ON A HARDY TYPE INEQUALITY AND A SINGULAR STURM-LIOUVILLE EQUATION

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ABSTRACT OF THE DISSERTATION

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In this dissertation, we first prove a Hardy type inequality for $u \in W_0^{m,1}(\Omega)$, where Ω is a bounded smooth domain in $\mathbb{R}^N$ and $m \geq 2$. For all $j \geq 0$, $1 \leq k \leq m-1$, such that $1 \leq j+k \leq m$, it holds that $\frac{\partial^j u(x)}{d(x)^{m-j-k}} \in W_0^{k,1}(\Omega)$, where d is a smooth positive function which coincides with $\text{dist}(x, \partial\Omega)$ near $\partial\Omega$, and ∂^l denotes any partial differential operator of order l.

We also study a singular Sturm-Liouville equation $-(x^{2\alpha}u')' + u = f$ on $(0,1)$, with the boundary condition $u(1) = 0$. Here $\alpha > 0$ and $f \in L^2(0,1)$. We prescribe appropriate (weighted) homogeneous and non-homogeneous boundary conditions at 0 and prove the existence and uniqueness of $H^2_{loc}(0,1]$ solutions. We study the regularity at the origin of such solutions. We perform a spectral analysis of the differential operator $\mathcal{L}u := -(x^{2\alpha}u')' + u$ under homogeneous boundary conditions.

Finally, we are interested in the equation $-(|x|^{2\alpha}u')' + |u|^{p-1}u = \mu$ on $(-1,1)$ with boundary condition $u(-1) = u(1) = 0$. Here $\alpha > 0$, $p \geq 1$ and μ is a bounded Radon measure on the interval $(-1,1)$. We identify an appropriate concept of solution for this equation, and we establish some existence and uniqueness results. We examine the limiting behavior of three approximation schemes. The isolated singularity at 0 is also investigated.

Preface

This dissertation is a compilation of the research papers written by the author during the course of his Ph. D. Each chapter in this dissertation contains one paper, while the references are collected at the end of this dissertation. Minor changes are made from the original papers in order to keep the consistency of the presentation style. Some chapters are collaborative work (with H. Castro for Chapter 1, 3 and 4, and with H. Castro and J. Dávlia for Chapter 2). Chapter 5 and 6 are written solely by the author.

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Chapter 1

A Hardy type inequality for $\mathbf{W^{m,1}(0,1)}$ functions

1.1 Introduction

It is well known ([31]) that if $u \in W^{1,p}(0,1)$ and $u(0) = 0$ then the Hardy inequality holds for $p > 1$, that is

$$\int_0^1 \left|\frac{u(x)}{x}\right|^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^1 \left|u'(x)\right|^p dx.$$

The constant $\frac{p}{p-1}$ is optimal for this inequality and it blows up as p goes to 1. This behaviour is confirmed by the fact that no such inequality can be proved when $p = 1$, as we can consider (see e.g. [8]) the non-negative function on $(0,1)$ defined by

$$v(x) = \frac{1}{1 - \log x}. \tag{1.1}$$

A simple computation shows that this function belongs to $W^{1,1}(0,1), u(0) = 0$, but $\frac{u(x)}{x}$ is not integrable.

When we turn to functions $u \in W^{2,p}(0,1),\ p \geq 1$, with $u(0) = u'(0) = 0$, there are three natural quantities to consider: $\frac{u(x)}{x^2}, \frac{u'(x)}{x}$ and $\left(\frac{u(x)}{x}\right)' = \frac{u'(x)}{x} - \frac{u(x)}{x^2}$. If $p > 1$, it is clear that both $\frac{u'(x)}{x}$ and $\frac{u(x)}{x^2} = \frac{u'(x)}{x} - \frac{1}{x^2}\int_0^x tu''(t)dt$ belong to $L^p(0,1)$. Thus $\left(\frac{u(x)}{x}\right)' \in L^p(0,1)$. If $p = 1$ one can no longer assert that $\frac{u(x)}{x^2}, \frac{u'(x)}{x}$ belong to $L^1(0,1)$, but surprisingly $\left(\frac{u(x)}{x}\right)' \in L^1(0,1)$. This reflects a "magic" cancellation of the non-integrable terms in the difference $\left(\frac{u(x)}{x}\right)' = \frac{u'(x)}{x} - \frac{u(x)}{x^2}$.

The same phenomenon remains valid when we keep increasing the number of derivatives, and this is the main result of this chapter.

Definition 1.1. *We say that u has the property (P_m) if*

$$u \in W^{m,1}(0,1) \text{ and } u(0) = Du(0) = \ldots = D^{m-1}u(0) = 0,$$

where $D^i u$ denotes the i-th derivative of u.

Theorem 1.2. *Assume u has the property (P_m) and j, k are non-negative integers.*

(i) If $k \geq 1$ and $1 \leq j+k \leq m$, then $\frac{D^j u(x)}{x^{m-j-k}}$ has the property (P_k) and

$$\left\| D^k \left(\frac{D^j u(x)}{x^{m-j-k}} \right) \right\|_{L^1(0,1)} \leq \frac{(k-1)!}{(m-j-1)!} \left\| D^m u \right\|_{L^1(0,1)} . \tag{1.2}$$

The constant is optimal.

(ii) There exists w having the property (P_m) such that

$$\frac{D^j w(x)}{x^{m-j}} \notin L^1(0,1), \ \forall j = 0, \ldots, m-1. \tag{1.3}$$

Remark 1.1. *For functions $u \in W^{2,p}(0,1)$, $p > 1$, with $u(0) = u'(0) = 0$, a slightly stronger result holds, namely, when we estimate the L^p norms of the three quantities $\frac{u(x)}{x^2}$, $\frac{u'(x)}{x}$ and $\left(\frac{u(x)}{x}\right)'$, we obtain*

$$\left\| \frac{u(x)}{x^2} \right\|_p \leq \alpha_p \left\| u'' \right\|_p , \quad \left\| \frac{u'(x)}{x} \right\|_p \leq \beta_p \left\| u'' \right\|_p , \text{ and } \left\| \left(\frac{u(x)}{x} \right)' \right\|_p \leq \gamma_p \left\| u'' \right\|_p , \tag{1.4}$$

with $\alpha_p, \beta_p, \gamma_p$ as the best possible constants. It is easy to see that $\alpha_p \to \infty$, $\beta_p \to \infty$ when $p \to 1$. However, a similar "magic" cancellation appears and γ_p remains bounded as $p \to 1$. A proof of this latter fact is presented in Section 1.3.

1.2 Proof of Theorem 1.2

We begin with the following observation.

Lemma 1.3 (Representation formula)**.** *If u has property (P_m), then*

$$u(x) = \frac{1}{(m-1)!} \int_0^x D^m u(s)(x-s)^{m-1} ds.$$

Proof. We proceed by induction. The case $m = 1$ is immediate since $u \in W^{1,1}(0,1)$ if and only if u is absolutely continuous. Now notice that

$$D^{m-1} u(x) = \int_0^x D^m u(s) ds.$$

If we use the induction hypothesis, we obtain

$$u(x) = \frac{1}{(m-2)!} \int_0^x \left(\int_0^s D^m u(t) dt \right) (x-s)^{m-2} ds.$$

The proof is completed after using Fubini's Theorem. ∎

Based on the function defined by (1.1), we have

Lemma 1.4. *There exists a function w having property (P_m), such that*

$$\frac{D^{m-1}w(x)}{x}, \frac{D^{m-2}w(x)}{x^2}, \ldots, \frac{Dw(x)}{x^{m-1}}, \frac{w(x)}{x^m} \notin L^1. \tag{1.5}$$

Proof. In order to construct the function w, we consider the function v defined in (1.1). As we said, v is a non-negative function on $(0,1)$, it has the property (P_1), but $\frac{v(x)}{x}$ does not belong to $L^1(0,1)$. Define $w(x)$ as

$$w(x) = \frac{1}{(m-2)!}\int_0^x v(s)(x-s)^{m-2}ds,$$

so w solves the equation $D^{m-1}w(x) = v(x)$, with initial condition $w(0) = Dw(0) = \ldots = D^{m-2}w(0) = 0$. Notice that w has the property (P_m), $D^k w(x) \geq 0$, $D^k w(1) < \infty$ and

$$\lim_{s\to 0} \frac{D^{m-k}w(s)}{s^{k-1}} = 0,$$

for all $k = 1, \ldots, m-1$. We now show that w satisfies (1.5). Notice that

$$\begin{aligned} +\infty &= \int_0^1 \frac{v(x)}{x}dx \\ &= \int_0^1 \frac{D^{m-1}w(x)}{x}dx \\ &= D^{m-2}w(1) + \int_0^1 \frac{D^{m-2}w(x)}{x^2}dx. \end{aligned}$$

Thus $\int_0^1 \frac{D^{m-2}w(x)}{x^2}dx = +\infty$. Similarly, if we keep integrating by parts we conclude that

$$\left\| \frac{D^{m-j}w(x)}{x^j} \right\|_{L^1(0,1)} = \int_0^1 \frac{D^{m-j}w(x)}{x^j} = \infty, \quad \forall\, j = 1, \ldots, m.$$

□

We can proceed to the

Proof of Theorem 1.2. The second part was proved in Lemma 1.4, so we will only prove the first part. Since the result is immediate when $j+k = m$, in the following we always assume that $j + k \leq m-1$.

To prove that $\frac{D^j u(x)}{x^{m-j-k}}$ has the property (P_k), we proceed by induction. For $k = 1$ and any $j = 0, \ldots, m-1$, $\frac{D^j u(x)}{x^{m-j-1}}$ has the property (P_1) because

$$\left.\frac{D^j u(x)}{x^{m-j-1}}\right|_{x=0} = (m-j-1)!D^{m-1}u(0) = 0.$$

Now assume the result holds for some k. Notice that if $j+k+1 \le m-1$ then

$$D\left(\frac{D^j u(x)}{x^{m-j-k-1}}\right) = \frac{D^{j+1}u(x)}{x^{m-(j+1)-k}} - (m-j-k-1)\frac{D^j u(x)}{x^{m-j-k}},$$

the righthand side of which has property (P_k) by the induction assumption. Thus we conclude that $D\left(\frac{D^j u(x)}{x^{m-j-k-1}}\right)$ has the property (P_k), completing the induction step.

Now we prove the estimate (1.2). Notice that

$$D^k\left(\frac{D^j u(x)}{x^{m-j-k}}\right) = \sum_{i=0}^{k}\binom{k}{i} D^{j+i}u(x) D^{k-i}\left(\frac{1}{x^{m-j-k}}\right), \tag{1.6}$$

and that

$$D^{k-i}\left(\frac{1}{x^{m-j-k}}\right) = (-1)^{k-i}\frac{(m-j-i-1)!}{(m-j-k-1)!}\frac{1}{x^{m-j-i}}. \tag{1.7}$$

Using the representation formula for u from Lemma 1.3, we obtain

$$D^{i+j}u(x) = \frac{1}{(m-j-i-1)!}\int_0^x D^m u(s)(x-s)^{m-j-i-1} ds. \tag{1.8}$$

By combining (1.6), (1.7) and (1.8) we obtain

$$\begin{aligned}
&D^k\left(\frac{D^j u(x)}{x^{m-j-k}}\right)\\
=&\sum_{i=0}^{k} z(-1)^{k-i}\frac{1}{(m-j-k-1)!}\int_0^x D^m u(s)\frac{(x-s)^{m-j-i-1}}{x^{m-j-i}} ds\\
=&\frac{1}{(m-j-k-1)!}\int_0^x D^m u(s)\frac{(x-s)^{m-j-1}}{x^{m-j}}\left(\sum_{i=0}^{k}\binom{k}{i}\left(\frac{x}{x-s}\right)^i (-1)^{k-i}\right) ds\\
=&\frac{1}{(m-j-k-1)!}\int_0^x D^m u(s)\frac{(x-s)^{m-j-1}}{x^{m-j}}\left(\frac{s}{x-s}\right)^k ds.\\
=&\frac{1}{(m-j-k-1)!}\int_0^x D^m u(s)\left(1-\frac{s}{x}\right)^{m-j-k-1}\left(\frac{s}{x}\right)^{k-1}\frac{s}{x^2} ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^1 \left|D^k\left(\frac{D^j u(x)}{x^{m-j-k}}\right)\right| dx\\
\le&\frac{1}{(m-j-k-1)!}\int_0^1 |D^m u(s)|\left(\int_s^1\left(1-\frac{s}{x}\right)^{m-j-k-1}\left(\frac{s}{x}\right)^{k-1}\frac{s}{x^2}dx\right) ds\\
=&\frac{1}{(m-j-k-1)!}\int_0^1 |D^m u(s)|\left(\int_s^1 (1-t)^{m-j-k-1} t^{k-1} dt\right) ds\\
\le&\frac{1}{(m-j-k-1)!}\|D^m u\|_{L^1(0,1)}\int_0^1 (1-t)^{m-j-k-1} t^{k-1} dt\\
=&\frac{(k-1)!}{(m-j-1)!}\|D^m u\|_{L^1(0,1)}.
\end{aligned}$$

The optimality of the constant is guaranteed by the optimality of Hölder's inequality. The proof of the theorem is now completed. □

1.3 The $W^{m,p}$ functions with $m \geq 2$ and $p > 1$

We begin by proving the result stated in Remark 1.1. Notice that for $u \in W^{2,p}(0,1)$ satisfying $u(0) = u'(0) = 0$, we can write

$$\left(\frac{u(x)}{x}\right)' = \frac{1}{x^2}\int_0^x su''(s)ds.$$

For $p > 1$, we can apply Hölder's inequality and Fubini's theorem to obtain,

$$\begin{aligned}\int_0^1 \left|\left(\frac{u(x)}{x}\right)'\right|^p dx &\leq \int_0^1 \frac{x^{\frac{p}{p'}}}{x^{2p}}\int_0^x s^p \left|u''(s)\right|^p dsdx \\ &= \int_0^1 s^p \left|u''(s)\right|^p \left(\int_s^1 \frac{1}{x^{p+1}}dx\right) ds \\ &\leq \frac{1}{p}\int_0^1 \left|u''(s)\right|^p ds,\end{aligned}$$

where p' and p are given by $\frac{1}{p} + \frac{1}{p'} = 1$. Hence

$$\left\|\left(\frac{u(x)}{x}\right)'\right\|_p \leq p^{-\frac{1}{p}} \left\|u''\right\|_p.$$

Thus, if we define γ_p as in (1.4), we have proved that $\gamma_p \leq p^{-\frac{1}{p}}$, i.e., γ_p remains bounded as $p \to 1$.

As one might expect, an analogue to Theorem 1.2 can be proved for $W^{m,p}$ functions. The result reads as follows

Theorem 1.5. *Let $m \geq 2$ and $p > 1$. If u belongs to $W^{m,p}(0,1)$ and satisfies $u(0) = Du(0) = \ldots = D^{m-1}u(0) = 0$, then for $k \geq 1$ and $1 \leq j + k \leq m$,*

$$\left\|D^k\left(\frac{D^j u(x)}{x^{m-j-k}}\right)\right\|_{L^p(0,1)} \leq \frac{B(pk, p(m-j-k-1)+1)^{\frac{1}{p}}}{(m-j-k-1)!}\left\|D^m u\right\|_{L^p(0,1)}, \tag{1.9}$$

where $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ denotes Euler's Beta function.

Proof. From the proof of Theorem 1.2, we have

$$D^k\left(\frac{D^j u(x)}{x^{m-j-k}}\right) = \frac{1}{(m-j-k-1)!}\int_0^x D^m u(s)\left(1-\frac{s}{x}\right)^{m-j-k-1}\left(\frac{s}{x}\right)^{k-1}\frac{s}{x^2}ds.$$

After applying Hölder's inequality, Fubini's theorem and a change of variables one obtains that

$$\begin{aligned}
&\int_0^1 \left| D^k \left(\frac{D^j u(x)}{x^{m-j-k}} \right) \right|^p dx \\
\leq &\left(\frac{1}{(m-j-k-1)!} \right)^p \int_0^1 |D^m u(s)|^p \left(\int_s^1 (1-t)^{p(m-j-k-1)} \, t^{pk-1} dt \right) ds \\
\leq &\left(\frac{1}{(m-j-k-1)!} \right)^p \int_0^1 |D^m u(s)|^p \left(\int_0^1 (1-t)^{p(m-j-k-1)} \, t^{pk-1} dt \right) ds \\
= &B(pk, p(m-j-k-1)+1) \left(\frac{1}{(m-j-k-1)!} \right)^p \int_0^1 |D^m u(s)|^p \, ds.
\end{aligned}$$

□

Chapter 2

A Hardy type inequality for $\mathbf{W_0^{m,1}}(\boldsymbol{\Omega})$ functions

2.1 Introduction

In this chapter, we prove the following result, which is the higher dimensional analogue of the Theorem 1.2 in Chapter 1.

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Given $x \in \Omega$, we denote by $\delta(x)$ the distance from x to the boundary $\partial\Omega$. Let $d : \Omega \to (0, +\infty)$ be a smooth function such that $d(x) = \delta(x)$ near $\partial\Omega$. Suppose $m \geq 2$ and let j, k be non-negative integers such that $1 \leq k \leq m-1$ and $1 \leq j+k \leq m$. Then for every $u \in W_0^{m,1}(\Omega)$, we have $\frac{\partial^j u(x)}{d(x)^{m-j-k}} \in W_0^{k,1}(\Omega)$ with*

$$\left\| \partial^k \left(\frac{\partial^j u(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)} \leq C \left\| u \right\|_{W^{m,1}(\Omega)}, \tag{2.1}$$

where ∂^l denotes any partial differential operator of order l and $C > 0$ is a constant depending only on Ω and m.

Remark 2.1. *We will see that the proof of Theorem 2.1 is different from the proof of Theorem 1.2 if we consider, for example, $N = 2$, $\Omega = \mathbb{R}^2_+ = \{(x_1, x_2);\ x_2 \geq 0, x_1 \in \mathbb{R}\}$, and $u \in C_c^\infty([0,1] \times [0,1])$. From Theorem 1.2 it is clear that*

$$\int_\Omega \left| \frac{\partial}{\partial x_2} \left(\frac{u(x_1, x_2)}{x_2} \right) \right| dx_1 dx_2 \leq C \int_\Omega \left| \frac{\partial^2 u(x_1, x_2)}{\partial x_2^2} \right| dx_1 dx_2.$$

However new technique (Lemma 2.6) will be needed to derive

$$\int_\Omega \left| \frac{\partial}{\partial x_1} \left(\frac{u(x_1, x_2)}{x_2} \right) \right| dx_1 dx_2 \leq C \left\| D^2 u \right\|_{L^1(\Omega)}.$$

The rest of this chapter is organized as the following. In Section 2.2 we introduce the notation used throughout this chapter and give some preliminary results. In order to present the main ideas used to prove Theorem 2.1, we begin in Section 2.3 with the proof of Theorem 2.1 for the special case $m = 2$. Then in Section 2.4 we provide the proof of Theorem 2.1 for the general case $m \geq 2$.

2.2 Notation and preliminaries

Throughout this work, we denote $\mathbb{R}^N_+ = \big\{(y_1, \ldots, y_{N-1}, y_N) \in \mathbb{R}^N;\ y_N > 0\big\}$, the upper half space, and $B^N_r(x_0) = \big\{x \in \mathbb{R}^N;\ |x - x_0| < r\big\}$. When $x_0 = 0$, we write $B^N_r = B^N_r(0)$.

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Given $x \in \Omega$, we denote by $\delta(x)$ the distance from x to the boundary $\partial\Omega$, that is

$$\delta(x) = \operatorname{dist}(x, \partial\Omega) = \inf\left\{|x - y|\,;\ y \in \partial\Omega\right\}.$$

For $\epsilon > 0$, the tubular neighborhood of $\partial\Omega$ in Ω is the set

$$\Omega_\epsilon = \left\{x \in \Omega;\ \delta(x) < \epsilon\right\}.$$

The following is a well known result (see e.g. Lemma 14.16 in [30]) and it shows that δ is smooth in some neighborhood of $\partial\Omega$.

Lemma 2.2. *Let Ω and $\delta : \Omega \to (0, \infty)$ be as above. Then there exists $\epsilon_0 > 0$ only depending on Ω, such that $\delta|_{\Omega_{\epsilon_0}} : \Omega_{\epsilon_0} \to (0, \infty)$ is smooth. Moreover, for every $x \in \Omega_{\epsilon_0}$ there exists a unique $y_x \in \partial\Omega$ so that*

$$x = y_x + \delta(x)\nu_{\partial\Omega}(y_x),$$

where $\nu_{\partial\Omega}$ denotes the unit inward normal vector field associated to $\partial\Omega$.

Since $\partial\Omega$ is smooth, for fixed $\tilde{x}_0 \in \partial\Omega$, there exists a neighborhood $\mathcal{V}(\tilde{x}_0) \subset \partial\Omega$, a radius $r > 0$ and a map

$$\tilde{\Phi} : B^{N-1}_r \to \mathcal{V}(\tilde{x}_0) \tag{2.2}$$

which defines a smooth diffeomorphism. Define

$$\mathcal{N}_+(\tilde{x}_0) = \left\{x \in \Omega_{\epsilon_0};\ y_x \in \mathcal{V}(\tilde{x}_0)\right\}, \tag{2.3}$$

where ϵ_0 and y_x are given in Lemma 2.2. We define $\Phi : B_r^{N-1} \times (-\epsilon_0, \epsilon_0) \to \mathbb{R}^N$ as

$$\Phi(\tilde{y}, t) = \tilde{\Phi}(\tilde{y}) + y_N \cdot \nu_{\partial\Omega}(\tilde{\Phi}(\tilde{y})), \tag{2.4}$$

where $\tilde{y} = (y_1, \ldots, y_{N-1})$, and we write

$$\mathcal{N}(\tilde{x}_0) = \Phi\left(B_r^{N-1} \times (-\epsilon_0, \epsilon_0)\right). \tag{2.5}$$

About the map Φ we have the following

Lemma 2.3. *The map* $\Phi|_{B_r^{N-1}\times(0,\epsilon_0)}$ *is a diffeomorphism and*

$$\mathcal{N}_+(\tilde{x}_0) = \Phi\left(B_r^{N-1} \times (0, \epsilon_0)\right).$$

Proof. This is a direct corollary of the definition of Φ through $\tilde{\Phi}$, and Lemma 2.2. □

Remark 2.2. *The map* $\Phi|_{B_r^{N-1}\times(0,\epsilon_0)}$ *gives a local coordinate chart which straightens the boundary near* $\tilde{x}_0$. *This type of coordinates are sometimes called* flow coordinates *(see e.g. [9] and [33]).*

From now on, $C > 0$ will always denote a constant only depending on Ω and possibly the integer $m \geq 2$. The following is a direct, but very useful, corollary.

Corollary 2.4. *Let* $f \in L^1(\mathcal{N}_+(\tilde{x}_0))$ *and* Φ *be given by* (2.4). *Then*

$$\begin{aligned}\frac{1}{C}\int_{B_r^{N-1}}\int_0^{\epsilon_0} |f(\Phi(\tilde{y}, y_N))|\, dy_N d\tilde{y} \leq &\int_{\mathcal{N}_+(\tilde{x}_0)} |f(x)|\, dx \\ \leq &C \int_{B_r^{N-1}}\int_0^{\epsilon_0} |f(\Phi(\tilde{y}, y_N))|\, dy_N d\tilde{y}\end{aligned}$$

Proof. Since $\Phi|_{B_r^{N-1}\times(0,\epsilon_0)}$ is a diffeomorphism, we know that for all $(\tilde{y}, y_N) \in B_r^{N-1} \times (0, \epsilon_0)$ we have

$$\frac{1}{C} \leq |\det D\Phi(\tilde{y}, y_N)| \leq C.$$

The result then follows from the change of variables formula. □

The following lemma provides us a partition of unity in $\mathbb{R}^N$, constructed from the neighborhoods $\mathcal{N}(\tilde{x}_0)$. Consider the open cover of $\partial\Omega$ given by $\{\mathcal{V}(\tilde{x});\ \tilde{x} \in \partial\Omega\}$, where $\mathcal{V}(\tilde{x}) \subset \partial\Omega$ is defined in (2.2). By the compactness of $\partial\Omega$, there exists $\{\tilde{x}_1, \ldots, \tilde{x}_M\} \subset \partial\Omega$, so that $\partial\Omega = \cup_{l=1}^M \mathcal{V}(\tilde{x}_l)$. Notice that by the definition of $\mathcal{N}(\tilde{x}_0)$ in (2.5) we also have that $\cup_{l=1}^{M}\mathcal{N}(\tilde{x}_l)$ is an open cover of $\partial\Omega$ in $\mathbb{R}^N$. The following is a classical result (see e.g. Lemma 9.3 in [8] and Theorem 3.15 in [1]).

Lemma 2.5 (partition of unity). *There exist functions $\rho_0, \rho_1, \ldots, \rho_M \in C^\infty(\mathbb{R}^N)$ such that*

(i) $0 \le \rho_l \le 1$ *for all* $l = 0, 1, \ldots, M$ *and* $\sum_{l=0}^{M} \rho_i(x) = 1$ *for all* $x \in \mathbb{R}^N$,

(ii) $\operatorname{supp}\rho_l \subset \mathcal{N}(\tilde{x}_l)$, *for all* $l = 1, \ldots, M$,

(iii) $\rho_0|_\Omega \in C_c^\infty(\Omega)$.

In order to simplify the notation, we will denote by ∂^l any partial differential operator of order l where l is a positive integer[1]. Also, ∂_i will denote the partial derivative with respect to the i-th variable, and $\partial_{ij}^2 = \partial_i \circ \partial_j$.

Remark 2.3. *We conclude this section by showing that, to prove Theorem 2.1, it is enough to prove estimate* (2.1) *for smooth functions with compact support. Suppose* $u \in W_0^{m,1}(\Omega)$, *then there exists a sequence* $\{u_n\} \subset C_c^\infty(\Omega)$, *so that* $\|u - u_n\|_{W^{m,1}(\Omega)} \to 0$ *as* $n \to \infty$. *In particular, after maybe extracting a subsequence, one can assume that*

$$\partial^l u_n \to \partial^l u \ \text{a.e. in } \Omega, \ \text{for all } 0 \le l \le m.$$

Since d is smooth, the above implies that for a.e. $x \in \Omega$ and all $j \ge 0$, $1 \le k \le m-1$ and $1 \le j+k \le m$:

$$\begin{aligned}
\partial^k\left(\frac{\partial^j u(x)}{d(x)^{m-j-k}}\right) &= \frac{\partial^{j+k}u(x)}{d(x)^{m-j-k}} + \partial^j u(x)\partial^k\left(\frac{1}{d(x)^{m-j-k}}\right) \\
&= \lim_{n\to\infty} \frac{\partial^{j+k}u_n(x)}{d(x)^{m-j-k}} + \partial^j u_n(x)\partial^k\left(\frac{1}{d(x)^{m-j-k}}\right) \\
&= \lim_{n\to\infty} \partial^k\left(\frac{\partial^j u_n(x)}{d(x)^{m-j-k}}\right).
\end{aligned}$$

Therefore, Fatou's Lemma applies and we obtain

$$\left\|\partial^k\left(\frac{\partial^j u(x)}{d(x)^{m-j-k}}\right)\right\|_{L^1(\Omega)} \le \liminf_{n\to\infty}\left\|\partial^k\left(\frac{\partial^j u_n(x)}{d(x)^{m-j-k}}\right)\right\|_{L^1(\Omega)}.$$

[1]In general, one would say: "For a given multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$, we denote by ∂^α the partial differential operator of order $l = |\alpha| = \alpha_1 + \ldots + \alpha_N$". Since we only care about the order of the operator, it makes sense to abuse the notation and identify α with its order $|\alpha| = l$.

Once (2.1) *were proved for* $u_n \in C_c^\infty(\Omega)$*, we get*

$$\left\| \partial^k \left(\frac{\partial^j u_n(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)} \leq C \left\| u_n \right\|_{W^{m,1}(\Omega)},$$

and thus we can conclude that

$$\left\| \partial^k \left(\frac{\partial^j u(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)} \leq C \liminf_{n\to\infty} \left\| u_n \right\|_{W^{m,1}(\Omega)} = C \left\| u \right\|_{W^{m,1}(\Omega)}.$$

Finally, the fact that $\frac{\partial^j u_n(x)}{d(x)^{m-j-k}} \in C_c^\infty(\Omega)$ *and* $\overline{C_c^\infty(\Omega)}^{W^{k,1}(\Omega)} = W_0^{k,1}(\Omega)$ *gives that* $\frac{\partial^j u(x)}{d(x)^{m-j-k}} \in W_0^{k,1}(\Omega)$.

2.3 The case $m = 2$

We begin this section by proving estimate (2.1) in Theorem 2.1 for $\Omega = \mathbb{R}^N_+$, $m = 2$, $j = 0$ and $k = 1$.

Lemma 2.6. *Suppose that* $u \in C_c^\infty(\mathbb{R}^N_+)$*. Then for all* $i = 1, \dots, N$

$$\left\| \partial_i \left(\frac{u(y)}{y_N} \right) \right\|_{L^1(\mathbb{R}^N_+)} \leq 2 \left\| u \right\|_{W^{2,1}(\mathbb{R}^N_+)}.$$

Proof. Consider first the case $i = N$. The proof is essentially the same as (1.2), but for the sake of completeness, we still provide the proof. Notice that we can write

$$\frac{\partial}{\partial y_N} \left(\frac{u(\tilde{y}, y_N)}{y_N} \right) = \frac{1}{y_N^2} \int_0^{y_N} \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) t dt.$$

Then integration by parts yields that

$$\begin{aligned}
\int_{\mathbb{R}^{N-1}} \int_0^\infty \left| \frac{\partial}{\partial y_N} \left(\frac{u(\tilde{y}, y_N)}{y_N} \right) \right| dy_N d\tilde{y} &\leq \int_{\mathbb{R}^{N-1}} \int_0^\infty \frac{1}{y_N^2} \int_0^{y_N} \left| \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) \right| t dt dy_N d\tilde{y} \\
&= \int_{\mathbb{R}^{N-1}} \int_0^\infty \left| \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) \right| t \int_t^\infty \frac{1}{y_N^2} dy_N dt d\tilde{y} \\
&= \int_{\mathbb{R}^{N-1}} \int_0^\infty \left| \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) \right| t \int_t^\infty \frac{1}{y_N^2} dy_N dt d\tilde{y} \\
&= \int_{\mathbb{R}^{N-1}} \int_0^\infty \left| \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) \right| dt d\tilde{y}.
\end{aligned}$$

Hence

$$\int_{\mathbb{R}^N_+} \left| \frac{\partial}{\partial y_N} \left(\frac{u(y)}{y_N} \right) \right| dy \leq \int_{\mathbb{R}^N_+} \left| \frac{\partial^2 u(y)}{\partial y_N^2} \right| dy. \tag{2.6}$$

When $1 \leq i \leq N-1$, we need to estimate $\int_{\mathbb{R}^N_+} \frac{1}{y_N} \left| \frac{\partial u}{\partial y_i}(y) \right| dy$. To do so, consider the change of variables $y = \Psi(x)$, where

$$\Psi(x_1, \ldots, x_i, \ldots, x_N) = (x_1, \ldots, x_i + x_N, \ldots, x_N). \tag{2.7}$$

Notice that $\det D\Psi(x) = 1$, so

$$\int_{\mathbb{R}^N_+} \frac{1}{y_N} \left| \frac{\partial u(y)}{\partial y_i} \right| dy = \int_{\mathbb{R}^N_+} \frac{1}{x_N} \left| \frac{\partial u}{\partial y_i}(\Psi(x)) \right| dx.$$

Observe that if we let $v(x) = u(\Psi(x))$, we can write

$$\frac{1}{x_N} \frac{\partial u}{\partial y_i}(\Psi(x)) = \frac{\partial}{\partial x_N} \left(\frac{v(x)}{x_N} \right) - \frac{\partial}{\partial y_N} \left(\frac{u(y)}{y_N} \right) \Bigg|_{y=\Psi(x)}. \tag{2.8}$$

Applying estimate (2.6) to u and v yields

$$\begin{aligned} \int_{\mathbb{R}^N_+} \frac{1}{x_N} \left| \frac{\partial u}{\partial y_i}(\Psi(x)) \right| dx &\leq \int_{\mathbb{R}^N_+} \left| \frac{\partial}{\partial x_N} \left(\frac{v(x)}{x_N} \right) \right| dx + \int_{\mathbb{R}^N_+} \left| \frac{\partial}{\partial y_N} \left(\frac{u(y)}{y_N} \right) \Bigg|_{y=\Psi(x)} \right| dx \\ &= \int_{\mathbb{R}^N_+} \left| \frac{\partial}{\partial x_N} \left(\frac{v(x)}{x_N} \right) \right| dx + \int_{\mathbb{R}^N_+} \left| \frac{\partial}{\partial y_N} \left(\frac{u(y)}{y_N} \right) \right| dy \\ &\leq \int_{\mathbb{R}^N_+} \left| \frac{\partial^2 v(x)}{\partial x_N^2} \right| dx + \int_{\mathbb{R}^N_+} \left| \frac{\partial^2 u(y)}{\partial y_N^2} \right| dy. \end{aligned}$$

Finally, notice that

$$\frac{\partial^2 v(x)}{\partial x_N^2} = \frac{\partial^2 u(y)}{\partial y_N^2} \Bigg|_{y=\Psi(x)} + 2 \frac{\partial^2 u(y)}{\partial y_i \partial y_N} \Bigg|_{y=\Psi(x)} + \frac{\partial^2 u(y)}{\partial y_i^2} \Bigg|_{y=\Psi(x)}. \tag{2.9}$$

Thus, after reversing the change of variables when needed, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N_+} \frac{1}{y_N} \left| \frac{\partial u(y)}{\partial y_i} \right| dy &= \int_{\mathbb{R}^N_+} \frac{1}{x_N} \left| \frac{\partial u}{\partial y_i}(\Psi(x)) \right| dx \\ &\leq 2 \int_{\mathbb{R}^N_+} \left| \frac{\partial^2 u(y)}{\partial y_N^2} \right| dy + 2 \int_{\mathbb{R}^N_+} \left| \frac{\partial^2 u(y)}{\partial y_i \partial y_N} \right| dy + \int_{\mathbb{R}^N_+} \left| \frac{\partial^2 u(y)}{\partial y_i^2} \right| dy \\ &\leq 2 \|u\|_{W^{2,1}(\mathbb{R}^N_+)}. \end{aligned}$$

□

Recall (see Section 2.2) that for every $\tilde{x}_0 \in \partial\Omega$, there exist the neighborhood $\mathcal{N}_+(\tilde{x}_0) \subset \Omega$ given by (2.3) and the diffeomorphism $\Phi : B_r^{N-1} \times (0, \epsilon_0) \to \mathcal{N}_+(\tilde{x}_0)$ given by (2.4). Moreover, we know that $\delta(x)$ is smooth over $\mathcal{N}_+(\tilde{x}_0)$. Hence we have

Lemma 2.7. *Let $\tilde{x}_0 \in \partial\Omega$ and $\mathcal{N}_+(\tilde{x}_0)$ be given by* (2.3)*, and suppose $u \in C_c^\infty(\mathcal{N}_+(\tilde{x}_0))$. Then for all $i = 1, \ldots, N$,*

$$\left\| \partial_i \left(\frac{u(x)}{\delta(x)} \right) \right\|_{L^1(\mathcal{N}_+(\tilde{x}_0))} \leq C \left\| u \right\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_0))}.$$

Proof. We first use Corollary 2.4 and obtain

$$\int_{\mathcal{N}_+(\tilde{x}_0)} \left| \partial_i \left(\frac{u(x)}{\delta(x)} \right) \right| dx \leq C \int_{B_r^{N-1}} \int_0^{\epsilon_0} \left| \partial_i \left(\frac{u(x)}{\delta(x)} \right) \Big|_{x=\Phi(\tilde{y},y_N)} \right| dy_N d\tilde{y}.$$

Let $v(\tilde{y}, y_N) = u(\Phi(\tilde{y}, y_N))$. We claim that

$$\int_{B_r^{N-1}} \int_0^{\epsilon_0} \left| \partial_i \left(\frac{u(x)}{\delta(x)} \right) \Big|_{x=\Phi(\tilde{y},y_N)} \right| dy_N d\tilde{y} \leq C \sum_{j=1}^{N} \int_{B_r^{N-1}} \int_0^{\epsilon_0} \left| \partial_j \left(\frac{v(\tilde{y}, y_N)}{y_N} \right) \right| dy_N d\tilde{y}. \tag{2.10}$$

We will prove (2.10) at the end, so that we can conclude the argument. Since $v \in C_c^\infty(B_r^{N-1} \times (0, \epsilon_0)) \subset C_c^\infty(\mathbb{R}_+^N)$, we can apply Lemma 2.6 and obtain

$$\int_{B_r^{N-1}} \int_0^{\epsilon_0} \left| \partial_j \left(\frac{v(\tilde{y}, y_N)}{y_N} \right) \right| dy_N d\tilde{y} \leq C \left\| v \right\|_{W^{2,1}(B_r^{N-1} \times (0,\epsilon_0))}.$$

Notice that by the chain rule and the fact that Φ is a diffeomorphism, we get that for all $1 \leq i, j \leq N$,

$$\left| \partial_{ij}^2 v(\tilde{y}, y_N) \right| \leq C \left(\sum_{p,q=1}^{N} \left| \partial_{pq}^2 u(x)|_{x=\Phi(\tilde{y},y_N)} \right| + \sum_{p=1}^{N} \left| \partial_p u(x)|_{x=\Phi(\tilde{y},y_N)} \right| \right),$$

so with the aid of Corollary 2.4, we can write

$$\begin{aligned} &\left\| v \right\|_{W^{2,1}(B_r^{N-1} \times (0,\epsilon_0))} \\ \leq &C \int_{B_r^{N-1}} \int_0^{\epsilon_0} \left(\sum_{p,q} \left| \partial_{pq}^2 u|_{x=\Phi(\tilde{y},y_N)} \right| + \sum_{p} \left| \partial_p u|_{x=\Phi(\tilde{y},y_N)} \right| \right) dy_N d\tilde{y} \\ \leq &C \int_{\mathcal{N}_+(\tilde{x}_0)} \left(\sum_{p,q} \left| \partial_{pq}^2 u(x) \right| + \sum_{p} \left| \partial_p u(x) \right| \right) dx \\ \leq &C \left\| u \right\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_0))}. \end{aligned}$$

To conclude, we need to prove (2.10). To do so, notice that $u(x) = v(\Phi^{-1}(x))$, and $\delta(x) = c(\Phi^{-1}(x))$, where $c(\tilde{y}, y_N) = y_N$. Thus, by using the chain rule we obtain

$$\partial_i \left(\frac{u(x)}{\delta(x)} \right) \Big|_{x=\Phi(\tilde{y},y_N)} = \sum_{j=1}^{N} \partial_j \left(\frac{v(y)}{c(y)} \right) \Big|_{y=(\tilde{y},y_N)} \cdot \partial_i (\Phi^{-1})_j (\Phi(\tilde{y}, y_N)),$$

and since Φ is a diffeomorphism, we obtain

$$\left|\partial_i\left(\frac{u(x)}{\delta(x)}\right)\Big|_{x=\Phi(\tilde{y},y_N)}\right| \le C\sum_{j=1}^{N}\left|\partial_j\left(\frac{v(y)}{c(y)}\right)\Big|_{y=(\tilde{y},y_N)}\right|.$$

Estimate (2.10) then follows by integrating the above inequality. □

We end this section with the proof of the main result when $m = 2$.

Proof of Theorem 2.1 when $m = 2$. When $j = 1$ and $k = 1$ the estimate (2.1) is trivial. Taking into account Remark 2.3, we only need to prove

$$\left\|\partial_i\left(\frac{u(x)}{d(x)}\right)\right\|_{L^1(\Omega)} \le C\left\|u\right\|_{W^{2,1}(\Omega)} \tag{2.11}$$

for $u \in C_c^\infty(\Omega)$ and $i = 1, 2, \ldots, N$. To do so, we use the partition of unity given by Lemma 2.5 to write $u(x) = \sum_{l=0}^{M} u_l(x)$ on Ω where $u_l(x) := \rho_l(x)u(x)$, $l = 0, 1, \ldots, M$. Now, without loss of generality, we can assume that $d(x) = \delta(x)$ for all $x \in \Omega_{\epsilon_0}$, and that $d(x) \ge C > 0$ for all $x \in \operatorname{supp}\rho_0 \cap \Omega$. Notice that in $\operatorname{supp}\rho_0 \cap \Omega$, we have

$$\frac{u_0}{d} \in C^\infty(\overline{\operatorname{supp}\rho_0 \cap \Omega}), \text{ with } \left\|\frac{u_0}{d}\right\|_{W^{1,1}(\operatorname{supp}\rho_0\cap\Omega)} \le C\left\|u_0\right\|_{W^{1,1}(\sup\rho_0\cap\Omega)}.$$

To take care of the boundary part, notice that $u_l \in C_c^\infty(\mathcal{N}_+(\tilde{x}_l))$ for $l = 1, \ldots, M$, so Lemma 2.7 applies and we obtain

$$\left\|\partial_i\left(\frac{u_l(x)}{\delta(x)}\right)\right\|_{L^1(\mathcal{N}_+(\tilde{x}_l))} \le C\left\|u_l\right\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_l))}, \text{ for all } l = 1, \ldots, M.$$

To conclude, notice that

$$\partial_i\left(\frac{u(x)}{d(x)}\right) = \sum_{l=1}^{M}\partial_i\left(\frac{u_l(x)}{\delta(x)}\right) + \partial_i\left(\frac{u_0(x)}{d(x)}\right)$$

on Ω and that $|\rho_l(x)|$, $|\partial_i\rho_l(x)|$ and $\left|\partial_{ij}^2\rho_l(x)\right|$ are uniformly bounded for all $l = 0, 1, \ldots, M$. Therefore

$$\begin{aligned}\left\|\partial_i\left(\frac{u(x)}{d(x)}\right)\right\|_{L^1(\Omega)} &\le \sum_{l=1}^{M}\left\|\partial_i\left(\frac{u_l(x)}{\delta(x)}\right)\right\|_{L^1(\mathcal{N}_+(\tilde{x}_l))} + \left\|\partial_i\left(\frac{u_0(x)}{d(x)}\right)\right\|_{L^1(\operatorname{supp}\rho_0\cap\Omega)}\\ &\le C\left(\sum_{l=1}^{M}\left\|u_l\right\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_l))} + \left\|u_0\right\|_{W^{1,1}(\operatorname{supp}\rho_0\cap\Omega)}\right)\\ &\le C\left\|u\right\|_{W^{2,1}(\Omega)},\end{aligned}$$

thus completing the proof. □

2.4 The general case $m \geq 2$

To prove the general case, we need to generalize Lemma 2.6 in the following way.

Lemma 2.8. *Suppose $u \in C_c^\infty(\mathbb{R}_+^N)$. Then for all $m \geq 1$ and $i = 1, \dots, N$ we have*

$$\left\| \partial_i \left(\frac{u(y)}{y_N^{m-1}} \right) \right\|_{L^1(\mathbb{R}_+^N)} \leq C \|u\|_{W^{m,1}(\mathbb{R}_+^N)} .$$

Proof. The case $m = 1$ is a trivial statement, whereas $m = 2$ is exactly what we proved in Lemma 2.6. So from now on we suppose $m \geq 3$. We first notice that when $i = N$, the result follows from the proof of Theorem 1.2 when $j = 0$ and $k = 1$.

When $1 \leq i \leq N - 1$, we can proceed the same as in the proof of Lemma 2.6. Define $v(x) = u(\Psi(x))$ where Ψ is given by (2.7). Notice that when $m \geq 3$, instead of equation (2.8) we have

$$\frac{1}{x_N^{m-1}} \frac{\partial u}{\partial y_i}(\Psi(x)) = \frac{\partial}{\partial x_N} \left(\frac{v(x)}{x_N^{m-1}} \right) - \frac{\partial}{\partial y_N} \left(\frac{u(y)}{y_N^{m-1}} \right) \bigg|_{y=\Psi(x)},$$

and instead of (2.9) we have

$$\frac{\partial^m v(x)}{\partial x_N^m} = \sum_{l=0}^{m} \binom{m}{l} \frac{\partial^m u(y)}{\partial y_i^{m-l} \partial y_N^l} \bigg|_{y=\Psi(x)} .$$

Hence the estimate is reduced to the result for $i = N$. We omit the details. □

We also have the analog of Lemma 2.7.

Lemma 2.9. *Let $\tilde{x}_0 \in \partial\Omega$ and $\mathcal{N}_+(\tilde{x}_0)$ as in Lemma 2.7. Let $u \in C_c^\infty(\mathcal{N}_+(\tilde{x}_0))$. Then for all $m \geq 1$ and $i = 1, \dots, N$ we have*

$$\left\| \partial_i \left(\frac{u(x)}{\delta(x)^{m-1}} \right) \right\|_{L^1(\mathcal{N}_+(\tilde{x}_0))} \leq C \|u\|_{W^{m,1}(\mathcal{N}_+(\tilde{x}_0))} .$$

Proof. The proof involves only minor modifications from the proof of Lemma 2.7, which we provide in the next few lines. Corollary 2.4 gives

$$\int_{\mathcal{N}_+(\tilde{x}_0)} \left| \partial_i \left(\frac{u(x)}{\delta(x)^{m-1}} \right) \right| dx \leq C \int_{B_r^{N-1}} \int_0^{\epsilon_0} \left| \partial_i \left(\frac{u(x)}{\delta(x)^{m-1}} \right) \bigg|_{x=\Phi(\tilde{y}, y_N)} \right| dy_N d\tilde{y}.$$

If $v(\tilde{y}, y_N) = u(\Phi(\tilde{y}, y_N))$, then

$$\int_{B_r^{N-1}} \int_0^{\epsilon_0} \left| \partial_i \left(\frac{u(x)}{\delta(x)^{m-1}} \right) \Big|_{x=\Phi(\tilde{y},y_N)} \right| dy_N d\tilde{y} \leq C \sum_{j=1}^{N} \int_{B_r^{N-1}} \int_0^{\epsilon_0} \left| \partial_j \left(\frac{v(\tilde{y}, y_N)}{y_N^{m-1}} \right) \right| dy_N d\tilde{y}. \quad (2.12)$$

Just as for (2.10), estimate (2.12) follows from the fact that Φ is a smooth diffeomorphism. Since $v \in C_c^\infty(B_r^{N-1} \times (0, \epsilon_0)) \subset C_c^\infty(\mathbb{R}_+^N)$, we can apply Lemma 2.8 and obtain

$$\int_{B_r^{N-1}} \int_0^{\epsilon_0} \left| \partial_j \left(\frac{v(\tilde{y}, y_N)}{y_N^{m-1}} \right) \right| dy_N d\tilde{y} \leq C \left\| v \right\|_{W^{m,1}(B_r^{N-1} \times (0,\epsilon_0))} .$$

Notice that by the chain rule and the fact that Φ is a smooth diffeomorphism, we get

$$|\partial^m v(\tilde{y}, y_N)| \leq C \sum_{l \leq m} \left| \partial^l u(x)|_{x=\Phi(\tilde{y},y_N)} \right| ,$$

where the left hand side is a fixed m-th order partial derivative, and in the right hand side the summation contains all partial derivatives of order $l \leq m$. Again with the aid of Corollary 2.4, we can write

$$\begin{aligned} \left\| v \right\|_{W^{m,1}(B_r^{N-1} \times (0,\epsilon_0))} &\leq C \sum_{l \leq m} \int_{B_r^{N-1}} \int_0^{\epsilon_0} \left(\left| \partial^l u|_{x=\Phi(\tilde{y},y_N)} \right| \right) dy_N d\tilde{y} \\ &\leq C \sum_{l \leq m} \int_{\mathcal{N}_+(\tilde{x}_0)} \left| \partial^l u(x) \right| dx \\ &\leq C \left\| u \right\|_{W^{m,1}(\mathcal{N}_+(\tilde{x}_0))} . \end{aligned}$$

□

And of course we have

Lemma 2.10. *Suppose $u \in C_c^\infty(\Omega)$. Then for all $m \geq 1$ and $i = 1, \ldots, N$ we have*

$$\left\| \partial_i \left(\frac{u(x)}{\delta(x)^{m-1}} \right) \right\|_{L^1(\Omega)} \leq C \left\| u \right\|_{W^{m,1}(\Omega)} .$$

We omit the proof of the above lemma, because it is almost a line by line copy of the proof of the estimate (2.11) in Section 2.3 using the partition of unity. We are now ready to prove Theorem 2.1.

Proof Theorem 2.1. For any fixed integer $m \geq 3$, just as what we did for the case $m = 2$, it is enough to prove the estimate (2.1) for $u \in C_c^\infty(\Omega)$. Notice that since

$$\left\|\partial^j u\right\|_{W^{m-j,1}(\Omega)} \leq \|u\|_{W^{m,1}(\Omega)} \text{ for all } 0 \leq j \leq m,$$

it is enough to show

$$\left\|\partial^k \left(\frac{u(x)}{d(x)^{m-k}}\right)\right\|_{L^1(\Omega)} \leq C \|u\|_{W^{m,1}(\Omega)}, \tag{2.13}$$

for $u \in C_c^\infty(\Omega)$ and $1 \leq k \leq m-1$. We proceed by induction in k. The case $k = 1$ corresponds exactly to Lemma 2.10. If one assumes the result for k, then we have to estimate for $i = 1, \ldots, N$,

$$\partial_i \partial^k \left(\frac{u(x)}{d(x)^{m-k-1}}\right) = \partial^k \left(\frac{\partial_i u(x)}{d(x)^{m-k-1}}\right) - (m-k-1)\partial^k \left(\frac{u(x)\partial_i d(x)}{d(x)^{m-k}}\right).$$

The induction hypothesis for $\tilde{m} = m-1$ yields

$$\left\|\partial^k \left(\frac{\partial_i u(x)}{d(x)^{(m-1)-k}}\right)\right\|_{L^1(\Omega)} \leq C \|\partial_i u\|_{W^{m-1,1}(\Omega)} \leq C \|u\|_{W^{m,1}(\Omega)}.$$

On the other hand, by using the induction hypothesis and the fact that d is smooth in $\overline{\Omega}$, we obtain

$$\left\|\partial^k \left(\frac{u(x)\partial_i d(x)}{d(x)^{m-k}}\right)\right\|_{L^1(\Omega)} \leq C \|u\partial_i d\|_{W^{m,1}(\Omega)} \leq C \|u\|_{W^{m,1}(\Omega)}.$$

Therefore

$$\left\|\partial_i \partial^k \left(\frac{u(x)}{d(x)^{m-k-1}}\right)\right\|_{L^1(\Omega)} \leq C \|u\|_{W^{m,1}(\Omega)},$$

thus concluding the proof. □

Chapter 3

A singular Sturm-Liouville equation under homogeneous boundary conditions

3.1 Introduction

This chapter concerns the following Sturm-Liouvile equation

$$\begin{cases} -\left(x^{2\alpha}u'(x)\right)' + u(x) = f(x) & \text{on } (0,1), \\ u(1) = 0, \end{cases} \tag{3.1}$$

where α is a positive real number and $f \in L^2(0,1)$ is given. We will study the existence, uniqueness and regularity of solutions of (3.1), under suitable *homogeneous* boundary data. We also discuss spectral properties of the differential operator $\mathcal{L}u := -\left(x^{2\alpha}u'\right)' + u$.

The classical ODE theory says that if for instance the right hand side f is a continuous function on $(0,1]$, then the solution set of (3.1) is a one parameter family of $C^2(0,1]$-functions. As we already mentioned, the first goal of this chapter is to select "distinguished" elements of that family by prescribing (weighted) *homogeneous* boundary conditions at the origin. In Chapter 4, we will study (3.1) under non-homogeneous boundary conditions at the origin.

When $0 < \alpha < \frac{1}{2}$, we have both a Dirichlet and a weighted Neumann problem. When $\alpha \geq \frac{1}{2}$, we only have a "Canonical" solution obtained by prescribing either a weighted Dirichlet or a weighted Neumann condition; as we are going to explain in Remark 3.20, the two boundary conditions yield the same solution.

Throughout this chapter $u \in H^2_{loc}(0,1]$ means $u \in H^2_{loc}(\epsilon,1)$ for all $\epsilon > 0$.

3.1.1 The case $0 < \alpha < \frac{1}{2}$

We first consider the Dirichlet problem.

Theorem 3.1 (Existence for Dirichlet Problem)**.** *Given* $0 < \alpha < \frac{1}{2}$ *and* $f \in L^2(0,1)$, *there exists a function* $u \in H^2_{loc}(0,1]$ *satisfying* (3.1) *together with the following properties:*

(i) $\lim_{x\to 0^+} u(x) = 0$.

(ii) $u \in C^{0,1-2\alpha}[0,1]$ *with* $\|u\|_{C^{0,1-2\alpha}} \leq C\|f\|_{L^2}$.

(iii) $x^{2\alpha}u' \in H^1(0,1)$ *with* $\|x^{2\alpha}u'\|_{H^1} \leq C\|f\|_{L^2}$.

(iv) $x^{2\alpha-1}u \in H^1(0,1)$ *with* $\|x^{2\alpha-1}u\|_{H^1} \leq C\|f\|_{L^2}$.

(v) $x^{2\alpha}u \in H^2(0,1)$ *with* $\|x^{2\alpha}u\|_{H^2} \leq C\|f\|_{L^2}$.

Here the constant C *only depends on* α.

Before stating the uniqueness result, we would like to give a few remarks of about this Theorem.

Remark 3.1. *There exists a function* $f \in C_c^\infty(0,1)$ *such that near the origin the solution given by Theorem 3.1 can be expanded in the following way*

$$u(x) = a_1x^{1-2\alpha} + a_2x^{3-4\alpha} + a_3x^{5-6\alpha} + \cdots \tag{3.2}$$

where $a_1 \neq 0$. *See Section 3.3.1 for the proof.*

Remark 3.2. *Theorem 3.1 only says* $(x^{2\alpha}u')' = x^{2\alpha}u'' + 2\alpha x^{2\alpha-1}u'$ *is in* $L^2(0,1)$. *A natural question is whether each term on the right-hand side belongs to* $L^2(0,1)$. *The answer is that, in general, neither of them is in* $L^2(0,1)$; *in fact, they are not even in* $L^1(0,1)$. *One can see this phenomenon in* (3.2), *where we have that* $x^{2\alpha-1}u'(x) \sim x^{2\alpha}u''(x) \sim x^{-1} \notin L^1(0,1)$.

Remark 3.3. *Part (iii) in Theorem 3.1 implies that* $u \in W^{1,p}(0,1)$ *for all* $1 \leq p < \frac{1}{2\alpha}$ *with* $\|u'\|_{L^p} \leq C\|f\|_{L^2}$, *where* C *is a constant only depending on* α. *However, one* cannot *expect that* $u \in W^{1,\frac{1}{2\alpha}}(0,1)$ *even if* $f \in C_c^\infty(0,1)$, *as the power series expansion* (3.2) *shows that* $u' \sim x^{-2\alpha}$ *near the origin.*

Remark 3.4. *Concerning the assertions in Theorem 3.1, we have the following implications: (i) and (iii) $\Rightarrow$ (iv); (iv) $\Rightarrow$ (ii); (iii) and (iv) $\Rightarrow$ (v). Those implications can be found in the proof of Theorem 3.1.*

Remark 3.5. *The assertions in Theorem 3.1 are optimal in the following sense: there exists $f \in L^2(0,1)$ such that $u \notin C^{0,\beta}[0,1]$ $\forall \beta > 1-2\alpha$; and one can find another $f \in L^2(0,1)$ such that $x^{2\alpha-1}u \notin H^2(0,1)$, $x^{2\alpha}u' \notin H^2(0,1)$, and $x^{2\alpha}u \notin H^3(0,1)$. See Section 3.3.1 for the counterexamples.*

Remark 3.6. *Theorem 3.1 tells us that both $x^{2\alpha}u'$ and $x^{2\alpha-1}u$ belong to $H^1(0,1)$, so in particular they are continuous up to the origin. It is natural to examine their values at the origin and how they are related to the right-hand side $f \in L^2(0,1)$. We actually have*

$$\lim_{x\to 0^+} x^{2\alpha}u'(x) = \int_0^1 f(x)g(x)dx, \tag{3.3}$$

and

$$\lim_{x\to 0^+} x^{2\alpha-1}u(x) = \frac{1}{1-2\alpha}\int_0^1 f(x)g(x)dx, \tag{3.4}$$

where the function g is the solution of

$$\begin{cases} -(x^{2\alpha}g'(x))' + g(x) = 0 & \text{on } (0,1), \\ g(1) = 0, \\ \lim_{x\to 0^+} g(x) = 1. \end{cases}$$

See Section 3.3.1 for the proof of this Remark. The existence of g will be given in Chapter 4. The uniqueness of g comes from Theorem 3.2 below.

Theorem 3.2 (Uniqueness for the Dirichlet problem)**.** *Let $0 < \alpha < \frac{1}{2}$. Assume that $u \in H^2_{loc}(0,1]$ satisfies*

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 & \text{on } (0,1), \\ u(1) = 0, \\ \lim_{x\to 0^+} u(x) = 0. \end{cases} \tag{3.5}$$

Then $u \equiv 0$.

In order to simplify the terminology, we denote by u_D the unique solution to (3.1) given by Theorem 3.1. Next we consider the regularity property of the solution u_D when the right-hand side f has a better regularity.

Theorem 3.3. *Let $0 < \alpha < \frac{1}{2}$ and $f \in W^{1,\frac{1}{2\alpha}}(0,1)$. Let u_D be the solution to (3.1) given by Theorem 3.1. Then $x^{2\alpha-1}u_D \in W^{2,p}(0,1)$ for all $1 \leq p < \frac{1}{2\alpha}$ with $\left\|x^{2\alpha-1}u_D\right\|_{W^{2,p}} \leq C\left\|f\right\|_{W^{1,p}}$, where C is a constant only depending on p and α.*

Remark 3.7. *One cannot expect that $x^{2\alpha-1}u_D \in W^{2,\frac{1}{2\alpha}}(0,1)$ even if $f \in C_c^\infty(0,1)$, as the power series expansion (3.2) shows that $(x^{2\alpha-1}u_D(x))'' \sim x^{-2\alpha}$ near the origin.*

Remark 3.8. *When $\alpha \geq \frac{1}{2}$, we* cannot *prescribe the Dirichlet boundary condition $\lim_{x\to 0^+} u(x) = 0$. Actually, for $\alpha \geq \frac{1}{2}$, there is no $H^2_{loc}(0,1]$-solution of*

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = f \qquad \text{on } (0,1), \\ u(1) = 0, \\ \lim\limits_{x\to 0^+} u(x) = 0, \end{cases} \tag{3.6}$$

for either $f \equiv 1$ or some $f \in C_c^\infty(0,1)$. See Section 3.3.1 for the proof.

Next we consider the case $0 < \alpha < \frac{1}{2}$ together with a weighted Neumann condition.

Theorem 3.4 (Existence for Neumann Problem)**.** *Given $0 < \alpha < \frac{1}{2}$ and $f \in L^2(0,1)$, there exists a function $u \in H^2_{loc}(0,1]$ satisfying (3.1) together with the following properties:*

(i) $u \in H^1(0,1)$ *with* $\left\|u\right\|_{H^1} \leq C\left\|f\right\|_{L^2}$.

(ii) $\lim_{x\to 0^+} x^{2\alpha-\frac{1}{2}}u'(x) = 0$.

(iii) $x^{2\alpha-1}u' \in L^2(0,1)$ *and* $x^{2\alpha}u'' \in L^2(0,1)$, *with* $\left\|x^{2\alpha-1}u'\right\|_{L^2} + \left\|x^{2\alpha}u''\right\|_{L^2} \leq C\left\|f\right\|_{L^2}$. *In particular,* $x^{2\alpha}u' \in H^1(0,1)$.

Here the constant C only depends on α.

Remark 3.9. *Notice the difference between Dirichlet and Neumann with respect to property (iii) of Theorem 3.4. See Remark 3.2.*

Remark 3.10. *The boundary behavior* $\lim_{x\to 0^+} x^{2\alpha-\frac{1}{2}}u'(x) = 0$ *is optimal in the following sense: for any* $0 < x \leq \frac{1}{2}$*, define*

$$K_\alpha(x) = \sup_{\|f\|_{L^2}\leq 1} \left| x^{2\alpha-\frac{1}{2}}u'(x) \right|.$$

Then $0 < \delta \leq K_\alpha(x) \leq 2$*, for some constant* δ *only depending on* α*. See Section 3.3.2 for the proof.*

Remark 3.11. *Theorem 3.4 implies that* $u \in C[0,1]$*, so it is natural to consider the dependence on* f *of the quantity* $\lim_{x\to 0^+} u(x)$*. One has*

$$\lim_{x\to 0^+} u(x) = \int_0^1 f(x)h(x)dx, \tag{3.7}$$

where h *is the solution of*

$$\begin{cases} -\left(x^{2\alpha}h'(x)\right)' + h(x) = 0 \quad \text{on } (0,1), \\ h(1) = 0, \\ \lim\limits_{x\to 0^+} x^{2\alpha}h'(x) = 1. \end{cases}$$

In particular, equation (3.7) *implies that the quantity* $\lim_{x\to 0^+} u(x)$ *is not necessarily 0. See Section 3.3.2 for the proof of this Remark. The existence of* h *will be given in Chapter 4. The uniqueness of* h *comes from Theorem 3.5 below.*

Theorem 3.5 (Uniqueness for the Neumann Problem)**.** *Let* $0 < \alpha < \frac{1}{2}$*. Assume that* $u \in H^2_{loc}(0,1]$ *satisfies*

$$\begin{cases} -\left(x^{2\alpha}u'(x)\right)' + u(x) = 0 \quad \text{on } (0,1), \\ u(1) = 0, \\ \lim\limits_{x\to 0^+} x^{2\alpha}u'(x) = 0. \end{cases} \tag{3.8}$$

Then $u \equiv 0$*.*

We denote by u_N the unique solution of (3.1) given by Theorem 3.4. We now state the following regularity result.

Theorem 3.6. *Let* $0 < \alpha < \frac{1}{2}$ *and* $f \in L^2(0,1)$*. Let* u_N *be the solution of* (3.1) *given by Theorem 3.4.*

(i) *If* $f \in W^{1,\frac{1}{2\alpha}}(0,1)$, *then* $u_N \in W^{2,p}(0,1)$ *for all* $1 \le p < \frac{1}{2\alpha}$ *with*

$$\|u_N\|_{W^{2,p}(0,1)} \le C\,\|f\|_{W^{1,p}}\,.$$

(ii) *If* $f \in W^{2,\frac{1}{2\alpha}}(0,1)$, *then* $x^{2\alpha-1}u'_N \in W^{2,p}(0,1)$ *for all* $1 \le p < \frac{1}{2\alpha}$, *with*

$$\left\|x^{2\alpha-1}u'_N\right\|_{W^{2,p}(0,1)} \le C\,\|f\|_{W^{2,p}}\,.$$

Here the constant C *depends only on* p *and* α.

Remark 3.12. *One cannot expect that* $u_N \in W^{2,\frac{1}{2\alpha}}(0,1)$ *nor* $x^{2\alpha-1}u'_N \in W^{2,\frac{1}{2\alpha}}(0,1)$. *Actually, there exists an* $f \in C_c^\infty(0,1)$ *such that,* $u_N \notin W^{2,\frac{1}{2\alpha}}(0,1)$ *and* $x^{2\alpha-1}u'_N \notin W^{2,\frac{1}{2\alpha}}(0,1)$. *See Section 3.3.2 for the proof.*

We now turn to the case $\alpha \ge \frac{1}{2}$. It is convenient to divide this case into three subcases. As we already pointed out, we only have a "Canonical" solution obtained by prescribing either a weighted Dirichlet or a weighted Neumann condition.

3.1.2 The case $\frac{1}{2} \le \alpha < \frac{3}{4}$

Theorem 3.7 (Existence for the "Canonical" Problem)**.** *Given* $\frac{1}{2} \le \alpha < \frac{3}{4}$ *and* $f \in L^2(0,1)$, *there exists* $u \in H^2_{loc}(0,1]$ *satisfying* (3.1) *together with the following properties:*

(i) $u \in C^{0,\frac{3}{2}-2\alpha}[0,1]$ *with* $\|u\|_{C^{0,\frac{3}{2}-2\alpha}} \le C\,\|f\|_{L^2}$ *and* $\lim_{x\to 0^+}(1-\ln x)^{-\frac{1}{2}}u(x) = 0$.

(ii) $\lim_{x\to 0^+} x^{2\alpha-\frac{1}{2}}u'(x) = 0$.

(iii) $x^{2\alpha-1}u' \in L^2(0,1)$ *and* $x^{2\alpha}u'' \in L^2(0,1)$, *with* $\left\|x^{2\alpha-1}u'\right\|_{L^2} + \left\|x^{2\alpha}u''\right\|_{L^2} \le C\,\|f\|_{L^2}$. *In particular,* $x^{2\alpha}u' \in H^1(0,1)$.

Here the constant C *depends only on* α.

Remark 3.13. *The same conclusions as in Remark 3.9–3.11 still hold for the solution given by Theorem 3.7.*

Theorem 3.8 (Uniqueness for the "Canonical" Problem)**.** *Let* $\frac{1}{2} \le \alpha < \frac{3}{4}$. *Assume* $u \in H^2_{loc}(0,1]$ *satisfies*

$$\begin{cases} -\left(x^{2\alpha}u'(x)\right)' + u(x) = 0 & \text{on } (0,1), \\ u(1) = 0. \end{cases}$$

If in addition one of the following conditions is satisfied

(i) $\lim_{x\to 0^+} x^{2\alpha}u'(x) = 0$,

(ii) $\lim_{x\to 0^+} (1-\ln x)^{-1}\, u(x) = 0$ *when* $\alpha = \frac{1}{2}$,

(iii) $u \in L^{\frac{1}{2\alpha-1}}(0,1)$ *when* $\frac{1}{2} < \alpha < \frac{3}{4}$,

(iv) $\lim_{x\to 0^+} x^{2\alpha-1}u(x) = 0$ *when* $\frac{1}{2} < \alpha < \frac{3}{4}$,

then $u \equiv 0$.

Again, to simplify the terminology, we call the unique solution of (3.1) given by Theorem 3.7 the "Canonical" solution and denote it by u_C. We now state the following regularity result.

Theorem 3.9. *Let* $\alpha = \frac{1}{2}$, k *be an positive integer, and* $f \in H^k(0,1)$. *Let* u_C *be the solution to* (3.1) *given by Theorem 3.7. Then* $u_C \in H^{k+1}(0,1)$ *and* $xu_C \in H^{k+2}(0,1)$ *with*

$$\|u_C\|_{H^{k+1}} + \|xu_C\|_{H^{k+2}} \le C\,\|f\|_{H^k},$$

where C *is a constant depending only on* k.

Remark 3.14. *A variant of Theorem 3.9 is already known. For instance in [23], the authors study the Legendre operator* $Lu = -\left((1-x^2)u'\right)'$ *in the interval* $(-1,1)$, *and they prove that the operator* $A = L + I$ *defines an isomorphism from* $D^k(A) := \left\{u \in H^{k+1}(-1,1);\ (1-x^2)u(x) \in H^{k+2}(-1,1)\right\}$ *to* $H^k(-1,1)$ *for all* $k \in \mathbb{N}$.

Theorem 3.10. *Let* $\frac{1}{2} < \alpha < \frac{3}{4}$ *and* $f \in W^{1,\frac{1}{2\alpha-1}}(0,1)$. *Let* u_C *be the solution to* (3.1) *given by Theorem 3.7. Then both* $u_C \in W^{1,p}(0,1)$ *and* $x^{2\alpha-1}u_C' \in W^{1,p}(0,1)$ *for all* $1 \le p < \frac{1}{2\alpha-1}$ *with*

$$\|u_C\|_{W^{1,p}} + \left\|x^{2\alpha-1}u_C'\right\|_{W^{1,p}} \le C\,\|f\|_{W^{1,p}},$$

where C *is a constant depending only on* p *and* α.

Remark 3.15. *One cannot expect that* $u_C \in W^{1,\frac{1}{2\alpha-1}}(0,1)$ *nor* $x^{2\alpha-1}u_C' \in W^{1,\frac{1}{2\alpha-1}}(0,1)$. *Actually, there exists an* $f \in C_c^\infty(0,1)$ *such that* $u_C \notin W^{1,\frac{1}{2\alpha-1}}(0,1)$ *and* $x^{2\alpha-1}u_C' \notin W^{1,\frac{1}{2\alpha-1}}(0,1)$. *See Section 3.3.2 for the proof.*

3.1.3 The case $\frac{3}{4} \le \alpha < 1$

Theorem 3.11 (Existence for the "Canonical" Problem)**.** *Given $\frac{3}{4} \le \alpha < 1$ and $f \in L^2(0,1)$, there exists a function $u \in H^2_{loc}(0,1]$ satisfying (3.1) together with the following properties:*

(i) $u \in L^p(0,1)$ *with* $\|u\|_{L^p} \le C\|f\|_{L^2}$, *where p is any number in $[1,\infty)$ if $\alpha = \frac{3}{4}$, and $p = \frac{2}{4\alpha-3}$ if $\frac{3}{4} < \alpha < 1$.*

(ii) $\lim_{x\to 0^+} (1-\ln x)^{-\frac{1}{2}} u(x) = 0$ *if* $\alpha = \frac{3}{4}$; $\lim_{x\to 0^+} x^{2\alpha-\frac{3}{2}} u(x) = 0$ *if* $\frac{3}{4} < \alpha < 1$.

(iii) $\lim_{x\to 0^+} x^{2\alpha-\frac{1}{2}} u'(x) = 0$.

(iv) $x^{2\alpha-1}u' \in L^2(0,1)$ *and* $x^{2\alpha}u'' \in L^2(0,1)$, *with* $\left\|x^{2\alpha-1}u'\right\|_{L^2} + \left\|x^{2\alpha}u''\right\|_{L^2} \le C\|f\|_{L^2}$. *In particular,* $x^{2\alpha}u' \in H^1(0,1)$.

Here the constant C depends only on α.

Remark 3.16. *The boundary behavior in assertion (ii) of Theorem 3.11 is optimal in the following sense: for any $0 < x \le \frac{1}{2}$ and $\frac{3}{4} \le \alpha < 1$, define*

$$\widetilde{K}_\alpha(x) = \begin{cases} \sup\limits_{\|f\|_{L^2}\le 1} \left|(1-\ln x)^{-\frac{1}{2}} u(x)\right|, & \text{when } \alpha = \dfrac{3}{4}, \\ \sup\limits_{\|f\|_{L^2}\le 1} \left|x^{2\alpha-\frac{3}{2}} u(x)\right|, & \text{when } \dfrac{3}{4} < \alpha < 1. \end{cases}$$

Then $0 < \delta \le \widetilde{K}_\alpha(x) \le C$, for some constants δ and C only depending on α. See Section 3.3.2 for the proof.

Remark 3.17. *The same conclusions as in Remark 3.9 and 3.10 hold for the solution given by Theorem 3.11.*

Theorem 3.12 (Uniqueness for the "Canonical" Problem)**.** *Let $\frac{3}{4} \le \alpha < 1$. Assume that $u \in H^2_{loc}(0,1]$ satisfies*

$$\begin{cases} -\left(x^{2\alpha}u'(x)\right)' + u(x) = 0 & \text{on } (0,1), \\ u(1) = 0. \end{cases}$$

If in addition one of the following conditions is satisfied *(i)* $\lim_{x\to 0^+} x^{2\alpha}u'(x) = 0$,

(ii) $\lim_{x\to 0^+} x^{2\alpha-1}u(x) = 0$,

(iii) $u \in L^{\frac{1}{2\alpha-1}}(0,1)$,

then $u \equiv 0$.

We still call the unique solution of (3.1) given by Theorem 3.11 the "Canonical" solution and denote it by u_C. Concerning the regularity of u_C for $\frac{3}{4} \le \alpha < 1$ we have the following

Theorem 3.13. *Let* $\frac{3}{4} \le \alpha < 1$ *and* $f \in W^{1,\frac{1}{2\alpha-1}}(0,1)$. *Let* u_C *be the solution to* (3.1) *given by Theorem 3.11. Then both* $u_C \in W^{1,p}(0,1)$ *and* $x^{2\alpha-1}u_C' \in W^{1,p}(0,1)$ *for all* $1 \le p < \frac{1}{2\alpha-1}$ *with*

$$\|u_C\|_{W^{1,p}} + \left\|x^{2\alpha-1}u_C'\right\|_{W^{1,p}} \le C\,\|f\|_{W^{1,p}}\,,$$

where C *is a constant depending only on* p *and* α.

Remark 3.18. *The same conclusion as in Remark 3.15 holds here.*

3.1.4 The case $\alpha \ge 1$

Theorem 3.14 (Existence for the "Canonical" Problem)**.** *Given* $\alpha \ge 1$ *and* $f \in L^2(0,1)$, *there exists a function* $u \in H^2_{loc}(0,1]$ *satisfying* (3.1) *together with the following properties:*

(i) $u \in L^2(0,1)$ *with* $\|u\|_{L^2} \le \|f\|_{L^2}$.

(ii) $\lim_{x\to 0^+} x^{\frac{\alpha}{2}}u(x) = 0$.

(iii) $\lim_{x\to 0^+} x^{\frac{3\alpha}{2}}u'(x) = 0$.

(iv) $x^{\alpha}u' \in L^2(0,1)$ *and* $x^{2\alpha}u'' \in L^2(0,1)$ *with* $\|x^{\alpha}u'\|_{L^2} + \left\|x^{2\alpha}u''\right\|_{L^2} \le C\,\|f\|_{L^2}$, *where* C *is a constant depending only on* α. *In particular,* $x^{2\alpha}u' \in H^1(0,1)$.

Remark 3.19. *The boundary behaviors in assertions (ii) and (iii) of Theorem 3.14 are optimal in the following sense: for $x \in \left(0, \frac{1}{2}\right)$ and $\alpha \geq 1$, define*

$$P_\alpha(x) = \sup_{\|f\|_{L^2} \leq 1} \left| x^{\frac{3\alpha}{2}} u'(x) \right|,$$

$$\widetilde{P}_\alpha(x) = \sup_{\|f\|_{L^2} \leq 1} \left| x^{\frac{\alpha}{2}} u(x) \right|.$$

Then $0 < \delta \leq P_\alpha(x) \leq C$ and $0 < \delta \leq \widetilde{P}_\alpha(x) \leq C$, where δ and C are constants depending only on α. See Section 3.3.2 for the proof.

Theorem 3.15 (Uniqueness for the "Canonical" Problem)**.** *Let $\alpha \geq 1$. Assume that $u \in H^2_{loc}(0, 1]$ satisfies*

$$\begin{cases} -\left(x^{2\alpha} u'(x)\right)' + u(x) = 0 & \text{on } (0, 1), \\ u(1) = 0. \end{cases}$$

If in addition one of the following conditions is satisfied

(i) $\lim_{x \to 0^+} x^{\frac{3+\sqrt{5}}{2}} u'(x) = 0$ *when* $\alpha = 1$,

(ii) $\lim_{x \to 0^+} x^{\frac{1+\sqrt{5}}{2}} u(x) = 0$ *when* $\alpha = 1$,

(iii) $\lim_{x \to 0^+} x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u'(x) = 0$ *when* $\alpha > 1$,

(iv) $\lim_{x \to 0^+} x^{\frac{\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u(x) = 0$ *when* $\alpha > 1$,

(v) $u \in L^1(0, 1)$,

then $u \equiv 0$.

As before, we call the solution of (3.1) given by Theorem 3.14 the "Canonical" solution and still denote it by u_C.

Remark 3.20. *For $\alpha \geq \frac{1}{2}$, the existence results (Theorem 3.7, 3.11, 3.14) and the uniqueness results (Theorem 3.8, 3.12, 3.15) guarantee that the weighted Dirichlet and Neumann conditions yield the same "Canonical" solution u_C.*

3.1.5 Connection with the variational formulation

Next we give a variational characterization of the unique solutions u_D, u_N and u_C given by Theorem 3.1, 3.4, 3.7, 3.11, 3.14. We begin by defining the underlying space

$$X^\alpha = \left\{u \in H^1_{loc}(0,1);\ u \in L^2(0,1) \text{ and } x^\alpha u' \in L^2(0,1)\right\},\ \alpha > 0. \tag{3.9}$$

For $u,\ v \in X^\alpha$, define

$$a(u,v) = \int_0^1 x^{2\alpha} u'(x) v'(x) dx + \int_0^1 u(x)v(x)dx$$

and

$$I(u) = a(u,u).$$

The space X^α becomes a Hilbert space under the inner product $a(\cdot,\cdot)$. See Section 3.6 for a detailed analysis of the space X^α.

Notice that the elements of X^α are continuous away from 0, so the following is a well-defined (closed) subspace

$$X^\alpha_0 = \{u \in X^\alpha;\ u(1) = 0\}. \tag{3.10}$$

Also, as it is shown in Section 3.6, when $0 < \alpha < \frac{1}{2}$, the functions in X^α are continuous at the origin, making

$$X^\alpha_{00} = \{u \in X^\alpha_0;\ u(0) = 0\} \tag{3.11}$$

a well defined subspace.

Let $0 < \alpha < \frac{1}{2}$ and $f \in L^2(0,1)$. Then the Dirichlet solution u_D given by Theorem 3.1 is characterized by the following property:

$$u_D \in X^\alpha_{00},\ \text{and}\ \min_{v \in X^\alpha_{00}} \left\{\frac{1}{2}I(v) - \int_0^1 f(x)v(x)dx\right\} = \frac{1}{2}I(u_D) - \int_0^1 f(x)u_D(x)dx. \tag{3.12}$$

The Neumann solution u_N given by Theorem 3.4 is characterized by:

$$u_N \in X^\alpha_0,\ \text{and}\ \min_{v \in X^\alpha_0} \left\{\frac{1}{2}I(v) - \int_0^1 f(x)v(x)dx\right\} = \frac{1}{2}I(u_N) - \int_0^1 f(x)u_N(x)dx. \tag{3.13}$$

Let $\alpha \geq \frac{1}{2}$ and $f \in L^2(0,1)$. Then the "Canonical" solution u_C given by Theorem 3.7, 3.11, or 3.14 is characterized by the following property:

$$u_C \in X^\alpha_0,\ \text{and}\ \min_{v \in X^\alpha_0} \left\{\frac{1}{2}I(v) - \int_0^1 f(x)v(x)dx\right\} = \frac{1}{2}I(u_C) - \int_0^1 f(x)u_C(x)dx. \tag{3.14}$$

The variational formulations (3.12), (3.13) and (3.14) will be established at the beginning of Section 3.3, which is the starting point for the proofs of all the existence results.

3.1.6 The spectrum

Now we proceed to state the spectral properties of the differential operator $\mathcal{L}u := -\left(x^{2\alpha}u'\right)' + u$. We can define two bounded operators associated with it: when $0 < \alpha < \frac{1}{2}$, we define the Dirichlet operator T_D,

$$\begin{aligned} T_D : L^2(0,1) &\longrightarrow L^2(0,1) \\ f &\longmapsto T_D f = u_D, \end{aligned} \tag{3.15}$$

where u_D is characterized by (3.12). We also define, for any $\alpha > 0$, the following "Neumann-Canonical" operator T_α,

$$\begin{aligned} T_\alpha : L^2(0,1) &\longrightarrow L^2(0,1) \\ f &\longmapsto T_\alpha f = \begin{cases} u_N \text{ if } 0 < \alpha < \dfrac{1}{2}, \\ u_C \text{ if } \alpha \geq \dfrac{1}{2}, \end{cases} \end{aligned} \tag{3.16}$$

where u_N and u_C are characterized by (3.13) and (3.14) respectively. By Theorem 3.35 in Section 3.6, we know that T_D is a compact operator for any $0 < \alpha < \frac{1}{2}$ while T_α is compact if and only if $0 < \alpha < 1$.

In what follows, for given $\nu \in \mathbb{R}$, the function J_ν: $(0, \infty) \longrightarrow \mathbb{R}$ denotes the *Bessel function of the first kind of parameter* ν. We use the positive increasing sequence $\{j_{\nu k}\}_{k=1}^{\infty}$ to denote all the positive zeros of the function J_ν (see e.g. [46] for a comprehensive treatment of Bessel functions). The results about the spectrum of the operators T_D and T_α read as:

Theorem 3.16 (Spectrum of the Dirichlet Operator)**.** *For* $0 < \alpha < \frac{1}{2}$*, define* $\nu_0 = \frac{\frac{1}{2}-\alpha}{1-\alpha}$*, and let* $\mu_{\nu_0 k} = 1 + (1-\alpha)^2 j_{\nu_0 k}^2$*. Then*

$$\sigma(T_D) = \{0\} \cup \left\{ \lambda_{\nu_0 k} := \frac{1}{\mu_{\nu_0 k}} \right\}_{k=1}^{\infty}.$$

For any $k \in \mathbb{N}$*, the functions defined by*

$$u_{\nu_0 k}(x) := x^{\frac{1}{2}-\alpha} J_{\nu_0}(j_{\nu_0 k} x^{1-\alpha})$$

is the eigenfunction of T_D *corresponding to the eigenvalue* $\lambda_{\nu_0 k}$. *Moreover, for fixed* $0 < \alpha < \frac{1}{2}$ *and* k *sufficiently large, we have*

$$\mu_{\nu_0 k} = 1 + (1-\alpha)^2 \left[\left(\frac{\pi}{2} \left(\nu_0 - \frac{1}{2} \right) + \pi k \right)^2 - \left(\nu_0^2 - \frac{1}{4} \right) \right] + O\left(\frac{1}{k} \right). \tag{3.17}$$

Theorem 3.17 (Spectrum of the "Neumann-Canonical" Operator)**.** *Assume* $\alpha > 0$ *and let* T_α *be the operator defined above.*

(i) *For* $0 < \alpha < 1$, *define* $\nu = \frac{\alpha - \frac{1}{2}}{1-\alpha}$, *and let* $\mu_{\nu k} = 1 + (1-\alpha)^2 j_{\nu k}^2$. *Then*

$$\sigma(T_\alpha) = \{0\} \cup \left\{ \lambda_{\nu k} := \frac{1}{\mu_{\nu k}} \right\}_{k=1}^{\infty}.$$

For any $k \in \mathbb{N}$, *the functions defined by*

$$u_{\nu k}(x) := x^{\frac{1}{2}-\alpha} J_\nu(j_{\nu k} x^{1-\alpha})$$

is the eigenfunction of T_α *corresponding to the eigenvalue* $\lambda_{\nu k}$. *Moreover, for fixed* $0 < \alpha < 1$ *and* k *sufficiently large, we have*

$$\mu_{\nu k} = 1 + (1-\alpha)^2 \left[\left(\frac{\pi}{2} \left(\nu - \frac{1}{2} \right) + \pi k \right)^2 - \left(\nu^2 - \frac{1}{4} \right) \right] + O\left(\frac{1}{k} \right). \tag{3.18}$$

(ii) *For* $\alpha = 1$, *the operator* T_1 *has no eigenvalues, and the spectrum is exactly* $\sigma(T_1) = \left[0, \frac{4}{5}\right]$.

(iii) *For* $\alpha > 1$, *the operator* T_α *has no eigenvalues, and the spectrum is exactly* $\sigma(T_\alpha) = [0, 1]$.

Recall that the discrete spectrum of an operator T is defined as

$$\sigma_d(T) = \{\lambda \in \sigma(T) : T - \lambda I \text{ is a Fredholm operator}\},$$

and the essential spectrum is defined as

$$\sigma_e(T) = \sigma(T) \backslash \sigma_d(T).$$

We have the following corollary about the essential spectrum.

Corollary 3.18 (Essential Spectrum of the "Neumann-Canonical" Operator)**.** *Assume that* $\alpha > 0$ *and let* T_α *be the operator defined above.*

(i) For $0 < \alpha < 1$, $\sigma_e(T_\alpha) = \{0\}$.

(ii) For $\alpha = 1$, $\sigma_e(T_1) = \left[0, \frac{4}{5}\right]$.

(iii) For $\alpha > 1$, $\sigma_e(T_\alpha) = [0, 1]$.

Remark 3.21. *This corollary follows immediately from the fact (see e.g. Theorem IX.1.6 of [24]) that, for any self-adjoint operator T on a Hilbert space, $\sigma_d(T)$ consists of the isolated eigenvalues with finite multiplicity. In fact, for Corollary 3.18 to hold, it suffices to prove that $\sigma_d(T) \subset EV(T)$, where $EV(T)$ is the set of all the eigenvalues. We present in Section 3.4.2 a simple proof of this inclusion.*

As the reader can see in Theorem 3.17, when $\alpha < 1$ the spectrum of the operator T_α is a discrete set and when $\alpha = 1$ the spectrum of T_1 becomes a closed interval, so a natural question is whether $\sigma(T_\alpha)$ converges to $\sigma(T_1)$ as $\alpha \to 1^-$ in some sense. The answer is positive as the reader can check in the following

Theorem 3.19. *Let $\alpha \le 1$. For the spectrum $\sigma(T_\alpha)$, we have*

(i) $\sigma(T_\alpha) \subset \sigma(T_1)$ *for all* $\frac{2}{3} < \alpha < 1$.

(ii) *For every $\lambda \in \sigma(T_1)$, there exists a sequence $\alpha_m \to 1^-$ and a sequence of eigenvalues $\lambda_m \in \sigma(T_{\alpha_m})$ such that $\lambda_m \to \lambda$ as $m \to \infty$.*

Remark 3.22. *Notice that in particular $\sigma(T_\alpha) \to \sigma(T_1)$ in the Hausdorff metric sense, that is*

$$d_H(\sigma(T_\alpha), \sigma(T_1)) \to 0, \ \text{as } \alpha \to 1^-,$$

where $d_H(X, Y) = \max\left\{\sup_{x\in X} \inf_{y\in Y} |x - y|, \sup_{y\in Y} \inf_{x\in X} |x - y|\right\}$ is the Hausdorff metric (see e.g. Chapter 7 of [34]).

Remark 3.23. *When $\alpha \le 1$, the spectrum of T_α has been investigated by C. Stuart [38]. In fact, he considered the more general differential operator $Nu = -(A(x)u')'$ under the conditions $u(1) = 0$ and $\lim_{x\to 0^+} A(x)u'(x) = 0$, with*

$$A \in C[0, 1]; \ A(x) > 0, \forall x \in (0, 1] \ \text{and} \ \lim_{x\to 0^+} \frac{A(x)}{x^{2\alpha}} = 1. \tag{3.19}$$

Notice that if $A(x) = x^{2\alpha}$*, we have the equality* $T_\alpha = (N+I)^{-1}$*, where the inverse is taken in the space* $L^2(0,1)$*. When* $\alpha < 1$*, C. Stuart proves that* $\sigma\left((N+I)^{-1}\right)$ *consists of isolated eigenvalues; this is deduced from a compactness argument. When* $\alpha = 1$*, C. Stuart proves that* $\max \sigma_e\left((N+I)^{-1}\right) = \frac{4}{5}$*. On the other hand, C. Stuart has constructed an elegant example of function* A *satisfying* (3.19) *with* $\alpha = 1$ *such that* $(N+I)^{-1}$ *admits an eigenvalue in the interval* $(\frac{4}{5}, 1]$*. Moreover, G. Vuillaume (in his thesis [43] under C. Stuart) used a variant of this example to get an arbitrary number of eigenvalues in the interval* $(\frac{4}{5}, 1]$*. However, we still have an*

Open Problem 1. *If* A *satisfies* (3.19) *for* $\alpha = 1$*, is it true that* $\sigma_e\left((N+I)^{-1}\right) = [0, \frac{4}{5}]$*?*

Similarly, when $\alpha > 1$*, one can still consider the differential operator* $Nu = -(A(x)u')'$ *under the conditions* $u(1) = 0$ *and* $\lim_{x\to 0^+} A(x)u'(x) = 0$*, where* A *satisfies* (3.19)*, and the operator* $(N+I)^{-1}$*, where the inverse is taken in the space* $L^2(0,1)$*, is still well-defined. By the same argument as in the case* $A(x) = x^{2\alpha}$ *(Theorem 3.17 (iii)) we know that* $\sigma\left((N+I)^{-1}\right) \subset [0,1]$*. However, we still have*

Open Problem 2. *Assume that* A *satisfies* (3.19) *for* $\alpha > 1$*.*

(i) Is it true that $\sigma\left((N+I)^{-1}\right) = [0,1]$*?*

(ii) Is it true that $\max \sigma_e\left((N+I)^{-1}\right) = 1$*, or more precisely* $\sigma_e\left((N+I)^{-1}\right) = [0,1]$*?*

The rest of the chapter is organized as the following. We begin by proving the uniqueness results in Section 3.2. We then prove the existence and regularity results in Section 3.3. The analysis of the spectrum of the operators T_α and T_D are performed in Sections 3.4 and 3.5 respectively. Finally we present in Section 3.6 some properties about weighted Sobolev spaces used throughout this work.

3.2 Proofs of all the uniqueness results

In this section we will provide the proofs of the uniqueness results stated in the Introduction.

Proof of Theorem 3.2. Since $u \in C(0,1]$ with $\lim_{x\to 0^+} u(x) = 0$, we have that $u \in$

$C[0,1]$. Notice that, for any $0 < x < 1$, we can write $x^{2\alpha}u'(x) = u'(1) - \int_x^1 u(s)ds$, which implies that $x^{2\alpha}u' \in C[0,1]$. Then we can multiply the equation (3.5) by u and integrate by parts over $[\epsilon, 1]$, and with the help of the boundary condition we obtain

$$\int_\epsilon^1 x^{2\alpha}u'(x)^2dx + \int_\epsilon^1 u(x)^2dx = x^{2\alpha}u'(x)u(x)|_\epsilon^1 \to 0, \text{ as } \epsilon \to 0^+.$$

Therefore, $u = 0$. □

Proof of Theorem 3.5. We first claim that $u \in C[0,1]$. Since $\lim_{x\to 0^+} x^{2\alpha}u'(x) = 0$, there exists $C > 0$ such that $-Cx^{-2\alpha} \leq u'(x) \leq Cx^{-2\alpha}$, which implies that $-Cx^{1-2\alpha} \leq u(x) \leq Cx^{1-2\alpha}$, hence $u \in L^\infty(0,1)$ because $0 < \alpha < \frac{1}{2}$. Write $u'(x) = \frac{1}{x^{2\alpha}}\int_0^x u(s)ds$ and deduce that $u' \in L^\infty(0,1)$, thus $u \in W^{1,\infty}(0,1)$. In particular $u \in C[0,1]$.

Then we can multiply the equation (3.8) by u and integrate by parts over $[\epsilon, 1]$, and with the help of the boundary condition we obtain

$$\int_\epsilon^1 x^{2\alpha}u'(x)^2dx + \int_\epsilon^1 u(x)^2dx = x^{2\alpha}u'(x)u(x)|_\epsilon^1 \to 0, \text{ as } \epsilon \to 0^+.$$

Therefore, $u \equiv 0$. □

Proof of (i) of Theorem 3.8 and (i) of Theorem 3.12. As in the proof of Theorem 3.5, it is enough to show that $u \in C[0,1]$. As before, the boundary condition implies that $u(x) \sim x^{1-2\alpha}$, which gives $u \in L^{\frac{1}{\alpha}}(0,1)$. To prove that $u \in C[0,1]$, we first write $x^{2\alpha-1}u'(x) = \frac{1}{x}\int_0^x u(s)ds$. Let $p_0 := \frac{1}{\alpha} > 1$. Since $u \in L^{p_0}(0,1)$, one can apply Hardy's inequality and obtain $\left\|x^{2\alpha-1}u'\right\|_{L^{p_0}} \leq C\left\|u\right\|_{L^{p_0}}$. Since $u(1) = 0$, this implies that $u \in X_{\cdot 0}^{2\alpha-1,p_0}(0,1)$. By Theorem 3.34 in Section 3.6, we have two alternatives

- $u \in L^q(0,1)$ for all $q < \infty$ when $\alpha \leq \frac{2}{3}$ or
- $u \in L^{p_1}(0,1)$ where $p_1 := \frac{1}{3\alpha-2} > p_0$ when $\frac{2}{3} < \alpha < 1$.

If the first case happens and $u \in L^q(0,1)$ for all $q < \infty$, then we apply Hardy's inequality and obtain $u \in X_{\cdot 0}^{2\alpha-1,q}(0,1)$ for all $q < \infty$, which embeds into $C[0,1]$ for q large enough. If the second alternative occurs and we apply Hardy's inequality once more, we conclude that $u \in X_{\cdot 0}^{2\alpha-1,p_1}(0,1)$. Therefore, either $u \in L^q(0,1)$ for all $q < \infty$ when $\alpha \leq \frac{4}{5}$ or $u \in L^{p_2}(0,1)$ where $p_2 = \frac{1}{5\alpha-4}$ when $\frac{4}{5} < \alpha < 1$. By repeating this argument finitely many times we can conclude that $u \in C[0,1]$. □

Proof of (ii) of Theorem 3.8. Let $\alpha = \frac{1}{2}$ and suppose that $u \in H^2_{loc}(0,1]$ satisfies

$$\begin{cases} -(xu'(x))' + u(x) = 0 \quad \text{ on } (0,1), \\ u(1) = 0, \\ \lim_{x \to 0^+} (1 - \ln x)^{-1} u(x) = 0. \end{cases}$$

Notice that $u \in C(0,1]$ together with $\lim_{x\to 0^+}(1-\ln x)^{-1}u(x) = 0$ and the integrability of $\ln x$, gives $u \in L^1(0,1)$. Define $w(x) = u(x)(1 - \ln x)^{-1}$. It is enough to show that $w = 0$. Notice that w solves

$$\begin{cases} (x(1-\ln x)w'(x))' = (1-\ln x)w(x) + w'(x) \quad \text{ on } (0,1), \\ w(0) = w(1) = 0. \end{cases} \tag{3.20}$$

We integrate equation (3.20) to obtain

$$x(1-\ln x)w'(x) = w'(1) - \int_x^1 (1-\ln s)w(s)dx = u'(1) - \int_x^1 u(s)ds.$$

Since $u \in L^1(0,1)$, the above computation shows that $x(1-\ln x)w'(x) \in C[0,1]$. Now we multiply (3.20) by w and we integrate by parts over $[\epsilon, 1]$ to obtain

$$\int_\epsilon^1 x(1-\ln x)w'(x)^2 dx + \int_\epsilon^1 (1-\ln x)w^2(x)dx = x(1-\ln x)w'(x)w(x)|_\epsilon^1 - \frac{1}{2}w^2(x)|_\epsilon^1 \to 0,$$

as $\epsilon \to 0^+$, proving that $w = 0$. □

At this point we would like to mention that the proof of (iii) of Theorem 3.8 and (iii) of Theorem 3.12 will be postponed to Proposition 3.23 of Section 3.3.2.

Proof of (iv) of Theorem 3.8 and (ii) of Theorem 3.12. Let $\frac{1}{2} < \alpha < 1$ and suppose that $u \in H^2_{loc}(0,1]$ satisfies

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 \quad \text{ on } (0,1), \\ u(1) = 0, \\ \lim_{x \to 0^+} x^{2\alpha-1} u(x) = 0. \end{cases}$$

Notice that $u \in C(0,1]$ together with $\lim_{x\to 0^+} x^{2\alpha-1}u(x) = 0$ and the integrability of $x^{1-2\alpha}$ for $\alpha < 1$, gives $u \in L^1(0,1)$. Define $w(x) = x^{2\alpha-1}u(x)$. We will show that

$w = 0$. Notice that w satisfies

$$\begin{cases} -(xw'(x))' + (2\alpha - 1)w'(x) + x^{1-2\alpha}w(x) = 0 & \text{on } (0,1), \\ w(0) = w(1) = 0. \end{cases} \tag{3.21}$$

Integrate (3.21) to obtain

$$xw'(x) = w'(1) - \int_x^1 s^{1-2\alpha}w(s)ds = u'(1) - \int_x^1 u(s)ds,$$

from which we conclude $xw'(x) \in C[0,1]$. Finally, multiply (3.21) by w and integrate by parts over $[\epsilon, 1]$ to obtain

$$\int_\epsilon^1 xw'(x)^2dx + \int_\epsilon^1 x^{1-2\alpha}w(x)^2dx = xw'(x)w(x)|_\epsilon^1 - \left(\alpha - \frac{1}{2}\right)w^2(\epsilon).$$

Letting $\epsilon \to 0^+$ and we conclude that $w = 0$. □

Proof of Theorem 3.15. Assume that (i) holds. Suppose that $u \in H^2_{loc}(0,1]$ satisfies

$$\begin{cases} -(x^2u'(x))' + u(x) = 0 & \text{on } (0,1), \\ u(1) = 0, \\ \lim_{x\to 0^+} x^{\frac{3+\sqrt{5}}{2}}u'(x) = 0. \end{cases}$$

Let $v(x) = x^{\frac{1+\sqrt{5}}{2}}u(x)$. Then $v \in H^2_{loc}(0,1]$ and it satisfies

$$\begin{cases} -(xv'(x))' + \sqrt{5}v'(x) = 0 & \text{on } (0,1), \\ v(1) = 0, \\ \lim_{x\to 0^+} \left(xv'(x) - \frac{1+\sqrt{5}}{2}v(x)\right) = 0, \end{cases} \tag{3.22}$$

from which we obtain that $xv' - \frac{1+\sqrt{5}}{2}v \in C[0,1]$ and $xv' - \sqrt{5}v \in H^1(0,1)$. Therefore $v \in C[0,1]$. Multiply (3.22) by v and integrate over $[\epsilon, 1]$ to obtain

$$\int_\epsilon^1 xv'(x)^2dx + \frac{1}{2}v^2(\epsilon) = \left(xv'(x) - \frac{1+\sqrt{5}}{2}v(x)\right)v(x)|_\epsilon^1 \to 0, \text{ as } \epsilon \to 0^+.$$

Therefore v is constant and thus $v(x) \equiv v(1) = 0$.

Assume that (ii) holds. Suppose that $u \in H^2_{loc}(0,1]$ satisfies

$$\begin{cases} -(x^2u'(x))' + u(x) = 0 & \text{on } (0,1), \\ u(1) = 0, \\ \lim_{x\to 0^+} x^{\frac{1+\sqrt{5}}{2}}u(x) = 0. \end{cases}$$

Let $w(x) = x^{\frac{1+\sqrt{5}}{2}} u(x)$. Then $w \in H^2_{loc}(0,1]$ and it satisfies

$$\begin{cases} -(xw'(x))' + \sqrt{5}w'(x) = 0 \quad \text{on } (0,1), \\ w(0) = w(1) = 0. \end{cases} \tag{3.23}$$

Therefore $xw' + \sqrt{5}w \in H^1(0,1)$, $w \in C[0,1]$, and $xw' \in C[0,1]$. Multiply (3.23) by w and integrate over $[\epsilon, 1]$ to obtain

$$\int_\epsilon^1 xw'(x)^2 dx = xw'(x)w(x)|_\epsilon^1 - \frac{\sqrt{5}}{2} w^2(x)|_\epsilon^1 \to 0, \text{ as } \epsilon \to 0^+.$$

Therefore w is constant, so $w(x) \equiv w(1) = 0$.

Assume that (iii) holds. Suppose that $u \in H^2_{loc}(0,1]$ satisfies

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 \quad \text{on } (0,1), \\ u(1) = 0, \\ \lim_{x\to 0^+} x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u'(x) = 0. \end{cases}$$

Define $g(x) = e^{\frac{x^{1-\alpha}}{1-\alpha}} u(x)$. Then $g \in H^2_{loc}(0,1]$ and it satisfies

$$\begin{cases} -(x^{2\alpha}g'(x))' + (x^\alpha g(x))' + x^\alpha g'(x) = 0 \quad \text{on } (0,1), \\ g(1) = 0, \\ \lim_{x\to 0^+} \left(x^{\frac{3\alpha}{2}} g'(x) - x^{\frac{\alpha}{2}} g(x) \right) = 0. \end{cases}$$

Multiply the above by g and integrate over $[\epsilon, 1]$ to obtain

$$\begin{aligned} \int_\epsilon^1 x^{2\alpha} g'(x)^2 dx &= x^{2\alpha} g'(x) g(x)|_\epsilon^1 - x^\alpha g^2(x)|_\epsilon^1 \\ &= \left(x^{\frac{3\alpha}{2}} g'(x) - x^{\frac{\alpha}{2}} g(x) \right) x^{\frac{\alpha}{2}} g(x)|_\epsilon^1. \end{aligned} \tag{3.24}$$

We now study the function $h(x) := x^{\frac{\alpha}{2}} g(x)$. We have

$$\begin{aligned} h(x) &= -\int_x^1 h'(s) ds \\ &= -\int_x^1 \left(\frac{\alpha}{2} s^{\frac{\alpha}{2}-1} g(s) + s^{\frac{\alpha}{2}} g'(s) \right) ds \\ &= \frac{\alpha}{2} \int_x^1 s^{\frac{3\alpha}{2}-1} g'(s) ds - \left(x^{\frac{3\alpha}{2}} g'(x) - x^{\frac{\alpha}{2}} g(x) \right) \\ &= -\frac{\alpha}{2} \left(\frac{3\alpha}{2} - 1 \right) \int_x^1 s^{\frac{3\alpha}{2}-2} g(s) ds - \frac{\alpha}{2} x^{\alpha-1} h(x) - \left(x^{\frac{3\alpha}{2}} g'(x) - x^{\frac{\alpha}{2}} g(x) \right). \end{aligned}$$

Hence we can write

$$h(x) = \left[1 + \frac{\alpha}{2}x^{\alpha-1}\right]^{-1}\left[-\frac{\alpha}{2}\left(\frac{3\alpha}{2}-1\right)\int_x^1 s^{\frac{3\alpha}{2}-2}g(s)ds - \left(x^{\frac{3\alpha}{2}}g'(x) - x^{\frac{\alpha}{2}}g(x)\right)\right].$$

We claim that there exists a sequence $\epsilon_n \to 0$ so that

$$\lim_{n\to\infty}\left|\int_{\epsilon_n}^1 s^{\frac{3\alpha}{2}-2}g(s)ds\right| < \infty.$$

Otherwise, assume that $\lim_{\epsilon\to 0^+}\int_\epsilon^1 s^{\frac{3\alpha}{2}-2}g(s)ds = \pm\infty$. Then

$$\lim_{x\to 0^+} x^{\frac{\alpha}{2}}e^{\frac{x^{1-\alpha}}{1-\alpha}}u(x) = \lim_{x\to 0^+} h(x) = \pm\infty.$$

This forces $\lim_{x\to 0^+} u(x) = \pm\infty$, so L'Hopital's rule applies to u and one obtains that

$$\lim_{x\to 0^+} x^{\frac{\alpha}{2}}e^{\frac{x^{1-\alpha}}{1-\alpha}}u(x) = \lim_{x\to 0^+}\frac{x^{\frac{3\alpha}{2}}e^{\frac{x^{1-\alpha}}{1-\alpha}}u'(x)}{-\frac{\alpha}{2}x^{\alpha-1}-1} = 0,$$

which is a contradiction. Therefore $\lim_{\epsilon_n\to 0^+} h(\epsilon_n)$ exists for some sequence $\epsilon_n \to 0$. Finally, use that sequence $\epsilon_n \to 0^+$ in (3.24) to obtain that $\int_0^1 x^{2\alpha}g'(x)^2dx = 0$, which gives g is constant, that is $g(x) \equiv g(1) = 0$.

Assume that (iv) holds. Suppose that $u \in H^2_{loc}(0,1]$ satisfies

$$\begin{cases} -\left(x^{2\alpha}u'(x)\right)' + u(x) = 0 \quad \text{on } (0,1), \\ u(1) = 0, \\ \lim_{x\to 0^+} x^{\frac{\alpha}{2}}e^{\frac{x^{1-\alpha}}{1-\alpha}}u(x) = 0. \end{cases}$$

Let $p(x) = e^{\frac{x^{1-\alpha}}{1-\alpha}}u(x)$, then w satisfies

$$\begin{cases} -\left(x^{2\alpha}p'(x)\right)' + \left(x^{\alpha}p(x)\right)' + x^{\alpha}p'(x) = 0 \quad \text{on } (0,1), \\ p(1) = 0, \\ \lim_{x\to 0^+} x^{\frac{\alpha}{2}}p(x) = 0. \end{cases} \tag{3.25}$$

We claim that $\lim_{x\to 0^+} x^{\frac{3\alpha}{2}}p'(x)$ exists, thus implying that $x^{\frac{3\alpha}{2}}p'(x)$ belongs to $C[0,1]$. Define $q(x) = x^{\frac{3\alpha}{2}}p'(x)$, then using (3.25) we obtain that, for $0 < x < 1$,

$$q'(x) = -\frac{\alpha}{2}x^{\frac{3\alpha}{2}-1}p'(x) + \alpha x^{\frac{\alpha}{2}-1}p(x) + 2x^{\frac{\alpha}{2}}p'(x).$$

A direct computation shows that, for $0 < x < 1$,

$$\int_x^1 q'(s)ds = \frac{\alpha}{2}\left(\frac{3\alpha}{2} - 1\right)\int_x^1 x^{\frac{3\alpha}{2}-2}p(s)ds + \frac{\alpha}{2}x^{\alpha-1}x^{\frac{\alpha}{2}}p(x) - 2x^{\frac{\alpha}{2}}p(x).$$

Since $x^{\frac{\alpha}{2}}p(x) \in C[0,1]$, we obtain that $x^{\frac{3\alpha}{2}-2}p(x) \in L^1(0,1)$. It implies that $x^{\frac{3\alpha}{2}}p'(x) = q(x) = -\int_x^1 q'(s)ds$ is continuous and that the $\lim_{x\to 0^+} q(x)$ exists. We now multiply (3.25) by $p(x)$ and integrate by parts to obtain

$$\int_0^1 x^{2\alpha}p'(x)^2 = x^{\frac{3\alpha}{2}}p'(x)x^{\frac{\alpha}{2}}p(x)|_0^1 = 0.$$

Thus proving that $p(x)$ is constant, i.e. $p(x) \equiv p(1) = 0$.

Finally assume that (v) holds. Define $k(x) = x^{2\alpha}u'(x)$. Notice that since $u \in L^1(0,1) \cap H^2_{loc}(0,1]$, from the equation we obtain that $k(x) = u'(1) - \int_x^1 u(s)ds$, so $k(x) \in C[0,1]$. We claim that $k(0) = 0$. Otherwise, near the origin $u'(x) \sim \frac{1}{x^{2\alpha}}$ and $u(x) \sim \frac{1}{x^{2\alpha-1}}$, which contradicts $u \in L^1(0,1)$. Therefore, $\lim_{x\to 0^+} x^{2\alpha}u'(x) = 0$. We are now in the case where (i) or (iii) applies, so we can conclude that $u = 0$. $\square$

3.3 Proofs of all the existence and the regularity results

Our proof of the existence results will mostly use functional analysis tools. We take the weighted Sobolev space X^α defined in (3.9) and its subspaces X^α_{00} and X^α_0 defined by (3.11) and (3.10). As we can see from Section 3.6, X^α equipped with the inner product given by

$$(u,v)_\alpha = \int_0^1 \left(x^{2\alpha}u'(x)v'(x) + u(x)v(x)\right)dx,$$

is a Hilbert space. X^α_{00} and X^α_0 are well defined closed subspaces. We define two notions of weak solutions as follows: given $0 < \alpha < \frac{1}{2}$ and $f \in L^2(0,1)$ we say u is a *weak solution of the first type* of (3.1) if $u \in X^\alpha_{00}$ satisfies

$$\int_0^1 x^{2\alpha}u'(x)v'(x)dx + \int_0^1 u(x)v(x)dx = \int_0^1 f(x)v(x)dx, \text{ for all } v \in X^\alpha_{00}; \tag{3.26}$$

and given $\alpha > 0$ and $f \in L^2(0,1)$ we say that u is a *weak solution of the second type* of (3.1) if $u \in X^\alpha_0$ satisfies

$$\int_0^1 x^{2\alpha}u'(x)v'(x)dx + \int_0^1 u(x)v(x)dx = \int_0^1 f(x)v(x)dx, \text{ for all } v \in X^\alpha_0. \tag{3.27}$$

The existence of both solutions are guaranteed by Riesz Theorem. Actually, (3.26) is equivalent to (3.12), while (3.27) is equivalent to (3.13) or (3.14) (see e.g. Theorem 5.6 of [8]). As we will see later, the weak solution of the first type is exactly the solution u_D mentioned in the Introduction, whereas the weak solution of the second type corresponds to either u_N when $0 < \alpha < \frac{1}{2}$ or u_C when $\alpha \geq \frac{1}{2}$.

3.3.1 The Dirichlet problem

Proof of Theorem 3.1. We will actually prove that the solution of (3.26) is the solution we are looking for in Theorem 3.1. Notice that by taking $v \in C_c^\infty(0,1)$ in (3.26) we obtain that $w(x) := x^{2\alpha}u'(x) \in H^1(0,1)$ with $(x^{2\alpha}u'(x))' = u(x) - f(x)$ and $\|w'\|_{L^2} \leq 2\|f\|_{L^2}$. Also since $u \in X_{00}^\alpha$ we have that $u(0) = u(1) = 0$.

Now we write

$$u(x) = \int_0^x u'(s)ds = -\frac{1}{1-2\alpha}\int_0^x \left(s^{2\alpha}u'(s)\right)' s^{1-2\alpha}ds + \frac{xu'(x)}{1-2\alpha},$$

where we have used that $\lim_{s\to 0^+} su'(s) = \lim_{s\to 0^+} s^{2\alpha}u'(s)\cdot s^{1-2\alpha} = 0$ for all $\alpha < \frac{1}{2}$. It implies that

$$x^{2\alpha-1}u(x) = \frac{x^{2\alpha}u'(x)}{1-2\alpha} + \frac{x^{2\alpha-1}}{2\alpha-1}\int_0^x \left(s^{2\alpha}u'(s)\right)' s^{1-2\alpha}ds,$$

and

$$\left(x^{2\alpha-1}u(x)\right)' = x^{2\alpha-2}\int_0^x \left(s^{2\alpha}u'(s)\right)' s^{1-2\alpha}ds.$$

From here, since $\alpha < \frac{1}{2}$, we obtain

$$\left|\left(x^{2\alpha-1}u(x)\right)'\right| \leq \frac{1}{x}\int_0^x \left(s^{2\alpha}u'(s)\right)' ds,$$

so Hardy's inequality gives

$$\left\|\left(x^{2\alpha-1}u\right)'\right\|_{L^2} \leq 2\left\|\left(x^{2\alpha}u'\right)'\right\|_{L^2} \leq 4\|f\|_{L^2}.$$

Therefore, $\left\|x^{2\alpha-1}u\right\|_{H^1} \leq C\|f\|_{L^2}$, where C is a constant depending only on α. Combining this result and the fact that $x^{2\alpha}u' \in H^1(0,1)$, we conclude that $x^{2\alpha}u \in H^2(0,1)$.

Also notice that $u \in C^{0,1-2\alpha}[0,1]$ is a direct consequence of $x^{2\alpha-1}u \in C[0,1] \cap C^1(0,1]$. The proof is finished. □

Proof of Remark 3.1. Take $f \in C_c^\infty(0,1)$. We know that $u(x) = A\phi_1(x) + B\phi_2(x) + F(x)$ where $\phi_1(x)$ and $\phi_2(x)$ are two linearly independent solutions of the equation $-(x^{2\alpha}u'(x))' + u(x) = 0$ and

$$F(x) = \phi_1(x)\int_0^x f(s)\phi_2(s)ds - \phi_2(x)\int_0^x f(s)\phi_1(s)ds.$$

Moreover, one can see that $\phi_i(x) = x^{\frac{1}{2}-\alpha} f_i\left(\frac{x^{1-\alpha}}{1-\alpha}\right)$ where $f_i(z)$'s are two linearly independent solutions of the Bessel equation

$$z^2\phi''(z) + z\phi'(z) - \left(z^2 + \left(\frac{\frac{1}{2}-\alpha}{1-\alpha}\right)^2\right)\phi(z) = 0.$$

By the properties of the Bessel function (see e.g. Chapter III of [46]), we know that near the origin,

$$\phi_1(x) = a_1x^{1-2\alpha} + a_2x^{3-4\alpha} + a_3x^{5-6\alpha} + \cdots, \text{ for } 0 < \alpha < \frac{1}{2},$$

and

$$\phi_2(x) = b_1 + b_2x^{2-2\alpha} + b_3x^{4-4\alpha} + b_4x^{6-6\alpha} + \cdots, \text{ for } 0 < \alpha < 1.$$

Also,

$$\phi_1(0) = 0,\ \phi_2(0) \neq 0,\ \phi_1(1) \neq 0, \text{ for } 0 < \alpha < \frac{1}{2},$$
$$\lim_{x\to 0^+} |\phi_1(x)| = \infty,\ \lim_{x\to 0^+} \phi_2(x) = b_1, \text{ for } \alpha \geq \frac{1}{2},$$

and

$$\lim_{x\to 0^+} x^{2\alpha}\phi_1'(x) \neq 0,\ \lim_{x\to 0^+} x^{2\alpha}\phi_2'(x) = 0,\ \phi_2(1) \neq 0, \text{ for } 0 < \alpha < 1.$$

Notice that $F(x) \equiv 0$ near the origin. Therefore, when imposing the boundary conditions $u(0) = u(1) = 0$, we obtain $u(x) = A\phi_1(x) + F(x)$ with $A = -\frac{F(1)}{\phi_1(1)}$. Take f such that

$$F(1) = \int_0^1 f(s)[\phi_2(s)\phi_1(1) - \phi_1(s)\phi_2(1)]ds \neq 0.$$

Then $u(x) \sim \phi_1(x)$ near the origin and we get the desired power series expansion. $\square$

Proof of Remark 3.3. From the proof of Theorem 3.1, we conclude that $w \in C[0,1]$ with $\|w\|_\infty \leq 2\|f\|_{L^2}$. From here we have

$$|u'(x)| = |w(x)x^{-2\alpha}| \leq \|w\|_\infty x^{-2\alpha}.$$

Thus, for $1 \le p < \frac{1}{2\alpha}$,

$$\left\|u'\right\|_{L^p} \le \|w\|_\infty \left\|x^{-2\alpha}\right\|_{L^p(0,1)} \le C(\alpha,p)\,\|f\|_2\,.$$

$\square$

Proof of Remark 3.5. If we take $f(x) := -(x^{2\alpha}u'(x))' + u(x)$, where $u(x) = x^{1-2\alpha}(x-1)$, we will see that $u \notin C^{0,\beta}[0,1]$, $\forall \beta > 1-2\alpha$. When $u(x) = x^{\frac{7}{4}-2\alpha}(x-1)$, we will see that $x^{2\alpha-1}u \notin H^2(0,1)$, $x^{2\alpha}u' \notin H^2(0,1)$, and $x^{2\alpha}u \notin H^3(0,1)$. $\square$

Proof of Remark 3.6. From Theorem 4.2 we know that the function g exists and $x^{2\alpha}g' \in L^\infty(0,1)$. Therefore, integration by parts gives

$$\int_0^1 f(x)g(x)dx = \int_0^1 -(x^{2\alpha}u'(x))'g(x) + u(x)g(x)dx = \lim_{x\to 0^+} x^{2\alpha}u'(x).$$

And the L'Hopital's rule immediately implies that

$$\lim_{x\to 0^+} x^{2\alpha-1}u(x) = \lim_{x\to 0^+} \frac{1}{1-2\alpha}x^{2\alpha}u'(x) = \frac{1}{1-2\alpha}\int_0^1 f(x)g(x)dx.$$

$\square$

Before we prove Theorem 3.3, we need the following lemma.

Lemma 3.20. *Let $0 < \alpha < \frac{1}{2}$ and $k_0 \in \mathbb{N}$. Assume $u \in W^{k_0+1,p}_{loc}(0,1)$ for some $p \ge 1$. If $\lim_{x\to 0^+} u(x) = 0$ and $\lim_{x\to 0^+} x^{k-2\alpha}\frac{d^{k-1}}{dx^{k-1}}\left(s^{2\alpha}u'(s)\right) = 0$ for all $1 \le k \le k_0$, then for $0 < x < 1$,*

$$\frac{d^k}{dx^k}\left(x^{2\alpha-1}u(x)\right) = x^{2\alpha-k-1}\int_0^x s^{k-2\alpha}\frac{d^k}{ds^k}\left(s^{2\alpha}u'(s)\right)ds, \quad \textit{for all } 1 \le k \le k_0.$$

Moreover

$$\left\|\frac{d^k}{dx^k}\left(x^{2\alpha-1}u\right)\right\|_{L^p} \le C\left\|\frac{d^k}{dx^k}\left(x^{2\alpha}u'\right)\right\|_{L^p},$$

where C is a constant depending only on p, α and k.

Proof. When $k_0 = 1$ we can write

$$\begin{aligned}\left(x^{2\alpha-1}u(x)\right)' &= \left(x^{2\alpha-1}\int_0^x s^{2\alpha}u'(s)\left(\frac{s^{1-2\alpha}}{1-2\alpha}\right)'ds\right)' \\ &= \left(\frac{x^{2\alpha-1}}{2\alpha-1}\int_0^x \left(s^{2\alpha}u'(s)\right)' s^{1-2\alpha}ds + \frac{x^{2\alpha}u'(x)}{1-2\alpha}\right)' \\ &= x^{2\alpha-2}\int_0^x \left(s^{2\alpha}u'(s)\right)' s^{1-2\alpha}ds.\end{aligned}$$

The rest of the proof is a straightforward induction argument. We omit the details. The norm bound is obtained by Fubini's Theorem when $p = 1$ and by Hardy's inequality when $p > 1$. □

Proof of Theorem 3.3. Notice that $\lim_{x\to 0^+} x^{2-2\alpha}\left(s^{2\alpha}u'(s)\right)'=0$ since both u and f are continuous. With the aid of Lemma 3.20 for $k_0 = 2$ we can write

$$\left(x^{2\alpha-1}u(x)\right)'' = x^{2\alpha-3}\int_0^x s^{2-2\alpha}\left(s^{2\alpha}u'\right)'' ds = x^{2\alpha-3}\int_0^x s^{2-2\alpha}\left(u(s)-f(s)\right)' ds.$$

The result is obtained by using the estimate in Lemma 3.20. □

Proof of Remark 3.8. We use the same notation as in the proof of Remark 3.1. We know that $u(x) = A\phi_1(x) + B\phi_2(x) + F(x)$ where $\phi_1(x)$ and $\phi_2(x)$ are two linearly independent solutions of the equation $-(x^{2\alpha}u'(x))' + u(x) = 0$ and

$$F(x) = 1, \text{ if } f \equiv 1,$$

or

$$F(x) = \phi_1(x)\int_0^x f(s)\phi_2(s)ds - \phi_2(x)\int_0^x f(s)\phi_1(s)ds, \text{ if } f \in C_c^\infty(0,1).$$

In either case we have $F \in C[0,1]$. We also know that

$$\lim_{x\to 0^+}|\phi_1(x)| = \infty, \quad \lim_{x\to 0^+}\phi_2(x) = b_1, \quad \text{for } \alpha \ge \frac{1}{2}.$$

Therefore, if one wants a continuous function at the origin, one must have $A = 0$. Then $u(x) = B\phi_2(x) + F(x)$. We see now that the conditions $u(1) = 0$ and $\lim_{x\to 0^+} u(x) = 0$ are incompatible. □

3.3.2 The Neumann problem and the "Canonical" problem

Proof of Theorems 3.4, 3.7, 3.11. For $0 < \alpha < 1$, let $u \in X_0^\alpha$ solving

$$\int_0^1 x^{2\alpha}u'(x)v'(x)dx + \int_0^1 u(x)v(x)dx = \int_0^1 f(x)v(x)dx, \text{ for all } v \in X_0^\alpha.$$

First notice that

$$\|u\|_{L^2} + \left\|x^\alpha u'\right\|_{L^2} \le \|f\|_{L^2}.$$

Also, if we take $v \in C_c^\infty(0,1)$, then $x^{2\alpha}u' \in H^1(0,1)$ with $(x^{2\alpha}u'(x))' = u(x) - f(x)$.

We now proceed to prove that $w(x) := x^{2\alpha}u'(x)$ vanishes at $x = 0$. Take $v \in C^2[0,1]$ with $v(1) = 0$ as a test function and integrate by parts to obtain

$$0 = \int_0^1 \left(-(x^{2\alpha}u'(x))' + u(x) - f(x)\right) v(x)dx = \lim_{x\to 0^+} x^{2\alpha}u'(x)v(x).$$

The claim is obtained by taking any such v with $v(0) = 1$.

The above shows that $w(x) := x^{2\alpha}u'(x) \in H^1(0,1)$ with $w(0) = 0$. Then, notice that for any function $w \in H^1(0,1)$ with $w(0) = 0$ one can write

$$|w(x)| = \left|\int_0^x w'(x)dx\right| \le x^{\frac{1}{2}} \left(\int_0^x w'(x)^2 dx\right)^{\frac{1}{2}},$$

thus

$$\lim_{x\to 0^+} x^{2\alpha-\frac{1}{2}}u'(x) = 0.$$

Also, Hardy's inequality implies that $\frac{w}{x} \in L^2(0,1)$ with $\left\|\frac{w}{x}\right\|_{L^2} \le 2\left\|w'\right\|_{L^2}$. Now recall that $w'(x) = (x^{2\alpha}u'(x))' = u(x) - f(x)$, so $\|w'\|_{L^2} \le \|u\|_{L^2} + \|f\|_{L^2} \le 2\|f\|_{L^2}$. Hence we have the estimate $\left\|x^{2\alpha-1}u'\right\|_{L^2} \le 4\|f\|_{L^2}$.

In order to prove $\left\|x^{2\alpha}u''\right\|_{L^2} \le C\|f\|_{L^2}$, one only need to apply the above estimates and notice that $x^{2\alpha}u''(x) = (x^{2\alpha}u'(x))' - 2\alpha x^{2\alpha-1}u'(x)$.

By Theorem 3.34, property (i) of Theorems 3.4, 3.7, 3.11 is a direct consequence of the fact that $u \in X_0^{2\alpha-1}$.

Finally we establish the property (ii) of Theorem 3.11. For $\alpha = \frac{3}{4}$, first notice that

$$\begin{aligned}\int_0^1 \frac{u^2(x)}{x(1-\ln x)}dx &\le -\int_0^1 x\left(\frac{2u(x)u'(x)}{x(1-\ln x)} - \frac{u^2(x)}{x^2(1-\ln x)} + \frac{u^2(x)}{x^2(1-\ln x)^2}\right)dx \\ &= -2\int_0^1 \frac{u(x)u'(x)}{1-\ln x}dx + \int_0^1 \frac{u^2(x)}{x(1-\ln x)}dx - \int_0^1 \frac{u^2(x)}{x(1-\ln x)^2}dx,\end{aligned}$$

thus

$$\int_0^1 \frac{u^2(x)}{x(1-\ln x)^2}dx \le 2\left|\int_0^1 \frac{u(x)}{x^{\frac{1}{2}}(1-\ln x)}x^{\frac{1}{2}}u'(x)dx\right|. \tag{3.28}$$

Now Holder's inequality gives $(1-\ln x)^{-1}x^{-\frac{1}{2}}u(x) \in L^2(0,1)$. Therefore

$$\left((1-\ln x)^{-1}u^2(x)\right)' = (1-\ln x)^{-2}x^{-1}u^2(x) + 2(1-\ln x)^{-1}x^{-\frac{1}{2}}u(x)x^{\frac{1}{2}}u'(x) \in L^1(0,1),$$

so $\lim_{x\to 0^+}(1-\ln x)^{-\frac{1}{2}}u(x)$ exists. If the limit is non-zero, then near the origin $(1-\ln x)^{-1}x^{-\frac{1}{2}}u(x) \sim (1-\ln x)^{\frac{1}{2}}x^{-\frac{1}{2}} \notin L^2(0,1)$, which is a contradiction. For $\frac{3}{4} < \alpha < 1$, notice that

$$x^{4\alpha-3}u^2(x) = -\int_x^1 \left(t^{4\alpha-3}u^2(t)\right)' dt = -(4\alpha-3)\int_x^1 t^{4\alpha-4}u^2(t)dt - 2\int_x^1 t^{4\alpha-3}u'(t)u(t)dt.$$

Since we know $x^{2\alpha-1}u' \in L^2(0,1)$, Theorem 3.33 implies that $x^{2\alpha-2}u \in L^2(0,1)$, hence $\lim_{x\to 0^+} x^{2\alpha-\frac{3}{2}}u(x)$ exists. If the limit is non-zero, then near the origin $u(x) \sim x^{\frac{3}{2}-2\alpha} \notin L^{\frac{2}{4\alpha-3}}(0,1)$, which is a contradiction. □

Proof of Remark 3.10 for all $0 < \alpha < 1$. First notice that $x^{2\alpha-\frac{1}{2}}u'(x) = \frac{1}{\sqrt{x}}\int_0^x (u(s) - f(s))ds$. Therefore, $\left|x^{2\alpha-\frac{1}{2}}u'(x)\right| \leq 2\|f\|_{L^2}$, i.e., $K(x) \leq 2$.

On the other hand, for fixed $0 < x \leq \frac{1}{2}$, define

$$f(t) = \begin{cases} x^{-\frac{1}{2}} & \text{if } 0 < t \leq x \\ 0 & \text{if } x < t < 1. \end{cases}$$

Then $\|f\|_{L^2} = 1$. Consider first the case when $\frac{3}{4} < \alpha < 1$. From Theorem 3.11 we obtain that $u \in X_0^{2\alpha-1}$, which embeds into L^{p_0} for $p_0 = \frac{2}{4\alpha-3} > 2$. Thus one obtains that $\left|\frac{1}{\sqrt{x}}\int_0^x u(s)ds\right| \leq x^{\frac{1}{2}-\frac{1}{p_0}}$. Then

$$K_\alpha(x) \geq \left|\frac{1}{\sqrt{x}}\int_0^x (u(s) - f(s))ds\right| \geq 1 - x^{\frac{1}{2}-\frac{1}{p_0}} \geq 1 - \left(\frac{1}{2}\right)^{\frac{1}{2}-\frac{1}{p_0}}.$$

Therefore $K_\alpha(x) \geq \delta_\alpha$ for $\delta_\alpha := 1 - \left(\frac{1}{2}\right)^{\frac{1}{2}-\frac{1}{p_0}}$. Notice that when $0 < \alpha \leq \frac{3}{4}$, then $u \in L^p$ for all $p > 1$, so the above argument remains valid. The proof is now finished. □

Proof of Remark 3.11 for all $\alpha < \frac{3}{4}$. To prove (3.7), first notice that, from Theorem 4.2, the function h exists and $x^{\frac{1}{2}}h \in L^\infty(0,1)$. Therefore, integration by parts gives

$$\int_0^1 f(x)h(x)dx = \int_0^1 (-(x^{2\alpha}u'(x))'h(x) + u(x)h(x))dx = \lim_{x\to 0^+} u(x).$$

□

In order to prove the further regularity results we need the following

Lemma 3.21. *Let* $\alpha > 0$ *be a real number and* $k_0 \geq 0$ *be an integer. Assume* $u \in W_{loc}^{k_0+2,p}(0,1)$ *for some* $p \geq 1$, *and* $\lim_{x\to 0^+} x^k \frac{d^k}{dx^k}\left(x^{2\alpha}u'(x)\right) = 0$ *for all* $0 \leq k \leq k_0$. *Then for* $0 < x < 1$,

$$\frac{d^k}{dx^k}\left(x^{2\alpha-1}u'(x)\right) = \frac{1}{x^{k+1}}\int_0^x s^k \frac{d^{k+1}}{ds^{k+1}}\left(s^{2\alpha}u'(s)\right)ds, \quad \textit{for all } 0 \leq k \leq k_0.$$

Moreover

$$\left\| \frac{d^k}{dx^k} \left(x^{2\alpha-1}u'\right) \right\|_{L^p} \le C \left\| \frac{d^{k+1}}{dx^{k+1}} \left(x^{2\alpha}u'\right) \right\|_{L^p},$$

where C is a constant depending only on p, α and k.

Proof. If $k_0 = 0$ then the statement is obvious. When $k_0 = 1$, the condition

$$x \left(x^{2\alpha}u'(x)\right)' \to 0$$

gives

$$\begin{aligned}\left(x^{2\alpha-1}u'(x)\right)' &= \left(\frac{1}{x}\int_0^x \left(s^{2\alpha}u'(s)\right)' ds\right)' \\ &= \left(-\frac{1}{x}\int_0^x s\left(s^{2\alpha}u'(s)\right)'' ds + \left(x^{2\alpha}u'(x)\right)'\right)' \\ &= \frac{1}{x^2}\int_0^x s\left(s^{2\alpha}u'(s)\right)'' ds.\end{aligned}$$

The rest of the proof is a straightforward induction argument. We omit the details. The norm bound is obtained by Fubini's Theorem when $p = 1$ and by Hardy's inequality when $p > 1$. □

Proof of Theorem 3.6. Assume that $f \in W^{1,\frac{1}{2\alpha}}(0,1)$. First notice that for $1 \le p < \frac{1}{2\alpha}$ we have $u' \in L^p$ since $x^{2\alpha}u' \in H^1(0,1)$. Also notice that $x(x^{2\alpha}u'(x))' = x(u-f) \to 0$ since both u and f are continuous. We use Lemma 3.21 for $k_0 = 1$ to conclude

$$\left\|(x^{2\alpha-1}u')'\right\|_{L^p} \le C \left\|(x^{2\alpha}u')''\right\|_{L^p} = C \left\|(u-f)'\right\|_{L^p} \le C \left\|f\right\|_{W^{1,p}},$$

where C is a constant only depending on p and α. Recall that $x^{2\alpha}u'' = u - 2\alpha x^{2\alpha-1}u' - f \in W^{1,p}(0,1)$. It implies

$$\left|u''(x)\right| = \left|x^{2\alpha}u''\right| x^{-2\alpha} \le C \left\|f\right\|_{W^{1,p}} x^{-2\alpha},$$

where C is a constant only depending on p and α. The above inequality gives that $u \in W^{2,p}(0,1)$ for all $1 \le p < \frac{1}{2\alpha}$, with the corresponding estimate.

Assume now $f \in W^{2,\frac{1}{2\alpha}}(0,1)$. We first notice that $x^2\left(x^{2\alpha}u'(x)\right)'' = x^2(u-f)' = x^{2\alpha}u'(x)x^{2-2\alpha} - x^2 f'(x) \to 0$ as $x \to 0^+$ since $f \in C^1[0,1]$. This allows us to apply Lemma 3.21 and obtain

$$\left(x^{2\alpha-1}u'(x)\right)'' = \frac{1}{x^3}\int_0^x s^2\left(s^{2\alpha}u'(s)\right)''' ds = \frac{1}{x^3}\int_0^x s^2\left(u(s)-f(s)\right)'' ds.$$

Lemma 3.21 also gives the desired estimate. □

Proof of Remark 3.12, 3.15, 3.18. It is enough to prove the following claim: there exists $f \in C_c^\infty(0,1)$ such that the solution u can be expanded near the origin as

$$u(x) = b_1 + b_2 x^{2-2\alpha} + b_3 x^{4-4\alpha} + b_4 x^{6-6\alpha} + \cdots \tag{3.29}$$

where $b_1 \neq 0$, $b_2 \neq 0$.

We use the same notation as the proof of Remark 3.1. Take $f \in C_c^\infty(0,1)$. We know that $u(x) = A\phi_1(x) + B\phi_2(x) + F(x)$ where $\phi_1(x)$ and $\phi_2(x)$ are two linear independent solutions of the equation $-(x^{2\alpha}u'(x))' + u(x) = 0$ and

$$F(x) = \phi_1(x)\int_0^x f(s)\phi_2(s)ds - \phi_2(x)\int_0^x f(s)\phi_1(s)ds.$$

Moreover,

$$\lim_{x\to 0^+} x^{2\alpha}\phi_1'(x) \neq 0,\ \lim_{x\to 0^+} x^{2\alpha}\phi_2'(x) = 0,\ \phi_2(1) \neq 0,\ \text{for } 0 < \alpha < 1.$$

Notice that $F(x) \equiv 0$ near the origin. Therefore, the boundary conditions

$$\lim_{x\to 0^+} x^{2\alpha}u'(x) = u(1) = 0$$

imply that we have $u(x) = B\phi_2(x) + F(x)$ with $B = -\frac{F(1)}{\phi_2(1)}$. Take f such that

$$F(1) = \int_0^1 f(s)[\phi_2(s)\phi_1(1) - \phi_1(s)\phi_2(1)]ds \neq 0.$$

Then $u(x) \sim \phi_2(x)$ near the origin and we get the desired power series expansion. □

Proof of Theorem 3.9. When $k = 0$ we have already established that $u \in X^0 = H^1(0,1)$. Also, we have that $xu'' \in L^2$, so $(xu)'' = (u + xu')' = 2u' + xu''$, that is $xu \in H^2(0,1)$.

When $k = 1$, notice that $x\,(xu'(x))' = x\,(u - f) \to 0$ since both f and u are in $H^1(0,1)$. we use Lemma 3.21 to write

$$u''(x) = \frac{1}{x^2}\int_0^x s\left(su'(s)\right)'' ds = \frac{1}{x^2}\int_0^x s\,(u(s) - f(s))'\, ds.$$

We conclude that $u'' \in L^2(0,1)$ using Lemma 3.21. The rest of the proof is a straightforward induction argument using Lemma 3.21. We omit the details. □

Lemma 3.22. *Suppose $0 < \alpha < 1$ and let $f \in L^\infty(0,1)$. If u is the solution of (3.27), then $u \in C[0,1]$ and $x^{2\alpha-1}u' \in L^\infty(0,1)$ with*

$$\|u\|_{L^\infty} + \left\|x^{2\alpha-1}u'\right\|_{L^\infty} \leq C\,\|f\|_{L^\infty},$$

where C is a constant depending only on α.

Proof. To prove $x^{2\alpha-1}u' \in L^\infty(0,1)$, it is enough to show that $u \in L^\infty(0,1)$ with $\|u\|_{L^\infty} \leq C\,\|f\|_{L^\infty}$. Indeed, if this is the case, by (3.27) we obtain that $x^{2\alpha}u' \in W^{1,\infty}(0,1)$ with $\lim_{x\to 0^+} x^{2\alpha}u'(x) = 0$. Hardy's inequality implies that $\left\|x^{2\alpha-1}u'\right\|_{L^\infty} \leq C_\alpha\,\|f\|_{L^\infty}$.

Now we proceed to prove that $u \in C[0,1]$. First notice that if $\alpha < \frac{3}{4}$ then $u \in C[0,1]$ by Theorem 3.7. So we only need to study what happens when $\frac{3}{4} \leq \alpha < 1$.

Suppose $\frac{3}{4} \leq \alpha < 1$. Since $u \in X^{2\alpha-1}$ we can use Theorem 3.34 to say that $u \in L^{p_0}(0,1)$ for $p_0 = \frac{2}{4\alpha-3}$, so $g := f - u \in L^{p_0}(0,1)$. From (3.27) we obtain that $\left(x^{2\alpha}u'(x)\right)' = g(x)$, therefore $x^{2\alpha}u' \in W^{1,p_0}(0,1)$. Since $p_0 > 1$ and $\lim\limits_{x\to 0^+} x^{2\alpha}u'(x) = 0$, we are allowed to use Hardy's inequality and obtain that $x^{2\alpha-1}u' \in L^{p_0}(0,1)$. Using Theorem 3.34 once more gives that either $u \in C[0,1]$ if $\alpha < \frac{7}{8}$, in which case we are done, or $u \in L^{p_1}(0,1)$ for $p_1 := \frac{2}{8\alpha-7}$ if $\frac{7}{8} \leq \alpha < 1$. If we are in the latter case, we repeat the argument. This process stops in finite time since $\alpha < 1$, thus proving that $u \in C[0,1]$. □

Proof of Theorem 3.10, 3.13. We begin by recalling from Lemma 3.22 that if $f \in L^\infty(0,1)$ then $x^{2\alpha-1}u' \in L^\infty(0,1)$, so $|u'(x)| \leq \left\|x^{2\alpha-1}u'(x)\right\|_{L^\infty} x^{1-2\alpha}$. This readily implies $u \in W^{1,p}(0,1)$. Now just as in the proof of Theorem 3.6 we can use Lemma 3.21 and write

$$(x^{2\alpha-1}u'(x))' = \frac{1}{x^2}\int_0^x s(s^{2\alpha}u'(s))''ds = \frac{1}{x^2}\int_0^x s(u(s) - f(s))'ds.$$

Notice that $|xu'(x)| \leq \left\|x^{2\alpha-1}u'\right\|_{L^\infty} x^{2-2\alpha}$. From here we obtain

$$\left|(x^{2\alpha-1}u'(x))'\right| \leq C\left(\left\|x^{2\alpha-1}u'\right\|_{L^\infty} x^{1-2\alpha} + \left\|f'\right\|_{L^p}\right).$$

The conclusion then follows by integration. □

Proof of Remark 3.16. First notice that, from the proof of (ii) of Theorem 3.11, when $\alpha = \frac{3}{4}$,

$$\left|(1-\ln x)^{-\frac{1}{2}}\, u(x)\right| \leq C \left\|x^{\frac{1}{2}} u'(x)\right\|_{L^2} \leq C \left\|f\right\|_{L^2},$$

and when $\frac{3}{4} < \alpha < 1$,

$$\left|x^{2\alpha-\frac{3}{2}} u(x)\right| \leq C_\alpha \left\|x^\alpha u'(x)\right\|_{L^2} \leq C_\alpha \left\|f\right\|_{L^2}.$$

That is, $\widetilde{K}_\alpha(x) \leq C_\alpha$.

On the other hand, we can write

$$\begin{aligned} u(x) =& \int_x^1 \frac{1}{t^{2\alpha}} \int_0^t (u(s)-f(s))dsdt \\ =& \frac{1}{1-2\alpha}\left(\frac{1}{x^{2\alpha-1}}\int_0^x f(t)dt + \int_x^1 \frac{f(t)}{t^{2\alpha-1}}dt\right) \\ &+ \frac{1}{1-2\alpha}\left(\int_0^1 (u(t)-f(t))dt - \frac{1}{x^{2\alpha-1}}\int_0^x u(t)dt - \int_x^1 \frac{u(t)}{t^{2\alpha-1}}dt\right). \end{aligned}$$

When $\alpha = \frac{3}{4}$, for fixed $0 < x \leq \frac{1}{2}$, take

$$f(t) = \begin{cases} 0 & \text{if } 0 < t \leq x \\ t^{-\frac{1}{2}}(-\ln x)^{-\frac{1}{2}} & \text{if } x < t < 1. \end{cases}$$

Then $\|f\|_{L^2} = 1$. Since $u \in L^p(0,1)$ for all $p < \infty$, we can say that, there exists $M_\alpha > 0$ independent of x such that

$$\left|\int_0^1 (u(t)-f(t))dt - \frac{1}{x^{2\alpha-1}}\int_0^x u(t)dt - \int_x^1 \frac{u(t)}{t^{2\alpha-1}}dt\right| \leq M_\alpha.$$

Then

$$\widetilde{K}_\alpha(x) \geq \frac{1}{2\alpha-1}\left(\frac{(-\ln x)^{\frac{1}{2}}}{(1-\ln x)^{\frac{1}{2}}} - \frac{M_\alpha}{(1-\ln x)^{\frac{1}{2}}}\right).$$

When $\frac{3}{4} < \alpha < 1$, for fixed $0 < x \leq \frac{1}{2}$, take

$$f(t) = \begin{cases} x^{-\frac{1}{2}} & \text{if } 0 < t \leq x \\ 0 & \text{if } x < t < 1. \end{cases}$$

Then $\|f\|_{L^2} = 1$. Since $u \in L^{p_0}(0,1)$ for $p_0 = \frac{2}{4\alpha-3} > 2$, we can say that, there exists $M_\alpha > 0$ and $\gamma_\alpha > 0$ such that

$$\left|x^{2\alpha-\frac{3}{2}}\int_0^1 (u(t)-f(t))dt - \frac{1}{\sqrt{x}}\int_0^x u(t)dt - x^{2\alpha-\frac{3}{2}}\int_x^1 \frac{u(t)}{t^{2\alpha-1}}dt\right| \leq M_\alpha x^{\gamma_\alpha}.$$

Then

$$\widetilde{K}_\alpha(x) \geq \frac{1}{2\alpha-1}\left(1 - M_\alpha x^{\gamma_\alpha}\right).$$

Now, for $\frac{3}{4} \leq \alpha < 1$, take $\epsilon_\alpha > 0$ such that $\widetilde{K}_\alpha(x) \geq \frac{1}{4}$ for all $0 < x < \epsilon_\alpha$. If $\epsilon_\alpha < x \leq \frac{1}{2}$, we take $f(t) = -2(3-2\alpha)t + 3(4-2\alpha)t^2 + t^{3-2\alpha} - t^{4-2\alpha}$, hence $u(t) = t^{3-2\alpha} - t^{4-2\alpha}$. Notice that $0 < \|f\|_{L^2} \leq 10$, so we obtain

$$\widetilde{K}_\alpha(x) \geq \frac{x^{\frac{3}{2}} - x^{\frac{5}{2}}}{10} \geq \frac{\epsilon_\alpha^{\frac{3}{2}} - \epsilon_\alpha^{\frac{5}{2}}}{10} > 0,$$

for all $\epsilon_\alpha \leq x \leq \frac{1}{2}$. The result follows when we take $\delta_\alpha := \min\left\{\frac{1}{4}, \frac{\epsilon_\alpha^{\frac{3}{2}} - \epsilon_\alpha^{\frac{5}{2}}}{10}\right\}$. □

Proof of Theorem 3.14. Let u be the solution of (3.27). By the definition of u, we have that $u \in L^2(0,1)$ and $x^\alpha u' \in L^2(0,1)$. As in the proof of Theorem 3.4, we have that u satisfies (3.1), $w(x) = x^{2\alpha}u'(x) \in H^1(0,1)$, $w(0) = 0$ and for any function v in X_0^α,

$$\lim_{x\to 0^+} x^{2\alpha}u'(x)v(x) = 0.$$

Take $v(x) = x^\alpha u'(x) - u'(1)$. Since $\alpha \geq 1$, we have

$$x^\alpha(x^\alpha u'(x))' = w'(x) - \alpha x^{\alpha-1}x^\alpha u'(x) \in L^2(0,1),$$

which means that $v \in X_0^\alpha$. Thus we obtain

$$\lim_{x\to 0^+} x^{3\alpha}u'^2(x) = 0.$$

To prove that $\lim_{x\to 0^+} x^{\frac{\alpha}{2}}u(x) = 0$, we first claim that $\lim_{x\to 0^+} x^{\frac{\alpha}{2}}u(x)$ exists. To do this, we write $x^\alpha u^2(x) = -\int_x^1 (s^\alpha u^2(s))'ds$. Notice that

$$(x^\alpha u^2(x))' = \alpha x^{\alpha-1}u^2(x) + 2x^\alpha u'(x)u(x) \in L^1(0,1).$$

Therefore

$$\lim_{x\to 0^+} x^\alpha u^2(x) = -\int_0^1 (s^\alpha u^2(s))'ds.$$

Now, we can conclude that $\lim_{x\to 0^+} x^{\frac{\alpha}{2}}u(x) = 0$. Otherwise, $u(x) \sim \frac{1}{x^{\frac{\alpha}{2}}} \notin L^2(0,1)$. □

Proof of Remark 3.19. Fix $\phi \in C^\infty(\mathbb{R})$ such that $\operatorname{supp}\phi = [-1,1]$, $\phi(0) = 1$, $\phi'(0) = 1$ and $0 \leq \phi \leq 2$. Denote $C = \|\phi'\|_{L^\infty(-1,1)} + \|\phi''\|_{L^\infty(-1,1)}$. For fixed $x \in \left(0, \frac{1}{2}\right)$, take

$$u(t) = \frac{1}{\left(\alpha C 2^{2\alpha+4} + 2\right)x^{\frac{\alpha}{2}}}\phi\left(2x^{-\alpha}(t-x)\right).$$

It is straightforward that $u \in C_c^\infty(0,1)$, $f := -(t^{2\alpha}u')' + u \in L^2(0,1)$ and $\|f\|_{L^2} \leq 1$. Moreover, $\left|x^{\frac{\alpha}{2}}u(x)\right| = \frac{1}{\alpha C 2^{2\alpha+4}+2}$ and $\left|x^{\frac{3\alpha}{2}}u'(x)\right| = \frac{2}{\alpha C 2^{2\alpha+4}+2}$. It follows that $P_\alpha(x) \geq \frac{2}{\alpha C 2^{2\alpha+4}+2}$ and $\widetilde{P}_\alpha(x) \geq \frac{1}{\alpha C 2^{2\alpha+4}+2}$. On the other hand, for all $x \in \left(0, \frac{1}{2}\right)$, note that

$$x^{3\alpha}(u'(x))^2 = 3\alpha \int_0^x s^{3\alpha-1}(u'(s))^2 ds + 2\int_0^x s^{3\alpha}u''(s)u'(s)ds,$$

$$x^\alpha u^2(x) = \alpha \int_0^x s^{\alpha-1}u^2(s)ds + 2\int_0^x s^\alpha u'(s)u(s)ds.$$

It follows that $P_\alpha(x) \leq 6\sqrt{\alpha}$ and $\widetilde{P}_\alpha(x) \leq 4\sqrt{\alpha}$. Therefore, the proof is complete. $\square$

Before we finish this section, we present a proposition which will be used when dealing with the spectral analysis of the operator T_α. Also, this proposition gives the postponed proof of (iii) of Theorem 3.8 and (iii) of Theorem 3.12.

Proposition 3.23. *Given $\frac{1}{2} \leq \alpha \leq 1$ and $f \in L^2(0,1)$, suppose that $u \in H^2_{loc}(0,1]$ solves*

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = f(x) \quad \text{on } (0,1), \\ u(1) = 0, \\ u \in L^{\frac{1}{2\alpha-1}}(0,1). \end{cases} \tag{3.30}$$

Then u is the weak solution obtained from (3.27).

Proof. We claim that $x^\alpha u' \in L^2(0,1)$. To do this, define $w(x) = x^{2\alpha}u'(x)$. Then $w \in H^1(0,1)$. If $w(0) \neq 0$, then without loss of generality one can assume that there exists $\delta > 0$ such that $0 < M_1 \leq w(x) \leq M_2$ for all $x \in [0,\delta]$. Therefore,

$$\int_x^\delta \frac{M_1}{t^{2\alpha}}dt \leq \int_x^\delta u'(t)dt \leq \int_x^\delta \frac{M_2}{t^{2\alpha}}dt, \ \forall x \in (0,\delta].$$

It implies that

$$M_1(\ln\delta - \ln x) \leq u(\delta) - u(x) \leq M_2(\ln\delta - \ln x), \ \forall x \in (0,\delta],$$

when $\alpha = \frac{1}{2}$, and

$$\frac{M_1}{2\alpha-1}\left(\frac{1}{x^{2\alpha-1}} - \frac{1}{\delta^{2\alpha-1}}\right) \leq u(\delta) - u(x) \leq \frac{M_2}{2\alpha-1}\left(\frac{1}{x^{2\alpha-1}} - \frac{1}{\delta^{2\alpha-1}}\right), \ \forall x \in (0,\delta],$$

when $\alpha > \frac{1}{2}$. In either situation, we reach a contradiction with $u \in L^{\frac{1}{2\alpha-1}}(0,1)$. Therefore, $w(0) = 0$, so Hardy's inequality gives

$$\left\|x^\alpha u'\right\|_2^2 = \int_0^1 \frac{w^2(x)}{x^{2\alpha}} \leq \int_0^1 \frac{w^2(x)}{x^2} < \infty.$$

Since $w \in H^1(0,1)$ satisfies $w(0) = 0$, we conclude, in the same way as in the proof of Theorem 3.7, that $\lim_{x\to 0^+} x^{-\frac{1}{2}} w(x) = 0$. Now, integrate (3.30) against any test function $v \in X_0^\alpha$ on the interval $[\epsilon, 1]$ and obtain

$$\int_\epsilon^1 x^{2\alpha} u'(x)v'(x)dx + \epsilon^{2\alpha} u'(\epsilon)v(\epsilon) + \int_\epsilon^1 u(x)v(x)dx = \int_\epsilon^1 f(x)v(x)dx.$$

Since $\frac{1}{2} \leq \alpha \leq 1$, we write

$$\epsilon^{2\alpha} u'(\epsilon)v(\epsilon) = \left[\epsilon^{2\alpha-\frac{1}{2}} w(\epsilon)\right] \left[\epsilon^{\frac{1}{2}} v(\epsilon)\right].$$

The estimate (3.46) in Section 3.6 tells us that $\left|x^{\frac{1}{2}} v(x)\right| \leq C_\alpha \left\|v\right\|_\alpha$, so we can send $\epsilon \to 0^+$ and obtain (3.27) as desired. □

3.4 The spectrum of the operator T_α

In this section we study the spectrum of the operator T_α. We divide this section into three parts. In subsection 3.4.1 we study the eigenvalue problem of T_α for all $\alpha > 0$. In subsection 3.4.2 we explore the rest of the spectrum of T_α for the non-compact case $\alpha \geq 1$. Finally, in subsection 3.4.3, we give the proof of Theorem 3.19.

3.4.1 The eigenvalue problem for all $\alpha > 0$

In this subsection, we focus on finding the eigenvalues and eigenfunctions of T_α. That is, we seek $(u, \lambda) \in L^2(0,1) \times \mathbb{R}$ such that $u \neq 0$ and $T_\alpha u = \lambda u$. By definition of T_α in Section 3.1.6, we have $\lambda \neq 0$ and the pair (u, λ) satisfies

$$\int_0^1 x^{2\alpha} u'(x)v'(x)dx + \int_0^1 u(x)v(x)dx = \frac{1}{\lambda} \int_0^1 u(x)v(x)dx, \ \forall v \in X_0^\alpha. \tag{3.31}$$

From here we see right away that if $\lambda > 1$ or $\lambda < 0$, then Lax-Milgram Theorem applies and equation (3.31) has only the trivial solution. Also, a direct computation shows that

$u \equiv 0$ is the only solution when $\lambda = 1$. This implies that all the eigenvalues belong to the interval $(0, 1)$. So we will analyze (3.31) only for $0 < \lambda < 1$.

As the existence and uniqueness results show, it amounts to study the following ODE for $\mu := \frac{1}{\lambda} > 1$,

$$-(x^{2\alpha} u'(x))' + u(x) = \mu u(x) \quad \text{on } (0,1), \tag{3.32}$$

under certain boundary behaviors. To solve (3.32), we will use Bessel's equation

$$y^2 f''(y) + y f'(y) + (y^2 - \nu^2) f(y) = 0 \quad \text{on } (0, \infty). \tag{3.33}$$

Indeed, we have the following

Lemma 3.24. *For $\alpha \neq 1$ and any $\beta > 0$, let f_ν be any solution of* (3.33) *with parameter $\nu^2 = \left(\frac{\alpha - \frac{1}{2}}{\alpha - 1}\right)^2$ and define $u(x) = x^{\frac{1}{2} - \alpha} f_\nu(\beta x^{1-\alpha})$. Then u solves*

$$-(x^{2\alpha} u'(x))' = \beta^2 (\alpha - 1)^2 u(x).$$

The proof of Lemma 3.24 is elementary, which we omit. We will also need a few known facts about Bessel functions, which we summarize in the following Lemmas (for the proofs see e.g. Chapter III of [46]).

Lemma 3.25. *For non-integer ν, the general solution to equation* (3.33) *can be written as*

$$f_\nu(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x). \tag{3.34}$$

The function $J_\nu(x)$ is called the Bessel function of the first kind of order ν. This function has the following power series expansion

$$J_\nu(x) = \frac{1}{\Gamma(\nu + 1)} \left(\frac{x}{2}\right)^\nu + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m + \nu + 1)} \left(\frac{x}{2}\right)^{2m+\nu}.$$

A similar expression can be obtained for $J'_\nu(x)$ by differentiating $J_\nu(x)$.

Lemma 3.26. *For non-negative integer ν, the general solution to equation* (3.33) *can be written as*

$$f_\nu(x) = C_1 J_\nu(x) + C_2 Y_\nu(x). \tag{3.35}$$

The function $J_\nu(x)$ *is the same as the one from Lemma 3.25, and the function* $Y_\nu(x)$ *is called the Bessel function of second kind which satisfies the following asymptotics: for* $0 < x << 1$,

$$Y_\nu(x) \sim \begin{cases} \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + \gamma\right] & \text{if } \nu = 0, \\ -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu & \text{if } \nu > 0, \end{cases}$$

where $\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{k} - \ln(n)\right)$ *is Euler's constant.*

Remark 3.24. *We have been using the notation* $f(x) \sim g(x)$*. This notation means that there exists constants* $c_1, c_2 > 0$ *such that*

$$c_1 |g(x)| \le |f(x)| \le c_2 |g(x)|.$$

Remark 3.25. *Suppose that* $\alpha \neq 1$*, and let* $\beta = \frac{\sqrt{\mu-1}}{|\alpha-1|}$*. Then Lemma 3.24-3.26 guarantee that the general solution of* (3.32) *is given by*

$$u(x) = \begin{cases} C_1 x^{\frac{1}{2}-\alpha} J_\nu(\beta x^{1-\alpha}) + C_2 x^{\frac{1}{2}-\alpha} J_{-\nu}(\beta x^{1-\alpha}) & \text{if } \nu \text{ is not an integer}, \\ C_1 x^{\frac{1}{2}-\alpha} J_\nu(\beta x^{1-\alpha}) + C_2 x^{\frac{1}{2}-\alpha} Y_\nu(\beta x^{1-\alpha}) & \text{if } \nu \text{ is an non-negative integer}. \end{cases} \tag{3.36}$$

Now the problem has been reduced to select the eigenfunctions from the above family.

We first study the eigenvalue problem for the compact case $0 < \alpha < 1$.

Proof of (i) of Theorem 3.17. We first consider the case when $0 < \alpha < \frac{1}{2}$. In this case notice that $\nu = \frac{\alpha - \frac{1}{2}}{1-\alpha}$ is negative and non-integer. From theorems 3.4 and 3.5, and equations (3.31), (3.32) and (3.36), we have that the eigenfunction is of the form

$$u(x) = C_1 x^{\frac{1}{2}-\alpha} J_\nu(\beta x^{1-\alpha}) + C_2 x^{\frac{1}{2}-\alpha} J_{-\nu}(\beta x^{1-\alpha})$$

with $\beta = \frac{\sqrt{\mu-1}}{|\alpha-1|}$, $\lim_{x\to 0^+} x^{2\alpha}u'(x) = 0$ and $u(1) = 0$. Then Lemma 3.25 gives that $x^{2\alpha}u'(x) \sim C_2 \frac{\beta^{-\nu}\left(\frac{1}{2}-\alpha\right)}{2^{-\nu}\Gamma(-\nu+1)}$. so the boundary condition $\lim_{x\to 0^+} x^{2\alpha}u'(x) = 0$ forces C_2 to vanish. Therefore $u(x) = C_1 x^{\frac{1}{2}-\alpha} J_\nu(\beta x^{1-\alpha})$. Now, the condition $u(1) = 0$ forces β to satisfy $J_\nu(\beta) = 0$, that is β must be a positive root of the the Bessel function J_ν, for $\nu = \frac{\alpha - \frac{1}{2}}{1-\alpha}$.

Therefore, we conclude that if we let $j_{\nu k}$ be the k-th positive root of $J_\nu(x)$, then

$$u_{\nu k}(x) = x^{\frac{1}{2}-\alpha} J_\nu(j_{\nu k} x^{1-\alpha}),\ k = 1, 2, \cdots$$

are the eigenfunctions and the corresponding eigenvalues are given by

$$\lambda_{\nu k} = \frac{1}{1 + (1-\alpha)^2 j_{\nu k}^2},\ k = 1, 2, \cdots .$$

Next, we investigate the case when $\frac{1}{2} \le \alpha < 1$. In this case, $\nu = \frac{\alpha - \frac{1}{2}}{1-\alpha}$ is non-negative and could be integer or non-integer. Using Lemma 3.25 and 3.26, we obtain the asymptotics of the general solution near the origin,

$$u(x) \sim \begin{cases} \frac{C_1 \beta^\nu}{\Gamma(\nu+1)2^\nu} + \frac{C_2 2^\nu}{\beta^\nu \Gamma(1-\nu)} x^{1-2\alpha} & \text{if } \alpha > \frac{1}{2}, \text{ and } \nu \text{ is not an integer}, \\ \frac{C_1 \beta^\nu}{\Gamma(\nu+1)2^\nu} - \frac{2^\nu \Gamma(\nu) C_2}{\beta^\nu \pi} x^{1-2\alpha} & \text{if } \alpha > \frac{1}{2}, \text{ and } \nu \text{ is an integer}, \\ \frac{C_1 \beta^\nu}{\Gamma(\nu+1)2^\nu} + \frac{2C_2}{\pi} [\ln(\beta\sqrt{x}) + \gamma] & \text{if } \alpha = \frac{1}{2}. \end{cases}$$

Now Proposition 3.23 says that it is enough to impose $u \in L^{\frac{1}{2\alpha-1}}(0,1)$ which forces $C_2 = 0$ and $u(x) = C_1 x^{\frac{1}{2}-\alpha} J_\nu(\beta x^{1-\alpha})$. Moreover, the condition $u(1) = 0$ forces β to satisfy $J_\nu(\beta) = 0$, that is β must be a positive root of the Bessel function J_ν, for $\nu = \frac{\alpha-\frac{1}{2}}{1-\alpha}$.

As before we conclude that

$$u_{\nu k}(x) = x^{\frac{1}{2}-\alpha} J_\nu(j_{\nu k} x^{1-\alpha}),\ k = 1, 2, \cdots$$

are the eigenfunctions and the corresponding eigenvalues are given by

$$\lambda_{\nu k} = \frac{1}{1 + (1-\alpha)^2 j_{\nu k}^2},\ k = 1, 2, \cdots .$$

Finally, the asymptotic behavior of $j_{\nu k}$ as $k \to \infty$ is well understood (see e.g. Chapter XV of [46]). We have

$$j_{\nu k} = k\pi + \frac{\pi}{2}\left(\nu - \frac{1}{2}\right) - \frac{4\nu^2 - 1}{8\left(k\pi + \frac{\pi}{2}\left(\nu - \frac{1}{2}\right)\right)} + O\left(\frac{1}{k^3}\right). \tag{3.37}$$

Using (3.37), we obtain that

$$\mu_{\nu k} = 1 + (1-\alpha)^2 \left[\left(\frac{\pi}{2}\left(\nu - \frac{1}{2}\right) + \pi k\right)^2 - \left(\nu^2 - \frac{1}{4}\right)\right] + O\left(\frac{1}{k}\right).$$

□

Next we consider the case $\alpha = 1$. In this case, the equation (3.36) is not the general solution for (3.32). However, as the reader can easily verify, the general solution for (3.32) when $\alpha = 1$ is given by

$$u(x) = \begin{cases} C_1 x^{-\frac{1}{2}+\sqrt{\frac{5}{4}-\mu}} + C_2 x^{-\frac{1}{2}-\sqrt{\frac{5}{4}-\mu}} & \text{for } \mu < \frac{5}{4}, \\ C_1 x^{-\frac{1}{2}} + C_2 x^{-\frac{1}{2}} \ln x & \text{for } \mu = \frac{5}{4}, \\ C_1 x^{-\frac{1}{2}} \cos\left(\sqrt{\mu - \frac{5}{4}} \ln x\right) + C_2 x^{-\frac{1}{2}} \sin\left(\sqrt{\mu - \frac{5}{4}} \ln x\right) & \text{for } \mu > \frac{5}{4}. \end{cases} \tag{3.38}$$

With equation (3.38) in our hands, we can prove the following:

Proposition 3.27. *If $\alpha = 1$, then T_α has no eigenvalues.*

Proof. For the general solution given by (3.38), we impose $u(1) = 0$, and obtain that any non-trivial solution has the form:

$$u(x) = \begin{cases} C x^{-\frac{1}{2}+\sqrt{\frac{5}{4}-\mu}} \left(1 - x^{-2\sqrt{\frac{5}{4}-\mu}}\right) & \text{for } \mu < \frac{5}{4}, \\ C x^{-\frac{1}{2}} \ln x & \text{for } \mu = \frac{5}{4}, \\ C x^{-\frac{1}{2}} \sin\left(\sqrt{\mu - \frac{5}{4}} \ln x\right) & \text{for } \mu > \frac{5}{4}, \end{cases}$$

for some $C \neq 0$. From here we see right away that if $\mu \geq \frac{5}{4}$ then $u \notin L^2(0,1)$. And when $\mu < \frac{5}{4}$, we obtain that

$$\int_0^1 u^2(x)dx = C^2 \int_0^1 x^{-1+2\sqrt{\frac{5}{4}-\mu}} \left(1 - x^{-2\sqrt{\frac{5}{4}-\mu}}\right)^2 dx.$$

Let $y = x^{2\sqrt{\frac{5}{4}-\mu}}$, so this integral becomes

$$\int_0^1 u^2(x)dx = C^2 \int_0^1 \left(1 - \frac{1}{y}\right)^2 dy \geq \frac{C^2}{4} \int_0^{\frac{1}{2}} \frac{1}{y^2} dy = +\infty.$$

This says that when $\alpha = 1$, there are no eigenvalues and eigenfunctions. □

Finally we investigate the case $\alpha > 1$. To investigate the eigenvalue problem in this case, we need the following fact about the Bessel's equation.

Lemma 3.28. *Assume that $f_\nu(t)$ is a non-trivial solution of Bessel's equation*

$$t^2 f_\nu''(t) + t f_\nu'(t) + (t^2 - \nu^2) f_\nu(t) = 0. \tag{3.39}$$

Then $\int_s^\infty t f_\nu^2(t)dt = \infty$, $\forall s > 0, \forall \nu > 0$.

Proof. We first define the function $g_\nu(t) = f_\nu(bt)$, for some $b \neq 1$. Then $g_\nu(t)$ satisfies the ODE

$$t^2 g_\nu''(t) + t g_\nu'(t) + (b^2t^2 - \nu^2) g_\nu(t) = 0. \tag{3.40}$$

From equation (3.39) and (3.40), we have

$$t^2(f_\nu''(t)g_\nu(t) - f_\nu(t)g_\nu''(t)) + t(f_\nu'(t)g_\nu(t) - f_\nu(t)g_\nu'(t)) + t^2(1-b^2)f_\nu(t)g_\nu(t) = 0,$$

or

$$t(f_\nu''(t)g_\nu(t) - f_\nu(t)g_\nu''(t)) + (f_\nu'(t)g_\nu(t) - f_\nu(t)g_\nu'(t)) + t(1-b^2)f_\nu(t)g_\nu(t) = 0,$$

i.e.

$$\frac{d}{dt}\left[t(f_\nu'(t)g_\nu(t) - f_\nu(t)g_\nu'(t))\right] + t(1-b^2)f_\nu(t)g_\nu(t) = 0.$$

Integrating the above equation we obtain

$$\begin{aligned}
&\int_s^N t f_\nu(t) g_\nu(t) dt \\
=&\frac{N(f_\nu'(N)g_\nu(N) - f_\nu(N)g_\nu'(N))}{b^2-1} - \frac{s(f_\nu'(s)g_\nu(s) - f_\nu(s)g_\nu'(s))}{b^2-1} \\
=&\frac{Nf_\nu'(N)f_\nu(bN) - bNf_\nu(N)f_\nu'(bN)}{b^2-1} - \frac{sf_\nu'(s)f_\nu(bs) - bsf_\nu(s)f_\nu'(bs)}{b^2-1} \\
\triangleq& A - B.
\end{aligned}$$

We then pass the limit as $b \to 1$. Notice that

$$\begin{aligned}
\lim_{b\to1} A &= \lim_{b\to1} \frac{Nf_\nu'(N)f_\nu(bN) - bNf_\nu(N)f_\nu'(bN)}{b^2-1} \\
&= \lim_{b\to1} \frac{N^2 f_\nu'(N)f_\nu'(bN) - Nf_\nu(N)f_\nu'(bN) - bN^2 f_\nu(N)f_\nu''(bN)}{2b} \\
&= \frac{N^2 f_\nu'(N)f_\nu'(N) - Nf_\nu(N)f_\nu'(N) - N^2 f_\nu(N)f_\nu''(N)}{2} \\
&= \frac{1}{2}\left(N^2 f_\nu'^2(N) + N^2 f_\nu^2(N) - \nu^2 f_\nu^2(N)\right),
\end{aligned}$$

and

$$\begin{aligned}
\lim_{b\to1} B &= \lim_{b\to1} \frac{sf_\nu'(s)f_\nu(bs) - bsf_\nu(s)f_\nu'(bs)}{b^2-1} \\
&= \frac{1}{2}\left(s^2 f_\nu'^2(s) + s^2 f_\nu^2(s) - \nu^2 f_\nu^2(s)\right).
\end{aligned}$$

Therefore

$$\int_s^N t f_\nu^2(t)dt = \frac{1}{2}\left(N^2 f_\nu'^2(N) + N^2 f_\nu^2(N) - \nu^2 f_\nu^2(N)\right) - \frac{1}{2}\left(s^2 f_\nu'^2(s) + s^2 f_\nu^2(s) - \nu^2 f_\nu^2(s)\right).$$

Sending $N \to \infty$, we deduce from the asymptotic behavior of the Bessel's function that $\int_s^\infty t f_\nu^2(t)dt = \infty$. □

Proposition 3.29. *If $\alpha > 1$, then T_α has no eigenvalues.*

Proof. We argue by contradiction. Suppose $\lambda = \frac{1}{\mu}$ is an eigenvalue and $u \in L^2(0,1)$ is the corresponding eigenfunction, then $\mu > 1$ and the pair (u, λ) satisfies (3.32). Lemma 3.24 says that $u(x) = x^{\frac{1}{2}-\alpha} f_\nu(\beta x^{1-\alpha})$ where $\beta = \frac{\sqrt{\mu-1}}{\alpha-1}$ and $f_\nu(t)$ is a non-trivial solution of

$$t^2 f_\nu''(t) + t f_\nu'(t) + (t^2 - \nu^2) f_\nu(t) = 0.$$

Applying the change of variable $\beta x^{1-\alpha} = t$ and Lemma 3.28 gives

$$\begin{aligned}\int_0^1 u^2(x)dx &= \int_0^1 x^{1-2\alpha} f_\nu^2(\beta x^{1-\alpha})dx \\ &= \frac{1}{\beta(\alpha-1)}\int_\beta^\infty \left(\frac{t}{\beta}\right)^{\frac{1-2\alpha}{1-\alpha}+\frac{1}{1-\alpha}-1} f_\nu^2(t)dt \\ &= \frac{1}{\beta^2(\alpha-1)}\int_\beta^\infty t f_\nu^2(t)dt = \infty,\end{aligned}$$

which is a contradiction. □

3.4.2 The rest of the spectrum for the case $\alpha \geq 1$

We have found the eigenvalues of T_α for all $\alpha > 0$. Next we study the rest of the spectrum for the non-compact case $\alpha \geq 1$. It amounts to study the surjectivity of the operator $T_\alpha - \lambda I$ in $L^2(0,1)$, that is, given $f \in L^2(0,1)$, we want determine whether there exists $h \in L^2(0,1)$ such that $(T - \lambda)h = f$. Since $\|T_\alpha\| \leq 1$, T_α is a positive operator, and T_α is not surjective, we can assume that $0 < \lambda \leq 1$. By letting $u = \lambda h + f$, the existence of the function $h \in L^2(0,1)$ is equivalent to the existence of the function $u \in L^2(0,1)$ satisfying

$$T_\alpha\left(\frac{u-f}{\lambda}\right) = u.$$

By the definition of T_α in Section 3.1.6, the above equation can be written as

$$\int_0^1 \left(x^{2\alpha} u'(x)v'(x) + \left(1 - \frac{1}{\lambda}\right) u(x)v(x) \right) dx = -\frac{1}{\lambda} \int_0^1 f(x)v(x)dx, \ \forall v \in X_0^\alpha. \quad (3.41)$$

Since we proved that there are no eigenvalues when $\alpha \geq 1$, a real number λ is in the spectrum of the operator T_α if and only if there exists a function $f \in L^2(0,1)$ such that (3.41) is not solvable. To study the solvability of (3.41) we introduce the following bilinear form,

$$a_\alpha(u,v) \triangleq \int_0^1 x^{2\alpha} u'(x)v'(x)dx + \left(1 - \frac{1}{\lambda}\right) \int_0^1 u(x)v(x)dx, \quad (3.42)$$

and we first study the coercivity of $a_1(u,v)$.

Lemma 3.30. *If $\lambda > \frac{4}{5}$, then $a_1(u,v)$ is coercive in X_0^1.*

Proof. We use Theorem 3.33 and obtain

$$\begin{aligned} a_1(u,u) &= \int_0^1 (xu'(x))^2 dx - \left(\frac{1}{\lambda} - 1\right) \int_0^1 u^2(x)dx \\ &\geq \int_0^1 (xu'(x))^2 dx - 4\left(\frac{1}{\lambda} - 1\right) \int_0^1 (xu'(x))^2 \\ &= \left(1 - 4\left(\frac{1}{\lambda} - 1\right)\right) \int_0^1 (xu'(x))^2 dx \\ &\geq \frac{1}{5}\left(1 - 4\left(\frac{1}{\lambda} - 1\right)\right) \|u\|_{X_0^1}^2 . \end{aligned}$$

Thus if $\lambda > \frac{4}{5}$, this bilinear form is coercive. □

Now we can prove the next

Proposition 3.31. *For $\alpha = 1$, the spectrum of the operator T_1 is exactly $\sigma(T_1) = \left[0, \frac{4}{5}\right]$.*

Proof. The coercivity of $a_1(u,v)$ gives immediately that $\sigma(T_1) \subset \left[0, \frac{4}{5}\right]$. To prove the reverse inclusion, we first claim that $(T_1 - \lambda)u = -\lambda$ is not solvable when $0 < \lambda \leq \frac{4}{5}$. Otherwise, by equation (3.41), there would exist $\mu = \frac{1}{\lambda}$ and $u \in L^2(0,1)$ such that

$$\begin{cases} -(x^2 u'(x))' + (1-\mu)u(x) = 1 \quad \text{on } (0,1), \\ u(1) = 0. \end{cases} \quad (3.43)$$

Equation (3.43) can be solved explicitly as

$$u(x) = \begin{cases} x^{-\frac{1}{2}}\left[C - \left(C + \frac{1}{1-\mu}\right)\ln x\right] + \frac{1}{1-\mu} & \text{for } \mu = \frac{5}{4}, \\ C_\mu x^{-\frac{1}{2}} \sin\left(A_\mu + \sqrt{\mu - \frac{5}{4}}\ln x\right) + \frac{1}{1-\mu} & \text{for } \mu > \frac{5}{4}, \end{cases}$$

where $C_\mu = \frac{C^2 + \frac{1}{(1-\mu)^2}}{\sqrt{\mu - \frac{5}{4}}}$, $\sin A_\mu = \frac{C}{C^2 + \frac{1}{(1-\mu)^2}}$ and C could be any real number. So we have that

$$\left\| u(x) - \frac{1}{1-\mu} \right\|^2_{L^2(0,1)} = \begin{cases} \int_{-\infty}^0 \left(C - \left(C + \frac{1}{1-\mu}\right) y\right)^2 dy & \text{for } \mu = \frac{5}{4}, \\ C_\mu \int_{-\infty}^0 \sin^2\left(A_\mu + y\right) dy & \text{for } \mu > \frac{5}{4}. \end{cases}$$

Notice that the right hand side above is $+\infty$ independently of C, thus proving that $u \notin L^2(0,1)$. Therefore $(T_1 - \lambda)h = -\lambda$ is not solvable in $L^2(0,1)$ for $0 < \lambda \leq \frac{4}{5}$. Also $0 \in \sigma(T_1)$, because T_1 is not surjective. This gives $\left[0, \frac{4}{5}\right] \subset \sigma(T_1)$ as claimed. □

Proposition 3.32. *For $\alpha > 1$, the spectrum of the operator T_α is exactly $\sigma(T_\alpha) = [0,1]$.*

Proof. As we already know, $\sigma(T_\alpha) \subset [0,1]$. So let us prove the converse. We first claim that the equation $(T_\alpha - \lambda)u = -\lambda$ is not solvable for $0 < \lambda < 1$. As before, this amounts to solve

$$-(x^{2\alpha}u'(x))' + (1-\mu)u(x) = 1,$$

where $\mu = \frac{1}{\lambda}$. Lemma 3.24 implies that $u(x) = x^{\frac{1}{2}-\alpha} f_\nu(\beta x^{1-\alpha}) + 1$ where $\beta = \frac{\sqrt{\mu - 1}}{\alpha - 1}$ and $f_\nu(t)$ is a non-trivial solution of

$$t^2 f_\nu''(t) + t f_\nu'(t) + (t^2 - \nu^2) f_\nu(t) = 0.$$

By Lemma 3.28 we conclude that $\|u\|_{L^2} = \infty$. So $(T_\alpha - \lambda)h = -\lambda$ is not solvable when $\lambda \in (0,1)$.

When $\lambda = 1$, take $f(x) = -\lambda x^{\epsilon - \frac{1}{2}}$, where $\epsilon > 0$ is to be determined, and try to solve $(T_\alpha - I)u = f$, which is equivalent to solve

$$\begin{cases} -(x^{2\alpha}u'(x))' = x^{\epsilon - \frac{1}{2}} & \text{on } (0,1), \\ u(1) = 0. \end{cases}$$

The general solution of this ODE is given by

$$u(x) = \frac{1}{(\frac{1}{2}+\epsilon)(\frac{3}{2}+\epsilon-2\alpha)} x^{\frac{3}{2}+\epsilon-2\alpha} + Cx^{-2\alpha+1} - C - \frac{1}{(\frac{1}{2}+\epsilon)(\frac{3}{2}+\epsilon-2\alpha)}.$$

We choose $0 < \epsilon < 2\alpha - 2$ so that $\frac{3}{2} + \epsilon - 2\alpha < -\frac{1}{2}$. Therefore, $\|u\|_{L^2} = \infty$ independently of C, thus $(T_\alpha - I)u = f$ is not solvable. Hence $(0, 1] \subset \sigma(T_\alpha)$. Also $0 \in \sigma(T_\alpha)$; thus the result is proved. □

Proof of Corollary 3.18. To prove (i), it is enough to notice that when $0 < \alpha < 1$ the operator T_α is compact and $R(T_\alpha)$ is not closed.

To prove (ii) and (iii), by the definition of essential spectrum and the fact that T_α has no eigenvalue when $\alpha \geq 1$, it is enough to show that $\sigma_d(T_\alpha) \subset EV(T_\alpha)$, where $EV(T_\alpha)$ is the set of the eigenvalues. Actually, for $\lambda \in \sigma_d(T_\alpha)$, we claim that $\dim N(T_\alpha - \lambda I) \neq 0$. Suppose the contrary, then $\dim N(T_\alpha - \lambda I) = 0$, and one obtains that

$$R(T_\alpha - \lambda I)^\perp = N(T_\alpha^* - \lambda I) = N(T_\alpha - \lambda I) = \{0\}.$$

Since $T_\alpha - \lambda I$ is Fredholm, it means that $R(T_\alpha - \lambda I)$ is closed and therefore $R(T_\alpha - \lambda I) = L^2(0, 1)$. That leads to the bijectivity of $T_\alpha - \lambda I$, which contradicts with $\lambda \in \sigma_d(T_\alpha)$. □

3.4.3 The proof of Theorem 3.19

Proof. To prove (i), it is equivalent to prove that $\mu_{\nu k} \geq \frac{5}{4}$ for all $k = 1, 2, \ldots$ and $\nu > \frac{1}{2}$. Indeed, since $\nu > \frac{1}{2}$, we have the following inequality (see [25]) for all $k = 1, 2, \ldots$,

$$j_{\nu k} > \nu + \frac{k\pi}{2} - \frac{1}{2} \geq \nu + \frac{\pi - 1}{2},$$

so

$$(1 - \alpha) j_{\nu k} = \frac{1}{2(\nu + 1)} j_{\nu k} \geq \frac{1}{2} + \frac{\pi - 3}{4(\nu + 1)} \geq \frac{1}{2}.$$

Thus $\mu_{\nu k} = 1 + (1 - \alpha)^2 j_{\nu k}^2 \geq \frac{5}{4}$.

To prove (ii), from [25] we obtain that for fixed $x > 0$, we have

$$\lim_{\nu \to \infty} \frac{j_{\nu, \nu x}}{\nu} = i(x), \tag{3.44}$$

where $i(x) := \sec\theta$ and θ is the unique solution in $\left(0, \frac{\pi}{2}\right)$ of $\tan\theta - \theta = \pi x$. Using this fact, and the definition of ν, we can write

$$\mu_{\nu k} = 1 + (1 - \alpha)^2 j_{\nu k}^2 = 1 + \left(\alpha - \frac{1}{2}\right)^2 \left(\frac{j_{\nu k}}{\nu}\right)^2.$$

Define $\nu_k = \frac{k}{x}$ (or equivalently, $\alpha_k = 1 - \frac{1}{2\left(\frac{k}{x}+1\right)}$), then (3.44) implies that

$$\mu_m := \mu_{\nu_m m} = 1 + \left(\alpha_m - \frac{1}{2}\right)^2 i^2(x)\,(1 + o(1))\,,$$

where $o(1)$ is a quantity that goes to 0 as $m \to \infty$. So for fixed $x > 0$ we find that (notice that $m \to \infty$ implies $\nu_m \to \infty$, which necessarily implies that $\alpha_m \to 1^-$)

$$\lambda_m := \frac{1}{\mu_m} \to \frac{1}{1 + \frac{1}{4} i^2(x)} =: \lambda(x).$$

It is clear from the definition of $i(x)$, that $i(x)$ is injective and that $i((0, +\infty)) = (1, +\infty)$, which gives that $\lambda(x)$ is injective and $\lambda((0, +\infty)) = \left(0, \frac{4}{5}\right)$. So we only need to take care of the endpoints, that is 0 and $\frac{4}{5}$. Firstly, consider $j_{\nu 1}$, the first root of $J_\nu(x)$. It is known that (see e.g. Chapter XV of [46])

$$j_{\nu 1} = \nu + O(\nu^{\frac{1}{3}}) \text{ as } \nu \to \infty.$$

Consider $\mu_m = \mu_{m1} = 1 + \left(\alpha_m - \frac{1}{2}\right)^2 (1 + o(1))\,,$ where $\alpha_m = 1 - \frac{1}{2(m+1)}$, and $o(1)$ goes to 0 as $m \to \infty$. This implies that

$$\lambda_m \to \frac{4}{5} \text{ as } \alpha_m \to 1^-.$$

To conclude the proof of (ii), recall that T_α is compact for all $\alpha < 1$ so $0 \in \sigma(T_\alpha)$. □

Proof of Remark 3.22. Notice that part (i) in Theorem 3.19 gives

$$\sup_{x \in \sigma(T_\alpha)} \inf_{y \in \sigma(T_1)} |x - y| = 0$$

for all $\frac{2}{3} < \alpha < 1$. Therefore, it is enough to prove

$$\lim_{\alpha \to 1^-} \sup_{x \in \sigma(T_1)} \inf_{y \in \sigma(T_\alpha)} |x - y| = 0.$$

Indeed, the compactness of $\sigma(T_1)$ implies that, for any $\epsilon > 0$, there exists $\{x_i\}_{i=1}^n \in \sigma(T_1)$ such that

$$\sup_{x \in \sigma(T_1)} \inf_{y \in \sigma(T_\alpha)} |x - y| \leq \max_{i=1,\ldots,n} d(x_i, \sigma(T_\alpha)) + \frac{\epsilon}{2}.$$

Then part (ii) in Theorem 3.19 gives the existence of $\alpha_\epsilon < 1$ such that $d(x_i, \sigma(T_\alpha)) \leq \frac{\epsilon}{2}$ for all $\alpha_\epsilon < \alpha < 1$ and all $i = 1, \ldots, n$. □

3.5 The spectrum of the operator T_D

Proof of Theorem 3.16. In order to find all the eigenvalues and eigenfunctions, we need the nontrivial solutions of

$$\begin{cases} -\left(x^{2\alpha}u'(x)\right)' + u(x) = \mu u(x) \quad \text{on } (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

Let $\nu_0 = \frac{\frac{1}{2}-\alpha}{1-\alpha}$, which is positive and never an integer. Equation (3.36) gives us its general solution

$$u(x) = C_1 x^{\frac{1}{2}-\alpha} J_{\nu_0}(\beta x^{1-\alpha}) + C_2 x^{\frac{1}{2}-\alpha} J_{-\nu_0}(\beta x^{1-\alpha}),$$

where $\beta = \frac{\sqrt{\mu-1}}{|\alpha-1|}$. The asymptotic of J_{ν_0} when $0 < x << 1$ yields

$$u(x) \sim \frac{C_1 k^{\nu_0}}{\Gamma(\nu_0+1)2^{\nu_0}} x^{1-2\alpha} + \frac{C_2 2^{\nu_0}}{k^{\nu_0}\Gamma(1-\nu_0)},$$

so imposing $u(0) = 0$ forces $C_2 = 0$. i.e. $u(x) = C_1 x^{\frac{1}{2}-\alpha} J_{\nu_0}(\beta x^{1-\alpha})$. Then $u(1) = 0$ forces β to satisfy $J_{\nu_0}(\beta) = 0$, that is β must be a positive root of the Bessel function J_{ν_0}, for $\nu_0 = \frac{\frac{1}{2}-\alpha}{1-\alpha}$.

Therefore, we conclude that

$$u_{\nu_0 k}(x) = x^{\frac{1}{2}-\alpha} J_{\nu_0}(j_{\nu_0 k} x^{1-\alpha}), \; k = 1, 2, \cdots$$

are the eigenfunctions and the corresponding eigenvalues are given by

$$\lambda_{\nu_0 k} = \frac{1}{1+(1-\alpha)^2 j_{\nu_0 k}^2}, \; k = 1, 2, \cdots.$$

The behavior of $\mu_{\nu_0 k}$ is then obtained from the asymptotic of $j_{\nu_0 k}$ just as we did in the study of the operators T_α. We omit the details. □

3.6 Appendix: a weighted Sobolev space

For $\alpha > 0$ and $1 \le p \le \infty$ define

$$X^{\alpha,p}(0,1) = \left\{ u \in W^{1,p}_{loc}(0,1); \;\; u \in L^p(0,1), x^\alpha u' \in L^p(0,1) \right\}.$$

Notice that the functions in $X^{\alpha,p}(0,1)$ are continuous away from 0. It makes sense to define the following subspace

$$X^{\alpha,p}_{\cdot 0}(0,1) = \{u \in X^{\alpha,p}(0,1);\ u(1) = 0\}\,.$$

When $p = 2$, we simplify the notation and write $X^{\alpha} := X^{\alpha,2}(0,1)$ and $X^{\alpha}_{0} := X^{\alpha,2}_{\cdot 0}(0,1)$. The space $X^{\alpha,p}(0,1)$ is equipped with the norm

$$\|u\|_{\alpha,p} = \|u\|_{L^p(0,1)} + \left\|x^{\alpha}u'\right\|_{L^p(0,1)}\,,$$

or sometimes, if $1 < p < \infty$, with the equivalent norm

$$\left(\|u\|^p_{L^p(0,1)} + \left\|x^{\alpha}u'\right\|^p_{L^p(0,1)}\right)^{\frac{1}{p}}\,.$$

The space X^{α} is equipped with the scalar product

$$(u,v)_{\alpha} = \int_0^1 \left(x^{2\alpha}u'(x)v'(x) + u(x)v(x)\right)dx,$$

and with the associated norm

$$\|u\|_{\alpha} = \left(\|u\|^2_{L^2(0,1)} + \left\|x^{\alpha}u'\right\|^2_{L^2(0,1)}\right)^{\frac{1}{2}}\,.$$

One can easily check that, for $\alpha > 0$ and $1 \leq p \leq \infty$, the space $X^{\alpha,p}(0,1)$ is a Banach space and $X^{\alpha,p}_{\cdot 0}(0,1)$ is a closed subspace. When $1 < p < \infty$ the space is reflexive. Moreover, the space X^{α} is a Hilbert space.

Weighted Sobolev spaces have been studied in more generality (see e.g. [35]). However, since our situation is more specific, we briefly discuss some properties which are relevant for our study.

Theorem 3.33. *For $1 \leq p \leq \infty$, let β be any real number such that $\beta + \frac{1}{p} > 0$. Assume that $u \in W^{1,p}_{loc}(0,1]$ and $u(1) = 0$. Then*

$$\left\|x^{\beta}u\right\|_{L^p} \leq C_{p,\beta}\left\|x^{\beta+1}u'\right\|_{L^p}\,, \tag{3.45}$$

where $C_{p,\beta} = \frac{p}{1+p\beta}$ for $1 \leq p < \infty$ and $C_{\infty,\beta} = \frac{1}{\beta}$. In particular, for $1 \leq p < \infty$ and $0 < \alpha \leq 1$, $|u|_{\alpha,p} := \|x^{\alpha}u'\|_{L^p}$ defines an equivalent norm for $X^{\alpha,p}_{\cdot 0}(0,1)$.

Proof. We first assume $1 \le p < \infty$ and write

$$\begin{aligned}\int_\epsilon^1 x^{p\beta}\left|u(x)\right|^p dx &= -\int_\epsilon^1 x\left(x^{p\beta}\left|u(x)\right|^p\right)' dx - \epsilon^{p\beta+1}\left|u(\epsilon)\right|^p \\ &\le -\int_\epsilon^1 x\left(x^{p\beta}\left|u(x)\right|^p\right)' dx \\ &= -p\beta\int_\epsilon^1 x^{p\beta}\left|u(x)\right|^p dx - p\int_\epsilon^1 x^{p\beta+1}\left|u(x)\right|^{p-2}u(x)u'(x)dx.\end{aligned}$$

Applying Holder's inequality, we obtain

$$(1+p\beta)\int_\epsilon^1 x^{p\beta}\left|u(x)\right|^p dx \le p\int_\epsilon^1 x^{p\beta}\left|u(x)\right|^p x^{\beta+1}\left|u'(x)\right| dx \le p\left\|x^\beta u\right\|_{L^p}^{p-1}\left\|x^{\beta+1}u'\right\|_{L^p}.$$

Then equation (3.45) is derived for $1 \le p < \infty$ and $C_{p,\beta} = \frac{p}{1+p\beta}$. When $p = \infty$, it is understood that $\frac{1}{p} = 0$ and $\beta > 0$, so we pass the limit for $p \to \infty$ in equation (3.45) and obtain

$$\left\|x^\beta u\right\|_{L^\infty} \le \frac{1}{\beta}\left\|x^{\beta+1}u'\right\|_{L^\infty}.$$

□

Theorem 3.34. *For $0 < \alpha \le 1$, $1 \le p \le \infty$, the space $X^{\alpha,p}(0,1)$ is continuously embedded into*

(i) $C^{0,1-\frac{1}{p}-\alpha}[0,1]$ if $0 < \alpha < 1 - \frac{1}{p}$ and $p \ne 1$,

(ii) $L^q(0,1)$ for all $q < \infty$ if $\alpha = 1 - \frac{1}{p}$,

(iii) $L^{\frac{p}{p\alpha-p+1}}(0,1)$ if $1 - \frac{1}{p} < \alpha \le 1$ and $p \ne \infty$.

Proof. For all $0 < x < y < 1$, we write $|u(y) - u(x)| \le \int_x^y |s^\alpha u'(s)|\, s^{-\alpha} ds$. Applying Holder's inequality, we obtain

$$|u(y)-u(x)| \le C_{\alpha,p}\left\|s^\alpha u'\right\|_{L^p}\begin{cases} x^{-\alpha} & \text{if } p = 1 \\ \left|y^{1-\frac{\alpha p}{p-1}} - x^{1-\frac{\alpha p}{p-1}}\right|^{\frac{p-1}{p}} & \text{if } 1 < p < \infty \text{ and } \alpha \ne 1 - \frac{1}{p} \\ \left|\ln y - \ln x\right|^{\frac{p-1}{p}} & \text{if } 1 < p < \infty \text{ and } \alpha = 1 - \frac{1}{p} \\ \left|y^{1-\alpha} - x^{1-\alpha}\right| & \text{if } p = \infty \text{ and } \alpha \ne 1 \\ \left|\ln y - \ln x\right| & \text{if } p = \infty \text{ and } \alpha = 1. \end{cases} \tag{3.46}$$

Then assertions (i) and (ii) of Theorem 3.34 follow directly from equation (3.46).

Next, we prove the assertion (iii) with $u \in X^{\alpha,p}_{\cdot 0}(0,1)$. That is, for $1 \le p < \infty$, $1 - \frac{1}{p} < \alpha \le 1$ and $u \in W^{1,p}_{loc}(0,1]$ with $u(1) = 0$, we claim

$$\|u\|_{L^{\frac{p}{p\alpha - p+1}}} \le \frac{p\alpha}{p\alpha - p + 1}\left(\frac{1}{\alpha}\right)^{\alpha} 2^{1-\alpha}\left\|x^{\alpha}u'\right\|_{L^p}. \tag{3.47}$$

If $\alpha = 1$, estimate (3.47) is a special case of (3.45). We now prove (3.47) for $p = 1$ and $0 < \alpha < 1$. Notice that, from equation (3.45),

$$\begin{aligned}\|x^{\alpha}u\|_{L^{\infty}} &\le \left\|(x^{\alpha}u)'\right\|_{L^1} \\ &\le \alpha\left\|x^{\alpha-1}u\right\|_{L^1} + \left\|x^{\alpha}u'\right\|_{L^1} \\ &\le 2\left\|x^{\alpha}u'\right\|_{L^1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^1 |u(x)|^{\frac{1}{\alpha}}\,dx &= -\frac{1}{\alpha}\int_0^1 x\,|u(x)|^{\frac{1}{\alpha}-2}\,u(x)u'(x)dx - \lim_{x\to 0^+} x\,|u(x)|^{\frac{1}{\alpha}} \\ &\le \frac{1}{\alpha}\left\|x^{\alpha}u'\right\|_{L^1}\left\|x^{1-\alpha}\,|u(x)|^{\frac{1}{\alpha}-1}\right\|_{L^{\infty}} \\ &\le \frac{1}{\alpha}2^{\frac{1-\alpha}{\alpha}}\left\|x^{\alpha}u'\right\|^{\frac{1}{\alpha}}_{L^1}.\end{aligned}$$

That is

$$\|u\|_{L^{\frac{1}{\alpha}}} \le \left(\frac{1}{\alpha}\right)^{\alpha} 2^{1-\alpha}\left\|x^{\alpha}u'\right\|_{L^1}. \tag{3.48}$$

Then we assume $1 < p < \infty$ and $1 - \frac{1}{p} < \alpha < 1$, we proceed as in the proof of the Sobolev-Gagliardo-Nirenberg inequality. That is, applying the inequality (3.48) to $u(x) = |v(x)|^{\gamma}$, for some $\gamma > 1$ to be chosen, it gives

$$\left(\int_0^1 |v(x)|^{\frac{\gamma}{\alpha}}\,dx\right)^{\alpha} \le \gamma\left(\frac{1}{\alpha}\right)^{\alpha} 2^{1-\alpha}\int_0^1 |v(x)|^{\gamma-1}\left|v'(x)\right| x^{\alpha}dx.$$

Using Holder inequality yields

$$\left(\int_0^1 |v(x)|^{\frac{\gamma}{\alpha}}\,dx\right)^{\alpha} \le \gamma\left(\frac{1}{\alpha}\right)^{\alpha} 2^{1-\alpha}\left\|x^{\alpha}v'\right\|_{L^p}\left(\int_0^1 |v(x)|^{\frac{p(\gamma-1)}{p-1}}\right)^{1-\frac{1}{p}}.$$

Let $\frac{\gamma}{\alpha} = \frac{p(\gamma-1)}{p-1}$. That is $\gamma = \frac{p\alpha}{p\alpha - p+1} > 1$ and the above inequality gives the desired result.

Finally, the assertion (iii) in the general case follows immediately from (3.47), because $\|u\|_{L^p} \le \|u - u(1)\|_{L^p} + |u(1)|$, while $u - u(1) \in X^{\alpha,p}_{\cdot 0}(0,1)$ and $|u(1)| \le (2^{p\alpha} + 1)\|u\|_{\alpha,p}$. □

We would like to point out that, by the assertion (i) in Theorem 3.34, we can define, for $1 < p \leq \infty$ and $0 < \alpha < 1 - \frac{1}{p}$,

$$X^{\alpha,p}_{00}(0,1) = \{u \in X^{\alpha,p}(0,1);\ u(0) = u(1) = 0\}.$$

Remark 3.26. *Notice that the inequalities* (3.45) *and* (3.47) *are particular cases of the inequalities proved by Caffarelli-Kohn-Nirenberg. For further reading on this topic we refer to their paper [15].*

Theorem 3.35. *Let $1 \leq p \leq \infty$. Then $X^{\alpha,p}(0,1)$ is compactly embedded into $L^p(0,1)$ for all $\alpha < 1$. On the other hand, the embedding is not compact when $\alpha \geq 1$.*

Proof. We first prove that, for $1 \leq p < \infty$ and $0 < \alpha < 1$, the space $X^{\alpha,p}_{\cdot 0}(0,1)$ is compactly embedded into $L^p(0,1)$. Let $\mathcal{F}$ be the unit ball in $X^{\alpha,p}_{\cdot 0}(0,1)$. It suffices to prove that $\mathcal{F}$ is totally bounded in $L^p(0,1)$. Notice that, by equation (3.46), $\forall \epsilon > 0$, there exists a positive integer m, such that

$$\|u\|_{L^p(0,\frac{2}{m})} < \epsilon,\ \forall u \in \mathcal{F}.$$

Define $\phi(x) \in C^\infty(\mathbb{R})$ with $0 \leq \phi \leq 1$ such that

$$\phi(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ 1 & \text{if } x \geq 2, \end{cases}$$

and take $\phi_m(x) = \phi(mx)$. Now $\phi_m \mathcal{F}$ is bounded in $W^{1,p}(0,1)$, and therefore is totally bounded in $L^p(0,1)$. Hence we may cover $\phi_m \mathcal{F}$ by a finite number of balls of radius ϵ in $L^p(0,1)$, say

$$\phi_m \mathcal{F} \subset \bigcup_i B(g_i, \epsilon),\ g_i \in L^p(0,1).$$

We claim that $\bigcup_i B(g_i, 3\epsilon)$ covers $\mathcal{F}$. Indeed, given $u \in \mathcal{F}$ there exists some i such that

$$\|\phi_m u - g_i\|_{L^p(0,1)} < \epsilon.$$

Therefore,

$$\begin{aligned} \|u - g_i\|_{L^p(0,1)} &\leq \|\phi_m u - g_i\|_{L^p(0,1)} + \|u - \phi_m u\|_{L^p(0,1)} \\ &< \epsilon + 2\|u\|_{L^p(0,\frac{2}{m})} \\ &\leq 3\epsilon. \end{aligned}$$

Hence we conclude that $\mathcal{F}$ is totally bounded in $L^p(0,1)$.

To prove the compact embedding for $X^{\alpha,p}(0,1)$ with $1 \leq p < \infty$ and $0 < \alpha < 1$, notice that for any sequence $\{v_n\} \subset X^{\alpha,p}(0,1)$ with $\|v_n\|_{\alpha,p} \leq 1$. One can define $u_n(x) = v_n(x) - v_n(1) \in X^{\alpha,p}_{\cdot 0}(0,1)$. Then

$$\|u_n\|_{\alpha,p} = \left\|x^\alpha u_n'\right\|_{L^p} = \left\|x^\alpha v_n'\right\|_{L^p} \leq 1.$$

What we just proved shows that there exists $u \in L^p(0,1)$ such that, up to a subsequence, $u_n \to u$ in L^p. Notice in addition that $|v_n(1)| \leq (2^{p\alpha}+1)\|v\|_{\alpha,p} \leq 2^{p\alpha}+1$, thus there exists $M \in \mathbb{R}$ such that, after maybe extracting a further subsequence, $v_n(1) \to M$. Then it is clear that $v_n(x) \to u(x) + M$ in L^p.

We now prove the embedding is not compact when $1 \leq p < \infty$ and $\alpha \geq 1$. To do so, define the sequence of functions

$$v_n(x) = \left(\frac{1}{nx(1-\ln x)^{1+\frac{1}{n}}}\right)^{\frac{1}{p}},$$

and

$$u_n(x) = v_n(x) - \left(\frac{1}{n}\right)^{\frac{1}{p}}, \ \forall n \geq 2.$$

Clearly $\|v_n\|_{L^p(0,1)} = 1$ and $1 - \left(\frac{1}{2}\right)^{\frac{1}{p}} \leq \|u_n\|_{L^p(0,1)} \leq 2$. Also $\|xu_n'\|_{L^p(0,1)} \leq \frac{6}{p}$. It means that $\{u_n(x)\}_{n=2}^{\infty}$ is a bounded sequence in $X^{\alpha,p}_{\cdot 0}(0,1)$ for $\alpha \geq 1$. However, it has no convergent subsequence in $L^p(0,1)$ since $u_n \to 0$ a.e. and $\|u_n\|_{L^p(0,1)}$ is uniformly bounded below.

If $p = \infty$ and $0 < \alpha < 1$, take $u \in X^{\alpha,\infty}(0,1)$ and equation (3.46) implies that

$$|u(x) - u(y)| \leq C_\alpha \left\|x^\alpha u'\right\|_{L^\infty} |x-y|^{1-\alpha}.$$

Therefore, the embedding is compact by the Ascoli-Arzela theorem. To prove that the embedding is not compact for $p = \infty$ and $\alpha \geq 1$, define the sequence of functions

$$\phi_n(x) = \begin{cases} -\frac{\ln x}{\ln n} & \text{if } \frac{1}{n} \leq x \leq 1 \\ 1 & \text{if } 0 \leq x < \frac{1}{n}. \end{cases}$$

We can see that ϕ_n is a bounded sequence in $X^{\alpha,\infty}(0,1)$ for $\alpha \geq 1$. However it has no convergent subsequence in $L^\infty(0,1)$ since $\phi_n \to 0$ a.e but $\|\phi_n\|_{L^\infty} = 1$. $\square$

We conclude this section with the following density result, which is not used in this chapter but is of independent interest.

Theorem 3.36. *Assume* $1 \le p < \infty$.

(i) *If* $p \neq 1$ *and* $0 < \alpha < 1 - \frac{1}{p}$, *we have that* $C^\infty[0,1]$ *is dense in* $X^{\alpha,p}(0,1)$ *and that* $C_c^\infty(0,1)$ *is dense in* $X_{00}^{\alpha,p}(0,1)$.

(ii) *If* $\alpha > 0$ *and* $\alpha \ge 1 - \frac{1}{p}$, *we have that* $C_c^\infty(0,1]$ *is dense in* $X^{\alpha,p}(0,1)$.

Proof. For any $1 \le p < \infty$, $\alpha > 0$ and $u \in X^{\alpha,p}(0,1)$, we first claim that there exists a sequence $\{\epsilon_n > 0\}$ with $\lim_{n\to\infty} \epsilon_n = 0$ such that:

- either $|u(\epsilon_n)| \le C$ uniformly in n, or
- $|u(\epsilon_n)| \le |u(x)|$ for all n and $0 < x < \epsilon_n$.

Indeed, if $|u(x)|$ is unbounded along every sequence converging to 0, we would have $\lim_{x\to 0^+} |u(x)| = +\infty$, in which case we can define $\epsilon_n > 0$ to be such that $|u(\epsilon_n)| = \min_{0<x\le\frac{1}{n}} |u(x)|$, thus completing the argument. In the rest of this proof, for any $u \in X^{\alpha,p}(0,1)$, sequence $\{\epsilon_n\}$ is chosen to have the above property.

We first prove (i). Assume $1 < p < \infty$ and $0 < \alpha < 1 - \frac{1}{p}$. To prove that $C^\infty[0,1]$ is dense in $X^{\alpha,p}(0,1)$, it suffices to show that $W^{1,p}(0,1)$ is dense in $X^{\alpha,p}(0,1)$. Take $u \in X^{\alpha,p}(0,1)$. Define

$$u_n(x) = \begin{cases} u(\epsilon_n) & \text{if } 0 < x \le \epsilon_n \\ u(x) & \text{if } \epsilon_n < x \le 1. \end{cases}$$

Then one can easily check that $u_n \in W^{1,p}(0,1)$ and that $u_n \to u$ in $X^{\alpha,p}(0,1)$ by the dominated convergence theorem. To prove that $C_c^\infty(0,1)$ is dense in $X_{00}^{\alpha,p}(0,1)$, it suffices to show that $W_0^{1,p}(0,1)$ is dense in $X_{00}^{\alpha,p}(0,1)$, to do so, we adapt a technique by Brezis (see the proof of Theorem 8.12 of [8], page 218): take $G \in C^1(\mathbb{R})$ such that $|G(t)| \le |t|$ and

$$G(t) = \begin{cases} 0 & \text{if } |t| \le 1 \\ t & \text{if } |t| > 2. \end{cases}$$

For $u \in X^{\alpha,p}_{00}(0,1)$, define $u_n = \frac{1}{n}G(nu)$. Then one can easily check that $u_n \in C_c(0,1) \cap X^{\alpha,p}(0,1) \subset W^{1,p}_0(0,1)$ and that $u_n \to u$ in $X^{\alpha,p}(0,1)$ by the dominated convergence theorem.

To prove the assertion (ii), we notice that it is enough to prove that $C^\infty_c(0,1)$ is dense in $X^{\alpha,p}_{\cdot 0}(0,1)$. Indeed, for any $u \in X^{\alpha,p}(0,1)$, define $\phi(x) \in C^\infty_c(0,1]$ such that $|\phi(x)| \le 1$ with

$$\phi(x) = \begin{cases} 1 & \text{if } \frac{2}{3} \le x \le 1 \\ 0 & \text{if } 0 \le x \le \frac{1}{3}. \end{cases}$$

Define $v(x) := u(x) - \phi(x)u(1)$, then $v \in X^{\alpha,p}_{\cdot 0}(0,1)$. If we can approximate v by $v_n \in C^\infty_c(0,1)$, then $u_n(x) = v_n(x) + \phi(x)u(1)$ belongs to $C^\infty_c(0,1]$ and it approximates u in $X^{\alpha,p}_{\cdot 0}(0,1)$. So let $\alpha > 1 - \frac{1}{p}$ and $1 \le p < \infty$, to prove that $C^\infty_c(0,1)$ is dense in $X^{\alpha,p}_{\cdot 0}(0,1)$, it suffices to show that $W^{1,p}_0(0,1)$ is dense in $X^{\alpha,p}_{\cdot 0}(0,1)$. To do so, for fixed $u \in X^{\alpha,p}_{\cdot 0}(0,1)$, define

$$u_n(x) = \begin{cases} \frac{u(\epsilon_n)}{\epsilon_n} x & \text{if } 0 \le x \le \epsilon_n \\ u(x) & \text{if } \epsilon_n < x \le 1. \end{cases}$$

Then $u_n \in W^{1,p}_0(0,1)$ and on the interval $(0,\epsilon_n)$ we have either $|u_n(x)| \le |u(x)|$ and $|u_n'(x)| \le \frac{|u(x)|}{x}$, or $|u_n(x)| \le C$ and $|u_n'(x)| \le \frac{C}{x}$ where C is independent of n. In both cases, since $\alpha > 1 - \frac{1}{p}$ and $x^{\alpha-1}u(x) \in L^p$ by Theorem 3.33, one can conclude that $u_n \to u$ in $X^{\alpha,p}(0,1)$ by the dominated convergence theorem.

For $\alpha = 1 - \frac{1}{p}$ and $1 < p < \infty$, again, it suffices to prove that $W^{1,p}_0(0,1)$ is dense in $X^{\alpha,p}_{\cdot 0}(0,1)$. For fixed $u \in X^{\alpha,p}_{\cdot 0}(0,1)$, define

$$u_n(x) = \begin{cases} \frac{u(\epsilon_n)(1-\ln \epsilon_n)}{1-\ln x} & \text{if } 0 \le x \le \epsilon_n \\ u(x) & \text{if } \epsilon_n < x \le 1. \end{cases}$$

One can easily check that $u_n \in C[0,1] \cap X^{\alpha,p}(0,1)$ and $u_n(0) = u_n(1) = 0$. On the interval $(0,\epsilon_n)$, we have either $|u_n(x)| \le |u(x)|$ and $|u_n'(x)| \le \frac{|u(x)|}{x(1-\ln x)}$, or $|u_n| \le C$ and $|u_n'(x)| \le \frac{C}{x(1-\ln x)}$ where C is independent of n. Notice that by using the same trick used in estimate (3.28), one can show that $x^{-\frac{1}{p}}(1-\ln x)^{-1}u \in L^p(0,1)$ for any $u \in X^{1-\frac{1}{p},p}_{\cdot 0}(0,1)$ with $1 < p < \infty$. Therefore, one can conclude that $u_n \to u$ in $X^{\alpha,p}(0,1)$.

The above shows that that $\{u \in C[0,1] \cap X^{\alpha,p}(0,1);\; u(0) = u(1) = 0\}$ is dense in $X^{\alpha,p}_{\cdot 0}(0,1)$. Finally, notice that by using the same argument used to prove (i), we obtain that $W^{1,p}_0(0,1)$ is dense in $\{u \in C[0,1] \cap X^{\alpha,p}(0,1);\; u(0) = u(1) = 0\}$, thus concluding the proof. $\square$

Chapter 4

A singular Sturm-Liouville equation under non-homogeneous boundary conditions

4.1 Introduction

In Chapter 3 we studied the equation (3.1), with (weighted) *homogeneous* Dirichlet and Neumann boundary conditions at the origin. In order to conclude that the boundary conditions used in Chapter 3 are the *only* appropriate boundary conditions, we investigate the existence of solutions for equation (3.1) under the corresponding (weighted) *non-homogeneous* boundary conditions at the origin.

Without loss of generality, we always assume that $f \equiv 0$ in (3.1). Consider the following (weighted) non-homogeneous Neumann problem,

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 \quad \text{on } (0,1), \\ u(1) = 0, \\ \lim_{x \to 0^+} \psi_\alpha(x) u'(x) = 1, \end{cases} \tag{4.1}$$

where

$$\psi_\alpha(x) = \begin{cases} x^{2\alpha} & \text{if } 0 < \alpha < 1, \\ x^{\frac{3+\sqrt{5}}{2}} & \text{if } \alpha = 1, \\ x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} & \text{if } \alpha > 1, \end{cases} \tag{4.2}$$

and the following (weighted) non-homogeneous Dirichlet problem,

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 \quad \text{on } (0,1), \\ u(1) = 0, \\ \lim_{x \to 0^+} \phi_\alpha(x) u(x) = 1, \end{cases} \tag{4.3}$$

where

$$\phi_\alpha(x) = \begin{cases} 1 & \text{if } 0 < \alpha < \frac{1}{2}, \\ (1-\ln x)^{-1} & \text{if } \alpha = \frac{1}{2}, \\ x^{2\alpha-1} & \text{if } \frac{1}{2} < \alpha < 1, \\ x^{\frac{1+\sqrt{5}}{2}} & \text{if } \alpha = 1, \\ x^{\frac{\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} & \text{if } \alpha > 1. \end{cases} \tag{4.4}$$

We have the following existence results for (4.1) and (4.3):

Theorem 4.1. *Given $\alpha > 0$, there exists a solution $u \in C^\infty(0,1]$ to the Neumann problem* (4.1)*.*

Theorem 4.2. *Given $\alpha > 0$, there exists a solution $u \in C^\infty(0,1]$ to the Dirichlet problem* (4.3)*.*

Remark 4.1. *The solutions given by theorems 4.1 and 4.2 are unique. This has already been proved in Chapter 3.*

Remark 4.2. *As one will see in the proof, when $\alpha \geq \frac{1}{2}$, the solution of* (4.3) *is a constant multiple of the solution of* (4.1) *and the constant only depends on α. Therefore, when $\alpha \geq \frac{1}{2}$, the boundary regularity of the solutions to both problems is automatically determined by the weight function ϕ_α given by* (4.4)*.*

Remark 4.3. *When $0 < \alpha < \frac{1}{2}$, by introducing a new unknown (e.g. $\tilde{u} = u - \frac{x^{1-2\alpha}-1}{1-2\alpha}$ for equation* (4.1) *and $\tilde{u} = u + (x^2 - 1)$ for equation* (4.3)*), both problems can be rewritten into the corresponding homogeneous problems with a right-hand side $f \in L^2(0,1)$, and therefore the existence, uniqueness and regularity results from Chapter 3 readily apply. However, in this case, we still provide a proof of independent interest for the Neumann problem via the Fredholm Alternative.*

4.2 Proof of the theorems

Proof of Theorem 4.1 when $0 < \alpha < 1$. Let $0 < \alpha < 1$ and $1 < p < \frac{1}{\alpha}$. We introduce the following functional framework. Recall the following functional space defined in

Chapter 3,

$$X^{\alpha,p}_{\cdot 0}(0,1) = \left\{ u \in W^{1,p}_{loc}(0,1);\ u \in L^p(0,1), x^\alpha u' \in L^p(0,1), u(1) = 0 \right\},$$

equipped with the (equivalent) norm $|u|_{\alpha,p} := \|x^\alpha u'\|_p$ (Theorem 3.33). Define $E = X^{\alpha,p}_{\cdot 0}(0,1)$ and $F = X^{\alpha,p'}_{\cdot 0}(0,1)$ and notice that since $1 < p < \infty$, both E and F are reflexive Banach spaces.

For $u \in E$ and $v \in F$, we define $B : E \longmapsto F^*$ by

$$B(u)v = \int_0^1 x^{2\alpha} u'(x) v'(x) dx.$$

We claim that B is an isomorphism. Clearly B is a linear bounded map with $\|B(u)\|_{F^*} \leq \|u\|_E$, so we only need to prove its invertibility.

To prove the surjectivity of B, consider the adjoint operator $B^* : F \longmapsto E^*$ given by $B^*(v)u = B(u)v$. It suffices to show that (see e.g. Theorem 2.20 in [8]) $\|v\|_F \leq \|B^*(v)\|_{E^*}$. Indeed, let g be any function in $L^p(0,1)$ with $\|g\|_p = 1$, and consider $u_g(x) := -\int_x^1 s^{-\alpha} g(s) ds$. Notice that $x^\alpha u_g'(x) = g$ and $u(1) = 0$, thus $\|u_g\|_E = \left\|x^\alpha u_g'\right\|_p = \|g\|_p = 1$. Therefore $u_g \in E$ and by definition we have

$$\begin{aligned} \|B^* v\|_{E^*} &\geq B^*(v) u_g \\ &= B(u_g) v \\ &= \int_0^1 x^{2\alpha} u_g'(x) v'(x) dx \\ &= \int_0^1 x^\alpha v'(x) g(x) dx. \end{aligned}$$

Since the above inequality holds for all $g \in L^p(0,1)$ with $\|g\|_p = 1$, taking supremum over all such g yields $\|v\|_F = \|x^\alpha v'\|_{p'} \leq \|B^* v\|_{E^*}$ as claimed.

To prove the injectivity of B, assume that $B(u) = \int_0^1 x^{2\alpha} u'(x) v'(x) dx = 0$ for all $v \in F$. Taking $v \in C_c^\infty(0,1) \subset F$ implies that $x^{2\alpha} u'(x) = C$ for some constant C. Furthermore, by taking $v \in C^\infty[0,1]$ with $v(0) = 1$ and $v(1) = 0$ gives that $C = 0$. Hence u is constant and it must be zero.

Next, we define $K : E \longmapsto F^*$ by

$$K(u)v = \int_0^1 u(x) v(x) dx.$$

Clearly this is a bounded linear map, with $\|K(u)\|_{F^*} \leq C\|u\|_E$. Also since the embedding $E \hookrightarrow L^p(0,1)$ is compact when $\alpha < 1$ (Theorem 3.35), we obtain that K is a compact operator.

Finally, consider the operator $A : E \longmapsto F^*$ defined by $A := B + K$. Then, the Fredholm Alternative theorem (see e.g. Theorem 6.6 in [8]) applies to the map $\tilde{A} : E \longmapsto E$ defined by $\tilde{A} := B^{-1} \circ A = Id + B^{-1} \circ K$ and we obtain

$$R(A) = R(\tilde{A}) = N(\tilde{A}^*)^\perp = N(A^*)^\perp.$$

We claim that $N(A^*) = \{0\}$. Indeed, $A^*v = 0$ is equivalent to

$$\int_0^1 x^{2\alpha} u'(x)v'(x)dx + \int_0^1 u(x)v(x)dx = 0,$$

for all $u \in E$. By taking $u \in C_c^\infty(0,1)$ we obtain that $(x^{2\alpha}v'(x))' = v(x)$. Taking u in $C^\infty[0,1]$ with $u(1) = 0$ and $u(0) = 1$ implies that $\lim_{x\to 0^+} x^{2\alpha}v'(x) = 0$. Since $v \in F$ we have that $v(1) = 0$. That is, v satisfies equation (3.1) with the homogeneous Neumann boundary condition as in Chapter 3. Hence the uniqueness result applies and we obtain $v \equiv 0$. This proves that $N(A^*) = \{0\}$, which implies $R(A) = F^*$. Therefore the equation $Au = \phi$ is uniquely solvable in E for all $\phi \in F^*$.

Using the above framework, take $\phi(v) = -v(0)$, $\forall v \in F$. Since $1 < p < \frac{1}{\alpha}$, we can apply Theorem 3.34, and obtain that the space F is continuously embedded into $C[0,1]$, so $\phi \in F^*$. Then a direct computation shows that the solution $u \in E$ of $Au = \phi$ is in fact in $C^\infty(0,1]$ and it satisfies (4.1). $\square$

Proof of Theorem 4.1 when $\alpha = 1$. One can directly check that $u(x) = -\frac{2}{1+\sqrt{5}} x^{\frac{-1-\sqrt{5}}{2}} + \frac{2}{1+\sqrt{5}} x^{\frac{-1+\sqrt{5}}{2}}$ solves

$$\begin{cases} -(x^2u'(x))' + u(x) = 0 & \text{on } (0,1), \\ u(1) = 0, \\ \lim\limits_{x\to 0^+} x^{\frac{3+\sqrt{5}}{2}} u'(x) = 1. \end{cases}$$

$\square$

Proof of Theorem 4.1 when $\alpha > 1$. Define[1]

$$I(x) := x^{1-2\alpha} \int_{-1}^{1} (1-t^2)^{\frac{\alpha}{2(\alpha-1)}} e^{\frac{tx^{1-\alpha}}{\alpha-1}} dt$$

and

$$A = -(\alpha-1)^{\frac{3\alpha-2}{2\alpha-2}} 2^{\frac{\alpha}{2(\alpha-1)}} \Gamma\left(\frac{3\alpha-2}{2\alpha-2}\right).$$

We claim that

$$\begin{cases} -(x^{2\alpha} I'(x))' + I(x) = 0 \text{ on } (0,1], \\ \lim_{x\to 0^+} x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} I'(x) = A. \end{cases}$$

Indeed, it is straightforward to check that $-(x^{2\alpha} I'(x))' + I(x) = 0$ on $(0,1]$. Moreover, the dominated convergence theorem implies that, as $x \to 0^+$,

$$\begin{aligned} & x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} I'(x) \\ =&(1-2\alpha)x^{\alpha-1}(\alpha-1)^{\frac{3\alpha-2}{2\alpha-2}} \int_{\frac{-2x^{1-\alpha}}{\alpha-1}}^{0} (-2r-(\alpha-1)r^2x^{\alpha-1})^{\frac{\alpha}{2(\alpha-1)}} e^r dr \\ &-(\alpha-1)x^{\alpha-1}(\alpha-1)^{\frac{3\alpha-2}{2\alpha-2}} \int_{\frac{-2x^{1-\alpha}}{\alpha-1}}^{0} r(-2r-(\alpha-1)r^2x^{\alpha-1})^{\frac{\alpha}{2(\alpha-1)}} e^r dr \\ &-(\alpha-1)^{\frac{3\alpha-2}{2\alpha-2}} \int_{\frac{-2x^{1-\alpha}}{\alpha-1}}^{0} (-2r-(\alpha-1)r^2x^{\alpha-1})^{\frac{\alpha}{2(\alpha-1)}} e^r dr \\ \to& -(\alpha-1)^{\frac{3\alpha-2}{2\alpha-2}} \int_{-\infty}^{0} (-2r)^{\frac{\alpha}{2(\alpha-1)}} e^r dr \\ =&A. \end{aligned}$$

From Theorems 3.14 and 3.15, we know that there exists a unique solution $w \in C^\infty(0,1]$ for the homogeneous equation

$$\begin{cases} -(x^{2\alpha} w'(x))' + w(x) = \dfrac{I(1)}{A} \quad \text{on } (0,1), \\ w(1) = 0, \\ \lim_{x\to 0^+} x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} w'(x) = 0. \end{cases}$$

Therefore, by linearity, $u(x) = w(x) + \frac{(I(x)-I(1))}{A} \in C^\infty(0,1]$ solves (4.1) for $\alpha > 1$. □

[1] A variant of this function can be found in Chapter III of [46], page 79.

Proof of Theorem 4.2 when $0 < \alpha < \frac{1}{2}$. From Theorems 3.1 and 3.2 we know that there is a unique function $w \in C^\infty(0,1]$ solving

$$\begin{cases} -(x^{2\alpha}w'(x))' + w(x) = -2(2\alpha+1)x^{2\alpha} + (x^2-1) & \text{on } (0,1), \\ w(0) = w(1) = 0. \end{cases}$$

Then by linearity, $u(x) = w(x) - (x^2 - 1)$ solves

$$\begin{cases} -(x^{2\alpha}w'(x))' + w(x) = 0 & \text{on } (0,1), \\ w(1) = 0, \\ w(0) = 1. \end{cases}$$

□

Proof of Theorem 4.2 when $\frac{1}{2} \le \alpha < 1$. We know from Theorem 4.1 that there exists $w \in C^\infty(0,1]$ solving the Neumann problem

$$\begin{cases} -(x^{2\alpha}w'(x))' + w(x) = 0 & \text{on } (0,1), \\ w(1) = 0, \\ \lim_{x\to 0^+} x^{2\alpha}w'(x) = 1. \end{cases} \tag{4.5}$$

Define

$$u(x) = \begin{cases} (1-2\alpha)w(x) & \text{when } \frac{1}{2} < \alpha < 1, \\ -w(x) & \text{when } \alpha = \frac{1}{2}. \end{cases}$$

We claim u solves

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 & \text{on } (0,1), \\ u(1) = 0, \\ \lim_{x\to 0^+} x^{2\alpha-1}u(x) = 1. \end{cases}$$

Indeed, from (4.5) we know that there exists $0 < \epsilon_0 < 1$ so that

$$\frac{1}{2x^{2\alpha}} \le w'(x) \le \frac{3}{2x^{2\alpha}}, \quad \forall 0 < x < \epsilon_0.$$

Since $\frac{1}{2} \le \alpha < 1$, by integrating the above inequality, we obtain that

$$\lim_{x\to 0^+} |u(x)| = \lim_{x\to 0^+} |w(x)| = \infty.$$

Therefore L'Hopital's rule applies, and we obtain that

$$\lim_{x\to 0^+} x^{2\alpha-1}u(x) = \lim_{x\to 0^+} \frac{x^{2\alpha}u'(x)}{1-2\alpha} = 1, \text{ when } \frac{1}{2} < \alpha < 1,$$

and

$$\lim_{x\to 0^+} \frac{u(x)}{1-\ln x} = -\lim_{x\to 0^+} xu'(x) = 1, \text{ when } \alpha = \frac{1}{2}.$$

□

Proof of Theorem 4.2 when $\alpha = 1$. One can directly check that $u(x) = x^{\frac{-1-\sqrt{5}}{2}} - x^{\frac{-1+\sqrt{5}}{2}}$ solves

$$\begin{cases} -(x^2u'(x))' + u(x) = 0 \quad \text{ on } (0,1), \\ u(1) = 0, \\ \lim_{x\to 0^+} x^{\frac{1+\sqrt{5}}{2}} u(x) = 1. \end{cases}$$

□

Proof of Theorem 4.2 when $\alpha > 1$. We know from Theorem 4.1 that there exists $w \in C^\infty(0,1]$ solving the Neumann problem

$$\begin{cases} -(x^{2\alpha}w'(x))' + w(x) = 0 \quad \text{ on } (0,1), \\ w(1) = 0, \\ \lim_{x\to 0^+} x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} w'(x) = 1. \end{cases}$$

Define $u(x) = -w(x)$. We claim that w solves

$$\begin{cases} -(x^{2\alpha}u'(x))' + u(x) = 0 \quad \text{ on } (0,1), \\ u(1) = 0, \\ \lim_{x\to 0^+} x^{\frac{\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u(x) = 1. \end{cases}$$

Indeed, from the boundary condition $\lim_{x\to 0^+} x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} w'(x) = 1$ we know that

$$\lim_{x\to 0^+} |u(x)| = \lim_{x\to 0^+} |w(x)| = \infty.$$

Therefore L'Hopital's rule applies, and we obtain that

$$\lim_{x\to 0^+} x^{\frac{\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u(x) = \lim_{x\to 0^+} \frac{x^{\frac{3\alpha}{2}} e^{\frac{x^{1-\alpha}}{1-\alpha}} u'(x)}{-\frac{\alpha}{2}x^{\alpha-1} - 1} = 1.$$

□

Chapter 5

A singular Sturm-Liouville equation involving measure data

5.1 Introduction

In this chapter, we consider the following singular Sturm-Liouville equation

$$\begin{cases} -(|x|^{2\alpha}u')' + u = \mu & \text{on } (-1,1), \\ u(-1) = u(1) = 0. \end{cases} \tag{5.1}$$

Here we assume that $\alpha > 0$ and $\mu \in \mathcal{M}(-1,1)$, where $\mathcal{M}(-1,1)$ is the space of bounded Radon measures on the interval $(-1,1)$. An equivalent way is to view a bounded Radon measure μ as a bounded linear functional on $C_0[-1,1]$. That is,

$$\mathcal{M}(-1,1) = (C_0[-1,1])^*, \tag{5.2}$$

where

$$C_0[-1,1] = \{\zeta \in C[-1,1];\ \zeta(-1) = \zeta(1) = 0\}.$$

By a *solution* u of (5.1), we mean a function u such that

$$u \in L^1(-1,1) \cap W^{1,1}_{loc}([-1,1]\setminus\{0\}),\ |x|^{2\alpha}u' \in BV(-1,1), \tag{5.3}$$

and u satisfies (5.1) in the usual sense (i.e., in the sense of measures).

We warn the reader that in the case when $0 < \alpha < \frac{1}{2}$, although $u' \in L^1(-1,1)$ (because $BV(-1,1) \subset L^\infty(-1,1)$), we *cannot* conclude that $u \in W^{1,1}(-1,1)$, since u' is *not* necessarily the distributional derivative of u on $(-1,1)$. In fact, denote by Du the distributional derivative of u on $(-1,1)$, it is easy to check that

$$Du = u' + \left(\lim_{x\to 0^+} u(x) - \lim_{x\to 0^-} u(x)\right)\delta_0, \tag{5.4}$$

where δ_0 is the Dirac mass at 0.

In this chapter we investigate the following questions about equation (5.1).

(i) *Existence of a solution.* As we are going to see, equation (5.1) admits a solution for every measure μ when $0 < \alpha < 1$. This is not true anymore when $\alpha \geq 1$; for this case we will present in Theorem 5.4 a necessary and sufficient condition on μ for the existence of a solution.

(ii) *Uniqueness of a solution.* As we are going to see, equation (5.1) admits plenty of solutions when $0 < \alpha < 1$ even for $\mu = 0$. Therefore it is natural to introduce a mechanism which will select among all solutions the most "regular" one. This solution will be called the *good solution* and we will establish its uniqueness in Section 5.2.

(iii) *Elliptic regularization.* For any $0 < \epsilon < 1$, we consider the following regularized equation

$$\begin{cases} -((|x|+\epsilon)^{2\alpha}u_\epsilon')' + u_\epsilon = \mu \quad \text{ on } (-1,1), \\ u_\epsilon(-1) = u_\epsilon(1) = 0. \end{cases} \tag{5.5}$$

Note that by the theorem of Lax-Milgram there exists a unique solution $u_\epsilon \in H_0^1(-1,1)$ with $u_\epsilon' \in BV(-1,1)$. We will study in Section 5.5 the limiting behavior of the family $\{u_\epsilon\}_{\epsilon>0}$ as $\epsilon \to 0$.

We start with the definition of the *good solution* for (5.1) when $0 < \alpha < 1$.

Definition 5.1. *Let $0 < \alpha < 1$. A solution u of (5.1) is called a good solution if it satisfies in addition*

$$\begin{cases} \lim_{x\to 0^+} u(x) = \lim_{x\to 0^-} u(x), \ \text{when } 0<\alpha<\frac{1}{2}, \\ \lim_{x\to 0^+} \left(1+\ln\frac{1}{|x|}\right)^{-1} u(x) = \lim_{x\to 0^-} \left(1+\ln\frac{1}{|x|}\right)^{-1} u(x), \ \text{when } \alpha = \frac{1}{2}, \\ \lim_{x\to 0^+} |x|^{2\alpha-1}u(x) = \lim_{x\to 0^-} |x|^{2\alpha-1}u(x), \ \text{when } \frac{1}{2} < \alpha < 1. \end{cases} \tag{5.6}$$

Our first result concerns the question of uniqueness.

Theorem 5.2. *Assume $\mu \equiv 0$ in (5.1). When $\alpha \geq 1$, the only solution of (5.1) is $u \equiv 0$. When $0 < \alpha < 1$, the only good solution of (5.1) is $u \equiv 0$.*

Remark 5.1. *When $0 < \alpha < 1$ we will prove in Section 5.3 that the class of all solutions of (5.1) with $\mu = 0$ is a one-dimensional space.*

The following two theorems are about the question of existence.

Theorem 5.3. *Assume $0 < \alpha < 1$. For each $\mu \in \mathcal{M}(-1,1)$, there exists a (unique) good solution of (5.1). Moreover, the good solution satisfies*

(i) $\lim_{x\to 0}\left(1+\ln\frac{1}{|x|}\right)^{-1}u(x) = -\lim_{x\to 0^+}|x|u'(x) = \lim_{x\to 0^-}|x|u'(x) = \frac{\mu(\{0\})}{2}$ *for* $\alpha = \frac{1}{2}$,

(ii) $\lim_{x\to 0}|x|^{2\alpha-1}u(x) = -\lim_{x\to 0^+}\frac{|x|^{2\alpha}u'(x)}{2\alpha-1} = \lim_{x\to 0^-}\frac{|x|^{2\alpha}u'(x)}{2\alpha-1} = \frac{\mu(\{0\})}{4\alpha-2}$ *for* $\frac{1}{2} < \alpha < 1$,

(iii) $\|u\|_{L^1} \le \|\mu\|_{\mathcal{M}}$ *and* $\|u^+\|_{L^1} \le \|\mu^+\|_{\mathcal{M}}$ *for all* $0 < \alpha < 1$.

Theorem 5.4. *Assume $\alpha \ge 1$. For each $\mu \in \mathcal{M}(-1,1)$, there exists a (unique) solution of (5.1) if and only if $\mu(\{0\}) = 0$. Moreover, if the solution exists, it satisfies*

(i) $\lim_{x\to 0}|x|^{\alpha}u(x) = \lim_{x\to 0}|x|^{2\alpha}u'(x) = 0$,

(ii) $\|u\|_{L^1} \le \|\mu\|_{\mathcal{M}}$ *and* $\|u^+\|_{L^1} \le \|\mu^+\|_{\mathcal{M}}$.

Remark 5.2. *Given $\alpha > 0$, denote*

$$k_\alpha = \begin{cases} \sup\left\{\|u\|_{L^1}\,;\ \mu \in \mathcal{M}(-1,1) \text{ and } \|\mu\|_{\mathcal{M}} \le 1\right\}, & \text{if } 0 < \alpha < 1, \\ \sup\left\{\|u\|_{L^1}\,;\ \mu \in \mathcal{M}(-1,1),\ \|\mu\|_{\mathcal{M}} \le 1 \text{ and } \mu(\{0\}) = 0\right\}, & \text{if } \alpha \ge 1, \end{cases}$$

where u is the solution of (5.1) identified in Theorems 5.3 and 5.4. These two theorems imply that $k_\alpha \le 1$. In fact, we can further prove that $k_\alpha < 1$ when $0 < \alpha < 1$. On the other hand, $k_\alpha = 1$ when $\alpha \ge 1$. See Section 5.4 for the proof of this remark.

Remark 5.3. *Assertion (i) in Theorem 5.4 is optimal in the following sense. Fix $x \in \left(-\frac{1}{2},\frac{1}{2}\right)\setminus\{0\}$ and define*

$$J_\alpha(x) = \sup\left\{|x|^{2\alpha}\left|u'(x)\right|\,;\ \mu \in \mathcal{M}(-1,1),\ \|\mu\|_{\mathcal{M}} \le 1 \text{ and } \mu(\{0\}) = 0\right\},$$

$$\widetilde{J}_\alpha(x) = \sup\left\{|x|^{\alpha}|u(x)|;\ \mu \in \mathcal{M}(-1,1),\ \|\mu\|_{\mathcal{M}} \le 1 \text{ and } \mu(\{0\}) = 0\right\},$$

where u is the solution of (5.1) corresponding to μ and we assume that $|x|^{2\alpha}u'$ is right-continuous (or left-continuous). Then $0 < \delta_\alpha \le J_\alpha(x) \le C_\alpha$, $\forall x \in \left(-\frac{1}{2},\frac{1}{2}\right)\setminus\{0\}$, and $0 < \delta_\alpha \le \widetilde{J}_\alpha(x) \le C_\alpha$, $\forall x \in \left(-\frac{1}{2},\frac{1}{2}\right)\setminus\{0\}$, where δ_α and C_α are constants depending only on α. See Section 5.4 for the proof of this remark.

Next, we consider the family $\{u_\epsilon\}_{\epsilon>0}$ where u_ϵ is the unique solution of the regularized equation (5.5) and our main results are the following two theorems.

Theorem 5.5. *Assume* $0 < \alpha < 1$. *Then as* $\epsilon \to 0$, $u_\epsilon \to u$ *uniformly on every compact subset of* $[-1,1]\backslash\{0\}$, *where* u *is the unique good solution of* (5.1).

Theorem 5.6. *Assume* $\alpha \geq 1$. *Then as* $\epsilon \to 0$, $u_\epsilon \to u$ *uniformly on every compact subset of* $[-1,1]\backslash\{0\}$, *where* u *is the unique solution of*

$$\begin{cases} -(|x|^{2\alpha}u')' + u = \mu - \mu(\{0\})\,\delta_0 & \text{on } (-1,1), \\ u(-1) = u(1) = 0. \end{cases} \tag{5.7}$$

Remark 5.4. *In Section 5.5 we will present further results about the mode of convergence in Theorems 5.5 and 5.6.*

Remark 5.5. *The stability of the good solution when* $\frac{1}{2} \leq \alpha < 1$ *is a delicate subject. For example, let* $\mu = \delta_0$ *and let* $f_n(x) = Cn\rho(nx-1)$, *where* $\rho(x) = \chi_{[|x|<1]}e^{\frac{1}{|x|^2-1}}$ *and* $C^{-1} = \int \rho$, *so that* $f_n \overset{*}{\rightharpoonup} \delta_0$ *in* $(C_0[-1,1])^*$. *Denote by* u_n *the unique good solution corresponding to* f_n. *Then* $u_n \to u$ *but* u *is not the good solution corresponding to* δ_0. *This subject will be discussed in Section 5.6.*

Remark 5.6. *Given* $\mu \in \mathcal{M}(0,1)$, *we can also study the equation*

$$\begin{cases} -(x^{2\alpha}u')' + u = \mu & \text{on } (0,1), \\ u(1) = 0. \end{cases} \tag{5.8}$$

Section 5.7 is devoted to equation (5.8) *under several appropriate boundary conditions at 0.*

In Chapter 6, we will study the above-mentioned questions for the following semilinear singular Sturm-Liouville equation with $\alpha > 0$ and $1 < p < \infty$,

$$\begin{cases} -(|x|^{2\alpha}u')' + |u|^{p-1}u = \mu & \text{on } (-1,1), \\ u(-1) = u(1) = 0. \end{cases} \tag{5.9}$$

Our study of (5.1) and (5.9) is motivated by various results about the (semilinear) elliptic equation

$$\begin{cases} -\Delta u + |u|^{p-1}u = \mu \quad \text{on } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{5.10}$$

where $1 \leq p < \infty$, Ω is a bounded smooth domain in $\mathbb{R}^N$ and μ is a bounded Radon measure on Ω. The linear case of (5.10) actually goes back to Stampacchia [36, 37] (see also Section 2 of Brezis-Strauss [13]).

For the semilinear case, the existence and uniqueness of an L^p-solution of (5.10) for all $1 < p < \infty$ and $\mu \in L^1(\Omega)$ is proved by Brezis-Strauss [13]. When μ is just a bounded Radon measure, the following two cases were studied separately:

(i) $1 < p < \frac{N}{N-2}$ if $N \geq 3$ and no restriction on p if $N = 1, 2$,

(ii) $p \geq \frac{N}{N-2}$ if $N \geq 3$.

Bénilan-Brezis proved the existence and uniqueness for case (i) and the nonexistence for case (ii) if $\mu = \delta_a$ for some $a \in \Omega$ (see, e.g., [4] and the references therein). For case (ii), a necessary and sufficient condition on μ for the existence of a solution was given by Baras-Pierre [2] (see an equivalent characterization by Gallouët-Morel [29]).

About the isolated (interior) singularity, Brezis-Véron [14] proved that the isolated singularity is removable for case (ii). For case (i), Véron [40] classified the asymptotic behavior of the solutions near the isolated singularity (a different proof was given by Brezis-Oswald [11]).

Brezis [7] observed that, for case (ii) with $\mu = \delta_a$ where $a \in \Omega$, a sequence of approximate solutions may converge to 0, which is obviously not the solution corresponding to $\mu = \delta_a$. This phenomenon was then studied by Brezis-Marcus-Ponce [10] in a more general setting.

We refer to Appendix A of Bénilan-Brezis [4] for a comprehensive review on this subject, and to the monographs of Véron [41, 42] for a variety of results about the singularities of solutions for more general classes of PDEs.

The rest of this chapter is organized as follows. We present in Section 5.2 some properties of the differential operator $(|x|^{2\alpha}u')'$, viewed as an unbounded linear operator on $L^1(-1,1)$. Theorem 5.2 will be a direct consequence of these properties. The non-uniqueness result when $0 < \alpha < 1$ will be established in Section 5.3. The existence results will be proved in Section 5.4. The elliptic regularization will be studied in Section 5.5. The lack of stability of the good solution when $\frac{1}{2} \leq \alpha < 1$ will be investigated in Section 5.6. Finally, equation (5.8) will be studied in Section 5.7.

5.2 An unbounded operator on $L^1(-1,1)$

In this section we consider the unbounded linear operator A_α: $D(A_\alpha) \subset L^1(-1,1) \to L^1(-1,1)$ where

$$A_\alpha u = -\left(|x|^{2\alpha} u'\right)', \tag{5.11}$$

$$\widetilde{D} = \left\{ u \in L^1(-1,1) \cap W^{2,1}_{loc}([-1,1] \setminus \{0\});\ u(1) = u(-1) = 0,\ |x|^{2\alpha} u' \in W^{1,1}(-1,1) \right\}, \tag{5.12}$$

and

$$D(A_\alpha) = \begin{cases} \widetilde{D} \cap C[-1,1], \text{ when } 0 < \alpha < \frac{1}{2}, \\ \widetilde{D} \cap \left\{ u;\ \left(1 + \ln \frac{1}{|x|}\right)^{-1} u \in C[-1,1] \right\}, \text{ when } \alpha = \frac{1}{2}, \\ \widetilde{D} \cap \left\{ u;\ |x|^{2\alpha - 1} u \in C[-1,1] \right\}, \text{ when } \frac{1}{2} < \alpha < 1, \\ \widetilde{D}, \text{ when } \alpha \geq 1. \end{cases} \tag{5.13}$$

We shall present several properties of the linear operator A_α which will be needed to establish the main results stated in the introduction.

Proposition 5.7. *The operator A_α satisfies the following properties.*

(i) For any $\alpha > 0$, the operator A_α is closed and its domain $D(A_\alpha)$ is dense in $L^1(-1,1)$.

(ii) For any $\lambda > 0$ and $\alpha > 0$, $I + \lambda A_\alpha$ maps $D(A_\alpha)$ one-to-one onto $L^1(-1,1)$ and $(I + \lambda A_\alpha)^{-1}$ is a contraction in $L^1(-1,1)$.

(iii) For any $\lambda > 0$, $\alpha > 0$ and $f \in L^1(-1,1)$, $\operatorname{ess\,sup}(I + \lambda A_\alpha)^{-1} f \leq \max\{0, \operatorname{ess\,sup} f\}$.

(iv) Let γ be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ containing the origin. For any $\alpha > 0$, let $u \in D(A_\alpha)$ and $g \in L^\infty(-1,1)$ be such that $g(x) \in \gamma(u(x))$ a.e. Then $\int_{-1}^{1} A_\alpha u(x) g(x) dx \geq 0$.

To prove Proposition 5.7, we start with two lemmas concerning the properties of the functions in the domain $D(A_\alpha)$.

Lemma 5.8. *Assume $0 < \alpha < \frac{1}{2}$. For $u \in D(A_\alpha)$ we have*

$$\lim_{x \to 0} |x|^{2\alpha} u'(x) = \frac{1}{2} \int_0^1 (A_\alpha u) \left(1 - s^{1-2\alpha}\right) ds - \frac{1}{2} \int_{-1}^0 (A_\alpha u) \left(1 - |s|^{1-2\alpha}\right) ds, \tag{5.14}$$

$$u(0) = \frac{1}{2(1-2\alpha)} \int_{-1}^1 (A_\alpha u) \left(1 - |s|^{1-2\alpha}\right) ds, \tag{5.15}$$

$$\left\| |x|^{2\alpha} u' \right\|_{L^\infty} \leq \frac{3}{2} \left\| A_\alpha u \right\|_{L^1}, \tag{5.16}$$

$$\|u\|_{W^{1,1}} \leq \frac{6}{1-2\alpha} \left\| A_\alpha u \right\|_{L^1}. \tag{5.17}$$

Proof. Given $u \in D(A_\alpha)$, we denote $K = \lim_{x \to 0} |x|^{2\alpha} u'(x)$. Then,

$$|x|^{2\alpha} u'(x) = -\int_0^x A_\alpha u(s) ds + K.$$

For $x \in (0,1)$, this implies that

$$u(x) = \frac{1 - x^{1-2\alpha}}{1-2\alpha} \int_0^x A_\alpha u(s) ds + \frac{1}{1-2\alpha} \int_x^1 A_\alpha u(s) \left(1 - s^{1-2\alpha}\right) ds - \frac{K(1 - x^{1-2\alpha})}{1-2\alpha}.$$

On the other hand, for $x \in (-1,0)$, we obtain that

$$u(x) = \frac{1 - |x|^{1-2\alpha}}{1-2\alpha} \int_x^0 A_\alpha u(s) ds + \frac{1}{1-2\alpha} \int_{-1}^x A_\alpha u(s) \left(1 - |s|^{1-2\alpha}\right) ds + \frac{K(1 - |x|^{1-2\alpha})}{1-2\alpha}.$$

Since $0 < \alpha < \frac{1}{2}$, the relation $u(0^+) = u(0^-)$ yields (5.14). The rest of the proof follows directly. ☐

Lemma 5.9. *Assume $\alpha \geq \frac{1}{2}$. Then*

$$D(A_\alpha) = \left\{ u \in \widetilde{D};\ \lim_{x \to 0} |x|^{2\alpha} u'(x) = 0 \right\}, \tag{5.18}$$

where $\widetilde{D}$ is defined by (5.12). For $u \in D(A_\alpha)$ we have

$$\left\| |x|^{2\alpha} u' \right\|_{L^\infty} \le \|A_\alpha u\|_{L^1}, \text{ when } \alpha \ge \frac{1}{2}, \tag{5.19}$$

$$\lim_{x\to 0} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u(x) = 0, \text{ when } \alpha = \frac{1}{2}, \tag{5.20}$$

$$\left\| \left(1 + \ln \frac{1}{|x|}\right)^{-1} u \right\|_{W^{1,1}} \le 4 \left\| A_{\frac{1}{2}} u \right\|_{L^1}, \text{ when } \alpha = \frac{1}{2}, \tag{5.21}$$

$$\lim_{x\to 0} |x|^{2\alpha-1} u(x) = 0, \text{ when } \alpha > \frac{1}{2}, \tag{5.22}$$

$$\left\| |x|^{2\alpha-1} u \right\|_{W^{1,1}} \le \frac{4}{2\alpha - 1} \|A_\alpha u\|_{L^1}, \text{ when } \alpha > \frac{1}{2}. \tag{5.23}$$

Proof. Given $u \in D(A_\alpha)$, we denote $K = \lim_{x\to 0} |x|^{2\alpha} u'(x)$. We claim that $K = 0$ if and only if $\left(1 + \ln \frac{1}{|x|}\right)^{-1} u \in C[-1,1]$ for $\alpha = \frac{1}{2}$ and $|x|^{2\alpha-1} u \in C[-1,1]$ for $\alpha > \frac{1}{2}$.

When $\alpha = \frac{1}{2}$, integration by parts yields

$$u(x) = \ln \frac{1}{x} \int_0^x A_\alpha u(s) ds + \int_x^1 A_\alpha u(s) \ln \frac{1}{s} ds - K \ln \frac{1}{x}, \ \forall x \in (0,1), \tag{5.24}$$

$$u(x) = \ln \frac{1}{|x|} \int_x^0 A_\alpha u(s) ds + \int_{-1}^x A_\alpha u(s) \ln \frac{1}{|s|} ds + K \ln \frac{1}{|x|}, \ \forall x \in (-1,0). \tag{5.25}$$

Notice that

$$\begin{aligned} &\lim_{x\to 0^+} \left(1 + \ln \frac{1}{x}\right)^{-1} \int_x^1 |A_\alpha u(s)| \ln \frac{1}{s} ds \\ &\le \lim_{x\to 0^+} \left(\frac{\ln(1 - \ln x)}{1 - \ln x} \int_{\frac{1}{1-\ln x}}^1 |A_\alpha u(s)|\, ds + \frac{-\ln x}{1 - \ln x} \int_x^{\frac{1}{1-\ln x}} |A_\alpha u(s)|\, ds \right) = 0. \end{aligned}$$

Similarly,

$$\lim_{x\to 0^-} \left(1 + \ln \frac{1}{|x|}\right)^{-1} \int_{-1}^x |A_\alpha u(s)| \ln \frac{1}{|s|} ds = 0.$$

Therefore,

$$-\lim_{x\to 0^+} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u(x) = \lim_{x\to 0^-} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u(x) = K.$$

Thus $K = 0$ if and only if $\left(1 + \ln \frac{1}{x}\right)^{-1} u \in C[-1,1]$.

When $\alpha > \frac{1}{2}$, the same computation implies that, for all $x \in (0,1)$,

$$u(x) = \frac{1 - x^{1-2\alpha}}{1 - 2\alpha} \int_0^x A_\alpha u(s) ds + \int_x^1 \frac{A_\alpha u(s) \left(1 - s^{1-2\alpha}\right)}{1 - 2\alpha} ds - \frac{K(1 - x^{1-2\alpha})}{1 - 2\alpha}, \tag{5.26}$$

and, for all $x \in (-1,0)$,

$$u(x) = \frac{1-|x|^{1-2\alpha}}{1-2\alpha}\int_x^0 A_\alpha u(s)ds + \int_{-1}^x \frac{A_\alpha u(s)\left(1-|s|^{1-2\alpha}\right)}{1-2\alpha}ds + \frac{K(1-|x|^{1-2\alpha})}{1-2\alpha}. \tag{5.27}$$

Notice that

$$\begin{aligned}&\lim_{x\to 0^+} x^{2\alpha-1}\int_x^1 |A_\alpha u(s)|\, s^{1-2\alpha}ds \\ \le &\lim_{x\to 0^+}\left(x^{\alpha-\frac{1}{2}}\int_{\sqrt{x}}^1 |A_\alpha u(s)|\, ds + \int_x^{\sqrt{x}} |A_\alpha u(s)|\, ds\right) = 0.\end{aligned}$$

Similarly,

$$\lim_{x\to 0^-} |x|^{2\alpha-1}\int_{-1}^x |A_\alpha u(s)|\, |s|^{1-2\alpha}ds = 0.$$

Therefore

$$-\lim_{x\to 0^+} |x|^{2\alpha-1}u(x) = \lim_{x\to 0^-} |x|^{2\alpha-1}u(x) = \frac{K}{2\alpha-1}. \tag{5.28}$$

Thus $K=0$ if and only if $|x|^{2\alpha-1}u \in C[-1,1]$.

Recall that $u \in D(A_\alpha)$ and thus $\left(1+\ln\frac{1}{|x|}\right)^{-1} u \in C[-1,1]$ when $\alpha = \frac{1}{2}$ and $|x|^{2\alpha-1}u \in C[-1,1]$ when $\frac{1}{2} < \alpha < 1$. When $\alpha \ge 1$, the fact that $u \in L^1(-1,1)$ together with the relation (5.28) implies that $K=0$. It completes the proof of (5.18). the rest of the proof follows easily. □

We now start to prove Proposition 5.7. The idea is similar as the one for Theorem 8 in Brezis-Strauss [13]. We denote

$$C_0^1[-1,1] = \left\{\zeta \in C^1[-1,1];\ \zeta(-1) = \zeta(1) = 0\right\}.$$

Proof of (i) of Proposition 5.7. It is clear that $D(A_\alpha)$ is dense in $L^1(-1,1)$. To prove that A_α is closed, we assume that there is a sequence $\{u_n\}_{n=1}^\infty$ in $D(A_\alpha)$ such that $u_n \to u$ in $L^1(-1,1)$ and $A_\alpha u_n \to f$ in $L^1(-1,1)$. We need to show that $u \in D(A_\alpha)$ and $A_\alpha u = f$. Denote $f_n = A_\alpha u_n$ and then $\{f_n\}_{n=1}^\infty$ is Cauchy in $L^1(-1,1)$. By (5.17), (5.21) and (5.23), we obtain

$$\|u_n - u_m\|_{W^{1,1}} \le \frac{6}{1-2\alpha}\|f_n - f_m\|_{L^1},\ \text{when } 0<\alpha<\frac{1}{2},$$

$$\left\|\left(1+\ln\frac{1}{|x|}\right)^{-1}(u_n - u_m)\right\|_{L^\infty} \le 4\|f_n - f_m\|_{L^1},\ \text{when } \alpha = \frac{1}{2},$$

$$\left\| |x|^{2\alpha-1} (u_n - u_m) \right\|_{L^\infty} \le \frac{4}{2\alpha - 1} \|f_n - f_m\|_{L^1}, \text{ when } \alpha > \frac{1}{2}.$$

These inequalities imply that $u_n \to u$ in $W_0^{1,1}(-1,1)$ if $0 < \alpha < \frac{1}{2}$, $(1 - \ln|x|)^{-1} u_n \to (1 - \ln|x|)^{-1} u$ in $C_0[-1,1]$ if $\alpha = \frac{1}{2}$ and $|x|^{2\alpha-1} u_n \to |x|^{2\alpha-1} u$ in $C_0[-1,1]$ if $\alpha > \frac{1}{2}$. To conclude that $u \in D(A_\alpha)$, we still need to show that $|x|^{2\alpha} u' \in W^{1,1}(-1,1)$. Notice that from (5.16) and (5.19) we obtain

$$\left\| |x|^{2\alpha} u_n' - |x|^{2\alpha} u_m' \right\|_{L^\infty} \le \frac{3}{2} \|f_n - f_m\|_{L^1}.$$

This implies that $|x|^{2\alpha} u_n' \to |x|^{2\alpha} u'$ in $C[-1,1]$. We can rewrite the identity $f_n = A_\alpha u_n$ as

$$\int_{-1}^{1} |x|^{2\alpha} u_n' \zeta' dx = \int_{-1}^{1} f_n \zeta dx, \ \forall \zeta \in C_0^1[-1,1].$$

Passing to the limit as $n \to \infty$, we obtain

$$\int_{-1}^{1} |x|^{2\alpha} u' \zeta' dx = \int_{-1}^{1} f \zeta dx, \ \forall \zeta \in C_0^1[-1,1].$$

Thus $|x|^{2\alpha} u' \in W^{1,1}(-1,1)$ and $A_\alpha u = f$. □

The key ingredients in the proof of (ii) of Proposition 5.7 are the duality of L^1 and L^∞, and the following maximum principle.

Lemma 5.10. *Let $\lambda > 0$ and $\alpha > 0$. For $g \in L^\infty(-1,1)$, there exists a function $u \in D(A_\alpha)$ such that $(I + \lambda A_\alpha) u = g$ and*

$$\min\{0, \operatorname{ess\,inf} g\} \le u \le \max\{0, \operatorname{ess\,sup} g\}. \tag{5.29}$$

Proof. Consider the Hilbert space

$$\begin{aligned} &X_0^\alpha(-1,1) \\ &= \left\{ u \in L^2(-1,1) \cap H^1_{loc}([-1,1] \setminus \{0\})\, ;\ u(1) = u(-1) = 0,\ |x|^\alpha u' \in L^2(-1,1) \right\}, \end{aligned}$$

with the inner product

$$(u,v)_\alpha = \int_{-1}^{1} |x|^{2\alpha} u' v' dx + \int_{-1}^{1} uv dx.$$

All the properties listed in Section 3.6 for the space $X_0^\alpha(0,1)$ can be inherited by $X_0^\alpha(-1,1)$ with some obvious changes. In particular, when $0 < \alpha < \frac{1}{2}$, Theorem

3.34 implies that every function u in $X_0^\alpha(-1,1)$ is continuous on the intervals $[-1,0]$ and $[0,1]$. As a consequence, we can define

$$H_\alpha = \begin{cases} X_0^\alpha(-1,1) \cap C[-1,1], \text{ when } 0 < \alpha < \frac{1}{2}, \\ X_0^\alpha(-1,1), \text{ when } \alpha \geq \frac{1}{2}. \end{cases} \tag{5.30}$$

It is closed in $X_0^\alpha(-1,1)$ and therefore it is a Hilbert space. Then the Lax-Milgram theorem yields that there exists an $u \in H_\alpha$ such that

$$\lambda \int_{-1}^{1} |x|^{2\alpha} u'v'dx + \int_{-1}^{1} uvdx = \int_{-1}^{1} gvdx, \; \forall v \in H_\alpha. \tag{5.31}$$

Choosing $v \in C_c^\infty(-1,1)$ it follows that $|x|^{2\alpha}u' \in H^1(-1,1)$ and $(I + \lambda A_\alpha)\, u = g$. When $0 < \alpha < \frac{1}{2}$, we obtain that $u' \in L^1(-1,1)$ since $|x|^{2\alpha}u' \in L^\infty(-1,1)$. Therefore $u \in D(A_\alpha)$. When $\alpha \geq \frac{1}{2}$, the existence results in Chapter 3 (Theorems 3.7, 3.11 and 3.14) imply that the solution given by (5.31) satisfies $\lim_{x\to 0} |x|^{2\alpha}u'(x) = 0$, so we deduce that $u \in D(A_\alpha)$ by (5.18). In order to prove (5.29), we use the Stampacchia's truncation method. Set $K = \max\{0, \operatorname{ess\,sup} g\}$ and take $v(x) = (u(x) - K)^+$ in (5.31). The rest of the proof is the same as the one for Theorem 8.19 in [8]. □

Proof of (ii) of Proposition 5.7. We first prove that $I + \lambda A_\alpha$ is one-to-one from $D(A_\alpha)$ to $L^1(-1,1)$. Assume $u \in D(A_\alpha)$ such that $(I + \lambda A_\alpha)\, u = 0$. We claim that $u = 0$.

For the case $0 < \alpha < \frac{1}{2}$, we argue by duality. Notice that

$$\lambda \int_{-1}^{1} |x|^{2\alpha} u'v'dx + \int_{-1}^{1} uvdx = 0, \; \forall v \in C_c^\infty(-1,1).$$

Since $C_c^\infty(-1,1)$ is dense in $W_0^{1,1}(-1,1)$, we find that

$$\lambda \int_{-1}^{1} |x|^{2\alpha} u'v'dx + \int_{-1}^{1} uvdx = 0, \; \forall v \in W_0^{1,1}(-1,1).$$

By Lemma 5.10, for any $g \in L^\infty(-1,1)$, there exists $v \in H_\alpha \subset W_0^{1,1}(-1,1)$ such that

$$\lambda \int_{-1}^{1} |x|^{2\alpha} w'v'dx + \int_{-1}^{1} wvdx = \int_{-1}^{1} gwdx, \; \forall w \in H_\alpha,$$

where H_α is defined in (5.30). Since $u \in D(A_\alpha) \subset H_\alpha$, take $w = u$ in the above identity. We deduce that $\int_{-1}^{1} gudx = 0$. As g is arbitrary in L^∞, u must be identically zero.

For the case $\alpha \geq \frac{1}{2}$, by (5.18) we obtain that $u \in C^{\infty}([-1,1]\setminus\{0\})$ and it satisfies

$$\begin{cases} -\lambda(|x|^{2\alpha}u')' + u = 0 \quad \text{on } (0,1), \\ \lim_{x\to 0^+} |x|^{2\alpha}u'(x) = u(1) = 0, \end{cases}$$

and

$$\begin{cases} -\lambda(|x|^{2\alpha}u')' + u = 0 \quad \text{on } (-1,0), \\ \lim_{x\to 0^-} |x|^{2\alpha}u'(x) = u(-1) = 0. \end{cases}$$

By the uniqueness results in Chapter 3 (Theorems 3.8, 3.12 and 3.15), we obtain that $u = 0$.

Next we prove that $I + \lambda A_\alpha$ is surjective from $D(A_\alpha)$ to $L^1(-1,1)$ and

$$\left\| (I + \lambda A_\alpha)^{-1} f \right\|_{L^1} \leq \|f\|_{L^1}.$$

Given $f \in L^1(-1,1)$, we take a sequence $\{f_n\}_{n=1}^{\infty}$ in $L^{\infty}(-1,1)$ such that $f_n \to f$ in $L^1(-1,1)$. For each f_n, by Lemma 5.10 and identity (5.31), there is a function $u_n \in H_\alpha \cap D(A_\alpha)$ such that

$$\lambda \int_{-1}^{1} |x|^{2\alpha} u_n' v' dx + \int_{-1}^{1} u_n v dx = \int_{-1}^{1} f_n v dx, \ \forall v \in H_\alpha, \tag{5.32}$$

where H_α is defined in (5.30). On the other hand, Lemma 5.10 and identity (5.31) also imply that for any $g \in L^{\infty}(-1,1)$ there exists $v \in H_\alpha$ such that

$$\lambda \int_{-1}^{1} |x|^{2\alpha} u_n' v' dx + \int_{-1}^{1} u_n v dx = \int_{-1}^{1} g u_n dx, \tag{5.33}$$

and $\|v\|_{L^\infty} \leq \|g\|_{L^\infty}$. Combining identities (5.32) and (5.33), we have

$$\left| \int_{-1}^{1} g u_n dx \right| = \left| \int_{-1}^{1} f_n v dx \right| \leq \|f_n\|_{L^1} \|v\|_{L^\infty} \leq \|f_n\|_{L^1} \|g\|_{L^\infty}.$$

Hence

$$\|u_n\|_{L^1} \leq \|f_n\|_{L^1}. \tag{5.34}$$

Notice that identity (5.32) yields that $A_\alpha u_n = \frac{1}{\lambda}(f_n - u_n)$. Taking into account (5.16), (5.19) and (5.34), we obtain

$$\left\| |x|^{2\alpha} u_n' \right\|_{L^\infty} \leq \frac{3}{\lambda} \|f_n\|_{L^1}, \ \forall \alpha > 0.$$

From (5.17) and (5.34), one deduces that

$$\|u_n\|_{W^{1,1}} \leq \frac{12}{\lambda(1-2\alpha)} \|f_n\|_{L^1}, \text{ when } 0 < \alpha < \frac{1}{2}.$$

It follows that:

(i) $\{u_n\}_{n=1}^{\infty}$ is Cauchy in $W_0^{1,1}(-1,1)$ when $0 < \alpha < \frac{1}{2}$,

(ii) $\{u_n\}_{n=1}^{\infty}$ is Cauchy in $L^1(-1,1)$ when $\alpha \geq \frac{1}{2}$,

(iii) $\left\{|x|^{2\alpha}u_n'\right\}_{n=1}^{\infty}$ is Cauchy in $C[-1,1]$ for all $\alpha > 0$.

Passing to the limit in (5.32) as $n \to \infty$, we have

$$\lambda \int_{-1}^{1} |x|^{2\alpha} u' \zeta' dx + \int_{-1}^{1} u\zeta dx = \int_{-1}^{1} f\zeta dx, \ \forall \zeta \in C_0^1[-1,1],$$

where $u \in W_0^{1,1}(-1,1)$ when $0 < \alpha < \frac{1}{2}$ and $\lim_{x\to 0} |x|^{2\alpha} u'(x) = 0$ when $\alpha \geq \frac{1}{2}$. Therefore $u \in D(A_\alpha)$, $(I + \lambda A_\alpha)\, u = f$ and $\left\|(I + \lambda A_\alpha)^{-1} f\right\|_{L^1} \leq \|f\|_{L^1}$. □

Proof of (iii) of Proposition 5.7. Let $f \in L^1(-1,1)$ and $u = (I + \lambda A)^{-1} f$. If $\operatorname{ess\,sup} f = +\infty$, there is nothing to prove, so we assume that $\operatorname{ess\,sup} f$ is finite. Define $f_n = \max\{f, -n\}$. Then $f_n \in L^\infty(-1,1)$ and, for n large enough, $\operatorname{ess\,sup} f_n = \operatorname{ess\,sup} f$. Take $u_n = (I + \lambda A)^{-1} f_n$ and Lemma 5.10 implies that

$$u_n \leq \max\{0, \operatorname{ess\,sup} f_n\} = \max\{0, \operatorname{ess\,sup} f\}.$$

Notice that

$$\|f_n - f\|_{L^1} = \int_{[f<-n]} (-n - f) \leq \int_{[f<-n]} |f| \to 0.$$

On the other hand, $\|u_n - u\|_{L^1} \leq \|f_n - f\|_{L^1}$ since $(I + \lambda A)^{-1}$ is a contraction. Therefore $u_n \to u$ in $L^1(-1,1)$ and $\operatorname{ess\,sup} u \leq \max\{0, \operatorname{ess\,sup} f\}$. □

Proof of (iv) of Proposition 5.7. Just apply Lemma 2 of Brezis-Strauss [13]. □

We conclude this section with the

Proof of Theorem 5.2. Assertion (ii) in Proposition 5.7 implies that the map $I + A_\alpha$ is one-to-one from $D(A_\alpha)$ to $L^1(-1,1)$. Therefore Theorem 5.2 follows. □

5.3 Non-uniqueness when $0 < \alpha < 1$

In this section we present a complete description of all solutions of (5.1) when $\mu = 0$ and $0 < \alpha < 1$. Throughout this section we assume $0 < \alpha < 1$.

We know from Theorem 4.1 that there exists a unique function $V \in C^\infty(0,1] \cap L^1(0,1)$ such that

$$\begin{cases} -(x^{2\alpha}V')' + V = 0 \quad \text{on } (0,1), \\ V(1) = 0, \\ \lim\limits_{x\to 0^+} x^{2\alpha}V'(x) = 1. \end{cases} \tag{5.35}$$

Set

$$U(x) = \begin{cases} V(x) \quad \text{for } x \in (0,1), \\ -V(-x) \quad \text{for } x \in (-1,0). \end{cases} \tag{5.36}$$

We claim that U is a solution of (5.1) with $\mu = 0$. Indeed, since $|x|^{2\alpha}U' \in C[-1,1]$, we obtain that

$$\left(|x|^{2\alpha}U'\right)' = U \quad \text{in } \mathcal{D}'(-1,1)$$

and thus $|x|^{2\alpha}U' \in W^{1,1}(-1,1)$ with $\left(|x|^{2\alpha}U'(x)\right)' = U(x)$. However, U is *not* a good solution. Otherwise we could apply Theorem 5.2 and conclude that $U \equiv 0$. This is impossible since $\lim\limits_{x\to 0} |x|^{2\alpha}U'(x) = 1$.

Using this function U we may now describe *all* solutions of (5.1) with $\mu = 0$.

Theorem 5.11. *A function u is a solution of (5.1) with $\mu = 0$ if and only if*

$$u = \tau U$$

for some $\tau \in \mathbb{R}$.

Proof. By linearity, since U is a solution of (5.1) with $\mu = 0$, then τU is a solution of (5.1) with $\mu = 0$. On the other hand, if u is a solution of (5.1) with $\mu = 0$, we have $|x|^{2\alpha}u' \in W^{1,1}(-1,1)$ and we denote $\lim\limits_{x\to 0} |x|^{2\alpha}u'(x) = \tau$. Then the function $v = u|_{(0,1)}$

satisfies

$$\begin{cases} -(x^{2\alpha}v')' + v = 0 \quad \text{on } (0,1), \\ v(1) = 0, \\ \lim_{x\to 0^+} x^{2\alpha}v'(x) = \tau. \end{cases}$$

By the uniqueness results in Chapter 3 (Theorems 3.5, 3.8 and 3.12), we deduce that

$$u = \tau U \quad \text{on } (0,1).$$

Similarly, $u = \tau U$ on $(-1,0)$. Thus $u = \tau U$. □

Remark 5.7. *The function V is strictly increasing and*

$$\lim_{x\to 0^+} V(x) = \begin{cases} V(0) < 0, \ \textit{if } 0 < \alpha < \frac{1}{2}, \\ -\infty, \ \textit{if } \frac{1}{2} \le \alpha < 1. \end{cases} \tag{5.37}$$

Proof. We first claim that $V' \ge 0$ on $(0,1)$. Indeed, integration by parts yields

$$\int_x^1 t^{2\alpha}\left(V'(t)\right)^2 dt + \int_x^1 V^2(t)dt = -\frac{1}{2}x^{2\alpha}\frac{d}{dx}\left(V(x)\right)^2, \ \forall x \in (0,1).$$

We deduce that $|V|$ is monotone and thus V doesn't change sign. Then V is also monotone. Recall that $\lim_{x\to 0^+} x^{2\alpha}V'(x) = 1$, we obtain that $V' \ge 0$ on $(0,1)$.

Next we claim that $V' > 0$ on $(0,1)$. Otherwise, denote

$$x_0 = \min\left\{x \in (0,1);\ V'(x) = 0\right\}.$$

We obtain that $x_0 \in (0,1)$ and $V(x_0) = V'(x_0) = 0$. The uniqueness of the initial value problem for V at x_0 implies that $V \equiv 0$ on some neighborhood of x_0. It contradicts the definition of x_0.

Then we prove (5.37). When $0 < \alpha < \frac{1}{2}$, the regularity results (Remark 4.3 and Theorem 3.4) imply that $V \in C[0,1]$. Obviously $V(0) < 0$. When $\frac{1}{2} \le \alpha < 1$, note that

$$1 = \lim_{x\to 0^+} x^{2\alpha}V'(x) = \begin{cases} -\lim_{x\to 0^+}\left(1 + \ln\frac{1}{x}\right)^{-1} V(x), \ \text{if } \alpha = \frac{1}{2}, \\ -(2\alpha - 1)\lim_{x\to 0^+} x^{2\alpha-1}V(x), \ \text{if } \frac{1}{2} < \alpha < 1. \end{cases}$$

Then (5.37) holds. □

5.4 Proof of the existence results

We start with the proof of Theorems 5.3 and 5.4. Given $\mu \in \mathcal{M}(-1,1)$, there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset L^\infty(-1,1)$ such that $f_n \overset{*}{\rightharpoonup} \mu$ in $(C_0[-1,1])^*$. By assertion (ii) in Proposition 5.7, there exists a unique $u_n \in D(A_\alpha)$ such that

$$\int_{-1}^{1} |x|^{2\alpha} u_n' \zeta' dx + \int_{-1}^{1} u_n \zeta dx = \int_{-1}^{1} f_n \zeta dx, \ \forall \zeta \in C_0^1[-1,1], \tag{5.38}$$

where $D(A_\alpha)$ is defined by (5.13).

Proof of Theorem 5.3. Notice that $\|f_n\|_{L^1} \leq C$, where C is independent of n. Then Lemma 5.8 and Lemma 5.9 imply that

$$\|u_n\|_{L^\infty} + \left\||x|^{2\alpha} u_n'\right\|_{L^\infty} + \left\|(|x|^{2\alpha} u_n')'\right\|_{L^1} \leq \widetilde{C}, \text{ if } 0 < \alpha < \frac{1}{2}, \tag{5.39}$$

$$\|u_n\|_{L^1} + \left\||x| u_n'\right\|_{W^{1,1}} + \left\|\left(1 + \ln \frac{1}{|x|}\right)^{-1} u_n\right\|_{W^{1,1}} \leq \widetilde{C}, \text{ if } \alpha = \frac{1}{2}, \tag{5.40}$$

$$\|u_n\|_{L^1} + \left\||x|^{2\alpha} u_n'\right\|_{W^{1,1}} + \left\||x|^{2\alpha-1} u_n\right\|_{W^{1,1}} \leq \widetilde{C}, \text{ if } \frac{1}{2} < \alpha < 1, \tag{5.41}$$

where $\widetilde{C}$ is independent of n. For all these three cases, there exists a subsequence n_k such that $u_{n_k} \to u$ in $L^1(-1,1)$ and $|x|^{2\alpha} u_{n_k}' \to |x|^{2\alpha} u'$ in $L^1(-1,1)$. Passing to the limit in (5.38) as $k \to \infty$, it follows that

$$\int_{-1}^{1} |x|^{2\alpha} u' \zeta' dx + \int_{-1}^{1} u \zeta dx = \int_{-1}^{1} \zeta d\mu, \ \forall \zeta \in C_0^1[-1,1].$$

When $0 < \alpha < \frac{1}{2}$, estimate (5.39) implies that $\|u_n'\|_{L^q} \leq \widetilde{C} \left\||x|^{-2\alpha}\right\|_{L^q}$ for some fixed $q \in (1, \frac{1}{2\alpha})$. Therefore the sequence u_n is bounded in $W^{1,q}(-1,1)$ and thus $u_{n_k} \to u$ in $C_0[-1,1]$. We conclude that u is a good solution of (5.1).

When $\alpha = \frac{1}{2}$, estimate (5.40) implies that $\left(1 + \ln \frac{1}{|x|}\right)^{-1} u \in BV(-1,1)$. Assume that

$$\lim_{x \to 0^+} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u(x) = A^+$$

and

$$\lim_{x \to 0^-} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u(x) = A^-.$$

If $A^+ = A^-$, then u is the good solution. Otherwise, we make the following "correction" by defining

$$\tilde{u} = u + \frac{A^+ - A^-}{2} U,$$

where U is given by (5.36). It is easy to check that

$$\lim_{x\to 0^+} \left(1 + \ln \frac{1}{|x|}\right)^{-1} \tilde{u}(x) = \lim_{x\to 0^-} \left(1 + \ln \frac{1}{|x|}\right)^{-1} \tilde{u}(x) = \frac{A^+ + A^-}{2}.$$

Therefore $\tilde{u}$ is the good solution of (5.1).

When $\frac{1}{2} < \alpha < 1$, estimate (5.41) implies that $|x|^{2\alpha-1}u \in BV(-1,1)$. By a similar "correction" one can obtain a good solution of (5.1).

Assertions (i), (ii) and (iii) will be proved in Section 5.5. □

Proof of Theorem 5.4. Suppose $\mu(\{0\}) = 0$. We claim that there exists a solution of (5.1). The same as the proof of Theorem 5.3, we apply Lemma 5.9 to obtain

$$\|u_n\|_{L^1} + \left\||x|^{2\alpha} u_n'\right\|_{W^{1,1}} + \left\||x|^{2\alpha-1} u_n\right\|_{W^{1,1}} \le \widetilde{C},$$

where $\widetilde{C}$ is independent of n. It follows that $|x|^{2\alpha} u_{n_k}' \to |x|^{2\alpha} u'$ in $L^1(-1,1)$ and that $u_{n_k} \to u$ uniformly on any closed interval $I \subset [-1,1]\setminus\{0\}$. The Fatou's lemma implies that $u \in L^1(-1,1)$. Passing to the limit in (5.38) as $k \to \infty$, we obtain

$$\int_{-1}^{1} |x|^{2\alpha} u' \zeta' dx + \int_{-1}^{1} u\zeta dx = \int_{-1}^{1} \zeta d\mu, \ \forall \zeta \in C_c^1((-1,1)\setminus\{0\}). \tag{5.42}$$

Here we use the same device as in Brezis-Véron [14]. Let $\varphi(x) \in C^\infty(\mathbb{R})$ be such that $0 \le \varphi \le 1$, $\varphi \equiv 0$ on $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\varphi \equiv 1$ on $\mathbb{R}\setminus(-1,1)$. Let $\varphi_n(x) = \varphi(nx)$. In (5.42), perform integration by parts and replace ζ by $\varphi_n\phi$ where $\phi \in C_c^2(-1,1)$. It follows that

$$-\int_{-1}^{1} u \left(|x|^{2\alpha}(\varphi_n\phi)'\right)' dx + \int_{-1}^{1} u\varphi_n\phi dx = \int_{-1}^{1} \varphi_n\phi d\mu, \ \forall \phi \in C_c^2(-1,1). \tag{5.43}$$

For each individual term on the left-hand side of (5.43), we obtain

$$\int_{-1}^{1} |x|^{2\alpha} u'(x)\varphi(nx)\phi''(x) dx \to \int_{-1}^{1} |x|^{2\alpha} u'(x)\phi''(x) dx,$$

$$2\alpha \int_{-1}^{1} u(x) \operatorname{sign} x |x|^{2\alpha-1} \varphi(nx)\phi'(x) dx \to 2\alpha \int_{-1}^{1} u(x) \operatorname{sign} x |x|^{2\alpha-1} \phi'(x) dx,$$

$$\int_{-1}^{1} u(x)\varphi(nx)\phi(x) dx \to \int_{-1}^{1} u(x)\phi(x) dx,$$

$$\left|2n\int_{-\frac{1}{n}}^{\frac{1}{n}}|x|^{2\alpha}u(x)\varphi'(nx)\phi'(x)dx\right|\leq\frac{2}{n^{2\alpha-1}}\left\|\varphi'\phi'\right\|_{L^\infty}\|u\|_{L^1(-\frac{1}{n},-\frac{1}{n})}\to 0,$$

$$\left|2\alpha n\int_{-\frac{1}{n}}^{\frac{1}{n}}u(x)\operatorname{sign}x|x|^{2\alpha-1}\varphi'(nx)\phi(x)dx\right|\leq\frac{2\alpha}{n^{2\alpha-2}}\left\|\varphi'\phi\right\|_{L^\infty}\|u\|_{L^1(-\frac{1}{n},\frac{1}{n})}\to 0,$$

$$\left|n^2\int_{-\frac{1}{n}}^{\frac{1}{n}}u(x)|x|^{2\alpha}\varphi''(nx)\phi(x)dx\right|\leq\frac{1}{n^{2\alpha-2}}\left\|\varphi''\phi\right\|_{L^\infty}\|u\|_{L^1(-\frac{1}{n},\frac{1}{n})}\to 0.$$

For the right-hand side of (5.43), notice that $\mu(\{0\})=0$ and therefore the Dominated Convergence Theorem implies that

$$\lim_{n\to\infty}\int_{-1}^{1}\varphi(nx)\phi(x)d\mu=\int_{-1}^{1}\phi(x)d\mu.$$

Thus

$$\int_{-1}^{1}|x|^{2\alpha}u'\phi'dx+\int_{-1}^{1}u\phi dx=\int_{-1}^{1}\phi d\mu,\ \forall\phi\in C_c^1(-1,1).$$

Therefore u is a solution of (5.1).

Conversely, assume that u is a solution of (5.1). We claim that $\mu(\{0\})=0$. Indeed, we have

$$-\int_{-1}^{1}u\left(|x|^{2\alpha}\zeta'\right)'dx+\int_{-1}^{1}u\zeta dx=\int_{-1}^{1}\zeta d\mu,\ \forall\zeta\in C_c^\infty(-1,1).\tag{5.44}$$

Take $\varphi\in C_c^\infty(\mathbb{R})$ such that $\varphi\equiv 1$ on $(-1,1)$, $\operatorname{supp}\varphi\subset(-2,2)$ and $0\leq\varphi\leq 1$. Replace $\zeta(x)$ by $\varphi(nx)$ in (5.44). Then for each individual term on the left-hand side of (5.44) we have

$$\left|n^2\int_{-\frac{2}{n}}^{\frac{2}{n}}u(x)|x|^{2\alpha}\varphi''(nx)dx\right|\leq 2^{2\alpha}\left\|\varphi''\right\|_{L^\infty}\|u\|_{L^1(-\frac{2}{n},\frac{2}{n})}\to 0,$$

$$\left|2\alpha n\int_{-\frac{2}{n}}^{\frac{2}{n}}u(x)|x|^{2\alpha-1}\varphi'(nx)\operatorname{sign}x dx\right|\leq\alpha 2^{2\alpha}\left\|\varphi'\right\|_{L^\infty}\|u\|_{L^1(-\frac{2}{n},\frac{2}{n})}\to 0,$$

$$\int_{-1}^{1}u(x)\varphi(nx)dx\to 0.$$

For the right-hand side of (5.44), we have

$$\int_{-1}^{1}\varphi(nx)d\mu=\mu(\{0\})+\int_{(0,\frac{2}{n}]}\varphi(nx)d\mu+\int_{[-\frac{2}{n},0)}\varphi(nx)d\mu.$$

Note that

$$\lim_{n\to\infty}\int_{(0,\frac{2}{n}]}\varphi(nx)d\mu=\lim_{n\to\infty}\int_{[-\frac{2}{n},0)}\varphi(nx)d\mu=0,$$

since

$$\lim_{n\to\infty} \mu\left(\left(0, \frac{2}{n}\right]\right) = \lim_{n\to\infty} \mu\left(\left[-\frac{2}{n}, 0\right)\right) = 0.$$

Therefore, $\mu(\{0\}) = 0$.

Now assume that the solution exists. We prove assertion (i). Notice that $|x|^{2\alpha}u^2 \in W^{1,1}_{loc}([-1,1]\setminus\{0\})$. Since $|x|^{2\alpha-1}u \in BV(-1,1)$, we have

$$\left(|x|^{2\alpha}u^2\right)' = 2|x|^{2\alpha}u'u + 2\alpha(\operatorname{sign} x)|x|^{2\alpha-1}u^2 \in L^1(-1,1).$$

That is $|x|^{2\alpha}u^2|_{(0,1)} \in W^{1,1}(0,1)$ and $|x|^{2\alpha}u^2|_{(-1,0)} \in W^{1,1}(-1,0)$. Therefore, the one-sided limits $\lim_{x\to 0^+}|x|^{\alpha}|u(x)|$ and $\lim_{x\to 0^-}|x|^{\alpha}|u(x)|$ exist. They must be zero. Otherwise, we obtain a contradiction with $u \in L^1(-1,1)$. The fact that $u \in L^1(-1,1)$ also forces $\lim_{x\to 0}|x|^{2\alpha}u'(x) = 0$. Assertion (ii) will be proved in Section 5.5. □

Proof of Remark 5.2. Given $0 < \alpha < 1$, from Chapter 3 we obtain that there exists a unique $\phi_\alpha \in W^{1,1}(0,1) \cap H^2_{loc}(0,1)$ such that $\phi_\alpha > 0$ on $[0,1]$ and

$$\begin{cases} -(x^{2\alpha}\phi_\alpha')' + \phi_\alpha = 0 \quad \text{on } (0,1), \\ \lim_{x\to 0^+} x^{2\alpha}\phi_\alpha'(x) = 0, \ \phi_\alpha(1) = 1. \end{cases}$$

Since $\phi_\alpha'(x) = \frac{1}{x^{2\alpha}}\int_0^x \phi_\alpha(t)dt > 0$, we deduce that $\phi_\alpha(0) \in (0,1)$. One can easily check that $G_\alpha(x) = 1 - \phi_\alpha(|x|) \in W^{1,1}_0(-1,1)$ satisfies

$$\begin{cases} -(|x|^{2\alpha}G_\alpha')' + G_\alpha = 1 \quad \text{on } (-1,1), \\ G_\alpha(-1) = G_\alpha(1) = 0. \end{cases}$$

Moreover, $\lim_{x\to 0}|x|^{\alpha}G_\alpha'(x) = 0$, $G_\alpha \geq 0$ and $\max_{x\in[-1,1]} G_\alpha = G_\alpha(0) = 1 - \phi_\alpha(0) \in (0,1)$.

When $0 < \alpha < 1$, we claim that $k_\alpha = G_\alpha(0)$. Indeed, for any $\mu \in \mathcal{M}(-1,1)$ and its corresponding good solution u, we have $\lim_{x\to 0}|x|^{2\alpha}G_\alpha'(x)u(x) = 0$. Therefore integration by parts yields

$$\begin{aligned} \int_{-1}^{1} G_\alpha d\mu &= \int_{-1}^{1} |x|^{2\alpha}u'G_\alpha' dx + \int_{-1}^{1} uG_\alpha dx \\ &= -\int_{-1}^{1} u(|x|^{2\alpha}G_\alpha')'dx + \int_{-1}^{1} uG_\alpha dx \\ &= \int_{-1}^{1} u dx. \end{aligned}$$

If $\mu \geq 0$, then $u \geq 0$ a.e. and $\|u\|_{L^1} \leq G_\alpha(0)\|\mu\|_{\mathcal{M}}$. For a general $\mu \in \mathcal{M}(-1,1)$, write $\mu = \mu^+ - \mu^-$. Let u_1 (resp. u_2) be the good solution corresponding to μ^+ (resp. μ^-). Then the linearity of equation (5.1) and the uniqueness of the good solution imply that $u = u_1 - u_2$. Therefore we obtain that $\|u\|_{L^1} \leq G_\alpha(0)\|\mu\|_{\mathcal{M}}$. On the other hand, take $\mu_n = \delta_{\frac{1}{n}}$, the Dirac mass at $\frac{1}{n}$, and let u_n be its corresponding good solution. Then

$$\lim_{n\to\infty} \|u_n\|_{L^1} = \lim_{n\to\infty} G_\alpha\left(\frac{1}{n}\right) = G_\alpha(0).$$

As a consequence, $k_\alpha = G_\alpha(0)$.

When $\alpha \geq 1$, take

$$u_n(x) = \begin{cases} |x|^{1-2\alpha} - 1, \text{ if } |x| \in \left(\frac{1}{n}, 1\right], \\ n^{2\alpha-1} - 1, \text{ if } |x| \in \left[0, \frac{1}{n}\right], \end{cases}$$

and define

$$\mu_n = u_n + (2\alpha - 1)\delta_{\frac{1}{n}} + (2\alpha - 1)\delta_{-\frac{1}{n}}.$$

It is easy to check that u_n solves

$$\begin{cases} -(|x|^{2\alpha}u_n')' + u_n = \mu_n \quad \text{ on } (-1,1), \\ u_n(-1) = u_n(1) = 0. \end{cases}$$

Since $\|u_n\|_{L^1} \to \infty$ as $n \to \infty$, we obtain that

$$k_\alpha \geq \sup_n \frac{\|u_n\|_{L^1}}{\|\mu_n\|_{\mathcal{M}}} = 1.$$

Recall that $k_\alpha \leq 1$ and therefore $k_\alpha = 1$. □

We conclude this section with the

Proof of Remark 5.3. For fixed $x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \setminus \{0\}$, take

$$\begin{aligned} u(t) =& \frac{2}{(5 + 2^{2\alpha+2})|x|^{2\alpha}} \left(t - x + \frac{1}{2}|x|^\alpha\right) \chi_{\left(x - \frac{1}{2}|x|^\alpha, x\right)} \\ &+ \frac{2}{(5 + 2^{2\alpha+2})|x|^{2\alpha}} \left(x + \frac{1}{2}|x|^\alpha - t\right) \chi_{\left(x, x + \frac{1}{2}|x|^\alpha\right)}. \end{aligned}$$

It is straightforward that $u \in W_0^{1,\infty}(-1,1)$, $\mu := -(|t|^{2\alpha}u')' + u \in \mathcal{M}(-1,1)$, $\|\mu\|_{\mathcal{M}} \leq 1$ and $\mu(\{0\}) = 0$. Moreover, $|x|^\alpha |u(x)| = \frac{1}{5+2^{2\alpha+2}}$ and $|x|^{2\alpha}|u'(x)| = \frac{2}{5+2^{2\alpha+2}}$. It follows that $J_\alpha(x) \geq \frac{2}{5+2^{2\alpha+2}}$ and $\widetilde{J}_\alpha(x) \geq \frac{1}{5+2^{2\alpha+2}}$. On the other hand, for all $x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \setminus \{0\}$, notice that

$$|x|^{2\alpha} u'(x) = \begin{cases} \int_0^x u(s)ds - \int_{(0,x]} d\mu, \ x > 0, \\ -\int_x^0 u(s)ds + \int_{(x,0)} d\mu, \ x < 0, \end{cases}$$

$$(|x|^\alpha u(x))^2 = -2\int_x^1 |s|^{2\alpha} u'(s)u(s)ds - 2\alpha \int_x^1 (\operatorname{sign} s)|s|^{2\alpha-1} u(s)u(s)ds.$$

Since $\|u\|_{L^1} \leq \|\mu\|_{\mathcal{M}}$, it follows that $J_\alpha(x) \leq 2$ and $\widetilde{J}_\alpha(x) \leq \left(2 + \frac{4\alpha}{2\alpha-1}\right)^{\frac{1}{2}}$. Therefore, the proof is complete. □

5.5 The elliptic regularization

For any $0 < \epsilon < 1$, by the Lax-Milgram theorem, since $\mathcal{M}(-1,1) \subset H^{-1}(-1,1)$, there exists a unique $u_\epsilon \in H_0^1(-1,1)$ such that $u'_\epsilon \in BV(-1,1)$ and

$$\int_{-1}^1 (|x| + \epsilon)^{2\alpha} u'_\epsilon v' dx + \int_{-1}^1 u_\epsilon v dx = \int_{-1}^1 v d\mu, \ \forall v \in H_0^1(-1,1). \tag{5.45}$$

In particular, take $v \in C_0^1[-1,1]$ and it follows that u_ϵ solves (5.5). Take $v_n = \varphi(nu_\epsilon)$ where $\varphi \in C^\infty(\mathbb{R})$ and $\varphi' \geq 0$ such that $\varphi \equiv 1$ on $[1,\infty)$, $\varphi \equiv -1$ on $(-\infty,-1]$ and $\varphi(0) = 0$. Notice that

$$\int_{-1}^1 (|x| + \epsilon)^{2\alpha} u'_\epsilon v'_n dx = \int_{-1}^1 (|x| + \epsilon)^{2\alpha} |u'_\epsilon|^2 \varphi'(nu_\epsilon) dx \geq 0.$$

Then

$$\|u_\epsilon\|_{L^1(-1,1)} = \lim_{n\to\infty} \int_{-1}^1 u_\epsilon v_n dx \leq \lim_{n\to\infty} \int_{-1}^1 v_n d\mu \leq \|\mu\|_{\mathcal{M}(-1,1)}. \tag{5.46}$$

We now examine the limiting behavior of the family $\{u_\epsilon\}_{\epsilon>0}$ and we are going to establish the following sharper form of Theorems 5.5 and 5.6.

Theorem 5.12. *Given $\alpha > 0$, As $\epsilon \to 0$, we have*

$$(|x| + \epsilon)^{2\alpha} u'_\epsilon \to |x|^{2\alpha} u' \ \text{in } L^p(-1,1), \ \forall p < \infty. \tag{5.47}$$

Moreover,

$$u_\epsilon \to u \ \text{in } C_0[-1,1], \ \text{if } 0 < \alpha < \frac{1}{2}, \tag{5.48}$$

$$\left(1 + \ln \frac{1}{|x| + \epsilon}\right)^{-1} u_\epsilon \to \left(1 + \ln \frac{1}{|x|}\right)^{-1} u \ \text{in } L^p(-1,1), \ \forall p < \infty, \ \text{if } \alpha = \frac{1}{2}, \tag{5.49}$$

$$(|x|+\epsilon)^{2\alpha-1}u_\epsilon \to |x|^{2\alpha-1}u \ in\ L^p(-1,1),\ \forall p<\infty,\ if\ \alpha>\frac{1}{2}. \tag{5.50}$$

Here u is the unique good solution of (5.1) *when* $0<\alpha<1$*, and u is the unique solution of* (5.7) *when* $\alpha \geq 1$*.*

Proof of Theorem 5.12 for $0<\alpha<\frac{1}{2}$*.* Take $v=u_\epsilon$ in (5.45) and it follows that

$$\left(\|u_\epsilon\|_{L^2}+\left\|(|x|+\epsilon)^\alpha u_\epsilon'\right\|_{L^2}\right)^2 \leq 2\|\mu\|_{\mathcal{M}}\|u_\epsilon\|_{L^\infty}.$$

Notice that $u_\epsilon(x)=\int_{-1}^x u_\epsilon'(t)dt$ and therefore

$$\|u_\epsilon\|_{L^\infty} \leq \left\|\frac{1}{(|x|+\epsilon)^\alpha}\right\|_{L^2}\left\|(|x|+\epsilon)^\alpha u_\epsilon'\right\|_{L^2} \leq \frac{2}{1-2\alpha}\left\|(|x|+\epsilon)^\alpha u_\epsilon'\right\|_{L^2}.$$

Thus

$$\|u_\epsilon\|_{L^2}+\left\|(|x|+\epsilon)^\alpha u_\epsilon'\right\|_{L^2} \leq \frac{4}{1-2\alpha}\|\mu\|_{\mathcal{M}}.$$

For a fixed $q\in\left(1,\frac{2}{2\alpha+1}\right)$, we have

$$\left\|u_\epsilon'\right\|_{L^q} \leq \left\|(|x|+\epsilon)^\alpha u_\epsilon'\right\|_{L^2}\left\|\frac{1}{(|x|+\epsilon)^\alpha}\right\|_{L^{\frac{2q}{2-q}}} \leq C\left\|(|x|+\epsilon)^\alpha u_\epsilon'\right\|_{L^2},$$

where C is independent of ϵ. Therefore the family $\{u_\epsilon\}_{\epsilon>0}$ is bounded in $W_0^{1,q}(-1,1)$ where $q\in\left(1,\frac{2}{2\alpha+1}\right)$. Taking into account (5.46) we obtain that $\left\|\left((|x|+\epsilon)^{2\alpha}u_\epsilon'\right)'\right\|_{\mathcal{M}} \leq 2\|\mu\|_{\mathcal{M}}$. Thus the family $\left\{(|x|+\epsilon)^{2\alpha}u_\epsilon'\right\}_{\epsilon>0}$ is bounded in $BV(-1,1)$. Then (5.47) and (5.48) hold for a subsequence $\{\epsilon_n\}_{n=1}^\infty$ with $\epsilon_n\downarrow 0$ as $n\to\infty$. Passing to the limit in (5.45) as $n\to\infty$, we obtain

$$\int_{-1}^1 |x|^{2\alpha}u'v'dx+\int_{-1}^1 uvdx=\int_{-1}^1 vd\mu,\ \forall v\in C_0^1[-1,1].$$

In particular, u is the good solution of (5.1). Notice that the above argument also shows that any convergent subsequence of $\{u_\epsilon\}_{\epsilon>0}$ converges to the good solution u. The uniqueness of the good solution and "the uniqueness of the limit" (see, e.g., page 392 of [8]) imply that (5.47) and (5.48) hold for the whole family $\{u_\epsilon\}_{\epsilon>0}$. □

Proof of Theorem 5.12 for $\alpha=\frac{1}{2}$*.* Since $((|x|+\epsilon)u_\epsilon')'=u_\epsilon-\mu$ and $(|x|+\epsilon)u_\epsilon'\in BV$, we denote $K_\epsilon^+=\lim_{x\to 0^+}u_\epsilon'(x)$ and $K_\epsilon^-=\lim_{x\to 0^-}u_\epsilon'(x)$. Without loss of generality, we can write

$$(|x|+\epsilon)u_\epsilon'(x)-\epsilon K_\epsilon^+=-\int_{(0,x)}d\mu+\int_0^x u_\epsilon(s)ds,\ \forall x\in(0,1),$$

and

$$-\left(|x|+\epsilon\right)u_\epsilon'(x)+\epsilon K_\epsilon^- = -\int_{(x,0)} d\mu + \int_x^0 u_\epsilon(s)ds,\ \forall x\in(-1,0).$$

Then integration by parts implies that, for $x\in(0,1)$,

$$\begin{aligned} u_\epsilon(x) = & \ln\left(\frac{1+\epsilon}{x+\epsilon}\right)\left(-\epsilon K_\epsilon^+ + \int_{(0,x)} d\mu - \int_0^x u_\epsilon(s)ds\right) \\ & - \int_x^1 u_\epsilon(s)\ln\left(\frac{1+\epsilon}{s+\epsilon}\right)ds + \int_{[x,1)} \ln\left(\frac{1+\epsilon}{s+\epsilon}\right)d\mu(s), \end{aligned} \tag{5.51}$$

and for $x\in(-1,0)$,

$$\begin{aligned} u_\epsilon(x) = & \ln\left(\frac{1+\epsilon}{|x|+\epsilon}\right)\left(\epsilon K_\epsilon^- + \int_{(x,0)} d\mu - \int_x^0 u_\epsilon(s)ds\right) \\ & - \int_{-1}^x u_\epsilon(s)\ln\left(\frac{1+\epsilon}{|s|+\epsilon}\right)ds + \int_{(-1,x]} \ln\left(\frac{1+\epsilon}{|s|+\epsilon}\right)d\mu(s). \end{aligned}$$

By the relation $u_\epsilon(0^+)=u_\epsilon(0^-)$, we have

$$\begin{aligned} \epsilon K_\epsilon^+ + \epsilon K_\epsilon^- = & \frac{1}{\ln\left(\frac{1+\epsilon}{\epsilon}\right)}\int_{(-1,0)\cup(0,1)} (\operatorname{sign} s)\ln\left(\frac{1+\epsilon}{|s|+\epsilon}\right)d\mu(s) \\ & - \frac{1}{\ln\left(\frac{1+\epsilon}{\epsilon}\right)}\int_{-1}^1 (\operatorname{sign} s)u_\epsilon(s)\ln\left(\frac{1+\epsilon}{|s|+\epsilon}\right)ds. \end{aligned}$$

Also recall the relation $\epsilon K_\epsilon^+ - \epsilon K_\epsilon^- = -\mu(\{0\})$, so we deduce that

$$\begin{aligned} \epsilon K_\epsilon^+ = & -\frac{1}{2}\mu(\{0\}) + \frac{1}{2\ln\left(\frac{1+\epsilon}{\epsilon}\right)}\int_{(-1,0)\cup(0,1)} (\operatorname{sign} s)\ln\left(\frac{1+\epsilon}{|s|+\epsilon}\right)d\mu(s) \\ & - \frac{1}{2\ln\left(\frac{1+\epsilon}{\epsilon}\right)}\int_{-1}^1 (\operatorname{sign} s)u_\epsilon(s)\ln\left(\frac{1+\epsilon}{|s|+\epsilon}\right)ds, \end{aligned} \tag{5.52}$$

and

$$\begin{aligned} \epsilon K_\epsilon^- = & \frac{1}{2}\mu(\{0\}) + \frac{1}{2\ln\left(\frac{1+\epsilon}{\epsilon}\right)}\int_{(-1,0)\cup(0,1)} (\operatorname{sign} s)\ln\left(\frac{1+\epsilon}{|s|+\epsilon}\right)d\mu(s) \\ & - \frac{1}{2\ln\left(\frac{1+\epsilon}{\epsilon}\right)}\int_{-1}^1 (\operatorname{sign} s)u_\epsilon(s)\ln\left(\frac{1+\epsilon}{|s|+\epsilon}\right)ds. \end{aligned}$$

It is easy to check that $|\epsilon K_\epsilon^+| \le \frac{3}{2}\|\mu\|_{\mathcal{M}}$ and $|\epsilon K_\epsilon^-| \le \frac{3}{2}\|\mu\|_{\mathcal{M}}$ since $\|u_\epsilon\|_{L^1} \le \|\mu\|_{\mathcal{M}}$. Moreover, by the above integral forms of $(|x|+\epsilon)u_\epsilon'$ and u_ϵ, a straightforward calculation implies that

$$\left\|\left(1+\ln\frac{1}{|x|+\epsilon}\right)^{-1}u_\epsilon\right\|_{W^{1,1}(-1,1)} + \left\|(|x|+\epsilon)u_\epsilon'\right\|_{BV(-1,1)} \le C,$$

where C is independent of ϵ. Then (5.47) and (5.49) hold for a subsequence $\{\epsilon_n\}_{n=1}^{\infty}$. Passing to the limit in (5.45) as $n \to \infty$, we obtain

$$\int_{-1}^{1} |x|u'v'dx + \int_{-1}^{1} uvdx = \int_{-1}^{1} vd\mu, \ \forall v \in C_0^1[-1,1].$$

We now show that $\lim_{x\to 0^+} \left(1+\ln\frac{1}{|x|}\right)^{-1} u(x) = \frac{1}{2}\mu(\{0\})$. We first claim that $\lim_{\epsilon\to 0} \epsilon K_\epsilon^+ = -\frac{1}{2}\mu(\{0\})$. Indeed, we have the following estimate,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\left|\frac{1}{\ln\left(\frac{1+\epsilon}{\epsilon}\right)}\int_0^1 u_\epsilon(s)\ln\left(\frac{1+\epsilon}{s+\epsilon}\right)ds\right| \\
=&\lim_{\epsilon\to 0}\left|\frac{1}{\ln\left(1+\frac{1}{\epsilon}\right)}\int_0^1 u_\epsilon(s)\ln\left(\frac{1}{s+\epsilon}\right)ds\right| \\
\le&\lim_{\epsilon\to 0}\left|\int_0^{\frac{1}{1+\ln\frac{1}{\epsilon}}-\epsilon} u_\epsilon(s)ds\right| + \lim_{\epsilon\to 0^+}\left|\frac{\ln\left(1+\ln\frac{1}{\epsilon}\right)}{\ln\left(1+\frac{1}{\epsilon}\right)}\int_0^1 u_\epsilon(s)ds\right| \\
=&0,
\end{aligned}$$

since $\|u_\epsilon\|_{L^2(-1,1)} \le C$ where C is independent of ϵ. All the other terms in (5.52) can be estimated in the same way. Therefore with the help of (5.51) we have

$$\begin{aligned}
\lim_{x\to 0^+}\left(1+\ln\frac{1}{|x|}\right)^{-1} u(x) &= \lim_{x\to 0^+}\lim_{n\to\infty}\left(1+\ln\frac{1}{|x|}\right)^{-1} u_{\epsilon_n}(x) \\
&= -\lim_{\epsilon\to 0}\epsilon K_\epsilon^+ = \frac{1}{2}\mu(\{0\}).
\end{aligned}$$

Similarly we can obtain that $\lim_{x\to 0^-}\left(1+\ln\frac{1}{|x|}\right)^{-1} u(x) = \lim_{\epsilon\to 0}\epsilon K_\epsilon^- = \frac{1}{2}\mu(\{0\})$. Therefore, u is the good solution of (5.1). Since the limit $\lim_{\epsilon\to 0}\epsilon K_\epsilon^+ = -\lim_{\epsilon\to 0}\epsilon K_\epsilon^- = -\frac{1}{2}\mu(\{0\})$ doesn't depend on the choice of the subsequence $\{\epsilon_n\}_{n=1}^{\infty}$, the above argument also shows that any convergent subsequence of $\{u_\epsilon\}_{\epsilon>0}$ converges to the good solution u. The uniqueness of the good solution and the uniqueness of the limit imply that (5.47) and (5.49) hold for the whole family $\{u_\epsilon\}_{\epsilon>0}$. □

Proof of Theorem 5.12 for $\frac{1}{2} < \alpha < 1$. We use the same method for the case $\alpha = \frac{1}{2}$. We denote $K_\epsilon^+ = \lim_{x\to 0^+} u_\epsilon'(x)$ and $K_\epsilon^- = \lim_{x\to 0^-} u_\epsilon'(x)$. We write

$$(|x|+\epsilon)^{2\alpha} u_\epsilon'(x) - \epsilon^{2\alpha}K_\epsilon^+ = -\int_{(0,x)} d\mu + \int_0^x u_\epsilon(s)ds, \ \forall x \in (0,1),$$

and

$$-(|x|+\epsilon)^{2\alpha} u_\epsilon'(x) + \epsilon^{2\alpha}K_\epsilon^- = -\int_{(x,0)} d\mu + \int_x^0 u_\epsilon(s)ds, \ \forall x \in (-1,0).$$

Then integration by parts yields, for $x \in (0,1)$,

$$\begin{aligned} u_\epsilon(x) = & \left(\frac{(x+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}}{2\alpha - 1} \right) \left(-\epsilon^{2\alpha} K_\epsilon^+ + \int_{(0,x)} d\mu - \int_0^x u_\epsilon(s) ds \right) \\ & - \int_x^1 u_\epsilon(s) \left(\frac{(s+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}}{2\alpha - 1} \right) ds \\ & + \int_{[x,1)} \frac{(s+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}}{2\alpha - 1} d\mu(s), \end{aligned}$$

and for $x \in (-1,0)$,

$$\begin{aligned} u_\epsilon(x) = & \left(\frac{(|x|+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}}{2\alpha - 1} \right) \left(\epsilon^{2\alpha} K_\epsilon^- + \int_{(x,0)} d\mu - \int_x^0 u_\epsilon(s) ds \right) \\ & - \int_{-1}^x u_\epsilon(s) \left(\frac{(|s|+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}}{2\alpha - 1} \right) ds \\ & + \int_{(-1,x]} \frac{(|s|+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}}{2\alpha - 1} d\mu(s). \end{aligned}$$

By the relation $u_\epsilon(0^+) = u_\epsilon(0^-)$, we have

$$\begin{aligned} \epsilon^{2\alpha} K_\epsilon^+ + \epsilon^{2\alpha} K_\epsilon^- = & \frac{\int_{(-1,0)\cup(0,1)} (\operatorname{sign} s) \left[(|s|+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha} \right] d\mu(s)}{\epsilon^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}} \\ & - \frac{\int_{-1}^1 (\operatorname{sign} s) u_\epsilon(s) \left[(|s|+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha} \right] ds}{\epsilon^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}}. \end{aligned}$$

Recall the relation $\epsilon^{2\alpha} K_\epsilon^+ - \epsilon^{2\alpha} K_\epsilon^- = -\mu(\{0\})$, so we deduce that

$$\begin{aligned} \epsilon^{2\alpha} K_\epsilon^+ = & -\frac{1}{2}\mu(\{0\}) - \frac{\int_{-1}^1 (\operatorname{sign} s) u_\epsilon(s) \left[(|s|+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha} \right] ds}{2[\epsilon^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}]} \\ & + \frac{\int_{(-1,0)\cup(0,1)} (\operatorname{sign} s) \left[(|s|+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha} \right] d\mu(s)}{2[\epsilon^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}]}, \end{aligned}$$

and

$$\begin{aligned} \epsilon^{2\alpha} K_\epsilon^- = & \frac{1}{2}\mu(\{0\}) - \frac{\int_{-1}^1 (\operatorname{sign} s) u_\epsilon(s) \left[(|s|+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha} \right] ds}{2[\epsilon^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}]} \\ & + \frac{\int_{(-1,0)\cup(0,1)} (\operatorname{sign} s) \left[(|s|+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha} \right] d\mu(s)}{2[\epsilon^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}]}. \end{aligned}$$

It is easy to check that $\left|\epsilon^{2\alpha} K_\epsilon^+\right| \leq \frac{3}{2} \|\mu\|_{\mathcal{M}}$ and $\left|\epsilon^{2\alpha} K_\epsilon^-\right| \leq \frac{3}{2} \|\mu\|_{\mathcal{M}}$ since $\|u_\epsilon\|_{L^1} \leq \|\mu\|_{\mathcal{M}}$. Moreover, the integral forms of $(|x|+\epsilon)^{2\alpha} u_\epsilon'$ and u_ϵ imply that

$$\left\| (|x|+\epsilon)^{2\alpha-1} u_\epsilon \right\|_{W^{1,1}(-1,1)} + \left\| (|x|+\epsilon)^{2\alpha} u_\epsilon' \right\|_{BV(-1,1)} \leq C, \qquad (5.53)$$

where C is independent of ϵ. Then (5.47) and (5.50) hold for a subsequence $\{\epsilon_n\}_{n=1}^{\infty}$. Passing to the limit in (5.45) as $n \to \infty$, we get

$$\int_{-1}^{1} |x|^{2\alpha} u'v'dx + \int_{-1}^{1} uvdx = \int_{-1}^{1} vd\mu, \ \forall v \in C_0^1[-1,1].$$

We now show that $\lim_{x\to 0^+} |x|^{2\alpha-1}u(x) = \frac{1}{2(2\alpha-1)}\mu(\{0\})$. We first claim that $\lim_{\epsilon\to 0}\epsilon^{2\alpha}K_\epsilon^+ = -\frac{1}{2}\mu(\{0\})$. Indeed, we have the following estimate,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\left|\frac{1}{\epsilon^{1-2\alpha}-(1+\epsilon)^{1-2\alpha}}\int_0^1 u_\epsilon(s)\left[(s+\epsilon)^{1-2\alpha}-(1+\epsilon)^{1-2\alpha}\right]ds\right| \\
=&\lim_{\epsilon\to 0}\left|\epsilon^{2\alpha-1}\int_0^1 u_\epsilon(s)(s+\epsilon)^{1-2\alpha}ds\right| \\
\le&\lim_{\epsilon\to 0}\left|\int_0^{\sqrt{\epsilon}-\epsilon} u_\epsilon(s)ds\right| + \lim_{\epsilon\to 0^+}\left|\frac{\epsilon^{2\alpha-1}}{\epsilon^{\alpha-\frac{1}{2}}}\int_{\sqrt{\epsilon}-\epsilon}^1 u_\epsilon(s)ds\right| \\
=&0,
\end{aligned}$$

since $\|u_\epsilon\|_{L^\theta(-1,1)} \le C$ for some fixed $\theta \in \left(1, \frac{1}{2\alpha-1}\right)$ and C is independent of ϵ. All the other terms in the identity for $\epsilon^{2\alpha}K_\epsilon^+$ can be estimated in the same way. Therefore

$$\begin{aligned}
\lim_{x\to 0^+} |x|^{2\alpha-1}u(x) &= \lim_{x\to 0^+}\lim_{n\to\infty} |x|^{2\alpha-1}u_{\epsilon_n}(x) \\
&= -\frac{1}{2\alpha-1}\lim_{\epsilon\to 0}\epsilon^{2\alpha}K_\epsilon^+ = \frac{1}{2(2\alpha-1)}\mu(\{0\}).
\end{aligned}$$

Similarly we can also get that $\lim_{\epsilon\to 0}\epsilon^{2\alpha}K_\epsilon^- = \frac{1}{2}\mu(\{0\})$ and $\lim_{x\to 0^-}|x|^{2\alpha-1}u(x) = \frac{\mu(\{0\})}{2(2\alpha-1)}$. Hence, u is the good solution of (5.1). The uniqueness of the limit then implies that (5.47) and (5.50) hold for the whole family $\{u_\epsilon\}_{\epsilon>0}$. □

Proof of Theorem 5.12 for $\alpha \ge 1$. In this case, we can still obtain (5.53) by the same computation from the previous case, so (5.47) and (5.50) hold for a subsequence $\{\epsilon_n\}_{n=1}^{\infty}$. In particular, it follows that $u_{\epsilon_n} \to u$ uniformly on any closed interval $I \subset [-1,1]\setminus\{0\}$. Passing to the limit in (5.45) as $n\to\infty$, we get

$$\int_{-1}^{1} |x|^{2\alpha}u'v'dx + \int_{-1}^{1} uvdx = \int_{-1}^{1} vd\mu, \ \forall v \in C_c^1((-1,1)\setminus\{0\}).$$

Since $\|u_\epsilon\|_{L^1} \le \|\mu\|_{\mathcal{M}}$, the Fatou's lemma yields that $u \in L^1(-1,1)$. The same argument from the proof of Theorem 5.4 (see Section 5.4) implies that

$$-\int_{-1}^{1} |x|^{2\alpha}u'\phi'dx + \int_{-1}^{1} u\phi dx = \int_{-1}^{1} \phi d(\mu - \mu(\{0\})\delta_0), \ \forall \phi \in C_c^1(-1,1).$$

Therefore u is the unique solution of (5.7). We can further deduce that (5.47) and (5.50) hold for the whole family $\{u_\epsilon\}_{\epsilon>0}$. □

Theorem 5.13. *If $\alpha \geq \frac{1}{2}$ and $\mu \in L^1(-1,1)$, the mode of convergence in* (5.49) *and* (5.50) *can be improved as*

$$\left(1+\ln\frac{1}{|x|+\epsilon}\right)^{-1} u_\epsilon \to \left(1+\ln\frac{1}{|x|}\right)^{-1} u \text{ in } C_0[-1,1], \text{ if } \alpha = \frac{1}{2}, \tag{5.54}$$

and

$$(|x|+\epsilon)^{2\alpha-1} u_\epsilon \to |x|^{2\alpha-1} u \text{ in } C_0[-1,1], \text{ if } \alpha > \frac{1}{2}. \tag{5.55}$$

Proof. We divide the proof into four steps.

Step 1. Assume $\mu \in L^1(-1,1)$. We claim that the family $\{u_\epsilon\}_{\epsilon>0}$ is equi-integrable. Here we use a device introduced by Gallouët-Morel [29]. Take a nondecreasing function $\varphi(x) \in C^\infty(\mathbb{R})$ such that $\varphi(x) = 0$ for $x \leq 0$, $\varphi(x) > 0$ for $x > 0$ and $\varphi(x) = 1$ for $x \geq 1$. For fixed $k \in \mathbb{N}$ and $t \in \mathbb{R}^+$, we define

$$P_{k,t}(x) = \operatorname{sign} x \, \varphi(k(|x|-t)).$$

It is clear that $P_{k,t}$ is a maximal monotone graph containing the origin. Moreover,

$$\{x;\ P_{k,t}(x) \neq 0\} = (-\infty,-t) \cup (t,+\infty),$$

$$|P_{1,t}(x)| \leq |P_{2,t}(x)| \leq \cdots |P_{k,t}(x)| \leq |P_{k+1,t}(x)| \cdots \leq 1,$$

and

$$\lim_{k\to\infty} |P_{k,t}| = \chi_{[|x|>t]}.$$

It is easy to see that

$$-\int_{-1}^{1} \left((|x|+\epsilon)^{2\alpha} u_\epsilon'\right)' P_{k,t}\left(u_\epsilon\right) dx \geq 0,$$

and therefore

$$\int_{-1}^{1} |P_{k,t}\left(u_\epsilon\right)| \, |u_\epsilon| dx \leq \int_{-1}^{1} |P_{k,t}\left(u_\epsilon\right)| \, |\mu| dx.$$

Passing to the limit as $k \to \infty$, the Monotone Convergence Theorem implies that

$$\int_{[|u_\epsilon|>t]} |u_\epsilon| dx \leq \int_{[|u_\epsilon|>t]} |\mu| dx, \ \forall t > 0 \text{ and } \forall \epsilon \in (0,1).$$

Then

$$\operatorname{meas}\{[|u_\epsilon|>t]\} \le \frac{1}{t}\int_{[|u_\epsilon|>t]} |u_\epsilon| dx \le \frac{1}{t}\|\mu\|_{L^1}.$$

For any $\tilde{\epsilon} > 0$, there exists $t_{\tilde{\epsilon}} > 0$ such that

$$\int_{[|u_\epsilon|>t_{\tilde{\epsilon}}]} |u_\epsilon| dx \le \int_{[|u_\epsilon|>t_{\tilde{\epsilon}}]} |\mu| dx \le \frac{\tilde{\epsilon}}{2},\ \forall \epsilon \in (0,1).$$

Take $\delta = \frac{\tilde{\epsilon}}{2t_{\tilde{\epsilon}}}$. Then for all $K \subset [-1,1]$ such that $\operatorname{meas} K < \delta$, we have

$$\begin{aligned}\int_K |u_\epsilon| dx &\le \int_{K\cap[|u_\epsilon|>t_{\tilde{\epsilon}}]} |u_\epsilon| dx + \int_{K\cap[|u_\epsilon|\le t_{\tilde{\epsilon}}]} |u_\epsilon| dx\\ &\le \int_{[|u_\epsilon|>t_{\tilde{\epsilon}}]} |u_\epsilon| dx + t_{\tilde{\epsilon}} \operatorname{meas} K\\ &\le \tilde{\epsilon}.\end{aligned}$$

Thus, the family $\{u_\epsilon\}_{\epsilon>0}$ is equi-integrable.

Step 2. Without loss of generality, assume $0 < \epsilon < \frac{1}{2}$. We claim that for $\alpha = \frac{1}{2}$ the the family $\left\{\left(1+\ln\frac{1}{|x|+\epsilon}\right)^{-1} u_\epsilon\right\}_{\epsilon>0}$ is equi-continuous on $[-1,1]$. Assume $0 \le x_1 < x_2 \le 1$. With the help of (5.51), we can write

$$\begin{aligned}&\left(1+\ln\frac{1}{x_1+\epsilon}\right)^{-1} u_\epsilon(x_1) - \left(1+\ln\frac{1}{x_2+\epsilon}\right)^{-1} u_\epsilon(x_2)\\ =&\ln(1+\epsilon)\left[\left(1+\ln\frac{1}{x_1+\epsilon}\right)^{-1} - \left(1+\ln\frac{1}{x_2+\epsilon}\right)^{-1}\right]\left(\int_0^1(\mu-u_\epsilon)ds - \epsilon K_\epsilon^+\right)\\ &+\left(\frac{\ln\frac{1}{x_1+\epsilon}}{1+\ln\frac{1}{x_1+\epsilon}} - \frac{\ln\frac{1}{x_2+\epsilon}}{1+\ln\frac{1}{x_2+\epsilon}}\right)\left(\int_0^{x_1}(\mu(s)-u_\epsilon(s))ds - \epsilon K_\epsilon^+\right)\\ &-\frac{\ln\frac{1}{x_2+\epsilon}}{1+\ln\frac{1}{x_2+\epsilon}}\int_{x_1}^{x_2}(\mu(s)-u_\epsilon(s))ds\\ &+\frac{1}{1+\ln\frac{1}{x_1+\epsilon}}\int_{x_1}^{x_2}(\mu(s)-u_\epsilon(s))\ln\frac{1}{s+\epsilon}ds\\ &+\left(\frac{1}{1+\ln\frac{1}{x_1+\epsilon}} - \frac{1}{1+\ln\frac{1}{x_2+\epsilon}}\right)\int_{x_2}^1(\mu(s)-u_\epsilon(s))\ln\frac{1}{s+\epsilon}ds.\end{aligned}$$

We claim that, $\forall\,\tilde{\epsilon} > 0$, there exists $\delta > 0$ such that if $|x_1 - x_2| < \delta$, then

$$\left|\left(\frac{1}{1+\ln\frac{1}{x_1+\epsilon}} - \frac{1}{1+\ln\frac{1}{x_2+\epsilon}}\right)\int_{x_2}^1(\mu(s)-u_\epsilon(s))\ln\frac{1}{s+\epsilon}ds\right| < \tilde{\epsilon},\ \forall\epsilon \in \left(0,\frac{1}{2}\right). \quad (5.56)$$

First of all, there exists $\delta_1 > 0$, such that if $0 < x < \delta_1$, then $\int_0^x |\mu - u_\epsilon| < \tilde{\epsilon}$ and $|x\ln x| < \tilde{\epsilon}$. For this δ_1, since the function $\eta(x) = \frac{1}{1+\ln\frac{1}{x}}$ is uniformly continuous on

$\left[0, \frac{3}{2}\right]$, there exists $\delta_2 > 0$, such that if $0 < x_2 < \delta_2$, then

$$\left| \frac{1}{1+\ln \frac{1}{x_2+\epsilon}} - \frac{1}{1+\ln \frac{1}{\epsilon}} \right| < \delta_1.$$

Hence if $0 < x_2 < \min\{\delta_1, \delta_2\}$, then

$$\left| \int_{x_2}^{\frac{1}{1+\ln \frac{1}{x_2+\epsilon}} - \frac{1}{1+\ln \frac{1}{\epsilon}}} |\mu(s) - u_\epsilon(s)| ds \right| < \tilde{\epsilon},$$

and

$$\begin{aligned}
&\left| \left(\frac{1}{1+\ln \frac{1}{x_1+\epsilon}} - \frac{1}{1+\ln \frac{1}{x_2+\epsilon}} \right) \int_{x_2}^{1} (\mu(s) - u_\epsilon(s)) \ln \frac{1}{s+\epsilon} ds \right| \\
\leq & \left(\frac{1}{1+\ln \frac{1}{x_2+\epsilon}} \right) \left(\ln \frac{1}{x_2+\epsilon} \right) \left| \int_{x_2}^{\frac{1}{1+\ln \frac{1}{x_2+\epsilon}} - \frac{1}{1+\ln \frac{1}{\epsilon}}} |\mu(s) - u_\epsilon(s)| ds \right| \\
& + \left(\frac{1}{1+\ln \frac{1}{x_2+\epsilon}} - \frac{1}{1+\ln \frac{1}{\epsilon}} \right) \left| \ln \left(\frac{1}{1+\ln \frac{1}{x_2+\epsilon}} - \frac{1}{1+\ln \frac{1}{\epsilon}} \right) \right| \int_0^1 |\mu - u_\epsilon| ds \\
\leq & (1 + 2\|\mu\|_{L^1})\tilde{\epsilon}.
\end{aligned}$$

Denote $\delta_3 = \{\delta_1, \delta_2\}$. If $1 \geq x_2 \geq \delta_3$, there exists $\delta_4 > 0$ such that if $|x_1 - x_2| < \delta_4$, then

$$\left| \frac{1}{1+\ln \frac{1}{x_2+\epsilon}} - \frac{1}{1+\ln \frac{1}{x_1+\epsilon}} \right| < \frac{\tilde{\epsilon}}{2|\ln \delta_3| \|\mu\|_{L^1}}.$$

Therefore take $\delta = \min\{\delta_3, \delta_4\}$ and one obtains (5.56). The rest of the proof for this claim follows in the same way.

Step 3. We claim that for $\alpha > \frac{1}{2}$ the family $\left\{(|x| + \epsilon)^{2\alpha-1} u_\epsilon\right\}_{\epsilon>0}$ is equi-continuous on $[-1, 1]$. Assume $0 \leq x_1 < x_2 \leq 1$. By the integral form of u_ϵ in the proof of Theorem 5.12, we can write

$$\begin{aligned}
&(2\alpha - 1)\left[(|x_1| + \epsilon)^{2\alpha-1} u_\epsilon(x_1) - (|x_2| + \epsilon)^{2\alpha-1} u_\epsilon(x_2)\right] \\
= & -\int_{x_1}^{x_2} (\mu(s) - u_\epsilon(s)) ds \\
& + (1+\epsilon)^{1-2\alpha} \left[(x_1+\epsilon)^{2\alpha-1} - (x_2+\epsilon)^{2\alpha-1}\right] \left(\epsilon^{2\alpha} K_\epsilon^+ - \int_0^1 (\mu - u_\epsilon) ds \right) \\
& + (x_1+\epsilon)^{2\alpha-1} \int_{x_1}^{x_2} (\mu(s) - u_\epsilon(s))(s+\epsilon)^{1-2\alpha} ds \\
& + \left[(x_1+\epsilon)^{2\alpha-1} - (x_2+\epsilon)^{2\alpha-1}\right] \int_{x_2}^{1} (\mu(s) - u_\epsilon(s))(s+\epsilon)^{1-2\alpha} ds.
\end{aligned}$$

Notice that, $\forall\tilde{\epsilon}>0$, there exists $\delta_1>0$ such that if $0<x_2<\delta_1$, then

$$
\begin{aligned}
&\left|\left[(x_1+\epsilon)^{2\alpha-1}-(x_2+\epsilon)^{2\alpha-1}\right]\int_{x_2}^{1}(\mu(s)-u_\epsilon(s))(s+\epsilon)^{1-2\alpha}ds\right|\\
\leq&\int_{x_2}^{x_2^{\min\left\{\frac{1}{2},\frac{1}{2(2\alpha-1)}\right\}}}|\mu(s)-u_\epsilon(s)|ds+2C_{2\alpha-1}\left\|\mu\right\|_{L^1}|x_2|^{\min\left\{\frac{1}{2},\alpha-\frac{1}{2}\right\}}\\
\leq&\tilde{\epsilon},
\end{aligned}
$$

where $C_{2\alpha-1}$ is the Hölder constant of the function $\eta(x)=x^{2\alpha-1}$ on $\left[0,\frac{3}{2}\right]$. When $1\geq x_2\geq\delta_1$, we have

$$
\begin{aligned}
&\left|\left[(x_1+\epsilon)^{2\alpha-1}-(x_2+\epsilon)^{2\alpha-1}\right]\int_{x_2}^{1}(\mu(s)-u_\epsilon(s))(s+\epsilon)^{1-2\alpha}ds\right|\\
\leq&2C_{2\alpha-1}\delta_1^{1-2\alpha}\left\|\mu\right\|_{L^1}|x_1-x_2|^{\min\{1,2\alpha-1\}}.
\end{aligned}
$$

Therefore, take

$$
|x_1-x_2|<\min\left\{\delta_1,\left(\frac{\tilde{\epsilon}}{2\delta_1^{1-2\alpha}\left\|\mu\right\|_{L^1}C_{2\alpha-1}}\right)^{\frac{1}{\min\{1,2\alpha-1\}}}\right\}.
$$

We obtain

$$
\left|\left[(x_1+\epsilon)^{2\alpha-1}-(x_2+\epsilon)^{2\alpha-1}\right]\int_{x_2}^{1}(\mu(s)-u_\epsilon(s))(s+\epsilon)^{1-2\alpha}ds\right|\leq\tilde{\epsilon}.
$$

The rest of the proof for this claim follows in the same way.

Step 4. The Ascoli-Arzelà theorem and the uniqueness of the limit imply (5.54) and (5.55). □

We conclude this section with the proof of assertions (i), (ii) and (iii) (resp. (ii)) in Theorem 5.3 (resp. Theorem 5.4).

Proof of (i) and (ii) of Theorem 5.3. The limiting function u in Theorem 5.12 when $0<\alpha<1$ is exactly the good solution satisfying assertions (i) and (ii). □

Proof of (iii) of Theorem 5.3 and (ii) of Theorem 5.4. For $\mu\in\mathcal{M}(-1,1)$, the estimate (5.46) implies that $\left\|u\right\|_{L^1}\leq\left\|\mu\right\|_{\mathcal{M}}$. Take a nondecreasing function $\varphi\in C^\infty(\mathbb{R})$ such that $\varphi\equiv 0$ on $(-\infty,0]$ and $\varphi\equiv 1$ on $[1,\infty)$. Replace v by $\varphi(nu_\epsilon)$ in (5.45) and pass to the limit as $n\to\infty$. It follows that $\int_{-1}^{1}u_\epsilon^+dx\leq\left\|\mu^+\right\|_{\mathcal{M}}$. Then the Fatou's lemma yields the desired result. □

5.6 The lack of stability of the good solution when $\frac{1}{2} \le \alpha < 1$

Recall that in Section 5.4, the stability of the good solution when $0 < \alpha < \frac{1}{2}$ and the stability of the solution when $\alpha \ge 1$ and $\mu(\{0\}) = 0$ have been established in the proof of Theorems 5.3 and 5.4. Here we only investigate the case when $\frac{1}{2} \le \alpha < 1$. In this case, as we pointed out in Remark 5.5, the stability of the good solution fails.

Assume $\frac{1}{2} \le \alpha < 1$. Given $\mu \in \mathcal{M}(-1,1)$, there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset L^\infty(-1,1)$ such that $f_n \overset{*}{\rightharpoonup} \mu$ in $(C_0[-1,1])^*$. Let u_n be the unique good solution of the following equation

$$\begin{cases} -(|x|^{2\alpha}u_n')' + u_n = f_n \quad \text{on } (-1,1), \\ u_n(-1) = u_n(1) = 0. \end{cases} \tag{5.57}$$

In fact, from assertion (ii) of Proposition 5.7 we know that $u_n \in D(A_\alpha)$ and

$$\int_{-1}^{1} |x|^{2\alpha}u_n'\zeta' dx + \int_{-1}^{1} u_n\zeta dx = \int_{-1}^{1} f_n\zeta dx, \ \forall \zeta \in C_0^1[-1,1], \tag{5.58}$$

where $D(A_\alpha)$ is defined by (5.13).

In this case, the limiting behavior of the sequence $\{u_n\}_{n=1}^{\infty}$ is rather *sensitive* to the choice of the sequence $\{f_n\}_{n=1}^{\infty}$ and in our main result we present a "good" choice and a "bad" choice.

Theorem 5.14. *Assume $\frac{1}{2} \le \alpha < 1$. Fix $\rho \in C(\mathbb{R})$ such that $\operatorname{supp}\rho = [-1,1]$, $\rho(x) = \rho(-x)$ and $\rho \ge 0$. Let $C^{-1} = \int \rho$.*

(i) Let $\rho_n(x) = Cn\rho(nx)$ and $f_n = \mu * \rho_n$ so that $f_n \overset{*}{\rightharpoonup} \mu$ in $(C_0[-1,1])^*$. Then as $n \to \infty$, we have

$$\left(1 + \ln\frac{1}{|x|}\right)^{-1} u_n \to \left(1 + \ln\frac{1}{|x|}\right)^{-1} u \text{ in } L^p(-1,1), \ \forall p < \infty, \ \text{if } \alpha = \frac{1}{2}, \tag{5.59}$$

$$|x|^{2\alpha-1}u_n \to |x|^{2\alpha-1}u \text{ in } L^p(-1,1), \ \forall p < \infty, \ \text{if } \frac{1}{2} < \alpha < 1, \tag{5.60}$$

where u is the unique good solution of (5.1).

(ii) Let $\rho_n(x) = Cn\rho(nx-1)$ and $f_n = \mu * \rho_n$ so that $f_n \overset{*}{\rightharpoonup} \mu$ in $(C_0[-1,1])^*$. Then as $n \to \infty$, (5.59) and (5.60) still hold. However the limiting function u is not

necessarily the good solution of (5.1). *In fact, when* $\alpha = \frac{1}{2}$,

$$\begin{cases} \lim_{x\to 0^+} \left(1+\ln\frac{1}{|x|}\right)^{-1} u(x) = \mu(\{0\}), \\ \lim_{x\to 0^-} \left(1+\ln\frac{1}{|x|}\right)^{-1} u(x) = 0, \end{cases}$$

and when $\frac{1}{2} < \alpha < 1$,

$$\begin{cases} \lim_{x\to 0^+} |x|^{2\alpha-1} u(x) = \frac{\mu(\{0\})}{2\alpha-1}, \\ \lim_{x\to 0^-} |x|^{2\alpha-1} u(x) = 0. \end{cases}$$

Remark 5.8. *By convolution* $f_n = \mu * \rho_n$, *we mean that* $f_n = \bar{f}_n|_{[-1,1]}$, *where*

$$\bar{f}_n(x) = (\rho_n * \bar{\mu})(x) = \int_{-\infty}^{+\infty} \rho_n(x-y)d\bar{\mu}(y) \tag{5.61}$$

and $\bar{\mu}$ *is the zero extension of* μ *on* $\mathbb{R}$, *i.e.,* $\bar{\mu}(A) = \mu(A\cap(-1,1))$, *for all Borel sets* $A \subset \mathbb{R}$.

Remark 5.9. *Even if we assume* $\mu \in L^1(-1,1)$, *it still cannot be guaranteed that the limiting function* u *is the good solution. Indeed, we can take* $f_n(x) = Cn\rho(nx-1) - Cn\rho(nx+1)$, *where* ρ *and* C *are given in Theorem 5.14. Then* $f_n \stackrel{*}{\rightharpoonup} 0$ *in* $(C_0[-1,1])^*$, *but the limiting function* $u \not\equiv 0$. *In fact, when* $\alpha = \frac{1}{2}$,

$$\begin{cases} \lim_{x\to 0^+} \left(1+\ln\frac{1}{|x|}\right)^{-1} u(x) = 1, \\ \lim_{x\to 0^-} \left(1+\ln\frac{1}{|x|}\right)^{-1} u(x) = -1, \end{cases}$$

and when $\frac{1}{2} < \alpha < 1$,

$$\begin{cases} \lim_{x\to 0^+} |x|^{2\alpha-1} u(x) = \frac{1}{2\alpha-1}, \\ \lim_{x\to 0^-} |x|^{2\alpha-1} u(x) = -\frac{1}{2\alpha-1}. \end{cases}$$

Remark 5.10. *For* $\frac{1}{2} \le \alpha < 1$, *the limiting function* u *is the good solution if and only if*

$$\begin{aligned} &\lim_{x\to 0^+} \lim_{n\to\infty} \left(\int_0^x f_n(s)ds + \left(\ln\frac{1}{|x|}\right)^{-1} \int_x^1 f_n(s)\ln\frac{1}{|s|}ds \right) \\ =&\lim_{x\to 0^-} \lim_{n\to\infty} \left(\int_x^0 f_n(s)ds + \left(\ln\frac{1}{|x|}\right)^{-1} \int_{-1}^x f_n(s)\ln\frac{1}{|s|}ds \right) \\ =&\frac{1}{2}\mu(\{0\}), \qquad \textit{when } \alpha = \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned}
&\lim_{x\to0^+}\lim_{n\to\infty}\left(\int_0^x f_n(s)ds+|x|^{2\alpha-1}\int_x^1 f_n(s)|s|^{1-2\alpha}ds\right)\\
=&\lim_{x\to0^-}\lim_{n\to\infty}\left(\int_x^0 f_n(s)ds+|x|^{2\alpha-1}\int_{-1}^x f_n(s)|s|^{1-2\alpha}ds\right)\\
=&\frac{1}{2}\mu(\{0\}),\qquad \textit{when } \frac{1}{2}<\alpha<1.
\end{aligned}$$

If $\mu\in L^1(-1,1)$ and the convergence is under the weak topology $\sigma(L^1,L^\infty)$, we can recover the stability of the good solution.

Theorem 5.15. *Assume that* $\frac{1}{2}\le\alpha<1$ *and* $\mu\in L^1(-1,1)$. *Let the sequence* $\{f_n\}_{n=1}^\infty\subset L^\infty(-1,1)$ *be such that* $f_n\rightharpoonup\mu$ *weakly in* $\sigma(L^1,L^\infty)$. *Let* u_n *be the unique good solution of* (5.57). *Then as* $n\to\infty$, *we have*

$$\left(1+\ln\frac{1}{|x|}\right)^{-1}u_n\to\left(1+\ln\frac{1}{|x|}\right)^{-1}u \ \textit{ in } C_0[-1,1],\ \textit{ if } \alpha=\frac{1}{2},\tag{5.62}$$

$$|x|^{2\alpha-1}u_n\to|x|^{2\alpha-1}u \ \textit{ in } C_0[-1,1],\ \textit{ if } \frac{1}{2}<\alpha<1,\tag{5.63}$$

where u *is the good solution of* (5.1).

Remark 5.11. *Under the assumption of Theorem 5.15, a functional analysis argument implies that* $u_n\rightharpoonup u$ *weakly in* $\sigma(L^1,L^\infty)$ *and* u *is always the unique good solution corresponding to* μ. *Indeed, recall the notation from Section 5.2 and denote*

$$\begin{aligned}
T: L^1(-1,1)&\to D(A_\alpha)\subset L^1(-1,1)\\
\mu&\mapsto(I+A_\alpha)^{-1}\mu.
\end{aligned}$$

Proposition 5.7 implies that T *is a bounded linear operator. Therefore, in view of Proposition 3.1 in [8], it is easy to check that* $Tf_n\rightharpoonup T\mu$ *weakly in* $\sigma(L^1,L^\infty)$. *Recall that* $Tf_n=u_n$, *so* $u_n\rightharpoonup u$ *weakly in* $\sigma(L^1,L^\infty)$ *where* $u=T\mu$. *The definition of* T *implies that* u *is the good solution corresponding to* μ. *A proof of* (5.62) *and* (5.63) *will be presented in the end of this section.*

We now start to prove Theorem 5.14; the proof relies on the following four lemmas.

Lemma 5.16. *Assume that* $\alpha = \frac{1}{2}$ *and* $\mu \in \mathcal{M}(-1,1)$. *Let* $\{f_n\}_{n=1}^{\infty} \subset L^{\infty}(-1,1)$ *be such that* $f_n \overset{*}{\rightharpoonup} \mu$ *in* $(C_0[-1,1])^*$. *Let* u_n *be the unique good solution of* (5.57). *Then there exists a subsequence* $\{n_k\}_{k=1}^{\infty}$ *such that*

$$\left(1 + \ln \frac{1}{|x|}\right)^{-1} u_{n_k} \to \left(1 + \ln \frac{1}{|x|}\right)^{-1} u \text{ in } L^p(-1,1),\ \forall p < \infty, \tag{5.64}$$

where u *is a solution of* (5.1), *such that* $\left(1 + \ln \frac{1}{|x|}\right)^{-1} u \in BV(-1,1)$ *and*

$$\begin{aligned} &\lim_{x\to 0^+} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u(x) \\ =& \lim_{x\to 0^+} \lim_{k\to\infty} \left(\int_0^x f_{n_k}(s)ds + \left(\ln \frac{1}{|x|}\right)^{-1} \int_x^1 f_{n_k}(s) \ln \frac{1}{|s|} ds \right), \end{aligned} \tag{5.65}$$

$$\begin{aligned} &\lim_{x\to 0^-} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u(x) \\ =& \lim_{x\to 0^-} \lim_{k\to\infty} \left(\int_x^0 f_{n_k}(s)ds + \left(\ln \frac{1}{|x|}\right)^{-1} \int_{-1}^x f_{n_k}(s) \ln \frac{1}{|s|} ds \right). \end{aligned} \tag{5.66}$$

Proof. The proof of Theorem 5.3 for the case $\alpha = \frac{1}{2}$ shows that there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that (5.64) holds, where u is a solution of (5.1) and $\left(1 + \ln \frac{1}{|x|}\right)^{-1} u \in BV(-1,1)$. We only need to establish (5.65) and (5.66). Notice that, with the help of (5.24), we have

$$\begin{aligned} &\lim_{x\to 0^+} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u(x) \\ =& \lim_{x\to 0^+} \lim_{k\to\infty} \left(\ln \frac{1}{|x|}\right)^{-1} u_{n_k}(x) \\ =& \lim_{x\to 0^+} \lim_{k\to\infty} \left(\int_0^x A_\alpha u_{n_k}(s)ds + \left(\ln \frac{1}{|x|}\right)^{-1} \int_x^1 A_\alpha u_{n_k}(s) \ln \frac{1}{|s|} ds \right). \end{aligned}$$

Write $A_\alpha u_{n_k} = f_{n_k} - u_{n_k}$. We can check that

$$\begin{aligned} &\lim_{x\to 0^+} \lim_{k\to\infty} \left(\int_0^x u_{n_k}(s)ds + \left(\ln \frac{1}{|x|}\right)^{-1} \int_x^1 u_{n_k}(s) \ln \frac{1}{|s|} ds \right) \\ =& \lim_{x\to 0^+} \left(\int_0^x u(s)ds + \left(\ln \frac{1}{|x|}\right)^{-1} \int_x^1 u(s) \ln \frac{1}{|s|} ds \right) \\ =&0. \end{aligned}$$

Therefore we obtain (5.65). One can perform the same computation to verify (5.66). ☐

Lemma 5.17. *Assume that* $\frac{1}{2} < \alpha < 1$ *and* $\mu \in \mathcal{M}(-1,1)$*. Let* $\{f_n\}_{n=1}^{\infty} \subset L^{\infty}(-1,1)$ *be such that* $f_n \overset{*}{\rightharpoonup} \mu$ *in* $(C_0[-1,1])^*$*. Let* u_n *be the unique good solution of* (5.57)*. Then there exists a subsequence* $\{n_k\}_{k=1}^{\infty}$ *such that*

$$|x|^{2\alpha-1}u_{n_k} \to |x|^{2\alpha-1}u \text{ in } L^p(-1,1),\ \forall p < \infty, \tag{5.67}$$

where u *is a solution of* (5.1)*, such that* $|x|^{2\alpha-1}u \in BV(-1,1)$ *and*

$$\begin{aligned}&\lim_{x\to0^+}|x|^{2\alpha-1}u(x)\\ =&\frac{1}{2\alpha-1}\lim_{x\to0^+}\lim_{k\to\infty}\left(\int_0^x f_{n_k}(s)ds + |x|^{2\alpha-1}\int_x^1 f_{n_k}(s)|s|^{1-2\alpha}ds\right),\end{aligned} \tag{5.68}$$

$$\begin{aligned}&\lim_{x\to0^-}|x|^{2\alpha-1}u(x)\\ =&\frac{1}{2\alpha-1}\lim_{x\to0^-}\lim_{k\to\infty}\left(\int_x^0 f_{n_k}(s)ds + |x|^{2\alpha-1}\int_{-1}^x f_{n_k}(s)|s|^{1-2\alpha}ds\right).\end{aligned} \tag{5.69}$$

Proof. Based on the proof of Theorem 5.3 for the case $\frac{1}{2} < \alpha < 1$, we only need to establish (5.68) and (5.69). Indeed,

$$\begin{aligned}&\lim_{x\to0^+}|x|^{2\alpha-1}u(x)\\ =&\lim_{x\to0^+}\lim_{k\to\infty}|x|^{2\alpha-1}u_{n_k}(x)\\ =&\lim_{x\to0^+}\lim_{k\to\infty}\left(\frac{1-|x|^{2\alpha-1}}{2\alpha-1}\int_0^x Au_{n_k}(s)ds + \frac{|x|^{2\alpha-1}}{2\alpha-1}\int_x^1 Au_{n_k}(s)(|s|^{1-2\alpha}-1)ds\right)\\ =&\frac{1}{2\alpha-1}\lim_{x\to0^+}\lim_{k\to\infty}\left(\int_0^x f_{n_k}(s)ds + |x|^{2\alpha-1}\int_x^1 f_{n_k}(s)|s|^{1-2\alpha}ds\right).\end{aligned}$$

One can perform the same computation to get (5.69). ☐

Lemma 5.18. *Fix* $\rho \in C(\mathbb{R})$ *such that* $\operatorname{supp}\rho = [-1,1]$*,* $\rho(x) = \rho(-x)$ *and* $\rho \geq 0$*. Let* $\rho_n(x) = Cn\rho(nx)$ *where* $C^{-1} = \int\rho$*. For* $\mu \in \mathcal{M}(-1,1)$*, let* $f_n = \mu * \rho_n$*. Then* $f_n \in C[-1,1]$*,* $\|f_n\|_{L^1} \leq \|\mu\|_{\mathcal{M}}$*, and* $f_n \overset{*}{\rightharpoonup} \mu$ *in* $(C_0[-1,1])^*$*. For any* $-1 < a < b < 1$

and $y \in [-1,1]$, *we have*

$$\lim_{n\to\infty}\int_{a-y}^{b-y}\rho_n(s)ds = \begin{cases} 0, & \text{for } y \in [-1,a), \\ \frac{1}{2}, & \text{for } y = a, \\ 1, & \text{for } y \in (a,b), \\ \frac{1}{2}, & \text{for } y = b, \\ 0, & \text{for } y \in (b,1]. \end{cases} \tag{5.70}$$

Moreover,

$$\lim_{x\to 0^+}\lim_{n\to\infty}\int_0^x f_n(s)ds = \lim_{x\to 0^-}\lim_{n\to\infty}\int_x^0 f_n(s)ds = \frac{1}{2}\mu(\{0\}). \tag{5.71}$$

Proof. Recall the notation (5.61). From Propositions 4.18 and 4.19 in [8], we know that $\bar{f}_n \in C_c(\mathbb{R})$, which in particular implies that $f_n \in C[-1,1]$. We can compute the L^1-norm of f_n as

$$\|f_n\|_{L^1} \le \int_{-\infty}^{+\infty}\left(\int_{-1}^{1}|\rho_n(x-y)|\,dx\right)d\,|\bar{\mu}|\,(y) \le \|\mu\|_{\mathcal{M}}\,.$$

For any $\zeta \in C_0[-1,1]$, define $\bar{\zeta} \in C_c(\mathbb{R})$ as the zero extension of ζ. Then

$$\lim_{n\to\infty}\int_{-1}^{1} f_n(x)\zeta(x)dx = \lim_{n\to\infty}\int_{-\infty}^{+\infty}\left(\rho_n * \bar{\zeta}\right)(y)d\bar{\mu}(y) = \int_{-1}^{1}\zeta d\mu,$$

since $\rho_n * \bar{\zeta} \to \bar{\zeta}$ uniformly as $n \to \infty$. It is straightforward to verify (5.70). Apply (5.70) for $a = 0$ and $b = x$ with $x \in (0,1)$, so the Dominated Convergence Theorem yields

$$\lim_{n\to\infty}\int_0^x f_n(s)ds = \lim_{n\to\infty}\int_{-1}^{1}\int_{-y}^{x-y}\rho_n(s)dsd\mu(y) = \mu((0,x)) + \frac{1}{2}\mu(\{x\}) + \frac{1}{2}\mu(\{0\}).$$

Similarly, for any $x \in (-1,0)$,

$$\lim_{n\to\infty}\int_x^0 f_n(s)ds = \mu((x,0)) + \frac{1}{2}\mu(\{x\}) + \frac{1}{2}\mu(\{0\}).$$

Notice that

$$\lim_{x\to 0^+}\mu((0,x)) = \lim_{x\to 0^+}\mu(\{x\}) = \lim_{x\to 0^-}\mu((x,0)) = \lim_{x\to 0^-}\mu(\{x\}) = 0.$$

Therefore identity (5.71) holds. $\square$

Lemma 5.19. *Let $\rho_n(x) = Cn\rho(nx-1)$ where C and ρ are specified in Lemma 5.18. For $\mu \in \mathcal{M}(-1,1)$, let $f_n = \mu * \rho_n$. Then $f_n \in C[-1,1]$, $\|f_n\|_{L^1} \le \|\mu\|_{\mathcal{M}}$ and $f_n \overset{*}{\rightharpoonup} \mu$ in $(C_0[-1,1])^*$. For any $-1 < a < b < 1$ and $y \in [-1,1]$, we have*

$$\lim_{n\to\infty} \int_{a-y}^{b-y} \rho_n(s)ds = \begin{cases} 0, \ \textit{for } y \in [-1,a), \\ 1, \ \textit{for } y \in [a,b), \\ 0, \ \textit{for } y \in [b,1]. \end{cases} \tag{5.72}$$

Moreover,

$$\begin{cases} \lim\limits_{x\to 0^+} \lim\limits_{n\to\infty} \int_0^x f_n(s)ds = \lim\limits_{x\to 0^+} \mu\left([0,x)\right) = \mu\left(\{0\}\right), \\ \lim\limits_{x\to 0^-} \lim\limits_{n\to\infty} \int_x^0 f_n(s)ds = \lim\limits_{x\to 0^-} \mu\left([x,0)\right) = 0. \end{cases} \tag{5.73}$$

The proof of Lemma 5.19 is the same as the one of Lemma 5.18. We now prove Theorem 5.14.

Proof of Theorem 5.14. We first prove (i). When $\alpha = \frac{1}{2}$, in view of Lemma 5.16 we only need to show

$$\begin{aligned} &\lim_{x\to 0^+} \lim_{n\to\infty} \left(\int_0^x f_n(s)ds + \left(\ln\frac{1}{|x|}\right)^{-1} \int_x^1 f_n(s)\ln\frac{1}{|s|}ds \right) \\ =&\lim_{x\to 0^-} \lim_{n\to\infty} \left(\int_x^0 f_n(s)ds + \left(\ln\frac{1}{|x|}\right)^{-1} \int_{-1}^x f_n(s)\ln\frac{1}{|s|}ds \right) \\ =&\frac{1}{2}\mu(\{0\}). \end{aligned}$$

Assume $x \in (0,1)$. We can write

$$\begin{aligned} \left| \left(\ln\frac{1}{|x|}\right)^{-1} \int_x^1 f_n(s)\ln\frac{1}{|s|}ds \right| &\le \int_x^{\frac{1}{1-\ln x}} |f_n(s)|ds + \frac{\ln(1-\ln x)}{-\ln x} \int_{\frac{1}{1-\ln x}}^1 |f_n(s)|ds \\ &\le \int_x^{\frac{1}{1-\ln x}} |f_n(s)|ds + \|\mu\|_{\mathcal{M}} \frac{\ln(1-\ln x)}{-\ln x}. \end{aligned}$$

Take $a = x$ and $b = \frac{1}{1-\ln x}$ in (5.70). We have

$$\begin{aligned} \lim_{n\to\infty} \int_x^{\frac{1}{1-\ln x}} |f_n(s)|ds &= \lim_{n\to\infty} \int_{-1}^1 \int_{x-y}^{\frac{1}{1-\ln x}-y} \rho_n(s)ds\, d|\mu|(y) \\ &= \frac{1}{2}|\mu|(\{x\}) + \frac{1}{2}|\mu|\left(\left\{\frac{1}{1-\ln x}\right\}\right) + |\mu|\left(\left(x, \frac{1}{1-\ln x}\right)\right). \end{aligned}$$

Therefore,

$$\lim_{x\to 0^+}\lim_{n\to\infty}\left(\ln\frac{1}{|x|}\right)^{-1}\int_x^1 f_n(s)\ln\frac{1}{|s|}ds=0.$$

Similarly,

$$\lim_{x\to 0^-}\lim_{n\to\infty}\left(\ln\frac{1}{|x|}\right)^{-1}\int_{-1}^x f_n(s)\ln\frac{1}{|s|}ds=0.$$

Thus (5.71) gives the desired result.

When $\frac{1}{2}<\alpha<1$, by Lemma 5.17 we only need to show

$$\begin{aligned}
&\frac{1}{2\alpha-1}\lim_{x\to 0^+}\lim_{n\to\infty}\left(\int_0^x f_n(s)ds+|x|^{2\alpha-1}\int_x^1 f_n(s)|s|^{1-2\alpha}ds\right)\\
=&\frac{1}{2\alpha-1}\lim_{x\to 0^-}\lim_{n\to\infty}\left(\int_x^0 f_n(s)ds+|x|^{2\alpha-1}\int_{-1}^x f_n(s)|s|^{1-2\alpha}ds\right)\\
=&\frac{1}{2(2\alpha-1)}\mu(\{0\}).
\end{aligned}$$

Assume $x\in(0,1)$. We can write

$$\begin{aligned}
\left||x|^{2\alpha-1}\int_x^1 f_n(s)|s|^{1-2\alpha}ds\right| &\le \int_x^{\sqrt{x}}|f_n(s)|ds+|x|^{2\alpha-1}\int_{\sqrt{x}}^1|f_n(s)||s|^{1-2\alpha}ds\\
&\le \int_x^{\sqrt{x}}|f_n(s)|ds+\|\mu\|_{\mathcal{M}}|x|^{\alpha-\frac{1}{2}}.
\end{aligned}$$

We have

$$\begin{aligned}
\lim_{x\to 0^+}\lim_{n\to\infty}\int_x^{\sqrt{x}}|f_n(s)|ds &= \lim_{x\to 0^+}\lim_{n\to\infty}\int_{-1}^1\int_{x-y}^{\sqrt{x}-y}\rho_n(s)ds\,d|\mu|(y)\\
&=\lim_{x\to 0^+}\left(\frac{1}{2}|\mu|(\{x\})+\frac{1}{2}|\mu|\left(\{\sqrt{x}\}\right)+|\mu|\left((x,\sqrt{x})\right)\right)\\
&=0.
\end{aligned}$$

Therefore (5.71) gives the desired result.

The proof of (ii) can be done in the same way, i.e., we can compute the one-sided limits (5.65), (5.66), (5.68) and (5.69) with the help of Lemma 5.19.

For both (i) and (ii), the convergence (5.64) and (5.67) can be recovered for the whole sequence $\{u_n\}_{n=1}^{\infty}$ by the uniqueness of the limit. □

Proof of Remark 6.9. From Exercise 4.37 of [8], we know that $f_n \overset{*}{\rightharpoonup} 0$ in $(C_0[-1,1])^*$. A direct computation for the one-sided limits (5.65), (5.66), (5.68) and (5.69) yields the conclusion. □

Proof of Remark 5.10. Just apply Lemma 5.16 and 5.17. □

We conclude this section with the

Proof of Theorem 5.15. In view of Remark 5.11, we only need to show (5.62) and (5.63). We divide the proof into four steps.

Step 1. Since $f_n \rightharpoonup \mu$ weakly in $\sigma(L^1, L^\infty)$, we obtain that the sequence $\{f_n\}_{n=1}^\infty$ is bounded in $L^1(-1,1)$ and it is equi-integrable.

Step 2. With the help of assertion (iv) in Proposition 5.7, the same argument from Step 1 in the proof of Theorem 5.13 shows that the sequence $\{u_n\}_{n=1}^\infty$ is equi-integrable.

Step 3. We claim that for $\alpha = \frac{1}{2}$ the sequence $\left\{\left(1+\ln\frac{1}{|x|}\right)^{-1} u_n\right\}_{n=1}^\infty$ is equi-continuous on $[-1,1]$ and for $\frac{1}{2} < \alpha < 1$ the sequence $\left\{|x|^{2\alpha-1}u_n\right\}_{n=1}^\infty$ is equi-continuous on $[-1,1]$. Here as an example we just show that the sequence $\left\{|x|^{2\alpha-1}u_n\right\}_{n=1}^\infty$ is equi-continuous on $[0,1]$ when $\frac{1}{2} < \alpha < 1$; all the other cases can be done in the same way.

Assume $0 \le x_1 < x_2 \le 1$. From (5.26) we can write

$$\begin{aligned}
&(1-2\alpha)\left[x_1^{2\alpha-1}u_n(x_1) - x_2^{2\alpha-1}u_n(x_2)\right] \\
&= \left(x_1^{2\alpha-1} - x_2^{2\alpha-1}\right)\int_0^1 (f_n(s)-u_n(s))ds - x_1^{2\alpha-1}\int_{x_1}^{x_2}(f_n(s)-u_n(s))s^{1-2\alpha}ds \\
&\quad + \int_{x_1}^{x_2}(f_n(s)-u_n(s))ds + \left(x_2^{2\alpha-1} - x_1^{2\alpha-1}\right)\int_{x_2}^1 (f_n(s)-u_n(s))s^{1-2\alpha}ds.
\end{aligned}$$

We claim that, for any $\epsilon > 0$, there exists $\delta > 0$ such that if $|x_1 - x_2| < \delta$, then

$$\left|\left(x_2^{2\alpha-1} - x_1^{2\alpha-1}\right)\int_{x_2}^1 (f_n(s)-u_n(s))s^{1-2\alpha}ds\right| < \epsilon. \tag{5.74}$$

Notice that, since the sequences $\{u_n\}_{n=1}^\infty$ and $\{f_n\}_{n=1}^\infty$ are equi-integrable and

$$C = \sup_n \left(\|f_n\|_{L^1} + \|u_n\|_{L^1}\right) < \infty,$$

there exists $\delta_1 > 0$, such that if $0 < x < \delta_1$ then

$$\begin{aligned}
&\left|x^{2\alpha-1}\int_x^1 (f_n(s)-u_n(s))s^{1-2\alpha}ds\right| \\
&\le \int_x^{\sqrt{x}} |f_n(s)-u_n(s)|ds + x^{\alpha-\frac{1}{2}}\int_0^1 |f_n - u_n|ds \le \epsilon.
\end{aligned}$$

Therefore, if $0 < x_2 < \delta_1$ then

$$\left|\left(x_2^{2\alpha-1} - x_1^{2\alpha-1}\right)\int_{x_2}^{1}(f_n(s) - u_n(s))s^{1-2\alpha}ds\right| \leq \epsilon.$$

If $1 \geq x_2 \geq \delta_1$, take $|x_1 - x_2| \leq \left(\frac{\epsilon}{CC_{2\alpha-1}}\right)^{\frac{1}{2\alpha-1}}\delta_1$, and it follows that

$$\left|\left(x_2^{2\alpha-1} - x_1^{2\alpha-1}\right)\int_{x_2}^{1}(f_n(s) - u_n(s))s^{1-2\alpha}ds\right| \leq C\delta_1^{1-2\alpha}C_{2\alpha-1}\left|x_1 - x_2\right|^{2\alpha-1} \leq \epsilon,$$

where $C_{2\alpha-1}$ is the Hölder constant for the function $\eta(x) = x^{2\alpha-1}$ on $[0,1]$. Therefore, take

$$\delta = \min\left\{\delta_1, \left(\frac{\epsilon}{CC_{2\alpha-1}}\right)^{\frac{1}{2\alpha-1}}\delta_1\right\}$$

and it leads to (5.74). Hence $\left\{|x|^{2\alpha-1}u_n\right\}_{n=1}^{\infty}$ is equi-continuous on $[0,1]$.

Step 4. The Ascoli-Arzelà theorem and the uniqueness of the limit imply (5.62) and (5.63). □

5.7 The problem on the interval $(0,1)$

In this section, we are going to discuss equation (5.8) under several appropriate boundary conditions at 0. By a *solution* of (5.8), we mean a function u such that

$$u \in W^{1,1}_{loc}(0,1],\ x^{2\alpha}u' \in BV_{loc}(0,1],$$

and u satisfies (5.8) in the usual sense.

When $\mu \equiv 0$, equation (5.8) under nonhomogeneous boundary conditions at 0 has been studied in Chapter 4. Therefore, for $\mu \in \mathcal{M}(0,1)$, we will focus on equation (5.8) under homogeneous boundary conditions. We have the following existence result.

Theorem 5.20. *Given $\mu \in \mathcal{M}(0,1)$, the following assertions hold.*

(i) When $0 < \alpha < \frac{1}{2}$, there exists a solution u of (5.8) *such that $u \in C[0,1]$, $x^{2\alpha}u' \in BV(0,1)$ and $\lim_{x\to 0^+} u(x) = 0$.*

(ii) When $0 < \alpha < \frac{1}{2}$, there exists a solution u of (5.8) *such that $u \in C[0,1]$, $x^{2\alpha}u' \in BV(0,1)$ and $\lim_{x\to 0^+} x^{2\alpha}u'(x) = 0$.*

(iii) *When* $\alpha = \frac{1}{2}$, *there exists a solution* u *of* (5.8) *such that* $\left(1 + \ln \frac{1}{x}\right)^{-1} u(x) \in C[0,1]$, $xu' \in BV(0,1)$ *and* $\lim_{x \to 0^+} xu'(x) = \lim_{x \to 0^+} \left(1 + \ln \frac{1}{x}\right)^{-1} u(x) = 0$.

(iv) *When* $\frac{1}{2} < \alpha < 1$, *there exists a solution* u *of* (5.8) *such that* $x^{2\alpha - 1} u(x) \in C[0,1]$, $x^{2\alpha} u' \in BV(0,1)$ *and* $\lim_{x \to 0^+} x^{2\alpha} u'(x) = \lim_{x \to 0^+} x^{2\alpha - 1} u(x) = 0$.

(v) *When* $\alpha \geq 1$, *there exists a solution* u *of* (5.8) *such that* $u \in L^1(0,1)$, $x^\alpha u(x) \in C[0,1]$, $x^{2\alpha} u' \in BV(0,1)$ *and* $\lim_{x \to 0^+} x^{2\alpha} u'(x) = \lim_{x \to 0^+} x^\alpha u(x) = 0$.

Note that the uniqueness result has been established in Theorems 3.2, 3.5, 3.8, 3.12 and 3.15.

Remark 5.12. *We have the following observations about the relation between equations* (5.1) *and* (5.8).

(i) *When* $0 < \alpha < \frac{1}{2}$, *for* $\mu \in \mathcal{M}(0,1)$, *let* $\bar{\mu} \in \mathcal{M}(-1,1)$ *be the even reflection of* μ *which doesn't charge the origin, i.e.,* $\bar{\mu}(A) = \mu(A \cap (0,1)) + \mu((-A) \cap (0,1))$, *for any Borel set* $A \subset (-1,1)$. *Let* v *be the good solution of* (5.1) *corresponding to* $\bar{\mu}$. *The uniqueness of the good solution implies that* $v(x) = v(-x)$. *Then it is straightforward that*

$$\begin{aligned}\lim_{x \to 0} |x|^{2\alpha} v'(x) =& \frac{1}{2} \int_{(-1,0) \cup (0,1)} (\text{sign}\, s) \left(1 - |s|^{1-2\alpha}\right) d\bar{\mu}(s) \\ & - \frac{1}{2} \int_{-1}^{1} (\text{sign}\, s) v(s) \left(1 - |s|^{1-2\alpha}\right) ds \\ =& 0.\end{aligned}$$

Therefore, $v|_{(0,1)}$ *is the solution of* (5.8) *satisfying (ii) of Theorem 5.20.*

(ii) *When* $\frac{1}{2} \leq \alpha < 1$, *for* $\mu \in \mathcal{M}(0,1)$, *let* $\bar{\mu} \in \mathcal{M}(-1,1)$ *be any extension of* μ *which doesn't charge the origin, i.e.,* $\bar{\mu}(\{0\}) = 0$, *and* $\bar{\mu}(A) = \mu(A)$, *for any Borel set* $A \subset (0,1)$. *Let* v *be the good solution of* (5.1) *corresponding to* $\bar{\mu}$. *Theorem 5.3 implies that* $u = v|_{(0,1)}$ *is the solution of* (5.8) *satisfying (iii) and (iv) of Theorem 5.20.*

(iii) *When* $\alpha \geq 1$, *for* $\mu \in \mathcal{M}(0,1)$, *let* $\bar{\mu} \in \mathcal{M}(-1,1)$ *be any extension of* μ *which doesn't charge the origin, i.e.,* $\bar{\mu}(\{0\}) = 0$, *and* $\bar{\mu}(A) = \mu(A)$, *for any Borel set*

$A \subset (0,1)$. Let v be the solution of (5.1) corresponding to $\bar{\mu}$. Theorem 5.4 implies that $u = v|_{(0,1)}$ is the solution of (5.8) satisfying (v) of Theorem 5.20.

Remark 5.13. *For all the five cases in Theorem 5.20, it always holds that $\|u\|_{L^1} \leq \|\mu\|_{\mathcal{M}}$ and $\|u^+\|_{L^1} \leq \|\mu^+\|_{\mathcal{M}}$. Moreover, for the solution satisfying (i) of Theorem 5.20, we have $\|u\|_{L^1} \leq \tilde{k}_\alpha \|\mu\|_{\mathcal{M}}$ for some $\tilde{k}_\alpha \in (0,1)$; for the solution satisfying (ii), (iii) and (iv) of Theorem 5.20, Remark 5.2 and 5.12 imply that $\|u\|_{L^1} \leq k_\alpha \|\mu\|_{\mathcal{M}}$.*

Remark 5.14. *We have the following observations about the optimality of Theorem 5.20.*

(i) When $0 < \alpha < \frac{1}{2}$, assertions (i) and (ii) of Theorem 5.20 imply that $u \in W^{1,p}(0,1)$, $\forall p < \frac{1}{2\alpha}$. In general, we cannot obtain that $u \in W^{1,\frac{1}{2\alpha}}(0,1)$.

(ii) When $\frac{1}{2} \leq \alpha < 1$, assertions (iii) and (iv) of Theorem 5.20 imply that $u \in L^p(0,1)$, $\forall p < \frac{1}{2\alpha-1}$ (define $\frac{1}{2\alpha-1} = +\infty$ if $\alpha = \frac{1}{2}$). In general, we cannot obtain that $u \in L^{\frac{1}{2\alpha-1}}(0,1)$.

(iii) The boundary behaviors listed in assertions (ii), (iii), (iv) and (v) of Theorem 5.20 are optimal in the following sense. Fix $x \in \left(0, \frac{1}{2}\right)$ and define

$$K_\alpha(x) = \sup_{\|\mu\|_{\mathcal{M}} \leq 1} \left|x^{2\alpha} u'(x)\right|, \ \text{when } \alpha > 0,$$

$$\widetilde{K}_\alpha(x) = \begin{cases} \sup\limits_{\|\mu\|_{\mathcal{M}} \leq 1} \left|\left(1 + \ln \frac{1}{x}\right)^{-1} u(x)\right|, & \text{when } \alpha = \frac{1}{2}, \\ \sup\limits_{\|\mu\|_{\mathcal{M}} \leq 1} \left|x^{2\alpha-1} u(x)\right|, & \text{when } \frac{1}{2} < \alpha < 1, \\ \sup\limits_{\|\mu\|_{\mathcal{M}} \leq 1} \left|x^{\alpha} u(x)\right|, & \text{when } \alpha \geq 1, \end{cases}$$

where u is the solution of (5.8) identified in (ii), (iii), (iv) and (v) of Theorem 5.20 and we assume that $x^{2\alpha}u'$ is right-continuous (or left-continuous). Then $0 < \delta_\alpha \leq K_\alpha(x) \leq C_\alpha$, $\forall x \in \left(0, \frac{1}{2}\right)$, and $0 < \delta_\alpha \leq \widetilde{K}_\alpha(x) \leq C_\alpha$, $\forall x \in \left(0, \frac{1}{2}\right)$, where δ_α and C_α are constants depending only on α.

Proof of Theorem 5.20. Note that assertions (ii), (iii), (iv) and (v) are consequences of Remark 5.12. We only need to prove assertion (i). Assume $0 < \alpha < \frac{1}{2}$ and recall the

Hilbert space

$$X_{00}^{\alpha}(0,1)=\left\{u\in H_{loc}^{1}(0,1);\ u\in L^{2}(0,1),\ x^{\alpha}u'\in L^{2}(0,1),\ u(0)=u(1)=0\right\},$$

defined in Section 3.6. We know that $X_0^{\alpha}(0,1)\subset C_0[0,1]$ for $0<\alpha<\frac{1}{2}$. In particular, $\mathcal{M}(0,1)\subset(X_{00}^{\alpha}(0,1))^{*}$. By the theorem of Lax-Milgram there exists $u\in X_{00}^{\alpha}(0,1)$ such that

$$\int_0^1 x^{2\alpha}u'v'dx+\int_0^1 uvdx=\int_0^1 vd\mu,\ \forall v\in X_{00}^{\alpha}(0,1). \tag{5.75}$$

This u satisfies assertion (i). □

Proof of Remark 5.13. We only need to prove this remark for u satisfying (i) of Theorem 5.20. To prove $\|u\|_{L^1}\le\|\mu\|_{\mathcal{M}}$ (resp. $\|u^{+}\|_{L^1}\le\|\mu^{+}\|_{\mathcal{M}}$), it is enough to take $v=\phi_n(u)$ in (5.75) where ϕ_n is the smooth approximation of $\operatorname{sign}x$ (resp. $\operatorname{sign}^{+}x$). To prove $\|u\|_{L^1}\le\tilde{k}_{\alpha}\|\mu\|_{\mathcal{M}}$, we only need to show that, for $0<\alpha<\frac{1}{2}$, we have $\tilde{k}_{\alpha}<1$, where $\tilde{k}_{\alpha}=\max_{x\in[0,1]}|F_{\alpha}(x)|$ and $F_{\alpha}\in C[0,1]\cap C^{\infty}(0,1)$ satisfying

$$\begin{cases}-\left(x^{2\alpha}F_{\alpha}'\right)'+F_{\alpha}=1 \quad \text{on } (0,1),\\ F_{\alpha}(0)=F_{\alpha}(1)=0.\end{cases}$$

Notice that $F_{\alpha}\ge 0$. Take $x_0\in(0,1)$ such that $F_{\alpha}(x_0)=\max_{x\in[0,1]}F_{\alpha}(x)$. Then $F_{\alpha}'(x_0)=0$ and $F_{\alpha}''(x_0)\le 0$. Since

$$\tilde{k}_{\alpha}=F_{\alpha}(x_0)=1+x_0^{2\alpha}F_{\alpha}''(x_0)+2\alpha x_0^{2\alpha-1}F_{\alpha}'(x_0),$$

it is enough to show that $F_{\alpha}''(x_0)<0$. Indeed, if $F_{\alpha}''(x_0)=0$, the uniqueness of the initial value problem for F_{α}' at x_0 implies that $F_{\alpha}'\equiv 0$ in a neighborhood of x_0. It follows that $F_{\alpha}\equiv 1$ in a neighborhood of x_0. We can write $F_{\alpha}=C_1f_1+C_2f_2+1$ for some constants C_1 and C_2, where f_1 and f_2 are the set of general solutions for the corresponding homogeneous equation. The linear independence of f_1 and f_2 forces $C_1=C_2=0$. We obtain a contradiction with $F_{\alpha}(0)=0$. □

Proof of Remark 5.14. We first prove assertion (i). We assume $0<\alpha<\frac{1}{2}$. Take $u(x)=x^{1-2\alpha}(x-1)$. It is easy to check that $\mu:=-(x^{2\alpha}u')'+u\in L^{\infty}(0,1)$ and $u(0)=u(1)=0$. However $u'\notin L^{\frac{1}{2\alpha}}(0,1)$. Take $v(x)=x^{1-2\alpha}\left(1+\ln\frac{1}{x}\right)^{-\alpha}-1$. It is easy to check that $\nu := -(x^{2\alpha}v')' + v \in L^1(0,1)$ and $\lim_{x\to 0^+} x^{2\alpha}v'(x) = v(1) = 0$. However, $v' \notin L^{\frac{1}{2\alpha}}(0,1)$.

Next we prove assertion (ii). When $\alpha = \frac{1}{2}$, take $u(x) = \left(1+\ln\frac{1}{x}\right)^{\frac{1}{2}} - 1$. When $\frac{1}{2} < \alpha < 1$, take $u(x) = x^{1-2\alpha}\left(1+\ln\frac{1}{x}\right)^{1-2\alpha} - 1$. It is easy to check that $\mu := -(x^{2\alpha}u')' + u \in L^1(0,1)$ and $\lim_{x\to 0^+} x^{2\alpha}u'(x) = u(1) = 0$. However, $u \notin L^{\frac{1}{2\alpha-1}}(0,1)$.

Then we prove assertion (iii). The case $\alpha \geq 1$ follows from Remark 5.3, so we focus on the case $0 < \alpha < 1$. For all $x \in \left(0, \frac{1}{2}\right)$, note that

$$x^{2\alpha}u'(x) = \int_0^x u(s)ds - \int_{(0,x]} d\mu,\ 0 < \alpha < 1,$$

$$u(x) = \ln\frac{1}{x}\left(\int_{(0,x)} d\mu - \int_0^x u(s)ds\right) - \int_x^1 u(s)\ln\frac{1}{s}ds + \int_{[x,1)} \ln\frac{1}{s}d\mu(s),\ \text{if } \alpha = \frac{1}{2},$$

$$\begin{aligned} u(x) =& \frac{x^{1-2\alpha}}{2\alpha-1}\left(\int_{(0,x)} d\mu - \int_0^x u(s)ds\right) - \frac{1}{2\alpha-1}\left(\int_{(0,1)} d\mu - \int_0^1 uds\right) \\ &- \int_x^1 u(s)\frac{s^{1-2\alpha}}{2\alpha-1}ds + \int_{[x,1)} \frac{s^{1-2\alpha}}{2\alpha-1}d\mu(s),\ \text{if } \frac{1}{2} < \alpha < 1. \end{aligned}$$

Also note that the solution u identified in assertions (ii), (iii) and (iv) of Theorem 5.20 satisfies $\|u\|_{L^1} \leq k_\alpha \|\mu\|_{\mathcal{M}}$, where $k_\alpha \in (0,1)$. Therefore, $\forall x \in \left(0, \frac{1}{2}\right)$, take $\mu = \delta_{\frac{x}{2}}$ and deduce that

$$K_\alpha(x) \geq 1 - k_\alpha > 0,\ \text{when } 0 < \alpha < 1,$$

$$\widetilde{K}_\alpha(x) \geq \frac{\ln 2}{1+\ln 2}(1-k_\alpha) > 0,\ \text{when } \alpha = \frac{1}{2},$$

$$\widetilde{K}_\alpha(x) \geq \frac{1}{2\alpha-1}\left[1 - \left(\frac{1}{2}\right)^{2\alpha-1}\right](1-k_\alpha) > 0,\ \text{when } \frac{1}{2} < \alpha < 1.$$

On the other hand, for all $x \in \left(0, \frac{1}{2}\right)$, it is easy to check that

$$K_\alpha(x) \leq 2,\ \text{if } 0 < \alpha < 1,$$

$$\widetilde{K}_\alpha(x) \leq 2,\ \text{if } \alpha = \frac{1}{2},$$

$$\widetilde{K}_\alpha(x) \leq \frac{2}{2\alpha-1},\ \text{if } \frac{1}{2} < \alpha < 1.$$

Therefore, the proof is complete. $\square$

Chapter 6

A semilinear singular Sturm-Liouville equation involving measure data

6.1 Introduction

In this chapter, we consider the following semilinear singular Sturm-Liouville equation

$$\begin{cases} -(|x|^{2\alpha}u')' + |u|^{p-1}u = \mu \quad \text{on } (-1,1), \\ u(-1) = u(1) = 0. \end{cases} \tag{6.1}$$

Here we assume that $\alpha > 0$, $p > 1$, and $\mu \in \mathcal{M}(-1,1)$, where $\mathcal{M}(-1,1)$ is the space of bounded Radon measures on the interval $(-1,1)$. (See (5.2)).

For the semilinear equation (6.1), we can adapt from Chapter 5 the notion of solution and the notion of good solution. Rewrite (6.1) as $-(|x|^{2\alpha}u')' + u = u - |u|^{p-1}u + \mu$. Then according to (5.3), a function u is a *solution* of (6.1) if

$$u \in L^p(-1,1) \cap W^{1,1}_{loc}([-1,1]\setminus\{0\}),\ |x|^{2\alpha}u' \in BV(-1,1), \tag{6.2}$$

and u satisfies (6.1) in the usual sense (i.e., in the sense of measures). When $0 < \alpha < 1$, a solution u of (6.1) is called a *good solution* if it satisfies in addition (5.6).

In this chapter, we are interested in the question of existence and uniqueness, the limiting behavior of three different approximation schemes, and the classification of the isolated singularity at 0.

It turns out that we need to investigate the following four cases separately:

$$0 < \alpha \le \frac{1}{2},\ p > 1, \tag{6.3}$$

$$\frac{1}{2} < \alpha < 1,\ 1 < p < \frac{1}{2\alpha - 1}, \tag{6.4}$$

$$\frac{1}{2} < \alpha < 1,\ p \ge \frac{1}{2\alpha - 1}, \tag{6.5}$$

$$\alpha \geq 1, \; p > 1. \tag{6.6}$$

As we are going to see, the notion of good solution is only necessary for case (6.3) and (6.4). In fact, for case (6.5), if the solution exists, it must be the good solution.

Our first result concerns the question of uniqueness.

Theorem 6.1. *If α and p satisfy* (6.3) *or* (6.4)*, then for every $\mu \in \mathcal{M}(-1,1)$ there exists at most one good solution of* (6.1)*. If α and p satisfy* (6.5) *or* (6.6)*, then for every $\mu \in \mathcal{M}(-1,1)$ there exists at most one solution of* (6.1)*.*

Remark 6.1. *In fact, for α and p satisfying* (6.3) *or* (6.4)*, there exist infinitely many solutions of* (6.1)*; all of them will be identified in Section 6.6.*

The next two theorems answer the question of existence.

Theorem 6.2. *Assume that α and p satisfy* (6.3) *or* (6.4)*. For every $\mu \in \mathcal{M}(-1,1)$, there exists a (unique) good solution of* (6.1)*. Moreover, the good solution satisfies*

(i) $\lim_{x\to 0}\left(1+\ln\frac{1}{|x|}\right)^{-1}u(x) = -\lim_{x\to 0^+}|x|u'(x) = \lim_{x\to 0^-}|x|u'(x) = \frac{\mu(\{0\})}{2}$ *when* $\alpha = \frac{1}{2}$ *and* $p > 1$,

(ii) $\lim_{x\to 0}|x|^{2\alpha-1}u(x) = -\lim_{x\to 0^+}\frac{|x|^{2\alpha}u'(x)}{2\alpha-1} = \lim_{x\to 0^-}\frac{|x|^{2\alpha}u'(x)}{2\alpha-1} = \frac{\mu(\{0\})}{4\alpha-2}$ *when* $\frac{1}{2} < \alpha < 1$ *and* $1 < p < \frac{1}{2\alpha-1}$,

(iii) $\left\||u|^{p-1}u - |\hat{u}|^{p-1}\hat{u}\right\|_{L^1} \leq \|\mu-\hat{\mu}\|_{\mathcal{M}}$, $\left\|\left(|u|^{p-1}u - |\hat{u}|^{p-1}\hat{u}\right)^+\right\|_{L^1} \leq \left\|(\mu-\hat{\mu})^+\right\|_{\mathcal{M}}$, *for $\mu, \hat{\mu} \in \mathcal{M}(-1,1)$ and their corresponding good solutions $u, \hat{u}$.*

Theorem 6.3. *Assume that α and p satisfy* (6.5) *or* (6.6)*. For each $\mu \in \mathcal{M}(-1,1)$, there exists a (unique) solution of* (6.1) *if and only if $\mu(\{0\}) = 0$. Moreover, if the solution exists, it satisfies*

(i) $\lim_{x\to 0}|x|^{2\alpha-1}u(x) = \lim_{x\to 0}|x|^{2\alpha}u'(x) = 0$,

(ii) $\left\||u|^{p-1}u - |\hat{u}|^{p-1}\hat{u}\right\|_{L^1} \leq \|\mu-\hat{\mu}\|_{\mathcal{M}}$, $\left\|\left(|u|^{p-1}u - |\hat{u}|^{p-1}\hat{u}\right)^+\right\|_{L^1} \leq \left\|(\mu-\hat{\mu})^+\right\|_{\mathcal{M}}$, *for $\mu, \hat{\mu} \in \mathcal{M}(-1,1)$ and their corresponding solutions $u, \hat{u}$.*

We now study (6.1) by three different approximation schemes. The first one is the elliptic regularization. Take $0 < \epsilon < 1$ and consider the following regularized equation

$$\begin{cases} -((|x|+\epsilon)^{2\alpha} u_\epsilon')' + |u_\epsilon|^{p-1} u_\epsilon = \mu \quad \text{on } (-1,1), \\ u_\epsilon(-1) = u_\epsilon(1) = 0. \end{cases} \tag{6.7}$$

Given $\alpha > 0$, $p > 1$ and $\mu \in \mathcal{M}(-1,1)$, note that the existence of $u_\epsilon \in H_0^1(-1,1)$ with $u_\epsilon' \in BV(-1,1)$ is guaranteed by minimizing the corresponding functional, and the uniqueness of u_ϵ is also standard. Our main results are the following two theorems.

Theorem 6.4. *Assume that α and p satisfy* (6.3) *or* (6.4)*. Then as $\epsilon \to 0$, $u_\epsilon \to u$ uniformly on every compact subset of* $[-1,1] \setminus \{0\}$*, where u is the unique good solution of* (6.1).

Theorem 6.5. *Assume that α and p satisfy* (6.5) *or* (6.6)*. Denote by δ_0 the Dirac mass at* 0*. Then as $\epsilon \to 0$, $u_\epsilon \to u$ uniformly on every compact subset of* $[-1,1] \setminus \{0\}$*, where u is the unique solution of*

$$\begin{cases} -(|x|^{2\alpha} u')' + |u|^{p-1} u = \mu - \mu(\{0\})\,\delta_0 \quad \textit{on } (-1,1), \\ u(-1) = u(1) = 0. \end{cases} \tag{6.8}$$

Remark 6.2. *In Section 6.3 we will present further results about the mode of convergence in Theorems 6.4 and 6.5.*

The second approximation scheme consists of truncating the nonlinear term. Fix $p > 1$ and $n \in \mathbb{N}$. Define $g_{p,n} : \mathbb{R} \to \mathbb{R}$ as

$$g_{p,n}(t) = (\operatorname{sign} t) \min\left\{ |t|^p, n^{1-\frac{1}{p}} |t| \right\}. \tag{6.9}$$

It is clear that

$$0 \le g_{p,1}(t) \le g_{p,2}(t) \le \cdots \le |t|^{p-1} t, \ \forall t > 0,$$

$$|t|^{p-1} t \le \cdots g_{p,2}(t) \le g_{p,1}(t) \le 0, \ \forall t < 0,$$

$$g_{p,n}(t) \to |t|^{p-1} t, \quad \text{as } n \to \infty.$$

Consider the equation

$$\begin{cases} -(|x|^{2\alpha}u_n')' + g_{p,n}(u_n) = \mu \quad \text{on } (-1,1), \\ u_n(-1) = u_n(1) = 0. \end{cases} \tag{6.10}$$

Rewrite (6.10) as $-(|x|^{2\alpha}u_n')' + u_n = u_n - g_{p,n}(u_n) + \mu$. Then according to (5.3), a function u_n is a *solution* of (6.10) if

$$u_n \in L^1(-1,1) \cap W^{1,1}_{loc}([-1,1]\setminus\{0\}),\ |x|^{2\alpha}u_n' \in BV(-1,1),$$

and u satisfies (6.10) in the usual sense. When $0 < \alpha < 1$, a solution u_n of (6.10) is called a *good solution* if it satisfies in addition (5.6).

We will see in Section 6.4 that when $0 < \alpha < 1$, for all $p > 1$ and $n \in \mathbb{N}$, there exists a unique good solution u_n of (6.10). When $\alpha \geq 1$, for all $p > 1$ and $n \in \mathbb{N}$, there exists a unique solution u_n of (6.10) if and only if $\mu(\{0\}) = 0$.

We have the following results concerning the sequence $\{u_n\}_{n=1}^{\infty}$.

Theorem 6.6. *Assume that α and p satisfy* (6.3) *or* (6.4)*. Then as $n \to \infty$, $u_n \to u$ uniformly on every compact subset of $[-1,1]\setminus\{0\}$, where u is the unique good solution of* (6.1)*.*

Theorem 6.7. *Assume that α and p satisfy* (6.5)*. Then as $n \to \infty$, $u_n \to u$ uniformly on every compact subset of $[-1,1]\setminus\{0\}$, where u is the unique solution of* (6.8)*.*

Theorem 6.8. *Assume that α and p satisfy* (6.6) *and $\mu(\{0\}) = 0$. Then as $n \to \infty$, $u_n \to u$ uniformly on every compact subset of $[-1,1]\setminus\{0\}$, where u is the unique solution of* (6.8)*.*

Remark 6.3. *The more precise mode of convergence in Theorems 6.6, 6.7 and 6.8 will be presented in Section 6.4.*

Remark 6.4. *The third approximation scheme consists of approximating the measure μ by a sequence of L^1-functions under the weak-star topology. This is a delicate subject. For example, for $\frac{1}{2} \leq \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$, let $\mu = \delta_0$ and $f_n = Cn\rho(nx-1)$, where $\rho(x) = \chi_{[|x|<1]}e^{\frac{1}{|x|^2-1}}$ and $C^{-1} = \int\rho$, so that $f_n \overset{*}{\rightharpoonup} \delta_0$ in $(C_0[-1,1])^*$. Let u_n*

be the good solution corresponding to f_n. Then $u_n \to u$ but u is not the good solution corresponding to δ_0. This subject will be discussed in Section 6.5.

Finally, we study the isolated singularity at 0. The next result asserts that for α and p satisfying (6.5) or (6.6), the isolated singularity at 0 is removable.

Theorem 6.9. *Assume that α and p satisfy (6.5) or (6.6). Given $f \in L^1(-1,1)$, assume that $u \in L^p_{loc}((-1,1)\backslash\{0\})$ satisfying*

$$-\int_{-1}^{1} u(|x|^{2\alpha}\zeta')'dx + \int_{-1}^{1} |u|^{p-1}u\zeta dx = \int_{-1}^{1} f\zeta dx, \ \forall\zeta \in C_c^\infty((-1,1)\backslash\{0\}).$$

Then $u \in L^p_{loc}(-1,1)$ and

$$-\int_{-1}^{1} u(|x|^{2\alpha}\zeta')'dx + \int_{-1}^{1} |u|^{p-1}u\zeta dx = \int_{-1}^{1} f\zeta dx, \ \forall\zeta \in C_c^\infty(-1,1). \tag{6.11}$$

Remark 6.5. *An easy consequence of Theorem 6.9 is that equation (6.1) does not have a solution if α and p satisfy (6.5) or (6.6) and $\mu = \delta_0$, which is a special case of Theorem 6.3.*

On the other hand, for α and p satisfying (6.3) or (6.4), the isolated singularity at 0 is not removable. In this case, we give a complete classification of the asymptotic behavior of the solutions.

Theorem 6.10. *Assume that α and p satisfy (6.3) or (6.4). Let $u \in C^2(0,1]$ be such that*

$$\begin{cases} -(x^{2\alpha}u')' + |u|^{p-1}u = 0 \qquad on\ (0,1), \\ u(1) = 0. \end{cases} \tag{6.12}$$

Then one of the following assertions holds.

(i) $u \equiv 0$.

(ii) $u \equiv u_c$ *for some constant $c \in (-\infty,0)\cup(0,+\infty)$, where u_c is the unique solution of (6.12) such that*

$$\lim_{x\to0^+}\frac{u_c(x)}{E_\alpha(x)} = c, \tag{6.13}$$

and

$$E_\alpha(x) = \begin{cases} 1, \text{ if } 0 < \alpha < \frac{1}{2}, \\ \ln\frac{1}{x}, \ \text{ if } \alpha = \frac{1}{2}, \\ \frac{1}{x^{2\alpha-1}}, \ \text{ if } \frac{1}{2} < \alpha < 1. \end{cases} \tag{6.14}$$

(iii) $u \equiv u_{+\infty}$*, where* $u_{+\infty}$ *is the unique solution of* (6.12) *such that*

$$\lim_{x\to 0^+} x^{\frac{2(1-\alpha)}{p-1}} u_{+\infty}(x) = l_{p,\alpha}, \tag{6.15}$$

and

$$l_{p,\alpha} = \left[(1-\alpha)^2\left(\frac{2}{p-1}\right)\left(\frac{2p}{p-1} - \frac{1}{1-\alpha}\right)\right]^{\frac{1}{p-1}}. \tag{6.16}$$

(iv) $u \equiv u_{-\infty}$*, where* $u_{-\infty} = -u_{+\infty}$*.*

Moreover, $u_{-c} = -u_c$*. If* $c > 0$ *or* $c = +\infty$*,* $u_c \geq 0$*. For* $c > 0$*,* $u_c \downarrow 0$ *and* $u_c \uparrow u_{+\infty}$*.*

Remark 6.6. *The solutions* $u_{+\infty}$ *and* $u_{-\infty}$ *are called the very singular solutions, which is a terminology introduced by Brezis-Peletier-Terman [12].*

Remark 6.7. *Given* $\mu \in \mathcal{M}(0,1)$*, we can also study the following equation,*

$$\begin{cases} -(x^{2\alpha}u')' + |u|^{p-1}u = \mu \quad \text{ on } (0,1), \\ u(1) = 0. \end{cases} \tag{6.17}$$

In Section 6.9, we discuss (6.17) *under appropriate boundary conditions at 0, and we will also compare the results about* (6.17) *with some well-known results about the semilinear elliptic equation.*

The rest of this chapter is organized as follows. The question of existence and uniqueness is studied in Section 6.2 where Theorems 6.1-6.3 are proved. The three approximation schemes mentioned in the introduction will be investigated respectively in Sections 6.3, 6.4 and 6.5. In Section 6.6, we describe all the solutions of (6.1) when α and p satisfy (6.3) or (6.4). The removability of the singularity is studied in Section 6.7 and the classification of the singularity is studied in Section 6.8. Finally, Section 6.9 is devoted to (6.17). Throughout this chapter, several lemmas in Chapter 5 are applied.

6.2 Proof of the uniqueness and existence results

We start with the proof of the uniqueness result.

Proof of Theorem 6.1. Fix $\mu \in \mathcal{M}(-1,1)$. If α and p satisfy (6.3) or (6.4), assume that u and $\hat{u}$ are two good solutions of (6.1) corresponding to μ. Then $u - \hat{u} \in D(A_\alpha)$ and $A_\alpha(u-\hat{u}) = |\hat{u}|^{p-1}\hat{u} - |u|^{p-1}u$, where $D(A_\alpha)$ is given by (5.13).

If α and p satisfy (6.5) or (6.6), assume that u and $\hat{u}$ are two solutions of (6.1) corresponding to μ. Then $-(|x|^{2\alpha}(u-\hat{u})')' = |\hat{u}|^{p-1}\hat{u} - |u|^{p-1}u$. We claim that $u - \hat{u} \in D(A_\alpha)$. For $\alpha \geq 1$, it is clear by the definition of $D(A_\alpha)$. For $\frac{1}{2} < \alpha < 1$ and $p \geq \frac{1}{2\alpha-1}$, by (5.18), it is enough to show that $\lim_{x\to 0} |x|^{2\alpha}(u-\hat{u})'(x) = 0$. Indeed, since $|x|^{2\alpha}(u-\hat{u})' \in BV(-1,1)$, the limits $\lim_{x\to 0^+} |x|^{2\alpha}(u-\hat{u})'(x)$ and $\lim_{x\to 0^-} |x|^{2\alpha}(u-\hat{u})'(x)$ exist. They have to be zero. Otherwise, it contradicts the fact that $u - \hat{u} \in L^p(-1,1)$ with $p \geq \frac{1}{2\alpha-1}$.

Then for all the cases, assertion (iv) of Proposition 5.7 implies that

$$\int_{-1}^{1} (|\hat{u}|^{p-1}\hat{u} - |u|^{p-1}u)\,\mathrm{sign}(u-\hat{u})dx = \int_{-1}^{1} A_\alpha(u-\hat{u})\,\mathrm{sign}(u-\hat{u})dx \geq 0.$$

On the other hand, $(|\hat{u}|^{p-1}\hat{u} - |u|^{p-1}u)\,\mathrm{sign}(u-\hat{u}) \leq 0$ a.e. Therefore $u = \hat{u}$ a.e. □

The basic idea in the proof of Theorems 6.2 and 6.3 is to approximate the measures by L^1-functions. Therefore, we start with the case when $\mu \in L^1(-1,1)$ in (6.1).

Proposition 6.11. *For every $\alpha > 0$, $p > 1$ and $f \in L^1(-1,1)$, there exists a unique $u \in D(A_\alpha) \cap L^p(-1,1)$ such that $A_\alpha u + |u|^{p-1}u = f$ a.e. on $(-1,1)$, where A_α and $D(A_\alpha)$ are given by* (5.11) *and* (5.13) *respectively. Moreover, $\||u|^p\|_{L^1} \leq \|f\|_{L^1}$ and $\|A_\alpha u\|_{L^1} \leq 2\|f\|_{L^1}$.*

To prove Proposition 6.11, we need the following result by Brezis-Strauss [13].

Lemma 6.12 (Theorem 1 in [13])**.** *Let β be a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ which contains the origin. Let Ω be any measure space. Let A be an unbounded linear operator on $L^1(\Omega)$ satisfying the following conditions.*

(i) The operator A is closed with dense domain $D(A)$ in $L^1(\Omega)$; for any $\lambda > 0$, $I+\lambda A$ maps $D(A)$ one-to-one onto $L^1(\Omega)$ and $(I+\lambda A)^{-1}$ is a contraction in $L^1(\Omega)$.

(ii) For any $\lambda > 0$ *and* $f \in L^1(\Omega)$, $\operatorname*{ess\,sup}_{\Omega}(I+\lambda A)^{-1}f \le \max\left\{0, \operatorname*{ess\,sup}_{\Omega} f\right\}$.

(iii) There exists $\delta > 0$ *such that* $\delta \|u\|_{L^1} \le \|Au\|_{L^1}$, $\forall u \in D(A)$.

Then for every $f \in L^1(\Omega)$*, there exists a unique* $u \in D(A)$ *such that* $Au(x) + \beta(u(x)) \ni f(x)$ *a.e. Moreover,* $\|f - Au\|_{L^1} \le \|f\|_{L^1}$ *and* $\|Au\|_{L^1} \le 2\|f\|_{L^1}$.

We now prove Proposition 6.11. We apply a device by Gallouët-Morel [29].

Proof of Proposition 6.11. We first assume $0 < \alpha < 1$. Applying Proposition 5.7 and the estimates (5.17) and (5.23), we deduce that A_α is an unbounded operator satisfying the conditions (i)-(iii) in Lemma 6.12. Consider $\beta(u) = |u|^{p-1}u$ as a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$. Then Lemma 6.12 implies the desired result.

We then assume $\alpha \ge 1$. For any $n \in \mathbb{N}$, consider the unbounded linear operator

$$A_{\alpha,n}u = -(|x|^{2\alpha}u')' + \frac{1}{n}u.$$

Take its domain $D(A_{\alpha,n}) = D(A_\alpha)$. Note that

$$A_{\alpha,n} = A_\alpha + \frac{1}{n}I,$$

$$\lambda A_{\alpha,n} + I = \left(\frac{\lambda}{n} + 1\right)\left(\frac{\lambda n}{\lambda + n}A_\alpha + I\right),$$

$$(\lambda A_{\alpha,n} + I)^{-1} = \left(\frac{\lambda n}{\lambda + n}A_\alpha + I\right)^{-1} \circ \frac{n}{\lambda + n}I.$$

It is clear that $A_{\alpha,n}$ satisfies the conditions (i)-(iii) in Lemma 6.12. Therefore, for every $\alpha \ge 1$, $p > 1$, $n \in \mathbb{N}$, and $f \in L^1(-1,1)$, there exists a unique $u_n \in D(A_\alpha) \cap L^p(-1,1)$ such that

$$-(|x|^{2\alpha}u_n')' + \frac{1}{n}u_n + |u_n|^{p-1}u_n = f \text{ on } (-1,1).$$

That is,

$$\int_{-1}^{1} |x|^{2\alpha}u_n'\zeta' dx + \int_{-1}^{1} \frac{1}{n}u_n\zeta dx + \int_{-1}^{1} |u_n|^{p-1}u_n\zeta dx = \int_{-1}^{1} f\zeta dx, \ \forall \zeta \in C_0^1[-1,1]. \quad (6.18)$$

Moreover, we have

$$\||u_n|^p\|_{L^1} + \frac{1}{n}\|u_n\|_{L^1} + \left\||x|^{2\alpha}u_n'\right\|_{L^\infty} + \left\|(|x|^{2\alpha}u_n')'\right\|_{L^1} \le C,$$

where C is independent of n. Therefore, passing to a subsequence if necessary, we can assume that there exists $u \in W^{1,1}_{loc}([-1,1]\backslash\{0\})$ such that $u_n(x) \to u(x), \forall x \in [-1,1]\backslash\{0\}$, and $|x|^{2\alpha}u_n' \to |x|^{2\alpha}u'$ in $L^1(-1,1)$. It implies that $u(-1) = u(1) = 0$ and $\frac{1}{n}u_n + |u_n|^{p-1}u_n \to |u|^{p-1}u$ a.e. on $(-1,1)$.

We now prove that the sequence $\left\{\frac{1}{n}u_n + |u_n|^{p-1}u_n\right\}_{n=1}^{\infty}$ is equi-integrable. For this purpose, take a nondecreasing function $\varphi(x) \in C^\infty(\mathbb{R})$ such that $\varphi(x) = 0$ for $x \leq 0$, $\varphi(x) > 0$ for $x > 0$ and $\varphi(x) = 1$ for $x \geq 1$. For fixed $k \in \mathbb{N}$ and $t \in \mathbb{R}^+$, define

$$P_{k,t}(x) = \operatorname{sign} x\, \varphi(k(|x| - t)).$$

It is clear that $P_{k,t}$ is a maximal monotone graph containing the origin. Moreover,

$$\{x : P_{k,t}(x) \neq 0\} = (-\infty, -t) \cup (t, +\infty),$$

$$|P_{1,t}(x)| \leq |P_{2,t}(x)| \leq \cdots |P_{k,t}(x)| \leq |P_{k+1,t}(x)| \cdots \leq 1,$$

$$\lim_{k\to\infty} |P_{k,t}| = \chi_{[|x|>t]}.$$

Then assertion (iv) in Proposition 5.7 implies that

$$-\int_{-1}^{1} (|x|^{2\alpha}u_n')' P_{k,t}\,(u_n)\, dx \geq 0.$$

Therefore

$$\int_{-1}^{1} |P_{k,t}\,(u_n)| \left(\frac{1}{n}|u_n| + |u_n|^p\right) dx \leq \int_{-1}^{1} |P_{k,t}\,(u_n)|\, |f| dx.$$

Passing to the limit as $k \to \infty$, the Monotone Convergence Theorem implies that

$$\int_{[|u_n|>t]} \left(\frac{1}{n}|u_n| + |u_n|^p\right) dx \leq \int_{[|u_n|>t]} |f| dx, \ \forall t > 0 \text{ and } \forall n \in \mathbb{N}.$$

Then

$$|[|u_n| > t]| \leq \frac{1}{t^p} \int_{[|u_n|>t]} |u_n|^p dx \leq \frac{C}{t^p}.$$

For any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that

$$\int_{[|u_n|>t_\epsilon]} \left(\frac{1}{n}|u_n| + |u_n|^p\right) dx \leq \int_{[|u_n|>t_\epsilon]} |f| dx \leq \frac{\epsilon}{2}, \ \forall n \in \mathbb{N}.$$

Take $\delta = \frac{\epsilon}{2(t_\epsilon^p + t_\epsilon)}$. Then for all $K \subset \mathbb{R}$ such that $|K| < \delta$, we have

$$\begin{aligned}
&\int_K \left(\frac{1}{n}|u_n| + |u_n|^p\right) dx \\
\leq &\int_{K\cap[|u_n|>t_\epsilon]} \left(\frac{1}{n}|u_n| + |u_n|^p\right) dx + \int_{K\cap[|u_n|\leq t_\epsilon]} \left(\frac{1}{n}|u_n| + |u_n|^p\right) dx \\
\leq &\int_{[|u_n|>t_\epsilon]} \left(\frac{1}{n}|u_n| + |u_n|^p\right) dx + (t_\epsilon^p + t_\epsilon)|K| \\
\leq &\epsilon.
\end{aligned}$$

Thus, the sequence $\{\frac{1}{n}u_n + |u_n|^{p-1}u_n\}_{n=1}^\infty$ is equi-integrable.

A theorem of Vitali implies that $\frac{1}{n}u_n + |u_n|^{p-1}u_n \to |u|^{p-1}u$ in $L^1(-1,1)$. Passing to the limit as $n \to \infty$ in (6.18), we obtain

$$\int_{-1}^1 |x|^{2\alpha}u'\zeta' dx + + \int_{-1}^1 |u|^{p-1}u\zeta dx = \int_{-1}^1 f\zeta dx, \ \forall \zeta \in C_0^1[-1,1].$$

Therefore, $u \in D(A_\alpha) \cap L^p(-1,1)$ and $A_\alpha u + |u|^{p-1}u = f$ a.e. on $(-1,1)$. The uniqueness follows from Theorem 6.1. □

We now start to prove Theorems 6.2 and 6.3. Given $\mu \in \mathcal{M}(-1,1)$, there exists a sequence $\{f_n\}_{n=1}^\infty \subset L^1(-1,1)$ such that $f_n \overset{*}{\rightharpoonup} \mu$ in $(C_0[-1,1])^*$. For each f_n, by Proposition 6.11, there exists a unique $u_n \in D(A_\alpha) \cap L^p(-1,1)$ such that

$$\int_{-1}^1 |x|^{2\alpha}u_n'\zeta' dx + \int_{-1}^1 |u_n|^{p-1}u_n\zeta dx = \int_{-1}^1 f_n\zeta dx, \ \forall \zeta \in C_0^1[-1,1]. \tag{6.19}$$

Lemma 6.13. *Assume that* $0 < \alpha < \frac{1}{2}$ *and* $p > 1$. *Let* $\{u_n\}_{n=1}^\infty$ *be the sequence satisfying* (6.19). *Then* $u_n \to u$ *in* $C[-1,1]$, *where* u *is the (unique) good solution of* (5.1).

Proof. Note that $\|f_n\|_{L^1} \leq C$, where C is independent of n. Then Lemma 5.8 implies that $\|u_n\|_{L^\infty} + \left\||x|^{2\alpha}u_n'\right\|_{W^{1,1}} \leq \widetilde{C}$, where $\widetilde{C}$ is independent of n. Therefore the sequence u_n is bounded in $W^{1,q}(-1,1)$ for some fixed $q \in (1, \frac{1}{2\alpha})$. By compactness, there exists a subsequence such that $u_{n_k} \to u$ in $C_0[-1,1]$ and $|x|^{2\alpha}u_{n_k}' \to |x|^{2\alpha}u'$ in $L^1(-1,1)$. Passing to the limit in (6.19) as $n_k \to \infty$, we obtain that

$$\int_{-1}^1 |x|^{2\alpha}u'\zeta' dx + \int_{-1}^1 |u|^{p-1}u\zeta dx = \int_{-1}^1 \zeta d\mu, \ \forall \zeta \in C_0^1[-1,1].$$

We conclude that u is a good solution of (6.1). The uniqueness of the good solution and "the uniqueness of the limit" imply that $u_n \to u$ in $C[-1,1]$. $\square$

Lemma 6.14. *Assume that* $\alpha = \frac{1}{2}$ *and* $p > 1$. *Let* $\{u_n\}_{n=1}^{\infty}$ *be the sequence satisfying* (6.19). *Then there exists a subsequence* $\{n_k\}_{k=1}^{\infty}$ *such that*

$$\left(1+\ln\frac{1}{|x|}\right)^{-1} u_{n_k} \to \left(1+\ln\frac{1}{|x|}\right)^{-1} u \text{ in } L^r(-1,1),\ \forall r<\infty, \tag{6.20}$$

where u *is a solution of* (6.1). *Moreover,* $\left(1+\ln\frac{1}{|x|}\right)^{-1} u \in BV(-1,1)$ *and*

$$\begin{aligned}&\lim_{x\to 0^+}\left(1+\ln\frac{1}{|x|}\right)^{-1} u(x)\\ =&\lim_{x\to 0^+}\lim_{k\to\infty}\left(\int_0^x f_{n_k}(s)ds + \left(\ln\frac{1}{|x|}\right)^{-1}\int_x^1 f_{n_k}(s)\ln\frac{1}{|s|}ds\right),\end{aligned} \tag{6.21}$$

$$\begin{aligned}&\lim_{x\to 0^-}\left(1+\ln\frac{1}{|x|}\right)^{-1} u(x)\\ =&\lim_{x\to 0^-}\lim_{k\to\infty}\left(\int_x^0 f_{n_k}(s)ds + \left(\ln\frac{1}{|x|}\right)^{-1}\int_{-1}^x f_{n_k}(s)\ln\frac{1}{|s|}ds\right).\end{aligned} \tag{6.22}$$

Proof. Lemma 5.9 implies that

$$\||x|u_n'\|_{W^{1,1}} + \left\|\left(1+\ln\frac{1}{|x|}\right)^{-1} u_n\right\|_{W^{1,1}} \le C,$$

where C is independent of n. As a consequence, we obtain (6.20). Moreover, $u_{n_k} \to u$ in $L^p(-1,1)$, $|x|u_{n_k}' \to |x|u'$ in $L^1(-1,1)$, and $\left(1+\ln\frac{1}{|x|}\right)^{-1} u \in BV(-1,1)$. Passing to the limit in (6.19) as $n_k \to \infty$, we obtain that u is a solution of (6.1). The proof of (6.21) and (6.22) is the same as the one of Lemma 5.16. $\square$

Lemma 6.15. *Assume that* $\frac{1}{2} < \alpha < 1$ *and* $1 < p < \frac{1}{2\alpha-1}$. *Let* $\{u_n\}_{n=1}^{\infty}$ *be the sequence satisfying* (6.19). *Then there exists a subsequence* $\{n_k\}_{k=1}^{\infty}$ *such that*

$$|x|^{2\alpha-1}u_{n_k} \to |x|^{2\alpha-1}u \text{ in } L^r(-1,1),\ \forall r<\infty, \tag{6.23}$$

where u *is a solution of* (6.1). *Moreover,* $|x|^{2\alpha-1}u \in BV(-1,1)$ *and*

$$\begin{aligned}&\lim_{x\to 0^+}|x|^{2\alpha-1}u(x)\\ =&\frac{1}{2\alpha-1}\lim_{x\to 0^+}\lim_{k\to\infty}\left(\int_0^x f_{n_k}(s)ds + |x|^{2\alpha-1}\int_x^1 f_{n_k}(s)|s|^{1-2\alpha}ds\right),\end{aligned} \tag{6.24}$$

$$\lim_{x\to 0^-} |x|^{2\alpha-1}u(x)$$
$$=\frac{1}{2\alpha-1}\lim_{x\to 0^-}\lim_{k\to\infty}\left(\int_x^0 f_{n_k}(s)ds + |x|^{2\alpha-1}\int_{-1}^x f_{n_k}(s)|s|^{1-2\alpha}ds\right). \tag{6.25}$$

Proof. Lemma 5.9 implies that $\left\||x|^{2\alpha}u_n'\right\|_{W^{1,1}} + \left\||x|^{2\alpha-1}u_n\right\|_{W^{1,1}} \le C$, where C is independent of n. As a consequence, we obtain (6.23). Moreover, $|x|^{2\alpha-1}u \in BV(-1,1)$, $u_{n_k} \to u$ in $L^p(-1,1)$ and $|x|^{2\alpha}u_{n_k}' \to |x|^{2\alpha-1}u'$ in $L^1(-1,1)$. Passing to the limit in (6.19) as $n_k \to \infty$, we obtain that u is a solution of (6.1). The proof of (6.24) and (6.25) is the same as the one of Lemma 5.17. □

Proof of Theorem 6.2. The existence of good solution for $0 < \alpha < \frac{1}{2}$ and $p > 1$ has been proved by Lemma 6.13.

Assume now that f_n is the sequence identified in Lemma 5.18. For $\alpha = \frac{1}{2}$ and $p > 1$, we claim that

$$\lim_{x\to 0^+}\lim_{n\to\infty}\left(\int_0^x f_n(s)ds + \left(\ln\frac{1}{|x|}\right)^{-1}\int_x^1 f_n(s)\ln\frac{1}{|s|}ds\right)$$
$$=\lim_{x\to 0^-}\lim_{n\to\infty}\left(\int_x^0 f_n(s)ds + \left(\ln\frac{1}{|x|}\right)^{-1}\int_{-1}^x f_n(s)\ln\frac{1}{|s|}ds\right)$$
$$=\frac{1}{2}\mu(\{0\}).$$

For $\frac{1}{2} < \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$, we claim that

$$\frac{1}{2\alpha-1}\lim_{x\to 0^+}\lim_{n\to\infty}\left(\int_0^x f_n(s)ds + |x|^{2\alpha-1}\int_x^1 f_n(s)|s|^{1-2\alpha}ds\right)$$
$$=\frac{1}{2\alpha-1}\lim_{x\to 0^-}\lim_{n\to\infty}\left(\int_x^0 f_n(s)ds + |x|^{2\alpha-1}\int_{-1}^x f_n(s)|s|^{1-2\alpha}ds\right)$$
$$=\frac{1}{2(2\alpha-1)}\mu(\{0\}).$$

The proof of these two claims is the same as their counterparts in the proof of (i) of Theorem 5.14. Therefore, in view of Lemmas 6.14 and 6.15, we proved the existence of good solution for $\frac{1}{2} \le \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$, as well as assertions (i) and (ii). Assertion (iii) will be proved in Section 6.3. □

Lemma 6.16. *Assume that α and p satisfy* (6.5) *or* (6.6)*. Let $\{u_n\}_{n=1}^\infty$ be the sequence satisfying* (6.19)*. Then $|x|^{2\alpha-1}u_n \to |x|^{2\alpha-1}u$ in $L^r(-1,1)$, $\forall r < \infty$, where u is the solution of* (6.8).

Proof. Lemma 5.9 implies that $\left\| |x|^{2\alpha} u_n' \right\|_{W^{1,1}} + \left\| |x|^{2\alpha-1} u_n \right\|_{W^{1,1}} \le C$, where C is independent of n. It follows that $|x|^{2\alpha} u_{n_k}' \to |x|^{2\alpha} u'$ and $|x|^{2\alpha-1} u_n \to |x|^{2\alpha-1} u$ in $L^r(-1,1)$, $\forall r < \infty$. Note that $\|u_n\|_{L^p} \le C$. Then Fatou's Lemma implies that $u \in L^p(-1,1)$. Passing to the limit in (6.19) as $n_k \to \infty$, we obtain

$$\int_{-1}^{1} |x|^{2\alpha} u' \zeta' dx + \int_{-1}^{1} |u|^{p-1} u \zeta dx = \int_{-1}^{1} \zeta d\mu, \ \forall \zeta \in C_c^1((-1,1) \backslash \{0\}). \tag{6.26}$$

Here we use the same device as in Brezis-Véron [14]. Let $\varphi(x) \in C^\infty(\mathbb{R})$ be such that $0 \le \varphi \le 1$, $\varphi \equiv 0$ on $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\varphi \equiv 1$ on $\mathbb{R} \backslash (-1,1)$. Let $\varphi_n(x) = \varphi(nx)$. In (6.26), perform integration by parts and replace ζ by $\varphi_n \phi$ where $\phi \in C_c^2(-1,1)$. It follows that

$$-\int_{-1}^{1} u(|x|^{2\alpha} (\varphi_n \phi)')' dx + \int_{-1}^{1} |u|^{p-1} u \varphi_n \phi dx = \int_{-1}^{1} \varphi_n \phi d\mu, \ \forall \phi \in C_c^2(-1,1). \tag{6.27}$$

For each term on the left-hand side of (6.27), we obtain

$$\int_{-1}^{1} |x|^{2\alpha} u'(x) \varphi(nx) \phi''(x) dx \to \int_{-1}^{1} |x|^{2\alpha} u'(x) \phi''(x) dx,$$

$$2\alpha \int_{-1}^{1} u(x) \operatorname{sign} x |x|^{2\alpha-1} \varphi(nx) \phi'(x) dx \to 2\alpha \int_{-1}^{1} u(x) \operatorname{sign} x |x|^{2\alpha-1} \phi'(x) dx,$$

$$\int_{-1}^{1} |u(x)|^{p-1} u(x) \varphi(nx) \phi(x) dx \to \int_{-1}^{1} |u(x)|^{p-1} u(x) \phi(x) dx,$$

$$\left| 2n \int_{-\frac{1}{n}}^{\frac{1}{n}} |x|^{2\alpha} u(x) \varphi'(nx) \phi'(x) dx \right| \le \frac{2}{n^{2\alpha-1}} \left\| \varphi' \phi' \right\|_{L^\infty} \|u\|_{L^1(-\frac{1}{n}, \frac{1}{n})} \to 0,$$

$$\left| 2\alpha n \int_{-\frac{1}{n}}^{\frac{1}{n}} u(x) \operatorname{sign} x |x|^{2\alpha-1} \varphi'(nx) \phi(x) dx \right| \le \frac{2\alpha}{n^{2\alpha-2}} \left(\frac{2}{n} \right)^{\frac{1}{p'}} \left\| \varphi' \phi \right\|_{L^\infty} \|u\|_{L^p(-\frac{1}{n}, \frac{1}{n})} \to 0,$$

$$\left| n^2 \int_{-\frac{1}{n}}^{\frac{1}{n}} u(x) |x|^{2\alpha} \varphi''(nx) \phi(x) dx \right| \le \frac{1}{n^{2\alpha-2}} \left(\frac{2}{n} \right)^{\frac{1}{p'}} \left\| \varphi'' \phi \right\|_{L^\infty} \|u\|_{L^p(-\frac{1}{n}, \frac{1}{n})} \to 0,$$

where p' is the Hölder conjugate of p, which satisfies $\frac{1}{p'} + 2\alpha - 2 \ge 0$. For the right-hand side of (6.27), the Dominated Convergence Theorem implies that

$$\lim_{n \to \infty} \int_{-1}^{1} \varphi(nx) \phi(x) d\mu = \int_{-1}^{1} \phi(x) d \left(\mu - \mu \left(\{0\} \right) \delta_0 \right).$$

Thus

$$\int_{-1}^{1} |x|^{2\alpha} u' \phi' dx + \int_{-1}^{1} |u|^{p-1} u \phi dx = \int_{-1}^{1} \phi d \left(\mu - \mu \left(\{0\} \right) \delta_0 \right), \ \forall \phi \in C_c^1(-1,1).$$

Therefore u is the solution of (6.8). $\square$

Proof of Theorem 6.3. Suppose $\mu(\{0\})=0$. Then Lemma 6.16 implies that (6.1) has a solution. Conversely, assume that u is a solution of (6.1). We claim that $\mu(\{0\})=0$. Indeed, we have

$$-\int_{-1}^{1} u(|x|^{2\alpha}\zeta')'dx+\int_{-1}^{1}|u|^{p-1}u\zeta dx=\int_{-1}^{1}\zeta d\mu,\ \forall\zeta\in C_c^\infty(-1,1). \tag{6.28}$$

Take $\varphi\in C_c^\infty(\mathbb{R})$ such that $\varphi\equiv 1$ on $(-1,1)$, $\operatorname{supp}\varphi\subset(-2,2)$ and $0\le\varphi\le 1$. Replace $\zeta(x)$ by $\varphi(nx)$ in (6.28). Then for each term on the left-hand side of (6.28), we have

$$\left|n^2\int_{-\frac{2}{n}}^{\frac{2}{n}} u(x)|x|^{2\alpha}\varphi''(nx)dx\right|\le\frac{2^{2\alpha+\frac{2}{p'}}}{n^{2\alpha-2+\frac{1}{p'}}}\left\|\varphi''\right\|_{L^\infty}\|u\|_{L^p(-\frac{2}{n},\frac{2}{n})}\to 0,$$

$$\left|2\alpha n\int_{-\frac{2}{n}}^{\frac{2}{n}} u(x)|x|^{2\alpha-1}\varphi'(nx)\operatorname{sign}x dx\right|\le\frac{2^{2\alpha+\frac{2}{p'}}\alpha}{n^{2\alpha-2+\frac{1}{p'}}}\left\|\varphi'\right\|_{L^\infty}\|u\|_{L^p(-\frac{2}{n},\frac{2}{n})}\to 0,$$

$$\int_{-1}^{1}|u(x)|^{p-1}u(x)\varphi(nx)dx\to 0.$$

For the right-hand side of (6.28), we have

$$\int_{-1}^{1}\varphi(nx)d\mu=\mu(\{0\})+\int_{(0,\frac{2}{n}]}\varphi(nx)d\mu+\int_{[-\frac{2}{n},0)}\varphi(nx)d\mu.$$

Note that

$$\lim_{n\to\infty}\int_{(0,\frac{2}{n}]}\varphi(nx)d\mu=\lim_{n\to\infty}\int_{[-\frac{2}{n},0)}\varphi(nx)d\mu=0.$$

Therefore, $\mu(\{0\})=0$.

Assume now that the solution exists. We prove assertion (i). Indeed, since $|x|^{2\alpha-1}u\in BV(-1,1)$, the one-side limits $\lim_{x\to 0^+}|x|^{2\alpha-1}u(x)$ and $\lim_{x\to 0^-}|x|^{2\alpha-1}u(x)$ exist. They must be zero. Otherwise, it contradicts $u\in L^p(-1,1)$. The same reason guarantees that $\lim_{x\to 0}|x|^{2\alpha}u'(x)=0$. Assertion (ii) will be proved in Section 6.3. □

6.3 The elliptic regularization

For any $0<\epsilon<1$, we consider the regularized equation (6.7). Since $\mathcal{M}(-1,1)\subset H^{-1}(-1,1)$, the solution u_ϵ of (6.7) is actually the minimizer of the following functional

$$I(u)=\frac{1}{2}\int_{-1}^{1}(|x|+\epsilon)^{2\alpha}|u'|^2dx+\frac{1}{p+1}\int_{-1}^{1}|u|^{p+1}dx-\int_{-1}^{1}ud\mu,\ \forall u\in H_0^1(-1,1).$$

It implies that u_ϵ satisfies the following weak formulation

$$\int_{-1}^{1} (|x|+\epsilon)^{2\alpha} u_\epsilon' v' dx + \int_{-1}^{1} |u_\epsilon|^{p-1} u_\epsilon v dx = \int_{-1}^{1} v d\mu, \ \forall v \in H_0^1(-1,1). \tag{6.29}$$

Take $v_n = \varphi(nu_\epsilon)$ where $\varphi \in C^\infty(\mathbb{R})$ and $\varphi' \geq 0$ such that $\varphi \equiv 1$ on $[1,\infty)$, $\varphi \equiv -1$ on $(-\infty,-1]$ and $\varphi(0)=0$. Notice that

$$\int_{-1}^{1} (|x|+\epsilon)^{2\alpha} u_\epsilon' v_n' dx = n \int_{-1}^{1} (|x|+\epsilon)^{2\alpha} |u_\epsilon'|^2 \varphi'(nu_\epsilon) dx \geq 0.$$

Then

$$\|u_\epsilon\|_{L^p(-1,1)}^p = \lim_{n\to\infty} \int_{-1}^{1} |u_\epsilon|^{p-1} u_\epsilon v_n dx \leq \lim_{n\to\infty} \int_{-1}^{1} v_n d\mu \leq \|\mu\|_{\mathcal{M}(-1,1)}. \tag{6.30}$$

We now examine the limiting behavior of the family $\{u_\epsilon\}_{\epsilon>0}$ and we are going to establish the following sharper form of Theorems 6.4 and 6.5.

Theorem 6.17. *Given $\alpha > 0$, as $\epsilon \to 0$, we have*

$$(|x|+\epsilon)^{2\alpha} u_\epsilon' \to |x|^{2\alpha} u' \ \text{in } L^r(-1,1), \ \forall r < \infty. \tag{6.31}$$

Moreover,

$$u_\epsilon \to u \ \text{in } C_0[-1,1], \ \text{if } 0 < \alpha < \frac{1}{2}, \tag{6.32}$$

$$\left(1 + \ln \frac{1}{|x|+\epsilon}\right)^{-1} u_\epsilon \to \left(1 + \ln \frac{1}{|x|}\right)^{-1} u \ \text{in } L^r(-1,1), \ \forall r < \infty, \ \text{if } \alpha = \frac{1}{2}, \tag{6.33}$$

$$(|x|+\epsilon)^{2\alpha-1} u_\epsilon \to |x|^{2\alpha-1} u \ \text{in } L^r(-1,1), \ \forall r < \infty, \ \text{if } \alpha > \frac{1}{2}. \tag{6.34}$$

Here u is the unique good solution of (6.1) *if α and p satisfy* (6.3) *or* (6.4)*; u is the unique solution of* (6.8) *if α and p satisfy* (6.5) *or* (6.6).

The proof for the case $0 < \alpha < \frac{1}{2}$ of Theorem 6.17 is the same as the proof for the case $0 < \alpha < \frac{1}{2}$ of Theorem 5.12, except some obvious modifications due to the nonlinear term. We omit the detail.

Proof of Theorem 6.17 for $\alpha = \frac{1}{2}$. Write $K_\epsilon^+ = \lim_{x\to 0^+} u_\epsilon'(x)$ and $K_\epsilon^- = \lim_{x\to 0^-} u_\epsilon'(x)$. One can perform integration by parts (the same as the proof of Theorem 5.12) and obtain, for $x \in (0,1)$,

$$\begin{aligned} u_\epsilon(x) =& \ln\left(\frac{1+\epsilon}{x+\epsilon}\right)\left(-\epsilon K_\epsilon^+ + \int_{(0,x)} d\mu - \int_0^x |u_\epsilon(s)|^{p-1} u_\epsilon(s) ds\right) \\ &- \int_x^1 |u_\epsilon(s)|^{p-1} u_\epsilon(s) \ln\left(\frac{1+\epsilon}{s+\epsilon}\right) ds + \int_{[x,1)} \ln\left(\frac{1+\epsilon}{s+\epsilon}\right) d\mu(s), \end{aligned}$$

and for $x \in (-1,0)$,

$$\begin{aligned} u_\epsilon(x) =& \ln\left(\frac{1+\epsilon}{|x|+\epsilon}\right)\left(\epsilon K_\epsilon^- + \int_{(x,0)} d\mu - \int_x^0 |u_\epsilon(s)|^{p-1}u_\epsilon(s)ds\right) \\ & - \int_{-1}^x |u_\epsilon(s)|^{p-1}u_\epsilon(s)\ln\left(\frac{1+\epsilon}{|s|+\epsilon}\right)ds + \int_{(-1,x]} \ln\left(\frac{1+\epsilon}{|s|+\epsilon}\right)d\mu(s). \end{aligned}$$

Taking into account the relations $u_\epsilon(0^+) = u_\epsilon(0^-)$ and $\epsilon K_\epsilon^+ - \epsilon K_\epsilon^- = -\mu(\{0\})$, we deduce that

$$\begin{aligned} \epsilon K_\epsilon^+ =& -\frac{1}{2}\mu(\{0\}) + \frac{1}{2\ln\left(\frac{1+\epsilon}{\epsilon}\right)}\int_{(-1,0)\cup(0,1)} (\operatorname{sign} s)\ln\left(\frac{1+\epsilon}{|s|+\epsilon}\right)d\mu(s) \\ & - \frac{1}{2\ln\left(\frac{1+\epsilon}{\epsilon}\right)}\int_{-1}^1 (\operatorname{sign} s)|u_\epsilon(s)|^{p-1}u_\epsilon(s)\ln\left(\frac{1+\epsilon}{|s|+\epsilon}\right)ds, \end{aligned}$$

and

$$\begin{aligned} \epsilon K_\epsilon^- =& \frac{1}{2}\mu(\{0\}) + \frac{1}{2\ln\left(\frac{1+\epsilon}{\epsilon}\right)}\int_{(-1,0)\cup(0,1)} (\operatorname{sign} s)\ln\left(\frac{1+\epsilon}{|s|+\epsilon}\right)d\mu(s) \\ & - \frac{1}{2\ln\left(\frac{1+\epsilon}{\epsilon}\right)}\int_{-1}^1 (\operatorname{sign} s)|u_\epsilon(s)|^{p-1}u_\epsilon(s)\ln\left(\frac{1+\epsilon}{|s|+\epsilon}\right)ds. \end{aligned}$$

It is easy to check that $|\epsilon K_\epsilon^+| \le \frac{3}{2}\|\mu\|_{\mathcal{M}}$ and $|\epsilon K_\epsilon^-| \le \frac{3}{2}\|\mu\|_{\mathcal{M}}$ since $\|u_\epsilon\|_{L^p}^p \le \|\mu\|_{\mathcal{M}}$. Therefore, we obtain that

$$\left\|\left(1+\ln\frac{1}{|x|+\epsilon}\right)^{-1} u_\epsilon\right\|_{W^{1,1}(-1,1)} + \|(|x|+\epsilon)u_\epsilon'\|_{BV(-1,1)} \le C,$$

where C is independent of ϵ. It follows that (6.31) and (6.33) hold for a subsequence $\{u_{\epsilon_n}\}_{n=1}^\infty$. Moreover, the sequence $\left\{|u_{\epsilon_n}|^{p-1}u_{\epsilon_n}\right\}_{n=1}^\infty$ is equi-integrable and $|u_{\epsilon_n}|^{p-1}u_{\epsilon_n} \to |u|^{p-1}u$ in $L^1(-1,1)$. Passing to the limit as $n \to \infty$ in (6.29), we obtain

$$\int_{-1}^1 |x|u'v'dx + \int_{-1}^1 |u|^{p-1}uvdx = \int_{-1}^1 vd\mu, \ \forall v \in C_0^1[-1,1].$$

Notice that $\|u_\epsilon\|_{L^{p+1}(-1,1)} \le C$. The same argument as in the proof of Theorem 5.12 implies that

$$-\lim_{\epsilon\to 0}\epsilon K_\epsilon^+ = \lim_{\epsilon\to 0}\epsilon K_\epsilon^- = \frac{1}{2}\mu(\{0\}),$$

and

$$\lim_{x\to 0^+}\left(1+\ln\frac{1}{|x|}\right)^{-1} u(x) = \lim_{x\to 0^-}\left(1+\ln\frac{1}{|x|}\right)^{-1} u(x) = \frac{1}{2}\mu(\{0\}).$$

Therefore, u is the good solution. The uniqueness of the good solution and the uniqueness of the limit imply that (6.31) and (6.33) hold for the family $\{u_\epsilon\}_{\epsilon>0}$. □

Proof of Theorem 6.17 for $\frac{1}{2} < \alpha < 1$. We denote $K_\epsilon^+ = \lim_{x\to 0^+} u_\epsilon'(x)$ and $K_\epsilon^- = \lim_{x\to 0^-} u_\epsilon'(x)$. Integration by parts yields, for $x \in (0,1)$,

$$\begin{aligned} u_\epsilon(x) = &\left(\frac{(x+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}}{2\alpha - 1}\right)\left(-\epsilon^{2\alpha}K_\epsilon^+ + \int_{(0,x)} d\mu - \int_0^x |u_\epsilon(s)|^{p-1}u_\epsilon(s)ds\right) \\ &- \int_x^1 |u_\epsilon(s)|^{p-1}u_\epsilon(s)\left(\frac{(s+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}}{2\alpha-1}\right)ds \\ &+ \int_{[x,1)} \frac{(s+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}}{2\alpha - 1} d\mu(s), \end{aligned}$$

and for $x \in (-1,0)$,

$$\begin{aligned} u_\epsilon(x) = &\left(\frac{(|x|+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}}{2\alpha - 1}\right)\left(\epsilon^{2\alpha}K_\epsilon^- + \int_{(x,0)} d\mu - \int_x^0 |u_\epsilon(s)|^{p-1}u_\epsilon(s)ds\right) \\ &- \int_{-1}^x |u_\epsilon(s)|^{p-1}u_\epsilon(s)\left(\frac{(|s|+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}}{2\alpha-1}\right)ds \\ &+ \int_{(-1,x]} \frac{(|s|+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}}{2\alpha - 1} d\mu(s). \end{aligned}$$

By the relations $u_\epsilon(0^+) = u_\epsilon(0^-)$ and $\epsilon^{2\alpha}K_\epsilon^+ - \epsilon^{2\alpha}K_\epsilon^- = -\mu(\{0\})$, we have

$$\begin{aligned} \epsilon^{2\alpha}K_\epsilon^+ = &-\frac{1}{2}\mu(\{0\}) - \frac{\int_{-1}^1 (\operatorname{sign} s)|u_\epsilon(s)|^{p-1}u_\epsilon(s)\left[(|s|+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}\right]ds}{2\left[\epsilon^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}\right]} \\ &+ \frac{\int_{(-1,0)\cup(0,1)} (\operatorname{sign} s)\left[(|s|+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}\right]d\mu(s)}{2\left[\epsilon^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}\right]}, \end{aligned}$$

and

$$\begin{aligned} \epsilon^{2\alpha}K_\epsilon^- = &\frac{1}{2}\mu(\{0\}) - \frac{\int_{-1}^1 (\operatorname{sign} s)|u_\epsilon(s)|^{p-1}u_\epsilon(s)\left[(|s|+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}\right]ds}{2\left[\epsilon^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}\right]} \\ &+ \frac{\int_{(-1,0)\cup(0,1)} (\operatorname{sign} s)\left[(|s|+\epsilon)^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}\right]d\mu(s)}{2\left[\epsilon^{1-2\alpha} - (1+\epsilon)^{1-2\alpha}\right]}. \end{aligned}$$

It is easy to check that $\left|\epsilon^{2\alpha}K_\epsilon^+\right| \le \frac{3}{2}\|\mu\|_{\mathcal{M}}$ and $\left|\epsilon^{2\alpha}K_\epsilon^-\right| \le \frac{3}{2}\|\mu\|_{\mathcal{M}}$ since $\|u_\epsilon\|_{L^p}^p \le \|\mu\|_{\mathcal{M}}$. Therefore, we obtain that

$$\left\|(|x|+\epsilon)^{2\alpha-1}u_\epsilon\right\|_{W^{1,1}(-1,1)} + \left\|(|x|+\epsilon)^{2\alpha}u_\epsilon'\right\|_{BV(-1,1)} \le C, \tag{6.35}$$

where C is independent of ϵ. It follows that (6.31) and (6.34) hold for a subsequence $\{u_{\epsilon_n}\}_{n=1}^\infty$.

If $1 < p < \frac{1}{2\alpha-1}$, there exists $\theta \in \left(p, \frac{1}{2\alpha-1}\right)$ such that $\|u_\epsilon\|_{L^\theta(-1,1)} \le C$. Thus the sequence $\left\{|u_{\epsilon_n}|^{p-1}u_{\epsilon_n}\right\}_{n=1}^\infty$ is equi-integrable and $|u_{\epsilon_n}|^{p-1}u_{\epsilon_n} \to |u|^{p-1}u$ in $L^1(-1,1)$.

Passing to the limit as $n \to \infty$ in (6.29), we obtain

$$\int_{-1}^{1} |x|^{2\alpha} u'v'dx + \int_{-1}^{1} |u|^{p-1} uvdx = \int_{-1}^{1} vd\mu, \ \forall v \in C_0^1[-1,1].$$

The same argument as in the proof of Theorem 5.12 implies that

$$-\lim_{\epsilon \to 0} \epsilon^{2\alpha} K_\epsilon^+ = \lim_{\epsilon \to 0} \epsilon^{2\alpha} K_\epsilon^- = \frac{1}{2}\mu(\{0\})$$

and

$$\lim_{x \to 0^+} |x|^{2\alpha-1} u(x) = \lim_{x \to 0^-} |x|^{2\alpha-1} u(x) = \frac{1}{2(2\alpha-1)}\mu(\{0\}).$$

Therefore, u is the good solution.

If $p \geq \frac{1}{2\alpha-1}$, a consequence of (6.35) is that $u_{\epsilon_n} \to u$ uniformly on any closed interval $I \subset [-1,1] \setminus \{0\}$. Passing to the limit as $n \to \infty$ in (6.29), we obtain

$$\int_{-1}^{1} |x|^{2\alpha} u'v'dx + \int_{-1}^{1} |u|^{p-1} uvdx = \int_{-1}^{1} vd\mu, \ \forall v \in C_c^1((-1,1) \setminus \{0\}).$$

Since $\|u_\epsilon\|_{L^p}^p \leq \|\mu\|_{\mathcal{M}}$, Fatou's lemma yields $u \in L^p(-1,1)$. The same argument as in the proof of Lemma 6.16 implies that u is the solution of (6.8). The uniqueness of the solution and the uniqueness of the limit imply that (6.31) and (6.34) hold for the family $\{u_\epsilon\}_{\epsilon>0}$. □

We omit the proof for the case $\alpha \geq 1$ of Theorem 6.17 since it is the same as the proof for the case $\frac{1}{2} < \alpha < 1$ and $p \geq \frac{1}{2\alpha-1}$.

If we assume the data to be L^1, we have a further result about the mode of convergence.

Theorem 6.18. *For* $\alpha \geq \frac{1}{2}$ *and* $\mu \in L^1(-1,1)$, *the mode of convergence in* (6.33) *and* (6.34) *can be improved as*

$$\left(1 + \ln \frac{1}{|x| + \epsilon}\right)^{-1} u_\epsilon \to \left(1 + \ln \frac{1}{|x|}\right)^{-1} u \text{ in } C_0[-1,1], \text{ if } \alpha = \frac{1}{2}, \tag{6.36}$$

and

$$(|x| + \epsilon)^{2\alpha-1} u_\epsilon \to |x|^{2\alpha-1} u \text{ in } C_0[-1,1], \text{ if } \alpha > \frac{1}{2}. \tag{6.37}$$

To prove Theorem 6.18, one can just perform the same argument as the proof of Theorem 5.13. We omit the detail.

As we indicated in the previous section, the following is the

Proof of (iii) of Theorem 6.2 and proof of (ii) of Theorem 6.3. For $\mu, \hat{\mu} \in \mathcal{M}(-1,1)$, denote by u_ϵ and $\hat{u}_\epsilon$ their corresponding solution of (6.7). From (6.29) we have

$$\int_{-1}^{1} (|x|+\epsilon)^{2\alpha}(u_\epsilon - \hat{u}_\epsilon)' v' dx + \int_{-1}^{1} (|u_\epsilon|^{p-1}u_\epsilon - |\hat{u}_\epsilon|^{p-1}\hat{u}_\epsilon) v dx$$
$$= \int_{-1}^{1} v d(\mu - \hat{\mu}), \ \forall v \in H_0^1(-1,1).$$

Take $v = \varphi_n (u_\epsilon - \hat{u}_\epsilon)$, where φ_n is the smooth approximation of either $\operatorname{sign} x$ or $(\operatorname{sign} x)^+$. We obtain

$$\left\| |u_\epsilon|^{p-1}u_\epsilon - |\hat{u}_\epsilon|^{p-1}\hat{u}_\epsilon \right\|_{L^1} \le \|\mu - \hat{\mu}\|_{\mathcal{M}},$$

and

$$\left\| \left(|u_\epsilon|^{p-1}u_\epsilon - |\hat{u}_\epsilon|^{p-1}\hat{u}_\epsilon \right)^+ \right\|_{L^1} \le \left\| (\mu - \hat{\mu})^+ \right\|_{\mathcal{M}}.$$

Then Fatou's lemma yields the desired result. □

6.4 The approximation via truncation

In this section, we consider the approximation scheme via the truncated problem (6.10). As we mentioned in the introduction, the following lemma ensures the sequence $\{u_n\}_{n=1}^{\infty}$ is well-defined.

Lemma 6.19. *Fix $p > 1$ and $n \in \mathbb{N}$. When $0 < \alpha < 1$, for each $\mu \in \mathcal{M}(-1,1)$, equation* (6.10) *has a unique good solution u_n. When $\alpha \ge 1$, for each $\mu \in \mathcal{M}(-1,1)$, equation* (6.10) *has a unique solution u_n if and only if $\mu(\{0\}) = 0$. Moreover, for both cases, $\|g_{p,n}(u_n)\|_{L^1} \le \|\mu\|_{\mathcal{M}}$ and $\left\|(|x|^{2\alpha}u_n')'\right\|_{\mathcal{M}} \le 2\|\mu\|_{\mathcal{M}}$.*

Proof. For $\mu \in \mathcal{M}(-1,1)$, take $f_m = \rho_m * \mu$, where ρ_m is specified in Lemma 5.18. Then $f_m \overset{*}{\rightharpoonup} \mu$ in $(C_0[-1,1])^*$ as $m \to \infty$. For fixed $m \in \mathbb{N}$, the same argument as in the proof of Proposition 6.11 implies that there exists $u_{n,m} \in D(A_\alpha)$ such that

$$\int_{-1}^{1} |x|^{2\alpha} u_{n,m}' \zeta' dx + \int_{-1}^{1} g_{p,n}(u_{n,m})\zeta dx = \int_{-1}^{1} f_m \zeta dx, \ \forall \zeta \in C_0^1[-1,1]. \tag{6.38}$$

Moreover,

$$\|g_{p,n}(u_{n,m})\|_{L^1} \le \|f_m\|_{L^1} \le \|\mu\|_{\mathcal{M}},$$

$$\left\|(|x|^{2\alpha}u'_{n,m})'\right\|_{L^1} \leq 2\left\|f_m\right\|_{L^1} \leq 2\left\|\mu\right\|_{\mathcal{M}}.$$

If $0 < \alpha < \frac{1}{2}$, then $\{u_{n,m}\}_{m=1}^{\infty}$ is a bounded sequence in $W^{1,q}(-1,1)$ for $1 < q < \frac{1}{2\alpha}$. Thus, passing to the limit as $m \to \infty$ in (6.38), we obtain

$$\int_{-1}^{1} |x|^{2\alpha}u'_n\zeta' dx + \int_{-1}^{1} g_{p,n}(u_n)\zeta dx = \int_{-1}^{1} \zeta d\mu,\ \forall \zeta \in C_0^1[-1,1], \tag{6.39}$$

where $u_n \in W^{1,1}(-1,1)$, $\left\|g_{p,n}(u_n)\right\|_{L^1} \leq \left\|\mu\right\|_{\mathcal{M}}$ and $\left\|(|x|^{2\alpha}u'_n)'\right\|_{\mathcal{M}} \leq 2\left\|\mu\right\|_{\mathcal{M}}$.

If $\frac{1}{2} \leq \alpha < 1$, as $m \to \infty$, we obtain $|x|^{2\alpha}u'_{n,m} \to |x|^{2\alpha}u'_n$ and $|x|^{2\alpha-1}u_{n,m} \to |x|^{2\alpha-1}u_n$ in $L^r(-1,1)$, $\forall r < \infty$. Then the Dominated Convergence Theorem implies that $g_{p,n}(u_{n,m}) \to g_{p,n}(u_n)$ in $L^1(-1,1)$. We again obtain (6.39). The same as the proof of Theorem 6.2, we can check that

$$\lim_{x\to 0^+}\left(1+\ln\frac{1}{|x|}\right)^{-1} u_n(x) = \lim_{x\to 0^-}\left(1+\ln\frac{1}{|x|}\right)^{-1} u_n(x) = \frac{1}{2}\mu\left(\{0\}\right),\ \text{if } \alpha = \frac{1}{2},$$

$$\lim_{x\to 0^+} |x|^{2\alpha-1}u_n(x) = \lim_{x\to 0^-} |x|^{2\alpha-1}u_n(x) = \frac{1}{2(2\alpha-1)}\mu\left(\{0\}\right),\ \text{if } \frac{1}{2} < \alpha < 1.$$

Therefore, u_n is a good solution of (6.10) with $\left\|g_{p,n}(u_n)\right\|_{L^1} \leq \left\|\mu\right\|_{\mathcal{M}}$ and $\left\|(|x|^{2\alpha}u'_n)'\right\|_{\mathcal{M}} \leq 2\left\|\mu\right\|_{\mathcal{M}}$.

If $\alpha \geq 1$, as $m \to \infty$, we obtain $|x|^{2\alpha}u'_{n,m} \to |x|^{2\alpha}u'_n$ in $L^r(-1,1)$, $\forall r < \infty$, and $u_{n,m} \to u_n$ uniformly on any closed interval $I \subset [-1,1]\backslash\{0\}$. Passing to the limit as $m \to \infty$, we have $\left\|g_{p,n}(u_n)\right\|_{L^1} \leq \left\|\mu\right\|_{\mathcal{M}}$ and

$$\int_{-1}^{1} |x|^{2\alpha}u'_n\zeta' dx + \int_{-1}^{1} g_{p,n}(u_n)\zeta dx = \int_{-1}^{1} \zeta d\mu,\ \forall \zeta \in C_c^1\left((-1,1)\backslash\{0\}\right).$$

The same as the proof of Theorem 5.4, we have that u_n is a solution of (6.10) if and only if $\mu\left(\{0\}\right) = 0$. If u_n is a solution, it clearly satisfies $\left\|g_{p,n}(u_n)\right\|_{L^1} \leq \left\|\mu\right\|_{\mathcal{M}}$ and $\left\|(|x|^{2\alpha}u'_n)'\right\|_{\mathcal{M}} \leq 2\left\|\mu\right\|_{\mathcal{M}}$.

We now proof the uniqueness. Assume that $u_n^{(1)}$ and $u_n^{(2)}$ are two solutions of (6.10) corresponding to μ. Then $u_n^{(1)} - u_n^{(2)} \in D(A_\alpha)$ and

$$-(|x|^{2\alpha}(u_n^{(1)} - u_n^{(2)})')' + g_{p,n}(u_n^{(1)}) - g_{p,n}(u_n^{(2)}) = 0.$$

Assertion (iv) of Proposition 5.7 implies that

$$-\int_{-1}^{1}(|x|^{2\alpha}(u_n^{(1)} - u_n^{(2)})')' \operatorname{sign}(u_n^{(1)} - u_n^{(2)})dx \geq 0.$$

Therefore, $g_{p,n}(u_n^{(1)}) = g_{p,n}(u_n^{(2)})$ and $u_n^{(1)} = u_n^{(2)}$ a.e. □

We now prove Theorems 6.6, 6.7 and 6.8. Actually, we will prove the following result with a more accurate mode of convergence.

Theorem 6.20. *As $n \to \infty$, we have*

$$|x|^{2\alpha} u_n' \to |x|^{2\alpha} u' \text{ in } L^r(-1,1),\ \forall r < \infty. \tag{6.40}$$

Moreover,

$$u_n \to u \text{ in } C_0[-1,1],\ \text{if } 0 < \alpha < \frac{1}{2}, \tag{6.41}$$

$$\left(1 + \ln \frac{1}{|x|}\right)^{-1} u_n \to \left(1 + \ln \frac{1}{|x|}\right)^{-1} u \text{ in } L^r(-1,1),\ \forall r < \infty,\ \text{if } \alpha = \frac{1}{2}, \tag{6.42}$$

$$|x|^{2\alpha-1} u_n \to |x|^{2\alpha-1} u \text{ in } L^r(-1,1),\ \forall r < \infty,\ \text{if } \alpha > \frac{1}{2}. \tag{6.43}$$

Here u is the unique good solution of (6.1) *if α and p satisfy* (6.3) *or* (6.4)*; u is the unique solution of* (6.8) *if α and p satisfy* (6.5) *or* (6.6).

Proof. Assume $0 < \alpha < \frac{1}{2}$. We obtain that the sequence $\{u_n\}_{n=1}^{\infty}$ is bounded in $W^{1,q}(-1,1)$ for $1 < q < \frac{1}{2\alpha}$. Hence, there exists a subsequence such that

(i) $u_{n_k} \to u$ in $C[-1,1]$,

(ii) $g_{p,n_k}(u_{n_k}) \to |u|^{p-1}u$ in $L^1(-1,1)$,

(iii) $|x|^{2\alpha} u_{n_k}' \to |x|^{2\alpha} u'$ in $L^r(-1,1)$, $\forall r < \infty$.

Passing to the limit as $n_k \to \infty$, we obtain that

$$\int_{-1}^{1} |x|^{2\alpha} u' \zeta' dx + \int_{-1}^{1} |u|^{p-1} u \zeta dx = \int_{-1}^{1} \zeta d\mu,\ \forall \zeta \in C_0^1[-1,1].$$

Thus, u is the good solution of (6.1).

Assume $\alpha = \frac{1}{2}$. Denote $K^+ = \lim_{x \to 0^+} |x| u_n'(x)$ and $K^- = \lim_{x \to 0^-} |x| u_n'(x)$. Integration by parts yields, for $x \in (0,1)$,

$$\begin{aligned} u_n(x) = \left(\ln \frac{1}{x}\right) &\left(-K^+ + \int_{(0,x)} d\mu - \int_0^x g_{p,n}(u_n(s)) ds\right) \\ &- \int_x^1 g_{p,n}(u_n(s)) \ln \frac{1}{s} ds + \int_{[x,1)} \ln \frac{1}{s} d\mu(s), \end{aligned}$$

and for $x \in (0,1)$,

$$u_n(x) = \left(\ln \frac{1}{|x|}\right) \left(K^- + \int_{(x,0)} d\mu - \int_x^0 g_{p,n}(u_n(s))ds\right)$$
$$- \int_{-1}^x g_{p,n}(u_n(s)) \ln \frac{1}{|s|} ds + \int_{(-1,x]} \ln \frac{1}{|s|} d\mu(s).$$

One can check that

$$\lim_{x \to 0^+} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u_n(x) = -K^+,$$
$$\lim_{x \to 0^-} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u_n(x) = K^-.$$

Since u_n is a good solution, then $K^+ + K^- = 0$. On the other hand, $K^- - K^+ = \mu(\{0\})$. Therefore, $K^+ = -\frac{1}{2}\mu(\{0\})$ and $K^- = \frac{1}{2}\mu(\{0\})$. Furthermore, a direct computation yields that

$$\left\| \left(1 + \ln \frac{1}{|x|}\right)^{-1} u_n \right\|_{W^{1,1}} + \left\| |x| u_n' \right\|_{BV} \leq C,$$

where C is independent of n. It implies that (6.40) and (6.42) hold for a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$. As a result, the sequence $\{g_{p,n_k}(u_{n_k})\}_{k=1}^{\infty}$ is equi-integrable and $g_{p,n_k}(u_{n_k}) \to |u|^{p-1}u$ in $L^1(-1,1)$. Passing to the limit as $n_k \to \infty$, we obtain that

$$\int_{-1}^1 |x| u' \zeta' dx + \int_{-1}^1 |u|^{p-1} u \zeta dx = \int_{-1}^1 \zeta d\mu, \ \forall \zeta \in C_0^1[-1,1].$$

Moreover, we can check that

$$\lim_{x \to 0^+} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u(x) = \lim_{x \to 0^+} \lim_{k \to \infty} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u_{n_k}(x) = -K^+ = \frac{1}{2}\mu(\{0\}),$$
$$\lim_{x \to 0^-} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u(x) = \lim_{x \to 0^-} \lim_{k \to \infty} \left(1 + \ln \frac{1}{|x|}\right)^{-1} u_{n_k}(x) = K^- = \frac{1}{2}\mu(\{0\}).$$

Thus, u is the good solution of (6.1).

Assume $\alpha > \frac{1}{2}$. Denote $K^+ = \lim\limits_{x \to 0^+} |x|^{2\alpha} u_n'(x)$ and $K^- = \lim\limits_{x \to 0^-} |x|^{2\alpha} u_n'(x)$. Integration by parts yields, for $x \in (0,1)$,

$$u_n(x) = \frac{x^{1-2\alpha} - 1}{2\alpha - 1} \left(-K^+ + \int_{(0,x)} d\mu - \int_0^x g_{p,n}(u_n(s))ds\right)$$
$$- \int_x^1 \frac{s^{1-2\alpha} - 1}{2\alpha - 1} g_{p,n}(u_n(s))ds + \int_{[x,1)} \frac{s^{1-2\alpha} - 1}{2\alpha - 1} d\mu(s),$$

and for $x \in (-1,0)$,

$$u_n(x) = \frac{|x|^{1-2\alpha}-1}{2\alpha-1}\left(K^- + \int_{(x,0)} d\mu - \int_x^0 g_{p,n}(u_n(s))ds\right) - \int_{-1}^x \frac{|s|^{1-2\alpha}-1}{2\alpha-1} g_{p,n}(u_n(s))ds + \int_{(-1,x]} \frac{|s|^{1-2\alpha}-1}{2\alpha-1} d\mu(s).$$

One can check that

$$\lim_{x\to 0^+} |x|^{2\alpha-1}u_n(x) = -\frac{K^+}{2\alpha-1},$$
$$\lim_{x\to 0^-} |x|^{2\alpha-1}u_n(x) = \frac{K^-}{2\alpha-1}.$$

When $\frac{1}{2} < \alpha < 1$, since u_n is the good solution, we have $K^+ + K^- = 0$. On the other hand, $K^- - K^+ = \mu(\{0\})$. Thus $K^+ = -\frac{1}{2}\mu(\{0\})$ and $K^- = \frac{1}{2}\mu(\{0\})$. When $\alpha \geq 1$, the fact that $u_n \in L^1(-1,1)$ implies that $K^+ = K^- = 0$. For either case, we have

$$\left\||x|^{2\alpha-1}u_n\right\|_{W^{1,1}} + \left\||x|^{2\alpha}u_n'\right\|_{BV} \leq C,$$

where C is independent of n. It implies that (6.40) and (6.43) hold for a subsequence $\{u_{n_k}\}_{k=1}^\infty$.

If α and p satisfy (6.4), it implies that $\{g_{p,n_k}(u_{n_k})\}_{n=1}^\infty$ is equi-integrable. Therefore $g_{p,n_k}(u_{n_k}) \to |u|^{p-1}u$ in $L^1(-1,1)$. Passing to the limit as $n_k \to \infty$, we obtain that

$$\int_{-1}^1 |x|^{2\alpha}u'\zeta' dx + \int_{-1}^1 |u|^{p-1}u\zeta dx = \int_{-1}^1 \zeta d\mu, \ \forall \zeta \in C_0^1[-1,1].$$

Moreover, we can check that

$$\lim_{x\to 0^+} |x|^{2\alpha-1}u(x) = \lim_{x\to 0^+}\lim_{k\to\infty} |x|^{2\alpha-1}u_{n_k}(x) = -\frac{1}{2\alpha-1}K^+ = \frac{1}{2(2\alpha-1)}\mu(\{0\}),$$
$$\lim_{x\to 0^-} |x|^{2\alpha-1}u(x) = \lim_{x\to 0^-}\lim_{k\to\infty} |x|^{2\alpha-1}u_{n_k}(x) = \frac{1}{2\alpha-1}K^- = \frac{1}{2(2\alpha-1)}\mu(\{0\}).$$

Thus, u is the good solution of (6.1).

If α and p satisfy (6.5) or (6.6), we obtain that $u_{n_k} \to u$ uniformly on any closed interval $I \subset [-1,1]\setminus\{0\}$. Therefore,

$$\int_{-1}^1 |x|^{2\alpha}u'\zeta' dx + \int_{-1}^1 |u|^{p-1}u\zeta dx = \int_{-1}^1 \zeta d\mu, \ \forall \zeta \in C_c^1((-1,1)\setminus\{0\}).$$

The same argument as in the proof of Lemma 6.16 implies that u is the solution of (6.8).

For all the above cases, the uniqueness of the limit implies that (6.40)-(6.43) hold for the whole sequence $\{u_n\}_{n=1}^{\infty}$. □

If we assume the data to be L^1, we have a further result about the mode of convergence.

Theorem 6.21. *For $\alpha \geq \frac{1}{2}$ and $\mu \in L^1(-1,1)$, the mode of convergence in* (6.42) *and* (6.43) *can be improved as*

$$\left(1+\ln\frac{1}{|x|}\right)^{-1} u_n \to \left(1+\ln\frac{1}{|x|}\right)^{-1} u \text{ in } C_0[-1,1], \text{ if } \alpha = \frac{1}{2},$$

$$|x|^{2\alpha-1}u_n \to |x|^{2\alpha-1}u \text{ in } C_0[-1,1], \text{ if } \alpha > \frac{1}{2}.$$

The proof of Theorem 6.21 is just the same as the one of Theorem 5.13, except some obvious modifications due to the nonlinear term. We omit the detail.

Remark 6.8. *The choice of $g_{p,n}$ can be more general than the one given by* (6.9). *In fact, assume that $g_{p,n}$ satisfies*

(i) $g_{p,n} \in C(\mathbb{R})$, *nondecreasing,*

(ii) $0 \leq g_{p,1}(t) \leq g_{p,2}(t) \leq \cdots \leq |t|^{p-1}t$, *for* $t \in (0,\infty)$,

(iii) $|t|^{p-1}t \leq \cdots g_{p,2}(t) \leq g_{p,1}(t) \leq 0$, *for* $t \in (-\infty, 0)$,

(iv) $g_{p,n}(t) \to |t|^{p-1}t$, *as* $n \to \infty$,

(v) for each $p > 1$ *and* $n \in \mathbb{N}$, *there exist constants* $C = C(p,n) > 0$ *and* $M = M(p,n) > 0$ *such that*

$$\begin{cases} |g_{p,n}(t)| \leq C|t|, \text{ for } |t| \in (M,\infty), \text{ if } 0 < \alpha < 1, \\ |g_{p,n}(t)| = C|t|, \text{ for } |t| \in (M,\infty), \text{ if } \alpha \geq 1. \end{cases}$$

Then all the results in this section still hold and the proof remains the same.

6.5 The lack of stability of the good solution for $\frac{1}{2} \le \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$

This section is devoted to the question of stability of the solution with respect to the perturbation of the measure μ under the weak-star topology. Recall that Lemma 6.13 implies that when $0 < \alpha < \frac{1}{2}$ and $p > 1$ the unique good solution is stable. Lemma 6.16 implies that when α and p satisfy (6.5) or (6.6) and $\mu(\{0\}) = 0$, the unique solution is stable. Therefore, we only investigate the stability of the good solution when $\frac{1}{2} \le \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$. In this case, as we pointed out in Remark 6.4, the stability of the good solution fails.

Assume $\frac{1}{2} \le \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$. Given $\mu \in \mathcal{M}(-1,1)$, there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset L^1(-1,1)$ such that $f_n \overset{*}{\rightharpoonup} \mu$ in $(C_0[-1,1])^*$. Let u_n be the unique good solution of the following equation

$$\begin{cases} -(|x|^{2\alpha}u_n')' + |u_n|^{p-1}u_n = f_n \quad \text{on } (-1,1), \\ u_n(-1) = u_n(1) = 0. \end{cases} \tag{6.44}$$

By Proposition 6.11, we know that $u_n \in D(A_\alpha) \cap L^p(-1,1)$ and

$$\int_{-1}^{1} |x|^{2\alpha}u_n'\zeta' dx + \int_{-1}^{1} |u_n|^{p-1}u_n\zeta dx = \int_{-1}^{1} f_n\zeta dx, \ \forall \zeta \in C_0^1[-1,1]. \tag{6.45}$$

The limiting behavior of the sequence $\{u_n\}_{n=1}^{\infty}$ is *sensitive* to the choice for the sequence $\{f_n\}_{n=1}^{\infty}$.

Theorem 6.22. *Assume that $\frac{1}{2} \le \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$. Take $\rho \in C(\mathbb{R})$ such that* $\operatorname{supp}\rho = [-1,1]$*, $\rho(x) = \rho(-x)$ and $\rho \ge 0$. Let $C^{-1} = \int \rho$ and $\rho_n(x) = Cn\rho(nx)$. For fixed $\tau \in \mathbb{R}$, take*

$$f_n = \mu * \rho_n + \tau\left(Cn\rho(nx-1) - Cn\rho(nx+1)\right). \tag{6.46}$$

Then $f_n \overset{*}{\rightharpoonup} \mu$ in $(C_0[-1,1])^*$. Let u_n be the unique good solution of (6.44)*. Then as $n \to \infty$, we have*

$$\left(1 + \ln\frac{1}{|x|}\right)^{-1} u_n \to \left(1 + \ln\frac{1}{|x|}\right)^{-1} u \ \textit{in } L^r(-1,1), \ \forall r < \infty, \ \textit{if } \alpha = \frac{1}{2}, \tag{6.47}$$

$$|x|^{2\alpha-1}u_n \to |x|^{2\alpha-1}u \ \textit{in } L^r(-1,1), \ \forall r < \infty, \ \textit{if } \frac{1}{2} < \alpha < 1, \tag{6.48}$$

where u is a solution of (6.1) *such that, if $\alpha = \frac{1}{2}$,*

$$\begin{cases} \lim_{x\to 0^+} \left(1+\ln\frac{1}{|x|}\right)^{-1} u(x) = -\lim_{x\to 0^+} |x|u'(x) = \frac{1}{2}\mu(\{0\}) + \tau, \\ \lim_{x\to 0^-} \left(1+\ln\frac{1}{|x|}\right)^{-1} u(x) = \lim_{x\to 0^-} |x|u'(x) = \frac{1}{2}\mu\left(\{0\}\right) - \tau, \end{cases} \tag{6.49}$$

and if $\frac{1}{2} < \alpha < 1$,

$$\begin{cases} \lim_{x\to 0^+} |x|^{2\alpha-1}u(x) = -\frac{1}{2\alpha-1}\lim_{x\to 0^+} |x|^{2\alpha}u'(x) = \frac{\mu(\{0\})}{2(2\alpha-1)} + \frac{\tau}{2\alpha-1}, \\ \lim_{x\to 0^-} |x|^{2\alpha-1}u(x) = \frac{1}{2\alpha-1}\lim_{x\to 0^-} |x|^{2\alpha}u'(x) = \frac{\mu(\{0\})}{2(2\alpha-1)} - \frac{\tau}{2\alpha-1}. \end{cases} \tag{6.50}$$

Remark 6.9. *A straightforward consequence of Theorem 6.22 is that the limiting function u is the good solution if and only if $\tau = 0$. This means that, in general, the stability of the good solution fails.*

Proof of Theorem 6.22. Note that we already have (6.20)-(6.25) by Lemmas 6.14 and 6.15. Also note that since u_{n_k} is the good solution of (6.44), we have

$$|x|^{2\alpha}u'_{n_k}(x) = \int_0^x \left(|u_{n_k}(s)|^{p-1}u_{n_k}(s) - f_{n_k}(s)\right) ds, \ \forall x \in (-1,1).$$

Therefore,

$$\lim_{x\to 0^+} |x|^{2\alpha}u'(x) = \lim_{x\to 0^+}\lim_{k\to\infty} |x|^{2\alpha}u'_{n_k}(x) = -\lim_{x\to 0^+}\lim_{k\to\infty}\int_0^x f_{n_k}(s)ds.$$

Similarly,

$$\lim_{x\to 0^-} |x|^{2\alpha}u'(x) = \lim_{x\to 0^-}\lim_{k\to\infty}\int_x^0 f_{n_k}(s)ds.$$

Then taking into account (6.46), one can obtain (6.49) and (6.50). Finally, the uniqueness of the limit implies (6.47) and (6.48). □

If $\mu \in L^1(-1,1)$ and the convergence is under the weak topology $\sigma(L^1, L^\infty)$, we can recover the stability of the good solution.

Theorem 6.23. *Assume that $\frac{1}{2} \le \alpha < 1$, $1 < p < \frac{1}{2\alpha-1}$ and $\mu \in L^1(-1,1)$. Let the sequence $\{f_n\}_{n=1}^\infty \subset L^1(-1,1)$ be such that $f_n \rightharpoonup \mu$ weakly in $\sigma(L^1, L^\infty)$. Let u_n be the unique good solution of* (6.44)*. Then as $n \to \infty$, we have*

$$\left(1+\ln\frac{1}{|x|}\right)^{-1} u_n \to \left(1+\ln\frac{1}{|x|}\right)^{-1} u \ \text{in } C_0[-1,1], \ \text{if } \alpha = \frac{1}{2}, \tag{6.51}$$

$$|x|^{2\alpha-1}u_n \to |x|^{2\alpha-1}u \textit{ in } C_0[-1,1], \textit{ if } \frac{1}{2} < \alpha < 1, \tag{6.52}$$

where u is the good solution of (6.1).

The proof of Theorem 6.23 is the same as the one of Theorem 5.15, except some obvious modifications due to the nonlinear term. We omit the detail.

6.6 The non-uniqueness for the case (6.3) and (6.4)

Throughout this section, we assume that α and p satisfy (6.3) and (6.4). We present a complete description of all the solutions of (6.1). Note that if u is a solution of (6.1), then we have

$$\lim_{x\to 0^+} |x|^{2\alpha}u'(x) - \lim_{x\to 0^-} |x|^{2\alpha}u'(x) = -\mu\left(\{0\}\right).$$

On the other hand, we have

Theorem 6.24. *Assume that α and p satisfy* (6.3) *and* (6.4)*. For any $\tau \in \mathbb{R}$ and any $\mu \in \mathcal{M}(-1,1)$, there exists a unique solution u of* (6.1) *such that*

$$\begin{cases} \lim\limits_{x\to 0^+} |x|^{2\alpha}u'(x) = \tau, \\ \lim\limits_{x\to 0^-} |x|^{2\alpha}u'(x) = \tau + \mu\left(\{0\}\right). \end{cases} \tag{6.53}$$

Proof. We first prove uniqueness. For any $\tau \in \mathbb{R}$ and any $\mu \in \mathcal{M}(-1,1)$, assume that both u_1 and u_2 are solutions of (6.1) satisfying (6.53). Then

$$-(|x|^{2\alpha}(u_1-u_2)')' + |u_1|^{p-1}u_1 - |u_2|^{p-1}u_2 = 0,$$

and $\lim\limits_{x\to 0} |x|^{2\alpha}(u_1-u_2)'(x) = 0$. When $0 < \alpha < \frac{1}{2}$, take $\phi \in C^\infty(\mathbb{R})$ such that $\phi(0) = 0$, $\phi' \geq 0$, $\phi > 0$ on $(0,+\infty)$, $\phi < 0$ on $(-\infty, 0)$, and $\phi = \text{sign}$ on $\mathbb{R}\backslash(-1,1)$. Since $u_1 - u_2 \in W^{1,1}(0,1)$, we have

$$\int_0^1 (|x|^{2\alpha}(u_1-u_2)')'\phi(u_1-u_2)dx = -\int_0^1 |x|^{2\alpha}((u_1-u_2)')^2\phi'(u_1-u_2)dx \leq 0.$$

Therefore,

$$\int_0^1 (|u_1|^{p-1}u_1 - |u_2|^{p-1}u_2)\phi(u_1-u_2)dx = 0.$$

It implies that $u_1 = u_2$ a.e. on $(0,1)$. The same argument implies that $u_1 = u_2$ a.e. on $(-1,0)$. When $\frac{1}{2} \le \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$, by Lemma 5.9, we have $u_1 - u_2 \in D(A_\alpha)$. Assertion (iv) of Proposition 5.7 implies that

$$\int_{-1}^{1} (|x|^{2\alpha}(u_1 - u_2)')' \operatorname{sign}(u_1 - u_2) dx \le 0.$$

Therefore, $u_1 = u_2$ a.e. on $(-1,1)$.

Next we prove the existence when $0 < \alpha < \frac{1}{2}$ and $p > 1$. We first claim that for every $\nu \in \mathcal{M}(0,1)$ and $\tau \in \mathbb{R}$, there exists $v \in W^{1,1}(0,1)$ such that $x^{2\alpha}v' \in BV(0,1)$ and

$$\begin{cases} -(x^{2\alpha}v')' + |v|^{p-1}v = \nu & \text{on } (0,1), \\ v(1) = 0, \\ \lim_{x\to 0^+} x^{2\alpha}v'(x) = \tau. \end{cases} \tag{6.54}$$

Indeed, define a nonlinear operator $A : C[0,1] \to C[0,1]$ as

$$\begin{aligned} Av(x) = & \frac{1 - x^{1-2\alpha}}{1-2\alpha} \int_0^x |v(s)|^{p-1}v(s)ds + \int_x^1 |v(s)|^{p-1}v(s)\frac{1-s^{1-2\alpha}}{1-2\alpha}ds \\ & - \int_x^1 \frac{1}{t^{2\alpha}} \int_{(0,t)} d\nu dt + \tau\frac{1-x^{1-2\alpha}}{1-2\alpha}. \end{aligned}$$

It is clear that A is continuous. Recall from Section 3.6 that X_0^α is compact in $C[0,1]$ when $0 < \alpha < \frac{1}{2}$. It is easy to check that $A\left(X_0^\alpha\right) \subset X_0^\alpha$. Therefore, the Schauder Fixed Point Theorem implies that there exists a fixed point $v \in X_0^\alpha$ such that $v = Av$. This fixed point v is precisely a solution of (6.54).

For any $\mu \in \mathcal{M}(-1,1)$, take $\mu_1 = \mu|_{(0,1)}$ and $\mu_2 = \mu|_{(-1,0)}$. For any $\tau \in \mathbb{R}$, we deduce from the above claim that there exist $u_1 \in W^{1,1}(0,1)$ and $u_2 \in W^{1,1}(-1,0)$ such that $x^{2\alpha}u_1' \in BV(0,1)$ and $|x|^{2\alpha}u_2' \in BV(-1,0)$, which satisfy

$$\begin{cases} -(x^{2\alpha}u_1')' + |u_1|^{p-1}u_1 = \mu_1 & \text{on } (0,1), \\ u_1(1) = 0, \lim_{x\to 0^+} x^{2\alpha}u_1'(x) = \tau, \end{cases}$$

and

$$\begin{cases} -(|x|^{2\alpha}u_2')' + |u_2|^{p-1}u_2 = \mu_2 & \text{on } (-1,0), \\ u_2(-1) = 0, \lim_{x\to 0^-} |x|^{2\alpha}u_2'(x) = \tau + \mu\left(\{0\}\right). \end{cases}$$

Take

$$u = \begin{cases} u_1 & \text{on } (0,1), \\ u_2 & \text{on } (-1,0). \end{cases}$$

Then u is a solution of (6.1) satisfying (6.53).

When $\frac{1}{2} \le \alpha < 1$ and $1 < p < \frac{1}{2\alpha-1}$, the existence of the solution of (6.1) with property (6.53) is a direct consequence of Theorem 6.22. □

6.7 Removable singularity

In this section, we prove Theorem 6.9. The idea of the proof is the same as Brezis-Véron [14] and Brezis [7].

Lemma 6.25. *Assume that $\alpha > 0$, $p > 1$ and $f \in L^1(-1,1)$. Let $u \in L^p_{loc}((-1,1)\backslash\{0\})$ be such that*

$$-\int_{-1}^{1} u(|x|^{2\alpha}\zeta')'dx + \int_{-1}^{1} |u|^{p-1}u\zeta dx = \int_{-1}^{1} f\zeta dx,\ \forall \zeta \in C_c^\infty((-1,1)\backslash\{0\}).$$

Then $u \in W^{2,1}_{loc}((-1,1)\backslash\{0\})$ and

$$-(|x|^{2\alpha}u')' + |u|^{p-1}u = f \quad \text{on } (a,b),\ \forall (a,b) \subset\subset (-1,1)\backslash\{0\}.$$

The proof of Lemma 6.25 is standard.

Lemma 6.26. *Assume that $\alpha > 0$, $p > 1$ and $f \in L^1(-1,1)$. Assume that $u \in W^{2,1}_{loc}((-1,1)\backslash\{0\})$ and*

$$-(|x|^{2\alpha}u')' + |u|^{p-1}u = f \quad \text{on } (a,b),\ \forall (a,b) \subset\subset (-1,1)\backslash\{0\}.$$

Then

$$-\int_{-1}^{1} u^+(|x|^{2\alpha}\zeta')'dx + \int_{-1}^{1} (u^+)^p\zeta dx \le \int_{-1}^{1} f^+\zeta dx,\ \forall \zeta \in C_c^\infty((-1,1)\backslash\{0\}) \text{ and } \zeta \ge 0. \tag{6.55}$$

Proof. Denote $\mathcal{L}u = (|x|^{2\alpha}u')'$. Fix an interval $(a,b) \subset\subset (-1,1)\backslash\{0\}$. We recall the following Kato's inequality (lemma A in [32]),

$$\mathcal{L}|u| \ge (\mathcal{L}u)\,\mathrm{sign}\, u \quad \text{in } \mathcal{D}'(a,b).$$

By the same argument as in Lemma 1 of [14], we obtain

$$\mathcal{L}(u^+) \geq (\mathcal{L}u)\,\mathrm{sign}^+ u \quad \text{ in } \mathcal{D}'(a,b), \tag{6.56}$$

where

$$\mathrm{sign}^+ x = \begin{cases} 1 & \text{when } x > 0, \\ \frac{1}{2} & \text{when } x = 0, \\ 0 & \text{when } x < 0. \end{cases}$$

Since $\mathcal{L}u = |u|^{p-1}u - f$ on (a,b), it implies that

$$\mathcal{L}(u^+) \geq |u|^{p-1}u\,\mathrm{sign}^+ u - f^+ = (u^+)^p - f^+ \quad \text{ in } \mathcal{D}'(a,b).$$

Therefore

$$-\int_a^b u^+(|x|^{2\alpha}\zeta')'dx + \int_{-1}^1 (u^+)^p\zeta dx \leq \int_{-1}^1 f^+\zeta dx,\ \forall \zeta \in C_c^\infty(a,b) \text{ and } \zeta \geq 0.$$

Since (a,b) is arbitrary in $(-1,1)\backslash\{0\}$, we derived (6.55). □

Lemma 6.27 (Maximum Principle)**.** *Let $\alpha > 0$. Assume that $(a,b) \subset\subset (-1,1)\backslash\{0\}$ and $u \in L^1(a,b)$ satisfying $u \geq 0$ a.e., $\mathrm{supp}\, u \subset\subset (a,b)$ and*

$$(|x|^{2\alpha}u')' \geq 0 \quad \textit{in } \mathcal{D}'(a,b).$$

Then $u = 0$ a.e. on (a,b).

Proof. Assume $\mathrm{supp}\, u \subset (\bar{\bar{a}}, \bar{\bar{b}}) \subset\subset (\bar{a}, \bar{b}) \subset\subset (a,b)$. Take the positive smooth mollifiers $\rho_n(x) = Cn\rho(nx)$ where $\rho(x) = \chi_{[|x|<1]}e^{\frac{1}{|x|^2-1}}$ and $C^{-1} = \int \rho$. Consider $u_n = u * \rho_n$ with n large enough such that $\left(\bar{a} - \frac{1}{n}, \bar{b} + \frac{1}{n}\right) \subset (a,b)$. Notice that $u_n \geq 0$ and $u_n \in C_c^\infty(a,b)$. We claim that

$$\int_a^b (|x|^{2\alpha}u_n')'\zeta dx \geq 0,\ \forall \zeta \in C_c^\infty(a,b) \text{ with } \zeta \geq 0. \tag{6.57}$$

Indeed, we have

$$\begin{aligned}\int_a^b (|x|^{2\alpha}u_n')'\zeta dx &= \int_a^b u_n(|x|^{2\alpha}\zeta')'dx \\ &= \int_{-\frac{1}{n}}^{\frac{1}{n}} \rho_n(y)\left(\int_{\bar{\bar{a}}}^{\bar{\bar{b}}} u(z)(|z+y|^{2\alpha}\zeta'(z+y))'dz\right)dy.\end{aligned}$$

It is enough to show

$$\int_{\bar{\bar{a}}}^{\bar{\bar{b}}} u(z)(|z+y|^{2\alpha}\zeta'(z+y))'dz \geq 0,\ \forall y \in (-\frac{1}{n}, \frac{1}{n}),\ \forall \zeta \in C_c^\infty(a,b) \text{ with } \zeta \geq 0.$$

We already know

$$\int_a^b u(z)(|z|^{2\alpha}\varphi'(z))'dz \geq 0,\ \forall \varphi \in C_c^\infty(a,b) \text{ with } \varphi \geq 0.$$

Given $y \in (-\frac{1}{n}, \frac{1}{n})$ and $\zeta \in C_c^\infty(a,b)$ with $\zeta \geq 0$, define

$$\bar{\varphi}(z) = \int_{\bar{a}}^{z} \frac{|t+y|^{2\alpha}}{|t|^{2\alpha}}\zeta'(t+y)dt + \int_{\bar{a}}^{\bar{b}} \frac{|t+y|^{2\alpha}}{|t|^{2\alpha}}|\zeta'(t+y)|dt \quad \text{ on } [\bar{a}, \bar{b}].$$

Take $\varphi = \bar{\varphi}h$ where h is the cut-off function such that $h \in C_c^\infty(a,b)$, $h \geq 0$, $h \equiv 1$ on $(\bar{\bar{a}}, \bar{\bar{b}})$ and $\operatorname{supp} h \subset (\bar{a}, \bar{b})$. Then $\varphi \in C_c^\infty(a,b)$ with $\varphi \geq 0$. Therefore

$$\int_{\bar{\bar{a}}}^{\bar{\bar{b}}} u(z)(|z+y|^{2\alpha}\zeta'(z+y))'dz = \int_a^b u(z)(|z|^{2\alpha}\varphi'(z))'dz \geq 0.$$

Thus we proved (6.57). It implies that $(|x|^{2\alpha}u_n')' \geq 0$ on (a,b). The classical Maximum Principle yields that $u_n = 0$. Since $u_n \to u$ in $L^1(a,b)$, we have $u = 0$ a.e. on (a,b). $\square$

Lemma 6.28 (Keller-Osserman Estimate)**.** *Assume that $\alpha > 0$, $p > 1$ and $f \in L^1(-1,1)$. Let $u \in W_{loc}^{2,1}((-1,1)\backslash\{0\})$ be such that*

$$-(|x|^{2\alpha}u')' + |u|^{p-1}u = f \quad \text{on } (a,b),\ \forall (a,b) \subset\subset (-1,1)\backslash\{0\}.$$

Then

$$u(x) \leq C(\alpha,p)|x|^{\frac{2\alpha-2}{p-1}} + u_0(x),\ \forall 0 < |x| \leq \frac{1}{2}, \tag{6.58}$$

where $C(\alpha,p)$ is a positive constant depending only on α and p, and $u_0 \in D(A_\alpha) \cap L^p(-1,1)$ is the unique solution of

$$\begin{cases} -(|x|^{2\alpha}u_0')' + u_0^p = |f| & \text{on } (-1,1), \\ u_0(-1) = u_0(1) = 0. \end{cases}$$

Proof. We fix x_0 such that $0 < |x_0| \leq \frac{1}{2}$. Consider the interval

$$I_{x_0} = \left(\operatorname{sign} x_0 \frac{|x_0|}{2}, \operatorname{sign} x_0 \frac{3|x_0|}{2}\right) \subset\subset (-1,1)\backslash\{0\}.$$

Define

$$v(x) = \lambda \left(\frac{|x_0|^2}{4} - (x - x_0)^2 \right)^{-\frac{2}{p-1}} \qquad \text{on } I_{x_0},$$

where $\lambda > 0$ is a constant to be determined so that

$$-(|x|^{2\alpha} v')' + v^p \geq 0 \qquad \text{on } I_{x_0}. \tag{6.59}$$

Indeed, we have

$$(|x|^{2\alpha} v')' = \frac{4\lambda}{p-1} \left(\frac{|x_0|^2}{4} - (x - x_0)^2 \right)^{-\frac{2}{p-1} - 2} \times J$$

where

$$J = \frac{2(p+1)}{p-1} (x - x_0)^2 |x|^{2\alpha} + \left(\frac{|x_0|^2}{4} - (x - x_0)^2 \right) \left(|x|^{2\alpha} + 2\alpha (x - x_0) |x|^{2\alpha - 1} \operatorname{sign} x \right).$$

Since $x \in I_{x_0}$, we have $|J| \leq A(\alpha) |x_0|^{2\alpha+2}$ where $A(\alpha)$ is a constant only depending on α. Notice that $-\frac{2}{p-1} - 2 = -\frac{2p}{p-1}$. Therefore,

$$-(|x|^{2\alpha} v')' + v^p \geq \left(-A(\alpha) \frac{4\lambda}{p-1} |x_0|^{2\alpha+2} + \lambda^p \right) \left(\frac{|x_0|^2}{4} - (x - x_0)^2 \right)^{-\frac{2p}{p-1}}.$$

Take λ such that

$$-A(\alpha) \frac{4\lambda}{p-1} |x_0|^{2\alpha+2} + \lambda^p = 0,$$

i.e.

$$\lambda = \left(\frac{4A(\alpha)}{p-1} |x_0|^{2\alpha+2} \right)^{\frac{1}{p-1}}.$$

Then the inequality (6.59) holds. Now take $\bar{v} = v + u_0$ which satisfies

$$-(|x|^{2\alpha} \bar{v}')' + \bar{v}^p \geq |f| \qquad \text{on } I_{x_0}.$$

Denote $\mathcal{L}u = (|x|^{2\alpha} u')'$. We have

$$\mathcal{L}(u - \bar{v}) \geq |u|^{p-1} u - \bar{v}^p \qquad \text{on } I_{x_0}.$$

Applying the revised Kato's inequality (6.56), we obtain

$$\mathcal{L}\left((u - \bar{v})^+ \right) \geq (|u|^{p-1} u - \bar{v}^p) \operatorname{sign}^+ (u - \bar{v}) \geq 0 \qquad \text{in } \mathcal{D}'(I_{x_0}).$$

Notice that $\lim_{x\to\partial I_{x_0}} \bar{v}(x) = +\infty$ and $u \in L^\infty(I_{x_0})$. It follows that $(u-\bar{v})^+ = 0$ near ∂I_{x_0}. Then Lemma 6.27 implies that $(u-\bar{v})^+ = 0$ on I_{x_0}. In particular,

$$u(x_0) \le \bar{v}(x_0) = \left(\frac{1}{4}\right)^{-\frac{2}{p-1}} \left(\frac{4A(\alpha)}{p-1}\right)^{\frac{1}{p-1}} |x_0|^{\frac{2\alpha-2}{p-1}} + u_0(x_0).$$

Let $C(\alpha,p) = \left(\frac{1}{4}\right)^{-\frac{2}{p-1}} \left(\frac{4A(\alpha)}{p-1}\right)^{\frac{1}{p-1}}$. Note that x_0 is arbitrary in $(0,\frac{1}{2}]$, so we obtain (6.58). □

Lemma 6.29. *Under the assumption of Theorem 6.9, we have $u \in L^p_{loc}(-1,1)$.*

Proof. We first prove that $u^+ \in L^p_{loc}(-1,1)$. Applying Lemma 6.25 and 6.26, we find

$$-\int_{-1}^1 u^+(|x|^{2\alpha}\zeta')'dx + \int_{-1}^1 (u^+)^p\zeta dx \le \int_{-1}^1 f^+\zeta dx,\ \forall\zeta\in C_c^\infty((-1,1)\backslash\{0\}) \text{ with } \zeta\ge 0.$$

Take $\varphi(x)\in C^\infty(\mathbb{R})$ such that $0\le\varphi\le 1$, $\varphi\equiv 0$ on $(-\frac{1}{2},\frac{1}{2})$ and $\varphi\equiv 1$ on $\mathbb{R}\backslash(-1,1)$. Define $\varphi_n(x)=\varphi(nx)\in C^\infty[-1,1]$. For any $\zeta\in C_c^\infty(-1,1)$ with $\zeta\ge 0$, we have

$$\int_{-1}^1 (u^+)^p\varphi_n\zeta dx \le \int_{-1}^1 u^+(|x|^{2\alpha}(\varphi_n\zeta)')'dx + \int_{-1}^1 f^+\varphi_n\zeta dx.$$

Notice that

$$\begin{aligned}
&\int_{-1}^1 u^+(|x|^{2\alpha}(\varphi_n\zeta)')'dx \\
=&2\alpha n\int_{-\frac{1}{n}}^{\frac{1}{n}} u^+ \operatorname{sign} x|x|^{2\alpha-1}\varphi'(nx)\zeta dx + 2\alpha\int_{-1}^1 u^+ \operatorname{sign} x|x|^{2\alpha-1}\varphi(nx)\zeta' dx \\
&+\int_{-1}^1 u^+|x|^{2\alpha}\varphi_n\zeta'' dx + 2n\int_{-\frac{1}{n}}^{\frac{1}{n}} u^+|x|^{2\alpha}\varphi'(nx)\zeta' dx + n^2\int_{-\frac{1}{n}}^{\frac{1}{n}} u^+|x|^{2\alpha}\varphi''(nx)\zeta dx.
\end{aligned}$$

In view of Lemma 6.28 and Proposition 6.11, we know

$$\left\|u^+|x|^{2\alpha-1}\right\|_{L^\infty(-\frac{1}{2},\frac{1}{2})} + \left\|nu^+|x|^{2\alpha}\right\|_{L^\infty(-\frac{1}{n},\frac{1}{n})} \le C,$$

where C is independent of n. Also notice that

$$\int_{-\frac{1}{n}}^{\frac{1}{n}} n|\varphi'(nx)|dx = \int_{-1}^1 |\varphi'(x)|dx$$

and

$$\int_{-\frac{1}{n}}^{\frac{1}{n}} n|\varphi''(nx)|dx = \int_{-1}^1 |\varphi''(x)|dx.$$

Therefore,

$$\int_{-1}^{1} u^+(|x|^{2\alpha}(\varphi_n\zeta)')'dx \le C,$$

where C is independent of n. It implies that

$$\int_{-1}^{1} (u^+)^p\varphi_n\zeta dx \le C.$$

Passing to the limit as $n \to \infty$, we have $(u^+)^p\zeta \in L^1(-1,1)$. Hence, $u^+ \in L^p_{loc}(-1,1)$. Similarly, $u^- \in L^p_{loc}(-1,1)$. □

Proof of Theorem 6.9. Take $\varphi(x) \in C^\infty(\mathbb{R})$ such that $0 \le \varphi \le 1$, $\varphi \equiv 0$ on $(-\frac{1}{2}, \frac{1}{2})$ and $\varphi \equiv 1$ on $\mathbb{R}\backslash(-1,1)$. Define $\varphi_n(x) = \varphi(nx) \in C^\infty[-1,1]$. Then we have

$$-\int_{-1}^{1} u(|x|^{2\alpha}(\varphi_n\zeta)')'dx + \int_{-1}^{1} |u|^{p-1}u\varphi_n\zeta dx = \int_{-1}^{1} f\varphi_n\zeta dx,\ \forall\zeta \in C_c^\infty(-1,1). \quad (6.60)$$

Note that $u \in L^p_{loc}(-1,1)$ by Lemma 6.29. Passing to the limit as $n \to \infty$ in (6.60), the same argument as in the proof of Lemma 6.16 implies (6.11). □

6.8 Classification of the singularity

In this section, we prove Theorem 6.10. The proof combines ideas by Véron [40, 41] and Brezis-Oswald [11].

Lemma 6.30. *Assume that $\alpha > 0$ and $p \ge 1$. Let $u \in C^2(0,1]$ satisfying (6.12). Then u can not change signs, i.e., either $u \ge 0$, or $u \le 0$ on $(0,1]$.*

Proof. For a fixed $t \in (0,1)$, multiply (6.12) on both sides by $u(x)$ and integrate by parts on the interval $(t,1)$. We obtain that

$$-\frac{1}{2}t^{2\alpha}\frac{d}{dt}(u^2(t)) = \int_t^1 x^{2\alpha}u'(x)u'(x)dx + \int_t^1 |u(x)|^{p+1}dx \ge 0.$$

It implies that $|u|$ is decreasing on $(0,1]$ and therefore changing sign is not permitted for u. □

Lemma 6.31. *Assume that $\alpha > 0$ and $p > 1$. Let $u \in C^2(0,1]$ be such that $u \ge 0$ and u satisfies (6.12). Let*

$$v(r) = \left(\frac{1}{1-\alpha}\right)^{\frac{2}{p-1}} u\left(r^{\frac{1}{1-\alpha}}\right) \in C^2(0,1]. \quad (6.61)$$

Then v solves

$$\begin{cases} -v''(r) - \left(\frac{\alpha}{1-\alpha}\right)\frac{1}{r}v'(r) + v^p(r) = 0 & \text{on } (0,1), \\ v(1) = 0. \end{cases} \tag{6.62}$$

Moreover $r^{\frac{2}{p-1}}v(r) \in L^\infty(0,1)$.

Proof. One can directly check that v solves (6.62). By Lemma 6.28, we have $x^{\frac{2-2\alpha}{p-1}}u(x) \in L^\infty(0,1)$. Therefore $r^{\frac{2}{p-1}}v(r) \in L^\infty(0,1)$. □

Lemma 6.32. *Assume that α and p satisfy* (6.3) *or* (6.4). *Assume that $v \in C^2(0,1]$, $v \geq 0$ and v solves* (6.62). *Denote*

$$\bar{l}_{p,\alpha} = \left[\left(\frac{2}{p-1}\right)\left(\frac{2p}{p-1} - \frac{1}{1-\alpha}\right)\right]^{\frac{1}{p-1}}. \tag{6.63}$$

Then one of the following assertions holds.

(i) $\lim_{r\to 0^+} r^{\frac{2}{p-1}}v(r) = \bar{l}_{p,\alpha}$.

(ii) $\lim_{r\to 0^+} r^{\frac{2}{p-1}}v(r) = 0$.

Moreover, if v satisfies (i), then

$$\left|v(r) - \bar{l}_{p,\alpha}r^{-\frac{2}{p-1}}\right| \leq \bar{l}_{p,\alpha}r^{\frac{2p}{p-1}-\frac{1}{1-\alpha}}, \ \forall r \in (0,1]. \tag{6.64}$$

Proof. Write $\bar{l}_{p,\alpha}^{-1}r^{\frac{2}{p-1}}v(r) = \phi(x)$ where $x = r^{\frac{2(p+1)}{p-1}-\frac{1}{1-\alpha}}$. It is easy to obtain that $\phi(x) \in L^\infty(0,1)$ and it solves

$$\begin{cases} x^2\phi''(x) = \dfrac{\bar{l}_{p,\alpha}^{p-1}}{\left(\frac{2(p+1)}{p-1} - \frac{1}{1-\alpha}\right)^2}(\phi^p(x) - \phi(x)) & \text{on } (0,1), \\ \phi(1) = 0. \end{cases}$$

We claim that $0 \leq \phi(x) \leq 1$. Indeed, if $\phi(x_0) > 1$ for some $x_0 \in (0,1)$, then ϕ is convex and increasing on $(0,x_0)$. Therefore $\phi''(x) \geq \frac{c}{x^2}$ on $(0,x_0)$, and thus $\phi(x) \geq \tilde{c} - c\ln x$, which contradicts $\phi \in L^\infty(0,1)$. Hence $0 \leq \phi(x) \leq 1$.

As a result, ϕ is concave and $\lim_{x\to 0^+}\phi(x)$ exists. If $0 < \lim_{x\to 0^+}\phi(x) < 1$, then $\phi''(x) \leq -\frac{c}{x^2}$ for x near 0, and thus $\phi(x) \leq -\tilde{c} + c\ln x$, which again contradicts $\phi \in L^\infty(0,1)$. Therefore either $\lim_{x\to 0^+}\phi(x) = 1$ or $\lim_{x\to 0^+}\phi(x) = 0$. If $\lim_{x\to 0^+}\phi(x) = 1$, since ϕ is concave, it implies that $1 \geq \phi(x) \geq 1 - x$, $\forall x \in (0,1]$, which is precisely (6.64). □

Lemma 6.33. *Assume that* $\frac{1}{2} < \alpha < 1$ *and* $1 < p < \frac{1}{2\alpha - 1}$*. Assume that* $v \in C^2(0,1]$*,* $v \geq 0$ *and* v *solves* (6.62)*. If* $\lim_{r \to 0^+} r^{\frac{2}{p-1}} v(r) = 0$*, then there exists* $\epsilon_0 > 0$ *such that* $r^{\frac{2}{p-1} - \epsilon_0} v(r) \in L^\infty(0,1)$*.*

In order to prove Lemma 6.33, we need the following lemma from [41], which is originally due to Chen-Matano-Véron [21].

Lemma 6.34 (Lemma 2.1 in Page 67 of [41])**.** *Let* $y(t) \in C[0,\infty)$ *be such that* $y \geq 0$ *and*

(i) $\lim_{t \to \infty} y(t) = 0$,

(ii) $\limsup_{t \to \infty} e^{\epsilon t} y(t) = +\infty, \ \forall \epsilon > 0$.

Then there exists $\eta \in C^\infty[0,\infty)$ *such that*

(i) $\eta > 0$, $\eta' < 0$, $\lim_{t \to \infty} \eta(t) = 0$,

(ii) $\lim_{t \to \infty} e^{\epsilon t} \eta(t) = +\infty, \ \forall \epsilon > 0$,

(iii) $0 < \limsup_{t \to \infty} \frac{y(t)}{\eta(t)} < \infty$,

(iv) $\left(\frac{\eta'}{\eta}\right)', \left(\frac{\eta'}{\eta}\right)'' \in L^1(0,\infty)$,

(v) $\lim_{t \to \infty} \frac{\eta'(t)}{\eta(t)} = \lim_{t \to \infty} \frac{\eta''(t)}{\eta(t)} = 0$.

Proof of Lemma 6.33. Write $v(r) = r^{-\frac{2}{p-1}} y(t)$ where $t = \ln \frac{1}{r}$ and $t \in [0,\infty)$. Denote $\beta = \frac{2(p+1)}{p-1} - \frac{1}{1-\alpha}$. Then $y(t) \in C^2[0,\infty)$, $\lim_{t \to \infty} y(t) = 0$ and $y(t)$ solves

$$\begin{cases} y''(t) + \beta y'(t) + \bar{l}_{p,\alpha}^{p-1} y(t) - y^p(t) = 0 \quad \text{on } (0,\infty), \\ y(0) = 0. \end{cases}$$

Assume $\limsup_{t \to \infty} e^{\epsilon t} y(t) = +\infty$, $\forall \epsilon > 0$. Denote $w(t) = \frac{y(t)}{\eta(t)}$ where η is given by Lemma 6.34. Then $w \in L^\infty(0,\infty) \cap C^2[0,\infty)$ and w satifies

$$w''(t) + \left(\beta + 2\frac{\eta'(t)}{\eta(t)}\right) w'(t) = f(t) \quad \text{on } (0,\infty), \tag{6.65}$$

where

$$f(t) = \eta^{p-1}(t) w^p(t) - \left(\bar{l}_{p,\alpha}^{p-1} + \frac{\eta''(t)}{\eta(t)} + \beta \frac{\eta'(t)}{\eta(t)}\right) w(t) \in L^\infty(0,\infty).$$

We claim that

$$\lim_{t\to\infty} w'(t) = \lim_{t\to\infty} w''(t) = 0. \tag{6.66}$$

We only show $\lim_{t\to\infty} w'(t) = 0$ since one can show the other part of (6.66) by the same idea. To show $\lim_{t\to\infty} w'(t) = 0$, it is enough to obtain that w' is uniformly continuous and $w' \in L^2(0,\infty)$. To do so, we first need $w' \in L^\infty(0,\infty)$. Indeed, from (6.65) we obtain

$$(\eta^2(t)e^{\beta t}w'(t))' = \eta^2(t)e^{\beta t}f(t).$$

That is,

$$w'(t) = \frac{\int_0^t \eta^2(s)e^{\beta s}f(s)ds}{e^{\beta t}\eta^2(t)} + \frac{w'(0)\eta^2(0)}{e^{\beta t}\eta^2(t)}.$$

Note that the Mean Value Theorem yields

$$\frac{\int_0^t \eta^2(s)e^{\beta s}f(s)ds}{e^{\beta t}\eta^2(t)-\eta^2(0)} = \frac{\eta^2(\xi)e^{\beta\xi}f(\xi)}{\beta e^{\beta\xi}\eta^2(\xi)+2e^{\beta\xi}\eta'(\xi)\eta(\xi)}, \tag{6.67}$$

where $\xi \in (0,t)$ and ξ depends on t. One can check that the right hand side of (6.67) is in $L^\infty(0,\infty)$. Therefore $w' \in L^\infty(0,\infty)$. As a consequence, w is uniformly continuous. To show the uniform continuity of w', note that (6.65) implies

$$\left(w'(t) + \left(\beta + 2\frac{\eta'(t)}{\eta(t)}\right)w(t)\right)' = f(t) + 2\left(\frac{\eta'(t)}{\eta(t)}\right)' w(t). \tag{6.68}$$

One can check that the right hand side of (6.68) is in $L^\infty(0,\infty)$. Therefore $w'(t) + \left(\beta + 2\frac{\eta'(t)}{\eta(t)}\right)w(t)$ is uniformly continuous and so is w'. Now, multiplying (6.65) by $w'(t)$, we obtain

$$\begin{aligned}
&\left(\beta + 2\frac{\eta'(t)}{\eta(t)}\right)(w'(t))^2 \\
&= -\frac{1}{2}\frac{d}{dt}(w'(t))^2 - \frac{1}{2}\frac{d}{dt}\left[\left(\bar{l}_{p,\alpha}^{p-1} + \frac{\eta''(t)}{\eta(t)} + \beta\frac{\eta'(t)}{\eta(t)}\right)w^2(t)\right] \\
&\quad + \frac{1}{2}\left(\frac{\eta''(t)}{\eta(t)} + \beta\frac{\eta'(t)}{\eta(t)}\right)' w^2(t) + \frac{d}{dt}\left(\frac{\eta^{p-1}(t)w^{p+1}(t)}{p+1}\right) - \frac{p-1}{p+1}\eta^{p-2}(t)\eta'(t)w^{p+1}(t).
\end{aligned}$$

Notice that $\eta^{p-2}\eta' w^{p+1} \in L^1(0,\infty)$ since

$$\int_0^n \left|\eta^{p-2}(s)\eta'(s)w^{p+1}(s)\right| ds \le \left|w^{p+1}(\xi)\right| \left|\eta^{p-1}(0) - \eta^{p-1}(n)\right| \le 2\left\|w\right\|_{L^\infty}^{p+1}\left\|\eta\right\|_{L^\infty}^{p-1},$$

where n is any integer, $\xi \in (0,n)$ and the choice of ξ depends on n. By lemma 6.34, there exists $t_n \to \infty$ such that $\lim_{n\to\infty} w(t_n) = \theta > 0$. Since $w' \in L^\infty(0,\infty)$, without loss of generality, one can assume that $\lim_{n\to\infty} w'(t_n)$ exists. As a result, we obtain that $\lim_{n\to\infty} \int_0^{t_n} (w'(t))^2 dt$ exists. Therefore $w' \in L^2(0,\infty)$.

Note that (6.65) and (6.66) imply $\lim_{t\to\infty} w(t) = 0$, which is a contradiction with $\lim_{n\to\infty} w(t_n) = \theta > 0$. Hence, there exists $\epsilon_0 > 0$ such that $e^{\epsilon_0 t} y(t) \in L^\infty(0,\infty)$, i.e., $r^{\frac{2}{p-1}-\epsilon_0} v(r) \in L^\infty(0,1)$. □

Lemma 6.35. *Assume that* $\frac{1}{2} < \alpha < 1$ *and* $1 < p < \frac{1}{2\alpha-1}$. *Assume that* $v \in C^2(0,1]$, $v \geq 0$ *and* v *solves* (6.62). *If* $r^{\frac{2\alpha-1}{1-\alpha}} v(r) \notin L^\infty(0,1)$, *then* $r^\theta v(r) \notin L^\infty(0,1)$, $\forall \theta < \frac{2}{p-1}$.

Proof. Fix $k \in \left[\frac{2\alpha-1}{1-\alpha}, \frac{2}{p-1}\right)$. Write $v(r) = M r^{-k} h(s)$ where $s = \frac{r^j}{j}$ with $j = 2k - \frac{2\alpha-1}{1-\alpha} > 0$ and M is a positive constant such that $M^{p-1} j^{\frac{2-k(p-1)}{j}-2} = 1$. Then $h(s) \in C^2(0,1/j]$, $h \geq 0$ and h solves

$$\begin{cases} h''(s) = s^{\frac{2-k(p-1)}{j}-2} h^p(s) - k\left(k - \frac{2\alpha-1}{1-\alpha}\right) j^{-2} s^{-2} h(s) & \text{on } (0,1/j), \\ h(1/j) = 0. \end{cases}$$

Integrating the above equation, we obtain, for $s \in (0,1/j)$,

$$\begin{aligned} & h(s) + k\left(k - \frac{2\alpha-1}{1-\alpha}\right) j^{-2} \int_s^{1/j} t^{-2} h(t)(t-s)dt \\ = & -h'(1/j)(1/j - s) + \int_s^{1/j} t^{\frac{2-k(p-1)}{j}-2} h^p(t)(t-s)dt. \end{aligned}$$

Therefore,

$$\left|h(s) + h'(1/j)(1/j - s)\right| \leq \int_s^{1/j} t^{\frac{2-k(p-1)}{2j}} h^p(t) t^{\frac{2-k(p-1)}{2j}-1} dt.$$

Assume $r^k v(r) \notin L^\infty(0,1)$. Then $h(s) \notin L^\infty(0,1/j)$. The above inequality then implies that

$$s^{\frac{2-k(p-1)}{2j}} h^p(s) \notin L^\infty(0,1/j).$$

The definition of h implies that $r^{k+\frac{2-k(p-1)}{2p}} v(r) \notin L^\infty(0,1)$. By induction, we obtain a sequence $k_n \in \left[\frac{2\alpha-1}{1-\alpha}, \frac{2}{p-1}\right)$ such that $r^{k_n} v(r) \notin L^\infty(0,1)$, $\forall n \in \mathbb{N}$, $k_0 = \frac{2\alpha-1}{1-\alpha}$ and

$$k_n = k_{n-1} + \frac{2 - k_{n-1}(p-1)}{2p}.$$

That is,

$$k_n = \frac{2}{p-1} - \left(\frac{p+1}{2p}\right)^n \left(\frac{2}{p-1} - \frac{2\alpha-1}{1-\alpha}\right).$$

Therefore, $r^\theta v(r) \notin L^\infty(0,1)$, $\forall \theta < \frac{2}{p-1}$. □

Lemma 6.36. *Assume that* $\frac{1}{2} \leq \alpha < 1$ *and* $1 < p < \frac{1}{2\alpha-1}$. *Let* $u \in C^2(0,1]$ *be such that* $u \geq 0$, $\frac{u}{E_\alpha} \notin L^\infty(0,1)$ *and* u *solves* (6.12), *where* E_α *is defined by* (6.14). *Then* $\lim_{x\to 0^+} x^{\frac{2(1-\alpha)}{p-1}} u(x) = l_{p,\alpha}$.

Proof. Since $\frac{u}{E_\alpha} \notin L^\infty$, it implies

$$\limsup_{x\to 0^+} \frac{u(x)}{E_\alpha(x)} = +\infty.$$

Consider v defined by (6.61). We have that

$$\limsup_{r\to 0^+} \frac{v(r)}{I_\alpha(r)} = +\infty,$$

where

$$I_\alpha(r) = \begin{cases} \ln\frac{1}{r}, \text{ if } \alpha = \frac{1}{2}, \\ r^{-\frac{2\alpha-1}{1-\alpha}}, \text{ if } \frac{1}{2} < \alpha < 1. \end{cases}$$

It is then equivalent to show that

$$\lim_{r\to 0^+} r^{\frac{2}{p-1}} v(r) = \bar{l}_{p,\alpha}, \tag{6.69}$$

where $\bar{l}_{p,\alpha}$ is given by (6.63). If $\alpha = \frac{1}{2}$, one can check that v is the radially symmetric and positive solution of the following equation

$$\begin{cases} -\Delta v + v^p = 0 \quad \text{ on } B_1 \backslash \{0\}, \\ v = 0 \quad \text{ on } \partial B_1, \end{cases}$$

where $B_1 \subset \mathbb{R}^2$ is the unit ball centered at the origin. Then Theorem 4.1 by Véron [40] implies (6.69). If $\frac{1}{2} < \alpha < 1$, Lemmas 6.32, 6.33 and 6.35 imply (6.69). □

Lemma 6.37. *Assume that* $\frac{1}{2} \leq \alpha < 1$ *and* $1 < p < \frac{1}{2\alpha-1}$. *Let* $u \in C^2(0,1]$ *be such that* $u \geq 0$, $\frac{u}{E_\alpha} \in L^\infty(0,1)$ *and* u *solves* (6.12), *where* E_α *is defined by* (6.14). *Then its even extension* $\bar{u}(x) := u(|x|)$ *is the good solution of the following equation*

$$\begin{cases} -(|x|^{2\alpha}\bar{u}')' + \bar{u}^p = c_0\delta_0 \quad \text{on } (-1,1), \\ \bar{u}(-1) = \bar{u}(1) = 0, \end{cases} \tag{6.70}$$

where c_0 *is some nonnegative constant.*

Proof. We first claim that there is a sequence $\{a_n\}_{n=1}^{\infty} \subset (0,1)$ such that $\lim_{n\to\infty} a_n = 0$ and that the sequence $\left\{a_n^{2\alpha}u'(a_n)\right\}_{n=1}^{\infty}$ is bounded. Otherwise, it means $\lim_{x\to 0^+} x^{2\alpha}u'(x) = -\infty$ since u is non-increasing. Then for all $M > 0$, there exists $a_M \in (0,1)$ such that $\lim_{M\to+\infty} a_M = 0$ and

$$u'(x) \le -\frac{M}{x^{2\alpha}},\ \forall x \in (0, a_M).$$

It follows that

$$\frac{u(a_M^2)}{E_\alpha(a_M^2)} \ge \frac{M}{2},\ \text{if } \alpha = \frac{1}{2},$$

and

$$\frac{u(a_M/2)}{E_\alpha(a_M/2)} \ge \frac{M}{2\alpha-1}\left[1-\left(\frac{1}{2}\right)^{2\alpha-1}\right],\ \text{if } \frac{1}{2} < \alpha < 1,$$

which contradicts $\frac{u}{E_\alpha} \in L^\infty(0,1)$. Therefore, such a sequence $\{a_n\}_{n=1}^{\infty}$ exists. Without loss of generality, assume $\lim_{n\to\infty} a_n^{2\alpha}u'(a_n) = -\frac{c_0}{2}$.

The assumptions $\frac{u}{E_\alpha} \in L^\infty(0,1)$ and $1 < p < \frac{1}{2\alpha-1}$ imply that $u \in L^p(0,1)$. For any $\zeta \in C_0^1[-1,1]$, from (6.12) one obtains

$$\int_{a_n}^1 |x|^{2\alpha}u'\zeta' dx + \int_{a_n}^1 u^p\zeta dx = -a_n^{2\alpha}u'(a_n)\zeta(a_n).$$

Passing to the limit as $n\to\infty$, it yields that $x^{2\alpha}u' \in L^1(0,1)$ and

$$\int_0^1 |x|^{2\alpha}u'\zeta' dx + \int_0^1 u^p\zeta dx = \frac{c_0}{2}\zeta(0).$$

A similar computation for $\bar{u}$ yields that $|x|^{2\alpha}\bar{u}' \in L^1(-1,1)$ and

$$\int_{-1}^1 |x|^{2\alpha}\bar{u}'\zeta' dx + \int_{-1}^1 \bar{u}^p\zeta dx = c_0\zeta(0),\ \forall \zeta \in C_0^1[-1,1].$$

Thus $|x|^{2\alpha}\bar{u}' \in BV(-1,1)$. Denote $\lim_{x\to 0^+} |x|^{2\alpha}\bar{u}'(x) = K^+$. We can check that

$$\lim_{x\to 0^+}\left(1+\ln\frac{1}{|x|}\right)^{-1}\bar{u}(x) = K^+,\quad \text{if } \alpha = \frac{1}{2},$$

$$\lim_{x\to 0^+} |x|^{2\alpha-1}\bar{u}(x) = \frac{K^+}{2\alpha-1},\quad \text{if } \frac{1}{2} < \alpha < 1.$$

Since $\bar{u}$ is an even function, we have

$$\lim_{x\to 0^+}\left(1+\ln\frac{1}{|x|}\right)^{-1}\bar{u}(x) = \lim_{x\to 0^-}\left(1+\ln\frac{1}{|x|}\right)^{-1}\bar{u}(x),\quad \text{if } \alpha = \frac{1}{2},$$

$$\lim_{x\to 0^+} |x|^{2\alpha-1}\bar{u}(x) = \lim_{x\to 0^-} |x|^{2\alpha-1}\bar{u}(x),\quad \text{if } \frac{1}{2} < \alpha < 1.$$

Then we can conclude that $\bar{u}$ is the good solution of (6.70). □

Proof of Theorem 6.10 for $0 < \alpha < \frac{1}{2}$. Lemma 6.30 implies that u does not change its sign. Therefore we only need to consider $u \geq 0$ in (6.12).

We first prove the uniqueness. For solutions of type (ii), if there are two solutions u_1 and u_2 solving (6.12) with $\lim_{x\to 0^+} u_i(x) = c$, $i = 1, 2$, then

$$\int_0^1 x^{2\alpha}((u_1-u_2)')^2\phi'(u_1-u_2)dx + \int_0^1 (u_1^p - u_2^p)\phi(u_1-u_2)dx = 0,$$

where $\phi \in C^\infty(\mathbb{R})$ such that $\phi(0) = 0$, $\phi' \geq 0$, $\phi > 0$ on $(0,\infty)$, $\phi < 0$ on $(-\infty, 0)$, and $\phi = \text{sign}$ on $\mathbb{R}\backslash(-1,1)$. It follows that $u_1 = u_2$ on $[0,1]$. For solutions of type (iii), if there are two solutions u_1 and u_2 solving (6.12) with $\lim_{x\to 0^+} x^{\frac{2(1-\alpha)}{p-1}} u_i(x) = l_{p,\alpha}$, $i = 1, 2$, then estimate (6.64) implies

$$|u_1(x) - u_2(x)| \leq 2l_{p,\alpha}x^{\sigma_0}, \ \forall x \in (0,1],$$

for some $\sigma_0 > 0$. Also notice that

$$-(x^{2\alpha}(u_1(x)-u_2(x))')' + c(x)(u_1(x)-u_2(x)) = 0 \quad \text{ on } (0,1),$$

where

$$c(x) = \begin{cases} \frac{u_1^p(x)-u_2^p(x)}{u_1(x)-u_2(x)}, \text{ if } u_1(x) \neq u_2(x), \\ pu_1^{p-1}(x), \text{ if } u_1(x) = u_2(x). \end{cases}$$

It is easy to check that $c \in C(0,1]$ and $c \geq 0$. A maximum principle on $(\epsilon, 1)$ implies

$$\max_{x\in(\epsilon,1)} |u_1(x)-u_2(x)| \leq |u_1(\epsilon)-u_2(\epsilon)| \leq 2l_{p,\alpha}\epsilon^{\sigma_0}.$$

Let $\epsilon \to 0^+$ and then $u_1 = u_2$ on $(0,1)$.

We now claim that, for $u \geq 0$ satisfying (6.12), one of the following assertions holds.

(i) $\lim_{x\to 0^+} x^{\frac{2(1-\alpha)}{p-1}} u(x) = l_{p,\alpha}$.

(ii) $\lim_{x\to 0^+} u(x) = c$, for some $c \geq 0$.

Indeed, denote

$$v(r) = \left(\frac{1-2\alpha}{1-\alpha}\right)^{\frac{p}{p-1}+\frac{3-4\alpha}{(p-1)(1-2\alpha)}} r^{\frac{1-2\alpha}{1-\alpha}} h\left(\frac{1-\alpha}{1-2\alpha} r^{-\frac{1-2\alpha}{1-\alpha}}\right), \tag{6.71}$$

where v is defined in (6.61). Then $h(s) \in C^2\left[\frac{1-\alpha}{1-2\alpha}, \infty\right)$ and h satisfies

$$h''(s) = s^{-p-2-\frac{1}{1-2\alpha}} h^p(s) \quad \text{ on } \left(\frac{1-\alpha}{1-2\alpha}, \infty\right).$$

A result of Fowler (Page 288 in [27]) implies that, as $s \to \infty$, either

$$h(s) = \left[\frac{(p(1-2\alpha)+1)(2-2\alpha)}{(p-1)^2(1-2\alpha)}\right]^{\frac{1}{p-1}} s^{\frac{p(1-2\alpha)+1}{(p-1)(1-2\alpha)}}(1+o(1)),$$

or

$$h(s) = As + B + \frac{A^p(1-2\alpha)^2}{2-2\alpha} s^{-\frac{1}{1-2\alpha}}(1+o(1)),$$

for some constants A and B. Therefore, the relation (6.71) implies our claim.

We then show the existence of the u_c and the $u_{+\infty}$. Consider the Hilbert space X_0^α given in Section 3.6. Note that $X_0^\alpha \subset C[0,1]$ since $0 < \alpha < \frac{1}{2}$. It is straightforward to check that there is a minimizer of the following constraint minimization problem,

$$\min_{u \in X_0^\alpha,\, u(0)=c} \left\{ \frac{1}{2} \int_0^1 x^{2\alpha}(u'(x))^2 dx + \frac{1}{p+1} \int_0^1 |u(x)|^{p+1} dx \right\},$$

and the minimizer is indeed the u_c. Moreover, a comparison principle implies that $u_{c_1} \geq u_{c_2}$ if $c_1 \geq c_2$. On the other hand, Lemma 6.28 implies that $u_c(x) \leq C(\alpha, p) x^{-\frac{2(1-\alpha)}{p-1}}$ for $0 < x \leq \frac{1}{2}$. Since u_c is decreasing, $u_c(x) \leq C(\alpha, p) 2^{\frac{2(1-\alpha)}{p-1}}$ for $\frac{1}{2} < x \leq 1$. Therefore $\lim_{c \to \infty} u_c(x) < \infty$ for all $x \in (0,1]$. We claim that $u_{+\infty}(x) = \lim_{c \to \infty} u_c(x)$. Indeed, since

$$\limsup_{x \to 0^+} u_{+\infty}(x) \geq \lim_{x \to 0^+} u_c(x) = c,$$

we have

$$\limsup_{x \to 0^+} u_{+\infty}(x) = +\infty.$$

Note that $u_{+\infty}$ is still a solution of (6.12). The previous claim implies that $u_{+\infty}$ satisfies (6.15).

Finally, denote $u_0(x) = \lim_{c \to 0^+} u_c(x)$. Then $\lim_{x \to 0^+} u_0(x) = 0$. Therefore $u_0 = 0$. □

Proof of Theorem 6.10 for $\frac{1}{2} \leq \alpha < 1$. The same as the case $0 < \alpha < \frac{1}{2}$, we only need to consider $u \geq 0$ in (6.12).

We first prove the uniqueness. Note that the even extension of u_c is the good solution of (6.70) with $c_0 = 2c$. The uniqueness of the good solution of (6.70) implies the uniqueness of u_c. The proof for the uniqueness of $u_{+\infty}$ is the same as the case $0 < \alpha < \frac{1}{2}$.

We now prove that, for $u \geq 0$ satisfying (6.12), one of the following three assertions holds.

(i) $u \equiv 0$.

(ii) $\lim\limits_{x \to 0^+} \frac{u(x)}{E_\alpha(x)} = c$, for some $c > 0$.

(iii) $\lim\limits_{x \to 0^+} x^{\frac{2(1-\alpha)}{p-1}} u(x) = l_{p,\alpha}$.

We consider $\limsup\limits_{x \to 0^+} \frac{u(x)}{E_\alpha(x)}$. If $\limsup\limits_{x \to 0^+} \frac{u(x)}{E_\alpha(x)} = 0$, Lemma 6.37 implies that $\bar{u}(x) := u(|x|)$ is the good solution of (6.70) with $c_0 = 0$. Therefore the uniqueness of the good solution of (6.70) forces $u \equiv 0$. If $0 < \limsup\limits_{x \to 0^+} \frac{u(x)}{E_\alpha(x)} < \infty$, then $\bar{u}$ satisfies (6.70) with $c_0 > 0$. Therefore by Theorem 6.2, we have $\lim\limits_{x \to 0^+} \frac{u(x)}{E_\alpha(x)} = c_0/2$. If $\limsup\limits_{x \to 0^+} \frac{u(x)}{E_\alpha(x)} = \infty$, Lemma 6.36 implies $\lim\limits_{x \to 0^+} x^{\frac{2(1-\alpha)}{p-1}} u(x) = l_{p,\alpha}$.

The existence of u_c is already given by Theorem 6.2. Note that the limits $\lim\limits_{c \to \infty} u_c(x)$ and $\lim\limits_{c \to 0^+} u_c(x)$ are well-defined for $x \in (0, 1]$. The same as the case $0 < \alpha < \frac{1}{2}$, we can check that $u_{+\infty}(x) = \lim\limits_{c \to \infty} u_c(x)$ and $0 = \lim\limits_{c \to 0^+} u_c(x)$. □

After Theorem 6.10 was done, the author was informed a recent work by Brandolini-Chiacchio-Cîrstea-Trombetti [6]. The authors in [6] studied the positive solutions of the following equation

$$-\operatorname{div}\left(\mathcal{A}(|x|)\nabla u\right) + u^p = 0 \quad \text{ on } B_1^* := B_1 \backslash \{0\},$$

where $B_1 \subset \mathbb{R}^N$ is the unit ball centered at the origin, $N \geq 3$, and $\mathcal{A}$ is a positive $C^1(0, 1]$-function such that

$$\lim_{t \to 0^+} \frac{t\mathcal{A}'(t)}{\mathcal{A}(t)} = \vartheta, \text{ for some } \vartheta \in (2 - N, 2).$$

For the special case when $\mathcal{A}(r) = r^\vartheta$ with $\vartheta \in (2 - N, 2)$, a consequence of the main result in [6] is

Theorem 6.38. *Assume* $1 < p < \frac{N}{N-2+\vartheta}$. *For a positive solution* $u \in C^2(0,1]$ *satisfying*

$$\begin{cases} u''(r) + (N-1+\vartheta)\frac{u'(r)}{r} = \frac{u^p(r)}{r^\vartheta} \quad \text{on } (0,1), \\ u(1) = 0, \end{cases} \tag{6.72}$$

one of the following cases occurs:

(i) $u \equiv 0$,

(ii) $\lim_{r\to 0^+} r^{N-2+\vartheta} u(r) = \lambda$, *for some* $\lambda \in (0,\infty)$,

(iii) $\lim_{r\to 0^+} r^{\frac{2-\vartheta}{p-1}} u(r) = \left[\frac{(N-(N-2+\vartheta)p)(2-\vartheta)}{(p-1)^2}\right]^{\frac{1}{p-1}}$.

Remark 6.10. *let* $\tilde{u}(x) = N^{-\frac{2}{p-1}} u(x^{1/N})$, *where* u *satisfies* (6.72). *Then* $\tilde{u}$ *satisfies*

$$\begin{cases} -(x^{2\alpha}\tilde{u}')' + \tilde{u}^p = 0 \quad \text{on } (0,1), \\ \tilde{u}(1) = 0, \end{cases}$$

where $\alpha = 1 - \frac{\vartheta-2}{N} \in \left(\frac{1}{2}, 1\right)$. *It is now easy to check that Theorem 6.38 coincides with the case* $\frac{1}{2} < \alpha < 1$ *of Theorem 6.10. However, the proofs of these two theorems are different.*

6.9 The equation on the interval $(0,1)$

In this section, we first consider the following equation,

$$\begin{cases} -(x^{2\alpha}u')' + |u|^{p-1}u = \mu \quad \text{on } (0,1), \\ \lim_{x\to 0^+} x^{2\alpha}u'(x) = \beta, \\ u(1) = 0, \end{cases} \tag{6.73}$$

where $\mu \in \mathcal{M}(0,1)$, $\alpha > 0$, $p > 1$ and $\beta \in \mathbb{R}$.

A function u is a *solution* of (6.73) if

$$u \in L^p(0,1) \cap W^{1,1}_{loc}(0,1],\ x^{2\alpha}u' \in BV(0,1), \tag{6.74}$$

and u satisfies (6.73) in the usual sense.

The following result concerns the existence and uniqueness of the solution of (6.73).

Theorem 6.39. *Let $\mu \in \mathcal{M}(0,1)$.*

(i) If α and p satisfy (6.3) *or* (6.4)*, then there exists a unique solution of* (6.73) *for all $\beta \in \mathbb{R}$. Moreover, this unique solution satisfies*

$$\lim_{x\to 0^+} \left(1+\ln\tfrac{1}{x}\right)^{-1} u(x) = -\lim_{x\to 0^+} xu'(x) = -\beta \text{ when } \alpha = \tfrac{1}{2} \text{ and } p > 1,$$
$$\lim_{x\to 0^+} x^{2\alpha-1}u(x) = -\lim_{x\to 0^+} \tfrac{x^{2\alpha}u'(x)}{2\alpha-1} = -\tfrac{\beta}{2\alpha-1} \text{ when } \tfrac{1}{2} < \alpha < 1 \text{ and } 1 < p < \tfrac{1}{2\alpha-1}.$$

(ii) If α and p satisfy (6.5) *or* (6.6)*, then there exists a solution of* (6.73) *if and only if $\beta = 0$. Moreover, if the solution exists, then it is unique and it satisfies*

$$\lim_{x\to 0^+} x^{2\alpha-1}u(x) = \lim_{x\to 0^+} x^{2\alpha}u'(x) = 0.$$

Proof. We first prove the existence in assertion (i). Take $\bar{\mu} \in \mathcal{M}(-1,1)$ as the zero extension of μ, i.e., $\bar{\mu}(A) = \mu(A \cap (0,1))$, where $A \subset (-1,1)$ is a Borel set. Then Theorem 6.24 implies that there exists a solution $\bar{u}$ satisfying

$$\begin{cases} -(|x|^{2\alpha}\bar{u}')' + |\bar{u}|^{p-1}\bar{u} = \bar{\mu} & \text{on } (-1,1), \\ \lim\limits_{x\to 0} |x|^{2\alpha}\bar{u}'(x) = \beta, \\ \bar{u}(-1) = \bar{u}(1) = 0. \end{cases}$$

Therefore, $u = \bar{u}|_{(0,1)}$ is a solution of (6.73).

We then prove the existence in assertion (ii). We still take $\bar{\mu}$ as the zero extension of μ. Notice that $\bar{\mu}(\{0\}) = 0$. Then Theorem 6.3 implies that there exists a solution $\bar{u}$ satisfying

$$\begin{cases} -(|x|^{2\alpha}\bar{u}')' + |\bar{u}|^{p-1}\bar{u} = \bar{\mu} & \text{on } (-1,1), \\ \lim\limits_{x\to 0} |x|^{2\alpha}\bar{u}'(x) = 0, \\ \bar{u}(-1) = \bar{u}(1) = 0. \end{cases}$$

Therefore, $u = \bar{u}|_{(0,1)}$ is a solution of (6.73) with $\beta = 0$. On the other hand, if (6.73) has a solution with $\beta \neq 0$, it implies that $u \sim \frac{1}{x^{2\alpha-1}}$ near $x = 0$. It is a contradiction with the fact that $u \in L^p(0,1)$.

We now prove the uniqueness for both cases. Assume that there are two solutions

u_1 and u_2. Then we have

$$\begin{cases} -(x^{2\alpha}(u_1-u_2)')' + |u_1|^{p-1}u_1 - |u_2|^{p-1}u_2 = 0 \quad \text{on } (0,1), \\ \lim_{x\to 0^+} x^{2\alpha}(u_1-u_2)'(x) = 0, \\ u_1(1) = u_2(1) = 0. \end{cases}$$

Define $\bar{u}_i \in W^{1,1}_{loc}([-1,1]\setminus\{0\})$, $i=1,2$, such that $\bar{u}_i = u_i$ on $(0,1)$ and $\bar{u}_i = 0$ on $(-1,0)$. Then the same argument for the uniqueness of Theorem 6.24 implies that $\bar{u}_1 = \bar{u}_2$. Thus, $u_1 = u_2$. □

Remark 6.11. *When $0<\alpha<\frac{1}{2}$, we can also consider the following equation,*

$$\begin{cases} -(x^{2\alpha}u')' + |u|^{p-1}u = \mu \quad \text{on } (0,1), \\ \lim_{x\to 0^+} u(x) = \beta, \\ u(1) = 0, \end{cases} \tag{6.75}$$

where $\mu\in\mathcal{M}(0,1)$, $p>1$ and $\beta\in\mathbb{R}$. Indeed, the uniqueness of the solution of (6.75) *has been proved in Theorem 6.10. The existence of the solution of* (6.75) *follows from the existence of the minimizer of the following minimization problem,*

$$\min_{u\in X_0^\alpha,\, u(0)=\beta}\left\{\frac{1}{2}\int_0^1 x^{2\alpha}(u'(x))^2dx + \frac{1}{p+1}\int_0^1 |u(x)|^{p+1}dx - \int_0^1 u(x)d\mu(x)\right\},$$

where X_0^α is given in Section 3.6. Moreover, a direct computation shows that this unique solution u satisfies

$$\lim_{x\to 0^+} x^{2\alpha}u'(x) = -\int_0^1 |u(s)|^{p-1}u(s)(1-s^{1-2\alpha})ds + \int_0^1 (1-s^{1-2\alpha})d\mu(s) - (1-2\alpha)\beta.$$

We now discuss the connections between Theorem 6.39 and the well-known existence results about the semilinear elliptic equation. Let $B_1\subset\mathbb{R}^N$ be the unit ball centered at the origin and $\mu\in\mathcal{M}(B_1)$. For $p>1$, consider the following equation,

$$\begin{cases} -\Delta u + |u|^{p-1}u = \mu \quad \text{on } B_1, \\ u = 0 \quad \text{on } \partial B_1. \end{cases} \tag{6.76}$$

Recall that a function u is a *weak solution* of (6.76) if $u\in L^p(B_1)\cap W_0^{1,1}(B_1)$ and

$$\int_{B_1} \nabla u\nabla\zeta dx + \int_{B_1} |u|^{p-1}u\zeta dx = \int_{B_1}\zeta d\mu,\ \forall\zeta\in C_0^\infty(B_1).$$

Although the general existence theory about (6.76) is well-known, the following Corollary provides a more precise information when μ is *rotationally invariant*, i.e., $\mu(A) = \mu(OA)$, where A is any Borel set in B_1 and O is any $N \times N$ orthogonal matrix.

Corollary 6.40. *Assume that $\mu \in \mathcal{M}(B_1)$ is rotationally invariant. Let $\left|\mathbb{S}^{N-1}\right|$ be the surface area of $\mathbb{S}^{N-1}$. Define $\tilde{\mu} \in \mathcal{M}(0,1)$ as*

$$\tilde{\mu}(A) = \mu\left(\left\{r\theta;\ r \in A,\ \theta \in \mathbb{S}^{N-1}\right\}\right),\ \forall A \subset (0,1) \text{ such that } A \text{ is a Borel set.} \tag{6.77}$$

Let $f_*\tilde{\mu}$ be the push-forward measure of $\tilde{\mu}$ under the map $f : [0,1] \to [0,1]$ with $f(r) = r^N$, i.e., $f_*\tilde{\mu}(A) = \tilde{\mu}(f^{-1}(A))$, $\forall A \subset (0,1)$, Borel set.

(i) Assume that $1 < p < \frac{N}{N-2}$ for $N \geq 3$, or $p > 1$ for $N = 2$. Then $u(x) = N^{\frac{2}{p-1}}\tilde{u}\left(|x|^N\right)$ is a weak solution of (6.76), where $\tilde{u}$ satisfies

$$\begin{cases} -(t^{2(1-\frac{1}{N})}\tilde{u}'(t))' + |\tilde{u}(t)|^{p-1}\tilde{u}(t) = N^{-\frac{2p}{p-1}}\left|\mathbb{S}^{N-1}\right|^{-1} f_*\tilde{\mu} \quad \text{on } (0,1), \\ \lim_{t\to 0^+} t^{2(1-\frac{1}{N})}\tilde{u}'(t) = N^{-\frac{2p}{p-1}}\left|\mathbb{S}^{N-1}\right|^{-1}\mu(\{0\}), \\ \tilde{u}(1) = 0. \end{cases} \tag{6.78}$$

(ii) Assume that $p \geq \frac{N}{N-2}$ for $N \geq 3$. Eq. (6.76) has a weak solution if and only if $\mu(\{0\}) = 0$. Moreover, if $\mu(\{0\}) = 0$, then $u(x) = N^{\frac{2}{p-1}}\tilde{u}\left(|x|^N\right)$ is a weak solution of (6.76), where $\tilde{u}$ satisfies

$$\begin{cases} -(t^{2(1-\frac{1}{N})}\tilde{u}'(t))' + |\tilde{u}(t)|^{p-1}\tilde{u}(t) = N^{-\frac{2p}{p-1}}\left|\mathbb{S}^{N-1}\right|^{-1} f_*\tilde{\mu} \quad \text{on } (0,1), \\ \lim_{t\to 0^+} t^{2(1-\frac{1}{N})}\tilde{u}'(t) = \tilde{u}(1) = 0. \end{cases} \tag{6.79}$$

To prove Corollary 6.40, we need the following lemma.

Lemma 6.41. *Assume that $\mu \in \mathcal{M}(B_1)$ is rotationally invariant. Assume that $u \in L^p(B_1) \cap W_0^{1,1}(B_1)$, u is radially symmetric, and*

$$\int_{B_1} \nabla u \nabla \zeta dx + \int_{B_1} |u|^{p-1}u\zeta dx = \int_{B_1} \zeta d\mu,\ \forall \zeta \in C_0^\infty(B_1) \text{ and } \zeta \text{ is radially syemmetric.}$$

Then u is a weak solution of (6.76).

Proof. We use the same idea as the proof of Proposition 5.1 by de Figueiredo-dos Santos-Miyagaki [28]. We first take $w \in L^p(B_1) \cap W_0^{1,1}(B_1)$ as a weak solution of

$$\Delta w = |u|^{p-1}u - \mu \quad \text{ on } B_1.$$

Then w is radially symmetric and

$$\int_{B_1} \nabla w \nabla \zeta dx + \int_{B_1} |u|^{p-1}u\zeta dx = \int_{B_1} \zeta d\mu, \ \forall \zeta \in C_0^\infty(B_1).$$

For any $\zeta \in C_0^\infty(B_1)$ such that ζ is radially symmetric, we have

$$\int_{B_1} w(\Delta\zeta) = \int_{B_1} u(\Delta\zeta).$$

Moreover, for any $\phi \in C_c^\infty(B_1)$ such that ϕ is radially symmetric, there exists $\zeta \in C_0^\infty(B_1)$ such that ζ is radially symmetric and $\Delta\zeta = \phi$ on B_1. It implies that

$$\int_\Omega (w-u)\phi dx = 0, \ \forall \phi \in C_c^\infty(B_1) \text{ and } \phi \text{ is radially syemmetric.}$$

Then

$$\int_0^1 (w(t) - u(t))\varphi(t) t^{N-1} dt = 0, \ \forall \varphi \in C_c^\infty(0,1).$$

Therefore $w = u$ a.e. □

Proof of Corollary 6.40. Note that Theorem 6.39 ensures the existence of $\tilde{u}$ in (6.78) and (6.79).

We first prove assertion (i). For any $\zeta \in C_0^\infty(B_1)$ such that ζ is radially symmetric, we denote $g(|x|^N) = \zeta(x)$. Then $g(t) \in C[0,1]$, $g(1) = 0$ and $g'(t) \in L^1(0,1)$. Therefore,

$$\begin{aligned} &\int_0^1 t^{2(1-\frac{1}{N})}\tilde{u}'(t)g'(t)dt + \int_0^1 |\tilde{u}(t)|^{p-1}\tilde{u}(t)g(t)dt \\ =&N^{-\frac{2p}{p-1}} \left|\mathbb{S}^{N-1}\right|^{-1} \int_0^1 g(t)d(f_*\tilde{\mu})(t) + N^{-\frac{2p}{p-1}} \left|\mathbb{S}^{N-1}\right|^{-1} g(0)\mu(\{0\}). \end{aligned} \tag{6.80}$$

Note that $\int_0^1 g(t)d(f_*\tilde{\mu})(t) = \int_0^1 g(r^N)d\tilde{\mu}(r)$ by Theorem 3.6.1 in Page 190 of [5]. Let $t = r^N$ in (6.80). We have

$$\begin{aligned} \int_0^1 g(r^N)d\tilde{\mu}(r) + g(0)\mu(\{0\}) =&N^{\frac{2p}{p-1}} \left|\mathbb{S}^{N-1}\right| \int_0^1 r^{2N-2}\tilde{u}'(r^N)g'(r^N)Nr^{N-1}dr \\ &+ N^{\frac{2p}{p-1}} \left|\mathbb{S}^{N-1}\right| \int_0^1 |\tilde{u}(r^N)|^{p-1}\tilde{u}(r^N)g(r^N)Nr^{N-1}dr. \end{aligned}$$

Let $u(x) = N^{\frac{2}{p-1}}\tilde{u}(|x|^N)$ with $x \in B_1$. Then $u \in L^p(B_1) \cap W_0^{1,1}(B_1)$. Moreover,

$$\int_{B_1} \nabla u \nabla \zeta dx = N^{\frac{2}{p-1}+2} \left|\mathbb{S}^{N-1}\right| \int_0^1 r^{2N-2}\tilde{u}'(r^N)g'(r^N)Nr^{N-1}dr,$$

$$\int_{B_1} |u|^{p-1}u\zeta dx = N^{\frac{2p}{p-1}} \left|\mathbb{S}^{N-1}\right| \int_0^1 |\tilde{u}(r^N)|^{p-1}\tilde{u}(r^N)g(r^N)Nr^{N-1}dr,$$

$$\int_{B_1} \zeta d\mu = \int g(r^N)d\tilde{\mu}(r) + g(0)\mu(\{0\}).$$

Therefore,

$$\int_{B_1} \nabla u \nabla \zeta dx + \int_{B_1} |u|^{p-1}u\zeta dx = \int_{B_1} \zeta d\mu,\ \forall \zeta \in C_0^\infty(B_1) \text{ and } \zeta \text{ is radially symmetric.}$$

By Lemma 6.41, u is a weak solution of (6.76).

We now prove assertion (ii). If $\mu(\{0\}) = 0$, then the same proof as the above shows that u is a weak solution of (6.76). On the other hand, if μ is rotationally invariant and (6.76) has a weak solution, then

$$\int_{B_1} \nabla u \nabla \zeta dx + \int_{B_1} |u|^{p-1}u\zeta dx = \int_{B_1} \zeta d\mu,\ \forall \zeta \in C_0^\infty(B_1) \text{ and } \zeta \text{ is radially symmetric.}$$

Write $g(r) = \zeta(x)$ where $r = |x|$. Then $g \in W^{1,\infty}(0,1)$ and $g(1) = 0$. Write $u(r) = u(x)$ where $r = |x|$. Then $|u|^p r^{N-1} \in L^1(0,1)$ and, by Theorem 2.3 in [28], $u \in W^{1,1}_{loc}(0,1)$ such that $r^{N-1}u' \in L^1(0,1)$. Therefore

$$\begin{aligned}&\left|\mathbb{S}^{N-1}\right| \int_0^1 r^{N-1}u'(r)g'(r)dr + \left|\mathbb{S}^{N-1}\right| \int_0^1 Nr^{N-1}|u(r)|^{p-1}u(r)g(r)dr\\ =& \int_0^1 g(r)d\tilde{\mu}(r) + g(0)\mu\left(\{0\}\right).\end{aligned}$$

That is

$$\lim_{r \to 0^+} r^{N-1}u'(r) = \left|\mathbb{S}^{N-1}\right|^{-1} \mu\left(\{0\}\right).$$

It forces $\mu\left(\{0\}\right) = 0$. Otherwise, $u \sim r^{-N+2}$ near $r = 0$. Therefore $|u|^p r^{N-1} \sim r^{-(N-2)p+N-1}$ near $r = 0$. Since $p \geq \frac{N}{N-2}$, it implies that $|u|^p r^{N-1} \notin L^1(0,1)$, which is a contradiction. □

The well-known result by Baras-Pierre [2] states that for $\mu \in \mathcal{M}(B_1)$, $p \geq \frac{N}{N-2}$ and $N \geq 3$, equation (6.76) has a weak solution if and only if

$$\mu(E) = 0,\ \forall E \subset B_1 \text{ such that } Cap_{2,p'}(E) = 0, \tag{6.81}$$

where $Cap_{2,p'}$ is the capacity associated with the $W^{2,p'}(\mathbb{R}^N)$-norm and p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Remark 6.12. *In the case when μ is rotationally invariant, the criterion* (6.81) *is equivalent to $\mu(\{0\}) = 0$. Therefore, the necessary and sufficient condition in assertion (ii) of Corollary 6.40 is consistent with* (6.81).

The proof of this remark relies on the following lemma.

Lemma 6.42. *let $\mu \in \mathcal{M}(B_1)$ be rotationally invariant, $\tilde{\mu}$ be defined by* (6.77)*, and $\mathcal{H}^{N-1}$ be the $(n-1)$-dimensional Hausdorff measure on $\mathbb{S}^{N-1}$. Then for any μ-integrable function f, we have*

$$\int_{B_1} f(x)d\mu(x) = \frac{1}{|\mathbb{S}^{N-1}|}\int_{(0,1)}\left(\int_{\mathbb{S}^{N-1}} f(r\theta)d\mathcal{H}^{N-1}(\theta)\right)d\tilde{\mu}(r) + f(0)\mu(\{0\}), \quad (6.82)$$

where $r = |x|$ and $\theta = \frac{x}{|x|}$, $\forall x \in B_1\setminus\{0\}$.

Proof. By a standard linearity and approximation argument, we only need to prove (6.82) for characteristic functions. Moreover, by a standard argument involving the properties of Borel algebra and Radon measure (see, e.g., the proof of Theorem 2.49 in [26]), we only need to show that

$$\mu((0,a]\times U) = \frac{1}{|\mathbb{S}^{N-1}|}\tilde{\mu}((0,a])\times\mathcal{H}^{N-1}(U),\ \forall a\in(0,1),\ \forall U\subset\mathbb{S}^{N-1} \text{ and } U \text{ is open.}$$

Apply once again the standard approximation argument. It is further reduced to show that

$$\int_{(0,a]\times\mathbb{S}^{N-1}}\phi\left(\frac{x}{|x|}\right)d\mu(x) = \frac{\tilde{\mu}((0,a])}{|\mathbb{S}^{N-1}|}\int_{\mathbb{S}^{N-1}}\phi(\theta)d\mathcal{H}^{N-1}(\theta),\ \forall\phi\in C(\mathbb{S}^{N-1}). \quad (6.83)$$

We use some ideas by Christensen [22] to show (6.83). For fixed $x \in \mathbb{S}^{N-1}$ and $\epsilon > 0$, denote

$$C(x;\epsilon) = \left\{y\in\mathbb{S}^{N-1};\ d(x,y)<\epsilon\right\},$$

the so-called spherical cap, where $d(\cdot,\cdot)$ is the standard distance on $\mathbb{S}^{N-1}$. Define

$$C(\epsilon) = \mu((0,a]\times C(x;\epsilon)).$$

Note that $C(\epsilon)$ is well-defined since μ is rotationally invariant and $\mu((0,a]\times C(x;\epsilon))$ is independent of $x\in\mathbb{S}^{N-1}$. Denote $B_a=(0,a]\times\mathbb{S}^{N-1}$. Define

$$K_\epsilon(x,y): B_a\times B_a\to\mathbb{R},$$

as

$$K_\epsilon(x,y)=\begin{cases}\frac{1}{C(\epsilon)}, \text{ if } d\left(\frac{x}{|x|},\frac{y}{|y|}\right)<\epsilon,\\ 0, \text{ otherwise.}\end{cases}$$

For any $x\in B_a$, write $\varphi(x)=\phi\left(\frac{x}{|x|}\right)$. Define

$$K_\epsilon\varphi(x)=\int_{B_a}K_\epsilon(x,y)\varphi(y)d\mu(y),\ \forall x\in B_a.$$

It is clear that $K_\epsilon\varphi(x)\to\varphi(x)$ as $\epsilon\to 0$ for all $x\in B_a$. Therefore, the Dominated Convergence Theorem implies that

$$\lim_{\epsilon\to 0}\int_{B_a}K_\epsilon\varphi(x)d(\mathcal{H}^{N-1}\times\bar{\mu})(x)=\int_{B_a}\varphi(x)d(\mathcal{H}^{N-1}\times\bar{\mu})(x).$$

Note that

$$\begin{aligned}\int_{B_a}K_\epsilon\varphi(x)d(\mathcal{H}^{N-1}\times\bar{\mu})(x)=&\int_{B_a}\varphi(y)\left(\int_{B_a}K_\epsilon(x,y)d(\mathcal{H}^{N-1}\times\bar{\mu})(x)\right)d\mu(y)\\ =&\frac{\bar{\mu}((0,a])\mathcal{H}^{N-1}(C(x;\epsilon))}{C(\epsilon)}\int_{B_a}\varphi(y)d\mu(y).\end{aligned}$$

Therefore, there exists $\lambda\in\mathbb{R}$ such that

$$\lim_{\epsilon\to 0}\frac{\bar{\mu}((0,a])\mathcal{H}^{N-1}(C(x;\epsilon))}{C(\epsilon)}=\lambda.$$

Take $\varphi\equiv 1$. It implies that $\lambda=\left|\mathbb{S}^{N-1}\right|$. Hence, identity (6.83) holds and the proof is complete. □

Proof of Remark 6.12. Assume that μ satisfies (6.81). Since $Cap_{2,p'}(\{0\})=0$, it is clear that $\mu(\{0\})=0$. On the other hand, assume that μ is rotationally invariant and $\mu(\{0\})=0$. For any $E\subset B_1$ such that $Cap_{2,p'}(E)=0$, it holds that $\dim_{\mathcal{H}}(E)\le N-2$, where $\dim_{\mathcal{H}}$ is the Hausdorff dimension. Therefore,

$$\int_{\mathbb{S}^{N-1}}\chi_E(r\theta)d\mathcal{H}^{N-1}(\theta)=0,\ \forall r\in(0,1).$$

Hence Lemma 6.42 implies that

$$\mu(E) = \frac{1}{|\mathbb{S}^{N-1}|} \int_{(0,1)} \left(\int_{\mathbb{S}^{N-1}} \chi_E(r\theta) d\mathcal{H}^{N-1}(\theta) \right) d\tilde{\mu}(r) + \mu(\{0\}) = 0.$$

□

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