AUTOMORPHIC DISTRIBUTIONS AND THE FUNCTIONAL EQUATION FOR THE STANDARD L-FUNCTION FOR G_2

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ABSTRACT OF THE DISSERTATION

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In this thesis we calculate a series expansion for automorphic distributions on the Lie group for the split real form of $\mathfrak{g}_2$. We then define distributional analogues of the θ function and the metaplectic Eisenstein series, which have many of the desirable properties of their smooth counterparts. In conclusion, we prove a functional equation for metaplectic Eisenstein distributions. It is believed that with these results, it should be possible to define a distributional version of the Rankin-Selberg integral given in [6], from which we should be able to derive the archimedean functional equation for the standard L-function of generic, cuspidal automorphic representations of the Lie group for the split real form of $\mathfrak{g}_2$.

Acknowledgements

At times I am tempted to think this thesis was a result of my intellect, determination, and overall character, but the truth is that I am not particularly strong in any of these traits. Instead, this thesis resulted from the commitment and patience of my teachers, the support of my family and friends, and the comfort I find in God. In particular, I wish to give heartfelt thanks to my wife Juliet who through her support, both emotional and practical, spurred me on towards completing this thesis. I would also like to thank my little "research assistant" Elijah, who constantly reminds me that there many things other than mathematics for us humans to investigate. In addition, I would like to thank my mother for her support and wisdom, and my sisters for providing a fun-filled, non-mathematical time whenever we come to visit. There are many other friends who have encouraged me throughout this thesis, and although I cannot list all their names, I would like to issue a general "thank you" to all of them nonetheless. Special thanks is also owed to my many mathematics teachers who played a pivotal role in encouraging me to pursue mathematics. In particular, I would like to thank Avner Ash, Benjamin Howard, Robert Gross, Wei Hu, Henryk Iwaniec, Jake Jacobson, Stephen Miller, and Jerrold Tunnell for their encouragement and instruction. In addition, I would like to thank Lisa Carbone and Brooke Feigon for their work in reviewing this thesis. I am grateful to all of you for your help in completing this thesis.

Dedication

To Juliet and Elijah.

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Introduction

In this thesis we present results for automorphic distributions on the split real form of the group G_2, as well as results for automorphic distributions on certain metaplectic groups. In chapter 1 we define the notion of an automorphic distribution associated to an automorphic representation, and reference a result of Casselman-Wallach which allows us to realize such distributions as elements of a distributional principal series representation space, or equivalently, as distributional sections of a vector bundle over G/B, where B is a Borel subgroup. If N is the unipotent radical of the global Cartan involution of B then N gives a dense open set in G/B. The restriction of automorphic distributions to N is well-defined, and hence motivates the study of $L^2(N_{\mathbb{Z}}\backslash N)$ where $N_{\mathbb{Z}} = G_{\mathbb{Z}} \cap N$ and $G_{\mathbb{Z}}$ is a discrete subgroup of G. Indeed, a series expansion of automorphic distributions can be deduced from a series expansion of elements belonging to $L^2(N_{\mathbb{Z}}\backslash N)$; we carry out this computation in chapter 2.

Two methods are available for computing Fourier series expansions for elements of $L^2(N_{\mathbb{Z}}\backslash N)$. One approach is to use the Kirillov orbit method [11] to compute irreducible $N_{\mathbb{Z}}$-invariant representations and then explicitly compute a certain class of $N_{\mathbb{Z}}$-invariant automorphic distributions on N in order to deduce the series expansion for elements of $L^2(N_{\mathbb{Z}}\backslash N)$ (see [15] for an explanation of how the latter step can be performed). The other method is more ad-hoc in nature, but uses only basic Fourier analysis results. We computed the series expansions using both methods and found (as one would hope) that both series expansions agree. In this thesis, we shall forgo presenting the Kirillov orbit method approach, and instead give a detailed account of the latter method in chapter 2.

In chapter 3, we calculate the "unbounded model" for distributional principal series representation spaces on $\widetilde{\mathrm{SL}}_2$ and $\widetilde{\mathrm{SL}_2^{\pm}}$. Such results will be needed for our work in chapter 4 with metaplectic Eisenstein distributions. In addition to this, we calculate the "unbounded model" for distributional principal series representation spaces on $\widetilde{J}$ and $\widetilde{J^{\pm}}$, which are groups that are closely related to a particular subgroup of G_2. We then conclude chapter 3 with defining a θ distribution on $\widetilde{J}$, and prove that that this distribution has many of the nice properties shared by its smooth counterparts.

In chapter 4 we define the metaplectic Eisenstein distribution, and prove the analytic continuation and functional equation of these distributions. In the context of principal series representations, the functional equation of metaplectic Eisenstein distributions is expressed in terms of other intertwined metaplectic Eisenstein distributions. Although the existence of such a functional equation is well-known, to the best of our knowledge, this is the first time its exact formulation has been recorded in the literature.

It is hoped that by using these various results, it will be possible to obtain the functional equation for the standard L-function for generic, cuspidal automorphic representations of G_2. Indeed, by utilizing the trilinear pairing of automorphic distributions defined in [16], it appears likely that a distributional analogue of the Rankin-Selberg integral in [6] should yield the functional equation for such L-functions.

Chapter 1

A Review of Representation Theory

1.1 Smooth Vectors and Distributions

Let G be a reductive Lie group and $\mathfrak{g}$ its corresponding Lie algebra.[1] Let V be a separable Hilbert space, $\mathrm{End}(V)$ the space of continuous (i.e., bounded) linear maps of V into V, and $\mathrm{GL}(V)$ the invertible elements of $\mathrm{End}(V)$. Let $\pi : G \to \mathrm{GL}(V)$ be a group homomorphism such that $(g, v) \mapsto \pi(g)v$ is a continuous map from $G \times V$ to V. In this case, the pair (π, V) is called a *representation* of G. If the image of π consists of unitary operators, then we say that (π, V) is a *unitary representation.*

In what follows, we shall assume that (π, V) is a (possibly non-unitary) representation of G. We shall wish to study a special subspace of V, and will need the following definitions in order to define this space. Let S be an open subset of $\mathbb{R}^n$ and $f : S \to V$. We say that f is *differentiable* at $x_0 \in S$ if there exists a linear map $L : \mathbb{R}^n \to V$ such that

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - L(x - x_0)\|}{|x - x_0|} = 0,$$

where $\|\cdot\|$ is the norm on V and $|\cdot|$ is the norm on $\mathbb{R}^n$. One can show that if such an L exists then L is unique [19, Theorem 2-1]. Furthermore, L is a continuous linear map since $\mathbb{R}^n$ is finite dimensional (i.e., $L \in \mathrm{End}(\mathbb{R}^n, V)$). We shall denote L by $f'(x_0)$.

If f is differentiable for all $x \in S$, then we define $f' : S \to \mathrm{End}(\mathbb{R}^n, V)$ in the aforementioned way. We call f' the *derivative* of f. If f' is continuous then we say that f is of class C^1. Under the operator norm, $\mathrm{End}(\mathbb{R}^n, V)$ is a Banach space [5, Proposition 5.4], and thus it makes sense to ask if f' is differentiable at points in S. Continuing in this manner, it is possible to define $f^{(k)}$ for all $k \in \mathbb{Z}_{>0}$. We say that f is of class C^k

[1] Following [23], we define real reductive groups to be finite covers of linear real reductive groups.

if $f^{(k)}$ is continuous. If f is of class C^k for all positive integers k, then we say that f is *smooth*, or equivalently, that f is of C^∞ class. We extend these definitions to functions defined over smooth manifolds in the usual way by use of coordinate charts. A *smooth vector* is a vector $v \in V$ such that $g \mapsto \pi(g)v$ is smooth. Let V^∞ denote the space of smooth vectors of V.

For $v \in V$ and $X \in \mathfrak{g}$, define

$$\pi(X)v = \lim_{t \to 0} \frac{\pi(\exp(tX))v - v}{t}. \tag{1.1}$$

For general v, this limit may not exist for every $X \in \mathfrak{g}$, but if $v \in V^\infty$, then one can prove that $\pi(X)v$ exists for every $X \in \mathfrak{g}$ and that $\pi(X)v \in V^\infty$. Furthermore, one can show that π is a Lie algebra homomorphism, and hence extends to an algebra homomorphism of the universal enveloping algebra $U(\mathfrak{g}^\mathbb{C})$ [13, Proposition 3.9]. For finite dimensional V, one can show that $V^\infty = V$, but this is not the case for V infinite dimensional. Nevertheless, one can show that V^∞ is dense in V [13, Theorem 3.15].

The space V^∞ can inherit a topology from the norm of V, but for many applications it is more useful to give a possibly finer topology defined by the following seminorms:

$$p_D(v) = \|\pi(D)v\|,$$

where $D \in U(\mathfrak{g})$. One can restrict to a countable number of such D by forming a vector space basis for $U(\mathfrak{g}^\mathbb{C})$. Notice that the seminorm for $D = I$ implies that this topology is indeed no coarser than that of V. One can show that V^∞ is a complete space under these seminorms, and hence V^∞ is a Fréchet space [23, Lemma 1.6.4].

Let V' denote the dual vector space of V, and let (π', V') denote the *dual representation* for (π, V). Recall that by definition,

$$(\pi'(g)f)(v) = f(\pi(g^{-1})v)$$

for $f \in V'$. Let $V^{-\infty}$ denote the space of continuous linear functionals on $(V')^\infty$. We say that $V^{-\infty}$ is the space of *distributions* for V. By fixing an inner-product on V' one can identify elements of V with elements of $(V')'$. Thus, as linear functionals on $(V')^\infty$, we have

$$V^\infty \subset V \subset V^{-\infty}.$$

In this way we are able to think of distributions as generalized functions in the sense of Gelfand et al.

We say that $\phi : G \to \mathrm{End}(W)$, where W is a Fréchet space, is a *smooth representation* if

(1) ϕ is a group homomorphism,

(2) $(g, w) \mapsto \phi(g)w$ is a continuous map from $G \times W$ to W, and

(3) $g \mapsto \phi(g)w$ is smooth for all $g \in G$.

Obviously (1) and (3) hold for π restricted to V^∞. One can show that (2) holds as well [23, Lemma 1.6.4]. Thus π restricted to V^∞ is a smooth representation. Likewise, the restriction of π' to $(V')^\infty$ is a smooth representation. The dual action of $(\pi', (V')^\infty)$ defines an action of G on $V^{-\infty}$, which we denote by π.

When working with the space of distributions $V^{-\infty}$, one typically gives $V^{-\infty}$ the weak* topology. However, we shall have need of a finer topology known as the *strong distribution topology* [21, §19]. Suppose we are given a Fréchet space W defined by semi-norms $\|\cdot\|_j$ for $j \in J$, where J is a countable set. We say that $B \subset W$ is *bounded* if there exists $(M_j)_{j\in J} \in \mathbb{R}^J_{>0}$ such that $\|v\|_j \leq M_j$ for all $v \in B$ and $j \in J$. We say that a sequence of distributions τ_n on W converges to a distribution τ on W in the strong distribution topology if for any $\epsilon > 0$ and bounded set $B \subset W$, there exists $N > 0$ such that

$$|\tau_n(\psi) - \tau(\psi)| < \epsilon$$

for all $n > N$ and $\psi \in B$. One can check that the action π on $V^{-\infty}$ is continuous with respect to the strong distribution topology.

1.2 Sections of Vector Bundles

Let B be a subgroup of G, and let (ω, V) be a finite-dimensional representation of B. We define an equivalence relation $\sim$ on $G \times V$:

$$(g_1, v_1) \sim (g_2, v_2) \text{ if there exists } b \in B \text{ such that } (g_1 b, \omega(b^{-1})v_1) = (g_2, v_2), \qquad (1.2)$$

where $g_1, g_2 \in G$ and $v_1, v_2 \in V$. Let $E_V = (G \times V)/\sim$. Observe that the map $(g, v) \mapsto g$ on $G \times V$ induces a well-defined map from E_V to G/B. Let $p : E_V \to G/B$ denote this map, and let $\mathcal{E}_\omega = \mathcal{E}(G, B, \omega)$ denote the fiber bundle with total space E_V, base space G/B, and projection map $p : E_V \to G/B$. For $gB \in G/B$, one can show that $p^{-1}(gB)$ is naturally isomorphic to V, and hence $\mathcal{E}_\omega$ is in fact a vector bundle. Furthermore, left inverse multiplication by G on $G \times V$ and G induces a well-defined action of G on E_V and G/B. With respect to this action, one can show that p is an equivariant map. Such vector bundles are commonly referred to as *equivariant vector bundles.*

Let $\Gamma^\infty(\mathcal{E}_\omega)$ denote the space of smooth sections of $\mathcal{E}_\omega$. We let π_ω denote the action of inverse left translation on elements of $\Gamma^\infty(\mathcal{E}_\omega)$. This action is well-defined since $\mathcal{E}_\omega$ is an equivariant vector bundle under this action. Since distributions can be defined in terms of local data, it follows that there also exists a space of vector-valued distribution sections of $\mathcal{E}_\omega$, which we shall denote by $\Gamma^{-\infty}(\mathcal{E}_\omega)$. As before, we let π_ω denote the action of inverse left translation on elements of $\Gamma^{-\infty}(\mathcal{E}_\omega)$.

Often times it will be helpful to view elements of $\Gamma^\infty(\mathcal{E}_\omega)$ as functions on G into $\mathbb{C}^m$ where m is the dimension of V over $\mathbb{C}$. To see how this is done, fix $s \in \Gamma^\infty(\mathcal{E}_\omega)$. For $g \in G$ there exists a unique $v_g \in V$ such that

$$s(gB) = \{(gb, \omega(b^{-1})v_g) : b \in B\} \in E_V.$$

From s we then define $f : G \to V$ by $f(g) = v_g$. Since $v_{gb} = \omega(b^{-1})v_g$ it follows that

$$f(gb) = \omega(b^{-1})f(g). \tag{1.3}$$

Conversely, for smooth $f : G \to V$ which satisfies (1.3), one can show that

$$s(gB) = \{(gb, \omega(b^{-1})f(g)) : b \in B\}$$

defines an element $\Gamma^\infty(\mathcal{E}_\omega)$. Consequently,

$$\Gamma^\infty(\mathcal{E}_\omega) \cong V_\omega^\infty(G) = \{f \in C^\infty(G, \mathbb{C}^m) : f(gb) = \omega(b^{-1})f(g) \text{ for all } g \in G, b \in B\}$$

where $C^\infty(G, \mathbb{C}^m)$ is the space of smooth functions from G to $\mathbb{C}^m$. Likewise, since

$\Gamma^\infty(\mathcal{E}_\omega)$ is a dense space in $\Gamma^{-\infty}(E_\omega)$, we see that

$$\Gamma^{-\infty}(E_\omega) \cong V_\omega^{-\infty}(G) = \{f \in C^{-\infty}(G, \mathbb{C}^m) : f(gb) = \omega(b^{-1})f(g) \text{ for all } g \in G, b \in B\}, \tag{1.4}$$

where $C^{-\infty}(G, \mathbb{C}^m)$ is the space of distribution vectors from G to $\mathbb{C}^m$, and where the equality in (1.4) is interpreted an equality between distributions on G. When given $f \in V_\omega^{-\infty}(G)$, we will let s_f denote the corresponding element of $\Gamma^{-\infty}(\mathcal{E}_\omega)$ given by the isomorphism in (1.4).

Let $h \in G$, and let N denote the unipotent radical of the Cartan involution of B, where B is a minimal parabolic subgroup of G. Observe that the map

$$hnB \mapsto (hnB, v_{hn})$$

is a local trivialization of the vector bundle $\mathcal{E}_\omega$ on $hNB \subset G/B$. We can restrict $s \in \Gamma^\infty(\mathcal{E}_\omega)$ to hNB, or more precisely, restrict s via the aforementioned local trivialization of the vector bundle. When we do, we obtain the function

$$hnB \mapsto v_{hn}$$

on hNB. We can likewise restrict distributional sections $s \in V_\omega^{-\infty}$ and shall do so often throughout this thesis.

In our applications, we will usually take B to be a minimal parabolic subgroup of G satisfying $G = BK$, where K is a maximal compact subgroup of G. Supposing that this is the case, for $f \in V_\omega^\infty$ define

$$\|f\| = \left(\int_K |f(k)|^2\right)^{1/2} dk$$

where dk is a Haar measure for K and $|\cdot|$ is the usual norm on $\mathbb{C}^m$. Under this norm, one show that V_ω^∞ is a pre-Hilbert space. Upon completion, we obtain a Hilbert space we denote by $V_\omega(G)$, with π_ω denoting the left regular representation as usual. One can show that V_ω^∞ is the space of smooth vectors for (π_ω, V_ω).

Consider the pairing

$$(f_1, f_2) = \int_K \langle f_1(k), f_2(k)\rangle dk,$$

where $f_1 \in V_\omega$, $f_2 \in V_{\omega'}$, ω' is the dual representation of ω, and $\langle \cdot, \cdot \rangle$ is the usual bilinear form on $\mathbb{C}^m \times \mathbb{C}^m$. One can show from the non-degeneracy of this pairing that the dual of V_ω can be identified with $V_{\omega'}$. Furthermore, one can show that if $V \cong V_\omega$ then $V^{-\infty} \cong V_\omega^{-\infty}$.

1.3 Automorphic Distributions

Let Γ be a discrete subgroup of G. The group G acts on $L^2(\Gamma \backslash G / Z_G)$ by the right regular representation, which we shall denote by r. One can prove that $(r, L^2(\Gamma\backslash G/Z_G))$ is in fact a unitary representation of G and that the space of smooth vectors for $(r, L^2(\Gamma\backslash G/Z_G))$ is contained in $C^\infty(\Gamma\backslash G/Z_G)$. Let (π, V) be an irreducible unitary representation of G which embeds as a direct summand of $L^2(\Gamma\backslash G/Z_G)$ and let

$$i : V \hookrightarrow L^2(\Gamma\backslash G/Z_G)$$

denote this embedding. If $v \in V^\infty$ then the function

$$\tau(v) = i(v)(e)$$

is well-defined; in fact, it can be shown that $\tau \in (V')^{-\infty}$. Furthermore, since $i(v) \in C^\infty(\Gamma\backslash G/Z_G)$, we have that τ is Γ-invariant. We signify this by writing $\tau \in ((V')^{-\infty})^\Gamma$. Elements of $((V')^{-\infty})^\Gamma$ which arise from such embeddings are called *automorphic distributions*. Since V^∞ is dense in V it follows that one can reconstruct i from τ.

A result of Casselman and Wallach ([3] and [22, Theorem 5.8]) states that there exists a (possibly non-unitary) representation ω of B such that:

$$V^\infty \hookrightarrow V_\omega^\infty, \quad V \hookrightarrow V_\omega, \text{ and } V^{-\infty} \hookrightarrow V_\omega^{-\infty}.$$

Therefore we can identify τ as an element of $V_\omega^{-\infty}$ for some (possibly non-unitary) representation ω of B.

Chapter 2

Automorphic Distributions on G_2

2.1 G_2 Preliminaries

Let $\mathfrak{g}$ denote the split real form of the Lie algebra $\mathfrak{g}_2$. We shall let $G = G_2$ denote the corresponding split real Lie group for $\mathfrak{g}$. We identify $\mathfrak{g}$ concretely as a Lie subalgebra of $\mathfrak{so}(4,3)$. This in turn allows us to identify G as a Lie subgroup of $\mathrm{SO}(4,3)$, the split real form of the Lie group $\mathrm{SO}(7)$. Ross Lawther has shown that the following root vectors of $\mathfrak{g}$ correspond to the positive roots (under the Bourbaki labeling) β, α, $\alpha+\beta$, $2\alpha+\beta$, $3\alpha+\beta$, $3\alpha+2\beta$ (respectively):

$$\mathcal{P}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{P}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{P}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{P}_4 = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{P}_5 = \begin{pmatrix} 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}, \qquad \mathcal{P}_6 = \begin{pmatrix} 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix},$$

and that the following root vectors of $\mathfrak{g}$ correspond to the negative roots $-\beta$, $-\alpha$, $-\alpha-\beta$, $-2\alpha-\beta$, $-3\alpha-\beta$, $-3\alpha-2\beta$ (respectively):

$$\mathcal{N}_1 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}, \qquad \mathcal{N}_2 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&-2&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0 \end{pmatrix},$$

$$\mathcal{N}_3 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&-2&0&0&0\\ 0&0&0&0&-1&0&0 \end{pmatrix}, \qquad \mathcal{N}_4 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&-2&0&0&0 \end{pmatrix},$$

$$\mathcal{N}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{N}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

By basic Lie theory we know that $\mathfrak{g}$ is generated by $\{\mathcal{P}_i\}$ and $\{\mathcal{N}_i\}$ as a Lie algebra and that G is generated by $\exp(x_i\mathcal{P}_i)$ and $\exp(y_i\mathcal{N}_i)$ where $x_i, y_i \in \mathbb{R}$. Let $\mathfrak{n}$ denote the Lie subalgebra generated by the $\{\mathcal{P}_i\}$ and let N denote its corresponding subgroup. Once again, by Lie theory we see that N is generated by $\exp(x_i\mathcal{P}_i)$ where $x_i \in \mathbb{R}$. The elements $\mathcal{P}_i$, $\mathcal{N}_i$, and $[\mathcal{P}_i, \mathcal{N}_i]$ for $i = 1, 2$ can be shown to form a Chevalley basis for $\mathfrak{g}_2$ over $\mathbb{Z}$. Let $\mathfrak{g}_\mathbb{Z}$ denote the $\mathbb{Z}$-span of this basis and let $G_\mathbb{Z}$ denote the subgroup of G which fixes the lattice $\mathfrak{g}_\mathbb{Z}$ under the adjoint action. One can show that $G_\mathbb{Z}$ is generated by $\exp(x_i\mathcal{P}_i)$ and $\exp(y_i\mathcal{N}_i)$ where $x_i, y_i \in \mathbb{Z}$ [2]. Let $N_\mathbb{Z} = G_\mathbb{Z} \cap N$. Notice that $N_\mathbb{Z}$ is generated by $\exp(x_i\mathcal{P}_i)$ where $x_i \in \mathbb{Z}$.

The groups N and $N_\mathbb{Z}$ will be primary objects of study throughout this thesis. Below we define the one parameter subgroups for N corresponding to the positive roots:

$$\begin{aligned} R_1(x) &= \exp(x\mathcal{P}_1), & R_2(x) &= \exp(x\mathcal{P}_2), & R_3(x) &= \exp(x\mathcal{P}_3), \\ R_4(x) &= \exp(x\mathcal{P}_4), & R_5(x) &= \exp(x\mathcal{P}_5), & R_6(x) &= \exp(x\mathcal{P}_6), \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}$. To streamline notation, we write $P_i = R_i(p_i)$, $Q_i = R_i(q_i)$, $T_i = R_i(t_i)$, $X_i = R_i(x_i)$, $Y_i = R_i(y_i)$ where $p_i, q_i, t_i, x_i, y_i \in \mathbb{R}$. Let $N_i = R_i(1)$, which we have essentially already shown to be generators of $N_\mathbb{Z}$.

Let G_β denote an embedded SL_2 of the Levi subgroup for the root β; specifically, we shall let G_β denote the group consisting of elements of the form

$$h = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & -b & 0 & 0 & 0 & 0 \\ 0 & -c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & -b & 0 \\ 0 & 0 & 0 & 0 & -c & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.2}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Let Γ_β denote the $\mathrm{SL}_2(\mathbb{Z})$ subgroup of G_β; that is to say, let Γ_β consists of elements h where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. Observe that for $h \in G_\beta$ we have

$$\begin{aligned} &hX_6h^{-1} = R_6(ax_6)R_5(cx_6), \quad hX_5h^{-1} = R_6(bx_5)R_5(dx_5), \quad hX_4h^{-1} = X_4, \\ &hX_3h^{-1} = R_6(-2a^2cx_3^3)R_5(-ac^2x_3^3)R_4(-acx_3^2)R_3(ax_3)R_2(cx_3), \\ &hX_2h^{-1} = R_6(-2b^2dx_2^3)R_5(-bd^2x_2^3)R_4(-bdx_2^2)R_3(bx_2)R_2(dx_2), \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &X_6 = hR_6(dx_6)R_5(-cx_6)h^{-1}, \quad X_5 = hR_6(-bx_5)R_5(ax_5)h^{-1}, \quad X_4 = hX_4h^{-1}, \\ &X_3 = hR_6(2cd^2x_3^3)R_5(-c^2dx_3^3)R_4(cdx_3^2)R_3(dx_3)R_2(-cx_3)h^{-1}, \\ &X_2 = hR_6(-2ab^2x_2^3)R_5(a^2bx_2^3)R_4(abx_2^2)R_3(-bx_2)R_2(ax_2)h^{-1}. \end{aligned} \tag{2.4}$$

For fixed $h \in G_\beta$, it follows from (2.3) and (2.4) that N is generated by X_1, hX_2h^{-1}, hX_3h^{-1}, hX_4h^{-1}, hX_5h^{-1}, and hX_6h^{-1} where $x_i \in \mathbb{R}$. Likewise, for fixed $\gamma \in \Gamma_\beta$, it follows from (2.3) and (2.4) that $N_\mathbb{Z}$ is generated by X_1, $\gamma X_2\gamma^{-1}$, $\gamma X_3\gamma^{-1}$, $\gamma X_4\gamma^{-1}$, $\gamma X_5\gamma^{-1}$, and $\gamma X_6\gamma^{-1}$ where $x_i \in \mathbb{Z}$; or more simply, we see that $N_\mathbb{Z}$ is generated by N_1, $\gamma N_2\gamma^{-1}$, $\gamma N_3\gamma^{-1}$, $\gamma N_4\gamma^{-1} = N_4$, $\gamma N_5\gamma^{-1}$, $\gamma N_6\gamma^{-1}$.

2.2 Basic Lemmas from Fourier Analysis

A principal aim of this chapter is to give a full description of functions $f : N \to \mathbb{C}$ which are $N_\mathbb{Z}$-invariant; that is to say, we wish to express such functions in terms of a general series expansion. We will accomplish this by using the lemmas given in this section.

Let $h : \mathbb{R} \times N \to \mathbb{C}$ be a continuous function. If K is a compact subset of N and K' is a compact subset of $\mathbb{R}$ then we define

$$\|h\|_{K' \times K} = \sup_{(x,n) \in K' \times K} |h(x,n)|.$$

Likewise, if $f : N \to \mathbb{C}$ is a continuous function then we define

$$\|f\|_K = \sup_{n \in K} |f(n)|.$$

If we have that $h(x+1,n) = h(x,n)$ for all $x \in \mathbb{R}$ and $n \in N$, then we define $h_m : N \to \mathbb{C}$ by

$$h_m(n) = \int_0^1 h(t,n) e(-mt)\, dt,$$

where $m \in \mathbb{Z}$, $n \in N$, and $e(z) = e^{2\pi i z}$.

Lemma 2.1. *Let $h : \mathbb{R} \times N \to \mathbb{C}$ be a smooth function such that $h(x+1,n) = h(x,n)$ for all $x \in \mathbb{R}$ and $n \in N$. For $m \in \mathbb{Z}_{\neq 0}$ and $j \in \mathbb{Z}_{>0}$,*

$$\|h_m\|_K \le (2\pi|m|)^{-j} \|\partial_x^j h\|_{[0,1]\times K}.$$

Furthermore,

$$\sum_{m \in \mathbb{Z}} \|h_m\|_K \le \|h\|_{[0,1]\times K} + \sum_{m \in \mathbb{Z}_{\neq 0}} \|h_m\|_K \le \|h\|_{[0,1]\times K} + \|\partial_x^2 h\|_{[0,1]\times K},$$

and for $j > 2$,

$$\sum_{m \in \mathbb{Z}} |m|^{j-2} \|h_m\|_K \le \|\partial_x^j h\|_{[0,1]\times K}.$$

Proof. For $j \in \mathbb{Z}_{\ge 0}$ it follows that $(\partial_x^j h)(x+1,n) = (\partial_x^j h)(x,n)$. Thus for $m \neq 0$, when we apply integration by parts we find that

$$\int_0^1 (\partial_x^j h)(t,n) e(-mt)\, dt = \left[(\partial_x^j h)(t,n) \frac{e(-mt)}{-2\pi i m} \right]_0^1 - \int_0^1 (\partial_x^{j+1} h)(t,n) \frac{e(-mt)}{-2\pi i m}\, dt$$
$$= \frac{1}{2\pi i m} \int_0^1 (\partial_x^{j+1} h)(t,n) e(-mt)\, dt.$$

By induction,

$$\|h_m\|_K \le (2\pi|m|)^{-j} \left\| \int_0^1 (\partial_x^j h)(t,n) e(-mt)\, dt \right\|_K \le (2\pi|m|)^{-j} \|\partial_x^j h\|_{[0,1]\times K}.$$

Since $\sum_{m \in \mathbb{Z}_{\neq 0}} m^{-2} = \frac{\pi^2}{3}$ it follows that

$$\sum_{m \in \mathbb{Z}} \|h_m\|_K = \|h_0\|_K + \sum_{m \in \mathbb{Z}_{\neq 0}} \|h_m\|_K \leq \|h\|_{[0,1]\times K} + \|\partial_x^2 h\|_{[0,1]\times K},$$

and for $j > 2$,

$$\sum_{m \in \mathbb{Z}} |m|^{j-2} \|h_m\|_K \leq \|\partial_x^j h\|_K.$$

□

Lemma 2.2. *Let $h : \mathbb{R} \times N \to \mathbb{C}$ be a smooth function such that $h(x+1, n) = h(x, n)$ for all $x \in \mathbb{R}$ and $n \in N$. For any compact subset $K \subset N$, the sum $\sum_{|m| \leq M} h_m(n) e(mx)$ converges uniformly to $h(x, n)$ on $\mathbb{R} \times K$ as $M \to \infty$. In particular, $\sum_{|m| \leq M} h_m(n)$ converges uniformly to $h(0, n)$ on K as $M \to \infty$.*

Proof. Observe that both h and $\partial_x^2 h$ are uniformly continuous on $\mathbb{R} \times K$. Thus for $\epsilon > 0$, there exists $\delta > 0$ and U an open subset of the identity element $e \in N$ such that

$$|h(x, n) - h(x_0, n_0)| < \epsilon \qquad \text{and} \qquad |\partial_x^2 h(x, n) - \partial_x^2 h(x_0, n_0)| < \epsilon \tag{2.5}$$

for $(x, n), (x_0, n_0) \in \mathbb{R} \times K$, $|x - x_0| < \delta$, and $n n_0^{-1} \in U$. Since K is compact, there exists a finite number of translates of U which cover K. Let $\{\gamma_1, \ldots, \gamma_\ell\}$ denote elements of N such that $\{U\gamma_1, \ldots, U\gamma_\ell\}$ covers K. It follows that for any point $n \in K$, there will exist a $\gamma_i \in \{\gamma_1, \ldots, \gamma_\ell\}$ such that $n\gamma_i^{-1} \in U$.

For each γ_i there exists $M_i \in \mathbb{Z}_{>0}$ such that

$$\sup_{x \in \mathbb{R}} \left| h(x, \gamma_i) - \sum_{|m| \leq M} h_m(\gamma_i) e(mx) \right| < \epsilon$$

for all $M > M_i$; this is simply the convergence of Fourier series on compacta for smooth functions in one variable. Let M_0 denote the maximum of these M_i. Fix $(x, n) \in \mathbb{R} \times K$

and let $\gamma_i \in \{\gamma_1, \ldots, \gamma_\ell\}$ such that $n\gamma_i^{-1} \in U$. Thus for $M > M_0$, we have

$$\begin{aligned}
&\left| h(x,n) - \sum_{|m|\leq M} h_m(n)e(mx) \right| \\
&\leq |h(x,n) - h(x,\gamma_i)| + \left| h(x,\gamma_i) - \sum_{|m|\leq M} h_m(\gamma_i)e(mx) \right| \\
&\quad + \left| \sum_{m\leq M} (h_m(\gamma_i) - h_m(n))e(mx) \right| \\
&\leq 2\epsilon + \left| \sum_{|m|\leq M} (h_m(\gamma_i) - h_m(n))e(mx) \right|. \qquad (2.6)
\end{aligned}$$

Let $p_n(x) = h(x,\gamma_i) - h(x,n)$. By Lemma 2.1 and (2.5) we have

$$\sum_{|m|\leq M} |\widehat{p}_n(m)| \leq \|p_n\|_{[0,1]} + \|p_n''\|_{[0,1]} < \epsilon + \|p_n''\|_{[0,1]}.$$

Since $p_n''(x) = \partial_x^2 h(x,\gamma_i) - \partial_x^2 h(x,n)$, it follows once more from (2.5) that

$$\sum_{|m|\geq M} |\widehat{p}_n(m)| \leq 2\epsilon.$$

Since $\widehat{p}_n(m) = h_m(\gamma_i) - h_m(n)$ it follows from (2.6) that

$$\left| h(x,n) - \sum_{|m|\leq M} h_m(n)e(nx) \right| \leq 5\epsilon$$

for $M > M_0$. Since this inequality holds for any $(x,n) \in \mathbb{R} \times K$, our lemma then follows. $\square$

Let $f : N \to \mathbb{C}$ be a locally integrable function. For $k_i, m_i \in \mathbb{Z}$, and $\gamma \in \Gamma_\beta$ we define the following functions:

$$f_{k_6}(n) = \int_0^1 f(T_6 n)e(-k_6 t_6)\, dt_6, \qquad (2.7a)$$

$$f_{k_6,k_5}(n) = \int_0^1 \int_0^1 f(T_6 T_5 n)e(-k_6 t_6 - k_5 t_5)\, dt_6\, dt_5, \qquad (2.7b)$$

$$f_{\gamma,m_5}(n) = \int_0^1 \int_0^1 f(\gamma T_6 T_5 \gamma^{-1} n)e(-m_5 t_5)\, dt_6\, dt_5, \qquad (2.7c)$$

$$f_{k_6,k_5,k_4}(n) = \int_0^1 \int_0^1 \int_0^1 f(T_6 T_5 T_4 n)e(-k_6 t_6 - k_5 t_5 - k_4 t_4)\, dt_6\, dt_5\, dt_4, \qquad (2.7d)$$

$$f_{\gamma,m_5,m_4}(n) = \int_0^1 \int_0^1 \int_0^1 f(\gamma T_6 T_5 T_4 \gamma^{-1} n) e(-m_5 t_5 - m_4 t_4)\, dt_6\, dt_5\, dt_4, \tag{2.7e}$$

$$f_{k_6,k_5,k_4,k_3}(n) = \int_0^1 \int_0^1 \int_0^1 \int_0^1 f(T_6 T_5 T_4 T_3 n) e(-k_6 t_6 - k_5 t_5 - k_4 t_4 - k_3 t_3)\, dt_6\, dt_5\, dt_4\, dt_3, \tag{2.7f}$$

$$f_{\gamma,m_5,m_4,m_3}(n) = \int_0^1 \int_0^1 \int_0^1 \int_0^1 f(\gamma T_6 T_5 T_4 T_3 \gamma^{-1} n) e(-m_5 t_5 - m_4 t_4 - m_3 t_3)\, dt_6\, dt_5\, dt_4\, dt_3, \tag{2.7g}$$

$$f_{k_6,k_5,k_4,k_3,k_1}(n) = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 f(T_6 T_5 T_4 T_3 T_1 n) e(-k_6 t_6 - k_5 t_5 - k_4 t_4) e(-k_3 t_3 - k_1 t_1)\, dt_6\, dt_5\, dt_4\, dt_3\, dt_1, \tag{2.7h}$$

$$f_{0,0,0,0,k_1,k_2}(n) = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 f(T_6 T_5 T_4 T_3 T_1 T_2 n) e(-k_1 t_1 - k_2 t_2)\, dt_6\, dt_5\, dt_4\, dt_3\, dt_1\, dt_2, \tag{2.7i}$$

where $n \in N$. We caution the reader about one aspect of our notation in (2.7). Notice that our notation for (2.7b) and (2.7c) are very similar in form; the only distinction between the two being that first index in (2.7b) is an integer and the first index in (2.7c) is an element of Γ_β. We hope that the reader will not be too confused by this similar notation; context will make clear which function we are referring to in our arguments. This same warning also applies to (2.7d), (2.7e), (2.7f), and (2.7g).

In section 2.3 and section 2.4, we will apply Lemmas 2.1 and 2.2 to obtain a series expansion for smooth, $N_{\mathbb{Z}}$-invariant $f : N \to \mathbb{C}$ in terms of the functions in (2.7) (see Theorem 2.10 for the end result). In order to apply Lemmas 2.1 and 2.2, we will need the following lemma.

Lemma 2.3. *If $f : N \to \mathbb{C}$ is an $N_{\mathbb{Z}}$-invariant, locally integrable function, then for all $k_i, m_i \in \mathbb{Z}$ and $\gamma \in \Gamma_\beta$, we have*

(a) $x_6 \mapsto f(\gamma X_6 \gamma^{-1} n)$ *is periodic with period 1,*

(b) $x_5 \mapsto \int_0^1 f(\gamma T_6 X_5 \gamma^{-1} n) e(-k_6 t_6)\, dt_6$ *is periodic with period 1,*

(c) $x_4 \mapsto \int_0^1 \int_0^1 f(\gamma T_6 T_5 X_4 \gamma^{-1} n) e(-m_5 t_5)\, dt_6\, dt_5$ *is periodic with period 1,*

(d) $x_3 \mapsto \int_0^1 \int_0^1 \int_0^1 f(\gamma T_6 T_5 T_4 X_3 \gamma^{-1} n) e(-m_5 t_5 - m_4 t_4)\, dt_6\, dt_5\, dt_4$ *is periodic with period 1,*

(e) $x_1 \mapsto \int_0^1 \int_0^1 \int_0^1 \int_0^1 f(T_6 T_5 T_4 T_3 X_1 n) e(-k_5 t_5 - k_4 t_4 - k_3 t_3)\, dt_6\, dt_5\, dt_4\, dt_3$ *is periodic with period 1,*

(f) $x_2 \mapsto \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 f(T_6 T_5 T_4 T_3 T_1 X_2 n) e(-k_1 t_1)\, dt_6\, dt_5\, dt_4\, dt_3\, dt_1$ *is periodic with period 1.*

Proof. Part (a) follows immediately from the $N_{\mathbb{Z}}$-invariance of f. Parts (b) and (c) follows from the $N_{\mathbb{Z}}$-invariance of f and the equalities

$$\gamma T_6 N_5 \gamma^{-1} = (\gamma N_5 \gamma^{-1})(\gamma T_6 \gamma^{-1}) \text{ and } \gamma T_6 T_5 N_4 \gamma^{-1} = (\gamma N_4 \gamma^{-1})(\gamma T_6 T_5 \gamma^{-1}),$$

which are seen to be true by either direct computation or by observing that X_4, X_5, and X_6 commute with each other for all $x_i \in \mathbb{R}$.

For part (d), observe that

$$\gamma T_6 T_5 T_4 N_3 \gamma^{-1} = (\gamma N_3 \gamma^{-1})(\gamma R_6(-3t_4 + t_6) T_5 T_4 \gamma^{-1}).$$

Thus

$$\begin{aligned}
&\int_0^1 \int_0^1 \int_0^1 f(\gamma T_6 T_5 T_4 N_3 X_3 \gamma^{-1} n) e(-m_5 t_5 - m_4 t_4)\, dt_6\, dt_5\, dt_4 \\
&= \int_0^1 \int_0^1 \left(\int_0^1 f(\gamma R_6(-3t_4 + t_6) T_5 T_4 X_3 \gamma^{-1} n) dt_6 \right) e(-m_5 t_5 - m_4 t_4)\, dt_5\, dt_4.
\end{aligned}$$

Part (d) then follows by performing the change of variables $t_6 \mapsto t_6 + 3t_4$ and applying part (a) to return the domain of integration in the t_6 variable to the interval $[0, 1]$.

For part (e), observe that $T_6 T_5 T_4 T_3 N_1 = N_1 R_6(-t_5 + t_6) T_5 T_4 T_3$. Thus

$$\begin{aligned}
&\int_0^1 \int_0^1 \int_0^1 \int_0^1 f(T_6 T_5 T_4 T_3 N_1 n) e(-k_5 t_5 - k_4 t_4 - k_3 t_3)\, dt_6\, dt_5\, dt_4\, dt_3 \\
&= \int_0^1 \int_0^1 \int_0^1 \left(\int_0^1 f(\gamma R_6(-t_5 + t_6) T_5 T_4 X_3 \gamma^{-1} n) dt_6 \right) e(-k_5 t_5 - k_4 t_4 - k_3 t_3) \\
&\qquad dt_5\, dt_4\, dt_3.
\end{aligned}$$

Part (e) then follows by performing the change of variables $t_6 \mapsto t_6 + t_5$ and applying part (a) to return the domain of integration in the t_6 variable to the interval $[0, 1]$.

For part (f), observe that

$$\begin{aligned}
&T_6T_5T_4T_3T_1N_2 \\
&= N_2R_6(t_1^2+3t_1t_3+3t_3^2+t_6)R_5(-t_1-3t_3-3t_4+t_5)R_4(t_1+2t_3+t_4)R_3(t_1+t_3)T_1.
\end{aligned}$$

By utilizing the $N_{\mathbb{Z}}$-invariance of f, performing the change of variables (in order)

$$\begin{aligned}
t_6 &\mapsto t_6-(t_1^2+3t_1t_3+3t_3^2)\\
t_5 &\mapsto t_5-(-t_1-3t_3-3t_4)\\
t_4 &\mapsto t_4-(t_1+2t_3)\\
t_3 &\mapsto t_3-t_1,
\end{aligned}$$

and apply parts (a)-(d), we are able to see that part (f) holds. □

For some results in section 2.4 we will need to be able to write f_{γ,m_5} in terms of f_{k_6,k_5}. Since $\gamma T_6T_5\gamma^{-1} = R_6(bt_5+at_6)R_5(dt_5+c_6)$ then

$$f_{\gamma,m_5}(n) = \int_0^1\int_0^1 f(R_6(bt_5+at_6)R_5(dt_5+ct_6)n)e(-m_5t_5)\,dt_6\,dt_5.$$

We perform the following simultaneous change of variables:

$$\begin{cases} t_5 \mapsto at_5-ct_6 \\ t_6 \mapsto -bt_5+dt_6, \end{cases} \tag{2.8}$$

which we can denote by the following map from $\mathbb{R}^2$ to $\mathbb{R}^2$:

$$\gamma_0 : (t_5,t_6) \mapsto (at_5-ct_6, -bt_5+dt_6).$$

If $\mathcal{G} = [0,1)^2$ then when we apply (2.8) we find that

$$f_{\gamma,m_5}(n) = \int_{\gamma_0^{-1}(\mathcal{G})} f(T_6T_5n)e(-am_5t_5+cm_5t_6)\,dt_6\,dt_5.$$

If we fix $n \in N$, then our integrand

$$(t_5,t_6) \mapsto f(T_6T_5n)e(-am_5t_5+cm_5t_6),$$

is a function on $N' = \langle T_6, T_5\rangle$ and is $N'_{\mathbb{Z}} = \langle N_6, m_5\rangle$ invariant.

Observe that the map $(t_5, t_6) \mapsto T_6T_5$ is a group isomorphism from $\mathbb{R}^2$ onto N'. When restricted to $\mathbb{Z}^2$ we obtain a group isomorphism onto $N'_{\mathbb{Z}}$. In light of this, we shall often identify $N'_{\mathbb{Z}}\backslash N'$ implicitly with $\mathbb{Z}^2\backslash\mathbb{R}^2$. From this, we see that $\mathcal{G}$ is a fundamental domain for $N'_{\mathbb{Z}}\backslash N'$. If we prove that $\gamma_0^{-1}(\mathcal{G})$ is also a fundamental domain for $N'_{\mathbb{Z}}\backslash N'$ then it would follow that

$$f_{\gamma,m_5}(n) = \int_0^1 \int_0^1 f(T_6T_5n)e(-am_5t_5 + cm_5t_6)\, dt_6\, dt_5 = f_{-cm_5,am_5}(n). \qquad (2.9)$$

The following lemma shows that this is indeed the case.

Lemma 2.4. *$\gamma_0(\mathcal{G})$ is a fundamental domain for $N'_{\mathbb{Z}}\backslash N'$, from which it follows that $\gamma_0^{-1}(\mathcal{G})$ is also a fundamental domain for $N'_{\mathbb{Z}}\backslash N'$.*

Proof. Observe that the union of all $\gamma_0([m_5, m_5+1) \times [m_6, m_6+1))$ where $m_5, m_6 \in \mathbb{Z}$, partition $\mathbb{R}^2$. Hence for $(t_5, t_6) \in \mathbb{R}^2$, we have that $(t_5, t_6) \in \gamma_0([m_5, m_5+1) \times [m_6, m_6+1))$ for some $m_5, m_6 \in \mathbb{Z}$. Thus for some $(e_5, e_6) \in [0, 1)^2$, we have

$$(t_5, t_6) = \gamma_0(m_5 + e_5, m_6 + e_6) = (am_5 - cm_6 + ae_5 - ce_6, -bm_5 + dm_6 - be_5 + de_6).$$

Hence

$$(t_5 - am_5 + cm_6, t_6 + bm_5 - dm_6) = (ae_5 - ce_6, -be_5 + de_6) \in \gamma_0(\mathcal{G}).$$

Therefore the coset of $\mathbb{Z}^2\backslash\mathbb{R}^2$ containing (t_5, t_6) has a representative in $\gamma_0(\mathcal{G})$.

Now suppose that there exists distinct $(x_5, x_6), (y_5, y_6) \in \gamma_0(\mathcal{G})$ such that

$$(x_5, x_6) = (y_5 + m_5, y_6 + m_6) \text{ where } m_5, m_6 \in \mathbb{Z}$$

(i.e. they represent the same coset of $\mathbb{Z}^2\backslash\mathbb{R}^2$). Since $(x_5, x_6), (y_5, y_6) \in \gamma_0(\mathcal{G})$, it follows that there exists $(u_5, u_6), (v_5, v_6) \in \mathcal{G}$ such that

$$\gamma_0(u_5, u_6) = (au_5 - cu_6, -bu_5 + du_6) = (x_5, x_6)$$
$$\gamma_0(v_5, v_6) = (av_5 - cv_6, -bv_5 + dv_6) = (y_5, y_6).$$

Thus

$$\gamma_0(u_5, u_6) = (y_5, y_6) + (m_5, m_6) = \gamma_0(v_5, v_6) + (m_5, m_6)$$

which implies

$$\gamma_0(u_5 - v_5, u_6 - v_6) = (m_5, m_6).$$

Hence $u_5 - v_5, u_6 - v_6 \in \mathbb{Z}$. Since $(u_5, u_6), (v_5, v_6) \in \mathcal{G}$ it follows that $u_5 = v_5$ and $u_6 = v_6$. Therefore each element of $\gamma_0(\mathcal{G})$ represents a distinct cosets of $\mathbb{Z}^2\backslash\mathbb{R}^2$. Thus $\gamma_0(\mathcal{G})$ is a fundamental domain as claimed. $\square$

Let $(\Gamma_\beta)_\infty$ denote the space of unipotent upper-triangular matrices of Γ_β and let $[\gamma]$ denote the coset of $\Gamma_\beta/(\Gamma_\beta)_\infty$ which contains $\gamma \in \Gamma_\beta$. Observe

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b + aq \\ c & d + cq \end{pmatrix}. \tag{2.10}$$

It follows from this computation and the theory of linear Diophantine equations that

$$\text{each } [\gamma] \text{ can be uniquely identified by } (a, c) \in \mathbb{Z}^2 \text{ such that } \gcd(a, c) = 1. \tag{2.11}$$

Therefore by (2.9), $f_{\gamma,m_5} = f_{\gamma',m_5}$ if $[\gamma] = [\gamma']$. As one might expect, we also have that

$$f_{\gamma,m_5,m_4,m_3} = f_{\gamma',m_5,m_4,m_3} \text{ if } [\gamma] = [\gamma']. \tag{2.12}$$

To see that this is indeed the case we apply the simultaneous change of variables (2.8) to f_{γ,m_5,m_4,m_3} and apply Lemma 2.4 to conclude that

$$f_{\gamma,m_5,m_4,m_3}(n) = \int_0^1 \int_0^1 \int_0^1 \int_0^1 f\big(R_6(t_6 - 2a^2cx_3^3)R_5(t_5 - ac^2x_3^3)R_4(t_4 - act_3^2)$$
$$R_3(at_3)R_2(ct_3)\big)e(cm_5t_6 - am_5t_5 - m_4t_4 - m_3t_3)\, dt_6\, dt_5\, dt_4\, dt_3.$$

Therefore, by this equality and (2.11) we have that (2.12) holds.

2.3 Preliminary Applications of Lemmas

In this section we will show that smooth, $N_{\mathbb{Z}}$-invariant $f : N \to \mathbb{C}$ can be closely approximated by finite sums with terms of the form (2.7). We repeatedly apply Lemmas 2.3 and 2.2 to accomplish this. To begin, let $\epsilon > 0$ and K a compact subset of N. By Lemma 2.3(a), Lemma 2.2, and Proposition 2.6 (a result we will prove in the next section) we have that there exists $M_0 \in \mathbb{Z}_{>0}$ such that

$$\Bigg\| \sum_{|k_6| \le M_6} f_{k_6} - f \Bigg\|_K < \epsilon, \tag{2.13}$$

and

$$\sum_{\substack{k_6\in\mathbb{Z}\\ |k_6|>M_6 \text{ or } |k_5|>M_6}}\sum_{k_5\in\mathbb{Z}} \|f_{k_6,k_5}\|_K < \epsilon \tag{2.14}$$

for all $M_6 > M_0$ where $M_6 \in \mathbb{Z}_{>0}$. We shall fix such a choice of M_6 in what follows. By Lemma 2.3(b) and Lemma 2.2 there exists $M_5 \in \mathbb{Z}_{>0}$ such that $M_5 \geq M_6$ and

$$\left\| \sum_{|k_5|\leq M_5} f_{k_6,k_5} - f_{k_6} \right\|_K < \frac{\epsilon}{2M_6+1}. \tag{2.15}$$

for $|k_6| \leq M_6$. By the triangle inequality, it follows from (2.14) and (2.15) that

$$\left\| \sum_{|k_6|\leq M_6} \sum_{|k_5|\leq M_5} f_{k_6,k_5} - f \right\|_K < 2\epsilon. \tag{2.16}$$

Observe

$$\begin{aligned} &\text{each } (k_6,k_5) \in \mathbb{Z}^2_{\neq(0,0)} \text{ can be written uniquely as } (-cm_5, am_5) \\ &\text{where } m_5 = \gcd(k_6,k_5), c = -\frac{k_6}{m_5}, a = \frac{k_5}{m_5}, \text{ and } \gcd(a,c) = 1. \end{aligned} \tag{2.17}$$

If $|k_6| \leq M_6$, $|k_5| \leq M_5$ then $m_5 = \gcd(k_6,k_5) \leq M_5$, $|a| = \left|\frac{k_5}{m_5}\right| \leq M_5$, and $|c| = \left|\frac{k_6}{m_5}\right| \leq M_6$. Therefore

$$\begin{aligned} &\{(k_6,k_5) \in \mathbb{Z}^2_{\neq(0,0)} : |k_6| \leq M_6, |k_5| \leq M_5\} \\ &\quad \subset \{(-cm_5, am_5) : 0 < m_5 \leq M_5, |a| \leq M_5, |c| \leq M_6, \gcd(a,c) = 1\}, \end{aligned}$$

and so by (2.9) and (2.14),

$$\left\| \sum_{0<m_5\leq M_5} \sum_{\substack{[\gamma]\in\Gamma_\beta/(\Gamma_\beta)_\infty\\ |a|\leq M_5, |c|\leq M_6}} f_{\gamma,m_5} + f_{0,0} - \sum_{\substack{|k_6|\leq M_6\\ |k_5|\leq M_5}} f_{k_6,k_5} \right\|_K \leq \sum_{\substack{k_6\in\mathbb{Z}\\ |k_6|>M_6 \text{ or } |k_5|>M_6}}\sum_{k_5\in\mathbb{Z}} \|f_{k_6,k_5}\|_K < \epsilon. \tag{2.18}$$

Thus by (2.16), (2.18), and the triangle inequality,

$$\left\| \sum_{0<m_5\leq M_5} \sum_{\substack{[\gamma]\in\Gamma_\beta/(\Gamma_\beta)_\infty\\ |a|\leq M_5, |c|\leq M_6}} f_{\gamma,m_5} + f_{0,0} - f \right\|_K < 3\epsilon. \tag{2.19}$$

By Lemma 2.3(c) and Lemma 2.2 there exists $M_4 \in \mathbb{Z}_{>0}$ such that $M_4 \geq M_5$,

$$\left\| \sum_{|m_4| \leq M_4} f_{\gamma,m_5,m_4} - f_{\gamma,m_5} \right\|_K < \frac{\epsilon}{(2M_6+1)M_5(2M_5+1)}, \text{ and}$$

$$\left\| \sum_{|k_4| \leq M_4} f_{0,0,k_4} - f_{0,0} \right\|_K < \epsilon$$

for $[\gamma] \in \Gamma_\beta/(\Gamma_\beta)_\infty$, $|a| \leq M_5$, $|c| \leq M_6$, $0 < m_5 \leq M_5$. By these inequalities, (2.19), and the triangle inequality,

$$\left\| \sum_{0<m_5 \leq M_5} \sum_{\substack{[\gamma]\in\Gamma_\beta/(\Gamma_\beta)_\infty \\ |a|\leq M_5, |c|\leq M_6}} \sum_{|m_4|\leq M_4} f_{\gamma,m_5,m_4} + \sum_{|k_4|\leq M_4} f_{0,0,k_4} - f \right\|_K < 5\epsilon. \tag{2.20}$$

By Lemma 2.3(d) and Lemma 2.2 there exists $M_3 \in \mathbb{Z}_{>0}$ such that $M_3 \geq M_4$,

$$\left\| \sum_{|m_3| \leq M_3} f_{\gamma,m_5,m_4,m_3} - f_{\gamma,m_5,m_4} \right\|_K < \frac{\epsilon}{(2M_6+1)M_5(2M_5+1)(2M_4+1)}, \text{ and}$$

$$\left\| \sum_{|k_3| \leq M_3} f_{0,0,k_4,k_3} - f_{0,0,k_4} \right\|_K < \frac{\epsilon}{2M_4+1},$$

for $[\gamma] \in \Gamma_\beta/(\Gamma_\beta)_\infty$, $|a| \leq M_5$, $|c| \leq M_6$, $0 < m_5 \leq M_5$, $|m_4| \leq M_4$, $|k_4| \leq M_4$. By these inequalities, (2.20), and the triangle inequality,

$$\left\| \sum_{0<m_5 \leq M_5} \sum_{\substack{[\gamma]\in\Gamma_\beta/(\Gamma_\beta)_\infty \\ |a|\leq M_5, |c|\leq M_6}} \sum_{|m_4|\leq M_4} \sum_{|m_3|\leq M_3} f_{\gamma,m_5,m_4,m_3} + \sum_{|k_4|\leq M_4} \sum_{|k_3|\leq M_3} f_{0,0,k_4,k_3} - f \right\|_K < 7\epsilon. \tag{2.21}$$

Let id denote the element h in (2.2) with $a = d = 1$ and $b = c = 0$. Likewise, let $-\mathrm{id}$ denote the element h in (2.2) with $a = d = -1$ and $b = c = 0$. Observe

$$(-\mathrm{id})T_6T_5T_4T_3(-\mathrm{id}) = R_6(-t_6)R_5(-t_5)T_4R_3(-t_3)$$

By performing the change of variables

$$t_6 \mapsto -t_6, \quad t_5 \mapsto -t_5, \quad t_3 \mapsto -t_3,$$

we find that

$$
\begin{aligned}
&f_{-\mathrm{id},m_5,m_4,m_3}(n)\\
&=\int_0^1\int_0^1\int_0^1\int_0^1 f(R_6(-t_6)R_5(-t_5)T_4R_3(-t_3)n)e(-m_5t_5-m_4t_4-m_3t_3)\\
&\qquad dt_6\,dt_5\,dt_4\,dt_3\\
&=-\int_0^{-1}\int_0^1\int_0^{-1}\int_0^{-1} f(T_6T_5T_4T_3n)e(m_5t_5-m_4t_4+m_3t_3)\,dt_6\,dt_5\,dt_4\,dt_3\\
&=\int_{-1}^0\int_0^1\int_{-1}^0\int_{-1}^0 f(T_6T_5T_4T_3n)e(m_5t_5-m_4t_4+m_3t_3)\,dt_6\,dt_5\,dt_4\,dt_3
\end{aligned}
$$

By Lemma 2.3 we are able to change the interval of integration from $[-1,0]$ to $[0,1]$ in the t_6, t_5, and t_3 variable. Thus $f_{-\mathrm{id},m_5,m_4,m_3}=f_{0,-m_5,m_4,-m_3}$. Likewise, one can see that $f_{\mathrm{id},m_5,m_4,m_3}=f_{0,m_5,m_4,m_3}$. Since $[\mathrm{id}]$ and $[-\mathrm{id}]$ are distinct elements in $\Gamma_\beta/(\Gamma_\beta)_\infty$ it follows that

$$
\begin{aligned}
&\sum_{0<m_5\le M_5}\sum_{\substack{[\gamma]\in\Gamma_\beta/(\Gamma_\beta)_\infty\\ |a|\le M_5,|c|\le M_6}}\sum_{|m_4|\le M_4}\sum_{|m_3|\le M_3} f_{\gamma,m_5,m_4,m_3}+\sum_{|k_4|\le M_4}\sum_{|k_3|\le M_3} f_{0,0,k_4,k_3} \qquad (2.22)\\
&=\sum_{0<m_5\le M_5}\sum_{\substack{[\gamma]\in\Gamma_\beta/(\Gamma_\beta)_\infty\\ [\gamma]\ne[\pm\mathrm{id}]\\ |a|\le M_5,|c|\le M_6}}\sum_{|m_4|\le M_4}\sum_{|m_3|\le M_3} f_{\gamma,m_5,m_4,m_3}+\sum_{|k_5|\le M_5}\sum_{|k_4|\le M_4}\sum_{|k_3|\le M_3} f_{0,k_5,k_4,k_3}.
\end{aligned}
$$

By Lemma 2.3(e) and Lemma 2.2 there exists $M_1\in\mathbb{Z}_{>0}$ such that $M_1\ge M_3$,

$$
\left\|\sum_{|k_1|\le M_1} f_{0,k_5,k_4,k_3,k_1}-f_{0,k_5,k_4,k_3}\right\|_K<\frac{\epsilon}{(2M_5+1)(2M_4+1)(2M_3+1)},
$$

for $k_5\le M_5$, $|k_4|\le M_4$, $|k_3|\le M_3$. By this inequality, (2.21), and (2.22), we find that

$$
\begin{aligned}
&\Bigg\|\sum_{0<m_5\le M_5}\sum_{\substack{[\gamma]\in\Gamma_\beta/(\Gamma_\beta)_\infty\\ [\gamma]\ne[\pm\mathrm{id}]\\ |a|\le M_5,|c|\le M_6}}\sum_{|m_4|\le M_4}\sum_{|m_3|\le M_3} f_{\gamma,m_5,m_4,m_3}\\
&\qquad+\sum_{|k_5|\le M_5}\sum_{|k_4|\le M_4}\sum_{|k_3|\le M_3}\sum_{|k_1|\le M_1} f_{0,k_5,k_4,k_3,k_1}-f\Bigg\|_K<8\epsilon, \qquad (2.23)
\end{aligned}
$$

By Lemma 2.3(f) and Lemma 2.2 there exists $M_2\in\mathbb{Z}_{>0}$ such that $M_2\ge M_1$,

$$
\left\|\sum_{|k_2|\le M_2} f_{0,0,0,0,k_1,k_2}-f_{0,0,0,0,k_1}\right\|_K<\frac{\epsilon}{(2M_1+1)},
$$

for $|k_1|\le M_1$. By this inequality and (2.23) we obtain the following proposition.

Proposition 2.5. *If $f:N\to\mathbb{C}$ is smooth, $N_{\mathbb{Z}}$-invariant, and $\epsilon>0$ then there exists $M_0\in\mathbb{Z}_{>0}$ such that for any $M_6>M_0$, where $M_6\in\mathbb{Z}_{>0}$, there exists M_5,M_4,M_3,M_1,*

$M_2 \in \mathbb{Z}_{>0}$ *such that* $M_6 \le M_5 \le M_4 \le M_3 \le M_1 \le M_2$ *and*

$$\Bigg\| \sum_{0<m_5\le M_5} \sum_{\substack{[\gamma]\in\Gamma_\beta/(\Gamma_\beta)_\infty \\ [\gamma]\ne[\pm\mathrm{id}] \\ |a|\le M_5,|c|\le M_6}} \sum_{|m_4|\le M_4} \sum_{|m_3|\le M_3} f_{\gamma,m_5,m_4,m_3}$$

$$+ \sum_{|k_5|\le M_5} \sum_{|k_4|\le M_4} \sum_{\substack{0<|k_3|\le M_3 \\ k_5\ne0 \text{ or } k_4\ne 0 \text{ or } k_3\ne 0}} \sum_{|k_1|\le M_1} f_{0,k_5,k_4,k_3,k_1} + \sum_{|k_1|\le M_1} \sum_{|k_2|\le M_2} f_{0,0,0,0,k_1,k_2} - f \Bigg\|_K < \epsilon,$$

where the terms in these sums are defined in (2.7).

2.4 Some Inequalities

Although Proposition 2.5 shows that smooth, $N_{\mathbb{Z}}$-invariant $f : N \to \mathbb{C}$ can be approximated by a finite sum of terms of the form (2.7), what we really desire is a series expansion for such f. In Theorem 2.10 we give an absolutely convergent series expansion for such f within the Banach space $C(K)$, the space of continuous functions on K equipped with the norm $\|\cdot\|_K$. In order to prove the absolute convergence of this series, we will need to prove various propositions in this section, but before we do, we prove a result we used in the previous section. Throughout this section we will let $I = [0, 1]$.

Proposition 2.6. *If* $f : N \to \mathbb{C}$ *is smooth,* $N_{\mathbb{Z}}$*-invariant, then* $\sum_{k_6\in\mathbb{Z}} \sum_{k_5\in\mathbb{Z}} \|f_{k_6,k_5}\|_K < \infty$.

Proof. Let

$$q_{k_6}(x_5, n) = \int_0^1 f(T_6X_5n)e(-k_6t_6)\,dt_6,$$
$$q(x_6, x_5, n) = f(X_6X_5n).$$

By Lemma 2.3(b) we see that q_{k_6} is periodic in x_5. Observe that f_{k_6,k_5} is the k_5-th Fourier coefficient in the x_5 variable of $q_{k_6}(x_5, n)$. Therefore by Lemma 2.1 we have

$$\sum_{k_6\in\mathbb{Z}} \sum_{k_5\in\mathbb{Z}} \|f_{k_6,k_5}\|_K \le \sum_{k_6\in\mathbb{Z}} \big(\|q_{k_6}\|_{I\times K} + \|\partial^2_{x_5}[q_{k_6}]\|_{I\times K}\big). \tag{2.24}$$

By Lemma 2.3(a) we see that q is periodic in x_6. Observe that q_{k_6} is the k_6-th Fourier coefficient in the x_6 variable of q and that $\partial^2_{x_5}[q_{k_6}]$ is the k_6-th Fourier coefficient in the x_6 variable of $\partial^2_{x_5}[q]$. Therefore by Lemma 2.1 we have

$$\sum_{k_6\in\mathbb{Z}} \|q_{k_6}\|_{I\times K} \le \|q\|_{I^2\times K} + \|\partial^2_{x_6}[q]\|_{I^2\times K}, \tag{2.25a}$$

$$\sum_{k_6\in\mathbb{Z}} \|\partial^2_{x_5}[q_{k_6}]\|_{I\times K} \le \|\partial^2_{x_5}[q]\|_{I^2\times K} + \|\partial^2_{x_6}\partial^2_{x_5}[q]\|_{I^2\times K}. \tag{2.25b}$$

Our proposition then follows from (2.24) and (2.25). □

The following propositions will be used the prove the absolute convergence stated in Theorem 2.10.

Proposition 2.7. *If $f : N \to \mathbb{C}$ is smooth, $N_{\mathbb{Z}}$-invariant, then*

$$\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} \|f_{0,0,0,0,k_1,k_2}\|_K < \infty.$$

Proof. Let

$$q_{k_1}(x_2,n) = \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 f(T_6T_5T_4T_3T_1X_2n)e(-k_1t_1)\,dt_6\,dt_5\,dt_4\,dt_3\,dt_1,$$
$$q(x_2,x_1,n) = \int_0^1\int_0^1\int_0^1\int_0^1 f(T_6T_5T_4T_3X_1X_2n)\,dt_6\,dt_5\,dt_4\,dt_3.$$

By Lemma 2.3(f) we see that q_{k_1} is periodic in the x_2 variable. Observe that $f_{0,0,0,0,k_1,k_2}$ is the k_2-th Fourier coefficient in the x_2 variable of q_{k_1}. Therefore by Lemma 2.1 we have

$$\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} \|f_{0,0,0,0,k_1,k_2}\|_K \le \sum_{k_1\in\mathbb{Z}} \|q_{k_1}\|_{I\times K} + \sum_{k_1\in\mathbb{Z}} \|\partial^2_{x_2}[q_{k_1}]\|_{I\times K}.$$

Next, observe that q is periodic in the x_1 variable by Lemma 2.3(e). Observe that q_{k_1} is the k_1-th Fourier coefficient in the x_1 variable of q. Likewise, $\partial^2_{x_2}[q_{k_1}]$ is the k_1-th Fourier coefficient in the x_1 variable of $\partial^2_{x_2}[q]$. Therefore by Lemma 2.1 we have

$$\begin{aligned}&\sum_{k_1\in\mathbb{Z}} \|q_{k_1}\|_{I\times K} + \sum_{k_1\in\mathbb{Z}} \|\partial^2_{x_2}[q_{k_1}]\|_{I\times K}\\ &\quad\le \|q\|_{I^2\times K} + \|\partial^2_{x_1}[q]\|_{I^2\times K} + \|\partial^2_{x_2}[q]\|_{I^2\times K} + \|\partial^2_{x_1}\partial^2_{x_2}[q]\|_{I^2\times K},\end{aligned}$$

and from this our proposition follows. □

Proposition 2.8. *If $f : N \to \mathbb{C}$ is smooth, $N_{\mathbb{Z}}$-invariant, then*

$$\sum_{k_5\in\mathbb{Z}}\sum_{k_4\in\mathbb{Z}}\sum_{k_3\in\mathbb{Z}}\sum_{k_1\in\mathbb{Z}} \|f_{0,k_5,k_4,k_3,k_1}\|_K < \infty.$$

Proof. Let

$$q(x_5,x_4,x_3,x_1,n) = \int_0^1 f(T_6X_5X_4X_3X_1n)\,dt_6,$$
$$q_{k_5}(x_4,x_3,x_1,n) = \int_0^1\int_0^1 f(T_6T_5X_4X_3X_1n)e(-k_5t_5)\,dt_6\,dt_5,$$
$$q_{k_5,k_4}(x_3,x_1,n) = \int_0^1\int_0^1\int_0^1 f(T_6T_5T_4X_3X_1n)e(-k_5t_5-k_4t_4)\,dt_6\,dt_5\,dt_4,$$
$$q_{k_5,k_4,k_3}(x_1,n) = \int_0^1\int_0^1\int_0^1\int_0^1 f(T_6T_5T_4T_3X_1n)e(-k_5t_5-k_4t_4-k_3t_3)\,dt_6\,dt_5\,dt_4\,dt_3.$$

Observe that q_{k_5,k_4,k_3} is periodic in x_1 by Lemma 2.3(e) and that f_{0,k_5,k_4,k_3,k_1} is the k_1-th Fourier coefficient of q_{k_5,k_4,k_3} in the x_1 variable. Therefore, by Lemma 2.1, we have that

$$\sum_{k_5\in\mathbb{Z}}\sum_{k_4\in\mathbb{Z}}\sum_{k_3\in\mathbb{Z}}\sum_{k_1\in\mathbb{Z}}\|f_{0,k_5,k_4,k_3,k_1}\|_K$$

is bounded by a finite linear combination of series of the form

$$\sum_{k_5\in\mathbb{Z}}\sum_{k_4\in\mathbb{Z}}\sum_{k_3\in\mathbb{Z}}\left\|D[q_{k_5,k_4,k_3}]\right\|_K \tag{2.26}$$

where $D = \mathrm{id}, \partial_{x_1}^2$.

Next observe that $D[q_{k_5,k_4}]$ is periodic in the x_3 variable by Lemma 2.3(d) and that $D[q_{k_5,k_4,k_3}]$ is the k_3-th Fourier coefficient of $D[q_{k_5,k_4}]$ in the x_3 variable where $D = \mathrm{id}, \partial_{x_1}^2$. By Lemma 2.1 we have that series of the form (2.26) are bounded by a finite linear combination of series of the form

$$\sum_{k_5\in\mathbb{Z}}\sum_{k_4\in\mathbb{Z}}\left\|D[q_{k_5,k_4}]\right\|_K \tag{2.27}$$

where $D = (\partial_{x_3})^{\ell_3}(\partial_{x_1})^{\ell_1}$ and $\ell_3, \ell_1 \in \mathbb{Z}_{\geq 0}$.[1]

Next observe that $D[q_{k_5}]$ is periodic in the x_4 variable by Lemma 2.3(c) and that $D[q_{k_5,k_4}]$ is the k_4-th Fourier coefficient of $D[q_{k_5}]$ in the x_4 variable where $D = (\partial_{x_3})^{\ell_3}(\partial_{x_1})^{\ell_1}$ and $\ell_3, \ell_1 \in \mathbb{Z}_{\geq 0}$. By Lemma 2.1 we have that series of the form (2.27) are bounded by a finite linear combination of series of the form

$$\sum_{k_5\in\mathbb{Z}}\left\|D[q_{k_5}]\right\|_K \tag{2.28}$$

where $D = (\partial_{x_4})^{\ell_4}(\partial_{x_3})^{\ell_3}(\partial_{x_1})^{\ell_1}$ and $\ell_4, \ell_3, \ell_1 \in \mathbb{Z}_{\geq 0}$.

Next observe that $D[q]$ is periodic in the x_5 variable by Lemma 2.3(b) and that $D[q_{k_5}]$ is the k_5-th Fourier coefficient in the x_5 variable of $D[q]$ where $D = (\partial_{x_4})^{\ell_4}(\partial_{x_3})^{\ell_3}\ (\partial_{x_1})^{\ell_1}$ and $\ell_4, \ell_3, \ell_1 \in \mathbb{Z}_{\geq 0}$. By Lemma 2.1 we have that series of the form (2.28) are bounded by a finite linear combination of terms of the form $D[q]$ where $D = (\partial_{x_5})^{\ell_5}(\partial_{x_4})^{\ell_4}(\partial_{x_3})^{\ell_3}(\partial_{x_1})^{\ell_1}$ and $\ell_5, \ell_4, \ell_3, \ell_1 \in \mathbb{Z}_{\geq 0}$. This in combination with the bounds given in (2.26), (2.27), and (2.28) prove our proposition. $\square$

Proposition 2.9. *If $f : N \to \mathbb{C}$ is smooth, $N_{\mathbb{Z}}$-invariant, then*

$$\sum_{m_5\in\mathbb{Z}}\sum_{[\gamma]\in\Gamma_\beta/(\Gamma_\beta)_\infty}\sum_{m_4\in\mathbb{Z}}\sum_{m_3\in\mathbb{Z}}\|f_{\gamma,m_5,m_4,m_3}\|_K < \infty.$$

[1] It should be noted that ℓ_3 and ℓ_1 are each bounded by 2. In this rest of this proof, it will be understood that such ℓ_j are always bounded by a quantity independent of the indices k_i. Likewise, when we state that a quantity is bounded by a linear combination of terms, it will be with the understanding that the coefficients of such a linear combination are independent of the indices k_i.

Proof. Observe that f_{γ,m_5,m_4,m_3} is the m_3-th Fourier coefficient in the x_3 variable of the function

$$\hbar_{\gamma,m_5,m_4}(x_3,n)=\int_0^1\int_0^1\int_0^1 f(\gamma T_6T_5T_4X_3\gamma^{-1}n)e(-m_5t_5-m_4t_4)\,dt_4\,dt_5\,dt_6;$$

recall that Lemma 2.3(d) shows that $\hbar_{\gamma,m_5,m_4}$ is indeed periodic in the x_3 variable. Therefore by Lemma 2.1, we have

$$\begin{aligned}
&\sum_{m_5\in\mathbb{Z}_{>0}}\ \sum_{[\gamma]\in\Gamma_\beta/(\Gamma_\beta)_\infty}\ \sum_{m_4\in\mathbb{Z}}\sum_{m_3\in\mathbb{Z}}\|f_{\gamma,m_5,m_4,m_3}\|_K\\
&\leq \sum_{m_5\in\mathbb{Z}_{>0}}\ \sum_{[\gamma]\in\Gamma_\beta/(\Gamma_\beta)_\infty}\ \sum_{m_4\in\mathbb{Z}}\|\hbar_{\gamma,m_5,m_4}\|_{I\times K}\\
&\quad+\sum_{m_5\in\mathbb{Z}_{>0}}\ \sum_{[\gamma]\in\Gamma_\beta/(\Gamma_\beta)_\infty}\ \sum_{m_4\in\mathbb{Z}}\left\|\partial_{x_3}^2[\hbar_{\gamma,m_5,m_4}]\right\|_{I\times K}.
\end{aligned}\tag{2.29}$$

Let

$$\hbar_{\gamma,m_5}(x_4,x_3,n)=\int_0^1\int_0^1 f(\gamma T_6T_5X_4X_3\gamma^{-1}n)e(-m_5t_5)\,dt_6dt_5.$$

Observe that $D[\hbar_{\gamma,m_5,m_4}]$ is the m_4-th Fourier coefficient in the x_4 variable of the function $D[\hbar_{\gamma,m_5}]$ where $D=\mathrm{id},\partial_{x_3}^2$; recall that Lemma 2.3(c) shows that $D[\hbar_{\gamma,m_5}]$ is indeed periodic in the x_4 variable. Therefore, by Lemma 2.1, we have

$$\begin{aligned}
&\sum_{m_5\in\mathbb{Z}_{>0}}\ \sum_{[\gamma]\in\Gamma_\beta/(\Gamma_\beta)_\infty}\ \sum_{m_4\in\mathbb{Z}}\|D[\hbar_{\gamma,m_5,m_4}]\|_{I\times K}\\
&\leq \sum_{m_5\in\mathbb{Z}_{>0}}\ \sum_{[\gamma]\in\Gamma_\beta/(\Gamma_\beta)_\infty}\|D[\hbar_{\gamma,m_5}]\|_{I^2\times K}+\sum_{m_5\in\mathbb{Z}_{>0}}\ \sum_{[\gamma]\in\Gamma_\beta/(\Gamma_\beta)_\infty}\left\|\partial_{x_4}^2D[\hbar_{\gamma,m_5}]\right\|_{I^2\times K}
\end{aligned}\tag{2.30}$$

where $D=\mathrm{id},\partial_{x_3}^2$. Thus by (2.29) and (2.30), we have that

$$\sum_{m_5\in\mathbb{Z}_{>0}}\ \sum_{[\gamma]\in\Gamma_\beta/(\Gamma_\beta)_\infty}\ \sum_{m_4\in\mathbb{Z}}\sum_{m_3\in\mathbb{Z}}\|f_{\gamma,m_5,m_4,m_3}\|_K\tag{2.31}$$

is bounded by a finite linear combination with terms of the form

$$\sum_{m_5\in\mathbb{Z}_{>0}}\ \sum_{[\gamma]\in\Gamma_\beta/(\Gamma_\beta)_\infty}\|D[\hbar_{\gamma,m_5}]\|_{I^2\times K}\tag{2.32}$$

where $D=\mathrm{id},\partial_{x_3}^2,\partial_{x_4}^2,\partial_{x_4}^2\partial_{x_3}^2$.

By applying the change of variables (2.8) and Lemma 2.4 we find that

$$\begin{aligned}
\hbar_{\gamma,m_5}(x_4,x_3,n)&=\int_0^1\int_0^1 f(\gamma T_6T_5X_4X_3\gamma^{-1}n)e(-m_5t_5)\,dt_6\,dt_5\\
&=\int_0^1\int_0^1 f(R_6(-2a^2cx_3^3+t_6)R_5(-ac^2x_3^3+t_5)R_4(-acx_3^2+x_4)R_3(ax_3)R_2(cx_3)n)\\
&\qquad e(-am_5t_5+cm_5t_6)\,dt_6\,dt_5\\
&=\not{p}_{-cm_5,am_5}(-2a^2cx_3^3,-ac^2x_3^3,-acx_3^2+x_4,ax_3,cx_3,n)
\end{aligned}$$

where

$$p_{k_6,k_5}(x_6,x_5,x_4,x_3,x_2,n) = \int_0^1 \int_0^1 f(T_6X_6T_5X_5X_4X_3X_2n)e(-k_6t_6 - k_5t_5)\,dt_6\,dt_5.$$

Let $\rho_\gamma : \mathbb{R}^2 \times N \to \mathbb{R}^5 \times N$ where

$$\rho_\gamma(x_4,x_3,n) = (-2a^2cx_3^3, -ac^2x_3^3, -acx_3^2 + x_4, ax_3, cx_3, n).$$

Observe that

$$\hbar_{\gamma,m_5}(x_4,x_3,n) = p_{-cm_5,am_5} \circ \rho_\gamma(x_4,x_3,n). \tag{2.33}$$

We will use this equality to compute useful bounds for $D[\hbar_{\gamma,m_5}]$ where $D = \mathrm{id}, \partial_{x_3}^2, \partial_{x_4}^2, \partial_{x_3}^2\partial_{x_4}^2$.

By (2.33) and the chain rule, observe

$$\begin{aligned}
&\partial_{x_3}[\hbar_{\gamma,m_5}](x_4,x_3,n)\\
&= \partial_{x_6}[p_{-cm_5,am_5}] \circ \rho_\gamma(x_4,x_3,n)\cdot(-6a^2cx_3^2) + \partial_{x_5}[p_{-cm_5,am_5}] \circ \rho_\gamma(x_4,x_3,n)\cdot(-3ac^2x_3^2)\\
&\quad + \partial_{x_4}[p_{-cm_5,am_5}] \circ \rho_\gamma(x_4,x_3,n)\cdot(-2acx_3) + \partial_{x_3}[p_{-cm_5,am_5}] \circ \rho_\gamma(x_4,x_3,n)\cdot a\\
&\quad + \partial_{x_2}[p_{-cm_5,am_5}] \circ \rho_\gamma(x_4,x_3,n)\cdot c.
\end{aligned}$$

Thus $\partial_{x_3}[\hbar_{\gamma,m_5}](x_4,x_3,n)$ can be written as a finite linear combination with terms of the form

$$(a^{\ell_1}c^{\ell_2}x_3^{\ell_3})\partial_{x_i}[p_{-cm_5,am_5}] \circ \rho_\gamma(x_4,x_3,n) \tag{2.34}$$

where $\ell_j \in \mathbb{Z}_{\geq 0}$, and $i \in \{2,\dots,6\}$.[2] When we apply ∂_{x_3} to (2.34), we find by the chain rule that (2.34) can be written as a finite linear combination with terms of the form

$$(a^{\ell_1}c^{\ell_2}x_3^{\ell_3})\partial_{x_{i_1}}\partial_{x_{i_2}}[p_{-cm_5,am_5}] \circ \rho_\gamma(x_4,x_3,n) \tag{2.35}$$

where $\ell_j \in \mathbb{Z}_{\geq 0}$, and $i_1, i_2 \in \{2,\dots 6\}$ (the ℓ_j in (2.34) will not necessarily be equal to the ℓ_j as in (2.35)). Therefore

$$\begin{aligned}
&\partial_{x_3}^2[\hbar_{\gamma,m_5}](x_4,x_3,n) \text{ can be written as a finite linear combination}\\
&\text{with terms of the form:}\quad (a^{\ell_1}c^{\ell_2}x_3^{\ell_3})\partial_{x_{i_1}}\partial_{x_{i_2}}[p_{-cm_5,am_5}] \circ \rho_\gamma(x_4,x_3,n)\\
&\text{where } \ell_j \in \mathbb{Z}_{\geq 0}, \text{ and } i_1, i_2 \in \{2,\dots,6\}.
\end{aligned} \tag{2.36}$$

Observe by the chain rule, that

$$\begin{aligned}
&\partial_{x_4}[(a^{\ell_1}c^{\ell_2}x_3^{\ell_3})D[p_{-cm_5,am_5}] \circ \rho_\gamma(x_4,x_3,n)]\\
&= (a^{\ell_1}c^{\ell_2}x_3^{\ell_3})\partial_{x_4}D[p_{-cm_5,am_5}] \circ \rho_\gamma(x_4,x_3,n)
\end{aligned} \tag{2.37}$$

[2] In this rest of this proof, it will be understood that such ℓ_j are always bounded by a quantity independent of the indices $[[\gamma]]$, m_i, and k_i. Likewise, when we state that a quantity is bounded by a linear combination of terms, it will be with the understanding that the coefficients of such a linear combination are independent of the indices $[[\gamma]]$, m_i, and k_i.

where $\ell_j \in \mathbb{Z}_{\geq 0}$, $D = \mathrm{id}, \partial_{x_{i_1}}, \partial_{x_{i_1}}\partial_{x_{i_2}}$, and $i_1, i_2 \in \{2, \dots 6\}$. From (2.33) and (2.37) we see that

$$\partial^2_{x_4}[\hbar_{\gamma,m_5}](x_4, x_3, n) = \partial^2_{x_4}[p_{-cm_5,am_5}] \circ \rho_\gamma(x_4, x_3, n), \tag{2.38}$$

and by (2.36) and (2.37),

$$\begin{aligned}&\partial^2_{x_4}\partial^2_{x_3}[\hbar_{\gamma,m_5}](x_4, x_3, n) \text{ can be written as a finite linear combination}\\ &\text{with terms of the form:} \quad (a^{\ell_1}c^{\ell_2}x_3^{\ell_3})\partial^2_{x_4}\partial_{x_{i_1}}\partial_{x_{i_2}}[p_{-cm_5,am_5}] \circ \rho_\gamma(x_4, x_3, n)\\ &\text{where } \ell_j \in \mathbb{Z}_{\geq 0}, \text{ and } i_1, i_2 \in \{2, \dots, 6\}.\end{aligned} \tag{2.39}$$

Therefore, by (2.36), (2.38), and (2.39), we have that $D'[\hbar_{\gamma,m_5}](x_4, x_3, n)$ for $D' = \mathrm{id}, \partial^2_{x_3}, \partial^2_{x_4}$, $\partial^2_{x_4}\partial^2_{x_3}$ can be written as a finite linear combination with terms of the form

$$(a^{\ell_1}c^{\ell_2}x_3^{\ell_3})D[p_{-cm_5,am_5}] \circ \rho_\gamma(x_4, x_3, n) \tag{2.40}$$

where $D = \partial^{j_1}_{x_4}\partial^{j_2}_{x_{i_1}}\partial^{j_3}_{x_{i_2}}$, $j_1 \in \{0, 2\}$, $j_2, j_3 \in \{0, 1\}$, $i_1, i_2 \in \{2, \dots 6\}$, and $\ell_1, \ell_2, \ell_3 \in \mathbb{Z}_{\geq 0}$. Thus we have that (2.31) is bounded by a finite linear combination of series of the form

$$\sum_{m_5 \in \mathbb{Z}_{>0}} \sum_{[\gamma] \in \Gamma_\beta/(\Gamma_\beta)_\infty} \left\| (a^{\ell_1}c^{\ell_2}x_3^{\ell_3})D[p_{-cm_5,am_5}] \circ \rho_\gamma(x_4, x_3, n) \right\|_{I^2 \times K} \tag{2.41}$$

where $D = \partial^{j_1}_{x_4}\partial^{j_2}_{x_{i_1}}\partial^{j_3}_{x_{i_2}}$, $j_1 \in \{0, 2\}$, $j_2, j_3 \in \{0, 1\}$, $i_1, i_2 \in \{2, \dots 6\}$, and $\ell_1, \ell_2, \ell_3 \in \mathbb{Z}_{\geq 0}$.

In what follows, we shall assume $D = \partial^{j_1}_{x_4}\partial^{j_2}_{x_{i_1}}\partial^{j_3}_{x_{i_2}}$ for some $j_1 \in \{0, 2\}$, $j_2, j_3 \in \{0, 1\}$, $i_1, i_2 \in \{2, \dots 6\}$. Observe

$$\begin{aligned}&\sum_{m_5 \in \mathbb{Z}_{>0}} \sum_{[\gamma] \in \Gamma_\beta/(\Gamma_\beta)_\infty} \left\| (a^{\ell_1}c^{\ell_2}x_3^{\ell_3})D[p_{-cm_5,am_5}] \circ \rho_\gamma(x_4, x_3, n) \right\|_{I^2 \times K}\\ &\leq \sum_{m_5 \in \mathbb{Z}_{>0}} \sum_{[\gamma] \in \Gamma_\beta/(\Gamma_\beta)_\infty} \left\| (a^{\ell_1}c^{\ell_2}x_3^{\ell_3}) \right\|_{I^2 \times K} \left\| D[p_{-cm_5,am_5}] \circ \rho_\gamma(x_4, x_3, n) \right\|_{I^2 \times K}\\ &\leq \sum_{(k_6,k_5) \in \mathbb{Z}^2_{\neq(0,0)}} |k_5^{\ell_1}k_6^{\ell_2}| \left\| D[p_{k_6,k_5}](x_6, x_5, x_4, x_3, x_2) \right\|_{\mathbb{R}^3 \times [-k_5,k_5] \times [-k_6,k_6] \times K}.\end{aligned} \tag{2.42}$$

If $k_i = 0$ then by $[-k_i, k_i]$ we mean the set $\{0\}$. In the last inequality we used that $|a| \leq |k_5|$, $|c| \leq |k_6|$ under the correspondence in (2.17). We also employed the inequality

$$\left\| D[p_{k_6,k_5}] \circ \rho_\gamma(x_4, x_3, n) \right\|_{I^2 \times K} \leq \left\| D[p_{k_6,k_5}](x_6, x_5, x_4, x_3, x_2) \right\|_{\mathbb{R}^3 \times [-k_5,k_5] \times [-k_6,k_6] \times K},$$

which is also justified by the inequalities $|a| \leq |k_5|$, $|c| \leq |k_6|$ and the definition of ρ_γ. Since X_6, X_5, X_4 all commute with each other, it follows that $p_{k_6,k_5}(x_6, x_5, x_4, x_3, x_2)$ is periodic with period 1 in x_6, x_5, and x_4. Since $D[p_{k_6,k_5}](x_6, x_5, x_4, x_3, x_2)$ inherits this periodicity in x_6, x_5, x_4, it follows that

$$\left\| D[p_{k_6,k_5}] \right\|_{\mathbb{R}^3 \times [-k_5,k_5] \times [-k_6,k_6] \times K} = \left\| D[p_{k_6,k_5}] \right\|_{I^3 \times [-k_5,k_5] \times [-k_6,k_6] \times K}. \tag{2.43}$$

Suppose $x_3 \in [-k_5, k_5]$. Then $x_3 = e_3 + r_3$ where $r_3 \in \mathbb{Z}$, $e_3 \in [0,1]$, which implies $|r_3| \leq |k_5|$. Since

$$T_6 X_6 T_5 X_5 X_4 X_3 X_2 = R_3(r_3) R_6(t_6 - 3r_3 x_4) X_6 T_5 X_5 X_4 R_3(e_3) X_2$$

it follows that when we change variables in t_6, apply Lemma 2.3(a), and invoke the $N_{\mathbb{Z}}$-invariance of f, we find that

$$p_{k_6,k_5}(x_6, x_5, x_4, x_3, x_2, n) = e(3k_6 r_3 x_4) p_{k_6,k_5}(x_6, x_5, x_4, e_3, x_2, n).$$

Thus

$$D[p_{0,k_5}](x_6, x_5, x_4, x_3, x_2, n) = D[p_{0,k_5}](x_6, x_5, x_4, e_3, x_2, n) \tag{2.44a}$$

$$D[p_{k_6,0}](x_6, x_5, x_4, x_3, x_2, n) = D[p_{k_6,0}](x_6, x_5, x_4, e_3, x_2, n) \tag{2.44b}$$

(if $k_5 = 0$ then $r_3 = 0$), and if $k_6 k_5 \neq 0$ then $D[p_{k_6,k_5}](x_6, x_5, x_4, x_3, x_2, n)$ can be written as a finite linear combination of terms of the form

$$(k_6 r_3)^{\ell'} e(3k_6 r_3 x_4) D'[p_{k_6,k_5}](x_6, x_5, x_4, e_3, x_2) \tag{2.45}$$

where $\ell' \in \mathbb{Z}_{\geq 0}$ and $D' = \partial_{x_4}^{j_1'} \partial_{x_{i_1'}}^{j_2'} \partial_{x_{i_2'}}^{j_3'}$, $j_1' \in \{0,1,2\}$, $j_2', j_3' \in \{0,1\}$, $i_1', i_2' \in \{2, \ldots 6\}$. Since $[-k_5, k_5]$ contains no more than $2|k_5| + 1 \leq 3|k_5|$ integers and $[-k_6, k_6]$ contains no more than $2|k_6| + 1 \leq 3|k_6|$ integers, then by (2.44),

$$\begin{aligned} &\left\| D[p_{0,k_5}](x_6, x_5, x_4, x_3, x_2) \right\|_{I^3 \times [-k_5,k_5] \times \{0\} \times K} \\ &\quad \leq 3|k_5| \cdot \left\| D[p_{0,k_5}](x_6, x_5, x_4, x_3, x_2) \right\|_{I^4 \times \{0\} \times K} \text{ for } k_5 \neq 0, \end{aligned} \tag{2.46a}$$

$$\begin{aligned} &\left\| D[p_{k_6,0}](x_6, x_5, x_4, x_3, x_2) \right\|_{I^3 \times \{0\} \times [-k_6,k_6] \times K} \\ &\quad \leq 3|k_6| \cdot \left\| D[p_{k_6,0}](x_6, x_5, x_4, x_3, x_2) \right\|_{I^3 \times \{0\} \times I \times K} \text{ for } k_6 \neq 0. \end{aligned} \tag{2.46b}$$

If $k_5 k_6 \neq 0$ then since $|r_3| \leq |k_5|$ and since $[-k_5, k_5]$ contains no more than $2|k_5| + 1 \leq 3|k_5|$ integers, it follows from (2.45) that

$$\left\| D[p_{k_6,k_5}](x_6, x_5, x_4, x_3, x_2) \right\|_{I^3 \times [-k_5,k_5] \times [-k_6,k_6] \times K} \tag{2.47}$$

is bounded by a finite linear combination with terms of the form

$$|k_5^{\ell'+1} k_6^{\ell'}| \cdot \left\| D'[p_{k_6,k_5}](x_6, x_5, x_4, x_3, x_2) \right\|_{I^4 \times [-k_6,k_6] \times K}. \tag{2.48}$$

Suppose $k_5 k_6 \neq 0$ and $x_2 \in [-k_6, k_6]$. Thus $x_2 = r_2 + e_2$ where $r_2 \in \mathbb{Z}$ and $e_2 \in [0,1)$, which implies $|r_2| \leq |k_6|$. Since

$$\begin{aligned} &T_6 X_6 T_5 X_5 X_4 X_3 X_2 \\ &= R_2(r_2) R_6(t_6 + 3r_2 x_3^2) X_6 R_5(t_5 - 3r_2^2 x_3 - 3r_2 x_4) X_5 R_4(2r_2 x_3 + x_4) X_3 R_2(e_2) \end{aligned}$$

it follows that when we change variables in t_6 and t_5, apply Lemma 2.3, and invoke the $N_{\mathbb{Z}}$-invariance of f, we find that

$$\begin{aligned} &p_{k_6,k_5}(x_6,x_5,x_4,x_3,x_2,n) \\ &= e(3k_6r_2x_3^2+3k_5r_2x_3+3k_5r_2x_4)p_{k_6,k_5}(x_6,x_5,2r_2x_2+x_4,x_3,e_2,n). \end{aligned}$$

Since $|r_2|\le|k_6|$, $[-k_6,k_6]$ contains no more than $2|k_6|+1\le 3|k_6|$ integers, and p_{k_6,k_5} is periodic in the x_4 variable, then (2.48) will be bounded by a finite linear combination with terms of the form

$$|k_6^{\ell_2'}k_5^{\ell_1'}|\cdot\left\|D''[p_{k_6,k_5}](x_6,x_5,x_4,x_3,x_2)\right\|_{I^5\times K} \tag{2.49}$$

where $\ell_1',\ell_2'\in\mathbb{Z}_{>0}$ and $D''=\partial_{x_4}^{j_1''}\partial_{x_{i_1''}}^{j_2''}\partial_{x_{i_2''}}^{j_3''}$, $j_1''\in\{0,1,2\}$, $j_2'',j_3''\in\{0,1\}$, $i_1'',i_2''\in\{2,\dots 6\}$. Consequently, (2.47) is bounded by a finite linear combination of terms of the form (2.49). By (2.41), (2.42), (2.43), (2.46), (2.47), and (2.49) it follows that (2.31) is bounded by a finite linear combination of series of the form

$$\sum_{k_5\in\mathbb{Z}}|k_5|^{\ell_1}\cdot\left\|D[p_{0,k_5}](x_6,x_5,x_4,x_3,x_2)\right\|_{I^4\times\{0\}\times K}, \tag{2.50a}$$

$$\sum_{k_6\in\mathbb{Z}}|k_6|^{\ell_2}\cdot\left\|D[p_{k_6,0}](x_6,x_5,x_4,x_3,x_2)\right\|_{I^3\times\{0\}\times I\times K}, \tag{2.50b}$$

$$\sum_{k_6\in\mathbb{Z}_{\neq0}}\sum_{k_5\in\mathbb{Z}_{\neq0}}|k_5^{\ell_1}k_6^{\ell_2}|\cdot\left\|D[p_{k_6,k_5}](x_6,x_5,x_4,x_3,x_2)\right\|_{I^5\times K}, \tag{2.50c}$$

where $D=\partial_{x_4}^{j_1}\partial_{x_{i_1}}^{j_2}\partial_{x_{i_2}}^{j_3}$, $j_1\in\{0,1,2\}$, $j_2,j_3\in\{0,1\}$, $i_1,i_2\in\{2,\dots 6\}$, and $\ell_1,\ell_2\in\mathbb{Z}_{>0}$ (we allow for ℓ_1, ℓ_2, and D to vary between (2.50a), (2.50b), (2.50c)).

Let

$$q_{k_6}(x_6,t_5,x_5,x_4,x_3,x_2,n)=\int_0^1 f(T_6X_6T_5X_5X_4X_3X_2n)e(-k_6t_6)\,dt_6.$$

Observe that $D[p_{k_6,k_5}]$ is the k_5-th Fourier coefficient for q_{k_6} in the t_5 variable; recall that q_{k_6} is indeed periodic in t_5 by Lemma 2.3(b). Therefore by Lemma 2.1, we have that

$$\begin{aligned} &\sum_{k_5\in\mathbb{Z}}|k_5|^{\ell_1}\left\|D[p_{0,k_5}](x_6,x_5,x_4,x_3,x_2,n)\right\|_{I^4\times\{0\}\times K} \\ &\qquad\le\left\|\partial_{x_5}^{\ell_1+2}D[q_0](x_6,t_5,x_5,x_4,x_3,x_2,n)\right\|_{I^5\times\{0\}\times K}, \end{aligned} \tag{2.51}$$

$$\begin{aligned} &\sum_{k_6\in\mathbb{Z}_{\neq0}}\sum_{k_5\in\mathbb{Z}_{\neq0}}|k_5^{\ell_1}k_6^{\ell_2}|\left\|D[p_{k_6,k_5}](x_6,x_5,x_4,x_3,x_2,n)\right\|_{I^5\times K} \\ &\qquad\le\sum_{k_6\in\mathbb{Z}_{\neq0}}|k_6^{\ell_2}|\left\|\partial_{x_5}^{\ell_1+2}D[q_{k_6}](x_6,t_5,x_5,x_4,x_3,x_2,n)\right\|_{I^6\times K}. \end{aligned} \tag{2.52}$$

Next, let

$$q(t_6,x_6,t_5,x_5,x_4,x_3,x_2,n)=f(T_6X_6T_5X_5X_4X_3X_2n).$$

Observe that $D[q_{k_6}]$ is the k_6-th Fourier coefficient of q in the t_6 variable; recall that q is indeed periodic in t_6 by Lemma 2.3(a). Therefore by Lemma 2.1, we have that

$$\begin{aligned}&\sum_{k_6\in\mathbb{Z}_{\neq 0}} |k_6^{\ell_2}| \left\| \partial_{x_5}^{\ell_1+2} D[q_{k_6}](x_6,t_5,x_5,x_4,x_3,x_2,n)\right\|_{I^6\times K}\\ &\leq \left\| \partial_{x_6}^{\ell_2+2}\partial_{x_5}^{\ell_1+2} D[q](t_6,x_6,t_5,x_5,x_4,x_3,x_2,n)\right\|_{I^7\times K} < \infty.\end{aligned}$$

Therefore, by (2.52) we have that

$$\begin{aligned}&\sum_{k_6\in\mathbb{Z}_{\neq 0}}\sum_{k_5\in\mathbb{Z}_{\neq 0}} |k_5^{\ell_1}k_6^{\ell_2}| \left\| D[p_{k_6,k_5}](x_6,t_5,x_5,x_4,x_3,x_2,n)\right\|_{I^5\times K}\\ &\leq \left\| \partial_{x_6}^{\ell_2+2}\partial_{x_5}^{\ell_1+2} D[q](t_6,x_6,t_5,x_5,x_4,x_3,x_2,n)\right\|_{I^7\times K} < \infty. \qquad (2.53)\end{aligned}$$

Lastly, consider

$$p(t_6,x_6,x_5,x_4,x_3,x_2,n) = \int_0^1 f(T_6X_6T_5X_5X_4X_3X_2n)\,dt_5.$$

Observe that $D[p_{k_6,0}]$ is the k_6-th Fourier coefficient of $D[p]$ in the t_6 variable; recall that p is indeed periodic in t_6 by Lemma 2.3(a). Therefore by Lemma 2.1, we have that

$$\begin{aligned}&\sum_{k_6\in\mathbb{Z}} |k_6|^{\ell_2}\cdot \left\| D[p_{k_6,0}](x_6,x_5,x_4,x_3,x_2,n)\right\|_{I^3\times\{0\}\times I\times K}\\ &\leq \left\| \partial_{x_6}^{\ell_2+2} D[p](t_6,x_6,x_5,x_4,x_3,x_2,n)\right\|_{I^4\times\{0\}\times I\times K} < \infty. \qquad (2.54)\end{aligned}$$

By (2.51), (2.53), (2.54) it follows that the terms of the form (2.50) are finite, and this proves our proposition. □

Recall that $C(K)$, the space of continuous functions on a compact set K, is a Banach spaces under the norm $\|\cdot\|_K$. Therefore, Propositions 2.7, 2.8, and 2.9 assert that

$$\begin{aligned}&\sum_{m_5\in\mathbb{Z}_{>0}}\sum_{\substack{[\gamma]\in\Gamma/\Gamma_\infty\\ [\gamma]\neq[\pm\mathrm{id}]}}\sum_{m_4\in\mathbb{Z}}\sum_{m_3\in\mathbb{Z}} f_{\gamma,m_5,m_4,m_3} + \sum_{\substack{k_5\in\mathbb{Z}\\ k_5\neq 0}}\sum_{\substack{k_4\in\mathbb{Z}\\ \text{or } k_4\neq 0}}\sum_{\substack{k_3\in\mathbb{Z}\\ \text{or } k_3\neq 0}}\sum_{k_1\in\mathbb{Z}} f_{0,k_5,k_4,k_3,k_1}\\ &+\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} f_{0,0,0,0,k_1,k_2} \qquad (2.55)\end{aligned}$$

converges absolutely in $C(K)$. Consequently, for any enumeration of the terms in (2.55), we have that the resulting sequence of partial sums converges in $C(K)$ and that the limit point of said sequence does not depend upon choice of enumeration (i.e. the series (2.55) converges unconditionally). Proposition 2.5 shows that there exists an enumeration of (2.55) whose sequence of partial sums contains a subsequence which converges to f. Therefore, by basic analysis, we are able to conclude that for any enumeration of (2.55), the sequence of partial sums converges to f. Thus the following proposition follows.

Theorem 2.10. *If $f : N \to \mathbb{C}$ is a smooth, $N_{\mathbb{Z}}$-invariant function and K is a compact subset of N then*

$$f = \sum_{m_5 \in \mathbb{Z}_{>0}} \sum_{\substack{[\gamma]\in\Gamma/\Gamma_\infty \\ [\gamma]\neq[\pm\mathrm{id}]}} \sum_{m_4\in\mathbb{Z}} \sum_{m_3\in\mathbb{Z}} f_{\gamma,m_5,m_4,m_3} + \sum_{\substack{k_5\in\mathbb{Z} \\ k_5\neq 0}} \sum_{\substack{k_4\in\mathbb{Z} \\ or\ k_4\neq 0}} \sum_{\substack{k_3\in\mathbb{Z} \\ or\ k_3\neq 0}} \sum_{k_1\in\mathbb{Z}} f_{0,k_5,k_4,k_3,k_1}$$
$$+ \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}} f_{0,0,0,0,k_1,k_2}$$

where convergence of the series is absolute (and therefore unconditional) in the Banach space $C(K)$. The terms of this series are defined in (2.7).

2.5 $N_{\mathbb{Z}}$-Invariant Functions

If $\ell \in \mathbb{Z}_{\neq 0}$ then let $[x]_\ell = [x]$ denote the coset of $\mathbb{Z}/\ell\mathbb{Z}$ which contains $x \in \mathbb{Z}$. As indicated in our definition, we will drop the subscript ℓ from $[x]_\ell$ when the value of ℓ is clear from context. Similarly, let $[[\gamma]]$ denote the double coset of $(\Gamma_\beta)_\infty\backslash\Gamma_\beta/(\Gamma_\beta)_\infty$ which contains $\gamma \in \Gamma_\beta$. Since

$$\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+cq & b+dq \\ c & d \end{pmatrix} \tag{2.56}$$

it follows from (2.11) that

$$\text{each } [[\gamma]],\ \text{other than } [[\mathrm{id}]] \text{ or } [[-\mathrm{id}]],\ \text{is uniquely identified by } c \text{ and } a (\bmod c); \tag{2.57}$$

notice that $[[\mathrm{id}]]$ and $[[-\mathrm{id}]]$ are distinct in $(\Gamma_\beta)_\infty\backslash\Gamma_\beta/(\Gamma_\beta)_\infty$.

For $q_i \in \mathbb{Z}$ and $\gamma \in \Gamma_\beta$ such that $[[\gamma]] \neq [[\pm\mathrm{id}]]$, we have

$$\begin{aligned}
&f_{\gamma,m_5,m_4,m_3}(Q_6Q_5Q_4Q_3Q_2Q_1n) \\
&= \int_0^1\int_0^1\int_0^1\int_0^1 f(\gamma T_6T_5T_4T_3\gamma^{-1}Q_6Q_5Q_4Q_3Q_2Q_1n)e(-m_5t_5 - m_4t_4 - m_3t_3) \\
&\quad dt_6\, dt_5\, dt_4\, dt_3 \\
&= \int_0^1\int_0^1\int_0^1\int_0^1 f\Big(Q_6Q_5Q_4Q_3Q_2Q_1(Q_1^{-1}\gamma)R_6(3abq_2^2t_3 + 6q_2q_3t_3 - 6adq_2q_3t_3 \\
&\quad + 3cdq_3^2t_3 + 3q_4t_3 + 3aq_2t_3^2 - 3cq_3t_3^2 + 3bq_2t_4 - 3dq_3t_4 + t_6)R_5(-3a^2q_2^2t_3 \\
&\quad + 6acq_2q_3t_3 - 3c^2q_3^2t_3 - 3aq_2t_4 + 3cq_3t_4 + t_5)R_4(2aq_2t_3 - 2cq_3t_3 + t_4) \\
&\quad T_3(Q_1^{-1}\gamma)^{-1}n\Big)e(-m_5t_5 - m_4t_4 - m_3t_3)dt_6\, dt_5\, dt_4\, dt_3.
\end{aligned}$$

By Lemma 2.3 , we can perform the following change of variables (in order):

$$\begin{aligned}
t_6 &\mapsto t_6 - (3abq_2^2t_3 + 6q_2q_3t_3 - 6adq_2q_3t_3 + 3cdq_3^2t_3 + 3q_4t_3 + 3aq_2t_3^2 \\
&\qquad\qquad - 3cq_3t_3^2 + 3bq_2t_4 - 3dq_3t_4) \\
t_5 &\mapsto t_5 - (-3a^2q_2^2t_3 + 6acq_2q_3t_3 - 3c^2q_3^2t_3 - 3aq_2t_4 + 3cq_3t_4) \\
t_4 &\mapsto t_4 - (2aq_2t_3 - 2cq_3t_3)
\end{aligned}$$

and utilize the $N_{\mathbb{Z}}$-invariance of f to conclude that

$$\begin{aligned}
&f_{\gamma,m_5,m_4,m_3}(Q_6Q_5Q_4Q_3Q_2Q_1n) \\
&= \int_0^1\int_0^1\int_0^1\int_0^1 f\big((Q_1^{-1}\gamma)T_6T_5T_4T_3(Q_1^{-1}\gamma)^{-1}n\big)e\big(-(m_3 - 2(aq_2 - cq_3)m_4 \\
&\quad - 3(aq_2 - cq_3)^2m_5)t_3 - (m_4 + 3(aq_2 - cq_3)m_5)t_4 - m_5t_5\big)\, dt_6\, dt_5\, dt_4\, dt_3 \\
&= f_{Q_1^{-1}\gamma,m_5,m_4+3(aq_2-cq_3)m_5,m_3-2(aq_2-cq_3)m_4-3(aq_2-cq_3)^2m_5}(n).
\end{aligned} \tag{2.58}$$

We formally define

$$\begin{aligned}
&f^{\Sigma}_{\gamma,m_5,m_4,m_3}(n) \\
&= \sum_{q_1\in\mathbb{Z}}\sum_{[q_2]\in\mathbb{Z}/c\mathbb{Z}}\sum_{q_3\in\mathbb{Z}} f_{\gamma,m_5,m_4,m_3}(Q_3Q_2Q_1n) \\
&= \sum_{q_1\in\mathbb{Z}}\sum_{[q_2]\in\mathbb{Z}/c\mathbb{Z}}\sum_{q_3\in\mathbb{Z}} f_{Q_1^{-1}\gamma,m_5,m_4+3(aq_2-cq_3)m_5,m_3-2(aq_2-cq_3)m_4-3(aq_2-cq_3)^2m_5}(n) \\
&= \sum_{q_1\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} f_{Q_1^{-1}\gamma,m_5,m_4+3jm_5,m_3-2jm_4-3j^2m_5}(n).
\end{aligned} \tag{2.59}$$

In the above equality we have used the fact that for $c \neq \pm 1$, each element of $\mathbb{Z}$ can be written uniquely as $aq_2 - cq_3$ where $q_3 \in \mathbb{Z}$ and $[q_2] \in \mathbb{Z}/c\mathbb{Z}$ (recall $c \neq 0$ since we have assumed $[[\gamma]] \neq [[\pm\mathrm{id}]]$), and for the case of $c = \pm 1$, we then have $q_2 = 0$ and obviously each element of $\mathbb{Z}$ can be written uniquely as $aq_2 - cq_3 = \mp q_3$ where $q_3 \in \mathbb{Z}$. One can see from (2.12) that the definition of $f^{\Sigma}_{\gamma,m_5,m_4,m_3}$ does not depend upon the choice of representative for $[[\gamma]]$. Lastly, observe that by (2.58), $f^{\Sigma}_{\gamma,m_5,m_4,m_3}$ is formally $N_{\mathbb{Z}}$-invariant.

The absolute convergence of (2.55) in $C(K)$ implies the absolute convergence of the series defining $f^{\Sigma}_{\gamma,m_5,m_4,m_3}$ in $C(K)$. Thus $f^{\Sigma}_{\gamma,m_5,m_4,m_3}$ is a well-defined element of $C(K)$. Of course, since our choice of K was arbitrary it follows that $f^{\Sigma}_{\gamma,m_5,m_4,m_3}$ is also defined as a continuous function on all of N. Furthermore, since the series defining $f^{\Sigma}_{\gamma,m_5,m_4,m_3}$ is formally $N_{\mathbb{Z}}$-invariant, it follows that $f^{\Sigma}_{\gamma,m_5,m_4,m_3}$ can be thought of as an $N_{\mathbb{Z}}$-invariant function on N.

Observe that formally,

$$\sum_{\substack{[[\gamma]]\in(\Gamma_\beta)_\infty\backslash\Gamma_\beta/(\Gamma_\beta)_\infty\\ [[\gamma]]\neq[[\pm\mathrm{id}]]}}\sum_{m_5\in\mathbb{Z}_{>0}}\sum_{[m_4]\in\mathbb{Z}/3m_5\mathbb{Z}}\sum_{m_3\in\mathbb{Z}} f^{\Sigma}_{\gamma,m_5,m_4,m_3}$$

$$=\sum_{\substack{[\gamma]\in\Gamma_\beta/(\Gamma_\beta)_\infty\\ [\gamma]\neq[\pm\mathrm{id}]}}\sum_{m_5\in\mathbb{Z}_{>0}}\sum_{m_4\in\mathbb{Z}}\sum_{m_3\in\mathbb{Z}} f_{\gamma,m_5,m_4,m_3}; \tag{2.60}$$

indeed, this follows from the fact that $[[\pm\mathrm{id}]]=[\pm\mathrm{id}]$ by (2.56) and the fact that $q_1\mapsto[Q_1^{-1}\gamma]$ is an injective map from $\mathbb{Z}$ into $\Gamma_\beta/(\Gamma_\beta)_\infty$ for $[[\gamma]]\neq[[\pm\mathrm{id}]]$ by (2.56) and (2.11). Both the right and left-hand sides of (2.60) can be seen to converge absolutely in $C(K)$ by noting that

$$\sum_{\substack{[[\gamma]]\in(\Gamma_\beta)_\infty\backslash\Gamma_\beta/(\Gamma_\beta)_\infty\\ [[\gamma]]\neq[[\pm\mathrm{id}]]}}\sum_{m_5\in\mathbb{Z}_{>0}}\sum_{[m_4]\in\mathbb{Z}/3m_5\mathbb{Z}}\sum_{m_3\in\mathbb{Z}} \|f^{\Sigma}_{\gamma,m_5,m_4,m_3}\|_K$$

$$\leq\sum_{\substack{[\gamma]\in\Gamma_\beta/(\Gamma_\beta)_\infty\\ [\gamma]\neq[\pm\mathrm{id}]}}\sum_{m_5\in\mathbb{Z}_{>0}}\sum_{m_4\in\mathbb{Z}}\sum_{m_3\in\mathbb{Z}} \|f_{\gamma,m_5,m_4,m_3}\|_K<\infty, \tag{2.61}$$

where finiteness follows from the absolute convergence of (2.55). One can then use basic analysis to show the left-hand side of (2.60) must converge (absolutely) in $C(K)$ to the right-hand side of (2.60).[3] Thus (2.60) is a well-defined identity in $C(K)$.

Let $L^2(N_{\mathbb{Z}}\backslash N)$ denote the space of $N_{\mathbb{Z}}$-invariant, square integrable, measurable functions on N modulo the space of functions which vanish almost everywhere. If $\mathcal{Q}$ is a fundamental domain for $N_{\mathbb{Z}}\backslash N$ then we can define an inner-product on $L^2(N_{\mathbb{Z}}\backslash N)$ by defining

$$\langle h_1,h_2\rangle=\int_{\mathcal{Q}} h_1(n)\overline{h_2(n)}\,dn \tag{2.62}$$

where $h_1,h_2\in L^2(N_{\mathbb{Z}}\backslash N)$ and dn is a fixed Haar measure for N. It is well-known that $L^2(N_{\mathbb{Z}}\backslash N)$ is a Hilbert space when equipped with the inner-product (2.62). From this inner-product we define a norm $\|\cdot\|_2$ for $L^2(N_{\mathbb{Z}}\backslash N)$ by

$$\|h\|_2=\langle h,h\rangle^{1/2}=\left(\int_{\mathcal{Q}}|h(n)|^2\,dn\right)^{1/2}, \tag{2.63}$$

where $h\in L^2(N_{\mathbb{Z}}\backslash N)$. If $h:N\to\mathbb{C}$ is a continuous, $N_{\mathbb{Z}}$-invariant function then it follows that $h\in L^2(N_{\mathbb{Z}}\backslash N)$; in particular, $f^{\Sigma}_{\gamma,m_5,m_4,m_3}\in L^2(N_{\mathbb{Z}}\backslash N)$ for smooth, $N_{\mathbb{Z}}$-invariant functions $f:N\to\mathbb{C}$. If K is the closure of $\mathcal{Q}$ then it is easy to see that $\|h\|_2\leq\|h\|_K$. Therefore by (2.61), the series

$$\sum_{\substack{[[\gamma]]\in\Gamma_\infty\backslash\Gamma/\Gamma_\infty\\ [[\gamma]]\neq[[\pm\mathrm{id}]]}}\sum_{m_5\in\mathbb{Z}_{>0}}\sum_{[m_4]\in\mathbb{Z}/3m_5\mathbb{Z}}\sum_{m_3\in\mathbb{Z}} f^{\Sigma}_{\gamma,m_5,m_4,m_3}$$

---

[3] In a Banach space, one can show that if $\sum_{i,j\in\mathbb{Z}_{>0}}a_{i,j}$ converges absolutely then the series $b_i=\sum_{j\in\mathbb{Z}_{>0}}a_{i,j}$ converges absolutely, the series $\sum_{i\in\mathbb{Z}_{>0}}b_i$ converges absolutely, and that $\sum_{i,j\in\mathbb{Z}_{>0}}a_{i,j}=\sum_{i\in\mathbb{Z}_{>0}}b_i$.

converges absolutely in $L^2(N_{\mathbb{Z}}\backslash N)$.

For $q_i \in \mathbb{Z}$, we have

$$\begin{aligned}
&f_{0,k_5,k_4,k_3,k_1}(Q_6Q_5Q_4Q_3Q_2Q_1n)\\
&= \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 f(T_6T_5T_4T_3T_1Q_6Q_5Q_4Q_3Q_2Q_1n)\\
&\quad e(-k_5t_5-k_4t_4-k_3t_3-k_1t_1)\,dt_6\,dt_5\,dt_4\,dt_3\,dt_1\\
&= \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 f\big(Q_6Q_5Q_4Q_3Q_2Q_1R_6(q_1q_2^3t_1+q_5t_1+q_2^3t_1^2+3q_1q_2^2t_3+3q_4t_3\\
&\qquad +3q_2^2t_1t_3+3q_2t_3^2+3q_1q_2t_4-3q_3t_4-q_1t_5+t_6)R_5(-q_2^3t_1-3q_2^2t_3-3q_2t_4+t_5)\\
&\quad R_4(q_2^2t_1+2q_2t_3+t_4)R_3(q_2t_1+t_3)T_1n\big)e(-k_5t_5-k_4t_4-k_3t_3-k_1t_1)\\
&\quad dt_6\,dt_5\,dt_4\,dt_3\,dt_1.
\end{aligned}$$

By Lemma 2.3 , we can perform the following change of variables (in order):

$$\begin{aligned}
t_6 &\mapsto t_6-(q_1q_2^3t_1+q_5t_1+q_2^3t_1^2+3q_1q_2^2t_3+3q_4t_3+3q_2^2t_1t_3\\
&\qquad +3q_2t_3^2+3q_1q_2t_4-3q_3t_4-q_1t_5)\\
t_5 &\mapsto t_5-(-q_2^3t_1-3q_2^2t_3-3q_2t_4)\\
t_4 &\mapsto t_4-(q_2^2t_1+2q_2t_3)\\
t_3 &\mapsto t_3-q_2t_1
\end{aligned}$$

and utilize the $N_{\mathbb{Z}}$-invariance of f to conclude that

$$\begin{aligned}
&f_{0,k_5,k_4,k_3,k_1}(Q_6Q_5Q_4Q_3Q_2Q_1n)\\
&= \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 f(T_6T_5T_4T_3T_1n)e\big(-(k_1-k_3q_2+k_4q_2^2+k_5q_2^3)t_1\\
&\quad -(k_3-2k_4q_2-3k_5q_2^2)t_3-(k_4+3k_5q_2)t_4-k_5t_5\big)\,dt_6\,dt_5\,dt_4\,dt_3\,dt_1\\
&= f_{k_5,k_4+3k_5q_2,k_3-2k_4q_2-3k_5q_2^2,k_1-k_3q_2+k_4q_2^2+k_5q_2^3}(n). \qquad (2.64)
\end{aligned}$$

Formally, we define

$$\begin{aligned}
f^{\Sigma}_{0,k_5,k_4,k_3,k_1}(n) &= \sum_{q_2\in\mathbb{Z}} f_{0,k_5,k_4,k_3,k_1}(Q_2n)\\
&= \sum_{q_2\in\mathbb{Z}} f_{k_5,k_4+3k_5q_2,k_3-2k_4q_2-3k_5q_2^2,k_1-k_3q_2+k_4q_2^2+k_5q_2^3}(n). \qquad (2.65)
\end{aligned}$$

Observe that $f^{\Sigma}_{0,k_5,k_4,k_3,k_1}$ is formally $N_{\mathbb{Z}}$-invariant. The absolute convergence of (2.55) in $C(K)$ implies the absolute convergence of the series defining $f^{\Sigma}_{0,k_5,k_4,k_3,k_1}$ in $C(K)$. Thus $f^{\Sigma}_{0,k_5,k_4,k_3,k_1}$ is a well-defined element of $C(K)$. Since our choice of K was arbitrary it follows that $f^{\Sigma}_{0,k_5,k_4,k_3,k_1}$ is also defined as a continuous function on all of N. Furthermore, since the series defining $f^{\Sigma}_{0,k_5,k_4,k_3,k_1}$ is formally $N_{\mathbb{Z}}$-invariant it follows that $f^{\Sigma}_{0,k_5,k_4,k_3,k_1}$ is $N_{\mathbb{Z}}$-invariant when thought of as a function on N, and thus can be identified as an element of $L^2(N_{\mathbb{Z}}\backslash N)$.

For $q_i \in \mathbb{Z}$, we have

$$\begin{aligned}
&f_{0,0,0,0,k_1,k_2}(Q_6Q_5Q_4Q_3Q_2Q_1n)\\
&= \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 f(T_6T_5T_4T_3T_1T_2Q_6Q_5Q_4Q_3Q_2Q_1n)e(-k_1t_1-k_2t_2)\\
&\quad dt_6\,dt_5\,dt_4\,dt_3\,dt_1\,dt_2\\
&= \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 f\Big(Q_6Q_5Q_4Q_3Q_2Q_1R_6(q_1q_2^3t_1+q_5t_1+q_2^3t_1^2-6q_1q_2q_3t_2\\
&\quad +3q_3^2t_2-3q_1q_4t_2+3q_4t_1t_2+3q_1q_3t_2^2+3q_1q_2t_1t_2^2-3q_3t_1t_2^2-2q_1^2t_2^3+q_1t_1t_2^3\\
&\quad +3q_1q_2^2t_3+3q_4t_3+3q_2^2t_1t_3-6q_3t_2t_3+3q_1t_2^2t_3+3q_2t_3^2+3q_1q_2t_4-3q_3t_4\\
&\quad -q_1t_5+t_6)R_5(-q_2^3t_1+6q_2q_3t_2+3q_4t_2-3q_3t_2^2+q_1t_2^3-3q_2^2t_3-3q_2t_4+t_5)\\
&\quad R_4(q_2^2t_1-2q_3t_2+q_1t_2^2+2q_2t_3+t_4)R_3(q_2t_1-q_1t_2+t_3)T_1T_2n\Big)\\
&\quad e(-k_1t_1-k_2t_2)\,dt_6\,dt_5\,dt_4\,dt_3\,dt_1\,dt_2.
\end{aligned}$$

By Lemma 2.3, we can perform the following change of variables (in order):

$$\begin{aligned}
t_6 &\mapsto t_6-(q_1q_2^3t_1+q_5t_1+q_2^3t_1^2-6q_1q_2q_3t_2+3q_3^2t_2-3q_1q_4t_2+3q_4t_1t_2+3q_1q_3t_2^2\\
&\quad +3q_1q_2t_1t_2^2-3q_3t_1t_2^2-2q_1^2t_2^3+q_1t_1t_2^3+3q_1q_2^2t_3+3q_4t_3+3q_2^2t_1t_3-6q_3t_2t_3\\
&\quad +3q_1t_2^2t_3+3q_2t_3^2+3q_1q_2t_4-3q_3t_4-q_1t_5)\\
t_5 &\mapsto t_5-(-q_2^3t_1+6q_2q_3t_2+3q_4t_2-3q_3t_2^2+q_1t_2^3-3q_2^2t_3-3q_2t_4)\\
t_4 &\mapsto t_4-(q_2^2t_1-2q_3t_2+q_1t_2^2+2q_2t_3)\\
t_3 &\mapsto t_3-(q_2t_1-q_1t_2)
\end{aligned}$$

and utilize the $N_{\mathbb{Z}}$-invariance of f to conclude that

$$f_{0,0,0,0,k_1,k_2}(Q_6Q_5Q_4Q_3Q_2Q_1n) = f_{0,0,0,0,k_1,k_2}(n).$$

Thus $f_{0,0,0,0,k_1,k_2}$ is a continuous, $N_{\mathbb{Z}}$-invariant function and can therefore be identified as an element of $L^2(N_{\mathbb{Z}}\backslash N)$.

Observe that formally,

$$\begin{aligned}
&\sum_{k_5\in\mathbb{Z}_{\neq 0}}\ \sum_{[k_4]\in\mathbb{Z}/3k_5\mathbb{Z}}\ \sum_{k_3\in\mathbb{Z}}\sum_{k_1\in\mathbb{Z}} f^{\Sigma}_{0,k_5,k_4,k_3,k_1} + \sum_{k_4\in\mathbb{Z}_{\neq 0}}\ \sum_{[k_3]\in\mathbb{Z}/2k_4\mathbb{Z}}\ \sum_{k_1\in\mathbb{Z}} f^{\Sigma}_{0,0,k_4,k_3,k_1}\\
&\quad + \sum_{k_3\in\mathbb{Z}_{\neq 0}}\ \sum_{[k_1]\in\mathbb{Z}/k_3\mathbb{Z}} f^{\Sigma}_{0,0,0,k_3,k_1} + \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} f_{0,0,0,0,k_1,k_2}\\
&= \sum_{\substack{k_5\in\mathbb{Z}\ k_4\in\mathbb{Z}\ k_3\in\mathbb{Z}\ k_1\in\mathbb{Z}\\ k_5\neq 0 \text{ or } k_4\neq 0 \text{ or } k_3\neq 0}} f_{0,k_5,k_4,k_3,k_1} + \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} f_{0,0,0,0,k_1,k_2} \qquad (2.66)
\end{aligned}$$

The left-hand side of (2.66) can be seen to converge absolutely in $C(K)$ by noting that

$$\begin{aligned}
&\sum_{k_5\in\mathbb{Z}_{\neq 0}}\ \sum_{[k_4]\in\mathbb{Z}/3k_5\mathbb{Z}}\ \sum_{k_3\in\mathbb{Z}}\sum_{k_1\in\mathbb{Z}} \left\|f^{\Sigma}_{0,k_5,k_4,k_3,k_1}\right\|_K + \sum_{k_4\in\mathbb{Z}_{\neq 0}}\ \sum_{[k_3]\in\mathbb{Z}/2k_4\mathbb{Z}}\ \sum_{k_1\in\mathbb{Z}} \left\|f^{\Sigma}_{0,0,k_4,k_3,k_1}\right\|_K\\
&\quad + \sum_{k_3\in\mathbb{Z}_{\neq 0}}\ \sum_{[k_1]\in\mathbb{Z}/k_3\mathbb{Z}} \left\|f^{\Sigma}_{0,0,0,k_3,k_1}\right\|_K + \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} \left\|f_{0,0,0,0,k_1,k_2}\right\|_K\\
&\leq \sum_{\substack{k_5\in\mathbb{Z}\ k_4\in\mathbb{Z}\ k_3\in\mathbb{Z}\ k_1\in\mathbb{Z}\\ k_5\neq 0 \text{ or } k_4\neq 0 \text{ or } k_3\neq 0}} \left\|f_{0,k_5,k_4,k_3,k_1}\right\|_K + \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} \left\|f_{0,0,0,0,k_1,k_2}\right\|_K < \infty, \qquad (2.67)
\end{aligned}$$

where finiteness follows from the absolute convergence of (2.55) in $C(K)$. One can then use basic analysis to show the left-hand side of (2.66) must converge (absolutely) in $C(K)$ to the right-hand side of (2.66). Thus (2.66) is a well-defined identity in $C(K)$.

Since $f^{\Sigma}_{0,k_5,k_4,k_3,k_1}, f^{\Sigma}_{0,0,k_4,k_3,k_1}, f^{\Sigma}_{0,0,0,k_3,k_1}, f_{0,0,0,0,k_1,k_2} \in L^2(N_{\mathbb{Z}}\backslash N)$ and since $\|h\|_2 \leq \|h\|_K$ for K the closure of a fundamental domain for $N_{\mathbb{Z}}\backslash N$, then it follows from (2.67) that

$$\begin{aligned}
&\sum_{k_5\in\mathbb{Z}_{\neq 0}}\ \sum_{[k_4]\in\mathbb{Z}/3k_5\mathbb{Z}}\ \sum_{k_3\in\mathbb{Z}}\sum_{k_1\in\mathbb{Z}} f^{\Sigma}_{0,k_5,k_4,k_3,k_1} + \sum_{k_4\in\mathbb{Z}_{\neq 0}}\ \sum_{[k_3]\in\mathbb{Z}/2k_4\mathbb{Z}}\ \sum_{k_1\in\mathbb{Z}} f^{\Sigma}_{0,0,k_4,k_3,k_1}\\
&\quad + \sum_{k_3\in\mathbb{Z}_{\neq 0}}\ \sum_{[k_1]\in\mathbb{Z}/k_3\mathbb{Z}} f^{\Sigma}_{0,0,0,k_3,k_1} + \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} f_{0,0,0,0,k_1,k_2}
\end{aligned}$$

converges to an element of $L^2(N_{\mathbb{Z}}\backslash N)$. Combining (2.60) and (2.66) with Theorem 2.10 and the inequality $\|h\|_2 \leq \|h\|_K$ yields the following theorem for smooth $f \in L^2(N_{\mathbb{Z}}\backslash N)$.

Theorem 2.11. *If $f \in L^2(N_{\mathbb{Z}}\backslash N)$ then*

$$\begin{aligned}
f = &\sum_{m_5\in\mathbb{Z}_{>0}}\ \sum_{\substack{[[\gamma]]\in(\Gamma_\beta)_\infty\backslash\Gamma_\beta/(\Gamma_\beta)_\infty\\ [[\gamma]]\neq[[\pm\mathrm{id}]]}}\ \sum_{[m_4]\in\mathbb{Z}/3m_5\mathbb{Z}}\ \sum_{m_3\in\mathbb{Z}} f^{\Sigma}_{\gamma,m_5,m_4,m_3}\\
&+ \sum_{k_5\in\mathbb{Z}_{\neq 0}}\ \sum_{[k_4]\in\mathbb{Z}/3k_5\mathbb{Z}}\ \sum_{k_3\in\mathbb{Z}}\sum_{k_1\in\mathbb{Z}} f^{\Sigma}_{0,k_5,k_4,k_3,k_1} + \sum_{k_4\in\mathbb{Z}_{\neq 0}}\ \sum_{[k_3]\in\mathbb{Z}/2k_4\mathbb{Z}}\ \sum_{k_1\in\mathbb{Z}} f^{\Sigma}_{0,0,k_4,k_3,k_1}\\
&+ \sum_{k_3\in\mathbb{Z}_{\neq 0}}\ \sum_{[k_1]\in\mathbb{Z}/k_3\mathbb{Z}} f^{\Sigma}_{0,0,0,k_3,k_1} + \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} f_{0,0,0,0,k_1,k_2}
\end{aligned}$$

where the sum converges absolutely in $L^2(N_{\mathbb{Z}}\backslash N)$.

In order to prove Theorem 2.11 for general $f \in L^2(N_{\mathbb{Z}}\backslash N)$ it will be necessary to define the terms $f^{\Sigma}_{\gamma,m_5,m_4,m_3}$, $f^{\Sigma}_{0,k_5,k_4,k_3,k_1}$ for general $f \in L^2(N_{\mathbb{Z}}\backslash N)$. To do this, we observe that f_{γ,m_5,m_4,m_3}, f_{0,k_5,k_4,k_3,k_1} are well-defined for general $f \in L^2(N_{\mathbb{Z}}\backslash N)$, and can therefore be thought of as elements of $L^2(\mathcal{Q})$ where $\mathcal{Q}$ is a fundamental domain for $N_{\mathbb{Z}}\backslash N$. We can formally define $f^{\Sigma}_{\gamma,m_5,m_4,m_3}$ and $f^{\Sigma}_{0,k_5,k_4,k_3,k_1}$ according to the series definitions we gave earlier in (2.59) and (2.65). In section 2.7 we will use results from section 2.6 to show that the series defining $f^{\Sigma}_{\gamma,m_5,m_4,m_3}$ and $f^{\Sigma}_{0,k_5,k_4,k_3,k_1}$ converge absolutely in $L^2(\mathcal{Q})$. Since the series defining $f^{\Sigma}_{\gamma,m_5,m_4,m_3}$ and $f^{\Sigma}_{0,k_5,k_4,k_3,k_1}$ are formally $N_{\mathbb{Z}}$-invariant, it would then follow that $f^{\Sigma}_{\gamma,m_5,m_4,m_3}$ and $f^{\Sigma}_{0,k_5,k_4,k_3,k_1}$ can be identified as elements of $L^2(N_{\mathbb{Z}}\backslash N)$. Once we have established that are $f^{\Sigma}_{\gamma,m_5,m_4,m_3}$ and $f^{\Sigma}_{0,k_5,k_4,k_3,k_1}$ are well-defined elements of $L^2(N_{\mathbb{Z}}\backslash N)$, we will then complete our proof of Theorem 2.11 for general $f \in L^2(N_{\mathbb{Z}}\backslash N)$.

2.6 Subspaces of $L^2(N_{\mathbb{Z}}\backslash N)$

By [12, Corollary 1.126], we see that the maps

$$\sigma_{\mathrm{id}} : \mathbb{R}^6 \to N, \qquad \sigma_{\mathrm{id}}(x_1, x_2, x_3, x_4, x_5, x_6) = X_6X_5X_4X_3X_2X_1, \tag{2.68}$$

$$\sigma_{\mathrm{alt}} : \mathbb{R}^6 \to N, \qquad \sigma_{\mathrm{alt}}(x_1, x_2, x_3, x_4, x_5, x_6) = X_1X_4X_3X_2X_5X_6, \tag{2.69}$$

are diffeomorphisms. Consequently, for any $n \in N$ there exists unique $x_i \in \mathbb{R}$ such that

$$n = X_6X_5X_4X_3X_2X_1;$$

a similar statement also for (2.69).

It will be important to integrate smooth functions $f : N \to \mathbb{C}$, and in order to do so, we will make reference to certain ideas in differential geometry [14]. We begin by selecting $\sigma_{\mathrm{id}}^{-1}$ to serve as a global coordinate chart for N. We then select an orientation for N, which in our case, will be given by an ordered global frame of the tangent bundle. We select the following smooth sections of the tangent bundle to form our global frame:

$$p \mapsto \partial^N_{x_i}[f]_p := \partial_{x_i}[f \circ \sigma_{\mathrm{id}}]_{\sigma_{\mathrm{id}}^{-1}(p)}$$

ordered from $i = 1, \ldots, 6$. Let dx_i^N denote the global frame of the cotangent bundle dual to this global frame. When we wish to integrate smooth $f : N \to \mathbb{C}$ of compact support, we shall do so by computing

$$\int_N f \cdot (dx_1^N \wedge dx_2^N \wedge dx_3^N \wedge dx_4^N \wedge dx_5^N \wedge dx_6^N).$$

For $g \in N$, let $(\ell(g)f)(n) = f(g^{-1}n)$. When we solve for p_i in the equation

$$P_6P_5P_4P_3P_2P_1 = Q_6Q_5Q_4Q_3Q_2Q_1X_6X_5X_4X_3X_2X_1 \tag{2.70}$$

we find that

$$p_6 = u_6 + x_6, \quad p_5 = u_5 + x_5, \quad p_4 = u_4 + x_4,$$
$$p_3 = u_3 + x_3, \quad p_2 = q_2 + x_2, \quad p_1 = q_1 + x_1,$$

where

$$\begin{aligned} u_6 &= q_6 - 6q_1q_2q_3x_2 - 3q_1^2q_2x_2^2 - 3q_1q_3x_2^2 - 2q_1^2x_2^3 - 6q_2q_3x_3 - 6q_1q_2x_2x_3 - 3q_1x_2^2x_3 \\ &\quad - 3q_2x_3^2 + 3q_3x_4 + q_1x_5, \\ u_5 &= q_5 - 3q_1q_2^2x_2 - 3q_1q_2x_2^2 - q_1x_2^3 - 3q_2^2x_3 + 3q_2x_4, \\ u_4 &= q_4 - 2q_1q_2x_2 - q_1x_2^2 - 2q_2x_3, \\ u_3 &= q_3 + q_1x_2. \end{aligned}$$

Thus

$$\begin{aligned} &\int_N \big(\ell((Q_6Q_5Q_4Q_3Q_2Q_1)^{-1})f\big) \cdot (dx_1^N \wedge dx_2^N \wedge dx_3^N \wedge dx_4^N \wedge dx_5^N \wedge dx_6^N) \\ &\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f \circ \sigma_{\mathrm{id}}\big(q_1 + x_1, q_2 + x_2, u_3 + x_3, u_4 + x_4, u_5 + x_5, \\ &\qquad u_6 + x_6\big) dx_6\, dx_5\, dx_4\, dx_3\, dx_2\, dx_1. \end{aligned}$$

Since dx_i are Haar measures for $\mathbb{R}$, we can perform the following change of variables (in order)

$$x_6 \mapsto x_6 - u_6, \qquad x_5 \mapsto x_5 - u_5, \qquad x_4 \mapsto x_4 - u_4, \qquad x_3 \mapsto x_3 - u_3,$$

to conclude that

$$\begin{aligned} &\int_N (\ell((Q_6Q_5Q_4Q_3Q_2Q_1)^{-1})f) \cdot (dx_1^N \wedge dx_2^N \wedge dx_3^N \wedge dx_4^N \wedge dx_5^N \wedge dx_6^N) \\ &\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f \circ \sigma_{\mathrm{id}}(x_1, x_2, x_3, x_4, x_5, x_6)\, dx_6\, dx_5\, dx_4\, dx_3\, dx_2\, dx_1 \\ &\quad = \int_N f\, (dx_1^N \wedge dx_2^N \wedge dx_3^N \wedge dx_4^N \wedge dx_5^N \wedge dx_6^N). \end{aligned}$$

Thus the differential form $dx_1^N \wedge dx_2^N \wedge dx_3^N \wedge dx_4^N \wedge dx_5^N \wedge dx_6^N$ induces a Haar measure for N. Furthermore, this induced Haar measure is already normalized to give fundamental domains of $N_{\mathbb{Z}}\backslash N$ a volume equal to 1, as can be seen from the following lemma.

Lemma 2.12.

(a) $N_{\mathbb{Z}} = \sigma_{\mathrm{id}}(\mathbb{Z}^6) = \sigma_{\mathrm{alt}}(\mathbb{Z}^6)$*, and*

(b) $\sigma_{\mathrm{id}}([0,1)^6)$ *and* $\sigma_{\mathrm{alt}}([0,1)^6)$ *are fundamental domains for* $N_{\mathbb{Z}}\backslash N$*.*

Proof. Clearly $\sigma_{\mathrm{id}}(\mathbb{Z}^6) \subset N_{\mathbb{Z}}$. Since $N_{\mathbb{Z}}$ is generated by $N_i = R_i(1)$ it follows from (2.70) that every element of $N_{\mathbb{Z}}$ can be written as $\sigma_{\mathrm{id}}(x)$ where $x \in \mathbb{Z}^6$, and thus $N_{\mathbb{Z}} = \sigma_{\mathrm{id}}(\mathbb{Z}^6)$. Observe

$$\begin{aligned} X_1X_4X_3X_2X_5X_6 = {} & R_6(-2x_1^2x_2^3 - 3x_1x_2^2x_3 + x_1x_5 + x_6)R_5(-x_1x_2^3 + x_5) \\ & R_4(-x_1x_2^2 + x_4)R_3(x_1x_2 + x_3)X_2X_1 \end{aligned} \tag{2.71}$$

implies that $\sigma_{\mathrm{alt}}(\mathbb{Z}^6) \subset N_{\mathbb{Z}}$. Conversely, observe

$$\begin{aligned} X_6X_5X_4X_3X_2X_1 = {} & X_1R_4(x_1x_2^2 + x_4)R_3(-x_1x_2 + x_3)X_2R_5(x_1x_2^3 + x_5) \\ & R_6(-2x_1^2x_2^3 + 3x_1x_2^2x_3 - x_1x_5 + x_6) \end{aligned}$$

implies that $N_{\mathbb{Z}} \subset \sigma_{\mathrm{alt}}(\mathbb{Z}^6)$, and hence $N_{\mathbb{Z}} = \sigma_{\mathrm{alt}}(\mathbb{Z}^6)$.

Next we prove that $\sigma_{\mathrm{id}}([0,1)^6)$ is a fundamental domain for $N_{\mathbb{Z}}\backslash N$. The proof for $\sigma_{\mathrm{alt}}([0,1)^6)$ is nearly identical. For $n \in N$, there exists $x_i \in \mathbb{R}$ such that $n = X_6X_5X_4X_3X_2X_1$. In (2.70) we can select integers q_1 and q_2 such that $0 \le q_1 + x_1, q_2 + x_2 < 1$. Proceeding, we can then select integers q_3, q_4, and q_5 such that

$$\begin{aligned} & 0 \le q_3 + q_1x_2 + x_3 < 1 \\ & 0 \le q_4 - 2q_1q_2x_2 - q_1x_2^2 - 2q_2x_3 + x_4 < 1 \\ & 0 \le q_5 - 3q_1q_2^2x_2 - 3q_1q_2x_2^2 - q_1x_2^3 - 3q_2^2x_3 + 3q_2x_4 + x_5 < 1. \end{aligned}$$

Lastly, we can select an integer q_6 such that

$$\begin{aligned} 0 \le {} & q_6 - 6q_1q_2q_3x_2 - 3q_1^2q_2x_2^2 - 3q_1q_3x_2^2 - 2q_1^2x_2^3 - 6q_2q_3x_3 \\ & - 6q_1q_2x_2x_3 - 3q_1x_2^2x_3 - 3q_2x_3^2 + 3q_3x_4 + q_1x_5 + x_6 < 1. \end{aligned}$$

This shows that each coset of $N_{\mathbb{Z}}\backslash N$ has a representative in $\sigma_{\mathrm{id}}([0,1)^6)$.

It remains to show that no two elements of $\sigma_{\mathrm{id}}([0,1)^6)$ are contained in the same coset of $N_{\mathbb{Z}}\backslash N$. To prove this, suppose that in (2.70) we have $q_i \in \mathbb{Z}$ and $x_i, p_i \in [0,1)$. By (2.70) it follows immediately that $q_1 = q_2 = 0$. Substituting $q_1 = q_2 = 0$ into (2.70) shows that we must have $q_3 = q_4 = q_5 = 0$. When we substitute $q_1 = \ldots = q_5 = 0$ into (2.70) we find that $q_6 = 0$ and hence it follows that $x_i = p_i$ for all i. $\square$

For $\sigma : \mathbb{R}^6 \to N$ a diffeomorphism, define

$$f^\sigma : \mathbb{R}^6 \to \mathbb{C} \text{ where } f^\sigma(x_1, \ldots, x_6) = f(\sigma(x_1, \ldots, x_6)).$$

For K a compact subset of N, define

$$K_{\mathrm{id}} = \sigma_{\mathrm{id}}^{-1}(K) \text{ and } K_{\mathrm{alt}} = \sigma_{\mathrm{alt}}^{-1}(K).$$

Lemma 2.13. *For K a compact set of N and $f : N \to \mathbb{C}$ a smooth function, we have*

$$\int_K f(n)dn = \int_{K_{\mathrm{id}}} f^{\sigma_{\mathrm{id}}}(x_1, \ldots, x_6)\, dx_6 \ldots dx_1 = \int_{K_{\mathrm{alt}}} f^{\sigma_{\mathrm{alt}}}(x_1, \ldots, x_6)\, dx_6 \ldots dx_1,$$

where $n \in N$ and dn is the Haar measure on N which gives fundamental domains of $N_{\mathbb{Z}}\backslash N$ a volume equal to 1.

Proof. Our discussion prior to Lemma 2.12 shows that

$$\int_K f(n)\, dn = \int_{K_{\mathrm{id}}} f^{\sigma_{\mathrm{id}}}(x_1, \ldots, x_6)\, dx_6 \ldots dx_1.$$

When we solve for p_i in the equation $X_1X_4X_3X_2X_5X_6 = P_6P_5P_4P_3P_2P_1$, it follows from (2.71) that p_i can be written in terms of x_i. It is then a straightforward matter to compute that $\det\left(\frac{d}{dx_i}(p_j)\right) = 1$. By [14, Corollary 14.3] we can then conclude the remaining equation in our lemma. □

Let $W = L^2(N_{\mathbb{Z}}\backslash N)$ and let r denote right-regular representation of N on W. Let

$$\begin{aligned} &W_{[[\gamma]],m_5,m_4,m_3} \text{ denote the closure of } \{f^{\Sigma}_{\gamma,m_5,m_4,m_3} : f \in L^2(N_{\mathbb{Z}}\backslash N), f \text{ smooth}\},\\ &W_{0,k_5,k_4,k_3,k_1} \text{ denote the closure of } \{f^{\Sigma}_{0,k_5,k_4,k_3,k_1} : f \in L^2(N_{\mathbb{Z}}\backslash N), f \text{ smooth}\},\\ &W_{0,0,0,0,k_1,k_2} \text{ denote the closure of } \{f_{0,0,0,0,k_1,k_2} : f \in L^2(N_{\mathbb{Z}}\backslash N), f \text{ smooth}\}, \end{aligned} \tag{2.72}$$

where all these closures are taken in W. One can check that these spaces are closed under r by referencing the definitions of $f^{\Sigma}_{\gamma,m_5,m_4,m_3}$ (2.59), $f^{\Sigma}_{0,k_5,k_4,k_3,k_1}$ (2.65), and $f_{0,0,0,0,k_1,k_2}$ (2.7i), all of which are defined by conditions and actions on the left.

For $f, h \in L^2(\mathcal{F})$ where $\mathcal{F} = [0,1)^6$, let

$$\langle f, h\rangle_{\mathcal{F}} = \int_0^1 \ldots \int_0^1 f(x_1, \ldots, x_6)\overline{h(x_1, \ldots, x_6)}\, dx_6 \ldots dx_1 \text{ and } \|f\|_{2,\mathcal{F}} = \langle f, f\rangle_{\mathcal{F}}^{1/2}. \tag{2.73}$$

By Lemma 2.13 it follows that if $f, h \in W = L^2(N_{\mathbb{Z}}\backslash N)$ then

$$\langle f, h\rangle = \langle f^{\sigma_{\mathrm{id}}}, h^{\sigma_{\mathrm{id}}}\rangle_{\mathcal{F}} = \langle f^{\sigma_{\mathrm{alt}}}, h^{\sigma_{\mathrm{alt}}}\rangle_{\mathcal{F}} \text{ and } \|f\|_2 = \|f^{\sigma_{\mathrm{id}}}\|_{2,\mathcal{F}} = \|f^{\sigma_{\mathrm{alt}}}\|_{2,\mathcal{F}};$$

recall that we defined $\langle \cdot, \cdot\rangle$ in (2.62) and $\|\cdot\|_2$ in (2.63). We shall use these equalities often throughout this section.

For the rest of this section we will suppose that $f \in W = L^2(N_{\mathbb{Z}}\backslash N)$ is a smooth function, unless indicated otherwise. We will also assume that $q_i \in \mathbb{Z}$.

2.6.1 Analysis of $W_{[[\gamma]],m_5,m_4,m_3}$

Suppose $[[\gamma]] \neq [[\pm\mathrm{id}]]$. In this subsection we will show that $W_{[[\gamma]],m_5,m_4,m_3}$ is isometric to $L^2(\mathbb{R}^2)$. Via this isometry we will construct a representation of N on $L^2(\mathbb{R}^2)$. We will then analyze the smooth vectors of $L^2(\mathbb{R}^2)$ under this representation, which will allow us to give an explicit description in section 2.7 of the $N_{\mathbb{Z}}$-invariant distributions on N. To begin this analysis, observe

$$\begin{aligned}
&\gamma T_6T_5T_4T_3\gamma^{-1}Q_3Q_2Q_1X_6X_5X_4X_3X_2X_1\\
&= R_6(bt_5+at_6+u_6)R_5(dt_5+ct_6+u_5)R_4(t_4+u_4)R_3(q_3+at_3+q_1x_2+x_3)\\
&\quad R_2(q_2+ct_3+x_2)R_1(q_1+x_1),
\end{aligned}$$

where

$$\begin{aligned}
u_6 &= -3cq_3^2t_3 - 6acq_3t_3^2 - 2a^2ct_3^3 - 6q_1q_2q_3x_2 - 6aq_1q_2t_3x_2 - 6cq_1q_3t_3x_2\\
&\quad - 6acq_1t_3^2x_2 - 3q_1^2q_2x_2^2 - 3q_1q_3x_2^2 - 3aq_1t_3x_2^2 - 3cq_1^2t_3x_2^2 - 2q_1^2x_2^3 - 6q_2q_3x_3\\
&\quad - 6aq_2t_3x_3 - 6cq_3t_3x_3 - 6act_3^2x_3 - 6q_1q_2x_2x_3 - 6cq_1t_3x_2x_3 - 3q_1x_2^2x_3\\
&\quad - 3q_2x_3^2 - 3ct_3x_3^2 + 3q_3x_4 + 3at_3x_4 + q_1x_5 + x_6,\\
u_5 &= -3c^2q_3t_3^2 - ac^2t_3^3 - 3q_1q_2^2x_2 - 6cq_1q_2t_3x_2 - 3c^2q_1t_3^2x_2 - 3q_1q_2x_2^2 - 3cq_1t_3x_2^2\\
&\quad - q_1x_2^3 - 3q_2^2x_3 - 6cq_2t_3x_3 - 3c^2t_3^2x_3 + 3q_2x_4 + 3ct_3x_4 + x_5,\\
u_4 &= -2cq_3t_3 - act_3^2 - 2q_1q_2x_2 - 2cq_1t_3x_2 - q_1x_2^2 - 2q_2x_3 - 2ct_3x_3 + x_4.
\end{aligned}$$

Recall that (2.58) and (2.7g) shows that

$$\begin{aligned}
&f_{Q_1^{-1}\gamma,m_5,m_4+3(aq_2-cq_3)m_5,m_3-2(aq_2-cq_3)m_4-3(aq_2-cq_3)^2m_5}(n)\\
&= f_{\gamma,m_5,m_4,m_3}(Q_3Q_2Q_1n)\\
&= \int_0^1\int_0^1\int_0^1\int_0^1 f(\gamma T_6T_5T_4T_3\gamma^{-1}Q_3Q_2Q_1n)(-m_5t_5-m_4t_4-m_3t_3)\,dt_6\,dt_5\,dt_4\,dt_3.
\end{aligned}$$

Therefore, when we perform the simultaneous change of variables (2.8) and then perform the following change of variables (in order)

$$t_6 \mapsto t_6 - u_6, \qquad t_5 \mapsto t_5 - u_5, \qquad t_4 \mapsto t_4 - u_4, \qquad t_3 \mapsto t_3 - \frac{q_2+x_2}{c},$$

it follows from Lemmas 2.3 and 2.4 that

$$
\begin{aligned}
&(f_{Q_1^{-1}\gamma,m_5,m_4+3(aq_2-cq_3)m_5,m_3-2(aq_2-cq_3)m_4-3(aq_2-cq_3)^2m_5})^{\sigma_{\mathrm{id}}}(x_1,x_2,x_3,x_4,x_5,x_6)\\
&= f_{\gamma,m_5,m_4,m_3}(Q_3Q_2Q_1X_6X_5X_4X_3X_2X_1)\\
&= e\Big(\frac{m_3q_2}{c}-\frac{am_4q_2^2}{c}-\frac{a^2m_5q_2^3}{c}+\frac{m_3-2m_4(aq_2-cq_3)-3m_5(aq_2-cq_3)^2}{c}x_2\\
&\quad -\frac{(a-cq_1)(m_4+3m_5(aq_2-cq_3))}{c}x_2^2-\frac{(a-cq_1)^2m_5}{c}x_2^3\\
&\quad +2(m_4+3m_5(aq_2-cq_3))x_2x_3+3m_5(a-cq_1)x_2^2x_3-3cm_5x_2x_3^2\\
&\quad +(m_4+3m_5(aq_2-cq_3))x_4+(am_5-cm_5q_1)x_5-cm_5x_6\Big)\\
&\quad \psi_{\gamma,m_5,m_4,m_3}(q_1+x_1,-cq_3+aq_2+ax_2-cq_1x_2-cx_3), \qquad (2.74)
\end{aligned}
$$

where

$$
\begin{aligned}
&\psi_{\gamma,m_5,m_4,m_3}(s_1,s_3)=\psi_{f;\gamma,m_5,m_4,m_3}(s_1,s_3)\\
&=\int_0^1\int_0^1\int_0^1\int_0^1 f^{\sigma_{\mathrm{id}}}\Big(s_1,ct_3,at_3-\frac{s_3}{c},t_4,t_5,t_6\Big)e(-m_3t_3-acm_4t_3^2+a^2c^2m_5t_3^3\\
&\quad -m_4t_4-am_5t_5+cm_5t_6+2m_4t_3s_3-3acm_5t_3^2s_3+3m_5t_3s_3^2)\,dt_6\,dt_5\,dt_4\,dt_3. \qquad (2.75)
\end{aligned}
$$

As indicated in the above definition, we will at times suppress writing f in the subscript of $\psi_{f;\gamma,m_5,m_4,m_3}$ when context is clear. Observe that by (2.59),

$$
\begin{aligned}
&\big(f^{\Sigma}_{\gamma,m_5,m_4,m_3}\big)^{\sigma_{\mathrm{id}}}(x_1,x_2,x_3,x_4,x_5,x_6)\\
&=\sum_{q_1\in\mathbb{Z}}\sum_{q_3\in\mathbb{Z}}\sum_{[q_2]\in\mathbb{Z}/c\mathbb{Z}}(f_{Q_1^{-1}\gamma,m_5,m_4+3(aq_2-cq_3)m_5,m_3-2(aq_2-cq_3)m_4-3(aq_2-cq_3)^2m_5})^{\sigma_{\mathrm{id}}}(x_1,...,x_6)\\
&=\sum_{q_1\in\mathbb{Z}}\sum_{q_3\in\mathbb{Z}}\sum_{[q_2]\in\mathbb{Z}/c\mathbb{Z}} e\Big(\frac{m_3q_2}{c}-\frac{am_4q_2^2}{c}-\frac{a^2m_5q_2^3}{c}\\
&\quad +\frac{m_3-2m_4(aq_2-cq_3)-3m_5(aq_2-cq_3)^2}{c}x_2-\frac{(a-cq_1)(m_4+3m_5(aq_2-cq_3))}{c}x_2^2\\
&\quad -\frac{(a-cq_1)^2m_5}{c}x_2^3+2(m_4+3m_5(aq_2-cq_3))x_2x_3+3m_5(a-cq_1)x_2^2x_3-3cm_5x_2x_3^2\\
&\quad +(m_4+3m_5(aq_2-cq_3))x_4+(am_5-cm_5q_1)x_5-cm_5x_6\Big)\\
&\quad \psi_{\gamma,m_5,m_4,m_3}(q_1+x_1,-cq_3+aq_2+ax_2-cq_1x_2-cx_3). \qquad (2.76)
\end{aligned}
$$

From (2.74) we see that

$$
\begin{aligned}
&\langle (f_{Q_1^{-1}\gamma,m_5,m_4+3(aq_2-cq_3)m_5,m_3-2(aq_2-cq_3)m_4-3(aq_2-cq_3)^2m_5})^{\sigma_{\mathrm{id}}},\\
&\qquad (f_{R_1(\ell_1)^{-1}\gamma,m_5,m_4+3(a\ell_2-c\ell_3)m_5,m_3-2(a\ell_2-c\ell_3)m_4-3(a\ell_2-c\ell_3)^2m_5})^{\sigma_{\mathrm{id}}}\rangle_{\mathcal{F}}=0
\end{aligned}
$$

if $q_1\neq \ell_1$ or $aq_2-cq_3\neq a\ell_2-c\ell_3$ (or equivalently, if $q_1\neq\ell_1$, $q_2\neq\ell_2$, or $q_3\neq\ell_3$); simply perform integration is the x_5 and x_4 variables in (2.73) to see why this is the case. Thus by

(2.76), the Pythagorean theorem, and changing variables, we find that

$$\begin{aligned}
&\Big(\big\|f^{\Sigma}_{\gamma,m_5,m_4,m_3}\big\|_2\Big)^2 = \Big(\big\|(f^{\Sigma}_{\gamma,m_5,m_4,m_3})^{\sigma_{\mathrm{id}}}\big\|_{2,\mathcal{F}}\Big)^2 \\
&= \sum_{q_1\in\mathbb{Z}}\sum_{q_3\in\mathbb{Z}}\sum_{[q_2]\in\mathbb{Z}/c\mathbb{Z}} \Big(\Big\|(f_{Q_1^{-1}\gamma,m_5,m_4+3(aq_2-cq_3)m_5,m_3-2m_4(aq_2-cq_3)-3m_5(aq_2-cq_3)^2})^{\sigma_{\mathrm{id}}}\Big\|_{2,\mathcal{F}}\Big)^2 \\
&= \sum_{q_1\in\mathbb{Z}}\sum_{q_3\in\mathbb{Z}}\sum_{[q_2]\in\mathbb{Z}/c\mathbb{Z}} \int_0^1\int_0^1\int_0^1 |\psi_{\gamma,m_5,m_4,m_3}(q_1+x_1, aq_2+ax_2-cq_1x_2-c(x_3+q_3))|^2 \\
&\quad dx_3\,dx_2\,dx_1 \\
&= \sum_{q_1\in\mathbb{Z}}\sum_{[q_2]\in\mathbb{Z}/c\mathbb{Z}} \int_0^1\int_0^1\int_{-\infty}^{\infty} |\psi_{\gamma,m_5,m_4,m_3}(q_1+x_1, aq_2+ax_2-cq_1x_2-cx_3)|^2\,dx_3\,dx_2\,dx_1 \\
&= \sum_{q_1\in\mathbb{Z}}\sum_{[q_2]\in\mathbb{Z}/c\mathbb{Z}} \int_0^1\int_0^1\int_{-\infty}^{\infty} |\psi_{\gamma,m_5,m_4,m_3}(q_1+x_1, -cx_3)|^2\,dx_3\,dx_2\,dx_1 \\
&= \sum_{[q_2]\in\mathbb{Z}/c\mathbb{Z}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi_{\gamma,m_5,m_4,m_3}(x_1, -cx_3)|^2\,dx_3\,dx_1 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi_{\gamma,m_5,m_4,m_3}(x_1, x_3)|^2\,dx_3\,dx_1.
\end{aligned} \tag{2.77}$$

This implies that $\psi_{\gamma,m_5,m_4,m_3}\in L^2(\mathbb{R}^2)$. Conversely, given $\psi\in L^2(\mathbb{R}^2)$, we can define

$$h_{\psi;[[\gamma]],m_5,m_4,m_3}(x_1,\ldots,x_6) = \sum_{\ell_1\in\mathbb{Z}}\sum_{\ell_3\in\mathbb{Z}}\sum_{[\ell_2]\in\mathbb{Z}/c\mathbb{Z}} r_{\ell_1,\ell_2,\ell_3}(x_1,x_2,x_3,x_4,x_5,x_6), \tag{2.78}$$

where

$$\begin{aligned}
&r_{\ell_1,\ell_2,\ell_3}(x_1,x_2,x_3,x_4,x_5,x_6) \\
&= e\Big(\frac{m_3\ell_2}{c} - \frac{am_4\ell_2^2}{c} - \frac{a^2m_5\ell_2^3}{c} + \frac{m_3-2m_4(a\ell_2-c\ell_3)-3m_5(a\ell_2-c\ell_3)^2}{c}x_2 \\
&\quad - \frac{(a-c\ell_1)(m_4+3m_5(a\ell_2-c\ell_3))}{c}x_2^2 - \frac{(a-c\ell_1)^2m_5}{c}x_2^3 \\
&\quad + 2(m_4+3m_5(a\ell_2-c\ell_3))x_2x_3 + 3m_5(a-c\ell_1)x_2^2x_3 - 3cm_5x_2x_3^2 \\
&\quad + (m_4+3m_5(a\ell_2-c\ell_3))x_4 + (am_5-cm_5\ell_1)x_5 - cm_5x_6\Big) \\
&\quad \psi(\ell_1+x_1, -c\ell_3+a\ell_2+ax_2-c\ell_1x_2-cx_3).
\end{aligned}$$

One can check that $h_{\psi;[[\gamma]],m_5,m_4,m_3}$ is a well-defined element of $L^2(\mathcal{F})$ by repeating the argument in (2.77) with $f^{\Sigma}_{\gamma,m_5,m_4,m_3}$ replaced by $h_{\psi;[[\gamma]],m_5,m_4,m_3}$. Furthermore, it can be shown that $h_{\psi;[[\gamma]],m_5,m_4,m_3}$ is $\sigma_{\mathrm{id}}^{-1}(N_{\mathbb{Z}})$-invariant. To see that this is the case, one solves for p_i in (2.70) and observes that

$$r_{\ell_1,\ell_2,\ell_3}(p_1,p_2,p_3,p_4,p_5,p_6) = r_{\ell_1+q_1,\ell_2+q_2,\ell_3+\ell_1q_2+q_3}(x_1,x_2,x_3,x_4,x_5,x_6).$$

Thus $h_{\psi;[[\gamma]],m_5,m_4,m_3}\circ\sigma_{\mathrm{id}}^{-1}$ is a well-defined element of $L^2(N_{\mathbb{Z}}\backslash N)$.

Let $\Phi_{[[\gamma]],m_5,m_4,m_3} : L^2(\mathbb{R}^2) \to W_{[[\gamma]],m_5,m_4,m_3}$ where

$$\Phi_{[[\gamma]],m_5,m_4,m_3}(\psi) = h_{\psi;[[\gamma]],m_5,m_4,m_3} \circ \sigma_{\mathrm{id}}^{-1},$$

and $\psi \in L^2(\mathbb{R}^2)$. Technically, we should prove that

$$h_{\psi;[[\gamma]],m_5,m_4,m_3} \circ \sigma_{\mathrm{id}}^{-1} = (h_{\psi;[[\gamma]],m_5,m_4,m_3} \circ \sigma_{\mathrm{id}}^{-1})^{\Sigma}_{[[\gamma]],m_5,m_4,m_3}$$

to justify that $W_{[[\gamma]],m_5,m_4,m_3}$ truly is the co-domain of $\Phi_{[[\gamma]],m_5,m_4,m_3}$. To see that this is the case, it suffices by (2.59) to show that

$$\begin{aligned}
&\int_0^1\int_0^1\int_0^1\int_0^1 r_{\ell_1,\ell_2,\ell_3} \circ \sigma_{\mathrm{id}}^{-1}((Q_1^{-1}\gamma)T_6T_5T_4T_3(Q_1^{-1}\gamma)^{-1}X_6X_5X_4X_3X_2X_1) \\
&\quad e\big(-m_5t_5 - (m_4 + 3m_5(aq_2 - cq_3))t_4 - (m_3 - 2m_4(aq_2 - cq_3) - 3m_5(aq_2 - cq_3)^2)t_3\big) \\
&\quad dt_6\, dt_5\, dt_4\, dt_3 \\
&= \begin{cases} 0 & \text{if } \ell_1 \neq q_1, \ell_2 \neq q_2, \text{ or } \ell_3 \neq q_3 \\ r_{q_1,q_2,q_3}(x_1,\dots,x_6) & \text{if } \ell_1 = q_1, \ell_2 = q_2, \text{ and } \ell_3 = q_3 \end{cases}. \qquad (2.79)
\end{aligned}$$

To show that (2.79) holds, one solves for p_i in the equation

$$P_6P_5P_4P_3P_2P_1 = (Q_1^{-1}\gamma)T_6T_5T_4T_3(Q_1^{-1}\gamma)^{-1}X_6X_5X_4X_3X_2X_1$$

and substitutes $r_{\ell_1,\ell_2,\ell_3}(p_1,p_2,p_3,p_4,p_5,p_6)$ in for

$$r_{\ell_1,\ell_2,\ell_3} \circ \sigma_{\mathrm{id}}^{-1}\big((Q_1^{-1}\gamma)T_6T_5T_4T_3(Q_1^{-1}\gamma)^{-1}X_6X_5X_4X_3X_2X_1\big)$$

on the left-hand side of (2.79). One then sees that the first case of (2.79) follows when we integrate in the t_6, t_5, and t_4 variables, and that the second case of (2.79) follows from a straightforward calculation. By (2.77), we see that $\Phi_{[[\gamma]],m_5,m_4,m_3}$ is an isometry; indeed, $\Phi_{[[\gamma]],m_5,m_4,m_3}$ is a surjection since $\Phi_{[[\gamma]],m_5,m_4,m_3}$ maps smooth functions in $L^2(\mathbb{R}^2)$ to a dense set in $W_{[[\gamma]],m_5,m_4,m_3}$.

We define a representation $\pi_{[[\gamma]],m_5,m_4,m_3}$ of N on $L^2(\mathbb{R}^2)$ by the equation

$$\pi_{[[\gamma]],m_5,m_4,m_3}(n)(\psi) = \Phi^{-1}_{[[\gamma]],m_5,m_4,m)3}(r(n)\Phi_{[[\gamma]],m_5,m_4,m_3}(\psi)),$$

where $\psi \in L^2(\mathbb{R}^2)$. We wish to give an explicit formula for $\pi_{[[\gamma]],m_5,m_4,m_3}$. To accomplish this, let

$$\begin{aligned}
&p_{\ell_1,\ell_2,\ell_3}(x_1,x_2,x_3,x_4,x_5,x_6) \\
&= e\Big(\frac{m_3\ell_2}{c} - \frac{am_4\ell_2^2}{c} - \frac{a^2m_5\ell_2^3}{c} + \frac{m_3 - 2m_4(a\ell_2 - c\ell_3) - 3m_5(a\ell_2 - c\ell_3)^2}{c}x_2 \\
&\quad - \frac{(a - c\ell_1)(m_4 + 3m_5(a\ell_2 - c\ell_3))}{c}x_2^2 - \frac{(a - c\ell_1)^2m_5}{c}x_2^3 \\
&\quad + 2(m_4 + 3m_5(a\ell_2 - c\ell_3))x_2x_3 + 3m_5(a - c\ell_1)x_2^2x_3 - 3cm_5x_2x_3^2 \\
&\quad + (m_4 + 3m_5(a\ell_2 - c\ell_3))x_4 + (am_5 - cm_5\ell_1)x_5 - cm_5x_6\Big),
\end{aligned}$$

and let

$$
\begin{aligned}
&q(x_1, x_3, y_1, \ldots, y_6) \\
&= e\Big(\frac{m_3 y_2}{c} - \frac{2m_4 x_3 y_2}{c} - \frac{3m_5 x_3^2 y_2}{c} - \frac{a m_4 y_2^2}{c} + m_4 x_1 y_2^2 - \frac{3a m_5 x_3 y_2^2}{c} + 3m_5 x_1 x_3 y_2^2 \\
&\quad - \frac{a^2 m_5 y_2^3}{c} + 2a m_5 x_1 y_2^3 - c m_5 x_1^2 y_2^3 + 2m_4 y_2 y_3 + 6m_5 x_3 y_2 y_3 + 3a m_5 y_2^2 y_3 \\
&\quad - 3c m_5 x_1 y_2^2 y_3 - 3c m_5 y_2 y_3^2 + m_4 y_4 + 3m_5 x_3 y_4 + a m_5 y_5 - c m_5 x_1 y_5 - c m_5 y_6\Big).
\end{aligned}
$$

When we solve for p_i in

$$
P_6 P_5 P_4 P_3 P_2 P_1 = X_6 X_5 X_4 X_3 X_2 X_1 Y_6 Y_5 Y_4 Y_3 Y_2 Y_1 \tag{2.80}
$$

we find that

$$
\begin{aligned}
&p_{\ell_1,\ell_2,\ell_3}(p_1, p_2, p_3, p_4, p_5, p_6) \\
&= p_{\ell_1,\ell_2,\ell_3}(x_1, x_2, x_3, x_4, x_5, x_6) q(x_1 + \ell_1, -c\ell_3 + a\ell_2 + a x_2 - c\ell_1 x_2 - c x_3, y_1, \ldots, y_6).
\end{aligned}
$$

Since

$$
\begin{aligned}
&r_{\ell_1,\ell_2,\ell_3}(x_1, x_2, x_3, x_4, x_5, x_6) \\
&= p_{\ell_1,\ell_2,\ell_3}(x_1, x_2, x_3, x_4, x_5, x_6)\psi(\ell_1 + x_1, -c\ell_3 + a\ell_2 + a x_2 - c\ell_1 x_2 + c x_3),
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&r_{\ell_1,\ell_2,\ell_3}(p_1, p_2, p_3, p_4, p_5, p_6) \qquad (2.81)\\
&= p_{\ell_1,\ell_2,\ell_3}(p_1, p_2, p_3, p_4, p_5, p_6)\psi(\ell_1 + p_1, -c\ell_3 + a\ell_2 + a p_2 - c\ell_1 p_2 + c p_3) \\
&= p_{\ell_1,\ell_2,\ell_3}(x_1, x_2, x_3, x_4, x_5, x_6) q(\ell_1 + x_1, -c\ell_3 + a\ell_2 + a x_2 - c\ell_1 x_2 - c x_3, y_1, \ldots, y_6) \\
&\quad \psi\big((\ell_1 + x_1) + y_1, (-c\ell_3 + a\ell_2 + a x_2 - c\ell_1 x_2 - c x_3) + (a y_2 - c(\ell_1 + x_1) y_2 - c y_3)\big).
\end{aligned}
$$

Observe $\big(r(Y_6 Y_5 Y_4 Y_3 Y_2 Y_1)\Phi_{[[\gamma]],m_5,m_4,m_3}(\psi)\big)(X_6 X_5 X_4 X_3 X_2 X_1)$ is simply a sum of $r_{\ell_1,\ell_2,\ell_3}(p_1, p_2, p_3, p_4, p_5, p_6)$. Thus by (2.81) we see that

$$
\big(r(Y_6 Y_5 Y_4 Y_3 Y_2 Y_1)\Phi_{[[\gamma]],m_5,m_4,m_3}(\psi)\big)(X_6 X_5 X_4 X_3 X_2 X_1)
$$

is of the form $h_{\widetilde{\psi};[[\gamma]],m_5,m_4,m_3}$, where $\widetilde{\psi} \in L^2(\mathbb{R}^2)$. When we solve for $\widetilde{\psi}$, we are then able to deduce that

$$
\begin{aligned}
&\big(\pi_{[[\gamma]],m_5,m_4,m_3}(Y_6 Y_5 Y_4 Y_3 Y_2 Y_1)\psi\big)(s_1, s_3) \\
&= q(s_1, s_3, y_1, \ldots, y_6)\psi(s_1 + y_1, s_3 + a y_2 - c s_1 y_2 - c y_3)
\end{aligned}
$$

$$= e\Big(\frac{m_3y_2}{c} - \frac{2m_4s_3y_2}{c} - \frac{3m_5s_3^2y_2}{c} - \frac{am_4y_2^2}{c} + m_4s_1y_2^2 - \frac{3am_5s_3y_2^2}{c} + 3m_5s_1s_3y_2^2$$
$$- \frac{a^2m_5y_2^3}{c} + 2am_5s_1y_2^3 - cm_5s_1^2y_2^3 + 2m_4y_2y_3 + 6m_5s_3y_2y_3 + 3am_5y_2^2y_3$$
$$- 3cm_5s_1y_2^2y_3 - 3cm_5y_2y_3^2 + m_4y_4 + 3m_5s_3y_4 + am_5y_5 - cm_5s_1y_5 - cm_5y_6\Big)$$
$$\psi(s_1 + y_1, s_3 + ay_2 - cs_1y_2 - cy_3). \tag{2.82}$$

Let $f \in W^{\infty}_{[[\gamma]],m_5,m_4,m_3}$; that is to say, let f be a smooth vector under the action of r. By definition, $\mathfrak{n}$ acts upon such f according to (1.1). Since $\Phi_{[[\gamma]],m_5,m_4,m_3}$ is an isometry it follows that $\psi = \psi_{f;[[\gamma]],m_5,[m_4],m_3}$ is also a smooth vector and that $\mathfrak{n}$ acts upon ψ via $\pi_{[[\gamma]],m_5,m_4,m_3}$ and (1.1). In particular, observe

$$\big(\pi_{[[\gamma]],m_5,m_4,m_3}(\mathcal{P}_1)\psi\big)(s_1,s_3) = \lim_{t_1\to 0} \frac{\big(\pi_{[[\gamma]],m_5,m_4,m_3}(T_1)\psi\big)(s_1,s_3) - \psi(s_1,s_3)}{t_1}$$
$$= \lim_{t_1\to 0} \frac{\psi(s_1+t_1,s_3) - \psi(s_1,s_3)}{t_1} = (\partial_{s_1}\psi)(s_1,s_3),$$

and

$$\big(\pi_{[[\gamma]],m_5,m_4,m_3}(\mathcal{P}_3)\psi\big)(s_1,s_3) = \lim_{t_3\to 0} \frac{\big(\pi_{[[\gamma]],m_5,m_4,m_3}(T_3)\psi\big)(s_1,s_3) - \psi(s_1,s_3)}{t_3}$$
$$= \lim_{t_3\to 0} \frac{\psi(s_1,s_3-ct_3) - \psi(s_1,s_3)}{t_3} = -c(\partial_{s_3}\psi)(s_1,s_3).$$

The above limits are in $L^2(\mathbb{R}^2)$ under the usual L^2 norm, and thus technically, $\partial_{s_1}\psi$ and $\partial_{s_3}\psi$ are weak L^2 derivative of ψ. By repeated application of the above argument we have that ψ has weak derivatives of all orders. By the Sobolev embedding theorem it follows that $\psi = \psi_{f;\gamma,m_5,m_4,m_3}$ is a smooth function on $\mathbb{R}^2$ which vanishes at infinity [5, Theorem 9.17].

Also observe that

$$\big(\pi_{[[\gamma]],m_5,m_4,m_3}(\mathcal{P}_4)\psi\big)(s_1,s_3) = \lim_{t_4\to 0} \frac{\big(\pi_{[[\gamma]],m_5,m_4,m_3}(T_4)\psi\big)(s_1,s_3) - \psi(s_1,s_3)}{t_4}$$
$$= \psi(s_1,s_3) \lim_{t_4\to 0} \frac{e(m_4t_4 + 3m_5t_4s_3) - 1}{t_4} = \psi(s_1,s_3)\frac{d}{dt_4}\big[e(m_4t_4 + 3m_5t_4s_3)\big]_{t_4=0}$$
$$= 2\pi i(m_4 + 3m_5s_3)\psi(s_1,s_3),$$

and

$$\big(\pi_{[[\gamma]],m_5,m_4,m_3}(\mathcal{P}_5)\psi\big)(s_1,s_3) = \lim_{t_5\to 0} \frac{\big(\pi_{[[\gamma]],m_5,m_4,m_3}(T_5)\psi\big)(s_1,s_3) - \psi(s_1,s_3)}{t_5}$$
$$= \psi(s_1,s_3) \lim_{t_5\to 0} \frac{e(am_5t_5 - cm_5s_1t_5) - 1}{t_5} = \psi(s_1,s_3)\frac{d}{dt_5}\big[e(am_5t_5 - cm_5s_1t_5)\big]_{t_5=0}$$
$$= 2\pi i(am_5 - cm_5s_1)\psi(s_1,s_3).$$

By repeated application of these arguments it follows that $|s_1|^{j_1}|s_3|^{j_3}\partial_{s_1}^{k_1}\partial_{s_3}^{k_3}[\psi] \in L^2(\mathbb{R}^2)$ for all $k_i, j_i \in \mathbb{Z}_{\geq 0}$. Once again, by the Sobolev Embedding Theorem it follows that $|s_1|^{j_1}|s_3|^{j_3}\partial_{s_1}^{k_1}\partial_{s_3}^{k_3}[\psi]$

is a smooth functions on $\mathbb{R}^2$ which vanish at infinity. Thus

$$\psi = \psi_{f;\gamma,m_5,m_4,m_3} \in \mathcal{S}(\mathbb{R}^2), \tag{2.83}$$

where $\mathcal{S}(\mathbb{R}^2)$ is the space of Schwartz functions on $\mathbb{R}^2$.

2.6.2 Analysis of W_{0,k_5,k_4,k_3,k_1}

Suppose $k_5 \neq 0$, $k_4 \neq 0$, or $k_3 \neq 0$. In this subsection we will show that W_{0,k_5,k_4,k_3,k_1} is isometric to $L^2(\mathbb{R})$. Via this isometry we will construct a representation of N on $L^2(\mathbb{R})$. We will then analyze the smooth vectors of $L^2(\mathbb{R})$ under this representation, which will allow us to give an explicit description in section 2.7 of the $N_{\mathbb{Z}}$-invariant distributions on N. To begin this analysis, observe

$$\begin{aligned} &T_6T_5T_4T_3T_1Q_2X_6X_5X_4X_3X_2X_1 \\ &= R_6(t_6+u_6)R_5(t_5+u_5)R_4(t_4+u_4)R_3(t_3+u_3)R_2(q_2+x_2)R_1(t_1+x_1), \end{aligned}$$

where

$$\begin{aligned} u_6 &= -2q_2^3t_1^2 - 3q_2^2t_1t_3 - 6q_2^2t_1^2x_2 - 6q_2t_1t_3x_2 - 6q_2t_1^2x_2^2 - 3t_1t_3x_2^2 - 2t_1^2x_2^3 \\ &\quad - 6q_2^2t_1x_3 - 6q_2t_3x_3 - 6q_2t_1x_2x_3 - 3t_1x_2^2x_3 - 3q_2x_3^2 + 3q_2t_1x_4 + 3t_3x_4 \\ &\quad + t_1x_5 + x_6, \\ u_5 &= -q_2^3t_1 - 3q_2^2t_1x_2 - 3q_2t_1x_2^2 - t_1x_2^3 - 3q_2^2x_3 + 3q_2x_4 + x_5, \\ u_4 &= -q_2^2t_1 - 2q_2t_1x_2 - t_1x_2^2 - 2q_2x_3 + x_4, \\ u_3 &= q_2t_1 + t_1x_2 + x_3. \end{aligned}$$

Recall that (2.64) and (2.7h) show that

$$\begin{aligned} &f_{0,k_5,k_4+3k_5q_2,k_3-2k_4q_2-3k_5q_2^2,k_1-k_3q_2+k_4q_2^2+k_5q_2^3}(n) \\ &= f_{0,k_5,k_4,k_3,k_1}(Q_2n) \\ &= \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 f(T_6T_5T_4T_3T_1Q_2n)e(-k_5t_5-k_4t_4-k_3t_3-k_1t_1) \\ &\quad dt_6\,dt_5\,dt_4\,dt_3\,dt_1. \end{aligned}$$

Therefore, when we perform the following change of variables (in order)

$$t_6 \mapsto t_6 - u_6, \quad t_5 \mapsto t_5 - u_5, \quad t_4 \mapsto t_4 - u_4, \quad t_3 \mapsto t_3 - u_3, \quad t_1 \mapsto t_1 - x_1,$$

it follows from Lemma 2.3 that

$$\begin{aligned}&(f_{0,k_5,k_4+3k_5q_2,k_3-2k_4q_2-3k_5q_2^2,k_1-k_3q_2+k_4q_2^2+k_5q_2^3})^{\sigma_{\mathrm{id}}}(x_1,x_2,x_3,x_4,x_5,x_6)\\&=f_{0,k_5,k_4,k_3,k_1}(Q_2X_6X_5X_4X_3X_2X_1)\\&=e\big((k_1-k_3q_2+k_4q_2^2+k_5q_2^3)x_1+(k_3-2k_4q_2-3k_5q_2^2)(x_3-x_1x_2)\\&\quad+(k_4+3k_5q_2)(x_4+x_1x_2^2)+k_5(x_5+x_1x_2^3)\big)\psi_{0,k_5,k_4,k_3,k_1}(x_2+q_2),\end{aligned}\tag{2.84}$$

where

$$\begin{aligned}&\psi_{0,k_5,k_4,k_3,k_1}(s_2)=\psi_{f;0,k_5,k_4,k_3,k_1}(s_2)\\&=\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 f^{\sigma_{\mathrm{id}}}(t_6,t_5,t_4,t_3,s_2,t_1)e(-k_1t_1-k_3t_3-k_4t_4-k_5t_5\\&\quad+k_3t_1s_2-k_4t_1s_2^2-k_5t_1s_2^3)\,dt_6\,dt_5\,dt_4\,dt_3\,dt_1.\end{aligned}\tag{2.85}$$

As indicated in the above equality, we will at times suppress writing f in the subscript of $\psi_{f;0,k_5,k_4,k_3,k_1}$ when context is clear. Observe that by (2.65),

$$\begin{aligned}&(f^{\Sigma}_{0,k_5,k_4,k_3,k_1})^{\sigma_{\mathrm{id}}}(x_1,x_2,x_3,x_4,x_5,x_6)\\&=\sum_{q_2\in\mathbb{Z}}(f_{0,k_5,k_4+3k_5q_2,k_3-2k_4q_2-3k_5q_2^2,k_1-k_3q_2+k_4q_2^2+k_5q_2^3})^{\sigma_{\mathrm{id}}}(x_1,x_2,x_3,x_4,x_5,x_6)\\&=\sum_{q_2\in\mathbb{Z}}e\big((k_1-k_3q_2+k_4q_2^2+k_5q_2^3)x_1+(k_3-2k_4q_2-3k_5q_2^2)(x_3-x_1x_2)\\&\quad+(k_4+3k_5q_2)(x_4+x_1x_2^2)+k_5(x_5+x_1x_2^3)\big)\psi_{0,k_5,k_4,k_3,k_1}(x_2+q_2).\end{aligned}\tag{2.86}$$

From (2.84) we see that

$$\begin{aligned}&\langle(f_{0,k_5,k_4+3k_5q_2,k_3-2k_4q_2-3k_5q_2^2,k_1-k_3q_2+k_4q_2^2+k_5q_2^3})^{\sigma_{\mathrm{id}}},\\&\qquad(f_{0,k_5,k_4+3k_5\ell_2,k_3-2k_4\ell_2-3k_5\ell_2^2,k_1-k_3\ell_2+k_4\ell_2^2+k_5\ell_2^3})^{\sigma_{\mathrm{id}}}\rangle_{\mathcal{F}}=0\end{aligned}$$

if $q_2\neq\ell_2$; simply perform integration is the x_4, x_3, and x_1 variables in (2.73) to see why this is the case. Thus by (2.86), the Pythagorean theorem, and changing variables, we find that

$$\begin{aligned}&\left(\left\|f^{\Sigma}_{0,k_5,k_4,k_3,k_1}\right\|_2\right)^2=\left(\left\|(f^{\Sigma}_{0,k_5,k_4,k_3,k_1})^{\sigma_{\mathrm{id}}}\right\|_{2,\mathcal{F}}\right)^2\\&=\sum_{q_2\in\mathbb{Z}}\left(\left\|(f_{0,k_5,k_4+3k_5q_2,k_3-2k_4q_2-3k_5q_2^2,k_1-k_3q_2+k_4q_2^2+k_5q_2^3})^{\sigma_{\mathrm{id}}}\right\|_{2,\mathcal{F}}\right)^2\\&=\sum_{q_2\in\mathbb{Z}}\int_0^1|\psi_{0,k_5,k_4,k_3,k_1}(x_2+q_2)|^2\,dx_2=\int_{-\infty}^{\infty}|\psi_{0,k_5,k_4,k_3,k_1}(x_2)|^2\,dx_2.\end{aligned}\tag{2.87}$$

This implies that $\psi_{0,k_5,k_4,k_3,k_1}\in L^2(\mathbb{R})$. Conversely, given $\psi\in L^2(\mathbb{R})$, we can define

$$h_{\psi;0,k_5,k_4,k_3,k_1}(x_1,x_2,x_3,x_4,x_5,x_6)=\sum_{\ell_2\in\mathbb{Z}}r_{\ell_2}(x_1,x_2,x_3,x_4,x_5,x_6),\tag{2.88}$$

where

$$
\begin{aligned}
&r_{\ell_2}(x_1, x_2, x_3, x_4, x_5, x_6)\\
&= e\big((k_1 - k_3\ell_2 + k_4\ell_2^2 + k_5\ell_2^3)x_1 + (k_3 - 2k_4\ell_2 - 3k_5\ell_2^2)(x_3 - x_1x_2)\\
&\quad + (k_4 + 3k_5\ell_2)(x_4 + x_1x_2^2) + k_5(x_5 + x_1x_2^3)\big)\psi(x_2 + \ell_2).
\end{aligned}
$$

One can check that $h_{\psi;0,k_5,k_4,k_3,k_1}$ is a well-defined element of $L^2(\mathcal{F})$ by repeating the argument in (2.87) with $f^{\Sigma}_{0,k_5,k_4,k_3,k_1}$ replaced by $h_{\psi;0,k_5,k_4,k_3,k_1}$. Furthermore, it can be shown that $h_{\psi;0,k_5,k_4,k_3,k_1}$ is $\sigma_{\mathrm{id}}^{-1}(N_{\mathbb{Z}})$-invariant. To see that this is the case, one solves for p_i in (2.70) and observes that

$$
r_{\ell_2}(p_1, p_2, p_3, p_4, p_5, p_6) = r_{\ell_2+q_2}(x_1, x_2, x_3, x_4, x_5, x_6).
$$

Thus $h_{\psi;0,k_5,k_4,k_3,k_1} \circ \sigma_{\mathrm{id}}^{-1}$ is a well-defined element of $L^2(N_{\mathbb{Z}}\backslash N)$.

Let $\Phi_{0,k_5,k_4,k_3,k_1} : L^2(\mathbb{R}) \to W_{0,k_5,k_4,k_3,k_1}$ where

$$
\Phi_{0,k_5,k_4,k_3,k_1}(\psi) = h_{\psi;0,k_5,k_4,k_3,k_1} \circ \sigma_{\mathrm{id}}^{-1}.
$$

Technically, we should prove that

$$
h_{\psi;0,k_5,k_4,k_3,k_1} \circ \sigma_{\mathrm{id}}^{-1} = (h_{\psi;0,k_5,k_4,k_3,k_1} \circ \sigma_{\mathrm{id}}^{-1})^{\Sigma}_{0,k_5,k_4,k_3,k_1}
$$

to justify that W_{0,k_5,k_4,k_3,k_1} truly is the co-domain of Φ_{0,k_5,k_4,k_3,k_1}. To see that this is the case, it suffices by (2.65) to show that

$$
\begin{aligned}
&\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 r_{\ell_2} \circ \sigma_{\mathrm{id}}^{-1}(T_6T_5T_4T_3T_1X_6X_5X_4X_3X_2X_1)e\big(-k_5t_5 - (k_4 + 3k_5q_2)t_4\\
&\quad - (k_3 - 2k_4q_2 - 3k_5q_2^2)t_3 - (k_1 - k_3q_2 + k_4q_2^2 + k_5q_2^3)t_1\big)\, dt_6\, dt_5\, dt_4\, dt_3\, dt_1\\
&= \begin{cases} 0 & \text{if } \ell_2 \neq q_2 \\ r_{q_2}(x_1, \ldots, x_6) & \text{if } \ell_2 = q_2 \end{cases}.
\end{aligned}
\tag{2.89}
$$

To show that (2.89) holds, one solves for p_i in the equation

$$
P_6P_5P_4P_3P_2P_1 = T_6T_5T_4T_3T_1X_6X_5X_4X_3X_2X_1 \tag{2.90}
$$

and substitutes $r_{\ell_2}(p_1, p_2, p_3, p_4, p_5, p_6)$ in for $r_{\ell_2} \circ \sigma_{\mathrm{id}}^{-1}(T_6T_5T_4T_3T_1X_6X_5X_4X_3X_2X_1)$ on the left-hand side of (2.89). One then see that the first case in (2.89) follows when we integrate in the t_4, t_3, t_1 variables, and that the second case in (2.89) follows from a straightforward calculation. By (2.87), we see that Φ_{0,k_5,k_4,k_3,k_1} is an isometry; indeed, Φ_{0,k_5,k_4,k_3,k_1} is a surjection since Φ_{0,k_5,k_4,k_3,k_1} maps smooth functions in $L^2(\mathbb{R})$ to a dense set in W_{0,k_5,k_4,k_3,k_1}.

We define a representation π_{0,k_5,k_4,k_3,k_1} of N on $L^2(\mathbb{R})$ by the equation

$$\pi_{0,k_5,k_4,k_3,k_1}(n)(\psi) = \Phi^{-1}_{0,k_5,k_4,k_3,k_1}(r(n)\Phi_{0,k_5,k_4,k_3,k_1}(\psi)),$$

where $\psi \in L^2(\mathbb{R})$. We wish to give an explicit formula for π_{0,k_5,k_4,k_3,k_1}. To accomplish this, let

$$\begin{aligned} &p_{\ell_2}(x_1,x_2,x_3,x_4,x_5,x_6) \\ &= e\big((k_1 - k_3\ell_2 + k_4\ell_2^2 + k_5\ell_2^3)x_1 + (k_3 - 2k_4\ell_2 - 3k_5\ell_2^2)(x_3 - x_1x_2) \\ &\quad + (k_4 + 3k_5\ell_2)(x_4 + x_1x_2^2) + k_5x_5\big) \end{aligned}$$

and let

$$\begin{aligned} &q(x_2,y_1,\ldots,y_6) \\ &= e\big(k_1y_1 - k_3x_2y_1 + k_4x_2^2y_1 + k_5x_2^3y_1 - k_3y_1y_2 + 2k_4x_2y_1y_2 + 3k_5x_2^2y_1y_2 + k_4y_1y_2^2 \\ &\quad + 3k_5x_2y_1y_2^2 + k_5y_1y_2^3 + k_3y_3 - 2k_4x_2y_3 - 3k_5x_2^2y_3 + k_4y_4 + 3k_5x_2y_4 + k_5y_5\big). \end{aligned}$$

When we solve for p_i in (2.80) we find that

$$p_{\ell_2}(p_1,p_2,p_3,p_4,p_5,p_6) = p_{\ell_2}(x_1,x_2,x_3,x_4,x_5,x_6)q(x_2+q_2).$$

Since

$$r_{\ell_2}(x_1,x_2,x_3,x_4,x_5,x_6) = p_{\ell_2}(x_1,x_2,x_3,x_4,x_5,x_6)\psi(\ell_2+x_2)$$

it follows that

$$\begin{aligned} &r_{\ell_2}(p_1,p_2,p_3,p_4,p_5,p_6) \\ &= p_{\ell_2}(p_1,p_2,p_3,p_4,p_5,p_6)\psi(\ell_2+p_2) \\ &= p_{\ell_2}(x_1,x_2,x_3,x_4,x_5,x_6)q(\ell_2+x_2,y_1,\ldots,y_6)\psi((\ell_2+x_2)+y_2). \end{aligned} \tag{2.91}$$

Observe $(r(Y_6Y_5Y_4Y_3Y_2Y_1)\Phi_{0,k_5,k_4,k_3,k_1}(\psi))(X_6X_5X_4X_3X_2X_1)$ is simply a sum of $r_{\ell_2}(p_1,p_2,p_3,p_4,p_5,p_6)$. Thus by (2.91) we see that

$$(r(Y_6Y_5Y_4Y_3Y_2Y_1)\Phi_{0,k_5,k_4,k_3,k_1}(\psi))(X_6X_5X_4X_3X_2X_1)$$

is of the form $h_{\widetilde{\psi};0,k_5,k_4,k_3,k_1}$, where $\widetilde{\psi} \in L^2(\mathbb{R})$. When we solve for $\widetilde{\psi}$, we are then able to deduce that

$$\begin{aligned} &(\pi_{0,k_5,k_4,k_3,k_1}(Y_6Y_5Y_4Y_3Y_2Y_1)\psi)(s_2) \\ &= q(s_2,y_1,\ldots,y_6)\psi(s_2+y_2) \\ &= e\big(k_1y_1 - k_3s_2y_1 + k_4s_2^2y_1 + k_5s_2^3y_1 - k_3y_1y_2 + 2k_4s_2y_1y_2 + 3k_5s_2^2y_1y_2 \\ &\quad + k_4y_1y_2^2 + 3k_5s_2y_1y_2^2 + k_5y_1y_2^3 + k_3y_3 - 2k_4s_2y_3 - 3k_5s_2^2y_3 + k_4y_4 \\ &\quad + 3k_5s_2y_4 + k_5y_5\big)\psi(s_2+y_2). \end{aligned} \tag{2.92}$$

Let $f \in W^{\infty}_{0,k_5,k_4,k_3,k_1}$. Since Φ_{0,k_5,k_4,k_3,k_1} is an isometry it follows that $\psi = \psi_{f;0,k_5,k_4,k_3,k_1}$ is also a smooth vector and that $\mathfrak{n}$ acts upon ψ via π_{0,k_5,k_4,k_3,k_1} and (1.1). In particular, observe

$$\begin{aligned}
(\pi_{0,k_5,k_4,k_3,k_1}(\mathcal{P}_2)\psi)(s_2) &= \lim_{t_2\to 0}\frac{(\pi_{0,k_5,k_4,k_3,k_1}(T_2)\psi)(s_2)-\psi(s_2)}{t_2}\\
&= \lim_{t_2\to 0}\frac{\psi(t_2+s_2)-\psi(s_2)}{t_2} = \psi'(s_2). \qquad (2.93)
\end{aligned}$$

The above limit is in $L^2(\mathbb{R})$ under the usual L^2 norm, and thus technically, ψ' is the weak L^2 derivative of ψ. By repeated application of the above argument we have that ψ has weak derivatives of all orders. By the Sobolev embedding theorem it follows that $\psi = \psi_{f;0,k_5,k_4,k_3,k_1}$ is a smooth function on $\mathbb{R}$ which vanishes at infinity.

Also observe that

$$\begin{aligned}
(\pi_{0,k_5,k_4,k_3,k_1}(\mathcal{P}_1)\psi)(s_2) &= \lim_{t_1\to 0}\frac{(\pi_{0,k_5,k_4,k_3,k_1}(T_1)\psi)(s_2)-\psi(s_2)}{t_1}\\
&= \psi(s_2)\lim_{t_1\to 0}\frac{e(k_1t_1-k_3t_1s_2+k_4t_1s_2^2+k_5t_1s_2^3)-1}{t_1}\\
&= \psi(s_2)\frac{d}{dt_1}\left[e(k_1t_1-k_3t_1s_2+k_4t_1s_2^2+k_5t_1s_2^3)\right]_{t_1=0}\\
&= 2\pi i(k_1-k_3s_2+k_4s_2^2+k_5s_2^3)\psi(s_2), \qquad (2.94)\\
(\pi_{0,k_5,k_4,k_3,k_1}(\mathcal{P}_3)\psi)(s_2) &= \lim_{t_3\to 0}\frac{(\pi_{0,k_5,k_4,k_3,k_1}(T_3)\psi)(s_2)-\psi(s_2)}{t_3}\\
&= \psi(s_2)\lim_{t_3\to 0}\frac{e(k_3t_3-2k_4t_3s_2-3k_5t_3s_2^2)-1}{t_3}\\
&= \psi(s_2)\frac{d}{dt_3}\left[e(k_3t_3-2k_4t_3s_2-3k_5t_3s_2^2)\right]_{t_3=0}\\
&= 2\pi i(k_3-2k_4s_2-3k_5s_2^2)\psi(s_2), \qquad (2.95)\\
(\pi_{0,k_5,k_4,k_3,k_1}(\mathcal{P}_4)\psi)(s_2) &= \lim_{t_4\to 0}\frac{(\pi_{0,k_5,k_4,k_3,k_1}(T_4)\psi)(s_2)-\psi(s_2)}{t_4}\\
&= \psi(s_2)\lim_{t_4\to 0}\frac{e(k_4t_4+3k_5t_4s_2)-1}{t_4}\\
&= \psi(s_2)\frac{d}{dt_4}[e(k_4t_4+3k_5t_4s_2)]_{t_4=0} = 2\pi i(k_4+3k_5s_2)\psi(s_2). \qquad (2.96)
\end{aligned}$$

By repeated application of these arguments it follows that $|s_2|^j\psi^{(k)} \in L^2(\mathbb{R})$ for all $k, j \in \mathbb{Z}_{\geq 0}$. Once again, by the Sobolev Embedding Theorem it follows that $|s_2|^j\psi^{(k)}$ are smooth functions on $\mathbb{R}$ which vanish at infinity. Thus

$$\psi = \psi_{f;0,k_5,k_4,k_3,k_1} \in \mathcal{S}(\mathbb{R}), \qquad (2.97)$$

where $\mathcal{S}(\mathbb{R})$ is the space of Schwartz functions on $\mathbb{R}$.

2.6.3 Analysis of $W_{0,0,0,0,k_1,k_2}$

In this subsection we will show that $W_{0,0,0,0,k_1,k_2}$ is isometric to $\mathbb{C}$. To begin this analysis, observe

$$\begin{aligned}
&T_6T_5T_4T_3T_1T_2X_6X_5X_4X_3X_2X_1 \\
&\quad = R_6(t_6+u_6)R_5(t_5+u_5)R_4(t_4+u_4)R_3(t_3+u_3)R_2(q_2+x_2)R_1(t_1+x_1)
\end{aligned}$$

where

$$\begin{aligned}
u_6 &= -2t_1^2t_2^3 - 3t_1t_2^2t_3 - 6t_1^2t_2^2x_2 - 6t_1t_2t_3x_2 - 6t_1^2t_2x_2^2 - 3t_1t_3x_2^2 - 2t_1^2x_2^3 - 6t_1t_2^2x_3 \\
&\quad - 6t_2t_3x_3 - 6t_1t_2x_2x_3 - 3t_1x_2^2x_3 - 3t_2x_3^2 + 3t_1t_2x_4 + 3t_3x_4 + t_1x_5 + x_6, \\
u_5 &= -t_1t_2^3 - 3t_1t_2^2x_2 - 3t_1t_2x_2^2 - t_1x_2^3 - 3t_2^2x_3 + 3t_2x_4 + x_5, \\
u_4 &= -t_1t_2^2 - 2t_1t_2x_2 - t_1x_2^2 - 2t_2x_3 + x_4, \\
u_3 &= t_1t_2 + t_1x_2 + x_3.
\end{aligned}$$

Therefore, when we perform the change of variables (in order)

$$\begin{aligned}
&t_6 \mapsto t_6 - u_6, \quad t_5 \mapsto t_5 - u_5, \quad t_4 \mapsto t_4 - u_4, \\
&t_3 \mapsto t_3 - u_3, \quad t_1 \mapsto t_1 - x_1, \quad t_2 \mapsto t_2 - x_2,
\end{aligned}$$

it follows from Lemma 2.3 that

$$f^{\sigma_{\mathrm{id}}}_{0,0,0,0,k_1,k_2} = c_{k_1,k_2}e(k_1x_1 + k_2x_2),$$

where

$$c_{k_1,k_2} = c_{f;k_1,k_2} = \int_0^1 \cdots \int_0^1 f^{\sigma_{\mathrm{id}}}(t_1,t_2,t_3,t_4,t_5,t_6)e(-k_1t_1 - k_2t_2)\, dt_6 \ldots dt_1. \tag{2.98}$$

As indicated in the above equality, we will at times suppress writing f in the subscript of $c_{f;k_1,k_2}$ when context is clear.

Observe

$$\|f_{0,0,0,0,k_1,k_2}\|_2 = \left(\int_0^1\int_0^1 |c_{k_1,k_2}e(k_1x_1+k_2x_2)|^2\, dx_2\, dx_1\right)^{1/2} = |c_{k_1,k_2}|.$$

Thus the map

$$f_{0,0,0,0,k_1,k_2} \mapsto c_{k_1,k_2}, \tag{2.99}$$

is an isometric injection from $W_{0,0,0,0,k_1,k_2}$ into $\mathbb{C}$. For $c \in \mathbb{C}$, let

$$h_{c;k_1,k_2}(x_1,x_2,x_3,x_4,x_5,x_6) = c\, e(k_1x_1 + k_2x_2).$$

One can easily show that $h_{c;k_1,k_2} \circ \sigma_{\mathrm{id}}^{-1}$ is $N_{\mathbb{Z}}$-invariant, and thus it follows that the map (2.99) is in fact a surjection onto $\mathbb{C}$. Thus $W_{0,0,0,0,k_1,k_2}$ is isometric to $\mathbb{C}$ via (2.99).

2.7 A Fourier Series for $N_{\mathbb{Z}}$-invariant Distributions on N

We will use the following lemma to prove Theorem 2.15, and from Theorem 2.15 we will deduce Theorem 2.11 for general $f \in W = L^2(N_{\mathbb{Z}}\backslash N)$.

Lemma 2.14.

$$\begin{aligned} W = &\left(\bigoplus_{m_5 \in \mathbb{Z}_{>0}} \bigoplus_{\substack{[[\gamma]] \in (\Gamma_\beta)_\infty \backslash \Gamma_\beta / (\Gamma_\beta)_\infty \\ [[\gamma]] \neq [[\pm \mathrm{id}]]}} \bigoplus_{[m_4] \in \mathbb{Z}/3m_5\mathbb{Z}} \bigoplus_{m_3 \in \mathbb{Z}} W_{[[\gamma]], m_5, m_4, m_3} \right) \\ &\oplus \left(\bigoplus_{k_5 \in \mathbb{Z}_{\neq 0}} \bigoplus_{[k_4] \in \mathbb{Z}/3k_5\mathbb{Z}} \bigoplus_{k_3 \in \mathbb{Z}} \bigoplus_{k_1 \in \mathbb{Z}} W_{0,k_5,k_4,k_3,k_1} \right) \\ &\oplus \left(\bigoplus_{k_4 \in \mathbb{Z}_{\neq 0}} \bigoplus_{[k_3] \in \mathbb{Z}/2k_4\mathbb{Z}} \bigoplus_{k_1 \in \mathbb{Z}} W_{0,0,k_4,k_3,k_1} \right) \oplus \left(\bigoplus_{k_3 \in \mathbb{Z}_{\neq 0}} \bigoplus_{[k_1] \in \mathbb{Z}/k_3\mathbb{Z}} W_{0,0,0,k_3,k_1} \right) \\ &\oplus \left(\bigoplus_{k_1 \in \mathbb{Z}} \bigoplus_{k_2 \in \mathbb{Z}} W_{0,0,0,0,k_1,k_2} \right) \end{aligned} \tag{2.100}$$

where the subspaces on the right-hand side of (2.100) *are defined in* (2.72).

Proof. In the previous section we showed that:

- if $u \in W_{[[\gamma]],m_5,m_4,m_3}$ then $u^{\sigma_{\mathrm{id}}}$ is of the form (2.78),
- if $u \in W_{0,k_5,k_4,k_3,k_1}$ then $u^{\sigma_{\mathrm{id}}}$ is of the form (2.88).

From these observations and the fact that $\langle u_1, u_2 \rangle = \langle u_1^{\sigma_{\mathrm{id}}}, u_2^{\sigma_{\mathrm{id}}} \rangle_{\mathcal{F}}$ for $u_1, u_2 \in W$, one can easily check that the subspaces listed on the right-hand side of (2.100) are pairwise orthogonal. Thus it follows that the right-hand side of (2.100) is indeed contained in W in a natural way.

Recall that we have proved Theorem 2.11 for smooth $f \in W = L^2(N_{\mathbb{Z}}\backslash N)$. This shows that smooth $f \in W$ are contained in the right-hand side of (2.100). Since both the left and right-hand sides of (2.100) are closed and since both the left and right-hand sides of (2.100) contain smooth f as a dense set, it follows by density that (2.100) holds. $\square$

Theorem 2.15. *For $f \in W = L^2(N_{\mathbb{Z}}\backslash N)$ we have*

$$f^{\sigma_{\mathrm{id}}}(x_1,x_2,x_3,x_4,x_5,x_6) = \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} c_{k_1,k_2} e(k_1x_1+k_2x_2) \tag{2.101a}$$

$$+\sum_{k_3\in\mathbb{Z}_{\neq 0}}\sum_{[k_1]\in\mathbb{Z}/k_3\mathbb{Z}}\sum_{q_2\in\mathbb{Z}} e\big((k_1-k_3q_2)x_1+k_3(-x_1x_2+x_3)\big)\psi_{0,0,0,k_3,k_1}(x_2+q_2) \tag{2.101b}$$

$$+\sum_{k_4\in\mathbb{Z}_{\neq 0}}\sum_{[k_3]\in\mathbb{Z}/2k_4\mathbb{Z}}\sum_{k_1\in\mathbb{Z}}\sum_{q_2\in\mathbb{Z}} e\big((k_1-k_3q_2+k_4q_2^2)x_1+(k_3-2k_4q_2)(-x_1x_2+x_3) \tag{2.101c}$$

$$+k_4(x_1x_2^2+x_4)\big)\psi_{0,0,k_4,k_3,k_1}(x_2+q_2)$$

$$+\sum_{k_5\in\mathbb{Z}_{\neq 0}}\sum_{[k_4]\in\mathbb{Z}/3k_5\mathbb{Z}}\sum_{k_3\in\mathbb{Z}}\sum_{k_1\in\mathbb{Z}}\sum_{q_2\in\mathbb{Z}} e\big((k_1-k_3q_2+k_4q_2^2+k_5q_2^3)x_1$$

$$+(k_3-2k_4q_2-3k_5q_2^2)(x_3-x_1x_2)+(k_4+3k_5q_2)(x_4+x_1x_2^2)+k_5(x_5+x_1x_2^3)\big)$$

$$\psi_{0,k_5,k_4,k_3,k_1}(x_2+q_2) \tag{2.101d}$$

$$+\sum_{m_5\in\mathbb{Z}_{>0}}\sum_{\substack{[[\gamma]]\in(\Gamma_\beta)_\infty\backslash\Gamma_\beta/(\Gamma_\beta)_\infty\\ [[\gamma]]\neq[[\pm\mathrm{id}]]}}\sum_{[m_4]\in\mathbb{Z}/3m_5\mathbb{Z}}\sum_{m_3\in\mathbb{Z}}\sum_{q_1\in\mathbb{Z}}\sum_{q_3\in\mathbb{Z}}\sum_{[q_2]\in\mathbb{Z}/c\mathbb{Z}} e\Big(\frac{m_3q_2}{c}-\frac{am_4q_2^2}{c}-\frac{a^2m_5q_2^3}{c}$$

$$+\frac{m_3-2m_4(aq_2-cq_3)-3m_5(aq_2-cq_3)^2}{c}x_2-\frac{(a-cq_1)(m_4+3m_5(aq_2-cq_3))}{c}x_2^2$$

$$-\frac{(a-cq_1)^2m_5}{c}x_2^3+(m_4+3m_5(aq_2-cq_3))(2x_2x_3+x_4)+(a-cq_1)m_5(x_2^2x_3+x_5)$$

$$-cm_5(3x_2x_3^2+x_6)\Big)\psi_{\gamma,m_5,m_4,m_3}(q_1+x_1,-cq_3+aq_2+ax_2-cq_1x_2-cx_3) \tag{2.101e}$$

where the above series converges absolutely in $L^2(\mathcal{F})$ (recall $\mathcal{F}=[0,1)^6$) and where $\psi_{\gamma,m_5,m_4,m_3}$, ψ_{0,k_5,k_4,k_3,k_1}, and c_{k_1,k_2} are defined according to (2.75), (2.85), *and* (2.98) *(respectively). Furthermore, it follows that $\psi_{\gamma,m_5,m_4,m_3}\in L^2(\mathbb{R}^2)$ and $\psi_{0,k_5,k_4,k_3,k_1}\in L^2(\mathbb{R})$. In particular, if $f\in W^\infty$ (a smooth vector of W under the right regular representation) then $\psi_{\gamma,m_5,m_4,m_3}\in\mathcal{S}(\mathbb{R}^2)$ and $\psi_{0,k_5,k_4,k_3,k_1}\in\mathcal{S}(\mathbb{R})$.*

Proof. Observe that Lemma 2.14 shows that $f\in W$ must be equal to an absolutely convergent series of the form (2.101), except that it is not immediately obvious that the $\psi_{\gamma,m_5,m_4,m_3}$, ψ_{0,k_5,k_4,k_3,k_1}, c_{k_1,k_2} in such a series expansion must be defined by (2.75), (2.85), and (2.98) (respectively); a priori, they are only known to be elements of $L^2(\mathbb{R}^2)$, $L^2(\mathbb{R})$, or $\mathbb{C}$. However, if in the definitions of $\psi_{f;\gamma,m_5,m_4,m_3}$, $\psi_{f;0,k_5,k_4,k_3,k_1}$, and $c_{f;k_1,k_2}$ given in (2.75), (2.85), and (2.98) (respectively), one replaces $f^{\sigma_{\mathrm{id}}}$ with the aforementioned series expansion and computes the resulting integrals, it becomes apparent that the $\psi_{\gamma,m_5,m_4,m_3}$, ψ_{0,k_5,k_4,k_3,k_1}, and c_{k_1,k_2} given by Lemma 2.14 agree with those defined in (2.75), (2.85), and (2.98) (respectively).

The statement regarding $f\in W^\infty$ follows immediately from (2.83) and (2.97). ☐

Recall that the principal obstruction to stating Theorem 2.11 for general $f\in L^2(N_{\mathbb{Z}}\backslash N)$ was our inability to define $f^{\Sigma}_{\gamma,m_5,m_4,m_3}$ and $f^{\Sigma}_{0,k_5,k_4,k_3,k_1}$ for general $f \in L^2(N_{\mathbb{Z}}\backslash N)$. Indeed, although we could define $f^{\Sigma}_{\gamma,m_5,m_4,m_3}$ and $f^{\Sigma}_{0,k_5,k_4,k_3,k_1}$ formally using series, it was not immediately obvious that such series should converge in $L^2(\mathcal{Q})$ where $\mathcal{Q}$ is a fundamental domain for $N_{\mathbb{Z}}\backslash N$. By Theorem 2.15 we see that the series defining $\left(f^{\Sigma}_{\gamma,m_5,m_4,m_3}\right)^{\sigma_{\mathrm{id}}}$ and $\left(f^{\Sigma}_{0,k_5,k_4,k_3,k_1}\right)^{\sigma_{\mathrm{id}}}$ converge absolutely in $L^2(\mathcal{F})$ (recall (2.76) and (2.86)). Since σ_{id} is a homeomorphism, it follows that the series defining $f^{\Sigma}_{\gamma,m_5,m_4,m_3}$ and $f^{\Sigma}_{0,k_5,k_4,k_3,k_1}$ must also converge absolutely in $L^2(\mathcal{Q})$ where $\mathcal{Q} = \sigma_{\mathrm{id}}(\mathcal{F})$. Since $\mathcal{Q}$ is a fundamental domain for $N_{\mathbb{Z}}\backslash N$ and since the series defining $f^{\Sigma}_{\gamma,m_5,m_4,m_3}$ and $f^{\Sigma}_{0,k_5,k_4,k_3,k_1}$ are formally $N_{\mathbb{Z}}$-invariant, it follows that $f^{\Sigma}_{\gamma,m_5,m_4,m_3}$ and $f^{\Sigma}_{0,k_5,k_4,k_3,k_1}$ are well-defined elements of $L^2(N_{\mathbb{Z}}\backslash N)$. Furthermore, Theorem 2.15 essentially shows that

$$
\begin{aligned}
f^{\sigma_{\mathrm{id}}} = &\sum_{m_5\in\mathbb{Z}_{>0}} \sum_{\substack{[[\gamma]]\in(\Gamma_\beta)_\infty\backslash\Gamma_\beta/(\Gamma_\beta)_\infty \\ [[\gamma]]\neq[[\pm\mathrm{id}]]}} \sum_{[m_4]\in\mathbb{Z}/3m_5\mathbb{Z}} \sum_{m_3\in\mathbb{Z}} \left(f^{\Sigma}_{\gamma,m_5,m_4,m_3}\right)^{\sigma_{\mathrm{id}}} \\
&+ \sum_{k_5\in\mathbb{Z}_{\neq 0}} \sum_{[k_4]\in\mathbb{Z}/3k_5\mathbb{Z}} \sum_{k_3\in\mathbb{Z}} \sum_{k_1\in\mathbb{Z}} \left(f^{\Sigma}_{0,k_5,k_4,k_3,k_1}\right)^{\sigma_{\mathrm{id}}} + \sum_{k_4\in\mathbb{Z}_{\neq 0}} \sum_{[k_3]\in\mathbb{Z}/2k_4\mathbb{Z}} \sum_{k_1\in\mathbb{Z}} \left(f^{\Sigma}_{0,0,k_4,k_3,k_1}\right)^{\sigma_{\mathrm{id}}} \\
&+ \sum_{k_3\in\mathbb{Z}_{\neq 0}} \sum_{[k_1]\in\mathbb{Z}/k_3\mathbb{Z}} \left(f^{\Sigma}_{0,0,0,k_3,k_1}\right)^{\sigma_{\mathrm{id}}} + \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}} \left(f_{0,0,0,0,k_1,k_2}\right)^{\sigma_{\mathrm{id}}},
\end{aligned}
$$

from which it follows that Theorem 2.11 holds for general $f \in L^2(N_{\mathbb{Z}}\backslash N)$, since σ_{id} is a homeomorphism.

It should be noted that our choice of coordinate chart σ_{id} for this Fourier series is in some sense arbitrary, and has some drawbacks with regards to certain computations we will need to perform. As we will see later, such computations are easily performed by using a Fourier series for $f^{\sigma_{\mathrm{alt}}}$ instead. Deducing a Fourier series for $f^{\sigma_{\mathrm{alt}}}$ is a simple matter now that we have established a Fourier series $f^{\sigma_{\mathrm{id}}}$. Indeed, by (2.71) and (2.101), we see that for $f \in W$,

$$f^{\sigma_{\mathrm{alt}}}(x_1,x_2,x_3,x_4,x_5,x_6) = \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}} c_{k_1,k_2} e(k_1x_1 + k_2x_2) \tag{2.102a}$$

$$+ \sum_{k_3\in\mathbb{Z}_{\neq 0}} \sum_{[k_1]\in\mathbb{Z}/k_3\mathbb{Z}} \sum_{q_2\in\mathbb{Z}} e\big((k_1 - k_3q_2)x_1 + k_3x_3\big)\psi_{0,0,0,k_3,k_1}(x_2+q_2) \tag{2.102b}$$

$$
\begin{aligned}
+ &\sum_{k_4\in\mathbb{Z}_{\neq 0}} \sum_{[k_3]\in\mathbb{Z}/2k_4\mathbb{Z}} \sum_{k_1\in\mathbb{Z}} \sum_{q_2\in\mathbb{Z}} e\big((k_1 - k_3q_2 + k_4q_2^2)x_1 + (k_3 - 2k_4q_2)x_3 + k_4x_4\big) \\
&\psi_{0,0,k_4,k_3,k_1}(x_2+q_2)
\end{aligned} \tag{2.102c}
$$

$$
\begin{aligned}
+ &\sum_{k_5\in\mathbb{Z}_{\neq 0}} \sum_{[k_4]\in\mathbb{Z}/3k_5\mathbb{Z}} \sum_{k_3\in\mathbb{Z}} \sum_{k_1\in\mathbb{Z}} \sum_{q_2\in\mathbb{Z}} e\big((k_1 - k_3q_2 + k_4q_2^2 + k_5q_2^3)x_1 \\
&+ (k_3 - 2k_4q_2 - 3k_5q_2^2)x_3 + (k_4 + 3k_5q_2)x_4 + k_5x_5\big)\psi_{0,k_5,k_4,k_3,k_1}(x_2+q_2)
\end{aligned} \tag{2.102d}
$$

$$+\sum_{m_5\in\mathbb{Z}_{>0}}\sum_{\substack{[[\gamma]]\in(\Gamma_\beta)_\infty\backslash\Gamma_\beta/(\Gamma_\beta)_\infty\\ [[\gamma]]\neq[[\pm\mathrm{id}]]}}\sum_{[m_4]\in\mathbb{Z}/3m_5\mathbb{Z}}\sum_{m_3\in\mathbb{Z}}\sum_{q_1\in\mathbb{Z}}\sum_{q_3\in\mathbb{Z}}\sum_{[q_2]\in\mathbb{Z}/c\mathbb{Z}} e\Big(\frac{m_3q_2}{c}-\frac{am_4q_2^2}{c}$$
$$-\frac{a^2m_5q_2^3}{c}+\frac{m_3-2m_4(aq_2-cq_3)-3m_5(aq_2-cq_3)^2}{c}x_2$$
$$-\frac{(a-cq_1)(m_4+3m_5(aq_2-cq_3))}{c}x_2^2-\frac{(a-cq_1)^2m_5}{c}x_2^3$$
$$+(m_4+3m_5(aq_2-cq_3))(x_1x_2^2+2x_2x_3+x_4)+(a-cq_1)m_5(2x_1x_2^3+3x_2^2x_3+x_5)$$
$$-cm_5(x_1^2x_2^3+3x_1x_2^2x_3+3x_2x_3^2+x_1x_5+x_6)\Big)$$
$$\psi_{\gamma,m_5,m_4,m_3}(q_1+x_1,aq_2-cq_3+ax_2-c(x_1x_2+x_3)-cq_1(x_1x_2+x_3)), \tag{2.102e}$$

where the above series converges absolutely in $L^2(\mathcal{F})$.

Since $N_{\mathbb{Z}}\backslash N$ is a compact set it follows that $W^\infty = C^\infty(N_{\mathbb{Z}}\backslash N)$ as sets. Furthermore, one can show via an application of Taylor's theorem that the usual Fréchet topology typically assigned to $C^\infty(N_{\mathbb{Z}}\backslash N)$ via the sup-norm agrees with the topology defined on W^∞. Consequently, we have that $W^{-\infty} = D'(N_{\mathbb{Z}}\backslash N)$ both as sets and as topological vector spaces.

In our applications we will need a Fourier series expansion for distributions $\tau \in D'(N_{\mathbb{Z}}\backslash N) = W^{-\infty}$. To do this, we will define some Sobolev spaces. Let $D \in U_{\mathbb{C}}(\mathfrak{g})$ be of order m. We say that $f \in W$ is *weakly D-differentiable* if there exists $h \in W$ such that

$$\int_{N_{\mathbb{Z}}\backslash N} f(n)(r(D)\phi)(n)\,dn = (-1)^m\int_{N_{\mathbb{Z}}\backslash N} h(n)\phi(n)\,dn$$

for all $\phi \in W^\infty$. We say that such $h \in W$ is the *weak D-derivative* of f. Let H^m denote the elements of W which have weak D-derivatives for all D of order $\leq m$. We equip H^m with the inner-product

$$\langle f, h\rangle_m = \sum_{D_i}\int_{N_{\mathbb{Z}}\backslash N}(r(D_i)f)(n)\overline{(r(D_i)h)(n)}\,dn$$

where $\{D_i\}$ is a basis for the subspace of elements of $U_{\mathbb{C}}(\mathfrak{g})$ of order $\leq m$. It is not too difficult to show that one obtains equivalent inner-products when different choices of basis are made. One can also show that H^m is a Hilbert space and that H_0^m, the closure of W^∞ in H^m, is also a Hilbert space. As usual, the topology of H^m (and H_0^m) is defined by the norm

$$\|f\|_m = \langle f, f\rangle_m^{1/2} = \left(\sum_{D_i}\|r(D_i)f\|_2\right)^{1/2},$$

where $f \in H^m$.

By [5, Proposition 5.15], it follows that if $\tau \in D'(N_{\mathbb{Z}}\backslash N)$, then τ is bounded with respect to finitely many of the semi-norms that define the topology on W^∞. As can be seen from the definition of these seminorms, it follows that there exists $m \in \mathbb{Z}_{>0}$ such that τ is bounded with respect to the Sobolev norm $\|\cdot\|_m$ on H_0^m. By the Riesz representation theorem, the map

$$v \mapsto \langle \cdot, v\rangle_m \tag{2.103}$$

from H_0^m onto $(H_0^m)'$ (the space of continuous linear functionals on H_0^m) is an isomorphism of topological vector spaces. The topology on H_0^m to which we refer to is the usual norm topology and the topology of $(H_0^m)'$ to which we refer to is the strong topology, which coincides with the usual operator norm topology. Since τ can be realized as an element of $(H_0^m)'$, there exists $v_\tau \in H_0^m$ which gives τ under the map (2.103); in particular,

$$\tau(\phi) = \langle \phi, v_\tau\rangle_m$$

for $\phi \in W^\infty$.

Observe that the series expansion given in Theorem 2.11 also holds in H_0^m since

$$\begin{aligned}
(r(D)f^{\Sigma}_{\gamma,m_5,m_4,m_3}) &= (r(D)f)^{\Sigma}_{[[\gamma]],m_5,m_4,m_3},\\
(r(D)f^{\Sigma}_{0,k_5,k_4,k_3,k_1}) &= (r(D)f)^{\Sigma}_{0,k_5,k_4,k_3,k_1},
\end{aligned} \tag{2.104}$$

for $D \in U_{\mathbb{C}}(\mathfrak{g})$ and $f \in H^m$. Thus we obtain the series expansion given in Theorem 2.11 for v_τ as an element of H_0^m. Since the injection of W^∞ into H_0^m is continuous and has dense image, when we pull back the resulting series expansion of v_τ to $W^{-\infty}$ we have that the corresponding series for τ converges unconditionally in $W^{-\infty}$ (which is equipped with the strong distribution topology) [21, §23]. In particular, we have the following theorem.

Theorem 2.16. *For $\tau \in W^{-\infty}$ we have*

$$\begin{aligned}
\tau = &\sum_{m_5\in\mathbb{Z}_{>0}} \sum_{\substack{[[\gamma]]\in(\Gamma_\beta)_\infty\backslash\Gamma_\beta/(\Gamma_\beta)_\infty\\ [[\gamma]]\neq[[\pm\mathrm{id}]]}} \sum_{[m_4]\in\mathbb{Z}/3m_5\mathbb{Z}} \sum_{m_3\in\mathbb{Z}} \tau^{\Sigma}_{[[\gamma]],m_5,m_4,m_3}\\
&+ \sum_{k_5\in\mathbb{Z}} \sum_{[k_4]\in\mathbb{Z}/3m_5\mathbb{Z}} \sum_{k_3\in\mathbb{Z}} \sum_{k_1\in\mathbb{Z}} \tau^{\Sigma}_{0,k_5,k_4,k_3,k_1} + \sum_{k_4\in\mathbb{Z}_{\neq 0}} \sum_{[k_3]\in\mathbb{Z}/2k_4\mathbb{Z}} \sum_{k_1\in\mathbb{Z}} \tau^{\Sigma}_{0,0,k_4,k_3,k_1}\\
&+ \sum_{k_3\in\mathbb{Z}_{\neq 0}} \sum_{[k_1]\in\mathbb{Z}/k_3\mathbb{Z}} \tau^{\Sigma}_{0,0,0,k_3,k_1} + \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}} \tau_{0,0,0,0,k_1,k_2},
\end{aligned}$$

where the series converges unconditionally in the strong distribution topology, and where

$$\begin{aligned}
\tau^{\Sigma}_{[[\gamma]],m_5,m_4,m_3}(\phi) &= \langle \phi, (v_\tau)^{\Sigma}_{[[\gamma]],m_5,m_4,m_3}\rangle_m,\\
\tau^{\Sigma}_{0,k_5,k_4,k_3,k_1}(\phi) &= \langle \phi, (v_\tau)^{\Sigma}_{0,k_5,k_4,k_3,k_1}\rangle_m,\\
\tau_{0,0,0,0,k_1,k_2}(\phi) &= \langle \phi, (v_\tau)_{0,0,0,0,k_1,k_2}\rangle_m,
\end{aligned} \tag{2.105}$$

for $\phi \in W^\infty$.

Corollary 2.17. *Let $\phi \in W^{\infty}$. If $\phi^{\sigma_{\mathrm{id}}}$ has the series expansion* (2.101) *then there exists tempered distributions $\rho_{\gamma,m_4,m_4,m_3} : \mathbb{R}^2 \to \mathbb{C}$, $\rho_{0,k_5,k_4,k_3,k_1} : \mathbb{R} \to \mathbb{C}$ and constants $d_{k_1,k_2} \in \mathbb{C}$ such that*

$$
\begin{aligned}
&\tau^{\Sigma}_{[[\gamma]],m_5,m_4,m_3}(\phi) = \rho_{\gamma,m_5,m_4,m_3}(\psi_{\gamma,m_5,m_4,m_3}),\\
&\tau^{\Sigma}_{0,k_5,k_4,k_3,k_1}(\phi) = \rho_{0,k_5,k_4,k_3,k_1}(\psi_{0,k_5,k_4,k_3,k_1}),\\
&\tau_{0,0,0,0,k_1,k_2}(\phi) = d_{k_1,k_2} c_{k_1,k_2}
\end{aligned}
\tag{2.106}
$$

Proof. We shall prove our corollary for the case of $\tau^{\Sigma}_{0,k_5,k_4,k_3,k_1}$. The other cases follow similarly. Observe

$$
\begin{aligned}
&\tau^{\Sigma}_{0,k_5,k_4,k_3,k_1}(\phi) = \langle \phi, (v_\tau)^{\Sigma}_{0,k_5,k_4,k_3,k_1}\rangle_m\\
&= \sum_{D_i} \int_0^1 \cdots \int_0^1 (r(D_i)\phi)^{\sigma_{\mathrm{id}}}(x_1,\ldots,x_6)\overline{((r(D_i)v_\tau)^{\Sigma}_{0,k_5,k_4,k_3,k_1})^{\sigma_{\mathrm{id}}}(x_1,\ldots,x_6)}\\
&\quad dx_6\ldots dx_1
\end{aligned}
$$

We replace $(r(D_i)\phi)^{\sigma_{\mathrm{id}}}$ and $((r(D_i)v_\tau)^{\Sigma}_{0,k_5,k_4,k_3,k_1})^{\sigma_{\mathrm{id}}}$ with their corresponding Fourier series of the form (2.101). Upon simplifying, we find that

$$
\begin{aligned}
&\tau^{\Sigma}_{0,k_5,k_4,k_3,k_1}(\phi) = \langle (v_\tau)^{\Sigma}_{0,k_5,k_4,k_3,k_1}, \phi\rangle_m\\
&= \sum_{D_i} \int_{-\infty}^{\infty} \psi_{r(D_i)\phi;0,k_5,k_4,k_3,k_1}(x_2)\overline{\psi_{r(D_i)v_\tau;0,k_5,k_4,k_3,k_1}(x_2)}\, dx_2.
\end{aligned}
$$

From the definition of π_{0,k_5,k_4,k_3,k_1} (2.92), it follows that

$$
\psi_{r(D)\phi;0,k_5,k_4,k_3,k_1} = \pi_{0,k_5,k_4,k_3,k_1}(D)(\psi_{\phi;0,k_5,k_4,k_3,k_1}).
$$

Observe that by (2.92), it follows that

$$
\begin{aligned}
&\pi_{0,k_5,k_4,k_3,k_1}(\mathcal{P}_5)\psi_{\phi;0,k_5,k_4,k_3,k_1} = (2\pi i)k_5\psi_{\phi;0,k_5,k_4,k_3,k_1},\\
&\pi_{0,k_5,k_4,k_3,k_1}(\mathcal{P}_6)\psi_{\phi;0,k_5,k_4,k_3,k_1} = 0.
\end{aligned}
$$

From this and (2.93), (2.94), (2.95), (2.96), it follows that $\psi_{r(D)\phi;0,k_5,k_4,k_3,k_1}$ is a linear combination of terms of the form $|x|^j\psi^{(k)}$ where $j,k \in \mathbb{Z}_{\geq 0}$. Thus we are left with an expression for $\tau_{0,k_5,k_4,k_3,k_1}(\phi)$ that is evidently a tempered distribution. We denote this distribution by ρ_{0,k_5,k_4,k_3,k_1}. $\square$

In later arguments it will be helpful to prove statements for $\tau \in W^{-\infty}$ by supposing $\tau \in W^{\infty}$ and then applying a density argument. In such instances we will replace $\psi_{\gamma,m_5,m_4,m_3}$, ψ_{0,k_5,k_4,k_3,k_1}, and c_{k_1,k_2} by $\rho_{\gamma,m_5,m_4,m_3}$, ρ_{0,k_5,k_4,k_3,k_1}, and d_{k_1,k_2}, respectively. This is justified since Corollary 2.17 shows that the aforementioned tempered distributions are induced by the Fourier components in (2.101).

2.8 Automorphic Distributions on G_2

Recall that in section 2.1, we concretely identified $G = G_2$ as a subgroup of $\mathrm{SO}(4,3)$. With regards to this concrete realization, let $B(G)$ denote the space of lower triangular matrices of G. Likewise, let

$$a_{u_1,u_2} = \begin{pmatrix} u_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_1u_2^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_1^{-1}u_2^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_1u_2^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_1^{-1}u_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_2^{-1} \end{pmatrix},$$

where $u_1, u_2 \in \mathbb{R}_{>0}$, and

$$m_{\epsilon_1,\epsilon_2} = \begin{pmatrix} \epsilon_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon_1\epsilon_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \epsilon_1\epsilon_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \epsilon_2 \end{pmatrix},$$

where $\epsilon_1, \epsilon_2 \in \{\pm 1\}$. Let

$$A(G) = \{a_{u_1,u_2} : u_1, u_2 \in \mathbb{R}_{>0}\},$$
$$M(G) = \{m_{\epsilon_1,\epsilon_2} : \epsilon_1, \epsilon_2 \in \{\pm 1\}\}.$$

Let $N_-(G)$ be the space of unipotent lower triangular matrices of G. Observe that $B(G) = M(G)A(G)N_-(G)$. We shall often times write $b \in B(G)$ as $b = man_-$ where $m \in M(G)$, $a \in A(G)$, and $n_- \in N_-(G)$.

Observe that every (possibly non-unitary) representation of $B(G)$ is of the form

$$\omega_{(\lambda_1,\lambda_2),(\delta_1,\delta_2)}(b) = \omega_{(\lambda_1,\lambda_2),(\delta_1,\delta_2)}(man_-) = u_1^{\lambda_1-1}\epsilon_1^{\delta_1}u_2^{\lambda_2-1}\epsilon_2^{\delta_2}$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\delta_1, \delta_2 \in \{0,1\}$. In our definition of $\omega_{(\lambda_1,\lambda_2),(\delta_1,\delta_2)}$, we have subtracted 1 in the exponents of u_i so as to maintain certain conventions in representation theory. With this convention, we have that the dual of $\omega_{(\lambda_1,\lambda_2),(\delta_1,\delta_2)}$ is $\omega_{(-\lambda_1,-\lambda_2),(\delta_1,\delta_2)}$.

Let (π, V) be a generic, irreducible, cuspidal representation of G and let i denote an embedding of (π, V) into $L^2(G_{\mathbb{Z}}\backslash G)$. Associated to (π, V) is the dual representation (π', V'), which we defined in section 1.1. For $v \in V^\infty$, define $\tau \in (V')^{-\infty}$ by $\tau(v) = i(v)(e)$, where $e \in G$ is the identity element. Since i is an embedding into $L^2(G_{\mathbb{Z}}\backslash G)$ it follows that τ is $G_{\mathbb{Z}}$-invariant. By a result of Casselman and Wallach ([3] and [22, Theorem 5.8]), we have that there exists $\lambda_1, \lambda_2 \in \mathbb{C}$, $\delta_1, \delta_2 \in \mathbb{Z}/2\mathbb{Z}$, and G-equivariant, topological vector space injections:

$$V^\infty \hookrightarrow V^\infty_{(-\lambda_1,-\lambda_2),(\delta_1,\delta_2)}(G) \quad \text{and} \quad (V')^{-\infty} \hookrightarrow V^{-\infty}_{(\lambda_1,\lambda_2),(\delta_1,\delta_2)}(G),$$

where we write $V^{\pm\infty}_{(\lambda_1,\lambda_2),(\delta_1,\delta_2)}(G)$ for $V^{\pm\infty}_{\omega(\lambda_1,\lambda_2),(\delta_1,\delta_2)}(G)$ (which we defined in section 1.2). We shall abuse notation by identifying $\tau \in (V')^{-\infty}(G)$ as an element of $V^{-\infty}_{(\lambda_1,\lambda_2),(\delta_1,\delta_2)}(G)$ rather than making explicit reference to the above injection.

By section 1.2, we can also identify τ with a distributional section of a line bundle over $G/B(G)$. Hence we can restrict τ to $N = N(G)$ since N gives a dense open set in $G/B(G)$. We shall occasionally abuse notation further by writing τ for this restriction to $N(G)$ when the context is clear, otherwise, we will write $\tau|_N$. Since τ is $G_{\mathbb{Z}}$-invariant, it follows that $\tau|_N$ is then $N_{\mathbb{Z}}$-invariant since $N_{\mathbb{Z}} = N \cap G_{\mathbb{Z}}$. Therefore we can identify $\tau|_N$ with an element of $D'(N_{\mathbb{Z}}\backslash N)$. By our comments following (2.102), we are able to identify τ with an element of $W^{-\infty}$ where $W = L^2(N_{\mathbb{Z}}\backslash N)$ is equipped with the right regular representation. Consequently, τ inherits the Fourier series given in Theorem 2.16. When $\tau \in V^\infty_{(\lambda_1,\lambda_2),(\delta_1,\delta_2)}$ one can show that $\tau|_N \in C^\infty(N_{\mathbb{Z}}\backslash N)$. In this case, $(\tau|_N)^{\sigma_{\mathrm{id}}}$ has the series expansion (2.101). Likewise, $(\tau|_N)^{\sigma_{\mathrm{alt}}}$ has the series expansion (2.102).

For U the unipotent radical of a parabolic subgroup of G, we let du denote a Haar measure for U. Recall that by the definition of cuspidality, we have

$$\int_{G_{\mathbb{Z}} \cap U \backslash U} i(v)(u)du = 0$$

for all U the unipotent radicals of proper parabolic subgroups of G and for all $v \in V^\infty$. Since $\langle \pi'(u)\tau, i(v)\rangle = \langle \tau, \pi(u^{-1})i(v)\rangle = i(v)(n)$, it follows that this cuspidality condition is equivalent to

$$\tau_U = \int_{G_{\mathbb{Z}} \cap U \backslash U} \pi'(u)\tau \, du = 0 \tag{2.107}$$

for all U that are unipotent radicals of a proper parabolic subgroup of G. The prior integral is well-defined since it takes values in $V^{-\infty}$ which is a complete, locally convex, Hausdorff topological vector space in which G acts continuously.

Lemma 2.18. *Let $\tau \in V^{-\infty}_{(\lambda_1,\lambda_2),(\delta_1,\delta_2)}(G)$, cuspidal, and $G_{\mathbb{Z}}$-invariant. Then*

(a) $d_{k_1,0} = d_{0,k_2} = 0$ *for all* $k_1, k_2 \in \mathbb{Z}$,

(b) $d_{k_1,k_2} = \mathrm{sgn}(\epsilon_1)^{\delta_1}\mathrm{sgn}(\epsilon_2)^{\delta_2} d_{\epsilon_2 k_1, \epsilon_1 k_2}$ *for all* $k_1, k_2 \in \mathbb{Z}$.

Proof. Let U_α denote the unipotent radical of the maximal parabolic subgroup of G which has trivial intersection with the one parameter subgroup for the α root. We can parametrize U_α by $n^{-1} = Y_6Y_5Y_4Y_3Y_2$, where $y_i \in \mathbb{R}$. By solving for p_i in

$$P_6P_5P_4P_3P_2P_1 = Y_6Y_5Y_4Y_3Y_2X_6X_5X_4X_3X_2X_1,$$

and utilizing (2.101) to evaluate $\tau^{\sigma_{\mathrm{id}}}(p_1, \ldots, p_6)$, we find that

$$\tau_{U_\alpha} = \int_0^1 \int_0^1 \sum_{k_1,k_2 \in \mathbb{Z}} d_{k_1,k_2} e(k_1x_1 + k_2(x_2 + y_2))\, dy_2\, dy_1 = \sum_{k_1 \in \mathbb{Z}} d_{k_1,0} e(k_1x_1).$$

Since we have assumed that $\tau_{U_\alpha} = 0$, it follows that $d_{k_1,0} = 0$ for all $k_1 \in \mathbb{Z}$. By a similar argument for U_β, the unipotent radical of the maximal parabolic subgroup of G which has trivial intersection with the one parameter subgroup for the β root, we obtain that $d_{0,k_2} = 0$ for all $k_2 \in \mathbb{Z}$. This proves part (a).

For part (b), it suffices by a density argument to suppose $\tau \in V^\infty_{(\lambda_1,\lambda_2),(\delta_1,\delta_2)}$. Observe

$$\begin{aligned} &m^{-1}_{\epsilon_1,\epsilon_2} X_6X_5X_4X_3X_2X_1 m_{\epsilon_1,\epsilon_2} \\ &\quad = R_6(\epsilon_1x_6)R_5(\epsilon_1\epsilon_2x_5)R_4(\epsilon_2x_4)R_3(\epsilon_1\epsilon_2x_3)R_2(\epsilon_1x_2)R_1(\epsilon_2x_1), \end{aligned}$$

and $\tau(gm^{-1}_{\epsilon_1,\epsilon_2}) = \mathrm{sgn}(\epsilon_1)^{\delta_1}\mathrm{sgn}(\epsilon_2)^{\delta_2}\tau(g)$. Since $M(G_2)$ is a subgroup of $G_\mathbb{Z}$ it follows that

$$\tau(g) = \pi'(m_{\epsilon_1,\epsilon_2})\tau(g) = \tau(m^{-1}_{\epsilon_1,\epsilon_2} g m_{\epsilon_1,\epsilon_2} m^{-1}_{\epsilon_1,\epsilon_2}) = \mathrm{sgn}(\epsilon_1)^{\delta_1}\mathrm{sgn}(\epsilon_2)^{\delta_2}\tau(m^{-1}_{\epsilon_1,\epsilon_2} g m_{\epsilon_1,\epsilon_2}).$$

Thus

$$\begin{aligned} &\tau^{\sigma_{\mathrm{id}}}(x_1, x_2, x_3, x_4, x_5, x_6) \\ &\quad = \mathrm{sgn}(\epsilon_1)^{\delta_1}\mathrm{sgn}(\epsilon_2)^{\delta_2}\tau^{\sigma_{\mathrm{id}}}(\epsilon_2x_1, \epsilon_1x_2, \epsilon_1\epsilon_2x_3, \epsilon_2x_4, \epsilon_1\epsilon_2x_5, \epsilon_1x_6). \end{aligned} \tag{2.108}$$

When we replace $\tau^{\sigma_{\mathrm{id}}}$ with its series representation (2.101), and integrate against the character

$$(x_1, x_2, x_3, x_4, x_5, x_6) \mapsto e(-k_1x_1 - k_2x_2)$$

we obtain part (b). $\square$

Let G_α denote an embedded SL_2 of the Levi subgroup for the root α; in particular, let G_α

be the group consisting of elements of the following form:

$$h = \begin{pmatrix} a & b & 0 & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a^2 & 2ab & -b^2 & 0 & 0 \\ 0 & 0 & ac & 1+2bc & -bd & 0 & 0 \\ 0 & 0 & -c^2 & -2cd & d^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & c & d \end{pmatrix}, \tag{2.109}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Let w_α denote the Weyl group reflection for the α root which comes from setting $a = d = 0$, $-b = c = 1$ for h in (2.109). Let w_β denote the Weyl group reflection for the β root which comes from setting $a = d = 0$, $-b = c = 1$ for h in (2.2).

In what follows, we will use the following normalization for the Fourier transform:

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e(-x\xi)\,dx,$$

where $f : \mathbb{R} \to \mathbb{C}$ in the above integral converges. Recall that $\rho_{0,0,1,0,0}$, $\rho_{0,0,0,1,0}$ are the distributional analogue of $\psi_{0,0,1,0,0}$, $\psi_{0,0,0,1,0}$ that occurs in (2.101c), (2.101b) and (2.102c), (2.102b). The following lemma gives us important equalities relating these terms to the distributional constant terms.

Lemma 2.19. *For* $\tau \in V^{-\infty}_{(\lambda_1,\lambda_2),(\delta_1,\delta_2)}(G)$, *we have*

(a) $\rho_{0,0,1,0,0}(x_2) = \operatorname{sgn}(-x_2)^{\delta_2}|x_2|^{\lambda_2-1}\rho_{0,0,0,1,0}(-x_2^{-1})$,

(b) $\widehat{\rho}_{0,0,0,1,0}(x_1) = \operatorname{sgn}(-x_1)^{\delta_1}|x_1|^{\lambda_1-1}\sum_{k_1\in\mathbb{Z}} c_{k_1,1}e\left(\frac{-k_1}{x_1}\right)$

as equalities between distributions on $\mathbb{R}_{\neq 0}$.

Proof. By a density argument it suffices to suppose $\tau \in V^{\infty}_{(\lambda_1,\lambda_2),(\delta_1,\delta_2)}$. Observe

$$\begin{aligned} w_\alpha^{-1}X_1X_4X_3X_2X_5X_6 = R_1(-x_5)&R_4(x_3)R_3(-x_4)R_2(-x_2^{-1})R_5(x_1) \\ &R_6(-3x_3x_4 + x_1x_5 + x_6)a_{1,|x_2|^{-1}}m_{1,\operatorname{sgn}(-x_2)}n_- \end{aligned}$$

where $n_- \in N_-$. By this and the $G_\mathbb{Z}$-invariance of f, we have that

$$\begin{aligned} &\tau^{\sigma_{\mathrm{alt}}}(x_1, x_2, x_3, x_4, x_5, x_6) \\ &= \operatorname{sgn}(-x_2)^{\delta_2}|x_2|^{\lambda_2-1}\tau^{\sigma_{\mathrm{alt}}}\left(-x_5, -x_2^{-1}, -x_4, x_3, x_1, -3x_3x_4 + x_1x_5 + x_6\right) \end{aligned}$$

as an equality between distributions on $\mathbb{R} \times \mathbb{R}_{\neq 0} \times \mathbb{R}^4$. By replacing both instances of $\tau^{\sigma_{\text{alt}}}$ in the above equation with (2.102) and integrating against $(x_1, x_3, x_4, x_5, x_6) \mapsto e(-x_4)$, we obtain part (a).

Next, observe

$$\begin{aligned} w_\beta^{-1} X_6 X_5 X_4 X_3 X_2 X_1 = R_6(3x_2^2 x_3 + x_5) R_5(-3x_2 x_3^2 - x_6) R_4(2x_2 x_3 + x_4) R_3(x_2) \\ R_2(-x_3) R_1(-x_1^{-1}) a_{|x_1|^{-1},1} m_{\operatorname{sgn}(-x_1),1} n_- \end{aligned}$$

where $n_- \in N_-$. By this, and the $G_{\mathbb{Z}}$-invariance of f, we have that

$$\begin{aligned} &\tau^{\sigma_{\text{id}}}(x_1, x_2, x_3, x_4, x_5, x_6) \\ &= \operatorname{sgn}(-x_1)^{\delta_1} |x_1|^{\lambda_1 - 1} \tau^{\sigma_{\text{id}}}\left(-x_1^{-1}, -x_3, x_2, 2x_2 x_3 + x_4, -3x_2 x_3^2 - x_6, 3x_2^2 x_3 + x_5\right) \end{aligned}$$

as an equality between distributions on $\mathbb{R}_{\neq 0} \times \mathbb{R}^5$. By replacing both instances of $\tau^{\sigma_{\text{id}}}$ in the above equation with (2.101) and then integrating against $(x_2, x_3, x_4, x_5, x_6) \mapsto e(-x_3)$, we find that

$$\int_0^1 \sum_{q_2 \in \mathbb{Z}} e(-x_1(x_2 + q_2)) \rho_{0,0,0,1,0}(x_2 + q_2)\, dx_2 = \operatorname{sgn}(-x_1)^{\delta_1} |x_1|^{\lambda_1 - 1} \sum_{k_1 \in \mathbb{Z}} c_{k_1,1} e\left(\frac{-k_1}{x_1}\right). \quad (2.110)$$

If we let $\widehat{\rho}_{0,0,0,1,0}$ denote the Fourier transform of $\rho_{0,0,0,1,0}$ then upon simplifying the left-hand side of (2.110), we obtain part (b). $\square$

Chapter 3
Distributions on Other Groups

All Lie groups are assumed to be over $\mathbb{R}$ in this chapter, unless stated otherwise.

3.1 A Double Cover of $\mathrm{SL}_2^{\pm}$

In this section we will analyze the distributional principal series representations for double cover groups of $\mathrm{SL}_2^{\pm}$ and SL_2. We begin by reviewing the multiplication laws for such groups and then proceed to analyze the *unbounded realizations* of these distributional principal series representation spaces. By studying these unbounded realizations, we can determine when certain distributional principal series representation spaces lie naturally within other distributional principal series representation spaces for larger groups.

Let $\mathrm{SL}_2^{\pm} = \{g \in \mathrm{GL}_2 : \det(g) = \pm 1\}$. As a set, let $\widetilde{\mathrm{SL}_2^{\pm}} = \mathrm{SL}_2^{\pm} \times \{\pm 1\}$. Recall that the Hilbert symbol for $\mathbb{R}$ is given by the following formula:

$$(x, y)_H = \begin{cases} -1 & \text{if } x < 0 \text{ and } y < 0 \\ 1 & \text{otherwise,} \end{cases}$$

where $x, y \in \mathbb{R}_{\neq 0}$. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2^{\pm}$, define

$$X\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} c & \text{if } c \neq 0 \\ d & \text{if } c = 0. \end{cases}$$

For $g_1, g_2 \in \mathrm{SL}_2^{\pm}$, define

$$\alpha(g_1, g_2) = \left(\frac{X(g_1 g_2)}{X(g_1)}, \frac{X(g_1 g_2)}{X(g_2)}\right)_H \left(\det(g_1), \frac{X(g_1 g_2)}{X(g_1)}\right)_H.$$

One can show that α is a 2-cocycle [10], and thus we give $\widetilde{\mathrm{SL}_2^{\pm}}$ the structure of a group by defining the following multiplication law:

$$(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1 g_2, \alpha(g_1, g_2)\epsilon_1 \epsilon_2),$$

where $g_1, g_2 \in \mathrm{SL}_2^{\pm}$ and $\epsilon_1, \epsilon_2 \in \{\pm 1\}$. In addition to identifying $\widetilde{\mathrm{SL}_2^{\pm}}$ as a group, one can give $\widetilde{\mathrm{SL}_2^{\pm}}$ the structure of a smooth manifold by requiring the map

$$\widetilde{g} = (g, \epsilon) \mapsto g \tag{3.1}$$

to be a smooth covering map from $\widetilde{\mathrm{SL}_2^{\pm}}$ onto $\mathrm{SL}_2^{\pm}$.

Let

$$\begin{aligned} m = m_{\epsilon_1,\epsilon_2} &= \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}, \quad a = a_u = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \\ n = n_x &= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \text{ and } n_- = n_{-,x} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \end{aligned}$$

where $\epsilon_1, \epsilon_2 \in \{\pm 1\}$, $u \in \mathbb{R}_{>0}$, and $x \in \mathbb{R}$. As indicated in the above definitions, sometimes we will suppress the variables ϵ_1, ϵ_2, u, and x in our notation. Let $B(\mathrm{SL}_2^{\pm})$ denote the subgroup of lower triangular matrices of $\mathrm{SL}_2^{\pm}$, and let

$$N(\mathrm{SL}_2^{\pm}) = \{n = n_x : x \in \mathbb{R}\}.$$

Under the Langlands decomposition, $B(\mathrm{SL}_2^{\pm}) = M(\mathrm{SL}_2^{\pm})A(\mathrm{SL}_2^{\pm})N_-(\mathrm{SL}_2^{\pm})$ where

$$\begin{aligned} M(\mathrm{SL}_2^{\pm}) &= \{m_{\epsilon_1,\epsilon_2} : \epsilon_1, \epsilon_2 \in \{\pm 1\}\}, \\ A(\mathrm{SL}_2^{\pm}) &= \{a_u : u \in \mathbb{R}_{>0}\}, \\ N_-(\mathrm{SL}_2^{\pm}) &= \{n_{-,x} : x \in \mathbb{R}\}. \end{aligned}$$

Next we define various subgroups of $\widetilde{\mathrm{SL}_2^{\pm}}$ analogous to the ones defined for $\mathrm{SL}_2^{\pm}$. Let

$$\begin{aligned} &\widetilde{m} = \widetilde{m}_{\epsilon_1,\epsilon_2,\epsilon_3} = (m_{\epsilon_1,\epsilon_2}, \epsilon_3), \quad \widetilde{a} = \widetilde{a}_u = (a_u, 1), \\ &\widetilde{n} = \widetilde{n}_x = (n_x, 1), \text{ and } \widetilde{n}_- = \widetilde{n}_{-,x} = (n_{-,x}, 1), \end{aligned} \tag{3.2}$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}$, $u \in \mathbb{R}_{>0}$, and $x \in \mathbb{R}$. Let

$$\begin{aligned} &M(\widetilde{\mathrm{SL}_2^{\pm}}) = \{\widetilde{m}_{\epsilon_1,\epsilon_2,\epsilon_3} : \epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}\}, \quad A(\widetilde{\mathrm{SL}_2^{\pm}}) = \{\widetilde{a}_u : u \in \mathbb{R}_{>0}\}, \\ &N_-(\widetilde{\mathrm{SL}_2^{\pm}}) = \{\widetilde{n}_{-,x} : x \in \mathbb{R}\}, \quad N(\widetilde{\mathrm{SL}_2^{\pm}}) = \{\widetilde{n}_x : x \in \mathbb{R}\}. \end{aligned}$$

One can check that $M(\widetilde{\mathrm{SL}_2^{\pm}})$, $A(\widetilde{\mathrm{SL}_2^{\pm}})$, $N_-(\widetilde{\mathrm{SL}_2^{\pm}})$, and $N(\widetilde{\mathrm{SL}_2^{\pm}})$ are subgroups of $\widetilde{\mathrm{SL}_2^{\pm}}$. Let $B(\widetilde{\mathrm{SL}_2^{\pm}}) = M(\widetilde{\mathrm{SL}_2^{\pm}})A(\widetilde{\mathrm{SL}_2^{\pm}})N_-(\widetilde{\mathrm{SL}_2^{\pm}})$.

For $\epsilon \in \{\pm 1\}$, let $\sigma_{\epsilon,1} : M(\widetilde{\mathrm{SL}_2^{\pm}}) \to \mathrm{GL}_2(\mathbb{C})$ denote the representation of $M(\widetilde{\mathrm{SL}_2^{\pm}})$ defined by the following equations:

$$\sigma_{\epsilon,1}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},\pm1\right)=\begin{pmatrix}\pm1&0\\0&\pm1\end{pmatrix},\qquad \sigma_{\epsilon,1}\left(\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\pm1\right)=\begin{pmatrix}0&\pm\epsilon\\\mp\epsilon&0\end{pmatrix},$$
$$\sigma_{\epsilon,1}\left(\begin{pmatrix}-1&0\\0&1\end{pmatrix},\pm1\right)=\begin{pmatrix}\pm\epsilon&0\\0&\mp\epsilon\end{pmatrix},\qquad \sigma_{\epsilon,1}\left(\begin{pmatrix}1&0\\0&-1\end{pmatrix},\pm1\right)=\begin{pmatrix}0&\mp1\\\mp1&0\end{pmatrix}. \tag{3.3}$$

Likewise, let $\sigma_{\epsilon,-1}:M(\widetilde{\mathrm{SL}_2^{\pm}})\to\mathrm{GL}_2(\mathbb{C})$ denote the representation of $M(\widetilde{\mathrm{SL}_2^{\pm}})$ defined by the following equations:

$$\sigma_{\epsilon,-1}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},\pm1\right)=\begin{pmatrix}\pm1&0\\0&\pm1\end{pmatrix},\qquad \sigma_{\epsilon,-1}\left(\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\pm1\right)=\begin{pmatrix}0&\pm\epsilon\\\mp\epsilon&0\end{pmatrix},$$
$$\sigma_{\epsilon,-1}\left(\begin{pmatrix}-1&0\\0&1\end{pmatrix},\pm1\right)=\begin{pmatrix}0&\mp1\\\mp1&0\end{pmatrix},\qquad \sigma_{\epsilon,-1}\left(\begin{pmatrix}1&0\\0&-1\end{pmatrix},\pm1\right)=\begin{pmatrix}\mp\epsilon&0\\0&\pm\epsilon\end{pmatrix}; \tag{3.4}$$

and for $\delta\in\mathbb{Z}/2\mathbb{Z}$, let $\varsigma_\delta:M(\widetilde{\mathrm{SL}_2^{\pm}})\to\mathrm{GL}_1(\mathbb{C})=\mathbb{C}^*$ denote the representation of $M(\widetilde{\mathrm{SL}_2^{\pm}})$ defined by the following equations:

$$\varsigma_\delta\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},\pm1\right)=1,\qquad \varsigma_\delta\left(\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\pm1\right)=(-1)^\delta,$$
$$\varsigma_\delta\left(\begin{pmatrix}-1&0\\0&1\end{pmatrix},\pm1\right)=(-1)^\delta,\qquad \varsigma_\delta\left(\begin{pmatrix}1&0\\0&-1\end{pmatrix},\pm1\right)=1. \tag{3.5}$$

For $\nu\in\mathbb{C}$, let

$$\mu_\nu(\widetilde{a}_u)=u^{\nu-1}. \tag{3.6}$$

Notice that μ_ν is a quasi-character of $A(\widetilde{\mathrm{SL}_2^{\pm}})$. For $\widetilde{b}=\widetilde{m}\widetilde{a}\widetilde{n}_-\in B(\widetilde{\mathrm{SL}_2^{\pm}})$, $\epsilon_i\in\{\pm1\}$, $\delta\in\{0,1\}$, we define

$$\omega_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{b})=\sigma_{\epsilon_1,\epsilon_2}(\widetilde{m})\mu_\nu(\widetilde{a}),$$
$$\omega_{\delta,\nu}(\widetilde{b})=\varsigma_\delta(\widetilde{m})\mu_\nu(\widetilde{a}),$$

both of which are representations of $B(\widetilde{\mathrm{SL}_2^{\pm}})$. In order to simplify notation, we will write $V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{\mathrm{SL}_2^{\pm}})$ for $V^{-\infty}_{\omega_{(\epsilon_1,\epsilon_2),\nu}}(\widetilde{\mathrm{SL}_2^{\pm}})$ and write $V^{-\infty}_{\delta,\nu}(\widetilde{\mathrm{SL}_2^{\pm}})$ for $V^{-\infty}_{\omega_{\delta,\nu}}(\widetilde{\mathrm{SL}_2^{\pm}})$ (see section 1.2 for the definition of these spaces). By using the standard basis for $\mathbb{C}^2$, we can write $f\in V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{\mathrm{SL}_2^{\pm}})$ as $f=(f_1,f_2)=\begin{pmatrix}f_1\\f_2\end{pmatrix}$ where $f_i\in C^{-\infty}(\widetilde{\mathrm{SL}_2^{\pm}},\mathbb{C})$.

As discussed in section 1.2, each $f=(f_1,f_2)\in V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{\mathrm{SL}_2^{\pm}})$ can be identified with a distributional section s_f of a vector bundle over $\widetilde{\mathrm{SL}_2^{\pm}}/B(\widetilde{\mathrm{SL}_2^{\pm}})$. Since the group $N(\widetilde{\mathrm{SL}_2^{\pm}})$ maps injectively into $\widetilde{\mathrm{SL}_2^\pm}/B(\widetilde{\mathrm{SL}_2^\pm})$, we shall abuse notation by writing this subset of $\widetilde{\mathrm{SL}_2^\pm}/B(\widetilde{\mathrm{SL}_2^\pm})$ as $N(\widetilde{\mathrm{SL}_2^\pm})$. Since as a subset of $\widetilde{\mathrm{SL}_2^\pm}/B(\widetilde{\mathrm{SL}_2^\pm})$, $N(\widetilde{\mathrm{SL}_2^\pm})$ is an open set, it follows that we can consider the restriction of s_f to $N(\widetilde{\mathrm{SL}_2^\pm})$. We denote this restriction by $f_0 = ((f_1)_0, (f_2)_0)$.[1] Since we can parameterize $N(\widetilde{\mathrm{SL}_2^\pm})$ by $x \mapsto \widetilde{n}_x$, we can also identify f_0 as a distribution on $\mathbb{R}$. Similarly, we let $f_\infty = ((f_1)_\infty, (f_2)_\infty)$ denote the restriction of s_f to $\widetilde{s}^{-1}N(\widetilde{\mathrm{SL}_2^\pm})$, where

$$\widetilde{s} = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1 \right), \tag{3.7}$$

and where $\widetilde{s}^{-1}N(\widetilde{\mathrm{SL}_2^\pm})$ is identified as a subset of $\widetilde{\mathrm{SL}_2^\pm}/B(\widetilde{\mathrm{SL}_2^\pm})$. Since we can parameterize $\widetilde{s}^{-1}N(\widetilde{\mathrm{SL}_2^\pm})$ by $x \mapsto \widetilde{s}^{-1}\widetilde{n}_x$, we can also identify f_∞ as a distribution $\mathbb{R}$. Since $N(\widetilde{\mathrm{SL}_2^\pm})$ and $\widetilde{s}^{-1}N(\widetilde{\mathrm{SL}_2^\pm})$ cover $\widetilde{\mathrm{SL}_2^\pm}/B(\widetilde{\mathrm{SL}_2^\pm})$, it follows that f is completely determined by f_0 and f_∞.

Since for $x \neq 0$,

$$\begin{aligned} &\widetilde{s}^{-1} \cdot \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1 \right) \\ &= \left(\begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix}, 1 \right) \cdot \left(\begin{pmatrix} |x|^{-1} & 0 \\ \mathrm{sgn}(x) & |x| \end{pmatrix}, 1 \right) \cdot \left(\begin{pmatrix} \mathrm{sgn}(-x) & 0 \\ 0 & \mathrm{sgn}(-x) \end{pmatrix}, 1 \right), \end{aligned} \tag{3.8}$$

then for $f \in V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{\mathrm{SL}_2^\pm})$ we have

$$f_\infty(x) = \begin{pmatrix} (f_1)_\infty(x) \\ (f_2)_\infty(x) \end{pmatrix} = \begin{cases} |x|^{\nu-1} \begin{pmatrix} -\epsilon_1 (f_2)_0(-x^{-1}) \\ \epsilon_1 (f_1)_0(-x^{-1}) \end{pmatrix} & \text{if } x > 0 \\ |x|^{\nu-1} \begin{pmatrix} (f_1)_0(-x^{-1}) \\ (f_2)_0(-x^{-1}) \end{pmatrix} & \text{if } x < 0, \end{cases} \tag{3.9}$$

as distributions on $\mathbb{R}_{\neq 0}$. Conversely, when given distributions $(f_1)_0, (f_2)_0, (f_1)_\infty, (f_2)_\infty \in C^{-\infty}(\mathbb{R})$ which satisfy (3.9), one can define a unique element $f \in V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{\mathrm{SL}_2^\pm})$. Thus

$$\begin{aligned} V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{\mathrm{SL}_2^\pm}) \cong \Big\{ &((f_1)_0, (f_2)_0, (f_1)_\infty, (f_2)_\infty) \in C^{-\infty}(\mathbb{R})^4 : \\ &(f_1)_0, (f_2)_0, (f_1)_\infty, \text{ and } (f_2)_\infty \text{ satisfy (3.9) as distributions on } \mathbb{R}_{\neq 0} \Big\}. \end{aligned} \tag{3.10}$$

This space on the right-hand side of (3.10) is known as the *unbounded model* for $V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{\mathrm{SL}_2^\pm})$.

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[1] We hope that the reader will not be too confused by this notation, but we felt that using a notation such as $(s_f)_0$ would become too cumbersome.

Statements analogous to (3.9) and (3.10) also hold for $V^{-\infty}_{\delta,\nu}(\widetilde{\mathrm{SL}_2^{\pm}})$. In particular, if we let f_0 denote the restriction of s_f to $N(\widetilde{\mathrm{SL}_2^{\pm}})$, and let f_∞ denote the restriction of s_f to $\widetilde{s}^{-1}N(\widetilde{\mathrm{SL}_2^{\pm}})$, then by (3.8),

$$f_\infty(x) = \mathrm{sgn}(-x)^{\delta}|x|^{\nu-1}f_0(-x^{-1}) \tag{3.11}$$

as distributions on $\mathbb{R}_{\neq 0}$. Hence

$$\begin{aligned} &V^{-\infty}_{\delta,\nu}(\widetilde{\mathrm{SL}_2^{\pm}}) \\ &\cong \Big\{(f_0, f_\infty) \in C^{-\infty}(\mathbb{R})^2 : f_0 \text{ and } f_\infty \text{ satisfy (3.11) as distributions on } \mathbb{R}_{\neq 0}\Big\}. \end{aligned} \tag{3.12}$$

Define $\widetilde{\mathrm{SL}}_2$ to be the inverse image of SL_2 under (3.1). We write the intersection of $M(\widetilde{\mathrm{SL}_2^{\pm}})$, $A(\widetilde{\mathrm{SL}_2^{\pm}})$, and $N(\widetilde{\mathrm{SL}_2^{\pm}})$ with $\widetilde{\mathrm{SL}}_2$ as $M(\widetilde{\mathrm{SL}}_2)$, $A(\widetilde{\mathrm{SL}}_2)$, and $N(\widetilde{\mathrm{SL}}_2)$, respectively.[2] For $\epsilon \in \{\pm 1\}$, let $\sigma_\epsilon : M(\widetilde{\mathrm{SL}}_2) \to \mathbb{C}^*$ denote the representation of $M(\widetilde{\mathrm{SL}}_2)$ defined by the following equations:

$$\sigma_\epsilon\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm 1\right) = \pm 1, \qquad \sigma_\epsilon\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \pm 1\right) = \mp \epsilon\, i, \tag{3.13}$$

We define

$$\omega_{(\epsilon),\nu}(\widetilde{b}) = \sigma_\epsilon(\widetilde{m})\mu_\nu(\widetilde{a}),[3] \tag{3.14}$$

which is a representation of $B(\widetilde{\mathrm{SL}}_2)$; recall that μ_ν is defined in (3.6). In order to simplify notation, we will write $V^{-\infty}_{(\epsilon),\nu}(\widetilde{\mathrm{SL}}_2)$ for $V^{-\infty}_{\omega_{(\epsilon),\nu}}(\widetilde{\mathrm{SL}}_2)$. Let

$$\begin{aligned} &\log(z) \text{ denote the branch cut of the logarithm whose imaginary} \\ &\text{part lies in } (-\pi, \pi], \text{ and let } z^t = \exp(\log(z)t) \text{ where } t \in \mathbb{C}, z \in \mathbb{C}^*. \end{aligned} \tag{3.15}$$

For $f \in V^{-\infty}_{(\epsilon),\nu}(\widetilde{\mathrm{SL}}_2)$, let f_0 denote the restriction of s_f to $N(\widetilde{\mathrm{SL}}_2) \cong \mathbb{R}$, and let f_∞ denote the restriction of s_f to $\widetilde{s}^{-1}N(\widetilde{\mathrm{SL}}_2) \cong \mathbb{R}$. Since $N(\widetilde{\mathrm{SL}}_2)$ and $\widetilde{s}^{-1}N(\widetilde{\mathrm{SL}}_2)$ cover $\widetilde{\mathrm{SL}}_2/B(\widetilde{\mathrm{SL}}_2)$, it follows that f is completely determined by f_0 and f_∞. By (3.8), we find that for $f \in V^{-\infty}_{(\epsilon),\nu}(\widetilde{\mathrm{SL}}_2)$,

$$f_\infty(x) = \mathrm{sgn}(-x)^{\epsilon/2}|x|^{\nu-1}f_0(-x^{-1}) \tag{3.16}$$

as distributions on $\mathbb{R}_{\neq 0}$. Conversely, when given distributions $f_0, f_\infty \in C^{-\infty}(\mathbb{R})$ which satisfy (3.16), one can define a unique element $f \in V^{-\infty}_{(\epsilon),\nu}(\widetilde{\mathrm{SL}}_2)$. Thus

$$\begin{aligned} &V^{-\infty}_{(\epsilon),\nu}(\widetilde{\mathrm{SL}}_2) \\ &\cong \{(f_0, f_\infty) \in C^{-\infty}(\mathbb{R})^2 : f_0 \text{ and } f_\infty \text{ satisfy (3.16) as distributions on } \mathbb{R}_{\neq 0}\}. \end{aligned} \tag{3.17}$$

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[2]This is somewhat redundant since $N(\widetilde{\mathrm{SL}}_2) = N(\widetilde{\mathrm{SL}_2^{\pm}})$, but is notationally consistent.

[3]The parenthesis around ϵ allow us to distiguish $\omega_{(\epsilon),\nu}$ from $\omega_{\delta,\nu}$

Let $\mathcal{L} : V^{-\infty}_{(\epsilon_1),\nu}(\widetilde{\mathrm{SL}}_2) \to V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{\mathrm{SL}_2^{\pm}})$ denote the map given by

$$\mathcal{L}\left(\begin{pmatrix} f_0 \\ f_\infty \end{pmatrix}\right) = \begin{pmatrix} f_0 \\ -if_0 \\ f_\infty \\ -if_\infty \end{pmatrix}, \tag{3.18}$$

where $f \in V^{-\infty}_{(\epsilon_1),\nu}(\widetilde{\mathrm{SL}}_2)$. Notice that we have used the isomorphisms in (3.10) and (3.17) in order to define this map. We see from (3.16) and (3.9) that $\mathcal{L}$ maps into $V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{\mathrm{SL}_2^{\pm}})$ as claimed. Furthermore, it follows that $\mathcal{L}$ is equivariant under the left regular representation of $\widetilde{\mathrm{SL}}_2$. Thus by (3.18), we are able to identify

$$V^{-\infty}_{(\epsilon_1),\nu}(\widetilde{\mathrm{SL}}_2) \subset V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{\mathrm{SL}_2^{\pm}}). \tag{3.19}$$

Let

$$\widetilde{m}_* = \left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, 1\right).$$

The following lemma allows us to describe $\mathcal{L}$ in terms of distributions on $\widetilde{\mathrm{SL}}_2$ and $\widetilde{\mathrm{SL}_2^{\pm}}$, as opposed to describing $\mathcal{L}$ in terms of restrictions of distributional sections of vector bundles.

Lemma 3.1. *Let $f \in V^{-\infty}_{(\epsilon_1),\nu}(\widetilde{SL}_2)$, and let $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{SL}_2)$ be the image of f under $\mathcal{L}$. If $\epsilon_2 = 1$, then*

$$(f_1)|_{\widetilde{SL}_2} = f, \quad (f_1)|_{\widetilde{SL}_2\widetilde{m}_*} = \epsilon_1 f, \quad (f_2)|_{\widetilde{SL}_2} = -if, \quad \textit{and} \quad (f_2)|_{\widetilde{SL}_2\widetilde{m}_*} = \epsilon_1 if;$$

and if $\epsilon_2 = -1$, then

$$(f_1)|_{\widetilde{SL}_2} = f, \quad (f_1)|_{\widetilde{SL}_2\widetilde{m}_*} = if, \quad (f_2)|_{\widetilde{SL}_2} = -if, \quad \textit{and} \quad (f_2)|_{\widetilde{SL}_2\widetilde{m}_*} = -f.$$

Proof. It suffices by a density argument to suppose that $f \in V^{\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{\mathrm{SL}}_2)$. As described in section 1.2, for $(f_1, f_2) \in V^{\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{\mathrm{SL}_2^{\pm}})$ there exists a corresponding section of a vector bundle $s_{(f_1,f_2)} : \widetilde{\mathrm{SL}_2^{\pm}}/B(\widetilde{\mathrm{SL}_2^{\pm}}) \to ((\widetilde{\mathrm{SL}_2^{\pm}}/B(\widetilde{\mathrm{SL}_2^{\pm}})) \times \mathbb{C}^2)/\sim$, where

$$s_{(f_1,f_2)}\Big(\widetilde{g}B(\widetilde{\mathrm{SL}_2^{\pm}})\Big) = \left\{\left(\widetilde{g}\widetilde{b}, \omega_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{b}^{-1})\begin{pmatrix} f_1(\widetilde{g}) \\ f_2(\widetilde{g}) \end{pmatrix}\right) : \widetilde{b} \in B(\widetilde{\mathrm{SL}_2^{\pm}})\right\}, \tag{3.20}$$

and where $\sim$ is an equivalence relation on $(\widetilde{\mathrm{SL}_2^{\pm}}/B(\widetilde{\mathrm{SL}_2^{\pm}})) \times \mathbb{C}^2$ defined in (1.2). If (f_1, f_2) is the image of f under $\mathcal{L}$, then it follows that

$$s_{(f_1,f_2)}\Big(\widetilde{g}B(\widetilde{\mathrm{SL}_2^{\pm}})\Big) = \left\{\left(\widetilde{g}\widetilde{b}, \omega_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{b}^{-1})\begin{pmatrix} f(\widetilde{g}) \\ -if(\widetilde{g}) \end{pmatrix}\right) : \widetilde{b} \in B(\widetilde{\mathrm{SL}_2^{\pm}})\right\}, \tag{3.21}$$

for $\widetilde{g} \in \widetilde{\mathrm{SL}}_2$; recall that every coset of $\widetilde{\mathrm{SL}_2^{\pm}}/B(\widetilde{\mathrm{SL}_2^{\pm}})$ has a representative of the form $\widetilde{g}B(\widetilde{\mathrm{SL}_2^{\pm}})$ where $\widetilde{g} \in \widetilde{\mathrm{SL}}_2$. By comparing (3.20) and (3.21), we see that

$$\begin{pmatrix} f_1(\widetilde{g}) \\ f_2(\widetilde{g}) \end{pmatrix} = \begin{pmatrix} f(\widetilde{g}) \\ -if(\widetilde{g}) \end{pmatrix},$$

for $\widetilde{g} \in \widetilde{\mathrm{SL}}_2$. This proves half of our lemma. To prove the other half, one uses the transformation law for $V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{\mathrm{SL}_2^{\pm}})$ to see that

$$\begin{pmatrix} f_1(\widetilde{g}\widetilde{m}_*) \\ f_2(\widetilde{g}\widetilde{m}_*) \end{pmatrix} = \begin{cases} \begin{pmatrix} \epsilon_1 f(\widetilde{g}) \\ \epsilon_1 i f(\widetilde{g}) \end{pmatrix} & \text{if } \epsilon_2 = 1 \\ & \\ \begin{pmatrix} i f(\widetilde{g}) \\ -f(\widetilde{g}) \end{pmatrix} & \text{if } \epsilon_2 = -1. \end{cases}$$

□

We conclude this section with an analysis of unbounded realizations of distributional principal series for SL_2. We write the intersection of $M(\mathrm{SL}_2^{\pm})$, $A(\mathrm{SL}_2^{\pm})$, and $N(\mathrm{SL}_2^{\pm})$ with SL_2 as $M(\mathrm{SL}_2)$, $A(\mathrm{SL}_2)$, and $N(\mathrm{SL}_2)$, respectively. Let $B(\mathrm{SL}_2) = M(\mathrm{SL}_2)A(\mathrm{SL}_2)N(\mathrm{SL}_2)$. For $\nu \in \mathbb{C}$, let $\mu_\nu(a_u) = u^{\nu-1}$, which is a quasi-character of $A(\mathrm{SL}_2)$. For $b = man_- \in B(\mathrm{SL}_2)$ we define,

$$\omega_{\delta,\nu}(b) = \varsigma_\delta(m)\mu_\nu(a),$$

which is a representation of $B(\mathrm{SL}_2)$. To simplify our notation, we will write $V^{-\infty}_{\delta,\nu}(\mathrm{SL}_2)$ for $V^{-\infty}_{\omega_{\delta,\nu}}(\mathrm{SL}_2)$. For $f \in V^{-\infty}_{\delta,\nu}(\mathrm{SL}_2)$, let f_0 denote the restriction of s_f to $N(\mathrm{SL}_2) \cong \mathbb{R}$, and let f_∞ denote the restriction of s_f to $s^{-1}N(\mathrm{SL}_2) \cong \mathbb{R}$ where

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since $N(\mathrm{SL}_2)$ and $s^{-1}N(\mathrm{SL}_2)$ cover $\mathrm{SL}_2/B(\mathrm{SL}_2)$, it follows that f is completely determined by f_0 and f_∞. Since for $x \neq 0$,

$$s^{-1} \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} |x|^{-1} & 0 \\ \mathrm{sgn}(x) & |x| \end{pmatrix} \cdot \begin{pmatrix} \mathrm{sgn}(-x) & 0 \\ 0 & \mathrm{sgn}(-x) \end{pmatrix}, \tag{3.22}$$

it follows for $f \in V^{-\infty}_{\delta,\nu}(\mathrm{SL}_2)$, we have that

$$f_\infty(x) = \mathrm{sgn}(-x)^\delta |x|^{\nu-1} f_0\big(-x^{-1}\big), \tag{3.23}$$

as distributions on $\mathbb{R}_{\neq 0}$. Conversely, when given distributions $f_0, f_\infty \in C^{-\infty}(\mathbb{R})$ which satisfy (3.23), one can define a unique element $f \in V_{\delta,\nu}^{-\infty}(\mathrm{SL}_2)$. Thus

$$\begin{aligned} &V_{\delta,\nu}^{-\infty}(\mathrm{SL}_2) \\ &\cong \{(f_0, f_\infty) \in C^{-\infty}(\mathbb{R})^2 : f_0 \text{ and } f_\infty \text{ satisfy (3.23) as distributions on } \mathbb{R}_{\neq 0}\}. \end{aligned} \tag{3.24}$$

We overload notation by defining $\mathcal{L} : V_{\delta,\nu}^{-\infty}(\mathrm{SL}_2) \to V_{\delta,\nu}^{-\infty}(\widetilde{\mathrm{SL}_2^{\pm}})$ to be the equivariant map (with respect to SL_2) given by

$$(f_0, f_\infty) \mapsto (f_0, f_\infty), \tag{3.25}$$

where $f \in V_{\delta,\nu}^{-\infty}(\mathrm{SL}_2)$. Notice that we used the isomorphisms (3.12) and (3.24) in our definition of $\mathcal{L}$. Since (3.11) and (3.23) agree it follows that $\mathcal{L}$ is indeed well defined. Thus we are able to identify

$$V_{\delta,\nu}^{-\infty}(\mathrm{SL}_2) \subset V_{\delta,\nu}^{-\infty}(\widetilde{\mathrm{SL}_2^{\pm}}). \tag{3.26}$$

3.2 A Double Cover of J

In this section we will analyze the distributional principal series representations for double cover groups of J and $J^{\pm}$, where J and $J^{\pm}$ are certain subgroups of G_2 we will define shortly. As in section 3.1, we will define the multiplication laws for such double cover groups, and then study the unbounded realizations of distributional principal series representation spaces for such groups. By studying these unbounded realizations, we can determine when certain distributional principal series representation spaces lie naturally within other distributional principal series representation spaces for larger groups.

One can show that

$$L_\beta = \left\{ \begin{pmatrix} ad-bc & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & -b & 0 & 0 & 0 & 0 \\ 0 & -c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{a}{ad-bc} & \frac{-b}{ad-bc} & 0 \\ 0 & 0 & 0 & 0 & \frac{-c}{ad-bc} & \frac{d}{ad-bc} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{ad-bc} \end{pmatrix} : \begin{array}{l} \text{where } a,b,c,d \in \mathbb{R} \\ \text{such that } ad-bc = \pm 1 \end{array} \right\}$$

is a subgroup of G_2 which is isomorphic to $\mathrm{SL}_2^{\pm}$. In fact, L_β is a subgroup of the Levi component for the β root, and hence L_β acts by conjugation on the subgroup $U_\beta = \{X_6X_5X_4X_3X_2 : x_i \in \mathbb{R}\}$. Let $J^{\pm} = U_\beta \cdot L_\beta$, which is a subgroup of G_2. Abstractly speaking, we see that $J^{\pm}$ is isomorphic to the semidirect product of L_β and U_β.

Via the covering map of $\widetilde{\mathrm{SL}_2^{\pm}}$ onto $\mathrm{SL}_2^{\pm} \cong L_\beta$, we are able to give the set $U_\beta \times \widetilde{\mathrm{SL}_2^{\pm}}$ the structure of a group. Specifically, the multiplication law is given by

$$(u_1\widetilde{g}_1) \cdot (u_2\widetilde{g}_2) = (u_1 g_1 u_2 g_1^{-1})\widetilde{g}_1\widetilde{g}_2$$

where $\widetilde{g}_1, \widetilde{g}_2 \in \widetilde{\mathrm{SL}_2^{\pm}}$, $u_1, u_2 \in U_\beta$, and g_1, g_2 are the respective images of $\widetilde{g}_1$ and $\widetilde{g}_2$ under the covering map of $\widetilde{\mathrm{SL}_2^{\pm}}$ onto $\mathrm{SL}_2^{\pm} \cong L_\beta$. We denote this group by $\widetilde{J^{\pm}}$ and we write elements of $\widetilde{J^{\pm}}$ as $u\widetilde{g}$ where $u \in U_\beta$ and $\widetilde{g} \in \widetilde{\mathrm{SL}_2^{\pm}}$. Let $\iota : \widetilde{\mathrm{SL}_2^{\pm}} \to \widetilde{J^{\pm}}$ denote the natural embedding of $\widetilde{\mathrm{SL}_2^{\pm}}$ into $\widetilde{J^{\pm}}$.

Let $B(J^{\pm})$ denote the space of lower triangular matrices of $J^{\pm}$. Under the Langlands decomposition, $B(J^{\pm}) = M(J^{\pm})A(J^{\pm})N_-(J^{\pm})$ where

$$M(J^{\pm}) = \iota(M(\mathrm{SL}_2^{\pm})), \quad A(J^{\pm}) = \iota(A(\mathrm{SL}_2^{\pm})), \text{ and } N_-(J^{\pm}) = \iota(N_-(\mathrm{SL}_2^{\pm})).$$

Likewise, let

$$M(\widetilde{J^{\pm}}) = \iota(M(\widetilde{\mathrm{SL}_2^{\pm}})), \quad A(\widetilde{J^{\pm}}) = \iota(A(\widetilde{\mathrm{SL}_2^{\pm}})), \quad N_-(\widetilde{J^{\pm}}) = \iota(N_-(\widetilde{\mathrm{SL}_2^{\pm}})),$$

and $B(\widetilde{J^{\pm}}) = M(\widetilde{J^{\pm}})A(\widetilde{J^{\pm}})N_-(\widetilde{J^{\pm}})$. We let $N(J^{\pm})$ denote the group of unipotent upper-triangular matrices of $J^{\pm}$. Since $N(J^{\pm}) = U_\beta \cdot \iota(N(\mathrm{SL}_2^{\pm}))$, we define $N(\widetilde{J^{\pm}}) = U_\beta \cdot \iota(N(\widetilde{\mathrm{SL}_2^{\pm}}))$. To simplify notation, we will write $\widetilde{m}$, $\widetilde{a}$, $\widetilde{n}$, $\widetilde{n}_-$, and $\widetilde{b}$ for the elements $\iota(\widetilde{m})$, $\iota(\widetilde{a})$, $\iota(\widetilde{n})$, $\iota(\widetilde{n}_-)$, and $\iota(\widetilde{b})$, respectively.

Observe that $\sigma_{\epsilon_1,\epsilon_2} \circ \iota^{-1}$ and $\varsigma_\delta \circ \iota^{-1}$ are representations of $M(\widetilde{J^{\pm}})$, and that $\mu_\nu \circ \iota^{-1}$ is a quasi-character of $A(\widetilde{J^{\pm}})$. For $\widetilde{b} = \widetilde{m}\widetilde{a}\widetilde{n}_- \in B(\widetilde{J^{\pm}})$ we define

$$\begin{aligned}
&\omega_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{b}) = (\sigma_{\epsilon_1,\epsilon_2} \circ \iota^{-1})(\widetilde{m}) \cdot (\mu_\nu \circ \iota^{-1})(\widetilde{a}), \text{ and} \\
&\omega_{\delta,\nu}(\widetilde{b}) = (\varsigma_\delta \circ \iota^{-1})(\widetilde{m}) \cdot (\mu_\nu \circ \iota^{-1})(\widetilde{a}),
\end{aligned}$$

both of which are representations of $B(\widetilde{J^{\pm}})$. In short, we are taking the representations we defined in section 3.1 and identifying them as representations on $\widetilde{J^{\pm}}$ via ι^{-1}. As usual, we shall write $V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{J^{\pm}})$ for $V^{-\infty}_{\omega_{(\epsilon_1,\epsilon_2),\nu}}(\widetilde{J^{\pm}})$, and write $V^{-\infty}_{\delta,\nu}(\widetilde{J^{\pm}})$ for $V^{-\infty}_{\omega_{\delta,\nu}}(\widetilde{J^{\pm}})$. By using the standard basis for $\mathbb{C}^2$, we shall write $f \in V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{J^{\pm}})$ as $f = (f_1, f_2) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ where $f_i \in C^{-\infty}(\widetilde{J^{\pm}}, \mathbb{C})$.

As discussed in section 1.2, $f = (f_1, f_2) \in V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{J^{\pm}})$ can be identified with a distributional section s_f of a vector bundle over $\widetilde{J^{\pm}}/B(\widetilde{J^{\pm}})$. Since the group $N(\widetilde{J^{\pm}})$ maps injectively into $\widetilde{J^{\pm}}/B(\widetilde{J^{\pm}})$, we shall abuse notation by writing this subset of $\widetilde{J^{\pm}}/B(\widetilde{J^{\pm}})$ as $N(\widetilde{J^{\pm}})$. Since as a subset of $\widetilde{J^{\pm}}/B(\widetilde{J^{\pm}})$, $N(\widetilde{J^{\pm}})$ is an open set, it follows that we can consider the restriction of s_f to $N(\widetilde{J^{\pm}})$. We denote this restriction by $f_0 = ((f_1)_0, (f_2)_0)$. Since we can parameterize

$N(\widetilde{J^{\pm}})$ by

$$(x_1, x_2, x_3, x_4, x_5, x_6) \mapsto X_6X_5X_4X_3X_2X_1,$$

we can also identify f_0 as a distribution vector on $\mathbb{R}^6$.[4] Likewise, we let $f_\infty = ((f_1)_\infty, (f_2)_\infty)$ denote the restriction of s_f to $\widetilde{s}^{-1}N(\widetilde{J^{\pm}})$, where $\widetilde{s}^{-1}N(\widetilde{J^{\pm}})$ is thought of as a subset of $\widetilde{\mathrm{SL}_2^{\pm}}/B(\widetilde{\mathrm{SL}_2^{\pm}})$. Here we abuse notation by writing $\widetilde{s}$ for $\iota(\widetilde{s})$; similarly, we write s for $\iota(s)$. Since we can parameterize $\widetilde{s}^{-1}N(\widetilde{J^{\pm}})$ by

$$(x_1, x_2, x_3, x_4, x_5, x_6) \mapsto \widetilde{s}^{-1}X_6X_5X_4X_3X_2X_1,$$

we can also identify f_∞ as a distribution vector on $\mathbb{R}^6$. Since $N(\widetilde{J^{\pm}})$ and $\widetilde{s}^{-1}N(\widetilde{J^{\pm}})$ cover $\widetilde{J^{\pm}}/B(\widetilde{J^{\pm}})$, it follows that f is completely determined by f_0 and f_∞.

Since

$$s^{-1}X_6X_5X_4X_3X_2s = R_6(3x_2^2x_3 + x_5)R_5(-3x_2x_3^2 - x_6)R_4(2x_2x_3 + x_4)R_3(x_2)R_2(-x_3),$$

it follows that

$$\begin{aligned}
&\widetilde{s}^{-1} \cdot X_6X_5X_4X_3X_2X_1 \\
&= R_6(3x_2^2x_3 + x_5)R_5(-3x_2x_3^2 - x_6)R_4(2x_2x_3 + x_4)R_3(x_2)R_2(-x_3)\widetilde{s}^{-1}\left(\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix}, 1\right).
\end{aligned}$$

Applying (3.8) then shows that that

$$\begin{aligned}
&\widetilde{s}^{-1} \cdot X_6X_5X_4X_3X_2X_1 \\
&= R_6(3x_3x_2^2 + x_5)R_5(-3x_2x_3^2 - x_6)R_4(2x_2x_3 + x_4)R_3(x_2)R_2(-x_3)R_1(-x_1^{-1}) \\
&\quad \cdot \left(\begin{pmatrix} |x_1|^{-1} & 0 \\ \mathrm{sgn}(x_1) & |x_1| \end{pmatrix}, 1\right) \cdot \left(\begin{pmatrix} \mathrm{sgn}(-x_1) & 0 \\ 0 & \mathrm{sgn}(-x_1) \end{pmatrix}, 1\right).
\end{aligned} \tag{3.27}$$

Thus for $f \in V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{J^{\pm}})$ we have that

$$f_\infty(x_1, x_2, x_3, x_4, x_5, x_6) = \begin{pmatrix} (f_1)_\infty(x_1, x_2, x_3, x_4, x_5, x_6) \\ (f_2)_\infty(x_1, x_2, x_3, x_4, x_5, x_6) \end{pmatrix}$$

---

[4]The parameterization of $N(\widetilde{J^{\pm}})$ which we have used is simply σ_{id}, which we defined in (2.68).

$$= \begin{cases} |x_1|^{\nu-1} \begin{pmatrix} -\epsilon_1 (f_2)_0(-x_1^{-1}, -x_3, x_2, 2x_2x_3 + x_4, -3x_2x_3^2 - x_6, 3x_2^2x_3 + x_5) \\ \epsilon_1 (f_1)_0(-x_1^{-1}, -x_3, x_2, 2x_2x_3 + x_4, -3x_2x_3^2 - x_6, 3x_2^2x_3 + x_5) \end{pmatrix} & \text{if } x_1 > 0 \\ \\ |x_1|^{\nu-1} \begin{pmatrix} (f_1)_0(-x_1^{-1}, -x_3, x_2, 2x_2x_3 + x_4, -3x_2x_3^2 - x_6, 3x_2^2x_3 + x_5) \\ (f_2)_0(-x_1^{-1}, -x_3, x_2, 2x_2x_3 + x_4, -3x_2x_3^2 - x_6, 3x_2^2x_3 + x_5) \end{pmatrix} & \text{if } x_1 < 0 \end{cases} \tag{3.28}$$

as distributions on $\mathbb{R}_{\neq 0} \times \mathbb{R}^5$. Conversely, when given distribution vectors $(f_1)_0, (f_2)_0, (f_1)_\infty, (f_2)_\infty \in C^{-\infty}(\mathbb{R}^6)$ which satisfy (3.28), one can define a unique element $f = (f_1, f_2) \in V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{J^\pm})$. Thus

$$\begin{aligned} V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{J^\pm}) \cong \Big\{ &((f_1)_0, (f_2)_0, (f_1)_\infty, (f_2)_\infty) \in C^{-\infty}(\mathbb{R}^6)^4 : \\ &(f_1)_0, (f_2)_0, (f_1)_\infty, \text{ and } (f_2)_\infty \text{ satisfy (3.28) as distributions on } \mathbb{R}_{\neq 0} \times \mathbb{R}^5 \Big\}. \end{aligned} \tag{3.29}$$

Statements analogous to (3.28) and (3.29) also hold for $f \in V^{-\infty}_{\delta,\nu}(\widetilde{J^\pm})$. In particular, if we let f_0 denote the restriction of s_f to $N(\widetilde{J^\pm}) \cong \mathbb{R}^6$, and let f_∞ denote the restriction of s_f to $\widetilde{s}^{-1}N(\widetilde{J^\pm}) \cong \mathbb{R}^6$, then

$$\begin{aligned} &f_\infty(x_1, x_2, x_3, x_4, x_5, x_6) \\ &= \operatorname{sgn}(-x_1)^\delta |x_1|^{\nu-1} f_0(-x_1^{-1}, -x_3, x_2, 2x_2x_3 + x_4, -3x_2x_3^2 - x_6, 3x_2^2x_3 + x_5) \end{aligned} \tag{3.30}$$

as distributions on $\mathbb{R}_{\neq 0} \times \mathbb{R}^5$. Consequently,

$$\begin{aligned} &V^{-\infty}_{\delta,\nu}(\widetilde{J^\pm}) \\ &\cong \Big\{ (f_0, f_\infty) \in C^{-\infty}(\mathbb{R}^6)^2 : f_0 \text{ and } f_\infty \text{ satisfy (3.30) as distributions on } \mathbb{R}_{\neq 0} \times \mathbb{R}^5 \Big\}. \end{aligned} \tag{3.31}$$

Let $J = \{g \in J^\pm : \det(g) = 1\}$. We define $\widetilde{J}$ to be the inverse image of J under the covering map $\widetilde{J^\pm} \to J^\pm$. We write the intersection of $M(\widetilde{J^\pm})$, $A(\widetilde{J^\pm})$, $N_-(\widetilde{J^\pm})$, and $N(\widetilde{J^\pm})$ with $\widetilde{J}$ as $M(\widetilde{J})$, $A(\widetilde{J})$, $N_-(\widetilde{J})$, and $N(\widetilde{J})$, respectively. Let $B(\widetilde{J}) = M(\widetilde{J})A(\widetilde{J})N_-(\widetilde{J})$. Observe that $\sigma_\epsilon \circ \iota^{-1}$ is a representation of $M(\widetilde{J})$ and that $\mu_\nu \circ \iota^{-1}$ is a quasi-character of $A(\widetilde{J})$. For $\widetilde{b} = \widetilde{m}\widetilde{a}\widetilde{n}_- \in B(\widetilde{J})$ we define

$$\omega_{(\epsilon),\nu}(\widetilde{b}) = (\sigma_\epsilon \circ \iota^{-1})(\widetilde{m}) \cdot (\mu_\nu \circ \iota^{-1})(\widetilde{a}),$$

which is a representation of $B(\widetilde{J})$. In order to simplify notation, we will write $V^{-\infty}_{(\epsilon),\nu}(\widetilde{J})$ for $V^{-\infty}_{\omega_{(\epsilon),\nu}}(\widetilde{J})$.

For $f \in V^{-\infty}_{(\epsilon),\nu}(\widetilde{J})$, let f_0 denote the restriction of s_f to $N(\widetilde{J}) \cong \mathbb{R}^6$, and let f_∞ denote the restriction of s_f to $\widetilde{s}^{-1}N(\widetilde{J}) \cong \mathbb{R}^6$. As earlier, we will also identify f_0 and f_∞ with the distributions $f_0 \circ \sigma_{\mathrm{id}}$ and $f_\infty \circ \sigma_{\mathrm{id}}$ on $\mathbb{R}^6$. Since $N(\widetilde{J})$ and $\widetilde{s}^{-1}N(\widetilde{J})$ cover $\widetilde{J}/B(\widetilde{J})$, it follows that f is completely determined by f_0 and f_∞. By (3.27), we find that for $f \in V^{-\infty}_{(\epsilon),\nu}(\widetilde{J})$ that

$$\begin{aligned}
&f_\infty(x_1,x_2,x_3,x_4,x_5,x_6) \\
&= \mathrm{sgn}(-x_1)^{\epsilon/2}|x_1|^{\nu-1} f_0\big(-x_1^{-1}, -x_3, x_2, 2x_2x_3+x_4, -3x_2x_3^2-x_6, 3x_2^2x_3+x_5\big)
\end{aligned} \tag{3.32}$$

as distributions on $\mathbb{R}_{\neq 0}\times\mathbb{R}^5$. Conversely, when given distributions $f_0, f_\infty \in C^{-\infty}(\mathbb{R}^6)$ which satisfy (3.32), one can define a unique element $f \in V^{-\infty}_{(\epsilon),\nu}(\widetilde{J})$. Thus

$$\begin{aligned}
&V^{-\infty}_{(\epsilon),\nu}(\widetilde{J^{\pm}}) \\
&\cong \{(f_0,f_\infty) \in C^{-\infty}(\mathbb{R}^6)^2 : f_0 \text{ and } f_\infty \text{ satisfy (3.32) as distributions on } \mathbb{R}_{\neq 0}\times\mathbb{R}^5\}.
\end{aligned} \tag{3.33}$$

We define an equivariant map (with respect to $\widetilde{J}$) from $f \in V^{-\infty}_{(\epsilon_1),\nu}(\widetilde{J})$ into $V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{J^{\pm}})$ by (3.18). Indeed, we see from (3.28), (3.29), (3.32), and (3.33), that (3.18) maps into $V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{J^{\pm}})$ as claimed. Thus by (3.18), we are able to identify

$$V^{-\infty}_{(\epsilon_1),\nu}(\widetilde{J}) \subset V^{-\infty}_{(\epsilon_1,\epsilon_2),\nu}(\widetilde{J^{\pm}}). \tag{3.34}$$

We write the intersection of $M(J^{\pm})$, $A(J^{\pm})$, and $N(J^{\pm})$ with J as $M(J)$, $A(J)$, and $N(J)$, respectively. Observe that $\varsigma_\delta \circ \iota^{-1}$ is a representation of $M(J)$ and that $\mu_\nu \circ \iota^{-1}$ is a quasi-character of $A(J)$. For $b = man_- \in B(J)$ we define

$$\omega_{\delta,\nu}(b) = (\varsigma_\delta \circ \iota^{-1})(m) \cdot (\mu_\nu \circ \iota^{-1})(a),$$

which is a representation of $B(J)$. In order to simplify notation, we will write $V^{-\infty}_{\delta,\nu}(J)$ for $V^{-\infty}_{\omega_{\delta,\nu}}(J)$. For $f \in V^{-\infty}_{\delta,\nu}(J)$, let f_0 denote the restriction of s_f to $N(J) \cong \mathbb{R}^6$, and let f_∞ denote the restriction of s_f to $s^{-1}N(J) \cong \mathbb{R}^6$. We shall also let f_0 and f_∞ denote the distributions $f_0 \circ \sigma_{\mathrm{id}}$ and $f_\infty \circ \sigma_\infty$, respectively. Since $N(J)$ and $s^{-1}N(J)$ cover $J/B(J)$, it follows that f is completely determined by f_0 and f_∞. By (3.27), we have that

$$\begin{aligned}
&s^{-1}\cdot X_6X_5X_4X_3X_2X_1 \\
&= R_6(3x_2^2x_3+x_5)R_5(-3x_2x_3^2-x_6)R_4(2x_2x_3+x_4)R_3(x_2)R_2(-x_3)R_1(-x_1^{-1}) \\
&\quad \cdot \begin{pmatrix} |x_1|^{-1} & 0 \\ \mathrm{sgn}(x_1) & |x_1| \end{pmatrix} \cdot \begin{pmatrix} \mathrm{sgn}(-x_1) & 0 \\ 0 & \mathrm{sgn}(-x_1) \end{pmatrix}.
\end{aligned} \tag{3.35}$$

Thus it follows that for $f \in V^{-\infty}_{\delta,\nu}(J)$,

$$\begin{aligned}
&f_\infty(x_1,x_2,x_3,x_4,x_5,x_6) \\
&= \mathrm{sgn}(-x_1)^{\delta}|x_1|^{\nu-1} f_0\big(-x_1^{-1}, -x_3, x_2, 2x_2x_3+x_4, -3x_2x_3^2-x_6, 3x_2^2x_3+x_5\big)
\end{aligned} \tag{3.36}$$

as distributions on $\mathbb{R}_{\neq 0} \times \mathbb{R}^5$. Conversely, when given distributions $f_0, f_\infty \in C^{-\infty}(\mathbb{R}^6)$ which satisfy (3.36), one can define a unique element $f \in V_{\delta,\nu}^{-\infty}(J)$. Thus

$$\begin{aligned} &V_{\delta,\nu}^{-\infty}(J) \\ &\cong \{(f_0, f_\infty) \in C^{-\infty}(\mathbb{R}^6)^2 : f_0 \text{ and } f_\infty \text{ satisfy (3.36) as distributions on } \mathbb{R}_{\neq 0}\}. \end{aligned} \tag{3.37}$$

For $f \in V_{\delta,\nu}^{-\infty}(J)$ we define an equivariant map into $V_{\delta,\nu}^{-\infty}(\widetilde{J^{\pm}})$ by $(f_0, f_\infty) \mapsto (f_0, f_\infty)$. Indeed, since (3.30) and (3.36) agree, it follows that this map is well-defined. Thus we are able to identify

$$V_{\delta,\nu}^{-\infty}(J) \subset V_{\delta,\nu}^{-\infty}(\widetilde{J^{\pm}}). \tag{3.38}$$

3.3 The Theta Distribution

In this section we will define a distribution $\theta \in V_{(-1),\frac{1}{2}}^{-\infty}(\widetilde{J})$, which has many of the properties of the more familiar automorphic theta functions. To deduce these properties for θ, we begin by considering the function

$$\Theta(z_1, x_2, x_3, x_4) = \sum_{m \in \mathbb{Z}} e(-m^2 z_1 + 2m(-z_1 x_2 + x_3) - (z_1 x_2^2 + x_4)),$$

where $z_1 = x_1 - iy_1$, $x_1, x_2, x_3, x_4 \in \mathbb{R}$, and $y_1 \in \mathbb{R}_{>0}$. Observe that Θ is a holomorphic function in the z_1 variable.

Proposition 3.2. *If $z_1 = x_1 - iy_1 \in \mathbb{C}$ and $y_1 > 0$ then*

(a) $\Theta(z_1, x_2, x_3, x_4) = \Theta(z_1 + 1, x_2, x_2 + x_3, -x_2^2 + x_4)$,

(b) $\Theta(z_1, x_2, x_3, x_4) = (2iz_1)^{-1/2}\Theta\left(\frac{-1}{4z_1}, -2x_3, \frac{x_2}{2}, 2x_2x_3 + x_4\right)$,

(c) $\Theta(z_1, x_2, x_3, x_4) = (1 - 4z_1)^{-1/2}\Theta\left(\frac{z_1}{-4z_1 + 1}, -x_2 + 4x_3, -x_3, 4x_3^2 + x_4\right)$.

Proof. Observe

$$\begin{aligned} &\Theta(z_1 + 1, x_2, x_3, x_4) \\ &= \sum_{m \in \mathbb{Z}} e(-m^2(z_1 + 1) + 2m(-(z_1 + 1)x_2 + x_3) - ((z_1 + 1)x_2^2 + x_4)) \\ &= \sum_{m \in \mathbb{Z}} e(-m^2 z_1 + 2m(-z_1 x_2 + (x_3 - x_2)) - (z_1 x_2^2 + (x_2^2 + x_4))) \\ &= \Theta(z_1, x_2, x_3 - x_2, x_2^2 + x_4). \end{aligned}$$

This proves part (a).

If we set $z_1 = -iy_1$ we find that

$$\begin{aligned}
&\Theta(-iy_1, x_2, x_3, x_4) \\
&= \sum_{m\in\mathbb{Z}} e(m^2 iy_1 + 2m(iy_1x_2 + x_3) - (-iy_1x_2^2 + x_4)) \\
&= \exp(-2\pi x_2^2 y_1)e(-x_4) \sum_{m\in\mathbb{Z}} \exp(2\pi(-m^2y_1 - 2mx_2y_1))e(2mx_3).
\end{aligned}$$

Let $f(t) = \exp(2\pi(-t^2y_1 - 2tx_2y_1))e(2tx_3)$. By changing variables in t and utilizing the fact that the Gaussian function is its own Fourier transform, we find that

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{-\infty}^{\infty} \exp(2\pi(-t^2y_1 - 2tx_2y_1))e(2tx_3 - t\xi)\,dt \\
&= \int_{-\infty}^{\infty} \exp(2\pi(-t^2y_1 + x_2^2y_1))e(2tx_3 - 2x_2x_3 - t\xi + x_2\xi)\,dt \\
&= e(-2x_2x_3 + x_2\xi)\exp(2\pi x_2^2y_1)\int_{-\infty}^{\infty} \exp(-2\pi y_1 t^2)e(-(\xi - 2x_3)t)\,dt \\
&= e(-2x_2x_3 + x_2\xi)\exp(2\pi x_2^2y_1)(2y_1)^{-1/2}\exp\left(\frac{-\pi(\xi - 2x_3)^2}{2y_1}\right).
\end{aligned}$$

By the Poisson summation formula, we obtain the following equality

$$\begin{aligned}
&\Theta(-iy_1, x_2, x_3, x_4) = \exp(-2\pi x_2^2y_1)e(-x_4)\sum_{m\in\mathbb{Z}} f(m) = \exp(-2\pi x_2^2y_1)e(-x_4)\sum_{m\in\mathbb{Z}} \widehat{f}(m) \\
&= \exp(-2\pi x_2^2y_1)e(-x_4)\sum_{m\in\mathbb{Z}} e(-2x_2x_3 + mx_2)\exp(2\pi x_2^2y_1)(2y_1)^{-1/2}\exp\left(\frac{-\pi(m - 2x_3)^2}{2y_1}\right) \\
&= (2y_1)^{-1/2}\sum_{m\in\mathbb{Z}} e(mx_2 - (2x_2x_3 + x_4))\exp\left(-\frac{\pi m^2}{2y_1} + \frac{2\pi m x_3}{y_1} - \frac{2\pi x_3^2}{y_1}\right) \\
&= (2y_1)^{-1/2}\sum_{m\in\mathbb{Z}} e\left(-m^2\frac{1}{4iy_1} + 2m\left(-\frac{1}{4iy_1}(-2x_3) + \left(\frac{x_2}{2}\right)\right)\right) \\
&\qquad \cdot e\left(-\left(\frac{1}{4iy_1}(-2x_3)^2 + (2x_2x_3 + x_4)\right)\right) \\
&= (2y_1)^{-1/2}\Theta\left(\frac{1}{4iy_1}, -2x_3, \frac{x_2}{2}, 2x_2x_3 + x_4\right).
\end{aligned}$$

Part (b) follows by analytic continuation since we have proven part (b) for $z_1 = -iy_1$ where $y_1 > 0$.

By applying parts (b), (a), and then (b) once more, we see that

$$\begin{aligned}
&\Theta(z_1, x_2, x_3, x_4)\\
&= (2iz_1)^{-1/2}\Theta\left(\frac{-1}{4z_1}, -2x_3, \frac{x_2}{2}, 2x_2x_3 + x_4\right)\\
&= (2iz_1)^{-1/2}\Theta\left(\frac{-1}{4z_1} + 1, -2x_3, \frac{x_2}{2} - 2x_3, -(-2x_3)^2 + 2x_2x_3 + x_4\right)\\
&= (2iz_1)^{-1/2}\left(2\left(\frac{-1}{4z_1} + 1\right)i\right)^{-1/2}\\
&\quad\Theta\left(\frac{-1}{4\left(\frac{-1}{4z_1} + 1\right)}, -2\left(\frac{x_2}{2} - 2x_3\right), \frac{-2x_3}{2}, 2(-2x_3)\left(\frac{x_2}{2} - 2x_3\right) + 2x_2x_3 - (-2x_3)^2 + x_4\right)\\
&= (2iz_1)^{-1/2}\left(2\left(\frac{-1}{4z_1} + 1\right)i\right)^{-1/2}\Theta\left(\frac{z_1}{-4z_1 + 1}, -x_2 + 4x_3, -x_3, 4x_3^2 + x_4\right).
\end{aligned}$$

If we set $z_1 = -iy_1$ in the above equality, we find that

$$\begin{aligned}
&\Theta(-iy_1, x_2, x_3, x_4)\\
&= (2y_1)^{-1/2}\left(2\left(\frac{1}{4iy_1} + 1\right)i\right)^{-1/2}\Theta\left(\frac{-iy_1}{-4(-iy_1) + 1}, -x_2 + 4x_3, -x_3, 4x_3^2 + x_4\right)\\
&= \left(4y_1\left(\frac{1}{4iy_1} + 1\right)i\right)^{-1/2}\Theta\left(\frac{-iy_1}{-4(-iy_1) + 1}, -x_2 + 4x_3, -x_3, 4x_3^2 + x_4\right)\\
&= (1 + 4iy_1)^{-1/2}\Theta\left(\frac{-iy_1}{-4(-iy_1) + 1}, -x_2 + 4x_3, -x_3, 4x_3^2 + x_4\right).
\end{aligned}$$

Since this proves part (c) for $z_1 = -iy_1$ where $y_1 > 0$, it follows by analytic continuation that part (c) holds for all z_1. $\square$

To define our θ distribution as an element of $V^{-\infty}_{(-1),\frac{1}{2}}(\widetilde{J})$, it will be helpful to consider an alternative unbounded realization for $V^{-\infty}_{(-1),\frac{1}{2}}(\widetilde{J})$. Let

$$\Omega = \widetilde{a}_2^{-1}\widetilde{s} = \widetilde{s}\widetilde{a}_2 = \left(\begin{pmatrix} 0 & -2^{-1} \\ 2 & 0 \end{pmatrix}, 1\right), \tag{3.39}$$

and let $f \in V^{-\infty}_{(-1),\frac{1}{2}}(\widetilde{J})$. Recall that associated to $f \in V^{-\infty}_{(-1),\frac{1}{2}}(\widetilde{J})$ is a distributional section s_f of a vector bundle over $\widetilde{J}/B(\widetilde{J})$. Let f_0 denote the restriction of s_f to $N(\widetilde{J})$, and let $f_{2\infty}$ denote the restriction of s_f to $\Omega^{-1}N(\widetilde{J})$. Since $N(\widetilde{J})$ and $\Omega^{-1}N(\widetilde{J})$ cover $\widetilde{J}/B(\widetilde{J})$, it follows that f is completely determined by f_0 and $f_{2\infty}$. Since for $x_1 \neq 0$, we have

$$\begin{aligned}
&\Omega^{-1}X_6X_5X_4X_3X_2X_1\\
&= R_6\left(\frac{1}{2}(3x_2^2x_3 + x_5)\right)R_5(-2(3x_2x_3^2 + x_6))R_4(2x_2x_3 + x_4)R_3\left(\frac{x_2}{2}\right)R_3(-2x_3)\\
&\quad R_1\left(\frac{-1}{4x_1}\right)\widetilde{a}_{2x_1}^{-1}\widetilde{m}_{\mathrm{sgn}(-x_1),\mathrm{sgn}(-x_1),1}\widetilde{n}_-,
\end{aligned}$$

where $\widetilde{n}_- \in N_-(\widetilde{J})$, it follows that

$$\begin{aligned} &f_{2\infty}(x_1,x_2,x_3,x_4,x_5,x_6) \\ &= \operatorname{sgn}(-x_1)^{-1/2}|2x_1|^{-1/2} f_0\left(\frac{-1}{4x_1}, -2x_3, \frac{x_2}{2}, 2x_2x_3+x_4, -2(3x_2x_3^2+x_6), \frac{1}{2}(3x_2^2x_3+x_5)\right), \end{aligned} \tag{3.40}$$

as an equality between distributions on $\mathbb{R}_{\neq 0}\times\mathbb{R}^5$. Conversely, when given distributions $f_0, f_{2\infty} \in C^{-\infty}(\mathbb{R})$ which satisfy (3.40), one can define a unique element $f \in V^{-\infty}_{(-1),\nu}(\widetilde{J})$. Thus

$$\begin{aligned} &V^{-\infty}_{(-1),\nu}(\widetilde{J}) \\ &\cong \{(f_0, f_{2\infty}) \in C^{-\infty}(\mathbb{R})^2 : f_0 \text{ and } f_{2\infty} \text{ satsify (3.40) as distributions on } \mathbb{R}_{\neq 0}\}. \end{aligned} \tag{3.41}$$

Let $\theta_0 \in C^{-\infty}(\mathbb{R}^6)$ denote the distribution

$$\theta_0(x_1,x_2,x_3,x_4,x_5,x_6) = \sum_{m\in\mathbb{Z}} e(-m^2x_1 + 2m(-x_1x_2+x_3) - (x_1x_2^2+x_4)). \tag{3.42}$$

Notice that θ_0 has no x_5 or x_6 variables in its definitions; we include these variables in our notation to make it clear that we can also identify θ_0 as a distribution on $N(\widetilde{J}) \cong \mathbb{R}^6$. Let $\theta_{2\infty} \in C^{-\infty}(\mathbb{R})$ denote the distribution

$$\theta_{2\infty} = e(-1/8)\theta_0. \tag{3.43}$$

We wish to show that the pair $(\theta_0, \theta_{2\infty})$ defines an element of $V^{-\infty}_{(-1),\frac{1}{2}}(\widetilde{J})$ via the isomorphism (3.41). To do this, observe that as $y_1 \to 0$ we have that $\Theta(z_1,x_2,x_3,x_4) \to \theta_0(x_1,x_2,x_3,x_4)$ for $z_1 = x_1 - iy_1$. Therefore by Proposition 3.2(b), when we let $y_1 \to 0$, we find that

$$\begin{aligned} &\theta_0(x_1,x_2,x_3,x_4,x_5,x_6) \\ &= \left(\lim_{y_1\to 0}(2z_1 i)^{-1/2}\right)\theta_0\left(\frac{-1}{4x_1}, -2x_3, \frac{x_2}{2}, x_4+2x_2x_3, -2(3x_2x_3^2+x_6), \frac{1}{2}(3x_2^2x_3+x_5)\right). \end{aligned}$$

Observe that for $x_1 \neq 0$,

$$\lim_{y_1\to 0}(2z_1 i)^{-1/2} = \begin{cases} |2x_1|^{-1/2}e(-1/8) & \text{if } x_1 > 0 \\ |2x_1|^{-1/2}e(1/8) & \text{if } x_1 < 0. \end{cases}$$

Thus for $x_1 \neq 0$,

$$\begin{aligned} &\theta_{2\infty}(x_1,x_2,x_3,x_4,x_5,x_6) = e(-1/8)\theta_0(x_1,x_2,x_3,x_4,x_5,x_6) \\ &= \operatorname{sgn}(-x_1)^{-1/2}|2x_1|^{-1/2}\theta_0\left(\frac{-1}{4x_1}, -2x_3, \frac{x_2}{2}, x_4+2x_2x_3, -2(3x_2x_3^2+x_6), \frac{1}{2}(3x_2^2x_3+x_5)\right). \end{aligned}$$

Therefore by (3.41), it follows that $(\theta_0, \theta_{2\infty})$ defines a unique element $\theta \in V^{-\infty}_{(-1),\nu}(\widetilde{J})$. Let s_θ denote the distributional section corresponding to θ. As our notation suggests, it follows that if we restrict s_θ to $N(\widetilde{J}) \cong \mathbb{R}^6$, we obtain our originally defined θ_0 given in (3.42). Likewise, if we restrict s_θ to $\Omega^{-1}N(\widetilde{J}) \cong \mathbb{R}^6$, we obtain our originally defined $\theta_{2\infty}$ given in (3.43). Also note that by (3.34), it follows that θ can also be thought of as an element of $V^{-\infty}_{(-1,1),\frac{1}{2}}(\widetilde{J^\pm})$.

Let

$$\widetilde{\Gamma}_1(4) = \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \left(\frac{c}{d}\right) \right) \in \widetilde{\mathrm{SL}_2^\pm} : a,b,c,d \in \mathbb{Z},\ a \equiv d \equiv 1(\mathrm{mod}\, 4),\ c \equiv 0(\mathrm{mod}\, 4) \right\}, \tag{3.44}$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol; see Proposition 4.2 for some properties of the Kronecker symbol. One can show that $\widetilde{\Gamma}_1(4)$ is a well-defined subgroup of $\widetilde{\mathrm{SL}_2}$. We also let $\widetilde{\Gamma}_1(4)$ denote $\iota\Big(\widetilde{\Gamma}_1(4)\Big) \subset \widetilde{J}$. Let

$$\widetilde{J}_{\mathbb{Z}} = U_{\mathbb{Z}}(4) \cdot \widetilde{\Gamma}_1(4), \text{ where } U_{\mathbb{Z}}(4) = \langle N_2^2, N_3, N_4, N_5^2, N_6 \rangle \tag{3.45}$$

(recall that $N_i = R_i(1)$).

Theorem 3.3. *θ is $\widetilde{J}_{\mathbb{Z}}$-invariant*

Proof. Recall that $h_{\psi;0,0,-1,0,0}$ is defined in (2.88). Observe that θ_0 is of the form $h_{\psi;0,0,-1,0,0}$ with $\psi = 1$. In subsection 2.6.2 we showed that such $h_{\psi;0,0,-1,0,0}$ is $N_{\mathbb{Z}}$-invariant.[5] Therefore θ_0 is $N_{\mathbb{Z}}$-invariant, and hence θ_0 is N_1 and $U_{\mathbb{Z}}(4)$-invariant.

Next, observe that for $x_1 \neq \frac{1}{4}$,

$$\begin{aligned}
&\Omega N_1 \Omega^{-1} X_6 X_5 X_4 X_3 X_2 X_1 \\
&= R_6(8x_3^3 + x_6) R_5(-16x_3^3 + x_5 - 4x_6) R_4(4x_3^2 + x_4) X_3 R_2(x_2 - 4x_3) \\
&\quad R_1\left(\frac{x_1}{-4x_1+1}\right) \widetilde{a}^{-1}_{|1-4x_1|} \widetilde{m}_{\mathrm{sgn}(1-4x_1),\mathrm{sgn}(1-4x_1),1} \widetilde{n}_-,
\end{aligned}$$

where $\widetilde{n}_- \in N_-(\widetilde{J})$. Therefore by the transformation law for $V^{-\infty}_{(-1),\frac{1}{2}}(\widetilde{J})$, we find that for $x_1 \neq \frac{1}{4}$,

$$\begin{aligned}
&(\pi(\Omega N_1 \Omega^{-1})^{-1}\theta)_0(x_1,x_2,x_3,x_4,x_5,x_6) \\
&= |1-4x_1|^{-1/2}\mathrm{sgn}(1-4x_1)^{-1/2} \\
&\quad \theta_0\left(\frac{x_1}{-4x_1+1}, x_2 - 4x_3, x_3, 4x_3^2 + x_4, -16x_3^3 + x_5 - 4x_6, 8x_3^3 + x_6\right).
\end{aligned}$$

Since $\Theta(z_1,x_2,x_3,x_4) = \Theta(z_1,-x_2,-x_3,x_4)$ (this can be seen directly from the definition of Θ by applying the change of index $m \mapsto -m$), it follows from part (c) of Proposition 3.2 that

$$\Theta(z_1,x_2,x_3,x_4) = (1-4z_1)^{-1/2}\Theta\left(\frac{z_1}{-4z_1+1}, x_2 - 4x_3, x_3, 4x_3^2 + x_4\right).$$

[5] Technically, we proved the $N_{\mathbb{Z}}$-invariance of $h_{\psi;0,0,-1,0,0}$ for $\psi \in L^2(\mathbb{R})$, but a similar argument also applies for $\psi \in \mathcal{S}'(\mathbb{R})$.

Since

$$\lim_{y_1\to 0}(1-4z_1)^{-1/2} = \begin{cases} |1-4x_1|^{-1/2} & \text{if } 1-4x_1>0 \\ (-i)|1-4x_1|^{-1/2} & \text{if } 1-4x_1<0, \end{cases}$$

then for $x_1 \neq \frac{1}{4}$,

$$\begin{aligned} &\theta_0(x_1,x_2,x_3,x_4,x_5,x_6) \\ &= |1-4x_1|^{-1/2}\mathrm{sgn}(1-4x_1)^{-1/2} \\ &\quad \theta_0\left(\frac{x_1}{1-4x_1}, x_2-4x_3, x_3, 4x_3^2+x_4, -16x_3^3+x_5-4x_6, 8x_3^3+x_6\right). \end{aligned}$$

Thus for $x_1 \neq \frac{1}{4}$,

$$(\pi(\Omega N_1\Omega^{-1})^{-1}\theta)_0(x_1,x_2,x_3,x_4,x_5,x_6) = \theta_0(x_1,x_2,x_3,x_4,x_5,x_6).$$

Since $\widetilde{\Gamma}_1(4)$ is generated by N_1 and $\Omega N_1\Omega^{-1}$, it follows that θ_0 is $\widetilde{J}_{\mathbb{Z}}$ invariant for $x_1 \neq \frac{1}{4}$.

In what follows we will let π denote the left regular representation on $V^{-\infty}_{(-1),\frac{1}{2}}(\widetilde{J})$. Suppose $Q_6Q_5Q_4Q_3Q_2 \in U_{\mathbb{Z}}(4)$. Since $\pi(Q_6Q_5Q_4Q_3Q_2)^{-1}\theta \in V^{-\infty}_{(-1),\frac{1}{2}}$ then there exists a distributional section $s_{\pi(Q_6Q_5Q_4Q_3Q_2)^{-1}\theta}$ of a line bundle over $\widetilde{J}/B(\widetilde{J})$ corresponding to $\pi(Q_6Q_5Q_4Q_3Q_2)^{-1}\theta$. Recall from section 1.2 that

$$s_{\pi(Q_6Q_5Q_4Q_3Q_2)^{-1}\theta}(\widetilde{j}B(\widetilde{J})) = s_\theta(Q_6Q_5Q_4Q_3Q_2\widetilde{j}B(\widetilde{J})),\ \text{where } \widetilde{j}B(\widetilde{J}) \in \widetilde{J}/B(\widetilde{J}),$$

and s_θ is the distributional section corresponding to θ. Recall that $(\pi(Q_6Q_5Q_4Q_3Q_2)^{-1}\theta)_{2\infty}$ is simply the restriction of $s_{\pi(Q_6Q_5Q_4Q_3Q_2)^{-1}\theta}$ to $\Omega^{-1}N(\widetilde{J})$, where we identify $\Omega^{-1}N(\widetilde{J})$ as a subset of $\widetilde{J}/B(\widetilde{J})$. Therefore, since

$$\begin{aligned} &Q_6Q_5Q_4Q_3Q_2\Omega^{-1} \\ &= \Omega^{-1}R_6\left(\frac{1}{2}\big(-3q_2^2q_3-q_5\big)\right)R_5\big(2\big(3q_2q_3^2+q_6\big)\big)R_4(2q_2q_3+q_4)R_3\left(-\frac{q_2}{2}\right)R_2(2q_3) \end{aligned}$$

it follows that

$$\begin{aligned} &(\pi(Q_6Q_5Q_4Q_3Q_2)^{-1}\theta)_{2\infty}(X_6X_5X_4X_3X_2X_1) \\ &= s_{\pi(Q_6Q_5Q_4Q_3Q_2)^{-1}\theta}(\Omega^{-1}X_6X_5X_4X_3X_2X_1) \\ &= s_\theta(Q_6Q_5Q_4Q_3Q_2\Omega^{-1}X_6X_5X_4X_3X_2X_1) \\ &= s_\theta\Bigg(\Omega^{-1}R_6\left(\frac{1}{2}\big(-3q_2^2q_3-q_5\big)\right)R_5\big(2\big(3q_2q_3^2+q_6\big)\big)R_4(2q_2q_3+q_4)R_3\left(-\frac{q_2}{2}\right)R_2(2q_3) \\ &\qquad X_6X_5X_4X_3X_2X_1\Bigg) \\ &= \theta_{2\infty}\Bigg(R_6\left(\frac{1}{2}\big(-3q_2^2q_3-q_5\big)\right)R_5\big(2\big(3q_2q_3^2+q_6\big)\big)R_4(2q_2q_3+q_4)R_3\left(-\frac{q_2}{2}\right)R_2(2q_3) \\ &\qquad X_6X_5X_4X_3X_2X_1\Bigg). \end{aligned}$$

Since $Q_6Q_5Q_4Q_3Q_1 \in U_{\mathbb{Z}}(4)$ then

$$R_6\left(\frac{1}{2}\left(-3q_2^2q_3 - q_5\right)\right)R_5\left(2\left(3q_2q_3^2 + q_6\right)\right)R_4(2q_2q_3 + q_4)R_3\left(-\frac{q_2}{2}\right)R_2(2q_3) \in N_{\mathbb{Z}}.$$

Therefore, by (3.43) and the $N_{\mathbb{Z}}$-invariance of θ_0 it follows that

$$\begin{aligned}
&(\pi(Q_6Q_5Q_4Q_3Q_2)^{-1}\theta)_{2\infty}(X_6X_5X_4X_3X_2X_1) \\
&= e(-1/8)\theta_0\Bigg(R_6\left(\frac{1}{2}\left(-3q_2^2q_3 - q_5\right)\right)R_5\left(2\left(3q_2q_3^2 + q_6\right)\right)R_4(2q_2q_3 + q_4)R_3\left(-\frac{q_2}{2}\right) \\
&\qquad R_2(2q_3)X_6X_5X_4X_3X_2X_1\Bigg) \\
&= e(-1/8)\theta_0(X_6X_5X_4X_3X_2X_1) = \theta_{2\infty}(X_6X_5X_4X_3X_2X_1).
\end{aligned}$$

Thus $\theta_{2\infty}$ is $U_Z(4)$-invariant.

Likewise, since

$$N_1\Omega^{-1} = \Omega^{-1}(\Omega N_1\Omega^{-1}) \text{ and } (\Omega N_1\Omega^{-1})\Omega^{-1} = \Omega^{-1}N_1,$$

it follows from (3.43) and the N_1 and $\Omega N_1\Omega^{-1}$-invariance of θ_0 (for $x_1 \neq \frac{1}{4}$), that

$$\begin{aligned}
&(\pi(N_1)^{-1}\theta)_{2\infty}(X_6X_5X_4X_3X_2X_1) = s_{\pi(N_1)^{-1}\theta}(\Omega^{-1}X_6X_5X_4X_3X_2X_1) \\
&= s_\theta(N_1\Omega^{-1}X_6X_5X_4X_3X_2X_1) = s_\theta(\Omega^{-1}(\Omega N_1\Omega^{-1})X_6X_5X_4X_3X_2X_1) \\
&= \theta_{2\infty}((\Omega N_1\Omega^{-1})X_6X_5X_4X_3X_2X_1) = e(-1/8)\theta_0((\Omega N_1\Omega^{-1})X_6X_5X_4X_3X_2X_1) \\
&= e(-1/8)\theta_0(X_6X_5X_4X_3X_2X_1) = \theta_{2\infty}(X_6X_5X_4X_3X_2X_1),
\end{aligned}$$

and

$$\begin{aligned}
&(\pi(\Omega N_1\Omega^{-1})^{-1}\theta)_{2\infty}(X_6X_5X_4X_3X_2X_1) = s_{\pi(\Omega N_1\Omega^{-1})^{-1}\theta}(\Omega^{-1}X_6X_5X_4X_3X_2X_1) \\
&= s_\theta(\Omega N_1\Omega^{-1}\Omega^{-1}X_6X_5X_4X_3X_2X_1) = s_\theta(\Omega^{-1}N_1X_6X_5X_4X_3X_2X_1) \\
&= \theta_{2\infty}(N_1X_6X_5X_4X_3X_2X_1) = e(-1/8)\theta_0(N_1X_6X_5X_4X_3X_2X_1) \\
&= e(-1/8)\theta_0(X_6X_5X_4X_3X_2X_1) = \theta_{2\infty}(X_6X_5X_4X_3X_2X_1),
\end{aligned}$$

for $x_1 \neq \frac{1}{4}$. Thus $\theta_{2\infty}$ is N_1 and $\Omega N_1\Omega^{-1}$-invariant for $x_1 \neq \frac{1}{4}$. Since

$$\begin{aligned}
&\{\Omega^{-1}X_6X_5X_4X_3X_2X_1 : (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}_{\neq\frac{1}{4}} \times \mathbb{R}^5\} \\
&\quad \bigcup \{X_6X_5X_4X_3X_2X_1 : (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}_{\neq\frac{1}{4}} \times \mathbb{R}^5\}
\end{aligned}$$

covers $\widetilde{J}/B(\widetilde{J})$, it follows that θ is $\widetilde{J}_{\mathbb{Z}}$-invariant. $\square$

Chapter 4

Metaplectic Eisenstein Distributions

4.1 The Metaplectic Eisenstein Distribution at ∞

In this chapter we will define distributional analogues of the metaplectic Eisenstein series, which we shall refer to as *metaplectic Eisenstein distributions.* Additionally, we will prove that such distributions have meromorphic continuation to $\mathbb{C}$, and prove that there exists a functional equation between these metaplectic Eisenstein distributions.

We will define our metaplectic Eisenstein distributions to be elements of the space $V^{-\infty}_{(\epsilon),\nu}(\widetilde{\mathrm{SL}}_2)$, where $\epsilon = \pm 1$. Recall that by definition,

$$\begin{aligned} &V^{-\infty}_{(\epsilon),\nu}(\widetilde{\mathrm{SL}}_2) \\ &= \{f \in C^{-\infty}(\widetilde{\mathrm{SL}}_2, \mathbb{C}) : f(\widetilde{g}\widetilde{b}) = \omega_{(\epsilon),\nu}(\widetilde{b}^{-1}) f(\widetilde{g}) \text{ for all } \widetilde{g} \in \widetilde{\mathrm{SL}}_2, \widetilde{b} \in B(\widetilde{\mathrm{SL}}_2)\}, \end{aligned} \tag{4.1}$$

where $\omega_{(\epsilon),\nu}$ is defined in (3.14). Also recall that $V^{-\infty}_{(\epsilon),\nu}(\widetilde{\mathrm{SL}}_2)$ comes equipped with the left regular representation, which we shall denote by π. In the following lemma we give explicit formulas for $(\pi(\widetilde{g})f)_0$ and $(\pi(\widetilde{g})f)_\infty$ in terms of f_0 and f_∞, where $f \in V^{-\infty}_{(\epsilon),\nu}(\widetilde{\mathrm{SL}}_2)$; see section 3.1 for the definitions of these terms. In the statement of the following lemma, we will use the following simple fact: if $a, b, c, d \in \mathbb{R}$, $\kappa \in \{\pm 1\}$ then there exists $\kappa' \in \{\pm 1\}$ such that

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \kappa \right) \widetilde{s}^{-1} = \widetilde{s}^{-1} \left(\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}, \kappa' \right), \tag{4.2}$$

where $\widetilde{s} = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1 \right)$ as defined in (3.7).

Lemma 4.1. *Let $f \in V^{-\infty}_{(\epsilon),\nu}(\widetilde{SL}_2)$ and $\widetilde{g}^{-1} = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \kappa \right) \in \widetilde{SL}_2$.*

(a) If $c \neq 0$ then

$$(\pi(\widetilde{g})f)_0(x) = \kappa(-c, cx + d)_H |cx + d|^{\nu - 1} \mathrm{sgn}(cx + d)^{\epsilon/2} f_0\left(\frac{ax + b}{cx + d} \right),$$

as an equality between distributions on $\mathbb{R}_{\neq \frac{-d}{c}}$.

(b) If $c = 0$ then

$$(\pi(\widetilde{g})f)_0(x) = \kappa|d|^{\nu-1}\mathrm{sgn}(d)^{\epsilon/2} f_0\left(\frac{x}{d^2} + \frac{b}{d}\right),$$

as an equality between distributions on $\mathbb{R}$.

(c) If $b \neq 0$ then

$$(\pi(\widetilde{g})f)_\infty(x) = \kappa'(b, -bx+a)_H|-bx+a|^{\nu-1}\mathrm{sgn}(-bx+a)^{\epsilon/2} f_\infty\left(\frac{dx-c}{-bx+a}\right),$$

as an equality between distributions on $\mathbb{R}_{\neq \frac{a}{b}}$.

(d) If $b = 0$ then

$$(\pi(\widetilde{g})f)_\infty(x) = \kappa'|a|^{\nu-1}\mathrm{sgn}(a)^{\epsilon/2} f_\infty\left(\frac{x}{a^2} - \frac{c}{a}\right),$$

as an equality between distributions on $\mathbb{R}$.

Recall that κ' is defined according to (4.2).

Proof. Observe that for $c \neq 0$ and $x \neq \frac{-d}{c}$, we have

$$\begin{aligned}
\widetilde{g}^{-1} \cdot \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1\right) &= \left(\begin{pmatrix} a & ax+b \\ c & cx+d \end{pmatrix}, 1\right) \cdot \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \kappa\right) \\
&= \left(\begin{pmatrix} 1 & \frac{ax+b}{cx+d} \\ 0 & 1 \end{pmatrix}, 1\right) \cdot \left(\begin{pmatrix} \mathrm{sgn}(cx+d) & 0 \\ 0 & \mathrm{sgn}(cx+d) \end{pmatrix}, (-c, cx+d)_H\right) \\
&\quad \cdot \left(\begin{pmatrix} |cx+d|^{-1} & 0 \\ 0 & |cx+d| \end{pmatrix}, 1\right) \cdot \left(\begin{pmatrix} 1 & 0 \\ \frac{c}{cx+d} & 1 \end{pmatrix}, 1\right) \cdot \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \kappa\right).
\end{aligned}$$

Notice that

$$\left(\begin{pmatrix} \mathrm{sgn}(cx+d) & 0 \\ 0 & \mathrm{sgn}(cx+d) \end{pmatrix}, (-c, cx+d)_H\right)$$

is an element of $M(\widetilde{\mathrm{SL}}_2)$. By utilizing the transformation law for $V^{-\infty}_{(\epsilon),\nu}(\widetilde{\mathrm{SL}}_2)$ (given in (4.1)), we find that

$$(\pi(\widetilde{g})f)_0(x) = \kappa(-c, cx+d)_H|cx+d|^{\nu-1}\mathrm{sgn}(cx+d)^{\epsilon/2} f_0\left(\frac{ax+b}{cx+d}\right),$$

as an equality between distributions on $\mathbb{R}_{\neq \frac{-d}{c}}$. Similarly, when $c = 0$ (which implies that $a = d^{-1} \neq 0$) we see that

$$\begin{aligned}
\widetilde{g}^{-1} \cdot \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1\right) &= \left(\begin{pmatrix} d^{-1} & \frac{x}{d} + b \\ 0 & d \end{pmatrix}, 1\right) \cdot \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \kappa\right) \\
&= \left(\begin{pmatrix} 1 & \frac{x}{d^2} + \frac{b}{d} \\ 0 & 1 \end{pmatrix}, 1\right) \cdot \left(\begin{pmatrix} \mathrm{sgn}(d) & 0 \\ 0 & \mathrm{sgn}(d) \end{pmatrix}, 1\right) \cdot \left(\begin{pmatrix} |d|^{-1} & 0 \\ 0 & |d| \end{pmatrix}, 1\right) \cdot \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \kappa\right).
\end{aligned}$$

Thus

$$(\pi(\widetilde{g})f)_0(x) = \kappa|d|^{\nu-1}\mathrm{sgn}(d)^{\epsilon/2} f_0\left(\frac{x}{d^2}+\frac{b}{d}\right),$$

as an equality between distributions on $\mathbb{R}$. This proves parts (a) and (b).

Recall that by (4.2) we have

$$\begin{aligned}\left(\pi\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix},\kappa\right)^{-1} f\right)_\infty(x) &= s_f\left(\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix},\kappa\right)\widetilde{s}^{-1}\widetilde{n}_x B(\widetilde{\mathrm{SL}_2})\right)\\ &= s_f\left(\widetilde{s}^{-1}\left(\begin{pmatrix} d & -c \\ -b & a\end{pmatrix},\kappa'\right)\widetilde{n}_x B(\widetilde{\mathrm{SL}_2})\right),\end{aligned}$$

where s_f is the distributional section of a vector bundle over $\widetilde{\mathrm{SL}_2}/B(\widetilde{\mathrm{SL}_2})$ which corresponds to f; see section 1.2 for more details on this correspondence. Thus we can repeat the argument in the prior paragraph with $\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix},\kappa\right)$ replaced by $\left(\begin{pmatrix} d & -c \\ -b & a\end{pmatrix},\kappa'\right)$ to prove the remainder of our lemma. □

Recall that in (3.44) we defined

$$\widetilde{\Gamma}_1(4) = \left\{\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix},\left(\frac{c}{d}\right)\right)\in\widetilde{\mathrm{SL}_2^{\pm}} : a,b,c,d\in\mathbb{Z},\ a\equiv d\equiv 1(\mathrm{mod}\,4),\ c\equiv 0(\mathrm{mod}\,4)\right\},$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol. One can show that $\widetilde{\Gamma}_1(4)$ is a well-defined subgroup of $\widetilde{\mathrm{SL}_2}$. Since we will be performing several computations involving the Kronecker symbol, we state (without proof) some of its important properties. More information about the Kronecker symbol can be found in [1, §3.4.3], [4, §5].

Proposition 4.2 (Properties of the Kronecker Symbol)**.** *Let $a,b,m,n\in\mathbb{Z}$.*

- *If p is an odd prime then*

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } a\equiv 0(\mathrm{mod}\,p),\\ 1 & \text{if there exists } x\in\mathbb{Z} \text{ such that } x^2\equiv a(\mathrm{mod}\,p),\\ -1 & \text{if there are no solutions to } x^2\equiv a(\mathrm{mod}\,p).\end{cases}$$

- $\left(\frac{a}{2}\right) = \begin{cases} 0 & \text{if } a\equiv 0(\mathrm{mod}\,2),\\ 1 & \text{if } a\equiv 1,7(\mathrm{mod}\,8)\cdot\\ -1 & \text{if } a\equiv 3,5(\mathrm{mod}\,8)\end{cases}$

- $\left(\frac{a}{1}\right)=1$, $\left(\frac{a}{-1}\right)=\begin{cases}-1 & \text{if } a<0,\\ 1 & \text{if } a\geq 0,\end{cases}$ *and* $\left(\frac{a}{0}\right)=\begin{cases}1 & \text{if } a=\pm 1\\ 0 & \text{otherwise.}\end{cases}$
- *If* $n=\pm p_1^{e_1}\cdots p_k^{e_k}$ *is the prime factorization of* n *then*

$$\left(\frac{a}{n}\right)=\left(\frac{a}{\pm 1}\right)\left(\frac{a}{p_1}\right)^{e_1}\cdots\left(\frac{a}{p_k}\right)^{e_k}.$$

- *If* $ab\neq 0$ *and* $n\neq -1$ *then* $\left(\frac{a}{n}\right)\left(\frac{b}{n}\right)=\left(\frac{ab}{n}\right)$.
- *If* $mn\neq 0$ *and* $a\neq -1$ *then* $\left(\frac{a}{m}\right)\left(\frac{a}{n}\right)=\left(\frac{a}{mn}\right)$.
- *If* $n>0$ *then* $\left(\frac{a}{n}\right)=\left(\frac{b}{n}\right)$ *if* $a\equiv b(\operatorname{mod} m)$ *where* $m=\begin{cases}4n & \text{if } n\equiv 2(\operatorname{mod} 4),\\ n & \text{otherwise.}\end{cases}$
- *If* $a\not\equiv 3(\operatorname{mod} 4)$ *and* $a\neq 0$ *then* $\left(\frac{a}{m}\right)=\left(\frac{a}{n}\right)$ *if* $m\equiv n(\operatorname{mod} c)$ *where* $c=\begin{cases}4a & \text{if } a\equiv 2(\operatorname{mod} 4),\\ a & \text{otherwise.}\end{cases}$
- *If* p,q *are positive odd integers then* $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)=(-1)^{\frac{(p-1)(q-1)}{4}}$.

Let δ_0 denote the Dirac distribution on $\mathbb{R}$ centered at zero. By (3.16), the pair $(\delta_0,0)$ defines an element of $V^{-\infty}_{(\epsilon),\nu}(\widetilde{\mathrm{SL}}_2)$. We abuse notation by writing this element of $V^{-\infty}_{(\epsilon),\nu}(\widetilde{\mathrm{SL}}_2)$ as δ_0. Let $\delta_\infty=\pi(\widetilde{s})\delta_0$, where

$$\widetilde{s}=\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},1\right).$$

Observe that

$$(\delta_\infty)_0=0 \text{ and } (\delta_\infty)_\infty=-\epsilon\, i\,\delta_0. \tag{4.3}$$

Lemma 4.3. *Let* $\widetilde{\gamma}^{-1}=\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix},\left(\frac{c}{d}\right)\right)\in\widetilde{\Gamma}_1(4)$.

(a) We have

$$(\pi(\widetilde{\gamma})\delta_\infty)_0=\begin{cases}\left(\frac{c}{d}\right)|c|^{-\nu-1}\mathrm{sgn}(c)^{\epsilon/2}\delta_{-d/c} & \text{if } c\neq 0,\\ 0 & \text{if } c=0,\end{cases}$$

as an equality between distributions on $\mathbb{R}$.

(b) We have

$$(\pi(\widetilde{\gamma})\delta_\infty)_\infty=\begin{cases}-\epsilon\, i\,(-c,d)_H\left(\frac{c}{d}\right)|d|^{-\nu-1}\mathrm{sgn}(d)^{\epsilon/2}\delta_{\frac{c}{d}} & \text{if } bc\neq 0,\\ -\epsilon\, i\,\delta_0 & \text{if } b\neq 0, c=0,\\ -\epsilon\, i\,\delta_c & \text{if } b=0,\end{cases}$$

as an equality between distributions on $\mathbb{R}$.

Proof. By Lemma 4.1(a,b) we have that

$$(\pi(\widetilde{\gamma})\delta_\infty)_0(x) = 0, \tag{4.4}$$

as an equality between distributions on $\mathbb{R}_{\neq \frac{-d}{c}}$ when $c \neq 0$, and as an equality between distributions on $\mathbb{R}$ when $c = 0$. Thus it remains to describe $(\pi(\widetilde{\gamma})\delta_\infty)_0$ about the point $\frac{-d}{c}$ when $c \neq 0$. To do this, first observe that $a, d \neq 0$ since $\widetilde{\gamma} \in \widetilde{\Gamma}_1(4)$. Next observe that

$$\widetilde{\gamma}\widetilde{s} = \left(\begin{pmatrix} -b & -d \\ a & c \end{pmatrix}, \left(\frac{c}{d}\right)(c,a)_H\right) = \left(\begin{pmatrix} c & d \\ -a & -b \end{pmatrix}, \left(\frac{c}{d}\right)(c,a)_H\right)^{-1}. \tag{4.5}$$

Thus by Lemma 4.1(a), we have that

$$\begin{aligned}(\pi(\widetilde{\gamma})\delta_\infty)_0(x) &= (\pi(\widetilde{\gamma}\widetilde{s})\delta_0)_0(x) \\ &= \left(\frac{c}{d}\right)(c,a)_H(a,-ax-b)_H|ax+b|^{\nu-1}\mathrm{sgn}(-ax-b)^{\epsilon/2}\delta_0\left(\frac{cx+d}{-ax-b}\right),\end{aligned} \tag{4.6}$$

as an equality between distributions on $\mathbb{R}_{\neq \frac{-b}{a}}$.

We can simplify our expression in (4.6). To do so, consider ϕ a test function on $\mathbb{R}_{\neq \frac{-b}{a}}$. By performing various changes of variables, we have that

$$\begin{aligned}&\int_{\mathbb{R}_{\neq \frac{-b}{a}}} (\pi(\widetilde{\gamma})\delta_\infty)_0(x)\phi(x)\,dx \\ &= \int_{\mathbb{R}_{\neq \frac{-b}{a}}} \left(\frac{c}{d}\right)(c,a)_H(a,-ax-b)_H|ax+b|^{\nu-1}\mathrm{sgn}(-ax-b)^{\epsilon/2}\delta_0\left(\frac{cx+d}{-ax-b}\right)\phi(x)\,dx \\ &= \int_{\mathbb{R}_{\neq 0}} \left(\frac{c}{d}\right)(c,a)_H(a,-ax)_H|ax|^{\nu-1}\mathrm{sgn}(-ax)^{\epsilon/2}\delta_0\left(\frac{cx - \frac{bc}{a} + d}{-ax}\right)\phi\left(x - \frac{b}{a}\right)dx \\ &= \int_{\mathbb{R}_{\neq 0}} \left(\frac{c}{d}\right)(c,a)_H(a,-ax)_H|ax|^{\nu-1}\mathrm{sgn}(-ax)^{\epsilon/2}\delta_0\left(\frac{-c}{a} - \frac{1}{a^2x}\right)\phi\left(x - \frac{b}{a}\right)dx \\ &= \int_{\mathbb{R}_{\neq 0}} x^{-2}\left(\frac{c}{d}\right)(c,a)_H(a,ax^{-1})_H|ax^{-1}|^{\nu-1}\mathrm{sgn}(ax^{-1})^{\epsilon/2}\delta_0\left(\frac{-c}{a} + \frac{x}{a^2}\right) \\ &\qquad \cdot \phi\left(-x^{-1} - \frac{b}{a}\right)dx \\ &= \int_{\mathbb{R}_{\neq 0}} (ax)^{-2}\left(\frac{c}{d}\right)(c,a)_H(a,(ax)^{-1})_H|(ax)^{-1}|^{\nu-1}\mathrm{sgn}((ax)^{-1})^{\epsilon/2}\delta_0\left(\frac{-c}{a} + x\right) \\ &\qquad \cdot \phi\left(-(a^2x)^{-1} - \frac{b}{a}\right)dx \\ &= c^{-2}\left(\frac{c}{d}\right)(c,a)_H(a,c^{-1})_H|c^{-1}|^{\nu-1}\mathrm{sgn}(c^{-1})^{\epsilon/2}\phi\left(-\frac{1}{ac} - \frac{b}{a}\right) \\ &= \left(\frac{c}{d}\right)|c|^{-\nu-1}\mathrm{sgn}(c)^{\epsilon/2}\phi\left(-\frac{d}{c}\right) \\ &= \left(\frac{c}{d}\right)|c|^{-\nu-1}\mathrm{sgn}(c)^{\epsilon/2}\int_{\mathbb{R}_{\neq \frac{-b}{a}}} \delta_{-d/c}(x)\phi(x)\,dx.\end{aligned}$$

In this last equality we have used the fact that $\frac{-b}{a} \neq \frac{-d}{c}$. Thus for $c \neq 0$, we have

$$(\pi(\widetilde{\gamma})\delta_\infty)_0 = \left(\frac{c}{d}\right)|c|^{-\nu-1}\mathrm{sgn}(c)^{\epsilon/2}\delta_{-d/c}, \tag{4.7}$$

as an equality between distributions on $\mathbb{R}_{\neq \frac{-b}{a}}$. Since $\frac{-b}{a} \neq \frac{-d}{c}$ then it follows from (4.4) that $(\pi(\widetilde{\gamma})\delta_\infty)_0$ vanishes about the point $\frac{-b}{a}$. Thus we conclude that (4.7) holds as an equality between distributions on $\mathbb{R}$. This proves part (a) of our lemma.

To prove part (b) of our lemma, recall that if $c = 0$ then we must have $a = d = 1$ since $\widetilde{\gamma} \in \widetilde{\Gamma}_1(4)$. Thus if $c = 0$, then

$$\widetilde{\gamma}\widetilde{s} = \left(\begin{pmatrix} -b & -1 \\ 1 & 0 \end{pmatrix}, 1\right) = \left(\begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix}, 1\right)^{-1}.$$

Therefore when $c = 0$, it follows from this equality and Lemma 4.1(c) that

$$(\pi(\widetilde{\gamma})\delta_\infty)_\infty = (\pi(\widetilde{\gamma}\widetilde{s})\delta_0)_\infty = 0, \tag{4.8}$$

as an equality between distributions on $\mathbb{R}_{\neq 0}$. If on the other hand we have $c \neq 0$, then it follows from (4.5) and Lemma 4.1(c) that

$$(\pi(\widetilde{\gamma})\delta_\infty)_\infty = (\pi(\widetilde{\gamma}\widetilde{s})\delta_0)_\infty = 0, \tag{4.9}$$

as an equality between distributions on $\mathbb{R}_{\neq \frac{c}{d}}$. Thus it remains to describe $(\pi(\widetilde{\gamma})\delta_\infty)_\infty$ about the point $\frac{c}{d}$, both for the case of $c = 0$ and the case of $c \neq 0$. To do this, we observe that for $bc \neq 0$ we have

$$\widetilde{\gamma}^{-1}\widetilde{s}^{-1} = \widetilde{s}^{-1}\left(\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}, (a,d)_H\left(\frac{c}{d}\right)\right),^1 \tag{4.10}$$

and for $bc = 0$ (which implies $a = d = 1$ since $\widetilde{\gamma} \in \widetilde{\Gamma}_1(4)$) we have

$$\widetilde{\gamma}^{-1}\widetilde{s}^{-1} = \widetilde{s}^{-1}\left(\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}, 1\right) \quad \text{if } b \neq 0, c = 0, \tag{4.11}$$

$$\widetilde{\gamma}^{-1}\widetilde{s}^{-1} = \widetilde{s}^{-1}\left(\begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}, 1\right) \quad \text{if } b = 0. \tag{4.12}$$

Notice that (4.10), (4.11), and (4.12) are equalities of the form (4.2). Thus by (4.10), (4.11),

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[1]Specifically, one calculates that $\widetilde{s}\widetilde{\gamma}^{-1}\widetilde{s}^{-1}$ is equal to $\left(\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}, (a,d)_H\left(\frac{c}{d}\right)\right)$. When performing such a computation with a computer algebra system, it is necessary to use the fact that $ad - bc = 1$ to deduce this equality.

and Lemma 4.1(c), it follows that for $b \neq 0$ we have

$$\begin{aligned}
&(\pi(\widetilde{\gamma})\delta_\infty)_\infty(x) \\
&= (a,d)_H\left(\frac{c}{d}\right)(b,-bx+a)_H|-bx+a|^{\nu-1}\mathrm{sgn}(-bx+a)^{\epsilon/2}(\delta_\infty)_\infty\left(\frac{dx-c}{-bx+a}\right), \\
&= -\epsilon i(a,d)_H\left(\frac{c}{d}\right)(b,-bx+a)_H|-bx+a|^{\nu-1}\mathrm{sgn}(-bx+a)^{\epsilon/2}\delta_0\left(\frac{dx-c}{-bx+a}\right), \qquad (4.13)
\end{aligned}$$

as an equality between distributions on $\mathbb{R}_{\neq\frac{a}{b}}$.[2] In the case where $b = 0$ (which implies $a = d = 1$ since $\widetilde{\gamma} \in \widetilde{\Gamma}_1(4)$), it follows from (4.12) and Lemma 4.1(d) that

$$(\pi(\widetilde{\gamma})\delta_\infty)_\infty(x) = -\epsilon\, i\, \delta_c(x)$$

as an equality between distributions on $\mathbb{R}$.

We can simplify our expression in (4.13). To do so, consider ϕ a test function on $\mathbb{R}_{\neq\frac{a}{b}}$. Upon performing various changes of variables, we find that

$$\begin{aligned}
&\int_{\mathbb{R}_{\neq\frac{a}{b}}} (\pi(\widetilde{\gamma})\delta_\infty)_\infty(x)\phi(x)\,dx \\
&= -\epsilon i\int_{\mathbb{R}_{\neq\frac{a}{b}}} (a,d)_H\left(\frac{c}{d}\right)(b,-bx+a)_H|-bx+a|^{\nu-1}\mathrm{sgn}(-bx+a)^{\epsilon/2}\delta_0\left(\frac{dx-c}{-bx+a}\right)\phi(x)\,dx \\
&= -\epsilon i\int_{\mathbb{R}_{\neq 0}} (a,d)_H\left(\frac{c}{d}\right)(b,-bx)_H|-bx|^{\nu-1}\mathrm{sgn}(-bx)^{\epsilon/2}\delta_0\left(\frac{dx+\frac{ad}{b}-c}{-bx}\right)\phi\left(x+\frac{a}{b}\right)dx \\
&= -\epsilon i\int_{\mathbb{R}_{\neq 0}} (a,d)_H\left(\frac{c}{d}\right)(b,-bx)_H|bx|^{\nu-1}\mathrm{sgn}(-bx)^{\epsilon/2}\delta_0\left(\frac{-d}{b}-\frac{1}{b^2x}\right)\phi\left(x+\frac{a}{b}\right)dx \\
&= -\epsilon i\int_{\mathbb{R}_{\neq 0}} (a,d)_H\left(\frac{c}{d}\right)\left(b,\frac{-x}{b}\right)_H|b|^{-\nu-1}|x|^{\nu-1}\mathrm{sgn}\left(\frac{-x}{b}\right)^{\epsilon/2}\delta_0\left(\frac{-d}{b}-\frac{1}{x}\right)\phi\left(\frac{x}{b^2}+\frac{a}{b}\right)dx \\
&= -\epsilon i\int_{\mathbb{R}_{\neq 0}} (a,d)_H\left(\frac{c}{d}\right)\left(b,\frac{1}{bx}\right)_H|b|^{-\nu-1}|x|^{-\nu-1}\mathrm{sgn}\left(\frac{1}{bx}\right)^{\epsilon/2}\delta_0\left(\frac{-d}{b}+x\right)\phi\left(\frac{-1}{b^2x}+\frac{a}{b}\right)dx \\
&= -\epsilon i(a,d)_H\left(\frac{c}{d}\right)\left(b,\frac{1}{d}\right)_H|d|^{-\nu-1}\mathrm{sgn}\left(\frac{1}{d}\right)^{\epsilon/2}\phi\left(\frac{-1}{bd}+\frac{a}{b}\right) \\
&= -\epsilon i(a,d)_H(b,d)_H\left(\frac{c}{d}\right)|d|^{-\nu-1}\mathrm{sgn}(d)^{\epsilon/2}\int_{\mathbb{R}_{\neq\frac{a}{b}}} \delta_{\frac{c}{d}}(x)\phi(x)\,dx.
\end{aligned}$$

In this last inequality we have used the fact that $\frac{a}{b} \neq \frac{c}{d}$. Thus for $b \neq 0$ we have

$$(\pi(\widetilde{\gamma})\delta_\infty)_\infty = -\epsilon i(a,d)_H(b,d)_H\left(\frac{c}{d}\right)|d|^{-\nu-1}\mathrm{sgn}(d)^{\epsilon/2}\delta_{\frac{c}{d}}, \qquad (4.14)$$

as an equality between distributions on $\mathbb{R}_{\neq\frac{a}{b}}$. Since $\frac{a}{b} \neq \frac{c}{d}$ it follows from (4.9) that $(\pi(\widetilde{\gamma})\delta_\infty)_\infty$ vanishes about the point $\frac{a}{b}$, and thus we can conclude that (4.14) holds as an equality between

[2] This equality does indeed hold when $b \neq 0$ and $c = 0$ since if $c = 0$ then $a = d = 1$, and thus $(a,d)_H\left(\frac{c}{d}\right) = 1$.

distributions on $\mathbb{R}$. Notice that the $b \neq 0$, $c = 0$ case in part (b) follows immediately from (4.14) since if $c = 0$ we have $a = d = 1$.

We can simplify our formula in (4.14) when $bc \neq 0$; in particular, $(a, d)_H(b, d)_H$ can be written more concisely. Observe that if $d > 0$ then it follows that $(a, d)_H(b, d)_H = (-c, d)_H$ since $(x, d)_H = 1$ whenever $d > 0$. We wish to prove that this is also the case for $d < 0$. To do so, recall that $(x, z)_H(y, z)_H = (xy, z)_H$ for $x, y, z \in \mathbb{R}_{\neq 0}$. Therefore $(a, d)(b, d)(-c, d) = (a, d)(-bc, d)$. If $d < 0$ and $a > 0$ then it follows that $ad < 0$, which implies that $-bc > 1$; thus $(a, d)(-bc, d) = 1$. If $a, d < 0$ then it follows that $ad > 0$, which implies that $-bc < 1$. Since we have assumed that $bc \neq 0$ we can conclude that $-bc < 0$ and thus $(a, d)(-bc, d) = 1$. Thus, we have that $(a, d)(b, d)(-c, d) = 1$, which implies that $(a, d)(b, d) = (-c, d)$. Therefore, when $bc \neq 0$ we have

$$(\pi(\widetilde{\gamma})\delta_\infty)_\infty = -\epsilon\, i\, (-c, d)_H \left(\frac{c}{d}\right) |d|^{-\nu-1} \operatorname{sgn}(d)^{\epsilon/2} \delta_{\frac{c}{d}} \tag{4.15}$$

as an equality between distributions on $\mathbb{R}$. This proves part (b) of our lemma. $\square$

Let

$$\widetilde{\Gamma}_\infty = \left\{ \left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, 1 \right) : n \in \mathbb{Z} \right\}.$$

For $\Re(\nu) > 1$, we define the *metaplectic Eisenstein distribution at* ∞ to be the following distribution in $V^{-\infty}_{(\epsilon),\nu}(\widetilde{\mathrm{SL}}_2)$:

$$\widetilde{E}^{(\infty)}_\nu(\widetilde{g}) = \zeta_2(2\nu + 1) \sum_{\widetilde{\gamma} \in \widetilde{\Gamma}_1(4)/\widetilde{\Gamma}_\infty} \pi(\widetilde{\gamma})\delta_\infty(\widetilde{g}), \tag{4.16}$$

where

$$\zeta_2(s) = \prod_{p \text{ odd prime}} (1 - p^{-s})^{-1}.$$

The summation over $\widetilde{\Gamma}_1(4)/\widetilde{\Gamma}_\infty$ is justified since by Lemma 4.3, we have that δ_∞ is $\widetilde{\Gamma}_\infty$-invariant under the left regular representation. Thus $\widetilde{E}_\nu$ is at least formally $\widetilde{\Gamma}_1(4)$-invariant. We shall justify the convergence of this series momentarily.

Observe that for $\widetilde{\gamma}^{-1} = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \left(\frac{c}{d}\right) \right)$ we have that

$$\widetilde{\gamma} \cdot \left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, 1 \right) = \left(\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \left(\frac{c}{d}\right) \right) \cdot \left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, 1 \right) = \left(\begin{pmatrix} d & dn - b \\ -c & -cn + a \end{pmatrix}, \left(\frac{c}{d}\right) \right).$$

Thus, for each coset of $\widetilde{\Gamma}_1(4)/\widetilde{\Gamma}_\infty$ there corresponds unique $(c, d) \in \mathbb{Z}^2$ such that $\gcd(c, d) = 1$, $c \equiv 0 (\operatorname{mod} 4)$, and $d \equiv 1 (\operatorname{mod} 4)$; indeed, such (c, d) is unique since the above equality shows that if $\widetilde{\gamma}\widetilde{\Gamma}_\infty = \widetilde{\gamma}'\widetilde{\Gamma}_\infty$ for some $\widetilde{\gamma}' \in \widetilde{\Gamma}_1(4)$, then

$$(\widetilde{\gamma}')^{-1} = \left(\begin{pmatrix} * & * \\ c & d \end{pmatrix}, \left(\frac{c}{d}\right) \right).$$

Conversely, when given $(c,d) \in \mathbb{Z}^2$ such that $\gcd(c,d) = 1$, $c \equiv 0 (\bmod 4)$, and $d \equiv 1 (\bmod 4)$, it follows that there exists $a, b \in \mathbb{Z}$ such that $ad - bc = 1$. Since $c \equiv 0 (\bmod 4)$ and $d \equiv 1 (\bmod 4)$ it follows that $a \equiv 1 (\bmod 4)$. Thus we are able to construct $\widetilde{\gamma}$ which corresponds to such (c,d). Therefore

$$\widetilde{\Gamma}_1(4)/\widetilde{\Gamma}_\infty \cong \{(c,d) \in \mathbb{Z}^2 : \gcd(c,d) = 1, c \equiv 0 (\bmod 4), d \equiv 1 (\bmod 4)\}.$$

Thus by Lemma 4.3, we have

$$\left(\widetilde{E}_\nu^{(\infty)}\right)_0(x) = \zeta_2(2\nu+1) \sum_{\substack{(c,d) \in \mathbb{Z}_{\neq 0} \times \mathbb{Z} \\ \gcd(c,d)=1 \\ d \equiv 1 (\bmod 4) \\ c \equiv 0 (\bmod 4)}} \left(\frac{c}{d}\right) |c|^{-\nu-1} \operatorname{sgn}(c)^{\epsilon/2} \delta_{-\frac{d}{c}}(x), \tag{4.17}$$

and

$$\left(\widetilde{E}_\nu^{(\infty)}\right)_\infty(x) = -\epsilon\, i\, \zeta_2(2\nu+1) \left(\delta_0(x) + \sum_{\substack{(c,d) \in \mathbb{Z}_{\neq 0} \times \mathbb{Z} \\ \gcd(c,d)=1 \\ d \equiv 1 (\bmod 4) \\ c \equiv 0 (\bmod 4)}} (-c,d)_H \left(\frac{c}{d}\right) |d|^{-\nu-1} \operatorname{sgn}(d)^{\epsilon/2} \delta_{\frac{c}{d}}(x) \right). \tag{4.18}$$

Notice that for $\Re(\nu) > 1$, the integrals of (4.17) and (4.18) converge uniformly and absolutely against compactly supported test functions on $\mathbb{R}$. Therefore, since $\widetilde{E}_\nu^{(\infty)}$ is determined completely by $\left(\widetilde{E}_\nu^{(\infty)}\right)_0$ and $\left(\widetilde{E}_\nu^{(\infty)}\right)_\infty$, it follows that $\widetilde{E}_\nu^{(\infty)}$ depends holomorphically on ν for $\Re(\nu) > 1$. Furthermore, it follows that our series expansion for $\widetilde{E}_\nu^{(\infty)}$ converges in the strong distribution topology.

4.2 Fourier Coefficients of $\widetilde{E}_\nu^{(\infty)}$

Since $\left(\widetilde{E}_\nu^{(\infty)}\right)_0$ is periodic, it has a Fourier series expansion. In particular,

$$\left(\widetilde{E}_\nu^{(\infty)}\right)_0(x) = \sum_{n \in \mathbb{Z}} a_n e(nx)$$

where

$$a_n = \int_0^1 \left(\widetilde{E}_\nu^{(\infty)}\right)_0(x) e(-nx)\, dx.$$

By carefully calculating a_n we will be able to show in section 4.3 that $\widetilde{E}^{(\infty)}$ has meromorphic continuation to all of $\mathbb{C}$. By (4.17), observe

$$\begin{aligned}
\frac{a_n}{\zeta_2(2\nu+1)} &= \sum_{\substack{(c,d)\in\mathbb{Z}_{\neq 0}\times\mathbb{Z}\\ \gcd(c,d)=1\\ d\equiv 1 (\mathrm{mod}\, 4)\\ c\equiv 0(\mathrm{mod}\, 4)}} \int_0^1 \left(\frac{c}{d}\right)|c|^{-\nu-1}\mathrm{sgn}(c)^{\epsilon/2}\delta_{-\frac{d}{c}}(x)e(-nx)\,dx \\
&= \sum_{\substack{(c,d)\in\mathbb{Z}_{>0}\times\mathbb{Z}\\ 0\le -d<c\\ d\equiv 1 (\mathrm{mod}\, 4)\\ c\equiv 0(\mathrm{mod}\, 4)}} \left(\frac{c}{d}\right)|c|^{-\nu-1}e\left(\frac{nd}{c}\right) + \epsilon\, i \sum_{\substack{(c,d)\in\mathbb{Z}_{<0}\times\mathbb{Z}\\ 0\ge -d>c\\ d\equiv 1 (\mathrm{mod}\, 4)\\ c\equiv 0(\mathrm{mod}\, 4)}} \left(\frac{c}{d}\right)|c|^{-\nu-1}e\left(\frac{nd}{c}\right) \\
&= \sum_{\substack{(c,d)\in\mathbb{Z}_{>0}\times\mathbb{Z}\\ 0\le -d<c\\ d\equiv 1 (\mathrm{mod}\, 4)\\ c\equiv 0(\mathrm{mod}\, 4)}} \left(\frac{c}{d}\right)|c|^{-\nu-1}e\left(\frac{nd}{c}\right) + \epsilon\, i \sum_{\substack{(c,d)\in\mathbb{Z}_{>0}\times\mathbb{Z}\\ 0\ge -d>-c\\ d\equiv 1 (\mathrm{mod}\, 4)\\ c\equiv 0(\mathrm{mod}\, 4)}} \left(\frac{-c}{d}\right)|c|^{-\nu-1}e\left(\frac{nd}{-c}\right) \\
&= \sum_{\substack{(c,d)\in\mathbb{Z}_{>0}\times\mathbb{Z}\\ 0\le d<c\\ d\equiv 3 (\mathrm{mod}\, 4)\\ c\equiv 0(\mathrm{mod}\, 4)}} \left(\frac{c}{-d}\right)|c|^{-\nu-1}e\left(\frac{-nd}{c}\right) + \epsilon\, i \sum_{\substack{(c,d)\in\mathbb{Z}_{>0}\times\mathbb{Z}\\ 0\le d<c\\ d\equiv 1 (\mathrm{mod}\, 4)\\ c\equiv 0(\mathrm{mod}\, 4)}} \left(\frac{-c}{d}\right)|c|^{-\nu-1}e\left(\frac{nd}{-c}\right)
\end{aligned}$$

For $c>0$ we have $\left(\frac{c}{d}\right) = \left(\frac{c}{-d}\right)$, and for $d\equiv 1(\mathrm{mod}\,4)$ we have $\left(\frac{-c}{d}\right) = \left(\frac{c}{d}\right)$. Thus

$$\begin{aligned}
a_n &= \epsilon\, i\zeta_2(2\nu+1) \sum_{\substack{(c,d)\in\mathbb{Z}_{>0}\times\mathbb{Z}\\ 0\le d<c\\ c\equiv 0(\mathrm{mod}\, 4)}} \Delta_d^{-\epsilon}\left(\frac{c}{d}\right)c^{-\nu-1}e\left(-\frac{nd}{c}\right) \\
&= \epsilon\, i\zeta_2(2\nu+1) \sum_{\substack{(c,d)\in\mathbb{Z}_{>0}\times\mathbb{Z}\\ 0\le d<4c}} \Delta_d^{-\epsilon}\left(\frac{4c}{d}\right)(4c)^{-\nu-1}e\left(-\frac{nd}{4c}\right),
\end{aligned}$$

where

$$\Delta_d = \begin{cases} 1 & \text{if } d\equiv 1(\mathrm{mod}\,4), \\ i & \text{if } d\equiv 3(\mathrm{mod}\,4), \\ 0 & \text{otherwise.}\end{cases}$$

Let

$$K_\kappa(n;4c) = \sum_{d\in\mathbb{Z}/(4c)\mathbb{Z}} \Delta_d^{-\kappa}\left(\frac{4c}{d}\right)e\left(\frac{nd}{4c}\right), \tag{4.19}$$

$$G(n;c) = \sum_{x\in\mathbb{Z}/c\mathbb{Z}} \left(\frac{x}{c}\right)e\left(\frac{nx}{c}\right), \tag{4.20}$$

where $\kappa\equiv 1(\mathrm{mod}\,2)$. Thus

$$a_n = \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1)\sum_{c\in\mathbb{Z}_{>0}} c^{-\nu-1}K_\epsilon(-n;4c).$$

Since $\overline{K_\kappa(n;c)} = K_{-\kappa}(-n;c)$ it follows that

$$a_n = \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1)\sum_{c\in\mathbb{Z}_{>0}} c^{-\nu-1}\mathcal{C}_{-\epsilon}(K_{-1}(\epsilon\, n;4c)),$$

where

$$\mathcal{C}_\epsilon(z) = \begin{cases} z & \text{if } \epsilon = 1, \\ \bar{z} & \text{if } \epsilon = -1. \end{cases} \tag{4.21}$$

For $c = 2^k c'$ where $\gcd(2,c') = 1$ and $k \in \mathbb{Z}_{\geq 0}$, it follows from [8, Lemma 2] that

$$K_{-1}(n;4c) = K_{-1}(n;2^{k+2}c') = K_{-c'}(n\overline{c'};2^{k+2})G(n\overline{2^{k+2}};c'),$$

where $\overline{c'}$ and $\overline{2^{k+2}}$ are integers such that $c'\overline{c'} \equiv 1 (\operatorname{mod} 2^{k+2})$ and $2^{k+2}\overline{2^{k+2}} \equiv 1(\operatorname{mod} c')$. Thus

$$\begin{aligned} a_n &= \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1)\sum_{c\in\mathbb{Z}_{>0}} c^{-\nu-1}\mathcal{C}_{-\epsilon}(K_{-1}(\epsilon\, n;4c)) \\ &= \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1)\sum_{\substack{c'\in\mathbb{Z}_{>0} \\ c' \text{ odd}}}\sum_{k\in\mathbb{Z}_{\geq 0}} (2^k c')^{-\nu-1}\mathcal{C}_{-\epsilon}\Big(K_{-c'}(\epsilon\, n\overline{c'};2^{k+2})G(\epsilon\, n\overline{2^{k+2}};c')\Big). \end{aligned} \tag{4.22}$$

For the remainder of this section, we will prove two lemmas which we will use in section 4.3 to see that the Fourier coefficients a_n can be expressed in terms of Dirichlet L-functions.

Lemma 4.4. *Let $c > 0$ and odd.*

(a) If $\gcd(n,c) = 1$ and c is prime then $G(n;c) = \left(\frac{n}{c}\right)\Delta_c c^{1/2}$

(b) If $c = pq$ and $\gcd(p,q) = 1$ then $G(n;c) = (-1)^{\frac{(p-1)(q-1)}{4}}G(n;p)G(n;q)$. Consequently, $\Delta_c G(n;c)$ is multiplicative since $\Delta_p\Delta_q = (-1)^{\frac{(p-1)(q-1)}{4}}\Delta_{pq}$.

(c) If c is square-free and $\gcd(n,c) = 1$ then $G(n;c) = \left(\frac{n}{c}\right)\Delta_c c^{1/2}$.

(d) If $k \geq 2$, p is prime, and $n = p^\ell n'$ where $\gcd(n',p) = 1$ and $\ell \in \mathbb{Z}_{\geq 0}$, then

$$G(n;p^k) = \begin{cases} 0 & \text{if } \ell < k-1, \\ -p^{k-1} & \text{if } \ell = k-1, k \text{ even}, \\ \left(\frac{n'}{p}\right)\Delta_p p^{k-1/2} & \text{if } \ell = k-1, k \text{ odd}, \\ p^k - p^{k-1} & \text{if } \ell \geq k, k \text{ even}, \\ 0 & \text{if } \ell \geq k, k \text{ odd}. \end{cases}$$

(e) If c is not square-free then $G(1;c) = 0$.

Proof. Part (a) is a classical result due to Gauss [7]. For part (b), observe that since c is odd then we must have p and q be odd. Therefore, the law of quadratic reciprocity for the Jacobi symbol shows that

$$1 = (-1)^{\frac{(p-1)(q-1)}{4}} \left(\frac{q}{p}\right)\left(\frac{p}{q}\right).$$

By this equality and the multiplicativity of the Jacobi symbol, it follows that

$$\begin{aligned}
&G(n;p)G(n;q) = \sum_{x(\bmod p)} \left(\frac{x}{p}\right) e\left(\frac{nx}{p}\right) \sum_{y(\bmod q)} \left(\frac{y}{q}\right) e\left(\frac{ny}{q}\right) \\
&= (-1)^{\frac{(p-1)(q-1)}{4}} \left(\frac{q}{p}\right)\left(\frac{p}{q}\right) \sum_{x(\bmod p)} \sum_{y(\bmod q)} \left(\frac{x}{p}\right)\left(\frac{y}{q}\right) e\left(\frac{nxq+nyp}{pq}\right) \\
&= (-1)^{\frac{(p-1)(q-1)}{4}} \sum_{x(\bmod p)} \sum_{y(\bmod q)} \left(\frac{xq}{p}\right)\left(\frac{yp}{q}\right) e\left(\frac{n(xq+yp)}{pq}\right) \\
&= (-1)^{\frac{(p-1)(q-1)}{4}} \sum_{x(\bmod p)} \sum_{y(\bmod q)} \left(\frac{xq+yp}{p}\right)\left(\frac{xq+yp}{q}\right) e\left(\frac{n(xq+yp)}{pq}\right) \\
&= (-1)^{\frac{(p-1)(q-1)}{4}} \sum_{x(\bmod p)} \sum_{y(\bmod q)} \left(\frac{xq+yp}{pq}\right) e\left(\frac{n(xq+yp)}{pq}\right) \\
&= (-1)^{\frac{(p-1)(q-1)}{4}} G(n;pq).
\end{aligned}$$

To prove part (c), suppose $c = q_1 \ldots q_k$ where the q_i are distinct odd primes. We will prove part (c) by performing induction on k. If $k = 1$ then part (c) follows immediately from part (a). Suppose that we know part (c) holds when $k = \ell - 1$. Thus if $k = \ell$, it follows from part (b) and part (a) that

$$\begin{aligned}
&\Delta_c G(n;c) = \Delta_{q_1\cdots q_{\ell-1}} G(n; q_1 \cdots q_{\ell-1}) \Delta_{q_\ell} G(n; q_\ell) \\
&= \left(\frac{n}{q_1 \cdots q_{\ell-1}}\right) \Delta^2_{q_1\cdots q_{\ell-1}} (q_1 \cdots q_{\ell-1})^{1/2} \left(\frac{n}{q_\ell}\right) \Delta^2_{q_\ell} (q_\ell)^{1/2} = \Delta^2_{q_1\cdots q_{\ell-1}} \Delta^2_{q_\ell} \left(\frac{n}{c}\right) c^{1/2}.
\end{aligned}$$

Since Δ_q^2 is a character, it follows that $\Delta^2_{q_1\cdots q_{\ell-1}} \Delta^2_{q_\ell} = \Delta_c^2$. Thus part (c) holds for $k = \ell$. Therefore by induction, part (c) holds in general.

To prove part (d), observe that since $\gcd(n', p) = 1$, it follows that observe that

$$G(n;p^k) = \sum_{x(\bmod p^k)} \left(\frac{x}{p^k}\right) e\left(\frac{n'x}{p^{k-\ell}}\right) = \left(\frac{n'}{p^k}\right) \sum_{x(\bmod p^k)} \left(\frac{x}{p^k}\right) e\left(\frac{x}{p^{k-\ell}}\right)$$

Since $\ell < k$, we have then that

$$\begin{aligned} G(n;p^k) &= \left(\frac{n'}{p^k}\right) \sum_{y (\operatorname{mod} p^\ell)} \sum_{x (\operatorname{mod} p^{k-\ell})} \left(\frac{x+p^{k-\ell}y}{p^k}\right) e\left(\frac{x+p^{k-\ell}y}{p^{k-\ell}}\right) \\ &= \left(\frac{n'}{p^k}\right) \sum_{y (\operatorname{mod} p^\ell)} \sum_{x (\operatorname{mod} p^{k-\ell})} \left(\frac{x+p^{k-\ell}y}{p}\right)^k e\left(\frac{x+p^{k-\ell}y}{p^{k-\ell}}\right) \\ &= \left(\frac{n'}{p^k}\right) \sum_{y (\operatorname{mod} p^\ell)} \sum_{x (\operatorname{mod} p^{k-\ell})} \left(\frac{x}{p^k}\right) e\left(\frac{x}{p^{k-\ell}}\right) \\ &= \left(\frac{n'}{p^k}\right) p^\ell \sum_{x (\operatorname{mod} p^{k-\ell})} \left(\frac{x}{p^k}\right) e\left(\frac{x}{p^{k-\ell}}\right). \end{aligned} \tag{4.23}$$

Now that we have established (4.23), we can proceed to prove part (d) on a case by case.

- If $\ell < k-1$ then by (4.23) it follows that

$$\begin{aligned} G(n;p^k) &= \left(\frac{n'}{p^k}\right) p^\ell \sum_{x (\operatorname{mod} p^{k-\ell})} \left(\frac{x}{p}\right)^k e\left(\frac{x}{p^{k-\ell}}\right) \\ &= \left(\frac{n'}{p^k}\right) p^\ell \sum_{x (\operatorname{mod} p)} \sum_{y (\operatorname{mod} p^{k-\ell-1})} \left(\frac{x+yp}{p}\right)^k e\left(\frac{x+yp}{p^{k-\ell}}\right) \\ &= \left(\frac{n'}{p^k}\right) p^\ell \sum_{x (\operatorname{mod} p)} \sum_{y (\operatorname{mod} p^{k-\ell-1})} \left(\frac{x}{p^k}\right) e\left(\frac{x}{p^{k-\ell}}\right) e\left(\frac{y}{p^{k-\ell-1}}\right) \\ &= \left(\frac{n'}{p^k}\right) p^\ell \sum_{x (\operatorname{mod} p)} \left(\frac{x}{p^k}\right) e\left(\frac{x}{p^{k-\ell}}\right) \sum_{y (\operatorname{mod} p^{k-\ell-1})} e\left(\frac{y}{p^{k-\ell-1}}\right) \\ &= 0. \end{aligned}$$

- If $\ell = k-1$ then by (4.23) it follows that

$$G(n;p^k) = \left(\frac{n'}{p^k}\right) p^\ell \sum_{x (\operatorname{mod} p)} \left(\frac{x}{p^k}\right) e\left(\frac{x}{p}\right).$$

  If k is even then $\left(\frac{x}{p^k}\right) = 1$ for $x \not\equiv 0 (\operatorname{mod} p)$, and since $\sum_{\substack{x (\operatorname{mod} p) \\ x \not\equiv 0 (\operatorname{mod} p)}} e\left(\frac{x}{p}\right) = -1$, we have in this case that

$$G(n;p^k) = -\left(\frac{n'}{p^k}\right) p^\ell.$$

  If k is odd then $\sum_{x (\operatorname{mod} p)} \left(\frac{x}{p^k}\right) e\left(\frac{x}{p}\right) = G(1;p)$, and so in this case we have that

$$G(n;p^k) = \left(\frac{n'}{p^k}\right) \Delta_p p^{\ell+1/2}.$$

- If $\ell \geq k$ then

$$G(n;p^k) = \sum_{x (\operatorname{mod} p^k)} \left(\frac{x}{p^k}\right) e\left(\frac{nx}{p^k}\right) = \sum_{x (\operatorname{mod} p^k)} \left(\frac{x}{p^k}\right).$$

If k is even then $G(n;p^k) = p^k - p^{k-1}$. If k is odd then

$$\begin{aligned} G(n;p^k) &= \sum_{x(\mathrm{mod}\, p^k)} \left(\frac{x}{p}\right) = \sum_{x(\mathrm{mod}\, p)} \sum_{y(\mathrm{mod}\, p^{k-1})} \left(\frac{x+py}{p}\right) \\ &= \sum_{x(\mathrm{mod}\, p)} \sum_{y(\mathrm{mod}\, p^{k-1})} \left(\frac{x}{p}\right) = 0. \end{aligned}$$

To prove part (e), observe that $c = q_1^{e_1} q_2^{e_1} \dots q_j^{e_m}$ where $e_j \in \mathbb{Z}_{>0}$ and the q_j are distinct primes. Since c is not square-free, we can arrange to have $e_1 \geq 2$. By part (b) it follows that

$$G(1;c) = (-1)^{\frac{(q_1^{e_1}-1)(q_2^{e_2}\dots q_j^{e_j}-1)}{4}} G(1;q_1^{e_1}) G(1; q_2^{e_2} \dots q_j^{e_j}).$$

By part (d), we have that $G(1,q_1^{e_1}) = 0$. Thus part (e) follows. $\square$

Let

$$\chi_4(c) = \begin{cases} 1 & \text{if } c \equiv 1 (\mathrm{mod}\, 4), \\ -1 & \text{if } c \equiv 3 (\mathrm{mod}\, 4), \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad \chi_8(c) = \begin{cases} 1 & \text{if } c \equiv 1 (\mathrm{mod}\, 8), \\ 1 & \text{if } c \equiv 3 (\mathrm{mod}\, 8), \\ -1 & \text{if } c \equiv 5 (\mathrm{mod}\, 8), \\ -1 & \text{if } c \equiv 7 (\mathrm{mod}\, 8), \\ 0 & \text{otherwise}. \end{cases}$$

Lemma 4.5. *Let $c > 0$ and odd.*

(a) If $n \equiv 1 (\mathrm{mod}\, 4)$ then

$$\Delta_c^{-1} K_{-c}(n\bar{c}; 4) = (1+i)\chi_4(c), \quad \textit{where } \bar{c} \in \mathbb{Z} \textit{ such that } c\bar{c} \equiv 1 (\mathrm{mod}\, 4),$$

$$\Delta_c^{-1} K_{-c}(n\bar{c}; 8) = 2^{3/2}(1+i)\chi_8(c), \quad \textit{where } \bar{c} \in \mathbb{Z} \textit{ such that } c\bar{c} \equiv 1 (\mathrm{mod}\, 8),$$

$$\Delta_c^{-1} K_{-c}(n\bar{c}; 2^{k+2}) = 0, \quad \textit{where } \bar{c} \in \mathbb{Z} \textit{ such that } c\bar{c} \equiv 1 (\mathrm{mod}\, 2^{k+2}),$$

for $k \geq 2$.

(b) If $n \equiv 3 (\mathrm{mod}\, 4)$ then

$$\Delta_c^{-1} K_{-c}(n\bar{c}; 4) = -(1+i)\chi_4(c), \quad \textit{where } \bar{c} \in \mathbb{Z} \textit{ such that } c\bar{c} \equiv 1 (\mathrm{mod}\, 4),$$

$$\Delta_c^{-1} K_{-c}(n\bar{c}; 8) = 0, \quad \textit{where } \bar{c} \in \mathbb{Z} \textit{ such that } c\bar{c} \equiv 1 (\mathrm{mod}\, 8),$$

$$\Delta_c^{-1} K_{-c}(n\bar{c}; 2^{k+2}) = 0, \quad \textit{where } \bar{c} \in \mathbb{Z} \textit{ such that } c\bar{c} \equiv 1 (\mathrm{mod}\, 2^{k+2}),$$

for $k \geq 2$.

(c) If $n = 2^\ell n'$, $\ell > 0$, *and* $\gcd(n', 2) = 1$, *then*

$$\Delta_c^{-1} K_{-c}(n\bar{c}; 4) = \begin{cases} (1+i)\chi_4(c) & \textit{if } \ell > 1, \\ -(1+i)\chi_4(c) & \textit{if } \ell = 1, \end{cases} \quad \textit{where } \bar{c} \in \mathbb{Z} \textit{ such that } c\bar{c} \equiv 1 (\operatorname{mod} 4)$$

$$\Delta_c^{-1} K_{-c}(n\bar{c}; 8) = 0, \quad \textit{where } \bar{c} \in \mathbb{Z} \textit{ such that } c\bar{c} \equiv 1 (\operatorname{mod} 8),$$

and for $k \geq 2$ *and* $\bar{c} \in \mathbb{Z}$ *such that* $c\bar{c} \equiv 1 (\operatorname{mod} 2^{k+2})$, *we have that:*

(1) if $\ell \geq k+2$ *then*

$$\Delta_c^{-1} K_{-c}(n\bar{c}; 2^{k+2}) = \begin{cases} (1+i)2^k \chi_4(c) & \textit{if } k \textit{ even}, \\ 0 & \textit{if } k \textit{ odd}, \end{cases}$$

(2) if $\ell = k+1$ *then*

$$\Delta_c^{-1} K_{-c}(n\bar{c}; 2^{k+2}) = \begin{cases} -(1+i)2^k \chi_4(c) & \textit{if } k \textit{ even}, \\ 0 & \textit{if } k \textit{ odd}, \end{cases}$$

(3) if $\ell = k$ *then*

$$\Delta_c^{-1} K_{-c}(n\bar{c}; 2^{k+2}) = \begin{cases} (1+i)2^k \chi_4(n'c) & \textit{if } k \textit{ even}, \\ 0 & \textit{if } k \textit{ odd}, \end{cases}$$

(4) if $\ell = k-1$ *then*

$$\Delta_c^{-1} K_{-c}(n\bar{c}; 2^{k+2}) = \begin{cases} 0 & \textit{if } k \textit{ even}, \\ \frac{1+i}{\sqrt{2}} 2^{k+1} \chi_8(n'c') & \textit{if } k \textit{ odd}, \end{cases}$$

(5) if $\ell \leq k-2$ *then*

$$\Delta_c^{-1} K_{-c}(n\bar{c}; 2^{k+2}) = 0.$$

Proof. Observe that for $k \geq 2$ we have that

$$\Delta_c^{-1} K_{-c}(n\bar{c}; 2^k) = \Delta_b^{-1} K_{-b}(m\bar{b}; 2^k)$$

if $c \equiv b (\operatorname{mod} 2^k)$ and $n \equiv m (\operatorname{mod} 2^k)$. Thus the evaluation of $\Delta_c^{-1} K_{-c}(n\bar{c}; 4)$ and $\Delta_c^{-1} K_{-c}(n\bar{c}; 8)$ in parts (a), (b), and (c) follows from a finite number of computations which are easy to perform. The evaluation of $\Delta_c^{-1} K_{-c}(n\bar{c}; 2^{k+2})$ for $k \geq 2$ in parts (a) and (b) are a consequence of part (c.5).

If $k \geq 1$ then

$$
\begin{aligned}
&K_{-c}(n\bar{c}; 2^{k+2}) \\
&= \sum_{d \pmod{2^{k+2}}} \Delta_d^c \left(\frac{2^{k+2}}{d}\right) e\left(\frac{n\bar{c}d}{2^{k+2}}\right) = \sum_{\substack{d \pmod{2^{k+2}} \\ d \text{ odd}}} \Delta_d^c \left(\frac{2^{k+2}}{d}\right) e\left(\frac{n\bar{c}d}{2^{k+2}}\right) \\
&= \sum_{\substack{d \pmod{2^{k+2}} \\ d \equiv 1 \pmod 8}} e\left(\frac{n\bar{c}d}{2^{k+2}}\right) + (-1)^k i^c \sum_{\substack{d \pmod{2^{k+2}} \\ d \equiv 3 \pmod 8}} e\left(\frac{n\bar{c}d}{2^{k+2}}\right) \\
&\quad + (-1)^k \sum_{\substack{d \pmod{2^{k+2}} \\ d \equiv 5 \pmod 8}} e\left(\frac{n\bar{c}d}{2^{k+2}}\right) + i^c \sum_{\substack{d \pmod{2^{k+2}} \\ d \equiv 7 \pmod 8}} e\left(\frac{n\bar{c}d}{2^{k+2}}\right) \\
&= \sum_{\substack{d \pmod{2^{k+2}} \\ d \equiv 1 \pmod 8}} e\left(\frac{n\bar{c}d}{2^{k+2}}\right) + (-1)^k i^c \sum_{\substack{d \pmod{2^{k+2}} \\ d \equiv 1 \pmod 8}} e\left(\frac{n\bar{c}(d+2)}{2^{k+2}}\right) \\
&\quad + (-1)^k \sum_{\substack{d \pmod{2^{k+2}} \\ d \equiv 1 \pmod 8}} e\left(\frac{n\bar{c}(d+4)}{2^{k+2}}\right) + i^c \sum_{\substack{d \pmod{2^{k+2}} \\ d \equiv 1 \pmod 8}} e\left(\frac{n\bar{c}(d+6)}{2^{k+2}}\right) \\
&= \left(1 + (-1)^k i^c e\left(\frac{2n\bar{c}}{2^{k+2}}\right) + (-1)^k e\left(\frac{4n\bar{c}}{2^{k+2}}\right) + i^c e\left(\frac{6n\bar{c}}{2^{k+2}}\right)\right) \sum_{\substack{d \pmod{2^{k+2}} \\ d \equiv 1 \pmod 8}} e\left(\frac{n\bar{c}d}{2^{k+2}}\right).
\end{aligned}
$$

Since we can write $n = 2^\ell n'$ where $\gcd(2, n') = 1$, this equation can also be written as

$$
\begin{aligned}
&K_{-c}(n\bar{c}; 2^{k+2}) \\
&= \left(1 + (-1)^k i^c e\left(\frac{2^{\ell+1} n'\bar{c}}{2^{k+2}}\right) + (-1)^k e\left(\frac{2^{\ell+2} n'\bar{c}}{2^{k+2}}\right) + i^c e\left(\frac{2^{\ell+1} \cdot 3n'\bar{c}}{2^{k+2}}\right)\right) \\
&\quad \cdot \sum_{\substack{d \pmod{2^{k+2}} \\ d \equiv 1 \pmod 8}} e\left(\frac{2^\ell n'\bar{c}d}{2^{k+2}}\right). \qquad (4.24)
\end{aligned}
$$

Along with this equation, we shall often use the identity $\Delta_c^{-1}(1+i^c) = (1+i)\chi_4(c)$ in evaluating $K_{-c}(n\bar{c}; 2^{k+2})$.

Suppose that $k \geq 2$.

- If $\ell \geq k+2$ then by (4.24) we have that

$$
\begin{aligned}
\Delta_c^{-1} K_{-c}(n\bar{c}; 2^{k+2}) &= \Delta_c^{-1}(1 + (-1)^k i^c + (-1)^k + i^c) 2^{k-1} \\
&= \begin{cases} \Delta_c^{-1}(1+i^c)2^k & \text{if } k \text{ even}, \\ 0 & \text{if } k \text{ odd}, \end{cases} = \begin{cases} (1+i)2^k \chi_4(c) & \text{if } k \text{ even}, \\ 0 & \text{if } k \text{ odd}. \end{cases}
\end{aligned}
$$

  This proves part (c.1).

- If $\ell = k+1$ then by (4.24) we have that

$$\Delta_c^{-1}K_{-c}(n\bar{c};2^{k+2}) = \Delta_c^{-1}(1+(-1)^k i^c + (-1)^k + i^c) \sum_{\substack{d (\mathrm{mod}\, 2^{k+2}) \\ d\equiv 1 (\mathrm{mod}\, 8)}} e\left(\frac{n'\bar{c}d}{2}\right)$$

$$= \begin{cases} -\Delta_c^{-1}(1+i^c)2^k & \text{if } k \text{ even,} \\ 0 & \text{if } k \text{ odd,} \end{cases} = \begin{cases} -(1+i)2^k\chi_4(c) & \text{if } k \text{ even,} \\ 0 & \text{if } k \text{ odd.} \end{cases}$$

This proves part (c.2).

- If $\ell = k$ then by (4.24) we have that

$$\Delta_c^{-1}K_{-c}(n\bar{c};2^{k+2}) = \Delta_c^{-1}(1-(-1)^k i^c + (-1)^k - i^c) \sum_{\substack{d (\mathrm{mod}\, 2^{k+2}) \\ d\equiv 1 (\mathrm{mod}\, 8)}} e\left(\frac{n'\bar{c}d}{4}\right).$$

Observe

$$\Delta_c^{-1}(1-(-1)^k i^c + (-1)^k - i^c) = \begin{cases} 2(1-i) & \text{if } k \text{ even,} \\ 0 & \text{if } k \text{ odd,} \end{cases}$$

since $\Delta_c^{-1}(1-i^c) = 1-i$ regardless of the value of c. Also observe that

$$\sum_{\substack{d (\mathrm{mod}\, 2^{k+2}) \\ d\equiv 1 (\mathrm{mod}\, 8)}} e\left(\frac{n'\bar{c}d}{4}\right) = \begin{cases} 2^{k-1}i & \text{if } c \equiv n' \equiv 1 (\mathrm{mod}\, 4), \\ -2^{k-1}i & \text{if } -c \equiv n' \equiv 1 (\mathrm{mod}\, 4), \\ -2^{k-1}i & \text{if } c \equiv -n' \equiv 1 (\mathrm{mod}\, 4), \\ 2^{k-1}i & \text{if } c \equiv n' \equiv 3 (\mathrm{mod}\, 4), \end{cases} = i2^{k-1}\chi_4(n'c).$$

Thus for k even,

$$\Delta_c^{-1}K_{-c}(n\bar{c};2^{k+2}) = (1+i)2^k\chi_4(n'c).$$

This proves part (c.3)

- If $\ell = k-1$ then by (4.24) we have that

$$\begin{aligned} &K_{-c}(n\bar{c};2^{k+2}) \\ &= \left(1+(-1)^k i^c e\left(\frac{n'\bar{c}}{4}\right) + (-1)^k e\left(\frac{n'\bar{c}}{2}\right) + i^c e\left(\frac{3n'\bar{c}}{4}\right)\right) \sum_{\substack{d (\mathrm{mod}\, 2^{k+2}) \\ d\equiv 1 (\mathrm{mod}\, 8)}} e\left(\frac{n'\bar{c}d}{8}\right) \\ &= \left(1+(-1)^k i^c e\left(\frac{n'\bar{c}}{4}\right) + (-1)^k e\left(\frac{n'\bar{c}}{2}\right) + i^c e\left(\frac{3n'\bar{c}}{4}\right)\right) \sum_{\substack{d (\mathrm{mod}\, 2^{k+2}) \\ d\equiv 1 (\mathrm{mod}\, 8)}} e\left(\frac{n'\bar{c}}{8}\right)^d \\ &= 2^{k-1}\left(1+(-1)^k i^c e\left(\frac{n'\bar{c}}{4}\right) + (-1)^k e\left(\frac{n'\bar{c}}{2}\right) + i^c e\left(\frac{3n'\bar{c}}{4}\right)\right) e\left(\frac{n'\bar{c}}{8}\right). \end{aligned}$$

If $n'\bar{c} \equiv 1 \pmod 4$ then

$$i^{c+3n'\bar{c}} = i^{c+3} = i^c(-i) = -i^{c+1} = -i^{c+n'\bar{c}}.$$

If $n'\bar{c} \equiv 3 \pmod 4$ then

$$i^{c+3n'\bar{c}} = i^{c+1} = -i^{c+3} = -i^{c+n'\bar{c}}.$$

Therefore, regardless of the values of n' and c, we have that

$$1 + (-1)^k i^{c+n'\bar{c}} - (-1)^k + i^{c+3n'\bar{c}} = 1 + (-1)^k i^{c+n'\bar{c}} - (-1)^k - i^{c+n'\bar{c}}$$
$$= \begin{cases} 0 & \text{if } k \text{ even} \\ 2(1 - i^{c+n'\bar{c}}), & \text{if } k \text{ odd.} \end{cases}$$

Thus $K_{-c}(n\bar{c}; 2^{k+2}) = 0$ if k is even, and if k is odd then

$$K_{-c}(n\bar{c}; 2^{k+2}) = 2^k(1 - i^{c+n'\bar{c}})e\left(\frac{n'\bar{c}}{8}\right).$$

If $n' \equiv 3 \pmod 4$ then $1 - i^{c+n'\bar{c}} = 0$ and hence $K_{-c}(n\bar{c}; 2^{k+2}) = 0$. Thus it remains to evaluate $K_{-c}(n\bar{c}; 2^{k+2})$ for $n \equiv 1 \pmod 4$. Towards this end, observe that

$$(1 - i^{c+n'\bar{c}})e\left(\frac{n'\bar{c}}{8}\right) = (1 - i^{c'+n''\overline{c'}})e\left(\frac{n''\overline{c'}}{8}\right)$$

if $c \equiv c' \pmod 8$ and $n' \equiv n'' \pmod 8$. In light of this, we give the following table for values of $\Delta_c^{-1} K_{-c}(n\bar{c}; 2^{k+2}) = \Delta_c^{-1} 2^k(1 - i^{c+n'\bar{c}})e\left(\frac{n'\bar{c}}{8}\right)$:

$n' \backslash c$	1	3	5	7
1	$\frac{1+i}{\sqrt{2}}2^{k+1}$	$\frac{1+i}{\sqrt{2}}2^{k+1}$	$-\frac{1+i}{\sqrt{2}}2^{k+1}$	$-\frac{1+i}{\sqrt{2}}2^{k+1}$
5	$-\frac{1+i}{\sqrt{2}}2^{k+1}$	$-\frac{1+i}{\sqrt{2}}2^{k+1}$	$\frac{1+i}{\sqrt{2}}2^{k+1}$	$\frac{1+i}{\sqrt{2}}2^{k+1}$

From this we see that

$$\Delta_c^{-1} K_{-c}(n\bar{c}; 2^{k+2}) = \frac{1+i}{\sqrt{2}} 2^{k+1} \chi_8(n'c).$$

This proves part (c.4).

- If $\ell \leq k - 2$ then

$$\sum_{\substack{d \pmod{2^{k+2}} \\ d \equiv 1 \pmod 8}} e\left(\frac{2^\ell n'\bar{c}d}{2^{k+2}}\right) = \sum_{\substack{d \pmod{2^{k+2}} \\ d \equiv 1 \pmod 8}} e\left(\frac{n'\bar{c}}{2^{k-\ell+2}}\right)^d.$$

If $\xi = e\left(\frac{1}{2^{k-\ell+2}}\right)$ and m is an odd integer, then since $\xi^{8m} \neq 1$ and since

$$\xi^{8m}\left(\sum_{\substack{d(2^{k+2}) \\ d \equiv 1 \pmod 8}} \xi^{md}\right) = \sum_{\substack{d \pmod{2^{k+2}} \\ d \equiv 1 \pmod 8}} \xi^{m(d+8)} = \sum_{\substack{d \pmod{2^{k+2}} \\ d \equiv 1 \pmod 8}} \xi^{md},$$

it follows that

$$\sum_{\substack{d (\bmod 2^{k+2}) \\ d \equiv 1 (\bmod 8)}} e\left(\frac{2^\ell n' \bar{c} d}{2^{k+2}}\right) = 0.$$

Hence by (4.24), we have that $\Delta_c^{-1} K_{-c}(n'\overline{c'}; 2^{k+2}) = 0$. This proves part (c.5).

□

4.3 The Meromorphic Continuation of Fourier Coefficients

To prove that $\widetilde{E}_\nu^{(\infty)}$ has a meromorphic continuation, we need to further simplify our Fourier coefficient calculations from the previous section. Let

$$\begin{aligned}
&\mathcal{G}_4(\epsilon, n, 2^k; \nu) \\
&= \left(1 - \left(\frac{\epsilon n}{2}\right) 2^{-\nu - \frac{1}{2}}\right) \prod_{\substack{p \text{ odd prime} \\ p | n}} \left(1 - p^{-2\nu-1}\right)^{-1} \sum_{j \in \mathbb{Z}_{\geq 0}} \chi_4(p^j) \mathcal{C}_{-\epsilon}\left(\Delta_{p^j} G(\epsilon n \overline{2^k}; p^j)\right) (p^j)^{-\nu-1},
\end{aligned}$$

$$\begin{aligned}
&\mathcal{G}_8(\epsilon, n, 2^k; \nu) \\
&= \left(1 - \left(\frac{\epsilon n}{2}\right) 2^{-\nu - \frac{1}{2}}\right) \prod_{\substack{p \text{ odd prime} \\ p | n}} \left(1 - p^{-2\nu-1}\right)^{-1} \sum_{j \in \mathbb{Z}_{\geq 0}} \chi_8(p^j) \mathcal{C}_{-\epsilon}\left(\Delta_{p^j} G(\epsilon n \overline{2^k}; p^j)\right) (p^j)^{-\nu-1},
\end{aligned}$$

and

$$L\left(\nu, \left(\frac{n}{\cdot}\right)\right) = \sum_{d \in \mathbb{Z}_{>0}} \left(\frac{n}{d}\right) d^{-\nu},$$

where $\left(\frac{n}{d}\right)$ is the Kronecker symbol. We will use the following lemma to expedite our simplification of a_n.

Lemma 4.6.

(a) For k even,

$$\zeta_2(2\nu + 1) \sum_{\substack{c' \in \mathbb{Z}_{>0} \\ c' \text{ odd}}} \chi_4(c') \mathcal{C}_{-\epsilon}\left(\Delta_{c'} G(\epsilon n \overline{2^k}; c')\right) (c')^{-\nu-1} = L\left(\nu + \frac{1}{2}, \left(\frac{\epsilon n}{\cdot}\right)\right) \mathcal{G}_4(\epsilon, n, 2^k; \nu).$$

(b) For k odd,

$$\zeta_2(2\nu + 1) \sum_{\substack{c' \in \mathbb{Z}_{>0} \\ c' \text{ odd}}} \chi_8(c') \mathcal{C}_{-\epsilon}\left(\Delta_{c'} G(\epsilon n \overline{2^k}; c')\right) (c')^{-\nu-1} = L\left(\nu + \frac{1}{2}, \left(\frac{\epsilon n}{\cdot}\right)\right) \mathcal{G}_8(\epsilon, n, 2^k; \nu).$$

Proof. By Lemma 4.4(b) it follows that $\chi_4(c') \mathcal{C}_{-\epsilon}\left(\Delta_{c'} G(\epsilon n \overline{2^k}; c')\right)$ is multiplicative. Therefore, by Lemma 4.4(a,d) we have that

$$\begin{aligned}
&\zeta_2(2\nu+1)\sum_{\substack{c'\in\mathbb{Z}_{>0}\\ c'\, odd}} \chi_4(c')\mathcal{C}_{-\epsilon}\Big(\Delta_{c'}G(\epsilon n\overline{2^k};c')\Big)(c')^{-\nu-1}\\
&=\zeta_2(2\nu+1)\prod_{p\text{ odd prime}}\sum_{j\in\mathbb{Z}_{\geq 0}}\chi_4(p^j)\mathcal{C}_{-\epsilon}\Big(\Delta_{p^j}G(\epsilon n\overline{2^k};p^j)\Big)(p^j)^{-\nu-1}\\
&=\zeta_2(2\nu+1)\prod_{\substack{p\text{ odd prime}\\ p\nmid n}}\left(1+\chi_4(p)\mathcal{C}_{-\epsilon}\left(\Delta_p^2\left(\frac{\epsilon n 2^k}{p}\right)\right)p^{-\nu-\frac{1}{2}}\right)\\
&\quad\cdot\prod_{\substack{p\text{ odd prime}\\ p\mid n}}\sum_{j\in\mathbb{Z}_{\geq 0}}\chi_4(p^j)\mathcal{C}_{-\epsilon}\Big(\Delta_{p^j}G(\epsilon n\overline{2^k};p^j)\Big)(p^j)^{-\nu-1}\\
&=\zeta_2(2\nu+1)\prod_{\substack{p\text{ odd prime}\\ p\nmid n}}\left(1+\left(\frac{\epsilon n}{p}\right)p^{-\nu-\frac{1}{2}}\right)\\
&\quad\cdot\prod_{\substack{p\text{ odd prime}\\ p\mid n}}\sum_{j\in\mathbb{Z}_{\geq 0}}\chi_4(p^j)\mathcal{C}_{-\epsilon}\Big(\Delta_{p^j}G(\epsilon n\overline{2^k};p^j)\Big)(p^j)^{-\nu-1}.
\end{aligned}$$

In the last we line we used the fact that $\chi_4(p)\Delta_p^2 = 1$ for all odd primes, and we used that $\left(\frac{2^k}{p}\right) = 1$ since k is even. Since

$$\left(1+\left(\frac{\epsilon n}{p}\right)p^{-\nu-\frac{1}{2}}\right) = \left(1-p^{-2\nu-1}\right)\cdot\left(1-\left(\frac{\epsilon n}{p}\right)p^{-\nu-\frac{1}{2}}\right)^{-1}, \tag{4.25}$$

it follows that

$$\begin{aligned}
&\zeta_2(2\nu+1)\sum_{\substack{c'\in\mathbb{Z}_{>0}\\ c'\, odd}} \chi_4(c')\mathcal{C}_{-\epsilon}\Big(\Delta_{c'}G(\epsilon n\overline{2^k};c')\Big)(c')^{-\nu-1}\\
&=\prod_{\substack{p\text{ odd prime}\\ p\mid n}}\left(1-p^{-2\nu-1}\right)^{-1}\prod_{\substack{p\text{ odd prime}\\ p\nmid n}}\left(1-\left(\frac{\epsilon n}{p}\right)p^{-\nu-\frac{1}{2}}\right)^{-1}\\
&\quad\cdot\prod_{\substack{p\text{ odd prime}\\ p\mid n}}\sum_{j\in\mathbb{Z}_{\geq 0}}\chi_4(p^j)\mathcal{C}_{-\epsilon}\Big(\Delta_{p^j}G(\epsilon n\overline{2^k};p^j)\Big)(p^j)^{-\nu-1}\\
&=L\left(\nu+\frac{1}{2},\left(\frac{\epsilon n}{\cdot}\right)\right)\left(1-\left(\frac{\epsilon n}{2}\right)2^{-\nu-\frac{1}{2}}\right)\prod_{\substack{p\text{ odd prime}\\ p\mid n}}\left(1-p^{-2\nu-1}\right)^{-1}\\
&\quad\cdot\prod_{\substack{p\text{ odd prime}\\ p\mid n}}\sum_{j\in\mathbb{Z}_{\geq 0}}\chi_4(p^j)\mathcal{C}_{-\epsilon}\Big(\Delta_{p^j}G(\epsilon n\overline{2^k};p^j)\Big)(p^j)^{-\nu-1}.
\end{aligned}$$

This proves part (a).

The proof for part (b) is very similar to the proof for part (a). By Lemma 4.4(b) we have that $\chi_8(c')\mathcal{C}_{-\epsilon}\Big(\Delta_{c'}G(\epsilon n\overline{2^k};c')\Big)$ is multiplicative. Therefore, by Lemma 4.4(a,d) we have that

$$\begin{aligned}
&\zeta_2(2\nu+1)\sum_{\substack{c'\in\mathbb{Z}_{>0}\\ c'\,odd}}\chi_8(c')\mathcal{C}_{-\epsilon}\Big(\Delta_{c'}G(\epsilon n\overline{2^k};c')\Big)(c')^{-\nu-1}\\
&=\zeta_2(2\nu+1)\prod_{p\text{ odd prime}}\sum_{j\in\mathbb{Z}_{\geq 0}}\chi_8(p^j)\mathcal{C}_{-\epsilon}\Big(\Delta_{p^j}G(\epsilon n\overline{2^k};p^j)\Big)(p^j)^{-\nu-1}\\
&=\zeta_2(2\nu+1)\prod_{\substack{p\text{ odd prime}\\ p\nmid n}}\left(1+\chi_8(p)\mathcal{C}_{-\epsilon}\left(\Delta_p\Delta_p\left(\frac{\epsilon n2^k}{p}\right)\right)p^{-\nu-\frac{1}{2}}\right)\\
&\quad\cdot\prod_{\substack{p\text{ odd prime}\\ p\mid n}}\sum_{j\in\mathbb{Z}_{\geq 0}}\chi_8(p^j)\mathcal{C}_{-\epsilon}\Big(\Delta_{p^j}G(\epsilon n\overline{2^k};p^j)\Big)(p^j)^{-\nu-1}\\
&=\zeta_2(2\nu+1)\prod_{\substack{p\text{ odd prime}\\ p\nmid n}}\left(1+\left(\frac{\epsilon n}{p}\right)p^{-\nu-\frac{1}{2}}\right)\\
&\quad\cdot\prod_{\substack{p\text{ odd prime}\\ p\mid n}}\sum_{j\in\mathbb{Z}_{\geq 0}}\chi_8(p^j)\mathcal{C}_{-\epsilon}\Big(\Delta_{p^j}G(\epsilon n\overline{2^k};p^j)\Big)(p^j)^{-\nu-1}.
\end{aligned}$$

In the last we line we used the fact that $\chi_8(p)\Delta_p^2\left(\frac{2^k}{p}\right)=\chi_8(p)\Delta_p^2\left(\frac{2}{p}\right)=1$ for all odd primes; this follows from the fact that $\left(\frac{2^k}{p}\right)=\left(\frac{2}{p}\right)$ since k is odd, and the fact that

$$\left(\frac{2}{p}\right)=\begin{cases}1 & \text{if } p\equiv 1,7 \pmod 8,\\ -1 & \text{if } p\equiv 3,5\pmod 8).\end{cases}$$

By (4.25) it follows that

$$\begin{aligned}
&\zeta_2(2\nu+1)\sum_{\substack{c'\in\mathbb{Z}_{>0}\\ c'\,odd}}\chi_8(c')\mathcal{C}_{-\epsilon}\Big(\Delta_{c'}G(\epsilon n\overline{2^k};c')\Big)(c')^{-\nu-1}\\
&=\prod_{\substack{p\text{ odd prime}\\ p\mid n}}\left(1-p^{-2\nu-1}\right)^{-1}\prod_{\substack{p\text{ odd prime}\\ p\nmid n}}\left(1-\left(\frac{\epsilon n}{p}\right)p^{-\nu-\frac{1}{2}}\right)^{-1}\\
&\quad\cdot\prod_{\substack{p\text{ odd prime}\\ p\mid n}}\sum_{j\in\mathbb{Z}_{\geq 0}}\chi_8(p^j)\mathcal{C}_{-\epsilon}\Big(\Delta_{p^j}G(\epsilon n\overline{2^k};p^j)\Big)(p^j)^{-\nu-1}\\
&=L\left(\nu+\frac{1}{2},\left(\frac{\epsilon n}{\cdot}\right)\right)\left(1-\left(\frac{\epsilon n}{2}\right)2^{-\nu-\frac{1}{2}}\right)\prod_{\substack{p\text{ odd prime}\\ p\mid n}}\left(1-p^{-2\nu-1}\right)^{-1}\\
&\quad\cdot\prod_{\substack{p\text{ odd prime}\\ p\mid n}}\sum_{j\in\mathbb{Z}_{\geq 0}}\chi_8(p^j)\mathcal{C}_{-\epsilon}\Big(\Delta_{p^j}G(\epsilon n\overline{2^k};p^j)\Big)(p^j)^{-\nu-1}.
\end{aligned}$$

This proves part (b). $\square$

Proposition 4.7. *Let $\widetilde{E}_\nu^{(\infty)} \in V_{(\epsilon),\nu}^{-\infty}(\widetilde{SL}_2)$ where $\Re(\nu) > 1$, with a_n the Fourier coefficients of $\left(\widetilde{E}_\nu^{(\infty)}\right)_0$. Recall that in (4.22) we showed that*

$$a_n = \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1)\sum_{\substack{c'\in\mathbb{Z}_{>0}\\ c'\ odd}}\sum_{k\in\mathbb{Z}_{\geq 0}}(2^k c')^{-\nu-1}\mathcal{C}_{-\epsilon}\Big(K_{-c'}(\epsilon\, n\overline{c'}; 2^{k+2})G(\epsilon\, n\overline{2^{k+2}}; c')\Big).$$

(a) If $\epsilon n \equiv 3 (\operatorname{mod} 4)$ then

$$a_n = -(1+\epsilon i)4^{-\nu-1}L\left(\nu+\frac{1}{2}, \left(\frac{\epsilon n}{\cdot}\right)\right)\mathcal{G}_4(\epsilon, n, 4; \nu).$$

(b) If $\epsilon n \equiv 1 (\operatorname{mod} 4)$ then

$$\begin{aligned} a_n &= (1+\epsilon i)4^{-\nu-1}L\left(\nu+\frac{1}{2}, \left(\frac{\epsilon n}{\cdot}\right)\right)\mathcal{G}_4(\epsilon, n, 4; \nu) \\ &\quad + (1+\epsilon i)2^{3/2}(1-i)8^{-\nu-1}L\left(\nu+\frac{1}{2}, \left(\frac{\epsilon n}{\cdot}\right)\right)\mathcal{G}_8(\epsilon, n, 8; \nu). \end{aligned}$$

(c) If $n = 2n'$ where $\gcd(n', 2) = 1$ then

$$a_n = -(1+\epsilon i)4^{-\nu-1}L\left(\nu+\frac{1}{2}, \left(\frac{\epsilon n}{\cdot}\right)\right)\mathcal{G}_4(\epsilon, n, 4; \nu).$$

(d) If $n = 2^\ell n'$ where $\ell > 1$ and $\gcd(n', 2) = 1$, then

$$\begin{aligned} a_n &= (1+\epsilon i)4^{-\nu-1}L\left(\nu+\frac{1}{2}, \left(\frac{\epsilon n}{\cdot}\right)\right)\mathcal{G}_4(\epsilon, n, 4; \nu) \\ &\quad + (1+\epsilon i)\sum_{\substack{k=2\\ k\ even}}^{\ell-2} 2^k\left(2^{k+2}\right)^{-\nu-1}L\left(\nu+\frac{1}{2}, \left(\frac{\epsilon n}{\cdot}\right)\right)\mathcal{G}_4(\epsilon, n, 2^{k+2}; \nu) \\ &\quad - \delta_{\ell\equiv 1(2)}(1+\epsilon i)2^{\ell-1}\left(2^{\ell+1}\right)^{-\nu-1}L\left(\nu+\frac{1}{2}, \left(\frac{\epsilon n}{\cdot}\right)\right)\mathcal{G}_4(\epsilon, n, 2^{\ell+1}; \nu) \\ &\quad + \delta_{\ell\equiv 0(2)}(1+\epsilon i)2^{\ell}\left(2^{\ell+2}\right)^{-\nu-1}\chi_4(\epsilon n')L\left(\nu+\frac{1}{2}, \left(\frac{\epsilon n}{\cdot}\right)\right)\mathcal{G}_4(\epsilon, n, 2^{\ell+2}; \nu) \\ &\quad + \delta_{\ell\equiv 0(2)}\frac{(1+\epsilon i)}{\sqrt{2}}2^{\ell+2}\left(2^{\ell+3}\right)^{-\nu-1}\chi_8(\epsilon n')L\left(\nu+\frac{1}{2}, \left(\frac{\epsilon n}{\cdot}\right)\right)\mathcal{G}_8(\epsilon, n, 2^{\ell+3}; \nu), \end{aligned}$$

where $\delta_{\ell\equiv j(2)} = \begin{cases} 1 & \textit{if } \ell \equiv j (\operatorname{mod} 2), \\ 0 & \textit{otherwise}, \end{cases}$ *for* $j \in \{0, 1\}$.

Proof. Suppose $\epsilon n \equiv 3 (\mathrm{mod}\, 4)$. By (4.22) and Lemma 4.5(b), we have that

$$\begin{aligned}
a_n &= \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c'\in\mathbb{Z}_{>0}\\ c' \text{ odd}}} \sum_{k\in\mathbb{Z}_{\geq 0}} (2^k c')^{-\nu-1}\mathcal{C}_{-\epsilon}\Big(\Delta_{c'}^{-1}K_{-c'}(\epsilon n\overline{c'}; 2^{k+2})\Delta_{c'}G(\epsilon n\overline{2^{k+2}}; c')\Big)\\
&= \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c'\in\mathbb{Z}_{>0}\\ c' odd}} (c')^{-\nu-1}\mathcal{C}_{-\epsilon}\big(\Delta_{c'}^{-1}K_{-c'}(\epsilon n\overline{c'}; 4)\Delta_{c'}G(\epsilon n\bar{4}; c')\big)\\
&= \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c'\in\mathbb{Z}_{>0}\\ c' odd}} (c')^{-\nu-1}\mathcal{C}_{-\epsilon}((-1-i)\chi_4(c'))\mathcal{C}_{-\epsilon}\big(\Delta_{c'}G(\epsilon n\bar{4}; c')\big)\\
&= \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c'\in\mathbb{Z}_{>0}\\ c' odd}} (c')^{-\nu-1}(-1+\epsilon i)\chi_4(c')\mathcal{C}_{-\epsilon}\big(\Delta_{c'}G(\epsilon n\bar{4}; c')\big)\\
&= -(1+\epsilon i) 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c'\in\mathbb{Z}_{>0}\\ c' odd}} \chi_4(c')\mathcal{C}_{-\epsilon}\big(\Delta_{c'}G(\epsilon n\bar{4}; c')\big)(c')^{-\nu-1}.
\end{aligned}$$

Part (a) then follows from Lemma 4.6(a).

Suppose $\epsilon n \equiv 1 (\mathrm{mod}\, 4)$. By (4.22) and Lemma 4.5(a) we have that

$$\begin{aligned}
a_n &= \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c'\in\mathbb{Z}_{>0}\\ c' \text{ odd}}} (c')^{-\nu-1}\mathcal{C}_{-\epsilon}\big(\Delta_{c'}^{-1}K_{-c'}(\epsilon n\overline{c'}; 4)\Delta_{c'}G(\epsilon n\bar{4}; c')\big)\\
&\quad + \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c'\in\mathbb{Z}_{>0}\\ c' \text{ odd}}} (2c')^{-\nu-1}\mathcal{C}_{-\epsilon}\big(\Delta_{c'}^{-1}K_{-c'}(\epsilon n\overline{c'}; 8)\Delta_{c'}G(\epsilon n\bar{8}; c')\big)\\
&= \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c'\in\mathbb{Z}_{>0}\\ c' \text{ odd}}} (c')^{-\nu-1}\mathcal{C}_{-\epsilon}((1+i)\chi_4(c'))\mathcal{C}_{-\epsilon}\big(\Delta_{c'}G(\epsilon n\bar{4}; c')\big)\\
&\quad + \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c'\in\mathbb{Z}_{>0}\\ c' \text{ odd}}} (2c')^{-\nu-1}\mathcal{C}_{-\epsilon}\Big(2^{3/2}(1+i)\chi_8(c')\Big)\mathcal{C}_{-\epsilon}\big(\Delta_{c'}G(\epsilon n\bar{8}; c')\big)\\
&= \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c'\in\mathbb{Z}_{>0}\\ c' \text{ odd}}} (c')^{-\nu-1}(1-\epsilon i)\chi_4(c')\mathcal{C}_{-\epsilon}\big(\Delta_{c'}G(\epsilon n\bar{4}; c')\big)\\
&\quad + \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c'\in\mathbb{Z}_{>0}\\ c' \text{ odd}}} (2c')^{-\nu-1}2^{3/2}(1-\epsilon i)\chi_8(c')\mathcal{C}_{-\epsilon}\big(\Delta_{c'}G(\epsilon n\bar{8}; c')\big)\\
&= (1+\epsilon i) 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c'\in\mathbb{Z}_{>0}\\ c' odd}} \chi_4(c')\mathcal{C}_{-\epsilon}\big(\Delta_{c'}G(\epsilon n\bar{4}; c')\big)(c')^{-\nu-1}\\
&\quad + (1+\epsilon i)2^{3/2} 8^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c'\in\mathbb{Z}_{>0}\\ c' \text{ odd}}} \chi_8(c')\mathcal{C}_{-\epsilon}\big(\Delta_{c'}G(\epsilon n\bar{8}; c')\big)(c')^{-\nu-1}.
\end{aligned}$$

Part (b) then follows from Lemma 4.6(a,b).

Suppose $n = 2n'$ with $\gcd(n', 2) = 1$. By (4.22) and Lemma 4.5(c), we have that

$$\begin{aligned}
a_n &= \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c' \in \mathbb{Z}_{>0} \\ c' \text{ odd}}} (c')^{-\nu-1} \mathcal{C}_{-\epsilon}\big(\Delta_{c'}^{-1} K_{-c'}(\epsilon n \overline{c'}; 4) \Delta_{c'} G(\epsilon n \bar{4}; c')\big) \\
&= \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c' \in \mathbb{Z}_{>0} \\ c' \text{ odd}}} (c')^{-\nu-1} \mathcal{C}_{-\epsilon}((-1-i)\chi_4(c')) \mathcal{C}_{-\epsilon}\big(\Delta_{c'} G(\epsilon n \bar{4}; c')\big) \\
&= -(1+\epsilon i) 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c' \in \mathbb{Z}_{>0} \\ c' \text{ odd}}} \chi_4(c') \mathcal{C}_{-\epsilon}\big(\Delta_{c'} G(\epsilon n \bar{4}; c')\big)(c')^{-\nu-1}.
\end{aligned}$$

Part (c) then follows from Lemma 4.6(a).

Suppose $n = 2^\ell n'$ with $\gcd(n', 2) = 1$ and $\ell > 1$. By (4.22) and Lemma 4.5(c), we have that

$$\begin{aligned}
a_n &= \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c' \in \mathbb{Z}_{>0} \\ c' \text{ odd}}} \sum_{k \in \mathbb{Z}_{\geq 0}} (2^k c')^{-\nu-1} \mathcal{C}_{-\epsilon}\Big(\Delta_{c'}^{-1} K_{-c'}(\epsilon n \overline{c'}; 2^{k+2}) \Delta_{c'} G(\epsilon n \overline{2^{k+2}}; c')\Big) \\
&= \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c' \in \mathbb{Z}_{>0} \\ c' \text{ odd}}} (c')^{-\nu-1} \mathcal{C}_{-\epsilon}((1+i)\chi_4(c')) \mathcal{C}_{-\epsilon}\big(\Delta_{c'} G(\epsilon n \bar{4}; c')\big) \\
&\quad + \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c' \in \mathbb{Z}_{>0} \\ c' \text{ odd}}} \sum_{\substack{k=2 \\ k \text{ even}}}^{\ell-2} (2^k c')^{-\nu-1} \mathcal{C}_{-\epsilon}\big((1+i)\chi_4(c') 2^k\big) \mathcal{C}_{-\epsilon}\Big(\Delta_{c'} G(\epsilon n \overline{2^{k+2}}; c')\Big) \\
&\quad + \delta_{\ell \equiv 1(2)} \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c' \in \mathbb{Z}_{>0} \\ c' \text{ odd}}} (2^{\ell-1} c')^{-\nu-1} \mathcal{C}_{-\epsilon}\big((-1-i)\chi_4(c') 2^{\ell-1}\big) \\
&\qquad \cdot \mathcal{C}_{-\epsilon}\Big(\Delta_{c'} G(\epsilon n \overline{2^{\ell+1}}; c')\Big) \\
&\quad + \delta_{\ell \equiv 0(2)} \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c' \in \mathbb{Z}_{>0} \\ c' \text{ odd}}} (2^\ell c')^{-\nu-1} \mathcal{C}_{-\epsilon}\big((1+i)\chi_4(\epsilon n' c') 2^\ell\big) \mathcal{C}_{-\epsilon}\Big(\Delta_{c'} G(\epsilon n \overline{2^{\ell+2}}; c')\Big) \\
&\quad + \delta_{\ell \equiv 0(2)} \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c' \in \mathbb{Z}_{>0} \\ c' \text{ odd}}} (2^{\ell+1} c')^{-\nu-1} \mathcal{C}_{-\epsilon}\left(\frac{1+i}{\sqrt{2}} 2^{\ell+2} \chi_8(\epsilon n' c')\right) \\
&\qquad \cdot \mathcal{C}_{-\epsilon}\Big(\Delta_{c'} G(\epsilon n \overline{2^{\ell+3}}; c')\Big) \\
&= (1+\epsilon i) 4^{-\nu-1}\zeta_2(2\nu+1) \sum_{\substack{c' \in \mathbb{Z}_{>0} \\ c' \text{ odd}}} \chi_4(c') \mathcal{C}_{-\epsilon}\big(\Delta_{c'} G(\epsilon n \bar{4}; c')\big)(c')^{-\nu-1} \\
&\quad + (1+\epsilon i) \sum_{\substack{k=2 \\ k \text{ even}}}^{\ell-2} 2^k \big(2^{k+2}\big)^{-\nu-1} \zeta_2(2\nu+1) \sum_{\substack{c' \in \mathbb{Z}_{>0} \\ c' \text{ odd}}} \chi_4(c') \mathcal{C}_{-\epsilon}\Big(\Delta_{c'} G(\epsilon n \overline{2^{k+2}}; c')\Big)(c')^{-\nu-1} \\
&\quad - \delta_{\ell \equiv 1(2)} (1+\epsilon i) 2^{\ell-1} \big(2^{\ell+1}\big)^{-\nu-1} \zeta_2(2\nu+1) \sum_{\substack{c' \in \mathbb{Z}_{>0} \\ c' \text{ odd}}} \chi_4(c') \mathcal{C}_{-\epsilon}\Big(\Delta_{c'} G(\epsilon n \overline{2^{\ell+1}}; c')\Big)(c')^{-\nu-1} \\
&\quad + \delta_{\ell \equiv 0(2)} (1+\epsilon i) 2^\ell \big(2^{\ell+2}\big)^{-\nu-1} \chi_4(\epsilon n') \zeta_2(2\nu+1) \sum_{\substack{c' \in \mathbb{Z}_{>0} \\ c' \text{ odd}}} \chi_4(c')
\end{aligned}$$

$$\cdot\, \mathcal{C}_{-\epsilon}\Big(\Delta_{c'} G(\epsilon n\overline{2^{\ell+2}}; c')\Big)(c')^{-\nu-1}$$

$$+\, \delta_{\ell\equiv 0(2)} \frac{1+\epsilon i}{\sqrt{2}} 2^{\ell+2}\big(2^{\ell+3}\big)^{-\nu-1} \chi_8(\epsilon n')\zeta_2(2\nu+1) \sum_{\substack{c'\in\mathbb{Z}_{>0}\\ c' \text{ odd}}} \chi_8(c')$$

$$\cdot\, \mathcal{C}_{-\epsilon}\Big(\Delta_{c'} G(\epsilon n\overline{2^{\ell+3}}; c')\Big)(c')^{-\nu-1}.$$

Part (d) then follows from Lemma 4.6(a,b). □

For $n \neq 0$, we wish to show that a_n has meromorphic continuation to all of $\mathbb{C}$. To do so, we write

$$\epsilon n = st \text{ where } s = \prod_{\substack{p^\ell || n\\ \ell \text{ even}}} p^\ell \text{ and } t = \epsilon \prod_{\substack{p^\ell || n\\ \ell \text{ odd}}} p^\ell. \tag{4.26}$$

Thus

$$\begin{aligned} L\Big(\nu+\frac{1}{2}, \Big(\frac{t}{\cdot}\Big)\Big) &= \prod_{p \text{ prime}} \Big(1 - \Big(\frac{t}{p}\Big)p^{-\nu-\frac{1}{2}}\Big)^{-1} \\ &= \prod_{\substack{p \text{ prime}\\ p|s}} \Big(1 - \Big(\frac{t}{p}\Big)p^{-\nu-\frac{1}{2}}\Big)^{-1} \prod_{\substack{p \text{ prime}\\ p\nmid s}} \Big(1 - \Big(\frac{t}{p}\Big)p^{-\nu-\frac{1}{2}}\Big)^{-1} \\ &= \prod_{\substack{p \text{ prime}\\ p|s}} \Big(1 - \Big(\frac{t}{p}\Big)p^{-\nu-\frac{1}{2}}\Big)^{-1} L\Big(\nu+\frac{1}{2}, \Big(\frac{\epsilon n}{\cdot}\Big)\Big), \end{aligned}$$

which implies

$$L\Big(\nu+\frac{1}{2}, \Big(\frac{\epsilon n}{\cdot}\Big)\Big) = \prod_{\substack{p \text{ prime}\\ p|s}} \Big(1 - \Big(\frac{t}{p}\Big)p^{-\nu-\frac{1}{2}}\Big) L\Big(\nu+\frac{1}{2}, \Big(\frac{t}{\cdot}\Big)\Big).$$

Therefore

$$\begin{aligned} &L\Big(\nu+\frac{1}{2}, \Big(\frac{\epsilon n}{\cdot}\Big)\Big)\mathcal{G}_4(\epsilon, n, 2^k; \nu) \\ &= L\Big(\nu+\frac{1}{2}, \Big(\frac{\epsilon n}{\cdot}\Big)\Big)\Big(1 - \Big(\frac{\epsilon n}{2}\Big)2^{-\nu-\frac{1}{2}}\Big) \\ &\quad \left(\prod_{\substack{p \text{ odd prime}\\ p|n}} \big(1-p^{-2\nu-1}\big)^{-1} \sum_{j\in\mathbb{Z}_{\geq 0}} \chi_4(p^j)\mathcal{C}_{-\epsilon}\Big(\Delta_{p^j} G(\epsilon n\overline{2^k}; p^j)\Big)(p^j)^{-\nu-1}\right) \\ &= \Big(1 - \Big(\frac{\epsilon n}{2}\Big)2^{-\nu-\frac{1}{2}}\Big) L\Big(\nu+\frac{1}{2}, \Big(\frac{t}{\cdot}\Big)\Big)\left(\prod_{\substack{p \text{ prime}\\ p|s}} \Big(1 - \Big(\frac{t}{p}\Big)p^{-\nu-\frac{1}{2}}\Big)\right) \\ &\quad \left(\prod_{\substack{p \text{ odd prime}\\ p|n}} \big(1-p^{-2\nu-1}\big)^{-1} \sum_{j\in\mathbb{Z}_{\geq 0}} \chi_4(p^j)\mathcal{C}_{-\epsilon}\Big(\Delta_{p^j} G(\epsilon n\overline{2^k}; p^j)\Big)(p^j)^{-\nu-1}\right) \end{aligned}$$

$$= p_{2,n} \cdot \left(1 - \left(\frac{\epsilon n}{2}\right) 2^{-\nu-\frac{1}{2}}\right) L\left(\nu + \frac{1}{2}, \left(\frac{t}{\cdot}\right)\right) \left(\prod_{\substack{p \text{ odd prime} \\ p|s}} \left(1 - \left(\frac{t}{p}\right) p^{-\nu-\frac{1}{2}}\right) \right)$$
$$\left(\prod_{\substack{p \text{ odd prime} \\ p|n}} \left(1 - p^{-2\nu-1}\right)^{-1} \sum_{j \in \mathbb{Z}_{\geq 0}} \chi_4(p^j) \mathcal{C}_{-\epsilon}\left(\Delta_{p^j} G(\epsilon n \overline{2^k}; p^j)\right) (p^j)^{-\nu-1} \right)$$

where

$$p_{2,n} = \begin{cases} \left(1 - \left(\frac{t}{2}\right) 2^{-\nu-\frac{1}{2}}\right) & \text{if } 2|s, \\ 1 & \text{otherwise.} \end{cases} \tag{4.27}$$

Since $\left(1 - \left(\frac{t}{p}\right) p^{-\nu-\frac{1}{2}}\right)\left(1 + \left(\frac{t}{p}\right) p^{-\nu-\frac{1}{2}}\right) = 1 - p^{-2\nu-1}$ it follows that

$$L\left(\nu + \frac{1}{2}, \left(\frac{\epsilon n}{\cdot}\right)\right) \mathcal{G}_4(\epsilon, n, 2^k; \nu)$$
$$= p_{2,n} \cdot \left(1 - \left(\frac{\epsilon n}{2}\right) 2^{-\nu-\frac{1}{2}}\right) L\left(\nu + \frac{1}{2}, \left(\frac{t}{\cdot}\right)\right)$$
$$\left(\prod_{\substack{p \text{ odd prime} \\ p|s}} \frac{\left(1 - \left(\frac{t}{p}\right) p^{-\nu-\frac{1}{2}}\right)}{(1 - p^{-2\nu-1})} \sum_{j \in \mathbb{Z}_{\geq 0}} \chi_4(p^j) \mathcal{C}_{-\epsilon}\left(\Delta_{p^j} G(\epsilon n \overline{2^k}; p^j)\right) (p^j)^{-\nu-1} \right)$$
$$\left(\prod_{\substack{p \text{ odd prime} \\ p|t}} \left(1 - p^{-2\nu-1}\right)^{-1} \sum_{j \in \mathbb{Z}_{\geq 0}} \chi_4(p^j) \mathcal{C}_{-\epsilon}\left(\Delta_{p^j} G(\epsilon n \overline{2^k}; p^j)\right) (p^j)^{-\nu-1} \right)$$
$$= p_{2,n} \cdot \left(1 - \left(\frac{\epsilon n}{2}\right) 2^{-\nu-\frac{1}{2}}\right) L\left(\nu + \frac{1}{2}, \left(\frac{t}{\cdot}\right)\right)$$
$$\left(\prod_{\substack{p \text{ odd prime} \\ p|s}} \left(1 + \left(\frac{t}{p}\right) p^{-\nu-\frac{1}{2}}\right)^{-1} \sum_{j \in \mathbb{Z}_{\geq 0}} \chi_4(p^j) \mathcal{C}_{-\epsilon}\left(\Delta_{p^j} G(\epsilon n \overline{2^k}; p^j)\right) (p^j)^{-\nu-1} \right)$$
$$\left(\prod_{\substack{p \text{ odd prime} \\ p|t}} \left(1 - p^{-2\nu-1}\right)^{-1} \sum_{j \in \mathbb{Z}_{\geq 0}} \chi_4(p^j) \mathcal{C}_{-\epsilon}\left(\Delta_{p^j} G(\epsilon n \overline{2^k}; p^j)\right) (p^j)^{-\nu-1} \right). \tag{4.28}$$

A nearly identical calculation shows that

$$
\begin{aligned}
&L\left(\nu+\frac{1}{2},\left(\frac{\epsilon n}{\cdot}\right)\right)\mathcal{G}_8(\epsilon,n,2^k;\nu)\\
&=p_{2,n}\cdot\left(1-\left(\frac{\epsilon n}{2}\right)2^{-\nu-\frac{1}{2}}\right)L\left(\nu+\frac{1}{2},\left(\frac{t}{\cdot}\right)\right)\\
&\quad\left(\prod_{\substack{p\text{ odd prime}\\ p|s}}\left(1+\left(\frac{t}{p}\right)p^{-\nu-\frac{1}{2}}\right)^{-1}\sum_{j\in\mathbb{Z}_{\geq 0}}\chi_8(p^j)\mathcal{C}_{-\epsilon}\left(\Delta_{p^j}G(\epsilon n\overline{2^k};p^j)\right)(p^j)^{-\nu-1}\right)\\
&\quad\left(\prod_{\substack{p\text{ odd prime}\\ p|t}}\left(1-p^{-2\nu-1}\right)^{-1}\sum_{j\in\mathbb{Z}_{\geq 0}}\chi_8(p^j)\mathcal{C}_{-\epsilon}\left(\Delta_{p^j}G(\epsilon n\overline{2^k};p^j)\right)(p^j)^{-\nu-1}\right).
\end{aligned}
\tag{4.29}
$$

The following lemma allows us to further simplify (4.28) and (4.29).

Lemma 4.8. *Let $n \neq 0$ and $\epsilon = \pm 1$ with $\epsilon n = st$ as above.*

(a) If p odd and $p^\ell || t$ then

$$\sum_{j\in\mathbb{Z}_{\geq 0}}\chi(p^j)\mathcal{C}_{-\epsilon}\left(\Delta_{p^j}G(\epsilon n\overline{2^k};p^j)\right)(p^j)^{-\nu-1}=(1-p^{-2\nu-1})\sum_{\substack{j=0\\ j\text{ even}}}^{\ell-1}p^{-j\nu},$$

where $\chi = \chi_4$ or χ_8.

(b) If p odd and $p^\ell || s$ then for even k we have

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}_{\geq 0}}\chi_4(p^j)\mathcal{C}_{-\epsilon}\left(\Delta_{p^j}G(\epsilon n\overline{2^k};p^j)\right)(p^j)^{-\nu-1}\\
&=\left(1+\left(\frac{t}{p}\right)p^{-\nu-\frac{1}{2}}\right)\left(\left(1-\left(\frac{t}{p}\right)p^{-\nu-\frac{1}{2}}\right)\sum_{\substack{j=0\\ j\text{ even}}}^{\ell-2}p^{-j\nu}+p^{-\ell\nu}\right),
\end{aligned}
$$

and for odd k we have

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}_{\geq 0}}\chi_8(p^j)\mathcal{C}_{-\epsilon}\left(\Delta_{p^j}G(\epsilon n\overline{2^k};p^j)\right)(p^j)^{-\nu-1}\\
&=\left(1+\left(\frac{t}{p}\right)p^{-\nu-\frac{1}{2}}\right)\left(\left(1-\left(\frac{t}{p}\right)p^{-\nu-\frac{1}{2}}\right)\sum_{\substack{j=0\\ j\text{ even}}}^{\ell-2}p^{-j\nu}+p^{-\ell\nu}\right).
\end{aligned}
$$

(c) If $p = 2$ then

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}_{\geq 0}}\chi_4(p^j)\mathcal{C}_{-\epsilon}\left(\Delta_{p^j}G(\epsilon n\overline{2^k};p^j)\right)(p^j)^{-\nu-1}=0,\\
&\sum_{j\in\mathbb{Z}_{\geq 0}}\chi_8(p^j)\mathcal{C}_{-\epsilon}\left(\Delta_{p^j}G(\epsilon n\overline{2^k};p^j)\right)(p^j)^{-\nu-1}=0.
\end{aligned}
$$

Proof. If $p|t$ and p odd, then there exists $n' \in \mathbb{Z}$ and $\ell \in \mathbb{Z}_{>0}$ such that $\epsilon n = p^\ell n'$, $\gcd(p, n') = 1$, and ℓ is odd. Therefore by Lemma 4.4(d), we have that

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}_{\geq 0}} \chi_4(p^j)\mathcal{C}_{-\epsilon}\Big(\Delta_{p^j} G(\epsilon n\overline{2^k}; p^j)\Big)(p^j)^{-\nu-1} \\
&= \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell-1} \chi_4(p^j)\mathcal{C}_{-\epsilon}\Big(\Delta_{p^j} G(\epsilon n\overline{2^k}; p^j)\Big)(p^j)^{-\nu-1} \\
&\quad + \chi_4(p^{\ell+1})\mathcal{C}_{-\epsilon}\Big(\Delta_{p^{\ell+1}} G(\epsilon n\overline{2^k}; p^{\ell+1})\Big)(p^{\ell+1})^{-\nu-1} \\
&= 1 + \sum_{\substack{j=2 \\ j \text{ even}}}^{\ell-1} (p^j - p^{j-1})(p^j)^{-\nu-1} - p^\ell(p^{\ell+1})^{-\nu-1} \\
&= 1 + \sum_{\substack{j=2 \\ j \text{ even}}}^{\ell-1} (p^{-j\nu} - p^{-j\nu-1}) - p^{-(\ell+1)\nu-1} \\
&= 1 - p^{-2\nu-1} + \sum_{\substack{j=2 \\ j \text{ even}}}^{\ell-3} (p^{-j\nu} - p^{-(j+2)\nu-1}) + p^{-(\ell-1)\nu} - p^{-(\ell+1)\nu-1} \\
&= (1 - p^{-2\nu-1}) + (1 - p^{-2\nu-1}) \sum_{\substack{j=2 \\ j \text{ even}}}^{\ell-3} p^{-j\nu} + (1 - p^{-2\nu-1})p^{-(\ell-1)\nu} \\
&= (1 - p^{-2\nu-1}) \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell-1} p^{-j\nu}.
\end{aligned}$$

This same argument also work when we replace χ_4 with χ_8, and thus we have proven part (a).

If $p|s$ and p odd, then $\epsilon n = p^\ell n'$ where $\gcd(p, n') = 1$ and ℓ is even. Therefore by Lemma 4.4(d), we have for even k that

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}_{\geq 0}} \chi_4(p^j)\mathcal{C}_{-\epsilon}\Big(\Delta_{p^j} G(\epsilon n\overline{2^k}; p^j)\Big)(p^j)^{-\nu-1} \\
&= \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell} \chi_4(p^j)\mathcal{C}_{-\epsilon}\Big(\Delta_{p^j} G(\epsilon n\overline{2^k}; p^j)\Big)(p^j)^{-\nu-1} \\
&\quad + \chi_4(p^{\ell+1})\mathcal{C}_{-\epsilon}\Big(\Delta_{p^{\ell+1}} G(\epsilon n\overline{2^k}; p^{\ell+1})\Big)(p^{\ell+1})^{-\nu-1} \\
&= 1 + \sum_{\substack{j=2 \\ j \text{ even}}}^{\ell} (p^j - p^{j-1})(p^j)^{-\nu-1} + \chi_4(p^{\ell+1})\mathcal{C}_{-\epsilon}\left(\Delta_{p^{\ell+1}}\left(\frac{2^k n'}{p}\right)\Delta_p p^{\ell+\frac{1}{2}}\right)(p^{\ell+1})^{-\nu-1} \\
&= 1 + \sum_{\substack{j=2 \\ j \text{ even}}}^{\ell} (p^j - p^{j-1})(p^j)^{-\nu-1} + \left(\frac{n'}{p}\right)p^{\ell+\frac{1}{2}}(p^{\ell+1})^{-\nu-1}.
\end{aligned}$$

Notice that we used the fact that $\chi_4(p^{\ell+1})\Delta_{p^{\ell+1}}\Delta_p = 1$ for all odd primes and that $\left(\frac{2^k}{p}\right) = 1$ since k is even. Likewise, since $\left(\frac{2^k}{p}\right) = \left(\frac{2}{p}\right)$ for odd k and $\chi_8(p^{\ell+1})\Delta_{p^{\ell+1}}\left(\frac{2}{p}\right)\Delta_p = 1$ for all primes, we have by Lemma 4.4(d) that for odd k,

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}_{\geq 0}} \chi_8(p^j)\mathcal{C}_{-\epsilon}\Big(\Delta_{p^j} G(\epsilon n\overline{2^k};p^j)\Big)(p^j)^{-\nu-1} \\
&= \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell} \chi_8(p^j)\mathcal{C}_{-\epsilon}\Big(\Delta_{p^j} G(\epsilon n\overline{2^k};p^j)\Big)(p^j)^{-\nu-1} + \chi_8(p^{\ell+1})\mathcal{C}_{-\epsilon}\Big(\Delta_{p^{\ell+1}} G(\epsilon n\overline{2^k};p^{\ell+1})\Big)(p^{\ell+1})^{-\nu-1} \\
&= 1 + \sum_{\substack{j=2 \\ j \text{ even}}}^{\ell} (p^j - p^{j-1})(p^j)^{-\nu-1} + \chi_8(p^{\ell+1})\mathcal{C}_{-\epsilon}\left(\Delta_{p^{\ell+1}}\left(\frac{2^k n'}{p}\right)\Delta_p p^{\ell+\frac{1}{2}}\right)(p^{\ell+1})^{-\nu-1} \\
&= 1 + \sum_{\substack{j=2 \\ j \text{ even}}}^{\ell} (p^j - p^{j-1})(p^j)^{-\nu-1} + \left(\frac{n'}{p}\right)p^{\ell+\frac{1}{2}}(p^{\ell+1})^{-\nu-1}.
\end{aligned}$$

Thus for part (b), it remains to simplify

$$1 + \sum_{\substack{j=2 \\ j \text{ even}}}^{\ell} (p^j - p^{j-1})(p^j)^{-\nu-1} + \left(\frac{n'}{p}\right)p^{\ell+\frac{1}{2}}(p^{\ell+1})^{-\nu-1}.$$

Observe

$$\begin{aligned}
&1 + \sum_{\substack{j=2 \\ j \text{ even}}}^{\ell} (p^j - p^{j-1})(p^j)^{-\nu-1} + \left(\frac{n'}{p}\right)p^{\ell+\frac{1}{2}}(p^{\ell+1})^{-\nu-1} \\
&= 1 + \sum_{\substack{j=2 \\ j \text{ even}}}^{\ell} (p^{-j\nu} - p^{-j\nu-1}) + \left(\frac{n'}{p}\right)p^{-(\ell+1)\nu-\frac{1}{2}} \\
&= 1 - p^{-2\nu-1} + \sum_{\substack{j=2 \\ j \text{ even}}}^{\ell-2} (p^{-j\nu} - p^{-(j+2)\nu-1}) + p^{-\ell\nu} + \left(\frac{n'}{p}\right)p^{-(\ell+1)\nu-\frac{1}{2}} \\
&= (1 - p^{-2\nu-1}) \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell-2} p^{-j\nu} + p^{-\ell\nu}\left(1 + \left(\frac{n'}{p}\right)p^{-\nu-\frac{1}{2}}\right) \\
&= \left(1 + \left(\frac{n'}{p}\right)p^{-\nu-\frac{1}{2}}\right)\left(\left(1 - \left(\frac{n'}{p}\right)p^{-\nu-\frac{1}{2}}\right) \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell-2} p^{-j\nu} + p^{-\ell\nu}\right).
\end{aligned}$$

Part (b) now follows since $\left(\frac{n'}{p}\right) = \left(\frac{t}{p}\right)$.

Part (c) follows directly from the fact that $\chi_4(p^j) = \chi_8(p^j) = 0$ for $j > 0$. $\square$

By Lemma 4.8, (4.28), and (4.29), we have for even k that

$$L\Big(\nu+\frac{1}{2},\Big(\frac{\epsilon n}{\cdot}\Big)\Big)\mathcal{G}_4(\epsilon,n,2^k;\nu)$$
$$= p_{2,n}\cdot\Big(1-\Big(\frac{\epsilon n}{2}\Big)2^{-\nu-\frac{1}{2}}\Big)L\Big(\nu+\frac{1}{2},\Big(\frac{t}{\cdot}\Big)\Big)$$
$$\Bigg(\prod_{\substack{p \text{ odd prime}\\ p^\ell||s}}\Bigg(\Big(1-\Big(\frac{t}{p}\Big)p^{-\nu-\frac{1}{2}}\Big)\sum_{\substack{j=0\\ j\text{ even}}}^{\ell-2}p^{-j\nu}+p^{-\ell\nu}\Bigg)\Bigg)\Bigg(\prod_{\substack{p \text{ odd prime}\\ p^\ell||t}}\sum_{\substack{j=0\\ j\text{ even}}}^{\ell-1}p^{-j\nu}\Bigg), \tag{4.30}$$

and for odd k, we have that

$$L\Big(\nu+\frac{1}{2},\Big(\frac{\epsilon n}{\cdot}\Big)\Big)\mathcal{G}_8(\epsilon,n,2^k;\nu)$$
$$= p_{2,n}\cdot\Big(1-\Big(\frac{\epsilon n}{2}\Big)2^{-\nu-\frac{1}{2}}\Big)L\Big(\nu+\frac{1}{2},\Big(\frac{t}{\cdot}\Big)\Big)$$
$$\Bigg(\prod_{\substack{p \text{ odd prime}\\ p^\ell||s}}\Bigg(\Big(1-\Big(\frac{t}{p}\Big)p^{-\nu-\frac{1}{2}}\Big)\sum_{\substack{j=0\\ j\text{ even}}}^{\ell-2}p^{-j\nu}+p^{-\ell\nu}\Bigg)\Bigg)\Bigg(\prod_{\substack{p \text{ odd prime}\\ p^\ell||t}}\sum_{\substack{j=0\\ j\text{ even}}}^{\ell-1}p^{-j\nu}\Bigg), \tag{4.31}$$

Observe that (4.30) and (4.31) are each the product of a Dirichlet polynomial and a Dirichlet L-function $L\big(\nu+\frac{1}{2},\big(\frac{t}{\cdot}\big)\big)$. If $t\neq 1$ then $L\big(\nu+\frac{1}{2},\big(\frac{t}{\cdot}\big)\big)$ has holomorphic continuation to all of $\mathbb{C}$. If $t=1$ then $L\big(\nu+\frac{1}{2},\big(\frac{t}{\cdot}\big)\big)$ has holomorphic continuation to all of $\mathbb{C}$ except for a simple pole $\nu=\frac{1}{2}$. Therefore, by Proposition 4.7, we see that for $n\neq 0$, a_n has holomorphic continuation to all of $\mathbb{C}$ except for a simple pole at $\nu=\frac{1}{2}$ if ϵn is a square.

In the following proposition we obtain a formula for a_0 which shows that a_0 also has holomorphic continuation to all of $\mathbb{C}$ except for a pole at $\nu=\frac{1}{2}$.

Proposition 4.9. *For $E_\nu^{(\infty)}\in V_{(\epsilon),\nu}^{-\infty}(\widetilde{SL}_2)$ with $\Re(\nu)>1$, we have that*

$$a_0=(1+\epsilon i)2^{-2\nu-2}\zeta(2\nu).$$

Proof. Recall that by (4.22) and (4.19), we have that

$$\begin{aligned} a_0 &= \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1)\sum_{c\in\mathbb{Z}_{>0}}c^{-\nu-1}\mathcal{C}_{-\epsilon}(K_{-1}(0;4c)) \\ &= \epsilon i 4^{-\nu-1}\zeta_2(2\nu+1)\sum_{c\in\mathbb{Z}_{>0}}c^{-\nu-1}\mathcal{C}_{-\epsilon}\Bigg(\sum_{d\in\mathbb{Z}/4c\mathbb{Z}}\Delta_d\Big(\frac{4c}{d}\Big)\Bigg). \end{aligned}$$

If c is a square then

$$\sum_{d\in\mathbb{Z}/4c\mathbb{Z}}\Delta_d\Big(\frac{4c}{d}\Big)=\sum_{\substack{d\in\mathbb{Z}/4c\mathbb{Z}\\ d\text{ odd}\\ \gcd(c,d)=1}}\Delta_d=\frac{1+i}{2}\phi(4c),$$

where ϕ is the Euler totient function. If c is not a square then we select $d' \in (\mathbb{Z}/4c\mathbb{Z})^*$ such that $\left(\frac{4c}{d'}\right) = -1$; since $\left(\frac{4c}{-1}\right) = 1$ it follows that we can always select d' such that $d' \equiv 1 (\operatorname{mod} 4)$. Since

$$\left(\frac{4c}{d'}\right) \sum_{\substack{d (\operatorname{mod} 4c) \\ d \equiv 1(4)}} \left(\frac{4c}{d}\right) = \sum_{\substack{d (\operatorname{mod} 4c) \\ d \equiv 1(4)}} \left(\frac{4c}{d}\right) \text{ it follows that } \sum_{\substack{d (\operatorname{mod} 4c) \\ d \equiv 1(4)}} \left(\frac{4c}{d}\right) = 0.$$

By the same argument it also follows that

$$\sum_{d (\operatorname{mod} 4c)} \left(\frac{4c}{d}\right) = 0, \text{ and thus } \sum_{\substack{d (\operatorname{mod} 4c) \\ d \equiv 3(4)}} \left(\frac{4c}{d}\right) = 0.$$

Therefore if c is not a square then we have

$$\sum_{d \in \mathbb{Z}/4c\mathbb{Z}} \Delta_d \left(\frac{4c}{d}\right) = 0.$$

Thus

$$\begin{aligned}
a_0 &= \epsilon i 4^{-\nu-1} \zeta_2(2\nu+1) \sum_{\substack{c \in \mathbb{Z}_{>0} \\ c \text{ a square}}} c^{-\nu-1} \frac{1-\epsilon i}{2} \phi(4c) \\
&= \frac{1+\epsilon i}{2} 4^{-\nu-1} \zeta_2(2\nu+1) \sum_{c \in \mathbb{Z}_{>0}} c^{-2\nu-2} \phi(4c^2) \\
&= \frac{1+\epsilon i}{2} 4^{-\nu-1} \zeta_2(2\nu+1) \left(\sum_{\substack{c \in \mathbb{Z}_{>0} \\ c \text{ odd}}} \sum_{k \in \mathbb{Z}_{\geq 0}} (2^k c)^{-2\nu-2} \phi(2^{2k+2} c^2) \right) \\
&= \frac{1+\epsilon i}{2} 4^{-\nu-1} \zeta_2(2\nu+1) \left(\sum_{\substack{c \in \mathbb{Z}_{>0} \\ c \text{ odd}}} \sum_{k \in \mathbb{Z}_{\geq 0}} (2^k)^{-2\nu-2} c^{-2\nu-2} 2^{2k+1} \phi(c^2) \right) \\
&= \frac{1+\epsilon i}{2} 4^{-\nu-1} \zeta_2(2\nu+1) \left(\sum_{\substack{c \in \mathbb{Z}_{>0} \\ c \text{ odd}}} 2 \sum_{k \in \mathbb{Z}_{\geq 0}} 2^{-2k\nu} c^{-2\nu-2} \phi(c^2) \right) \\
&= (1+\epsilon i) 2^{-2\nu-2} (1-2^{-2\nu})^{-1} \zeta_2(2\nu+1) \left(\sum_{\substack{c \in \mathbb{Z}_{>0} \\ c \text{ odd}}} c^{-2\nu-2} \phi(c^2) \right).
\end{aligned}$$

Since $\phi(c^2)$ is multiplicative, it follows that

$$a_0 = (1+\epsilon i) 2^{-2\nu-2} \left(1-2^{-2\nu}\right)^{-1} \zeta_2(2\nu+1) \prod_{p \text{ odd prime}} \sum_{k \in \mathbb{Z}_{\geq 0}} (p^k)^{-2\nu-2} \phi(p^{2k}).$$

Observe that for odd primes p we have that

$$\sum_{k\in\mathbb{Z}_{\geq 0}} (p^k)^{-2\nu-2}\phi(p^{2k}) = 1 + \sum_{k\in\mathbb{Z}_{\geq 1}} p^{-2k\nu-2k}(p^{2k} - p^{2k-1})$$
$$= 1 + \sum_{k\in\mathbb{Z}_{\geq 1}} (p^{-2k\nu} - p^{-2k\nu-1}) = 1 - p^{-2\nu-1} + \sum_{k\in\mathbb{Z}_{\geq 1}} (p^{-2k\nu} - p^{-2(k+1)\nu-1})$$
$$= 1 - p^{-2\nu-1} + (1 - p^{-2\nu-1}) \sum_{k\in\mathbb{Z}_{\geq 1}} p^{-2k\nu} = (1 - p^{-2\nu-1}) \sum_{k\in\mathbb{Z}_{\geq 0}} p^{-2k\nu}$$
$$= \frac{1 - p^{-2\nu-1}}{1 - p^{-2\nu}}.$$

Using this expression and the Euler product expansion for $\zeta_2(2\nu+1)$ gives us that

$$a_0 = (1+\epsilon i)2^{-2\nu-2}(1-2^{-2\nu})^{-1}\zeta_2(2\nu+1) \prod_{p \text{ odd prime}} \frac{1-p^{-2\nu-1}}{1-p^{-2\nu}}$$
$$= (1+\epsilon i)2^{-2\nu-2}(1-2^{-2\nu})^{-1} \prod_{p \text{ odd prime}} (1-p^{-2\nu})^{-1}$$
$$= (1+\epsilon i)2^{-2\nu-2}\zeta(2\nu).$$

□

Now that we have established the meromorphic continuation of the Fourier coefficients a_n, it remains to prove that $\left(\widetilde{E}_\nu^{(\infty)}\right)_0(x) = \sum_{n\in\mathbb{Z}} a_n e(nx)$ also has meromorphic continuation. In particular, we wish to show that the series $\sum_{n\in\mathbb{Z}} a_n e(nx)$ converges for all $\nu \neq \frac{1}{2}$. We begin by establishing bounds for

$$L\left(\nu + \frac{1}{2}, \left(\frac{\epsilon n}{\cdot}\right)\right)\mathcal{G}_4(\epsilon, n, 2^k; \nu),$$

when k even, and for

$$L\left(\nu + \frac{1}{2}, \left(\frac{\epsilon n}{\cdot}\right)\right)\mathcal{G}_8(\epsilon, n, 2^k; \nu),$$

when k odd. To do this, we will compute our bounds using the formulas given (4.30) and (4.31).

For $\Re(\nu) \geq 0$, we have the following (crude) bounds (recall that $\mathfrak{p}_{2,n}$ was defined in (4.27)):

$$\left|\mathfrak{p}_{2,n} \cdot \left(1 - \left(\frac{\epsilon n}{2}\right)2^{-\nu-\frac{1}{2}}\right)\right| \leq (1 + 2^{-\Re(\nu)-\frac{1}{2}})^2 \leq (1+2^{-\frac{1}{2}})^2 < 4,$$

$$\left| \prod_{\substack{p \text{ odd prime} \\ p^\ell || t}} \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell-1} p^{-j\nu} \right| \leq \prod_{\substack{p \text{ odd prime} \\ p^\ell || t}} \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell-1} 1 < \prod_{\substack{p \text{ odd prime} \\ p^\ell || t}} p^\ell \leq |n|,$$

$$\left| \prod_{\substack{p \text{ odd prime} \\ p^\ell || s}} \left(\left(1 - \left(\frac{t}{p}\right)p^{-\nu-\frac{1}{2}}\right) \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell-2} p^{-j\nu} + p^{-\ell\nu} \right) \right|$$

$$= \left| \prod_{\substack{p \text{ odd prime} \\ p^\ell || s}} \left(\sum_{\substack{j=0 \\ j \text{ even}}}^{\ell} p^{-j\nu} - \left(\frac{t}{p}\right) p^{-\nu-\frac{1}{2}} \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell-2} p^{-j\nu} \right) \right|$$

$$\leq \prod_{\substack{p \text{ odd prime} \\ p^\ell || s}} \left(\left| \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell} p^{-j\nu} \right| + \left| \left(\frac{t}{p}\right) p^{-\nu-\frac{1}{2}} \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell-2} p^{-j\nu} \right| \right)$$

$$\leq \prod_{\substack{p \text{ odd prime} \\ p^\ell || s}} \left(\sum_{\substack{j=0 \\ j \text{ even}}}^{\ell} 1 + p^{-\frac{1}{2}} \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell-2} 1 \right)$$

$$< \prod_{\substack{p \text{ odd prime} \\ p^\ell || s}} \left(p^\ell + p^\ell \right) < \prod_{\substack{p \text{ odd prime} \\ p^\ell || s}} p^{\ell+1} \leq |n|^2.$$

For $\Re(\nu) \leq 0$, we also have the following bounds:

$$\left| \not{p}_{2,n} \cdot \left(1 - \left(\frac{\epsilon n}{2}\right) 2^{-\nu-\frac{1}{2}} \right) \right| \leq (1 + 2^{-\Re(\nu)-\frac{1}{2}})^2 < (2 \cdot 2^{-\Re(\nu)})^2 = 4^{-\Re(\nu)-1},$$

$$\left| \prod_{\substack{p \text{ odd prime} \\ p^\ell || t}} \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell-1} p^{-j\nu} \right| < \prod_{\substack{p \text{ odd prime} \\ p^\ell || t}} \ell \cdot p^{-\ell \Re(\nu)} < \prod_{\substack{p \text{ odd prime} \\ p^\ell || t}} p^{\ell(-\Re(\nu)+1)}$$

$$\leq |n|^{-\Re(\nu)+1},$$

$$\left| \prod_{\substack{p \text{ odd prime} \\ p^\ell || s}} \left(\left(1 - \left(\frac{t}{p}\right) p^{-\nu-\frac{1}{2}} \right) \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell-2} p^{-j\nu} + p^{-\ell\nu} \right) \right|$$

$$= \left| \prod_{\substack{p \text{ odd prime} \\ p^\ell || s}} \left(\sum_{\substack{j=0 \\ j \text{ even}}}^{\ell} p^{-j\nu} - \left(\frac{t}{p}\right) p^{-\nu-\frac{1}{2}} \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell-2} p^{-j\nu} \right) \right|$$

$$\leq \prod_{\substack{p \text{ odd prime} \\ p^\ell || s}} \left(\left| \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell} p^{-j\nu} \right| + \left| \left(\frac{t}{p}\right) p^{-\nu-\frac{1}{2}} \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell-2} p^{-j\nu} \right| \right)$$

$$\leq \prod_{\substack{p \text{ odd prime} \\ p^\ell || s}} \left(\sum_{\substack{j=0 \\ j \text{ even}}}^{\ell} p^{-\ell\Re(\nu)} + p^{-\Re(\nu)-\frac{1}{2}} \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell-2} p^{-\ell\Re(\nu)} \right)$$

$$\leq \prod_{\substack{p \text{ odd prime} \\ p^\ell || s}} \left(2p^{-\Re(\nu)} \sum_{\substack{j=0 \\ j \text{ even}}}^{\ell} p^{-\ell\Re(\nu)} \right) < \prod_{\substack{p \text{ odd prime} \\ p^\ell || s}} p \cdot p^{-\Re(\nu)} \ell p^{-\ell\Re(\nu)}$$

$$< \prod_{\substack{p \text{ odd prime} \\ p^\ell || s}} p \cdot p^{-\Re(\nu)} p^\ell p^{-\ell\Re(\nu)} = \prod_{\substack{p \text{ odd prime} \\ p^\ell || s}} p^{(\ell+1)(-\Re(\nu)+1)} \le |n^2|^{-\Re(\nu)+1}.$$

By these inequality, it follows from (4.30) for even k that

$$\begin{aligned}
&\left| L\left(\nu + \frac{1}{2}, \left(\frac{\epsilon n}{\cdot}\right)\right) \mathcal{G}_4(\epsilon, n, 2^k; \nu)\right| \\
&= \begin{cases} 4|n|^3 \left|L\left(\nu + \frac{1}{2}, \left(\frac{t}{\cdot}\right)\right)\right| & \text{if } \Re(\nu) \ge 0 \\ 4^{-\Re(\nu)-1}|n|^{-\Re(\nu)+1}|n^2|^{-\Re(\nu)+1}\left|L\left(\nu + \frac{1}{2}, \left(\frac{t}{\cdot}\right)\right)\right| & \text{if } \Re(\nu) < 0 \end{cases} \\
&= \begin{cases} 4|n|^3 \left|L\left(\nu + \frac{1}{2}, \left(\frac{t}{\cdot}\right)\right)\right| & \text{if } \Re(\nu) \ge 0 \\ |4n^3|^{-\Re(\nu)+1}\left|L\left(\nu + \frac{1}{2}, \left(\frac{t}{\cdot}\right)\right)\right| & \text{if } \Re(\nu) < 0. \end{cases}
\end{aligned} \tag{4.32}$$

Similarly, it follows from (4.31) for odd k that

$$\begin{aligned}
&\left| L\left(\nu + \frac{1}{2}, \left(\frac{\epsilon n}{\cdot}\right)\right) \mathcal{G}_8(\epsilon, n, 2^k; \nu)\right| \\
&= \begin{cases} 4|n|^3 \left|L\left(\nu + \frac{1}{2}, \left(\frac{t}{\cdot}\right)\right)\right| & \text{if } \Re(\nu) \ge 0 \\ |4n^3|^{-\Re(\nu)+1}\left|L\left(\nu + \frac{1}{2}, \left(\frac{t}{\cdot}\right)\right)\right| & \text{if } \Re(\nu) < 0. \end{cases}
\end{aligned} \tag{4.33}$$

We now show that there exists $K : \mathbb{C}_{\neq \frac{1}{2}} \to \mathbb{R}_{>0}$ such that

$$\left| L\left(\nu + \frac{1}{2}, \left(\frac{t}{\cdot}\right)\right)\right| \le K(\nu), \tag{4.34}$$

for all t and $\Re(\nu) > \frac{1}{2}$. First observe that for $\Re(\nu) > \frac{1}{2}$ we have that

$$\left| L\left(\nu + \frac{1}{2}, \left(\frac{t}{\cdot}\right)\right)\right| \le \left|\zeta\left(\Re(\nu) + \frac{1}{2}\right)\right|.$$

Thus such $K(\nu)$ exists for $\Re(\nu) > \frac{1}{2}$. By utilizing the functional equation for $L\big(\nu + \frac{1}{2}, \left(\frac{t}{\cdot}\right)\big)$ for $t \neq 1$, we see that for $\Re(\nu) < -\frac{1}{2}$ we have

$$\begin{aligned}
\left| L\left(\nu + \frac{1}{2}, \left(\frac{t}{\cdot}\right)\right)\right| &\le \max_{\delta \in \{0,1\}} \left(\left| \frac{\Gamma\left(\frac{1-2\nu+2\delta}{4}\right)}{\Gamma\left(\frac{1+2\nu+2\delta}{4}\right)} \right| \right) \left| \frac{t}{\pi} \right|^{-\Re(\nu)} \left| L\left(\frac{1}{2} - \nu, \left(\frac{t}{\cdot}\right)\right)\right| \\
&\le \max_{\delta \in \{0,1\}} \left(\left| \frac{\Gamma\left(\frac{1-2\nu+2\delta}{4}\right)}{\Gamma\left(\frac{1+2\nu+2\delta}{4}\right)} \right| \right) \left| \frac{t}{\pi} \right|^{-\Re(\nu)} \left| \zeta\left(\frac{1}{2} - \Re(\nu)\right)\right|.
\end{aligned}$$

Finding a bound for when $t = 1$ is easy since $L\big(\nu + \frac{1}{2}, \left(\frac{t}{\cdot}\right)\big) = \zeta\big(\nu + \frac{1}{2}\big)$, and clearly such a function is bounded by $\left|\zeta\big(\nu + \frac{1}{2}\big)\right|$. Thus we have shown that $K(\nu)$ which satisfies (4.34) does indeed exist for $|\Re(\nu)| > \frac{1}{2}$. On $|\Re(\nu)| \le \frac{1}{2}$, $\nu \neq \frac{1}{2}$ it is well-known that there exists $E(\nu)$ which bounds all $L\big(\nu, \left(\frac{t}{\cdot}\right)\big)$ continuously in ν and polynomially in t away from $\nu \neq \frac{1}{2}$.

In Proposition 4.7 we see that a_n for $n \neq 0$, consist of finite sums with terms of the form

$$2^k (2^{k+2})^{-\nu-1} L\left(\nu, \left(\frac{t}{\cdot}\right)\right) \mathcal{G}_j(\epsilon, n, 2^k, \nu),$$

where $j = 4, 8$. The number of such summands for a given a_n are bounded polynomially in n. Therefore, since each factor in such summands is seen to be polynomially bounded in n, it follows that the a_n for $n \neq 0$ are polynomially bounded in n. This suffices to show that $\left(\widetilde{E}_\nu^{(\infty)}\right)_0(x) = \sum_{n\in\mathbb{Z}} a_n e(nx)$ converges on $\mathbb{C}_{\neq \frac{1}{2}}$. With some extra work one can also see that $\left(\widetilde{E}_\nu^{(\infty)}\right)_0(x) = \sum_{n\in\mathbb{Z}} a_n e(nx)$ has a simple pole at $\nu = \frac{1}{2}$: indeed, one can easily check that $(\nu - \frac{1}{2})a_n$ are holomorphic and that $(\nu - \frac{1}{2}) \sum_{n\in\mathbb{Z}} a_n e(nx)$ converges on all of $\mathbb{C}$.

Now that we have established that $\left(\widetilde{E}_\nu^{(\infty)}\right)_0$ has meromorphic continuation to $\mathbb{C}$ with a simple pole at $\nu = \frac{1}{2}$, it then follows that $\widetilde{E}_\nu^{(\infty)}$ also has meromorphic continuation to $\mathbb{C}$, provided that $\left(\widetilde{E}^{(\infty)}\right)_\infty$ has meromorphic continuation to $\mathbb{C}$. One means of doing this is to compute a Fourier series expansion for $\left(\widetilde{E}^{(\infty)}\right)_\infty$ and establish the meromorphic continuation of its Fourier coefficients just as we have done for $\left(\widetilde{E}_\nu^{(\infty)}\right)_0$. Instead, we will deduce the meromorphic continuation of $\left(\widetilde{E}_\nu^{(\infty)}\right)_\infty$ in large part from the meromorphic continuation of $\left(\widetilde{E}^{(\infty)}\right)_0$.

Since $\left(\widetilde{E}_\nu^{(\infty)}\right)_0$ and $\left(\widetilde{E}_\nu^{(\infty)}\right)_\infty$ are themselves the restrictions of $s_{\widetilde{E}_\nu^{(\infty)}}$ to $N(\widetilde{\mathrm{SL}}_2)B(\widetilde{\mathrm{SL}}_2)$ and $\widetilde{s}^{-1}N(\widetilde{\mathrm{SL}}_2)B(\widetilde{\mathrm{SL}}_2)$ respectively, and since the complement of $N(\widetilde{\mathrm{SL}}_2)B(\widetilde{\mathrm{SL}}_2)\cap\widetilde{s}^{-1}N(\widetilde{\mathrm{SL}}_2)B(\widetilde{\mathrm{SL}}_2)$ in $\widetilde{\mathrm{SL}}_2/B(\widetilde{\mathrm{SL}}_2)$ is simply $\widetilde{s}^{-1}B(\widetilde{\mathrm{SL}}_2)$, it follows that we have already established the meromorphic continuation of $\left(\widetilde{E}_\nu^{(\infty)}\right)_\infty\Big|_{\mathbb{R}_{\neq 0}}$. Since by (4.18) we have that

$$\left(\widetilde{E}_\nu^{(\infty)}\right)_\infty - \left(\widetilde{E}_\nu^{(\infty)}\right)_\infty\Big|_{\mathbb{R}_{\neq 0}} = \epsilon i \zeta_2(2\nu+1)\delta_0, \tag{4.35}$$

it follows that $\widetilde{E}_\nu^{(\infty)}$ has meromorphic continuation to all of $\mathbb{C}$, with a simple poles at $\nu = 0, \frac{1}{2}$ and $\nu = -\frac{\pi i m}{\log(2)} - \frac{1}{2}$ where $m \in \mathbb{Z}$.

4.4 The Metaplectic Eisenstein Distribution at 0

Recall that $\widetilde{E}_\nu^{(\infty)}$ is the distributional analogue of the usual metaplectic Eisenstein series based at the cusp ∞. Next we shall define $\widetilde{E}_\nu^{(0)}$, which will be a distributional analogue of the metaplectic Eisenstein series based at the cusp 0. To do this, recall that in (3.39) we defined

$$\Omega = \widetilde{a}_2^{-1}\widetilde{s} = \widetilde{s}\widetilde{a}_2 = \left(\begin{pmatrix} 0 & -2^{-1} \\ 2 & 0 \end{pmatrix}, 1\right), \tag{4.36}$$

where $\widetilde{s} = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1\right)$ and $\widetilde{a}_2 = \left(\begin{pmatrix} 2 & 0 \\ 0 & 2^{-1} \end{pmatrix}, 1\right)$, as defined in (3.7) and (3.2) respectively. Let $\widetilde{\Gamma}_{(0)} = \Omega\widetilde{\Gamma}_\infty\Omega^{-1}$. Following [9], we define the *metaplectic Eisenstein distribution at* 0 to be the following distribution in $V^{-\infty}_{(\epsilon),\nu}(\widetilde{\mathrm{SL}_2})$:

$$\widetilde{E}^{(0)}_\nu = \zeta_2(2\nu+1) \sum_{\widetilde{\gamma}\in\widetilde{\Gamma}_1(4)/\widetilde{\Gamma}_{(0)}} \pi(\widetilde{\gamma}\Omega)\delta_\infty, \tag{4.37}$$

where $\Re(\nu) > 1$. Since δ_∞ is invariant under left translation by $\widetilde{\Gamma}_\infty$, it follows that $\pi(\Omega)\delta_\infty$ is invariant under left translation by $\widetilde{\Gamma}_{(0)}$. Thus the summation over $\widetilde{\Gamma}_1(4)/\widetilde{\Gamma}_{(0)}$ in the definition of $\widetilde{E}^{(0)}_\nu$ is justified. By construction, we see that $\widetilde{E}^{(0)}_\nu$ is formally $\widetilde{\Gamma}_1(4)$-invariant. We will justify the convergence of the series defining $\widetilde{E}^{(0)}_\nu$ momentarily.

Since $\widetilde{\Gamma}_1(4)$ is generated by $\left(\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}, 1\right)$ and $\Omega\left(\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}, 1\right)\Omega^{-1} = \left(\begin{pmatrix} 1 & 0 \\ -4 & 1\end{pmatrix}, 1\right)$, it follows that $\Omega\widetilde{\Gamma}_1(4)\Omega^{-1} = \widetilde{\Gamma}_1(4)$. Since $\Omega\widetilde{\Gamma}_1(4)\Omega^{-1} = \widetilde{\Gamma}_1(4)$ and $\Omega\widetilde{\Gamma}_\infty\Omega^{-1} = \widetilde{\Gamma}_{(0)}$, we have that

$$\widetilde{E}^{(0)}_\nu = \zeta_2(2\nu+1) \sum_{\widetilde{\gamma}\in\widetilde{\Gamma}_1(4)/\widetilde{\Gamma}_\infty} \pi(\Omega\widetilde{\gamma})\delta_\infty = \pi(\Omega)\widetilde{E}^{(\infty)}_\nu.$$

Therefore the series defining $\widetilde{E}^{(0)}_\nu$ converges for $\Re(\nu) > 1$, and we have that $\widetilde{E}^{(0)}_\nu$ has meromorphic continuation to $\mathbb{C}$ with a simple pole at $\nu = \frac{1}{2}$. Since $\pi(\Omega)\widetilde{E}^{(\infty)}_\nu$ is $\widetilde{\Gamma}_\infty$-invariant, it follows that $\left(\widetilde{E}^{(0)}_\nu\right)_0$ is periodic. Thus $\left(\widetilde{E}^{(0)}_\nu\right)_0$ has a Fourier series expansion:

$$\left(\widetilde{E}^{(0)}_\nu\right)_0(x) = \sum_{n\in\mathbb{Z}} b_n e(nx),$$

where

$$b_n = \int_0^1 \left(\widetilde{E}^{(0)}_\nu\right)_0(x) e(-nx)\, dx.$$

In section 4.5, we will establish a functional equation between $\widetilde{E}^{(\infty)}_\nu$ and $\widetilde{E}^{(0)}_\nu$. We will do this by computing an explicit formula for the Fourier coefficient b_0. The derivation of this formula will make considerable use of the Kronecker symbol. We refer the reader back to Proposition 4.2, where the relevant properties of the Kronecker symbol are stated.

Proposition 4.10. *For $\widetilde{E}^{(0)}_\nu \in V^{-\infty}_{(\epsilon),\nu}(\widetilde{SL}_2)$ with $\Re(\nu) > 1$, we have that*

$$b_0 = \epsilon i 2^{-\nu-1}\zeta_2(2\nu).$$

Proof. Suppose $\widetilde{\gamma}^{-1} = \left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}, \left(\frac{c}{d}\right)\right) \in \widetilde{\Gamma}_1(4)$. If $c \neq 0$ then $ad \equiv 1 (\operatorname{mod} c)$, which implies that $\left(\frac{c}{a}\right)\left(\frac{c}{d}\right) = \left(\frac{c}{ad}\right) = \left(\frac{c}{1}\right) = 1$. Thus $\left(\frac{c}{d}\right) = \left(\frac{c}{a}\right)$, both for $c \neq 0$ and $c = 0$. Next we seek to evaluate $\left(\frac{-b}{a}\right)\left(\frac{c}{a}\right) = \left(\frac{-bc}{a}\right)$ in order to express $\left(\frac{c}{d}\right) = \left(\frac{c}{a}\right)$ in terms of $\left(\frac{-b}{a}\right)$. Since $a \equiv 1 (\operatorname{mod} 4)$ it follows that $\left(\frac{-bc}{a}\right) = \left(\frac{-1}{a}\right)\left(\frac{bc}{a}\right) = \left(\frac{bc}{a}\right)$. Therefore when $b \neq 0$, we have that the sign of b is inconsequential when it comes to evaluating $\left(\frac{-bc}{a}\right)$. Observe that for $b > 0$ we have that

$$\begin{aligned}
\left(\frac{-bc}{a}\right) &= \left(\frac{1}{a}\right) = 1 && \text{if } a > 0,\\
\left(\frac{-bc}{a}\right) &= \left(\frac{-bc}{-1}\right)\left(\frac{-bc}{-a}\right) = \operatorname{sgn}(-c)\left(\frac{1}{-a}\right) = \operatorname{sgn}(-c) && \text{if } a < 0.
\end{aligned}$$

Hence $\left(\frac{-b}{a}\right)\left(\frac{c}{a}\right) = \left(\frac{-bc}{a}\right) = (a,-c)_H$ for $b \neq 0$. Thus for $b \neq 0$, we have that

$$\left(\frac{c}{d}\right) = \left(\frac{c}{a}\right) = \left(\frac{-b}{a}\right)(a,-c)_H = \left(\frac{b}{a}\right)(a,-c)_H; \tag{4.38}$$

the last equality follows since $a \equiv 1 (\operatorname{mod} 4)$. We shall make reference to this equality later in our proof.

Observe that for $c \neq 0$, we have

$$\widetilde{\gamma}\Omega\widetilde{s} = \left(\begin{pmatrix} -\frac{d}{2} & 2b \\ \frac{c}{2} & -2a \end{pmatrix}, \operatorname{sgn}(c)\left(\frac{c}{d}\right)\right) = \left(\begin{pmatrix} -2a & -2b \\ -\frac{c}{2} & -\frac{d}{2} \end{pmatrix}, \operatorname{sgn}(c)\left(\frac{c}{d}\right)\right)^{-1}.$$

Thus by Lemma 4.1(a), we have for $c \neq 0$,

$$\begin{aligned}
&(\pi(\widetilde{\gamma}\Omega)\delta_\infty)_0(x) = (\pi(\widetilde{\gamma}\Omega\widetilde{s})\delta_0)_0(x)\\
&= \operatorname{sgn}(c)\left(\frac{c}{d}\right)\left(\frac{c}{2}, -\frac{c}{2}x - \frac{d}{2}\right)_H \left|\frac{-c}{2}x + \frac{-d}{2}\right|^{\nu-1} \operatorname{sgn}\left(-\frac{c}{2}x - \frac{d}{2}\right)^{\epsilon/2} \delta_0\left(\frac{-2ax-2b}{-\frac{c}{2}x - \frac{d}{2}}\right)\\
&= \operatorname{sgn}(c)\left(\frac{c}{d}\right)(c, -cx-d)_H 2^{-\nu+1}|cx+d|^{\nu-1}\operatorname{sgn}(-cx-d)^{\epsilon/2}\delta_0\left(4\left(\frac{ax+b}{cx+d}\right)\right),
\end{aligned}$$

as an equality between distributions on $\mathbb{R}_{\neq \frac{-d}{c}}$. Let ϕ a test function of compact support on $\mathbb{R}_{\neq \frac{-d}{c}}$. Observe that

$$\begin{aligned}
&\int_{\mathbb{R}_{\neq \frac{-d}{c}}} \operatorname{sgn}(c)\left(\frac{c}{d}\right)(c,-cx-d)_H 2^{-\nu+1}|cx+d|^{\nu-1}\operatorname{sgn}(-cx-d)^{\epsilon/2}\delta_0\left(4\left(\frac{ax+b}{cx+d}\right)\right)\phi(x)\,dx\\
&= \int_{\mathbb{R}_{\neq 0}} \operatorname{sgn}(c)\left(\frac{c}{d}\right)(c,-cx)_H 2^{-\nu+1}|cx|^{\nu-1}\operatorname{sgn}(-cx)^{\epsilon/2}\delta_0\left(4\left(\frac{ax - \frac{ad}{c} + b}{cx}\right)\right)\phi\left(x - \frac{d}{c}\right)dx\\
&= \int_{\mathbb{R}_{\neq 0}} \operatorname{sgn}(c)\left(\frac{c}{d}\right)(c,-cx)_H 2^{-\nu+1}|cx|^{\nu-1}\operatorname{sgn}(-cx)^{\epsilon/2}\delta_0\left(\frac{4a}{c} - \frac{4}{c^2x}\right)\phi\left(x - \frac{d}{c}\right)dx\\
&= \int_{\mathbb{R}_{\neq 0}} \operatorname{sgn}(c)\left(\frac{c}{d}\right)\left(c, \frac{-4x}{c}\right)_H 2^{-\nu+1}4^{\nu}|c|^{-\nu-1}|x|^{\nu-1}\operatorname{sgn}\left(\frac{-4x}{c}\right)^{\epsilon/2}\delta_0\left(\frac{4a}{c} - \frac{1}{x}\right)\phi\left(\frac{4x}{c^2} - \frac{d}{c}\right)dx\\
&= \int_{\mathbb{R}_{\neq 0}} \operatorname{sgn}(c)\left(\frac{c}{d}\right)\left(c, \frac{4}{cx}\right)_H 2^{\nu+1}|c|^{-\nu-1}|x|^{-\nu-1}\operatorname{sgn}\left(\frac{4}{cx}\right)^{\epsilon/2}\delta_0\left(\frac{4a}{c} + x\right)\phi\left(\frac{-4}{c^2x} - \frac{d}{c}\right)dx\\
&= \operatorname{sgn}(c)\left(\frac{c}{d}\right)\left(c, \frac{-1}{a}\right)_H 2^{\nu+1}|4a|^{-\nu-1}\operatorname{sgn}\left(\frac{-1}{a}\right)^{\epsilon/2}\phi\left(\frac{1}{ac} - \frac{d}{c}\right)\\
&= \operatorname{sgn}(c)\left(\frac{c}{d}\right)(c,-a)_H|2a|^{-\nu-1}\operatorname{sgn}(-a)^{\epsilon/2}\phi\left(\frac{-b}{a}\right).
\end{aligned} \tag{4.39}$$

If $b \neq 0$, then it follows from (4.38) that

$$\begin{aligned}\operatorname{sgn}(c)\left(\frac{c}{d}\right)(c,-a)_H \operatorname{sgn}(-a)^{\epsilon/2} &= \operatorname{sgn}(c)\left(\frac{b}{a}\right)(a,-c)_H(c,-a)_H \operatorname{sgn}(-a)^{\epsilon/2} \\ = \operatorname{sgn}(c)\left(\frac{b}{a}\right)\operatorname{sgn}(ac)\operatorname{sgn}(-a)^{\epsilon/2} &= \operatorname{sgn}(a)\left(\frac{b}{a}\right)\operatorname{sgn}(-a)^{\epsilon/2} = -\left(\frac{b}{a}\right)\operatorname{sgn}(-a)^{-\epsilon/2}.\end{aligned}$$

By this equality and (4.39), we have for $b \neq 0$, that

$$(\pi(\widetilde{\gamma}\Omega)\delta_\infty)_0 = -\left(\frac{b}{a}\right)|2a|^{-\nu-1}\operatorname{sgn}(-a)^{-\epsilon/2}\delta_{\frac{-b}{a}}, \tag{4.40}$$

as an equality between distributions on $\mathbb{R}_{\neq \frac{-d}{c}}$. Conveniently, (4.40) also holds for $b = 0$ since $b = 0$ implies $a = d = 1$ and $\left(\frac{0}{1}\right) = 1$; indeed, observe that in this case we have

$$\operatorname{sgn}(c)\left(\frac{c}{d}\right)(c,-a)_H \operatorname{sgn}(-a)^{\epsilon/2} = \operatorname{sgn}(c)\operatorname{sgn}(c)\operatorname{sgn}(-1)^{\epsilon/2} = \epsilon i = -\left(\frac{0}{1}\right)\operatorname{sgn}(-1)^{-\epsilon/2}.$$

Thus it remains to describe $(\pi(\widetilde{\gamma}\Omega)\delta_\infty)_0$ about the point $\frac{-d}{c}$ for when $c \neq 0$. To do this, observe that for $c \neq 0$ we have that

$$\widetilde{\gamma}\Omega = \left(\begin{pmatrix} -2b & -\frac{d}{2} \\ 2a & \frac{c}{2}\end{pmatrix}, \left(\frac{c}{d}\right)(a,c)_H\right) = \left(\begin{pmatrix} \frac{c}{2} & \frac{d}{2} \\ -2a & -2b\end{pmatrix}, \left(\frac{c}{d}\right)(a,c)_H\right)^{-1}.$$

Thus by Lemma 4.1(a) we have that

$$(\pi(\widetilde{\gamma}\Omega)\delta_\infty)_0 = 0,$$

as an equality between distributions on $\mathbb{R}_{\neq \frac{-b}{a}}$. Since $\frac{-d}{c} \neq \frac{-b}{a}$, it follows that $(\pi(\widetilde{\gamma}\Omega)\delta_\infty)_0$ vanishes about the point $\frac{-d}{c}$, and thus (4.40) holds as an equality between distributions on $\mathbb{R}$.

With (4.40) established as equality between distributions on $\mathbb{R}$, it remains for us to index the cosets of $\widetilde{\Gamma}_1(4)/\widetilde{\Gamma}_{(0)}$ in some natural way. Observe

$$\begin{aligned}\widetilde{\gamma}\cdot\left(\begin{pmatrix} 1 & 0 \\ -4n & 1\end{pmatrix}, 1\right) &= \left(\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}, \left(\frac{c}{d}\right)\right)\cdot\left(\begin{pmatrix} 1 & 0 \\ -4n & 1\end{pmatrix}, 1\right) \\ &= \left(\begin{pmatrix} d+4bn & -b \\ -c-4an & a\end{pmatrix}, \left(\frac{-c-4an}{d+4bn}\right)\right).\end{aligned}$$

From this equality we see that to each coset of $\widetilde{\Gamma}_1(4)/\widetilde{\Gamma}_{(0)}$ there corresponds $(a,b) \in \mathbb{Z}^2$ such that $\gcd(a,b) = 1$ and $a \equiv 1 (\operatorname{mod} 4)$. Furthermore, this correspondence is unique, for if both $\widetilde{\gamma}\widetilde{\Gamma}_{(0)}$ and $\widetilde{\gamma}'\widetilde{\Gamma}_{(0)}$ correspond to the same $(a,b) \in \mathbb{Z}$, then it follows that $\widetilde{\gamma}^{-1}\widetilde{\gamma}' \in \widetilde{\Gamma}_{(0)}$. Conversely, when given $(a,b) \in \mathbb{Z}^2$ such that $\gcd(a,b) = 1$ and $a \equiv 1 (\operatorname{mod} 4)$, it follows that $\gcd(a, 4b) = 1$. Thus there exists $c', d \in \mathbb{Z}$ such that $ad - 4bc' = ad - b(4c') = 1$. Since $a \equiv 1 (\operatorname{mod} 4)$ then

$d \equiv 1 (\operatorname{mod} 4)$. If we let $c = 4c'$ then $c \equiv 0 (\operatorname{mod} 4)$. Thus we are able to construct $\widetilde{\gamma}$ which corresponds to such (a, b). Therefore

$$\widetilde{\Gamma}_1(4)/\widetilde{\Gamma}_{(0)} \cong \{(a,b) \in \mathbb{Z}^2 : \gcd(a,b) = 1, a \equiv 1 (\operatorname{mod} 4)\}. \tag{4.41}$$

By (4.41), we have that

$$\left(\widetilde{E}_\nu^{(0)}\right)_0 = -\zeta_2(2\nu+1) \sum_{\substack{(a,b)\in\mathbb{Z}^2 \\ \gcd(a,b)=1 \\ a\equiv 1 (\operatorname{mod} 4)}} \left(\frac{b}{a}\right) |2a|^{-\nu-1} \operatorname{sgn}(-a)^{-\epsilon/2} \delta_{\frac{-b}{a}}$$

Observe

$$\begin{aligned}
\frac{b_0}{\zeta_2(2\nu+1)} &= -\int_0^1 \sum_{\substack{(a,b)\in\mathbb{Z}^2 \\ \gcd(a,b)=1 \\ a\equiv 1 (\operatorname{mod} 4)}} \left(\frac{b}{a}\right) |2a|^{-\nu-1} \operatorname{sgn}(-a)^{-\epsilon/2} \delta_{\frac{-b}{a}}(x)\, dx \\
&= \epsilon i \sum_{\substack{(a,b)\in\mathbb{Z}_{>0}\times\mathbb{Z} \\ 0\le -b<a \\ a\equiv 1 (\operatorname{mod} 4)}} \left(\frac{b}{a}\right) |2a|^{-\nu-1} - \sum_{\substack{(a,b)\in\mathbb{Z}_{<0}\times\mathbb{Z} \\ 0\ge -b>a \\ a\equiv 1 (\operatorname{mod} 4)}} \left(\frac{b}{a}\right) |2a|^{-\nu-1} \\
&= \epsilon i \sum_{\substack{(a,b)\in\mathbb{Z}_{>0}\times\mathbb{Z} \\ 0\le b<a \\ a\equiv 1 (\operatorname{mod} 4)}} \left(\frac{-b}{a}\right) |2a|^{-\nu-1} - \sum_{\substack{(a,b)\in\mathbb{Z}_{>0}\times\mathbb{Z} \\ 0\le b<a \\ a\equiv 3 (\operatorname{mod} 4)}} \left(\frac{b}{-a}\right) |2a|^{-\nu-1}.
\end{aligned}$$

Observe that if $a \equiv 1 (\operatorname{mod} 4)$ then $\left(\frac{-b}{a}\right) = \left(\frac{-1}{a}\right)\left(\frac{b}{a}\right) = \left(\frac{b}{a}\right)$, and if $b \ge 0$ and $a \ne 0$ then $\left(\frac{b}{-a}\right) = \left(\frac{b}{-1}\right)\left(\frac{b}{a}\right) = \left(\frac{b}{a}\right)$. Thus

$$\frac{b_0}{\zeta_2(2\nu+1)} = \epsilon i \sum_{a\in\mathbb{Z}_{>0}} \left(\sum_{b\in\mathbb{Z}/a\mathbb{Z}} \left(\frac{b}{a}\right) \right) \Delta_a^\epsilon |2a|^{-\nu-1}.$$

Observe that if a is a square then

$$\sum_{b\in\mathbb{Z}/a\mathbb{Z}} \left(\frac{b}{a}\right) = \phi(a),$$

but if a is not a square then

$$\sum_{b\in\mathbb{Z}/a\mathbb{Z}} \left(\frac{b}{a}\right) = 0.$$

The latter case follows since if a is not a square then there exists $b' \in \mathbb{Z}$ such that $\gcd(a, b') = 1$ and $\left(\frac{b'}{a}\right) = -1$, and thus

$$-\left(\sum_{b\in\mathbb{Z}/a\mathbb{Z}} \left(\frac{b}{a}\right) \right) = \left(\frac{b'}{a}\right) \left(\sum_{b\in\mathbb{Z}/a\mathbb{Z}} \left(\frac{b}{a}\right) \right) = \left(\sum_{b\in\mathbb{Z}/a\mathbb{Z}} \left(\frac{b'b}{a}\right) \right) = \left(\sum_{b\in\mathbb{Z}/a\mathbb{Z}} \left(\frac{b}{a}\right) \right).$$

Therefore

$$b_0 = \epsilon i \zeta_2(2\nu+1) \sum_{a\in\mathbb{Z}_{>0}} \Delta_{a^2}^\epsilon \phi(a^2) |2a^2|^{-\nu-1} = \epsilon i \zeta_2(2\nu+1) 2^{-\nu-1} \sum_{\substack{a\in\mathbb{Z}_{>0} \\ a \text{ odd}}} \phi(a^2) |a|^{-2\nu-2}.$$

Since $\phi(a^2)$ is multiplicative, it follows just as in the proof of Proposition 4.9 that

$$\begin{aligned} b_0 &= \epsilon i \zeta_2(2\nu+1)2^{-\nu-1} \prod_{p \text{ odd prime}} \sum_{k \in \mathbb{Z}_{\geq 0}} \phi(p^{2k})(p^k)^{-2\nu-2} \\ &= \epsilon i \zeta_2(2\nu+1)2^{-\nu-1} \prod_{p \text{ odd primes}} \frac{1-p^{-2\nu-1}}{1-p^{-2\nu}} = \epsilon i 2^{-\nu-1}\zeta_2(2\nu). \end{aligned}$$

□

4.5 The Eisenstein Distribution Functional Equation

Recall that in (3.2) and (3.7) we defined the following elements of $\widetilde{\mathrm{SL}}_2$:

$$\widetilde{m}_{\epsilon_1,\epsilon_2,\epsilon_3} = \left(\begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}, \epsilon_3 \right), \quad \widetilde{a}_u = \left(\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, 1 \right),$$

$$\widetilde{n}_x = \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1 \right), \quad \widetilde{n}_{-,x} = \left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, 1 \right), \text{ and } \widetilde{s} = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1 \right),$$

where $\epsilon_i \in \{\pm 1\}$, $u \in \mathbb{R}_{>0}$, and $x \in \mathbb{R}$. Throughout this section, we will suppose that $f \in V^{\infty}_{(\epsilon),-\nu}(\widetilde{\mathrm{SL}}_2)$ where $\epsilon = \pm 1$; notice that we now have $-\nu$ as our complex parameter instead of ν. Recall that by definition,

$$\begin{aligned} & V^{\infty}_{(\epsilon),\nu}(\widetilde{\mathrm{SL}}_2) \\ & = \{ f \in C^{\infty}(\widetilde{\mathrm{SL}}_2, \mathbb{C}) : f(\widetilde{g}\widetilde{b}) = \omega_{(\epsilon),\nu}(\widetilde{b}^{-1}) f(\widetilde{g}) \text{ for all } \widetilde{g} \in \widetilde{\mathrm{SL}}_2, \widetilde{b} \in B(\widetilde{\mathrm{SL}}_2)\}, \end{aligned} \tag{4.42}$$

where $\omega_{(\epsilon),\nu}$ is defined in (3.14).

For $\Re(\nu) > 0$, let $I_\nu : V^{\infty}_{(\epsilon),-\nu}(\widetilde{\mathrm{SL}}_2) \to V^{\infty}_{(\epsilon),\nu}(\widetilde{\mathrm{SL}}_2)$ where

$$(I_\nu f)(\widetilde{g}) = \int_{-\infty}^{\infty} f(\widetilde{g}\,\widetilde{s}\,\widetilde{n}_t)\, dt. \tag{4.43}$$

In a moment we will show that the integral defining I_ν does indeed converge for $\Re(\nu) > 0$, and that the codomain of I_ν is in indeed $V^{\infty}_{(\epsilon),\nu}(\widetilde{\mathrm{SL}}_2)$. With regards to justifying the convergence of the integral, observe that by the transformation law given in (4.42), we have that

$$\begin{aligned} (I_\nu f)_0(x) &= \int_{-\infty}^{\infty} f(\widetilde{n}_x\,\widetilde{s}\,\widetilde{n}_y)\, dy = \int_{-\infty}^{\infty} f\left(\widetilde{n}_{(-y^{-1}+x)}\widetilde{a}_{|y|^{-1}}\widetilde{m}_{\mathrm{sgn}(y),\mathrm{sgn}(y),\mathrm{sgn}(y)}\widetilde{n}_{-,y^{-1}}\right) dy \\ &= \int_{-\infty}^{\infty} |y|^{-\nu-1}\mathrm{sgn}(y)^{-\epsilon/2} f\left(\widetilde{n}_{(-y^{-1}+x)}\right) dy = \int_{-\infty}^{\infty} |y|^{\nu-1}\mathrm{sgn}(-y)^{-\epsilon/2} f_0(x+y)\, dy, \end{aligned} \tag{4.44}$$

with the latter integral converging absolutely for $\Re(\nu) > 0$. By an almost identical argument it also follows that

$$(I_\nu f)_\infty(x) = \int_{-\infty}^{\infty} |y|^{\nu-1}\mathrm{sgn}(-y)^{-\epsilon/2} f_\infty(x+y)\, dy, \tag{4.45}$$

with the integral converging absolutely for $\Re(\nu) > 0$. Thus the integral defining $I_\nu f$ converges absolutely for $\Re(\nu) > 0$. As for justifying the codomain statement, observe

$$\begin{aligned}(I_\nu f)(\widetilde{g}\widetilde{a}_u) &= \int_{-\infty}^{\infty} f(\widetilde{g}\widetilde{a}_u\widetilde{s}\widetilde{n}_t)\,dt = \int_{-\infty}^{\infty} f(\widetilde{g}\widetilde{s}\widetilde{n}_{\frac{t}{u^2}}\widetilde{a}_u^{-1})\,dt \\ &= u^2\int_{-\infty}^{\infty} f(\widetilde{g}\widetilde{s}\widetilde{n}_t\widetilde{a}_u^{-1})\,dt = u^{-\nu+1}\int_{-\infty}^{\infty} f(\widetilde{g}\widetilde{s}\widetilde{n}_t)\,dt = \omega_{(\epsilon),\nu}\big(\widetilde{a}_u^{-1}\big)(I_\nu f)(\widetilde{g}).\end{aligned}$$

From this equality and the fact that $M(\widetilde{\mathrm{SL}}_2)$ is the center of $\widetilde{\mathrm{SL}}_2$, it follows that I_ν does indeed have $V^\infty_{(\epsilon),\nu}(\widetilde{\mathrm{SL}}_2)$ as its codomain.

Observe that I_ν is an intertwining operator between the spaces $V^\infty_{(\epsilon),-\nu}(\widetilde{\mathrm{SL}}_2)$ and $V^\infty_{(\epsilon),\nu}(\widetilde{\mathrm{SL}}_2)$. A well-known result from representation theory states that I_ν can be meromorphically continued to all of $\mathbb{C}$ [13]. Our goal for this section is to describe $I_\nu\Big(\widetilde{E}^{(\infty)}_{-\nu}\Big)$ in terms of $\widetilde{E}^{(\infty)}_\nu$ and $\widetilde{E}^{(0)}_\nu$, and to thus obtain a distributional analogue of the functional equation for metaplectic Eisenstein series. To accomplish this, we need to define the following *Gamma factors*. For $\delta \in \{0,1\}$, let

$$G_\delta(\nu) = \begin{cases} 2\cos\big(\frac{\pi\nu}{2}\big)(2\pi)^{-\nu}\Gamma(\nu) & \text{if } \delta = 0, \\ 2i\sin\big(\frac{\pi\nu}{2}\big)(2\pi)^{-\nu}\Gamma(\nu) & \text{if } \delta = 1; \end{cases} \tag{4.46}$$

and for $\epsilon_1, \epsilon_2 \in \{\pm 1\}$, let

$$\begin{aligned} G_{\epsilon_1,\epsilon_2}(\nu) &= \left(e\Big(\frac{\epsilon_2\nu}{4}\Big) + \epsilon_1 i\, e\Big(\frac{-\epsilon_2\nu}{4}\Big)\right)(2\pi)^{-\nu}\Gamma(\nu) \\ &= \sqrt{2}(1+\epsilon_1 i)\cos\Big(\frac{\pi}{2}\Big(\nu - \frac{\epsilon_1\epsilon_2}{2}\Big)\Big)(2\pi)^{-\nu}\Gamma(\nu). \end{aligned} \tag{4.47}$$

The following lemma gives some integral representations for these gamma factors.

Lemma 4.11. *For $0 < \Re(\nu) < 1$,*

(a) $G_\delta(\nu) = \displaystyle\int_{-\infty}^{\infty} \mathrm{sgn}(x)^\delta |x|^{\nu-1} e(x)dx,$

(b) $G_{\epsilon_1,\epsilon_2}(\nu) = \displaystyle\int_{-\infty}^{\infty} \mathrm{sgn}(\epsilon_2 x)^{\frac{\epsilon_1}{2}} |x|^{\nu-1} e(x)dx,$

where the above integrals converge conditionally.

Proof. For $0 < \Re(\nu) < 1$, one can show that

$$\int_0^\infty |x|^{\nu-1} e(\pm x)\,dx = e\Big(\frac{\pm\nu}{4}\Big)(2\pi)^{-\nu}\Gamma(\nu). \tag{4.48}$$

Thus for $0 < \Re(\nu) < 1$ we have that

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{sgn}(x)^{\delta}|x|^{\nu-1}e(x)\,dx = \int_{0}^{\infty} |x|^{\nu-1}e(x)\,dx + (-1)^{\delta}\int_{-\infty}^{0} |x|^{\nu-1}e(x)\,dx \\
&= \int_{0}^{\infty} |x|^{\nu-1}e(x)\,dx + (-1)^{\delta}\int_{0}^{\infty} |x|^{\nu-1}e(-x)\,dx \\
&= e\Big(\frac{\nu}{4}\Big)(2\pi)^{-\nu}\Gamma(\nu) + (-1)^{\delta}e\left(\frac{-\nu}{4}\right)(2\pi)^{-\nu}\Gamma(\nu) \\
&= \Big(\Big(\cos\Big(\frac{\nu}{4}\Big) + i\,\sin\Big(\frac{\nu}{4}\Big)\Big) + (-1)^{\delta}\Big(\cos\Big(\frac{\nu}{4}\Big) - i\,\sin\Big(\frac{\nu}{4}\Big)\Big)\Big)(2\pi)^{-\nu}\Gamma(\nu) \\
&= G_{\delta}(\nu),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{sgn}(\epsilon_2 x)^{\frac{\epsilon_1}{2}}|x|^{\nu-1}e(x)dx = \int_{-\infty}^{\infty} \mathrm{sgn}(x)^{\frac{\epsilon_1}{2}}|x|^{\nu-1}e(\epsilon_2 x)dx \\
&= \int_{0}^{\infty} \mathrm{sgn}(x)^{\frac{\epsilon_1}{2}}|x|^{\nu-1}e(\epsilon_2 x)dx + \int_{-\infty}^{0} \mathrm{sgn}(x)^{\frac{\epsilon_1}{2}}|x|^{\nu-1}e(\epsilon_2 x)dx \\
&= \int_{0}^{\infty} |x|^{\nu-1}e(\epsilon_2 x)dx + \epsilon_1 i\int_{0}^{\infty} |x|^{\nu-1}e(-\epsilon_2 x)dx \\
&= \left(e\Big(\frac{\epsilon_2\nu}{4}\Big) + \epsilon_1 i e\left(\frac{-\epsilon_2\nu}{4}\right)\right)(2\pi)^{-\nu}\Gamma(\nu) \\
&= G_{\epsilon_1,\epsilon_2}(\nu).
\end{aligned}$$

$\square$

We can define the intertwining operator I_ν on $V^{-\infty}_{(\epsilon),-\nu}(\widetilde{\mathrm{SL}}_2)$ in two equivalent ways. One approach is to extend I_ν by continuity on the dense subset of smooth functions in $V^{-\infty}_{(\epsilon),-\nu}(\widetilde{\mathrm{SL}}_2)$. Alternatively, one can define I_ν on $V^{-\infty}_{(\epsilon),-\nu}(\widetilde{\mathrm{SL}}_2)$ by first observing that the pairing on $V^{\infty}_{(\epsilon),-\nu}(\widetilde{\mathrm{SL}}_2)\times V^{\infty}_{(-\epsilon),\nu}(\widetilde{\mathrm{SL}}_2)$ described in section 1.2 extends continuously to $V^{-\infty}_{(\epsilon),-\nu}(\widetilde{\mathrm{SL}}_2)\times V^{\infty}_{(-\epsilon),\nu}(\widetilde{\mathrm{SL}}_2)$. Thus for $\tau \in V^{-\infty}_{(\epsilon),-\nu}(\widetilde{\mathrm{SL}}_2)$ and $f \in V^{\infty}_{(-\epsilon),\nu}$, we define $I_\nu\tau$ by the equality

$$(I_\nu\tau, f) = (\tau, I_{-\nu}f)$$

that is to say, we define I_ν on $V^{-\infty}_{(\epsilon),-\nu}$ to be the adjoint of $I_{-\nu}$ on $V^{\infty}_{(-\epsilon),\nu}$. We use this latter definition of I_ν for when working with elements of $V^{-\infty}_{(\epsilon),-\nu}$.

It is well-known that $I_{-\nu}\circ I_\nu$ is a scalar operator. The following lemma, which follows from [20, p. 89], gives us an exact description of this scalar operator.

Lemma 4.12. *For $f \in V^{-\infty}_{(\epsilon),-\nu}(\widetilde{SL}_2)$,*

$$(I_{-\nu}I_\nu f) = \epsilon\frac{2\pi i\cot(\pi\nu)}{\nu}f.$$

Let $\mathbb{1}_\nu$ denote the element of $V^{-\infty}_{(\epsilon),\nu}(\widetilde{\mathrm{SL}_2})$ defined by

$$(\mathbb{1}_\nu)_0(x) = 1 \text{ and } (\mathbb{1}_\nu)_\infty(0) = 0.$$

Similarly, to avoid confusion, we will write $\delta_{\infty,\nu}$ for the δ_∞ element of $V^{-\infty}_{(\epsilon),\nu}(\widetilde{\mathrm{SL}_2})$.

Lemma 4.13. *For $\delta_{\infty,-\nu}, \mathbb{1}_{-\nu} \in V^{-\infty}_{(\epsilon),-\nu}(\widetilde{SL_2})$ we have*

(a) $I_\nu(\delta_{\infty,-\nu}) = -\epsilon i \mathbb{1}_\nu$,

(b) $I_\nu(\mathbb{1}_{-\nu}) = -\dfrac{2\pi \cot(\pi\nu)}{\nu}\delta_{\infty,\nu}$.

Proof. For $f \in V^{\infty}_{(-\epsilon),\nu}(\widetilde{\mathrm{SL}_2})$, observe that $\int_{-\infty}^{\infty} (\delta_\infty)_0(x)(I_{-\nu}f)_0(x)\,dx = 0$. By (4.3) and (4.43), we have for $\Re(\nu) < 0$ that

$$\int_{-\infty}^{\infty} (\delta_\infty)_\infty(x)(I_{-\nu}f)_\infty(x) = -\epsilon i (I_{-\nu}f)_\infty(0) = -\epsilon i \int_{-\infty}^{\infty} f(\widetilde{s}^{-1}\widetilde{s}\,\widetilde{n}_t)\,dt$$
$$= -\epsilon i \int_{-\infty}^{\infty} f(\widetilde{n}_t)\,dt = -\epsilon i \int_{-\infty}^{\infty} \mathbb{1}_0(x) f_0(x)\,dx.$$

Thus $I_\nu \delta_{\infty,-\nu} = -\epsilon i \mathbb{1}_\nu$ for $\Re(\nu) < 0$. To see that this equality also holds for $\Re(\nu) \geq 0$, observe that for $h \in C_c^\infty(\mathbb{R})$, we have that

$$\nu \mapsto \int_{\mathbb{R}} (-\epsilon i \mathbb{1}_\nu)_0(x) h(x)\,dx$$

is holomorphic on $\mathbb{C}$, and that

$$\nu \mapsto \int_{\mathbb{R}} (I_\nu \delta_{\infty,-\nu})_0(x) h(x)\,dx = \int_{\mathbb{R}} (\delta_{\infty,-\nu})_0(x) I_{-\nu}(h)(x)\,dx$$

is meromorphic on $\mathbb{C}$. The uniqueness of meromorphic continuation then asserts that

$$\int_{\mathbb{R}} (I_\nu \delta_{\infty,-\nu})_0(x) h(x)\,dx = \int_{\mathbb{R}} (-\epsilon i \mathbb{1}_\nu)_0(x) h(x)\,dx$$

as meromorphic functions on $\mathbb{C}$. Since this equality holds for any $h \in C_c^\infty(\mathbb{R})$, it follows that $(I_\nu \delta_{\infty,-\nu})_0 = (-\epsilon i \mathbb{1}_\nu)_0$. A similar argument shows that $(I_\nu \delta_{\infty,-\nu})_\infty = (-\epsilon i \mathbb{1}_\nu)_\infty$, and thus part (a) follows.

Lemma 4.12 applied to $\delta_{\infty,-\nu}$ shows that $I_{-\nu} I_\nu \delta_{\infty,-\nu} = c(\nu)\delta_{\infty,-\nu}$, where $c(\nu) = \epsilon \frac{2\pi i \cot(\pi\nu)}{\nu}$. Therefore by part (a), $-\epsilon i I_{-\nu} \mathbb{1}_\nu = c(\nu)\delta_{\infty,-\nu}$, which becomes

$$I_\nu \mathbb{1}_{-\nu} = \epsilon i c(-\nu)\delta_{\infty,\nu} = (\epsilon i)\epsilon \frac{2\pi i \cot(-\pi\nu)}{-\nu}\delta_{\infty,\nu} = -\frac{2\pi \cot(\pi\nu)}{\nu}\delta_{\infty,\nu}$$

when we replace ν with $-\nu$ and solve for $I_\nu \mathbb{1}_{-\nu}$. $\square$

Recall that a_n denotes the Fourier coefficients of $\left(\widetilde{E}_\nu^{(\infty)}\right)_0$ and that b_n denotes the Fourier coefficients of $\left(\widetilde{E}_\nu^{(0)}\right)_0$. To show the dependence of these coefficients on ν, we will also write $a_n(\nu)$ and $b_n(\nu)$ for a_n and b_n, respectively. Let $a_\infty(\nu)$ denote the coefficient of δ_0 in

$$\left(\widetilde{E}_\nu^{(\infty)}\right)_\infty - \left(\widetilde{E}_\nu^{(\infty)}\right)_\infty\Big|_{\mathbb{R}_{\neq 0}}.$$

By (4.35) we see that

$$a_\infty(\nu) = \epsilon i \zeta_2(2\nu+1).$$

Likewise, let $b_\infty(\nu)$ denote the coefficient of δ_0 in

$$\left(\widetilde{E}_\nu^{(0)}\right)_\infty - \left(\widetilde{E}_\nu^{(0)}\right)_\infty\Big|_{\mathbb{R}_{\neq 0}}.$$

We shall refer to $a_\infty(\nu)$ and $b_\infty(\nu)$ as *Fourier coefficients at* ∞. Observe

$$\begin{aligned}\left(\widetilde{E}_\nu^{(0)}\right)_\infty &= \left(\pi(\Omega)\widetilde{E}_\nu^{(\infty)}\right)_\infty = \left(\pi(\widetilde{a}_2^{-1}\widetilde{s})\widetilde{E}_\nu^{(\infty)}\right)_\infty = \left(\pi(\widetilde{a}_2^{-1}\widetilde{s}\widetilde{s})\widetilde{E}_\nu^{(\infty)}\right)_0 \\ &= \left(\pi(\widetilde{a}_2^{-1})\pi((-\mathrm{id},-1))\widetilde{E}_\nu^{(\infty)}\right)_0 = -\epsilon i\left(\pi(\widetilde{a}_2^{-1})\widetilde{E}_\nu^{(\infty)}\right)_0,\end{aligned}$$

where Ω is defined in (4.36). Therefore, since $\left(\widetilde{E}_\nu^{(\infty)}\right)_0$ has no delta distribution at 0 by (4.17) it follows that $\left(\widetilde{E}_\nu^{(0)}\right)_\infty$ vanishes about 0. Thus

$$b_\infty(\nu) = 0.$$

In what follows, we shall use the following identities:

$$\frac{\pi\cot(\pi\nu)}{\nu}G_0(2\nu+1) = -G_0(2\nu), \tag{4.49}$$

$$\zeta(1-s) = G_0(s)\zeta(s). \tag{4.50}$$

Recall that by Proposition 4.9, we have that

$$a_0(\nu) = (1+\epsilon i)2^{-2\nu-2}\zeta(2\nu).$$

Therefore by Lemma 4.13, (4.49), and (4.50), we have that

$$\begin{aligned}I_\nu(a_0(-\nu)\mathbb{1}_{-\nu}) &= -\frac{2\pi\cot(\pi\nu)}{\nu}a_0(-\nu)\delta_{\infty,\nu} = -\frac{\pi\cot(\pi\nu)}{\nu}(1+\epsilon i)2^{2\nu-1}\zeta(-2\nu)\delta_{\infty,\nu} \\ &= -\frac{\pi\cot(\pi\nu)}{\nu}(1+\epsilon i)2^{2\nu-1}G_0(2\nu+1)\zeta(2\nu+1)\delta_{\infty,\nu} \\ &= (1+\epsilon i)2^{2\nu-1}G_0(2\nu)\zeta(2\nu+1)\delta_{\infty,\nu} \\ &= \left(-\epsilon i(1+\epsilon i)2^{2\nu-1}G_0(2\nu)(1-2^{-2\nu-1})^{-1}\right)(\epsilon i\zeta_2(2\nu+1))\delta_{\infty,\nu} \\ &= \left((1-\epsilon i)2^{2\nu-1}(1-2^{-2\nu-1})^{-1}G_0(2\nu)\right)a_\infty(\nu)\delta_{\infty,\nu},\end{aligned} \tag{4.51}$$

and

$$
\begin{aligned}
&I_\nu(a_\infty(-\nu)\delta_{\infty,-\nu}) = -\epsilon i a_\infty(-\nu)\mathbb{1}_\nu = (1-2^{2\nu-1})\zeta(-2\nu+1)\mathbb{1}_\nu \\
&= (1-2^{2\nu-1})G_0(2\nu)\zeta(2\nu)\mathbb{1}_\nu \\
&= \Big((1-\epsilon i)2^{2\nu-1}(1-2^{-2\nu-1})^{-1}G_0(2\nu)\Big) \\
&\quad \cdot \Big((1-\epsilon i)^{-1}2^{-2\nu+1}(1-2^{-2\nu-1})(1-2^{2\nu-1})\zeta(2\nu)\Big)\mathbb{1}_\nu.
\end{aligned} \tag{4.52}
$$

We wish to write $(1-\epsilon i)^{-1}2^{-2\nu+1}(1-2^{-2\nu-1})(1-2^{2\nu-1})\zeta(2\nu)$ as $a_0(\nu)+d(\nu)b_0(\nu)$ where $d:\mathbb{C}\to\mathbb{C}$. Since by Proposition 4.9 and Proposition 4.10, we know that

$$
\begin{aligned}
&a_0(\nu) = (1+\epsilon i)2^{-2\nu-2}\zeta(2\nu), \\
&b_0(\nu) = \epsilon i 2^{-\nu-1}\zeta_2(2\nu) = \epsilon i 2^{-\nu-1}(1-2^{-2\nu})\zeta(2\nu),
\end{aligned}
$$

it follows that when we solve for $d(\nu)$ we find that

$$d(\nu) = (1-\epsilon i)2^{-\nu}(1-2^{2\nu}).$$

Theorem 4.14. *For $\widetilde{E}_\nu^{(\infty)} \in V_{(\epsilon),\nu}^{-\infty}(\widetilde{SL}_2)$ we have*

$$I_\nu\Big(\widetilde{E}_{-\nu}^{(\infty)}\Big) = \Big((1-\epsilon i)2^{2\nu-1}(1-2^{-2\nu-1})^{-1}G_0(2\nu)\Big)\Big(\widetilde{E}_\nu^{(\infty)} + (1-\epsilon i)2^{-\nu}(1-2^{2\nu})\widetilde{E}_\nu^{(0)}\Big).$$

Proof. We have established that the 0-th Fourier coefficient of

$$I_\nu\Big(\widetilde{E}_{-\nu}^{(\infty)}\Big) - \Big((1-\epsilon i)2^{2\nu-1}(1-2^{-2\nu-1})^{-1}G_0(2\nu)\Big)\Big(\widetilde{E}_\nu^{(\infty)} + d(\nu)\widetilde{E}_\nu^{(0)}\Big) \tag{4.53}$$

is equal to zero, and that the Fourier coefficient at ∞ of (4.53) is also equal to zero. In light of [17, (2.17)], we have then that (4.53) is cuspidal at ∞. Indeed, in classical terms, the series $\Big(\widetilde{E}_\nu^{(0)}\Big)_0(x) = \sum_{n\in\mathbb{Z}} b_n(\nu)e(nx)$ is seen to be a series expansion of $\widetilde{E}_\nu^{(0)}$ based at the cusp at ∞. If one can establish that both $\widetilde{E}_\nu^{(\infty)}$ and $\widetilde{E}^{(0)}$ are cuspidal at the cusp $\frac{1}{2}$, then it follows from the general theory of the metaplectic Eisenstein series that we must also have that (4.53) is cuspidal at the cusp 0. Since for almost all $\nu \in \mathbb{C}$, 0 is the only automorphic distribution which is cuspidal at all these cusps, it follows by meromorphic continuation that (4.53) holds for all ν at which $\widetilde{E}_\nu^{(\infty)}$ and $\widetilde{E}_\nu^{(0)}$ are defined.

In order to determine if $\widetilde{E}_\nu^{(\infty)}$ is cuspidal at $\frac{1}{2}$, we let

$$\Theta = \left(\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, 1\right),$$

and let $\widetilde{E}^{(\infty)}|_\Theta(\widetilde{g}) = \widetilde{E}^{(\infty)}(\Theta\widetilde{g})$. One can check that the minimal $n \in \mathbb{Z}_{>0}$ for which $\left(\widetilde{E}^{(\infty)}_\nu|_\Theta\right)_0(x+n) = \left(\widetilde{E}^{(\infty)}_\nu|_\Theta\right)_0(x)$ is $n = 4$. Indeed, this is the case since

$$\Theta\widetilde{n}_1 = \left(\begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}, 1\right)\Theta, \quad \Theta\widetilde{n}_2 = \left(\begin{pmatrix} -3 & 2 \\ -8 & 5 \end{pmatrix}, 1\right)\Theta,$$
$$\Theta\widetilde{n}_3 = \left(\begin{pmatrix} -5 & 3 \\ -12 & 7 \end{pmatrix}, 1\right)\Theta, \quad \text{and} \quad \Theta\widetilde{n}_4 = \left(\begin{pmatrix} -7 & 4 \\ -16 & 9 \end{pmatrix}, 1\right)\Theta,$$

and since

$$\left(\begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}, 1\right), \quad \left(\begin{pmatrix} -3 & 2 \\ -8 & 5 \end{pmatrix}, 1\right), \quad \text{and} \quad \left(\begin{pmatrix} -5 & 3 \\ -12 & 7 \end{pmatrix}, 1\right)$$

are seen to not be elements of $\widetilde{\Gamma}_1(4)$ (either for failing the congruence conditions, or for having a second coordinate incompatible with the corresponding Kronecker symbol associated to the matrix coordinate), while

$$\left(\begin{pmatrix} -7 & 4 \\ -16 & 9 \end{pmatrix}, 1\right) = \left(\begin{pmatrix} -7 & 4 \\ -16 & 9 \end{pmatrix}, \left(\frac{-16}{9}\right)\right)$$

is an element of $\widetilde{\Gamma}_1(4)$. Thus when we calculate Fourier coefficients of $\left(\widetilde{E}^{(\infty)}_\nu|_\Theta\right)_0$, we do so by integrating over the interval $[0, 4)$.

Since

$$\Theta \cdot \left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, 1\right) = \left(\begin{pmatrix} -3 & 2 \\ -8 & 5 \end{pmatrix}, 1\right) \cdot \Theta,$$

and since

$$\left(\begin{pmatrix} -3 & 2 \\ -8 & 5 \end{pmatrix}, 1\right) \cdot \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, -1\right) = \left(\begin{pmatrix} -3 & 2 \\ -8 & 5 \end{pmatrix}, -1\right) = \left(\begin{pmatrix} -3 & 2 \\ -8 & 5 \end{pmatrix}, \left(\frac{-8}{5}\right)\right) \in \widetilde{\Gamma}_1(4),$$

it follows from our transformation law for $\widetilde{E}^{(\infty)}_\nu$ that

$$\begin{aligned}
&\left(\widetilde{E}^{(\infty)}|_\Theta\right)_0(x+2) = \widetilde{E}^{(\infty)}\left(\Theta\widetilde{n}_2\widetilde{n}_x B(\widetilde{\mathrm{SL}}_2)\right) \\
&= \widetilde{E}^{(\infty)}\left(\left(\begin{pmatrix} -3 & 2 \\ -8 & 5 \end{pmatrix}, 1\right)\Theta\widetilde{n}_x B(\widetilde{\mathrm{SL}}_2)\right) = -\widetilde{E}^{(\infty)}\left(\left(\begin{pmatrix} -3 & 2 \\ -8 & 5 \end{pmatrix}, -1\right)\Theta\widetilde{n}_x B(\widetilde{\mathrm{SL}}_2)\right) \\
&= -\widetilde{E}^{(\infty)}\left(\Theta\widetilde{n}_x B(\widetilde{\mathrm{SL}}_2)\right) = -\left(\widetilde{E}^{(\infty)}|_\Theta\right)_0(x).
\end{aligned}$$

Therefore,

$$\int_0^4 \left(\widetilde{E}^{(\infty)}_\nu|_\Theta\right)_0(x)\,dx = \int_0^2 \left(\widetilde{E}^{(\infty)}_\nu|_\Theta\right)_0(x)\,dx + \int_0^2 \left(\widetilde{E}^{(\infty)}_\nu|_\Theta\right)_0(x+2)\,dx = 0.$$

By an identical argument, it also follows that

$$\int_0^4 \left(\widetilde{E}_\nu^{(\infty)}|_\Theta\right)_0(x)\,dx = 0.$$

To show that $\widetilde{E}_\nu^{(\infty)}$ is cuspidal at $\frac{1}{2}$, it remains to show that the Fourier coefficient at ∞ (i.e. the coefficient of δ_0 of $\left(\widetilde{E}^{(\infty)}|_\Theta\right)_\infty$) is equal to zero. To see that this is the case, observe that for $\widetilde{\gamma}^{-1} = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \left(\frac{c}{d}\right)\right)$ we have that

$$\Theta^{-1}\widetilde{\gamma}\widetilde{s} = \left(\begin{pmatrix} -b & -d \\ a+2b & c+2d \end{pmatrix}, *\right) = \left(\begin{pmatrix} c+2d & d \\ -a-2b & -b \end{pmatrix}, *\right)^{-1}.$$

Therefore by Lemma 4.1(c), we have that

$$((\pi(\widetilde{\gamma})\delta_\infty)|_\Theta)_\infty(x) = ((\pi(\widetilde{\gamma}\widetilde{s})\delta_0)|_\Theta)_\infty(x) = \left((\pi(\Theta^{-1}\widetilde{\gamma}\widetilde{s})\delta_0)\right)_\infty(x) = 0,$$

as an equality between distributions on $\mathbb{R}_{\neq \frac{c+2d}{d}}$. Since $c+2d \neq 0$ (since $c+2d \equiv 2 (\bmod 4)$), it follows from (4.16) that the Fourier coefficient at ∞ for $\widetilde{E}^\infty|_\Theta$ is zero. Therefore, $\widetilde{E}_\nu^{(\infty)}$ is cuspidal at the cusp $\frac{1}{2}$.

Likewise, for the case of $\widetilde{E}_\nu^{(0)}$, we observe that

$$\Theta^{-1}\widetilde{\gamma}\Omega\widetilde{s} = \left(\begin{pmatrix} -\frac{d}{2} & 2b \\ \frac{c}{2}+d & -2a-4b \end{pmatrix}, *\right) = \left(\begin{pmatrix} -2a-4b & -2b \\ -\frac{c}{2}-d & -\frac{d}{2} \end{pmatrix}, *\right)^{-1},$$

and therefore by Lemma 4.1(c,d), we have that

$$((\pi(\widetilde{\gamma}\Omega)\delta_\infty)|_\Theta)_\infty(x) = ((\pi(\widetilde{\gamma}\Omega\widetilde{s})\delta_0)|_\Theta)_\infty(x) = \left((\pi(\Theta^{-1}\widetilde{\gamma}\Omega\widetilde{s})\delta_0)\right)_\infty(x) = 0,$$

as an equality between distributions on $\mathbb{R}_{\neq 2+\frac{a}{b}}$ when $b \neq 0$, and as an equality between distributions on $\mathbb{R}$ when $b = 0$. In this latter case we see immediately from (4.37) that the Fourier coefficient at ∞ is zero, and in the former case we have that the Fourier coefficient at ∞ is zero since $2 + \frac{a}{b} = \frac{a+2b}{b} \neq 0$ (since $a+2b \equiv 1, 3 (\bmod 4)$). Therefore, $\widetilde{E}_\nu^{(0)}$ is cuspidal at the cusp $\frac{1}{2}$. $\square$

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