on the number of blue cubes contained within the tower.

## Rebecca and Francesca C. - Summary

Rebecca and Francesca C. were videotaped only when they explained their solution to the class. That is, they were not filmed building their solution. They explained, inductively, the reason the towers doubled each time the height increased. They created the one-, two-, three-, and four-tall towers when choosing from two colors. They demonstrated how the towers were created from the previous height by adding a blue or an orange cube. Francesca C. and Rebecca had seen this problem in a previous course. Their towers were arranged by cases based on the number of a certain colored cube. There is evidence of the use of the elevator pattern and pair-wise opposites in their towers that are four-tall. Their towers containing one cube of a certain color and three cubes of
another are arranged in the elevator pattern. Their towers containing two cubes of a certain color are paired as opposites.

## Towers Revisited

The class revisited the problem a week later. Only Jessica, Jamie, Kim and Francesca S. were present. At this time, they all indicated that they understood the formula to be $2^{n}$. The instructor asked them to explain why the formula was $2^{n}$. Jessica was the only student who could reiterate part of Rebecca and Francesca C.'s inductive argument from a week earlier. She said, "... the only thing I can remember truthfully is that you have two choices. You can either add a white one on or you can add a blue one on." [Line 2.2.73] The instructor discussed with the class that the fact that you have two choices means that the formula doubles, or is multiplied by two. After the instructor revisited the explanation with the class, Jessica and Jamie were able to explain to the class the inductive rule for towers when choosing from three colors.

### 5.2.2 Analysis of Strategies Compared to the Existing Research

Jessica, Jamie, Kim and Francesca S. started building their towers using opposites. There is evidence of students in the third, fourth, eleventh, and college beginning the problem by building opposites (Alston \& Maher, 1993; Maher \& Martino, 1998; Maher \& Martino, 1996a; Maher \& Martino, 1996b; Martino, 1992; Martino \& Maher, 1999; Tarlow, 2004; Glass, 2001). Jessica and Jamie also employed the strategy of cousins which was used by Stephanie and Dana in the third grade and fourth grade, Brandon in
the fourth grade, and Melinda in college (Maher \& Martino, 1998; Maher \& Martino, 1996a; Maher \& Martino, 1996b; Martino, 1992; Glass, 2001).

Jessica and Jamie initially organized their towers in groups of opposites similar to the students in grades three and four (Martino, 1992; Martino \& Maher, 1999). They rearranged their 18 towers into the organizational structure that included the staircase pattern, the elevator pattern, and remaining towers grouped as opposites. After the instructor asked them to explain their organizational strategy, they realized they had duplicates and they rearranged their towers into the pattern of staircase and pair-wise opposites (abandoning the elevator pattern). Kim and Francesca S. also arranged their towers in the staircase pattern and pair-wise opposites. After a discussion with the instructor, they decided to rearrange their towers by cases based on the number of certain colored cube. When Rebecca and Francesca C. presented their findings, their towers are arranged by cases based on the number of a certain colored cube. In the cases with one orange cube and three orange cubes, they arranged the towers in the elevator pattern. In the case with two oranges, they arranged these six towers as pair-wise opposites.

There is evidence of students in grade school, high school, and college employing the patterns of opposites, cousins, staircase and elevator to build the towers. Furthermore, the use of these patterns to initially organize the towers, like the methods employed by Jessica and Jamie and Kim and Francesca S., was also found at a range of grade levels. For example, there is evidence of four college students (Melinda, Wesley, Elizabeth, and Stephanie) organizing their towers with a mixture of the staircase pattern and/or the
elevator pattern with the remaining towers as pair-wise opposites (Glass, 2001). In eleventh grade, three students (Michelle, Sherly, and Ali) organized their towers in pairwise opposites (Tarlow, 2004). Tarlow also explains that Sherly and Ali, working together, organized the towers with one blue cube in the elevator pattern. Four fourth graders (Brandon, Justin, Stephanie, and Dana), and four third graders (Michael, Jamie, Meredith, and Jackie) used one or a combination of opposites, cousins, staircase, and/or the elevator pattern to initially organize their sixteen towers (Maher \& Martino, 1998; Maher \& Martino, 1996a; Maher \& Martino, 1996b; Martino, 1992; Martino \& Maher, 1999, Maher, Sran, \& Yankelewitz, 2010).

Table 5.1
Towers problem - initial building/organizing towers

|  | Students in this Study |  |  | Existing Research |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Jessica <br> Jamie | Kim <br> Francesca S. | Rebecca <br> Francesca C. | Grade <br> School <br> $\left(3^{\text {rd }}, 4^{\text {th }}, 5^{\text {th }}\right)$ | High <br> School <br> $\left(11^{\text {th }}\right.$ grade $)$ | College |
| Opposites | x | x | x | x | x | x |
| Cousins | x |  |  | x |  | x |
| Elevator | x | x | x | x | x | x |
| Staircase | x | x |  | x |  | x |

At first, Jessica and Jamie suggested that the reason the answer was 16 was because four squared is 16 and they hypothesized that for three-tall towers, the answer should be nine. There is evidence of an initial conjecture that the formula was equal to the square of the height of the tower by other college students and two high school students in the research. In the eleventh grade, Sherly and Ali predicted that if the towers were three-tall, there would be nine towers (Tarlow, 2004). About half of the college students in the
study by Glass (2001) predicted that if the towers were five-tall, the solution would be 25. Some of the college students that made this prediction did not investigate further because, not only did they have a formula, but they believed they were finished because everyone else had the same answer and/or they couldn't find any more towers. One college student, Stephanie, justified that this theory could not work because she explained that the answer had to be even because the towers could be built in pairs.

After Jamie and Jessica made this prediction, the instructor suggested they build the towers three-tall to test their conjecture. They found the answer to be eight and realized the total number of towers doubled each time the height of the tower increased. After looking at the doubling pattern, they were able to find the formula to be $2^{n}$. They understood two to represent the number of colors and the exponent $n$ to represent the height of the towers. Furthermore, they found the general formula for the towers problem, any number of colors and any height of towers to be $m^{n}$. When they revisited the problem a week later, they were able to explain inductively why the number of towers tripled if there were three colors from which to choose. Rebecca and Francesca C., in their demonstration of the inductive rule, also explained that the number of total towers doubled as the height of the towers increased.

Of the 19 students reported by Glass (2001), nine students recognized the doubling pattern and of those nine only five could explain that, as the tower height increased, the number of towers doubled because you are able to add a choice of two cubes to the top of the tower. Using this understanding, many of the students were able to find the correct
number of five-tall towers when choosing from two colors and the correct number of four-tall towers when choosing from three colors. However, Glass does not indicate if the college students found the formula $2^{n}$ or the general formula $m^{n}$.

In grade 11, Robert explained the doubling rule inductively (Tarlow, 2004). The eleventh graders (Robert, Michelle, Angela, and Magda) discovered the general formula of $m^{n}$ but they were not able to explain the reason for this formula (Tarlow, 2004). The younger students did not find the general formula for the towers problem. However, there is evidence of some of these students recognizing the doubling pattern and justifying the doubling pattern inductively. In fourth grade, Stephanie and Milan recognized that the formula doubles and Milan is able to explain, inductively the reason for the doubling pattern (Maher, 1998; Maher \& Martino, 1996a; Maher \& Martino, 1996b, Maher \& Martino, 1997). In fifth grade, Stephanie is able to explain the reason for the doubling pattern inductively as well (Maher \& Martino, 1997). Stephanie and Milan had revisited this problem several times over an extended time period.

To justify that they have found all of the towers, Kim and Francesca S. organize their 16 towers by cases based on the number of blue cubes. Seven of the college students in the study by Glass (2001) organized their towers by cases. Angela, Magda, and Robert (eleventh grade) also organized by case based on a certain colored cube (Tarlow, 2004). There is evidence of third graders (Meredith and Jackie) and fourth graders (Brandon and Stephanie) organizing their solutions by cases (Maher, 1998; Maher \& Martino, 1998; Maher \& Martino, 1997; Maher \& Martino, 1996a; Maher \& Martino, 1996b; Martino \&

Maher, 1999). Unlike the students in this study, some of the younger students, as well as some college students justified that they were finished by such reasons as 1) they could not find anymore towers or 2) everyone else got the same answer (Alston \& Maher, 1993; Martino, 1992; Glass, 2001).

Table 5.2
Towers problem - pattern recognition/formulas

|  | Students in this Study |  |  | Existing Research |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Jessica <br> Jamie | Kim <br> Francesca S. | Rebecca <br> Francesca <br> C. | Grade <br> School <br> $\left(3^{\text {rd }}, 4^{\text {th }}, 5^{\text {th }}\right)$ | High <br> $\left(11^{\text {th }}\right.$ grade $)$ | College |
| $n$ squared | x |  |  |  | x | x |
| Doubling <br> Pattern | x |  | x | x | x | x |
| 2 to the $n$ | x |  |  |  | x |  |
| $m$ to the $n$ | x |  |  |  | x |  |

Table 5.3
Towers problem - justifications

|  | Students in this Study |  |  | Existing Research |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Jessica <br> Jamie | Kim <br> Francesca S. | Rebecca <br> Francesca <br> C. | Grade <br> School <br> $\left(3^{\text {rd }}, 4^{\text {th }}, 5^{\text {th }}\right)$ | High <br> School <br> $\left(11^{\text {th }}\right.$ grade $)$ | College |
| Can't find <br> Anymore |  |  |  | x |  | x |
| Cases |  | x |  | x | x | x |
| Inductive <br> Argument | x |  | x | x | x | x |

### 5.3 The Pizza Problem

A week after working on the towers problem, the students worked on the pizza problem. They began the pizza problem after 20 minutes of class. During the first 20
minutes of class, they had revisited the towers problem. Only Jessica, Jamie, Kim and Francesca S. were present and all four organized their answer by cases based on the number of toppings.

### 5.3.1 Results

## Jessica and Jamie - Summary

They began the problem by listing the number of pizzas based on the number of toppings. For a plain pizza they agreed that there would be one pizza and for the pizza containing all of the toppings, they agreed there would only be one pizza. For the onetopping pizzas, they agreed that there would be four different pizzas. For the two- and three-topping pizzas, Jamie said that there would be 12 pizzas each. She explained to make a two-topping pizza, each topping would be paired with the remaining three toppings, creating three pizzas. Therefore three pizzas would be created for each of the four toppings. That is, 3 times 4 which is equal to 12 pizzas. For the three-topping pizzas, they listed three pizza combinations that contained sausage. Jamie explained that the answer would also be 12 for the three-topping pizzas because there would be three more lists of three that contained the remaining three toppings (mushroom, pepperoni, and peppers). Jessica indicated she was not convinced and she suggested that the answer might be similar to the towers problem. They decided to use the Unifix cubes to create the pizzas for the two- and three-topping pizzas.

They decided to use a specific color cube to represent a topping and used the yellow cubes for sausage, blue cubes for peppers, white cubes for pepperoni, and black cubes for mushroom. They worked on creating the three-topping pizzas first. They controlled for variables by keeping one color cube constant on the top of the tower while switching the other two colored cubes. Within the subgroups of the constant colored top cube, they were careful not to make duplicates. That is, in the group that contained yellow as the top cube, they made YBW (yellow, black, white) and were careful not to make YWB (yellow, white, black). According to their key, this tower would represent a pizza that contained sausage, pepperoni, and mushroom. By following this process, they immediately eliminated half of the solutions. The number of permutations of four colors taken three at a time is 24 . However, using this process, they created 12 towers. Next, they eliminated any duplicates from these 12 resulting in four three-topping pizzas. To find the number of two-topping pizzas, they followed the same strategy. They found all of the towers that contained two of a color ( 12 towers) and then eliminated duplicates. They found six two-tall towers that represented six two-topping pizzas.

They did not list all of their pizzas. Instead, they listed the number of pizzas for the zero-, one-, two-, three-, and four-topping pizzas. They noticed that the pizza problem and the towers problem resulted in the same number pattern $(1,4,6,4,1)$. They indicated that they believed they had the correct answer to the problem because the answer to the pizza problem followed this sequence of numbers. However, Jessica indicated that she was confused because, as she said, in the towers problem the order of the colored cubes
mattered and in the pizza problem the order of the toppings does not matter. That is, a pizza with mushroom and peppers is the same as a pizza with peppers and mushroom. However, a RRRW tower is different than a RWRR tower. They tried to justify the total number of pizzas for each number of topping with a formula but were unsuccessful. For the one-topping pizza, they believed the formula to be 4 to the first power. They explained that the base of this formula is four because there are four toppings to choose from and that the exponent is one because the choice of toppings is one. However, they indicated that they were very confused when it came to the two-topping pizzas because, as they explained, they could not find a formula, using exponents, which would give them an answer of six. It was not until the class discussed the isomorphism between the two problems that they understood the connection.

## Kim and Francesca S. - Summary

Kim and Francesca S. worked separately but checked with each other occasionally. They both organized their solutions by cases based on the number of pizza toppings. Kim found her pizzas using a combination of a modified tree diagram and a list. This tree diagram is "modified" because the branches stem off of a circle. Kim labeled her toppings using the full topping name. Francesca S . listed her pizzas and used the full topping name, abbreviations, or the initial of the topping. There is evidence that Francesca S. controlled for variables when checking her two-topping pizzas.

With her pencil, Francesca S. silently verified her two-topping pizzas. She checked these pizzas by starting with sausage and running through her one-topping pizzas. That is,
she pointed to " $S$ " (sausage) and then pointed to "Pep" (peppers). She pointed to " $S$ " and then pointed to "pepperoni." She pointed to " S " and then pointed to "mushroom." She then moved to "Pep" (peppers) and pointed to "pepperoni." She pointed to "Pep" and pointed to "mushroom."

When both girls agreed the answer to the problem was 16 , they told the instructor that they were finished but did not explain how they knew they were finished. The instructor suggested that they write down the total number of pizzas for each case. After seeing that the solution to followed the number sequence of $1,4,6,4,1$, Francesca $S$. suggested that it might be connected to the formula $m^{n}$ but then suggested that it might be four squared where four is the number of toppings and said that if there were three toppings, the answer might be nine. The instructor suggested that they find the pizzas when choosing from three toppings. Francesca S. found the total number of pizzas when choosing from three toppings to be eight. She proceeded to look at the total number of pizzas when choosing from two toppings and one topping. She and Kim discussed that the total number of pizzas doubled each time a topping choice was added. They indicated that they believed the formula to be the same as the towers problem, $2^{n}$, where $n$ was equal to the number of toppings. However, they were not able to explain why they understood this to be the formula until they investigated the connections between the pizza and towers problem.

### 5.3.2 Analysis of Strategies Compared to the Existing Research

There is evidence of students solving this problem in elementary school $\left(3^{\text {rd }}, 4^{\text {th }}\right.$, and $5^{\text {th }}$ grades), high school ( $10^{\text {th }}$ and $11^{\text {th }}$ grades), and college (Bellisio, 1999; Glass, 2001; Muter, 1999; Muter \& Uptegrove, 2010; Martino \& Maher, 1999; Tarlow, 2004). In each of these studies, every solution was organized by cases. The majority of the cases were based on the number of toppings although, at both the younger grades and the older grades, some students organized their cases based on a specific topping. The methods with which the students built their pizzas within each case and the representations used varied. However, there is evidence that a form of controlling for variables was used by some students at every age when creating the two-topping pizzas. The three solutions in this study are no exception. All three student solutions were organized by cases based on the number of toppings. Their strategies for creating the pizzas were different from each other but each was similar to one or more previous solutions found in the existing research.

Kim found her pizzas using a combination of a modified tree diagram and a list. The branches of her tree diagram stem off of a circle. It is an assumption this is a circle because it resembles a pizza. Kim labeled her toppings using the full topping name. The only other evidence of students using a tree diagram to solve this problem occurred in the eleventh grade (Tarlow, 2004). At the younger ages, many students used pictures and symbols to represent the pizzas. The use of circles to represent pizzas was often used. For example, a group of fourth graders (Kevin and Steve) and a group of fifth graders
(Marcel and Frederick) listed their pizzas within a giant circle (Bellisio, 1999). There was no other evidence, besides Kim, of high school or college students using circles to represent pizzas.

Francesca S. listed her pizzas based on the number of toppings and used the full topping name, abbreviations, or the initial of the topping. The use of lists was seen at every age. For example, fourth grader, Jamie, and fifth graders, Kersa and Ebonie, listed their pizza combinations based on the number of toppings and used the full topping names and/or abbreviations (Bellisio, 1999). Eleventh graders, Angela, Magda, Michelle, and Sherly, listed their pizzas based on the number of toppings (Tarlow, 2004) and seven of the 19 college students reported by Glass (2001) listed their pizzas based on the number of toppings and used the full topping name, abbreviations, and/or an initial.

Jessica and Jamie did not list all of the pizzas but instead, listed the number of pizzas. Jamie conjectured that the number of two-topping pizzas should be 12 because each of the four toppings could be combined with the remaining three toppings. Errol, a college student, also suggested the total number of two-topping pizzas was 12 and justified this total in the same manner as Jamie (Glass, 2001). The students in this study did not show any other evidence of using formulas to solve this problem. There is evidence of some of the high school and college students suggesting the use of formulas that they had previously learned but most were unsuccessful in applying the formulas (Glass, 2001; Muter, 1999, Tarlow, 2004).

To find the total number of two- and three-topping pizzas, Jessica and Jamie used Unifix cubes. The only other evidence of students solving this problem using Unifix cubes occurred in New Brunswick in the fifth grade (Bellisio, 1999). Like the students in the fifth grade, Jamie and Jessica used a specific color to represent a topping. However, unlike the students in the fifth grade, they did not create all 16 pizzas using towers. They only used the cubes to understand the number of pizzas that contained two toppings and the number of pizzas that contained three toppings.

When Jessica and Jamie created their three-topping pizzas using Unifix cubes, they controlled for variables by keeping the top cube in the three-tall tower a constant color while switching the remaining two cubes. There is evidence of Francesca S. also controlling for variables when she checks that she has created all of the two-topping pizzas. All of the college students in the study by Glass $(2001,2004,2010)$ were systematic when creating the two-topping pizzas. As Glass (2001) explains, "They held one topping fixed and paired with the each of the other toppings. They then moved to the next topping on the list" (p. 287). Some of the college students only paired toppings that were not previously paired similar to the techniques of Kim and Francesca S. The other students listed all two-topping pizzas and then eliminated the duplicates (Glass, 2004, 2010). Even though Jessica and Jamie used cubes, they also created all of the two-topping pizzas and then eliminated the duplicates.

There is also evidence of students in grade school and high school controlling for variables when creating the two-topping pizzas. Meredith, a third grader, combined
sausage with every other topping before moving to the next topping (Martino and Maher, 1999). Bellisio (1999) explains how three fourth graders (Jamie, Colin, and Brandon) and eight fifth graders (Artesia, Bhupur, Victor, Ronald, Ivan, Romina, Stephanie, and Matt) controlled for variables when creating their two-topping pizzas. There is evidence of eleventh grader, Robert, controlling for variables when creating his two-topping pizzas (Tarlow, 2004).

After they found all of the pizzas, Jessica and Jamie recognized the sequence of numbers that made up the total $(1,4,6,4,1)$ to be the same pattern of numbers as the answer to the towers problem. They indicated that they believed they had the correct answer to the problem because the answer to the pizza problem followed this sequence of numbers. However, Jessica explained how she didn't understand the connection because, as she said, in the towers problem the order of the colored cubes mattered and in the pizza problem the order of the toppings does not matter. Tenth grader, Ankur, and eleventh grader, Angela, also explained to their groups that in the pizza problem the order of the toppings doesn't matter but in the towers problem the order of the colored cubes matters (Muter, 1999; Tarlow, 2004). Two college students in the study by Glass (2001), Brian and Melinda, also expressed this difference between the two problems.

Jessica and Jamie were not able to justify why the total number of pizzas was 16 when choosing from four toppings until they investigated the isomorphism between the towers and the pizza problem. Francesca S. and Kim discovered that the total number of pizzas doubled each time a topping choice was added and indicated that they believed the
formula to be the same as the towers problem, $2^{n}$, where $n$ is equal to the number of toppings. However, they were not able to explain why until they investigate the connections with the towers problem.

The only other evidence of a student discovering the formula to be $2^{n}$, before investigating the isomorphism, is by tenth grader Michael. Muter (1999) and Muter \& Uptegrove (2010) explain how Michael found his pizzas by using a binary coding system. He used ones and zeros to indicate that a topping was included on the pizza or not. Many of the high school and college students connected their solution to Pascal's triangle or recognized that the solution doubled each time another topping choice was presented (Glass, 2001; Muter, 1999; Tarlow, 2004). However, not until they investigated deeper by looking at extensions of the four-topping pizza problem, Pascal's triangle, and/or the connection between the towers and the pizzas problem, were they able to justify the solution of 16 pizzas.

Some of the elementary grade students gave reasons such as "we checked with each other and got the same answer," "we couldn't find anymore," and "anytime we created another pizzas, it was already on our list" when asked how they knew that they had found all of the possible pizzas (Bellisio, 1999). Only Brandon (fourth grade) could thoroughly and systematically explain how he accounted for all possible pizza combinations using his chart (Bellisio, 1999).

Table 5.4
Pizza problem - building/organizing pizzas

|  | Students in this Study |  | Existing Research |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Jessica <br> Jamie | Kim | Francesca <br> S. | Grade <br> School <br> $\left(3^{\text {rd }}, 4^{\text {th }}, 5^{\text {th }}\right)$ | High School <br> $\left(10^{\text {th }} \& 11^{\text {th }}\right)$ | College |
| Tree Diagram |  | x |  | x | x |  |
| Used Pictures |  | x |  | x |  | x |
| Listed Pizzas |  |  | x | x | x | x |
| Unifix Cubes | x |  |  | x |  |  |
| Cases by \# of <br> Toppings | x | x | x | x | x | x |
| Cases - Other |  |  |  | x |  | x |
| Controlled for <br> Variables | x |  | x | x | x | x |
| Two to the $n$ |  |  | x |  | x |  |
| Order Doesn't <br> Matter | x |  |  |  | x | x |

### 5.4 Connections Between the Towers Problem and the Pizza Problem

After the students worked on the pizza problem, they all indicated that they believed there was a connection between the pizza problem and the towers problem. At this point, they had a class discussion and the instructor writes on the board what both groups have discovered. Both groups of students found that the numbers for the zero-, one-, two-, three-, and four-topping pizzas matched the numbers for the towers containing zero, one, two, three, and four blue cubes $(1,4,6,4,1)$. They all agreed that the formula for both problems is $2^{n}$ where $n$ represents the number of toppings in the pizza problem and $n$ represents the height in the towers problem. They explained that the two represented the number of colors in the towers problem. At this point, neither group was able to explain what the two means for the pizza problem. The instructor explained to them that she
wanted them to be able to explain the complete isomorphism between the problems. That is, they were to explain what the base of two meant in terms of the pizza formula and she explained that she wanted them to be able to match a specific tower with a specific pizza. She suggested that they look at the three-tall towers and the pizzas when choosing from three toppings.

### 5.4.1 Results

Jessica and Jamie - Summary

Jessica and Jamie created the eight three-tall towers using black and white cubes and organized them in cases based on the number of white cubes contained within the tower. They also listed the eight pizzas when choosing from three toppings. They organized their pizzas by cases based on the number of toppings. They agreed that the tower without white (all black) represented the pizza without any toppings. Jessica suggested that the towers with one white represented the pizzas with one topping, the towers with two whites represented the pizzas with two toppings, and the solid white tower represented the three-topping pizza. They both agreed that the white cube represented the toppings but, at this point, they did not indicate that the position of the white cube represented a specific topping. Jessica remarked, "It just bothers me because the towers are positional and these aren't." [Line 2.2.618]

They explained what they have found to the instructor. Pointing to the towers, the instructor replied, "Okay, that's perfect. But you know what I'm gonna ask next which is

- which pizza is this and this and this?" [Line 2.2.628] Jessica replied by saying, "This one, we like decided it would be sausage [points to the white cube in WBB], pepperoni [points to the white cube in $B W B$ ], mushroom [points to the white cube in BBW]." [Line 2.2.630] This was the first time she indicated that a specific position in the tower indicated a specific topping. She proceeded to explain which two-topping pizzas were represented by the towers.


## Kim and Francesca S. - Summary

They built the eight three-tall towers using blue and orange cubes and organized them by cases based on the number of blue cubes. They referred to the list of eight pizzas that Francesca S. has already listed by number of toppings. Francesca S. connected the tower with all orange cubes with the plain pizza. She and Kim discussed that the tower with all blue cubes would be the pizza with all of the toppings. Francesca S. stated that the towers with one blue would be the one topping pizzas and the towers with two blue would be the two topping pizzas. They decided that the blue cubes represented the toppings.

They explained to the instructor what they have found and the instructor asked them to explain which pizza is represent by the BOO tower. They eventually indicated that the BOO tower represented a pizza with sausage. The instructor left them so that they could connect the remaining towers with pizzas. Francesca S. indicated that she understood that a specific position in the tower matches with a specific pizza topping. She told Kim, "Yeah, so blue on top is sausage. Blue in the second, I put, is peppers and then the third one's mushroom." [Line 2.1.618] However, they could not connect the tower with two
blue cubes to a specific two-topping pizza. They instructor joined them again and they discussed the OBB tower (this tower is peppers and mushroom). Kim replied that it is sausage-peppers and Francesca S. explained that it was sausage-mushroom or peppersmushroom because mushroom was the bottom cube. [Lines 2.1.628-2.1.630]

Francesca S. explained to Kim that she thought OBB is pepper and mushrooms because if the first cube is sausage and the first cube is orange, then there is no sausage on that pizza. The instructor says "So you're saying that the position makes a difference?" [Line 2.1.680] Francesca S. agreed and she pointed to the BOB tower and correctly identified this tower as the sausage and mushroom pizza. Kim expressed that she was completely confused. [Lines 2.1.650-2.1.689]

The instructor explained that they have already told her that the position of each cube represented a specific topping. She asked them to explain the difference between a blue cube and an orange cube. They could not. The instructor proceeded to ask them to identify certain towers in terms of pizzas. Francesca S. and Kim could correctly name the pizzas that corresponded with each tower. However, when asked to explain, in general, the difference between an orange cube and a blue cube, they could not. Eventually, Kim said that the blue meant that the topping was there and the orange meant that the topping was not there. [Lines 2.1.705-2.1.770] The instructor asked them to write the complete isomorphism for homework.

### 5.4.2 Analysis of Strategies Compared to the Existing Research

Both groups of students in this study were able to explain the complete isomorphism between the pizza problem and the towers problem. That is, they explained that the two represents the colors in the towers problem and the inclusion or exclusion of the topping in the pizza problem. The exponent represents the height of the tower and the number of toppings. Furthermore, they were able to connect the position of a cube in a tower with a pizza topping and explained that if the cube was a certain color the topping would be included. However, Kim and Francesca S. were able make these connections only after the instructor had worked with them.

There is evidence of students in third and fourth grade making connections between the two problems but only with teacher intervention. Maher and Martino (1998) describe how Brandon, a fourth grader, was able to make the connection between the two problems after the teacher had encouraged him to focus on one specific color in the towers problem. After he rearranged his towers into groups based on the number of a specific color cube, he was able to match a specific pizza to a specific tower. Furthermore, he was able to explain that it didn't matter if you focused on the yellow cube or the red cube to make the connection.

Martino and Maher (1999) describe the connections made to the towers and the pizza problem by two third graders, Meredith and Sarah. Without the teacher intervention, the girls consistently wanted to make the towers using different colored cubes. By asking the
girls if they could make the towers with two colors and showing them Brandon's solution to the towers problem, the girls could match the specific towers with specific pizzas.

Muter (1999) describes students in tenth grade explaining the complete isomorphism between the towers problem and the pizza problem. These students began the class by discussing the binary coding system that Michael had used to solve the pizza problem. In Michael's coding system, each position represented a topping and a one indicated that the topping was on the pizza and a zero indicated that the topping was not on the pizza. They were able to explain that the base of two in the pizza problem represented the two choices of whether to include the topping or not.

Tarlow (2004) explains how two eleventh graders, Robert and Stephanie, were able to explain the isomorphism between the two problems. Robert explained that the answer to both questions was determined by the formula $2^{n}$ where $n$ was equal to the height of the tower or the number of toppings. The base, 2, which represented two colors in the towers problem, also indicated, in terms of pizzas, the two choices: with or without toppings. Furthermore, Stephanie explained how a particular position in the tower represents a particular topping.

Glass (2001) explains how four college students, Wesley, Rob2, Mike, and Errol were able to explain the isomorphism during the class session. Three other students, Jeff, Lisa, and Elizabeth were able to describe the isomorphism a week later during an interview. Two students, Rob1 and Donna, could not explain the isomorphism while two students,

Melinda and Stephanie, partially made the connection. That is, Melinda could match all but the two-topping pizzas to a specific tower. Stephanie connected the color of the cube with the fact that the topping would be off or on the pizza but did not match a specific cube position with the topping. In eleventh grade, one group of students (Anglea, Michelle, Sherly, and Magda) could not make the connections either (Tarlow, 2004).

Table 5.5
Towers and pizza problem - connections

|  | Students in this Study |  |  | Existing Research |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Jessica <br> Jamie | Kim <br> Francesca S. | Grade <br> School <br> $\left(3^{\text {rd }}, 4^{\text {th }}, 5^{\text {th }}\right)$ | High School <br> $\left(10^{\text {th }} \& 11^{\text {th }}\right)$ | College |  |  |
| Explained each <br> component of the <br> Towers Formula | x | x | x | x |  |  |  |
| Explained each <br> component of the <br> Pizza Formula | x | x | x | x |  |  |  |
| Connected <br> Specific Tower <br> with Pizza | x | x | x | x | x |  |  |

### 5.5 Ankur's Challenge

### 5.5.1 Results

Jessica and Jamie - Summary

Jamie and Jessica solved this problem directly using a cases approach. They first realized that each tower must contain exactly two of the same color (Jessica refers to this as the "dominant" color) and one cube of each of the remaining two colors. They focused
on creating all of the towers with a specific dominant color, a direct approach to the problem. Jessica and Jamie, using paper and pencil, wrote all of the towers with brown as the dominant color using the first initial of the color name to symbolize the colored cubes. They found 12 towers within this case and concluded that this would be the same for the remaining two dominant colors. Therefore, they concluded the answer would be 12 times 3 which would be equal to 36 .

In creating the 12 towers, Jessica and Jamie used a controlling for variables strategy by keeping the two brown cubes together and systematically moving them up the tower. One of each of the green and maroon cubes would fill in the remaining two cubes. For each of these towers, they created the opposite tower by switching the green and maroon cubes (keeping the brown cubes in the same positions). They created their next six towers by separating the brown cubes and filling in the remaining cubes with one green and one maroon cube. Again, they created the opposite tower after creating a new tower.

After Jessica and Jamie found their 36 towers, Jessica expressed that they needed to understand why the answer was 36 and she suggested that they focus on the 45 remaining towers. They were able to understand 27 of the 45 towers (the three solid towers plus 24 towers which were composed of three cubes of the same color and one cube of another color). However, they were not able to find the remaining 18 towers. Jessica indicated that she understood the 18 towers they were missing only after Rebecca and Francesca C. presented their solution involving the complement. Jessica says "Now I understand what we were missing - it's their doubles." [Line 1.1.597]

Kim and Francesca S. - Summary

Kim and Francesca S. solved the problem similarly to Jessica and Jamie. They solved the problem directly by a cases approach. They also first realized that each tower must contain exactly two of the same color (a "dominant" color) and one cube of each of the remaining two colors. They built their towers using Unifix cubes and created all of the towers with white as the top cube. They found 12 towers within this case and concluded that this would be the same for the remaining two colors. Therefore, the final answer would be 12 times 3 which would be equal to 36 .

In creating these 12 towers, Kim and Francesca S. also used a controlling for variables approach. They built their towers by keeping the top cube white. They had twelve towers with white on top. These 12 towers were separated into three sub-cases where each of these sub-cases was based on the color of the second cube. The first subcase contained two towers that had white on top and white as the second cube. The second sub-case contained five towers that had white on top and blue as the second cube. The third sub-case contained five towers that had white on top and yellow as the second cube.

## Rebecca and Francesca C. - Summary

Rebecca and Francesca C. solved the complement of the problem. That is, they found all of the towers that did not have at least one of each color. Once they found the total number of towers in this set, they subtracted this number from the total number of
possible four-tall towers when selecting from three colors. They organized the 45 towers that were in the complement into cases. They wrote their solution on the chalkboard using the first letter of each of the color names to represent the cubes.

They broke their solution up into three sub-cases. The first sub-case, the "doubles," contained the towers with two cubes of one color and two cubes of another color. They created six towers in this group and explained that these six would be multiplied by three for each of the three colors to get a total of 18. The next sub-case, the "solids," contained the towers where all cubes were the same color. They had one solid tower on the board and explained that this would be multiplied by three to get a total of three solid towers. The last sub-case, "the triples," was comprised of the towers containing three cubes of one color and one cube of another. They found eight towers in this case for each color. They multiplied this number by 3 to get a total of 24 towers in this sub-case. They explained that the sum of 18,3 , and 24 is equal to 45 towers that are excluded from the list and that the answer is 36 because 81 minus 45 is 36 .

### 5.5.2 Analysis of Strategies Compared to the Existing Research

In a study by Glass and Maher (2004), the solutions of 22 students to Ankur's Challenge were analyzed and categorized. Glass and Maher organized the 22 solutions into four categories: (1) Justification by Cases, (2) Inductive Arguments, (3) Elimination Arguments, and (4) Analytic Method. These students were either in high school, undergraduate, or graduate school. The report by Glass and Maher includes the five students from the study by Glass (2001) and one of the high school students, Romina,
reported by Muter (1999). In addition to these 22 solutions, the solutions of two tenth graders, Ankur and Mike, reported by Muter (1999) will also be compared in this section.

Nine of the 22 solutions reported by Glass and Maher (2004) are classified as Justifications by Cases. The solutions of Kim, Francesca S., Jessica and Jamie would also be placed in this category. Of these nine solutions, three undergraduates (April, Bernadette, and Rob1) and one graduate student (Traci) controlled for variables in a similar way to Kim and Francesca S. April and Rob1 kept the top cube a constant color to create their 12 towers while Bernadette and Traci, using the same logic, kept a constant colored cube on the bottom of the towers to create their 12 towers.

Joanne and her partner Donna (both undergraduate students), described each of the six possible positions for two cubes of the same color in a four-tall tower. This is similar to how Jessica and Jamie created their 12 towers. The difference is that Jessica and Jamie wrote all possible 12 towers on their paper. Joanne's group explained that for each of the six possible arrangements for two cubes of the same color, two towers can be created by alternating the other two colors in the unfilled positions. As they explain, there are three different colors that could be the dominant color so there are six color combinations (3 dominant colors times 2 options per tower) for each for each of the six towers. Therefore, the answer is 36 (six times six) possible towers. Rob2 (undergraduate) also used a similar approach to Jessica and Jamie by keeping the dominant cubes together and systematically moving them down the tower filling in the remaining cubes with one of each of the
remaining colors. Then, like Jessica and Jamie, he created the remaining six towers by separating the dominant colored cubes.

Romina, a high school student, used a cases approach very similar to the four students in this study. However, like Joanne and Donna, she only created six towers. Using a chart, she demonstrated all of the possible positions for two cubes of the same color in a four-tall tower. She used a one to represent this dominant color and found six possible towers. For each of these six towers, two towers can be created by alternating the other two colors in the unfilled positions. She used X's and O's to symbolize the remaining two colors. This method created a total of 12 towers for each dominant color for a total of 36 towers.

As described by Muter (1999), Mike and Ankur (in the same session as Romina) solved this problem by focusing on the complement. They quickly figured the total number of four-tall towers choosing from three colors to be 81. After seeing Romina's proof, they agreed that the answer to the problem was 36. Like Jessica, they explained that to be convinced of this answer, they needed to understand the 45 remaining towers. They eventually created, on paper, the 45 towers using a series of numbers to represent the colors. They broke these 45 towers into three sub-cases: 24 towers with three of one color and one cube of another color, three towers where each tower contained all cubes of the same color, and 18 towers with two cubes of the same color and two cubes of another color.

Rebecca and Francesca C. solved the problem by looking at the complement similar to Ankur and Mike. Unlike Ankur and Miker, they used letters to symbolize the cubes and did not create all 45 towers. They broke their solution up into the same three subcases as Mike and Ankur and named these sub-cases the "doubles," "solids," and "triples." Mike and Ankur wrote all 18 towers in the sub-case "doubles" while Rebecca and Francesca C. created six towers in this group and explained that these six would be multiplied by three for each of the three colors to get a total of 18 . For "solids," they drew one tower on the board and explained that this tower would be multiplied by three and for "triples," they drew eight towers on the board and explained that for each of the three colors, there would be eight towers so there would be a total of 24 towers in this subcase. Like Mike and Ankur, they explained there was a total of 45 towers that are excluded from the list and that the answer is 36 because 81 minus 45 is 36 .

In the analysis by Glass and Maher (2004) there is not a category for the complement. The remaining solutions are categorized in one of three remaining categories. These categories are Inductive Arguments, Elimination Arguments, and Analytic Method. Finding the complement is most similar to the solutions contained in the category of elimination. Four students' solutions, Robert (undergraduate), Penny (undergraduate), Liz (graduate), and Mary (graduate) fall into this category. Penny used a tree diagram and created all of the possible 81 towers and then crossed out the towers that did not have at least one of each color. The remaining students found the total number of four-tall towers to be 81 and subtracted the number of towers that did not have at least one of each color.

This method is similar to the method used by the high school students, Mike and Ankur and, from this study, Rebecca and Francesca C. However, unlike Mike and Rebecca's groups, these three students used formulas to find the total number of towers in the complement, as opposed to creating subsets of the towers. Although the complement solution requires subtracting the 45 towers from the total number of towers at the end, the focus of the investigation is to justify the 45 towers that form the complement. For this reason, it is suggested that this type of solution be categorized separately from the elimination argument.

Table 5.6
Ankur's Challenge - initial building/organizing towers

|  | Students in This Study |  |  | Existing Research |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Jessica <br> Jamie | Kim <br> Francesca S. | Rebecca <br> Francesca C. | High School <br> $\left(10^{\text {th }}\right.$ grade $)$ | College | Graduate <br> School |
| Contr. for <br> Variables | x | x |  | x | x | x |
| Justification <br> by Cases | x | x |  | x | x | x |
| Inductive <br> Argument |  |  |  |  | x | x |
| Elimination <br> Argument |  |  |  |  | x | x |
| Analytic <br> Method |  |  |  |  |  | x |
| Complement |  |  | x | x |  |  |

### 5.6 The Role of the Teacher

Martino and Maher (1999) proposed four types of teacher questioning that have been shown to foster student understanding. These include questioning that: (1) facilitates
justification; (2) offers opportunities for generalization; (3) invites opportunities to make connections; and (4) facilitates awareness of solutions presented by other students. These four categories will be explored in the sections that follow. However, in studying the teacher moves, additional categories emerged. These include, first, allowing ample time for initial exploration of the problem and, second, questions that promote the communication of ideas. It was found that these categories of questions led to students' clarification of ideas. A third questioning category that emerged included the instructor's drawing attention to a particular element in a student's solution; a fourth was questioning that invited the student to test conjectures. Although these last two categories would be considered under the heading of "facilitates justification" and "facilitates generalization" respectively, they are emphasized because of the value they played in extending understanding.

## Allowing Ample Time for Initial Exploration

Martino and Maher (1999) stress that students must have time to explore the problem without any teacher intervention. The teacher should intervene only "after students have built a solution, consulted with each other and posed a solution that they believe is valid" (p. 56). It is at this point that the students are ready to be challenged to explain their reasoning and justify their solutions. There is evidence in this study of the instructor giving time for the students to explore. Furthermore, her initial questioning is very openended, asking the class to explain what they have done or are in the process of doing.

In the towers problem, after Jamie and Jessica have built and organized 18 towers, the instructor asks, "Explain to me where you are and what you've got..." [Line 1.1.87] They explain what they have found and the instructor listens. At the end of the explanation, Jessica says, "But we're not sure if that's it - we want to keep thinking - but at the moment we have 18." [Line 1.1.106] The teacher replies "Ok" and leaves them to work. The instructor visits Kim and Francesca S. after they have built 16 towers and organized them. She begins their conversation by saying "You guys are going to explain to me your strategy for building them. So, tell me what you did." [Line 1.1.123]

During Ankur's Challenge, the instructor gives many opportunities for Jessica and Jamie to explore the problem. After they have found the answer to be 36 and are trying to understand which towers make up the 45 towers that are not in the set, the teacher first approaches. She says "How's it going?" [Line 1.1.485] They explain to her how they found the 36 towers. After their explanation, she leaves them alone to explore for another nine minutes until Jessica calls her over.

During the pizza problem, the instructor approaches Kim only after four minutes. However, Kim has called her over because the videographer has implied that she had done something wrong. At this point Kim has found 11 pizzas and asks the instructor if she had done something wrong. The instructor replies, "You know I don't answer questions like that. Okay. So, what did you do?" [Line 2.1.182] After Kim explains how she found 11 pizzas, the instructor replies "Okay. So, it doesn't sound like you have any
questions, seem to be doing everything okay." [Line 2.1.199] She leaves her to explore further.

The instructor has left Francesca S. alone for about eight minutes. She says to her: "Ok, so, I looked at Kim's. Now tell me how you're doing it." [Line 2.1.215] Francesca S. has found 12 pizzas at this point and explains what she has found. After the explanation, the instructor leaves her to explore further. After about 15 minutes, the instructor first approaches Jessica and Jamie. She says to them, "So explain to me what you're doing." [Line 2.2.315] At this point, the girls have figured out the number of pizzas for all but the two-topping pizzas. They indicate that they believe the answer to be similar to the towers but they explain to the instructor that 1) they need to find the number of two-tall towers and 2) they want to understand why they are the same. She reiterates what they have said by saying, "Well, you think you're getting the same numbers but you still don't exactly have the six two-topping - the six that you think you going to get for two toppings. Is that right?" [Line 2.2.325] They reply "yes" and explain to her that they are working on that. She leaves them to continue working.

Table 5.7
Teacher questioning - initial question after student exploration

|  | First Question after <br> Initial Exploration | Time Elapsed Before <br> First Question | Transcript <br> Line |
| :--- | :--- | :--- | :--- |
| Towers <br> Jessica/Jamie | "Explain to me where you are and <br> what you've got..." | 6 minutes | 1.1 .87 |
| Towers <br> Kim/Francesca <br> S. | "You guys are going to explain to <br> me your strategy for building them. <br> So, tell me what you did." | 10 minutes | 1.1 .123 |
| Ankur's <br> Jessica/Jamie | "How's it going?" | 8 minutes | 1.1 .485 |
| Pizza <br> Kim | "So, what did you do?" | 4 minutes | 2.1 .182 |
| Pizza <br> Francesca S. | "Now tell me how you're doing it." | 8 minutes | 2.1 .215 |
| Pizza <br> Jessica/Jamie | "So explain to me what you're <br> doing." | 15 minutes | 2.2 .315 |

As the data indicate, the instructor only spoke to the students after they had worked on the problem for some time and had devised their own way of approaching the problem. Her initial questions were open-ended and promoted the communication of ideas. Maher et al. (2010) explain that "a central component of the learning process is encouraging students to communicate their ideas" (p. 3). Furthermore, as explained by Webb (2013), the process of presenting the ideas might expose "contradictions or incompleteness of ideas that are recognized by the explainer or are pointed out by others" (p. 20). There is evidence of clarification or extensions of ideas by these students after the teacher has asked them to explain their solutions to her.

Questions that Promote the Communication of Ideas - Presenting Ideas that Lead to

## Clarification

In the initial stages of the towers problem, Jessica and Jamie found 18 towers. The instructor returns to their group after they had some time to think about their 18 towers. She says to them, "Ok, so, alright, explain your groupings one more time." [Line 1.1.159] They have organized their towers into five groups. Within these groupings, they have duplicates. In explaining their organizational structure, Jessica moves a tower from the elevator grouping to complete the staircase grouping and the duplicate is exposed.

Later, after Jessica and Jamie discover the doubling pattern to the towers, they explain to the teacher that they believe the answer is powers of twos. In explaining their discovery to the instructor, they are able to extend their thinking to understand that, not only is the solution a power of two, but that the formula is $2^{n}$ where $n$ is equal to the height of the towers.

| 1.1 .260 | $21: 15$ | Instructor | Now I am ready to hear what you guys have to say. <br> [Camera turns and focuses on Jamie and Jessica's <br> group.] |
| :--- | :--- | :--- | :--- |
| 1.1 .261 | $21: 22$ | Jamie | We think we figured it out. We think it's the powers <br> of two. |
| 1.1 .262 | $21: 27$ | Jessica | Yeah, because we, we... when you told us to do <br> [inaudible, pointing to 8 towers, 3-tall each] we only <br> got eight. So we were like let's go down to two and <br> see what we get there and we got 4. [Indicating group <br> of 4 towers - each two-tall] So you have two raised to |


|  |  |  | the second power. |
| :--- | :--- | :--- | :--- |
| 1.1 .263 | $21: 38$ | Jamie | Do you know what it is? It's whatever number of <br> towers - |
| 1.1 .264 | $21: 40$ | Jessica | That's the power. |
| 1.1 .265 | $21: 41$ | Jamie | That's the power. |
| 1.1 .266 | $21: 43$ | Instructor | Oh.... |
| 1.1 .267 | $21: 44$ | Jamie | Two squared, two to the third, two to the fourth. |
| 1.1 .268 | $21: 46$ | Instructor | So you could tell me how many there's gonna be five- <br> tall - without doing it? |
| 1.1 .269 | $21: 50$ | Jessica | That's 32. |

During the pizza problem, Kim calls the instructor over because she indicates that she believes that she had done something wrong. The instructor says, "So what did you do?" [Line 2.1.182] As Kim explains, her plain pizza, her one-topping pizzas, and her twotopping pizzas, she realizes she is not finished. Kim says "Oh, I forgot I have to do the three toppings." [Line 2.1.198]

Later, during the pizza problem, Francesca S. explores her conjecture that since they found 16 pizzas when choosing from four toppings, that there might be nine pizzas when choosing from three toppings. She lists all of the pizzas when choosing from three toppings. She explains to the instructor that she found nine pizzas. The instructor replies "Show me." [Line 2.1.321] While explaining her findings to the instructor, Francesca S. realizes that she only had eight pizzas on her list. Francesca says: "Okay, wait. That
doesn't even make sense. Wait. How did I count nine? Okay. It's eight." [Lines 2.1.326-
2.1.327]

Table 5.8
Teacher questioning that led to clarification

|  | Teacher Questioning | Clarification Revealed | Transcript Line |
| :---: | :---: | :---: | :---: |
| Towers Jessica/Jamie | "Ok, so, alright, explain your groupings one more time." | They found two towers that were duplicates. | 1.1.159 |
| Towers Jessica/Jamie | "Now I am ready to hear what you guys have to say." | By explaining the pattern, they realize the formula is $2^{n}$. | 1.1.260 |
| Pizza <br> Kim | "So what did you do?" | Kim realizes that she is not finished. | 2.1.182 |
| Pizza <br> Francesca S. | "Show me." | Francesca S. realizes that she has eight pizzas, not nine. | 2.1.321 |

It is interesting that the instructor did not point out the errors or the incompleteness in the students' solutions. Instead, she asked them to explain what they have found. During their explanations, the students, themselves, were able to find their errors or extend their thinking.

## Communicating Ideas - Drawing Attention to Other Students Solutions

One of the four types of teacher questioning proposed by Martino and Maher (1999) is questioning that "facilitates awareness of solutions presented by other students" (p. 70). As Martino and Maher suggest, "To extend present individual knowledge, it is sometimes useful to ask a student to consider a justification produced by another student so that similarities and differences between the two approaches can be considered" (p. 57).

Webb (2013) explained that listening to others' solutions is an important part of the learning process because by listening, students might "generate new connections between their own ideas or between their own and others' ideas" (p. 20).

There are obvious instances of the teacher encouraging the sharing of ideas with each other. For example, during the pizza problem, Kim has found six pizzas with two toppings and Francesca S. has found five. The instructor says, "So compare, right? Okay." [Line 2.1.250] After comparing, Francesca S. finds the pizza she was missing. Later, the instructor again encourages the students to communicate their ideas to each other. She says: "Ok, I'll just let you go. Then tell me when you both think that you're in agreement that you're done and what you got." [Line 2.1.255]

Another obvious way to get students to listen to each other's solutions is to have them present their findings to each other. At the end the towers problem and Ankur's Challenge, the instructor has each group present their findings. What is not so obvious is what appears to be a planned move by this instructor. It appears as though she strategically picked the order in which the students presented their solutions to maximize their learning potential from each other.

The order in which she chose the students to present their solutions to the towers problem is as follows: 1) Kim and Francesca S., 2) Rebecca and Francesca C., and 3) Jessica and Jamie. Kim and Francesca S. explained that they organized their solution by cases based on the number of blue cubes. After they finish their explanation, the instructor says: "And so your total numbers were one [indicating YYYY], four [indicating

Set 2]...." [Line 1.1.342] Kim replies, "four [indicating Set 2], six [indicating Set 3], four [indicating Set 4], and one [indicating BBBB]" [Line 1.1.343] Next, Rebecca and Francesca C. explain how the towers can be built inductively. After their explanation, the instructor says "So, now, many of you discovered the rule - it doubles - and this is the explanation. Right? This is the reason why it doubles. So there's more than just, yeah, we see a pattern its times two. Here's the reason why its times two. Right? Inductive reasoning, right? See, that wasn't too bad. Okay. This is why it goes times two every time. And so now you know - just like with the induction, like with the dominos, right? If you have any height tower, you know what the next one's gonna be because of this inductive rule she talked about. Ok - questions?" [Line 1.1.358]

Jessica and Jamie present last. They explain how they found the formula to be $2^{n}$ and that the general formula is $m^{n}$ where $m$ represents the number of colors and $n$ is the height of the tower. They demonstrate their general formula with the nine towers that are two-tall when choosing from three colors. The instructor asks them for a reason why the formula is $3^{n}$. The instructor says, "Well, see now, can you follow up with what Rebecca and Francesca did - the one-tall when you've got three colors to choose from?" [Line 1.1.386] Jessica and Jamie explain that for one-tall towers choosing from three colors the answer would be three. She replies, "Right, now remember what you said, Rebecca, come in closer and point." [Line 1.1.389] Rebecca explains, the inductive rule using three colors, "For each one-tall tower you can, for this one [indicating B] you can add either a brown, a green, or a maroon. For this one, [indicating G] you can add either a brown, a
green, or a maroon. So, for each one, there's three possible towers you can make to create it two tall. So, you add a green, you know, you can add a green, you can add a maroon, or you can add a brown. So you end up with, you know, three more from what you already have." [Line 1.1.390] Francesca C. replies, "So the answers triple." [Line 1.1.392] The instructor replies, "That's right - the other one was doubled and this one now is tripled." [Line 1.1.393]

In presenting the solutions to Ankur's Challenge, the instructor chose the following order: (1) Jessica and Jamie, (2) Kim and Francesca S., and (3) Rebecca and Francesca C.

Jessica and Jamie explain how they found 12 towers that contained two brown cubes and one of each of the other colored cubes. They explain that there would be three sets of 12 for each of the three colors that would be the "dominant" color (two of the same colored cube). Kim and Francesca S. had a similar solution but they found the 12 towers by keeping the top cube white while changing the remaining three cubes. They explained that they would get three sets of 12 by changing the top colored cube. The strategy by Rebecca and Francesca S. is different. They explain how they found the 45 towers that do not have one of each color. They broke these 45 towers into three cases. The first case, which they named the "doubles," contained 18 towers that contained two cubes of one color and two cubes of another color. The second case, named the "solids," contained the three solid colored towers. The third case, "triples," contained the 24 towers that have three cubes of one color and one cube of another color. After hearing the explanation by Rebecca and Francesca S., Jessica says, "Now I understand what we were missing - it's
their doubles." [Line 1.1.597] Jamie and she had been working on understanding the complement to justify the 36 towers that they had found. In trying to understand the complement, they found all of the towers except for the "doubles."

In Ankur's Challenge, it is apparent that Jessica has discovered the incompleteness of her justification by listening to the solution of Rebecca and Francesca S. because she expresses this by saying, "Now I understand what we were missing - it's their doubles." [Line 1.1.597] A week later, the students are asked to justify why the formula for the towers problem choosing from two colors is $2^{n}$. During this discussion, the instructor asks Jamie and Jessica if they recall the explanation by Rebecca. Jessica says, "Well, the thing is... the only thing I can remember truthfully is that you have two choices. You can either add a white one on or you can add a blue one on. [Holds up a single blue cube and a single white cube.]" [Line 2.2.73] This leads to a discussion about induction and, in explaining the formula $m^{n}$, Jessica says, "Yeah, you have that many choices; you have to keep multiplying by that." [Line 2.2.79]

In the towers problem, it appears the instructor choose the order based on the complexity of the solutions. Kim and Francesca justify the answer of 16 by organizing their towers based on cases. The instructor pointed out that number of towers in each case is $1,4,6,4$ and 1 . At this point, there is no further discussion about these particular numbers but perhaps the instructor is highlighting this sequence of numbers for future explorations in connections to the pizza problem and Pascal's triangle. Next, she has Francesca C. and Rebecca present. They explain that they have also found 16 towers, the
formula to be $2^{n}$, and they justified their solution inductively. Lastly, she has Jessica and Jamie explain the general formula, $m^{n}$, and she asks the class to explain, using Rebecca and Francesa C.'s inductive argument, why when choosing from three colors, the formula is $3^{n}$. It appears that she strategically choose this order so that the students can build on the previously presented solutions.

In Ankur's Challenge, their solutions do not appear to be at a different level of complexity but are two different types of justifications. The first two groups have similar solutions of finding 12 towers and multiplying these 12 towers by three to find the solution of 36 . The instructor highlighted how the way in which the two groups found these solutions were both by cases but slightly different cases. The last group also broke their solution into cases but they concentrated on the 45 towers that were not in the set. Perhaps by allowing the two "similar" solutions to go back-to-back allowed for a closer examination of the similarities as well as the differences. Regardless, there was evidence of Jessica's deeper understanding by listening to the solutions of others.

The next three categories of teacher moves require the teacher to know, not only the mathematical content well, but her students and their solutions. She must also have a vision as to the direction in which she wants the students to explore the problems. This will require acute listening on the part of the teacher to understand the current level of understanding of her students. The first category is teacher questioning that is intended to have the students focus on a particular element of their solution. As Martino and Maher (1999) explain, these questions might assist "the student in focusing on that aspect of
his/her argument which may ultimately resolve difficulty" (p. 56). Under the organizational structure by Martino and Maher, these questions would be categorized as "questions that facilitate justification." The second category contains questions or suggestions to allow the student to conjecture and/or test their conjectures to possibly bring them closer to a solution. These questions would be categorized as "questions that facilitate generalization." The third category involves questions that "invite learners to make connections" (Martino \& Maher, 1999, p. 66).

## Questions that Draw Attention to a Particular Aspect of a Student's Solution

Initially, Kim and Francesca S. found 16 towers and organized them by pair-wise opposites along with groups that contain the staircase pattern. The instructor asks them to explain their organizational structure. Francesca S. explains, "These are alternating and then three blue, two blue, one blue [pointing to relevant cubes in YBBB, YYBB, YYYB]." [Line 1.1.136] The instructor replies, "16. Ok. Well, you said this one is like three blue, two blue, one blue [Pointing to YBBB, YYBB, YYYB in Set 5]. There's also a three blue here and there's a two blue here [Points to Set 3]. So I don't quite see a total [inaudible]" [Line 1.1.139]. The students discuss how some of the towers have one blue cube, some have two blue cubes, and some have three blue cubes. Using the fact that the students have referred to their towers by the number of blue cubes contained in the tower, the instructor suggests organizing them based on the number of blue cubes. She says, "So can you organize them - put all the twos together and the ones and threes together and see if you can find something that jumps out that they are all there?" [Line 1.1.156] Eventually,
they justify the solution by cases based on the number of blue cubes contained within the tower. The teacher focused on the words that the students had used to describe the towers to suggest another organizational structure that led to clarification.

A week later, when Jessica and Jamie are trying to make the connection between the solution to the pizza problem and the solution to the towers problem, the instructor emphasizes the individual answers to the different cases within the towers and pizza problem. That is, she encourages the students to focus on the individual totals for the plain, one-topping, two-topping, and three-topping pizzas as well as the individual totals for the zero blue, one blue, two blue, and three blue towers. She says, "Okay, now why do I like that? Because I see $1,3,3,1$. Right? Now you've got pizzas also that you organized by number of toppings, at least they did. Don't make them, write them. There's the 1 , the 3 , the 3 , the 1 . The no toppings, the one topping, the two topping, the three toppings. So, list them out and they will match up under here and then you will just figure it out. You will look at it and think hard and you will see what you see." [Line 2.2.582] It is after she directs their focus that they are able to explain the isomorphism between the pizza and towers problem.

The instructor applies the same teaching strategy to Kim and Francesca S. She points at their notebooks and says, "I'm looking for- here, yeah, here's what you said about the towers [Reading from Francesca's notebook on towers problem] - you said 1, 4, 6, 4, 1. Here's what you said about the pizzas. [Pointing to Kim's notebook]" [Line 2.1.387] She then says, "And there's the one - the plain one $-1,4,6,4,1$. Not only are you getting
two to the $n$, you're getting exactly the same numbers." [Line 2.1.389] Kim asks if that is supposed to happen and the instructor replies by saying, "You're gonna tell me why it happens." [Line 2.1.391]

Later, in discussing the isomorphism between the two problems, the class is discussing the reason for the base of two in the formula $2^{n}$ for the pizza problem. Francesca S. proposes an explanation for the two. She says, "Like, one plain or either one that has toppings on it." [Line 2.1.441] But then Francesca S. indicates that she is confused. The instructor says, "Yeah, but you're getting there." [Line 2.1.454] It could be interpreted that the instructor is encouraging Francesca S. to continue with this thought. That is, it suggests that she does not want Francesca S. to lose her focus.

The instructor works with Kim and Francesca S. to try to understand the isomorphism between the towers and the pizza problem. They are trying to understand how the position of the cube in the tower represents a topping and how the color of the cube indicates if the topping is on the pizza or not. Throughout their conversation, the instructor reiterates what the students have said to draw attention to the important features of their conversation. For example, Francesca S. explains "This one would be pepper and mushroom [pointing to the OBB tower] because it doesn't have a sausage on top [points to the orange cube on top of the OBB tower]. So we know it's gonna have to be pepper and mushroom." [Line 2.1.679] The teacher replies, "So you're saying the position makes a difference?' [Line 2.1.680] Francesca S. replies, "Yes."

Later, in an attempt to understand the meaning of the color cube in terms of pizza, Francesca S. says, "The order matters not the colors, I don't think." [Line 2.1.733] The teacher focuses on two towers that have a different colored cube in the first position.

| 2.1 .734 | $1: 07: 10$ | Instructor | Well, the color... Now, how do you know that <br> there is sausage on this pizza [holds up the OBB <br> tower] and that there isn't sausage on this pizza? <br> [Holds up the BOO tower.] |
| :--- | :--- | :--- | :--- |
| 2.1 .735 | $1: 07: 18$ | Kim | Because of the top. |
| 2.1 .736 | $1: 07: 18$ | Francesca S. | Yeah, because the tops are switched. |
| 2.1 .737 | $1: 07: 19$ | Instructor | Yeah, so what specifically does blue mean and <br> what specifically does orange mean? |
| 2.1 .738 | $1: 07: 23$ | Francesca S. | Okay, blue means there's sausage on the pizza; <br> orange means there's no sausage. |

By focusing on just the top cube and reiterating the question, Francesca $S$. is able to answer what the colored cube represents in terms of pizzas.

As Freudenthal (1991) explains, "guiding means striking a delicate balance between the force of teaching and the freedom of learning" (p.55). The instructor did not tell the students the information they were missing. Instead, by encouraging them to focus on a particular aspect of their solution, she is guiding them with their problem-solving.

## Making Conjectures/Testing Conjectures

As expressed earlier, mathematics instruction should mimic the way mathematics is achieved and mathematical thinking occurs (Freudenthal, 1991; Pólya, 1945, 1954; Schoenfeld, 1992). Part of the exploration process of mathematical thinking is making conjectures and testing these conjectures. As Fendel and Resek (1990) explain, testing these conjectures might involve examining simpler examples. Pólya (1945) also suggests the heuristic of looking at "simpler analogous problem" to help solve a mathematical problem (p.38). There is evidence of the instructor modeling this type of behavior by suggesting the students look at examples that involve smaller numbers. Many of these instances enabled the students to recognize a pattern and formulate a solution.

During the towers problem, after Jessica and Jamie find the answer to be 16, they indicate that they believe the mathematical formula has to do with squares. Jamie says, "Maybe it had something to do with the squares." [Line 1.1.212] The instructor replies "Make a prediction for three." [Line 1.1.215] They predict for three-tall towers problem choosing from two colors, the answer would be nine. The instructor replies: "Well why don't you try three and see how that works out." [Line 1.1.217] They build the three-tall towers and realize the answer is eight. On their own, they build the two-tall towers and realize the answer is four. They see the doubling pattern and predict the formula to be $2^{n}$ where $n$ is the height of the towers.

Later, Jessica says, "And two is the, the amount of um, the colors. I'm thinking." [Line 1.1.282] The teacher replies "Suppose there was three colors - what's it gonna be?" [Line 1.1.283] Jamie replies "three to the..." and Jessica finishes her sentence "to however tall it is." [Line 1.1.285] "So, why don't you get a third color?" [Line 1.1.286] The instructor suggests: "Do three colors and just do them two-tall because you will get too many if you.... [Line 1.1.289]

In the pizza problem, Francesca $S$. conjectures that the reason there are 16 pizzas is because of the formula four squared. The instructor asks her, "So, if that's true, what do you think you would get if there was only three toppings to choose from?" [Line 2.1.298] She replies, "Nine." The instructor then says, "Well, why don't you do a three-topping case and see what you get?" [Line 2.1.300] She finds that there are eight pizzas. She and Kim decide to find how many pizzas there are when there are two toppings, one topping and no toppings from which to choose. This leads Kim to conjecture that the formula is $2^{n}$ 。

When the students explore the connection between the pizza and towers problems, they look at Pascal's triangle. The instructor suggests they look at row three (as opposed to row four). She says, "Now, suggestion. Row four is kind of a pain because you have sixteen. Row three, you only got eight. I would suggest to work with row three because eight is easier. Now, not only do you have the numbers that go there, you can actually write each of the eight pizzas and each of the eight towers." [Line 2.2.539]

Table 5.9
Teacher suggestions to investigate smaller examples

|  | Student's Conjecture | Teacher's Suggestion | Transcript <br> Line |
| :--- | :--- | :--- | :--- |
| Towers <br> Jessica/Jamie | Hypothesized the formula <br> to be four squared. | "Make a prediction for three." | 1.1 .215 |
| Towers <br> Jessica/Jamie | Hypothesized the base is <br> equal to the number of <br> colors. | "Suppose there was three colors - <br> what's it gonna be?" | 1.1 .283 |
| Towers <br> Jessica/Jamie | Hypothesized the base is <br> equal to the number of <br> colors. | "Do three colors and just do them <br> two-tall because you will get too <br> many if you...." | 1.1 .289 |
| Pizza | Hypothesized the formula <br> to be four squared. | "So, if that's true, what do you <br> think you would get if there was <br> Kim/Francesca | 2.1 .298 |
| only three toppings to choose <br> from?" | Connections <br> Whole Class | Trying to understand the <br> connections. | I would suggest to work with <br> row three because eight is easier." |

Table 5.10
Teacher suggestions to test conjecture

|  | Student's Conjecture | Teacher's Suggestion | Transcript <br> Line |
| :--- | :--- | :--- | :--- |
| Towers <br> Jessica/Jamie | Hypothesized there should <br> be nine towers three-tall. | "Well why don't you try three <br> and see how that works out." | 1.1 .217 |
| Towers <br> Jessica/Jamie | Hypothesized the base is <br> equal to the number of <br> colors. | "So, why don't you get a third <br> color?" | 1.1 .286 |
| Pizza | Hypothesized there should <br> be nine pizzas when <br> choosing from three <br> toppings. | "Well, why don't you do a three <br> topping case and see what you <br> get?" | 2.1 .300 |

By suggesting the students look at a smaller example, make a conjecture, and then test the conjecture the instructor is modeling behavior to that of a mathematician. These suggestions led the girls to see the pattern and model this behavior on their own. Once they saw that there were eight towers (or pizzas), they all decided to look at the towers
two-tall (or two-topping pizzas). They discovered the formulas by employing these strategies which were all initiated by the suggestion of the instructor.

Another reason for the importance of highlighting these suggestions by the instructor is because the conjecture that the formula is four squared is not unique to this class. There is evidence of other students making similar predictions. In the eleventh grade, Sherly and Ali predict that there should be nine towers three-tall when choosing from two colors (Tarlow, 2004). In college, Stephanie, Lisa, Errol, and Wesley all predict there should be 25 five-tall towers when choosing from two colors based on the fact that they believe there are 16 four-tall towers because four squared is 16 (Glass, 2001). Future teachers may benefit from understanding these teacher moves if they are to engage their students in this type of problem-solving.

## Questions that Promote Connections

To understand underlying mathematical structures, it may be beneficial to make connections between mathematical problems. Martino and Maher (1999) suggest questions that are "aimed at generalization can enable this student to form mathematical connections between classes of problems" (p. 57). In this study, the students explained the isomorphism between the towers problem and the pizza problem. There are some specific moves the instructor made to promote the connections.

As explained earlier, after Kim and Francesca S.'s solution to the towers problem, the instructor highlights the fact that the number of towers in each of their cases follows the
number sequence $1,4,6,4,1$. Although they do not discuss this further, she emphasizes this aspect of the solution, perhaps for future possible connections. After Kim and Francesca S. have found the formula for the pizza problem, the instructor says "Two to the $n$-does that look familiar? Where have you seen that before?" [Line 2.1.382] She asks a similar question when they are looking at Pascal's triangle and focus on row four $1,4,6,4,1$. She asks, "And where did you see that?" [Line 2.2.485] After they reply that they saw this number sequence in both the pizza problem and the towers problem, she asks them to explain what each of the numbers represents in terms of pizzas and towers. She begins the conversation by saying, "Well what did the one represent?" [Line 2.2.491] These types of questions enable the students to begin the process of understanding the connections.

## Conclusion

It has been shown that the teacher played an important role in these students' discovery of mathematics. She did not lecture to the students and point out relationships. Instead, she gave them ample time to explore the problems and intervened in a timely manner with questions/suggestions that were open-ended but purposeful. As explained earlier, an important part of mathematical learning is justifying solutions. There were many instances of the instructor reiterating that they are to justify their solutions and asking for reasons why. For example, when Kim and Francesca S. realize that the solutions to the pizza problem and the towers problem are similar, Kim asks, "Is that
supposed to happen?" [Line 2.1.390] The teacher replies, "You're gonna tell me why it happens." [Line 2.1.391]

Furthermore, there are instances of the students telling each other that they are to find the reason for their solutions. For example, during Ankur's Challenge after they had found the answer to be 36, Jessica says to Jamie, "But we need reasoning why this works. Why we know that that's all." [Line 1.1.513] All of the instances of the questions that promoted justification were not highlighted in this discussion because the need to justify the solutions was embedded in each of the tasks that the instructor presented.

However, there is one move by this teacher that promotes justification for a generalization that should be emphasized. There is evidence that most of the older students (high school and college), like the students in this study, found the formula to the four-tall towers problem when choosing from two colors to be $2^{n}$ (Glass, 2001; Tarlow, 2004). Most students are able to explain how the two represents the colors in the towers and that $n$ is equal to the height of the tower. The difficulty lies in understanding why the formula is $2^{n}$.

When faced with this difficulty, the instructor in this study says to her students, "Yeah, but how come it's two to the $n$ ? How come it's not 2 times $n$ ? How come it's not $n$ squared or some other thing? How come it's two to the $n$ ? What is it that makes that..." [Line 2.1.380] She highlights the need to understand why it is that particular formula. It is examples such as this and the ones presented in this study that should be
noted so that when other teachers decide to engage their students in similar problemsolving, they can be equipped with examples of effective teacher questioning.

### 5.7 Conclusions

This study compared the solutions and strategies of elementary, high school, and twoyear college students to the solutions of math majors in the junior year of college. It was shown that these tasks tended to elicit the same kind of organizational structures, patterns, and forms of reasoning across contexts and age levels. The math majors in this study, who were all pre-service teachers, approached all of the problems in the same manner as the younger students including incorporating the use of manipulatives. Preservice teachers should be exposed to the types of problem-solving activities they will bring to their future classrooms and this study demonstrates that these problems can engage students of all age levels. Although the types of solution strategies were similar, there are two important differences that are of note.

First, similar to the other college students in the study by Glass, these students generally solved the problems faster than the younger elementary-grade students. The length of the class period was one hour and 15 minutes. Within one class period, these undergraduate students were successfully able to solve both the towers problem and Ankur's Challenge. The following week, they solved the pizza problem and investigated the isomorphism between the two problems within the hour and 15 minutes.

Second, unlike the younger students, they sought to justify the solution by trying to find the underlying formula for the solution and tried to generalize the formula. In general, the students in this study were able to generalize and justify the solution to the towers problem rather quickly. However, generalizing and justifying the pizza problem proved to be more difficult. Not unlike the results of the other studies involving the high school and college students, these students were able to justify and generalize a solution to the pizza problem after investigating the isomorphism with the towers problem. This study further demonstrates the value of including the investigation of the isomorphism between two mathematical problems that are, on the surface, very different but have the same underlying mathematical structures.

One of the most valuable observations of this study is that these pre-service teachers were engaged in solving the same mathematical tasks that were presented to students at the elementary and high school levels. As explained and probably expected, they solved the problem quicker and delved deeper. It is significant that these students were engaged in investigating the underlying mathematical concepts. These problems can lead to further discussion about mathematical concepts including Pascal's triangle, Pascal's identity, isomorphism, and mathematical induction. If these pre-service teachers are to bring these tasks into their future classrooms, understanding the underlying problem structures as well as connections that could be made to the curriculum will be important. This study provides mathematical tasks to math educators of prospective teachers that could eventually be used by these prospective teachers in their future classrooms.

Not only is this study significant for pre-service teachers, this study is significant for undergraduate mathematics education in general because incorporating alternative learning styles to undergraduate education is stressed. This study demonstrates that mathematical learning can take place in a college classroom that supports mathematical exploration and justification. These students found patterns, made conjectures, and engaged in justifying those conjectures. Some key elements to a setting that fosters exploration and justification include well-chosen mathematical problems, collaboration with peers, and strategic interventions from the teacher.

As shown, the instructor played a critical role in the learning process. There were instances in which her moves were specific and related to the students' progress. For example, the instructor often suggested that the students make and test their conjectures after they hypothesized a solution. These suggestions often led to the discovery of patterns that led to generalization. But more often than not, her moves were more general and encouraged explanation and collaboration. They often took the form of "tell me what you have done" which encouraged the communication of ideas. Although these moves were often subtle, they appear to be deliberate, and they were very effective. There were many instances in which the students themselves found clarity to their solutions during their own explanations. Furthermore, by encouraging the students to listen to each other, it appears that the instructor deliberately encouraged the students to rely on each other as opposed to relying on her expertise.

If an active learning style is to be incorporated into the undergraduate classroom, understanding the effective moves of the instructor is essential. Not only will this information be valuable for the undergraduate instructor but since teachers tend to teach the way they are taught, these moves will be beneficial to future teachers. In summary, this study demonstrates tasks that could be used in an undergraduate mathematics course that highlight exploration and justification. Furthermore, it was shown that these tasks engage students of all ages and therefore can be used as valuable problem-solving sessions for prospective teachers to learn from and model in their own classrooms. Lastly, this study highlights valuable moves made by the teacher for educators that may want to incorporate an active learning experience into their classroom.

### 5.7.1 Implications

This study adds to the existing research by analyzing how a group of pre-service teachers in their junior year of college solve and justify their solutions to the towers problem, the pizza problem, and Ankur's Challenge. Not only is it important to understand how students of various ages solve these problems, this knowledge could prove to be very beneficial to teachers who may bring these tasks to future classrooms. Pólya (1954) explains that the best way a teacher can help a student is by encouraging his/her already existing ideas. According to Pólya, "The best is, however, to help the student naturally. The teacher should put himself in the student's place, he should see the student's case, he should try to understand what is going on in the student's mind, and ask a question or indicate a step that could have occurred to the student himself" (p.1).

Encouraging the communication of ideas can allow the teacher to understand what is occurring in her students' minds. However, the ideas might not always be articulated well, if at all. If the teacher is equipped with an understanding of the strategies, justifications, and misconceptions that often occur by other students when solving these problems, the teacher may be better equipped to guide her own students in their problem solving.

Martino and Maher (1999) also explain that knowledge of how one learns mathematics is very important in the success of a teacher. As they explain, there is an art to teacher questioning which involves this knowledge and it takes time to develop. Martino and Maher explain, "The art of questioning may take years to develop for it requires an in-depth knowledge of both mathematics and children's learning of mathematics. Once acquired, the teacher has available a powerful tool to support students in their building of mathematical ideas" (p. 54). Perhaps, by understanding the patterns and justification that often arise, the time to develop this expertise can be lessened.

Although more research is needed in the college classroom, this study demonstrates that college students can learn and be involved with problems that are exploratory in nature. Furthermore, Senk, Keller, and Ferrini-Mundy (2004) explain that it is essential that the mathematics courses taken by pre-service teachers develop "understanding of both mathematical content and mathematical process such as defining, conjecturing and proving" (p. 148). This study provides three tasks and examples of solutions to these
tasks found by pre-service teachers in a mathematics course that provided opportunities for conjecturing, generalizing, and justifying.

### 5.7.2 Limitations

The class size was small. There were six students in this class. On the day that the pizza problem was implemented, there were only four students in the classroom. The results may differ in a larger classroom setting. Furthermore, as a case study, it is hard to generalize the results. However, the longitudinal study as well as the study by Glass (2001) obtained similar results.

### 5.7.3 Suggestions for Further Study

More research is needed on how students in college solve these combinatoric tasks. It would also be of great interest not only to study how the students built their solutions, but to analyze the role of the instructor as well. The moves by the instructor in this study appeared to be deliberate and were an integral part of the learning process. However, the classroom size in this study was small. The instructor had many opportunities to work with all of the students. Perhaps, also, since there were not many people in the room, she had better opportunities to hear their discussions. Clearly, an undergraduate classroom size of six is unusual. It would be beneficial to replicate this study in a larger college classroom. Furthermore, it would be of great interest to analyze the moves of the teachers in the existing research on these tasks. If future teachers were to implement these tasks in their classrooms, a collection of effective teacher questioning would be a valuable tool.

## APPENDIX A

Outline of Course Schedule

Math 380 - Mathematics Reasoning and Assessment
Felician College
Class Met WF from 2:35-3:50
Spring 2011

|  | Topic | Date |  |
| :--- | :--- | :--- | :--- |
| Friday, <br> January 21 | First Day - <br> Introductions | Beliefs assessment <br> Counting strand pre-assessment (the <br> committee problem) <br> Gang of Four video - pre-assessment for <br> homework | Francesca <br> C. was <br> absent |
| Wednesday, <br> January 26 | Class <br> Cancelled due <br> to Snow |  | Mixture of <br> Topics |
| Friday, <br> January 28 | Handed in the fraction pre-assessment <br> Discussed quadratic and exponential <br> functions. <br> Discussed patterns and deduction. <br> Discussed triangular and Fibonacci <br> numbers <br> Worked on the Handshake Problem. | Francesca <br> C. was <br> absent |  |
| Wednesday, <br> February 2 | Class <br> Cancelled due <br> to Snow | Mixture of <br> Topics | Discussed homework questions <br> Focused on Triangular numbers and the <br> Chessboard problem |
| Friday, <br> February 4 | Kim was <br> absent |  |  |
| Wednesday, <br> February 9 | Induction | The instructor did proofs that <br> demonstrated the steps for induction | All there. |
| Friday, <br> February 11 <br> Videotaped | Combinatorics <br> Intervention | Towers 4-tall choosing from 2 colors <br> Ankur's Challenge | All there |
| Wednesday, <br> February 16 | Induction | Spent all of the class time going over <br> induction homework. | All there |
| Friday, <br> February 18 <br> Videotaped | Combinatorics <br> Intervention | The towers problem - 4 tall, 2 colors. <br> The pizza problem - 4 toppings. <br> Isomorphism between the towers and the <br> pizza problems. | Becca and <br> Francesca <br> C. were <br> absent |
| Wednesday, <br> February 23 <br> Videotaped | Combinatorics <br> Intervention | Discussed the isomorphism between the <br> pizza, the towers, and Pascal's triangle. <br> Isomorphism between the binomial <br> expansion and the towers and pizza | Jamie and <br> Francesca <br> C. were <br> absent |


|  |  | problems. <br> Family with four children. |  |
| :--- | :--- | :--- | :--- |
| Friday, <br> February 25 | Formal Proofs | The instructor explains proof by <br> contradiction, proof by cases, and <br> induction. <br> Watched the Brandon video and they were <br> asked to see what types of informal proofs <br> they saw in the video. | All there |
| Wednesday, <br> March 2 | Proofs and <br> Fibonacci <br> numbers |  | All there |
| Friday, <br> March 4 <br> Videotaped | Combinatorics <br> Intervention | Addition rule for pascal's triangle using <br> towers and pizzas. <br> Taxi Cab Problem. | Jamie, <br> Francesca <br> C. and <br> Rebecca <br> were |
| absent |  |  |  |$|$| Wednesday, |
| :--- | Spring Break | March 9 |
| :--- | :--- | :--- | :--- |


|  |  |  | data |
| :--- | :--- | :--- | :--- |
| Wednesday, <br> April 6 | Number <br> Theory | Fundamental theorem of arithmetic <br> Prime Factorization and abundant <br> numbers | Don't have <br> attendance <br> data |
| Friday, <br> April 8 | Number <br> Theory/ <br> Introduction to <br> Fraction <br> intervention | Conjectures and proofs <br> Prime Factorization and abundant <br> numbers. <br> Introduced Cuisenaire rods | Don't have <br> attendance <br> data |
| Wednesday, <br> April 13 <br> Videotaped | Fraction <br> Intervention | Upper and lower bound video watched | All there |
| Friday, <br> April 15 | Fraction <br> Intervention |  | All there |
| Wednesday, <br> April 20 <br> Videotaped | Fraction <br> Intervention | No Class - <br> holiday | Only four <br> students |
| Friday, <br> April 22 | Fractions | Fraction |  |
| Wednesday, <br> April 27 | Intervention | Friday, <br> April 29 <br> Videotaped | Mixture of <br> Topics |
| Wednesday <br> May 4 <br> Videotaped | Signed numbers <br> Taxicab problems <br> Some fraction problems | All there |  |
| Friday, <br> May 6 <br> Videotaped | Mixture of <br> Topics | Signed numbers <br> Taxicab problems <br> Some fraction problems | Only three <br> students <br> there |
| Wednesday, <br> May 11 | No Class - <br> reading day | Kim absent <br> finals | 2 take home essays <br> beliefs post-assessment <br> fractions post-assessment |
| Friday, <br> May 13 | Kast day absent |  |  |

## APPENDIX B

Transcript 1.1 - February 11, 2011
Camera View One

## February 11, 2011

The tape begins with the instructor displaying a powerpoint that explains the problem for the day. The problem is the towers problem and extensions of the tower problem.

How many towers 4 tall can be made from 2 colors?

- Part one - What is the answer?
- Part two - Convince me that your answer is correct.

They break into three groups, two people each, and work on the problem.

- One group is composed of Jessica and Jamie (Red and White cubes)
- One group is composed of Kim and Francesca S. (Blue and Yellow cubes)
- One group is composed of Francesca C. and Rebecca

There is only one camera on February, 11, 2011. The camera focuses on Jessica and Jamie building their towers. The camera switches to Kim and Francesca S. but quickly switches back to Jessica and Jamie. Sometimes, you can hear Kim and Francesca S. in the background or they interject. Their conversation is only added when relevant. [When the towers are described, the first color written is the top cube of the tower.]

| 1 | 1:35 | Group 1 <br> (Jessica and Jamie) | [construct two 4-tall towers: RRRR, WWWW] |
| :---: | :---: | :---: | :---: |
| 2 | 1:46 | Jessica | Well, first, right now, we're just trying to figure out, what, exactly we're gonna do. |
| 3 | 1:52 | Jamie | Want to do that? [constructs one 4-tall tower, WRWR] |
| 4 | 1:54 | Jessica | Alright and then the other one would be like this, right? Because that's the opposite one? [constructs one 4-tall tower, $R W R W$, places it next to $W R W R$ ] |
| 5 | 2:00 | Jamie | Yep, and then if we do two red and two white [constructs one 4-tall tower, RRWW] |
| 6 | 2:05 | Jessica | And then two white and two red - with the white on top [constructs one 4-tall tower, WWRR, places it next to RRWW] |
| 7 | 2:08 | Jamie | Got anymore white? Oh and then? |
| 8 | 2:10 | Jessica | Yeah, yeah, you got a whole bunch. |
| 9 | 2:14 | Jamie | We could do three red and one white? Well, like put them like that? [Camera is focused on the other group so we |


|  |  |  | are unable to see what they are building] |
| :---: | :---: | :---: | :---: |
| 10 | 2:20 | Jessica | Yeah, alright. And then there 's..... |
| 11 | 2:28 | Jessica | All three, what, on top? [The camera focuses back on this group and we are able to see what they have built. <br> They have constructed three more 4-tall towers: <br> RRRW,RWWW, WWWR, places them next to each other] |
| 12 | 2:30 | Jamie | Uh, hum. |
| 13 | 2:33 | Jessica | Okay, so you have three white on top, three red on top and... [Looks at the group of three towers: RRRW, $R W W W, W W W R$. Places the RRRW and RWWW next to each other leaving the WWWR by itself.] No, but you did three red on top, wait a second, oh. |
| 14 | 2:38 | Jamie | Do you know what I mean, and then you could put the red on the bottom. Same thing, right? [Jessica rearranges the three towers so that $R W W W$ and $W W W R$ are next to each other. The RRRW tower stands alone.] Yeah. |
| 15 | 2:42 | Jessica | Like that and then we could do this and then. [Knocks over the $R R R W$ tower] Too fast obviously! |
| 16 | 2:49 | Jessica | [Stands the RRRW tower and creates WWWR tower and places it next to $R R R W$ ] Like that because that is the opposite one. |
| 17 | 2:51 | Jamie | Right. |
| 18 | 2:53 | Jessica | Well, this is what we should do. Because we have uh... [Lays two towers on their sides next to each other, RRRR, RRRW] |
| 19 | 2:58 | Jamie | Now you need the three. [Jessica grabs the RRWW and places it next to the RRRR, RRRW] Now you have that. Okay and now you need the one, okay. [Jessica takes the $R W W W$ and places it next to the $R R W W]$ There you go. |
| 20 | 3:03 |  | [There are four towers grouped together resembling a |



| 28 | 3:38 | Jessica | [Switches the order of the two towers in Set 3 to RWRW, $W R W R]$ I'm like OCD, I swear. |
| :---: | :---: | :---: | :---: |
| 29 | 3:49 | Jessica | Oh! In the middle, remember? [Constructs another tower, RWWR] |
| 30 | 3:56 | Jessica | Yeah, they're just starting to get all the colors together. [Indicating the other group.] |
| 31 | 3:58 | Jamie | Alright, so I will do this one. [constructs another tower, WRRW] |
| 32 | 4:00 | Jessica | Like that. [She places the RWWR tower in the first position in Set 3] |
| 33 | 4:05 | Jessica | Alright so that goes there. [Points to the middle set (Set 3) and Jamie places the tower she has created, WRRW, in the fourth position in Set 3] |
| 34 | 4:10 | Jessica | Hum... [They stare at the 12 towers they have laid before them: <br> Set 1, Set 3, and Set 2] |
| 35 | 4:16 | Jamie | [pointing each tower in the set farthest to the right, beginning with $W W W W]$ We have four white, three, two, and one. |
| 36 | 4:19 | Jamie | And the same way there [Pointing at the set farthest to the left] |
| 37 | 4:21 | Jessica | Yeah. Right and then we have [pointing, in succession at the middle set of towers, WRRW, RWWR, WRWR, $R W R W]$ two, two, two, two. |


| 38 | 4:31 | Jessica | Hmm. Is there anything else? Yeah there is, um.... [Pulls out the fourth tower in the first set, $R W W W]$ |
| :---: | :---: | :---: | :---: |
| 39 | 4:39 | Jamie | Yeah, you can do one on top and one, right? |
| 40 | 4:43 | Jessica | Like this. Did we do this yet? We didn't do this yet. [Constructs another tower, WRWW] |
| 41 | 4:51 | Jessica | Remember, move it down the line? [Places WRWW tower to the left of RWWW tower in a separate pair (Set 4)] |
| 42 | 4:53 | Jamie | Yeah, okay. |
| 43 | 4:54 | Jessica | So then after that would be two whites. |
| 44 | 4:57 | Jamie | And two reds |
| 45 | 4:58 | Jessica | [Pointing to the right of Set 4] Yeah, if you want, if you want to work that way [Pointing to the left of Set 4] and I'll work this way out. |
| 46 | 5:00 | Jamie | Alright. |
| 47 | 5:01 | Jessica | Or, or... we'll figure something out. Yeah, I think we are going to run out. |
| 48 | 5:08 | Jamie | Oh, I get it. |
| 49 | 5:09 | Jessica | All three, yeah. Three whites and one red. [Constructs a tower, WWRW, places it to the extreme left in Set 4] |
| 50 | 5:12 | Jamie | There you go. [Constructs a tower, WWWR, places it to the extreme left in Set 4] |
| 51 | 5:15 | Jessica | Yeah. Alright, so this is the three whites [Picks up Set 4 which contains all the towers with three red and places it above the other sets] Now you got to do the three reds. |
| 52 | 5:24 | Jamie | [Constructs another tower, WRRR, places it upside down by itself (Set 5)] |
| 53 | 5:26 | Jessica | [Turns WRRR right side up] Alright so you have it with |


|  |  |  | the white on top? |
| :---: | :---: | :---: | :---: |
| 54 | 5:28 | Jamie | Yep. |
| 55 | 5:29 | Jessica | Alright, so the one after that would be two red on the bottom, a white in the middle, a white on top. [Constructs another tower, RWRR, places it to the left of WRRR in Set 5] |
| 56 | 5:35 | Jamie | And it would be.... |
| 57 | 5:38 | Jessica | The white one would go there - the next one for you. [Pointing to the left of RWRR in Set 5] |
| 58 | 5:43 | Jamie | [Constructs another tower, RRWR, places it to the left of RWRR in Set 5] |
| 59 | 5:48 | Jessica | [Constructs another tower, RRWR, places it to the left of $R R W R$ in Set 5] |
| 60 | 5:53 | Jamie | [Pointing to Set 3 , which has 3 towers RRRR, RRRW, $R R W W$ ] Are we missing one here? |
| 61 | 5:56 | Jessica | I think so. Yeah, we are. I think I moved it. |
| 62 | 5:59 | Jessica | Oh look I have two of the same. [Laughing, picks up $R R W R$ from Set 5, changes it to $R R R W$ - puts it back in same position]. |
| 63 | 6:07 | Jamie | We're missing the one with ... |
| 64 | 6:11 | Jessica | The three white... |
| 65 | 6:12 | Jamie | The three white... |
| 66 | 6:13 | Jessica | Three white and a red? Yeah, one red. How did we do that? I think I might have moved it. [Constructs another tower, $R W W W$, places it to the extreme right, Set 5] |
| 67 | 6:20 | Jamie | I think you did. |
| 68 | 6:21 | Jessica | Yeah. I'm thinking [inaudible] |


| 69 | 6:31 | Jessica | [Holding the RRRR tower] Alright, well this is all the possible combinations for four red. Three red [Picks up the $R R R W$ tower and puts it down.] I've done this. |
| :---: | :---: | :---: | :---: |
| 70 | 6:38 | Jessica | [Pointing to Set 5] That's our 3 reds. |
| 71 | 6:40 | Jessica | [Pointing to Set 4] Our 3 whites. So we've done this. [Separates RRRR and RRRW from Set 1. Separates $W W W W$ and $W W W R$ from Set 2.] |
| 72 | 6:44 | Jessica | These are our twos, like this. [Puts RRWW and WWRR with Set 3] |
| 73 | 6:50 | Jamie | Those are our threes, here, right? |
| 74 | 6:52 | Jessica | Yeah, these are the threes. [Indicating RRRW and $W W W R]$ Maybe we need more threes? No cuz, cuz, its up there. |
| 75 | 7:01 | Jamie | Yeah. |
| 76 | 7:02 | Jessica | Hmm. |
| 77 | 7:05 | Jamie | We did the alternating ones. |
| 78 | 7:06 | Jessica | Oh, you know what? See, we did this twice. [Identifies WRRR, WRRR and RWWW, RWWW. Removes WRRR and $R W W W]$ |
| 79 | 7:10 | Jamie | Okay, so then we have to take one out. That's why you took it out. |
| 80 | 7:13 | Jessica | Yeah, alright, so these... |
| 81 | 7:14 | Jamie | So these go there. Do we have any other ones that are....? |
| 82 | 7:19 | Jessica | I don't think so. |
| 83 | 7:21 | Jessica | Yeah, so that was the reason we took it out of these sets because it's the same up there. [Blocks are ordered as follows: top left: Set 5: RRRW, RRWR, RWRR, WRRR; Top right, Set 4: WWWR, WWRW, WRWW, RWWW; |


|  |  |  | bottom left, Set 1: RRRR, RRRW, RRWW; bottom middle, Set 3: RWWR, RWRW, WRWR, WRRW; bottom right, Set 2: $W W R R, W W W R, W W W W]$ I think that's it. |
| :---: | :---: | :---: | :---: |
| 84 |  |  | [The sets appear as follows: <br> Set 5 and Set 4 $\begin{array}{cccccccccc} R & R & R & R & R & W & W & W & W & W \\ R & R & R & W & W & R & R & W & W & W \\ R & R & W & W & R & W & R & R & W & W \\ R & W & W & R & W & R & W & R & R & W \end{array}$ <br> Set 1, Set 3, and Set 2] |
| 85 | 7:30 | Jessica | That's so smart. I would have never thought of that. |
| 86 | 7:34 | Jessica | [To Group 2] The way you guys have it like building and we have ours like this. |
| 87 | 7:55 | Instructor | Explain to me where you are and what you've got... |
| 88 | 7:56 | Jessica | Oh, okay. Well, we started here. We started with...um, well no, what did we start with actually? I think we started with these [holding up RWRW, WRWR (Set 3)] Because we thought of pepper... they're like peppermint sticks. |
| 89 | 8:06 | Instructor | Okay, so there's two and two. Okay. |
| 90 | 8:09 | Jessica | And then we knew that we needed twos. [Pointing to $W R R W, R W W R(S e t ~ 3)]$ So there's two on the inside and |


|  |  |  | the two whites on the outside. And then two whites on the inside, two red on the outside. |
| :---: | :---: | :---: | :---: |
| 91 | 8:15 | Instructor | Ok. |
| 92 | 8:16 | Jessica | [Indicates Set 3: RWWR, RWRW, WRWR, WRRW] So those are our twos. |
| 93 | 8:18 | Instructor | Ok. |
| 94 | 8:19 | Jessica | And then we did um, one, our ones [Pointing to Set 5 (one white each tower)] |
| 95 | 8:22 | Instructor | That's the one white case. |
| 96 | 8:24 | Jessica | One white case and this is the one red case. [Points to each single red case in each tower of Set 4] |
| 97 | 8:26 | Instructor | Okay, and then what else do you have? |
| 98 | 8:28 | Jessica | And then we did the four [Indicating each tower of Sets 1 and 2 that are all the same color] and we have three of them, then you have two of them [Indicating the remaining towers of Set $1(R R R W$ and $R R W W)$ and Set 2 ( $W W W R$ and $W W R R$ )] |
| 99 | 8:35 | Jamie | And the one case is already up here. |
| 100 | 8:37 | Jessica | Yeah, the last case is up there. [pointing to Sets 4 and 5] |
| 101 | 8:40 | Instructor | Ok. So what's your final answer? |
| 102 | 8:43 | Jessica | Our final is 18. |
| 103 | 8:45 | Instructor | 18? Okay. |
| 104 | 8:49 | Jessica | Yeah, 18. |
| 105 | 8:50 | Instructor | Okay, you're good with that? But you're not sure? |
| 106 | 8:52 | Jessica | But we're not sure if that's it - we want to keep thinking - but at the moment we have 18 . |


| 107 | 8:56 | Instructor | Ok - I - do I have a question? Think about it some more before I come back with my question. |
| :---: | :---: | :---: | :---: |
| 108 | 9:02 | Jessica/Jamie | Ok. |
| 109 | 9:10 | Jessica | Well, let's see... you can put them together or separate that's the only way you can do it - and we've done it both ways. [Points to Set 3 containing $R W W R, R W R W$, WRWR, WRRW]. |
| 110 | 9:17 | Jamie | Can you like separate them so that you have...[inaudible] And this is over here, right? [Points to the RRRW in Set 1 and points to RRRW in Set 5] |
| 111 | 9:24 | Jessica | Yeah. |
| 112 | 9:24 | Jamie | We're not doubling that? |
| 113 | 9:25 | Jessica | No. |
| 114 | 9:25 | Jamie | Alright. |
| 115 | 9:26 | Jessica | Cause that would just be the same thing and we're.... we're trying to do it without repeating. |
| 116 | 9:29 | Jamie | Alright. |
| 117 | 9:34 | Jamie | I guess we should just start like [inaudible] [Picks up more blocks and starts building.] |
| 118 | 9:36 | Jessica | [Picks up more blocks and starts building] And seeing if we can come up with anything else? |
| 119 | 9:38 | Jamie | Yeah. |
| 120 | 9:44 | Jessica | I don't think there is because it doesn't seem like it cause.... |
| 121 | 9:48 | Jamie | Cause if you move any of them.... |
| 122 | 9:51 | Jessica | Cause...we did it every way it looks like. |


| 123 | 9:55 | Instructor | [Instructor is speaking to group 2. The camera focuses on Kim and Francesca S.] You guys are going to explain to me your strategy for building them. So, tell me what you did. |
| :---: | :---: | :---: | :---: |
| 124 | 10:02 | Kim | We kinda each did it. Like, I would do one thing and then she would do the opposite. <br> [Bottom left Set 1: BYBB, YBYY, Bottom right Set 2: BYYB, YBBY, YYBY, BBYB, Middle left Set 3: BYYY, <br> BBYY, BBBY, Middle right Set 4: YYYY, BBBB, Top, Set <br> 5: $Y B Y B, B Y B Y, Y B B B, Y Y B B, Y Y Y B]$ |
| 125 | 10:07 | Instructor | Explain opposite. |
| 126 | 10:10 | Kim | Meaning, like, for this one there's a blue, yellow, blue, blue |
| 127 | 10:13 | Instructor | Okay |
| 128 | 10:14 | Kim | So then the opposite is yellow, blue, yellow, yellow. |
| 129 | 10:16 | Instructor | Okay. |
| 130 | 10:17 | Francesca S. | We tried to do that [inaudible] |
| 131 | 10:19 | Instructor | Okay, so besides building opposites, do you have any grouping strategy here that you should tell me about? |
| 132 | 10:25 | Kim | What do you mean by grouping? |
| 133 | 10:25 | Instructor | How come these are down here [indicating Sets 1 and 2] and those are up there? [indicating Set 5] And this is a group [indicating Set 4] and this is a group kind of, so? [indicating Set 3] |
| 134 | 10:32 | Francesca S. | [Pointing to Set 4] These are all the..... these are both the colors. [Pointing to Set 3] These are like...One blue, two blue, three blue. |
| 135 | 10:38 | Instructor | Okay. |


| 136 | 10:39 | Francesca S. | These are alternating and then three blue, two blue, one blue [Set 5: pointing to relevant blocks in YBBB, YYBB, YYYB]. |
| :---: | :---: | :---: | :---: |
| 137 | 10:41 | Instructor | Okay, so.... And how many did you get total? |
| 138 | 10:46 | Kim/Francesca S. | 16 |
| 139 | 10:47 | Instructor | 16. Ok. Well, you said this one is like three blue, two blue, one blue [Pointing to YBBB, YYBB, YYYB in Set 5]. There's also a three blue here and there's a two blue here [Points to Set 3]. So I don't quite see a total [inaudible] |
| 140 | 11:01 | Kim | [Moves YBBB, YYBB, YYYB to Set 3, containing BYYY, $B B Y Y, B B B Y]$. Could you group those together? |
| 141 | 11:03 | Instructor | Well, you can group them any way you want - but I want it to jump out at me that, "This is it. And there isn't anymore." And I'm not quite sure that I see that yet. Um... [inaudible] |
| 142 | 11:18 | Instructor | [Pointing to Set 3] Because like here's a three-blue and here's a three-blue - but then there's a three-blue down here on this case [indicating Set 2] and a three blue in this case [indicating Set 1] And then these are two blues [indicating Set 1] but there's two blues up there [indicating Set 5]. |
| 143 | 11:27 | Kim | It depends on the order. |
| 144 | 11:28 | Francesca S. | Yeah. |
| 145 | 11:29 | Instructor | Okay. |
| 146 | 11:29 | Kim | There's a specific order. |
| 147 | 11:30 | Instructor | Okay.... So explain the order. Explain why this goes here [pointing to Set 2] and not with those over there [pointing to Set 5] |


| 148 | 11:38 | Francesca S. | [Points to Set 5] Because these are alternating. These are like [inaudible, pointing to Set 2]. |
| :---: | :---: | :---: | :---: |
| 149 | 11:41 | Instructor | Okay, so... so these are blue. You mean, so this is two... you know, this is sort of like what she said in the video these are two took apart [indicating $Y B Y B, B Y B Y$ ] and these are two stuck together, kind of? [Indicating $B Y Y B$, $Y B B Y]$. |
| 150 | 11:52 | Kim | Yeah, yeah. |
| 151 | 11:53 | Instructor | Okay. But these are all twos [indicating $B Y Y B, Y B B Y$ ] whereas this is not twos [indicating YYBY, BBYB] |
| 152 | 11:58 | Instructor | So, I could see these could go up there, sort of, as two different kinds of twos. [Moving BYYB, YBBY alongside Set 5, YBYB, BYBY] But then... |
| 153 | 12:03 | Francesca S. | [Pointing to $B Y B B, Y B Y Y$ ] This one too kinda is kind of twos, right? I mean ones. It kinda goes like that. [Pointing to $Y Y B Y, B B Y B]$ |
| 154 | 12:08 | Instructor | Ok, cause this is.... Well, explain why this goes with that. |
| 155 | 12:13 | Francesca S. | Because this is like, only one of them is different in the whole thing. [Pointing to BYBB and YBYY] |
| 156 | 12:16 | Instructor | Ok, ok. Ok, so that's an interesting organization. So, how about if you try that. These are a two's [Pointing to $Y B Y B, B Y B Y, B Y Y B, Y B B Y]$ - these are a one and three [Pointing to $B Y B B, Y B Y Y, Y Y B Y, B B Y B$ ], right? One and three. And these are special because they're all one color. [Pointing to BBBB and YYYY.] So can you organize them - put all the twos together and the ones and threes together and see if you can find something jumps out that they are all there? |
| 157 | 12:37 | Instructor | Okay, now I am going to ask you guys what you have. So you can move over here [talking to the videographer]. <br> What do you got? [Talking to Jessica and Jamie.] |


| 158 | 12:42 | Group 1 <br> Jessica | We still have the same thing. We were trying to think to see if there's any other way but it doesn't look like there is. |
| :---: | :---: | :---: | :---: |
| 159 | 12:48 | Instructor | Ok, so, alright, explain your groupings one more time. |
| 160 | 12:51 | Jessica | Alright. So this one is, we have three reds and one white in all of these. [Indicating Set 1, RRRW, RRWR, RWRR, WRRR] |
| 161 | 12:55 | Instructor | Okay. |
| 162 | 12:56 | Jessica | So we just went up the line [pointing to each individual white block in Set 1] to where each one of them could be/so/look different. |
| 163 | 12:58 | Instructor | Ok, three reds and one white. I believe that's the only way to do three reds and one white. |
| 164 | 13:02 | Jessica | [Pointing to Set $2-W W W R$, WWRW, WRWW, RWWW] And then we did the opposite with three whites and one red. |
| 165 | 13:05 | Instructor | Okay. |
| 166 | 13:06 | Jessica | With this one [Indicating Set 3 RRRR, RRRW, RRWW]. <br> Let me just pull this down so you can see [moving tower RRRW from Set 1 to Set 3]. Oh maybe not cause... <br> [Putting RRRW back in Set 1] |
| 167 | 13:12 | Instructor | I see a problem now that you pulled that one down. Pull that one back down again. |
| 168 | 13:16 | Jessica | [Putting RRRW back into Set 3] We've have two of the same. |
| 169 | 13:15 | Instructor | Yes you do. |
| 170 | 13:19 | Jessica | Oh no, oh no.... [Putting RRRW back to Set 1.] |
| 171 | 13:27 | Jamie | Oh, you get it? You see? It's the same....[Picking up $R W W W$ tower from Set 2, flipping it, and moving it |


|  |  |  | adjacent to the tower WWWR in the set containing WWWW, WWWR, WWRR] |
| :---: | :---: | :---: | :---: |
| 172 | 13:32 | Jessica | Well this ones the same as that one because the red's on top on that one. [Picking up WWWR tower from Set 2 and moving it adjacent to the set containing WWWW, WWWR, WWRR] |
| 173 | 13:41 | Jessica | [Picks up duplicates, RRRW and WWWR, and places them to the side.] Alright, so these two - so it's not 16 then - cuz these two are extra - it's 14 - right? |
| 174 | 13:46 | Jamie | Was it 18? |
| 175 | 13:47 | Jessica | Did we have 18? |
| 176 | 13:48 | Jamie | I think we had 18. |
| 177 | 13:49 | Jessica | Yeah, we had 18 so now it's down to 16 , which is what they have [pointing to Group 2] |
| 178 | 13:53 | Jamie | [inaudible] Is it like, something wrong with this? [Pointing to the towers in the middle, WRRW, RWWR, RWRW, WRWR] |
| 179 | 14:00 | Jessica | No, because, umm, I think it's because these, like this one's odd and this is odd. [Pointing to the towers with one red and the towers with one white.] These are even and there's only solid colors [Pointing to the towers with two red and two white: WRRW, RWWR, RWRW, WRWR ]. Yeah, you can either separate them or keep them together. |
| 180 | 14:14 | Jessica | Yeah, because if you keep them together [inaudible] |
| 181 | 14:19 | Jamie | And these are like...This is like this. [inaudible] |
| 182 | 14:42 | Jessica | Well, if you wanted to do the three here [pointing to $W W R R]$, the one up here that would make these white, that's that one [pointing to $R W W W$ ]. |


| 183 | 14:47 | Jamie | Right. |
| :---: | :---: | :---: | :---: |
| 184 | 14:47 | Jessica | But other than that... Yeah, if we had the two up there [Points to $W W W R$ ] cause there would be.... This one. [Points to RRWW] |
| 185 | 15:04 | Jessica | See, we had extras. [Laughing] |
| 186 | 15:11 | Jessica | Yeah, so 16. [inaudible] |
| 187 | 15:18 | Jamie | Maybe it has something to do with four being the [inaudible] |
| 188 | 15:23 | Jessica | Maybe. [inaudible] Sixteen must be it then. The only reason we got 18 though is because [inaudible] without even noticing. |
| 189 | 16:03 | Jessica | Because this is part of that. [moving the RWWW tower to another group to compare, then moving it back] |
| 190 | 16:10 | Jamie | And this is part of this one, right? |
| 191 | 16:13 | Jessica | Yeah, and this one's part of that one [moving the WRRR tower to another group and then moving it back]. |
| 192 | 16:18 | Jamie | Now if you put them there. Like put this one here [moves the WRRR tower and places it next to WWRR, WWWR, and $W W W W$ ]. And put that one. [Points to $R W W W$ ] Which one? This one? [Points to $R W W W$ ] |
| 193 | 16:24 | Jessica | Yeah, there. |
| 194 | 16:24 | Jamie | There. [Picks up RWWW and places it next to $R R W W$, RRRW, RRRR] |
| 195 | 16:28 | Jamie | If you think about it, it is all 4 by 4. |
| 196 | 16:31 | Jessica | Yeah. |
| 197 | 16:36 | Jamie | Right? |
| 198 | 16:38 | Jessica | Yeah. |


| 199 | 16:38 | Jessica and Jamie | [The sets appear as follows: $\begin{array}{cccc} R & R & W & W \\ R & W & W & R \\ W & R & R & W \\ R & R & W & W \end{array}$ <br> Set 4 $\begin{array}{cccccccccccc} R & R & R & R & W & R & R & W & W & W & W & W \\ R & R & R & W & R & W & W & R & R & W & W & W \\ R & R & W & W & R & W & R & W & R & R & W & W \\ R & W & W & W & W & R & W & R & R & R & R & W \end{array}$ <br> Set 1, Set 3, and Set 2] |
| :---: | :---: | :---: | :---: |
| 200 | 16:38 | Instructor | I think you said something interesting. So I want you to say it again now that I'm here. |
| 201 | 16:43 | Jamie | If you put them all like this... |
| 202 | 16:45 | Jessica | Like the steps, like that. [indicating Set 1] |
| 203 | 16:46 | Instructor | Okay. |
| 204 | 16:47 | Jamie | It's all 4 by 4 . |
| 205 | 16:50 | Instructor | Okay. |
| 206 | 16:50 | Jessica | And we got 16 - we have 4 sets of 4 . [Indicating each individual set of 4] |
| 207 | 16:57 | Instructor | So, alright, you see a pattern, it sounds like. |
| 208 | 17:00 | Jessica | I see a pattern here and here [indicating Sets 1 and 2 (step ladder pattern)] - well, that's the most obvious pattern. And then... |


| 209 | 17:06 | Jessica | Keeping the two apart, keeping the two together [Indicating Set 3] |
| :---: | :---: | :---: | :---: |
| 210 | 17:08 | Instructor | Okay, I saw a pattern also. Okay. But when you said 16, I also thought of a numerical pattern. I don't know if that's what you were thinking of. |
| 211 | 17:15 | Jessica | We were trying to think of how to, like a mathematical problem like... four... |
| 212 | 17:21 | Jamie | Maybe it had something to do with the squares. |
| 213 | 17:23 | Jessica | Yeah like the fact that it was four to begin with... |
| 214 | 17:25 | Jamie | And we ended up with 16. |
| 215 | 17:26 | Instructor | Make a prediction for 3. |
| 216 | 17:29 | Jessica | If we had 16 for 4, maybe 3 would be 9 . |
| 217 | 17:32 | Instructor | Well why don't you try 3 and see how that works out. |
| 218 | 17:36 | Jessica | Ok. Let's just push these up there. [moving 4 Sets of towers out of the way] |
| 219 | 17:38 | Instructor | Yeah, keep them there. Keep them there and be ready to reorganize them if you have to. |
| 220 | 17:41 | Jessica | That's fine. Okay. We're probably gonna need more blocks. I have a feeling we're gonna need more blocks. Can I grab another bag of blocks? [Gets up to get more blocks.] |
| 221 | 17:48 | Instructor | Sure. Take whatever you want. |
| 222 | 17:50 | Jessica | Is there one color or two in there? |
| 223 | 18:01 | Jessica | No, we got some white left over. [Returns with a bag of black and white blocks] |
| 224 | 18:05 | Jessica | Alright, I got... I got more - so maybe we can use black for the three. So we'll use black because it has to be 3- |


|  |  |  | tall. |
| :---: | :---: | :---: | :---: |
| 225 | 18:13 | Jamie | Can we use the white? |
| 226 | 18:15 | Jessica | Yeah, the whites we're gonna use because we have white in here too. [Constructs 3 3-tall towers using black ( $B$ ) and white ( $W$ ) cubes. $B B B, W B B, W W B$, lays them down side by side (Set 1]] |
| 227 | 18:29 | Jamie | [Construct 4-tall tower, WWWW, and lays it on its side] |
| 228 | 18:32 | Jessica | There three tall. [Laughs and takes the white cube off of the top of the tower Jamie has just created. She places this three tall tower WWW next to the other towers she had created.] |
| 229 | 18:35 | Jessica | Alright, see? [Points to the group of four towers: BBB, $W B B, W W B, W W W$. |
| 230 | 18:36 | Jamie | Yeah. |
| 231 | 18:43 | Jessica | Alright, so now.... |
| 232 | 18:50 | Jamie | [Constructs tower, WWB, places it underneath the first set of four] |
| 233 | 18:52 | Jessica | Okay, which means it would be the black on the top. |
| 234 | 18:54 | Jamie | Do you want to separate, no, separate this one? [Separating WWW] And start with this one [indicating $W W B]$ - because they're gonna be the same - you know what I mean? |
| 235 | 19:03 | Jessica | Oh, yeah, cuz we already started - ok [constructs tower, $B W B]$. |
| 236 | 19:10 | Jessica | This is a copy of that [indicating WWB, WWB] |
| 237 | 19:15 | Jessica | All right we need two white on the bottom and a black on top like that. [Constructs tower BWW, places it to the left of $W W W$ (Set 2)] Then we need a white on the bottom and |


|  |  |  | two black on top. Like that. |
| :---: | :---: | :---: | :---: |
| 238 | 19:23 | Jamie | [Constructs BBW tower, places it to the extreme left in Set 2] |
| 239 | 19:24 | Jessica | Like that, yeah. Then we have this [indicating BWB tower. Constructs WBW tower, places it to the right of BWB (Set 3)] |
| 240 | 19:36 | Jamie | Should there be one more? There has to be one more. [There sets appear as follows: $\begin{array}{cccccc} B & W & W & B & B & W \\ B & B & W & B & W & W \\ B & B & B & W & W & W \end{array}$ <br> Set 1 and Set 2 <br> B $W$ <br> $W \quad B$ <br> B $W$ <br> Set 3] |
| 241 | 19:40 | Jessica | Well, if our thing's going to be correct. |
| 242 | 19:43 | Jamie | Maybe there's not and this is the answer. |
| 243 | 19:47 | Jamie | Did we repeat any of those? |
| 244 | 19:48 | Jessica | No, but notice this is 8 [pointing to the 3 -tall towers.] and that was 16 [pointing to the 4 -tall towers.] |
| 245 | 19:55 | Jessica | Maybe we will have 4 for 2 and it can be like colors squared, not colors um.. two, yeah, two raised to a certain power. |
| 246 | 20:04 | Jamie | Yeah. |


| 247 | 20:05 | Jessica | Like two raised to the two would be four. Cause you could have either that...[constructs 2 sets of two-tall towers - Set 1: $B B, W W$, Set $2: W B, B W$, pointing to the 2 sets] |
| :---: | :---: | :---: | :---: |
| 248 | 20:10 | Jamie | That's what it is. |
| 249 | 20:10 | Jessica | That's what it is. The powers of 2. Wasn't that uhmm that was one of my classes. That's what it is - powers of 2. |
| 250 | 20:20 | Jamie | [inaudible] |
| 251 | 20:23 | Jessica | [Indicating all of the 2-tall blocks (4 total)] That's 2 squared. |
| 252 | 20:27 | Jessica | [Indicating all of the 3-tall blocks (8 total)] That's two, two to the third. Yeah. |
| 253 | 20:33 | Jessica | [Pointing to the red and white blocks 4-tall towers (16 total towers)] Two to the $4^{\text {th }}$ is 16.2 to the $5^{\text {th }}$ is 32 . Yeah, so..... I think we figured it out. I'm so proud. [laughing] |
| 254 | 20:42 | Instructor | Okay, okay. Now, I'll get back to you in a sec because they are still explaining their organization here to me. |
| 255 | 20:48 | Instructor | So. Alright. So, I can see that there's only, you know, I can see that's the answer for the three blue and I can see, oh, for the yellow blue, and I can see that's the answer for the three blue because there is no place else. [Camera turns to focus on Kim and Francesca S.] |
| 256 | 20:54 | Group 2: <br> Kim and <br> Francesca S. | [Blocks arrange in 5 Sets, left to right: YYYY (set 1), Set 2: $B Y Y Y, Y B Y Y, ~ Y Y B Y, ~ Y Y Y B, ~ S e t ~ 3: ~ Y B Y B, ~ B Y B Y, ~ Y Y B B, ~$ $B B Y Y, Y B B Y, B Y Y B$, Set 4: BBBY, BBYB, BYBB, YBBB, Set 5: BBBB] |
| 257 | 20:59 | Instructor | [pointing to Set 3: six towers with two blues and two yellows] This one just doesn't jump out at me that that's the only way to do two and two. You know what I mean? Like, it sort of jumps out here. [pointing to Set 2] But it |


|  |  |  | just doesn't quite jump out there [pointing to Set 3] Is there a way to make it like jump out like "That's it." You know what I mean? |
| :---: | :---: | :---: | :---: |
| 258 | 21:10 | Kim | Like a pattern? |
| 259 | 21:11 | Instructor | Yeah, or something, you know. |
| 260 | 21:15 | Instructor | Now I am ready to hear what you guys have to say. [Camera turns and focuses on Jamie and Jessica's group.] |
| 261 | 21:22 | Jamie | We think we figured it out. We think it's the powers of two. |
| 262 | 21:27 | Jessica | Yeah, because we, we... when you told us to do [inaudible, pointing to 8 towers, 3-tall each] we only got eight. So we were like let's go down to two and see what we get there and we got 4. [Indicating group of 4 towers - each two-tall] So you have two raised to the second power. |
| 263 | 21:38 | Jamie | Do you know what it is? It's whatever number of towers |
| 264 | 21:40 | Jessica | That's the power. |
| 265 | 21:41 | Jamie | That's the power. |
| 266 | 21:43 | Instructor | Oh.... |
| 267 | 21:44 | Jamie | Two squared, two to the third, two to the fourth. |
| 268 | 21:46 | Instructor | So you could tell me how many there's gonna be five-tall - without doing it? |
| 269 | 21:50 | Jessica | That's 32. |
| 270 | 21:54 | Instructor | Okay, okay. That's very nice but there's always more. Okay? |
| 271 | 21:57 | Instructor | [Points to 4-tall towers] Now, now you had patterns there |


|  |  |  | but this looks like it's easier to do patterns [Indicating 3tall towers]. So.... How do you know that there isn't any more here? |
| :---: | :---: | :---: | :---: |
| 272 | 22:05 | Jessica | [Pointing to Set $1(B B B, W B B, W W B)$ ] Well, because the one after this one would have to be 3 -white and we already have that there [indicating set 2] |
| 273 | 22:10 | Instructor | Okay. |
| 274 | 22:11 | Jessica | So, we just, we pulled this basically $[W B B]$ we looked at this and we made one looks... copied, just opposite colors [ $B W W]$. Cuz if you try to go down any further, it's just gonna be what we already have. |
| 275 | 22:23 | Instructor | Okay, okay. So, that is like proof by contradiction - if you can't go any further down because there's no place else to go, right? |
| 276 | 22:31 | Jessica | Yeah. |
| 277 | 22:31 | Instructor | And um....Well, you didn't quite give me an inductive proof but you sort of did. You gave me an explicit proof, actually, an explicit answer. Right? You said, tell me again what you said the answer is. |
| 278 | 22:46 | Jamie | The power of two - so if you're building them three tall it would be two raised to the third. |
| 279 | 22:52 | Instructor | So the height is the power? |
| 280 | 22:53 | Jessica | Yeah, the height is the power. |
| 281 | 22:55 | Instructor | Ok. Ok. |
| 282 | 22:58 | Jessica | And two is the, the amount of um, the colors. I'm thinking. |
| 283 | 23:04 | Instructor | Ok, so maybe you want, might need a piece of paper for this. Suppose there was three colors - what's it gonna be? |


| 284 | 23:11 | Jamie | Three raised to the.... |
| :---: | :---: | :---: | :---: |
| 285 | 23:12 | Jessica | To however tall it is. |
| 286 | 23:15 | Instructor | So, why don't you get a third color? |
| 287 | 23:19 | Jessica | Black, red, and white? |
| 288 | 23:20 | Instructor | [Points to original towers] But leave everything you already got. Leave everything you already got. |
| 289 | 23:35 | Instructor | Do three colors and just do them 2-tall because you will get too many if you.... [Hands them bag of blocks] |
| 290 | 23:42 | Jessica | Three colors two tall? |
| 291 | 23:43 | Instructor | Yeah. |
| 292 | 23:44 | Jessica | Hum, that's going to be interesting. |
| 293 | 23:46 | Instructor to Group 2 | Now, while they're building them. I want you to focus over here [to the videographer] and I want you guys to explain to me what you've got. Do you have anything else? [YYYY, Set 1: BYYY,YBYY,YYBY,YYYB, Set 2: BBYY,YYBB, YBBY, BYYB, YBYB, BYBY, Set 3: BBBY, BBYB, BYBB, YBBB, and Set 4: BBBB] |
| 294 | 23:55 | Kim | We kind of tried to do it in a pattern like this one. [Pointing to Set 3: BBBY, BBYB, BYBB, YBBB] |
| 295 | 23:58 | Instructor | Okay, and? So what you were asked - you had the zero, the one, the three, and the four. But we're working on organizing the two. |
| 296 | 24:07 | Francesca | [Pointing at first two towers in Set 2: BBYY and YYBB] Yeah. The only way I can see is this is like 2 blue, 2 blue, then this one's [Pointing to $3^{\text {rd }}$ and fourth towers in Set 2] the blue, then the yellow. |
| 297 | 24:13 | Kim | [Pointing at first two towers in Set 2: BBYY and YYBB] Like these are kinda like stuck together. |


| 298 | 24:16 | Instructor | Yeah, okay. |
| :---: | :---: | :---: | :---: |
| 299 | 24:17 | Francesca S. | If anything they will be in groups like this. But this is the large group. [Pointing to Set 2: $B B Y Y, Y Y B B, Y B B Y, B Y Y B$, YBYB, BYBY] |
| 300 | 24:21 | Kim | Like those are our subgroups. |
| 301 | 24:21 | Instructor | Okay, yeah, okay. So this is a proof by cases and these are your subcases. |
| 302 | 24:26 | Francesca S. | Yeah. |
| 303 | 24:27 | Instructor | Um... I also just happened to notice something when you said two together [Pointing at first two towers in Set 2, switches position of $Y Y B B, Y B B Y]$ I might put this one in the two together case cause that would show that there isn't any other way to do two together. Does that make sense? |
| 304 | 24:39 | Francesca S. | Yeah. |
| 305 | 24:42 | Instructor | But what you said was also reasonable. And then the two apart..... So if this is the two together and this is the two apart. Let's see... [inaudible] [Moves BYYB in between $Y B Y B$ and $B Y B Y$ ] |
| 306 | 24:55 | Kim | [inaudible] Like you can have like another subgroup of the two yellows instead of the two blues together. |
| 307 | 25:00 | Instructor | Yeah [inaudible] |
| 308 | 25:04 | Instructor | However, however, this case [picks up the BBYY tower] and this case [points to the YYBB tower] would also work for the two yellows together. So you would have a problem there in that you would have one tower that's in both groups. Which is why this one's nicer because its got together and apart. |
| 309 | 25:22 | Instructor | Ok. I like this organization. And what I want you to do is write down - get your notebook or something and write |


|  |  |  | down how many are like this, how many like this, how many like this, how many like that. [Pointing to each set in succession] |
| :---: | :---: | :---: | :---: |
| 310 | 26:00 | Instructor | While they're working here [indicating group 2], we can look to what they are having to say here [indicating group 1] and then we are going to all talk about our stuff. |
| 311 | 26:08 | Instructor to Group 1 | So three colors, two-tall, what did you get? |
| 312 | 26:09 | Jessica | Nine. |
| 313 | 26:10 | Instructor | Nine. Ok, does that fit in with your prediction? |
| 314 | 26:12 | Jessica | Yes. Three colors, which is the base. Three is the base. Two tall is the exponent so three squared is nine. |
| 315 | 26:20 | Instructor | Ok, so, okay, so you have an extension and I'm going to pull the next thing up. [The instructor goes to the board. The powerpoint slide says: <br> More on towers <br> What is your prediction for <br> Towers that are 3 cubes tall <br> Towers that are 5 cubes tall <br> Towers that are $n$ cubes tall <br> Convince me that your answer is correct!] |
| 316 | 26:33 | Instructor | I already asked you... the extensions was if there $n$ cubes tall and you got two colors to choose from. You know the answers for that. Now you got $m$ colors to choose from, I want that equation. |
| 317 | 26:46 | Jessica | Oh, 'cause we figured out when you have 2 colors to the $n$. So now it's going to be $m \ldots$ so however many total.... Oh, $m$ to the $n$. |
| 318 | 26:59 | Jamie | Right. |
| 319 | 27:00 | Jessica | Yeah, cause $m$ is the color, the amount of colors. And $n$ is how tall. Yeah... height. Yeah. Cause that was two. <br> Yeah. So $m$ is the base and $n$ is the exponent. [Writes $m^{n}$ in her notebook. Writes $m=$ colors base; $n=h e i g h t$ |


|  |  |  | exponent] |
| :--- | :--- | :--- | :--- |
| $\mathbf{3 2 0}$ | $27: 34$ | Jessica | Alright, so we got that. |
| $\mathbf{3 2 1}$ | $27: 35$ | Jamie | So that is [inaudible] |
| $\mathbf{3 2 2}$ | $27: 47$ | Jessica | You were saying if there is $m$ amount of colors, right? <br> [asking the instructor] |
| $\mathbf{3 2 3}$ | $27: 49$ | Instructor | Yes. |
| $\mathbf{3 2 4}$ | $27: 50$ | Jessica | Yeah, it is $m$ to the $n$. |$|$| 325 |
| :--- |
| $\mathbf{2 7 : 5 2}$ | Instructor to | Okay, okay. Now I want everybody to present to |
| :--- |
| everyone else. And we are going to start with this group |
| right here [indicating Kim and Francesca S]. Okay. So, |
| Francesca and Rebecca hustle on over here. I know you |
| started something else but I want everybody to watch. |
| Okay. |


|  |  |  | yellow. |
| :---: | :---: | :---: | :---: |
| 334 | 28:43 | Instructor | Ok, now notice that they have [addresses Group 1] I know you guys are trying something else but I want you to watch their organization too. [Pointing to Set 3] The two and two that you have two sub-groups, right? |
| 335 | 28:52 | Kim | Oh, yeah. [Set 3: separates BBYY, YBBY, YYBB from $B Y B Y, B Y Y B, Y B Y B]$ Which can be broken down to that. |
| 336 | 28:56 | Kim | [Indicating BBYY] Cuz these two blues are stuck together. |
| 337 | 29:00 | Francesca S. | And those are separated blues. |
| 338 | 29:02 | Instructor | It's pretty convincing that there's the only way to do two blues together. Okay. And these are... |
| 339 | 29:06 | Kim | [indicating BYBY, $\left.\mathrm{BYYB}^{\text {a }}, Y B Y B\right]$ These are apart. |
| 340 | 29:08 | Instructor | Two blues apart. For a total of.....? Six. |
| 341 | 29:13 | Kim | Six, yeah. And then these two are three blue and one yellow [Indicating Set 4]. And then the other group is all blue [Indicating Set 5]. |
| 342 | 29:23 | Instructor | And so your total numbers were one [indicating $Y Y Y Y$ ], four [indicating Set 2]... |
| 343 | 29:28 | Kim | four [indicating Set 2], six [indicating Set 3], four [indicating Set 4], and one [indicating BBBB] |
| 344 | 29:31 | Instructor | Okay. |
| 345 | 29:33 | Instructor | Now, you guys did a slightly different organization for yours [pointing to the group of Rebecca and Francesca $C$.] - at least for your groups of two. So, I want to focus everybody go on over and look at what Rebecca and Francesca did. You're gonna tell us about your organization and you're gonna tell us about your inductive rule. And then last we're gonna do Jamie and |


|  |  |  | Jessica for their rule. Ok? So, wait till everybody gets here so they can see what you're doing. Go. |
| :---: | :---: | :---: | :---: |
| 346 | 29:46 | Group 3 | [Three different tower sizes, 4-tall, 3-tall, and 2-tall using blue and orange cubes. 4-tall Set 1: BBBB, Set 2: OBBB, BOBB, $B B O B, B B B O$, Set 3: BBOO, OOBB, Set 4: BOBO, OBOB, Set 5: BOOB, OBBO, Set 6: OOOB, OOBO, OBOO, BOOO, OOOO; 3-tall, Set 1: BBB, Set 2: BBO, BOB, OBB, Set 3, OOB, OBO, BOO, Set 4: OOO; 2-tall: $B B, O B, B O, O O]$ |
| 347 | 30:00 | Group 3 <br> Francesca C. | We'll start with the 4-tall [indicating Set 2] - with the 4 tall, what we did was... we also had a group of the three blue and the three orange [indicating Set 6]. And there were four of each of those. [Removes OOOO from Group 6, placing it by itself] |
| 348 | 30:12 | Francesca C. | But the way we made sub-groups of the two blue and two orange, is I put them into groups of two. And I did two blue on top, two orange on top, two blue on the bottom, two orange on the bottom [indicating $O O B B$ and BBOO ]. |
| 349 | 30:24 | Francesca C. | Then we had, like each of these [indicating $B O B O / O B O B$ and $B O O B / O B B O$ ] are opposites. We had the every other color [indicating $B O B O / O B O B$ ] and then we had the two in the middle [indicating $B O O B / O B B O$ ]. |
| 350 | 30:30 | Francesca C. | And then there's the one and the one [indicating $B B B B$ and $O O O O]$. So they came out to the same amount. |
| 351 | 30:34 | Instructor | Okay. And then [to Rebecca] you were gonna talk about the pattern that you saw - I asked them to build - I asked them for the prediction for 3-tall and 5-tall - and like most of you they thought maybe it would be nine. And, tell us what you found. |
| 352 | 30:47 | Rebecca | So, if you have just 1-tall tower, you only have two [indicating one blue block and one orange block] And then in order to get the second one with yellow, you can add a blue and you'll get this one [ BO ]. Or you can add |


|  |  |  | another orange and you'll get this one $[O O]$. So, for each tower, you add one or the other to get the next group. To double it. |
| :---: | :---: | :---: | :---: |
| 353 | 31:06 | Rebecca | See with this one, you can either add a blue to get that one, or an orange to get the next one. |
| 354 | 31:12 | Instructor | Now, stop right there. Everybody get that? |
| 355 | 31:14 | Class | Yeah. |
| 356 | 31:15 | Instructor | Okay, go. |
| 357 | 31:16 | Rebecca | So then for each tower....to make it another block higher, you can add either an orange or a blue, so it would essentially double what you have. |
| 358 | 31:27 | Instructor | So, now, many of you discovered the rule - it doubles and this is the explanation. Right? This is the reason why it doubles. So there's more than just, yeah, we see a pattern its times two. Here's the reason why its times two. Right? Inductive reasoning, right? See, that wasn't too bad. Okay. This is why it goes times two every time. And so now you know - just like with the induction, like with the dominos, right? If you have any height tower, you know what the next one's gonna be because of this inductive rule she talked about. Ok - questions? |
| 359 | 31:58 | Instructor | Okay, we are going to do the last group over here [indicating Group 1]. They investigated an explicit formula for $n$ cubes tall - and, then, well tell us about what you found. |
| 360 | 32:14 | Instructor | [Addresses Group 1] You found a relationship between the height and number of towers. |
| 361 | 32:18 | Jessica | And the number of colors. |
| 362 | 32:20 | Instructor | Okay, well first start with the number....just we're doing the two colors, right? And you found out - what's the relationship between the height and the total number of |


|  |  |  | towers you can build? |
| :---: | :---: | :---: | :---: |
| 363 | 32:28 | Jamie | Well, whatever the height is - so say if it's two-tall, and your using two colors, two is the base and two is the exponent. So it would be like two raised to the second power. |
| 364 | 32:42 | Instructor | Ok, that's what you found over there, that there were four of them. Okay. |
| 365 | 32:45 | Jessica | So the height of one of these - [holding tower WWWW] Um... I'm sorry, I forgot what I was going to say... I had it. |
| 366 | 32:52 | Instructor | The height is four, right? |
| 367 | 32:53 | Jessica | Yeah, four-tall. But I was trying to remember what this was [points to notebook] |
| 368 | 32:57 | Instructor | Well, we're not quite up there yet. But your right - 4 -tall, and Jamie you said its two to the height power. |
| 369 | 33:04 | Jessica | That's what it was. Two colors [holding WWRR], four tall would be two to the fourth power. And then when you put them all together you've got two of each kind, like you have two here [points to $R R W W, W W R R$ ] like they [Group 3] had in the separate thing. Then we have the two full ones [indicating RRRR, WWWW] |
| 370 | 33:21 | Instructor | You had a slightly different organization. |
| 371 | 33:22 | Jessica | We have a slightly different organization but we have exactly what everybody else had with that. |
| 372 | 33:27 | Instructor | And we got 16. |
| 373 | 33:28 | Jessica | Yeah, we got 16 with all that... cause if you put them all together.... |
| 374 | 33:31 | Instructor | So you investigated also when there were three colors to choose from. |


| 375 | 33:35 | Jessica | Yeah, we noticed that the three colors [points to 3 sets using Brown $(B)$, Green $(G)$ and Maroon $(M)$ cubes: $B B$, $B G, B M$; $M M, M G, M B ; G G, G M, G B]$ so three would be our base number - like we said for the two for the fourhigh. And since we had three colors, it was two raised to the third power. |
| :---: | :---: | :---: | :---: |
| 376 | 33:54 | Instructor | Backwards. |
| 377 | 33:34 | Jamie | Backwards, three raised to the... |
| 378 | 33:36 | Jessica | Oh, three raised to the second power - sorry - brain's not working quite right today. |
| 379 | 34:00 | Instructor | Ok so $m$ is the base and so you gave us for $m$ colors and the height of $n$ ??? Tell us what you got and I'll write it up here. |
| 380 | 34:07 | Jessica | The height of $n \ldots$ |
| 381 | 34:09 | Instructor | Three colors you said it was three squared is nine towers that you can make two-tall, which you showed us the nine towers [Writes on board: 3 squared $=9$.] That's for three colors and height two. [Writes on the board: 3 colors, two height $]$ Now I asked you, what if there's $m$ colors and the height $n$. |
| 382 | 34:29 | Jessica | It's $m$ to the $n$. |
| 383 | 34:32 | Instructor | And you said its $m$ to the $n$ power. [Writes on the board $m$ colors, $n$ height; $m$ raised to the $n$ ] Ok. |
| 384 | 34:38 | Instructor | Do we have a reason for that? |
| 385 | 34:43 | Jessica | Just by what we did before. |
| 386 | 34:44 | Instructor | Well, see now, can you follow up with what Rebecca and Francesca did - the one-tall when you've got three colors to choose from? |
| 387 | 34:56 | Jessica | One-tall - three colors [places 3 cubes - M, G, B on |


|  |  |  | table] |
| :---: | :---: | :---: | :---: |
| 388 | 35:01 | Jamie | So it would be three to the one. |
| 389 | 35:02 | Instructor | [Addresses Group 3] Right, now remember what you said, Rebecca, come in closer and point. |
| 390 | 35:07 | Rebecca | For each one-tall tower you can, for this one [indicating $B]$ you can add either a brown, a green, or a maroon. For this one, [indicating $G$ ] you can add either a brown, a green, or a maroon. So, for each one, there's three possible towers you can make to create it two tall. So, you add a green, you know, you can add a green, you can add a maroon, or you can add a brown. So you end up with, you know, three more from what you already have. |
| 391 | 35:33 | Instructor | Does that make sense to everybody? |
| 392 | 35:35 | Francesca C. | So the answers triple. |
| 393 | 35:36 | Instructor | That's right - the other one was doubled and this one now is tripled. |
| 394 | 35:40 | Francesca C. | Because it's three colors. |
| 395 | 35:42 | Instructor | Because there's three colors. Ok, now you guys finished pretty fast. So of course, we're never done. Let's see, yes, we have time. We have plenty of time. So I am going to give you the challenge problem that came out of this for students who had actually been doing this a really long time. So this will be a real challenge but I know you can do it. |
| 396 | 36:02 | Instructor | Okay, this one... this one is based on some of the stuff that you just investigated now. They call it Ankur's challenge because the kid who made up the problem's name is Ankur. So.... |
| 397 | 36:19 | Instructor | Okay... and this builds on something that Rebecca noticed which is when you only got two tall and three colors, some of them don't have all three colors in them. |


|  |  |  | So here you got four tall and three colors to choose from. <br> Each cube must have at least one of each color. So how <br> many can you make? |
| :--- | :--- | :--- | :--- |
| $\mathbf{3 9 8}$ | $36: 48$ | Jessica | It's three to the fourth minus three. |
| $\mathbf{3 9 9}$ | $36: 51$ | Instructor | Well... I want something with some justification there. <br> Okay. But that was an interesting, original thought. <br> What's the minus three, what are you subtracting? |
| $\mathbf{4 0 0}$ | $37: 01$ | Jessica | We're subtracting the full, the full color ones. Because <br> you said that it has to one, it has to be every color. |
| $\mathbf{4 0 1}$ | $37: 09$ | Instructor | Yeah, okay, but see you want to subtract those but think <br> about that some more. Okay, that was a good start <br> though. |
| $\mathbf{4 0 2}$ | $37: 15$ | Jessica | [inaudible] There's probably more.... |
| $\mathbf{4 0 3}$ | $37: 16$ | Jamie | There's probably more that fit both cases. |


| 412 | 37:50 | Instructor | Eighty-one. |
| :---: | :---: | :---: | :---: |
| 413 | 37:50 | Jessica | Eighty-one. |
| 414 | 37:51 | Instructor | So... that's a lot of towers to build. So you might want to try to figure it out rather than trying to draw them, uh, build them all. Or you can just try to build the ones that must have one of each color because there's not so many of them. Okay, but it's up to you. |
| 415 | 38:12 | Group1 <br> Jessica | Alright, so we'll have brown, green, and maroon, as they said. Alright. And they have to be four tall. I really don't want to draw all of these cause I know...[She has written $B-G-M$ in her notebook and starts to draw a tower.] |
| 416 | 38:28 | Jamie | Just draw like a... |
| 417 | 38:30 | Jessica | I'm probably just going to draw a line next to them. So, we will start with blue on the bottom? Or brown on the bottom? |
| 418 | 38:36 | Jamie | Brown. |
| 419 | 38:36 | Instructor | You guys are drawing pictures so if she wants to build she can borrow your whites, for example. [Indicating Francesca S.] |
| 420 | 38:42 | Jessica | Yeah, I was actually going to ask you if we were done with this. Okay, so we have blue on the bottom. Let's start with three blues. It's probably easier to work our way down. [She writes on her paper a tower of GBBB] |
| 421 | 38:54 | Jamie | But then we will have to take that one out though. Cause you can't have...[inaudible] |
| 422 | 39:01 | Jessica | One of each color. Okay, so the three... So we know there can't be three of anything. So it would have to be like, maroon. [She erases the second blue and replaces it with a maroon. She now has GMBB] |
| 423 | 39:10 | Jamie | That's why it is going to be less [inaudible] You end up |


|  |  |  | taking out three. I think you have to take out four and then whatever. |
| :---: | :---: | :---: | :---: |
| 424 | 39:19 | Jessica | So if we do that one - we can do blue, blue, green, maroon. And then we can separate them or we can move them up. [She now has another tower written on her paper MGBB] |
| 425 | 39:33 | Jamie | Right. |
| 426 | 39:34 | Jessica | Yeah, so we will move them up next. So put the green here, the maroon here, and the two blue in the middle. <br> [She now has another tower written on her paper MBBG] |
| 427 | 39:39 | Jamie | Uh-huh. |
| 428 | 39:40 | Jessica | And then you would have to do the opposite. Which would be the green here, the maroon here, and the two blue in the middle. And then you would move it up once more. [She now has another tower written on her paper GBBM] |
| 429 | 39:48 | Jamie | Right. So then that's for every color. |
| 430 | 39:51 | Jessica | Yeah, so we can just take this and then just... |
| 431 | 39:52 | Jamie | Multiply it. |
| 432 | 39:53 | Jessica | Multiply it. |
| 433 | 39:54 | Jessica | But then we would have to make sure it's that. And then... blue, blue, maroon, green. So we have one, two, three, four, five, six. We have six for two blues. Yeah, cause there's always gotta have to be two of one color. [She now has six towers written on her paper. They are as follows: |




| 458 | 41:52 | Jamie | So if there's three, six. So if there's $12,24,36$. |
| :---: | :---: | :---: | :---: |
| 459 | 42:00 | Jessica | Well, it's... yeah, you know what, we did take a lot out because if you can't use three, you took all the versions of three blues, three maroons, three things out - that's a lot. |
| 460 | 42:09 | Jamie | Yeah, so that's why [inaudible]. What's the difference between 81 and 36 ? |
| 461 | 42:17 | Jessica | 81 minus 36 is.....Your asking me to do basic math? [writes in notebook] |
| 462 | 42:26 | Jamie | 45 |
| 463 | 42:31 | Jessica | Maybe take 45 out - probably it is - if you think about it. |
| 464 | 42:35 | Jamie | You're taking 15 out from each color. |
| 465 | 42:40 | Jessica | Yeah |
| 466 | 42:41 | Jamie | Write that down. |
| 467 | 42:43 | Jessica | Fifteen out. If you want to do evenly, yeah, its 15 out of each. |
| 468 | 42:49 | Jamie | Of each color, 15, 30, 45. |
| 469 | 42:54 | Jessica | Yeah, cause it would have to be even of how many you take out. |
| 470 | 42:57 | Jamie | So that's it. |
| 471 | 43:02 | Jessica | [Counting number of towers in notebook] That will be $1,2,3,4,5,6,7,8,9,10,11,12$. |
| 472 | 43:06 | Jamie | So why are you taking 15 out from each one of them? Because you're eliminating five - you're taking out 5 from each color - from each case. See what I'm saying there's three colors - you're taking out five blocks. |
| 473 | 43:22 | Jessica | For each color. |


| 474 | 43:23 | Jamie | For each, like, section. |
| :---: | :---: | :---: | :---: |
| 475 | 43:27 | Jessica | Oh, ok. |
| 476 | 43:28 | Jamie | You know what I mean like five times three - so five from the blue, five from the green, five from the maroon or brown, whatever. |
| 477 | 43:34 | Jessica | We're taking 15 towers out? |
| 478 | 43:36 | Jamie | Out of each color - not each color - each section- so, like this is one, right? And then we're gonna start with another one starting with like, two green on the bottom. And then two maroon on the bottom. So, you're taking 15 out of each case. |
| 479 | 43:53 | Jessica | Oh, yeah, yeah, yeah because you can't do the three. |
| 480 | 43:55 | Jamie | So what you're doing is you're taking five towers out from each color, from each case. |
| 481 | 44:01 | Jessica | Oh, 'cause you're taking like three. |
| 482 | 44:04 | Jamie | You're taking five out of the blue. |
| 483 | 44:07 | Jessica | Yeah, cause, like, what I'm saying is like you can't have three - sets of three - and you can't have sets of four. Even though there's only one of those. [She writes 4-1, 4-1,4-1 at the top of her notebook.] |
| 484 | 44:16 | Jamie | That's why you're taking out five. [inaudible] case, which is 15 . That's why it's 45 . |
| 485 | 44:31 | Instructor | How's it going? |
| 486 | 44:32 | Jessica | We're assuming that 36 is the total amount of them if you have to have one of each color in each. |
| 487 | 44:39 | Instructor | Well, you're not assuming, you must of proved it, right? Or convinced yourself? |
| 488 | 44:43 | Jamie | What we did was we took the original formula... |


| 489 | 44:45 | Jessica | [interrupting] 81 |
| :---: | :---: | :---: | :---: |
| 490 | 44:45 | Jamie | which was three to the fourth and then we did it for one case using two blues and.... |
| 491 | 44:56 | Jessica | Two browns. |
| 492 | 44:57 | Jamie | Two browns, yeah. So it's... so we came up with 12, so 12 times three - three colors is 36 . We subtracted that [points to notebook 81-36] to see how much the difference was and it was 45 . So we realized you were taking out 15 from each case. Which is, you're taking out five from each separate color because five time three is fifteen. |
| 493 | 45:23 | Instructor | Ok, I sort of follow you, except up to that part. |
| 494 | 45:25 | Jessica | The last part. The last part we're still trying to like figure out exactly. |
| 495 | 45:30 | Instructor | Okay, okay, but back up to the... you said there was 12 . There was 12 that had the brown duplicated and so now you're telling me there's gonna to be twelve that have... |
| 496 | 45:39 | Jessica | The maroon [inaudible] |
| 497 | 45:43 | Instructor | Ok. Ok. Ok. Now, those are convincing words, but I'd like to see all that convincing stuff written on the paper too. Okay. Just so you have, you know, a record for yourselves. |
| 498 | 46:01 | Jessica | Alright, so. We will use two... |
| 499 | 46:03 | Jamie | Can we make them all? |
| 500 | 46:05 | Jessica | Yeah, but the thing is like... do we have enough? |
| 501 | 46:10 | Jamie | No, I don't think so. |
| 502 | 46:11 | Jessica | I don't think we have enough. Thirty -six different columns. |


| 503 | 46:18 | Jamie | Why don't you draw it out for the green, on the bottom? |
| :---: | :---: | :---: | :---: |
| 504 | 46:20 | Jessica | Okay, so what we do? Start with green on the bottom? [She starts writing the towers in her notebook.] So it would be green, green, maroon, brown. Green, green.... And then you move them up. Green, green, with a maroon on top. Then we do the opposite [inaudible] Move it to the top.... Green, green....And then you separated them? Top and bottom? |
| 505 | 47:07 | Jamie | Right. And then switch the inside. |
| 506 | 47:18 | Jessica | And then separate them like that and separate them like that. [The new set of 12 towers are as follows: |
| 507 | 47:47 | Jessica | Alright so this is green dominant and this is blue dominant. |
| 508 | 47:50 | Jamie | So now we just have to do the maroon. |
| 509 | 47:51 | Jessica | Maroon. I'm going to have to draw it smaller so I can fit it all on this page. Alright...so maroon dominant. [Writing in her notebook.] Maroon, maroon, brown, |



|  |  |  | two. [Writes down 3 to the fourth equals 81 in notebook] So, um... |
| :---: | :---: | :---: | :---: |
| 516 | 50:20 | Jessica | If you have one green, then you have one maroon, you're gonna have two brown. [Writes $G-1, M-1, B-2$ in notebook] But if you have one maroon and one brown, you are gonna have two green. [Writes $M-1, B-1, G-2$ in notebook] And if you have one brown and one green, [Writes $B-1, G-1, M-2$ in notebook] |
| 517 | 50:37 | Jamie | That's how you get the three cases. |
| 518 | 50:38 | Jessica | Yeah, that's how you get the three different cases. Because you can't have three of anything. |
| 519 | 50:45 | Jamie | Can you have one of anything? |
| 520 | 50:48 | Jessica | What do you mean? |
| 521 | 50:49 | Jamie | No, you can't because there are four. |
| 522 | 50:51 | Jessica | Yeah, it's four-tall. Yeah. You can't do one, one, and one because that's only equal to three. And you can't have three of one color because then you won't have the other two. [Writing in notebook] |
| 523 | 51:06 | Jamie | Okay, right. |
| 524 | 51:08 | Jessica | The color would be $m$. |
| 525 | 51:12 | Jamie | Right, because there could only be two [inaudible] Right? |
| 526 | 51:17 | Jessica | Yeah, because if you have three of one color then you can't have one of each. But other than that like... Ok, so now we go back to this, where we said we had 81 - we found out that there's 36 of them - so we had 45 . <br> [Writing in notebook. She writes 81-36=45] |
| 527 | 51:39 | Jessica | Which meant there was 15 from the maroon dominant case. [Writes in notebook M-15] |


| 528 | 51:44 | Jamie | The green and the brown. |
| :---: | :---: | :---: | :---: |
| 529 | 51:47 | Jessica | Yeah, the green and the brown [Writes in notebook $G-$ $15, B-15]$ One of the browns is because it's a set of four browns. Four browns. [Writes $4 B$ ] |
| 530 | 52:00 | Jessica | One is from four greens [Writes $4 G$ ] and one is from four maroons [Writes 4 M] |
| 531 | 52:06 | Jamie | And then you do three blue... |
| 532 | 52:07 | Jessica | If you have three blue, how many cases of three blues can you have though? |
| 533 | 52:13 | Jamie | You can have two. Right? Three blue on the bottom or three blue on the top. |
| 534 | 52:18 | Jessica | Yeah, but then you have..., you can have it where... I keep saying blue. I'm going to start using blue soon, I swear. |
| 535 | 52:25 | Jessica | Cause if you have three, right? You can either have that [Builds GBBB] or you can have that [Builds MBBB] or need more colors. Or you can have it with it on the bottom [Builds BBBM places it next to MBBB; builds $B B B G$, places it next to $G B B B$ ] |
| 536 | 52:42 | Jamie | So maybe there are four of them [inaudible] |
| 537 | 52:45 | Jessica | Well also the fact if you separate them. |
| 538 | 52:52 | Jessica | All right, so. All right, so, separate it. [Builds BMBB and $B B M B]$ Then there's the one with the red two down, it's the third one down. Right? And the same with the green, so that's eight so far. |
| 539 | 53:18 | Jamie | [Counting towers] No, four, eight, twelve plus the three is fifteen. Get it? Look this is for one color...[pointing to the set of four towers that contain three browns and one maroon] |


| 540 | 53:28 | Jessica | [Points to notebook] This is for... this is for the brown, brown dominant. Three brown. |
| :---: | :---: | :---: | :---: |
| 541 | 53:34 | Jamie | Three brown. So this, this total, this plus this plus this is three. You have three that you're taking out. [She refers to the line in the notebook that says $4 M+4 G+4 B=3]$ Okay? Now if you have three brown you get... |
| 542 | 53:45 | Jessica | Eight. Eight each because you have the green and the red. You have to take eight out because you can't...cause there's no red in this. Remember you had this? |
| 543 | 53:55 | Jamie | Yeah, but isn't this for the green case? |
| 544 | 53:58 | Jessica | No, this is for the same brown case. The three browns. Notice three browns. [Holding GBBB] |
| 545 | 54:02 | Jamie | Right. But you have [points to $G$ of $G B B B$ ] green up here. Right. I'm suggesting that, this whole thing is three - this line. Right? So now if you have three brown - with - no wait. |
| 546 | 54:15 | Jessica | See - but there's two sets of three browns. There's the three browns with the red and the three browns with greens. |
| 547 | 54:21 | Jamie | So that's six, right? |
| 548 | 54:23 | Jessica | So that's uh.., actually, no cause we have four of them. We have the four of the three browns with the red and we have four of the [inaudible] |
| 549 | 54:30 | Jamie | So that's eight? |
| 550 | 54:31 | Jessica | So you have eight here... [points to notebook] Which is 8, 16, 32. |
| 551 | 54:37 | Jamie | 8 is the total. |
| 552 | 54:39 | Jessica | Yeah, eight. 8 here, 8 here, and 8 here [points to notebook]. Cause look brown dominant, I just need a few |


|  |  |  | more. If you have brown dominant. [Constructs $B G B B$, places in between $G B B B, B B B G$. Constructs $B B G B$, places in between $G B B B, B G B B]$ |
| :---: | :---: | :---: | :---: |
| 553 | 54:59 | Jamie | Okay, so then it's 8,8 , and 8 . |
| 554 | 55:01 | Jessica | Yeah, that's what I want to show that it's eight, eight, and eight. |
| 555 | 55:03 | Jamie | So that's eight, eight, and eight. So that's 24. Right? |
| 556 | 55:12 | Jessica | Yeah. Yeah, that's 24 , so we're up to 27 . We need to get to 45 . So would we do the other? What else would it be? [Addressing the instructor] We're trying to figure out why thirty-six. |
| 557 | 55:24 | Instructor | Ok, so you figure your counting 45 that are not... |
| 558 | 55:27 | Jessica | Yeah, we're trying to figure out how the 45 - like what the 45 of them are that we're not using. We've only gone up to 27 . |
| 559 | 55:37 | Instructor | Okay. I'm gonna ask for a break now from what you're doing though because I want everyone to present what they've got. Okay? Now you do have a reason why you got the 36 right? |
| 560 | 55:46 | Jessica | [Pointing to notebook] Yeah, cause its 12, 12, and 12. |
| 561 | 55:48 | Instructor | Ok, so you start presenting your reason and then their group is gonna do their reason [Group 2] and then you guys are gonna do your reason [Group 3]. |
| 562 | 55:55 | Jessica | You got thirty-six? |
| 563 | 55:56 | Instructor | Yes. |
| 564 | 55:56 | Jessica | Yes! |
| 565 | 55:57 | Instructor | And you guys got? |
| 566 | 55:59 | Kim (Group 2) | $36!$ |


| 567 | 56:00 | Instructor | Yeah! [Addresses Group 1] Ok, so what did you... Explain, take it from the top - how did you know there's 36? Alright. How do you know there's 36? Yes, you. [Addresses Group 1] Either one of you can tell us. |
| :---: | :---: | :---: | :---: |
| 568 | 56:10 | Jessica | [To Jamie] You want me to do this since you did most of the last one? All right, since I wrote all of it [indicating notebook]. Yeah, probably turn it like that so everyone can see. What we started with was - we started with two brown because you said you had to have one of each color. |
| 569 | 56:28 | Instructor | So brown is the one you're gonna duplicate [inaudible] |
| 570 | 56:30 | Jessica | Yeah, we decided to make brown the more dominant of the towers. It would always be two brown in all of these. So we have two brown with the maroon on the bottom and the green on top. And then we had two brown with the green and then the maroon. |
| 571 | 56:44 | Jessica | And then from there, we moved them up to the middle [meaning the two browns] and then either maroon had to be on the top or on the bottom and the green had to be on the top or the bottom. So that's four. [points to notebook at the towers where the there are two browns next to each other: $G M B B, M G B B, M B B G, G B B M, B B G M, B B M G]$ |
| 572 | 56:52 | Instructor | So that's [inaudible] |
| 573 | 56:54 | Jessica | And then you had the two brown on top. So it had to be either green and maroon right under that. And under that either green or maroon being the opposite [points to the towers with two browns on top: BBGM, BBMG] |
| 574 | 57:02 | Jessica | Then we split it - so you have a brown on top and bottom and then you decide if you want it like that or like that. [points to the following towers: BMGB, BGMB] |
| 575 | 57:12 | Jessica | And then we put them spaced out in the... [points to $G B M B, M B G B]$ And you can see, right here and there |


|  |  |  | [points to four B cubes in the two towers]. |
| :---: | :---: | :---: | :---: |
| 576 | 57:19 | Instructor | Positions 2 and 4. |
| 577 | 57:21 | Jessica | Yeah, there you go. Positions 2 and 4. And then like you said before, then you would have the green on top and the maroon on bottom or the other way around. And then [points to $B M B G, B G B M$ ] we just switched it so that the brown would be in spot 1 and spot 3 and then maroon and green or green and maroon. |
| 578 | 57:35 | Instructor | So every configuration you doubled because the green and the maroon could change places. |
| 579 | 57:40 | Jessica | Yes. |
| 580 | 57:40 | Instructor | And you got how many for two browns? |
| 581 | 57:44 | Jessica | 12 |
| 582 | 57:45 | Instructor | Ok so you got 12 for two browns - so how do you know the answer is 36 ? |
| 583 | 57:49 | Jessica | Well, you could... There's no more ways that we could figure out on here - because there was nowhere else that you could put the browns that was different than [inaudible] |
| 584 | 57:58 | Jamie | And you can't have three. |
| 585 | 57:59 | Jessica | And you can't have three of them. And you can't have all four of it. |
| 586 | 58:02 | Jessica | So what we did was we took this [points to the number 12 , the total for the BB combination] and we noticed there's three different colors - so since you have twelve of this one [Brown] you had to have twelve of that [points to set with two greens] and you had to have twelve of that [points to set with two maroons] so 12 times 3 is 36 . |
| 587 | 58:12 | Instructor | Ok, now, any questions about that? Now I think I want |


|  |  |  | you guys [Group 2] to go next because you also did a proof by cases but you did a different organization. And your organization... their organization started with white on top. So, case 1, white on top. Go for it. |
| :---: | :---: | :---: | :---: |
| 588 | 58:34 | Kim | [They have 3 sets: Set 1:WWYB,WWBY; Set 2: <br> WBYW,WBWY,WBYB,WBBY,WBYY; SET 3: <br> $W Y W B, W Y B W, W Y Y B, W Y B Y, W Y B B]$ Okay, so. Here are... The whites are on top. And then we have three different groups. Like you can have white with white like stuck together [points to Set 1]. Then you can have white with blue [points to Set 2]. And then white with yellow [points to Set 3]. So then, once we got the first two blocks [indicates all the sets] then we just figured out the third row and then the fourth row. |
| 589 | 58:56 | Francesca S. | And then we figured out it was twelve so we figured it would be the same for blue on top and the same for yellow on top. |
| 590 | 59:02 | Instructor | So they also found 12 times 3 but it's a totally different organization. Right? As you can see this is the first subgroup [Set 1] white and then white. Right? So if you have to have two whites on top, this is the only way you can have two of the same color. Right? And then they worked their way with blue and yellow and found five in each of those sub-cases. So, totally different way to get twelve, right? But it's still times three. |
| 591 | 59:21 | Instructor | Now, [Group 3] these guys, now go up and take a picture of the front of the board. [inaudible] And tell us... They did a subtraction. Right? They started with 81, go for it. |
| 592 | 59:34 | Rebecca | We started with the ones that you can include. [Addresses Francesca C.] Do you want to talk about the doubles and I'll do the triples? |
| 593 | 59:37 | Francesca C. | Sure, each of the ones that you can exclude because there needs to be every color [The have written on the board: 3 |


|  |  |  | different towers! Doubles x 3 BBYY, BYYB, YYBB, BBRR, $B R R B, R B B R$ 18] all three of the colors in each group. We were able to exclude each of these. The doubles is where there's two colors of each one. So let's say like there's like two blue of one and then two yellows [Points to the BBYY tower]. This is only two colors, not all three colors, in each of these [inaudible] And it turns out that for just the blue, like cause we did blue and yellow and then blue and red. For just the blue, there was six uh... six different ones that had to be excluded. And then we times that by three because there's three [inaudible] |
| :---: | :---: | :---: | :---: |
| 594 | 1:00:06 | Francesca C. | And we did the same thing for solids [on board: solid x 3 YYYY 3] and triples [on board: triples x 3 BYYY, YBYY, YYBY, YYYB, RYYY, YRYY, YYRY, YYYR 24]. |
| 595 | 1:00:09 | Rebecca | Yeah, for triples, we just wrote up there all the yellows. So, if you have three yellows and one of the other colors, there is eight different ways that you can do it. And then if you have the three colors, you have yellow, blue and red, so you can multiply it by three. And then you added them all up, which equals 45 and 81 minus 45 is 36 . [On board: 45 excluded 36 stay!] |
| 596 | 1:00:30 | Instructor | Ok, any questions? |
| 597 | 1:00:33 | Jessica | No but I like their's. Now I understand what we were missing - it's their doubles. |

## APPENDIX C

Transcript 2.1 - February 18, 2011
Camera View One

## February 18, 2011 - Camera View 1

The tape begins with the instructor displaying a powerpoint. The powerpoint slide says: Summarize our previous results:

- Two colors, four cubes tall: 16
- You organized the towers by number of blue cubes
- How many towers for 0 blue, 1 blue, 2 blue, 3 blues, and 4 blues?
- Two colors, $n$ cubes tall: $2^{n}$
- Why is it $2^{n}$ ?
- $m$ colors, $n$ cubes tall: $m^{n}$
- Why is it $m^{n}$ ?

They break into two groups, two people each, and work on the problem.

- One group is composed of Jessica and Jamie
- One group is composed of Kim and Francesca S.

This camera view focuses on to Kim and Francesca S. (Blue and Orange cubes). [When the towers are described, the first color written is the top cube of the tower.]

| $\mathbf{1}$ | $01: 26$ | Francesca S. | So what do we have to do? |
| :--- | :---: | :--- | :--- |
| $\mathbf{2}$ | $01: 28$ | Kim | The same thing as... Like you know how, like, we <br> had like, three and then orange on top. [She builds <br> the tower OBBB.] |
| $\mathbf{3}$ | $01: 33$ | Francesca S. | Oh, so we have to do that again. |
| $\mathbf{4}$ | $01: 35$ | Kim | Yeah, we are doing the same thing. Right? We are <br> doing the same thing we did last week? |
| $\mathbf{5}$ | $01: 38$ | Instructor | Uh-huh. I just want you to reconstruct the <br> organization. |
| $\mathbf{6}$ | $01: 47$ | Kim | Well, here's the... Do you have the all blue one? We <br> can use that one [She grabs the BBBB tower that <br> Francesca S. has created.] Do we have the same <br> ones? Make sure that we don't repeat. [She has <br> created the following towers: BBBB, OOBB, OOOB, |
| OBBB.] |  |  |  |


| 7 | 02:03 | Francesca S. | No cause mine are on the bottom. [She has created BOOO, BBOO, BBBO.] |
| :---: | :---: | :---: | :---: |
| 8 | 02:05 | Kim | Okay, so I'll do the orange on the top. |
| 9 | 02:07 | Francesca S. | Yeah. |
| 10 | 02:07 | Kim | Alright. |
| 11 | 02:13 | Kim | Is that it? Oh, I need an all orange one. |
| 12 | 02:33 | Francesca S. | Is that all? [They have eight towers in two groups. Set 1: OOOO, OOOB, OOBB, OBBB and Set 2: BBBB, BBBO, BBOO, BOOO] |
| 13 | 02:35 | Kim | Wasn't there sixteen? |
| 14 | 02:36 | Francesca S. | Yeah, yeah. |
| 15 | 02:36 | Kim | There was sixteen. Two and two. Two there and two over there. |
| 16 | 02:44 | Francesca S. | No, now we have to interchange them. |
| 17 | 02:47 | Kim | Like..... |
| 18 | 02:53 | Francesca S. | Yeah, like this. [She has created BOBO ] |
| 19 | 02:59 | Kim | There's one. [She has created $O B B O$ ] Now we have to do two blue on the outside and two orange in the middle. |
| 20 | 03:04 | Francesca S. | Yeah. |
| 21 | 03:09 | Kim | And then orange, blue, orange, blue. |
| 22 | 03:12 | Francesca S. | Yeah. |
| 23 | 03:19 | Kim | Um....So we have four more... What else? There were sixteen. We're missing four somehow. [They now have 12 towers separated into three groups. Set 1: OOOO, OOOB, OOBB, OBBB; Set 2: BBBB, BBBO, BBOO, BOOO; Set 3: BOOB, OBBO, BOBO, |


|  |  |  | OBOB] |
| :---: | :---: | :---: | :---: |
| 24 | 03:48 | Francesca S. | Yeah. |
| 25 | 04:00 | Kim | Two orange... |
| 26 | 04:05 | Francesca S. | Yeah, we already did those. |
| 27 | 04:13 | Kim | Did we do this? [She holds $B O B B$ ] |
| 28 | 04:19 | Francesca S. | No. Yeah, we didn't do those. |
| 29 | 04:26 | Kim | And then two blue. [She creates $B B O B$ ] |
| 30 | 04:32 | Francesca S. | Yeah. |
| 31 | 04:33 | Instructor | Okay, you are missing something. [They now have 14 towers separated into three groups. Set 1: OOOO, $O O O B, O O B B, O B B B$; Set 2: BBBB, BBBO, BBOO, BOOO; Set 3: BOBB, BBOB, OBOB, BOBO, BOOB, OBBO] |
| 32 | 04:34 | Kim | I know. We're missing... |
| 33 | 04:36 | Instructor | So, I want you to switch over to the other organization. Just 'cause I want to compare it to what they're doing. And the other organization is... First, there's zero blue. How many do you have like that? One. Now you're going to do all the ones that have one blue. So... |
| 34 | 04:57 | Kim | One blue. [She picks up the BOOO tower.] |
| 35 | 05:00 | Instructor | Now, that's important because that should tell you what are you missing? |
| 36 | 05:07 | Kim | One in the middle? |
| 37 | 05:08 | Instructor | And? Yeah, so do that one. Do the one that you are missing. |
| 38 | 05:10 | Kim | Oh, okay. And then orange. [She creates an $O O B O$ |


|  |  |  | tower.] And then there would be the blue one there. |
| :---: | :---: | :---: | :---: |
| 39 | 05:25 | Instructor | No, that's the opposite. |
| 40 | 05:25 | Francesca S. | Yeah, that's the opposite I'm saying. [Points to $B O B B]$. |
| 41 | 05:26 | Instructor | Right. Yeah, the opposite of that one. Exactly right. |
| 42 | 05:27 | Francesca S. | Yeah, we're missing those. |
| 43 | 05:27 | Instructor | Okay. And now, did you see when we pulled those out it was immediately obvious what was missing? |
| 44 | 05:28 | Francesca S. | Yeah. |
| 45 | 05:31 | Instructor | Alright. Okay, so you can organize them. How are you going to organize them to show me that you have them all now? Okay, so there's your one blue case. Right? That's a good proof that there's nothing else. [She points to the towers that they have created with one blue: $B O O O, O B O O, O O B O, O O O B]$ Okay, there's zero blue, the one blue. Now do the two blue, the three blue, and the four blue. |
| 46 | 05:54 | Kim | Okay, here's two blue. [She moves $O O B B$.] |
| 47 | 05:55 | Francesca S. | Two blue [She hands Kim two towers: OBBO and BOOB.] |
| 48 | 05:59 | Francesca S. | Yeah, but these are two orange too. [She holds $O B O B$ and $B O B O$.] |
| 49 | 06:02 | Kim | Does that matter? |
| 50 | 06:07 | Francesca S. | Well, any of them that have two blue would have... |
| 51 | 06:08 | Kim | Yeah. |
| 52 | 06:17 | Kim | What if they have... I have a question. |
| 53 | 06:23 | Kim | Alright and then we have three blue. |


| 54 | 06:30 | Kim | Okay, I have a question. |
| :---: | :---: | :---: | :---: |
| 55 | 06:31 | Instructor | Yes. |
| 56 | 06:32 | Kim | So you know how there's two blues? |
| 57 | 06:33 | Instructor | Yes. |
| 58 | 06:34 | Kim | Do we do the same case for two oranges? Or just two blues? |
| 59 | 06:35 | Instructor | Well, you tell me. If you were focusing on orange, would the two orange case .... What would the two orange case look like compared to what this looks like. |
| 60 | 06:48 | Kim | Oh, they're the same. |
| 61 | 06:50 | Instructor | Okay, so you don't have to do it. |
| 62 | 06:51 | Kim | Oh. |
| 63 | 06:52 | Instructor | Okay, so, here's the... No blues, one blue, two blues, what's next? [She indicates the first three sets that Kim and Francesca have created: Set 1: OOOO; Set 2: BOOO, OBOO, OOBO, OOOB; Set 3: OOBB, OBBO, $B O O B, B O B O, O B O B, B B O O]$ |
| 64 | 06:59 | Kim | Three blues. |
| 65 | 07:00 | Instructor | Three blues. Organize it so we can see that you know you have it all. |
| 66 | 07:00 | Francesca S. | [She takes the BBBO tower and the OBBB tower and places them next to each other.] |
| 67 | 07:05 | Instructor | You've got some more. |
| 68 | 07:07 | Francesca S. | Oh, yeah. [Laughs] [Take BBOB and BOBB and places them in between $B B B O$ and $O B B B$ ] |
| 69 | 07:10 | Instructor | Now this sort of relates to what you said here [points |


|  |  |  | to the set of six towers that have two orange cubes <br> and two blue cubes]. The three blue case [points to <br> the four towers with three blue cubes and one orange <br> cube], could also be called? If you were focusing on <br> orange, instead of blue, you could call this? |
| :--- | :--- | :--- | :--- |
| $\mathbf{7 0}$ | $07: 18$ | Francesca S. | One orange. |


| 80 | 08:51 | Kim | Do we have to separate? Cause didn't we do it last time were we separated them? |
| :---: | :---: | :---: | :---: |
| 81 | 08:55 | Instructor | Yeah, you proved that there were only six by making two like sub-cases. |
| 82 | 09:01 | Kim | Oh, 'cause the blues are stuck together. |
| 83 | 09:02 | Instructor | Yeah, so you can separate them that way. But for now I just wanted the answer how many of each kind did you have. |
| 84 | 09:08 | Kim | Oh, okay. |
| 85 | 09:28 | Kim | Sixteen. [The camera focuses on Kim's paper. She has written: Blue Case; 0 blue 1; 1 blue 4; 2 blue 6; 3 blue 4; 4 blue 1; 16] |
| 86 | 09:30 | Instructor | Okay, you wrote it down, you wrote it down. |
| 87 | 09:32 | Francesca S. | Yeah. |
| 88 | 09:35 | Instructor | Okay, okay. And you said all orange which is the same as all blue. Right? |
| 89 | 09:39 | Francesca S. | Yeah. |
| 90 | 09:40 | Instructor | I mean when you wrote that case down. Okay. And it adds up to sixteen. And that was just... I wanted you to write that down. We're going to think about that later on. But for now I just wanted you to re-iterate, in your papers, the reason why the answer turned out to be two to the $n$. |
| 91 | 9:59 | Instructor | Remember um... Rebecca was talking about it and other people were. So, you know. Give it your best shot as to a good explanation. |
| 92 | 10:07 | Kim/ <br> Francesca S. | [They continue to write in their notebooks. The camera does not show what they are writing.] |


| $\mathbf{9 3}$ | $10: 17$ | Jessica | We did "why is it two to the $n$ ?" It's two because the <br> two represents the number of colors. Two is the base. <br> [The camera is focused on Kim and Francesca S. but <br> we hear Jessica state this in the background.] |
| :--- | :--- | :--- | :--- |
| $\mathbf{9 4}$ | $10: 24$ | Kim | That's true. There are two colors. [laughing] |$|$| $10: 29$ |
| :--- |
| $\mathbf{9 5}$ |


| 107 | 12:27 | Instructor | Um, well, they discussed it over there. I think your group didn't do it but they decided to move on to that. And.. I'm not sure if... did the whole class discuss that? |
| :---: | :---: | :---: | :---: |
| 108 | 12:36 | Francesca S. | No. |
| 109 | 12:36 | Instructor | Okay, then I will ask them to explain to you and you don't have to write it down. Just listen to their explanation. You don't have to write down ahead of time. After they explain it to you, I'll want you to write it down. |
| 110 | 12:45 | Francesca S. | Okay. |
| 111 | 12:45 | Kim/ <br> Francesca S. | [Kim and Francesca S. sit silently.] |
| 112 | 13:06 | Kim | I think we briefly said it last time. |
| 113 | 13:08 | Francesca S. | Yeah, but I don't remember. |
| 114 | 13:10 | Kim | Weren't they like three to the $n$ ? |
| 115 | 13:12 | Instructor | While we are waiting for them, to do $m$ to the $n$. Two to the $n$. |
| 116 | 13:18 | Kim | Okay. |
| 117 | 13:19 | Instructor | And you said what? I heard them say. Two is the number of colors and four is the... |
| 118 | 13:23 | Francesca S. | And then four is the height. |
| 119 | 13:23 | Instructor | ... height of the towers. Now there was a little more to it than that, which we recalled that Rebecca said. |
| 120 | 13:30 | Instructor | Do you remember when she was doing her little proof? She started out with something like... I'm going to take this away and put it back later. [She takes apart one of the towers previously built.] She started out with there's one tall towers, right? [She |

$\left.\begin{array}{|l|l|l|l|}\hline & & & \begin{array}{l}\text { puts a single blue cube down and a single orange } \\ \text { cube down.] You actually missed this discussion in } \\ \text { class on Wednesday so it's good to go over. [Talking } \\ \text { to Francesca S.] That's it, right? [pointing to the } \\ \text { single orange and single blue cube] This is it. }\end{array} \\ \hline \mathbf{1 2 1} & 13: 46 & \text { Kim } & \begin{array}{l}\text { Oh, then don't you add one and then it would be like } \\ \text { one.... }\end{array} \\ \hline \mathbf{1 2 2} & 13: 50 & \text { Instructor } & \begin{array}{l}\text { You add... Well, sort of, yes. }\end{array} \\ \hline \mathbf{1 2 3} & 14: 53 & \text { Instructor } & \begin{array}{l}\text { Now this is... How does this relate to what we are } \\ \text { doing induction? Here's step one - } n \text { tall towers. One } \\ \text { tall towers, right? Step one, you pick some low }\end{array} \\ \text { number. } \\ \hline \mathbf{1 2 4} & & & \text { Instructor }\end{array} \begin{array}{l}\text { Step two: you say "I'm at some height." We don't } \\ \text { have to think about that too much, but. What do you } \\ \text { do for each one of these? You started to say it... You } \\ \text { can either do what or what? }\end{array}\right\}$

| 129 | 14:44 | Instructor | Exactly. Right, each one is twice as many. Okay? So that's the idea. |
| :---: | :---: | :---: | :---: |
| 130 | 14:52 | Instructor | Now I'd like you to go over. You guys are ready to explain? |
| 131 | 14:55 | Jessica | Yes. |
| 132 | 14:56 | Instructor | $M$ to the $n$ for $m$ equals three. So they're going to talk about three colors. [Francesca S., Kim, and the instructor move over to Jamie and Jessica's group.] |
| 133 | 14:59 | Kim | Three colors! |
| 134 | 15:01 | Instructor | Yes, now we're doing towers where you got three colors to choose from. And they're going to explain their answer and you're going to ask questions and make sure that they explain it to your total satisfaction. |
| 135 | 15:12 | Jamie | Okay, so you have three colors: yellow, blue and white. So $m$ represents the colors. So you have three. And if you want to make it three tall... [There are three single cubes in a row: yellow, blue, and white. In front of each of these single cubes, there are three single cubes: yellow, blue, and white.] |
| 136 | 15:23 | Instructor | Well, start with one tall. |
| 137 | 15:24 | Jamie | One tall, okay, one tall would be three to the one which is three. |
| 138 | 15:28 | Jessica | Because you only have three choices. |
| 139 | 15:28 | Jamie | 'Cause you can only have three choices. |
| 140 | 15:32 | Jessica | Then when you get to two tall, you have three choices per the one that you already have. So you have the yellow can either be yellow-yellow, yellowblue, or yellow-white. Blue can be blue-yellow, blueblue, or blue-white. And white can be white-yellow, |


|  |  |  | white-blue, or white-white. So, since you have three choices each time, it's three squared this time... um.. I'm trying to think... |
| :---: | :---: | :---: | :---: |
| 141 | 15:59 | Instructor | That's it! Three squared is... |
| 142 | 16:01 | Jessica | Three squared is nine. So you have nine total choices, nine total ways that you can do it. |
| 143 | 16:09 | Instructor | And then... Now I am going to you guys [Talking to Kim and Francesca S.]. And if you were going to start off with those three tall, now she explained how you got nine of those, right? So what are you going to do with those nine to get to the next level? |
| 144 | 16:17 | Francesca S. | Nine squared. |
| 145 | 16:19 | Instructor | Well, it's nine squared. It's..... |
| 146 | 16:20 | Kim | It would be nine cubed. |
| 147 | 16:21 | Instructor | No it would be... |
| 148 | 16:23 | Kim | I mean three cubed. |
| 149 | 16:24 | Instructor | Each one of those gets [inaudible] |
| 150 | 16:29 | Jessica | So like if you did that...It basically becomes like the whole tree-type thing. Because then for this yellow, you would have to have yellow, white, and blue. [Placing a blue, white, and yellow cube above the yellow, yellow cubes.] And for that blue you would have do the same thing that we did here with the yellow, white, and blue. |
| 151 | 16:45 | Instructor | So you can see that if you had to build them it would get complicated but once you see the pattern, you don't need to build them. |
| 152 | 16:50 | Jessica | Yeah. That goes there, that goes there, and this one goes here. [Placing a blue, white, and yellow cube |


|  |  |  | above the yellow, blue cubes.] And like it just continues to go on and on forever. |
| :---: | :---: | :---: | :---: |
| 153 | 16:58 | Instructor | Okay, so, that's great! There's your induction, right? No matter where you are at, you can always go to the next step with times three. Okay. So are you satisfied with that? |
| 154 | 17:06 | Francesca S. | Yes. |
| 155 | 17:06 | Instructor | Okay, so now we're going to move on to the next problem. Okay, so.... We're going to leave this for a moment and move on to the next combinatorics question, which is on the next slide. Here it is. The pizza problem. [The slide says: <br> The Pizza Problem <br> - There are four possible pizza toppings: <br> Sausage <br> Peppers <br> Pepperoni <br> Mushrooms <br> - You can have a plain pizza (no toppings), or a pizza with any combination of the above toppings. How many pizzas is it possible to make? <br> Part 1: What's the answer? <br> Part 2: Convince me that your answer is correct. ] |
| 156 | 17:22 | Kim | Pizza! [Kim and Francesca S. return to their seats.] |
| 157 | 17:26 | Class | [There is a conversation with the instructor, the class, and the videographer. They discuss that they had already did this problem in another class but they realize that this problem is different than the problem in the other class. This conversation is not transcribed.] |
| 158 | 18:16 | Instructor | This is what we want... the usual. Four possible toppings. So you can choose a plain pizza, right? Or you can put any of the toppings. You can put all of |


|  |  |  | them; you can put some of them... How many pizzas <br> can you make? [inaudible] |
| :--- | ---: | :--- | :--- |
| $\mathbf{1 5 9}$ | $18: 29$ | Kim | Okay. |
| $\mathbf{1 6 0}$ | $18: 30$ | Instructor | Any questions? |
| $\mathbf{1 6 1}$ | $18: 35$ | Kim | [Talking to the videographer.] It might take awhile. <br> I'm just warning you. So alright. Well, we have to <br> use the plain pizza. |
| $\mathbf{1 6 2}$ | $18: 50$ | Kim | Do you want to do a tree first? Will that help? |
| $\mathbf{1 6 3}$ | $18: 52$ | Francesca S. | Yeah. |
| $\mathbf{1 6 5}$ | $18: 53$ | Kim | Okay. Maybe I'll make a pizza. |
| $\mathbf{1 6 6}$ | $19: 05$ | Kim | Francesca S. <br> It's probably easier for me to just list it. Like... |
| $\mathbf{1 6 9}$ | $19: 59$ | Instructor camera focuses on Kim's paper. She is writing |  |
| in her notebook.] So you can have sausage....you |  |  |  |
| can have peppers... you can have pepperoni. I don't |  |  |  |
| even know how to spell pepperoni. [Videographer |  |  |  |
| says "That's a lot of p's."] Like Mississippi. |  |  |  |
| Mushroom.... [Kim has drawn a big circle on her |  |  |  |
| paper. Off of this circle are three branches, each one |  |  |  |
| of these branches is labeled as sausage, peppers, |  |  |  |$|$| pepperoni, or mushroom.] |
| :--- |


| 170 | 20:00 | Francesca S. | Or they could have four? |
| :---: | :---: | :---: | :---: |
| 171 | 20:01 | Instructor | Think of all the different... if that's a pizza place you're going to, what are the choices that you could have? You could have with two toppings or three toppings... [inaudible]. |
| 172 | 20:08 | Francesca S. | Oh, so you want us to do all of them? |
| 173 | 20:10 | Instructor | All the different ways that you can do it - including plain. |
| 174 | 20:12 | Francesca S. | Okay. |
| 175 | 20:15 | Kim | And then peppers and mushrooms. [Moves to next option (branch): peppers. From this branch, she creates two more branches: pepperoni and mushrooms] |
| 176 | 20:28 | Kim | Pepperoni and... mushroom. [Moves to next option (branch): pepperoni. Creates one branch from pepperoni-mushrooms] And then by the time you get to mushroom, there's like nothing. Like they've already been in all the groups. |
| 177 | 20:45 | Kim | Then you have plain pepperoni, then you have plain sausage, plain peppers. [Writes down, in a separate list, pepperoni, sausage, peppers] |
| 178 | 21:02 | Kim | Okay, so that's one, two, three, four, five, six, seven, eight, nine, and then.... [Labels the options identified: 1: Plain pizza, 2: pepperoni, 3: sausage, 4: peppers, 5: sausage-pepperoni, 6: sausagepeppers, 7: sausage-mushrooms, 8: pepperspepperoni, 9: peppers-mushrooms, 10: pepperonimushrooms, 11: mushrooms]. I have eleven. Have you gotten that? |
| 179 | 21:21 | Francesca S. | No, I'm still doing it. |


| 180 | 21:26 | Instructor | [The videographer asks if she can help. The instructor responds to the videographer.] No. You can't say anything. |
| :---: | :---: | :---: | :---: |
| 181 | 21:30 | Kim | [Addressing Instructor] I have a question then. Did I do it wrong? |
| 182 | 21:35 | Instructor | You know I don't answer questions like that. Okay. So, what did you do? |
| 183 | 21:41 | Kim | Ok, well, I have to count for the plain pizza. [Points to \#1 - plain pizza] |
| 184 | 21:43 | Instructor | Now, first, you two were talking it over? Or are you doing it separately? |
| 185 | 21:46 | Francesca S | We kind of doing it separately. |
| 186 | 21:47 | Kim | [Points to Francesca S.] Yeah. |
| 187 | 21:47 | Instructor | Okay, fine. That's fine. So, plain pizza, that's one. Okay. |
| 188 | 21:51 | Kim | Then, I have plain pepperoni. |
| 189 | 21:53 | Instructor | Okay, which means pepperoni is the only topping on it? |
| 190 | 21:54 | Kim | Yeah, and then plain sausage and then plain peppers. So those are my four, like, you know, like? |
| 191 | 22:01 | Instructor | Another question - how come you left out mushroom? |
| 192 | 22:04 | Kim | Oh! Ok. Mushroom. [Adds mushroom to list of single-topping pizzas, labels it \#5] |
| 193 | 22:07 | Instructor | Alright, so those are like, this is like your one topping pizzas? You're telling me? [Indicating list of single-topping pizzas] |
| 194 | 22:11 | Kim | Yeah. And then two toppings. [Points to sausage - |


|  |  |  | peppers, sausage - pepperoni, and sausage peppers] |
| :---: | :---: | :---: | :---: |
| 195 | 22:16 | Instructor | Sausage with all those three things. Okay, that's very nice. And peppers with those three things, ok. |
| 196 | 22:20 | Kim | So, those are all two. |
| 197 | 22:22 | Instructor | Ok. |
| 198 | 22:23 | Kim | Oh, I forgot I have to do the three toppings. |
| 199 | 22:25 | Instructor | Okay. So, it doesn't sound like you have any questions, seem to be doing everything okay. |
| 200 | 22:28 | Kim | Ok. Renumber... [She erases the numbers she has placed in front of each pizza.] |
| 201 | 22:45 | Francesca S. | [The camera focuses on her paper. Listed in her notebook: 1) Plain, 1 Topping: 2) S, 3) Pep, 4) Pepperoni, 5) Mushrooms, 2 Toppings: 6) $S, P$, 7) $S$, Pepperoni, 8) S, Mush, 9) Pep, Pepperoni, 10) Pep, mushrooms] I'm just doing with the one topping, then two topping, then three topping, then four topping. Like each different - that's like the easiest way to do it. |
| 202 | 23:08 | Francesca S. | [Adds 11) Pepperoni to list in group containing 2 toppings. With her pencil, she silently goes over her two topping pizzas. She checks these pizzas by starting with sausage and running through her one topping pizzas. That is, she points to sausage (" $S$ ") and then points to peppers "Pep". She points to sausage (" $S$ ") and then points to pepperoni. She points to sausage (" $S$ ") and then points to mushroom. She then moves to peppers "Рep" and points to pepperoni. She points to peppers "Pep" and points to mushroom.] |
| 203 | 23:47 | Francesca S. | [Erases Pepperoni as 11 on list for two toppings] |


|  |  |  | [Writes Mushroom of \#11 and then erases it.] |
| :---: | :---: | :---: | :---: |
| 204 | 23:58 | Kim | How many do you have so far? |
| 205 | 23:59 | Francesca S. | Ten. [Erases the number 11.] |
| 206 | 24:06 | Francesca S. | [Starts a new list - heading: 3 topping] |
| 207 | 24:14 | Francesca S. | [Writes down in new list: 11) S, Pep, Pepperoni] |
| 208 | 24:21 | Kim | So when you get to three toppings, there would only be one right? You really can't.... |
| 209 | 24:28 | Francesca S. | Yeah, but you have to put like mushroom. Cause there's sausage, peppers, and pepperoni, but where do you stick the mushrooms? You know what I mean? Like, you have to make like a new one. |
| 210 | 24:40 | Kim | You mean for the four toppings? |
| 211 | 24:42 | Francesca S. | No for the three topping. |
| 212 | 24:43 | Kim | Oh, I forgot yeah, I forgot there's four. Okay. |
| 213 | 24:47 | Francesca S. | Yeah. |
| 214 | 24:49 | Francesca S. | [Continues with list; Writes down: 12) S, Pep, Mushrooms] |
| 215 | 25:07 | Instructor | [To Francesca S.] Ok, so, I looked at Kim's. Now tell me how you're doing it. |
| 216 | 25:09 | Francesca S. | I'm just doing it by like, one-topping, two-topping, three-topping. And then seeing if I miss any. |
| 217 | 25:15 | Instructor | Ok. |
| 218 | 25:16 | Instructor | [Points to Plain (\#1)] Alright. And then you started with zero toppings, right? |
| 219 | 25:18 | Francesca S. | Yeah, plain topping. |
| 220 | 25:19 | Instructor | Ok. |


| 221 | 25:23 | Francesca S. | [Continues with list; Writes down: 13) Pep, Pepe, Mushroom] |
| :---: | :---: | :---: | :---: |
| 222 | 25:42 | Kim | How many do you got for three? |
| 223 | 25:45 | Francesca S. | [Writes down 14 on list] I'm on the 14th. |
| 224 | 25:46 | Kim | Alright, so you've got pepperoni, peppers, mushroom, right? |
| 225 | 25:50 | Francesca S. | Wait, say that again. |
| 226 | 25:51 | Kim | Pepperoni, peppers, and then mushroom. |
| 227 | 24:53 | Francesca S. | Yeah, I have that. |
| 228 | 25:55 | Kim | Sausage, pepperoni, peppers. |
| 229 | 25:57 | Francesca S. | Yeah. |
| 230 | 25:58 | Kim | Mushroom, sausage, peppers. |
| 231 | 26:01 | Francesca S. | Yeah, I have that. |
| 232 | 26:02 | Kim | Or did I repeat it? |
| 233 | 26:03 | Francesca | Wait. Mushroom, sausage, pepper - yeah, I have that. |
| 234 | 26:06 | Kim | And then mushroom, pepperoni, I didn't finish the third one. Mushroom, pepperoni and.... peppers did I repeat that? Yeah, that was the first one. |
| 235 | 26:18 | Francesca S. | [Continues with list; Writes down: 14) S, Pepe, Mushrooms] |
| 236 | 26:21 | Kim | Mushroom, pepperoni.... Sausage. Okay. |
| 237 | 26:31 | Francesca S. | I got four of them. |
| 238 | 26:32 | Kim | Yeah, that's four, right? |
| 239 | 26:33 | Francesca S. | Yeah. How much did you get for the two? |


| $\mathbf{2 4 0}$ | $26: 37$ | Kim | Two, I got - one, two, three, four, five, six, seven? |
| :--- | :---: | :--- | :--- |
| $\mathbf{2 4 1}$ | $26: 45$ | Francesca S. | I got six, wait, I only got five. |
| $\mathbf{2 4 2}$ | $26: 52$ | Francesca S. | [Clarifying \#12 and \#14 to the videographer.] That's <br> sausage, pepper, mushroom [\#12 S, Pep, <br> Mushrooms] That's sausage, pepperoni, mushroom <br> [\#14 S, Pepe, Mushroom]. |
| $\mathbf{2 4 3}$ | $27: 00$ | Kim | Alright, so I got pepperoni and then...Alright, so, <br> mushroom...Oh... wait... <br> [Written in Kim's notebook: 1: Plain pizza, 2: <br> pepperoni, 3: sausage, 4: peppers, 5: mushrooms. <br> Then around the circle is 4 groups - 6: sausage- <br> pepperoni, 7: sausage-peppers, 8: sausage- <br> mushrooms, 9: peppers-pepperoni, 10: peppers- <br> mushrooms, 11: pepperoni-mushrooms, 12: <br> mushrooms] |
| $\mathbf{2 4 4}$ | $27: 11$ | Kim | This one isn't a two-topping. [Crosses out \#12 - <br> mushrooms] |
| $\mathbf{2 4 5}$ | $27: 20$ | Kim | Because mushroom that would just be by itself, so, <br> you really can't do mushroom by itself. So, I crossed |
| that one out. Alright. |  |  |  |


|  |  |  | peppers, sausage and mushrooms, peppers and pepperoni, peppers and mushroom and then pepperoni and mushroom. |
| :---: | :---: | :---: | :---: |
| 252 | 27:57 | Francesca S. | [Francesca S. agreed to each pizza on Kim's list except for the last one.] Oh, okay, yeah, that is the one I missed. [Adds Pepe, Mushrooms to the 2topping list] |
| 253 | 28:02 | Instructor | So, does that mean you're done? |
| 254 | 28:03 | Kim | I haven't thought about it. I wasn't... |
| 255 | 28:07 | Instructor | Ok, I'll just let you go. Then tell me when you're both think that you're in agreement that you're done and what you got. |
| 256 | 28:13 | Kim | Ok, so the first one is plain pizza because we've got to account for the plain one. |
| 257 | 28:17 | Francesca S. | Yeah. |
| 258 | 28:18 | Kim | So, that's one. And then, all pepperoni, all sausage, all peppers, all mushroom. |
| 259 | 28:25 | Kim | And then sausage-pepperoni, sausage-peppers, sausage-mush... |
| 260 | 28:30 | Kim | How many do you have all together actually? |
| 261 | 28:32 | Francesca S. | 16. |
| 262 | 28:34 | Kim | 1,2,3,4,5,6,7,8,9,10,11,12,13, 14, 15,16. Ok. |
| 263 | 27:53 | Kim | I think we're done. [To Instructor] I think we are done. |
| 264 | 28:58 | Instructor | Ok. |
| 265 | 29:00 | Kim | [To Francesca S.] Right? |
| 266 | 29:00 | Francesca S. | Yeah. |


| $\mathbf{2 6 7}$ | $29: 01$ | Kim | You would agree that we're done? |
| :--- | ---: | :--- | :--- |
| $\mathbf{2 6 8}$ | $29: 03$ | Videographer | What did you guys get? |
| $\mathbf{2 6 9}$ | $29: 04$ | Kim | 16. |
| $\mathbf{2 7 0}$ | $29: 05$ | Instructor | Sixteen. And you organized them by number of <br> toppings. Am I right? |
| $\mathbf{2 7 1}$ | $29: 10$ | Kim | Yeah even though mine's all sloppy. |
| $\mathbf{2 7 2}$ | $29: 12$ | Instructor | Well, you'll have time to do it neat, but let's have a <br> table just like you did for the other one. How many <br> plains did you get? How many two-topping, one- <br> topping, how many two-topping all the way up. Do <br> you know what I'm saying? And then think of a |
| convincing argument that you have them all. |  |  |  |


| 283 | 30:27 | Francesca S. | Yeah, 16. |
| :---: | :---: | :---: | :---: |
| 284 | 30:30 | Kim | Ok, now we have to make an argument on why we're right. |
| 285 | 30:43 | Francesca S. | [Pointing in her notebook] It probably has something to do with this. |
| 286 | 30:46 | Kim | Oh, $m$ to the $n$ ? |
| 287 | 30:48 | Francesca S. | Something like that. |
| 288 | 30:54 | Kim | If you have one pizza, wouldn't it be like... [Inaudible] |
| 289 | 30:58 | Francesca S. | No, it would have to do with the toppings. |
| 290 | 30:59 | Kim | Oh. |
| 291 | 31:01 | Francesca S. | So it's like 4 toppings. |
| 292 | 31:04 | Instructor | What are you saying? |
| 293 | 31:07 | Francesca S. | No, I'm trying to think of something that has to do with like $m$ to the $n$. |
| 294 | 31:10 | Instructor | Ah, okay. |
| 295 | 31:11 | Francesca S. | Like, if that's the reason for... [inaudible] |
| 296 | 31:14 | Instructor | Well, then. My next question was.. .So what do you think the answer would be if you had five toppings or if you had three toppings? |
| 297 | 31:22 | Francesca S. | Yeah, that's what I'm not sure 'cause I don't know like the only way to get 16 would be four to the second, four to the two. |
| 298 | 31:29 | Instructor | So, if that's true, what do you think you would get if there was only three toppings to choose from? |
| 299 | 31:36 | Francesca S. | Would it be nine? I don't know if it would be nine. |


| 300 | 31:38 | Instructor | Well, why don't you do a three topping case and see what you get? |
| :---: | :---: | :---: | :---: |
| 301 | 31:42 | Kim | Three toppings. Wait, didn't we already... |
| 302 | 31:45 | Francesca S. | If its three toppings, then... |
| 303 | 31:47 | Instructor | No, I mean you only have three toppings to choose from. |
| 304 | 31:49 | Kim | Oh. |
| 305 | 31:50 | Instructor | So leave off the pepperoni or something. Alright. Sausage, peppers and mushroom. What if you only had three, how many pizzas could you get? Alright, [to Francesca] I think that's what you were saying, right? |
| 306 | 31:59 | Francesca | Yeah. |
| 307 | 31:59 | Instructor | Something to do with the toppings, so the question is what do you do with the toppings. Okay? |
| 308 | 32:05 | Kim | Maybe it's the toppings like you are raising it to the toppings. It's like, 'cause you're having one pizza but like you can raise it to however many toppings there are. |
| 309 | 32:16 | Francesca S. | Yeah, something like that. |
| 310 | 32:25 | Kim | [Looks out the window] Sorry, I'm a little distracted right now. |
| 311 | 32:27 | Francesca S. | I know what is going on out there? |
| 312 | 32:37 | Kim | If you raise the one pizza to the... cause there's 16 ways of doing it. But you wouldn't raise one to the sixteenth - that wouldn't make sense. |
| 313 | 32:47 | Francesca S. | I know. I want to see what I get for three. |
| 314 | 32:51 | Kim | But then you would get sixteen different pizzas - |


|  |  |  | does that make sense? 'Cause you can have the <br> sixteen pizzas, like, at one time. |
| :--- | ---: | :--- | :--- |
| $\mathbf{3 1 5}$ | $33: 02$ | Francesca S. | So, that's four, so wait, four toppings, 16 pizzas. <br> So... |
| $\mathbf{3 1 6}$ | $33: 08$ | Kim | Four.... |, | So would that mean three toppings is nine pizzas? |
| :--- |, | 33:11 |
| :--- |
| $\mathbf{3 1 8}$ |


| 326 | 34:04 | Francesca S. | Okay, wait. That doesn't even make sense. Wait. How did I count nine? |
| :---: | :---: | :---: | :---: |
| 327 | 34:10 | Francesca S. | [Counting the entire list] Okay. It's eight. I think I counted that. |
| 328 | 34:14 | Instructor | Okay. I think so. |
| 329 | 34:18 | Instructor | [To Kim] Are you ok with that or not or are you not sure? |
| 330 | 34:20 | Kim | I don't even know what I'm doing right now [laughing]. |
| 331 | 34:22 | Instructor | So, I know, it's late, it's Friday. [To Francesca] All right, you look pretty confident about what you're doing, so you explain to Kim. |
| 332 | 34:30 | Francesca S. | No, I don't know now. Now it's wrong. |
| 333 | 34:33 | Instructor | Okay. Okay. Well, are you confident that it stopped at eight, that there really aren't nine? |
| 334 | 34:37 | Francesca S. | Yeah. |
| 335 | 34:38 | Instructor | So, you can explain your organization to her and convince her that there really are eight and then think about what your gonna do with that information. |
| 336 | 34:44 | Francesca S. | Yeah. |
| 337 | 34:46 | Instructor | So, then you think well for three its eight, for four its sixteen - can you think of a different kind of pattern? |
| 338 | 34:55 | Francesca S. | Okay, so it has to do with 8 and 16. Like something like that cause four is 16 . [Writes in notebook 4-16; 3-8] |
| 339 | 35:04 | Kim | But are we using the three toppings or are we using like? [Inaudible] |
| 340 | 35:07 | Francesca S. | No, we are doing.... I just wanted to see if like there |


|  |  |  | was a pattern, do you know what I mean? |
| :---: | :---: | :---: | :---: |
| 341 | 35:12 | Kim | Oh. What would have happened if there was two? Would that be four? |
| 342 | 35:17 | Francesca S. | Probably, yes. |
| 343 | 35:29 | Francesca S. | [Writes in notebook: S, P; Plain] There would only be two. |
| 344 | 35:33 | Kim | So, then if you put five. |
| 345 | 35:34 | Francesca S. | No, no, no, there would be four. [She erases what she had written previously] 'Cause you could have sausage, peppers, and then sausage/peppers. No actually there would be three, no, wait. Four, there would be four. <br> [Writes a new list that reads: <br> Plain-1 <br> S <br> P - 2 <br> $S, P-1]$ |
| 346 | 35:48 | Kim | So then four [inaudible], then two goes to four so then one topping... |
| 347 | 35:59 | Francesca S. | Was two. |
| 348 | 36:01 | Kim | Two. Then zero toppings has to be one. And then what happens if you go to five? Does that automatically go to 32 ? [laughing] Do you want to count that? |
| 349 | 36:15 | Francesca S. | No. My brain hurts. [Writes in notebook 5-32 above 4-16; 3-8] |
| 350 | 36:24 | Kim | [To Instructor] We think we found a pattern. |
| 351 | 36:26 | Francesca S. | [Writes in notebook 1-2; 0-1; below 4-16; 3-8; 5-32] |
| 352 | 36:32 | Instructor | [The videographer asks if she can help. The instructor responds to the videographer.] No. You are not going to. [A conversation between the |


|  |  |  | videographer and the instructor occurs. Their conversation is not transcribed.] |
| :---: | :---: | :---: | :---: |
| 353 | 36:42 | Instructor | Ok, so, what have you got here? |
| 354 | 36:45 | Kim | Well the fifth one we kinda guessed. |
| 355 | 36:47 | Francesca S. | Yeah, we guessed it goes down by - you divide it by two [Indicating pattern for toppings: 32, 16, 8, 4, 2, 1] |
| 356 | 36:51 | Instructor | You divide it by two when you're going down what do you do when you're going up? |
| 357 | 36:54 | Francesca S. | Multiply by two. |
| 358 | 36:54 | Kim | Multiply? |
| 359 | 36:55 | Instructor | That wasn't a question, was it? |
| 360 | 36:57 | Kim | Multiply! |
| 361 | 36:58 | Instructor | Ok, so it's times two when you're going up. Okay. Okay, so this means [Pointing to 0-1] if you have no toppings, you can only make one pizza. That would be which pizza? |
| 362 | 37:09 | Kim | The plain one. |
| 363 | 37:11 | Instructor | [Pointing to 1-2] The plain one, okay. Okay, and if you have one topping... |
| 364 | 37:11 | Kim | You get to make two pizzas. |
| 365 | 37:12 | Instructor | Which is... |
| 366 | 37:13 | Kim | Whatever topping - two toppings? |
| 367 | 37:16 | Instructor | Whatever topping it is. Or... |
| 368 | 37:18 | Francesca S. | Plain. |
| 369 | 37:18 | Instructor | Plain, okay. And then with two toppings - [Pointing |


|  |  |  | to 2-4] I've seen the three, so you can go ahead and explain two to me. So two you've got mushrooms and you've got sausage. What are the four possible things you can do? |
| :---: | :---: | :---: | :---: |
| 370 | 37:30 | Francesca S. | Yeah, we did that already here. |
| 371 | 37:31 | Instructor | [Pointing to Plain - 1; 1 topping S; P; 2; 2 topping - $S, P-1]$ Oh, yeah. Plain, both, and one of each, very nice. Okay. So, you have the inductive rule here, right? It looks like to me. To add another topping, you multiply by two. Right? |
| 372 | 37:46 | Kim | So that's $n$ squared? I mean not $n$ squared. Two to the $n$. |
| 373 | 37:51 | Instructor | Two to the $n$. Okay. You have an explicit formula. So, now you're telling me that when you have $n$ toppings, the number of possible pizzas is...Ok, ok, now the next question is why? |
| 374 | 38:07 | Kim | Why? |
| 375 | 38:07 | Francesca S. | Wait, $n$ equals the number of toppings? |
| 376 | 38:10 | Instructor | Well, that's what you told me. Isn't that what you said? |
| 377 | 38:13 | Kim | Yeah. |
| 378 | 38:14 | Instructor | Okay, well, why is it two to the $n$ different pizzas and... Okay, so there's more questions coming up... |
| 379 | 38:20 | Kim | Because the $n$ represents the toppings. |
| 380 | 38:22 | Instructor | $N$ represents how many toppings there are. Yeah, but how comes it's two to the $n$ ? How come it's not 2 times $n$ ? How come it's not $n$ squared or some other thing? How come it's two to the $n$ ? What is it that makes that... |


| $\mathbf{3 8 1}$ | $38: 30$ | Kim | That's a good question. |
| :--- | ---: | :--- | :--- |
| $\mathbf{3 8 2}$ | $38: 32$ | Instructor | Now, now, I wanted to give you an example, but <br> before I do that, let's talk about something that these <br> guys [indicating other group] found out. Um, there's <br> a couple things they noticed that you may have <br> noticed also. But, no, before I do that you tell me <br> what you noticed. Two to the $n-$ does that look <br> familiar? Where have you seen that before? |
| $\mathbf{3 8 3}$ | $38: 51$ | Francesca S. | The tower problem. |
| $\mathbf{3 8 4}$ | $38: 53$ | Instructor | The tower problem. How is that - pizzas - like <br> towers? Did you notice you got 16 also? |
| $\mathbf{3 8 5}$ | $39: 02$ | Kim | Yeah, I noticed that too, actually. |
| $\mathbf{3 8 6}$ | $39: 04$ | Instructor | Okay, how is it like that? In fact, let's see. [A small <br> conversation occurs between the videographer, Kim, <br> and the instructor which is not transcribed.] |
| $\mathbf{3 8 7}$ | $39: 18$ | Instructor | I'm looking for- here, yeah, here's what you said <br> about the towers [Reading from Francesca's |
| notebook on towers problem] - you said 1, 4, 6, 4, 1. |  |  |  |$|$| Here's what you said about the pizzas. [Pointing to |
| :--- |
| Kim's notebook] |


| 393 | 39:49 | Instructor | Yeah, so. But now you gotta tell me why. What is it about pizzas and toppings that you keep getting the same answers? Pizzas and towers, rather... |
| :---: | :---: | :---: | :---: |
| 394 | 39:57 | Francesca S. | It can't be the number of pizzas. Is it because you can have a pizza that is plain and a pizza that has toppings? Is that what the two stands for? |
| 395 | 40:04 | Instructor | Work on that. You work on that. You talk about it and write it up and what I want you to tell me is exactly how these are the same. Okay? |
| 396 | 40:15 | Kim | Okay, so in the other problem, it was two to the nine. [Inaudible] |
| 397 | 40:21 | Francesca S. | This was two because of the number of colors and four was because the height. |
| 398 | 40:28 | Kim | The height, yeah. Right. |
| 399 | 40:32 | Kim | But there's always gonna be... so there's always gonna be a plain pizza. |
| 400 | 40:35 | Francesca S. | Yeah, I think it has something to do with that. There's always gonna be a plain one and one that has a topping. [They both write in their notebooks.] |
| 401 | 41:07 | Kim | Ok, she wants us to expand on... |
| 402 | 41:09 | Francesca S. | I know. |
| 403 | 41:15 | Kim | [To the instructor] Ok, after, I have a question. |
| 404 | 41:17 | Instructor | Go ahead. |
| 405 | 41:18 | Kim | After we're done taping, can she tell us the answer? [Indicating the videographer.] |
| 406 | 41:20 | Instructor | No. Cuz we're not done yet. [A conversation between the instructor, videographer, and Kim occurs but this conversation is not transcribed.] |


| 407 | 42:20 | Kim | Alright, well, I'm kind of stuck. |
| :---: | :---: | :---: | :---: |
| 408 | 42:21 | Francesca S. | Yeah, me too. |
| 409 | 42:26 | Kim | Maybe she'll put the answer on the board, what do you think? [laughing] |
| 410 | 42:31 | Instructor | This is what you guys told me, right? |
| 411 | 42:33 | Kim | Yeah. |
| 412 | 42:35 | Instructor | So what else do you have to say about this? |
| 413 | 42:41 | Instructor | Now this we said... <br> [The camera focuses on the board and the instructor has written the following: |
| 414 | 42:50 | Kim | So the two is the number of pizzas. |
| 415 | 42:54 | Francesca S. | Yeah, cause you [inaudible] |
| 416 | 42:57 | Instructor | Okay, well wait a minute. Two is... |
| 417 | 42:58 | Kim | You always have to have one plain and [inaudible] |
| 418 | 43:00 | Francesca S. | $N$ is the toppings, $n$ is the toppings... |
| 419 | 43:02 | Instructor | $N$ is like... What do you mean by the toppings? |
| 420 | 43:04 | Francesca S. | By the four toppings. |
| 421 | 43:05 | Instructor | Okay, so the number of toppings to choose from. [She writes on the board $n=\#$ toppings to choose] |
| 422 | 43:05 | Francesca S. | Yeah. |


| 423 | 43:18 | Instructor | Okay, two is... what would you say about the two? |
| :---: | :---: | :---: | :---: |
| 424 | 43:24 | Kim | Well, because that one had two colors. |
| 425 | 43:26 | Francesca S. | Yeah, the types of pizza - you can either have plain or one with toppings. |
| 426 | 43:29 | Kim | Oh, I have another question. Can it be thin crust or thick crust? That's another [laughing] |
| 427 | 43:37 | Instructor | Do you know what? We can do half pizzas. But we are not going to do that. |
| 428 | 43:38 | Jessica | We can't even figure out this! [laughing] |
| 429 | 43:44 | Instructor | You guys sort of started to say some stuff about the two but I didn't hear it exactly. So, when you're ready come up here and write what the two equals here. [on the board] |
| 430 | 43:52 | Francesca S. | Is it because it is one plain pizza and one pizza with toppings? |
| 431 | 43:56 | Instructor | Okay, so... |
| 432 | 43:58 | Francesca S. | There's only like two types. |
| 433 | 43:59 | Instructor | So what words do you want me to put up here? Two represents what? |
| 434 | 44:03 | Francesca S. | Yeah, like two types... I don't know. |
| 435 | 44:05 | Jessica | Two toppings. |
| 436 | 44:06 | Francesca S. | Yeah, like... |
| 437 | 44:08 | Kim | What about deep dish pizza? See now I'm in the pizza mode. |
| 438 | 44:10 | Instructor | Equals two types. [She writes on the board $2=$ two types] |


| 439 | 44:12 | Francesca S. | Yeah, like two types of pizzas. |
| :---: | :---: | :---: | :---: |
| 440 | 44:15 | Instructor | Two types. |
| 441 | 44:15 | Francesca S. | Like, one plain or either one that has toppings on it. |
| 442 | 44:18 | Instructor | Okay. [She continues to write on the board $2=$ two types; plain or topp.] |
| 443 | 44:20 | Jessica | We didn't even get that far. |
| 444 | 44:26 | Instructor | Okay, well if you didn't.... But you guys did see some of this so let's talk about it together before we move on. Those guys noticed and you noticed too, I thought, the $1,4,6,4,1$. |
| 445 | 44:36 | Jessica | Yeah, that we got. |
| 446 | 44:37 | Instructor | Okay, so and the word...Kim, what was the word we were talking about there? |
| 447 | 44:41 | Kim | Isomorphism. |
| 448 | 44:42 | Instructor | Yeah, so the isomorphism is you keep getting the same answers to the two different problems. Right? And the question is... I want to know more about isomorphism, which means that you got to tell me exactly why you are getting the same answers. |
| 449 | 44:56 | Instructor | And you are sort of getting there, right? They told me that this is two to the $n$ because two means two colors and $n$ is the height. <br> [She points to the formula for the towers problem on the board: $\begin{aligned} & 2^{n} \quad 2=2 \text { colors } \\ & n=\text { height }] \end{aligned}$ |
| 450 | 45:03 | Instructor | And over here you said $n$ is the number of toppings. So this is one component of the isomorphism. You're telling me... What are you telling me about the relationship between height and number of |


|  |  |  | toppings? <br> [She points to the formula for the pizza problem on the board: $\begin{aligned} & 2^{n} \quad 2=2 \text { types }- \text { plain or topp. } \\ & n=\# \text { of toppings to choose from }] \end{aligned}$ |
| :---: | :---: | :---: | :---: |
| 451 | 45:16 | Francesca S. | They equal each other? |
| 452 | 45:17 | Instructor | The height is the same as the number of toppings, for some reason, right? When you got a height of three, you get eight and when you have the number of toppings is three, you get eight. So that's part of the isomorphism. Okay? And now you see two's in both places. So there's something with two's also. This one is very straight forward, two colors. And this is ... I didn't quite get this. Two types? |
| 453 | 45:39 | Francesca S. | I don't know. |
| 454 | 45:40 | Instructor | Yeah, but you're getting there. |
| 455 | 45:43 | Jessica | It's a good idea. At least you guys had the idea. |
| 456 | 45:45 | Instructor | And you guys were actually... you saw there was something to do with the towers too. And you were saying that the different colors equaled the different toppings. |
| 457 | 45:53 | Jessica | The different toppings |
| 458 | 45:54 | Instructor | However, you didn't get two colors that way you got lots of colors. So you were actually saying... what were you saying? |
| 459 | 46:02 | Jessica | The difference between them or....? |
| 460 | 46:03 | Instructor | You were saying... you were saying it wasn't the height equal to the number of toppings, you said something else was equal to the number of toppings. |


| 461 | 46:15 | Jessica | What did we say was equal to the number of toppings? [To Jamie] The colors? |
| :---: | :---: | :---: | :---: |
| 462 | 46:20 | Instructor | Yeah, yeah, it was the colors. |
| 463 | 46:27 | Instructor | It was colors, right? You had four colors representing four different toppings. And then the heights did not represent the number of toppings. And you guys found something else that seemed a little more straightforward, actually. The height and the number of toppings seemed to have something to do with each other. |
| 464 | 46:43 | Instructor | Okay. So that's the next thing to be working on. And, let's see....we have almost 20 minutes so we have some time to think about that. So there's some relationship here and they confirmed it actually because they did the three topping pizzas, right? And you got eight. In fact, you did the two topping pizzas and got four and so on. So my suggestion would be maybe.... |
| 465 | 47:10 | Jessica | You have eight? |
| 466 | 47:11 | Instructor | For the three topping pizzas. For three... when there's three toppings to choose from. Right? We'll leave off mushrooms so there's only pepper, pepperoni and sausage to choose from. When you got three choices, there's only eight possible pizzas, right? |
| 467 | 47:21 | Francesca S. | Yeah. |
| 468 | 47:22 | Instructor | And in fact, you should have made it very clear when there's only two toppings. |
| 469 | 47:26 | Francesca S. | There's four |
| 470 | 47:27 | Instructor | Two toppings, there's four. And what are the four? |


| 471 | 47:31 | Francesca S. | Plain, and then there's sausage and peppers, and then there's...sausage and peppers separately, and then there's sausage and peppers together. So, four. |
| :---: | :---: | :---: | :---: |
| 472 | 47:40 | Instructor | Right so there's your four. One plain, one with both of them, one with one and one with the other one. There's your four. So that was an example that it's working just the way we thought it was working. It is the doubling pattern. |
| 473 | 47:54 | Instructor | Okay, so but I want something a little more specific. Like exactly. Those words don't quite do it although you are heading in the right direction. |
| 474 | 48:03 | Instructor | Okay, so exactly what is the isomorphism? To say that you have to be able say, well number of towers equals number of pizzas. That's one thing. Height equals number of toppings, that's another thing. What does that two have to do with it? And... and eventually you should be able to tell me - here's a pizza, here's a tower, they map onto each other. Right? Isomorphisms - the elements of one set have to map on to the elements of the other set. |
| 475 | 48:30 | Jessica | I only have one problem with it because I can't figure out anything.... If two is our base, there's nothing in the $n$, that's a whole number, that can give us six. |
| 476 | 48:46 | Instructor | Yeah, you're right two to the something doesn't give you six, here, either. But two to the something gives you the total number. Okay. |
| 477 | 48:57 | Jamie | Does it have anything to do with the triangle? |
| 478 | 49:00 | Instructor | Pascal's Triangle? Um... |
| 479 | 49:03 | Jessica | We were trying to see if it did at all... and we didn't see it. |


| 480 | 49:10 | Instructor | [inaudible] Well, you can write Pascal's triangle on the board. [Talking to Jessica] |
| :---: | :---: | :---: | :---: |
| 481 | 49:25 | Jessica | [Jessica goes to the board and writes Pascal's Triangle.] I don't even need a notebook [laughing]. <br> [She has written the following on the board: ```l 11 121 1331 14641 15101051]``` |
| 482 | 49:49 | Instructor | Yeah, you can stop with that row actually. Okay, now, you will look at your papers, you will look up there, and you will tell me what you see. |
| 483 | 49:55 | Kim | Patterns! |
| 484 | 49:56 | Instructor | Yeah, now look at your paper again and what else do you see up there? |
| 485 | 50:01 | Kim | Oh, 1, 4, 6, 1! |
| 486 | 50:02 | Instructor | Write it up there. Come on up and show us what you see. |
| 487 | 50:07 | Kim | Do you want me to do it? |
| 488 | 50:07 | Instructor | Yeah, I want you to do it - you 're the one that saw that. Draw a little arrow to what you were just talking about. |
| 489 | 50:12 | Kim | Can I use blue? |
| 490 | 50:13 | Instructor/ Jessica | Yes. |


| 491 | 50:15 | Kim | Okay. This one. [She draws an arrow to the row that contains 1, 4, 6, 4, 1] |
| :---: | :---: | :---: | :---: |
| 492 | 50:19 | Instructor | And where did you see that? |
| 493 | 50:21 | Kim | In the toppings and the... |
| 494 | 50:23 | Instructor | The toppings and the.... |
| 495 | 50:24 | Jessica | The towers. |
| 496 | 50:25 | Instructor | The towers. So what does that $1,4,6,4,1$ represent? |
| 497 | 50:31 | Jessica | Our total.... |
| 498 | 50:32 | Instructor | Well what did the one represent? You're sitting there by it Francesca so you can tell us. |
| 499 | 50:36 | Francesca S. | The one is um... the plain or the one with the four colors. |
| 500 | 50:41 | Jessica | With everything. |
| 501 | 50:42 | Francesca S. | Yeah, with everything. |
| 502 | 50:42 | Instructor | Okay, alright. So that's interesting 'cause you saw it in both cases. So that one is either, in terms of pizzas, it was the plain pizza. [Writes on the board. $1=$ plain] And in terms of towers, which tower was it? |
| 503 | 50:54 | Jessica | It was the one with that had all the colors. All the same colors. All blue or all white for us. |
| 504 | 50:58 | Instructor | All ... let's say all white which is no blue. [Writes on the board $1=$ all white ( 0 blue)] |
| 505 | 51:06 | Instructor | And the four for pizzas was which pizzas was that? |
| 506 | 51:10 | Jessica | Was one of each type of topping. The one topping pizzas. |


| 507 | 51:13 | Instructor | The one topping pizzas. [Writes on the board $4=1$ topping] And for the towers? |
| :---: | :---: | :---: | :---: |
| 508 | 51:21 | Jessica | The towers was the... um...was the single... |
| 509 | 51:30 | Instructor | How did we describe those in two words? |
| 510 | 51:32 | Jessica | In two words? I'm trying to think - it's not working. [laughs] |
| 511 | 51:38 | Instructor | A number and a color. |
| 512 | 51:42 | Jessica | For every.... |
| 513 | 51:48 | Francesca S. | It's like ... It would be like three orange, one white. |
| 514 | 51:51 | Instructor | So it's - so since we're focusing on the... the one blue. [Writes on the board $1=1$ blue] And six equals... What is the six? |
| 515 | 52:03 | Jessica | The six is... |
| 516 | 52:04 | Instructor | In terms of pizzas? |
| 517 | 52:08 | Jessica | Two topping pizzas. |
| 518 | 52:12 | Instructor | Okay. [Writes $6=$ two toppings; $6=$ ] |
| 519 | 52:13 | Jessica | And that's with the two blue. |
| 520 | 52:13 | Instructor | Two blue [Writes $6=$ two blue] |
| 521 | 52:14 | Jessica | I think it is supposed to be four equals one blue, by the way. |
| 522 | 52:15 | Instructor | [She changes the one in $1=1$ blue to $4=1$ blue.] |
| 523 | 52:18 | Kim | I found another pattern. |
| 524 | 52:20 | Instructor | Okay, let's finish this and tell me the other pattern. |
| 525 | 52:23 | Instructor | So $1,4,6$, the second four is? |


| 526 | 52:25 | Jessica | Three toppings. |
| :---: | :---: | :---: | :---: |
| 527 | 52:31 | Instructor | And over here? [The instructor writes $4=3$ toppings; 4=3 blue] |
| 528 | 52:31 | Jessica | It's the three blues and the last one is four toppings and the other one is the four blues. [The instructor writes $l=4$ toppings; $1=4$ blue] |
| 529 | 52:49 | Instructor | Okay. Look at this and think about it and Kim will tell us what other pattern she sees. |
| 530 | 52:56 | Kim | Okay, well can I show you my paper to explain it better? |
| 531 | 52:59 | Instructor | Yes. |
| 532 | 53:07 | Kim | Oh, but can you go back to the? Yeah. [Indicating the slide that contains Pascal's triangle.] Okay, so you know how like if you add across you get like 1 , $2,4,8,16,32$. And it's like backwards of what I did before. |
| 533 | 53:15 | Instructor | Yeah, what do you mean by if you add across? |
| 534 | 53:18 | Kim | Like, you know how like you have one and then 1 and 1 , so that's two. |
| 535 | 53:22 | Instructor | So the sum is two. |
| 536 | 53:25 | Kim | And the sum is four. The sum is... |
| 537 | 53:29 | Instructor | Eight. |
| 538 | 53:30 | Kim | Yeah. |
| 539 | 53:31 | Jessica | I think that was part of Pascal's triangle. |
| 540 | 53:32 | Kim | Well, yeah but that relates to what we did before. |
| 541 | 53:36 | Instructor | And that's our answer for four also. So if you start with row zero, row $n$ of Pascal's triangle gives you |


|  |  |  | all the cases for towers and all the cases for pizzas. That's what you told me up here. |
| :---: | :---: | :---: | :---: |
| 542 | 53:49 | Kim | Yeah, that's what... |
| 543 | 53:53 | Instructor | Okay, we still want to hear... So what is the exact isomorphism? And for that... well that's what you are going to think about. I don't think we have time for this now. That's what you are going to think about for homework. Well, we got 10 minutes to think about it now. Yes, right, this is for example, so when you figure this out... [Take the WWWB tower out of Jessica's hand] Sorry, I took this out of your hand. |
| 544 | 54:05 | Jessica | That's fine, I was just thinking. |
| 545 | 54:05 | Instructor | When you figure it out you're going to be able to take this one and say 'this is the something pizza'. And then when I give you a pizza, I'm going to say "the pepperoni/sausage pizza build me the tower that's isomorphic to that." That's what you are going to be able to do when you figure it out. Okay? |
| 546 | 54:22 | Instructor | Now, suggestion. Row four is kind of a pain because you have sixteen. Row three, you only got eight. I would suggest to work with row three because eight is easier. Now, not only do you have the numbers that go there, you can actually write each of the eight pizzas and each of the eight towers. |
| 547 | 54:44 | Instructor | The towers that goes with that one. The pizza that goes with that one. The three towers, list them. The three pizzas, list them. Write them all eight things down in their groups and see if you can just look at them and see exactly how they're related to each other. Do you know what I am saying? You're going to have three towers in this group, you're going to have three pizzas in this group. How can you match |


|  |  |  | them up? Okay? |
| :--- | ---: | :--- | :--- |
| $\mathbf{5 4 8}$ | $55: 07$ | Kim | We're pretending that these are pizzas? |
| $\mathbf{5 4 9}$ | $55: 10$ | Instructor | Yeah, you are pretending those are pizzas. |
| $\mathbf{5 5 0}$ | $55: 11$ | Francesca S. | Each color block would be a different topping, right? <br> And we could test out eight? |
| $\mathbf{5 5 1}$ | $55: 16$ | Instructor | Well, that's what they were doing but they had some <br> issues with it because then they couldn't get the <br> height right. My suggestion would be build your <br> eight three tall towers, write down your eight three <br> topping pizzas which you already did. And just look. |
| $\mathbf{5 5 2}$ | $55: 29$ | Francesca S. | Okay. |
| $\mathbf{5 5 3}$ | $55: 32$ | Kim | I had pizza for lunch and now I'm [inaudible] |


| 563 | 56:08 | Kim | Blues stuck together. [Building towers] |
| :---: | :---: | :---: | :---: |
| 564 | 56:09 | Francesca S. | Oh, you got that one? [Building towers] |
| 565 | 56:14 | Kim | Well you can do the blue on the top. Well, I kind of already did it but.... |
| 566 | 56:17 | Francesca S. | Yeah. |
| 567 | 56:29 | Kim | Oh, what about blue on... oh. Blue on top, like that. |
| 568 | 56:36 | Francesca S. | Wait we're missing one, I think. [They have built the following seven towers: $\begin{array}{lllllll} O & O & O & O & B & B & B \\ O & B & O & B & O & B & B \\ O & O & B & B & O & O & B \\ ] & & & & & & \end{array}$ |
| 569 | 56:37 | Instructor | Yeah, I would be happier if you put this one in with the other one blues. [She pulls out BOO ] Where do you think I'm gonna want you to put that one? |
| 570 | 56:44 | Kim | This one? |
| 571 | 56:44 | Instructor | Yeah. |
| 572 | 56:47 | Kim | Right there, where it was? |
| 573 | 56:48 | Instructor | Nope. |
| 574 | 56:49 | Kim | Before I moved it? |
| 575 | 56:50 | Instructor | Nope. Why do you think I like having it here? [She rearranges the towers so they now appear as follows: $\begin{array}{ccccccc} O & B & O & O & O & B & B \\ O & O & B & O & B & B & B \end{array}$ |


|  |  |  | $\begin{array}{lllllll} O & O & O & B & B & O & B \\ ] & & & & & & \\ \hline \end{array}$ |
| :---: | :---: | :---: | :---: |
| 576 | 56:55 | Kim | Oh, because it's a pattern! |
| 577 | 56:56 | Instructor | Yeah, and not only that those are the three that have exactly one blue. |
| 578 | 56:59 | Francesca S. | We are missing a one orange, right? |
| 579 | 57:01 | Instructor | Yes, you are. |
| 580 | 57:02 | Kim | In the middle. |
| 581 | 57:03 | Francesca S. | Yeah, in the middle. [She builds BOB towers and places it with the one orange towers.] |
| 582 | 57:11 | Instructor | Okay, so now how many, what were those numbers in Pascal's triangle for the third row? |
| 583 | 57:20 | Francesca S. | 1,3 , yeah, $1,3,3,1$ |
| 584 | 57:22 | Instructor | So, alright, did you hear what she said? |
| 585 | 57:23 | Kim | Oh, 1, 3, 3, oh.... |
| 586 | 57:24 | Francesca S. | That's crazy. |
| 587 | 57:25 | Instructor | Okay. 1, 3, 3, 1 . Now you're going to do the $1,3,3$, 1 pizzas and you're going to look at it and see what you can figure out. |
| 588 | 57:35 | Kim | Three pizzas. |
| 589 | 57:38 | Francesca S. | Which is right here [She is pointing to her notebook]. So this one's plain [points to the OOO tower] |
| 590 | 57:57 | Kim | One plain. So then this would be sausage. |
| 591 | 58:02 | Francesca S. | So what would the blue be in here? [Pointing to the blue cube in BOO ] |

$\left.\begin{array}{|l|r|l|l|}\hline \mathbf{5 9 2} & 58: 08 & \text { Kim } & \begin{array}{l}\text { I don't know. Like a random topping and then just... } \\ \text { like the orange, the all orange is plain, right? }\end{array} \\ \hline \mathbf{5 9 3} & 58: 21 & \text { Francesca S. } & \text { Yeah. } \\ \hline \mathbf{5 9 4} & 58: 24 & \text { Kim } & \begin{array}{l}\text { Right? But then what would the all blue be? Is that } \\ \text { the same thing as... }\end{array} \\ \hline \mathbf{5 9 5} & 58: 26 & \text { Francesca S. } & \text { All blue would be the three toppings. } \\ \hline \mathbf{5 9 6} & 58: 27 & \text { Kim } & \begin{array}{l}\text { Oh, gotcha. Oh. }\end{array} \\ \hline \mathbf{5 9 7} & 58: 29 & \text { Francesca S. } & \begin{array}{l}\text { And then this one would be one topping [Points to } \\ \text { the three towers with one blue cube]. Oh, my God - } \\ \text { Okay. Look - this is one topping because it has one } \\ \text { blue. [Points to the three towers with one blue cube.] } \\ \text { Two toppings because it has two blue. [Points to the }\end{array} \\ \text { three towers with two blue cubes.] }\end{array}\right\}$

|  |  |  | this tower goes with plain [holds up the all orange <br> tower]. This tower goes with everything [holds up <br> the all blue tower]. What does this tower go with? <br> [Holds up a tower with one blue - BOO] |
| :--- | ---: | :--- | :--- |
| $\mathbf{6 0 6}$ | $1: 00: 11$ | Francesca S. | One topping. |
| $\mathbf{6 0 7}$ | $1: 00: 12$ | Instructor | Which one? |
| $\mathbf{6 0 8}$ | $1: 00: 13$ | Francesca S. | Like sausage. |
| $\mathbf{6 0 9}$ | $1: 00: 14$ | Instructor | Alright, this goes with sausage. |
| $\mathbf{6 1 0}$ | $1: 00: 15$ | Kim | Oh, you want, like, a specific? |
| $\mathbf{6 1 1}$ | $1: 00: 16$ | Instructor | Yeah. And then this one goes with what [points to <br> OBO]? And then this one goes with what [points to <br> OOB]? And explain to me why it goes with that. <br> Okay? And the same thing with these [points to the <br> three towers with two blue cubes]. So you have to <br> start - okay, this one is sausage, fine. See what else |
| you can figure out. |  |  |  |


|  |  |  | put, is peppers and then the third one's mushroom. |
| :--- | :---: | :--- | :--- |
| $\mathbf{6 1 9}$ | $1: 01: 00$ | Kim | What did you put for the middle one? |
| $\mathbf{6 2 0}$ | $1: 01: 02$ | Francesca S. | Peppers. |
| $\mathbf{6 2 1}$ | $1: 01: 03$ | Kim | Peppers. But then for this one. [She points to the <br> three towers with two blue cubes.] |
| $\mathbf{6 2 2}$ | $1: 01: 06$ | Francesca S. | Yeah, this one we need to like correspond to that - <br> you know what I mean? |
| $\mathbf{6 2 3}$ | $1: 01: 18$ | Instructor | Alright, so tell me what you got. |
| $\mathbf{6 2 4}$ | $1: 01: 19$ | Francesca S. | So we just did this one - sausage, peppers, and <br> mushrooms. |
| $\mathbf{6 2 5}$ | $1: 01: 23$ | Instructor | Okay, so the blue on top is sausage [points to the <br> BOO tower]. The blue in the middle is [points to the |
| $\mathbf{6 2 6}$ | $1: 01: 28$ | Francesca S. | OBO tower], what was it? Peppers? |
| $\mathbf{6 2 7}$ | $1: 01: 29$ | Instructor | Yeah, peppers. <br> And the blue on the bottom is mushroom [points to <br> the OOB tower].Okay, so now what are you going to <br> do with these? [She points to the three towers with |
| $\mathbf{6 3 2}$ | $1: 01: 56$ | Francesca S. | 'Cause it is on the bottom. |
| two blue cubes.] |  |  |  |


| $\mathbf{6 3 3}$ | $1: 01: 57$ | Instructor | So what are you saying? Mushrooms on the bottom? |
| :--- | :---: | :--- | :--- |
| $\mathbf{6 3 4}$ | $1: 02: 00$ | Francesca S. | Yeah. |
| $\mathbf{6 3 5}$ | $1: 02: 01$ | Kim | So then this has to be mushroom? [Points to the <br> bottom blue cube in OBB] |
| $\mathbf{6 3 6}$ | $1: 02: 02$ | Francesca S. | I don't know but this one's on the bottom. [Points to <br> the bottom blue cube in BOB] |
| $\mathbf{6 3 7}$ | $1: 02: 04$ | Instructor | So this one's mushroom, and what's the middle one? <br> [Points to the orange cube in OOB] |
| $\mathbf{6 3 8}$ | $1: 02: 10$ | Francesca S. | Yeah, peppers. |
| $\mathbf{6 3 9}$ | $1: 02: 12$ | Instructor | And the top one is sausage. Alright. So you're saying <br> this one has mushrooms on it [points to OOB tower]. <br> But what about sausage and peppers? It doesn't have <br> sausage and peppers, you're telling me. |
| $\mathbf{6 4 0}$ | $1: 02: 20$ | Francesca S. | Yeah. |
| $\mathbf{6 4 1}$ | $1: 02: 21$ | Instructor | Okay, and this one has [points to the OBO tower]... <br> which one - this one has peppers but it doesn't have <br> mushroom or sausage. And this one [points to the |
| BOO tower] doesn't have mushroom, doesn't have |  |  |  |
| peppers but it does have sausage. |  |  |  |


| 649 | 1:02:44 | Instructor | Okay. Alright. So you designated something here [points the three towers with one blue]. See if you can designate something similar here [points to the three towers with two blues]. And see what you come up with. I'll leave it at that. But you should.... |
| :---: | :---: | :---: | :---: |
| 650 | 1:02:57 | Francesca S. | I think this one is pepper and mushroom. [Points to $O B B]$ |
| 651 | 1:03:00 | Kim | Pepper and mushroom? |
| 652 | 1:03:01 | Instructor | Explain to Kim why you think it is pepper and mushroom. |
| 653 | 1:03:03 | Francesca S. | Right? Because if this is sausage [points to the top blue cube in the BOO tower]. |
| 654 | 1:03:07 | Kim | Oh, so that one's sausage [points to the top blue cube in BOO ], that one doesn't necessarily have to be sausage [points to the top orange cube in OBB]. |
| 655 | 1:03:10 | Francesca S. | Yeah. I guess, I don't know. |
| 656 | 1:03:14 | Kim | Okay, so it's basically the opposite of whatever we have for that. |
| 657 | 1:03:20 | Francesca S. | I guess, I don't know. |
| 658 | 1:03:24 | Kim | Or, since we said that's mushroom [points to the bottom blue cube in $O O B$ ] this one would be mushroom? [Points to the bottom blue cube in OBB] |
| 659 | 1:03:30 | Francesca S. | Yeah. |
| 660 | 1:03:31 | Kim | And then this one would be mushroom? [Points to the BOB tower.] Right? And then that's mushroom [points to the BBO tower]. |
| 661 | 1:03:34 | Francesca S. | No. That would be peppers. [Points to the $O B O$ tower] |


| 662 | 1:03:39 | Kim | Oh, because, oh... that's smart. Okay. |
| :---: | :---: | :---: | :---: |
| 663 | 1:03:44 | Francesca S. | Wait. So this one's.... So this one's mushroom. [Points to the OBB tower.] |
| 664 | 1:04:00 | Kim | And then... Do we have to explain what that is? [Points to the orange cube in OBB.] |
| 665 | 1:04:04 | Francesca S. | Yeah, that would... this one would be peppers and mushroom because it has no sausage on it. |
| 666 | 1:04:12 | Kim | So that's mushroom [points the bottom blue cube of $O B B$ ], peppers or the other one [points to the middle blue cube in $O B B]$ ? |
| 667 | 1:04:16 | Francesca S. | Yeah. |
| 668 | 1:04:23 | Francesca S. | But I think this whole thing is mushrooms. [Points to both blue cubes in OBB.] |
| 669 | 1:04:27 | Kim | But why would all of them? Because they are different. |
| 670 | 1:04:30 | Francesca S. | No, this thing. The two blues. [Points to the two blues in the OBB tower.] |
| 671 | 1:04:32 | Kim | Oh, the two blue together. |
| 672 | 1:04:34 | Francesca S. | Yeah. |
| 673 | 1:04:37 | Kim | So then, these two would be the same thing [points to the two blue cubes in $B O B$ ]. |
| 674 | 1:04:41 | Francesca S. | Yeah. |
| 675 | 1:04:43 | Kim | Okay, alright, I have a question. [Speaking to the instructor.] |
| 676 | 1:04:51 | Instructor | Okay. |
| 677 | 1:04:51 | Kim | Okay, so these two together, they would be one topping or are they different toppings? [Points to the |


|  |  |  | two blues in the OBB tower.] Because they're the <br> same color. |
| :--- | ---: | :--- | :--- |
| $\mathbf{6 7 8}$ | $1: 04: 55$ | Instructor | Oh, you have to tell me that. |
| $\mathbf{6 7 9}$ | $1: 04: 58$ | Francesca S. | This one would be pepper and mushroom [pointing <br> to the $O B B$ tower] because it doesn't have a sausage <br> on top [points to the orange cube on top of the OBB <br> tower]. So we know it's gonna have to be pepper and <br> mushroom. |
| $\mathbf{6 8 0}$ | $1: 05: 04$ | Instructor | So you're saying the position makes a difference? |
| $\mathbf{6 8 1}$ | $1: 05: 06$ | Francesca S. | Yeah. |
| $\mathbf{6 8 2}$ | $1: 05: 06$ | Instructor | This position is sausage. [Points to the top orange <br> cube in OBB.] |
| $\mathbf{6 8 3}$ | $1: 05: 08$ | Francesca S. | Yeah. |
| $\mathbf{6 8 4}$ | $1: 05: 08$ | Instructor | And when you have... What does a blue in this <br> position mean? [Points to the top orange cube in |
| $\mathbf{6 8 5}$ | $1: 05: 12$ | Francesca S. | Sausage. <br> $\mathbf{6 8 6}$ |
| $\mathbf{6 8 9}$ | $1: 05: 13$ | Instructor | And a yellow in that position means no sausage. |
| $\mathbf{6 8 9}$ | $1: 05: 14$ | Francesca S. | So this one would be sausage and mushroom. [Points <br> to the BOB tower.] |
| $\mathbf{6 8 9}$ | $1: 05: 20$ | Instructor | Okay, alright. You said the bottom row is <br> mushroom, the top row is sausage, and the middle |
| row is peppers. |  |  |  |


|  |  |  | orange? What does the blue mean and what does the orange mean? |
| :---: | :---: | :---: | :---: |
| 693 | 1:05:37 | Francesca S. | Um... We didn't do that. We just did the order of them. |
| 694 | 1:05:40 | Instructor | Okay, but you did say it implicitly when you were telling me things. |
| 695 | 1:05:43 | Francesca S. | Oh, yeah. |
| 696 | 1:05:44 | Instructor | How did you know that this one meant mushroom and peppers? [Pointing to the $O B B$ tower.] |
| 697 | 1:05:47 | Francesca S. | Because this one's sausage. [Points to the top blue cube in BOO.] We named this one sausage and sausage is the first one. |
| 698 | 1:05:50 | Instructor | Okay, so. |
| 699 | 1:05:51 | Francesca S. | So blue's not there so... |
| 700 | 1:05:52 | Instructor | So blue means what specifically? |
| 701 | 1:05:56 | Francesca S. | Sausage. Blue on the first.... [Points to the top blue cube in BOO.] |
| 702 | 1:06:00 | Instructor | Blue on the first thing means sausage is there [points to the top blue cube in BOO ]. Yellow on the first thing means sausage....? [Points to the top orange cube in $O B B$.] |
| 703 | 1:06:06 | Francesca S. | Is not there. |
| 704 | 1:06:07 | Instructor | Is not there. |
| 705 | 1:06:08 | Kim | So sausage is on this one and this one. [Points to the top blue cubes in the two towers $B O B$ and $B B O$.] |
| 706 | 1:06:10 | Instructor | Yeah. |
| 707 | 1:06:10 | Kim | Because there's sausage on top. |


| 708 | 1:06:12 | Instructor | Yeah, now you tell me what does this indicate? <br> [Points to the bottom orange cube in $B B O$.] |
| :---: | :---: | :---: | :---: |
| 709 | 1:06:16 | Kim | There's sausage on it. |
| 710 | 1:06:17 | Francesca S. | Yeah, this one's sausage, this one's sausage. [Points to the BBO tower.] |
| 711 | 1:06:18 | Instructor | This particular one I am pointing to on the bottom. [Points to the bottom orange cube in $B B O$.] |
| 712 | 1:06:20 | Francesca S. | Peppers. |
| 713 | 1:06:21 | Instructor | Peppers or mushrooms? I thought the bottom was mushrooms. |
| 714 | 1:06:24 | Instructor | Oh yes, peppers. Yes, but what does this particular one on the bottom mean? [Points to the bottom orange cube in BBO.] |
| 715 | 1:06:27 | Kim | Mushrooms? |
| 716 | 1:06:27 | Francesca S. | Mushrooms. |
| 717 | 1:06:28 | Instructor | And what does it say about mushrooms? Does this pizza have mushrooms or not? |
| 718 | 1:06:31 | Kim | No. |
| 719 | 1:06:32 | Francesca S. | Yes. |
| 720 | 1:06:32 | Instructor | Yes, no? |
| 721 | 1:06:33 | Francesca S. | Yeah, it does sausage and peppers. Sausage and mushrooms. |
| 722 | 1:06:37 | Kim | But I thought this one meant mushrooms. [Points to the bottom blue cube in $O O B$.] |
| 723 | 1:06:40 | Instructor | Yeah, I thought mushroom was on the bottom too. |
| 724 | 1:06:43 | Francesca S. | Yeah. |


| 725 | 1:06:44 | Kim | Yeah, that's mushroom. So mushroom isn't on that one. |
| :---: | :---: | :---: | :---: |
| 726 | 1:06:47 | Instructor | Because? |
| 727 | 1:06:49 | Kim | Because... |
| 728 | 1:06:49 | Francesca S. | Oh, two blue and one orange -this one's sausage and peppers. [Points to the BBO tower.] |
| 729 | 1:06:51 | Instructor | Okay, so specifically blue means and yellow means, orange means what? Blue means? You told me, I think. |
| 730 | 1:07:00 | Francesca S. | Blue is sausage. |
| 731 | 1:07:02 | Instructor | No it isn't. [Kim laughs.] |
| 732 | 1:07:05 | Instructor | You told me the top is sausage. |
| 733 | 1:07:06 | Francesca S. | Yeah, top is sausage. The order matters not the colors, I don't think. |
| 734 | 1:07:10 | Instructor | Well, the color... Now, how do you know that there is sausage on this pizza [holds up the OBB tower] and that there isn't sausage on this pizza? [Holds up the BOO tower.] |
| 735 | 1:07:18 | Kim | Because of the top. |
| 736 | 1:07:18 | Francesca S. | Yeah, because the tops are switched. |
| 737 | 1:07:19 | Instructor | Yeah, so what specifically does blue mean and what specifically does orange mean? |
| 738 | 1:07:23 | Francesca S. | Okay, blue means there's sausage on the pizza; orange means there's no sausage. |
| 739 | 1:07:29 | Instructor | Yeah, but down here, but down here what does this blue and this orange mean? [She holds up the $B O B$ tower and the BBO tower.] |


| 740 | 1:07:33 | Kim | Well, they both have sausage on them. |
| :---: | :---: | :---: | :---: |
| 741 | 1:07:34 | Instructor | No, I mean the ones at the bottom. |
| 742 | 1:07:35 | Kim | Oh, um... |
| 743 | 1:07:37 | Francesca S. | Mushroom. |
| 744 | 1:07:37 | Kim | One has mushroom and one doesn't. |
| 745 | 1:07:39 | Instructor | Yeah, which one doesn't? |
| 746 | 1:07:40 | Kim | That one. [Points to the BBO tower.] |
| 747 | 1:07:41 | Instructor | So what does yellow, orange mean in isolation without referring to a specific topping? What does orange tell you both about sausage and about mushroom? |
| 748 | 1:07:51 | Francesca S. | That there's both on it? |
| 749 | 1:07:55 | Instructor | Well, maybe you want to think about it. |
| 750 | 1:07:56 | Francesca S. | My brain's hurting. |
| 751 | 1:07:58 | Instructor | That's great! |
| 752 | 1:08:00 | Francesca S. | I think I actually got it. |
| 753 | 1:08:01 | Instructor | Yeah, you did actually get it. |
| 754 | 1:08:03 | Kim | [Laughing] [inaudible] |
| 755 | 1:08:05 | Instructor | Yes, you told me that this pizza does not have mushroom [pointing to the BBO tower] but this pizza does have mushroom [pointing to the BOB tower], just looking at the bottom cube. |
| 756 | 1:08:12 | Francesca S. | Yeah. |
| 757 | 1:08:13 | Instructor | And you told me that looking at the top cube, this has sausage [pointing to the BOO tower] and this doesn't have sausage [pointing to the OBB tower]. |


| 758 | 1:08:19 | Kim | Yeah. |
| :---: | :---: | :---: | :---: |
| 759 | 1:08:20 | Instructor | Right? Now, what does this tell you about peppers? <br> [Holding the $O B O$ and $O O B$ towers.] |
| 760 | 1:08:25 | Kim | That one does and one doesn't. |
| 761 | 1:08:26 | Instructor | Yeah. And how do you know that one does and one doesn't? |
| 762 | 1:08:29 | Kim | Because the blue means... |
| 763 | 1:08:31 | Francesca S. | 'Cause peppers is the second. |
| 764 | 1:08:31 | Instructor | The blue means? |
| 765 | 1:08:32 | Francesca S. | Peppers. |
| 766 | 1:08:33 | Kim | That it's there. |
| 767 | 1:08:35 | Instructor | Say it again. Repeat yourself. |
| 768 | 1:08:37 | Kim | That it's there. |
| 769 | 1:08:38 | Instructor | The blue means that it's there. And what does the orange mean? |
| 770 | 1:08:40 | Kim | That it's not there. |
| 771 | 1:08:41 | Instructor | Bingo! |
| 772 | 1:08:45 | Kim | I had no idea what you were asking. [laughing] |
| 773 | 1:08:48 | Instructor | Isn't that what you already knew? You knew that, right? |
| 774 | 1:08:55 | Francesca S. | Yeah. |
| 775 | 1:08:56 | Instructor | You weren't coming out with it but you knew that. That's what I want you to write up. |
| 776 | 1:08:59 | Instructor | We can be done now. |

## APPENDIX D

Transcript 2.2 - February 18, 2011
Camera View Two

## February 18, 2011 - Camera View 2

The tape begins with the instructor displaying a powerpoint. The powerpoint slide says: Summarize our previous results:

- Two colors, four cubes tall: 16
- You organized the towers by number of blue cubes
- How many towers for 0 blue, 1 blue, 2 blue, 3 blues, and 4 blues?
- Two colors, $n$ cubes tall: $2^{n}$
- Why is it $2^{n}$ ?
- $m$ colors, $n$ cubes tall: $m^{n}$
- Why is it $m^{n}$ ?

They break into two groups, two people each, and work on the problem.

- One group is composed of Jessica and Jamie
- One group is composed of Kim and Francesca S.

This camera view focuses on Jessica and Jamie. (Blue and White cubes).
[When the towers are described, the first color written is the top cube of the tower.]

| 1 | 00:23 | Jessica | [Creates five towers using blue and white cubes. She creates the solid blue tower. Then she creates the blue towers with one white. The first tower she creates with one white cube has the white cube in the top position. The next tower has the white cube in the second position, then the third position, and the last tower has a white cube in the bottom position. Her towers are as follows: $\begin{array}{ccccc} B & W & B & B & B \\ B & B & W & B & B \\ B & B & B & W & B \\ B & B & B & B & W \\ ] & & & & \end{array}$ |
| :---: | :---: | :---: | :---: |
| 2 | 00:23 | Jamie | [Creates five towers using blue and white cubes. She creates the solid white tower. Then she creates the |


|  |  |  | white towers with one blue. The first tower she creates with one blue cube has the blue cube in the top position. The next tower has the blue cube in the second position, then the third position, and the last tower has a blue cube in the bottom position. Her towers are as follows: <br> $\begin{array}{lllll}W & B & W & W & W\end{array}$ <br> $\begin{array}{lllll}W & W & B & W & W\end{array}$ <br> $\begin{array}{lllll}W & W & W & B & W\end{array}$ <br> $W \quad W \quad W \quad W \quad B$ ] |
| :---: | :---: | :---: | :---: |
| 3 | 01:12 | Jessica | I got the four blue, then I got the three blue, and now I am working on the two blue - which is actually going to be, both of ours are going to be the same for that though. |
| 4 | 01:21 | Instructor | Okay, so yeah, you're doing the no blue and one blue and then you are going to two blue and then there's three and four. |
| 5 | 01:28 | Jessica | We'll just take out whatever we have extras of. |
| 6 | 01:29 | Jamie | Yeah. That one and that one are the same [compares WWBB tower that she has created with the WWBB tower that Jessica has created.] |
| 7 | 01:34 | Jessica | Yeah, we will just take it out later. [Builds six towers with two blues. She builds the following towers: $\begin{array}{cccccc} W & B & B & W & B & W \\ B & W & B & W & W & B \\ B & W & W & B & B & W \\ W & B & W & B & W & B \end{array}$ |


|  |  |  | ] |
| :---: | :---: | :---: | :---: |
| 8 | 02:13 | Instructor | Okay, you got six and you got four. So... |
| 9 | 02:16 | Jamie | I'm missing that one. [She points to the BWWB tower that Jessica has created. Jamie has the following four towers: $\begin{array}{llll} B & W & W & B \\ B & W & B & W \\ W & B & W & B \\ W & B & B & W \\ ] & & & \end{array}$ |
| 10 | 02:21 | Jessica | But either way [inaudible] either way [inaudible]... |
| 11 | 02:22 | Instructor | Yeah, put them together because you're doing one group so...just put - make sure that you agree that that's all of them. |
| 12 | 02:26 | Jessica | I think six is all, right? Cause you got this together, that one together. [Picks up the BWWB tower and the WBBW tower.] |
| 13 | 02:34 | Instructor | Also together [inaudible] |
| 14 | 02:35 | Jessica | Together but... |
| 15 | 02:38 | Jamie | So we're taking this one out, right? [Indicating the $B W W B$ and WBBW towers.] |
| 16 | 02:39 | Jessica | We take this one out and that one out [picks up the $B W W B$ and $W B B W$ tower]. And we got that and that which are the same thing [indicating the BWBW and WBWB towers that Jamie also has] - which is six. |
| 17 | 02:49 | Instructor | Okay, so... So you're going to write down - you have to make a little table. Right? How many, on the x -axis number, the first number is how many blue cubes. |


|  |  |  | Zero, one, two, three, four. And then, you know... |
| :---: | :---: | :---: | :---: |
| 18 | 03:01 | Jamie | Do you want us all to do it or? |
| 19 | 03:03 | Instructor | Why don't you do it too so you have it in your own notes. |
| 20 | 03:05 | Jamie | Okay, sure. [She starts to write in her notebook.] |
| 21 | 03:11 | Jessica | I'm just putting four blue here and how many towers. [She is writing in her notebook.] |
| 22 | 03:14 | Jamie | Okay. |
| 23 | 03:15 | Jessica | The amount of colors and then the towers. |
| 24 | 03:17 | Jamie | So you're going to put a different column for towers? |
| 25 | 03:25 | Jessica | Yeah. See it's the number of blues and the number of towers. [Writing in her notebook.] The number of blues and the number of towers. Alright, so four blue can only have one tower, three blue we have four towers, and then two blue is six towers. And I have to do one blue with the whites. [She takes the cubes and starts to build towers.] |
| 26 | 04:02 | Jamie | So which one are you doing? Three? |
| 27 | 04:04 | Jessica | I'm doing the single blue now and you're doing the single white. |
| 28 | 04:22 | Jamie | [Inaudible] |
| 29 | 04:24 | Jessica | Uh-hum. And then we'll just pull the extras. [Starts to build towers.] |
| 30 | 04:51 | Instructor | And what are you doing? Oh, you're building? |
| 31 | 04:53 | Jessica | Continue to build [inaudible] number of blues, number of towers [inaudible] number of whites, number of towers. |



|  |  |  |  $B$ $W$ $W$ $W$ $W$ $B$ $B$ $B$ <br>    $W$    $B$  <br>          <br>    $W$   $B$   <br>    $W$   $B$   <br>       $B$   |
| :---: | :---: | :---: | :---: |
| 41 | 05:56 | Jessica | Alright so. [Writes in her notebook] One, four, six, four, one. And the same thing for white. |
| 42 | 06:23 | Jamie | It's the same thing [writing in her notebook]. |
| 43 | 06:24 | Jessica | Yeah. Same exact thing. |
| 44 | 06:33 | Jessica | [Looking at the board.] Oh, now we need to do the whole "why is it two to the $n$ ?" |
| 45 | 06:37 | Jamie | Oh, because of the two color thing? Did you write that down? |
| 46 | 06:43 | Jessica | I may have. [Looks through her notebook.] I don't think I wrote down why though. No, we just wrote why it's, like how we put it together. We didn't say why it is two. Because it's two colors. |
| 47 | 07:04 | Instructor | Okay, so you just said something important that you're going to write down, right? |
| 48 | 07:07 | Jessica/Jamie | Yes. |
| 49 | 07:12 | Jessica | [Writing in her notebook] Why is it two to the $n$ ? And the two is because of the number of colors. Oops, I almost wrote blue instead of colors. |
| 50 | 07:43 | Jessica | Why is it $m$ to the $n$ ? Because $m$ represents the number of colors. |
| 51 | 08:02 | Jamie | Amount of colors. |
| 52 | 08:03 | Jessica | Different colors, maybe? |


| 53 | 08:03 | Jamie | Uh-huh. And $n$ represents the height of the tower? |
| :---: | :---: | :---: | :---: |
| 54 | 08:19 | Jessica | Yeah. [Jamie and Jessica write in their notebooks.] |
| 55 | 08:58 | Instructor | Okay, and where are you guys at? |
| 56 | 08:59 | Jessica | The answer. |
| 57 | 09:00 | Instructor | You got your answers? You're both happy with your answers to both questions? |
| 58 | 09:03 | Jamie | Yes. |
| 59 | 09:03 | Instructor | Okay, tell me what they are. |
| 60 | 09:05 | Jessica | Well, we did why is it two to the $n$ ? It's two because the two represents the number of colors. |
| 61 | 09:11 | Instructor | Okay. |
| 62 | 09:12 | Jessica | Two is the base. |
| 63 | 09:13 | Instructor | Okay. And $n$ ? |
| 64 | 09:18 | Jessica | Oh, $n$ is the tall/height of the towers. Like $N$ amount of blocks is the height of the towers. So four blocks. And then $m$ to the $n$, is the other one. We just said $m$ is the different number of colors and $n$ is the height of the tower. |
| 65 | 09:34 | Instructor | Okay, now there was something else that I think Rebecca said the last time that sort of goes along with it. I understand two different colors but you didn't exactly give me $100 \%$ reason why it's two to the $n$ as opposed to, say, two times $n$. "Why is it two to the $n$ power?" is the question. |
| 66 | 09:53 | Jessica | Oh, it's because... |
| 67 | 09:55 | Jamie | Because two is... |
| 68 | 09:59 | Instructor | Yeah, and you don't have to explain it on the fly right |


|  |  |  | now but think it over and see if you can come up with something. |
| :---: | :---: | :---: | :---: |
| 69 | 10:02 | Jamie | Does it have something to do with how like....? |
| 70 | 10:05 | Jessica | The choice... |
| 71 | 10:06 | Jamie | Yeah, the choices that you are allowed and you can't have the same thing so that eliminates like the extra choices? |
| 72 | 10:13 | Instructor | Oh, I'm not sure what you are saying but... We'll hear what you had to say about choices. [To Jessica] |
| 73 | 10:17 | Jessica | Well, the thing is... the only thing I can remember truthfully is that you have two choices. You can either add a white one on or you can add a blue one on. <br> [Holds up a single blue cube and a single white cube.] |
| 74 | 10:27 | Instructor | Okay, we said this last time. That's sort of like induction. Right? So no matter where you're starting from, like if you're starting from these [points to the four towers with three blue cubes and one white cube], each one of these, say it again, you can.... |
| 75 | 10:38 | Jessica | You can either have a white one or a blue one. |
| 76 | 10:40 | Instructor | And the fact that you have two choices means you're multiplying by two. So that's what I'm getting at multiplying by two, multiplying by two, means two to the $n$. |
| 77 | 10:48 | Jessica | Oh, that's right. |
| 78 | 10:51 | Instructor | And so similarly for $m$ to the $n$, the $m$ choices. Which means every single time you know you have [inaudible]. |
| 79 | 10:58 | Jessica | Yeah, you have that many choices; you have to keep multiplying by that. |


| $\mathbf{8 0}$ | $11: 01$ | Instructor | Okay, so write out all of those things in your own <br> words so you can, you know. |
| :--- | :---: | :--- | :--- |
| $\mathbf{8 1}$ | $11: 09$ | Jamie | [Writing in her notebook.] Alright, so... Two choices. |
| $\mathbf{8 2}$ | $11: 14$ | Jessica | Two to the $n$ is uh... you have, every time that you <br> make a tower, you have two choices of the color. |
| $\mathbf{8 3}$ | $11: 23$ | Jamie | You can either add a blue or a white. |
| $\mathbf{8 4}$ | $11: 24$ | Jessica | Yes, that's what we have to write down. So every time <br> you have a tower... |
| $\mathbf{8 6}$ | $11: 25$ | Jamie | Instructor |
| $\mathbf{8 7}$ | $11: 51$ | Jessica | So you guys are writing down your explanations. And <br> they didn't do the $m$ choices. So I'm gonna have them <br> come over and you can explain to them $m$ to the $n$. <br> Explain for three, m equals three, say. |
| $\mathbf{8 8}$ | $11: 59$ | Jamie | Okay. |
| $\mathbf{8 9}$ | $12: 02$ | Jessica | To add.... |
| $\mathbf{9 4}$ |  | Two choices - either blue or white. Well, in our case <br> blue. |  |
| $\mathbf{9 4}$ | $12: 06$ | Jamie | To add either blue or white. To formulate the new <br> tower? It's like it's a new tower each time. Right? |
| $\mathbf{9 4}$ | $12: 49$ | Jessica | Yeah, yeah. To um... yeah, to formulate the new |
| Alright she said [inaudible] [Talking in a whisper to |  |  |  |
| Jamie.] |  |  |  |
| tower. |  |  |  |


| 95 | 13:06 | Jessica | Yeah, cause with this one you can do this, this or this. [She lays a down a single yellow cube and puts three single cubes above this cube - yellow, blue, and white.] And then this, same thing. [She lays a down a single white cube and puts three single cubes above this cube - yellow, blue, and white.] |
| :---: | :---: | :---: | :---: |
| 96 | 13:18 | Jamie | And then the same thing with the [inaudible]. Why didn't you [inaudible]? |
| 97 | 13:21 | Jessica | Cause I don't want to take our stuff apart. [She lays a down a single blue cube and puts three single cubes above this cube - yellow, blue, and white.] |
| 98 | 13:29 | Jessica | So one tall you have three choices which is three to the first power. |
| 99 | 13:35 | Jessica | Two tall, you have three more choices per block. |
| 100 | 13:39 | Jamie | Which equals nine. |
| 101 | 13:39 | Jessica | Which equals nine. |
| 102 | 13:40 | Instructor | Now I'd like you to go over. You guys are ready to explain? |
| 103 | 13:43 | Jessica | Yes. |
| 104 | 13:43 | Instructor | $M$ to the $n$ for $m$ equals three. So they're going to talk about three colors. [Francesca S., Kim, and the instructor move over to Jamie and Jessica's group.] |
| 105 | 13:47 | Kim | Three colors! |
| 106 | 13:48 | Instructor | Yes, now we're doing towers where you got three colors to choose from. And they're going to explain their answer and you're going to ask questions and make sure that they explain it to your total satisfaction. |
| 107 | 14:01 | Jamie | Okay, so you have three colors: yellow, blue and |


|  |  |  | white. So $m$ represents the colors. So you have three. And if you want to make it three tall... [There are three single cubes in a row: yellow, blue, and white. In front of each of these single cubes, there are three single cubes: yellow, blue, and white.] |
| :---: | :---: | :---: | :---: |
| 108 | 14:10 | Instructor | Well, start with one tall. |
| 109 | 14:11 | Jamie | One tall, okay, one tall would be three to the one which is three. |
| 110 | 14:16 | Jessica | Because you only have three choices. |
| 111 | 14:17 | Jamie | 'Cause you can only have three choices. |
| 112 | 14:20 | Jessica | Then when you get to two tall, you have three choices per the one that you already have. So you have the yellow can either be yellow-yellow, yellow-blue, or yellow-white. Blue can be blue-yellow, blue-blue, or blue-white. And white can be white-yellow, whiteblue, or white-white. So, since you have three choices each time, it's three squared this time... um.. I'm trying to think... |
| 113 | 14:47 | Instructor | That's it! Three squared is... |
| 114 | 14:49 | Jessica | Three squared is nine. So you have nine total choices, nine total ways that you can do it. |
| 115 | 14:56 | Instructor | And then... Now I am going to ask you guys [Talking to Kim and Francesca S.]. And if you were going to start off with those three tall, now she explained how you got nine of those, right? So what are you going to do with those nine to get to the next level? |
| 116 | 15:05 | Francesca S. | Nine squared. |
| 117 | 15:07 | Instructor | Well, it's nine squared. It's..... |
| 118 | 15:08 | Kim | It would be nine cubed. |


| 119 | 15:09 | Instructor | No it would be... |
| :---: | :---: | :---: | :---: |
| 120 | 15:11 | Kim | I mean three cubed. |
| 121 | 15:12 | Instructor | Each one of those gets [inaudible] |
| 122 | 15:17 | Jessica | So like if you did that...It basically becomes like the whole tree-type thing. Because then for this yellow, you would have to have yellow, white, and blue. [Placing a blue, white, and yellow cube above the yellow, yellow cubes.] And for that blue you would have do the same thing that we did here with the yellow, white, and blue. |
| 123 | 15:33 | Instructor | So you can see that if you had to build them it would get complicated but once you see the pattern, you don't need to build them. |
| 124 | 15:38 | Jessica | Yeah. That goes there, that goes there, and this one goes here. [Placing a blue, white, and yellow cube above the yellow, blue cubes.] And like it just continues to go on and on forever. |
| 125 | 15:45 | Instructor | Okay, so, that's great! There's your induction, right? No matter where you are at, you can always go to the next step with times three. Okay. So are you satisfied with that? |
| 126 | 15:53 | Francesca S. | Yes. |
| 127 | 15:54 | Instructor | Okay, so now we're going to move on to the next problem. Okay, so.... We're going to leave this for a moment and move on to the next combinatorics question, which is on the next slide. Here it is. The pizza problem. [The slide says: <br> The Pizza Problem <br> - There are four possible pizza toppings: <br> Sausage <br> Peppers <br> Pepperoni |


|  |  |  | Mushrooms <br> - You can have a plain pizza (no toppings), or a pizza with any combination of the above toppings. How many pizzas is it possible to make? <br> Part 1: What's the answer? <br> Part 2: Convince me that your answer is correct.] |
| :---: | :---: | :---: | :---: |
| 128 | 16:10 | Kim | Pizza! [Kim and Francesca S. return to their seats.] |
| 129 | 16:14 | Class | [There is a conversation with the instructor, the class, and the videographer. They discuss that they had already did this problem in another class but they realize that this problem is different than the problem in the other class. This conversation is not transcribed.] |
| 130 | 17:04 | Instructor | This is what we want... the usual. Four possible toppings. So you can choose a plain pizza, right? Or you can put any of the toppings. You can put all of them; you can put some of them... How many pizzas can you make? [inaudible] Questions? |
| 131 | 17:20 | Jessica | [The camera focuses back on Jessica and Jamie.] Did you want us to keep these together or we can take them apart? [Holding up the previously built towers.] |
| 132 | 17:23 | Instructor | Just leave them for now. I don't think you will need them for what you are doing here. Just leave them on the side and work on that one [inaudible]. |
| 133 | 17:41 | Jamie | Do you want to write it as one? Work on one? |
| 134 | 17:48 | Jessica | Oh, yeah, work on one paper. Yeah. |
| 135 | 17:50 | Jamie | So sausage. |
| 136 | 17:53 | Jessica | We'll have that as S for sausage. Yeah, let's move this stuff away. [They move the towers out of the way and Jessica starts to write in her notebook.] |


| 137 | 17:59 | Jessica | Alright so S is sausage. |
| :---: | :---: | :---: | :---: |
| 138 | 18:04 | Jamie | You're gonna have to have some something for peppers and pepperoni. |
| 139 | 18:06 | Jessica | Well, we'll do Pe for peppers and Pi for pepperoni. And then M for mushroom. |
| 140 | 18:13 | Jamie | And then plain - Pl? |
| 141 | 18:16 | Jessica | Well that would be, that would be a lack of a topping. |
| 142 | 18:19 | Jamie | Okay. |
| 143 | 18:21 | Jessica | So. So the first choice is with um, number of toppings. So if you had zero toppings, that's one, you only have one type of pizza. Then if you have one topping, that's four types of pizzas because you have four different toppings. [Her paper reads as follows: <br> S-Sausage <br> Pe <br> Pi <br> M <br> ] |
| 144 | 18:38 | Jamie | Right. |
| 145 | 18:39 | Jessica | And then if you have two.... |
| 146 | 18:41 | Jamie | Two toppings, that would be - it would be like sausage-peppers, sausage-pepperoni, sausagemushroom. So that's three for each one so it's twelve. Isn't it twelve? |
| 147 | 18:53 | Jessica | Yeah, yeah, okay, cause you.... Sausage and pepperoni which is one. Sausage and, oh, peppers is two. And then sausage and mushrooms is three - and |


|  |  |  | then you times four. |
| :---: | :---: | :---: | :---: |
| 148 | 19:02 | Jamie | Times four is twelve. |
| 149 | 19:04 | Jessica | Alright. Let me just put three times four which is twelve. [Her paper reads as follows: <br> S-Sausage <br> Pe <br> Pi <br> M <br> ] |
| 150 | 19:09 | Jamie | And then if you have three toppings. It's... |
| 151 | 19:12 | Jessica | Sausage, peppers, pepperoni. Sausage, peppers, mushroom. Sausage, pepperoni, mushroom. I think that's it. It can't be. Sausage, pepperoni, yeah. |
| 152 | 19:24 | Jessica | Sausage, peppers, pepperoni. Then you have sausage, pepperoni, mushroom. Sausage, peppers, mushroom. Is everything being used together? [She has written the following on her paper: <br> S Pe Pi <br> S Pi M <br> $S P e M]$ |
| 153 | 19:43 | Jessica | And then, oh, wait, so this is the sausage case. |
| 154 | 19:47 | Jamie | So then it would be the same for... |
| 155 | 19:49 | Jessica | Three? |
| 156 | 19:51 | Jamie | No, for four total. Sausage, pepperoni, pi. |
| 157 | 19:54 | Jessica | Yeah, but that's still three times four. |


| $\mathbf{1 5 8}$ | $19: 56$ | Jamie | Maybe it's the same -that's hard to believe. |
| :--- | :--- | :--- | :--- |
| $\mathbf{1 5 9}$ | $19: 59$ | Jessica | We'll go back to this one. Right now we have three <br> times four. |
| $\mathbf{1 6 0}$ | $20: 03$ | Jamie | And then the four is just one. |
| $\mathbf{1 6 1}$ | $20: 06$ | Jessica | Yeah, cause you can't change them. |
| $\mathbf{1 6 2}$ | $20: 08$ | Jamie | So maybe those two are the same. |
| $\mathbf{1 6 3}$ | $20: 11$ | Jessica | I don't know I have a feeling that it should be like the <br> four again because when we did the blocks problem, <br> remember? It was one, four, [looks through her |
| $\mathbf{1 6 4}$ | $20: 28$ | Jessica | notebook]. It was one, four, six, four, one. Or that's <br> just maybe something else. |
| $\mathbf{1 6 5}$ | $20: 36$ | Jamight, do you want to try like see if we can figure it |  |
| out with this? Alright. We have to figure out which |  |  |  |
| color is going to be which. |  |  |  |, | Jamie it down next to it. Sausage say yellow. |
| :--- |
| $\mathbf{1 6 6}$ |
| $20: 39$ |
| Jessica |


| 176 | 20:51 | Jessica | We're not supposed to use those. [inaudible] I don't know how much. |
| :---: | :---: | :---: | :---: |
| 177 | 20:57 | Jamie | We might not have enough. |
| 178 | 20:58 | Jessica | No, we will. |
| 179 | 21:01 | Jessica | Alright so three types. Three... |
| 180 | 21:04 | Jamie | [inaudible] |
| 181 | 21:07 | Jessica | We might. |
| 182 | 21:08 | Jamie | You know? You know what I mean like how you have three choices? |
| 183 | 21:12 | Jessica | Yeah, it's like the towers. |
| 184 | 21:12 | Jamie | Maybe it's something to the - maybe you can express it as $m$ to the $\ldots$. or something. See what I mean? Instead of colors, make it choices. |
| 185 | 21:21 | Jessica | Yeah. |
| 186 | 21:22 | Jamie | So like if you have four choices it would be like your colors. Right? |
| 187 | 21:30 | Jessica | Yeah, I think we should just try to work it out first to figure out if we can see the pattern. |
| 188 | 21:33 | Jamie | Yeah, okay. We'll do the sausage-yellow and then Pe is blue? |
| 189 | 21:41 | Jessica | You're gonna do... |
| 190 | 21:45 | Jessica | And then white |
| 191 | 21:46 | Jamie | White. [She builds a 3-tall tower - YBW. B represents blue and Bl represents Black] |
| 192 | 21:46 | Jessica | Cause we're only doing three tall. |
| 193 | 21:50 | Jessica | And then I'll do the next one which should be |


|  |  |  | sausage, pepperoni, and mushroom. [She builds a 3tall tower - YBlW] |
| :---: | :---: | :---: | :---: |
| 194 | 22:02 | Jamie | Mushroom. |
| 195 | 22:06 | Jessica | Alright and that's the only two that you could have. Because like if you have sausage and mushroom - the only choices are pepperoni and peppers. So let's start with... [They have two 3-tall towers -YBW and YBlW] |
| 196 | 22:19 | Jamie | So is it six? |
| 197 | 22:22 | Jessica | Well let's see, if you have the two here. |
| 198 | 22:26 | Jamie | Do you know what I mean? So this is which one? [Pointing to the YBW tower] This is sausage? |
| 199 | 22:30 | Jessica | This is sausage, peppers, and mushroom [pointing to the YBW tower]. And this is sausage, pepperoni, and mushroom [pointing to the YBlW tower]. |
| 200 | 22:38 | Jamie | So which one are we missing? [Jamie points to Jessica's notebook.] |
| 201 | 22:39 | Jessica | Now we need to move on to - without sausage. |
| 202 | 22:44 | Jamie | But why do we have three here [pointing to Jessica's notebook] and two here? [Pointing to towers.] |
| 203 | 22:50 | Jessica | Oh, because we didn't put these two together [pointing to the blue and the black cube in each of the two towers]. We didn't do sausage, plus - oh, you already made that. That's the third one we needed. [She has another 3-tall tower which is $Y B B l$ ] |
| 204 | 23:00 | Jamie | Yeah, I was like. |
| 205 | 23:01 | Jessica | Yeah, that was the sausage, peppers, pepperoni. [Pointing to the 3-tall tower - YBBl] Right? |
| 206 | 23:05 | Jamie | Yeah. Now look at this though. |


| 207 | 23:07 | Jessica | No, wait, wait. That's not - it's sausage, peppers, and mushroom [pointing to each individual cube in the 3tall tower - YBBl]. Is what we have. |
| :---: | :---: | :---: | :---: |
| 208 | 23:15 | Jamie | Sausage, peppers, and mushroom. [Pointing at the notebook.] |
| 209 | 23:16 | Jessica | Yep. Sausage, pepperoni, pepperoni... Oh, white, white is pepperoni. [Picks up the YBlW tower.] |
| 210 | 23:23 | Jamie | Just switch the white and the black. |
| 211 | 23:25 | Jessica | Sausage, peppers, and pepperoni which is white. Which is this one [picks up and places down the YBW tower]. Which is the first one we had. Then sausage, pepperoni, and mushroom [places the YWBl tower next to the YBW tower]. Then we had sausage, peppers, and mushroom. Yeah, that was right. [Places the YBBl tower next the other two towers.] Right? So those were the three. |
| 212 | 23:54 | Jamie | Now you can do the same thing for... |
| 213 | 23:56 | Jessica | For the other four. |
| 214 | 23:58 | Jamie | For the other, for the other three. |
| 215 | 24:00 | Jessica | Oh yeah, that's right. The other three. |
| 216 | 24:04 | Jamie | Three. |
| 217 | 24:05 | Jessica | There's not one other choice? |
| 218 | 24:07 | Jamie | Well let's try it. |
| 219 | 24:11 | Jessica | You can either have... |
| 220 | 24:12 | Jamie | What about the black in the middle? |
| 221 | 24:15 | Jessica | It doesn't matter placement. Like this one... |
| 222 | 24:16 | Jamie | Well, that's true. |


| $\mathbf{2 2 3}$ | $24: 17$ | Jessica | Placement doesn't matter. |
| :--- | :--- | :--- | :--- |
| $\mathbf{2 2 4}$ | $24: 18$ | Jamie | Maybe it's the same then. |
| $\mathbf{2 2 5}$ | $24: 21$ | Jessica | Well, let's see. You can have it without mushrooms or <br> with mushrooms. You can have it without peppers or <br> with peppers. Or you can have it without pepperoni or <br> with pepperoni. |
| $\mathbf{2 2 6}$ | $24: 41$ | Jamie | That's it. |
| $\mathbf{2 2 7}$ | $24: 42$ | Jessica | Yeah, and then..... What about, um, those... this one <br> is going to be - can I have a black please? Because <br> you can do one - like this one is obviously going to be <br> different so it won't be part of that group. But you can <br> have the one that has one sausage. [She builds a WBBl <br> tower.] |
| $\mathbf{2 2 8}$ | $25: 04$ | Jamie | Right but.... |
| $\mathbf{2 2 9}$ | $25: 05$ | Jessica | But that's going to be part of the other group. |
| $\mathbf{2 3 0}$ | $25: 06$ | Jamie | Exactly. |
| $\mathbf{2 3 1}$ | $25: 08$ | Jessica | It probably is twelve. |
| $\mathbf{2 3 2}$ | $25: 08$ | Jamie | Why don't we make them all? |
| $\mathbf{2 3 3}$ | $25: 11$ | Jessica | Let's make them all and see... |
| $\mathbf{2 3 4}$ | $25: 12$ | Jamie | Alright, I'll do this one and you do, um... |
| $\mathbf{2 3 5}$ | $25: 14$ | Jessica | So you are going to start with pepperoni? |
| $\mathbf{2 3 5}$ | $25: 16$ | Jessica | Alright, so, okay, so yours is pepperoni and I'll do the <br> peppers. |
| $25: 27$ | Jessica | Here, put this in between us. [Places the notebook, on <br> the table, in between the two of them.] |  |
| Yeah. |  |  |  |


| 239 | 25:28 | Jamie | Yeah, I'm confused. |
| :---: | :---: | :---: | :---: |
| 240 | 25:29 | Jessica | So that you can read it. So you're doing pepperoni and I'm doing peppers. |
| 241 | 25:34 | Jamie | Right. |
| 242 | 25:40 | Jessica | It's not going to be twelve because we are going to have to take some out, like, like that. Alright. |
| 243 | 25:46 | Jamie | Yeah, because see in this problem the order doesn't matter. That's the problem. |
| 244 | 25:50 | Jessica | Yeah. Hate that... |
| 245 | 25:53 | Jamie | 'Cause the other one you could do every single combination. |
| 246 | 25:56 | Jessica | Yeah, because you didn't have to take... |
| 247 | 25:56 | Jamie | Because you added them. Right. |
| 248 | 26:00 | Jessica | Alright, we'll make them all and then take out anything that has more than the same. |
| 249 | 26:02 | Jamie | Okay. Okay, so we got pepperoni. |
| 250 | 26:12 | Jessica | If you have pepperoni, peppers, and mushroom. |
| 251 | 26:14 | Jamie | Okay, I'll write this down. |
| 252 | 26:19 | Jessica | Oh, yeah, we're going to have to write everything down. Alright, so mine's going to be peppers, sausage, and pepperoni. Peppers, sausage and mushroom. Peppers with pepperoni, mushroom. Alright. [She writes in her notebook the following: Pe S Pi <br> PeSM <br> Pe Pi M] |


| 253 | 26:43 | Jamie | [The camera focuses on Jamie's notebook and she had the following written down in her notebook: |
| :---: | :---: | :---: | :---: |
| 254 | 26:53 | Jamie | Why am I getting four? |
| 255 | 26:56 | Jessica | What did you get? Pepperoni, peppers, mushroom. |
| 256 | 27:00 | Jamie | Pepperoni, mushroom, peppers. That's the same as that. |
| 257 | 27:05 | Jessica | Yeah. |
| 258 | 27:05 | Jamie | That's why. [She crosses off Pepp-Mush-Peppers from her list (the second column)] Pepperoni, sausage, mushroom. And then pepperoni, sausage, peppers. |
| 259 | 27:12 | Jessica | Alright. I keep going to take apart our other stuff [laughing]. |
| 260 | 27:19 | Jamie | That's, that's up there. So... |
| 261 | 27:19 | Jessica | Peppers, sausage, mushroom - that's not mushroom. [Builds a BYBl tower.] |
| 262 | 27:33 | Jamie | Pepperoni is white. |
| 263 | 27:35 | Jessica | We are going to have a lot less of.... |
| 264 | 27:37 | Jamie | Is yellow and mushroom is black. [She has built two towers - WBBl and WYBl.] |
| 265 | 27:42 | Jessica | Pepperoni, peppers.... |
| 266 | 27:50 | Jamie | And white... I found a pattern. |


| 267 | $27: 53$ | Jessica | You found a pattern? |
| :--- | :--- | :--- | :--- | :--- |
| $\mathbf{2 6 8}$ | $27: 55$ | Jamie | Let me just figure this out. Pepperoni, white. Sausage <br> is yellow. Peppers.... I found a pattern. I think. Maybe <br> not. Is that right? |
| $\mathbf{2 6 9}$ | $28: 16$ | Jessica | And then we need the last one which is just mushroom <br> which is just black, yellow, blue. Black, yellow, <br> white. And black, blue, white. [She builds three more <br> towers - BlYB, BlYW, BlBW. Together, they have a <br> total of twelve towers as follows: |


| 276 | 29:02 | Jamie | That should be it. |
| :---: | :---: | :---: | :---: |
| 277 | 29:03 | Jessica | Well let's just check. One black. One black. |
| 278 | 29:05 | Jamie | This is the same one. [Holding up a tower that has a black, yellow and white cube.] |
| 279 | 29:09 | Jessica | Which one? This one. [Holding up the other tower that has a black yellow and white cube.] |
| 280 | 29:10 | Jamie | Yeah. |
| 281 | 29:10 | Jessica | Alright. So take that one out [Jamie puts one of the towers with a black, yellow, and white cube to the side]. Here, let's do one black first and see if there is anything alike. We got black, blue, white. Black, blue, yellow. Black, blue, yellow. |
| 282 | 29:23 | Jamie | That's the same. [Jessica puts to the side one of the two towers that has a black, blue and yellow cube.] |
| 283 | 29:25 | Jessica | Black white yellow. Okay, now the yellows. These two are the same. [She places one of the two towers with a blue, yellow, and white cube to the side. ] |
| 284 | 29:33 | Jamie | It came out to four. [They have four towers as follows: <br> ] |
| 285 | 29:36 | Jessica | Blue with yellow and white. Blue with yellow and black. Blue with white and black. Yeah, you can only have four choices. Which makes me doubt this one [pointing to her notebook]. |
| 286 | 29:46 | Jamie | That one's wrong. |
| 287 | 29:47 | Jessica | Yeah. So this is...Well, it probably... |


| $\mathbf{2 8 8}$ | $29: 51$ | Jamie | That makes sense though. |
| :--- | :--- | :--- | :--- |
| $\mathbf{2 8 9}$ | $29: 52$ | Jessica | This is going to be six. Because notice it was six with <br> the towers - it's the same. |
| $\mathbf{2 9 0}$ | $30: 00$ | Jamie | But why is it the same? |
| $\mathbf{2 9 1}$ | $30: 02$ | Jessica | I don't know. |
| $\mathbf{2 9 2}$ | $30: 03$ | Jamie | So let's do this one if this one's six then we know <br> that... |
| $\mathbf{2 9 3}$ | $30: 07$ | Jessica | Then we know it's the same. Alright. So it's going to <br> be two colors. Do you want to do these two with the <br> two with the two....? |
| $\mathbf{2 9 4}$ | $30: 15$ | Jamie | Yeah. |
| $\mathbf{2 9 5}$ | $30: 16$ | Jessica | And I'll do the bottom two. |
| $\mathbf{2 9 6}$ | $30: 17$ | Jamie | Yeah. |
| $\mathbf{2 9 7}$ | $30: 18$ | Jessica | We're still going to need all of the four colors. So let's <br> just ... |
| $\mathbf{2 9 8}$ | $30: 21$ | Jamie | Yellow and blue, right? |
| $\mathbf{2 9 9}$ | $30: 23$ | Jessica | I'm just putting everything in... |
| $\mathbf{3 0 0}$ | $30: 24$ | Jamie | Alright, so I'm doing the sausage one? |
| $\mathbf{3 0 1}$ | $30: 25$ | Jessica | You're doing sausage and peppers. I'm doing <br> pepperoni and mushrooms. |
| $\mathbf{3 0 4}$ | $30: 36$ | Jamie | Sausage and peppers and then... |
| $30: 39$ | Jessica | You can use, yeah, you can use all of the colors still <br> but it like... |  |
| 305 | Jamie | Oh, just starting with... |  |
|  | Jessica | Like it starts with sausage or it starts with peppers. <br> That's what I meant. It's easier to do it. Yeah, I know, |  |


|  |  |  | I wasn't making sense technically. |
| :---: | :---: | :---: | :---: |
| 306 | 30:49 | Jamie | It is six. |
| 307 | 30:51 | Jessica | It is? |
| 308 | 30:53 | Jamie | It is six. Yep. It's definitely six. I don't know where we got that from. Sausage-peppers, sausagepepperoni, sausage-mushrooms. [She has made three 2-tall towers: $\begin{array}{ccc} Y & Y & Y \\ B & W & B l \\ ] & & \end{array}$ |
| 309 | 31:04 | Jessica | Alright. Alright and I have my pepperoni and mushroom. Pepperoni and peppers. And then pepperoni and sausage. [She has made three 2-tall towers: $\begin{array}{llll} W & W & W \\ B l & B & Y \\ ] & & \end{array}$ |
| 310 | 31:18 | Jamie | That's it. So it's six. |
| 311 | 31:20 | Jessica | Well this is for... this is sausage [holding the three towers with yellow as the top cube] and this pepperoni [holding the three towers with white as the top cube]. But notice right here [compares the WY tower with the YW tower]. |
| 312 | 31:28 | Jamie | That one's the same. |
| 313 | 31:29 | Jessica | These two are the same. So take that one out [puts the WY tower to the side] and then when we do the other ones, we're still going to have to taking one out for each one. |


| 314 | 31:36 | Jamie | Right. So is it five? |
| :---: | :---: | :---: | :---: |
| 315 | 31:38 | Instructor | So explain to me what you're doing. |
| 316 | 31:40 | Jessica | Right now we're trying to figure out - because we couldn't figure out three without actually doing it. And we noticed that it's one, four and we think this is wrong, we think this is supposed to be six, not twelve. |
| 317 | 31:50 | Instructor | Okay. |
| 318 | 31:51 | Jessica | And then 4, 1 which is exactly like our... |
| 319 | 31:54 | Jamie | Towers problem. |
| 320 | 31:54 | Jessica | Our towers problem which was one, four, six, four one. |
| 321 | 31:57 | Instructor | Alright so you're trying to sort of match it up with towers, somehow. |
| 322 | 32:00 | Jessica | We're trying to see if it's like towers. |
| 323 | 32:02 | Instructor | Okay. You're getting the same numbers anyhow. |
| 324 | 32:06 | Jessica | Exactly. That's why we're trying to see if it was like that. |
| 325 | 32:08 | Instructor | Well, you think you're getting the same numbers but you still don't exactly have the six two topping - the six that you think you going to get for two toppings. Is that right? |
| 326 | 32:15 | Jessica | Yeah, we're working on that now. |
| 327 | 32:16 | Instructor | Okay. |
| 328 | 32:17 | Jessica | That's what we're working on now. Alright. I just want to see it because I need to see it for it to work for me. Black and blue. And black and white. [She builds three towers all with a black cube on top $-B l Y, B l B$, and $B l W$.] And then the last one is with the blues on |


|  |  |  | top which is this, that, and that. [She builds three towers all with a blue cube on top - BY, BW, and WBl.] |
| :---: | :---: | :---: | :---: |
| 329 | 32:43 | Jessica | Alright. So let's do all of the ones like we did the last time. [She lines up all of the two tall towers that have one black cube.] All the ones that have one black. Oh, wow. Got to take two out there. [She takes two of the three towers that have one white and one black cube.] |
| 330 | 32:55 | Jamie | That's four. |
| 331 | 32:58 | Jessica | And... [She takes out one of the two towers that have a yellow and a black cube.] |
| 332 | 32:59 | Jamie | That's three. |
| 333 | 33:01 | Jessica | So those are all different. Now we do it with the one white. [She lines up the five towers that have one white cube.] Which is taking those two out. [She takes out two towers - one has a white and a blue cube and the other has a white and a yellow cube.] Because these two are the same as those two. And then the one with the yellow. [She lines up the towers with one yellow cube.] We still didn't get six. |
| 334 | 33:20 | Jamie | We got five. |
| 335 | 33:21 | Jessica | Oh wait. Six. [Takes out a duplicate tower that has one blue cube and one yellow cube.] |
| 336 | 33:26 | Jamie | We got six. |
| 337 | 33:26 | Jessica | We got six, like we wanted. [The have six two tall towers as follows: $\begin{array}{llllll} W & W & B & B l & B l & B l \\ B & Y & Y & Y & B & W \\ ] & & & & & \end{array}$ |


| 338 | 33:28 | Jamie | It's the same. |
| :---: | :---: | :---: | :---: |
| 339 | 33:29 | Jessica | Yeah, but why is it the same? |
| 340 | 33:31 | Jamie | Alright, let's think about this. See now, why I don't understand why it's the same is because in the towers problem... |
| 341 | 33:39 | Jessica | It was positional. And this one isn't positional |
| 342 | 33:43 | Jamie | It's not. |
| 343 | 33:45 | Jessica | That's why it doesn't make sense of why it's the same numbers, right? |
| 344 | 33:48 | Jamie | We got the same numbers. [Talking to the instructor.] |
| 345 | 33:49 | Jessica | [Talking to the instructor.] We got the same numbers but the thing is we can't make a connection on why it would be the same. Seeing as this one is positional and this one isn't. 'Cause we had to take all these, here, out, because they happened more than once. |
| 346 | 34:02 | Instructor | Okay, so what you're saying is black represents what? |
| 347 | 34:08 | Jessica | Black represents mushrooms. And we have white representing pepperoni, blue representing peppers, and yellow representing sausage. |
| 348 | 34:18 | Instructor | So that means the two topping pizzas are going to be two tall towers. And the four topping pizza - there's only going to be one of them that's four tall. |
| 349 | 34:28 | Jessica | Yeah. |
| 350 | 34:29 | Instructor | So the heights of your towers are changing with this mapping. We could say, right? |
| 351 | 34:36 | Jessica | If they're two tall and you can only have two toppings, our toppings are now our colors and our tall is now - it's what our $n$ was. It's still $n$. So it does... |


| $\mathbf{3 5 2}$ | $34: 50$ | Jamie | It has a connection. |
| :--- | :--- | :--- | :--- |
| $\mathbf{3 5 3}$ | $34: 51$ | Jessica | It has a connection. Even though it's not positional. <br> It's non-positional. |
| $\mathbf{3 5 4}$ | $34: 55$ | Instructor | So you should write all of that down as carefully as <br> you can to explain, say to me, if I wanted to <br> reconstruct what you did so that I could reconstruct it <br> perfectly. |
| $\mathbf{3 5 5}$ | $35: 05$ | Jessica | Okay, alright. | | $\mathbf{3 5 6}$ | $35: 08$ | Jamie | I think you're better with the wording. |
| :--- | :--- | :--- | :--- |


|  |  |  | topping. Which would be... wait, oh, wait, that's the colors. So you have one topping....Wait. Why did this work for two toppings and not for one? |
| :---: | :---: | :---: | :---: |
| 365 | 36:54 | Jamie | One topping, you have four choices. It's just the choices. |
| 366 | 36:58 | Jessica | Just choices. |
| 367 | 36:58 | Jamie | You know, it's not like you're building the tower. |
| 368 | 37:02 | Jessica | Oh, that's right because it's only one tall. |
| 369 | 37:04 | Jamie | That's like induction. That's like that first step, $n$ equals1, you're just testing it, right? |
| 370 | 37:11 | Jessica | Well yeah, cause you only have the.... |
| 371 | 37:13 | Jamie | [inaudible] |
| 372 | 37:14 | Jessica | Oh, duh - the colors is the base, right? |
| 373 | 37:17 | Jamie | Yeah. |
| 374 | 37:17 | Jessica | You have four colors. |
| 375 | 37:18 | Jamie | Four to the zero. |
| 376 | 37:20 | Jessica | Four to the... |
| 377 | 37:21 | Jamie | Four to the one, I'm sorry |
| 378 | 37:22 | Jessica | Yeah, four to the one. |
| 379 | 37:23 | Jamie | Is four. |
| 380 | 37:24 | Jessica | Equals four because you can only have one of each topping. I'm gonna have to write this out again but.... If you do two toppings, you get six pizzas... you have.... You have two because this is um... what is the $n$ before? Our $n$ was what? I'm trying to remember. [She has written the following in her |


|  |  |  | notebook: <br> zero toppings - one pizza - plain (common <br> knowledge) <br> one topping - four pizzas $-4^{1}=4$ <br> two topping - six pizzas - 2] |
| :---: | :---: | :---: | :---: |
| 381 | 37:58 | Jamie | $N$ was the height, remember? $M$ was the color; $n$ was the height. |
| 382 | 38:03 | Jessica | Yes, that's right. Alright, so we have two colors to two tall - but that doesn't make sense. And we got six. |
| 383 | 38:14 | Jamie | How did you get six on the other one though? Go back. How did you get six then? |
| 384 | 38:26 | Jessica | [Jessica flips her notebook back to a page that contains their results to the towers problem.] Because... it was the two.... Difference is this one wasn't positional. Oh, do you know what, we were only using two colors in this and we got six because it's not positional. And we were using four colors in this but it is positional. |
| 385 | 38:51 | Jamie | But two to the what gave us six? |
| 386 | 38:54 | Jessica | There's nothing two to the anything that could give you six. So it would have to be... |
| 387 | 38:59 | Jamie | So that didn't even fit...? |
| 388 | 39:00 | Jessica | That doesn't even fit a mathematical formula. |
| 389 | 39:02 | Jamie | Okay. That's confusing though. |
| 390 | 39:06 | Jessica | Yeah, now I'm confused again. |
| 391 | 39:08 | Jamie | We're confused. |
| 392 | 39:13 | Jessica | The three topping was four pizzas. And then four |


|  |  |  | toppings.... [inaudible] was one pizza. Oh, that's just common knowledge. Because you can't change even if you, it doesn't matter what order you put them in. Hum....[She has written the following in her notebook: <br> zero toppings - one pizza - plain (common knowledge) <br> one topping - four pizzas $-4^{1}=4$ <br> two topping - six pizzas - <br> three toppings - four pizzas <br> four toppings - one pizza (common knowledge)] |
| :---: | :---: | :---: | :---: |
| 393 | 40:15 | Jessica | I'm not even sure. |
| 394 | 40:28 | Jessica | I'm going to look at Pascal's triangle - maybe it has something that has something with that. |
| 395 | 40:33 | Jamie | Alright, I have it written down if you want to look at this. [She opens her notebook to Pascal's triangle and places the notebook next to Jamie's notebook.] |
| 396 | 40:45 | Jessica | Three toppings is four. |
| 397 | 40:49 | Jamie | Ah-ha, wait a minute, I see something. I see something. |
| 398 | 40:52 | Jessica | When you add up... |
| 399 | 40:54 | Jamie | Two, three, four. |
| 400 | 40:58 | Jessica | Yeah, but what about... |
| 401 | 41:00 | Jamie | Keep going.... |
| 402 | 41:01 | Jessica | Two is six. |
| 403 | 41:03 | Jamie | Two is six? Two, three, four, five. Are you sure it's |


|  |  |  | not six and not five. |
| :---: | :---: | :---: | :---: |
| 404 | 41:10 | Jessica | Yeah, look - the three whites, they're all different. The three yellows, they're all different. [Pointing to the two tall towers.] |
| 405 | 41:18 | Jamie | And the three blues. |
| 406 | 41:19 | Jessica | And the three blacks, they're all different. And the three blues are all different. |
| 407 | 41:27 | Jamie | There's gotta be something. It just like looks too... Maybe since we have three - three, four, five, six. That's six. [Pointing to Pascal's triangle.] |
| 408 | 41:38 | Jessica | This is all six? |
| 409 | 41:40 | Jamie | No. |
| 410 | 41:40 | Jessica | No? It's not because this is three, wait... |
| 411 | 41:42 | Jamie | This is three, four, five, six. |
| 412 | 41:46 | Jessica | But the whole thing is why... |
| 413 | 41:47 | Jamie | Two, three, four, five, six. It's gotta be something. It looks too familiar to not be anything. Do you know what I mean? But now if you had four. |
| 414 | 42:07 | Kim | Oh, I have another question. |
| 415 | 42:09 | Jessica | I don't know. |
| 416 | 42:09 | Kim | Can it be thin crust or thick crust? That's another [laughing] |
| 417 | 42:13 | Instructor | [The camera focuses on the board and the instructor has written the following: |


|  |  |  | $\begin{array}{cc} 2^{n} \quad 2=2 \text { colors } & 2^{n} \quad 2= \\ n=\text { height } & n=\# r \text { of toppings to choose }] \end{array}$ |
| :---: | :---: | :---: | :---: |
| 418 | 42:16 | Instructor | Do you know what? We can do half pizzas. But we are not going to do that. |
| 419 | 42:20 | Jessica | We can't even figure out this! [laughing] |
| 420 | 42:22 | Instructor | You guys sort of started to say some stuff about the two but I didn't hear it exactly. So, when you're ready come up here and write what the two equals here. [on the board] |
| 421 | 42:30 | Francesca S. | Is it because it is one plain pizza and one pizza with toppings? |
| 422 | 42:34 | Instructor | Okay, so... |
| 423 | 42:36 | Francesca S. | There's only like two types. |
| 424 | 42:37 | Instructor | So what words do you want me to put up here? Two represents what? |
| 425 | 42:40 | Francesca S. | Yeah, like two types... I don't know. |
| 426 | 42:42 | Jessica | Two toppings. |
| 427 | 42:43 | Francesca S. | Yeah, like... |
| 428 | 42:44 | Kim | What about deep dish pizza? See now I'm in the pizza mode. |
| 429 | 42:48 | Instructor | Equals two types. [She writes on the board $2=$ two types] |
| 430 | 42:50 | Francesca S. | Yeah, like two types of pizzas. |
| 431 | 42:52 | Instructor | Two types. |
| 432 | 42:53 | Francesca S. | Like, one plain or either one that has toppings on it. |
| 433 | 42:55 | Instructor | Okay. [She continues to write on the board $2=$ two types; plain or topp.] |


| 434 | 42:57 | Jessica | We didn't even get that far. |
| :---: | :---: | :---: | :---: |
| 435 | 43:03 | Instructor | Okay, well if you didn't.... But you guys did see some of this so let's talk about it together before we move on. Those guys noticed and you noticed too, I thought, the $1,4,6,4,1$. |
| 436 | 43:13 | Jessica | Yeah, that we got. |
| 437 | 43:14 | Instructor | Okay, so and the word...Kim, what was the word we were talking about there? |
| 438 | 43:18 | Kim | Isomorphism. |
| 439 | 43:19 | Instructor | Yeah, so the isomorphism is you keep getting the same answers to the two different problems. Right? And the question is... I want to know more about isomorphism, which means that you got to tell me exactly why you are getting the same answers. |
| 440 | 43:32 | Instructor | And you are sort of getting there, right? They told me that this is two to the $n$ because two means two colors and $n$ is the height. <br> [She points to the formula for the towers problem on the board: $\begin{aligned} & 2^{n} \quad 2=2 \text { colors } \\ & n=\text { height }] \end{aligned}$ |
| 441 | 43:40 | Instructor | And over here you said $n$ is the number of toppings. So this is one component of the isomorphism. You're telling me... What are you telling me about the relationship between height and number of toppings? <br> [She points to the formula for the pizza problem on the board: $2^{n} \quad 2=2 \text { types }- \text { plain or topp. }$ <br> $n=\#$ of toppings to choose from] |
| 442 | 43:53 | Francesca S. | They equal each other? |
| 443 | 43:54 | Instructor | The height is the same as the number of toppings, for some reason, right? When you got a height of three, |


|  |  |  | you get eight and when you have the number of toppings is three, you get eight. So that's part of the isomorphism. Okay? And now you see two's in both places. So there's something with two's also. This one is very straight forward, two colors. And this is ... I didn't quite get this. Two types? |
| :---: | :---: | :---: | :---: |
| 444 | 44:16 | Francesca S. | I don't know. |
| 445 | 44:17 | Instructor | Yeah, but you're getting there. |
| 446 | 44:20 | Jessica | It's a good idea. At least you guys had the idea. |
| 447 | 44:22 | Instructor | And you guys were actually... you saw there was something to do with the towers too. And you were saying that the different colors equaled the different toppings. |
| 448 | 44:30 | Jessica | The different toppings |
| 449 | 44:31 | Instructor | However, you didn't get two colors that way you got lots of colors. So you were actually saying... what were you saying? |
| 450 | 44:38 | Jessica | The difference between them or....? |
| 451 | 44:40 | Instructor | You were saying... you were saying it wasn't the height equal to the number of toppings, you said something else was equal to the number of toppings. |
| 452 | 44:51 | Jessica | What did we say was equal to the number of toppings? [To Jamie] The colors? |
| 453 | 44:57 | Instructor | Yeah, yeah, it was the colors. |
| 454 | 45:04 | Instructor | It was colors, right? You had four colors representing four different toppings. And then the heights did not represent the number of toppings. And you guys found something else that seemed a little more straightforward, actually. The height and the number of toppings seemed to have something to do with each |

$\left.\begin{array}{|l|l|l|l|}\hline & & & \text { other. } \\ \hline \mathbf{4 5 5} & 45: 20 & \text { Instructor } & \begin{array}{l}\text { Okay. So that's the next thing to be working on. And, } \\ \text { let's see....we have almost 20 minutes so we have } \\ \text { some time to think about that. So there's some } \\ \text { relationship here and they confirmed it actually } \\ \text { because they did the three topping pizzas, right? And } \\ \text { you got eight. In fact, you did the two topping pizzas } \\ \text { and got four and so on. So my suggestion would be } \\ \text { maybe.... }\end{array} \\ \hline \mathbf{4 5 6} & 45: 47 & \text { Jessica } & \text { You have eight? } \\ \hline \mathbf{4 5 7} & 45: 48 & \text { Instructor } & \begin{array}{l}\text { For the three topping pizzas. For three... when there's } \\ \text { three toppings to choose from. Right? We'll leave off } \\ \text { mushrooms so there's only pepper, pepperoni and }\end{array} \\ \text { sausage to choose from. When you got three choices, } \\ \text { there's only eight possible pizzas, right? }\end{array}\right\}$

|  |  |  | you are heading in the right direction. |
| :---: | :---: | :---: | :---: |
| 465 | 46:40 | Instructor | Okay, so exactly what is the isomorphism? To say that you have to be able say, well number of towers equals number of pizzas. That's one thing. Height equals number of toppings, that's another thing. What does that two have to do with it? And... and eventually you should be able to tell me - here's a pizza, here's a tower, they map onto each other. Right? Isomorphisms - the elements of one set have to map on to the elements of the other set. |
| 466 | 47:07 | Jessica | I only have one problem with it because I can't figure out anything.... If two is our base, there's nothing in the $n$ that's a whole number that can give us six. |
| 467 | 47:23 | Instructor | Yeah, you're right two to the something doesn't give you six, here, either. But two to the something gives you the total number. Okay. |
| 468 | 47:34 | Jamie | Does it have anything to do with the triangle? |
| 469 | 47:38 | Instructor | Pascal's Triangle? Um... |
| 470 | 47:40 | Jessica | We were trying to see if it did at all... and we didn't see it. |
| 471 | 47:52 | Instructor | [inaudible] Well, you can write Pascal's triangle on the board. [Talking to Jessica] |
| 472 | 48:10 | Jessica | [Jessica goes to the board and writes Pascal's Triangle.] I don't even need a notebook [laughing]. <br> [She has written the following on the board: $\begin{gathered} 1 \\ 11 \\ 121 \\ 1331 \end{gathered}$ |


|  |  |  | 14641 <br> $1510105 l]$ |
| :--- | :--- | :--- | :--- |
| $\mathbf{4 7 3}$ | $48: 26$ | Instructor | Yeah, you can stop with that row actually. Okay, now, <br> you will look at your papers, you will look up there, <br> and you will tell me what you see. |
| $\mathbf{4 7 4}$ | $48: 33$ | Kim | Patterns! |
| $\mathbf{4 7 5}$ | $48: 34$ | Instructor | Yeah, now look at your paper again and what else do <br> you see up there? |
| $\mathbf{4 7 6}$ | $48: 38$ | Kim | Oh, 1, 4, 6, 1! |


| 489 | 49:02 | Instructor | The towers. So what does that $1,4,6,4,1$ represent? |
| :---: | :---: | :---: | :---: |
| 490 | 49:09 | Jessica | Our total.... |
| 491 | 49:11 | Instructor | Well what did the one represent? You're sitting there by it Francesca so you can tell us. |
| 492 | 49:14 | Francesca S. | The one is um... the plain or the one with the four colors. |
| 493 | 49:19 | Jessica | With everything. |
| 494 | 49:20 | Francesca S. | Yeah, with everything. |
| 495 | 49:21 | Instructor | Okay, alright. So that's interesting 'cause you saw it in both cases. So that one is either, in terms of pizzas, it was the plain pizza. [Writes on the board. $1=$ plain] And in terms of towers, which tower was it? |
| 496 | 49:31 | Jessica | It was the one with that had all the colors. All the same colors. All blue or all white for us. |
| 497 | 49:35 | Instructor | All ... let's say all white which is no blue. [Writes on the board $1=$ all white ( 0 blue)] |
| 498 | 49:43 | Instructor | And the four for pizzas was which pizzas was that? |
| 499 | 49:47 | Jessica | Was one of each type of topping. The one topping pizzas. |
| 500 | 49:50 | Instructor | The one topping pizzas. [Writes on the board $4=1$ topping] And for the towers? |
| 501 | 49:59 | Jessica | The towers was the... um...was the single... |
| 502 | 50:08 | Instructor | How did we describe those in two words? |
| 503 | 50:10 | Jessica | In two words? I'm trying to think - it's not working. [laughs] |
| 504 | 50:16 | Instructor | A number and a color. |


| $\mathbf{5 0 5}$ | $50: 20$ | Jessica | For every.... |
| :--- | :--- | :--- | :--- |
| $\mathbf{5 0 6}$ | $50: 26$ | Francesca S. | It's like ... It would be like three orange, one white. |
| $\mathbf{5 0 7}$ | $50: 29$ | Instructor | So it's - so since we're focusing on the... the one <br> blue. [Writes on the board $1=1$ blue] And six <br> equals... What is the six? |
| $\mathbf{5 0 8}$ | $50: 38$ | Jessica | The six is... |
| $\mathbf{5 0 9}$ | $50: 39$ | Instructor | In terms of pizzas? |
| $\mathbf{5 1 0}$ | $50: 42$ | Jessica | Two topping pizzas. |
| $\mathbf{5 1 1}$ | $50: 47$ | Instructor | Okay. [Writes $6=$ two toppings; $6=$ ] |$|$| Ind |
| :--- |


| 523 | 51:33 | Kim | Okay, well can I show you my paper to explain it better? |
| :---: | :---: | :---: | :---: |
| 524 | 51:35 | Instructor | Yes. |
| 525 | 51:38 | Kim | Oh, but can you go back to the? Yeah. [Indicating the slide that contains Pascal's triangle.] Okay, so you know how like if you add across you get like 1, 2, 4, $8,16,32$. And it's like backwards of what I did before. |
| 526 | 51:52 | Instructor | Yeah, what do you mean by if you add across? |
| 527 | 51:55 | Kim | Like, you know how like you have one and then 1 and 1 , so that's two. |
| 528 | 52:00 | Instructor | So the sum is two. |
| 529 | 52:02 | Kim | And the sum is four. The sum is... |
| 530 | 52:06 | Instructor | Eight. |
| 531 | 52:07 | Kim | Yeah. |
| 532 | 52:08 | Jessica | I think that was part of Pascal's triangle. |
| 533 | 52:09 | Kim | Well, yeah but that relates to what we did before. |
| 534 | 52:13 | Instructor | And that's our answer for four also. So if you start with row zero, row $n$ of Pascal's triangle gives you all the cases for towers and all the cases for pizzas. That's what you told me up here. |
| 535 | 52:24 | Kim | Yeah, that's what... |
| 536 | 52:26 | Instructor | Okay, we still want to hear... So what is the exact isomorphism? And for that... well that's what you are going to think about. I don't think we have time for this now. That's what you are going to think about for homework. Well, we got 10 minutes to think about it now. Yes, right, this is for example, so when you figure this out... [Take the WWWB tower out of |


|  |  |  | Jessica's hand] Sorry, I took this out of your hand. |
| :---: | :---: | :---: | :---: |
| 537 | 52:48 | Jessica | That's fine, I was just thinking. |
| 538 | 52:51 | Instructor | When you figure it out you're going to be able to take this one and say 'this is the something pizza'. And then when I give you a pizza, I'm going to say "the pepperoni/sausage pizza build me the tower that's isomorphic to that." That's what you are going to be able to do when you figure it out. Okay? |
| 539 | 53:07 | Instructor | Now, suggestion. Row four is kind of a pain because you have sixteen. Row three, you only got eight. I would suggest to work with row three because eight is easier. Now, not only do you have the numbers that go there, you can actually write each of the eight pizzas and each of the eight towers. |
| 540 | 53:29 | Instructor | The towers that goes with that one. The pizza that goes with that one. The three towers, list them. The three pizzas, list them. Write them all eight things down in their groups and see if you can just look at them and see exactly how they're related to each other. Do you know what I am saying? You're going to have three towers in this group, you're going to have three pizzas in this group. How can you match them up? Okay? |
| 541 | 53:54 | Jessica | [The camera focuses back on Jessica and Jamie's group.] Alright. First, I'm going to write down that the toppings equals height of towers. |
| 542 | 54:00 | Jamie | Toppings equals height of towers. [Writing in her notebook.] |
| 543 | 54:08 | Jessica | Which, okay, so we need... $1,3,3,1$. We're going to have eight pizzas and eight towers. Alright. So, let's just do the towers first so we're only going to have three colors. Eight towers two colors. |


| 544 | 54:35 | Jamie | Two cubed. Two cubed equals eight. She's trying to get to eight towers. |
| :---: | :---: | :---: | :---: |
| 545 | 54:43 | Jessica | Yeah, but how tall are we doing it? |
| 546 | 54:47 | Jamie | Three. |
| 547 | 54:48 | Jessica | We're doing three high. |
| 548 | 54:49 | Jamie | Two colors. |
| 549 | 54:50 | Jessica | Three high. |
| 550 | 54:51 | Jamie | Two cubed. |
| 551 | 54:53 | Jessica | Okay, so what do you want to use? Put black and white. We haven't done just black and white before. Alright so we're doing three high. |
| 552 | 55:08 | Jamie | Two high. No, three high, I'm sorry. |
| 553 | 55:12 | Jessica | Pick one. Alright, so we're doing three high. Alright. [inaudible] |
| 554 | 55:28 | Jessica | Alright, so we're doing three high, two colors. |
| 555 | 55:33 | Jamie | Does it matter positional? |
| 556 | 55:34 | Jessica | Not with the towers it doesn't. But with the... There's supposed to be eight of them? |
| 557 | 55:48 | Jessica | Oh, well, let's make the obvious one. [She builds the all black tower. She has built two other 3-tall towers $B W W$ and WBW.] |
| 558 | 55:51 | Jamie | And I'll take these. |
| 559 | 55:51 | Jessica | I'm gonna have to start taking those apart. Unless you don't want to use white. Unless you want to use yellow since you have more yellow. |
| 560 | 56:00 | Jamie | That's fine. [inaudible] [She has built two towers - |


|  |  |  | $B W W$ and BWB.] |
| :---: | :---: | :---: | :---: |
| 561 | 56:06 | Jessica | But we don't have enough whites. Taking stuff apart. |
| 562 | 56:12 | Jamie | Oh, wait - I'm doing the same one as you. |
| 563 | 56:14 | Jessica | No you're doing two black. I'm doing two white. |
| 564 | 56:18 | Jamie | Oh, but this one and this one are the same. [She compares the BWW tower she has created with Jessica's BWW tower.] |
| 565 | 56:21 | Jessica | We'll take it out later like we did before. |
| 566 | 56:25 | Jamie | Alright - that, that and that. [Referring to the three towers she has created - WWW, BWB, BWW] Which other one should I do? |
| 567 | 56:37 | Jessica | This one [points to the BWW tower]. You need two black together and one white on the bottom. These two are the same, so we will just say that's one. And these are all different? Yes. And that means we should have - three, six, seven. We need one more. |
| 568 | 57:01 | Jessica | What about? |
| 569 | 57:02 | Instructor | Alright, group them. |
| 570 | 57:03 | Jessica | Oh, I know one more - give me two white. I need a white. [She creates a BWW tower.] |
| 571 | 57:08 | Instructor | Okay, if this is a duplicate, I can take this away, right? |
| 572 | 57:10 | Jessica | Yeah. |
| 573 | 57:11 | Instructor | Okay. So now... |
| 574 | 57:13 | Jessica | Oh, no, that's that one. Yeah, it is. |
| 575 | 57:18 | Instructor | Alright so, yeah, I want to see that same organization or else I'm not going to be able to figure out if we got them all. So here's your zero white. Now show me the |


|  |  |  | one white cases. |
| :---: | :---: | :---: | :---: |
| 576 | 57:30 | Jessica | One white? These - there's this, there's this. There should be one more. [She puts the BWB tower and the $B B W$ tower together.] That's the one we didn't make. |
| 577 | 57:39 | Jamie | The top one. |
| 578 | 57:41 | Jessica | The top one is what we didn't make. [She creates the $W B B$ tower and places it with BWB and BBW.] |
| 579 | 57:43 | Instructor | Okay, so there's the no white, the one white, now where's the two white cases? |
| 580 | 57:49 | Jessica | Two white cases. |
| 581 | 57:50 | Instructor | And there's the three white case. |
| 582 | 57:54 | Instructor | Okay, now why do I like that? Because I see $1,3,3,1$. Right? Now you've got pizzas also that you organized by number of toppings, at least they did. Don't make them, write them. There's the 1 , the 3 , the 3 , the one. The no toppings, the one topping, the two topping, the three toppings. So, list them out and they will match up under here and then you will just figure it out. You will look at it and think hard and you will see what you see. |
| 583 | 58:18 | Jamie | So this is no topping. |
| 584 | 58:20 | Jessica | Yeah, we'll do this as plain. |
| 585 | 58:22 | Jamie | Plain. So you're going to write this under this - plain. |
| 586 | 58:28 | Jessica | Now we need to write what the three topping, what they are. |
| 587 | 58:33 | Jamie | Okay. |
| 588 | 58:34 | Jessica | Which ones are we going to use? Sausage, pepperoni, and mushroom? |


| 589 | 58:37 | Jamie | Yeah, that's fine. |
| :---: | :---: | :---: | :---: |
| 590 | 58:38 | Jessica | Alright so you have - sausage, pepperoni, mushroom. Sausage.... Oh wait. Are we still using all four? Okay, we're still using all four. [Writing in her notebook.] |
| 591 | 58:55 | Jamie | Yeah, you have to. |
| 592 | 58:56 | Jessica | Yeah, so sausage, pepperoni, mushroom. Sausage, pepperoni, |
| 593 | 58:59 | Jamie | Sausage... no peppers. |
| 594 | 59:01 | Jessica | Well, I was going to do pepperoni, peppers. |
| 595 | 59:02 | Jamie | Oh, okay. Sausage, pepperoni, and peppers? |
| 596 | 59:07 | Jessica | Yeah. And then you need sausage, peppers, and mushrooms. |
| 597 | 59:18 | Jamie | This has to go down here. |
| 598 | 59:19 | Jessica | Yeah. And then with this three. Oh, wait. We're doing - wait. No, we're not doing it right. You're supposed to do one topping, two toppings, three toppings. We're only supposed to be using the three, remember? [She erases what she has written in her notebook.] |
| 599 | 59:36 | Jamie | Oh. |
| 600 | 59:38 | Jessica | So you have...sausage, peppers, mushrooms. Let's just use peppers, it's easier. And then you have sausage and peppers. |
| 601 | 59:48 | Jamie | Now this is the next one. |
| 602 | 59:48 | Jessica | Sausage and mushrooms. And peppers and mushrooms. |
| 603 | 59:53 | Jamie | Sausage and peppers. |
| 604 | 59:54 | Jessica | Sausage and mushrooms. And then peppers and |


|  |  |  | mushrooms. And then with the last one is sausage, peppers, mushrooms. [She has written the following on her paper: $\begin{aligned} & \text { toppings }=\text { height of towers } \\ & \begin{array}{llll} 1 & 3 & 3 & 1 \\ \text { plain } & S & S P & S P M \\ & P & S M & \\ & & M & P M \end{array} \\ & \\ & \\ & \end{aligned}$ |
| :---: | :---: | :---: | :---: |
| 605 | 1:00:03 | Jamie | That's why. |
| 606 | 1:00:06 | Jessica | But what kind of connection are we supposed to be making? Is what I'm trying to figure out. I can't get the... |
| 607 | 1:00:12 | Jamie | Alright, um... let's figure this out here. $1,3,3,1$, six seven eight. |
| 608 | 1:00:20 | Jessica | Well look at this. [She holds up the BBB tower.] Right? Our plain is without toppings, this is without white. So without white so... |
| 609 | 1:00:34 | Jamie | This is all black. |
| 610 | 1:00:36 | Jessica | Then for this... |
| 611 | 1:00:38 | Jamie | You're gonna take... Yep, that's it. |
| 612 | 1:00:42 | Jessica | Sausage, pepper, mushroom. Alright so - Sausage pepper, mushroom. [She points to the white cube in the WBB, BWB, and BBW towers.] And then this one for our white is sausage and peppers [she points to the two white cubes in BWW]. Sausage and mushrooms [she points to the two white cubes in WBW]. Peppers and mushrooms [she points to the two white cubes in |


|  |  |  | $W W B]$. And this one is sausage, peppers, mushrooms [pointing to the tower that contains all white].So our white is our toppings. |
| :---: | :---: | :---: | :---: |
| 613 | 1:01:04 | Jamie | The white are the topping. |
| 614 | 1:01:05 | Jessica | For us, white is toppings. |
| 615 | 1:01:14 | Jamie | [inaudible] done. |
| 616 | 1:01:15 | Jessica | Yeah, but... I'm just trying to figure out other than like the obvious - like yeah, they're the same. There's no... |
| 617 | 1:01:27 | Jamie | I can't... a specific reason. |
| 618 | 1:01:33 | Jessica | It just bothers me because the towers are positional and these aren't. |
| 619 | 1:01:37 | Jamie | So how are they the same? |
| 620 | 1:01:42 | Jessica | We'll because, you know what when we did - notice how we did, when we did this and we got six [she holds six blue and white, 4-tall towers in her hand]. We did take two of them out because it used to be eight. We're taking stuff out. |
| 621 | 1:01:54 | Jamie | That's why. |
| 622 | 1:01:54 | Jessica | That's why it's the same. |
| 623 | 1:01:57 | Instructor | Okay, we got one minute to tell me what you got with the last minute. |
| 624 | 1:02:00 | Jamie | White equals topping. |
| 625 | 1:02:01 | Jessica | Yeah, we have our white is toppings. So, the first one - we used this to help us figure out. [Holding the black tower - BBB.] Got one tower, without white you got a plain because no toppings. Three towers with one white - three pizzas with one topping. Three towers with two whites, three pizzas with three |


|  |  |  | toppings. |
| :--- | :--- | :--- | :--- |
| $\mathbf{6 2 6}$ | $1: 02: 27$ | Instructor | Two toppings. |
| $\mathbf{6 2 7}$ | $1: 02: 28$ | Jessica | Two toppings, sorry. And then the three whites in the <br> last tower is the three toppings on one pizza. |
| $\mathbf{6 2 8}$ | $1: 02: 36$ | Instructor | Okay, that's perfect. But you know what I'm gonna <br> ask next which is - which pizza is this and this and <br> this? [She points to the following towers: WBB, BWB, <br> and BBW.] Exactly which pizza? |
| $\mathbf{6 2 9}$ | $1: 02: 46$ | Jamie | Okay, so, this is... it's that one right? |
| $\mathbf{6 3 0}$ | $1: 02: 49$ | Jessica | This one, we like decided it would be sausage [points <br> to the white cube in WBB], pepperoni [points to the <br> white cube in BWB], mushroom [points to the white <br> cube in BBW]. |
| $\mathbf{6 3 1}$ | $1: 02: 53$ | Instructor | Sausage, pepperoni, mushroom. Okay. So then what's <br> this one? [Points to the BWW tower.] |
| $\mathbf{6 3 2}$ | $1: 02: 57$ | Jessica | Um, let me put them in the correct order. [She <br> rearranges the 3-tall towers with two white cubes as - <br> WWB, WBW, and BWW] So it's sausage and |
| $\mathbf{6 3 3}$ | $1: 03: 11$ | Instructor | pepperoni [she points to the two white cubes in WWB]. <br> Sausage and mushrooms [she points to the two white <br> cubes in WBW]. Pepperoni and mushrooms [she <br> points to the two white cubes in BWW]. Notice that it's <br> sausage, pepperoni’s in the second place, and <br> mushroom is always in the third place. |
|  | Okay, alright, so there's your homework. Complete <br> description of the isomorphism. |  |  |
|  | Jessica | Okay. |  |

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[^0]:    ${ }^{1}$ Videos and related metadata of students solving these problems used in the research at Rutgers University can be found at The Video Mosaic Collaborative website (http://videomosaic.org).

[^1]:    Figure 2.2. Example of a pair of opposites.

[^2]:    ${ }^{2}$ Bob in Glass \& Maher (2004) is the same student as Rob2 in Glass (2001)

