THE NONEQUILIBRIUM DYNAMICS OF QUANTUM INTEGRABLE MODELS

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The primary goals of this dissertation are to describe and elucidate a new formalism to study the out of equilibrium dynamics of integrable models, and to apply this formalism to some specific problems. In particular, I describe the “quench dynamics” of a gas of interacting bosons in one dimension, and provide details of some preliminary work on the isotropic Heisenberg chain. The formalism, based on earlier work by Yudson, provides a way of getting around some of the difficulties involved in calculating the time evolution of arbitrary initial states evolved with integrable Hamiltonians.

In addition to this, I aim to discuss some of the general aspects of quench dynamics in quantum systems. The description of the nonequilibrium dynamics of a given system depends significantly on initial states, and the time scales at which the system is probed compared with the various inherent time scales in the system. I present the experimental context and motivation for these studies, and survey the existing techniques and efforts at understanding the relaxation of systems that are far from their equilibrium states or ground states.
Preface

Parts of this thesis, in particular a large part of Chapter 5 have been published in Refs. [71] and [72]. The main content of Chapters 5 and 6 is work done in collaboration with my advisor, Natan Andrei.
Acknowledgements

The last several years of my life in graduate school have been influenced in ways beyond my own comprehension, and possibly knowledge, by the large number of wonderful people I have had the opportunity to meet and get to know. It would therefore be unfair for me to acknowledge only a subset of these people. That said, it is obviously not possible to produce such a list, even in principle, though I will try to be as complete as possible.

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Dedication

To my parents, for all the opportunities.
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Chapter 1

Introduction

1.1 “A Quantum Newton’s Cradle” ¹

Almost everyone is familiar the Newton’s cradle, a popular toy that adorns many desks. It consists of a few steel balls suspended by V-shaped strings that constrain them to travel in a straight line. When one ball is raised and dropped, it hits the pack, stops, and causes a ball from the other end to get kicked due to conservation of momentum, and to a very good extent, energy.

![Figure 1.1: A Classical Newton’s Cradle. When one ball is raised and dropped, it hits the pack, stops, and causes a ball from the other end to get kicked due to conservation of momentum and energy.](image_url)

One doesn’t typically expect this kind of behavior in the quantum realm, but in Ref. [84], Kinoshita, Wenger, and Weiss essentially showed just that. In the experiment, a gas of Rubidium atoms was loaded into an optical lattice, a grid of standing light waves made by lasers (more on this in Section 2.1). The atoms are localized near the nodes or antinodes. By using light of different frequencies, the gas was split into two pieces and then released under the influence of a harmonic trap, as shown in Figure 1.2.

Normally, one would expect such packets to spread and the system to settle, or

¹Title of Ref. [84]
The steel balls in the original toy are replaced with wave packets made of hundreds of Rubidium atoms, and collide in the center. This is a schematic of what the experiment observes. Image provided by David Weiss.

However, they noticed that after several thousand collisions the system still oscillated back and forth, and momenta didn’t reach an equilibrium distribution at long time. Absorption images of the experiment taken at different times reveal this oscillation (see Figure 1.3).

The strange behavior observed was ascribed to the so-called *integrability* of such a one dimensional gas of neutral integer spin atoms\(^2\). The existence of several additional conservation laws, a property of such models, was conjectured to be the reason for the behavior; these conservation laws seemed to protect the system from decoherence and mixing. Indeed this has been nearly conclusively established by numerous subsequent experiments and theory. See Ref. [26] for a detailed account.

This experiment was a remarkable breakthrough on multiple fronts. First, the fact that such a purely quantum phenomenon can be observed in a completely artificially constructed experiment is remarkable, and speaks to the technological progress scientists and engineers have made in the fields of laser optics, cold atoms, and imaging. Second, the phenomenon itself is rather unexpected and new to the realm of quantum dynamics, and the long life time of the system needs some explanation. Third, it

\(^2\)We’ll have a lot more to say about integrability in the other chapters.
Figure 1.3: Absorption images of the gas clearly showing the two packets oscillating back and forth. Image provided by David Weiss.

gives a tremendous boost to the study of nonequilibrium phenomena theoretically by providing a trustworthy experimental basis to test theories. The progress since then has been manifold. We’ll explore this in more detail in Chapter 2. For now, as an indicator, Figure 1.4 shows the growth in the number of articles containing the words “nonequilibrium” (and variations) and “quantum” in the abstract. The blue curve tracks articles only on the cond-mat arXiv, while the red curve tracks all other physics arXivs. Notwithstanding the rather incomplete nature of this plot, it still indicates that nonequilibrium quantum phenomena have gained significant traction in recent years.
1.2 Quantum nonequilibrium physics

Experiments like the one described in the previous section, and many others \(^3\) provide not only the motivation to theoretically examine quantum nonequilibrium problems, but also a context in which to study these phenomena. Before we move on I’ll describe this context, namely that of quench dynamics.

1.2.1 What is a quench?

In this work, I focus on a specific way of throwing a system out of equilibrium called a “quantum quench”. A quantum quench, like its metallurgical counterpart, involves suddenly changing a parameter of the system. The quantum analogs of quenching a hot piece of metal in cold water are for example, suddenly changing the size of the system, changing an external electric or magnetic field, or changing a coupling strength. In general, any sudden change affecting the system will be called a quench. As we will see in Chapter 2, a lot of these protocols are achievable in experiments.

Theoretically, a quantum quench is usually modeled literally as a sudden change. For instance, if the quench involves adding a term \(H_I\) to the initial Hamiltonian \(H_0\) of

\(^3\)See e.g., Refs. [20, 21, 29, 49, 102, 110, 120, 123, 135] for a collection of recent experiments and Refs. [17, 18, 42] for reviews.
a system, then we quite literally write down the full Hamiltonian as

$$H = H_0 + \theta(t)H_1$$  \hspace{1cm} (1.1)

where we use the standard Heaviside $\theta$ function. Experimentally, the word sudden doesn’t mean anything and it becomes important to characterize what we mean by “sudden”. Usually, this means that the time over which the quench occurs, $\tau_Q$, is much shorter than all other time scales in the system. In the experimental context, this is usually the fastest relaxation time in the experiment, which is on the order of milliseconds.

It is easy to see why a quantum quench is a good protocol to study nonequilibrium dynamics. First, a system in or near equilibrium is in its ground state and possibly a few low lying excited states. Such a system is characterized by a small finite number of eigenstates which are nearby in energy. In a global quench however, we put a finite amount of energy density into the system, and potentially excite very high energy states. More precisely, starting with an initial state $|\Psi_0\rangle$, the quench evolves this with a Hamiltonian $H$,

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle = \sum_n e^{-iE_nt}|n\rangle\langle n|\Psi_0\rangle,$$  \hspace{1cm} (1.2)

and the eigenstates (of the final Hamiltonian) that participate in the time evolution depend only on the overlaps with the initial state. In general, an arbitrarily large number of eigenstates participate, corresponding to arbitrarily large energies, only determined by the initial state and the final Hamiltonian. The system is therefore far from an equilibrium state. This can also be viewed as expanding a particular vector in the Hilbert space of a quantum system in an arbitrary orthonormal basis. Naturally, for a generic initial state, one expects a linear combination of a large number of vectors from the new basis. The initial state clearly plays a very important role, and this will be discussed in further detail. Eq. (1.2) also gives us an idea as to why calculating the quench dynamics is a hard problem – it requires knowledge of the entire spectrum, and the ability to carry out the summation over oscillating phases.

There are also other protocols for a quench. These aren’t sudden but proceed over some finite amount of time and typically occur when a coupling constant or external
field is changed to drive the system through a phase transition. For example, in the
anisotropic Heisenberg chain, the anisotropy $\Delta$ can be changed from less than one to
greater that one, driving the system from the gapless phase into the gapped phase.
Driving a system through a critical point at a steady rate is also considered a quench.
This is because as the parameter reaches its critical value, the relaxation time of the
system diverges, and any finite rate becomes a rapid quench. In other words, it is not
possible to maintain adiabaticity. This leads to the appearance of defects (domains of
different phases) and goes under the name of the Kibble-Zurek mechanism [27, 83,
145]. We do not study these quench protocols.

1.2.2 A note on “nonequilibrium”

Colloquially speaking, equilibrium usually refers to situations that are static in time
and nonequilibrium refers to situations that are changing in time, or dynamic 4. The
title of this dissertation then looks a little redundant, for either “dynamics” or “non-
equilibrium” should suffice to describe the situations we are interested in. I use non-
equilibrium dynamics to indicate that indeed these systems are away from any equi-
librium state but also fluctuate appreciably in time, as opposed to systems that are
away from their equilibrium state but slowly changing in time, or systems that are
indeed dynamic, for example, traveling waves, but are in a nonequilibrium steady state
in the sense that they are solutions of time-independent equations of motion. We are
interested in the entire gamut of nonequilibrium and dynamical processes, and steady
states are certainly part of that.

1.2.3 Integrable models

As mentioned earlier, integrable models have properties that makes their dynamics
different from generic models, mainly having to do with the presence of additional
conservation laws. In this work, we are primarily interested in the quench dynamics
of such models.

4Like more or less every scientifically rigorous statement, this one needs to be qualified with ifs and
buts, but these will become clearer as we go on, and I will avoid getting tied up here.
A rigorous definition of an integrable model is not essential here. Instead, I will use a working definition. For our purposes an integrable model is one that can be solved exactly using the Bethe ansatz. We’ll discuss the properties that a particular model must have for integrability and we’ll describe integrability in more detail in Chapter 3. For now, an integrable model is one for which we can explicitly write down the multiparticle eigenstates. As far as this work is concerned, these are all one dimensional models obtained from physical systems via some approximations and simplifications, which will be discussed in detail in Chapter 3. As seen in Section 1.1, these models describe certain experiments very well. To reiterate, integrable models differ from nonintegrable ones in fundamental ways, and signatures of this can be seen in experiments.

1.3 An overview of nonequilibrium phenomena

The success of thermodynamics as a broad framework to explain a wide array of equilibrium phenomena ranging from the specific heat of solids to phase transitions, quantum statistical properties via the partition function [127], and in general the notion of statistical ensembles has led to equilibrium phenomena becoming the norm. It is of course entirely the other way around — in nature we are surrounded by nonequilibrium phenomena. The Earth’s motion around the Sun, the Moon’s motion around the Earth, the flow of rivers and the periodic nature of tides are all dynamical phenomena. It is therefore surprising that so many of these phenomena can be described by a rather simple set of laws and equations. The fact that something as dynamic as water becoming steam can be described in terms of a few numbers (the critical exponents [92, 127]) is a remarkable consequence of what we now understand as renormalization and the emergence of effective theories. The same framework can be used to understand the ferromagnetic transition in some metals, a very different physical system, but surprisingly the underlying effective theories look very similar.

Unfortunately, truly nonequilibrium phenomena have not been amenable to such global laws, although a variety of methods have been developed to tackle different
types of nonequilibrium problems [38]. While notions of temperature and pressure go back to the early 1600s, the foundations for a statistical or probabilistic treatment of many particle systems began with Bernoulli and was further developed by Gibbs, Boltzmann and others [75]. Bernoulli’s work on fluid dynamics can in fact be considered one of the earliest attempts at understanding the nonequilibrium problem of fluid flow. However, early notions of nonequilibrium thermodynamics appeared in the form of irreversible processes. The most successful formulation was the balance equation approach, which was essentially a flow equation for entropy that took into account entropy production due to irreversibility in the system. The type of nonequilibrium phenomena we study are different in that in a quench, energy and entropy are pumped into the system during the quench, but the system evolves unitarily, and depending on the particular system at hand can be invariant under time reversal. It is only systems connected to external baths that see continuous entropy production in our context. This is for example required to sustain a current through the system.

Several natural phenomena are dynamical in nature. Weather systems, traffic patterns, the flow of rivers, the responses of a bridge to people walking and wind, the continuous effect of tides on a shoreline form a small subset of the macroscopic phenomena that one would call dynamical. The time scales of the dynamics vary vastly — some slow and some fast with respect to time scales of our observation of these phenomena. A growing tree doesn’t seem to be doing much over what would be a long time by our standards, but this is merely because what we observe is limited.

On a microscopic scale, one can argue that every system is dynamical. Some of these systems do not deviate appreciably on a macroscopic scale from some equilibrium, and some do not deviate appreciable even microscopically. These are the types of system that one would call equilibrium systems, and can be described by the laws of thermodynamics and certain emergent macroscopic observables. A gas in a container for example can be described by its pressure, volume, and temperature to a very good approximation, and even though the microscopic description is dynamic and the gas particles are in continuous motion, collision, and vibration, the three macroscopic quantities do not vary appreciably. This quasi-equilibrium behavior of systems is due
in part to them being in an environment, the so-called thermodynamic bath, and any coherent nonequilibrium behavior is rapidly quelled by dephasing of states due to interaction with the environment and dissipation of energy.

The one nonequilibrium phenomenon that is commonplace is the existence of currents. If we connect a piece of resistive wire like a light bulb to a battery, then the potential difference drives a steady current through the bulb, until all the charge of the battery equilibrates, i.e., the battery discharges. The nonequilibrium phenomenon in question here is a steady state produced by a bias in the boundary conditions. It is similar to a ball rolling down an inclined plane. The steady state phenomenon occurs after any transients have died out, and overcomes dissipation since energy is continuously pumped into the system by maintaining a potential energy bias.

Steady state currents are a staple of transport problems in several quantum models, e.g. quantum hall states [70], transport through impurities [46, 47, 100], and currents in one dimensional wires. This area has also attracted a lot of attention lately with the technological advances in fabricating quantum dots and controlling tunneling barriers and quantum point contacts [54]. It is also possible to establish currents in a quench experiment, as we will describe in Chapter 6. Recently transport has been experimentally observed in the ultracold atom context [101, 30]. Steady state currents in the Anderson model, that shows the Kondo effect [68] is currently a theoretically unsolved problem in spite of the model being integrable. The two lead model, necessary to establish a current poses complications due to the so-called string states and has eluded a fully exact treatment [28].

1.4 Outline of the thesis

The remainder of the thesis is organized as follows.

In Chapter 2 I discuss some features of quench dynamics in closed one-dimensional systems. I then describe the specific experimental and theoretical motivation to study non-equilibrium phenomena. In Section 2.1, I focus on the specific experiments that are of relevance in the context of this work. I will describe what models can be used to
describe these experiments, and the different phenomena that have been observed. In Section 2.2, I describe the commonly used tools to study non-equilibrium phenomena, their advantages and disadvantages, and provide reasons as to why new tools are required for a complete understanding on such phenomena. In Section 2.4, I survey some of the literature in the field, and describe some of the major results. Section 2.5 provides an overview of the open questions in the field, and also describe which of these questions, the present work addresses.

In Chapter 3, I provide a brief review of integrability and the Bethe Ansatz (Sections 3.1 and 3.2). Specifically, I describe only those aspects of the Bethe Ansatz that are useful for the purposes of this work. In Section 3.3, I show some examples of how the Bethe Ansatz can be used to obtain eigenstates of models that we will study the dynamics of. In Section 3.4, I discuss the role of the Bethe Ansatz in the context of quantum quenches, the advantages, and the difficulties involved in a direct application to dynamics.

Carrying on along these lines, in Chapter 4 I review Yudson’s integral representation for arbitrary states in an infinite volume system. I also derive the completeness of the “Yudson” basis for a specific case. In Section 4.3 I provide an explicit example of how the Yudson integral representation can be used to obtain the quench dynamics using the Resonant Level Model. I will derive finite time behavior and asymptotics. Section 4.4 describes how the Yudson representation can be extended to other integrable models, what the challenges are, and how we can perhaps overcome them.

Chapters 5 and 6 form the main body of this work. I derive the Yudson representation for the Lieb-Liniger model and the Heisenberg chain, and use it to study the time evolution of these two models. In the Lieb-Liniger case, we show how the dynamics is controlled by fixed points at long time that either lead to fermionization of the bosons, or formation of bound state correlations. Chapter 6 presents ongoing research.

In Chapters 7 and 8 I present possible future directions of this work, and provide a summary and conclusion to the thesis.
Chapter 2
Quantum nonequilibrium dynamics

Quench experiments have come to the forefront of studies in quantum nonequilibrium phenomena due to tremendous technological progress over the last decade or so. Before we venture into a theoretical description of specific quenches in specific systems, it is necessary to provide the experimental context in which this work is based, namely that of ultracold atoms. I will also briefly introduce the various theoretical models and approaches to understanding these phenomena.

2.1 Experiments to probe out of equilibrium phenomena

2.1.1 Experiments with cold atoms

The first big success of cold atoms takes us to the 2001 Physics Nobel prize awarded to Cornell, Ketterle, and Wiemann “for the achievement of Bose-Einstein condensation in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates” [105] for work done in 1995 [34, 76, 37]. They for the first time explicitly showed a gas of bosons (Sodium atoms) forming a condensate, a prediction made by Bose and Einstein in the mid-twenties [19, 41]. In these experiments, a gas of Rubidium or Sodium atoms was trapped and cooled using lasers leading to the formation of a condensate. Figure 2.1 shows a photograph of a trapped cold gas.

Fast-forward about 18 years, and we have the state-of-the-art experiments in which bosonic and fermionic gases can be trapped and cooled, their interactions can be controlled, and phase transitions can be driven by applying external electric and magnetic

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1Superfluidity, essentially a condensation phenomenon, was observed in the late 30s but it was never observed in a gas before this.
It is this control and precision that has made ultracold atoms a perfect setting for the study of isolated quantum systems. Most of the theoretical studies about the out of equilibrium behavior, especially quench dynamics is in the context of cold atom experiments, and it is indeed the experimental arena to which our calculations apply. The systems are formed by trapping a gas of atoms using standing light waves made by lasers. The gases are cooled evaporatively and are well isolated from any thermal baths making them ideal for studying relaxation and thermalization in isolated quantum systems. The interactions between the particles, the potentials, and their statistics can be controlled by the use of external magnetic and electric fields, tuning the optical lattice, and loading different atoms into the traps. Systems with mobile impurities can also be studied by loading two or more different species of atoms into the lattices. Lattices can be three dimensional or can be made quasi-1d or 2d by using confining potentials. The typical relaxation and evaporation time scales in these systems are in
the milliseconds. This makes measurement easier than in solid state systems. It also allows for sudden quenches. Disorder is also largely absent, unless introduced.

Tuning the parameters allows the study of superfluid behavior, Mott insulators, spin chains and so on. Such a gas trapped by lasers and cooled to nano-Kelvin temperatures can be quenched by suddenly changing the interaction between the atoms, and the external trapping potential. Evolution can be globally observed by imaging the gas, and the time evolution of densities and correlation functions can be obtained from these images [18, 57, 84, 102].

In one dimension, which is of particular interested to us, the typical models that are used to study these systems are the Bose-Hubbard model, the XXZ model, the Sine-Gordon model and the Lieb-Liniger model. Each of these models studies a different regime of the gas. The Bose-Hubbard model is optimal for atoms hopping on a one dimensional lattice. A particular limit of the Bose-Hubbard model can be mapped to the XXZ spin chain [8] which is integrable. The continuum gas is captured by the Lieb-Liniger model.

We are primarily concerned with integrable models. In this context, it is an important question as to how “integrable” a particular experimental realization is. Often the experimental setup maintains an external trapping potential and including such a potential in an integrable Hamiltonian may render the system non-integrable. Experimentally, such potentials need to be eliminated to the extent possible. This can be partially achieved by using a flat potential profile and concentrating on the center of the trap. As has been shown in Ref. [84], and discussed in Section. 1.1, the dynamics in certain experiments very closely resembles what we expect from an integrable model, and it is believed that we can indeed create integrable systems to a close approximation. This also opens up the question of how far from integrability do we need to be in order to see the effects of integrability breaking.

2.1.2 Optical lattices

Optical lattices form the basis of trapping ultracold atoms in various configurations (see Refs. [16, 73] for reviews). The concept is simple. As shown in Figure 2.2, an
optical lattice consists of opposing laser beams. In one dimension, two opposing coherent laser beams whose phases are appropriately tuned can set up a standing wave. The electric field interacts with the induced atomic dipoles creating a potential for the atoms,

\[ V_{\text{dipole}} = -d \cdot E(r) \sim aI \]  

(2.1)

where \( a \) is the frequency dependent polarizability and \( I \) is the intensity of light. By tuning the frequency away from any atomic resonances, atomic transitions and energy transfer are avoided. By tuning the frequency to be greater than or less than the resonance frequency, an attractive or repulsive potential can be obtained depending on the sign of the detuning. Laser light of wavelength \( \lambda \) produces standing waves having \( \lambda/2 \) periodicity – this sets the lattice constant. Three laser beams in perpendicular directions can trap atoms in all three directions and make a 3d lattice.

![Figure 2.2: An optical lattice formed by standing waves of laser light. The frequencies can be tuned to localize the atoms at the nodes or antinodes.](image)

Atoms localized in potential wells formed by the optical lattice can tunnel into neighboring wells, and the tunneling rate determines the hopping probability in a tight-binding type of Hamiltonian, and is proportional to the height of the well and
consequently the intensity of the laser.

\[ t \propto V_0 \propto I. \] (2.2)

By increasing the intensity, we can strongly localize the atoms, creating an insulator, or lower it creating a fluid.

On top of the underlying lattice that localizes the atoms, additional laser profiles can create traps of various shapes, a typical one being parabolic. These traps can also be used in making a three dimensional gas quasi-2d or quasi-1d by making the traps in some directions very steep so that the kinetic energy is suppressed in those directions. Once atoms are loaded into an optical lattice and cooled, we are ready to turn on interactions, let the gas expand, and make measurements!

### 2.1.3 Feshbach resonances

The tunability of the interactions between ultracold atoms is a very useful feature as it can be used to realize several different regimes of behavior like Bose-Einstein condensate (BECs), superfluidity, Mott insulators, and so on [16, 110, 73, 59]. In the approximation that the only important interactions between the atoms are two-body \( s \)-wave interactions (as assumption that is usually valid in ultracold atoms [31]), the interaction potential, or coupling strength depends only on the corresponding scattering length,

\[ V = \frac{4\pi \hbar^2 a}{m} \] (2.3)

where \( m \) is the atomic mass, and \( a \) is the \( s \)-wave scattering length. This scattering length can be varied from \(-\infty < a < \infty\) by exploiting the so-called Feshbach resonance, which is essentially a resonance between a bound state and a scattering state. A positive binding energy leads to a repulsive interaction, while a negative binding energy leads to an attractive interaction. The position of the resonance, and therefore the binding energy can be tuned using an external magnetic field.

The spin structure of the atoms play a crucial role in Feshbach resonances. The three spins involved are the electron spin, the orbital angular momentum, and the nuclear spin. For the alkali metal atoms like \(^7\text{Li}\) typically used in ultracold setups,
the ground states are usually zero orbital angular momentum states. The only relevant quantum numbers then are the nuclear spin and its projection, and the electron spin, which is always 1/2. The atom then behaves as a fermion or boson depending on the nuclear spin quantum number. States corresponding to different nuclear spin can further be split by an external magnetic field. In a collision of two atoms, several of these can be accessed – the potentials corresponding to different states are called collision channels. The interaction between two collision channels can produce a Feshbach resonance, described below.

Consider the simple case of two atoms interacting via the potentials shown in Figure 2.3. The two potentials usually correspond to different hyperfine states split by an external magnetic field. At large distance $R$, the atoms are essentially free. When they come close however, if the bound state energy of the closed channel is similar to the scattering energy, then a resonance occurs. The scattering length corresponding to this resonance depends on the splitting between the two states, and can be controlled by changing the external magnetic field,

$$a(B) = a_0 \left( 1 - \frac{\Delta}{B - B_0} \right).$$

(2.4)

In the above equation, $a_0$ is the scattering length off-resonance associated with the background potential of Figure 2.3. $B_0$ is the resonant value of magnetic field, which is a property of the particular species used and the particular hyperfine state involved in the resonance, and $\Delta$ is the width of the resonance. Figure 2.4 shows a plot of the
scattering length as a function of the magnetic field.

![Graph of scattering length vs. magnetic field](image)

Figure 2.4: Scattering length as a function of magnetic field near a Feshbach resonance. The scattering length can be made positive or negative, corresponding to repulsive or attractive interactions.

The scattering length has a divergence and a discontinuity \( B = B_0 \); on either side of this divergence the sign of the scattering length is different. Tuning the resonance through this allows us to go from repulsive to attractive interactions and vice versa. There also exists a zero crossing, or a point where the scattering length goes to zero. It is easy to see that this point is

\[
B = B_0 + \Delta.
\]

At this point, the particles become noninteracting.

### 2.1.4 Recent developments

Over the last decade, the use of the Feshbach resonance to establish fine control over ultracold gases has become commonplace [31]. Recently more exotic properties are being realized [17], as ultracold gases slowly achieve the ideal of Feynman’s quantum simulator [48].

**Artificial gauge fields and topological properties**

The atoms used in ultracold experiments are neutral and we cannot examine their interaction with electric or magnetic fields in order to understand the behavior of
charged particles. In particular, producing the quantum Hall effect seems impossible. The physics of the quantum Hall effect is a simple example of so called topological properties. The Hall conductance is quantized, and the quantization filling factor corresponds to the Chern number, the coefficient of a total derivative term in the Lagrangian. Vorticity in a superfluid is also such a topological property emerging from the boundaries or a singularity in the flow, and as such is a global phenomenon.

One ingenious method of simulating vorticity or a Lorentz force in an ultracold atom experiment is to rotate the gas maintaining a balance between centrifugal and trapping potentials. The Coriolis force then provides the “Lorentz force” since it has the same velocity dependence and form,

\[ F_{\text{Coriolis}} = -2m\Omega \times v, \quad F_{\text{Lorentz}} = -qB \times v. \] (2.6)

The mass \( m \) plays the role of the charge \( q \), and the angular velocity \( \Omega \) plays the role of the magnetic field \( B \).

There are also other optical means of producing vorticity. See Ref. [17] for a review and the references therein for details.

**Atom scale imaging**

Absorption imaging is the common way in which a gas of ultracold atoms is measured. Near resonant light is shone on the sample, regions of high density absorb and scatter more light and regions of low density, less. The resulting image is captured on a Charge Couple Device (CCD) sensor, similar to the ones in digital cameras, albeit through microscope lenses. This however does not provide atom scale resolution.

Recently, fluorescence imaging with high resolution low noise CCDs has made it possible to resolve single atoms by maintaining a very low intensity of fluorescence [17] – each atom scatters only a few thousand photons. Further, the gas can be imaged at any point by freezing it momentarily by increasing the lattice depth to prevent tunneling. The primary difference is that in absorption imaging, the scattered light is lost, and only the “shadow” is recorded. However, in fluorescence imaging
each atoms displays its presence via fluorescence from resonant light. This new technique allows in-situ atom resolved measurements.

2.2 Theoretical methods and tools

In Physics, experiments are always modeled, and one aims to provide the simplest possible model that explains the findings of the experiment, and perhaps more \(^2\). In the field of nonequilibrium quantum dynamics, experiments using ultracold atoms have turned out to be unusually clean (i.e. disorder free) and well controlled, and are proving to be an alternate means of understanding the physics. Instead of studying a model, one simulates it, but this time with a real quantum system, as opposed to a simulation on a classical computer, implementing a point of view going back to Feynman [48]. However, whereas making measurements on an experiment in various different conditions gives us tremendous insight, predicting new behavior or developing a broader understanding requires models, and we need theoretical tools to calculate the predictions of these models.

The cold atom experiments that we are interested in are on one dimensional systems of bosons or fermions. There are a few effective models that can be used to understand different regimes of the gas.

2.2.1 Models

The most basic model describing these experiments is the Hubbard model,

\[
H_{\text{Hubbard}} = t \sum_{\langle ij \rangle} c_i^\dagger c_j + U (c_i^\dagger c_j)^2,
\]

which describes either bosons or fermions hoping between nearest neighbors on a lattice with on-site interactions. Additional terms can be included to account for next-nearest-neighbor hopping, longer range interactions, and external potentials. Various limits of the couplings \(U/t\) lead to different effective models. By changing lattice parameters or considering excitations on top of the lattice, a further set of effective models

\(^2\)Occam's razor
can be realized. For instance, for bosons in the regime where \( u/t \gg 1 \), double occupancy is suppressed, and we get a model of hard core bosons hopping on a lattice. A Jordan-Wigner transformation [52] can be used to transform the bosons to free fermions. It essentially maps the bosonic creation operators to fermionic creation operators. However, in order to get the commutation relations correct (bosons commute, while fermions anticommute), the mapping is a little more complicated:

\[
\begin{align*}
&b_j^\dagger = e^{\pi \sum_{<ij} c_i^\dagger c_j^\dagger}, \\
&b_j = c_j e^{-i\pi \sum_{<ij} c_i^\dagger c_i} , \\
&b_j^\dagger b_j = c_j^\dagger c_j.
\end{align*}
\] (2.8)

A similar transformation can be used to go between spin-1/2 operators and fermions in one dimension, and therefore this model can also be mapped to the interacting XXZ model,

\[
H_{XXZ} = -J \sum_j \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta (\sigma_j^z \sigma_{j+1}^z - 1)
\] (2.9)

Chapter 6 presents some preliminary work on the dynamics of the critical (isotropic, \( \Delta = 1 \)) Heisenberg model.

The “lattice” is usually formed by a deep laser standing wave which localizes the atoms at the nodes or antinodes. First, the lattice spacing can be controlled by using different laser frequencies, and the depth can be controlled by varying the amplitude. Making the amplitude small allows the atoms to tunnel easily, transitioning to a continuum model of free bosons, or with \( \delta \)-function interaction, the Lieb-Liniger model, described in Chapters 3 and 5. The intermediate regime between the lattice and the continuum can be captured by adding a sinusoidally varying potential to a continuum bosonic model, giving us the sine-Gordon model.

Of the above models, all other than the Bose-Hubbard model are integrable and can be solved exactly. Such models are of special interest to us. It is indeed remarkable that such specialized models can be experimentally realized. We do however, neglect three body losses and next-nearest-neighbor (NNN) hopping or long range interactions. These processes typically render models nonintegrable and one typically needs
numerical techniques to study them. Fortunately, neglecting these is a reasonable assumption in cold atom experiments.

Inspite of these simplifications, the quench dynamics of these models is not easy to compute. As explained in Section 2.3, a quench potentially excites arbitrarily high energy eigenstates of the final (i.e., post-quench) Hamiltonian. Most theoretical tools are designed to deal with either ground state properties and small excitations above the ground state, or to study the effect of interactions perturbatively, and especially to deal with equilibrium situations. We need a more general set of tools, and some new ones to deal with the far from equilibrium, highly excited, and strongly interacting systems in quench experiments. This is especially true of one dimensional systems, which is the subject of this thesis. In one dimension, due to geometric hindrance, interactions are always strong. The best example of this is that the Fermi liquid breaks down in one dimension and we are forced to use a more complicated picture, namely the Luttinger liquid. Fortunately the Luttinger liquid is exactly solvable via bosonization, and indeed displays physics that is unique to one dimension, namely spin-charge separation, the idea that the spin and charge degrees of freedom propagate at different velocities. See Ref. [52] for more on the various methods to solve the Luttinger liquid.

2.2.2 Field theoretic methods

Renormalization group based methods have been tremendously useful in all of condensed matter physics in understanding low energy effective theories of a variety of phenomena [5]. Out of equilibrium, the standard technology of calculating expectation values fails, and one has to employ the Schwinger-Keldysh formalism [128, 82]. See Ref. [4] for a review.

In order to apply perturbative techniques the model must have a small parameter. Usually the coupling constant of the interaction terms are treated as small, i.e., weak. Basically, the perturbation around the solvable model must be small. It is for example possible to perturb in an external electric field, when the model without the electric field is exactly solvable. A series expansion can then be developed in the external field, and time dependent expectation values can be evaluated as a series. This is especially
useful when the question of interest is the linear response of a system [108, 107]. In other cases it is possible to develop a perturbation series in the inverse number of flavors or colors (internal symmetries), the so-called large-$N$ limit. These expansions are exact in all orders of the appropriately scaled coupling constant but approximate in $1/N$ [115].

2.2.3 Numerical methods

With the development of powerful algorithms and fast computers, it has become feasible to study the dynamics of quantum systems purely numerically. There are several commonly used techniques that each have their advantages and disadvantages. Some numerical methods aim to exactly solve a given system, while others use various approximations. Methods like exact diagonalization, Density Matrix Renormalization Group (DMRG), Dynamical Mean Field Theory (DMFT), Numerical Renormalization Group (NRG) have been implemented numerically. I will briefly describe two of these that have been the most popular with the field of nonequilibrium quantum dynamics.

Exact diagonalization

The method is precisely that. Given a Hamiltonian, one solves the matrix eigenvalue problem. For this to be feasible, the volume of the system is a fixed finite number, say $L$, and the number of particles is fixed as well, say $N$. One can then write down a basis for the Hilbert space, and in this basis, the Hamiltonian is a matrix. This matrix is then diagonalized, giving the eigenvectors and eigenvalues. If only the ground state properties and perhaps a few excitations are desired, the task is computationally a little simpler. However, in order to calculate time evolution after a quench, where all the eigenstates that have a nonzero overlap with the initial state are relevant, one has to carry out a full diagonalization.

Full diagonalization algorithms are computationally intensive and it is not feasibly to study very large systems. Sizes of $L \sim 20$ are about as large as possible [126]. Sometimes it is possible to decrease the effective size of the Hilbert space by exploiting
other conserved quantities. For instance, a spin model with \( N \) spins has a Hilbert space dimension of \( 2^N \). However, if the model conserves either the total spin or \( S_z \), then one can work in a particular spin sector with a fixed \( S_z \). If out of the \( N \) spins, \( M \) are up and \( N - M \) are down, then the \( z \)-component of the spin is fixed at \( S_z = \frac{1}{2}(2M - N) \), and the dimension of the Hilbert space is now \( \binom{N}{M} \). As an example, for \( N = 20 \), without restricting to the spin sector, the dimension is \( 2^{20} \cdot 10^6 \). For \( N = 20 \) and \( M = 10 \), we instead have a Hilbert space dimension of about \( 10^5 \), nearly a factor of 10 improvement. It also makes more physical sense to treat each spin sector separately if there is no mixing.

The advantage of exact numerical diagonalization is that once the eigenvalue equation is solved, we can extract anything we want. Expectation values of operators in arbitrary states, time evolution, entropy calculation, correlation functions are all easily accessible once the entire spectrum is known. See Refs. [116, 117, 118, 65, 119] for example.

### Density Matrix Renormalization Group

The Density Matrix Renormalization Group (DMRG) [137, 36] has become a very popular method recently because of its versatility and ease of application to a wide variety of one dimensional models, and initial states. In its equilibrium avatar, the method can effectively find the ground state of a system without having to diagonalize the full Hamiltonian, a problem that is exponential in the size of the system, due to the way the Hilbert space of quantum problems scales. The DMRG method overcomes this by using representative states to start with, and improving the guess for the true ground state iteratively, each step involving a much less accurate diagonalization of the Hamiltonian to find the lowest energy states. Further, in each step, only a subset of the system can be considered, thus speeding up the process.

This method can be extended to calculate time dependence by using some approximation of the time evolution operator, and directly acting on an initial state. The main speed up is obtained by using a suitably truncated Hilbert space instead of the full Hilbert space. As the entanglement increases with time evolution, more and more
states in the exponentially large Hilbert space need to be retained, and therefore eventually the method slows down to impractical speeds. Further, it is not straightforward to use this method on models with long range interactions. See [63, 125, 126] for reviews.

### 2.2.4 Exact methods

These methods are applicable to a specific set of models. They allow the exact calculation of correlation functions, ground state properties etc. Exact in this context means that we either have an analytical result, or a set of integral or differential (or both) equations whose numerical solution provides us with the numbers. It is different from exact diagonalization, since that method, while also being exact within numerical error, applies to direct numerical solution of the Schrödinger equation, and is restricted to finite sized lattice models.

In this Section, we will outline the three most commonly used exact methods, and they all apply to models in one spatial (and one time) dimension.

**Conformal field theory**

Conformal Field Theories are a class of field theories that have no length scale (are scale invariant), and typically describe critical points. The field theory describing the critical point is an effective field theory, constructed from the quasi-particle excitations of the critical point [144]. The key aspect here is that conformal invariance (Lorentz invariance plus scale invariance plus inversion symmetry) in one dimension strongly constrains all correlation functions to a fixed form determined by the field content of the theory. For more details, see Refs. [144, 51]

Conformal field theory methods have been successfully employed to understand nonequilibrium problems, such as the steady state current through quantum point contacts [47, 46], in quantum hall systems [70] and Ising models [23].
Bosonization

Bosonization is a method where collective excitations in fermion densities are treated as the quasi-particle. The resulting effective field theory contains bosonic fields, and four-point local interactions in the fermions are reduced to two-point interactions, thus making them quadratic. This transformation renders, for example, the Luttinger liquid, which is interacting in the fermions, into a free field theory in terms of the bosons [52]. One can also bosonize more complicated fermionic models to effective interacting bosonic models, sometimes with topological terms. Ref. [98] provides an excellent introduction to bosonization and the connection with Conformal Field Theory.

Bosonization is similar to other methods like Jordan-Wigner transformations where an appropriate field transformation (possible nonlocal) provides a more convenient representation of the model. All these methods are more conducive to treating initial states which are eigenstates of the model, and quenching the coupling constant. Starting with a physical initial state in fermions and diagonalizing the model in terms of bosons requires one to represent the initial state in an appropriate language, and this is not always straightforward. The fermionic representation of hardcore bosons is one such example [32, 33, 87].

Bethe Ansatz

The Bethe Ansatz is yet another technique that is applicable to a subset of models that are defined as being integrable. Since this thesis primarily deals with integrable models, a detailed exposition on the Bethe Ansatz as applicable here is provided in Chapter 3.

2.3 Dynamics of $1d$ isolated many-body systems

While studying the thermodynamic properties of a quantum system, one needs to enumerate and classify the eigenstates of the Hamiltonian in order to construct the partition function. To achieve this, some finite volume boundary conditions are typically
imposed — either periodic boundary conditions to maintain translation invariance or hard wall boundary conditions when the system has physical ends. One can then identify the ground state and the low lying excitations that dominate the low-temperature physics. In the \textit{thermodynamic limit} \( \lim_{N \to \infty} \frac{N}{L} = \rho \), the limit of very large systems with large number of particles at finite density, the effect of the boundary condition is negligible and we expect the results to be universally valid.

When the system is out of equilibrium, a different set of issues arises. We shall consider here the process of a “quantum quench” where one studies the time evolution of a system after a sudden change in the parameters of the Hamiltonian governing the system. To be precise, one assumes that the system starts in some stationary state \( |\Psi_0\rangle \). This stationary state can be thought of as the ground state of some (interacting) Hamiltonian \( H_0 \). Following the quench at \( t = 0 \), the system evolves in time under the influence of a new Hamiltonian \( H \) which may differ from \( H_0 \) in many ways. One may add an interaction, change an interaction coupling constant, apply or remove an external potential or increase the size of the system. Further, the quench can be sudden, i.e., over a time window much shorter than other time scales in the system, driven at a constant rate or with a time dependent ramp.

In this paper we shall concentrate on sudden quenches where the initial state \( |\Psi_0\rangle \) describes a system in a finite region of space with a particular density profile: lattice-like, or condensate-like (see Fig. 2.5). Under the effect of the quenched Hamiltonian the system evolves as \( |\Psi_0, t\rangle = e^{-iHt}|\Psi_0\rangle \).

To compute the evolution, it is convenient to expand the initial state in the eigenbasis of the evolution Hamiltonian,

\[
|\Psi_0\rangle = \sum_{\{n\}} C_n |n\rangle, \quad (2.10)
\]

where \( |n\rangle \) are the eigenstates of \( H \) and \( C_n = \langle n|\Psi_0 \rangle \) are the overlaps with the initial state, determining the weights with which different eigenstates contribute to the time evolution:

\[
|\Psi_0, t\rangle = \sum_{\{n\}} e^{-i\epsilon_n t} C_n |n\rangle. \quad (2.11)
\]
Figure 2.5: Initial states. (a) For $\frac{a}{\sigma} \gg 1$, we have a lattice like state, $|\Psi_{\text{latt}}\rangle$. (b) For $a = 0$, we have a condensate like state $|\Psi_{\text{cond}}\rangle$, $\sigma$ determines the spread.

The evolution of observables is then given by,

$$
\langle \hat{O}(t) \rangle_{\Psi_0} = \langle \Psi_0, t | \hat{O} | \Psi_0, t \rangle = \sum_{\{n,m\}} e^{-i(\epsilon_m - \epsilon_n)t} C_n^* C_m \langle n | \hat{O} | m \rangle,
$$

(2.12)

with observables that may be local operators, correlation functions, currents or global quantities such as entanglements.

The time evolution is characterized by the energy of the initial state,

$$
\epsilon_{\text{quench}} = \langle \Psi_0 | H | \Psi_0 \rangle = \sum_{\{n\}} \epsilon_n |C_n|^2
$$

(2.13)

which is conserved throughout the evolution, specifying the energy surface on which the system moves. This surface is determined by the initial state through the overlaps $C_n$. Unlike the situation in thermodynamics where the ground state and low-lying excitations play a central role, this is not the case out-of-equilibrium. A quench puts energy into the system which the isolated system cannot dissipate and it cannot relax to its ground state. Rather, the eigenstates that contribute to the dynamics depend strongly on the initial state via the overlaps $C_n$ (see Fig. 2.6).

A vivid illustration comes from comparing quenches in systems that differ in the sign of the interaction. In the Bose-Hubbard model and the XXZ model it has been observed that the sign of the interaction plays no role in the quench dynamics [89, 8], even though the ground states that correspond to the different signs are very different. For example, for the XXZ magnet, the ground state is either ferromagnetic or
Néel ordered (RVB in 1d) depending on the sign of the anisotropy \( \Delta \). In Appendix A, we show results for the Bose-Hubbard model, and provide an argument for this observation. The Lieb-Liniger model, whose quench dynamics we describe here, on the other hand shows very different behavior, reaching an long time equilibrium state that depends mainly on the sign of the interaction.

![Diagram](image)

**Figure 2.6:** Difference between quench dynamics and thermodynamics. After a quench, the system probes high energy states and does not necessarily relax to the ground state. In thermodynamics, we minimize the energy (or free energy) of a system and probe the region near the ground state.

In the experiments that we seek to describe, a system of \( N \) bosons is initially confined to a region of space of size \( L \) and then allowed to evolve on the infinite line while interacting with short range interactions. It is important to first understand the time-scales of the phenomena that we are studying. There are two main types of time scales here. One is determined by the initial condition (spatial extent, overlap of nearby wave-functions), and the other by the parameters of the quenched system (mass, interaction strength).

For an extended system, i.e., where we start with a locally uniform density (see Fig. 2.5a), we expect the dynamics to be in the constant density regime as long as \( t \ll \frac{L}{v}, v \) being the characteristic velocity of propagation. Although the low energy thermodynamics of a constant density Bose gas can be described by a Luttinger liquid [52], we expect the collective excitations of the quenched system to behave as a
highly excited Liquid since the initial state is far from the ground state. It is also possible that depending on the energy density $\epsilon_{\text{quench}}/L$, the Luttinger liquid description may break down altogether.

The other time scale that enters the description of nonequilibrium dynamics is the interaction time scale, $\tau$, a measure of the time it takes the interactions to develop fully: $\tau \sim \frac{1}{\epsilon}$ for the Lieb-Liniger model $^3$. Assuming $L$ is large enough so that $\tau \ll \frac{L}{\nu}$, we expect a fully interacting regime to be operative at times beyond the interactions scale until $t \gg \frac{L}{\nu}$ and the density of the system can no longer be considered constant, diminishing with time as the system expands. In the Lieb-Liniger model, this leads to an effective increase in the coupling constant which manifests itself as fermionization for repulsive interaction and bound-state correlations in the case of attractive interactions. Thus the main operation of the interaction occurs in the time range $\tau \lesssim t \lesssim \frac{L}{\nu}$, over which the wave function rearranges and after which the system is dilute and freely expands. For the case $L = \infty$, free expansion is not present. Figure 2.7 summarizes the different time-scales involved in a dynamical situation.

$^3$One can refine the estimate for the interaction time setting $\tau \sim \frac{1}{\pi\epsilon}$, with $\delta E = \langle \Phi_0 | H_I | \Phi_0 \rangle$. Also if we start from a lattice-like state, $\tau$ will include a short time scale $\tau_a \sim \frac{\pi}{\delta}$ before which the system only expands as a non-interacting gas, until neighboring wave-functions overlap sufficiently.
2.4 Established results

2.4.1 Eigenstate thermalization

An isolated quantum system evolves unitarily and a priori there is no reason why the long time behavior should be independent of the initial states. In other words there is no reason why the system should thermalize, and indeed we have seen examples of integrable models that do not. However, for systems that do, we need to understand possible mechanisms by which this can occur. The time evolution of an observable $\mathcal{O}$ in a state $|\Psi_0\rangle$ is given by

$$\langle \mathcal{O}(t) \rangle = \sum_{m,n} O_{mn} c_m^* c_n e^{-i(E_n - E_m)t}$$

(2.14)

where the $c_m$ are as usual the overlaps of the eigenstates of the Hamiltonian and the initial state. If $t \rightarrow \infty$ limit exists, then it must correspond to $E_m = E_n$ \footnote{While fairly intuitive, this statement is rigorous and a result of the Riemann-Lebesgue lemma [58].}. In the absence of degeneracies, this gives

$$\langle \mathcal{O}(t) \rangle = \sum_{m} O_{mm} |c_m|^2$$

(2.15)

which is an average in the so-called diagonal ensemble.

Deutsch and Srednicki proposed a mechanism by which the expectation value of the operator $\mathcal{O}(t)$ thermalizes at long time, i.e., reaches a value independent of the particular initial state [39, 130]. Rigol et al propose a couple of other mechanisms for the thermalization [117]. Together they amount to the following:

1. For physical initial state, the overlaps $c_n$ have a flat distribution and so we reach the microcanonical distribution obtained by assuming a priori equal probabilities;

2. The fluctuations in the operator matrix elements and the overlaps from one eigenstate to the next are uncorrelated, so the sum is essentially an unbiased sampling;

3. The operator matrix elements have a flat distribution, and factor out of the sum.
Each of these scenarios can be realized, but it is not obvious that all thermalization necessarily occurs in this fashion. Further, as we see in the following section, dependence on the initial state can be partial.

### 2.4.2 Generalized Gibbs Ensemble

The idea of a thermodynamic ensemble goes back to Boltzmann and Gibbs, and the roots lie in the probabilistic or phase space approach to the equilibrium configuration of a gas. The basic notion is that the system maximizes the number of configurations available for a particular macroscopic state. For a microcanonical description, this is rule is satisfied by having equal a priori probabilities for the different configurations. The quantity being maximized is the phase space volume, or the log of it, which up to a factor is the entropy. The true power of this approach comes when the same notion of maximizing entropy is taken to describe a system connected to a bath, giving us the so-called canonical ensemble or Gibbs ensemble.

The notion of a Generalized Gibbs Ensemble goes back to Jaynes [74, 75] who treated the Gibbs ensemble (or the canonical ensemble) as the simple outcome of maximizing the entropy of a system subject to constraints. The Gibbs ensemble is obtained by requiring that the expectation value of the energy is fixed by the thermodynamic bath,

$$\rho_{\text{Gibbs}} = \frac{e^{-\beta H}}{\text{Tr} e^{-\beta H}}. \quad (2.16)$$

The Lagrange multiplier for this constraint is $\beta$, the inverse temperature. It was a natural extension to him that if the system has multiple constraints, then each of these constraints must be taken into account. Accounting for a chemical potential and a fixed expectation for the particle number $N$ leads to the Grand Canonical Ensemble,

$$\rho_{\text{GC}} = \frac{e^{-\beta H - \mu N}}{\text{Tr} e^{-\beta H - \mu N}}. \quad (2.17)$$

For a system with several constraints that can be written as expectation values of observables (or in general operators), the entropy must me maximized subject to all these constraints giving

$$\rho_{\text{GGE}} = \frac{e^{\sum \lambda_j l_j}}{\text{Tr} e^{\sum \lambda_j l_j}}. \quad (2.18)$$
This generalized ensemble was christened the Generalized Gibbs Ensemble (GGE) in a paper by Rigol et al. [118]. The authors applied the GGE to explain the observation that the asymptotic (in time) state reached by an integrable model (they studied hard core bosons on a lattice) could not be described by a Gibbs ensemble. The insight that indeed integrable models possess a large number of additional conservation laws lead the authors to postulate the above ensemble of Jaynes and indeed they showed that this sufficiently described the asymptotic state.

The notion of conserved quantities in quantum systems is not as simple as it is in classical systems. In contrast to classical systems, every quantum system has as many conserved quantities (charges, currents) as the dimension of the Hilbert space. This can easily be seen by setting

\[ P_n = |n\rangle \langle n|. \tag{2.19} \]

\( P_n \) are projectors on to the \( n \)-th eigenstate and are conserved, since

\[ [H, P_n] = H|n\rangle \langle n| - |n\rangle \langle n|H = E_n |n\rangle \langle n| - |n\rangle \langle n|E_n = 0. \tag{2.20} \]

For a generic quantum system in the thermodynamic limit, the Hilbert space is infinite dimensional, and therefore every quantum system trivially has an infinite number of conserved charges, and if we include all of these in a GGE, then trivially we obtain a valid ensemble, but this does not make the description of the state any simpler, since it involves the full spectrum of the Hamiltonian. In other words, the description is moot.

Integrable models on the other hand possess several local conservation laws on the order of the size of the system and not the dimension of the Hilbert space. This is therefore a much more restrictive set of conservation laws, and it is believe that indeed these are the conserved quantities that must enter a GGE [118, 25]. Indeed this has been studied in several models, mostly integrable models that can be mapped to noninteracting models [25, 44, 62, 32, 33]. Recently some results have been obtained for interacting integrable models as well [45, 113].

The general consensus is that for translationally invariant initial states or in general for initial states that don’t break the symmetries of the post-quench Hamiltonian, a GGE description of the asymptotic state is valid, provided we are in a well-defined
thermodynamic limit. However, most of the results are for models that can be mapped to noninteracting models for initial states being ground states of these models. There are few results on quenches from inhomogeneous free bosons to a Tonks-Girardeau gas for example, and it is not entirely clear that in general the GGE holds true [91].

2.5 Open questions

As I began to discuss in the previous section, the last several years of studies about the nonequilibrium behavior of quantum systems has given us tremendous insight into the nature of relaxation, time evolution of correlations, the effects of interactions, and so on [93, 112, 26, 18, 17]. At the same time, it has opened up several new directions to explore and several new questions that need answers before we can gain a complete understanding.

2.5.1 Role of initial states

The time evolution of an isolated quantum system after a quench process is completely governed by the final Hamiltonian, and occurs unitarily without loss of energy. The set of eigenstates of the final Hamiltonian that participate in the time evolution is determined by overlaps with the initial state. In a lot of quench studies, one starts with the ground state of a particular Hamiltonian, and quenches the Hamiltonian to a new one, typically by changing a coupling constant or adding an external potential. This has the potential to mislead that the initial Hamiltonian has a role to play. However, the notion of an initial Hamiltonian is not well defined. In other words, the quench knows nothing about the initial Hamiltonian, only the initial state. One can therefore ask how the long time state after the quench depends on the initial state.

In Chapter 5 we show one instance where some features of the asymptotic state (in our case, a signature of fermionic correlations) are independent of the details, while the structure of the wave function retains memory of the entire initial state. In Appendix A, we show that the initial state determines whether the asymptotic state resembles free fermions or a bound state of bosons.
In general, one needs a good way to characterize initial states in order to be able to make general statements regarding their evolution after a quench. It is known that the symmetries of the initial state play an essential role (see discussion on independence of sign of interaction in Appendix A), but it is also known that there are other factors at play. See Refs. [121, 122, 65, 119] for recent work regarding “chaotic” initial states and the onset of chaos in quench dynamics.

This is a difficult question to answer in general partly because of the wide range of possible initial configurations, and no obvious way to classify them. Further, the theoretical tools at our disposal each have their limitations and this necessitates a piecewise picture reconstructed from several different approaches. This is largely an open question that is being intensively studied at present.

### 2.5.2 Role of integrability and its breaking

Integrable models have special properties that makes their quench dynamics different from generic models. The experiment described in Section 1.1 indicates the features expected from an integrable model, while it is almost certain that there exist three body losses and other nonintegrable interactions in the gas. The fact that it can still be well approximated by an integrable model seems to indicate that integrability is not broken appreciably. This, however, is not easy to quantify. Recent work [60] indicates that some mechanisms of breaking integrability\(^5\) retain most of the properties while others do not. It is an open question as to how this works in quantum integrable systems in general, and what effects a particular breaking mechanism will give rise to. In Chapter 5.5, we conjecture a flow in a parameter space of integrable and near-integrable models.

---

\(^5\)The so-called *soft* or *weak integrability breaking* breaks integrability by making the parameters of the integrable model weakly time dependent.
Chapter 3

The Bethe Ansatz

Hans Bethe, who won the Nobel prize in 1967 “for his contributions to the theory of nuclear reactions, especially his discoveries concerning the energy production in stars” [104] is popularly known as the person who taught us why stars shine. He spent a significant part of his career studying nuclear reactions and understanding the mechanism responsible for the energy produced in bright stars. Refs. [14, 12, 97] form the main body of this work. See also Ref. [15, 13].

In 1931, Bethe published a paper (Ref. [11]) that studied the problem of isotropic spins in a one dimensional lattice. By converting the problem to a first quantized version, he successfully obtained the eigenstates of the so-called Heisenberg model [66] analytically. This allowed the exact calculation of the spectrum of the model. The method was soon extended to the anisotropic XXZ model by Hulthén and Orbach [69, 109]. Over the next several years, his method, that came to be known as the Bethe Ansatz was successfully applied to exactly solve several quantum one dimensional problems and obtain the ground state and excitation spectra. These include the Kondo model and Gross-Neveu model [6], the Anderson impurity model [138], the massive Thirring model, and others [133, 136]. A huge breakthrough in this field was due to the work of C. N. Yang and C. P. Yang [139, 140]. They developed a technique to extend the Bethe Ansatz to models with internal symmetries like the Yang-Gaudin model. The method involves introducing an auxiliary set of rapidities, the so-called spin rapidities and setting up a second Bethe Ansatz. The solution of this gives the complete solution of the problem. This method was crucial to the solution of the Anderson and Kondo model.

The notion of solving a system exactly was not new. The famous Korteweg-de Vries
(KdV) equation, a nonlinear equation predicted excitations that propagated without decay. This and other such exactly solvable Hamiltonian systems lead to the development of the theory of classical integrability. There were different classes of partial differential equations that could be solved exactly and described interesting physical phenomena like solitons, propagating waves that did not dissipate. The solutions could be obtained via the inverse scattering approach, which was developed to understand properties of a scatterer based on scattering data. These differential equations being exactly integrable, the physical models that produced them were called integrable models. Ref [43] provides a detailed account of classical integrability and the inverse scattering approach.

The connection between classical integrability and the Bethe Ansatz, which was a method to exactly solve certain quantum models was made by the work of Faddeev, Sklyanin, and Takhtajan [86, 129]. The inverse scattering method was generalized to use quantum operators in place of the classical matrices (of the Lax representation) and the Bethe Ansatz could entirely be cast in this new language. This approach became known as the Algebraic Bethe Ansatz and provided a general approach to the Yang-Yang problem.

For our purposes, we will not require the sophistication of the Algebraic Bethe Ansatz. We will only be interested with the traditional Bethe Ansatz, and in particular, how we can obtain eigenstates. We will also require an understanding of the two-particle S-matrix and the appearance of complex eigenvalues.

### 3.1 Integrable models

As a working definition, we will call models that can be solved via the Bethe Ansatz integrable models. These models are indeed special, and there are several conditions that need to be met in order for a model to be integrable, which I discuss in the following section. Here, I am interested in some general properties, or implications of a model being integrable.
3.1.1 Conservation laws

All physical systems have some conservation laws, or conserved quantities. For example, Hamiltonian \(^1\) systems conserve energy. Conservation laws are usually related to symmetries of the Hamiltonian and the Hamilton equations of motion. Energy conservation is a result of time-translation invariance – the principle that there exists no absolute time. Conservation of momentum is a result of spatial translational invariance – the principle that the system looks the same from every point and there is no absolute position. Time-translation symmetry can be broken by terms in the Hamiltonian that explicitly depend on time, and spatial translational invariance can be broken in systems with hard boundaries for example. In such systems, momentum is not conserved. A particle can hit a wall and change direction. Angular momentum conservation comes from rotational symmetry. This relation between the symmetries of a physical system and conservation laws discovered by Emmy Noether [106] has been one of the most significant discoveries in theoretical physics. When we make the transition to quantum systems, this allows to represent translation, rotations, etc. via the action of unitary operators constructed from the conserved quantities. So, time translation by \( t \), or time evolution is generated by \( e^{-iHt} \) where \( H \) is the Hamiltonian operator, whose expectation value, the energy is conserved. Spatial translations are generated by the momentum operator, \( e^{-ipa} \) and so on.

Integrable systems, it turns out, have several additional conservation laws. Take for example the two body central force problem. It is a simple example of a classical integrable model, and indeed we have one additional conservation law corresponding to the \textit{Laplace-Runge-Lenz vector} [56] shown in Figure 3.1. These additional conservation laws naturally constrain the dynamics of the system. The classical-quantum analogs found in the elevation of Poisson brackets to commutators can be used to obtain the spectrum of the Hydrogen atom, the quantum analog of the two body central force problem, by assuming a conserved Runge-Lenz vector without recourse to the Schrödinger equation [111].

\(^1\)For time independent Hamiltonians.
Figure 3.1: The Laplace-Runge-Lenz vector, given by $A = p \times L - mk\hat{r}$ is an additional conserved quantity in the two-body central force problem. Reproduced from Ref. [35].

As mentioned in Section 2.4.2, all quantum systems have an infinite (i.e., dimension of Hilbert space) number of conserved quantities—the projectors onto the eigenstates. Integrable systems are different in that they have a nontrivial set of local conserved quantities. The number of these is linear in the system size (or number of particles) instead of exponential, not unlike classical integrable systems. These conserved charges can be explicitly obtained for some models like the Lieb-Liniger model and some spin chains, but in general, it is a difficult problem [86].

The extra conservation laws are expected to constrain the dynamics of integrable systems, and indeed this has been observed in phenomena that can well be described by a thermodynamic ensemble that incorporates these conservation laws. See Section 2.4.2 for more.

### 3.2 An overview of the Bethe Ansatz

The Bethe Ansatz is a method of obtaining the eigenstates of a quantum integrable model. It is an Ansatz in the sense that we assume a particular form for the multi-particle wave functions. In order for these to be eigenstates of the system, they have to satisfy certain properties. In our cases, we will only be interested in obtaining the eigenstates, and not the eigenvalues under a set of boundary conditions. The solution of the eigenvalue problem is in general complicated, and the existence of solutions is not always obvious. The completeness of the eigenstates is also not straightforward.
to establish in some cases. For our purposes, the Bethe Ansatz provides us with the
eigenstates, and we will not impose any quantization conditions (usually in the form
of (anti)periodic boundary conditions) on the eigenvalues, as is typically done to find
the ground state. We will take up completeness when we discuss the Yudson represen-
tation in Chapter 4. For a detailed tutorial on the Bethe Ansatz and several examples,
see Refs. [81, 79, 80]. For the thermodynamics of integrable systems, see Ref. [132]. In
this section, we outline the method, and in the next section, we provide some exam-
ple.

One can approach the problem of finding the eigenstates using the Bethe Ansatz
algorithmically. It consists of the following steps:

1. Write the Hamiltonian in a first quantized form, and find the one-particle eigen-
states. These are necessarily noninteracting.

2. Assume an $S$-matrix based form for the two particle wave functions—we’ll see
what this means shortly.

3. Formulate the Bethe Ansatz, i.e., write down multiparticle eigenstates using the
$S$-matrix language, and prove the Yang-Baxter relations for the $S$-matrix. There
are some extra conditions which need to be met (existence of solutions, etc.), but
it will not be required for our purposes.

Already, it is clear that we are assuming that the Fock space of the problem can be
divided into Hilbert spaces with fixed particle number. This is not true of several inter-
esting models. For instance, the QED vertex coupling electrons to photons involves
a particle production process, and the $\phi^4$ field theory allows a $1 \rightarrow 3$ process. In these
systems, particle number is not a good quantum number since it isn’t conserved, and
consequently we cannot use the Bethe Ansatz to solve these models exactly.

The method is as follows. Given a second quantized Hamiltonian $H$, we can split
it as

$$H = H_0 + H_1$$

where $H_0$ is the noninteracting (Gaussian) part, and the interactions are in $H_1$. The
field operators in the Hamiltonian, and their internal symmetries describe the Fock
space of the problem. We first obtain the single particle solutions to the problem, i.e., what is the eigenstate corresponding to a single field excitation above the vacuum. Denoting the vacuum by \( |0\rangle \), and a generic single-particle creation operator by \( \alpha^\dagger(x) \), we are interested in the solutions of

\[
H \int_x f(\lambda, x) \alpha^\dagger(x) |0\rangle = E(\lambda) \int_x f(\lambda, x) \alpha^\dagger(x) |0\rangle.
\]

Once we obtain \( f(\lambda, x) \), the solution to the \( N \)-particle noninteracting problem can be trivially written as

\[
|\vec{\lambda}\rangle = \int \prod_j f(\Lambda_j, x_j) \alpha^\dagger(x_j) |0\rangle.
\]

In order to account for the interactions, we start with a two particle state which we write in general as

\[
|\lambda_1, \lambda_2\rangle = \int_{x_1, x_2} f(\lambda_1, x_1) f(\lambda_2, x_2) Z(\lambda_1, x_1; \lambda_2, x_2) \alpha^\dagger(x_1) \alpha^\dagger(x_2) |0\rangle.
\]

For the noninteracting problem, \( Z = 1 \). For the models we are interested in, we will only encounter short range interactions. Let us assume that \( H_I \) is such that for \( x_1 \neq x_2 \), the particles are noninteracting, and they are only aware of each other when they meet. In this situation, it makes sense to define two sectors, \( x_1 > x_2 \) and \( x_1 < x_2 \). We then assume that

\[
Z(\lambda_1, x_1; \lambda_2, x_2) = A(\lambda_1, \lambda_2) \theta(x_1 - x_2) + B(\lambda_1, \lambda_2) \theta(x_2 - x_1).
\]

If we can satisfy the Schrödinger equation with this ansatz, then we have a two particle eigenstate. I reiterate that this ansatz works only for point interactions\(^2\). The ratio \( B/A \) becomes the two particle \( S \)-matrix.

Once the two particle state is established, the ansatz involves writing down the multiparticle states in terms of the two particle \( S \)-matrices,

\[
|\vec{\lambda}\rangle = \int \prod_{i<j} Z_{ij} \prod_j f(\Lambda_j, x_j) \alpha^\dagger(x_j) |0\rangle
\]

\(^2\)For models with long range interactions, different prescriptions are required. Examples of these are the Calogero-Sutherland model, the Haldane-Shastry model and the \( 1/t \)-Hubbard model. See Ref. [131] for more
where we've used the shorthand \( Z_{ij} \) to denote \( Z(\lambda_i, x_i; \lambda_j, x_j) \). A general term in this expression involves the product of several \( S \)-matrix factors depending on which sector it belongs to. Obviously, there is an ambiguity in the ordering of the product, and the \( S \)-matrix factors don't necessarily commute\(^3\). In order for these products to be consistent, they must satisfy what is known as the Yang-Baxter equation [132], which establishes the equivalence of the multiple ways of factorizing a multiparticle interaction into a product of two particle interactions.

![Diagram](image)

**Figure 3.2:** Factoring a three particle interaction. If the interaction is factorizable as products of two particle interactions, then the multiple ways of doing so must be equivalent.

Consider the example shown in Figures 3.3 and 3.2. There are two ways to go from 123 to 321, and they should be equivalent. That is equivalent to saying

\[
S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}. \tag{3.7}
\]

It is sufficient to prove the three particle condition for it to be valid in general.

With this, we have the multiparticle eigenstates. Typically, the next step is to impose some quantization conditions on these states and obtain a spectrum, the ground state, excitations and so on. Since we will be interested in time evolution on the infinite line, this step is not required. This has an important consequence which will be discussed in Chapter 4.

\(^3\)Specifically, when internal symmetries are present, the \( S \)-matrices could contain operators that generate the symmetries
3.3 Examples

In this section, I will obtain the Bethe Ansatz eigenstates of two models whose quench dynamics will be studied in Chapters 4 and 5. These derivations can also be found in several text books on the subject (see e.g., Refs. [132, 86, 131]).

3.3.1 Resonant Level Model

The Resonant Level Model (RLM) describes a bath of non-interacting spinless electrons coupled to a quantum dot, which serves as a local two-level (occupied or unoccupied) impurity. Figure 3.4 shows a schematic of this model. Experimentally, such a system can be constructed by etching leads on a substrate and using a tunneling coupling to a quantum dot, which is a highly confined two-dimensional electron gas. The strong confinement causes a large gap between energy levels, and therefore it can be considered as a two-level system. Figure 3.5 shows a scanning electron image of such a device.

We start with a model of free spinless electrons attached to a quantum dot, which
we model as a local two level (occupied or unoccupied) atom

\[ H = \sum_k \epsilon_k c_k^\dagger c_k + V_k (c_k^\dagger d + d^\dagger c_k) + \epsilon_d d^\dagger d. \]  
(3.8)

The “dot” has no internal degrees of freedom and is described by a canonical fermionic field \( d \), \( \{d^\dagger, d\} = 1 \). Experimentally, the dot is a highly confined 2d electron gas. The strong spatial confinement creates a large gap between neighboring energy levels and we can essentially consider it as a two level system with an occupied and an unoccupied state having energy \( \epsilon_d \). The lead electrons are non-interacting and spinless, and are also described by canonical fermionic fields \( c_k^\dagger, c_{k'} \). For electrons on a lattice, the dispersion relation is \( \epsilon_k \sim \cos k \) and for nonrelativistic electrons on in the
continuum, $\epsilon_k \sim k^2$ as shown in Fig. 3.6. We consider a filled Fermi sea for the leads

![Figure 3.6: Possible dispersion relations for the lead fermions. The horizontal line indicates the Fermi level. Near the Fermi level, the dispersion is linear.](image)

and are only interested in modeling the low energy behavior. This corresponds to fluctuations near the Fermi surface. All the scales in the problem are considered much smaller than the bandwidth. Further, for a dot with no internal structure, we can expand the electron wave functions in a partial wave expansion and preserve only the $s$-wave or zero angular momentum component. In one-dimension the Fermi-surface is disconnected. The small fluctuations around the Fermi-surface therefore do not mix.

For Fermi energy $\epsilon_F$, we can expand the energy of the electron as

$$
\epsilon_k = \frac{k^2}{2m} - \epsilon_F \approx v_F (k - k_F)
$$

where we choose one of the two points on the Fermi surface, and $v_F = k_F/m$ is the Fermi velocity. Calling the $s$-wave component of the expansion $\psi^\dagger(k)$, we have

$$
\psi^\dagger_{L,R} = \int_{-\Lambda}^{\Lambda} e^{\pm i(k-k_F)r} \psi_0(k) = \int_{-\Lambda}^{\Lambda} e^{\pm ikr} \psi_0(k + k_F)
$$

where $|k - k_F| \ll 2\Lambda$, $\Lambda$ being twice the bandwidth, or the momentum cutoff. We now have right moving waves $\psi_R \sim e^{ikr}$ and left moving waves $\psi_L \sim e^{-ikr}$, the physical picture is one of right moving incoming waves hitting the impurity at the origin and getting reflected into left moving waves, and the Hamiltonian, described on a half
space, is given by

\[ H = -i \int_{-\infty}^{0} dx \left[ \psi_R^\dagger(x) \partial_x \psi_R(x) - \psi_L^\dagger(x) \partial_x \psi_L(x) \right] + H_{\text{dot}} + H_{\text{tunneling}} \quad (3.11) \]

We can “unfold” the lead and obtain a Hamiltonian described on the full space with only right moving electrons. The transformation takes the form

\[ \psi_L^\dagger(x < 0) = \psi_R^\dagger(x > 0). \quad (3.12) \]

This is schematically shown in Figure 3.7. At the end of this procedure, we obtain a Hamiltonian with a linear spectrum (only excitations near the Fermi energy) with right moving chiral fermions (unfolded leads with only s-wave scattering). Putting back the tunneling and dot Hamiltonians, we get the Hamiltonian of the Resonant Level Model.

\[ H_{\text{RLM}} = -i \int dx \psi^\dagger(x) \partial_x \psi(x) + t [\psi^\dagger(0) d + d^\dagger \psi(0)] + \epsilon_d d^\dagger d. \quad (3.13) \]

We will now use the Bethe Ansatz to obtain the eigenstates of this model. Note that the RLM is a Gaussian model and we can obtain the full propagator or Green functions exactly by using Dyson equations for the dressed propagator. However, it is instructive to use the Bethe Ansatz as we will see the “scattering” form of the wave function. In Section 3.3.2, we will obtain the eigenstates of an interacting model.

We will first find the single particle eigenstates. The Hilbert space of the one particle case has two basis vectors \( \psi^\dagger(x) \) and \( d^\dagger \). We expect the one particle eigenstate to be a linear combination of these. We can assume the form

\[ |k\rangle = \left[ \int dx f_k(x) \psi^\dagger(x) + g_k d^\dagger \right] |0\rangle. \quad (3.14) \]

In the regions \( x < 0 \) and \( x > 0 \), the Schrödinger equation for the right moving (recall that we only have right movers) lead fermions has plane-wave solutions. So we
expect $f_k(x) \sim e^{ikx}$. However, we do expect the solutions to differ for the $x < 0$ and $x > 0$ regions since hopping through the dot will potentially modify the wave function by a phase. So we will assume the following scattering form for the wave function:

$$f_k(x) = e^{ikx}[A\theta(-x) + B\theta(x)].$$ \hspace{1cm} (3.15)

Acting on our guess with the Hamiltonian, we get

$$H|k\rangle = E_k|k\rangle = \int dx \left\{ -i\partial_x e^{ikx}[A\theta(-x) + B\theta(x)] + t g_k \delta(x) \right\} \psi^\dagger(x)|0\rangle + \left[t f_k(0) + \epsilon_d g_k\right] d^\dagger|0\rangle. \hspace{1cm} (3.16)$$

Before we can solve this equation, we need to establish a regularization convention for the $\theta$-functions. We define

$$\partial_x \theta(\pm x) = \pm \delta(x), \hspace{1cm} \theta(x)\delta(x) = \frac{1}{2}\delta(x). \hspace{1cm} (3.17)$$

This regularization can be shown to be consistent with other methods of solution of the problem, and with any continuous approximations to step functions and $\delta$-functions. With this, the solution of (3.16) is given by

$$E_k = k, \hspace{1cm} \frac{B}{A} = \frac{k - \epsilon_d + \frac{i\epsilon_d}{2}}{k - \epsilon_d - \frac{i\epsilon_d}{2}}, \hspace{1cm} g_k = \frac{t}{k - \epsilon_d}. \hspace{1cm} (3.18)$$

Given that the model is non-interacting, we can immediately write down the multi-particle states as constructed from a product of single particle states.

$$|\bar{k}\rangle = \prod_j |k_j\rangle \hspace{1cm} (3.19)$$

In Section 4.3, we carry out a quench where we couple the lead electrons to the dot at $t = 0$ and calculate the time evolution of the dot occupation and transient current as an example of the Yudson representation.

### 3.3.2 Lieb-Liniger model

The previous model was non-interacting and the $S$-matrix is trivial, i.e., $S = 1$. It doesn’t need the Bethe Ansatz and can be treated by any other means, e.g., the full
Green’s functions of the model can easily be obtained, therefore giving access to all correlation functions and observables. We mention it here since it serves as a simple example of the Bethe Ansatz method, and more importantly, in Chapter 4, as a simple example of the Yudson technique.

We now move on to a more complicated interacting integrable model of bosons in a one dimensional continuum. A brief comment is in order here. The reader might be aware of several integrable models whose solutions come in the form of using a transformed set of fields (by Jordan-Wigner transformation, Bogoliubov rotations, etc.), which become the quasi-particles of the model. In the quasi-particle basis, the model is non-interacting, and therefore trivially solvable. Examples of this include the Tonks-Girardeau model [53], the transverse field and longitudinal field Ising models [25], hard-core bosons on a lattice [32, 33], and the Luttinger liquid [50]. This of course does not mean that the models are not-interesting or trivial. Especially in the case of quench dynamics, when the initial state is not best represented in the quasi-particle basis. We will not be discussing such models in this thesis.

The Lieb-Liniger model cannot be written in terms of non-interacting quasi particles. In this section, we will derive the multi-particle eigenstates of this model. We will not be interested in the ground state and excitation spectrum of the model. For more on this, please see Refs. [94, 132, 86].

The model describes bosons on a line interacting via a \( \delta \)-function potential:

\[
H_{\text{LL}} = \int \mathcal{d}x \left[ \frac{1}{2} \partial_x b^\dagger(x) \partial_x b(x) + c [b^\dagger(x) b(x)]^2 \right].
\] (3.20)

The mass of the bosons has been set to \( \frac{1}{2} \). For the \( c > 0 \), the potential is repulsive, and for \( c < 0 \), it is attractive. The following applies to either case. The Shrödinger equation that one obtains for the above Hamiltonian is

\[
[-\partial_x^2 + 2c b^\dagger(x) b(x)] b^\dagger(x) = \partial_t b^\dagger(x).
\] (3.21)

\(^4\)By non-interacting, we mean that the quasi-particles must have canonical commutation relations. It is always possible to write the eigenstates in terms of new creation operators (or fields) that have non-standard commutation relations that are pseudo-momentum dependent. This is precisely the role of the \( S \)-matrix.
This is called the *Non-linear Schrödinger equation* and has been studied extensively both classically and quantum mechanically [43, 86].

Let us begin by assuming a form for the single particle wave functions:

\[ |k⟩ = \mathcal{N}(k) \int dx \, e^{ikx} b^+(x)|0⟩ \tag{3.22} \]

where we have defined $|0⟩$ as the vacuum containing no particles, and $\mathcal{N}(k)$ is a normalization factor. Acting on this state with the Hamiltonian (3.20), we get

\[ H_{LL} |k⟩ = -\partial_x^2 \mathcal{N}(k) e^{ikx} = E(k) \mathcal{N}(k) e^{ikx} \tag{3.23} \]

giving the stationary solution to the Schrödinger equation with energy $E(k) = k^2$. We now assume a "Bethe form" for the two-particle wave-functions:

\[ |k_1, k_2⟩ = \mathcal{N}(k_1, k_2) \int dx_1 dx_2 \, e^{ik_1 x_1 + ik_2 x_2} [\theta(x_1 - x_2) + S_{12} \theta(x_2 - x_1)] b^+(x_1) b^+(x_2)|0⟩. \tag{3.24} \]

Note that as written, the state is not manifestly symmetric in $x_1, x_2$, as it should be for bosonic fields. We will deal with this momentarily. The important point to note is the form of the wave function. Since the interaction is active only when the two particles are on top of each other, in each region $x_1 > x_2$ and $x_1 < x_2$, there is no interaction and the Schrödinger equation is solved by plane waves. However, the interaction is active when the two particle cross – this is captured via the $S$-matrix $S_{12}$ that we need to calculate. This equivalently amounts to a boundary condition when $x_1 = x_2$, and is commonly known as the *cusp condition*. This approach can be found in Ref. [86]. The Schrödinger equation for the above state yields

\[ \int dx_1 dx_2 \left[ (-\partial_{x_1}^2 - \partial_{x_2}^2) \left\{ e^{ik_1 x_1 + ik_2 x_2} [\theta(x_1 - x_2) + S_{12} \theta(x_2 - x_1)] \right\} + c(1 - S_{12}) \delta(x_1 - x_2) e^{ik_1 x_1 + ik_2 x_2} b^+(x_1) b^+(x_2)|0⟩ \right] \]

\[ = E(k_1, k_2) \int dx_1 dx_2 \, e^{ik_1 x_1 + ik_2 x_2} [\theta(x_1 - x_2) + S_{12} \theta(x_2 - x_1)] b^+(x_1) b^+(x_2)|0⟩ \tag{3.25} \]

Simplifying this using the regularization in (3.17) gives

\[ -i(k_1 - k_2)(1 - S_{12}) + c(1 + S_{12}) = 0, \tag{3.26} \]
which gives

\[ S_{12} = \frac{k_1 - k_2 + ic}{k_1 - k_2 - ic} \quad (3.27) \]

The energy \( E(k_1, k_2) = k_1^2 + k_2^2 \).

The next step in the process is to write down the multiparticle states. These are written in general as follows

\[
|\vec{k}\rangle = \mathcal{N}(\vec{k}) \int d\vec{x} \sum Q \theta(x_Q) A_Q(\vec{\lambda}) \prod_j e^{ik_j x_j} b^\dagger(x_j) |0\rangle.
\]

where \( A_Q \) is the phase acquired in going from the ordering \( x_1, x_2, \ldots, x_N \) to the ordering \( x_{Q1}, x_{Q2}, \ldots, x_{QN} \). It is essentially a product of several two particle \( S \)-matrix factors. In this model, since the \( S \)-matrices are \( c \)-numbers, this condition is trivially satisfied. It is sufficient to satisfy the three particle condition in order for the multi-particle condition to be satisfied. This is due to the factorizability of all interactions into products of two particle interactions, a requirement for the integrability of this model. In other words, the number of particles \( N = \int_x b^\dagger(x)b(x) \), has to be a conserved quantum number in order to be able to write down eigenstates with fixed particle number. In Chapter 5 we write this state in a slightly different form, more conducive for application to the Yudson formalism.

The above derivation is valid for both the attractive \((c < 0)\) and the repulsive \((c > 0)\) model since at point we assumed a sign for \( c \). The sign is however crucial and determines both the nature of the ground state and excitations, as well as the quench dynamics as we will see in Chapter 5. For \( c > 0 \), the spectrum consists of real valued momenta and the ground state is essentially a Fermi sphere, with the particles obeying an exclusion principle [94]. The attractive case, however is more interesting as the spectrum permits momenta in complex conjugate pairs. These correspond to bound state, or McGuire clusters [94, 139, 99], and we will see these emerge naturally in the time evolution.

### 3.4 Bethe Ansatz and quenches

To carry out the computation of the quench dynamics we need to know the eigenstates of the propagating Hamiltonian. The Bethe Ansatz approach is helpful in this respect.
as it provides us with the eigenstates of a large class of interacting one dimensional Hamiltonians. Many of the Hamiltonians that can be thus be solved are of fundamental importance in condensed matter physics and have been proposed to study various experimental situations. A partial list includes the Heisenberg chain (magnetism), the Hubbard model (strong-correlations), the Lieb-Liniger model (cold atoms in optical traps), the Kondo model, and the Anderson model (impurities in metals, quantum dots) [11, 96, 94, 6, 7, 136]. For a Hamiltonian to possess eigenstates that are given in the form of a Bethe Ansatz it must have the property that multi-particle interactions can be consistently factorized into series of two particle interactions, all of them being equivalent\(^5\).

While originally formulated to understand the Heisenberg chain [11], the technology has been studied extensively and recast into a more sophisticated algebraic formulation [129, 86]. The usual focus of the Bethe Ansatz approach has been on the thermodynamic properties of the system: determining the spectrum, the free energy, and susceptibilities. Also, considerable efforts were made to compute correlation functions [133, 86]. In particular, the Algebraic Bethe Ansatz has been used in conjunction with numerical methods to calculate so-called form factors which allows access to the dynamical structure functions [103, 24, 61].

Using the Bethe Ansatz to extract dynamics one encounters a triply complicated problem — the first being to obtain the full spectrum of the Hamiltonian, the second to calculate overlaps, and the third to carry out the sum, which in some cases involves sums over large sets of different “configurations” of states. The overlaps are particularly difficult to evaluate due to the complicated nature of the the Bethe eigenstates and their normalization. The problem is more pronounced in the far from equilibrium case of a quench when the state we start with suddenly finds itself far away from the eigenstates of the new Hamiltonian [see eq. (2.11)], and all the eigenstates have non-trivial weights in the time-evolution. In all but the simplest cases, the problem is non-perturbative and the existing analytical techniques are not suited for a direct

\(^5\)This set of conditions is known as the Yang-Baxter equation.
application to such a situation.

Often quench calculations are carried out in finite volume and the infinite volume limit is taken at the end of the calculation. This may be necessary for thermodynamic calculations, as noted, but not for quench dynamics (see sec. 2.3). We shall instead carry out the quench directly in the infinite volume limit in which case the Yudson representation allows us to carry out the calculations in an efficient way, doing away with some of the difficulties mentioned above by not requiring any information about the spectrum, and by using integration as opposed to discrete summation.

More explicitly, the Schrodinger equation for $N$ bosons $H\langle \vec{\lambda} \rangle = \epsilon(\vec{\lambda})\langle \vec{\lambda} \rangle$ is satisfied for any value of the momenta $\{\lambda_j, j = 1, \cdots, N\}$ if no boundary conditions are imposed. The initial state can then be written as $|\Psi_0\rangle = \int d\lambda_1 \cdots d\lambda_N C_{\vec{\lambda}} |\vec{\lambda}\rangle$, with the integration over $\vec{\lambda}$ replacing the summation. This is equivalent to summing over an over-complete basis, the relevant elements in the sum being automatically picked up by the overlap with the physical initial state.

It is important to note, however, that while the spectrum of an infinite system is continuous, it can still be very complicated. This is indeed the case with several integrable models, including the Lieb-Liniger model, where the analytic structure of the $S$-matrix (the momentum dependent phase picked up when two particles cross) may permit momenta in the form of complex-conjugate pairs signifying bound states. In the formalism we employ, such states are taken into account by appropriately choosing the contours of integration in the complex plane. We have now developed the base to go on and understand this formalism.
Chapter 4
The Yudson representation

4.1 A complete basis in infinite volume

As seen in Section 3.4, time evolving arbitrary initial states is complicated even in the case of an integrable model where we know the eigenstates. Our goal is to try and circumvent some of the issues that make this process complicated. In view of the discussion in Section 2.3, it is meaningful to talk about quenches directly in the infinite volume limit. This leads to one simplification that there are no boundary conditions, and consequently, no Bethe Ansatz Equations to solve. In other words, the spectrum is not quantized, and it makes sense to talk about eigenstates of the Hamiltonian for any values of the momentum\(^1\).

Our goal is to find a representation for arbitrary states in terms of a complete basis directly in the infinite volume limit. For a generic state \(|\Psi_0\rangle\), we expect to write down something like

\[
|\Psi_0\rangle = \int \mathcal{D}\lambda \; |\lambda\rangle \langle \lambda | \Psi_0\rangle, \tag{4.1}
\]

where \(\mathcal{D}\lambda\) denotes both the integration measure and the domain. We have replaced the sum over eigenstates by an integral since the eigenvalues are no longer constrained by boundary conditions or finite size\(^2\).

As an example, let’s see how this works in a model of free fermions on an infinite line. The Hamiltonian is

\[
H_{\text{FF}} = \int_k \epsilon_k c_k^\dagger c_k \tag{4.2}
\]

\(^1\)More correctly, the rapidity, which coincides with the physical momentum for some models.

\(^2\)In the thermodynamic limit of an infinite number of particles, even with periodic boundary conditions, the discrete sum becomes an integral over an appropriately defined weighting function [140]. Here, we are referring to a finite number of particles and infinite volume, and so this is a different situation. We can write an integral since the momenta are not quantized by boundary conditions.
and the eigenstates are

$$|k⟩ = \prod_j c_j^† |0⟩ = \int \prod_j dy_j e^{ik_j y_j} c_j^† (y_j) |0⟩. \quad (4.3)$$

If we write a generic initial state as $|\Psi_0⟩ = \prod_j c_j^† (x_j) |0⟩$, we see that

$$\frac{1}{N!} \int \left[ \prod_j \frac{dk_j}{2\pi} \right] |k⟩ ⟨k|Ψ_0⟩$$

$$= \frac{1}{N!} \sum_P (-1)^P \int_{y_j}^{∞} \prod_j \delta(y_j - x_{Pj}) e^{ik_j (x_j - x_{Pj})} c_j^† (y_j) |0⟩$$

$$= \frac{1}{N!} \sum_P (-1)^P \int_{y_j}^{∞} \prod_j e^{ik_j (y_j - x_{Pj})} c_j^† (y_j) |0⟩$$

$$= \frac{1}{N!} \sum_P (-1)^P \prod_j e^{ik_j (x_{Pj})} |0⟩$$

$$= |Ψ_0⟩ \quad (4.4)$$

On the lattice, the integral over $k$ will be restricted to the first Brillouin Zone, $k ∈ [-π, π]$, and the Dirac delta function above gets replaced by a Kronecker delta function. While the free fermion case might seem rather trivial, we will see that more complicated models can also be dealt with in this fashion.

The central aspect of the above formula is determining the measure of integration and the domain, or integration paths. While it might seem that since the momenta are not quantized, we can integrate over the entire allowed range of momenta, a complication arises for integrable models. Recall that for many integrable models, when we solve for the spectrum (or momentum eigenvalues) of a finite sized model, we end up with solutions in the form of complex conjugate pairs. These are the so-called string solutions, and in the Lieb-Liniger model, e.g., correspond to bound states (more on this in Chapter 5). In order for (4.1) to be a valid representation, these complex values must be taken into account. The integration paths must therefore be allowed to explore the complex plane, in a controlled fashion. The key, as we will see, is to exploit the poles of the $S$-matrix and choose contours such that they give zero residues when integrated.
4.2 The Yudson integral representation

The integral representation described here was originally developed by Yudson [142, 143] to understand superradiance in the Dicke model [40], when he noticed that the sum over the eigenstates fell into a certain pattern that could be obtained by calculating residues at appropriate poles [141]. These poles are dictated by the two-particle $S$-matrix (see Section 3.2) of the Dicke model, which is integrable.

Depending on the particular model, certain changes have to be made to the representation, in particular, the integration measure, and the paths or “contours” of integration. Here, I describe some generalities of the representation, and then we’ll see some examples.

First, in order to simplify the overlap of a Bethe eigenstate and the initial state, we write the initial state in an ordered way as

$$|\Psi_0\rangle = \theta(x_1 > x_2 > \cdots > x_N) \prod_j c^\dagger(x_j) |0\rangle. \quad (4.5)$$

This can be done in general:

$$\prod_j c^\dagger(x_j) = \sum_P \theta(x_{P_1} > x_{P_2} > \cdots > x_{P_N}) \prod_j c^\dagger(x_j)$$

$$= \sum_P \theta(x_1 > x_2 > \cdots > x_N) \prod_j c^\dagger(x_{P_j}) \quad (4.6)$$

$$= \sum_P \zeta^P \theta(x_1 > x_2 > \cdots > x_N) \prod_j c^\dagger(x_j)$$

with $\zeta = \pm$ depending of whether the particles are bosons or fermions.

Further, we modify (4.1) to

$$|\Psi_0\rangle = \int D\lambda \, |\lambda\rangle \langle\lambda| \Psi_0\rangle \quad (4.7)$$

where $|\lambda\rangle$ corresponds to a modified eigenstate, where again, we order the particles. That is, if the eigenstate of the model were given by

$$|\lambda\rangle = \int_g \phi(y, \lambda) \prod_j c^\dagger(y_j) |0\rangle, \quad (4.8)$$

then,

$$|\lambda\rangle = \int_g \theta(y_1 > y_2 > \cdots > y_N) \phi(y, \lambda) \prod_j c^\dagger(y_j) |0\rangle. \quad (4.9)$$
In the above case of free fermions, the main simplification this achieves is to remove the sum over permutations in (4.4). However, in more complex interacting models, this step achieves a greater simplification, by eliminating the $S$-matrix in $|\lambda\rangle$.

The next important step is determining the integration measure, and a set of contours of integration. Not explicitly visible in the representation is the fact that one has to write the Bethe Ansatz eigenstates in an appropriate parametrization. The reason for this is that the choice of contours depends on the analytic structure of the $S$-matrix. As mentioned in Chapter 3, the analytic structure of the $S$-matrix determines whether a particular model admits complex valued momenta, or bound states. We will see that the Yudson representation will automatically tell us this. However, in order to write down a useful Yudson representation, the choice of contours has to be reasonable and that puts some constraints on the pole-structure of the $S$-matrix. Parametrizing for a simple pole structure typically complicates the single particle states, however\(^3\). Each model will therefore require a separate proof of the validity of (4.7), but below I will outline the general idea.

As discussed in Section 3.2, Bethe Ansatz eigenstates have a typical form, where the wave-functions are separated into different configuration sectors, and the sectors are related to each other via phases. These phases are products of two particle $S$-matrices that correspond to the phase acquired when two particles cross each other. The fact that any two sectors are related via such a product is due to the factorizability of the interactions and is a hallmark of Bethe Ansatz integrable models.

Writing a generic eigenstate as

$$|\lambda\rangle = \int_y \sum_Q \theta(y_Q) A_Q(\{\lambda_j\}) \prod_j \phi(y_j, \lambda_j) c^\dagger(y_j)|0\rangle$$

(4.10)

where $c^\dagger(y_j)$ are either bosonic or fermionic\(^4\). We set $A_I = 1$, so that

$$|\lambda\rangle = \int_y \theta(y_I) \prod_j \phi(y_j, \lambda_j) c^\dagger(y_j)|0\rangle$$

(4.11)

\(^3\)See p.523-527 of Ref. [136] for the parametrizations that provide a simple $S$-matrix pole structure for various models.

\(^4\)For impurity models, or models with multiple species, the single particle creation operators are more complicated, but the method is straightforward to generalize. We study an impurity model in detail in the next section.
If the initial state is $|\Psi_0\rangle$, then we have to find a set of contours for $\lambda$ such that (4.7) is satisfied, i.e., we have to show that

$$\int D\lambda \sum_Q \theta(y_Q) A_Q(\{\lambda_j\}) \prod_j \phi(y_j, \lambda_j) \phi^t(x_j, \lambda_j) = \prod_j \delta(y_j - x_j). \quad (4.12)$$

This is assuming that we are already in a parametrization that makes the poles of the $S$-matrix simple enough to treat. In order to see explicitly how this works, we have to chose a specific model. The next section shows a simple example and Chapter 5 discusses in detail a fully interacting model, and contains the main results of this dissertation.

### 4.3 The Resonant Level model - an example

The RLM was introduced in Section 3.3.1. As we saw there, the RLM being noninteracting, it can be trivially diagonalized, though we used it as a simple example of Bethe Ansatz states. Here, we choose to use the Yudson representation to obtain the time evolution of a given initial state. In particular, we calculate the time evolution of the dot occupation after a quench involving coupling a free 1D fermi sea (linearized to right moving chiral fermions) to the dot, schematically shown in Figure 4.1.

![Figure 4.1: Schematic of the quench process. The figure shows the fermi sea in the lead being coupled to the quantum impurity at $t = 0$.](image)

Let us begin with the linearized RLM Hamiltonian.

$$H_{RLM} = -i \int dx \frac{\partial \psi^t}{\partial x} \frac{\partial \psi}{\partial x} + \sqrt{2\Gamma} (\psi^t(0) d + d^t \psi(0)) + \epsilon_d d^t d. \quad (4.13)$$

We would like to start with the dot lead coupling $\Gamma = 0$. We can have an eigenstate of
$N$ right moving electrons in the lead:

$$|k\rangle = \int d^N x \prod_j e^{ik_j x_j} \psi^\dagger (x_j) |0\rangle$$

(4.14)

One can check that the above is indeed an eigenstate of the uncoupled lead for any set \{\(k_j\)\} with \(E = \sum_j k_j\).

At \(t = 0\), we couple the lead to the dot. The above state then becomes the initial state that we want to evolve in time. Before we move on to time-evolve, let’s rewrite the initial state in a way that’s in line with the form of the Yudson representation. Inserting unity as a sum over different orderings of the \(x_j\), and some manipulation, we obtain

$$|k\rangle = \int d^N x \sum_{\{P\}} \text{sgn}(P) \prod_j e^{ik_j x_j} |\vec{x}\rangle,$$

(4.15)

where

$$|x\rangle = \theta(x_1 > x_2 > \cdots > x_N) \prod_j \psi^\dagger (x_j) |0\rangle.$$ 

(4.16)

The state \(|x\rangle\) is now in a form that can be plugged into the Yudson representation, which we will now proceed to obtain.

### 4.3.1 Yudson representation for the RLM

The RLM being a noninteracting model, the Yudson representation is in some sense trivial. However, the approach of directly evolving an initial state gives us complete control over the process, rather than having to start with a ground state. In particular, it gives us the ability to start with states with different symmetries, entanglement, and so on. Since we are using the RLM as an example of time evolution using the Yudson representation, we will study the simple case described above.

Consider the initial state \(|x\rangle\) described above. It is straightforward to check that

$$|x\rangle = \frac{1}{(2\pi)^N} \int d^N y \int d^N k \prod_j e^{ik_j (y_j - x_j)} \frac{k_j - \epsilon_d - i\Gamma \text{sgn}(y_j)}{k_j - \epsilon_d + i\Gamma} \frac{k_j - \epsilon_d + i\Gamma \text{sgn}(x_j)}{k_j - \epsilon_d - i\Gamma} \psi^\dagger (y_j) |0\rangle.$$ 

(4.17)

For any \(x_j < 0\) there is a pole at \(k_j^* = \epsilon_d - i\Gamma\) for \(y_j > 0\). Since \(y_j - x_j > 0\), we can carry out the integration by closing the contour in the upper half plane giving us a residue
of zero, since there is no pole in this region. For \( y_j < 0 \) all the poles are canceled, and we can carry out the integration to obtain \( 2\pi\delta(y_j - x_j) \). For \( x_j > 0 \), the pole is at \( k_j^* = \epsilon_d + i\Gamma \) for \( y_j < 0 \). Again these two constraints imply that \( y_j < x_j \), and closing the contour in the lower half plane give a zero residue. For \( y_j > 0 \), there are no poles, and we can carry out the integral to obtain \( 2\pi\delta(y_j - x_j) \). We have therefore shown that the above identity is indeed true. We have ignored the “dot” part of the eigenstate, but it is easy to show that it produces no contribution.

The lead part of state at time \( t \) is therefore given by

\[
|x, t\rangle_l = e^{-iH_{RLM}t}|x\rangle
\]

\[
= \frac{1}{(2\pi)^N} \int d^Ny \int d^Nk \prod_j e^{ik_j(y_j-x_j-t)} \frac{k_j - \epsilon_d - i\Gamma \text{sgn}(y_j)}{k_j - \epsilon_d + i\Gamma} \psi^\dagger(y_j)|0\rangle.
\]

Using arguments above, we can show that the above integral is nonzero for \( y_j < x_j + t \), which we interpret as a causality constraint. In this region, the integration can easily be carried out via the residue theorem, and is provided below.

### 4.3.2 Time evolution of the dot occupation

The state at arbitrary time \( t \) is given by

\[
|x, t\rangle = \theta(x_1 > \cdots > x_N) \int d^Ny \prod_{j=1}^N(F(y_j, x_j, t)\psi^\dagger(y_j) + G(x_j, t)\delta(y_j)d^\dagger) |0\rangle,
\]

where

\[
F(y, x, t) = \delta(y - x - t) - 2\Gamma\theta(x < 0 < y < x + t)e^{(\Gamma + i\epsilon_d)(y - x - t)},
\]

\[
G(x, t) = -i\sqrt{2\Gamma}\theta(x < 0 < x + t)e^{-(\Gamma + i\epsilon_d)(x + t)}.
\]

Now we can go back to our initial state and find the out state corresponding to it as

\[
|k, t\rangle = \int d^N x \sum_{\{P\}} \text{sgn}(P) \prod_j e^{ik_k x_j} |x, t\rangle.
\]

There are two things to note here. One is that the state \( |x, t\rangle \) can be simplified, since terms in the product with multiple \( d^\dagger \)s are identically zero (Pauli exclusion). We can
thus expand the product and write down

\[ |x, t\rangle = \theta(x_1 > \cdots > x_N) \int d^N y \left( \prod_{j=1}^{N} F(y_j, x_j, t) \psi^\dagger(y_1) \cdots \psi^\dagger(y_N) + \right. \]

\[ \left. + \sum_{k=1}^{N} \prod_{j \neq k} F(y_j, x_j, t) G(x_k, t) \delta(y_k) \psi^\dagger(y_1) \cdots \psi^\dagger(y_N) \right) |0\rangle. \]  

(4.23)

The subscript \( k \) on \( d^\dagger \) indicates that it occurs in the \( k \)-th position. This can be further simplified by rearranging the terms in the sum so \( d^\dagger \) always appears in the end. One can then swap \( y_k \) and \( y_N \) in each term (since the \( y \) are dummy variables that we integrate over, this is just a change of variables). For all \( k \neq N \) this swapping of the positions \( k \) and \( N \) gives us a minus sign due to the complete antisymmetry of the state. We get,

\[ |x, t\rangle = \theta(x_1 > \cdots > x_N) \int d^N y \left( \mathfrak{F}(y, x, t) \psi^\dagger(y_1) \cdots \psi^\dagger(y_N) + \right. \]

\[ \left. + \mathfrak{G}(y, x, t) \delta(y_N) \psi^\dagger(y_1) \cdots \psi^\dagger(y_{N-1}) d^\dagger |0\rangle, \right. \]

(4.24)

where

\[ \mathfrak{F}(y, x, t) = \prod_{j=1}^{N} F(y_j, x_j, t), \]  

(4.25)

\[ \mathfrak{G}(y, x, t) = \prod_{j=1}^{N-1} F(y_j, x_j, t) G(x_N, t) - \]

\[ - \sum_{k=1}^{N-1} F(y_1, x_1, t) \cdots F(y_k, x_N, t) \cdots F(y_{N-1}, x_{N-1}, t) G(x_k). \]  

(4.26)

(4.27)

Recall that we eventually Fourier transform the \( x_j \), so they are also dummy variables and we can do a change of variables. To be more precise we can reverse the steps used to write the initial state in a particular sector. We end up anti-symmetrizing the \( \mathfrak{F} \) and \( \mathfrak{G} \) in \( x \).

The second point to note is that after the Fourier transform, we get a multiparticle state, where the momenta of each particle are localized by a delta function. In other words,

\[ \langle k|k\rangle = \sum_{P} \text{sgn}(P) \prod_{j} \delta(k_{P_j} - k_j). \]  

(4.28)

These are bad states to work with when we take expectation values since we get \( \delta(0) \) type of infinities. Further, the integrals that appear in the Yudson representation would
diverge in a norm without suitable regularization. We would therefore, instead, like to work with Gaussian wave packets localized in position space, so we have normalizable wave functions, and then take the limit of the width of the Gaussian going to infinity. Consider the following as an incoming state:

\[ |k\rangle = \prod_{j=1}^{N} \frac{1}{(2\pi)^{1/2}} \int dx_j e^{ik_j x_j} - \frac{x_j^2}{2\sigma^2} \psi^\dagger(x_j) |0\rangle \] (4.29)

The norm of the state is

\[ \langle k | k \rangle = \sum_P \text{sgn}(P) \prod_j \frac{\sigma}{\sqrt{2\pi}} e^{-(k_P - k_j)^2 \sigma^2 / 2}. \] (4.30)

In the limit of \( \sigma \to \infty \), we get back the above result for plane waves.

We are interested in the specific case of the initial momenta being chosen from a filled Fermi sea (think about periodic boundary conditions on a circle of length \( L \)). The momenta, therefore are,

\[ k_j = -\frac{2\pi j}{L} \] (4.31)

In the \( \sigma \to \infty \) limit, the above norm reduces to \( \delta(k_j - k_P) = \frac{L}{2\pi} \delta_{jP} \).

We will work with finite \( \sigma \), and make series expansion in \( 1/\sigma \) where applicable and retain only first order terms. We will then choose the momenta from a Fermi sea and take the limit of \( \sigma \to \infty \). Now that we have well defined states, we can go ahead and ask questions about expectation values. In particular, we will be interested in computing the expectation value for the dot occupation,

\[ \langle n_d \rangle = \langle k, t | d^\dagger d | k, t \rangle. \] (4.32)

What we're therefore interested in computing is the quantity \( \langle x', t | d^\dagger d | x, t \rangle \). We can then carry out the Fourier transform w.r.t \( x_i \) and \( x'_i \) with momenta \( k_j \). We will take the spread of the Gaussian weight, \( \sigma \) to be very large wherever necessary. From eq. (4.24), we get,

\[ \langle x', t | d^\dagger d | x, t \rangle = \theta(x_1 > \cdots > x_N) \theta(x'_1 > \cdots > x'_N) \]

\[ \int d^{N-1}y \sum_{\{P\}} \text{sgn}(P) \mathcal{G}^*(Py, x', t) \mathcal{G}(y, x, t), \] (4.33)
where $P$ in the above equation denotes permutation on $1, \ldots, N - 1$.

One could alternatively first Fourier transform the state at time $t$. This is a better approach if we can carry out the $x$ integrals analytically. There is one glitch due to the ordering $\theta(x_1 > x_2 > \cdots > x_N)$. We'll sort this out below. Consider just the lead part of the state at time $t$:

$$|k, t\rangle_l = \frac{1}{(2\pi)^{N/2}} \int d^N y d^N x \theta(x_1 > \cdots > x_N) \sum_{\{P\}} \text{sgn}(P) \prod_j e^{ik_j x_j - x_j^2/4\sigma^2} \mathcal{F}(x, y, t) \prod_j \psi^\dagger(y_j) |0\rangle.$$ (4.34)

The $\mathcal{F}(x, y, t)$ can be anti-symmetrized in $y$. Let us call

$$\mathcal{F}^A(x, y, t) = \frac{1}{N!} \sum_{\{P\}} \text{sgn}(P) \mathcal{F}(x, Py, t).$$ (4.35)

The structure of $\mathcal{F}$ in the case of free particles (and we believe in general) is such that anti-symmetrizing in $y$ automatically makes it antisymmetric in $x$ also. For the non-interacting case, this is obvious as $\mathcal{F}$ is just a product of 1 particle wave functions. Thus in eq (4.34) above, the whole integrand (without the $\theta$ function) is now symmetric in $x$. It therefore simplifies to

$$|k, t\rangle_l = \frac{1}{(2\pi)^{N/2}N!} \int d^N y d^N x \sum_{\{P\}} \text{sgn}(P) \prod_j e^{ik_j x_j - x_j^2/4\sigma^2} \mathcal{F}^A(x, y, t) \prod_j \psi^\dagger(y_j) |0\rangle.$$ (4.36)

For the free particle case, the integrals over $x$ now factorize and can be carried out exactly. Once this is done, the overlaps will be easier to evaluate ($y$ will be the remaining integration variable, and $t$ is the parameter. $k_j$ will have to be specified.)

So, we will now try to carry out the above Gaussian integration explicitly. Writing out eq. (4.36) more explicitly, we have,

$$|k, t\rangle_l = \frac{1}{(2\pi)^{N/2}(N!)^2} \sum_{\{P\}, \{Q\}} \text{sgn}(P) \text{sgn}(Q) \int d^N y I(k_P, y_Q, t) \prod_j \psi^\dagger(y_j) |0\rangle,$$ (4.37)
where
\[ I(k_P, y_Q, t) = \prod_j \int d x_j \exp \left( \frac{i k_P x_j - x_j^2 / 4 \sigma^2}{j} \right) F(x_j, y_Q, t). \] (4.38)

The integral over \( x_j \) above is trivial to carry out. The result is
\[ I(k_P, y_Q, t) = \prod_j \left( \exp \left( i k_P (y_Q - t) - (y_Q - t)^2 / 4 \sigma^2 \right) - 2 \sqrt{\pi \sigma^2} \theta(0 < y_Q < t) \right) \]
\[ e^{-\left( \Gamma + i \epsilon_d \right)(t-y_Q) - (k_P - \epsilon_d + i \Gamma)^2 \sigma^2} \]
\[ \left( \text{erf} \left( (t-y_Q) / 2 \sigma + i(k_P - \epsilon_d + i \Gamma) \sigma \right) - \text{erf} \left( i(k_P - \epsilon_d + i \Gamma) \sigma \right) \right) \] (4.39)

We also have
\[ \langle k, t | d^\dagger d | k, t \rangle = \langle k, 0 | k, 0 \rangle - l \langle k, t | k, t \rangle l. \] (4.40)

We have one last formula to obtain, i.e., \( I(k, t | k, t) \). The overlap introduces some redundancy in the permutations, and we can easily get rid of it with a factor of \( N! \). We get,
\[ l \langle k, t | k, t \rangle l = \frac{1}{(2\pi)^N N!} \sum_{P, P', Q} \text{sgn}(P) \text{sgn}(P') \text{sgn}(Q) \int d^N y \ I^*(k_P', y_Q, t) I(k_P, y, t). \] (4.41)

For the above form of \( I \), antisymmetrizing in \( k \) automatically antisymmetrizes in \( y \). We thus end up with,
\[ l \langle k, t | k, t \rangle l = \frac{1}{(2\pi)^N N!} \sum_{P, P'} \text{sgn}(P) \text{sgn}(P') \int d^N y \ I^*(k_{P'}, y, t) I(k_P, y, t). \] (4.42)

However, we can now move the anti-symmetrization to the \( y \), and get
\[ l \langle k, t | k, t \rangle l = \frac{1}{(2\pi)^N N!} \sum_{P, P'} \text{sgn}(P) \text{sgn}(P') \int d^N y \ I^*(k, y_{P'}, t) I(k, y_P, t). \] (4.43)

But now we’re back to having redundancy since \( y \) is just an integration variable. We finally get
\[ l \langle k, t | k, t \rangle l = \frac{1}{(2\pi)^N} \sum_p \text{sgn}(p) \int d^N y \ I^*(k, y_P, t) I(k, y, t). \] (4.44)
Going back to our initial state at finite $\sigma$, we can write

$$\langle k, 0 | k, 0 \rangle = \det A,$$  \hspace{1cm} (4.45)

where $A_{pk} = \frac{e}{\sqrt{2\pi}} e^{-(k-p)^2 \sigma^2 / 2}$. In the limit $\sigma \to \infty$, $A_{pk} \to \delta(p - k)$. For the time evolved state we have,

$$|k, t\rangle_I = \prod_{j=1}^{N} \int dy_j f(k_j, y_j, t) \psi^\dagger(y_j) |0\rangle. \hspace{1cm} (4.46)$$

where $f$ is just shorthand for the wave function above.

As noted above, we’re interested in computing the norm of the above wave function. We have,

$$\langle n_I(t) \rangle = \det(A + \Gamma DD^\dagger) \hspace{1cm} (4.47)$$

where $D$ is a column vector where the index is the momentum $^5$. After some simplification (involving taking leading order in $1/\sigma$ and some algebraic manipulation), it is given by

$$D_k = 2e^{i(k+i\Gamma)t / 2} \text{sinc}((k - \epsilon_d + i\Gamma)t / 2) \hspace{1cm} (4.48)$$

It is also important that the entries in $DD^\dagger$ (a square matrix) have a well defined $\sigma \to \infty$ limit - in fact they are independent of $\sigma$. The matrix determinant lemma [22] allows us to write

$$\langle n_I(t) \rangle = \det(A)(1 + D^\dagger A^{-1}D). \hspace{1cm} (4.49)$$

In the large $\sigma$ limit, $A$ becomes the identity matrix (times $L/2\pi$), and we have a very simple expression for $n_I$. We now assign $k_j = -2\pi j / L$, where $L$ is the length of system where we choose the initial state from. After some manipulation we get the evolution of the dot occupation after a quench:

$$\langle n_d(t) \rangle = \frac{2\Gamma}{L} \sum_j \left( \frac{1 - 2e^{-\Gamma} \cos(t(k_j - \epsilon_d)) + e^{-2\Gamma}}{\Gamma^2 + (k_j - \epsilon_d)^2} \right). \hspace{1cm} (4.50)$$

\textsuperscript{5}Note that since the $k$ are going to be chosen from a Fermi sea with a finite number of electrons, $N$, we can always think in terms of finite matrices. The thermodynamic limit will be taken in the end. When we say index, we mean the $j$ in $k = -2\pi j / L$. 
This can be converted to an integral in the thermodynamic limit, and we can also take the cut-off $D \sim N/L$ to infinity. This gives us

$$\langle n_d(t) \rangle = \frac{e^{-t\Gamma}}{2\pi} \left( \cosh(t\Gamma) \left( \pi - 2 \arctan(\epsilon_d/\Gamma) - 
\text{Ci}(t(\epsilon_d - i\Gamma)) + \text{Ci}(t(\epsilon_d + i\Gamma)) \right)
+ \sinh(t\Gamma) \left( \pi - 
\text{Si}(t(\epsilon_d + i\Gamma)) + i\text{Si}(t(-\epsilon_d + i\Gamma)) \right) \right) \quad (4.51)$$

Figure 4.2 shows the dot occupation as a function of time, and Figure 4.3 shows the dot occupation starting with two different initial states, one having the dot occupied. We see that starting with the dot occupied has no effect on the final state, due to the infinite lead acting as a thermal reservoir for the dot.

![Figure 4.2: Time evolution of dot occupation. The different curves correspond to different values of $\Gamma$ and $\epsilon_d$. The dot is easier to occupy when the energy level is below the fermi level of the leads. The time scale to reach the asymptotic value is $\tau \sim 1/\Gamma$.](image)

4.3.3 Note on finite systems

In the discussion in Section 4.3.2, we implicitly managed to get results for finite volume, and then take a thermodynamic limit. This is interesting, given that the Yudson representation is valid only in the infinite volume limit. Of course, we can perfectly well take the initial condition to have some finite extent. We started out with an initial
state that was a free Fermi sea

\[ |\Psi_0\rangle = \prod_k c_k^\dagger |0\rangle. \] (4.52)

If we start with the initial state defined on a ring of length \(L\) with \(N\) particles, then

\[ k_j = -\frac{2\pi j}{L}, \quad j = 1, \ldots, N. \] (4.53)

What is interesting, however, is that inspite of the eigenstates used for time evolution being from the infinite volume basis, the observable calculated, \(\langle n_d\rangle\), picks out the eigenvalues corresponding to a finite length \(L\) system. We suspect that this might be a feature specific to this model because of the noninteracting nature, i.e., different momentum modes do not mix, and the quantum impurity has an \(O(1/L)\) effect on the physics of the leads. Further, since we can carry out the calculation for finite time, it is meaningful to first take the large \(N\) limit, giving us a true thermodynamic limit of the problem.

### 4.4 Application to other integrable models

The previous section showed the Yudson method applied to a noninteracting model. Everything could have been achieved without recourse to it. In the next two Chapters, we’ll see the Yudson representation applied to interacting integrable models which...
cannot be mapped to a noninteracting model. Here I would like to outline the Yudson representation for some other models that are of interest.

The key to the Yudson representation is finding an appropriate set of integration contours. In principle the representation should be possible for any integrable model. It would therefore be very interesting to generalize the approach to the Sine-Gordon model (bosons with a periodic potential), Anderson impurity model (interacting impurity model that displays the Kondo effect), the Kondo model, the fermionic Hubbard model, etc. Work is currently underway on a few of these. I will briefly outline the specific issues that emerge in using the approach for the Anderson model.

4.4.1 Anderson impurity model

The Anderson impurity model is a simple model of Kondo physics in one dimension. A noninteracting lead of spinful electrons is coupled to a two level impurity that has an onsite Coulomb interaction. The linearized and unfolded (see Section 3.3.1) Hamiltonian is given by

\[
H_{\text{AIM}} = -i \int dx \psi_{al}^{\dagger} \partial_x \psi_{al} + t_l \left[ \psi_{al}^{\dagger} d_a + d^{\dagger}_a \psi_{al} \right] + \epsilon d^{\dagger}_a d_a + U d^{\dagger}_a d^\downarrow d^\uparrow d^\downarrow. \quad (4.54)
\]

\(a\) is a spin index and \(l\) is a lead or flavor index. The interaction only couples up and down spins due to Pauli exclusion. The model is exactly solvable via the Bethe Ansatz [138]. In the limit of infinite Coulomb repulsion, or the strongly interacting limit, the dot occupation is restricted to 0 or 1, and the \(S\)-matrix simplifies to

\[
S_{ij} = \frac{k_i - k_j + i\hat{P}_{ij}}{k_i - k_j + i}. \quad (4.55)
\]

It is trivial for two particles in a triplet (or spin symmetric) state. As such, it is straightforward to come up with a set of Yudson contours for this problem, and they are similar to the case of the Dicke model [142] or the attractive Lieb-Liniger model that we study in the next chapter. The complication arises from the fact that the \(S\)-matrices are not \(c\)-numbers and a multiparticle \(S\)-matrix cannot be written as a generic product of two particle \(S\)-matrices, since the ordering is important. A further complication is produced by the fact that for generic initial states, the spin permutation operators
generate new states with permuted spin indices, and the book keeping is highly complicated. We are currently trying to study alternative formulations in order to deal with this issue.

It must however be mentioned that in principle the problem is solvable. If an efficient numerical algorithm is developed to keep track of the matrices and carry out the contour integrations, then we can indeed calculate relevant physical quantities. In that sense, it is an improvement over other analytical techniques where the difficulties are in the formulation stage.
Chapter 5
Quench dynamics of the Lieb-Liniger model

This Chapter studies the time-evolution of interacting bosons following a quench from a few different initial states. Most of the material here appears in Refs. [71] and [72].

5.1 The Lieb-Liniger model

Bosons in one dimensional traps interact via short range potentials, which can be well approximated by a $\delta$-function interaction. This model was solved in 1963 by Lieb and Liniger [94] who originally introduced it to overcome shortcomings of other models and approaches for understanding quantum gases and liquids, and to provide a rigorous result to test perturbation theory against. In particular, they sought to improve upon Girardeau’s hard-core boson model [53] by providing a tunable parameter and better model a low density gas, perhaps extensible to higher dimensions. The Schrödinger equation for the model is also commonly known as the Non-linear Schrödinger equation and has been extensively studied both classically and quantum mechanically.

The Hamiltonian is given by

$$H = \int_x [\partial b^+(x) \partial b(x) + c b^+(x) b(x) b^+(x) b(x)],$$

where $b$ is a bosonic field and $c$ is the interaction strength. The mass has been set to $1/2$. The action for the model $\sim \int_{x,t} [\partial_t - \partial_x^2]$. Time therefore has the dimension of $(\text{length})^2$. The coupling constant $c$ has the dimensions of length.

The model is integrable and the eigenstates take the Bethe Ansatz form,

$$|\vec{\lambda}\rangle = N(\vec{\lambda}) \int \prod_{i<j} Z_{ij}(\lambda_i - \lambda_j) \prod_j e^{i\lambda_j x_j} b^+(x_j)|0\rangle,$$

where $\vec{\lambda}$ is a vector of Bethe roots and $N(\vec{\lambda})$ is a normalization factor.
where \( N(\lambda) \) is a normalization factor determined by a particular solution, and
\[
Z_{ij}^z(z) = \frac{z - ic \text{sgn}(x_i - x_j)}{z - ic}
\]
(5.3)

incorporates the two particle \( S \)-matrix, \( S_{ij} = \frac{\lambda_i - \lambda_j + ic}{\lambda_i - \lambda_j - ic} \) occurring when two bosons cross.

The above state satisfies
\[
H |\vec{\lambda}\rangle = \sum_j \lambda_j^2 |\vec{\lambda}\rangle
\]
(5.4)

for any value of the momenta \( \vec{\lambda} \), which, depending on the sign of \( c \), may be pure real or form complex pairs. In our work, we study the evolution dynamics of the model on the infinite line and need not solve for explicit distributions of the \( \vec{\lambda} \) that characterize the low lying energy eigenstates, as discussed in section 2.3.

### 5.2 Yudson representation

In 1985, V. I. Yudson presented a new approach to time evolve the Dicke model (a model for superradiance in quantum optics [40]) considered on an infinite line [142]. The dynamics in certain cases was extracted in closed form with much less work than previously required, and in some cases where it was even impossible with earlier methods. The core of the method is to bypass the laborious sum over momenta using an appropriately chosen set of contours and integrating over momentum variables in the complex plane. It is applicable in its original form to models with a particular pole structure in the two particle \( S \)-matrix, and a linear spectrum. We generalize the approach to the case of the quadratic spectrum and apply it to the study of quantum quenches.

As discussed earlier, in order to carry out the quench of a system given at \( t = 0 \) in a state \( |\Psi_0\rangle \) one naturally proceeds by introducing a “unity” in terms of a complete set of eigenstates and then apply the evolution operator,
\[
|\Psi_0, t\rangle = e^{-iHt} \sum_{\{\lambda\}} |\vec{\lambda}\rangle \langle \vec{\lambda}|\Psi_0\rangle = \sum_{\{\lambda\}} e^{-ie(\vec{\lambda})t} |\vec{\lambda}\rangle \langle \vec{\lambda}|\Psi_0\rangle
\]
(5.5)

The Yudson representation overcomes the difficulties in carrying out this sum by using an integral representation for the complete basis directly in the infinite volume limit.
In the following two sections, we will discuss the representation for the repulsive and attractive models. Each will require a separate set of contours of integrations in order for the representation to be valid. We will notice that in the repulsive case, it is sufficient to integrate over the real line. The attractive case will require the use of contours separated out in the imaginary direction (to be qualified below) consistent with the fact that the spectrum consists of "strings" with momenta taking values as complex conjugate pairs [94].

5.2.1 Repulsive case

We begin by discussing the repulsive case, \( c > 0 \). For this case, a similar approach has been independently developed in Ref. [134] and has been used by Lamacraft [90] to calculate noise correlations in the repulsive model. We will start with a generic initial state given by

\[
|\Psi_0\rangle = \int \Phi_s(\vec{x}) \prod_j b^\dagger(x_j)|0\rangle.
\] (5.6)

with \( \Phi_s \) symmetrized. Using the symmetry of the bosonic operators, we can rewrite this state in terms of N-boson coordinate basis states,

\[
|\Psi_0\rangle = N! \int \Phi_s(\vec{x})|\vec{x}\rangle
\] (5.7)

where,

\[
|\vec{x}\rangle = \theta(\vec{x}) \prod_j b^\dagger(x_j)|0\rangle.
\] (5.8)

with \( \theta(\vec{x}) = \theta(x_1 > x_2 > \cdots > x_N) \). It suffices therefore to show that we can express any coordinate basis state as an integral over the Bethe Ansatz eigenstates

\[
|\vec{x}\rangle = \theta(\vec{x}) \int \prod_j \frac{d\lambda_j}{2\pi} A(\vec{\lambda}, \vec{x})|\vec{\lambda}\rangle
\] (5.9)

with appropriately chosen contours of integration \( \{\gamma_j\} \) and \( A(\lambda, \vec{x}) \), which plays a role similar to the overlap of the eigenstates and the initial state.

We claim that in the repulsive case equation (5.9) is realized with

\[
A(\vec{\lambda}, \vec{x}) = \prod_j e^{-i\lambda_j x_j}.
\] (5.10)
and the contours $\gamma_j$ running along the real axis from minus to plus infinity. In other words eqn. (5.9) takes the form,

$$|\vec{x}\rangle = \theta(\vec{x}) \int \prod_j \frac{d\lambda_j}{2\pi} \prod_{i < j} Z_{ij}^y(\lambda_i - \lambda_j) \prod_j e^{i\lambda_j(y_j-x_j)} b^\dagger(y_j) |0\rangle. \quad (5.11)$$

Equivalently, we claim that the $\lambda$ integration above produces $\prod_j \delta(y_j - x_j)$.

We shall prove this in two stages. Consider first $y_N > x_N$. To carry out the integral using the residue theorem, we have to close the integration contour in $\lambda_N$ in the upper half plane. The poles in $\lambda_N$ are at $\lambda_j - ic, j < N$. These are all below $\gamma_N$ and so the result is zero. This implies that any non-zero contribution comes from $y_N \leq x_N$. Let us now consider $y_{N-1} > x_{N-1}$. The only pole above the contour is $\lambda_{N-1}^* = \lambda_N + ic$. However, we also have, $y_{N-1} > x_{N-1} > x_N \Rightarrow y_{N-1} > y_N$. This causes the only contributing pole to get canceled. The integral is again zero unless $y_{N-1} \leq x_{N-1}$. We can proceed in this fashion for the remaining variables thus showing that the integral is non-zero only for $y_j \leq x_j$.

Now consider $y_1 < x_1$. We have to close the contour for $\lambda_1$ below. There are no poles in that region, and the residue is zero. Thus the integral is non-zero for only $y_1 = x_1$. Consider $y_2 < x_2$. The only pole below, at $\lambda_2^* = \lambda_1 - ic$ is canceled as before since we have $y_2 < x_2 < x_1 = y_1$. Again we get that the integral is only non-zero for $y_2 = x_2$. Carrying this on, we end up with

$$\theta(\vec{x}) \int \prod_j \delta(y_j - x_j) b^\dagger(y_j) |0\rangle = |\vec{x}\rangle. \quad (5.12)$$

In order to time evolve this state, we can act on it with the unitary time evolution operator. Since the integrals are well-defined we can move the operator inside the integral signs to obtain,

$$|\vec{x}, t\rangle = \theta(\vec{x}) \int \prod_j \frac{d\lambda_j}{2\pi} e^{-ic(\lambda_j)t} A(\vec{\lambda}, \vec{x}) |\vec{\lambda}\rangle. \quad (5.13)$$

### 5.2.2 Attractive case

We now consider an attractive interaction, $c < 0$. As mentioned earlier, this changes the spectrum of the Hamiltonian, allowing complex, so-called string solutions, which in
this model, correspond to many-body bound states. In fact, the ground state at $T = 0$
consists of one $N$-particle bound state. We will see how the Yudson integral representation takes this into account. Similar properties are seen to emerge in [114], where
the authors obtain a propagator for the attractive Lieb-Liniger model by analytically
continuing the results obtained by Tracy and Widom [134] for the repulsive model.

One will immediately notice from the eigenstates that the pole structure of the $S$-
matrix is altered. This change prevents the proof of the previous section from working.
In particular, the poles in the variable $\lambda_N$ are at $\lambda_j + i|c|$ for $j < N$, and the residue of
the contour closed in the upper half plane is not zero any more. We need choose a
contour to avoid this pole. This can be achieved by separating the contours in the
imaginary direction such that adjacent $\text{Im}[\lambda_j - \lambda_{j-1}] > |c|$. At first sight, this seems
to pose a problem as the quadratic term in exponent diverges at large positive $\lambda$ and
positive imaginary part. There are two ways around this. We can tilt the contours as
shown in Fig. 5.1 so that they lie in the convergent region of the Gaussian integral. The
pieces towards the end, that join the real axes though essential for the proof to work
at $t = 0$, where we evaluate the integrals using the residue theorem, do not contribute
at finite time as the integrand vanishes on them as they are taken to infinity. Another
more natural means of doing this is to use the finite spatial support of the initial state.
The overlaps of the eigenstates with the initial state effectively restricts the support for
the $\lambda$ integrals, making them convergent.

The proof of equation (5.9) now proceeds as in the repulsive case. We start by assuming that $y_N > x_N$ requiring us to close the contour in $\lambda^*_N$ in the upper half plane. This encloses no poles due to the choice of contours and the integral is zero unless $y_N \leq x_N$. Now assume $y_{N-1} > x_{N-1}$. Closing the contour above encloses one pole at $\lambda^*_{N-1} = \lambda_N - i|c|$, however since $y_{N-1} > x_{N-1} > x_N \geq y_N$, this pole is canceled by the numerator and again we have $y_{N-1} \leq x_{N-1}$. We proceed in this fashion and then backwards to show that the integral is non-zero only when all the poles cancel, giving us $\prod_j \delta(y_j - x_j)$, as required.

5.3 Two particle dynamics

We begin with a detailed discussion of the quench dynamics of two bosons. As we saw, it is convenient to express any initial state in terms of an ordered coordinate basis, $|\vec{x}\rangle = \theta(x_1 > x_2 > \cdots > x_N) \prod_j b^\dagger(x_j)|0\rangle$. At finite time, the wave function of bosons initially localized at $x_1$ and $x_2$ and subsequently evolved by a repulsive Lieb-Liniger Hamiltonian is given by,

\[
|\vec{x},t\rangle = e^{-iHt} \theta(x_1 - x_2)b^\dagger(x_1)b^\dagger(x_2)|0\rangle
\]

\[
= \int_{y,\lambda} Z_{12}^y(\lambda_1 - \lambda_2)e^{-i\lambda_1^2t - i\lambda_2^2t + i\lambda_1(y_1-x_1) + i\lambda_2(y_2-x_2)}
\]

\[
\times b^\dagger(y_1)b^\dagger(y_2)|0\rangle
\]

\[
= \int_y \frac{e^{(y_2-x_2)^2} + i(y_2-x_2)^2}{4\pi i t}
\]

\[
\times \left[ 1 - c\sqrt{\pi it} \theta(y_2 - y_1)e^{\frac{ia^2}{4}} \text{erfc}\left(\frac{i - 1}{4 \sqrt{i}}\right) \right]
\]

\[
\times b^\dagger(y_1)b^\dagger(y_2)|0\rangle.
\]

where $\alpha = 2ct - i(y_1 - x_1) - i(y_2 - x_2)$. The above expression retains the Bethe form of wave functions defined in different configuration sectors. The only scales in the problem are the interaction strength $c$ and the scale from the initial condition, the separation between the particles at $t = 0$.

In order to get physically meaningful results we need to start from a physical initial
state. We choose first the state $|\Psi(0)_{\text{latt}}\rangle$ where bosons are trapped in a periodic trap forming initially a lattice-like state (see fig. 2.5a),

$$|\Psi(0)_{\text{latt}}\rangle = \prod_j \left[ \frac{1}{(\pi \sigma^2)^{1/4}} \int \mathcal{d}^x e^{-\frac{(x_j + (j-1)a)^2}{2\sigma^2}} b^\dagger(x_j) \right] |0\rangle. \quad (5.15)$$

If we assume that the wave functions of neighboring bosons do not overlap significantly, i.e., $e^{-\frac{a^2}{2\sigma^2}} \ll 1$, then the ordering of the initial particles needed for the Yudson representation is induced by the non-overlapping support and it becomes possible to carry out the integral analytically.

We now calculate the evolution of some observable in the state $|\Psi(0)_{\text{latt}}\rangle$. Consider first the evolution of the density $\rho(x) = b^\dagger(x) b(x)$ at $x = 0$. Fig. 5.2 shows $\langle \Psi_{\text{latt}}, t | \rho(0) | \Psi_{\text{latt}}, t \rangle$ for repulsive, attractive and non-interacting bosons. No difference is discernible between the three cases. The reason is obvious: the local interaction is operative only when the wave functions of the particles overlap. As we have taken $\sigma \ll a$ this will occur only after a long time when the wave-function is spread out and overlap is negligible.

Figure 5.2: $\langle \rho(x = 0, t) \rangle$ vs. $t$, after the quench from $|\Psi_{\text{latt}}\rangle$. $\sigma/a \sim 0.1$. The curves appear indistinguishable (i.e. lie on top of each other) since the particles start out with non-significant overlap. The interaction effects would show up only when they have propagated long enough to have spread sufficiently to reach a significant overlap, at which time the density is too low.

Consider now an initial state where we set the separation $a$ to zero, starting with maximal initial overlap between the bosons $|\Psi(0)_{\text{cond}}\rangle$. We refer to this state as a condensate (in position-space). Fig. 5.3 shows the density evolution for attractive,
repulsive and no interaction. The decay of the density is slower for attractive model than the for the non-interacting which in turn is slower than for the repulsive model - indeed, unlike before, the interaction is operative from the beginning. Still, the density

\[
\langle \rho(x = 0, t) \rangle \text{ vs. } t, \text{ after the quench from } |\Psi_{\text{cond}} \rangle. \sigma \sim 0.5, a = 0. \text{ As the bosons overlap interaction effects show up immediately. Lower line: } c = 1, \text{ Upper line: } c = -1, \text{ Middle line: } c = 0.
\]

does not show much difference between repulsive and attractive interactions in this case. However, a drastic difference will appear when we study the noise correlations \( \langle \Psi_0, t | \rho(x_1) \rho(x_2) | \Psi_0, t \rangle \), as will be shown below.

A comment about the attractive case is in order here. Recall that the contours of integration are separated in the imaginary direction. In order to carry out the integration over \( \lambda \), we shift the contour for \( \lambda_2 \) to the real axis, and add the residue of the pole at \( \lambda_2 = \lambda_1 + i|c| \). The two particle finite time state can be written as

\[
|\vec{x}, t \rangle_2 = \int_y \gamma_c \prod_{i < j = 1, 2} Z_{ij}^y(\lambda_i - \lambda_j) \times \prod_{j = 1, 2} e^{-i\lambda_2^j t + i\lambda_j(y_j - x_j)} b^\dagger(y_j)|0\rangle \\
= \int_y \left[ \int_{\gamma_r} \prod_{i < j = 1, 2} Z_{ij}^y(\lambda_i - \lambda_j) \prod_{j = 1, 2} e^{-i\lambda_2^j t + i\lambda_j(y_j - x_j)} b^\dagger(y_j)|0\rangle \\
+ \theta(y_2 - y_1) I(\lambda_2 = \lambda_1 + i|c|, t) b^\dagger(y_1) b^\dagger(y_2)|0\rangle \right] 
\]

(5.16)

\( \gamma_c \) refers to contours that are separated in imaginary direction, \( \gamma_r \) refers to all \( \lambda \) integrated along real axis. \( I(\lambda_2 = \lambda_1 + i|c|, t) \) is the residue obtained by shifting the \( \lambda_2 \) contour to the real axis from the pole at \( \lambda_2 \) at \( \lambda_1 + i|c| \). This second term corresponds
to a two-particle bound state. It is given by

\[ I(\lambda_2 = \lambda_1 + i|c|, t) = \]
\[ -2c \int_y \int_{\lambda_1} e^{i\lambda_1(y_1 - x_1) + i(\lambda_1 + i|c|)(y_2 - x_2) - \lambda_1^2 t - i(\lambda_1 + i|c|)^2 t} \]
\[ = -2c \int_y \int_{\lambda_1} e^{i\lambda_1(y_1 - x_1 + y_2 - x_2) - \lambda_1^2 (y_2 - y_1)} \]
\[ \times e^{-i(\lambda_1 - i|c|/2)t - i(\lambda_1 + i|c|/2)t} \]
\[ = -c \int_y \int_{\lambda_1} e^{i\lambda_1(y_1 - x_1 + y_2 - x_2) - \lambda_1^2 (y_2 - y_1)} \]
\[ \times e^{-2\lambda_1^2 t + i|c|^2 / 2}. \]

This contribution corresponds to the particles propagating as a bound state, \( e^{-|c|^2 |y_2 - y_1|/2} \), with kinetic energy \( E_k = 2\lambda_1^2 \), and binding energy \( E_b = -c^2 / 2 \). The rest of the expression, \( e^{i\lambda_1(y_1 - x_1 + y_2 - x_2) - \lambda_1^2 (y_2 - y_1)} \), yields the overlap of the bound state with the initial state \( |\vec{x}\rangle \). Note that the overlap decays exponentially as the distance \( |x_1 - x_2| \) between the initial positions is increased.

Such bound states appear for any number of particles involved. For instance, for three particles, the Yudson representation with complex \( \lambda \)s automatically produces multiple bound-states coming from the poles, i.e. \( I(\lambda_2 = \lambda_1 + i|c|), I(\lambda_3 = \lambda_1 + i|c|) \), etc. They give rise to two and three particle bound states, the latter being of the form \( e^{-|c|^2 |y_1 - y_2| + |y_1 - y_3| + |y_2 - y_3|} \) with binding energy \( E_b = -2c^2 \). It is important to remark that these bound states were not put in by hand, but arise straightforwardly from the contour representation. The binding energy of an \( N \)-particle bound state thus appearing in the time evolution is \( E_b = -c^2 N(N^2 - 1)/12 \) as expected from the spectrum of the Hamiltonian [139].

Finally, we combine the bound state contribution discussed above and the “real axis” term, \( \int_y \int_{\gamma} \prod_{i<j} Z_{ij}(\lambda_i - \lambda_j) \prod_{i} e^{-i\lambda_i^2 t + i\lambda_i(y_i - x_i)} b^+(y_i) |0\rangle \), which corresponds to the non-bound propagating states in (5.16). The resulting wave function is,

\[ |\vec{x}, t\rangle_2 = \int_y \frac{e^{i(y_1 - x_1)^2 + i(y_2 - x_2)^2}}{4\pi it} \]
\[ \left[ 1 + |c|\sqrt{\frac{\pi i\theta(y_2 - y_1)}{4\pi it}} e^{\frac{\pi i\theta}{4\sqrt{t}}} \right] \]
\[ \times e^{\frac{i - 1}{4\sqrt{t}}} \]
\[ \times b^+(y_1) b^+(y_2) |0\rangle \] (5.18)
where $\tilde{\alpha} = -2|c|t - i(y_1 - x_1) - i(y_2 - x_2)$. Surprisingly, the wave function maintains its form and we only need to replace $c \rightarrow -c$. This simple result is not valid for more than two particles.

We now compute the noise correlation in the evolving state. We expect the interaction to have a significant effect as the geometry of the set up measures the interference of “direct” and “crossed” measurements as shown in fig. 5.4a. In contrast, the

\[
\langle \rho(x) \rho(x - d) \rangle
\]

Figure 5.4: (a) The Hanbury-Brown Twiss effect [64], where two detectors are used to measure the interference of the direct (big dashes) and the crossed waves (small dashes). The $S$-matrix enters explicitly. (b) The density measurement is not directly sensitive to the $S$-matrix. The thick black line shows the wave-function amplitude, the dotted lines show time propagation.

density measurements do not see the $S$-matrix, as shown in fig. 5.4b.

This is the famous Hanbury-Brown Twiss experiment [64] where for free bosons or fermions, the crossing produces a phase of $\pm 1$ and causes destructive or constructive interference. In our case the set up is generalized to multiple time dependent sources with the phase given by the two particle $S$-matrix capturing the interactions between the particles. In Fig. 5.5 we present the two point correlation matrix $\langle \rho(x_1) \rho(x_2) \rangle$ for the repulsive gas, attractive gas, and the non-interacting gas, shown at different times,
starting with the lattice initial state. Figure 5.6 shows the same for the condensate initial state. In both these, we note that the repulsive gas develops fermionic correlations

\[
\begin{align*}
    c = 0 \\
    c > 0 \\
    c < 0
\end{align*}
\]

(i.e., strong anti-bunching), and the attractive gas retains bosonic correlations at long time, showing strong bunching.

It is interesting to compare this result with the time evolution after a quench on the lattice by the Bose-Hubbard model, the lattice counterpart of the Lieb-Liniger model, as we shall see in Appendix A. The results for continuum model differs strongly from those of the lattice model.

We expect the results to be qualitatively similar for higher particle number. In order to go beyond two particles however, the integrations cannot be carried out exactly. However, we can extract the asymptotic behavior of the wavefunction analytically, as we show below.
Figure 5.6: Time evolution of density-density correlation matrix ($\langle \rho(x)\rho(y) \rangle$) for the $|\Psi_{\text{cond}}\rangle$ initial state. Blue is zero and red is positive. The repulsive model shows antibunching, i.e., fermionization at long times, while the attractive model shows bunching.

5.4 Multiparticle dynamics at long times

In this section, we derive an expression for the multiparticle wavefunction evolution at long times. The number of particles $N$ is kept fixed in the limiting process, hence, as discussed in the introduction we are in the low density limit where interactions are expected to be dominant. The other regime where $N$ is sent to infinity first will be discussed in a separate report. We first deal with the repulsive model, for which no bound states exist and the momentum integrations can be carried out over the real line, and then proceed to the attractive model. In a separate sub-section, we examine the effect of starting with a condensate-like initial state.
5.4.1 Repulsive interactions - Asymptotics

From (5.9) we can see by scaling $\lambda \rightarrow \lambda \sqrt{t}$, we get

$$Z_{ij}^\theta(\lambda_i - \lambda_j) \rightarrow \text{sgn}(y_i - y_j) + O\left(\frac{1}{\sqrt{t}}\right), \quad (5.19)$$

yielding to leading order,

$$|\Psi_0, t\rangle \rightarrow \int_x \int_y \int_\lambda \theta(\vec{x}) \Psi_0(\vec{x}) \times \prod_j \frac{1}{\sqrt{t}} e^{-i\lambda_j^2 t + i\lambda_j (y_j - x_j)} \prod_{i<j} \text{sgn}(y_i - y_j) b^\dagger(y_j) |0\rangle$$

$$= \int_{x,y,\lambda,k} \theta(\vec{x}) \Psi_0(\vec{x}) \prod_j e^{-i\lambda_j^2 t + i\lambda_j (y_j - x_j)} e^{-ik_j y_j} c^\dagger_k |0\rangle$$

$$= e^{-iH_0^f t} \int_x \mathcal{A}_y \theta(\vec{x}) \Psi_0(\vec{x}) \prod_j c^\dagger(x_j) |0\rangle,$$

c$^\dagger(y)$ being fermionic creation operators replacing the “fermionized” hardcore bosonic operators, $\prod_j c^\dagger(y_j) = \prod_{i<j} \text{sgn}(y_i - y_j) b^\dagger(y_j)$. We denote $H_0^f = \int_x \partial c^\dagger(x) \partial c(x)$ the free fermionic Hamiltonian and $\mathcal{A}_y$ is an anti-symmetrizer acting on the $y$ variables. Thus, the repulsive Bose gas, for any value of $c > 0$, is governed in the long time by the $c = \infty$ hard core boson limit (or its fermionic equivalent) [78, 53], and the system equilibrates with an asymptotic momentum distribution, $n_k = \langle \Psi_0 | c_k^\dagger c_k | \Psi_0 \rangle$, determined by the antisymmetric wavefunction $\tilde{\Psi}_0(\vec{y}) = A_y \theta(\vec{y}) \Psi_0(\vec{y})$ and the total energy, $E_{\Psi_0} = \langle \Psi_0 | H | \Psi_0 \rangle$.

We will now derive the corrections to the infinite time limit. At large time, we use the stationary phase approximation to carry out the $\lambda$ integrations. The phase oscillations come primarily from the exponent $e^{-i\lambda_j^2 t + i\lambda_j (y_j - x_j)}$. At large $t$ (i.e., $t \gg \frac{1}{c^2}$), the oscillations are rapid, and the stationary point is obtained by solving

$$\frac{d}{d\lambda_j} [ -i\lambda_j^2 t + i\lambda_j (y_j - x_j) ] = 0. \quad (5.20)$$

Note that typically one would ignore the second term above since it doesn’t oscillate faster with increasing $t$, but here we cannot since the integral over $y$ produces a non-zero contribution for $y \sim t$ at large time. Doing the Gaussian integral around this point
(and fixing the S-matrix prefactor to its stationary value), we obtain for the repulsive case,

$$|\vec{x}, t⟩ \rightarrow \int \prod_{i<j} Z_{ij}^{y_i y_j} \left( \frac{y_i - y_j - x_i + x_j}{2t} \right) \times \prod_j \frac{1}{\sqrt{4\pi it}} e^{-i\frac{(y_j - x_j)^2}{4t}} e^{-i\frac{(y_j - x_j)^2}{4t}} b^\dagger(y_j) |0⟩.$$  \(5.21\)

In the above expression, the wavefunction has support mainly from regions where \(y_j/t\) is of order one. In an experimental setup, one typically starts with a local finite density gas, i.e., a finite number of particles localized over a finite length. With this condition, at long time, we can neglect \(x_j/t\) in comparison with \(y_j/t\), giving

$$|\vec{x}, t⟩ \rightarrow \int \prod_{i<j} Z_{ij}(\xi_i - \xi_j) \prod_j \frac{1}{\sqrt{4\pi it}} e^{-i\frac{(\xi_j - x_j)^2}{4t}} e^{-i\frac{(\xi_j - x_j)^2}{4t}} b^\dagger(y_j) |0⟩.$$  \(5.22\)

where \(\xi = \frac{y}{2t}\).

We turn now to calculate the asymptotic evolution of some observables. To compute the expectation value of the density we start from the coordinate basis states, \(|\vec{x}', t|\rho(z)|\vec{x}, t⟩\) which we then integrate with the chosen initial state,

$$\langle \vec{x}', t|\rho(z)|\vec{x}, t⟩ = \sum_{\{P\}} \int y \left( \sum_j \delta(y_j - z) \right) \times \prod_{i<j} Z_{ij}(\xi_i - \xi_j) Z_{P_i P_j}^\dagger(\xi_{P_i} - \xi_{P_j}) \prod_j \frac{1}{4\pi t} e^{-i(\xi_j - \xi_{P_j})}.$$  \(5.23\)

Note that the above product of S-matrices is actually independent of the ordering of the \(y\). First, only those terms appear in the product for which the permutation \(P\) has an inversion. For example, say for three particles, if \(P = 312\), then the inversions are 13 and 23. It is only these terms which give a non-trivial S-matrix contribution. For the non-inverted terms, here 12, we get

$$\frac{\xi_1 - \xi_2 - ic \text{sgn}(y_1 - y_2)}{\xi_1 - \xi_2 + ic} \frac{\xi_1 - \xi_2 + ic \text{sgn}(y_1 - y_2)}{\xi_1 - \xi_2 - ic}$$  \(5.24\)

which is always unity irrespective of the ordering of \(y_1, y_2\). For a term with an inversion, say 23, we get,

$$\frac{\xi_2 - \xi_3 - ic \text{sgn}(y_2 - y_3)}{\xi_2 - \xi_3 + ic} \frac{\xi_3 - \xi_2 + ic \text{sgn}(y_3 - y_2)}{\xi_3 - \xi_2 - ic}$$  \(5.25\)

which is always equal to

$$\frac{\xi_2 - \xi_3 + ic}{\xi_2 - \xi_3 - ic} = S(\xi_2 - \xi_3)$$  \(5.26\)
irrespective of the sign of $y_2 - y_3$. This allows us to carry out the integration over the $y_j$.

**Lattice initial state**

In order to calculate physical observables, we have to choose initial states. We choose two different initial states for the problem, one with $N$ particles distributed with uniform density in a series of harmonic traps given by,

$$|\Psi_{\text{latt}}\rangle = \int x \prod_{j=1}^{N} \frac{1}{(\pi \sigma^2)^{\frac{3}{2}}} e^{-\frac{(x_j + (j-1)a)^2}{2\sigma^2}} b^\dagger(x_j)|0\rangle,$$

(5.27)

such that the overlap between the wave functions of two neighboring particles is negligible. In this particular case, the ordering of the particles is induced by the limited non-overlapping support of the wave function.

In the lattice-like state, the initial wave function starts out with the neighboring particles having negligible overlap. At small time (as seen from (5.14)), the particle repel each other, but they never cross due to the repulsive interaction. So at large time, the interaction does not play a role since the wave functions are sufficiently non-overlapping. It is only the $P = 1$ contribution then that survives, and we get for the density

$$\langle x', t | \rho(z) | x, t \rangle = \sum_j \prod_{k \neq j} \frac{\delta(x_k - x_j) e^{-i \frac{\pi}{2} (x_j - x'_j)}}{4\pi t}.$$

(5.28)

We need to integrate the position basis vectors $|\vec{x}\rangle$ over some initial condition. We do this here for the lattice state (5.27) This gives

$$\langle \Psi_{\text{latt}}, t | \rho(z) | \Psi_{\text{latt}}, t \rangle = \rho_{\text{latt}}(\xi z) = \frac{N \sigma}{2 \sqrt{\pi t}} e^{-\frac{\xi^2 z^2}{2\sigma^2}}$$

(5.29)

Mathematically, any $S$-matrix factor that appears will necessarily have zero contribution from the pole - this is easy to see from the pole structure, and the ordering of the coordinates. In order to get a non-zero result, we need to fix at least two integration variables (i.e., the $y_j$). Thus the first non-trivial contribution comes from the two-point correlation function.

We now proceed to calculate the evolution of the noise, i.e., the two body correlation function $\rho_2(z, z'; t)_{\text{latt}} = \langle \Psi_{\text{latt}}, t | \rho(z) \rho(z') | \Psi_{\text{latt}}, t \rangle$. The contributions can be
grouped in terms of number of crossings, which corresponds to a grouping in terms of the coefficient $e^{-ca}$ [90]. The leading order term can be explicitly evaluated and we show below which terms contribute. In general we have

$$
\rho_{2\text{ latt}}(z, z'; t) = \sum_{\{P\}} \int y \left( \sum_{j, K} \delta(y_j - z) \delta(y_k - z') \right) \times \prod_{i < j, (ij) \in P} S(\xi_i - \xi_j) \prod_j \frac{1}{4\pi t} e^{i(\xi_j x_j - \xi_{j'} x_{j'})}. \quad (5.30)
$$

The above shorthand in the $S$-matrix product means that only the $(ij)$ that belong to the inversions in $P$ are included. We will now determine which terms contribute in this sum. First note that for integration over a particular $\xi_j$, the residue depends on the sign of $x_j - x_{P_j}$. Let us consider a specific example. Consider the three particle case with the term $P = 321$. All three $S$-matrix factors appear in this term. $\xi_3$ has a pole at $\xi_1 - ic$ and $\xi_2 - ic$. Thus integrating over $y_3$ will give a non-zero residue only if $x_3 > x'_1$ which is however not satisfied by the initial conditions we choose. So, everything is zero, unless we do not integrate over $y_3$, implying it has to be one of the measured variables. Similarly for $\xi_1$, the poles are at $\xi_2 + ic$ and $\xi_3 + ic$. In order to get a non-zero residue we need $x_1 < x'_3$ which again is not satisfied by the initial conditions. We get a non-zero result if we pin $\xi_1$. As for $\xi_2$ it has poles both above and below the real line and so this always gives a non-zero contribution irrespective of the sign of $x_2 - x'_2$.

One can see that this argument can be extended to the case with more particles. Depending on what coordinates we are measuring at, we’ll get a specific contribution from the sum over permutations. The next simplification comes from not allowing any crossings among the unmeasured coordinates. It can be shown that allowing for these gives us a higher order contribution in $e^{-ca}$. In other words, the leading order contribution comes from terms such as $P = 21, 32, 321, 4231, 5342, 52341, \ldots$. The only exchanges are on the ends. A general term will therefore look like (for $l < k$),

$$
\int y' \delta(y_l - z) \delta(y_k - z') \prod_{j \neq \{l, k\}} S(\xi_i - \xi_j) S(\xi_j - \xi_k) \times \frac{e^{-i\xi_i (x_j - x')} \xi_{l} - \xi_k + ic e^{-i\xi_j (x_j - x')} - i\xi_k (x_k - x')}{4\pi t} \frac{\xi_l - \xi_k - ic}{(4\pi t)^2}. \quad (5.31)
$$
We have to sum over $l, k$, which will automatically sum over the number of intermediate $j$’s appearing. We’ll integrate the above general term, since the $y_j$ integrals factor anyway. This gives

\[
\prod_{i \neq k, l, j} \delta(x_i - x'_i) S(\xi_l - \xi_k) e^{-i \xi_l (x_l - x'_l) - i \xi_k (x_k - x'_k)} (4\pi t)^2 \times \prod_j \left[ \delta(x_j - x'_j) - S(\xi_l - \xi_k - i\epsilon) \right] \times \left\{ \theta(x_j > x'_j) e^{-i(\xi_l - i\epsilon)(x_j - x'_j)} + \theta(x_j < x'_j) e^{-i(\xi_k + i\epsilon)(x_j - x'_j)} \right\}
\] (5.32)

We can sum the different contributions now. Note that the number of terms appearing the product over $j$ is given by $k - l - 1$. So for a given $l$, we have to sum over all the $k$.

Using a shorthand notation, the sum can be written as (note that it is understood that $y_l$ and $y_k$ are integrated over using the delta functions. We retain the indices to keep track of the terms. We actually have $\xi_l = 2zt$ and $\xi_k = 2z't$.

\[
\sum_{l, k} \prod_{i \neq k, l, j} \delta f_{lk} \prod_{j=1}^{k-1} g_{jlk}
\] (5.33)

In order to proceed with the summation, we have to integrate over the $x$. We use the initial conditions described by (5.27), i.e., the lattice-like state. We’ll do it term by term.

\[
\int_{x_l, x'_l} \delta(x_l - x'_l) e^{-\frac{(x_l + (i-1)a)^2}{2\sigma^2} - \frac{(x'_l + (i-1)a)^2}{2\sigma^2}} \sqrt{\pi} \sigma = 1
\] (5.34)

\[
\int_{x_l, x'_l, x_k, x'_k} f_{lk} e^{-\frac{(x_l + (i-1)a)^2}{2\sigma^2} - \frac{(x'_l + (i-1)a)^2}{2\sigma^2}} \pi \sigma^2 = \frac{\sigma^2}{4\pi t^2} S(\xi_l - \xi_k) e^{-(\xi_l^2 + \xi_k^2)\sigma^2} e^{ia(k-l)(\xi_l - \xi_k)}
\] (5.35)

\[
\int_{x_l, x'_l} g_{lk} e^{-\frac{(x_l + (i-1)a)^2}{2\sigma^2} - \frac{(x'_l + (i-1)a)^2}{2\sigma^2}} \pi \sigma^2 = 1 - 2c \sqrt{\pi} \sigma \\
\times S(\xi_l - \xi_k - i\epsilon) \left[ e^{(c + i\xi_l)^2\sigma^2} \text{erfc} \{(c + i\xi_l)\sigma\} + e^{(c - i\xi_k)^2\sigma^2} \text{erfc} \{(c - i\xi_k)\sigma\} \right]
\] (5.36)
Note that last expression has no $j$ dependence. So, the sum over the product over $j$ is just a geometric series. Recall that $\xi_l$ and $\xi_k$ are fixed at $z$ and $z'$ respectively. This series can be summed. Note that the number of terms in the product is equal to the $k - l$, and summing over them is effectively summing over $k$. The previous term therefore needs to be taken into account. Writing $\xi_l = \xi_z$ and $\xi_k = \xi_{z'}$, we can write the sum as

$$\frac{\sigma^2}{4\pi l^2} S(\xi_z - \xi_{z'}) e^{-(\xi_z^2 + \xi_{z'}^2)\sigma^2} e^{ia(k-l)(\xi_z - \xi_{z'})} \frac{g^{k-l-1}_{zz'}}{g_{zz'}}$$

(5.37)

where

$$g_{zz'} = 1 - 2c \sqrt{\pi} \sigma S(\xi_z - \xi_{z'} - ic) \left[ e^{(c+i\xi_z)^2/\sigma^2} \text{erfc}\{(c+i\xi_z)\sigma\} \right. \left. + e^{(c-i\xi_{z'})^2/\sigma^2} \text{erfc}\{(c-i\xi_{z'})\sigma\} \right]$$

(5.38)

Finally, we have to account for the $k > l$ case which is equivalent to setting $\xi_l = \xi_{z'}$ and $\xi_k = \xi_z$. Doing this is further equivalent to adding the complex conjugate. We also have to take into account the term with no permutations. So, finally we have,

$$\rho_{2 \text{ latt}}(z, z') = \frac{N^2 \sigma^2}{4\pi l^2} e^{-(\xi_z^2 + \xi_{z'}^2)\sigma^2} \left[ 1 + \frac{2}{N^2} \text{Re} S(\xi_z - \xi_{z'}) \right. \times e^{ia(z-z')} \frac{N(1 - e^{ia(z-z')}g_{zz'}) + e^{iaN(z-z')}g_{zz}^N - 1}{[1 - g_{zz'} e^{ia(z-z')}]^2}$$

(5.39)

To compare with the Hanbury-Brown Twiss result, we calculate the normalized spatial noise correlations, given by $C_2(z, z') \equiv \frac{\rho_{2 \text{ latt}}(z,z')}{\rho_2(z)\rho_2(z')} - 1 \equiv C_2(z, z')$. In the non-interacting case, i.e., $c = 0$, $S(\xi) = 1$ and $g_{zz'} = 0$ and we recover the HBT result for $N = 2$,

$$C_2^0(\xi_z, \xi_{z'}) = \frac{1}{2} \cos(a(\xi_z - \xi_{z'}))$$

(5.40)

One can also check that the limit of $c \to \infty$ gives the expected answer for free fermions, namely,

$$C_2^\infty(\xi_z, \xi_{z'}) = -\frac{1}{2} \cos(a(\xi_z - \xi_{z'}))$$

(5.41)

At finite $c$ we can see a sharp fermionic character appear that broadens with increasing $c$ as shown in Fig. 5.7. The large time behavior is captured in a small window around $\xi = 0$. One can see that at any finite $c$, the region near zero develops a strong fermionic character, thus indicating that irrespective of the value of the coupling that we start
Figure 5.7: Normalized noise correlation function $C_2(\xi, -\xi)$. Fermionic correlations develop on a time scale $\tau \sim c^{-2}$, so that for any $c$ we get a sharp fermionic peak near $\xi = 0$, i.e., at large time. The key shows values of $ca$.

with, the model flows towards an infinitely repulsive model at large time, that can be described in terms of free fermions. We also obtained this result “at” $t = \infty$ at the beginning of this section.

For higher particle number, we see “interference fringes” corresponding to the number of particles, that get narrower and more numerous with an increase, memory of the initial lattice state. However, the asymptotic fermionic character does not disappear. Figures 5.8 and 5.9 show the noise correlation function for five and ten particles respectively. The large peaks are interspersed by smaller peaks and so on. This reflects the character of the initial state.

Figure 5.8: Normalized noise correlation function for five particles released for a Mott-like state for $c > 0$
Figure 5.9: Normalized noise correlation function for ten particles released for a Mott-like state for $c > 0$.

**Quenching from a bound state**

In this section our initial state is the ground state of the attractive Lieb-Liniger Hamiltonian (with interaction strength $-c_0 < 0$. For two bosons, this take the form\cite{94},

$$|\Psi_{\text{bound}}\rangle = \int e^{-c_0|x_1-x_2| - \frac{\sigma^2}{2\varphi^2}} \frac{x_1^3}{3\varphi^2} b^\dagger(x_1) b^\dagger(x_2)|0\rangle,$$  \hspace{1cm} (5.42)

and we quench it with a repulsive Hamiltonian. The long time noise correlations are displayed in Fig. 5.10. We see that while the initial state correlations are preserved over most of the evolution, in the asymptotic long time limit the characteristic fermionic dip. We expect similar effects for any number of bosons.

Figure 5.10: Normalized noise correlation function for two particle quenched from a bound state into the repulsive regime. The legend indicates the values of $\epsilon$ that the state is quenched into. We start with $c_0\varphi^2 = 3$, $\sigma = 1$, $c_0$ being the interaction strength of the initial state Hamiltonian. Again, we see the fermionic dip, but the rest of the structure is determined by the initial state.
5.4.2 Attractive interactions

For the attractive case, since the contours of integration are spread out in the imaginary direction, we have the contributions from the poles in addition to the stationary phase contributions at large time. The stationary phase contribution is picked up on the real line, but as we move the contour, it stays pinned above the poles and we need to include the residue obtained from going around them, leading to sum over several terms. Fig. 5.11 shows an example of how this works.

![Figure 5.11: Contribution from stationary phase and pole at large time in the attractive model. The blue (lower) contour represents the shifted contour.](image)

In Ref. [71], a formula was provided for the asymptotic state. Here we give a more careful treatment by taking into account that the fixed point of the approximation moves for terms that come from a pole of the $S$-matrix. It is therefore necessary to first shift the contours of integration, and then carry out the integral at long time. We carry this out below.

Shifting a contour over a pole leads to an additional term from the residue:

$$
\int_{\gamma_2} \frac{d\lambda_2}{2\pi} \rightarrow \int_{\gamma^R_2} \frac{d\lambda_2}{2\pi} - i\mathcal{R}(\lambda_2 \rightarrow \lambda_1 + i|c|) \quad (5.43)
$$

where $\mathcal{R}(x)$ indicates that we evaluate the residue given by the pole $x$. $\gamma_j$ indicates the original contour of integration and $\gamma^R_j$ indicates that integration is carried out over the real axis. Proceeding with the other variables we end up with

$$
\int_{\gamma_1,\gamma_2,\cdots,\gamma_N} \rightarrow \int_{\gamma^R_1} \left[ \int_{\gamma^R_2} + i\mathcal{R}(\lambda_2 \rightarrow \lambda_1 + i|c|) \right] \left[ \int_{\gamma^R_3} + i\mathcal{R}(\lambda_3 \rightarrow \lambda_1 + i|c|) + i\mathcal{R}(\lambda_3 \rightarrow \lambda_2 + i|c|) \right] \cdots \\
\times \left[ \int_{\gamma^R_N} + i\mathcal{R}(\lambda_N \rightarrow \lambda_1 + i|c|) + i\mathcal{R}(\lambda_N \rightarrow \lambda_2 + i|c|) + \cdots + i\mathcal{R}(\lambda_N \rightarrow \lambda_{N-1} + i|c|) \right] \quad (5.44)
$$
The integrals can now be evaluated using the stationary phase approximation. The correction produced by the above procedure does not affect the qualitative features observed in Ref. [71].

**Lattice initial state**

We now calculate the evolution of the density and the two body correlation function in order to compare with the repulsive case. We will first study the two particle case. Although we have a finite time expression for this case from which we can directly take a long time limit, we will study the asymptotics using the above scheme for an $N$-particle state, since we have an analytical expression to go with. We get two terms, the first being the stationary phase contribution, and is just like the repulsive case with $c \rightarrow -c$. The second is the contribution from the pole. It contains the bound state contribution which brings about another interesting feature of the attractive case. While the asymptotic dynamics of the repulsive model is solely dictated by the new variables $\xi_j \equiv \frac{y_j}{t} + ic$, and all the time dependence of the wave function enters through this “velocity” variable, this is not the case in the attractive model. While it is true that the system is naturally described in terms of $\xi$ variables, there still exists non-trivial time dependence.

First, we integrate out the $x$ dependence assuming an initial lattice-like state. This gives,

$$|\Psi_{\text{latt}}(t)\rangle = \int y \sum_{\xi_j^+} \prod_{i<j} S_{ij}(\xi_j^+ - \xi_j^-) \prod_j \frac{(4\pi \sigma^2)^{1/2}}{\sqrt{4\pi i t}}$$

$$\times e^{-(\sigma^2/2+it)(\xi_j^+)^2 + it(2\xi_j^- + \delta j + \delta (j-1))} b^\dagger(y_j)|0\rangle. \quad (5.45)$$

Defining $\phi(\xi, t)$ from $|\Psi_{\text{latt}}(t)\rangle = \int y \phi(\xi, t) \prod_j b^\dagger(y_j)|0\rangle$, we have for the density evolution under attractive interactions, $c < 0$,

$$\rho_{\text{latt}}(z; t) = \sum_{(p),j} \int y \delta(y_j - z) \phi^*(\xi_p, t) \phi(\xi, t) \quad (5.46)$$

We can show numerically (the expressions are a bit unwieldy to write here), that asymptotically, the density shows the same Gaussian profile that we expect from a uniformly diffusing gas, namely, $e^{-\xi^2 \sigma^2}$. 
With this, we can proceed to compute the noise correlation function. The two particle case is easy, as there are no more integrations to carry out. We get,

\[ \rho_{2\text{latt}}(z, z'; t) = \sum_{\{p\}, j, k} \int y \delta(y_j - z) \delta(y_k - z') \phi^*(\xi_p, t) \phi(\xi, t) \]

\[ = |\phi_s(\xi_z, \xi_z')|^2, \]

where \(\phi_s\) is the symmetrized wavefunction. Fig.5.12 shows the normalized noise correlations for different values of \(t\).

![Figure 5.12: Variation of \(C_2\) for the attractive case with time. Note the growth of the central peak. At larger times, the correlations away from zero fall off. \(ta^2 = 20, 40, 60\) for blue (top), magenta (middle) and yellow (bottom) respectively.]

For more particles, we see interference fringes similar to the repulsive case. We note that the central peak increases and sharpens with time, indicating increasing contribution from bound states to the correlations (see Fig. 5.13 for an example).

### 5.4.3 Starting with a condensate - attractive and repulsive interactions

In this section, we study the evolution of the Bose gas after a quench from an initial state where all the bosons are in a single level of a harmonic trap. For \(t < 0\), the state is described by

\[ |\Psi_{\text{cond}}\rangle = \int_x S_x \prod_j \frac{e^{-\frac{x_j^2}{2\sigma^2}}}{(\pi \sigma^2)^{\frac{3}{4}}} b^\dagger(x_j) |0\rangle. \]

(5.48)
Recall that in order to use the Yudson representation, the initial state needs to be ordered. We can rewrite the above state as

\[
|\Psi_{\text{cond}}\rangle = \int_x \theta(x_1 > \ldots > x_N) \times S_x \prod_j e^{-\frac{x_j^2}{2\sigma^2}} b^\dagger(x_j) |0\rangle \tag{5.49}
\]

where \( S \) is a symmetrizer. The time evolution can be carried out via the Yudson representation, and again, we concentrate on the asymptotics. For the repulsive model, the stationary phase contribution is all that appears, and we get

\[
|\vec{x}\rangle = \int \prod_{j<i} S_{ij} \left( \frac{y_i - y_j - x_i + x_j}{2t} \right) \prod_j \frac{1}{\sqrt{2\pi i t}} e^{-i \left( \frac{(y_j - x_j)^2}{2t} + i \frac{(y_i - x_i)^2}{2t} \right)} b^\dagger(y_j) |0\rangle. \tag{5.50}
\]

At large time \( t \), we therefore have

\[
|\Psi_{\text{cond}}(t)\rangle = \int \theta(x_1 > \ldots > x_N) \phi_2(x) I(y, x, t) \times \prod_j b^\dagger(y_j) |0\rangle \tag{5.51}
\]

\( \phi_2(x) \) is symmetric in \( x \). \( I(y, x, t) \) is symmetric in the \( y \) but not in the \( x \). Therefore we have to carry out the \( x \) integration over the wedge \( x_1 > \ldots > x_N \). This is not straightforward to carry out. If \( I(y, x, t) \) was also symmetric in \( x \), then we can add the other wedges to rebuild the full space in \( x \). However, due to the \( S \)-matrix factors, symmetrizing in \( y \) does not automatically symmetrize in \( x \). The exponential factors on the other hand are automatically symmetric in both variables if one of them is symmetrized because their functional dependence is of the form \( f(y_j - x_j) \). It is however
possible to make the $S$-matrix factors approximately symmetric in $x$, and we will define what we mean by approximately shortly. What is important is to obtain a $y_j - x_j$ dependence. As of now, the $S$-matrix that appears in the above expression is

$$ S_{ij}^y \left( \frac{y_i - y_j - x_i + x_j}{2t} \right) = \frac{y_i - y_j - x_i + x_j}{2t} - ic \text{sgn}(y_i - y_j) $$

(5.52)

First, we can change $\text{sgn}(y_i - y_j)$ to $\text{sgn} \left( \frac{y_i - y_j}{2t} \right)$ since $t > 0$. Next, note that asymptotically in time, the stationary phase contribution comes from $\frac{y}{2t} \sim \mathcal{O}(1)$. However, since $x$ has finite extent, at large enough time, $\frac{x}{2t} \sim 0$. We are therefore justified in writing $\text{sgn} \left( \frac{y_i - x_i}{2t} - \frac{y_j - x_j}{2t} \right)$. The only problem could arise when $y_i \sim y_j$. However, if this occurs, then the $S$-matrix is approximately $\text{sgn}(y_i - y_j)$ which is antisymmetric in $ij$. With this prefactor the particles are effectively fermions, and therefore at $y_i \sim y_j$, the wave-function has an approximate node. At large time therefore, we do not have to be concerned with the possibility of particles overlapping, and including the $x_i$ inside the $\text{sgn}$ function is valid. With this change the $S$-matrix also becomes a function of $y_j - x_j$ and symmetrizing over $y$ one automatically symmetrizes over $x$.

In short, we have established that the wave function asymptotically in time can be made symmetric in $x$. This allows us to rebuild the full space. We get

$$ |\Psi_{\text{cond}}(t)\rangle = \int_{x,y} \theta(x) \phi_2(x) I^x(y, x, t) \prod_j b^\dagger_j(y_j) |0\rangle $$

$$ = \int_{x,y} \phi_2(x) I^x(y, x, t) \prod_j b^\dagger_j(y_j) |0\rangle 

(5.53)$$

where the $s$ superscript indicates that we have established that $I(y, x, t)$ is also symmetric in $x$. With this in mind, we can do away with the ordering when we’re integrating over the $x$ if we symmetrize the initial state wave function and the final wave function. Note that when we calculate the expectation value of a physical observable, the symmetry of the wavefunction is automatically enforced, and thus taken care of automatically.

Recall that when we calculated the noise correlations of the repulsive gas, in order to get an analytic expression for $N$ particles, we considered the leading order term, i.e., the HBT term. We did this by showing that higher order crossings produced terms
higher order in $e^{-2ca}$ which we claimed was a small number. Now, however, $a = 0$, and although the calculation is essentially the same with our approximate symmetrization, this simplification does not occur. The two and three particle results remain analytically calculable, but for higher numbers, we have to resort to numerical integration.

Fig. 5.14 shows the noise correlation for two and three repulsive bosons starting from a condensate. For non-interacting particles, we expect a straight line $C_2 = \frac{1}{2}$. When repulsive interactions are turned on, we see the characteristic fermionic dip develop. The plots for the attractive Bose gas are shown in Figs. 5.15 and 5.16. As expected from the non-interacting case the oscillations arising from the interference of particles separated spatially does not appear. The attractive however does show the oscillations near the central peak that are also visible in the case when we start from a lattice-like state. It is interesting to note that for three particles we do not see any additional structure develop in the attractive case.

Figure 5.14: $C_2(\xi, -\xi)$ for two (blue, bottom) and three (magenta, top) repulsive bosons starting from a condensate. Unlike the attractive case, there is no explicit time dependence asymptotically. $ca = 3$

5.5 Conclusions and the dynamic RG hypothesis

We have shown that the Yudson contour integral representation for arbitrary states can indeed be used to understand aspects of the quench dynamics of the Lieb-Liniger model, and obtain the asymptotic wave functions exactly. The representation overcomes some of the major difficulties involved in using the Bethe-Ansatz to study the
Figure 5.15: Noise correlation for two attractive bosons starting from a condensate - as
time increases from blue (top) to yellow (bottom), the central peak dominates.

Figure 5.16: $C_2(\xi, -\xi)$ for three attractive bosons starting from a condensate. Note that
the side peak structure found in fig. 5.13 is missing due to the initial condition. We
show the evolution at three times. As time increases, the oscillations near the central
peak die out. Times from top to bottom $tc^2 = 20, 40, 60$.

dynamics of some integrable systems by automatically accounting for complicated
states in the spectrum.

We see some interesting dynamical effects at long times. The infinite time limit of
the repulsive model corresponds to particles evolving with a free fermionic Hamiltonian.
It retains, however, memory of the initial state and therefore is not a thermal state. The correlation functions approach that of hard core bosons at long time indicating a dynamical increase in interaction strength. The attractive model also shows a dynamic strengthening of the interaction and the long time limit is dominated by a
multiparticle bound state. This of course does not mean that it condenses. In fact the
state diffuses over time, but remains strongly correlated.

We may interpret our results in terms of a “dynamic RG” in time. The asymptotic evolutions of the model both for $c > 0$ and for $c < 0$ are given by the Hamiltonians $H^*_\pm$ with $c \to \pm \infty$ respectively. Accepting the RG logic behind the conjecture one would expect that there would be basins of attraction around the Lieb-Liniger Hamiltonian with models whose long time evolution would bring them close to the “dynamic fixed points” $H^*_\pm$. One such Hamiltonian would have short range potentials replacing the $\delta$-function interaction that renders the Lieb-Liniger model integrable. Perhaps, lattice models could be also found in this basin whose time asymptotics would be close, in the repulsive case, to that given by a free fermionic model on the lattice. Clearly, as discussed earlier, the Bose-Hubbard model is not such a model since it has a lattice symmetry that is not present in the Lieb-Liniger model. This could be however overcome by adding such terms as the next nearest hopping or interactions that break this symmetry, or as shown in Appendix A, with an appropriate choice of initial state.

We have to emphasize, however, that as these models are not integrable, we do not expect that they would actually flow to $H^*_\pm$. Instead, starting close enough in the “basin”, they would flow close to $H^*_\pm$ and spend much time in its neighborhood, eventually evolving into another, thermal state. We thus conjecture that away from integrability, a system would approach the corresponding non-thermal equilibrium, where the dynamics will slow down leading to a “prethermal” state [10]. Fig. 5.17 shows a schematic of this. Such prethermalization behavior has indeed been observed in lattice models [85]. The system is expected to eventually find a thermal state. It is therefore of interest to characterize different ways of breaking integrability to see when a system is “too far” from integrability to see this effect and in what regimes a system can be considered as close to integrability. For a review and background on this subject, see Ref. [112].

Further, the flow diagram in Fig. 5.17 might have another axis that represents initial states. Studying the Bose-Hubbard model shows an interesting initial state dependence. Whereas the sign of the interaction does not affect the quench dynamics, the asymptotic state depends strongly on the initial state, with a lattice-like state leading to
fermionization, and a condensate-like state retaining bosonic correlations. The strong
dependence on the initial state in the quench dynamics is evident from eq. (2.11) and
is subject of much debate, in particular as relating to the Eigenstate Thermalization
Hypothesis [117, 130, 39].

\[ \begin{align*}
\text{state} & \quad \text{pre-thermalization} \\
\text{thermal state} & \quad \text{non-integrable model}
\end{align*} \]

Figure 5.17: Schematic showing pre-thermalization of states in a non-integrable model

This work also opens up several new questions. It provides a prediction for exper-
iments that can be carried out in the context of continuum cold atom systems (though
the experiments we are aware of are carried out on the lattice and therefore described
by the Bose-Hubbard model) Theoretically, while the representation is provable math-
ematically, further investigation is required to understand, physically, how it achieves
the tedious sum over eigenstates, while automatically accounting for the details of the
spectrum. This would allow us to extend the approach to other models with a more
complicated S-matrix structure. It would also be useful to tie this approach to other
means of calculating overlaps in the Algebraic Bethe Ansatz, i.e., the form-factor ap-
proach. The representation can essentially be thought of as a different way of writing
the identity operator. From that standpoint, it could serve as a new way of evaluating
correlation functions using the Bethe Ansatz. We are currently studying generaliza-
tions of this approach to other models that can be realized in optical lattices.
Chapter 6

Quench dynamics of the Heisenberg chain

In this Chapter, we show some preliminary results regarding the quench dynamics of the isotropic Heisenberg chain, or the XXX model. We’ll start with the noninteracting version, the XX model and try to understand the time evolution of the spin current starting from a domain wall state. Before we do that, we’ll work out the Yudson representation for the Heisenberg model, starting with the noninteracting case of the XX model.

6.1 XX model

We’ll start with the XX Heisenberg chain in one dimension. The Hamiltonian is

\[ H = -J \sum_{j=-M}^{M} \sigma^x_j \sigma^x_{j+1} + \sigma^y_j \sigma^y_{j+1} \]  

(6.1)

It describes spins hopping on a lattice of \( M \) sites. We will take \( M \) to be very large, and consequently, will not worry about the boundary conditions at this point. Our goal is to work in an open system. The above model is non-interacting since there is no \( \sigma^z \) term. The eigenstates can be written by inspection:

\[ |k\rangle = \sum_{P} \sum_{m_i=-M}^{M} \prod_{j=1}^{N} e^{ik_j m_j} \sigma^-_{m_j}|0\rangle \]  

(6.2)

where \( k_j \) are the momenta, and the vacuum state \( |0\rangle \) is defined as the state with all spins up, i.e., \( \sigma^+_j |0\rangle = 0 \). Acting on the above state with the Hamiltonian (6.1), we get,

\[ H|k\rangle = -2J \sum_{p} \sum_{m_i} \sum_{l \neq 1} \prod_{j \neq l} e^{ik_p m_j + ik_j m_l} (\sigma^-_{m_j+1} + \sigma^-_{m_j-1}) \sigma^-_{m_l}|0\rangle \]

\[ = -2J \sum_{p} \sum_{m_i} \sum_{l} \prod_{j} e^{ik_p m_j} 2 \cos(k_P) \sigma^-_{m_j}|0\rangle = -4J \left( \sum_{l} \cos k_l \right) |k\rangle. \]  

(6.3)
The energy eigenvalues are given by
\[ E = -4J \sum_{j=1}^{N} \cos k_j, \quad k \in [-\pi, \pi]. \tag{6.4} \]

The spin of the above state,
\[ S_z = \frac{M}{2} - N, \quad N \leq M. \tag{6.5} \]

The above can also be alternatively be carried out in the language of fermions using the Jordan-Wigner transformation. We will however, stick to the spin operator notation.

### 6.1.1 The Yudson representation in the natural parametrization

We are interested in a Yudson-like integral representation for an arbitrary state. Let’s say we start with a state that has \( N \) spins flipped. A general state can be written as,
\[ |\Psi_0\rangle = \prod_{j=1}^{N} \sigma_{n_j}^- |0\rangle. \tag{6.6} \]

We can also assume without loss of generality that \( n_1 > n_2 > \cdots > n_N \). Our goal is to show that
\[ |\Psi_0\rangle = \int \{ \vec{k} \} |\vec{k}\rangle \langle \vec{k}| \Psi_0 \rangle, \tag{6.7} \]
where we are yet to determine the meaning of the “Yudson state” \( |\vec{k}\rangle \) and the contours of integration. Given that this is a non-interacting model, these are fairly obvious. We chose
\[ |\vec{k}\rangle = \sum_{m_j} \theta(m_1 > \cdots > m_N) \prod_{j=1}^{N} e^{ik_j m_j} \sigma_{m_j}^- |0\rangle \tag{6.8} \]
and integrate all the \( k_j \) from \([-\pi, \pi]\). We get,
\[ |\Psi_0\rangle = \sum_{m_j} \prod_{j=1}^{N} \int_{-\pi}^{\pi} \frac{dk_j}{2\pi} e^{ik_j (m_j - n_j)} \sigma_{m_j}^- |0\rangle \]
\[ = \sum_{m_j} \prod_{j=1}^{N} \frac{\sin(\pi (m_j - n_j))}{\pi (m_j - n_j)} \sigma_{m_j}^- |0\rangle. \tag{6.9} \]

Since \( m_j - n_j \) is an integer, the above function is zero unless \( m_j = n_j \) for all \( j \), thus proving the claim. We can now carry out the time evolution by acting on it with \( e^{-iHt} \).
The finite time state is given by,

\[ |\Psi(t)\rangle = \sum_{m_i} \prod_{j=1}^{N} \int_{-\pi}^{\pi} \frac{d k_j}{2\pi} e^{i 4J t \cos k_j} e^{i k_j (m_j - n_j)} \sigma^-_{m_j} |0\rangle \]

\[ = \sum_{m_i} \prod_{j} J^B_{m_j - n_j} (-4Jt) \sigma^-_{m_j} |0\rangle. \]  

(6.10)

where \( J^B_n \) is the Bessel function of the first kind. The expectation value of the spin at some site \( m \), e.g., evolves as

\[ \langle \sigma^z_m \rangle = \langle \Psi(t)|\sigma^z_m|\Psi(t)\rangle \]

\[ = \sum_{m_i, m'_i} \prod_{j} J^B_{m_j - n_j} (-4Jt) J^B_{m'_j - n_j} (-4Jt) \langle 0|\sigma^z_{m_j} \sigma^z_{m_j} |0\rangle \]

\[ = \sum_{m_i, m'_i} \prod_{j} J^B_{m_j - n_j} (-4Jt) J^B_{m'_j - n_j} (-4Jt) (1 - 2\delta_{m, m_j}) \langle 0|\sigma^z_{m_j} \sigma^-_{m_j} |0\rangle \]

\[ = \sum_{m_i} \prod_{j} J^B_{m_j - n_j} (-4Jt) J^B_{m_j - n_j} (-4Jt) (1 - 2\delta_{m, m_j}) \]

\[ = \sum_{m_i} \prod_{j} J^B_{m_j - n_j} (-4Jt) J^B_{m_j - n_j} (-4Jt) (1 - 2\delta_{m, m_j}). \]  

(6.11)

### 6.1.2 The \( \lambda \) Parametrization

We’ll also work out the above in another parametrization of the momentum. In view of having eventually to deal with the XXX model, we will reparametrize the momenta in terms of rapidities,

\[ k = 2 \tan^{-1} 2\lambda. \]  

(6.12)

For \( k \in [-\pi, \pi] \), \( \lambda \in [-\infty, \infty] \). The energy becomes

\[ E = -4J \sum_j \cos k_j = -2J \sum_j \left( \frac{1}{\lambda_j^2 + \frac{1}{4}} - 2 \right). \]  

(6.13)

We can always shift the energies to eliminate the constant term. The measure of the integral in the Yudson representation transforms as well:

\[ dk \rightarrow \frac{d\lambda}{\lambda^2 + \frac{1}{4}} \]

(6.14)

and \( \lambda \in [-\infty, \infty] \). In this language, the Yudson representation becomes

\[ |\Psi_0\rangle = \sum_{y_i} \prod_{j} \int_{-\infty}^{\infty} \frac{d\lambda_j}{2\pi(\lambda_j^2 + \frac{1}{4})} e^{i2\tan^{-1}(2\lambda_j)(m_j - n_j)} \sigma^-_{m_j} |0\rangle \]

\[ = \sum_{y_i} \prod_{j} \int_{-\infty}^{\infty} \frac{d\lambda_j}{2\pi(\lambda_j^2 + \frac{1}{4})} \left( -\frac{\lambda_j - i/2}{\lambda_j + i/2} \right)^{m_j - n_j} \sigma^-_{m_j} |0\rangle. \]  

(6.15)
Our reparametrization has therefore introduced a new analytic structure on the integrand. The measure now has poles as $\pm i$. The integrand has poles at $+i/2$ or $-i/2$ depending on whether $m > n$ or $n > m$. Since in this simple case, the integrand factorizes, let us consider the $j$-th integral. If $m_j > n_j$, then the integrand has a pole (in general, multiple, but no branch-cuts) at $\lambda_j = -i/2$. We want to avoid this pole since it is a multiple pole. In order to carry out the integral we are allowed to close the contour in either the upper or the lower half plane and use the residue theorem. We choose to close the contour in the upper half plane picking up the pole at $\lambda_j = i/2$ coming from the measure. However, the integrand vanishes at $\lambda_j = i/2$, and so we must have $m_j \leq n_j$. For $m_j < n_j$, the integrand has a pole at $\lambda_j = i/2$, and so to avoid it, we close the contour in the lower half plane picking up the pole at $-i/2$ from the measure. But, again, at $-i/2$, the integrand vanishes. So, we must have that $m_j = n_j$. In order to calculate the value of the integral, we can use the residue theorem again. We have,

$$\langle \Psi_0 \rangle = \sum_{y_i} \prod_j \int_{-\infty}^{\infty} d\lambda_j \frac{d\lambda_j}{2\pi(\lambda_j^2 + 1/4)} \delta_{m_j, n_j} \sigma_{m_j}^- |0\rangle$$

$$= \prod_{j=1}^{N} \theta(x_j) \langle \sigma_{m_j}^- |0\rangle.$$  

(6.16)

thus proving the identity.

At finite time, it might seem that the exponent is badly behaved at the poles, but a straightforward series expansion of the exponent, followed by integration, and resummation will give us the Bessel function obtained in the previous section. At any rate, the fact that poles appear in the “plane wave” part of the wave function is unusual. In Appendix B, we show how these poles physically correspond to hopping produced by the time evolution.

One can show that after a long time evolution, the system becomes quadratic free fermions, which is expected as the low-energy physics of fermions on a lattice: $E_k \sim \cos k = 1 - \frac{k^2}{2} + \ldots$.

We will now shift our attention to understanding the ferromagnetic XXX model, which has a non-trivial two particle S-matrix. The structure of the S-matrix will determine the integration contours. We follow the description of this model provided in [86].
6.1.3 Long time asymptotics

While we can obtain the finite time wavefunction explicitly in the free model, it is instructive to study the asymptotic behavior directly in both the $k$ and the $\lambda$ languages.

In the $k$ language, the integral we have is

$$
\int_k e^{4iJt \cos k} e^{ik(m-n)}
$$

(6.17)

At long time, we can do a stationary phase approximation:

$$
\frac{d}{dk} [4iJt \cos k + ik(m-n)] = 0 \implies \sin k = \frac{m-n}{4Jt}.
$$

(6.18)

The value of the integral is therefore (up to prefactors)

$$
\exp \left[ \sqrt{1 - \left( \frac{m-n}{4Jt} \right)^2} \right] \left\{ \sqrt{1 - \left( \frac{m-n}{4Jt} \right)^2 + i \frac{m-n}{4Jt}} \right\}.
$$

(6.19)

For the energy to be a real number, we require

$$
\left| \frac{m-n}{4Jt} \right| \leq 1
$$

(6.20)

which is like a Lieb-Robinson [95] bound.

In the $\lambda$ language, we have

$$
\int_\lambda e^{\frac{2iJt}{(\lambda^2 + \frac{1}{4})} + i2 \tan^{-1}(2\lambda)(m-n)}
$$

(6.21)

The stationary phase approximation yields

$$
\frac{d}{d\lambda} \left[ \frac{2iJt}{(\lambda^2 + \frac{1}{4})} + i2 \tan^{-1}(2\lambda)(m-n) \right] \implies \lambda = \frac{1 \pm \sqrt{1 - \left( \frac{m-n}{4Jt} \right)^2}}{(m-n)/2Jt}.
$$

(6.22)

This also produces the same bound as above.

6.2 XXX model

The XXX Hamiltonian is given by

$$
H = -J \sum_j \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z - 1 \right).
$$

(6.23)
The model has two vacua, one with all spins pointing up, and the other with all down. We’ll use the “up” vacuum.

\[ \sigma_j^+ |0\rangle \equiv 0. \quad (6.24) \]

Also, we introduce the spin raising and lowering operators,

\[ \sigma_j^\pm \equiv \frac{\sigma_j^x \pm i \sigma_j^y}{2}. \quad (6.25) \]

Their commutation relations are,

\[ [\sigma_j^+, \sigma_m^-] = [\sigma_m^+, \sigma_j^-] = 0, \quad [\sigma_j^z, \sigma_m^\pm] = \pm 2 \sigma_m^\pm \delta_{mn}. \quad (6.26) \]

Rewriting the Hamiltonian (6.23) using the above operators, we get,

\[ H = -J \sum_j \left( 2 \left( \sigma_j^- \sigma_{j+1}^+ + \sigma_{j+1}^- \sigma_j^+ \right) + \sigma_j^z \sigma_{j+1}^z - 1 \right). \quad (6.27) \]

6.2.1 Eigenstates

The one particle eigenstates of this model are:

\[ |\lambda_1\rangle = \sum_{m_1} \left( \frac{\lambda_1 + i/2}{\lambda_1 - i/2} \right)^{m_1} \sigma_m^- |0\rangle. \quad (6.28) \]

We can check this explicitly.

\[ H|\lambda_1\rangle = -2J \sum_{m_1} \left( \frac{\lambda_1 + i/2}{\lambda_1 - i/2} \right)^{m_1} (\sigma_{m_1+1}^- + \sigma_{m_1-1}^- - 2 \sigma_{m_1}^-) |0\rangle \]

\[ = -2J \sum_{m_1} \left( \frac{\lambda_1 + i/2}{\lambda_1 - i/2} \right)^{m_1} \left( \frac{\lambda_1 + i/2}{\lambda_1 - i/2} + \frac{\lambda_1 - i/2}{\lambda_1 + i/2} - 2 \right) \sigma_{m_1}^- |0\rangle \]

\[ = -2J \frac{-1}{\lambda_1^2 + \frac{i}{4}} |\lambda_1\rangle. \quad (6.29) \]

So for one spin down, the energy is \( E(\lambda_1) \equiv \frac{2J}{\lambda_1 + \frac{i}{4}}. \)

We now need the two particle eigenstates of this model. In what follows, we’ll only check if the state below is a solution of the Schrödinger equation. This generally follows the approach in [109].

\[ |\lambda_1, \lambda_2\rangle = \sum_{m_1, m_2} \frac{\lambda_1 - \lambda_2 - i \text{sgn}(m_1 - m_2)}{\lambda_1 - \lambda_2 - i} \left( \frac{\lambda_1 + i/2}{\lambda_1 - i/2} \right)^{m_1} \left( \frac{\lambda_2 + i/2}{\lambda_2 - i/2} \right)^{m_2} \sigma_{m_1}^z \sigma_{m_2}^- |0\rangle. \quad (6.30) \]
$m_1$ and $m_2$ are summed over $Z$. It is important that we work on an infinite line, since making it finite will necessitate the use of boundary conditions, which will constrain the momenta. We will first consider the case when the two spins are not neighboring. The sgn function in the S-matrix is not affected as the hopping cannot interchange spins that aren’t neighboring. So,

$$H|\lambda_1, \lambda_2\rangle_{\text{far}} = -2J \sum_{m_1,m_2} \sum_i \frac{\lambda_1 - \lambda_2 - i \text{sgn}(m_1 - m_2)}{\lambda_1 - \lambda_2 - i} \left( \frac{\lambda_1 + i/2}{\lambda_1 - i/2} \right)^{m_1} \left( \frac{\lambda_2 + i/2}{\lambda_2 - i/2} \right)^{m_2} \times \left( \sigma_j^{-} \sigma_{j+1}^{+} + \sigma_j^{+} \sigma_{j+1}^{-} + \frac{1}{2} \sigma_j^{+} \sigma_{j+1}^{+} \sigma_{m_1}^{\sigma} \sigma_{m_2}^{\sigma} |0\rangle \right)$$

$$= -2J \sum_{m_1,m_2} \frac{\lambda_1 - \lambda_2 - i \text{sgn}(m_1 - m_2)}{\lambda_1 - \lambda_2 - i} \left( \frac{\lambda_1 + i/2}{\lambda_1 - i/2} \right)^{m_1} \left( \frac{\lambda_2 + i/2}{\lambda_2 - i/2} \right)^{m_2} \times \left( \sigma_{m_1}^{-} \sigma_{m_2}^{-} + \sigma_{m_1}^{\sigma} \sigma_{m_2}^{\sigma} \right)$$

$$= -2J \sum_{m_1,m_2} \frac{\lambda_1 - \lambda_2 - i \text{sgn}(m_1 - m_2)}{\lambda_1 - \lambda_2 - i} \left( \frac{\lambda_1 + i/2}{\lambda_1 - i/2} \right)^{m_1} \left( \frac{\lambda_2 + i/2}{\lambda_2 - i/2} \right)^{m_2} \times \left( \delta_{m_1,m_2+1} \frac{\lambda_1 + i/2}{\lambda_1 - i/2} + \delta_{m_1+1,m_2} \frac{\lambda_2 + i/2}{\lambda_2 - i/2} \right)$$

$$+ \delta_{m_1,m_2+1} \frac{\lambda_2 - i/2}{\lambda_2 + i/2} \sigma_{m_1}^{\sigma} \sigma_{m_2}^{\sigma} |0\rangle.$$

(6.31)

In the above expression, “far” indicates that the next to nearest neighbor state that appears in it is incomplete. The remaining piece will come from the below state. We now consider the case of two neighboring spins. This is the only case where the interaction
is active. The state we consider is,

\[ |\lambda_1, \lambda_2\rangle_{\text{near}} = \sum_{m_1, m_2} (\delta_{m_1+1, m_2} + \delta_{m_1, m_2+1}) \frac{\lambda_1 - \lambda_2 - i \text{sgn}(m_1 - m_2)}{\lambda_1 - \lambda_2 - i} \]

\[ \times \left( \frac{\lambda_1 + i/2}{\lambda_1 - i/2} \right)^{m_1} \left( \frac{\lambda_2 + i/2}{\lambda_2 - i/2} \right)^{m_2} \sigma_m^+ \sigma_{m+1}^- |0\rangle \]

\[ = \sum_{m_1} \left( \frac{\lambda_1 + i/2}{\lambda_1 - i/2} \right)^{m_1} \left( \frac{\lambda_2 + i/2}{\lambda_2 - i/2} \right)^{m_1} \left[ \lambda_1 - \lambda_2 + i \frac{\lambda_2 + i/2}{\lambda_1 - i/2} + \frac{\lambda_1 + i/2}{\lambda_1 - i/2} \right] \sigma_m^+ \sigma_{m+1}^- |0\rangle \]

For two neighboring down spins, the kinetic terms can either put the spins on top of each other, a process which is forbidden in the spin-\( \frac{1}{2} \) model, or it can separate the spins by a single up spin. This changes the energy of the system. For antiferromagnetic coupling, this decreases the energy.

\[ H|\lambda_1, \lambda_2\rangle_{\text{near}} = -2J \sum_j \sum_{m_1, m_2} (\delta_{m_1+1, m_2} + \delta_{m_1, m_2+1}) \frac{\lambda_1 - \lambda_2 - i \text{sgn}(m_1 - m_2)}{\lambda_1 - \lambda_2 - i} \]

\[ \times \left( \frac{\lambda_1 + i/2}{\lambda_1 - i/2} \right)^{m_1} \left( \frac{\lambda_2 + i/2}{\lambda_2 - i/2} \right)^{m_2} \sigma_j^- \sigma_{j+1}^+ + \sigma_{j+1}^- \sigma_j^+ + \frac{1}{2} \sigma_j^\sigma_{j+1}^\dagger \sigma_{m_1}^+ \sigma_{m_2}^- |0\rangle \]

\[ = -2J \sum_j \sum_{m_1} \left[ \frac{\lambda_1 - \lambda_2 + i}{\lambda_1 - \lambda_2 - i} \left( \frac{\lambda_1 + i/2}{\lambda_1 - i/2} \right)^{m_1} \left( \frac{\lambda_2 + i/2}{\lambda_2 - i/2} \right)^{m_1+1} \right] \sigma_j^- \sigma_{j+1}^+ + \sigma_{j+1}^- \sigma_j^+ + \frac{1}{2} \sigma_j^\sigma_{j+1}^\dagger \sigma_{m_1}^+ \sigma_{m_2}^- |0\rangle \]

\[ = -2J \sum_{m_1} \left[ \frac{\lambda_1 - \lambda_2 + i}{\lambda_1 - \lambda_2 - i} \left( \frac{\lambda_1 + i/2}{\lambda_1 - i/2} \right)^{m_1+1} \right] \sigma_{m_1}^- \sigma_{m_1+1}^+ + \sigma_{m_1+1}^- \sigma_{m_1}^+ - 2\sigma_{m_1}^\sigma_{m_1+1}^\dagger \sigma_{m_1}^\sigma_{m_1+1}^\dagger |0\rangle \]

\[ + \left. \left[ \frac{\lambda_1 + i/2}{\lambda_1 - i/2} \right] \left( \frac{\lambda_2 + i/2}{\lambda_2 - i/2} \right)^{m_1-1} \right] \sigma_{m_1-1}^- \sigma_{m_1}^+ + \sigma_{m_1}^- \sigma_{m_1+1}^+ - 2\sigma_{m_1}^\sigma_{m_1+1}^\dagger \sigma_{m_1}^\sigma_{m_1+1}^\dagger |0\rangle \]

\[ (6.33) \]
Adding (6.33) and (6.31), we get

\[
H(|\lambda_1, \lambda_2\rangle_{\text{far}} + |\lambda_1, \lambda_2\rangle_{\text{near}})
= [E(\lambda_1) + E(\lambda_2)]|\lambda_1, \lambda_2\rangle_{\text{far'}} - 2J \sum_{m_1, m_2} \frac{\lambda_1 - \lambda_2 - i \text{sgn}(m_1 - m_2)}{\lambda_1 - \lambda_2 - i} \left( \frac{\lambda_1 + i/2}{\lambda_1 - i/2} \right)^{m_1} \left( \frac{\lambda_2 + i/2}{\lambda_2 - i/2} \right)^{m_2} |\delta_{m_1, m_2 + 1}\rangle
\times \left( \frac{\lambda_1 + i/2 + \lambda_2 - i/2}{\lambda_1 - i/2 + \lambda_2 - 2i} + \delta_{m_1 + 1, m_2} \left( \frac{\lambda_1 - i/2 + \lambda_2 + i/2}{\lambda_1 + i/2 + \lambda_2 - i/2 - 2} \right) \right) \sigma_{m_1}^{-} \sigma_{m_2}^{-} |0\rangle
- 2J \sum_{m_1} \left[ \frac{\lambda_1 - \lambda_2 + i}{\lambda_1 - \lambda_2 - i} \left( \frac{\lambda_1 + i/2}{\lambda_1 - i/2} \right)^{m_1} \left( \frac{\lambda_2 + i/2}{\lambda_2 - i/2} \right)^{m_1 + 1} \right] \left\{ \sigma_{m_1}^{-1} \sigma_{m_1 + 1}^{-} \right\} + \left( \frac{\lambda_1 + i/2}{\lambda_1 - i/2} \right)^{m_1} \left( \frac{\lambda_2 + i/2}{\lambda_2 - i/2} \right)^{m_1 - 1} \left\{ \sigma_{m_1 + 1}^{-1} \sigma_{m_1 - 1}^{-} + \sigma_{m_1}^{-1} \sigma_{m_1 - 2}^{-} \right\} |0\rangle
= [E(\lambda_1) + E(\lambda_2)]|\lambda_1, \lambda_2\rangle_{\text{far}} - 2J \sum_{m_1} \left( \frac{\lambda_1 + i/2}{\lambda_1 - i/2} \right)^{m_1} \left( \frac{\lambda_2 + i/2}{\lambda_2 - i/2} \right)^{m_1}
\times \left[ \frac{\lambda_1 - \lambda_2 + i}{\lambda_1 - \lambda_2 - i} \left( \frac{\lambda_1 - i/2 + \lambda_2 + i/2}{\lambda_1 + i/2 + \lambda_2 - i/2 - 2} \right) \right] \sigma_{m_1}^{-} \sigma_{m_1 + 1}^{-} |0\rangle
= [E(\lambda_1) + E(\lambda_2)]|\lambda_1, \lambda_2\rangle.
\]

(6.34)

In the first equation above, the first term and the third term combine to complete the non-neighboring states, and the remaining piece can be checked to produce the required term.

We can therefore write the \( N \) particle eigenstates of the model:

\[
|\bar{\lambda}\rangle = \sum_{m_i, j = 1 \ldots N} \prod_{i < j} \frac{\lambda_i - \lambda_j - i \text{sgn}(m_i - m_j)}{\lambda_i - \lambda_j - i} \prod_{j} \left( \frac{\lambda_j + i/2}{\lambda_j - i/2} \right)^{m_j} \sigma_{m_j}^{-} |0\rangle,
\]

(6.35)

with energy \( \sum_{j=1}^{N} E(\lambda_j) \). The Yang-Baxter relations are trivially satisfied.

The above can also be written using the parametrization shown in the first section. They are equivalent.

### 6.2.2 Yudson representation

In this section, we’ll try to use a different set of contours and try to prove the identity. This set of contours is identical to the ones that seem to work for the XXZ model in the
−1 < Δ < 1 regime. Fig. 6.1 shows the contour of integration and the poles from the S-matrix. We will show that

\[
\int \left[ \prod_j \frac{d\lambda_j}{2\pi} \right] \prod_{i<j} \frac{\lambda_i - \lambda_j - i \text{sgn}(m_i - m_j)}{\lambda_i - \lambda_j - i} \prod_j \left( \frac{\lambda_j + i/2}{\lambda_j - i/2} \right)^{m_j - n_j} \frac{1}{\lambda_j^2 + 1/4} = \prod_j \delta_{m_j n_j}
\]

(6.36)

First note that if we indeed integrate along the red contour, the S-matrix poles don’t contribute since they’re outside the contour. Given that, for \( m_j > n_j \), the pole coming from the “plane-wave” is at \( \lambda_j = i/2 \) and therefore is also outside this contour. The red contour therefore encloses no poles and its residue is zero. We therefore have that for all \( j \), \( m_j \leq n_j \).

It remains to show that for \( m_j < n_j \), the integral is again zero. Assume \( m_1 < n_1 \). \( \lambda_1 \) has the following poles:

\[
\lambda_1^* = \begin{cases} 
-i/2 \\
\lambda_j + i, \forall j > 1
\end{cases}
\]

(6.37)

We want to avoid calculating the residue from the pole at \(-i/2\) since it is a multiple
pole. To do this, we deform the contour for $\lambda_1$ as shown in Fig. 6.2. It is important

to ensure that during this deformation, we are not intersecting any poles. The poles
that could potentially create a problem are those from the $S$-matrices. These are $\lambda_j^* = \lambda_1 - i, \forall j > 1$. The integration contours for $\lambda_{j>1}$ are still along the red path in Fig. 6.1.
The deformation of the $\lambda_1$ contour does not bring any new poles into the the contours
for $\lambda_{j>1}$, and is therefore allowed. We deform it all the way to $-i\infty$ where of course
the integrand vanishes. We’re then left with an integral along the real axis. We now,
close the contour in the upper half plane, thus capturing all the $S$-matrix poles. We
evaluate the residue to get

$$\int \left[ \prod_{j>1} \frac{1}{2\pi} \right] \sum_{j>1} \left\{ [i - i \text{sgn}(m_1 - m_j)] \left( \frac{\lambda_j + \frac{3i}{2}}{\lambda_j + \frac{i}{2}} \right)^{m_1 - n_1} \frac{1}{(\lambda_j + i)^2 + \frac{1}{4}} \right\} \times \prod_{k \neq j} \frac{\lambda_j - \lambda_k + i - i \text{sgn}(m_1 - m_k)}{\lambda_j - \lambda_k} \prod_{i<k, i \neq j} \frac{\lambda_i - \lambda_k - i \text{sgn}(m_i - m_k)}{\lambda_i - \lambda_k - i} \times \prod_{j \neq 1} \left( \frac{\lambda_j + i/2}{\lambda_j - i/2} \right)^{m_j - n_j} \frac{1}{\lambda_j^2 + 1/4} \quad (6.38)$$

The above is a sum of $N - 1$ terms. Consider the $\lambda_j$ integral. The integral is zero unless there’s a pole at $\lambda_j = -i/2$. In the integrand, this term appears as

$$\frac{1}{(\lambda_j + i)^{m_1 - n_1 + n_j - m_j + 2}} \quad (6.39)$$

We also have $m_1 < m_j$ and $n_1 > n_j$, i.e., $m_1 - m_j \leq -1$ and $n_j - n_1 \leq -1$. Therefore, $m_1 - m_j + n_j - n_1 + 2 \leq 0$, and this isn’t a pole. The pole at $\lambda_j^* = \lambda_k$ is now a concern. If we started out with contours that were concentric rather than on top of each other, then depending on whether $j > k$ or $j < k$ we’ll have a pole at this point. Let’s assume that this pole indeed exists. In order to evaluate the $\lambda_j$ integral, we just calculate the residue at the pole. This sets $\lambda_j = \lambda_k$. Let’s say $j < k$, then there is another $S$-matrix term of the form

$$\frac{\lambda_j - \lambda_k - i \text{sgn}(m_j - m_k)}{\lambda_j - \lambda_k - i} \bigg|_{\lambda_j = \lambda_k} = \text{sgn}(m_j - m_k). \quad (6.40)$$

Note however that the rest of the integrand is now symmetric in $m_j \leftrightarrow m_k$, including the spin operators. The $\text{sgn}$ therefore kills the state, and the result is zero.

The integral over $\lambda_j$ is therefore 0. This argument holds true for all $j$. The integral therefore is zero for $m_1 < n_1$ leaving $m_1 = n_1$ as the only possibility.

Let’s now move to integrating $\lambda_2$ and assume that $m_2 < n_2$. The only pole below which would prevent us from deforming the contour is $\lambda_2^* = \lambda_1 - i$. However, $m_2 < n_2 < n_1 = m_1 \implies m_2 < m_1$, and this pole is canceled. For the poles above, we use an argument similar to the above case to show that the integral is zero, implying that $m_2 = n_2$. It is now obvious that this can be carried out for all the $\lambda_j$. We have therefore
proved that the integral is non-zero only for \( m_j = n_j \) for all \( j \). With this condition, the integral becomes

\[
\int \left[ \prod_j \frac{d\lambda_j}{2\pi} \right] \prod_j \frac{1}{\lambda_j^2 + 1/4} = \prod_j \int \frac{d\lambda_j}{2\pi} \frac{1}{(\lambda_j + i/2)(\lambda_j - i/2)} = 1.
\]  

(6.41)

\[
\square
\]

### 6.3 Quench from a domain wall - XX model

The initial state is

\[
|\Psi_0\rangle = \prod_{j=-\infty}^{0} \sigma_j^- |0\rangle
\]  

(6.42)

shown in Fig. 6.3. The vacuum \(|0\rangle\) is the state with all spins up. We evolve this state with the XX Hamiltonian

\[
H_{\text{XX}} = -J \sum_j (\sigma_j^+ \sigma_{j+1}^- + \sigma_{j+1}^+ \sigma_j^-).
\]  

(6.43)

The eigenstates are given by

\[
|k\rangle = \sum_{m_j} \prod_j e^{ik_j m_j} |\sigma_j^- |0\rangle
\]  

(6.44)

with energy \( E = -2J \sum_j \cos k_j \). The above states are also eigenstates of the total spin operator, \( S^z \) given by

\[
S^z = \frac{1}{2} \sum_j \sigma_j^z.
\]  

(6.45)

From here on, we will set \( J = \frac{1}{2} \).

The spin current operator will be defined by

\[
f^\sigma (t) = \frac{dN_L}{dt}
\]  

(6.46)
where $N_L = \sum_{j=-\infty}^{0} \langle \sigma_j^{\uparrow} \rangle$. It measures the rate at which the spin flows into the region $j > 0$. We measure the expectation value of this operator in the finite time state, given by

$$|\Psi(t)\rangle = \sum_{m_j} \prod_j (\sigma_j^{m_j}) \langle m_j-\eta_j(t) | \sigma_j^{\uparrow} | 0 \rangle$$

(6.47)

where $n_j$ are the initial positions of the flipped spins. In our case, this is from $-\infty$ to 0. It is easy to show that

$$\dot{j}^\sigma(t) = 2 \sum_{j=0}^{\infty} J_j(t) \dot{J}_{j+1}(t).$$

(6.48)

The current is plotted in Fig. 6.4 - it saturates at $\frac{2}{n}$. Fig. 6.5 shows the time evolution

![Figure 6.4: Time evolution of spin current $\dot{j}^\sigma(t)$ after a quench from a domain wall situated at the origin.](image1)

![Figure 6.5: Time evolution of spins after a quench from a domain wall situated at the origin.](image2)
Let us now calculate this current in an eigenstate. Since we expect a steady state current, we will evaluate it using the continuity equation:

\[ \frac{d\sigma_i^z}{dt} = i[\sigma_i^z, H] = 2(j_l - j_{l-1}) \]  \hspace{1cm} (6.49)

where

\[ j_l = i(\sigma_{l+1}^+ \sigma_{l+1}^- - \sigma_{l+1}^+ \sigma_{l}^-) \]  \hspace{1cm} (6.50)

The current \( j^\sigma \) in Eq. (6.46) is given by

\[ j^\sigma(t) = \sum_{l \leq 0} \Bigl\langle \frac{d\sigma_i^z}{dt} \Bigr\rangle = 2 \langle j_0 \rangle = 2i \langle \sigma_0^+ \sigma_1^- - \sigma_1^+ \sigma_0^- \rangle \]  \hspace{1cm} (6.51)

### 6.3.1 Spin current in the XX model - directly obtaining the steady state result

For the XX case, the S-matrix is 1, and the sums over the \( m_j \) run from \( -\infty \) to \( +\infty \), except for \( m_k \) which is zero. We get,

\[
\langle j^\sigma(t) \rangle = -4\Im \sum_k \sum_p \left[ \prod_{l \neq k} \int \frac{dk_l}{2\pi} e^{ik_la_{p_j - n_j}} \right] 
\times \int \frac{dk_k}{2\pi} \frac{dq_k}{2\pi} e^{-ik_k n_k + iq_k (n_k - 1)} e^{2it\cos k_k - \cos q_k}
\]

\[
= -4\Im \sum_k \int \frac{dk}{2\pi} \frac{dq}{2\pi} e^{-i(k-q)n_k - qk} e^{2it\cos k - \cos q}
\]

\[
= -4\Im \sum_{n_k=0}^{\infty} \int \frac{dk}{2\pi} \frac{dq}{2\pi} e^{-i(k-q)n_k - qk} e^{2it\cos k - \cos q}
\]

\[
= -4\Im \int \frac{dk}{2\pi} \frac{dq}{2\pi} \frac{e^{-i(qk + 2it consolidated terms) \cos k - \cos q}}{1 - e^{i(k-q)}}
\]

\[
= -4\Im \int \frac{dk}{2\pi} \frac{dq}{2\pi} \frac{\sin^{\frac{k+q}{2}} \sin^{\frac{k-q}{2}}}{1 - e^{i(k-q)}}
\]

\[
= -4\Im \int \frac{dk}{2\pi} \frac{dq}{2\pi} \frac{e^{-4it \sin^{\frac{k+q}{2}} \sin^{\frac{k-q}{2}}}}{2\sin^{\frac{k-q}{2}} \sin^{\frac{k+q}{2}}} - 2it \sin^{\frac{k-q}{2}} \cos^{\frac{k+q}{2}}
\]

\[
= 2 \times \int \frac{dk}{2\pi} \frac{dq}{2\pi} \sin \left[ 4t \sin^{\frac{k-q}{2}} \sin^{\frac{k+q}{2}} \right] \sin^{\frac{k+q}{2}} \sin^{\frac{k-q}{2}}
\]
We now take the long time limit, \( t \to \infty \). This gives us

\[
\langle j^\sigma(t) \rangle = 2\pi \int \frac{dk}{2\pi} \frac{dq}{2\pi} \delta \left[ \sin \left( \frac{k-q}{2} \right) \sin \left( \frac{k+q}{2} \right) \right] \sin^2 \left( \frac{k+q}{2} \right)
\]

\[
= 4\pi \int \frac{dk}{2\pi} \frac{dq}{2\pi} \frac{\delta(k-q) \sin^2 \left( \frac{k+q}{2} \right)}{\left| \sin \left( \frac{k+q}{2} \right) \right|}
\]

\[
= 2 \int \frac{dq}{2\pi} \left| \sin q \right|
\]

\[
= \frac{4}{\pi}
\]

which is indeed the correct limit obtained from the earlier calculation.

For short times we can expand the \( \sin \) to linear order in \( t \) giving

\[
\langle j^\sigma(t) \rangle = 2 \int \frac{dk}{2\pi} \frac{dq}{2\pi} \sin \left[ 4t \sin \left( \frac{k-q}{2} \right) \sin \left( \frac{k+q}{2} \right) \right] \sin \left( \frac{k+q}{2} \right)
\]

\[
= 4t \int \frac{dk}{2\pi} \frac{dq}{2\pi} 4t \sin^2 \left( \frac{k+q}{2} \right)
\]

\[
= 4t \int \frac{dk}{2\pi} \frac{dq}{2\pi} [1 - \cos(k+q)]
\]

\[
= 4t
\]

showing a linear increase.

We can make a series expansion of the \( \sin \) and evaluate the integrals leading to

\[
\langle j^\sigma(t) \rangle = \sum_{n=0}^{\infty} (-1)^n \frac{4(2n)!t^{2n+1}}{(n!)^3(n+1)!}
\]

\[
= 4t \, {}_2F_1(1/2; 1, 2; -4t^2)
\]

(6.55)

which has been plotted numerically elsewhere.

### 6.4 Quench from a domain wall - XXX model

The initial state is

\[
|\Psi_0\rangle = \prod_{j=-\infty}^{0} \sigma_j^- |0\rangle
\]

(6.56)

shown in Fig. 6.3. The vacuum \(|0\rangle\) is the state with all spins up. We evolve this state with the XXX Hamiltonian

\[
H_{XXX} = -\frac{1}{2} \sum_j [2(\sigma_j^+ \sigma_{j+1}^- + \sigma_{j+1}^+ \sigma_j^-) + (\sigma_j^z \sigma_{j+1}^- - 1)].
\]

(6.57)
The eigenstates are given by the Bethe Ansatz

\[ |\bar{\lambda}\rangle = \sum_{m, \lambda} \prod_{i<j} \frac{\lambda_i - \lambda_j - i \text{sgn}(m_i - m_j)}{\lambda_i - \lambda_j - i} \prod_{j} \left( \frac{\lambda_i + i/2}{\lambda_j - i/2} \right)^{m_j} \sigma_{m_j}^- |0\rangle, \]

with energy \( \sum_{j=1}^{N} E(\lambda_j) \) with \( E(\lambda) = \frac{2f}{\lambda + \frac{1}{2}} \). The Yang-Baxter relations are trivially satisfied. The above states are also eigenstates of the total spin operator, \( S^z \) given by

\[ S^z = \frac{1}{2} \sum_j \sigma_j^z. \]

From here on, we will set \( J = \frac{1}{2} \).

The spin current is given by

\[ j^\sigma(t) = i \int_{-\infty}^{\infty} \sigma_l^z H = 2i(\sigma_l^- \sigma_l^+ - \sigma_l^- \sigma_l^+) = -4 \Im(\sigma_l^+ \sigma_l^-) \]

We’ll use Yudson to evaluate time-dependent expectation values later. For now, let’s obtain general expressions for these quantities. Let’s say that the finite-time state after the quench is given by

\[ |\Psi(t)\rangle = \sum_{m,j} \phi(\{m_j\}, \{n_j\}, t) \prod_j \sigma_{m_j}^- |0\rangle. \]

The expectation value of the spin-current operator is

\[ \langle j^\sigma(t) \rangle = -4 \Im(\langle \Psi(t) | \sigma_l^+ \sigma_l^- | \Psi(t) \rangle) = 4 \Im \sum_P \sum_k \sum_{m_j} \phi^*(\{m_{P_j} + \delta_{P,j}\}, \{n_j\}, t) \phi(\{m_j\}, \{n_j\}, t) \delta_{m_0,0} \]

Before we get there, let’s evaluate the norm of the finite time wave function to ensure that all is good.

### 6.4.1 Conservation of the norm of the wave function

Before we try to evaluate the expectation value of any operators, it will be instructive to see how we obtain \( \langle \Psi(t) | \Psi(t) \rangle = 1 \). We need to show that

\[ \sum_P \sum_{m_j} \phi^*(\{m_{P_j}\}, \{n_j\}, t) \phi(\{m_j\}, \{n_j\}, t) = 1 \]
Writing this out in full glory, we have,

\[ \langle \Psi(t) | \Psi(t) \rangle \]

\[ = \sum_{\mathcal{D}} \sum_{\mathcal{m}_{ji}} \int D^* \eta \, D \lambda \prod_{i < j} \frac{\eta_i - \eta_j + i \text{sgn}(m_{Pi} - m_{Pj})}{\eta_i - \eta_j + i} \prod_{j} \left( \frac{\eta_j - i/2}{\eta_j + i/2} \right)^{m_{Pj} - n_j} e^{\frac{\mu}{\eta_j^2 + 1/4}} \]
\[ \times \prod_{i < j} \frac{\lambda_i - \lambda_j - i \text{sgn}(m_i - m_j)}{\lambda_i - \lambda_j - i} \prod_{j} \left( \frac{\lambda_j + i/2}{\lambda_j - i/2} \right)^{m_j - n_j} e^{\frac{-\mu}{\lambda_j^2 + 1/4}} \]

where the \( \mathcal{D}^* \) is there to remind us that the contours of integration have to be complex-conjugated as well. Since all the \( \eta \) have the same integration contours, we can relabel them. In each term in the sum over permutations \( P \), we then rename the \( \eta_j \rightarrow \eta_{Pj} \),

\[ \langle \Psi(t) | \Psi(t) \rangle \]

\[ = \sum_{\mathcal{D}} \sum_{\mathcal{m}_{ji}} \int D^* \eta \, D \lambda \prod_{i < j} \frac{\eta_{Pj} - \eta_{Pj} + i \text{sgn}(m_{Pi} - m_{Pj})}{\eta_{Pj} - \eta_{Pj} + i} \prod_{j} \left( \frac{\eta_{Pj} - i/2}{\eta_{Pj} + i/2} \right)^{m_{Pj} - n_j} e^{\frac{\mu}{\eta_{Pj}^2 + 1/4}} \]
\[ \times \prod_{i < j} \frac{\lambda_i - \lambda_j - i \text{sgn}(m_i - m_j)}{\lambda_i - \lambda_j - i} \prod_{j} \left( \frac{\lambda_j + i/2}{\lambda_j - i/2} \right)^{m_j - n_j} e^{\frac{-\mu}{\lambda_j^2 + 1/4}} \]

\[ = \sum_{\mathcal{D}} \sum_{\mathcal{m}_{ji}} \int D^* \eta \, D \lambda \prod_{i < j} \frac{\eta_{Pj} - \eta_{Pj} + i \text{sgn}(m_{Pi} - m_{Pj}) \lambda_i - \lambda_j - i \text{sgn}(m_i - m_j)}{\eta_{Pj} - \eta_{Pj} + i \lambda_i - \lambda_j - i} \]
\[ \times \prod_{j} \left( \frac{\eta_j - i/2}{\eta_j + i/2} \right)^{m_j - n_j} \left( \frac{\lambda_j + i/2}{\lambda_j - i/2} \right)^{m_j - n_j} e^{\frac{\mu}{\eta_j^2 + 1/4}} e^{\frac{-\mu}{\lambda_j^2 + 1/4}} \]

(6.64)

We expect the sum over the \( m_j \) to produce something like \( \delta(\eta_j - \lambda_j) \). In order to prove this, we have to regularize the sum over \( m_j \) appropriately. If we expand the S-matrix terms above, we get an expression with several terms, each in a specific sector. Let us look at one specific sector with the ordering \( m_{Q_1} > m_{Q_2} > \cdots \). The ordering will correspond to a specific combination of S-matrix factors, that are independent of the \( m_j \). Let’s call this \( A_{Q,P} \). Writing only the sum over the \( m_j \) explicitly for this particular term (and leaving out the parts that don’t have any \( m \)-dependence), we get

\[ A_{Q,P} \sum_{m_{Q_1} = -\infty}^{\infty} \sum_{m_{Q_2} = -\infty}^{m_{Q_1}} \cdots \sum_{m_{Q_N} = -\infty}^{m_{Q_{N-1}}} \prod_{j} \left( \frac{\eta_j - i/2}{\eta_j + i/2} \frac{\lambda_j + i/2}{\lambda_j - i/2} \right)^{m_j} e^{-\epsilon|m_j|} \]

(6.65)
where we have included a regulator. The limit $\epsilon \to 0$ will be taken later. We can rewrite the summand using the “$k$ language” ¹.

\[
A_{Q,P} \sum_{m_{Q_1}=-\infty}^{m_{Q_1}} \sum_{m_{Q_2}=-\infty}^{m_{Q_2}} \cdots \sum_{m_{Q_N}=-\infty}^{m_{Q_N}} \prod_j e^{i(k_j - q_j)m_j}e^{-\epsilon|m_j|} \tag{6.67}
\]

One can check that the above sum is 0 in the limit of $\epsilon \to 0$ unless all the $k_j = q_j$ in which case it blows up. The sum is therefore $\sim \prod 2\pi \delta(k_j - q_j)$. One easy way to see this, is to assume that all but one of the $k_j = q_j$. In that case the sum goes to zero prior to taking the $\epsilon \to 0$ limit. The only way it is non-zero is if all were equal. A careful calculation gives us the prefactor which is $2\pi$ for each momentum, giving a total factor of $(2\pi)^N$. There is however also a $1/N!$ that comes from the fact that the sum is over only of of $N!$ possible sectors. This gives a total factor of $\frac{(2\pi)^N}{N!}$. The $(2\pi)^N$ cancels out the $\frac{1}{\pi}$ in the momentum integrals. This can be easily checked by looking at the 1, 2 and 3 particle cases. Further,

\[
\delta(k - q) = \delta(2 \arctan 2\lambda - 2 \arctan 2\eta) = \left(\eta^2 + \frac{1}{4}\right) \delta(\lambda - \eta) \tag{6.68}
\]

Note on the integration contours

It might seem like the delta function now restricts the integration to only the real axis. However precisely since this means that the contribution from the rest of the Yudson contour is zero, we are free to keep it. Another way to look at it this is to choose the conjugate set of contours for the “$\eta$” ket state so that when it gets conjugated (in the bra), both momentum variables are integrated along the same contours.

Eq. (6.65) then becomes

\[
\langle \Psi(t)|\Psi(t) \rangle = \frac{1}{N!} \sum_P \int \mathcal{D}\lambda \sum_Q A_{Q,P} \prod_j \left(\frac{\lambda_j + i/2}{\lambda_j - i/2}\right)^{n_j - n_j} \frac{1}{(\lambda_j^2 + 1/4)} \tag{6.69}
\]

We need to understand the interplay of the permutations $P$ and $Q$ now. The factor $A_{Q,P}$ is composed of several S-matrix factors. The $\lambda$ factors directly correspond to the ordering $Q$. However, the $\eta$ factors corresponding to the ordering $P$. Let us consider

¹Discussed elsewhere. The transformation is $k = 2 \arctan 2\lambda$. 
the simplest non-trivial case of 2 particles. For \( Q = Q_{12} \), i.e., \( m_2 > m_1 \) and for \( P = P_{12} \),

\[
A_{Q_{12}P_{12}} = \frac{\eta_2 - \eta_1 + i \lambda_1 - \lambda_2 + i}{\eta_2 - \eta_1 + i \lambda_1 - \lambda_2 - i} = \frac{\lambda_1 - \lambda_2 + i}{\lambda_1 - \lambda_2 - i}
\]  

(6.70)

For \( P = 1 \), we get

\[
A_{Q_{12}1} = \frac{\eta_2 - \eta_1 - i \lambda_1 - \lambda_2 + i}{\eta_2 - \eta_1 + i \lambda_1 - \lambda_2 - i} = 1
\]  

(6.71)

For \( Q = 1 \), we have the following two possibilities,

\[
A_{1P_{12}} = \frac{\eta_2 - \eta_1 - i \lambda_1 - \lambda_2 - i}{\eta_2 - \eta_1 + i \lambda_1 - \lambda_2 - i} = \frac{\lambda_1 - \lambda_2 - i}{\lambda_1 - \lambda_2 + i}
\]  

(6.72)

\[
A_{11} = \frac{\eta_2 - \eta_1 + i \lambda_1 - \lambda_2 - i}{\eta_2 - \eta_1 + i \lambda_1 - \lambda_2 + i} = 1
\]  

(6.73)

This case is quite straightforward. The important point to note here is that we either have 1 or \( S_{12} \). Consider an ordering given by \( Q \), i.e., \( m_{Q_1} > m_{Q_2} > \cdots > m_{Q_N} \). We get an \( S \)-matrix \( S_{ik} \) for every \( Q_i > Q_k \). Consider one such inverted pair \((i,j)\), i.e., the permutation \( Q \) has made \( m_j > m_i \), \( i < j \). Consider now some permutation \( P \) in the double sum over \( P \) and \( Q \). Suppose \( i = P_r, j = P_s \). If \( r < s \), then the factors in \( A_{Q,P} \) corresponding to the pair \( i,j \) is

\[
\frac{\eta_i - \eta_j + i \text{sgn}(m_i - m_j) \lambda_i - \lambda_j + i}{\eta_i - \eta_j + i \lambda_i - \lambda_j - i} = \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}
\]  

(6.74)

If \( r > s \), then we get

\[
\frac{\eta_i - \eta_j + i \text{sgn}(m_i - m_j) \lambda_i - \lambda_j + i}{\eta_i - \eta_j + i \lambda_i - \lambda_j - i} = \frac{\lambda_i - \lambda_j - i}{\lambda_i - \lambda_j + i}
\]  

(6.75)

It can be easily checked that if \((i,j)\) was not a pair that was inverted by \( Q \), then the outcome is again either 1 or \( \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i} \). This shows that the factor \( A_{Q,P} \) is indeed composed of only \( S \) and not \( S^* \). In fact it doesn’t even depend on \( Q \). This can be easily seen as follows.

If for a given pair of \((i,j)\), the permutation \( P \) does not invert the pair, then the \( S \)-matrix factor is 1 irrespective of \( Q \). If the pair is inverted, then we have an \( S \)-matrix factor corresponding to that inversion, \( S_{ij} = \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i} \), and this is easily seen to be independent of \( Q \). The sum over \( Q \) then produces an \( N! \) canceling out the \( \frac{1}{N!} \) coming from the sum over \( m \). We therefore have

\[
\langle \Psi(t) | \Psi(t) \rangle = \sum_P \int \mathcal{D} \lambda \ A_P \prod_j \left( \frac{\lambda_j + i/2}{\lambda_j - i/2} \right)^{n_j - n_i} \frac{1}{\lambda_j^2 + 1/4}
\]  

(6.76)
Consider a particular permutation \( P \) that is represented by a set of inversions. The \( n_i \) are an ordered set: \( n_1 > n_2 > \cdots \). So let us consider the integral over a particular \( \lambda_k \) chosen for the permutation \( P \) such that it does not have a pole at \(-i/2\). Since the integral over \( \lambda_k \) is essentially the residue at this pole, we get 0. This can be done for every permutation except the identity, for which all the plane-wave factors are 1. For the identity permutation, the \( S \)-matrix is also 1. Doing the integral gives us
\[
\langle \Psi(t)|\Psi(t) \rangle = 1 
\] (6.77)

### 6.4.2 Spin current in the XXX model

For the XXX model, via Yudson, we know that
\[
\phi(\{m_j\}, \{n_j\}, t) = \int_{\{\gamma_j\}} \left[ \prod_j \frac{d\lambda_j}{2\pi} \right] \prod_{i<j} \frac{\lambda_i - \lambda_j - isgn(m_i - m_j)}{\lambda_i - \lambda_j - i} \lambda_i - \lambda_j - i^{-i/2} \prod_j \left( \frac{\lambda_j + i/2}{\lambda_j - i/2} \right)^{m_j - n_j} e^{-\frac{\lambda_j^2}{\lambda_j^2 + 1/4}} \] (6.78)

where the contours \( \{\gamma_j\} \) are discussed elsewhere.

Now, we can go back to the calculation of the spin current. Consider, as above a general term which is ordered as \( m_{Q_1} > \cdots > m_{Q_N} \). In the above case, the calculation of the norm was simplified because the sum was over all integers. In the spin current calculation however, the sum over the \( m \) is split by \( m_k \) which is pinned to 0. So for all \( m_j < m_k \), the sum is from \(-\infty \rightarrow 0 \) and for \( m_j > m_k \), the sum is from \( 0 \rightarrow \infty \).

The sum can still be carried out in general, but it does not give us a delta function, so no obvious simplification occurs. There is a delta-function piece, but the sum is finite non-zero when \( k_j \neq q_j \). We’ll do the sum first.

Consider a particular sector \( Q \) with the order \( m_{Q_1} > m_{Q_2} > \cdots > m_{Q_{l-1}} > m_{Q_l} = k \) > \( m_{Q_{l+1}} > \cdots > m_{Q_N} \). Like before, this corresponds to some \( S \)-matrix factor \( A_{Q,P} \). The sum we’re interested in is
\[
A_{Q,P} \sum_{m_{Q_1}=1}^{m_{Q_1}} \cdots \sum_{m_{Q_l}=1}^{m_{Q_l}} e^{\delta m_0} \prod_{m_{Q_{l+1}}=-\infty}^{m_{Q_l}-1} \sum_{m_{Q_N}=-\infty}^{m_{Q_l}} \prod_j e^{(k_j-q_j)m_j} \] (6.79)

We can either regularize the sum by using a weighted tail or by doing a finite sum and taking the limit at the end. As we can see the sum is split into two parts by the
measurement. Carrying out the sum, we get
\[
\frac{e^{i(k_0_1 - q_0_1)} e^{i(k_0_2 - q_0_2)} \ldots e^{i(k_0_{N-1} - q_0_{N-1})}}{[1 - e^{i(k_0_1 - q_0_1)}] [1 - e^{i(k_0_1 - q_0_1) + i(k_0_2 - q_0_2)}] \ldots [1 - e^{i(k_0_1 - q_0_1) + \ldots + i(k_0_{N-1} - q_0_{N-1})}]} 
\times e^{-iq_0_1} e^{-i(k_{0_1+1} - q_0_{0_1+1})} e^{-i(k_{0_2+2} - q_0_{0_2+2})} \ldots e^{-i(k_{0_{N+1}} - q_0_{0_{N+1}})} \right]
\times \frac{1}{[1 - e^{-i(k_{0_{N+1}} - q_0_{0_{N+1}})}] \ldots [1 - e^{-i(k_{0_{N+1}} - q_0_{0_{N+1}}) + \ldots + i(k_{0_{N+1}} - q_0_{0_{N+1}})}]}
\times \prod_j e^{it \left( \frac{1}{\eta_j^2 + 1/4} - \frac{1}{\eta_j + 1/4} \right)}
\]
\[(6.80)\]

The limits of the regulators have been taken, so the above expression is assuming that none of the denominators vanish.

Writing the full expression this far, we get
\[
\langle j^y(t) \rangle
\]
\[
= 43 \sum_P \sum_k \sum_{m_j} \phi^\ast (\{ m_P + \delta_P k \}, \{ n_j \}, t) \phi (\{ m_j \}, \{ n_j \}, t) \delta_{m_0 0}
\]
\[
= 43 \sum_P \sum_Q \sum_l \int \mathcal{D} \eta \mathcal{D} \lambda \prod_j e^{i[q_l n_j - ik_j n_j]} A_{Q,P} (\eta_P, \lambda)
\]
\[
\times e^{i(k_0 - q_0)} e^{i(k_0 - q_0)} \ldots e^{i(k_{0_{N-1}} - q_{0_{N-1}})} 
\times e^{-i(k_{0_{1+1}} - q_{0_{0_{1+1}}})} e^{-i(k_{0_{2+2}} - q_{0_{0_{2+2}}})} \ldots e^{-i(k_{0_{N+1}} - q_{0_{0_{N+1}}})} 
\times \prod_j e^{it \left( \frac{1}{\eta_j^2 + 1/4} - \frac{1}{\eta_j + 1/4} \right)}
\]
\[(6.81)\]

Recall that the \( A_{Q,P} (\eta_P, \lambda) \) is the product of \( S \)-matrices produced by the ordering \( m_{Q_1} > m_{Q_2} > \ldots > m_{Q_N} \). We also need to keep in mind that for the \( \eta \)'s, the \( S \)-matrix term containing \( m_{Q_i} \) has \( m_{Q_i} + 1 \). We are also going to freely use the \( k \) and \( q \) language inside the integral.

How do we treat the infinite time case? As \( t \to \infty \) we expect the integral to be zero, unless the oscillating parts vanish. However it is important that this limit be taken after the limit of \( N \)
\( t \to \infty \). In order to do this, we have to carry out the sum over \( k \) (the measured spin), which is equivalent to a sum over \( l \). At \( t \to \infty \), the integral is non-zero for
\[
\sum_l \frac{1}{l^2 + 1/4} - \frac{1}{l + 1/4} = 0.
\]
\[(6.82)\]

This equation obviously has several solutions, the most obvious being \( \eta_{P_j} = \lambda_j \). The solution in general defines a hyperplane in the \( 2N \) dimensional momentum space.
Any solution which keeps the integrand finite is therefore suppressed by the zero integration region for the dimension perpendicular to this hyperplane. We are therefore interested in solutions where the integrand blows up. This occurs for $k_j = q_j \forall j \neq k$. This will potentially give us the infinite time results, but one still needs to evaluate the contour integrals. At finite time, we will have to evaluate these integrals numerically. This is currently work in progress.
Chapter 7
Directions for future work

The work described in this thesis opens up more questions than it answers. The Yudson representation provides some simplifications in calculating the time evolution of physical, experimentally realizable initial states with integrable Hamiltonians. In the following sections, I outline possible directions of further exploration.

7.1 Extensions to other models and situations

The most immediate requirement is to understand the quench dynamics of other integrable models like the Sine-Gordon model [77], interacting resonant level model (IRLM), the Anderson impurity model, Kondo model and so on [136]. The reason is that as mentioned in Chapter 2, the behavior of interacting integrable models under a variety of quench experiments is still far from understood. Our broader understanding of the role of integrability relies on not only general formal results [23, 25, 117], but also on understanding the behavior of a variety of local and nonlocal observables under time evolution from a variety of initial conditions, especially mixed states and states that are not translationally invariant.

Transport

Of the above models, the Anderson model, IRLM, and Kondo model being impurity models, their transport properties are also of interest. Earlier attempts using the Scattering Bethe Ansatz framework provided some insights into steady state currents [100]. With the Yudson representation, we have access to every piece, at least in principle. Currently work is underway to examine the long-time states of the two lead Anderson impurity model in order to get insight into the nature of the eigenstates of the
nonequilibrium problem, i.e., the model with a finite voltage bias. The existence of bound states (complex momentum eigenstates) significantly complicates the problem.

**Initial state dependence**

There is sufficient evidence that the long time behavior of integrable models after a quench depends on the initial state. In fact, even in the Bose-Hubbard model, a non-integrable model, it is the initial state that controls the asymptotic state, and not the interaction sign as explained in Appendix A. Since the initial state is up to our choice, we can arrange that it does not respect the symmetries of the Hamiltonian, or is not translationally invariant, has a particular momentum distribution and so on. We can also start with states that are not entangled or highly entangled, and all of this seems to affect the outcome. (need refs!!)

The Yudson representation works for arbitrary initial states per se, but is naturally suited to states defined in position space with minimal interparticle overlap. Within these constraints, we can still create inhomogeneous and mixed states. For example we can choose

$$|\Psi_0\rangle \sim (\sigma_{-1}^- + \sigma_0^-)^2|0\rangle$$

as an initial state for a Heisenberg (XXZ) quench. This state breaks the bipartite lattice symmetry, and so we expect that the sign of the interaction will play a role in its evolution (see Appendix A).

**7.2 Finite volume**

As it stands, the Yudson representation applies directly in infinite volume, and this indeed eliminates several complications while also being experimentally meaningful. However, this raises an order of limits issue when we consider long time behavior. If we take the limit of $t \to \infty$ with finite number of particles (or flipped spins) $N$, then we are automatically in the zero density limit since the systems boundaries have undergone expansion bounded by the Lieb-Robinson velocity [95]. This is a particular regime of physics which is interesting, but in order to gain access to finite density
situations, the limit of $N \to \infty$ must be taken before the long time limit. Only then can we gain access to statistical distributions and nontrivial asymptotic quantities. This is not straightforward since we are dealing with full wave functions in the particle number basis, and not densities. It would therefore be advantageous to be able to study the finite volume limit in order to get access to finite density cases.

In order to write a Yudson representation in finite volume, we are forced to replace the integral over pseudomomenta or rapidities by sums constrained by the Bethe Ansatz equation. Starting with an ordered initial state still allows for some simplification. Work in this direction is currently in progress [55]. It also ties the Yudson representation with the Algebraic Bethe Ansatz approach.

7.3 Thermodynamic ensembles and asymptotic states

As discussed in Section 2.4.2, the generalized Gibbs Ensemble has been shown to be a valid description for the long time states in quenches with several integrable models. However, since most of these claims have been verified for local observables and homogeneous initial conditions, and mostly with integrable models mappable to non-interacting models, it is interesting to study the behavior of the asymptotic state from an ensemble point of view. In principle, the Yudson representation provides us with a way of doing this.

For a particular operator $\mathcal{O}$, we can write the finite time expectation value using the Yudson representation as

$$\langle \mathcal{O} \rangle = \int_{\lambda,\eta} D\lambda D\eta \ e^{iE(\lambda) t - iE(\eta) t} \langle \Psi_0|\eta \rangle \langle \eta|\mathcal{O}|\lambda \rangle \langle \lambda|\Psi_0 \rangle. \quad (7.2)$$

We can then study the infinite time limit of this expression directly. The complication is that, the thermodynamic limit of $N \to \infty$ and $t \to \infty$ do not commute since we are in infinite volume. It would be interesting to study this limit and try to make a direct connection with an ensemble formulation, which would look like:

$$\langle \mathcal{O} \rangle_E = \frac{\text{Tr} e^{\int \lambda(\theta) \hat{I}(\theta) \mathcal{O}}}{\text{Tr} e^{\lambda(\theta) \hat{I}(\theta)}} \quad (7.3)$$

where the $\hat{I}$ are some operators that in the GGE we expect to be the local conserved
charges of the theory. The question we would like to address is if the long time limit can be written in terms of some ensemble and what the operators $\hat{I}$ would be.

We have carried out directly the large time limit for one case of the spin current in the XX model in Section 6.1.3, after first taking the thermodynamic limit. We have setup the problem for the interacting case in Section 6.2.2. The primary issues are making sure that the limits are correctly taken, and then being able to do the contour integrals. This is currently work in progress.

### 7.4 Technical aspects

Beyond the above mentioned aspects, there are mathematical aspects of interest. Intuitively speaking, the representation accounts for all the possible rapidity eigenvalues that the spectrum could contain. Being infinite volume, the values lie of a space that is partly a continuum (the real part in the Lieb-Liniger case), and partly discrete (the imaginary parts). However, one does not need any prior knowledge in order to make the representation work. As we have seen, it is possible to guess the contours of integration. This process can be informed by a knowledge of the spectrum, or conversely, the Yudson representation can tell us what eigenstates contribute to the time evolution.

**Integration contours**

The choice of integration contours is the central aspect of the Yudson representation. We have by no means shown that the choice of contour or the momentum parametrization is unique. In fact the possibility of multiple feasible parametrizations, contours, and integration measures opens up the chance of finding a useful set of contours for other models as well.

A good example is the Heisenberg model. From studying the Lieb-Liniger case, it seems that the contours have to be chosen in order to “avoid” capturing the poles during integration. However, this strategy fails in the Heisenberg model, and we need a completely different approach to the contours. As of now, we have no general method of guessing the right contours, and have used trial and error. It would be of interest
to understand this part of the problem better. The solution will potentially provide an understanding the relation of the Yudson representation and other well known aspects of integrable modes such as conservation laws and the nature of solutions.

**Connection with finite volume approaches**

Other than the Yudson representation and some works in the mathematics literature regarding the Lieb-Liniger model [134, 114] that don’t allude to the Bethe Ansatz explicitly, the other methods are based on the Bethe Ansatz for finite systems, either the thermodynamic approach or the algebraic approach. Both of these require a solution of the Bethe Ansatz equations to pin down the eigenvalues. However, in the infinite volume limit, we must recover the Yudson representation, including the string states. This connection would be interesting to establish.
Chapter 8

Conclusion

Put simply, the Yudson representation provides a resolution of the identity, directly applicable in infinite volume:

$$1 = \int_\lambda |\lambda\rangle\langle\lambda|.$$  \hspace{1cm} (8.1)

The primary advantage comes from the presence of an integral instead of a sum, and the occurrence of the “Yudson eigenstate” $|\lambda\rangle$. Note that the above formula is valid in all generality, as long as we remember to apply it on ordered states in the position basis. Given this, the representation lends itself quite naturally to a variety of initial states localized in a position basis. One could of course perfectly well take a Fourier transform to get states localized in momentum space, or even more complicated initial states. This property allows us to find expectation values of operators in any state time evolved with an integrable Hamiltonian. It is important to contrast this with other methods based on the Bethe Ansatz where the initial state is typically the ground state of a similar model with a different coupling constant. Further, the ground state solution is typically obtained with periodic boundary conditions and the thermodynamic limit is then taken. It is not straightforward to deal with a large finite number of particles in the system, since in finite volume, the pseudo-momenta or rapidities must satisfy the Bethe Ansatz equations. In any case, experimentally, it is meaningful to talk about a local initial state expanding into infinite volume, and therefore the Yudson representation is also a natural way of thinking about the experimental system.

A note about the practical value of the generic representation is in order here. It has been shown through a few different examples (see Chapters 4,5,6) that the contours of integration in the Yudson representation hold the key to accounting for the entire space of eigenvalues of a particular integrable model. It is obvious that we have
made no attempt to claim any general means to obtain the correct contours for a given problem other than to guess what it might be with some prior knowledge of the structure of eigenvalues (i.e., whether complex solutions exist or not). Even then, as in the case of the Heisenberg model, it is not obvious what the choice is. Thus far, the choice has involved making (hopefully educated) guesses and trying to prove the validity of the representation. If it fails, we start over. In general it is not obvious that we can find such a set of contours for any given integrable model, and even if we do, one still has to do several integrals that are not straightforward. However, it does reduce the problem to one of finding an efficient analytical or numerical method to carry out integrations. It greatly simplifies the calculation of overlaps, one of the biggest challenges of algebraic Bethe ansatz based approaches, and we believe it can be suitably generalized for the finite density case. We have seen some preliminary work towards this in Section 6.4. The primary constraint here is to ensure that we take the limit of large number of particles before taking the long time limit.

1The more obvious choice of contour, which excludes the single particle poles in the $\lambda$ parametrization does not work
Appendix A

Quenching the Bose-Hubbard model

We compare the results obtained in Section 5.3 with those from the lattice version of the Lieb-Liniger model - the Bose-Hubbard model,

\[ H_{\text{BH}} = \sum_i \left[ (t b_i^\dagger b_{i+1} + \text{h.c.}) + Un_i(n_i - 1) \right] \] (A.1)

It describes bosons hopping on a 1d lattice with on-site interaction \( U \) and is non-integrable since it allows multiparticle interactions on the same site. It has been extensively studied in many contexts and much is known about its equilibrium properties (see e.g., Ref. [88]). For \( 0 < U/t \ll 1 \), the model is a superfluid, and for \( U/t \gg 1 \) it is a Mott insulator. For negative \( U \), the model is attractive and the ground state is a Bose condensate. A non-equilibrium phase diagram of the Bose-Hubbard model is given in Ref. [85].

We study here the two boson quench dynamics and contrast it with the corresponding dynamics of the Lieb-Liniger model. Contrary to what one may expect, the introduction of the lattice modifies the dynamics in an essential way even at long times and distances. The calculations of density correlations as a function of time after a sudden quench have been carried out using the Algorithms and Libraries for Physics Simulations (ALPS) code [1, 9, 3] and the Open source TEBD package [2] after making the necessary modifications to accommodate the initial states we are interested in. Our results confirm some results obtained in Ref. [89].

In Fig. A.1 we show the time evolution of the correlation matrix defined as \( \langle n_i n_j \rangle \) after a sudden quench from an initial state \( b_0^\dagger b_0^\dagger |0\rangle \), and in Fig. A.2 the evolution from initial state \( b_0^\dagger b_i^\dagger |0\rangle \). We quench into the interacting regime, where \( |U|/t = 10 \). There are a couple of interesting features: (1) Unlike the situation in the Lieb-Liniger model where the bunching or anti-bunching effect is independent of the initial state, here, quenching
a lattice-like state leads to anti-bunching Fig. A.2, while quenching a condensate-like state leads to bunching Fig. A.1. It is also interesting to compare the anti-bunching evolution of the bosons with the evolution of free fermions in Fig. A.3. (2) The sign of

![Figure A.1: Time evolution of the correlation matrix after a sudden quench from a state containing two bosons on the same site. The values increase from blue (0) to red. The correlations remain strong in the center indicating strong bunching.](image)

the interaction plays no role in the evolution in the Bose-Hubbard model, as seen from either figures. This is unlike the situation in the continuum model where for repulsive interactions anti-bunching (fermionization) occurs independently of the initial state, while bunching will take place for attractive interactions.

This non-dependence on the sign of the interaction is due to a particle-hole symmetry that is present on the lattice, but not in the continuum. The 1$d$ lattice, being bipartite, allows the the transformation $b_j \to e^{i\pi j} b_j$, with $j$ the site index, under which the hopping terms pick up a minus sign, while the on-site interaction terms are unaffected, thus $U/t \to -U/t$. In terms of $U$ the corresponding unitary operator we have,

$$U H_{BH}(t, -U) U^\dagger = -H_{BH}(t, U).$$

(A.2)

Denoting eigenstates and eigenvalues of $H_{BH}(t, U) \equiv H_{BH}$ by $|m\rangle$ and $\epsilon_m$ and the corresponding eigenstates and eigenvalues of $H_{BH}(t, -U) \equiv \tilde{H}_{BH}$ by $|\tilde{m}\rangle$ and $\tilde{\epsilon}_m$ we can relate the states by: $|\tilde{m}\rangle = U |m\rangle$, and the eigenvalues by: $\tilde{\epsilon}_m = -\epsilon_m$. The time
Figure A.2: Time evolution of the correlation matrix after a sudden quench from a state containing bosons on two neighboring sites. The values increase from blue (0) to red. The off diagonal correlations indicate antibunching, as can be seen from free fermion evolution in Fig. A.3.

The time evolution of an operator $O$ under the action of $H_{BH}(t, U)$ from an initial state $|\Psi_0\rangle$

$$\langle O(t) \rangle_H = \sum_{m, m'} \langle \Psi_0 | m' \rangle \langle m | \Psi_0 \rangle \langle O | m \rangle e^{-i(\epsilon_m - \epsilon_{m'})t} \quad (A.3)$$

Under $\hat{H}_{BH}$,

$$\langle O(t) \rangle_{\hat{H}} = \sum_{\hat{n}, \hat{n}'} \langle \Psi_0 | \hat{n}' \rangle \langle \hat{n} | \Psi_0 \rangle \langle O | \hat{n}' \rangle e^{-i(\epsilon_{\hat{n}} - \epsilon_{\hat{n}'})t}$$

$$= \sum_{m, m'} \langle \Psi_0 | U | m' \rangle \langle m | U^\dagger | \Psi_0 \rangle \langle O | m \rangle e^{i(\epsilon_m - \epsilon_{m'})t} \quad (A.4)$$

Both initial state we considered, $|\Psi_{\text{latt}}\rangle = b_0^\dagger b_1^\dagger 0\rangle$ and $|\Psi_{\text{cond}}\rangle = (b_0^\dagger)^2 0\rangle$ are simply transformed, $U|\Psi_0\rangle = \pm |\Psi_0\rangle$ and as they occur twice in the overlaps the transformation leaves no effect. Similarly the operators we have considered (density-density correlations) are bilinear in the site operators and are therefore not affected, $U^\dagger O U = O$.

This gives

$$\langle O(t) \rangle_{\hat{H}} = \sum_{m, m'} \langle \Psi_0 | m' \rangle \langle m | \Psi_0 \rangle \langle O | m \rangle e^{i(\epsilon_m - \epsilon_{m'})t} \quad (A.5)$$

Next, we note that the Bose-Hubbard Hamiltonian is invariant under time reversal, i.e., $[H_{BH}, T] = 0$ where $T$ is the anti-unitary time reversal operator:

$$T t T^{-1} = -t, \quad T i T^{-1} = -i. \quad (A.6)$$
Figure A.3: Time evolution of free fermions on a lattice. Notice how off diagonal correlations develop. The values increase from blue (0) to red.

Applying the time reversal operator to the expectation value above, we get

\[
\langle O(t) \rangle_{\tilde{H}} = T \langle O(t) \rangle_{\tilde{H}} T^{-1}
\]

\[
= \sum_{m, m'} \langle \Psi_0 | m' \rangle^* \langle m | \Psi_0 \rangle^* \langle m' | O | m \rangle e^{i(\epsilon_m - \epsilon_{m'}) t}
\]

\[
= \sum_{m, m'} \langle \Psi_0 | m \rangle \langle m' | \Psi_0 \rangle \langle m | O | m' \rangle e^{i(\epsilon_m - \epsilon_{m'}) t}
\]

\[
= \sum_{m, m'} \langle \Psi_0 | m' \rangle \langle m | \Psi_0 \rangle \langle m' | O | m \rangle e^{-i(\epsilon_m - \epsilon_{m'}) t}
\]

\[
= \langle O(t) \rangle_{H}
\]

thus indeed, the time evolution looks the same for both signs of the interaction. Note that with initial states or operators that are not invariant (up to a sign) under the transformation \( \mathcal{U} \), we should see a difference in the time evolution of the attractive and repulsive models.

A similar symmetry exists in the XXZ model or the Hubbard model in 1d (or higher dimensional bipartite lattices). For the magnet, the

sign of the anisotropy \( \Delta \) leads to either ferromagnetic or antiferromagnetic ground states for negative or positive anisotropy. However, it does not influence the quench dynamics [8], as can be seen from arguments like the above. Similarly in the Hubbard model, the quench dynamics is unaffected by the change of sign of \( U \) [124].
Appendix B

The meaning of the single particle poles in the Heisenberg model

As seen in Chapter 6, in the $\lambda$-parametrization which is convenient for the Yudson representation, seemingly spurious single particle poles appear:

$$e^{ikm} \sim \frac{\lambda + i/2^m}{\lambda - i/2}.$$  \hspace{1cm} (B.1)

These poles have a very simple interpretation once we look at the time evolution.

In order to keep things simple, we’ll study the one flipped spin case in the XX model. The finite time state for an initial state $|\Psi_0\rangle = \sigma^+_n |0\rangle$ is given by

$$|\Psi(t)\rangle = \sum_m \int_{\lambda} \left( \frac{\lambda + i/2}{\lambda - i/2} \right)^{m-n} e^{-\frac{2ilt}{\lambda^2 + 1/4}} \sigma^+_m |0\rangle.$$  \hspace{1cm} (B.2)

The singularity in the exponent poses an infinite series of poles of increasing order. The $k$-parametrization has no such issue. In order to better understand what these poles do, let us write the exponent as a series. Since each term has a fixed integer-order pole (i.e., no branch cuts), we can carry out the integrals. Writing only the integral over $\lambda$, we get

$$\sum_p \frac{1}{p!} \int_{\lambda} \left( \frac{\lambda + i/2}{\lambda - i/2} \right)^{m-n} \frac{(-2iJt)^p}{(\lambda^2 + 1/4)^{p+1}}.$$  \hspace{1cm} (B.3)

Our first goal now is to find out when this integral is nonzero. Let’s do this for a term of the $p$-th order in the expansion. The box contours described in Section 6.2.2 give zero unless we have poles at both $\pm i/2$. The absence of either of these poles allows us to shift the contour to wrap around the absent pole giving zero. In order to have both poles, we must have positive powers of these factors in the denominator. That

\[1\text{For the XX model, we could just as well integrate along the real axis, and the arguments here remain unchanged.} \]
implies:
\[ m - n + p + 1 > 0, \quad \text{and} \quad n - m + p + 1 > 0 \implies |m - n| \leq p. \]  
(B.4)

\(|m - n|\) is the distance of propagation of a spin from its starting position. Thus each increasing order in the sum allows the spins to propagate out further. At very short time, the terms with small \(p\) dominate, and therefore the spins don’t propagate far. At large time more and more terms in the sum contribute, making the spins propagate out further. It is also important to note that the above is a nonzero contribution only if the spin propagates to a position where the spin is not flipped already.

Thus, we see explicitly how the time evolution operator makes the spins hop via the poles in the plane wave term, also showing that indeed these terms in the \(\lambda\) representation correspond to propagating waves. Interactions will play the role of the “direct” and “crossed” hoppings to have different phases via the \(S\)-matrices, and therefore different weights in the time evolution.
Bibliography


