UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "The Implications of the work of Popper, Polya, and Lakatos for a Model of Mathematics Instruction," submitted by Alexander James Dawson in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

Thomas E. Kieren
Supervisor

R. S. Mottlock

Harry Garfruble

R. G. Sinclair

M. A. Hay

Kenneth O. May
External Examiner

Date May 28, 1969

ABSTRACT

The purpose of the study was to describe the instructional applications of a philosophically based model of mathematical inquiry to the teaching of mathematics. The first phase of the study developed the model of inquiry of mathematics based on the philosophical position known as Critical Fallibilism. In the second phase of the study, stratagems of teaching were derived from the Fallibilistic model of mathematical inquiry. This phase of the study also included an assessment of the Madison Project as a Fallibilistic approach to the teaching of mathematics.

The philosophical portion of the study included a description of Critical Fallibilism as this position has been developed by Karl Popper. Since Popper's description of Fallibilism is tied to a theory of the growth of scientific knowledge, it was necessary to undertake an application of Fallibilism to mathematics. This was done by analysing the recent results obtained by Imre Lakatos. Moreover, the work of George Polya was considered and some of his ideas incorporated into the model of mathematical inquiry in so far as they applied to the description of the plausible reasoning aspects of mathematical inquiries. On the basis of this description, the growth of mathematics was characterized as being a conjecture and refutation process.

The model of inquiry in mathematics developed produced a description of the processes of the growth of mathematical knowledge characterized by two heuristic patterns which were called the naive and the deductive heuristic. Within each heuristic pattern three phases were

identified: origination, testing, and the proving phases of mathematical inquiry. The distinction between the two heuristic patterns results from the order in which passage is obtained through these phases.

In the next phase of the study, a model of instruction was derived from the paradigm of mathematical inquiry. The model of instruction was seen to be composed of two basic stratagems: the naive stratagem and the deductive stratagem. The naive stratagem was composed of two substrategies which were designated as the TP strategy (denoting that the testing phase preceded the proving phase), and the PT strategy (in which the proving phase precedes the testing phase). The deductive stratagem differed from the naive stratagem in the area of conjecture origination. The various phases of each of these strategies were discussed at length and examples were given which illustrated how a teacher and student could come to function Fallibilistically. These illustrations gave both the short-range view of the use of the stratagems as well as the global view. The latter illustrations attempted to show how a learning situation could be created which depicted the instructional organization of a unit of mathematics.

The model of instruction was then utilized to assess the ways in which the Madison Project can be characterized as Fallibilistic in its approach to the teaching of mathematics. It was concluded that the Madison Project exhibits strong Fallibilistic tendencies.

In general, approaching problems in the teaching of mathematics by means of a philosophically based model of inquiry seems to be a fruitful avenue of research.

ACKNOWLEDGEMENTS

I am grateful to the many persons who assisted me in completing this study.

In particular, I wish to express my sincere appreciation to the following:

Sir K. R. Popper and Dr. I. Lakatos, who generously and graciously supplied copies of their papers and manuscripts without which the study could not have been culminated;

Dr. S. E. Sigurdson, my supervisor for the major portion of the study, whose sensitivity, analytical ability, guidance, and understanding encouraged me to see the study to its conclusion;

Dr. T. E. Kieren, my supervisor in the final stages of the project, whose quick grasp and sharp insight into the nature of the problem increased the quality of the finished product;

Dr. G. Eastwood, whose interest in Fallibilism aided me in the identification and preliminary development of the study;

Dr. H. Garfinkle, whose probing questions and critical attitude in the latter stages of the study aided me in interpreting the role of Fallibilism for the description of a model of mathematical inquiry;

Dr. R. G. Sinclair and Dr. M. Nay, whose attention to the mathematical and educational content of the study made the relationship between the two more evident;

Dr. W. A. Riddell, Principal, University of Saskatchewan, Regina Campus, whose financial support for the production of the study is greatly appreciated;

Dr. J. O. Burgess and the faculty and staff of the Department of Elementary Education, University of Saskatchewan, Regina Campus, whose cooperation and assistance made the completion of the project less difficult;

Miss R. Doege and Mrs. B. Wrigley, whose high quality of work under difficult circumstances made the final product pleasing to the eye;

My wife Sandra-Jean and children Jeffrey and Susan, whose continuous support, understanding, and love motivated me through the years and made the entire project feasible and worthwhile.

A. J. D.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

THE NATURE AND SIGNIFICANCE OF THE PROBLEM

I. INTRODUCTION

A discussion which currently pervades much of the educational literature is that of what constitutes the body of knowledge of a discipline, and how each of the disciplines acquires its body of knowledge. In addition, Bruner,[1] Schwab,[2] and others are leaders of the dialogue as to what are the processes of inquiry, the pathways to knowing, characteristic of each of the disciplines. Schwab, among others, suggests that the curriculum of the schools, in so far as it focuses on the disciplines, should reflect both of the above mentioned aspects of any discipline. This discussion gives rise to many questions some of which are the following: What is the structure of the body of knowledge in each discipline? What are the processes or patterns of inquiry in each discipline? Of central importance to this study is the question: What is the pattern of discovery, the logic of discovery, utilized by mathematicians in creating mathematics?

Although many writers including Russell and Whitehead have identified the logical or deductive feature of mathematical inquiry,

---

[1]Jerome S. Bruner, The Process of Education (Harvard University Press, 1962).

[2]Joseph J. Schwab, "Structuring of the Disciplines: Meaning and Significance," in G. W. Ford and L. Pugno, editors, The Structure of Knowledge and the Curriculum (Rand McNally and Company, Chicago, 1964).

and Polya, most notably, has clarified somewhat the role of plausible reasoning in mathematics, no one has yet explicated and clarified how these two features come together to form a logic of discovery in mathematics. The question of how these two features either interact or function separately in the processes of the creation of mathematics does not seem to have been considered. The need for research in this area is evident. Moreover, such research is concerned with the methodological aspects of a discipline; that is, with the methods utilized by the practitioner in some discipline in endeavoring to expand the body of knowledge peculiar to his discipline.

Scheffler, for example, identifies several questions pertaining to knowledge and its acquisition. Of particular note is the following methodological question: "How ought the search for knowledge to be conducted?" In order to answer this question, Scheffler states that it is necessary ". . . to offer some conception of the proper methods to be employed in inquiry, together with a justification of these methods."[3] Benacerraf and Putnam state the problem in a slightly different way (and relevant specifically to mathematics). They identify two groups of individuals who are interested in the philosophy and methodology of mathematics. The first group ". . . wants to PROMULGATE certain mathematical methods as ACCEPTABLE. . ." whereas the second group ". . . wants to DESCRIBE the ACCEPTED ones."[4]

---

[3]Israel Scheffler, Conditions of Knowledge (Scott, Foresman and Company, Chicago, 1965), p. 5.

[4]Paul Benacerraf and Hilary Putnam, Philosophy of Mathematics (Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1964), p. 3.

The present study falls into the latter category; that is, it is an attempt to describe the nature of mathematical inquiry. The approach pursued in the study is to first adopt a particular philosophical position. Although the adoption of a philosophical position is not a necessity it does provide a basis for a consistent approach to the task of attempting to obtain a model of the nature of mathematical inquiry. At the same time, however, the adoption of a philosophical position prescribes to a certain degree the components of a model of mathematical inquiry. As a consequence, the report constitutes a description of the nature of mathematical inquiry from a particular point of view. Furthermore, the philosophical orientation taken must be concerned with the problem of the growth of knowledge for in attempting to describe the nature of the processes of inquiry in mathematics the focus is on the description of the growth of mathematical knowledge. Once a philosophical position has been taken and described, it becomes possible to develop a model of mathematical inquiry and to draw out the implications such a model has for the design and application of teaching strategies.

It is not only for philosophical reasons that such a model should be developed. Nor is it a purely academic exercise. The development of such a model has definite implications for the mathematics curriculum of today's schools. The major basis of these implications arises from the earlier discussion of a discipline as being viewed as a body of knowledge <u>and</u> a mode of inquiry.[5] With such a view, then, it is only

---

[5]This point of view is expanded and defended in Chapter II.

once that a philosophy of mathematical inquiry is decided upon that this aspect of mathematics may be reflected consistently in the mathematics curriculum of the schools. Moreover, if patterns of procedure, or pathways to knowing, utilized by mathematicians can be described, then teaching strategies exemplifying these patterns can be created with the result of perhaps having children view mathematics as a living and vital discipline.

In slightly different terms, the study is designed to develop a model of mathematical inquiry and draw out the implications for teaching strategies or instruction models. These strategies in turn have implications for the activities taking place in mathematics classrooms. This hierarchy can be thought of in terms of three levels: level (1) being a philosophically based model of mathematical inquiry; level (2) being curriculum and instruction models or strategies of teaching; and level (3) being classroom activities. The literature indicates some research work progressing from level (3) to the second level, such as the Madison Project, but almost none from the second level to the first level. In addition, for the discipline of mathematics there is scant evidence of any research directed from the first to second and subsequently third levels. The research being reported here flows from level (1) to level (2) and finally to an appraisal of current curriculum materials which are concerned with educational activities on the third level. This pattern of research has not been attempted before; at least, not so far as the discipline of mathematics is concerned. As a result, this pattern is one reason for the instigation and culmination of the study.

This discussion gives rise to the problems towards which the study is directed. Moreover, it provides initial insights into the significance of the problem being investigated.

II. THE PROBLEM AND ITS SIGNIFICANCE

The major concern of the study is to describe the practical applications of a model of mathematical inquiry to the derivation and utilization of a model of instruction in school mathematics. The study is divided into two phases. The first phase develops a model of the mode of inquiry of mathematics with Critical Fallibilism as a philosophical foundation. The second phase shows the practical application of such a model by deriving a model of instruction, identifying the strategies of teaching suggested by this model, and by assessing the ways in which the Madison Project is Fallibilistically oriented. One way of viewing these two phases is in terms of models _of_ something and models _for_ something. In the former case, mathematics and Fallibilism are taken as given. The problem is to describe how a mathematician goes about creating mathematics, what strategies he employs, when these processes and strategies are seen from the adopted viewpoint. In the latter case, the model developed is taken as given and provides the basis for the derivation of a model of instruction. This latter model is then used to determine the ways in which the Madison Project exemplifies a Fallibilistic approach to the teaching of mathematics. This distinction between models _of_ and models _for_ is one drawn by Brodbeck. She writes:

> . . . when an area about which we already know a good deal is used to suggest laws for an area about which little is known, then the familiar area providing the form of the laws may be called a model for the new area.[6]

In the present case, the model of instruction serves as a model for the appraisal of the Madison Project as a Fallibilistic approach to the teaching of mathematics, whereas the creation of the model of the mode of inquiry represents an attempt to provide a paradigm of the nature of mathematical inquiry. The objective of these two phases is to show the usefulness and fruitfulness of a philosophical approach to developmental problems in the teaching and learning of mathematics and to the appraisal of curriculum materials currently being produced.

If the suggestion voiced earlier that the curriculum of the schools, in so far as it focuses on the disciplines, should reflect both the substantive structure (body of knowledge) and the patterns of inquiry of the discipline is a valid objective of the schools, then it is imperative that a clear and concise conceptualization of the logic of discovery is available. This is necessary so that the mathematics curriculum can reflect this aspect of the discipline of mathematics. Until a description of the heuristics peculiar to mathematics is available, it is unreasonable to expect teachers to make the basic strategies of mathematics a central focus of their teaching. But until this can be done--

---

[6]May Brodbeck, "Models, Meaning, and Theories," in L. Gross, editor, Symposium on Sociological Theory. (Harper and Row, Publishers, New York, 1959), p. 379. Underscoring is mine.

until a clearly delineated model of the pathways to knowing in mathematics is available--the teaching of mathematics suffers. It is possible to continue without the advantage of this knowledge. Indeed, this is being done today. Polya argues, however, that this lack of the teaching of the mode of inquiry, mathematical 'know-how' in his terminology, is the biggest gap in the present teaching of mathematics, and, indeed, in the education of future teachers of mathematics. Moreover, as Polya points out, this situation is to be expected since in the past the education of teachers of mathematics has not focused at all on the heuristics of mathematical inquiry, on mathematical 'know-how', with the result that any treatment by teachers of the heuristics of mathematics is a random occurrence and not the result of a systematic approach by teachers to this aspect of mathematics.[7]

The body of knowledge, or the substantive structure to use current terminology, and the mode of inquiry are two complementary aspects of any discipline. The substantive structure is generated by means of the mode of inquiry. Moreover, the existence of gaps and contradictions in the substantive structure fosters further research and hence applications of the mode of inquiry. The inclusion of one aspect without the other in the curriculum can only lead to either dull, factual courses (substantive structure alone), or to methodological courses with little or

---

[7]George Polya, Mathematical Discovery (John Wiley and Son, Inc., New York, 1962), Vol. 1, p. viii. Hereafter referred to as MD.

no factual content. Schwab in writing about the teaching of science agrees with this view when he states:

> Unless we intend to impart all knowledge as true dogma we shall need to impart to our students some idea of the degrees and kinds of validation (procedures) which exist.[8]

Hence, the substantive structure and the mode of inquiry should not be separated, nor should either component be completely absent from the curriculum. Moreover, when the mode of inquiry is lacking, there is not the sense of excitement, of discovery, present in the handling of the substantive structure which is desirable for effective and meaningful teaching. As a consequence, there is need for the description of the mode of inquiry of mathematics NOT so the emphasis in the mathematics curriculum can swing like a pendulum to this aspect of mathematics, but so the focus of the mathematics curriculum is on the complementary nature of these two features of mathematics.

The second phase of the problem is designed to provide at the very least prima facie evidence of the fruitfulness of a model of mathematical inquiry. Since the study is not basically a philosophical one, although the model is developed from a particular philosophical position, its ultimate utility is for teachers of mathematics, and the education of potential teachers of mathematics. Hence, if the paradigm is to have relevance for the teaching and learning of mathematics, then it is imperative that some of the educational implications of the model be

---

[8] Joseph J. Schwab, "Problems, Topics, and Issues," in Stanley Elam, editor, Education and the Structure of Knowledge (Rand McNally and Company, Chicago, 1964), p. 11.

considered. As a result, the schema or model is utilized as a means of appraising modern curriculum materials designed for use in mathematics classrooms.

The study can be characterized by a series of three if-then statements. If a particular philosophical position is adopted, then the mode of inquiry of mathematics may be characterized in a certain way. If this particular model of mathematical inquiry is taken as given, then a certain instructional model may be derived. If this model of instruction is taken as a point of reference, then it suggests certain stratagems of teaching which can then be used to assess modern mathematics materials. The study adopts the philosophical position known as Critical Fallibilism. It then develops the model of mathematical inquiry based on this position, derives the model of instruction, identifies the stratagems of teaching, and finally utilizes these stratagems in determining in which ways the Madison Project exhibits Fallibilistic tendencies.

This represents the opposite plan of attack taken by some curriculum projects where teaching strategies are developed from particular learning situations and subsequently a small 'p' philosophical basis is derived for these strategies.[9] The present study is an attempt to proceed in the opposite direction by first adopting a philosophical basis and then deriving the educational consequences for the teaching of mathematics.

Specifically, the study is designed to fill a void now existing

---

[9]The Madison Project is an extremely good example of this way of developing teaching strategies.

in the mathematics curricula of the schools. This void is created by the absence of an adequate description of the mode of inquiry of mathematics, and consequently there is an inadequate treatment of this aspect of mathematics. In summary, the specific goals of the study are to (1) develop a model of a mode of inquiry of mathematics using Critical Fallibilism as a philosophical basis, (2) derive a model of instruction and develop teaching stratagems from this latter model, and (3) assess in which ways the Madison Project exhibits Fallibilistic tendencies.

III. THE MODE OF INQUIRY

The substantive structure of a discipline is developed by the practitioners of that discipline. In doing so, these individuals exemplify a process of invention, of discovery, of creativity. They follow certain patterns or pathways, utilize certain strategies, and exhibit certain behaviors, not always logical, in attempting to expand the state of knowledge in their area. These patterns may or may not be strictly deductive. Moreover, what constitutes the basis of valid conclusions in a discipline is discernable from the activities of the researcher in his field. What is and is not acceptable as evidence in a particular discipline is generally a function of agreement among the investigators in that discipline, although it is unwise to assume that this agreement is always unanimous.

The mode of inquiry of mathematics, then, is defined as the patterns, the methods, or the procedures utilized by the creative mathematician. The phrase of 'mode of inquiry' is used to identify or to describe the pathways to knowing pursued by mathematicians in attempting to create the finished products of mathematics. The term is meant to denote the manner or method of developing mathematical knowledge.

Webster's New Collegiate Dictionary defines 'mode' as the "manner of doing or being, method. . ." and 'inquiry' as "a search for truth, information, or knowledge; research, investigation."[10] Hence, mode of inquiry can be defined as the manner or method of searching for knowledge, or the method or manner of doing research. Consequently, the mode of inquiry of mathematics is defined as the manner of searching for mathematical knowledge, or the method or manner of doing mathematical research.

A model of the mode of inquiry of mathematics is a diagrammatic description, a paradigm, of the methods pursued by mathematicians in searching for mathematical knowledge, in investigating and researching mathematical problems. A model of these activities should describe the major stratagems utilized by mathematicians in proceeding from a problem to its solution. Moreover, in the case of mathematics at least, it should indicate the relationship between the roles of deductive reasoning and non-deductive reasoning in mathematical inquiry. In short, the mode of inquiry attempts to describe the logic of discovery which a mathematician uses in the processes of generating mathematical knowledge.

---

[10] Webster's New Collegiate Dictionary, 2nd Edition, (Thomas Allen, Limited, Toronto, 1953).

IV. THE METHOD

The model of the mode of inquiry of mathematics to be developed is based on a Critical Fallibilistic orientation to the growth of knowledge. Critical Fallibilism is a philosophical position created and advanced during the past four decades by Karl Raimund Popper.[11] It represents Popper's attempt to describe the patterns or logic of discovery as practiced in the expansion of scientific knowledge.

The Fallibilistic position is one which characterizes the growth of knowledge as being a conjecture and refutation process. As such, the view propagated by advocates of this position is that all knowledge is tentative and subject to constant and never-ending criticism. As a result, there seems to be a certain amount of agreement between the Fallibilists and writers in the field of education (Schwab, for example) as to the nature of scientific knowledge.[12] However, the actual educational implications of a Fallibilistic position have not been explored. Hence, the present study uses Critical Fallibilism as a philosophical basis for the model of mathematical inquiry to be developed. The choice of Critical Fallibilism represents a view of the growth of knowledge in a particular discipline that has not been explored and whose fruit-

---

[11]See, for example, Karl R. Popper, Conjectures and Refutations (Basic Books Inc., New York, 1962), and The Logic of Scientific Discovery (Basic Books Inc., New York, 1958). Hereafter these are referred to as C & R and L.Sc.D. respectively.

[12]These results are expanded and supported in Chapter II.

fulness has not been examined. It represents an alternative to, for example, such programmes as the Euclidean, Empiricist, and Inductivist programmes as described by Lakatos and reported in Chapter III of this study.

However, Critical Fallibilism as expounded by Popper concerns itself with scientific knowledge only--the growth of mathematical knowledge is not considered. Consequently, it is necessary to apply this position to mathematical knowledge; that is to say, the question of how mathematical knowledge grows if a Fallibilistic orientation is adopted has to be answered. Imre Lakatos has recently begun to explore this question.[13] The application of Fallibilism to the growth of mathematical knowledge reported in this study is derived mainly from Lakatos' results.

As a result, Chapter III of the present study is a description of Critical Fallibilism and its applications to mathematics. The writings of Popper and Lakatos are analyzed and reported on with the objective of obtaining some clues, so to speak, as to the nature of mathematical inquiry as seen from the Fallibilist's point of view. The rationale for selecting Critical Fallibilism rests on its unexamined potential as a possibly fruitful means of characterizing the growth of mathematical knowledge. That it is an alternate way of viewing the growth of knowledge is discussed in Chapter III when the various programmes for des-

---

[13]Imre Lakatos, "Proofs and Refutations," The British Journal for the Philosophy of Science, May, 1963, hereafter referred to as P & R.

cribing the growth of knowledge are considered.

Once the Fallibilist position is clarified and initial steps are taken in applying this position to the growth of mathematical knowledge, it is possible to expand, describe, and explain the Fallibilistic view of the nature of mathematical inquiry. Moreover, it is here that the role of plausible reasoning stratagems are considered. Consequently, the recent work of George Polya on the revival of the heuristics of mathematics is considered next with the objective of ascertaining the relationship of the plausible reasoning aspects of mathematical inquiry to the processes of discovery as a whole in mathematics. Furthermore, it then is necessary to define the role of deductive reasoning in mathematical investigations and to determine the relationship of the plausible and deductive aspects of inquiry. Here the work of Lakatos is considered again in order to explain his results concerning the role of deductive reasoning in mathematical research. Finally, once the description of mathematical inquiry is completed, it is possible to develop the model of the mode of inquiry of mathematics.

Therefore, Chapter IV includes the analysis of Polya's writing and the implications to mathematics of Fallibilism as developed by Lakatos. As a final section of Chapter IV, the model of the mode of inquiry is given, described, and explained.

The next phase of the study is to derive from the model, strategies of teaching exemplifying the Fallibilistic view of the process of inquiry in mathematics. These derivations are given with the objective of describing in a concrete fashion the applicability of Critical Fallibilism to the teaching and learning of mathematics. The strategies derived are then used to appraise the instructional models developed by the Madison Project. The objectives and teaching strategies or models created by the Madison Project are presented and described with a comparison being provided between these goals and strategies and those derived from the Fallibilist position. The reason for the selection of these materials stems from the fact that the Madison Project represents a new and challenging approach to the teaching of mathematics--a modern view of the nature of the learning and teaching process in mathematics.

Hence, the procedure of the study is to (1) describe and analyse the Fallibilist position; (2) examine and clarify the application of Fallibilism to the growth of mathematical knowledge; (3) develop the model of the mode of inquiry of mathematics; (4) derive teaching strategies from this model; and (5) to describe and appraise the philosophy and instructional models of the Madison Project.

V. DELIMITATIONS OF THE STUDY

The study deals with only one of the many problems which surround the structure of knowledge. No attempt is made to make a case, so to speak, for mathematics as being a distinct discipline. The assumption is made that mathematics is a distinct discipline. This does not preclude, however, the possibility of teaching mathematics in conjunction with some other discipline, although if a clearly distinct mode of inquiry of mathematics can be identified then this possibility might not be desirable. No attempt is made to identify the relationship between mathematics and the other disciplines. The study is not concerned explicitly with the substantive domain of mathematics, although this area is dealt with to a certain limited extent. Hence, the study focuses on one aspect of one discipline--the mode of inquiry of mathematics. Moreover, the study deals with only the features of fluid inquiry of mathematics and not with stable inquiry. This distinction between fluid and stable inquiry is dealt with in Chapter II.

Furthermore, from all the philosophical positions available, only Critical Fallibilism is used as a basis for the model to be developed. This in effect produces a particular view of the mode of inquiry of mathematics. If some other philosophical orientation were adopted, such as Positivism or Rationalism, it could be expected that a different model might be derived in which different points of emphasis would appear. The choice of Fallibilism, however, constitutes a potentially fruitful orientation to the investigation of the nature of mathematical inquiry. It is true, nevertheless, that this study is only one possible way to depict and describe the nature of mathematical inquiry. But, at the same time, it is one view which has not been considered. Traditionally, mathematical inquiry has been viewed as Euclidean in nature; that is, mathematical inquiry as being purely deductive and flowing logically from a set of axioms with truth inundating the system from the top. Critical Fallibilism represents an alternative to this view.

Part of the reason for the selection of Fallibilism stems from this situation of its unexplored potential as well as the possibility of

Fallibilism being an alternative to some of the historical ways of depicting the growth of knowledge; that is, such programmes as the Euclidean, Empiricist, and Inductivist programmes which are designed to describe the growth and nature of scientific knowledge. Hence, Fallibilism as a basis for a model of mathematical inquiry is proposed as an alternative: an alternative which has not been investigated and whose potential has not been assessed.

The derivation of teaching strategies is only one possible application of the model. Obviously, more tests could be carried out in order to assess the fruitfulness of the model. The decision was made, however, to limit the range of applications in this study to the development of strategies of instruction. Without further applications at a future time, the fruitfulness or usefulness of the model remains in doubt, and, hence, such tests should be applied at a later date.

Finally, the choice of the Madison Project as the one modern project to be appraised as a possible application of Fallibilism is to be noted. The decision to focus on these materials was made on the basis of the Madison Project's stated objectives and the creative nature of their instructional models.

VI. OUTLINE OF THE REPORT

The present chapter is an introduction to and a description of the nature of the study. Chapter II is composed of a review of the literature that deals with the definition and description of the mode of inquiry of any discipline, and specifically with the nature of the processes of mathematical inquiry. Chapter III is a description of Critical Fallibilism and its application to the field of mathematics. As such Chapter III provides the philosophical basis for Chapter IV which includes the analysis of the writings of Polya and Lakatos as well as the development and description of the model of the mode of inquiry of mathematics. Chapter V contains the derived strategies of teaching and the appraisal of the instructional models developed by the Madison Project. Finally, Chapter VI is devoted to a summary of the study along with the conclusions, limitations of the study, implications for the design and execution of mathematics teaching strategies, and implications for further research.

CHAPTER II

REVIEW OF THE LITERATURE

I. INTRODUCTION

The present chapter reviews the literature relevant to the problem under consideration. In order to facilitate such a review the chapter is divided into two major sections. The first section is devoted to reviewing the literature concerned with the logic of discovery or mode of inquiry in any subject matter area. In this section the conclusions and points of view advanced by Schwab, Downey, Bruner and others are considered as they apply to the generation of a paradigm of the processes of inquiry. This section is designed not only to expand the rationale behind the study given in Chapter I, but also to provide some general indications of the nature and features of a mode of inquiry in any particular discipline. Furthermore, since the model to be developed is based on a Fallibilistic view of the growth of knowledge, some consideration is given to relating this position to Schwab's view of the growth of knowledge.

In the second major section of the present chapter, the relevant literature dealing specifically with the mode of inquiry of mathematics is considered. Here the writings of Polya, Easley, and Lakatos are reviewed. The objective of this section is to provide clues as to some of the features which characterize a mode of inquiry in mathematics. Also, a brief description of Lakatos' work is given in order to explicate the relationship of his work to the study as a whole.

These two major sections together are designed to provide a résumé of the history and present status of the problem being reported in this study. Moreover, these two sections represent an expanded version of the rationale behind the study. As such, they set the stage for the remainder of the investigation in that it is here that the theoretical background is given which explicates the relevance and importance of the problem towards which the study is directed.

II. THE MODE OF INQUIRY--A GENERAL VIEW

The mode of inquiry of a discipline has most recently been identified by Schwab. He cites three problems which are deserving of immediate study by educators. The three problems are those ". . . of identifying the significantly different disciplines, and of locating their relations to one another; . . . of identifying the substantive structure of each discipline; . . . and of identifying the syntactical structure of the discipline."[1] The present study is directed at an examination of the last of these problems relative to the discipline of mathematics.

Relative to this problem, Schwab goes on to say that:

> There is, then, the problem of determining for each discipline what it does by way of discovery and proof, what criteria it uses for

---

[1]Joseph J. Schwab, "Structuring of the Disciplines: Meaning and Significance" in G. W. Ford and L. Pugno, editors, The Structure of Knowledge and the Curriculum. (Rand McNally and Company, Chicago, 1964) pp. 11-14.

the relationship of his work to the study as a whole.

These two major sections together are designed to provide a résumé of the history and present status of the problem being reported in this study. Moreover, these two sections represent an expanded version of the rationale behind the study. As such, they set the stage for the remainder of the investigation in that it is here that the theoretical background is given which explicates the relevance and importance of the problem towards which the study is directed.

II. THE MODE OF INQUIRY--A GENERAL VIEW

The mode of inquiry of a discipline has most recently been identified by Schwab. He cites three problems which are deserving of immediate study by educators. The three problems are those ". . . of identifying the significantly different disciplines, and of locating their relations to one another; . . . of identifying the substantive structure of each discipline; . . . and of identifying the syntactical structure of the discipline."[1] The present study is directed at an examination of the last of these problems relative to the discipline of mathematics.

Relative to this problem, Schwab goes on to say that:

> There is, then, the problem of determining for each discipline what it does by way of discovery and proof, what criteria it uses for

---

[1]Joseph J. Schwab, "Structuring of the Disciplines: Meaning and Significance" in G. W. Ford and L. Pugno, editors, The Structure of Knowledge and the Curriculum. (Rand McNally and Company, Chicago, 1964), pp. 11-14.

> measuring the quality of its data, how strictly it can apply canons of evidence, and in general, of determining the route or pathway by which the discipline moves from its raw data through a longer or shorter process of interpretation to its conclusions.[2]

Furthermore, Schwab sees the mode of inquiry of a discipline as being composed of two complementary and interacting modes of inquiry, two modes which he calls STABLE inquiry and FLUID inquiry. "It is the function of stable inquiry", according to Schwab, "to accumulate what a doctrinal education teaches us to conceive as the whole of scientific knowledge."[3] And further, "It is the business of the stable enquirer, in short, to construct an edifice, not to question its plan. Each stable enquiry is conceived to fill a particular blank space in a growing body of knowledge."[4]

On the other hand, fluid inquiry is characterized by trial and error--conjectures and refutations. In contrasting stable and fluid inquiry, Schwab contends that:

> . . . fluid enquiry operates through miscarriage and tends towards frustration . . . . In short, fluid enquiry is engaged in invention, hence in the test of the previously untested. Hence failures are among its normal expectations.[5]

The alternation between these two forms of inquiry is characteristic of

---

[2] Ibid., p. 14.

[3] Joseph J. Schwab, "The Teaching of Science as Enquiry," in J. J. Schwab and P. Brandwein, The Teaching of Science (Harvard University Press, 1964), p. 15.

[4] Ibid., pp. 15-16.

[5] Ibid., p. 17.

scientific progress. However, Schwab argues that in recent decades the main emphasis in the acquisition of knowledge has shifted from a concern with stable inquiry to a concern with fluid inquiry. The reason for this shift, Schwab argues, is the change in the social role of science from that of an almost leisure time activity to an activity which is coming to dominate society. As a result, fluid inquiry now takes a central place in the pursuit of knowledge.[6]

Moreover, because of the nature of fluid inquiry--its conjecture and refutation orientation--the status of scientific knowledge has altered from knowledge thought to be absolutely valid to a situation in which knowledge is characterized as being relative. Schwab supports this point of view when he states:

> It is this base in a conceptual principle of enquiry which renders scientific knowledge fragile, dubitable, subject to change. For research does not proceed indefinitely on the basis of the principles which guided its first enquiries. On the contrary, the same enquiries which accumulate knowledge by the aid of certain principles of enquiry also test these principles. As selected principles are used, the consequences ensue. Knowledge of the subject unfolds. Experimental techniques are refined and invented. The new knowledge lets us envisage new, more adequate, more telling conceptions of the subject matter. The growth of techniques permits us to put the new conceptions into practice as guiding principles of a renewed enquiry.
>
> The effect of these perennial renewals of enquiry I have called the revisionary character of scientific knowledge. With each change in conceptual system, the older knowledge gained through use of the older principles sinks into limbo. The facts embodied are salvaged, reordered, and reused, but the knowledge which formerly embodied these facts is replaced. There is, then, a continuing revision of scientific knowledge as principles of enquiry are used, tested thereby, and supplanted.[7]

---

[6] Ibid., pp. 18-20.

[7] Ibid., pp. 14-15.

Mario Bunge in a preface to a volume dedicated to Karl Popper echoes Schwab's view:

> The progress of knowledge--the perfecting of truths--is not a linear accumulation of definitive acquisitions but a zigzagging process in which counter-examples and unfavorable evidence ruin generalizations and prompt the invention of more comprehensive and sometimes deeper generalizations, to be criticized in their turn. A CRITICAL APPROACH to problems, procedures, and results in every field of inquiry is, therefore, a necessary condition for the continuance of progress.[8]

It can be seen from these two statements that there is a close affinity between Schwab's view of the growth of scientific knowledge and the Fallibilist's view of this process as embodied in Bunge's statement. They both express the tentative nature of scientific knowledge and the importance of tests in the creation of knowledge. They both point out the constant improvement and deepening of generalizations or conceptual schema, but at the same time noting the necessity of the testing of these new results. Although it would be unfair to classify Schwab as a Fallibilist on the basis of this evidence, it can be contended that he at least 'tends' towards this view.

Schwab uses the terminology of short-term syntax and long-term syntax as being equivalent to stable inquiry and fluid inquiry respectively. Long-term syntax or fluid inquiry arises when the basic assumptions used in the processes of short-term syntax are brought into question. The prime mover in long-term syntax is the desire for ever increasing richness and pervasiveness of the subject-matter content of

---

[8]Mario Bunge, (Ed.) The Critical Approach to Science and Philosophy (The Free Press of Glencoe, Collier-MacMillan Limited, London, 1964). p. viii.

a discipline. Schwab lists four aims of fluid inquiry:

> The aims of fluid enquiry are four in number. They are, first, to detect among stable enquiries the incoherencies of data, the failures of subject matter to respond to the questions put under the aegis of the extant structures, the conflict of conclusions, which indicate the inadequacies of the substantive structure used. The second problem of fluid enquiry is to obtain clues from current stable enquiries as to the specific weakness or inadequacy which characterizes the principle in question. The third problem of fluid enquiry is, of course, to devise a modification of the existing structure or a wholly new structure to replace it. The fourth problem of fluid enquiry is to test proposed new structures by submitting them to the community of the discipline for debate, defense, and attack.[9]

These four aims characterize the growth of knowledge as a process of analysis of existent structures, identification of their weaknesses, the proposal of possibly better structures, and finally the testing of these new proposals. Moreover, this process is a continual one, for ultimate schema though perhaps possible are not probable. Schwab contends that the public, and here educators whether teachers, professors of teacher-education institutions, or educational research workers must be included, should become cognizant of the growth of knowledge as being a product of FLUID inquiry.

> What is required is that in the very near future a substantial segment of our publics become cognizant of science as a product of <u>fluid enquiry</u>, understand that it is a mode of investigation which rests on conceptual innovation, proceeds through uncertainty and failure, and eventuates in knowledge which is contingent, dubitable, and hard to come by.[10]

---

[9]Joseph J. Schwab, "Problems, Topics, and Issues", in Stanley Elam, <u>Education and the Structure of Knowledge</u> (Rand McNally & Company, Chicago, 1964), p. 33.

[10]Schwab, <u>op</u>. <u>cit</u>., "Science As Enquiry", p. 5.

Moreover, if the mathematics curriculum of today's schools is to reflect this long-term syntax of the discipline of mathematics, then a description of the processes of fluid inquiry in mathematics must be obtained. However, such descriptions are not presently available. Schwab contends;

> It is equally clear, I hope that adequate syntactical descriptions for purposes of instruction are almost wholly lacking in the available school literature. They are most wanting and most needed in history and in science.[11]

It may be added that the need is no less urgent in mathematics for the teaching of mathematics has been characterized for too long as a consideration of the finished products of a mathematician's work and no consideration has been given to the processes of creation in mathematics.

The implications for the teaching and learning of a discipline are clear. It is necessary to have adequate descriptions of both the substantive structure and the syntactical structure of any discipline in order to be able to convey the complete nature of the discipline to the student. Once these are available, it becomes possible to:

> . . . convey to students: (a) the revisionary character of bodies of scientific knowledge; (b) the extent to which knowledge is yet knowledge, though provisional; (c) some idea of the enhancement of knowledge which accrues from the reflexive testing and replacement of principles.[12]

Downey approaches the problem of a logic of discovery from a slightly different view. He focuses attention on the relationahip between the mode of inquiry of a discipline and the discipline as a

---

[11]Schwab, op. cit., "Problems, Topics, and Issues", p. 31.

[12]Ibid., p. 34.

whole. He contends that ". . . each discipline is distinctive not only in the particular domain of knowledge with which it is concerned, but also in the unique 'way of life' it imposes on the scholar in the field. . . ."[13] Indeed, Downey claims that the logic of discovery of any particular discipline is perhaps more important than the body of knowledge of the subject area. In considering priorities, Downey argues that:

> Study of any particular area should result, first, in a facility for the mode of inquiry appropriate for dealing with the phenomena of the field, and, second, in the acquisition of a structured fund of information about the field.[14]

Hence, it seems that every discipline consists of two aspects--a body of knowledge and a mode of inquiry. Moreover, Schwab is seen as arguing that the mode of inquiry can be subdivided into two types of inquiry--stable and fluid. Furthermore, not only do these types of inquiry complement each other, but the body of knowledge and the mode of inquiry of any discipline are also complementary in that the body of knowledge is produced by the processes of inquiry and the processes of inquiry are fostered by gaps, contradictions, and errors in the existent body of knowledge.

The present study is concerned with the identification of the characteristics of fluid inquiry in mathematics. The explication of such characteristics would provide a basis for the development of

---

[13]Lawrence W. Downey, The Secondary Phase of Education (Blaisdell Press, Toronto, 1965), p. 51.

[14]Ibid., p. 81.

teaching strategies designed to acquaint students with the nature of fluid inquiry in mathematics. Bruner supports such an endeavor when he states his belief that it would:

> be wise to assess what attitudes or heuristic devices are most pervasive and useful, and that an effort should be made to teach children a rudimentary version of them that might be further refined as they progress through school.[15]

Several inferences may be drawn from this discussion. First, each discipline consists of two complementary aspects--a mode of inquiry and a body of knowledge. Second, the mode of inquiry of any discipline can be divided into two types of processes, namely, a stable inquiry and a fluid inquiry. Third, if the curriculum developed for a particular discipline is to accurately reflect the essence of that discipline, then the teacher must include as a major objective the development of the student's understanding of the logic of discovery of that discipline. Fourth, in order for teachers to be able to do this, the mode of inquiry must be clearly delineated and understood by the teacher. And finally, as Schwab has pointed out, such a description of the mode of inquiry is virtually non-existent. The need for research in this area is evident and urgent. The present study is an attempt to begin to meet this need.

---

[15]Jerome S. Bruner, <u>The Process of Education</u> (Harvard University Press, 1961), p. 7.

III. MATHEMATICS AND A MODE OF INQUIRY

Although the literature contains scant evidence of the description of the logic of discovery, the mode of inquiry, of mathematics and virtually no evidence of attempts to develop paradigms of such a process, there has recently been some attempts to at least identify in general terms some of the features which a model of fluid inquiry in mathematics might encompass. These efforts mark a beginning in the search for an adequate description of the processes of mathematical inquiry. These first attempts are reviewed below.

At a conference held in Athens, Georgia, during September of 1967 one of the topics discussed at considerable length is that of needed research in the teaching of mathematics.[16] As a subtopic of this discussion, the question of whether mathematics should be viewed as a process or as a product was considered. What is of importance relevant to this study is not that the question was considered at all, but rather that it implies a view of mathematics as something more than a finished product and admits of mathematics as possibly being a process. This alternate way of viewing mathematics is one which has been advocated by Polya, Luchins and Luchins, and Easley among others.

However, if there is one thing that can be said of mathematics it is that very few people agree as to what mathematics is--agree as to the

---

[16]For a report of this conference see the Journal of Research and Development in Education, Volume 1, Number 1, (College of Education, University of Georgia, Athens, Georgia, Fall, 1967).

nature of mathematics.[17] In recent years mathematics has been thought of by some writers to be synonymous with axiomatics. However, as d'Abro points out, one of the major contributions of axiomatics is that:

> It has shown that, in a study of the nature of mathematics we cannot be satisfied with axiomatics alone but must go behind the postulates and discover their origin. In this way we are led to trace the historical development of a doctrine. . . .[18]

Most attempts to describe the heuristics of mathematical inquiry have focused on the deductive or logical aspects of mathematics.

Luchins and Luchins, for example, contend that mathematics has for too long been characterized as a purely logical deductive activity and that the emphasis in the teaching of mathematics has focused in a disproportionate way on the end-product of mathematics, that is, on the written expression of mathematics. They write:

> Moreover, as a living activity mathematics is neither exhausted by nor typified by logic. The contention that mathematics is logic tends to focus on the written expression of mathematics rather than on the activity whose end product is the written expression. It tends to stress, therefore, the postulational-deductive nature of mathematics and the formal verification or proof of theorems. Aspects which tend to be overlooked are the spirit of inquiry and the modes of discovery that lead to theorems.[19]

And further:

> The thesis that mathematical reasoning is adequately accounted for by logic seems to rest on the assumption that mathematical reasoning is largely confined to the formal verification of the logical validity or logical correctness of mathematical proofs. This assumption seems

---

[17] A. d'Abro, "The Controversies on the Nature of Mathematics", in R. W. Marks, The Growth of Mathematics, (Bantam Matrix Edition, 1964), pp. 45 - 74.

[18] Ibid., p. 67.

[19] A. S. Luchins and E. H. Luchins, Logical Foundations of Mathematics for Behavioral Scientists (Holt, Rinehart, & Winston, Inc., New York, 1965), p. 129.

> also to underlie the thesis that mathematics reduces to logic or that mathematics is logic. Yet some of the writers who consider that mathematics reduces to logic also admit that logic plays a relatively small part in the mathematical activities that lead to the discovery of theorems and of their proofs.
>
> . . . .
>
> It would seem that mathematics was not and is not discovered through logical reasoning *per se*, although the fruits of mathematical inquiry may be put into logical form and its correctness checked through criteria established with logic. But the activities of mathematicians and of mathematics involve inquiries which lead to discovery or invention, in addition to verification of logical correctness.[20]

From their standpoint, then, Luchins and Luchins contend that there is more to mathematics than logic; that is, mathematics cannot be characterized as consisting of deductive reasoning alone. Other types of reasoning or patterns of procedure must be utilized by mathematicians in the act of discovering or inventing mathematics.

Polya is one writer who attempts to show what this other aspect of mathematical inquiry may be. He attempts to show the relevance of plausible reasoning to mathematics. Although he makes a convincing case for the presence of plausible reasoning in the teaching of mathematics, he does not clearly delineate the relationship between deductive and plausible reasoning.

However, Polya makes an important point when he distinguishes between two features of mathematics. Moreover, this distinction closely parallels the one drawn previously by Luchins and Luchins:

> Mathematics as a FINISHED SCIENCE appears quite otherwise than mathematics IN THE MAKING. In finished mathematics only axioms,

---

[20] *Ibid.*, p. 260.

> definitions, and rigorous demonstrative arguments should find a place. In mathematics research, however, in the unfinished mathematics growing in the head and under the pencil of a mathematician, great or small, it is a different matter: there we find blind groping, guesses, occasional false steps, and many merely plausible arguments. Here is the point which is of critical importance for our thesis: MERELY PLAUSIBLE ARGUMENTS PLAY A DECISIVE ROLE IN DEVELOPING MATHEMATICS.[21]

Moreover, Polya views plausible reasoning as a part of a heuristic approach to the teaching of mathematics. Indeed, Polya's book How To Solve It is "an attempt to revive heuristic in a modern and modest form."[22] Polya describes heuristic reasoning as follows:

> Heuristic reasoning is reasoning not regarded as final and strict but as provisional and plausible only, whose purpose is to discover the solution of the present problem. We are often obliged to use heuristic reasoning. We shall attain complete certainty when we shall have obtained the complete solution, but before obtaining certainty we must often be satisfied with a more or less plausible guess. We may need the provisional before we attain the final. We need heuristic reasoning when we construct a strict proof as we need scaffolding when we erect a building.[23]

Hence, Polya's suggestion of the plausible reasoning aspect of mathematical inquiry seems to be a first step in providing a description of mathematics which does not focus completely on the logico-deductive nature of mathematical investigations. Consequently, this provides a first clue as to at least two features of mathematical inquiry, that is, the plausible and deductive aspects of mathematical studies.

---

[21] George Polya, "Mathematics as a subject for learning plausible reasoning", The Mathematics Teacher, Vol. 52, No. 1, January, 1959, pp. 7-8.

[22] George Polya, How To Solve It, (Doubleday Anchor Books, 1945), p. 113, hereafter referred to as HTSI.

[23] Ibid.

As a result, any attempt to describe the mode of inquiry of mathematics must take into account the role of heuristics and plausible reasoning in the creation of mathematical knowledge. However, this must not be done to the exclusion of the deductive aspects of mathematics. Rather such a description should explain the relationships between the plausible and deductive aspects of mathematical inquiry.

Imre Lakatos is making some initial steps in endeavoring to describe this relationship. He views mathematics as not being characterized as either a dispassionate and bleak application of deductive logic or an irrational approach of blind guessing. Rather, he contends that for live and vital mathematics:

> . . . an investigation of INFORMAL mathematics will yield a rich situational logic for working mathematicians, a situational logic which is neither mechanical nor irrational. . . .[24]

Such a situational logic as Lakatos develops presents insights into the relationship between the deductive and plausible reasoning aspects of mathematical inquiry. His results are reported in Chapters III and IV.

Furthermore, the ultimate objective of the development of a model of the processes of mathematical inquiry is to generate testable teaching strategies. As a consequence, any model developed must not become so aesthetic as to defy its fruitfulness or usefulness as a basis for creating testable hypotheses: hypotheses which can be experimentally evaluated. Hence, even though the present study adopts the

---

[24]Imre Lakatos, P & R, p. 5.

philosophical position of Fallibilism and studies the implications of this view as a means of developing a model of mathematical inquiry, it is still possible to relate this epistemological framework to actual classroom situations. Easley supports this view when he states:

> While epistemological studies do not ordinarily determine how knowledge should be taught, they provide conceptual frameworks which have some prima facie value in formulating educational problems for empirical investigation.[25]

Moreover, Easley concludes that an approach to the teaching of mathematics which focuses on the heuristics of mathematical inquiry may reap untold benefits in terms of student interest and performance:

> . . . we may also learn that the really new venture for mathematics curriculum reform lies in the area of heuristic procedures. If mathematics educators learn to apply the insights into mathematical inquiry which Polya and Lakatos have set forth, at the level in the teaching of mathematics on which the growing edge of the student's understanding happens to lie, the interest and achievement of students may be expected to increase markedly.[26]

Hence, the shift to the emphasis of mathematical heuristics as distinct from the end-products of mathematics seems quite strong. However, this shift has not been accompanied by corresponding developments in the theory of mathematical heuristic. The present study attempts to provide at least one such theoretical framework, though admittedly it takes one philosophical position only.

Several inferences may be drawn from this discussion. First,

---

[25]Jack A. Easley, Jr., "Logic and Heuristic in Mathematics Curriculum Reform", in Imre Lakatos, editor, Problems in the Philosophy of Mathematics (North-Holland Publishing Company, Amsterdam, 1967), p. 216.

[26]Ibid., p. 228.

the mode of inquiry of mathematics has not been clearly delineated. Second, in attempting to describe the processes of mathematical inquiry, attention must be given to the interrelationship between deductive and plausible types of reasoning. Moreover, the situation is not an either-or one, but rather one in which these two forms of reasoning complement and interact with each other. For as Polya states:

> A serious student of mathematics, intending to make it his life's work, must learn demonstrative reasoning; it is his profession and the distinctive mark of his science. Yet for real success he must also learn plausible reasoning; this is the kind of reasoning on which his creative work will depend.[27]

It may be added that the above statement can be applied not just to the potential professional mathematician, but also to all students studying mathematics. If it is not applied to all students--if these students only see the deductive end-products of mathematics--then the curriculum is impoverished as is the student's conception of mathematics. Indeed, it may be speculated that some students' distaste of mathematics can be traced to the conceptualization and teaching of mathematics as a 'dead' deductive discipline.

A third inference which may be drawn from this review is that, to use Schwab's terminology, the focus of inquiry in mathematics has shifted from that of a stable inquiry to a fluid inquiry. Indeed, the past few years has seen the study of the foundations and methods of mathematics revived. The discussions between the three schools

---

[27]George Polya, Mathematics and Plausible Reasoning (Princeton University Press, Princeton, New Jersey, 1954), Vol. 1, p. vi., hereafter referred to as MPR.

of mathematical thought--logicism, formalism, and intuitionism--are again active today.[28] This renewed activity comes after a period of "pessimistic stagnation".[29] Moreover, Lakatos has recently documented the view of a renaissance of empiricism in mathematical foundations.[30] All of these studies are directed at the basic processes of mathematical inquiry; that is, they are themselves inquiries into the processes of mathematical inquiry. Kalmar has posed the question:

> Why do we not confess that mathematics, like other sciences, is ultimately based upon, and has to be tested in practice? Many respectable sciences have excellent reputations, without claiming that they are 'pure deductive sciences'.[31]

Hence, the foundations of mathematics is again becoming a topic of lively discussion. Moreover, the focus of this discussion seems to be on the processes of inquiry of mathematics rather than on the foundations of mathematics, a distinction which Popper and Lakatos make and which is discussed in Chapter III. Whether such an approach is fruitful can only be determined by applying it and testing it to determine its consequences.

---

[28]See, for example, the books Problems in the Philosophy of Mathematics by Lakatos and Philosophy of Mathematics by Benacerraf and Putnam both of which have been quoted earlier.

[29]Laszlo Kalmar, "Foundations of Mathematics--Whither Now?" in Imre Lakatos, editor, Problems in the Philosophy of Mathematics (North Holland Publishing Company, Amsterdam, 1967), p. 193.

[30]Imre Lakatos, "A renaissance of empiricism in the recent philosophy of mathematics" in Imre Lakatos, editor, Problems in the Philosophy of Mathematics (North Holland Publishing Company, Amsterdam, 1967).

[31]Kalmar, op. cit., p. 193.

IV. CHAPTER SUMMARY

The review of the literature presented in this chapter has served to marshall theoretical support for the need of delineating the logic of discovery of a discipline; in particular, the discipline of mathematics.

Schwab's description of the two types of inquiry have been identified and described. The distinction between stable inquiry and fluid has been noted. As well, the recent shift of emphasis from stable inquiry to fluid inquiry served to indicate the need for a fresh look at the processes of inquiry in any discipline. Furthermore, this fresh look must be taken from a pedagogical standpoint; that is, from the standpoint of the implications a description of the logic of discovery may have for obtaining testable hypotheses about strategies of teaching.

With respect to the discipline of mathematics the recent works of Polya, Luchins and Luchins, Easley and Lakatos have been mentioned with the view of ascertaining some of the features a paradigm of mathematical inquiry must consider. It was noted that the deductive aspects and plausible reasoning aspects of mathematical inquiry should not be viewed as separate and distinct entities, but rather as complementary and interacting aspects of a live situational logic of mathematics.

CHAPTER III

THE CRITICAL PHILOSOPHY

I. INTRODUCTION

Critical philosophy is the particular philosophical position advanced by K. R. Popper. It was developed by him during the period from 1919 to 1935 and culminated in the publication of his book The Logic of Scientific Discovery.[1] Popper has continued to expand and defend his position since that time.[2] Popper's main concern is the problem of how knowledge grows for he sees this as the central concern of epistemology. It is also Popper's contention that this problem can be most fruitfully studied by recourse to the study of the growth of scientific knowledge. Popper's results centre about five main areas: the problem of induction, conjectures and refutations, the falsifiability thesis, degrees of testability, and degrees of corroboration. The first section of this chapter describes his conclusions in each of these five areas and resultant implications for a theory of the growth of knowledge.

Imre Lakatos has recently studied the implications of the Critical Philosophy as embodied in Popper's writings for the growth

---

[1]Karl R. Popper, The Logic of Scientific Discovery (Basic Books Inc., New York, 1958). This was originally published in German in 1934 as Logik der Forschung.

[2]See, for example, Karl R. Popper, Conjectures and Refutations (Basic Books Inc., New York, 1962).

of mathematical knowledge. Lakatos has attempted to determine what, if any, touchpoints exist between the Criticalist position and the ways in which mathematical knowledge seems to grow. Lakatos is also concerned with the heuristic aspects of the teaching of mathematics. Indeed, his series of four articles entitled "Proofs and Refutations" are dedicated to Karl Popper and George Polya, an indication of Lakatos' interest with, first, the Critical philosophy of Popper and, second, Polya's revival of mathematical heuristic. In the introduction to his series of articles Lakatos states that "the purpose of these essays is to approach some problems of the METHODOLOGY OF MATHEMATICS."[3] Lakatos deals with Popper's central themes as they apply to mathematical knowledge and its acquisition. Lakatos' results are described and discussed in the second section of this chapter.

The position advanced by the Critical Fallibilist described in this chapter is to be the foundation of a model of mathematical inquiry. The description of Fallibilism provided simply presents the Fallibilist view of the growth of knowledge as honestly and completely as possible: no attempt is made to criticize this view, although this could and has been done. The Fallibilist view is taken as given and utilized as a philosophical basis for the development of just one model of mathematical inquiry.

There should be no implication derived that the researcher feels

---

[3] Imre Lakatos, "Proofs and Refutations," The British Journal for the Philosophy of Science, May, 1963.

that this is THE only philosophical position which could be adopted. Moreover, the views advanced concerning the growth of knowledge are those of Popper, although obviously the investigator takes full responsibility for the interpretation and description of them.

Consequently, in this chapter the researcher describes and analyses, first, the Critical Fallibilistic position with respect to the growth of knowledge as espoused by Popper, and, second, Lakatos' analysis of the Critical viewpoint as it applies to the generation of mathematical knowledge.

II. THE GROWTH OF KNOWLEDGE

The Problem of Induction

Popper's view of induction, his attitude towards induction and conclusions concerning induction, may be summed up in three words: Popper rejects induction. Before we consider what Popper's view is it would perhaps be helpful to understand what Popper contends the problem of induction is, and what its proponents contend induction is and what can be done with induction.

There are at least two points at which induction may be utilized in the creation of knowledge. The first of these is in the origination of hypotheses; the second is in the confirmation of hypotheses which have been originated. In this first area, these conjectures are unproven, unverified, or uncorroborated statements put forth in an attempt to provide an explanation of some phenomenon. For example, that this is THE only philosophical position which could be adopted. Moreover, the views advanced concerning the growth of knowledge are those of Popper, although obviously the investigator takes full responsibility for the interpretation and description of them.

Consequently, in this chapter the researcher describes and analyses, first, the Critical Fallibilistic position with respect to the growth of knowledge as espoused by Popper, and, second, Lakatos' analysis of the Critical viewpoint as it applies to the generation of mathematical knowledge.

II. THE GROWTH OF KNOWLEDGE

The Problem of Induction

Popper's view of induction, his attitude towards induction and conclusions concerning induction, may be summed up in three words: Popper rejects induction. Before we consider what Popper's view is it would perhaps be helpful to understand what Popper contends the problem of induction is, and what its proponents contend induction is and what can be done with induction.

There are at least two points at which induction may be utilized in the creation of knowledge. The first of these is in the origination of hypotheses; the second is in the confirmation of hypotheses which have been originated. In this first area, these conjectures are unproven, unverified, or uncorroborated statements put forth in an attempt to provide an explanation of some phenomenon. For example, Goldbach's famous conjecture has sometimes been argued as being a classic example of inductive inference. In this realm, inductive inference works from singular statements to a universal statement. The argument that Goldbach's conjecture is an example of inductive inference may be reconstructed somewhat as follows: Given the singular instances of addends and sums below,

n	sum
6	3 + 3
8	3 + 5
10	3 + 7, 5 + 5
12	5 + 7
14	7 + 7, 3 + 11

it would seem that we could make the inductive generalization that any even number greater than four could be expressed as the sum of two odd prime numbers; this was Goldbach's conjecture. The proponents of inductive inference would argue that this is an outstanding example of inductive generalization. Popper, on the other hand, rejects induction and he would argue that this is not an example of an inductive generalization, but rather an example of a guess, a conjecture, a leap from the unknown.

Popper dismisses induction because it cannot be logically justified. His rejection of induction stems from Hume's refutation of induction, which Popper found to be "clear and conclusive",[4] and which

---

[4]Popper, C & R, p. 42.

arises from the fact that any:

> attempt to justify the practice of induction by an appeal to experience must lead to INFINITE REGRESS. As a result, we can say that theories can never be inferred from observation statements, or rationally justified by them.[5]

By infinite regress is meant the withdrawal or going back without limits or without end--the backward progression an infinite number of steps--an indeterminate number of steps. Furthermore, to accept induction, Popper claims, is to say that:

> we obtain our knowledge by repetition and induction, and therefore by a logically invalid and rationally unjustifiable procedure, so that all apparent knowledge is merely a kind of belief--belief based on habit. This would imply that even scientific knowledge is irrational, so that rationalism is absurd and must be given up.[6]

Few people would disagree with Popper and with his conclusions concerning the unjustifiable aspect of induction. But even Hume who destroyed induction as a logically sound form of reasoning still clung to induction as a PSYCHOLOGICAL PRINCIPLE. As a psychological principle, induction is described as "OUR HABIT OF BELIEVING IN LAWS (which) IS THE PRODUCT OF FREQUENT REPETITION--of the repeated observations that things of a certain kind are constantly conjoined with things of another kind."[7] But Popper rejects induction as a psychological principle as well. He bases his argument on logical grounds although he also presents empirical evidence in support of his rejection. To paraphrase Popper's argument, the type of repetition postulated by Hume

---

[5]Ibid.

[6]Ibid., p. 45.

[7]Ibid., p. 43.

can never be perfectly the same: similar yes, but not perfectly the same. It follows that there can be repetitions only from a particular viewpoint. But this implies logically that there must always be a point of view, a set of expectations, BEFORE there can be repetition. But this leads to an infinite regress since we may then ask how we obtained this set of expectations and so on back infinitely.[8]

Thus Popper dismisses induction both as a logical procedure and as a psychological principle, and he rejects them both for the same reason; that is to say, because induction leads to infinite regress.

The second point at which induction may be utilized in the creation of knowledge is in the area of the confirmation of a conjecture. The problem of induction in this area, according to Popper, is that of how one can:

> establish the truth of universal statements which are based on experience, such as the hypotheses and theoretical systems of the empirical sciences.[9]

This process of justifying a universal statement by offering singular statements in support of it amounts to induction. Popper argues that:

> . . . people who say of a universal statement that we know its truth from experience usually mean that the truth of this universal statement can somehow be reduced to the truth of singular ones, and that these singular ones are known by experience to be true; which amounts to saying that the universal statement is based on inductive inference.[10]

---

[8] Ibid., pp. 44-45.

[9] Popper, L.Sc.D., p. 27.

[10] Ibid., p. 28.

Popper dismisses this characterization of the processes of the growth of knowledge, because he rejects induction. He rejects induction in this second case for the same reasons as before; induction cannot be logically justified. Moreover, in Popper's view, induction may lead to logical inconsistencies and falsehoods; inductive inference cannot establish the 'truth' of any conjecture.

Popper summarizes his views with respect to induction as follows:

> (1) Induction, i.e. inference based on many observations, is a myth. It is neither a psychological fact, nor a fact of ordinary life, nor one of scientific procedure.
>
> (2) The actual procedure of science is to operate with conjectures: to jump to conclusions--often after one single observation. . . .[11]

Hence, Popper's rejection of induction stems from purely logical considerations, and it was his conclusions regarding induction which led him to formulate his theory of conjectures and refutations:

> Thus I was led by purely logical considerations to replace the psychological theory of induction by the following view. Without waiting, passively, for repetitions to impress or impose regularities upon us, we actively try to impose regularities upon the world. We try to discover similarities in it, and to interpret it in terms of laws invented by us. Without waiting for premises we jump to conclusions. These may have to be discarded later, should observation show that they are wrong.
>
> This was a theory of trial and error--of CONJECTURES AND REFUTATIONS.[12]

Thus, Popper's rejection of induction is intimately related to his theory of conjectures and refutations as well as to his falsifiability thesis. These two areas will be considered next.

---

[11] Popper, C & R, p. 53.

[12] Ibid., p. 46.

Conjectures and Refutations

"The central problem of epistemology," according to Popper, "has been and still is the problem of the growth of knowledge."[13] It is also Popper's contention that this problem can be most fruitfully studied by a study of the growth of scientific knowledge. He argues that in dealing with the traditional problems of epistemology, problems related to the growth of knowledge, the two standard methods of linguistic analysis are not adequate to do the job since these problems transcend these methods. A paraphrase of his arguments in support of these contentions is given below.

The first of the linguistic methods is that of the analysis of ordinary or common-sense knowledge. Those philosophers who adopt this approach contend that scientific knowledge is simply an extension, or an out-growth, of common-sense knowledge. But they conclude as a result that common-sense knowledge must be the easier of the two to analyse.[14] Popper disagrees with this conclusion, for he contends that by studying common-sense knowledge only, these philosophers eliminate a host of problems from consideration because their methods are not powerful enough to deal with them.[15] As an example, he cites the problem of the growth of knowledge; it is impossible for philosophers studying common-sense knowledge to deal with this problem since the most important way in which common-sense knowledge grows is

---

[13]Popper, L.Sc.D., p. 15. [14]Ibid., p. 18.

[15]Ibid., pp. 18-19.

by turning into scientific knowledge, an area of study not considered by these philosophers. Hence, this method of linguistic analysis is inadequate to study the problem of the growth of knowledge since its focus remains on common-sense knowledge.

The second linguistic approach is via the construction of artificial model languages; "that is to say, the construction of what they believe to be models of 'the language of science'."[16] The difficulty with this approach, Popper contends, is that the results obtained, these constructed model languages, are conceptually poorer than even ordinary languages. Popper concludes:

> It is a result of their poverty that they yield only the most crude and the most misleading model of the growth of knowledge--the model of an accumulating heap of observation statements.[17]

Hence, according to Popper, analysis of scientific knowledge remains the one way of attacking the problem of how does knowledge grow.

The Criticalist method of analyzing scientific knowledge is that of rational discussion. By rational discussion Popper means the method of stating clearly one's problem and then the critical examination of all proposed solutions to this problem. Popper equates the critical attitude and the rational attitude. He explains his position in this way:

> The point is that whenever we try to propose a solution to a problem, we ought to try as hard as we can to overthrow our solution, rather than defend it. . . . Yet criticism will be fruitful only

---

[16] Ibid., p. 20.

[17] Ibid., p. 22.

> if we state our problem as clearly as we can and put our solution in a sufficiently definite form--a form in which it can be critically discussed.[18]

But how is growth in scientific knowledge determined? By growth or progress, do we mean the accumulation of facts, whatever they are, and of observations? Popper answers these questions in the negative when he states:

> You will have noticed from this formulation that it is not the accumulation of observations which I have in mind when I speak of the growth of scientific knowledge, but the repeated overthrow of scientific theories and their replacement by better or more satisfactory ones.[19]

Moreover, this growth is absolutely necessary if science is to maintain its rational and empirical character. It is Popper's conclusion that the rationality and empirical quality of science is determined by the way it grows:

> The way, that is, in which scientists discriminate between available theories and choose the better one or (in the absence of a satisfactory theory) the way they give reasons for rejecting all the available theories, thereby suggesting some of the conditions with which a satisfactory theory should comply.[20]

The problem still remains of how does one originate a solution to some problem? Moreover, what does it mean to examine proposed solutions critically? Popper answers these questions by proposing that the growth of knowledge is predicated on a system of conjectures and refutations. One makes a conjecture, a conjecture which is put forth as a

---

[18] Ibid., p. 16.

[19] Popper, C & R, p. 215.

[20] Ibid.

solution to some problem. This proposed solution is then studied rationally and critically; that is, it is subjected to severe tests designed to refute the proposed conjecture. If the conjecture survives its tests, if it is not refuted, then we tentatively accept the conjecture as <u>one</u> possible solution to the problem.

Popper relates that his theory of conjecture and refutation arose out of a dissatisfaction felt by him towards Marx's theory of history, Freud's psycho-analysis, and Adler's so-called individual psychology.[21] Popper's dissatisfaction with these three theories stemmed from the fact that no matter what evidence or observations were produced, the advocates of these theories could <u>always</u> interpret such evidence as confirming their theories. Popper summarizes his conclusions regarding these theories as follows:

> It was precisely this fact--that they always fitted, that they were always confirmed--which in the eyes of their admirers constituted the strongest argument in favour of these theories. It began to dawn on me that this apparent strength was in fact their weakness.[22]

This was not the case with another theory Popper was interested in at that time--Einstein's theory of relativity.[23] Here, Popper felt, was a theory which was dramatically different from the three mentioned previously. What struck Popper was the fact that Einstein's theory could be falsified; that is to say, Einstein's theory forbid certain

---

[21] <u>Ibid</u>., pp. 33-35.

[22] <u>Ibid</u>., p. 35.

[23] Circa, 1919-1920.

things to happen. If it turned out that they did occur, then Einstein's theory would have been refuted. It was this very situation--that Einstein's theory could be refuted whereas the others could not--which led Popper to formulate his criterion of the scientific status of a theory. The criterion which he proposed was that of testability, refutability, or falsifiability. Popper concluded that for a theory to be called scientific it must be possible to test the theory in such a way that it could be refuted or falsified.

As a consequence, Popper proposed that scientific knowledge grows by a system of conjecture and refutation--of trial and error. We propose a solution to a problem, but rather than attempting to find confirming instances to this conjecture, which would be an activity of self-fulfilling prophecy, we attempt to design severe, risky tests which could refute the theory. Popper cites the example of Eddington's experiment as being a severe test of Einstein's theory.[24]

It should be pointed out however that the method of trial and error is not identical with the method of conjectures and refutations. For as Popper states:

> The method of trial and error is not, of course, simply identical with the scientific and critical approach--with the method of conjecture and refutation. . . . The difference lies not so much in the trials as in a critical and constructive attitude towards errors, errors which the scientist consciously and cautiously tries to uncover in order to refute his theories with searching arguments, including appeals to the most severe experimental tests his theories and his ingenuity permit him to design.[25]

---

[24]Popper, C & R, p. 36.

[25]Ibid., p. 52.

As a result of the above considerations Popper reformulates the conclusions he reached as follows:

> (1) It is easy to obtain confirmations, or verifications, for nearly every theory--if we look for confirmations.
>
> (2) Confirmations should count only if they are the result of RISKY PREDICTIONS; that is to say, if, unenlightened by the theory in question, we should have expected an event which was incompatible with the theory--an event which would have refuted the theory.
>
> (3) Every 'good' scientific theory is a prohibition: it forbids certain things to happen. The more a theory forbids, the better it is.
>
> (4) A theory which is not refutable by any conceivable event is non-scientific. Irrefutability is not a virtue of a theory (as people often think) but a vice.
>
> (5) Every genuine TEST of a theory is an attempt to falsify it, or to refute it. Testability is falsifiability; but there are degrees of testability: some theories are more testable, more exposed to refutation, than others; they take, as it were, greater risks.
>
> (6) Confirming evidence should not count EXCEPT WHEN IT IS THE RESULT OF A GENUINE TEST OF A THEORY; and this means that it can be presented as a serious but unsuccessful attempt to falsify the theory.[26]

Moreover, it is possible within the field of science, according to Popper, to have a criterion of progress. He states that:

> . . . even before a theory has ever undergone an empirical test we may be able to say whether, provided it passes certain specified tests, it would be an improvement on other theories with which we are acquainted. . . . To put it a little differently, I assert that we know what a good scientific theory should be like, and--even before it has been tested--what kind of theory would be better still, provided it passes certain crucial tests. And it is this (meta-scientific) knowledge which makes it possible to speak of progress in science, and of a rational choice between them.[27]

---

[26] Ibid., p. 36.

[27] Ibid., p. 217.

This thesis implies that even before a new theory is tested empirically it is possible to determine its 'potential'. That a theory's potential is a relative matter is obvious from the fact that the preferred theory is the one which can be most severely tested. The theory to be preferred is the one which is not trivial, but one which is daring, interesting and informative. Popper summarizes these results as follows:

> This criterion of relative potential satisfactoriness. . . characterizes as preferable the theory which tells us more; that is to say, the theory which contains the greater amount of empirical information or CONTENT; which is logically stronger; which has the greater explanatory and predictive power; and which can therefore be MORE SEVERELY TESTED by comparing predicted facts with observations.[28]

But what does Popper mean by 'empirical information' or 'content'? The answer to this question is given and illustrated below.

Consider two statements 'a' and 'b' and their conjunction 'ab'. Suppose that 'a' is the statement: "It will snow on Tuesday"; that 'b' is the statement: "The snow will melt on Wednesday"; and that 'ab' is the statement: "It will snow on Tuesday and the snow will melt on Wednesday". It is clear that the informative content or logical content of 'ab' is greater than or at least equal to the logical content of either 'a' or 'b'. It is also obvious that the probability of 'ab' will be less than or equal to the probability of either of its components. Symbolically, these results may be given as follows:

---

[28]<u>Ibid</u>.

(1) $I_c(a) \leq I_c(ab)$ and $I_c(b) \leq I_c(ab)$ where $I_c(i)$ means the information content of the statement 'i';

(2) $p(a) \geq p(ab)$ and $p(b) \geq p(ab)$ where $p(i)$ means the probability of the statement 'i' being true.

This means that as the content of a theory increases the probability of a theory being true decreases and vice versa. From this one may conclude, as did Popper, that:

> if the growth of knowledge means that we operate with theories of increasing content, it must also mean that we operate with theories of decreasing probability. . . . Thus if our aim is the advancement or growth of knowledge, then a high probability . . . cannot possibly be our aim as well: THESE TWO AIMS ARE INCOMPATIBLE.
>
> . . . .
>
> Thus if we aim, in science, at a high informative content. . . then we have to admit that we also aim at a low probability.
>
> And since a low probability means a high probability of being falsified, it follows that a high degree of falsifiability, or refutability, or testability is one of the aims of science--in fact, precisely the same aim as a high informative content.[29]

Popper concludes, therefore, that the criterion of 'progress' or 'potential satisfactoriness' is testability, or improbability, or falsifiability, or refutability.

With respect to the basic epistemological question of how does knowledge grow, Popper concludes that there are three requirements which must be met in order for knowledge to grow.

---

[29] Ibid., pp. 218-219.

The first requirement is that any proposed theory "should proceed from some simple, new, and powerful, unifying idea about some connection or relation. . . between hitherto unconnected things. . . or facts. . . or new 'theoretical entities'."[30] The second requirement is that of testability. The new theory must be independently testable by virtue of the fact that it (the theory) proposes new and testable hypotheses. If this requirement is met by a new theory, then this new theory is potentially a progressive step forward since it proposes new testable consequences which any previous theory explaining the same phenomenon did not.

These first two requirements are, however, only logical requirements. An old and a new theory could be compared logically in order to determine if the new theory met the first two requirements to a greater degree than the old theory. But there is a third requirement which Popper proposes and this is a material requirement. His third requirement is that "the new theory should pass some new, and severe, tests."[31]

This third requirement is different from the first two requirements. Popper contends that the first two requirements:

> are indispensable for deciding whether the theory in question should be accepted as a serious candidate for examination by empirical tests; or in other words, whether it is an interesting and promising theory.[32]

---

[30]<u>Ibid</u>., p. 241.

[31]<u>Ibid</u>., p. 242.

[32]<u>Ibid</u>.

Popper further contends that his third requirement is indispensable in a different sense, because:

> further progress in science would become impossible if we did not reasonably often manage to meet the third requirement; thus if the progress of science is to continue, and its rationality not to decline, we need not only successful refutations, but also positive successes.[33]

Hence, Popper's Critical philosophy is not a negative approach to the problem of the growth of knowledge. Although he contends that scientific knowledge progresses by means of conjectures and attempted refutations, and that we should endeavor to refute our conjectures, he realizes that science could not grow without having corroborated conjectures.

Briefly and in summary, then, Popper takes the view that scientific knowledge grows by means of conjectures and refutations; that for scientific knowledge to grow a new theory must proceed from some 'new, and powerful, unifying idea'; that the new theory must be testable, and the new theory must have positive successes. Moreover, to be classified as scientific, the new theory must be testable, or falsifiable, and its potential satisfactoriness depends on its empirical content.

Falsifiability Thesis

Popper's falsifiability thesis has been alluded to earlier when the question of the scientific status of a theory was considered. The

---

[33]Ibid., p. 243.

criterion Popper proposes for determining whether a theory should be classified as scientific or not is that of testability, refutability, or falsifiability.[34] Popper's criterion of falsifiability is an attempt to provide a line of separation between scientific and non-scientific statements or systems of statements. Popper calls this the problem of demarcation between scientific and non-scientific theories. It is his contention that falsifiability or refutability provides an adequate line of demarcation. He states that:

> According to this view, which I still uphold, a system is to be considered as scientific only if it makes assertions which may clash with observations; and a system is, in fact, tested by attempts to produce such clashes, that is to say by attempts to refute it. Thus testability is the same as refutability, and can therefore likewise be taken as a criterion of demarcation.[35]

By proposing this criterion of demarcation Popper is attempting to distinguish between theories which may be called scientific and theories which must be called metaphysical. He does not, as does Carnap for example, attempt to distinguish between meaningful and meaningless theories, for Popper contends that metaphysical theories can be meaningful, and, indeed, could at some time in the future become scientific.[36]

Moreover, this "criterion of demarcation cannot be an absolutely

---

[34]Popper uses these three terms interchangeably as does the present writer.

[35]Popper, C & R, p. 256.

[36]Ibid., p. 257. See also L.Sc.D., section 85.

sharp one but will itself have degrees."[37] Some theories will be very well testable, some theories hardly testable, and yet other theories will not be testable at all. It is this last class of theories, the non-testable theories, which the Criticalist regards as non-scientific and of no interest to the empirical scientist.[38]

Popper's conceptualization of the criterion of demarcation could be represented diagrammatically as follows:

Scientific - testable	Class of all statements of a language in which we intend to formulate a science.[39]
Metaphysical--non-testable	

FIGURE I

CRITERION OF DEMARCATION

The fact that the criterion of demarcation is not absolutely sharp leads one to the concept of the degrees of testability of a theory.

It is advisable to first draw a distinction between falsifiability and falsification. Falsifiability is a criterion; a criterion which separates theories into two classes--the scientific and the non-scientific. Falsification however is concerned with the conditions

---

[37] Ibid.

[38] Ibid.

[39] Ibid.

under which some potentially falsifiable theory, a scientific theory, may be regarded as having been falsified. Popper offers the following answer to the question of when is a theory to be considered falsified:

> We say that a theory is falsified only if we have accepted basic statements which contradict it. . . . This condition is necessary, but not sufficient; for we have seen that non-reproducible single occurrences are of no significance to science. Thus a few stray basic statements contradicting a theory will hardly induce us to reject it as falsified. We shall take it as falsified only if we discover a REPRODUCIBLE EFFECT which refutes the theory. In other words, we only accept the falsification if a low-level empirical hypothesis which describes such an effect is proposed and corroborated.[40]

Moreover, a theory is regarded as falsifiable if it splits the class of all possible basic statements into two subclasses. The first subclass consists of all those statements which are inconsistent with the theory. Popper calls this the class of potential falsifiers of the theory.[41] The other subclass consists of those basic statements which the theory does not contradict. Hence, a theory is falsifiable or scientific if its class of potential falsifiers is not empty.

Again, a diagram may serve to illustrate more clearly what Popper has in mind. Suppose that the circle of Figure 2 represents the class of all possible basic statements. The area of the circle could be thought of as representing the totality of all possible empirical worlds. Then an empirical event is represented by one of the radii of the circle (or a sector or wedge of the circle).[42]

---

[40]Popper, L.Sc.D., p. 86.

[41]Ibid.

[42]Ibid., p. 90.

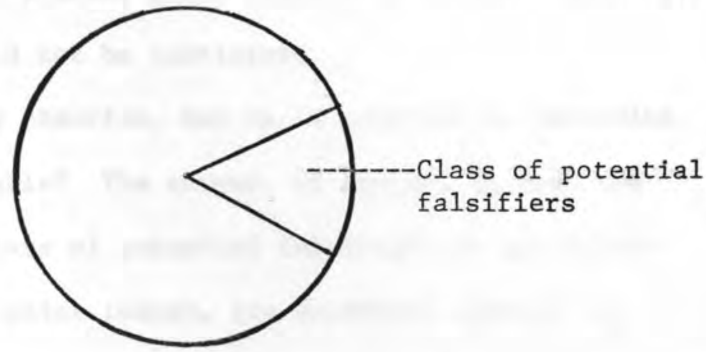

FIGURE 2

CLASS OF POTENTIAL FALSIFIERS

The falsifiability thesis states that for a theory to be regarded as scientific there must be at least one radius or very narrow sector which the theory forbids--the area of potential falsifiers must not be empty. As a result, every scientific theory has associated with it a class of potential falsifiers, a class of statements concerning events, which the theory forbids. It follows that the larger this class of potential falsifiers is the riskier the theory becomes; the probability of the theory decreases. But it was shown earlier (see page 48ff.) that this is the same as increasing the empirical content of the theory. Consequently, from the Criticalist point of view the aim of science is to create theories with the greatest possible empirical content and this amounts to the same thing as increasing the size of the class of potential falsifiers until it is as large as possible. In terms of Figure 2, this means that the goal of science is to make the sector representing the class of potential falsifiers as large as possible and reducing the class of permitted basic statements to a minimum. The former sector could never comprise the whole circle for if it did the theory would not be consistent.

Given two competing theories, how is it possible to determine which is the better testable? The answer, of course, is that the theory with the larger class of potential falsifiers is the better testable for it is the riskier theory, its empirical content is greater, it restricts more. But how is it possible to compare classes of potential falsifers when these classes are infinitely large? Popper proposes several means of doing this: the most sensitive of these methods is described below.

Degrees of Testability

Since the class of potential falsifiers is infinitely large and denumerable, the concept of the cardinality of a class of falsifiers is insufficient to solve the problem of the comparison of two classes of falsifiers because they will have the same cardinality.

The subclass relation is proposed by Popper as being the most sensitive way of comparing classes of potential falsifiers of a theory. Consider two statements 'x' and 'y'. The statement 'x' is said to be better testable or falsifiable in a higher degree than the statement 'y' (symbolically $Fsb(x) > Fsb(y)$) if and only if the class of potential falsifiers of 'y' is a proper subclass of the class of potential falsifiers of 'x'.

If, however, the classes of potential falsifiers of 'x' and 'y' are equivalent, then we may write $Fsb(x) = Fsb(y)$ and say that 'x' and 'y' have the same degree of falsifiability. Moreover, if neither of the two classes of potential falsifiers of 'x' and 'y' includes the other as a proper subclass, then the two statements are said to be non-comparable and we can write $Fsb(x)//Fsb(y)$.

Furthermore, if 't' denotes a tautological statement, it follows that $Fsb(t) = 0$ since a tautological statement is not falsifiable. On the other hand, a self-contradictory statement 'c' is everywhere falsifiable, and we may write $Fsb(c) = 1$. As a consequence, we obtain the result that for an empirical statement 'e', which is potentially falsifiable, that $Fsb(c) > Fsb(e) > Fsb(t)$, or $1 > Fsb(e) > 0$. Hence the degree of falsifiability of an empirical statement 'e' falls within the open interval bounded by one and zero. It should be noted that a metaphysical statement 'm', which by the criterion of falsifiability is not testable, has the same degree of falsifiability as that of a tautological statement, namely $Fsb(m) = 0$.[43]

Although Popper suggests other ways of describing the varying degrees of testability of a theory, he concluded that the subclass relation is the most sensitive way so far discovered.[44] The important point however is that falsifiability is a relative concept; that is to say, there is not an absolutely falsifiable theory, only theories which are more or less falsifiable.

---

[43]Ibid., pp. 114-118.

[44]Ibid., p. 130.

As a result, theories are also only more or less corroborated. Degrees of testability and degrees of corroboration are closely related concepts for "a theory can be better corroborated the better testable it is."[45]

Degrees of Corroboration

"Theories are not verifiable," according to Popper, "but they can be 'corroborated'.[46] Theories are not verifiable because the principle of induction breaks down; it leads to infinite regress or to 'a priorism'. It is not possible to assert that a theory is completely verified on the basis of confirming instances because the very next instance may refute the theory. But this is the Inductivist programme; the attempt to inject truth from the bottom.

Theories can be corroborated, however. Corroboration, in the Popperian sense, simply means that a theory has stood up to certain specified tests. Corroboration simply asserts that certain statements are compatible with a theory (corroborating instances) and that other statements are incompatible with a theory (falsifying instances). If a theory passes its tests, then we may say that it is corroborated; if a theory does not pass its tests, then we say that it is falsified and we begin to look for a new theory.

But Popper contends that this is not enough, for we want to be

---

[45] Ibid., p. 269.

[46] Ibid., p. 251.

able to say, for example, that theory A is better corroborated than theory B. Do we count the number of corroborating instances of each theory (assuming, of course, that neither theory has met with falsifying instances) and regard the theory with the greater number of corroborating instances as being the one which is better corroborated? Popper contends that:

> The DEGREE OF CORROBORATION of a theory can surely not be established simply by counting the number of the corroborating instances. . . . For it may happen that one theory appears to be far less well corroborated than another one, even though we have derived very many basic statements with its help, and only a few with the help of the second. As an example we might compare the hypothesis "All crows are black' with the hypothesis . . . 'the electronic charge has the value determined by Millikan'. Although in the case of a hypothesis of the former kind, we have presumably encountered many more corroborative basic statements, we shall nevertheless judge Millikan's hypothesis to be the better corroborated of the two.[47]

From a fallibilistic point of view, rather than the number of corroborating instances determining the degree of corroboration of a theory, it is the severity of the tests which determines how well a theory has been corroborated. But this in turn leads one to the concept of the degree of testability of a theory for the severity of a test is dependent on the degree of testability of a theory. It follows, since testability and falsifiability are equivalent in Popperian terms, that the degree of corroboration of a theory depends on the degree of falsifiability of the theory. But it must be noted that the degree of corroboration is not only dependent on falsifiability, for a theory

---

[47]Ibid., p. 267.

may be highly falsifiable but very poorly corroborated, or indeed it may even be falsified. It still remains however that any appraisal of the degree of corroboration of a theory must take into account the severity of the tests to which the theory has been applied.

Popper contends that we reason somewhat as follows when attempting to appraise the degree of corroboration of a theory:

> Its degree of corroboration will increase with the number of its corroborating instances. Here we usually accord to the first corroborating instances far greater importance than to later ones: once a theory is well corroborated, further instances raise its degree of corroboration only very little. This rule however does not hold good if these new instances are very different from the earlier ones, that is if they corroborate the theory in a NEW FIELD OF APPLICATION. In this case, they may increase the degree of corroboration very considerably. The degree of corroboration of a theory which has a higher degree of universality can thus be greater than that of a theory which has a lower degree of universality (and therefore a lower degree of falsifiability). In a similar way, theories of a higher degree of precision can be better corroborated than less precise ones.[48]

Although it is possible to conclude from this statement that induction has crept back into Popper's philosophy since what he seems to be saying is that the degree of corroboration increases as one adds corroborating instances in an inductive way, this conclusion is not justifiable, Popper argues, for it fails to take into consideration the relationship between degrees of testability and degrees of corroboration. For as Popper states:

> According to my view, the corroborability of a theory--and also the degree of corroboration of a theory which has in fact passed severe tests, stand both. . . in inverse ratio to its logical

---

[48]Ibid., p. 269.

> probability; they both increase with its degree of testability and simplicity. BUT THE VIEW IMPLIED BY PROBABILITY LOGIC (inductive logic) IS THE PRECISE OPPOSITE OF THIS. Its upholders let the probability of a hypothesis increase in DIRECT PROPORTION to its logical probability--although there is no doubt that they INTEND their 'probability of a hypothesis' to stand for much the same thing that I try to indicate by 'degree of corroboration'.[49]

And further:

> . . . it is the tendency of inductive logic to make scientific hypotheses as CERTAIN as possible. Scientific significance is assigned to the various hypotheses only to the extent to which they can be justified by experience. A theory is regarded as scientifically valuable only because of the close LOGICAL PROXIMITY. . . between the theory and empirical statements.[50]

In other words, the Inductivist attempts to obtain absolute certainty; he attempts to do this by 'collecting' more and more confirming instances for his theory. The inductivist wants hypotheses which have a high probability of being 'true'. This is not the Criticalist approach. The Criticalist wants highly risky hypotheses, highly falsifiable conjectures, and he does not go about 'collecting' corroborating instances in the hope of confirming his conjectures. Indeed, corroborating instances do increase the degree of corroboration of a theory, but not in an inductive fashion. Corroborating instances increase the degree of corroboration of a theory because these instances are the result of severe tests of the theory, tests which have proven unsuccessful in refuting the theory. At least, that is the Critical Fallibilist's view concerning corroboration.

In summary Critical Fallibilism is a philosophical position

---

[49]Ibid., p. 270.

[50]Ibid., p. 272.

which characterizes knowledge as growing by conjectures and refutations. According to the Criticalist, knowledge is never certain, only provisional; we can never know, we can only guess. But conjectures can be improved. They can be improved by taking into account the refutations of previous conjectures, by strengthening the conjecture by utilizing the failures of the past. The goal is not to rescue conjectures which have met refutation, but to create conjectures which take these refutations into account and attempt to eliminate the refutation by proposing a new conjecture. Lakatos delineates some of the methods available in mathematics for the strengthening of conjectures and also some of the pitfalls that are involved.

A Global View

The preceding subsections have described Popper's results in five areas. These areas represent the primary thrust of Popper's position. However, a description of Popper's views would be incomplete without taking a global view of how scientific knowledge grows according to Popper's formulation. Consequently, this subsection is designed to give a more complete picture of Popper's conceptualization of how scientific knowledge grows. In doing so, two main points are considered: levels of universality and the empirical basis for scientific theories.

Within any theoretical system various levels of universality may be identified. The statements with the highest level of universality would be the axioms or postulates of the system. A statement would be classified as being on a lower level of universality if it is deducible from the axioms of the system. As a result, a theoretical system may be pictured as a pyramid with the axioms at the apex (the highest level of universality) and the theorems of the system occupying lower levels on the pyramid (lower levels of universality). The axioms or postulates at the apex of the system are of course hypothetical statements--hypotheses--relative to the lower level statements. Moreover, even the lower level statements are hypothetical, in Popper's view, since they may ultimately be falsified by even lower level statements utilizing the principle of the retransmission of falsity. (This principle is discussed later.) The fact that higher level statements can be falsified by lower level statements gives rise to Popper's view on the empirical basis of theoretical systems which is discussed subsequently.

Theoretical systems are constructed by seeking even higher level universal statements. Systems grow by the introduction of more primitive axioms--more risky hypotheses in Popper's terminology--from which the "old" system may then be deduced. It becomes clear, then, how the axioms of one theoretical system may become theorems in another system if more universal statements are found from which the former axioms are derivable. For example, Peano's axioms have become theorems deducible from even more primitive axioms in the field of set theory. Set theory would have, in Popper's view, statements of higher universality than the system deducible from Peano's axioms. How this growth of theoretical systems applies to mathematics is discussed in the third section of the present chapter when Lakatos' results are described.

The question of how theoretical systems are falsified leads one to a consideration of the empirical basis of Popper's description of the growth of scientific knowledge. The means of falsifying a theoretical system is according to Popper the modus tollens of classical logic, a deductive reasoning pattern which is described in the third section of this chapter and again in Chapter IV. Briefly, however, modus tollens requires that the falsification of a consequence 'c' derived from a set of premises 'p' implies the falsification of 'p'. It should be noted that in the case of theoretical systems (as opposed to isolated singular implications) the falsification of a consequence of the system may or may not falsify the entire system. Whether the falsifying instance does in fact falsify the entire system depends on whether the counterexample is global or local in nature. The concept of local and global counterexamples has been developed by Lakatos and is discussed in the next section.

The empirical basis for Popper's conceptualization of the growth of scientific knowledge functions as follows. From a set of high level generalization (axioms) consequences are deduced in a purely logical fashion. These deductions are examined for internal consistency, the entire system is studied in order to determine if it is empirical in nature as opposed to being tautological, and this new system is compared with older formulations in order to determine if it might represent an advance on the older theory. Finally, the new theory must be subjected to empirical tests. Experiments are designed which serve to 'test' the derived consequences. If the experimental results do not agree with the derived consequences (the consequences are refuted), then part or all of the new system has been refuted. If, however, the experimental results and the derived consequences agree within the limits of experimental error, Popper would hold that the theory has been corroborated. Note, however, that the experimental test in Popper's view is designed to refute the derived consequences. Furthermore, if the derived consequences and the experimental results agree, Popper would argue that such corroboration does not 'justify' the theoretical system. Such results only corroborate the theory as far as the tests have gone. Further tests are possible, and, indeed, we could have an infinite regress of tests. The decision to halt the process of testing is dogmatic, but such dogmatism is innocent since the tests may be renewed if necessary. Finally, Popper contends that psychologism has not re-entered his system since corroborations are not an attempt to justify a theoretical system. Perceptual experiences motivate a decision to stop the testing process, but they do not, according to Popper, justify a statement or theory.

III. THE GROWTH OF MATHEMATICAL KNOWLEDGE

Imre Lakatos has recently studied the applicability and implications of the results of Critical Fallibilism as embodied in Popper's writings for the growth of mathematical knowledge. Indeed, Lakatos' articles cannot be fully understood unless they are considered in the light of the influence of Critical philosophy. Lakatos states in his paper "Proofs and Refutations" that the results obtained must be "seen against the background of . . . Popper's critical philosophy."[51]

---

[51] Lakatos, P & R, p. 1.

Moreover, this paper extends some of the work done by Polya and hence must be seen in the light of Polya's revival of mathematical heuristic. In the introduction to this series of articles Lakatos states that "the purpose of these essays is to approach some problems of the METHODOLOGY OF MATHEMATICS."[52] The aim of an earlier paper written by Lakatos "IS TO EXHIBIT MODERN MATHEMATICAL PHILOSOPHY AS DEEPLY EMBEDDED IN GENERAL EPISTEMOLOGY AND AS ONLY TO BE UNDERSTOOD IN THIS CONTEXT."[53]

These two purposes or aims are dealt with in reverse order; that is to say, first the discussion of Lakatos' attempts to embed mathematical philosophy in general epistemology, and, secondly, a description of Lakatos' efforts and results in dealing with some of the problems of mathematical methodology. Further to this second purpose, it is Lakatos' goal:

> to elaborate the point that informal, quasi-empirical, mathematics does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations.[54]

Obviously, Lakatos is influenced by Popper's Critical Philosophy.[55] This second purpose is dealt with in Chapter IV when the investigator considers as well Polya's conclusions regarding a 'logic of discovery' for mathematics.

---

[52]Ibid., p. 4.

[53]Imre Lakatos, "Infinite Regress and Foundations of Mathematics," The Aristotelian Society, Supplementary Volume, 34:157, 1962, hereafter referred to as "IRFM".

[54]Lakatos, op. cit., p. 6.

[55]See page 39ff.

In the next subsection the problem of the relationship of mathematical philosophy to general epistemology is considered first. The characterization of mathematical knowledge as quasi-empirical is described as a sequel to this discussion.

<u>Mathematical Foundations and General Epistemology</u>

The aim of this subsection as stated above is to discuss Lakatos' attempt to embed mathematical philosophy in general epistemology; and, moreover, to embed it in Popper's critical orientation to epistemology. In order to do this, it is first necessary to explicate the nature and function of four rationalistic programmes which are designed to deal with a problem enunciated by sceptical philosophers. The problem posed by sceptical philosophy is that of infinite regress; the sceptics used infinite regress to show that it was impossible to find foundations of knowledge. The rationalists could not accept this result and as a consequence put forward three programmes, as Lakatos calls them, in an attempt to answer the sceptics. The fourth programme, the Critical programme, is a different kind of response to this problem and is dealt with last.

These programmes constitute four distinct ways of depicting the growth of mathematical knowledge. Hence they must be included in any discussion of how mathematical knowledge grows. The Fallibilist programme constitutes yet another way of describing the growth of mathematical knowledge. It is not presented here as the only possible orientation to the nature of mathematical inquiry, but as an alternative which needs to be explored and which may meet at least some of the sceptics criticism.

Each of the first three programmes is described in terms of a deductive system. A deductive system in this context is considered to be a system composed of a set of axioms, a set of definitions, a set of undefined terms, and a set of theorems which follow logically from the set of axioms. The crucial question with respect to such a deductive system is that of the direction of the flow of 'truth-value'; whether it is from the axioms to theorems or conclusions or vice versa.

Four possibilities exist for the flow of 'truth-value'. Truth may flow downward from axioms to theorems or upward from theorems to axioms. Furthermore, falsity may flow upward from theorems to axioms or it may flow downward from the axioms to the theorems. However, this last possibility cannot exist for "if the truth-value at the top was FALSE, there would of course be no current of truth-value in the system."[56] The other three cases comprise the three rationalistic programmes as described by Lakatos.

<u>The Euclidean Programme</u>. The characteristic of the Euclidean Programme is truth-value injection at the top of the system, at the level of the axioms. If such axioms are considered to have the truth-value 'true', or in other words, to be trivially true or self-evident truths, then the truth-value flows downwards pervading the whole system. Hence, all results or theorems in a Euclidean system are true since falsity cannot be deduced from true premises. As a result, ". . . a

---

[56]Lakatos, "IRFM," p. 158.

Euclidean theory contains only indubitably true propositions, neither conjectures nor refutations are in it."[57]

The difficulty with this programme lies, of course, in the self-evidency of the premises. The problem arises as to whether there exists a final set of primitive self-evident truths. This proves to be a great difficulty in physics in the search for elementary particles. Instead of obtaining even more elementary particles, more trivial basic premises, "modern science led to terms ever more theoretical and to propositions ever more unlikely . . . ."[58] This whole problem, moreover, leads one back to infinite regress which is precisely the concept the programme was designed to overcome. Hence the Euclidean Programme fails, according to Lakatos, because it could not rid itself of infinite regress.

It is to be noted that despite this failure, it is never necessary for "a Euclidean . . . to admit defeat; his programme is irrefutable. One can never refute the pure existential statement that there exists a set of trivial first principles from which all truth follows."[59] Hence, a Euclidean can always rescue a Euclidean theory which is in danger by arguing that the set of axioms in the system are not really primitive ones. However, this adhocness results in infinite regress again for it amounts to trying to push the set of axioms back to even more self-evident terms.

---

[57] Ibid.

[58] Ibid., p. 160.

[59] Ibid., p. 161.

Euclideanism in science has been on constant retreat for several centuries, but this is not so in mathematics. As recently as the publication of Principia Mathematica by Russell and Whitehead, mathematicians have attempted to establish mathematics on a Euclidean foundation. But even Russell is forced to admit that this is an impossibility.[60] For no Euclidean theory, such as Russell constructed, can ever survive the criticism of the sceptics. Moreover, those individuals who attempt to Euclideanize mathematics are their own worst sceptics for they are constantly troubled by the devastating question: "Have we REALLY reached the primitive terms? Have we REALLY reached the axioms?"[61] So even the truly great philosopher and mathematician Bertrand Russell fails in his attempt to Euclideanize mathematics, and no one has endeavored to do so since for their efforts are predestined to failure.

The Empiricist Programme. Lakatos characterizes a deductive system as an empiricist theory in the following way:

> I call a deductive system an 'empiricist theory' if the propositions at the bottom (basic statements) consist of perfectly well known terms (empirical terms) and there is a possibility of INFALLIBLE TRUTH-VALUE-INJECTION at this bottom which, if the truth-value is FALSE, flows upwards through the deductive channels (explanations) and inundates the whole system. . . . Thus an empiricist theory is either conjectural (except possibly for true statements at the very bottom) or consists of conclusively false propositions.[62]

---

[60]Bertrand Russell, My Philosophical Development (George Allen and Unwin Ltd., London, 1959), p. 212.

[61]Lakatos, op. cit., p. 167.

[62]Ibid., pp. 158-159.

Lakatos argues that this is the programme which science switched to when it became evident that the Euclidean Programme is impossible in science. Scientific theories can never be true or even probable. They are conjectural or they are falsified, the point of view advocated by Popper. Hence it seems that Popper is essentially an empiricist, but this conclusion must be tempered by the fact that Popper's work was initiated in response to the Inductivist Programme. Moreover, this conclusion must also be tempered by the fact that Popper contends that we can never know, we can only guess. The Empiricist Programme however was created to offset the sceptics who stated that we could not 'know'; hence, their Programme is so constituted as to provide foundations for knowledge, foundations for knowing. Popperian Fallibilism does not make any such claims.

Scientists could not accept this strict empiricism; this characterization of scientific knowledge as having no foundations. Even some mathematicians such as Russell and Whitehead could not accept this Popperian view. As a result they opted for the Inductivist Programme, a programme which ultimately led to a probabilistic view of the foundations of knowledge, a view which Popper effectively destroyed but which refuses to die.

<u>The Inductivist Programme</u>. The Inductivist Programme which Popper attacks is, according to Lakatos:

> . . . a desperate effort to build a channel through which truth flows <u>upward</u> from the basic statements, thus establishing an additional logical principle, the PRINCIPLE OF RETRANSMISSION OF

TRUTH. Such a principle would enable the inductivist to inundate the whole system with truth from below.[63]

Inductivists do not accept the claim that knowledge is only conjectural or false. They attempt to re-establish the certainty of knowledge by their programme. Lakatos appeals to Fallibilism in arguing against this approach:

> Popper showed, in his criticism of the probabilistic version of the theory of inductive inference, that there cannot even be a partial transference of meaning and truth upwards. But then he showed that injection of meaning and of truth-value at the bottom level are far from being trivial; that there are no "empirical" terms, but ONLY "theoretical" ones, and that there is nothing conclusive about the truth-value of basic statements thus refurbishing the old Greek criticism of sense-experience.[64]

The reader is directed to section two of this chapter for the reason why Popper rejected induction. Suffice it here to say that as the result of Popper's criticism, it is Lakatos' conclusion that the Inductivist Programme does not escape the sceptics problem of infinite regress.

The Critical Programme. The Critical Programme is a different kind of response to the sceptics than the three previous programmes in that it accepts the sceptics contentions of the fallibility of any truth-value injection of truth. Lakatos summarizes Popper's position in this way:

> Popperian CRITICAL FALLIBILISM takes the infinite regress in proofs and definitions seriously, does not have illusions about "stopping" them, accepts the sceptic criticism of an infallible truth-injection. In this approach there are no Foundations of

---

[63] Ibid., p. 162.

[64] Ibid., p. 165.

> Knowledge, either at the top or at the bottom of theories, but there can be tentative truth-injection and tentative meaning-injections at any point. An "empiricist theory" is either false or conjectural. A "Popperian theory" can only be conjectural. We never KNOW, we only guess. We can, however, turn our guesses into criticisable ones, and criticise and improve them.[65]

The central question then becomes: How do you improve your guesses?[66] Returning to the sceptics for a moment, they could then ask: How do you know that you have improved your guesses? The answer is, of course, that one does not know, we only guess that we have. As Lakatos states: "There is nothing wrong with infinite regress in guesses."[67]

These four programmes then are all attempts to answer the sceptics' problem of infinite regress. Two of them, the Euclidean and the Inductivist Programmes, have failed according to Lakatos. The Empiricist Programme is successful to a degree, but again Lakatos concludes that it could not establish the certainty of knowledge. The Critical Programme accepts the sceptics criticism and responds by contending that we can never 'know', we can only guess.

But these four programmes are created to describe the nature of scientific knowledge, not mathematical knowledge, although Russell's attempt to Euclideanize mathematics was described. One other attempt to establish foundations for mathematics should be described; that is the Hilbertian Programme. It is designed to retrieve knowledge, this

---

[65]Ibid.

[66]How one responds to a philosophy which is predicated on guessing poses some difficult but interesting problems for all educators.

[67]Lakatos, op. cit.

time mathematical knowledge, from the criticism of the sceptics.

The Hilbertian Programme. The goal of Hilbert's programme is to rescue mathematics from contradictions and the criticisms of sceptics. Luchins and Luchins outline the aims of this programme as follows:

> 1. To establish each branch of classical mathematics . . . as an AXIOMATIC theory.
>
> 2. To show that each such axiomatic system is CONSISTENT in the sense that it is FREE FROM CONTRADICTIONS.
>
> 3. To show that the axiomatic system is COMPLETE in the sense that any true proposition of the system is deducible from the axioms.[68]

Luchins and Luchins list three more aims of Hilbert's programme but these need not concern us here for this programme was effectively destroyed when Godel showed the impossibility of achieving simultaneously both aims two and three of this plan.

In unsophisticated terminology, Godel's incompleteness theorem states that if a system is consistent and sufficiently rich to be a formal number-theoretic system, then it must be incomplete. Hence, both the second and third aims of Hilbert's programme cannot be achieved for "the price of consistency is incompleteness."[69] Therefore, the Hilbertian Programme collapses and with it the final attempt to Euclideanize mathematics.

Lakatos argues, then, that the Hilbertian and Euclidean Programmes have failed; they failed in the first case because of Godel's results and

---

[68] Abraham S. Luchins and Edith H. Luchins, Logical Foundations of Mathematics for Behavioral Scientists (Holt, Rinehart, and Winston Inc., Toronto, 1965), p. 142.

[69] Ibid., p. 150.

in the second case because of the sceptics criticism. But if, as Lakatos argues, mathematics cannot be characterized by any of these programmes, how can it be described? Is mathematics empirical? Is mathematics fallible? How does mathematical knowledge grow? The results obtained by Lakatos in an effort to provide at least partial answers to these questions are dealt with next. Moreover, in attempting to provide answers to these questions, Lakatos is advancing the Fallibilist's programme as an alternative to the programmes described and rejected by him.

Mathematics is quasi-empirical

Lakatos contends that modern mathematics is quasi-empirical. But what does he mean by the phrase 'quasi-empirical'? Moreover, what does it mean to say that mathematics is quasi-empirical? Lakatos' answers to these problems are paraphrased below.

A deductive system is quasi-empirical, according to Lakatos, if the pattern of truth-value flow in the system ". . . is (the) retransmission of falsity from the false basic statements 'upwards' towards the 'hypotheses' . . . ."[70] It is to be noted, however, that a quasi-empirical system may or may not be empirical. It is empirical if "its basic theorems are spatio-temporally singular basic statements."[71] The term quasi-empirical describes the nature of the truth-value transmission

---

[70]Imre Lakatos, "A Renaissance of Empiricism in the Recent Philosophy of Mathematics," (Mimeographed copy, limited distribution), p. 11. A condensed version of this paper appears in Lakatos' book Problems in the Philosophy of Mathematics.

[71]Ibid., p. 10.

in a particular deductive system, not whether this system is, in fact, empirical in the scientific sense. Scientific theories are quasi-empirical meaning that the characteristic flow of truth value is the retransmission of falsity; they are empirical meaning that their potential falsifiers are intersubjectively testable basic statements. With relation to Popper's demarcation criterion, it is possible for a theory to be quasi-empirical in the Lakatosean sense but non-testable in the Popperian sense in that the class of potential falsifiers of the theory may be empty. This may indeed be the case for mathematics; that is to say, mathematics may be quasi-empirical but non-testable.

Lakatos firmly contends that mathematics is quasi-empirical. As a result, he sees mathematics as being conjectural and mathematical systems--at their best--being well corroborated.[72] This is in opposition to the view that mathematics is Euclidean in nature which means of course that mathematics is true and in no way conjectural. But according to Lakatos, the efforts of principally Russell and Hilbert to Euclideanize mathematics failed: "the GRANDE LOGIQUES cannot be proved true--not even consistent; they can only be proved false--or even inconsistent."[73]

Thus this leaves mathematics being quasi-empirical, conjectural, and speculative: at least this is what Lakatos contends. He goes on to argue that mathematics, like science, seems to grow by means of conjectures and refutations (Popper's terminology) or proofs and refutations (Lakatos'

---

[72] Ibid.

[73] Ibid., p. 15.

terminology). But "if mathematics and science are both quasi-empirical, the crucial difference between them, if any, must be in the nature of their 'basic statements' or 'potential falsifiers'."[74] Lakatos goes on to say that:

> NOBODY WILL CLAIM THAT MATHEMATICS IS EMPIRICAL IN THE SENSE THAT ITS POTENTIAL FALSIFIERS WOULD BE SINGULAR SPATIO-TEMPORAL STATEMENTS. BUT THEN WHAT IS THE NATURE OF MATHEMATICS? OR, WHAT IS THE NATURE OF THE POTENTIAL FALSIFIERS OF MATHEMATICAL THEORIES?[75]

In answering his own question Lakatos puts forth the concept of a heuristic falsifier which he explains in the following way:

> . . . if we insist that a formal theory should be the formalisation of some informal theory, then a formal theory may be said to be 'refuted' if one of its theorems is negated by the corresponding theorem of the informal theory. One could call such an informal theorem a HEURISTIC FALSIFIER of the formal theory.[76]

It must be kept in mind that a potential falsifier, whether in science or mathematics, is really a rival hypothesis; a hypothesis which contradicts the theory being tested. The question then becomes just what is considered to be a rival hypothesis in mathematics.

This problem has not been solved, and, indeed, is beyond the scope of this study. But what is of interest is Lakatos' contention that mathematics is quasi-empirical but not empirical in the scientific sense; not empirical in the sense that the potential falsifiers in mathematics are not empirically testable statements. This raises another problem since if mathematics is quasi-empirical but its falsifiers are not

---

[74] Ibid., p. 23.

[75] Ibid.

[76] Ibid., p. 24.

empirical, what are they? Lakatos contends that this raises the question of a ". . . demarcation between testable and untestable metaphysical theories with regard to the basic statements."[77] Furthermore, Lakatos seems to be postulating that metaphysical theories may be testable and, if they are, their testability depends on the rather slippery concept of a heuristic falsifier.

In summary of this section, Lakatos' arguments have been presented that all attempts to Euclideanize mathematics have failed. Three choices are then open as to how to characterize mathematics in endeavoring to place it in a general epistemological framework. Popper argues that the Inductivist approach will not suffice. Moreover, Empiricism does not seem to be applicable to mathematics for surely the justification of mathematical statements does not rest upon empirical evidence. That would seem to leave only the approach taken by the Critical Fallibilists: an approach which contends that mathematics is conjectural and not capable of ultimate certainty. It is an approach which focuses on the way mathematics grows, not on the foundations of mathematics. Moreover, it represents an alternate view; a view whose potential fruitfulness is in need of further assessment. It is the view adopted in this study as a basis for a model of mathematical inquiry.

Lakatos' contention derived from Popper's explanation of the growth of scientific knowledge is that mathematics grows by a process of proofs and refutations--by guessing and testing. The question of how

---

[77]<u>Ibid</u>., p. 33.

one can improve his guesses is dealt with in Chapter IV when the writings of George Polya are considered.

The remainder of this study is based on Critical Fallibilism and consequently the view that mathematics is fallibilistic. As a result, it is taken as given that mathematics grows by means of conjectures and refutations. The next chapter endeavors to shed some light on some of the methodological problems of mathematics when it is conceived of in this way.

IV. CHAPTER SUMMARY

This chapter attempts to establish a theoretical foundation for the construction of a model of mathematical inquiry. The theoretical foundation is based on Popper's Critical Fallibilism. Hence, this chapter included, first, a description of the Fallibilist position, and, second, an analysis of the applicability of this philosophical position for the growth of mathematical knowledge.

Critical Fallibilism arose from Popper's dissatisfaction with the Inductivist explanation of the growth of knowledge, an approach which characterizes knowledge as growing by acquiring an ever increasing number of confirming instances and hypotheses generated by inductive generalization. As an alternative to this view, Critical Fallibilism proposes that knowledge grows by conjectures and refutations. Moreover, confirming instances only corroborate a conjecture if they are the result of honest severe tests of the proposed conjecture. The degree of corroboration of a conjecture was seen to depend at least partially on the degree of testability of a conjecture. Furthermore, the degree of testability is a function of the theory's falsifiability.

Contrary to the Inductivist approach, the Criticalist contends that the goal of science is the proposing of highly risky conjectures; conjectures which have a high informative content which means also that it has a low probability of survival. A high level of informative content is equivalent to a theory having a large class of potential falsifiers. A theory which has no class of potential falsifiers is classified as being non-scientific.

The Criticalist approach to the growth of knowledge is not a negative one for it was seen that one requirement for science to grow is that it must meet with positive successes.

The second major section of the chapter dealt with the implications of Critical Fallibilism for the growth of mathematical knowledge. Three rationalistic approaches to the explanation of the growth of knowledge were considered. It was argued by Lakatos that these approaches were inadequate to explain how mathematics grows. The Criticalist position was then advanced by Lakatos as an alternate way of depicting the expansion of mathematical knowledge.

From this point of view mathematical knowledge is conjectural. The only logical principal which applies is the retransmission of falsity. As a result, mathematics cannot hope to obtain absolutely true conjectures. This is a characterization of mathematics as being quasi-empirical. The question of the nature of potential falsifiers in mathematics is still open and in need of further research.

Mathematics is seen as starting with problems, mathematical problems. A conjecture is then made in an attempt to solve the problem. The conjecture may then be tested severely, and, if the conjecture survives its tests, it may be retained tentatively. The corroborating instances SUGGEST that it may be possible to prove the conjecture. Hence conjectures precede proofs. One must guess a theorem before it can be proved. In mathematics, unlike science, a methodological step beyond the testing phase is possible; a step which involves attempting to prove the conjecture. This is the picture of mathematical growth suggested by an application of Critical Fallibilism to the problem of the growth of mathematical knowledge.

It must be noted again, however, that by taking Fallibilism as a basis for the model of mathematical inquiry, a biased view of the processes of mathematical investigation will result. However, due to the weaknesses pointed out by Lakatos in other programmes designed to describe the growth of knowledge, it seems potentially profitable to explore the possibilities offered by Fallibilism as an alternate way of viewing the growth of mathematical knowledge. Furthermore, because of the unique view of the growth of mathematical knowledge posed by Fallibilism, it generates a unique description of the mode of inquiry of mathematics. It generates a mode of inquiry which classifies the growth of knowledge as being the tentative proposal and acceptance of conjectures. But, this is dealt with extensively in Chapter IV and, hence, any further discussion of it here would be premature.[78]

---

[78]For criticism of the Fallibilistic position see Mario Bunge, editor, The Critical Approach to Science and Philosophy (The Free Press of Glencoe, Collier-MacMillan Limited, London, 1964).

CHAPTER IV

THE METHODOLOGY OF MATHEMATICS

I. INTRODUCTION

The present chapter of the study is concerned with ultimately the development of a model of the mode of inquiry of mathematics--a paradigm of the ways in which mathematical knowledge grows. Such a model is designed to depict the nature of the 'logic of discovery' existent in mathematics. But this at once raises questions concerning the methodology of mathematical inquiry. Hence some of the problems relating to mathematical methodology are considered; problems such as how are mathematical conjectures obtained? Are there patterns which may be fruitful in attempting to arrive at a mathematical conjecture? Furthermore, once a conjecture is created, what is the next step? How is a mathematical conjecture corroborated? and, if it is corroborated, what are the strategies involved in the proving of the conjecture?

Throughout this discussion, the philosophical position advanced by the Critical Fallibilists is taken as the basis for the construction of the model of mathematical inquiry; that is, mathematics is assumed to function or to grow by a process of conjectures and refutations. With this philosophical position as a basis, the chapter is logically divided into three main sections.

The first section deals with the nature of the conjecturing process in mathematics. Here the first consideration is given to the 'heuristics' of mathematics, a topic recently revived by George Polya. Because of Polya's revival of this topic, and, equally, due to his interest in and recommendations for the teaching of secondary school mathematics, the investigator chose to study and report on Polya's results with the view of obtaining insights into the logic of discovery of mathematics, the heuristics of mathematics. Polya views heuristics as "the study of means and methods of problem solving."[1] He presents some patterns of plausible reasoning that he believes are suggestive in attempting to make and corroborate mathematical conjectures. He does not, however, discuss the strategies of proving a conjecture once it has been made and corroborated.

Lakatos, on the other hand, continues his study from the point at which Polya leaves off; Polya ends his discussion at the point of where a conjecture is corroborated. Lakatos proceeds from this point to outline some strategies of proving conjectures. In doing so, he illustrates some of the methods of 'saving' a conjecture; techniques designed to rescue a conjecture from refutation. These techniques form part of the 'logic of discovery' of mathematics in the area of the attempted refutation of conjectures and is dealt with in section three of the chapter.

These two sections develop the details of the model. In section four, these details are utilized to provide a fully developed model of mathematical inquiry. The model is then utilized to derive some strategies of teaching; these results are reported in Chapter V.

---

[1] George Polya, MD, Vol. 1, p. vi.

II. THE ORIGINATION AND CORROBORATION OF MATHEMATICAL CONJECTURES

Polya's revival of mathematical heuristic is contained in his books entitled How To Solve It, Mathematics and Plausible Reasoning, in two volumes, and Mathematical Discovery, also in two volumes. Of the many purposes these books are intended to serve, the following are noted as particularly appropriate to the present study.

First, Polya wishes to distinguish between two types of reasoning: deductive reasoning and plausible reasoning. On the one hand, deductive reasoning is "safe, beyond controversy, and final."[2] On the other hand, plausible reasoning is "hazardous, controversial, and provisional."[3] His purpose in drawing this distinction is that he contends that although mathematics is traditionally considered to be a deductive science, plausible reasoning has a role to play in the origination and corroboration of mathematical conjectures. He states his case in the following way:

> Mathematics is regarded as a demonstrative science. Yet this is only one of its aspects. Finished mathematics presented in a finished form appears as purely demonstrative, consisting of proofs only. Yet mathematics in the making resembles any other human knowledge in the making. You have to guess a mathematical theorem before you prove it; you have to guess the idea of the proof before you carry through the details. You have to combine observations and follow analogies; you have to try and try again. The result of the mathematician's creative work is demonstrative reasoning, a proof; but the proof is discovered by plausible reasoning, by guessing. If the learning of mathematics reflects to any degree the invention

---

[2]George Polya, MPR, Vol. 1, p. v.

[3]Ibid.

> of mathematics, it must have a place for guessing, for plausible inference.[4]

Hence Polya contends that mathematical conjectures are obtained by means of guessing; that it is only the final polished product of mathematics which may be characterized as deductive. Mathematics in the process of being created does not follow deductive patterns, but rather, it follows plausible reasoning patterns; patterns which Polya has identified and which are discussed below. Therefore, Polya aims at trying to describe some patterns of plausible inference which are utilized in mathematics; patterns which are utilized mainly for the corroboration of mathematical conjectures.

Polya also explicates some of the means of originating a mathematical conjecture. In this area, Polya is an advocate of inductive generalization, a form of generalization criticized by Popper but which Polya feels is used in mathematics and moreover is a particular case of plausible inference. Polya contends:

> . . . that inductive reasoning is a particular case of plausible reasoning. Observe also (what modern writers almost forgot, but some older writers, such as Euler and Laplace, clearly perceived) that the role of inductive evidence in mathematical investigation is similar to its role in physical research.[5]

In his later work, Mathematical Discovery, Polya wishes to ". . . study . . . the means and methods of problem solving."[6] This, of course, is the study of mathematical heuristic, a topic which Polya has resur-

---

[4]Ibid., p. vi.

[5]Ibid., pp. vii-viii.

[6]Polya, MD, Vol. 1, p. vi.

rected, a topic which is ". . . half-forgotten and half-discredited nowadays. . . ."[7] Polya assumes a case history approach to his discussion of mathematical heuristic.

However, overriding these theoretical aims, the study of heuristic and of plausible reasoning, Polya has a more practical aim; an aim which permeates all of his books in this area. As Polya states this aim, it is ". . . to improve the preparation of high school mathematics teachers,"[8] a preparation which Polya found "insufficient". In order to accomplish this aim Polya argues that it is necessary for any prospective teacher to be thoroughly acquainted with mathematical 'know-how'. He writes:

> Our knowledge about any subject consists of INFORMATION and of KNOW-HOW. If you have genuine BONA FIDE experience of mathematical work on any level, elementary or advanced, there will be no doubt in your mind that, in mathematics, know-how is much more important than mere possession of information. Therefore, in the high school, as on any other level, we should impart, along with a certain amount of information, a certain degree of KNOW-HOW to the student.
> . . . .
>
> The teacher should know what he is supposed to teach. He should show his students how to solve problems--but if he does not know, how can he show them? The teacher should develop his students' know-how, their ability to reason; he should recognize and encourage creative thinking--but the curriculum he went through paid insufficient attention to his mastery of the subject matter and no attention at all to his know-how, to his ability to reason, to his ability to solve problems, to his creative thinking. Here is, in my opinion, the worst gap in the present preparation of high school mathematics teachers.[9]

---

[7] Ibid.

[8] Ibid., p. vii.

[9] Ibid., pp. vii-viii.

Mathematical know-how is simply an alternate expression for mathematical heuristic. Hence it follows that Polya's theoretical and practical aims are compatible; indeed, they reinforce one another.

The reason for the utilization of Polya's work in the present study should now be clear, for the present study is designed to develop a model of mathematical inquiry, a model which attempts to explicate the nature of the origination process in mathematics, the corroboration phase of mathematical inquiry, and the proving strategies existent in mathematics.

Origination of Mathematical Conjectures

The question towards which this section is directed is the following: How is a mathematical conjecture originated? It is to be noted at the outset that no final definitive answer to this question is possible. Many writers including Descartes and Leibnitz have attempted to outline such a method but without success. What is possible, however, is to provide some suggestive patterns, patterns which may have a wide range of applicability even if they do not have universal applicability and success. Hence in what follows there is not an explicit, definitive description of a 'method' of originating a mathematical conjecture.

Following the Fallibilist position, conjectures are merely guesses, leaps from the unknown. These guesses may be guided by certain expectations; that is, certain psychological tendencies to expect that certain things may go together. Popper uses this argument in his criticism of induction. He argues that one does not simply leap to a conclusion inductively on the basis of a few empirical statements, but, rather, one guesses and then tests his conjectures, his guesses.

All conjectures arise from problems. Conjectures are proposed solutions to problems. Hence conjectures are means of solving problems. "Solving a problem means finding a way out of a difficulty, a way around an obstacle, attaining an aim which was not immediately attainable."[10] Hence in order to originate a conjecture there first must exist a problem in need of a solution. Problems present one with an unknown situation, a situation for which a response has not been developed, an action has not been arrived at, or an explication has not been provided. Mathematical inquiry, indeed any form of inquiry, starts from the unknown and attempts by conjecturing and attempted refutations to arrive at a known situation. The path of inquiry is from the unknown to the known and NOT THE OPPOSITE as is usually contended. Moreover, before the 'known' area can be reached, conjectures or guesses about the 'unknown' region must be made. As a result, conjecturing precedes corroboration and proof, and, in turn, conjectures arise out of a problematic situation.

Polya agrees with Popper that a mathematical conjecture must be guessed before it can be proven or, indeed, corroborated. However, Polya argues that this process can be an inductive one. Moreover, Polya feels that there are certain patterns or tentative approaches which are utilized to aid the inductive leap to a generalization.

---

[10] Ibid., p. v.

The first orientation Polya discusses is that of looking for ANALOGIES. To illustrate his argument Polya presents a case history of the origination of Goldbach's conjecture. He argues as follows:

> We started from observing the ANALOGY of the three relations $3 + 7 = 10$, $3 + 17 = 20$, and $13 + 17 = 30$. . .[11]

"Analogy", Polya contends, "is a sort of similarity."[12] He goes on to say that ". . . two SYSTEMS are analogous, if they AGREE IN CLEARLY DEFINABLE RELATIONS TO THEIR RESPECTIVE PARTS."[13] With respect to the three Goldbachian cases given above, Polya contends there is an analogy among these three arithmetic sums. They are analogous because the sums are all even numbers and multiples of ten. Furthermore, the addends are all prime numbers. He then argues that these analogies led to the inductive generalization known as Goldbach's conjecture. Popper, however, would argue that the very fact that these analogies are looked for and, indeed, expected, is evidence of a prior expectancy and hence not an inductive leap to a generalization. This disagreement between the positions taken by Polya and Popper is considered later. What is of importance here is Polya's contention that one way of originating a conjecture is to look for and seek out analogies. What is analogous to what is often very unclear and ambiguous. "Analogy is often vague."[14] But this does not diminish the usefulness of reasoning by analogy for both Polya

---

[11]Polya, op. cit., MPR, Vol. 1, p. 12.

[12]Ibid., p. 13.

[13]Ibid.

[14]Ibid., p. 28.

and Popper state that no matter how the conjecture is obtained, what is of crucial importance is that it be tested. Hence even if analogy is vague, even if it may on occasion lead nowhere (which is probably the case more often than not), it is a fruitful means of obtaining a conjecture.

Polya identifies three forms of clarified though not perfect analogies. The first of these clarified analogies is that of ". . . SYSTEMS OF OBJECTS SUBJECT TO THE SAME FUNDAMENTAL LAWS (or axioms). . ."[15] Multiplication, if zero is deleted from the system, and addition of rational numbers, for example, are analogous for they both satisfy the same set of rules or axioms. Indeed, they are both commutative groups. The second type of clarified analogy is that of ISOMORPHISM; that is, a ". . . one-to-one CORRESPONDENCE THAT PRESERVES THE LAWS OF CERTAIN RELATIONS."[16] In this realm, the addition of real numbers is analogous to the multiplication of POSITIVE numbers. The final form of clarified analogy is that of HOMOMORPHISM which according to Polya ". . . is a kind of SYSTEMATICALLY ABRIDGED TRANSLATION."[17]

Hence what Polya identifies are three sorts of clarified analogies, analogical models if you like, which can serve as a pattern to be watched for, to be utilized, in attempting to arrive at a conjecture. The fruitfulness of an analogical approach to the origination of mathematical

---

[15]<u>Ibid</u>.

[16]<u>Ibid</u>., p. 29.

[17]<u>Ibid</u>.

conjectures is then to be noted.

Another pattern of plausible generation which Polya identifies is that of specialization. Polya considers the operation of specialization is that of ". . . passing from the consideration of a given set of objects to that of a smaller set, contained in the given one."[18] For example, if a problem concerning polygons is being considered, the problem could be specialized to a consideration of triangles. Hence in attempting to solve some problem, it may be fruitful to look at a special case of the problem, to limit its scope, and to make the conjecture relative to the specialization rather than the whole problem at once. In this way some clues as to the solution of the more general problem may be obtained.

Polya, then, identifies two general approaches of possibly arriving at a conjecture; those of analogy and of specialization. Three different types of clarified analogies were considered; isomorphic, homomorphic, and systems which are subject to the same axioms which may be termed axiomatically analogous.

These patterns are suggestive; suggestive in the sense they may be useful in obtaining a conjecture, but which do not guarantee success. They are psychological expectations which may or may not be fulfilled. Their fruitfulness arises from the fact that as Polya's case histories of mathematical discoveries show they have been utilized with great results in the past. This is not to say that they will be successful in

---

[18] Ibid., p. 13.

the future, but then as was mentioned previously no certain method of originating interesting conjectures can be hoped for.

Polya's contribution is depicted in Figure 3 below:

Origination

PROBLEM--CONJECTURE

(1) Analogy

(a) axiomatic

(b) isomorphic

(c) homomorphic

(2) Specialization

FIGURE 3

ORIGINATION OF CONJECTURES

From a problem a conjecture is to be formulated in an attempt to solve the problem. Strategies denoted by analogy and specialization may be fruitful in obtaining a conjecture; a conjecture which is then tested severely in order to determine its adequacy as a solution to the problem.

There is no attempt at finality here; that is to say, it is not contended that the list of suggestive strategies given above is in any way exhaustive. Other patterns may be proposed in the future. Indeed, this is likely to be the case.

Once a conjecture has been brought forth, what is the next step in the logic of discovery of mathematics? Following again the Fallibilist position, it is then necessary to test the conjecture. If the tests fail, the conjecture is falsified. But if it is not refuted even after severe tests, then it becomes corroborated. Are there identifiable corroborative patterns in mathematics? Polya thinks so and has identified three such patterns.

Testing and Corroborating a Mathematical Conjecture

The testing and corroboration of a mathematical conjecture arises when the mathematician finds it impossible to obtain a deductive proof of his conjecture. In such a situation the mathematician tests his conjecture. If he refutes his conjecture, if he finds a counterexample to his conjecture, then he has utilized the deductive pattern known as MODUS TOLLENS. But what if he does not refute his conjecture? How is the corroborative evidence to be evaluated?

For example, it was once conjectured that the quadratic expression $x^2 + x + 41$, where x is a natural number, would yield all the prime numbers greater than 37.[19] No formal deductive proof of this conjecture was ever discovered, or indeed, could have been discovered, for the conjecture was refuted. Consider the case x = 40; on substitution into the quadratic expression, the result is obtained that 1600 + 40 + 41 = 1681 which equals forty-one squared and hence is not a prime number. Indeed, it can be shown that no polynomial of the form $p(x) = \sum_{n=1}^{\infty} a_n x^n$ can yield all the prime numbers.

---

[19]A prime number is a positive integer whose only divisors are one and the number itself. One is not considered to be a prime. Two is the only even prime number.

For a further example which yields somewhat different results, consider again Goldbach's conjecture. Stated simply it is as follows: Any even number greater than four can be expressed as the sum of two odd prime numbers. Table I below gives corroborating instances for the conjecture for values of n from 6 to 40.

TABLE I

CORROBORATING INSTANCES OF GOLDBACH'S CONJECTURE

n	sum of odd primes	n	sum of odd primes
6	3 + 3	24	5 + 19, 7 + 17, 11 + 13
8	3 + 5	26	3 + 23, 7 + 19, 13 + 13
10	3 + 7, 5 + 5	28	5 + 23, 11 + 17
12	5 + 7	30	7 + 23, 11 + 19, 13 + 17
14	7 + 7, 3 + 11	32	3 + 29, 13 + 19
16	3 + 13, 5 + 11	34	3 + 31, 5 + 29, 11 + 23, 17 + 17
18	5 + 13, 7 + 11	36	5 + 31, 7 + 29, 13 + 23, 17 + 19
20	3 + 17, 7 + 13	38	7 + 31, 19 + 19
22	3 + 19, 5 + 17, 11 + 11	40	3 + 37, 11 + 19, 17 + 23

However, this conjecture is still not deductively proven. Moreover, no matter how many confirming instances of the type given in Table I one may list, this listing cannot serve to prove the conjecture.

Hence it may be seen that testing arises in mathematics when a mathematician is unable to provide a deductive proof of some conjecture. Further, this testing takes the form of attempted refutations of the conjecture and corroboration is secondary to testing. The conjecture only becomes corroborated if the tests fail.

Polya's patterns of plausible inference are designed as patterns which provide psychological credibility to a conjecture. Polya is emphatic in stating that these patterns cannot establish the truth of a conjecture, but he contends they can be and are utilized to determine if the conjecture is worthy of consideration. But he nevertheless, in spite of this heuristic evidence, views these corroborated conjectures as tentative, tentative since they are neither proven or refuted.

The patterns described below are drawn from Polya's book Patterns of Plausible Inference which is the second volume of his Mathematics and Plausible Reasoning. In these patterns Polya assumes the principle of induction as a valid way of obtaining knowledge.

Confirming a Consequence[20] The first pattern considered may be labelled as the confirmation of a consequence. This is the pattern which is utilized above in examining Goldbach's conjecture. Put in a syllogistic format, the pattern would appear as follows:

$$
(1) \qquad \begin{array}{l} A \dashrightarrow B \\ B \text{ true} \\ \hline A \text{ more credible} \end{array}
$$

This pattern may be compared to the classical deductive reasoning pattern, modus tollens, which has the following syllogistic form:

---

[20]Polya, op. cit., MPR, Vol. 2, p. 3.

$$\begin{array}{ll} & A \dashrightarrow B \\ (2) & B \text{ false} \\ \hline & A \text{ false} \end{array}$$

The goal, of course, is to establish the truth or falsity of conjecture A. At least two comments are pertinent at this point. First, pattern one can never prove the conjecture A. No matter how many consequences B which are determined to be true, the truth of A is not obtained. This is obviously not the case for the deductive pattern two. Second, the mathematician may utilize both of these patterns when he is testing a conjecture. If the test of the conjecture fails, pattern one is being utilized. However, if the test of the conjecture is successful, that is, the conjecture is refuted, then pattern two was used. The first pattern, Polya contends, does not prove a given conjecture, but it does render the conjecture more credible.

Pattern one may be extended to the case of successive confirmations of the proposed conjecture. If several confirming instances of a conjecture are obtained, as was done for Goldbach's conjecture, then with each successive confirmation the conjecture becomes somewhat more credible. However, if some confirming instance that was quite different from all the preceding confirming instances was discovered, then the credibility of the conjecture would be strengthened to a greater degree. Polya sums up the situation when he states that ". . . the strength of the additional verification increases when the analogy of the newly verified consequence with the previously verified consequence decreases."[21]

---

[21] Ibid., p. 30.

Consider now an illustration, due to Polya, which exemplifies the above pattern.[22] Let 'f' denote the number of prime factors of the integer n, and call n 'evenly' factorized or 'oddly' factorized according as f is even or odd. For example, 20 = 2 x 2 x 5 is oddly factorized whereas 36 = 2 x 2 x 3 x 3 is evenly factorized. All the prime numbers will be oddly factorized, the perfect squares such as 4, 9, 16, 25, . . . are evenly factorized and the number one is taken as evenly factorized. Table II lists the f's as even or odd for the first twenty-four positive numbers. In studying this table, it is seen that the sequence of e's and o's do not display any simple pattern. There does not seem to be any regularity in the occurrence of an 'e' or an 'o'. The e's and o's seem to occur, as it were, at random. It has been proven that among the first n integers about as many will be evenly factorized as will be oddly factorized, if n is large.[23]

Hence, it would seem that the conjecture of randomness is corroborated. This would seem to imply that the evenly and oddly factorized integers would follow each other in a random succession. However, upon further consideration of Table II it may be observed that there are eleven evenly factorized numbers and thirteen oddly factorized numbers. Furthermore, it has been empirically verified that for n up to six thousand this relationship obtains; that is, the number of evenly factorized numbers is

---

[22]<u>Ibid</u>., p. 49.

[23]This proof is long and quite difficult.

TABLE II

FACTORIZATION OF THE FIRST TWENTY-FOUR POSITIVE INTEGERS CHARACTERIZED AS EVEN OR ODD*

n	f	n	f
1	e	13	o
2	o	14	e
3	o	15	e
4	e	16	e
5	o	17	o
6	e	18	o
7	o	19	o
8	o	20	o
9	e	21	e
10	e	22	e
11	o	23	o
12	o	24	e

*An integer is characterized as being 'evenly' factorized if it has an even number of prime factors, and as being 'oddly' factorized if it has an odd number of prime factors.

less than or equal to the number of oddly factorized numbers.[24] This empirical evidence was sufficient to lead Polya to the conjecture that "For $n > 2$, the evenly factorized integers are never in the majority among the first n integers."[25] According to pattern one given above it is possible at this point to go on and attempt to refute or confirm the conjecture for larger values of n. But this procedure would add little to the credibility of the conjecture. If, however, it was possible to prove one or two quite improbable consequences of this conjecture, then the credibility of the conjecture would be increased according to the pattern. This is what Polya did. He assumed, for the moment, that his conjecture was true. He was able to show that two quite improbable consequences whose truth had already been established were derivable from his conjecture. This would seem to lend a greater degree of credibility to his original conjecture. But, the conjecture as far as the investigator can determine has not yet been proven deductively.

Popper and Polya would seem to be in agreement as to pattern one. Popper was quoted earlier (see page 62) as saying that greater importance is given to the first corroborating instances of a conjecture than to later ones. Furthermore, this generalization does not hold if the new corroboration is very different from the earlier ones: ". . . that is if they corroborate the theory in a NEW FIELD OF APPLICATION."[26] This is

---

[24]Polya verified his conjecture for n up to 1500.

[25]Polya, op. cit., MPR, Vol. 2, p. 50.

[26]Popper, L.Sc.D., p. 269.

essentially what Polya is arguing in putting forth pattern one. Moreover, as the last example illustrates, a corroboration in a new field, in an area where the conjecture is highly improbable, increases the credibility, the degree of corroboration, of the conjecture.

<u>Examining a Possible Ground</u>. Polya labels his second pattern of plausible inference the "Examining (of) a possible ground."[27] This pattern in a syllogistic format appears as follows:

$$\begin{array}{ll} & A \longleftarrow\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!- B \\ (3) & B \text{ false} \\ \hline & A \text{ less credible} \end{array}$$

This pattern is obviously related to the deductive pattern, modus ponens, which has the following syllogistic form:

$$\begin{array}{ll} & A \longleftarrow\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!- B \\ (4) & B \text{ true} \\ \hline & A \text{ true} \end{array}$$

Moreover, pattern four is the Euclidean pattern which Lakatos identifies. Similar comments can be made with respect to pattern three as were made for pattern one. Using pattern three can never lead to a final proof of the proposed conjecture A. Nor does pattern three falsify A although it makes A less credible. Truth-value cannot flow in this pattern as was pointed out earlier (see page 70) by Lakatos. Polya argues, however, that the falsification of a possible ground for a conjecture can only

---

[27] Polya, <u>op</u>. <u>cit</u>., p. 19.

diminish the credibility of the conjecture.

What does this pattern three mean? It is desired to prove or disprove the conjecture A. By some process, it has been determined that B implies A. Attention is turned to B as perhaps being more promising, more susceptible to a deductive proof. If B were proved, then A would follow as a logical consequence. B is a possible ground for A. But suppose it turns out that B is false. How does this effect the conjecture A? With respect to this question, Polya states that ". . . our confidence in a conjecture can only diminish when a possible ground for the conjecture has been exploded."[28] Hence, even though no truth-value can flow in this pattern, Polya contends that the psychological credibility of the conjecture must decrease as the result of the destruction of a possible basis for the conjecture.

<u>Examination</u> <u>of</u> <u>a</u> <u>Conflicting</u> <u>Conjecture</u>. The third and last pattern of plausible inference which Polya identifies is that of the examination of a conflicting conjecture.[29] Again, in a syllogistic format, this pattern appears as follows:

$$\begin{array}{ll} & \text{A incompatible with B} \\ (5) & \text{B false} \\ \hline & \text{A more credible} \end{array}$$

The corresponding deductive pattern is given below:

---

[28] <u>Ibid</u>., p. 123.

[29] <u>Ibid</u>.

A incompatible with B

(6) B true

A false

By incompatible it is meant that the truth of one of either A or B implies the falsity of the other. Concern is again with establishing the truth or falsity of A.

The main difference between pattern five and patterns one and three is that both of the conjectures A and B could be false. Hence, the falsity of one of these conjectures does not prove the other. If, on the other hand, either of the two conjectures is proven, then the falsity of the other is established. Polya concludes that "our confidence in a conjecture can only increase when an incompatible rival conjecture is exploded."[30]

The Fallibilist point of view regarding conflicting conjectures is that the one with the higher degree of testability, the riskier one, is the one to be desired. However, once the conjectures have actually been tested and the one is refuted, it does not necessarily mean that the degree of corroboration, the credibility of the other, is increased; it would only be increased IF it has passed severe tests which the other conjecture did not pass. The fact that one of two conflicting conjectures is refuted does not increase the credibility, in the Fallibilist view, of the other UNLESS it has survived some test. Polya does not seem to draw this distinction. He seems to conclude that the explosion of a

---

[30]*Ibid.*, p. 20.

conflicting conjecture automatically increases the credibility of the other conjecture, whether or not it has been tested.

Credibility and Induction. Two additional points need to be mentioned in connection with Polya's patterns of plausible reasoning. These points relate to what exactly does Polya mean by credibility and in what manner does Polya see the appraisal of the credibility of a conjecture as being an inductive process.

Polya views credibility as the weight of the evidence in support of a conjecture. Consequently, Polya's concept of credibility serves the same function as degrees of corroboration do for Popper. Polya defines his concept of credibility in relation to some conjecture, say A, as being ". . . the reliability of this conjecture A, the strength of evidence in favor of A, our confidence in A, the degree of credence we should give to A, in short the CREDIBILITY OF THE CONJECTURE A."[31]

However, Polya like Popper does not contend that the strength of evidence in support of a conjecture can be given a numerical value. The calculus of probability cannot be utilized to assess in quantitative terms either the credibility of a conjecture to follow Polya's terminology or the degree of corroboration utilizing Popper's terminology. All that Polya does contend is that his patterns indicate the DIRECTION of support of evidence for a conjecture but not the STRENGTH of the evidence. For example, Polya argues that each confirming instance of

[31] Ibid., pp. 116-117.

Goldbach's conjecture strengthens the conjecture in a positive fashion, but it is not possible to say by 'how much' the conjecture is strengthened.

Polya summarizes his own views concerning the status of his patterns of plausible reasoning by drawing a comparison between these patterns and the syllogistic patterns of deductive reasoning such as modus tollens. Polya characterizes the deductive syllogisms as being ". . . IMPERSONAL, UNIVERSAL, SELF-SUFFICIENT, and DEFINITIVE."[32] They are impersonal in the sense that they are not dependent on the personality of the user; universal in that they apply to all fields of knowledge; self-sufficient in that once the premises of a syllogism are accepted, the conclusion follows automatically; and definitive in the sense that the conclusion obtained in a deductive syllogism is final, provided, of course, that the premises are accepted.

In contrast to this, Polya contends that his plausible reasoning patterns are impersonal, universal, self-sufficient and provisional,[33] but with some very important qualifications. These patterns are impersonal in that the DIRECTION of the support the evidence offers is independent of the observer, but the STRENGTH of such support is not. As a result, the patterns of plausible reasoning proposed by Polya are 'one-sided'; they are restricted to the direction only, and not the strength of evidence associated with a conjecture. Hence, the patterns are

---

[32] Ibid., p. 112.

[33] Ibid., pp. 114-115.

impersonal, but this 'impersonality' applies only to one aspect of these patterns.

Plausible reasoning patterns Polya contends are also universal in the sense given above. But as with the impersonality of these patterns, universality is only applied to one aspect of plausible reasoning; the aspect of the direction of the evidence relating to some conjecture.

Again, the same qualification must be applied to the self-sufficiency of these patterns. The premises entail the conclusions in such patterns, but the conclusions are not durable since only the direction and not the strength of the evidence is entailed in the premises. An exception has to be noted with the case of pattern five where the additional qualification must be added that the alternate conjecture must be tested and survive before the conclusion is valid.

Finally, plausible reasoning patterns simply are not definitive; they are provisional. This is obvious if one realizes that the very next test of the conjecture which if successful corroborates the conjecture may fail and as a result refute the conjecture. No matter how many corroborating instances of a conjecture there are the next intersubjectively testable application of the conjecture may refute it in which case all the evidence which has supported the conjecture is nullified.

Therefore, Polya's patterns by his own admission are not definitive and only impersonal, universal, and self-sufficient because these patterns are one-sided; they indicate a direction but not the strength of evidence in relation to some conjecture. This, however, does not make these patterns any less useful or fruitful for the acquisition of knowledge is a human endeavor. A computer can be programmed to reason deductively; so can a human being. But a computer cannot be programmed to reason plausibly whereas a human being probably functions in this mode of reasoning more than in the deductive mode. The human being functions in the plausible reasoning mode because the creation of knowledge, the acquisition and organization of knowledge, is a conjectural process and as such is characterized by patterns of plausible rather than deductive reasoning.

The first step across the bridge to reach the island of knowledge is a risky one--it is a guess. In attempting to cross the bridge many wrong steps may be taken. The bridge itself may collapse. It is like attempting to build a bridge across a very wide river which is constantly covered by heavy fog. Many false girders may be put in place and some may be weak and cause the bridge to collapse. Moreover, even if the bridge does reach land on the other side of the river, the constructors may never know because the fog would be too thick to see below. But this need not deter one from building bridges for perhaps the river has islands and perhaps a foundation for the bridge can be constructed which provides positive support for the bridge; a bridge designed to transport one across the river of ignorance.

Popper and Polya do not disagree that the beginnings of knowledge, even mathematical knowledge, depends on guesses--conjectures. However, they do disagree on as to how the guesses are formulated. Polya contends that an inductive process is involved; Popper does not. It is the investigator's view that this disagreement is one based on different interpretations of case histories, and hence lies in the realm of psychological investigation. Indeed, Popper states that:

> . . . how it happens that a new idea occurs to a man--whether it is a musical theme, a dramatic conflict, or a scientific theory--may be of great interest to empirical psychology; but it is irrelevant to the logical analysis of scientific knowledge.[34]

Popper does, however, condescend to give his view as to how new ideas might be conceived:

> However, my view of the matter, for what it is worth, is that there is no such thing as a logical method of having new ideas, or a logical reconstruction of this process. My view may be expressed by saying that every discovery contains 'an irrational element' or 'a creative intuition'. . .[35]

For Polya this process is an inductive one. But little can be gained from the point of view of this study in pursuing this disagreement for it seems to depend entirely on personal interpretation. Whether this process is to be called inductive or not is not an important point since it seems to hinge on how one defines induction. Hence, words seem to be getting in the way of an idea, and it is the idea that is important, not the name attached to the idea. At least in the area of the origination of a conjecture, the disagreement does not seem to be a fruitful one to pursue.

However, in the area of the confirmation of a conjecture, the disagreement between these two men is of importance. In this area the lines of opposition are not quite as clear-cut and they are in need of

---

[34]Popper, L.Sc.D., p. 31.

[35]Ibid., p. 32.

clarification. First, as has been noted earlier, Popper rejects induction as a means of establishing or justifying a conjecture. Polya would agree. For Polya, as for all mathematicians, a conjecture can only be established definitively by deductive logic. An argument based on plausible reasoning cannot decide the truth of any conjecture, be it mathematical or scientific. But Polya argues (see pattern one above) that inductively obtained evidence can be utilized to assess the credibility of an as yet unproven conjecture. Popper, on the other hand, argues that such evidence does indeed increase the degree of corroboration of a conjecture but NOT in an inductive fashion. Here these two positions would seem to be in disagreement. But are they? In analysing Polya's position more closely, it is to be noted that he argues that the "more danger" there is of a consequence being refuted, the "more honor" there is attached to that consequence if it is not refuted. Or: "If a conjecture escapes the danger of refutation it shall be esteemed in proportion to the risk involved."[36] But this is precisely the basis of Popper's degrees of corroboration of a theory. Note also that in Polya's view:

> . . . THE INCREASE IN OUR CONFIDENCE BROUGHT ABOUT BY THE CONFIRMATION OF A NEW CONSEQUENCE. . . VARIES INVERSELY AS THE CREDIBILITY OF THE NEW CONSEQUENCE, APPRAISED. . . IN THE LIGHT OF THE PREVIOUSLY VERIFIED CONSEQUENCES.[37]

---

[36]Polya, op. cit., MPR, Vol. 2, p. 126.

[37]Ibid., p. 125.

Compare this to Popper's statement that:

> . . . the degree of corroboration of a theory which has in fact passed severe tests, stand both, . . . , in inverse ratio to its logical probability.[38]

Moreover, a low logical probability means for Popper the same as it does for Polya; a high probability of being refuted. Hence Polya and Popper do not really seem to be in disagreement. But is this too a battle over words; in this case, the word induction. Not at all, for although Polya seems to agree with Popper that the greater the risk, the higher the credibility or degree of corroboration, Polya argues that each corroborating instance INCREASES in an inductive fashion the logical probability of the hypothesis. This is the Inductivist's goal; a high degree of certainty for his hypothesis. Popper rejects this because to hold such a view means that corroboration DECREASES with testability. Hence, it would seem that Polya is holding contradictory viewpoints. Either corroboration increases testability or it does not. If it does, then corroborating instances cannot increase the logical probability of a conjecture. Polya's difficulty seems to arise in trying to characterize his patterns as inductive. If this feature of these patterns is dropped, then the contradiction is removed and he and Popper end up being in full agreement.

Polya's patterns of plausible inference can then be seen as forming part of the mode of inquiry of mathematics. They come into play in the confirmation phase of the logic of mathematical inquiry. Figure 4

---

[38]Popper, op. cit., p. 270.

depicts Polya's contribution in this area.

Testing

Conjecture--Corroboration

(1) Examining a consequence

(2) Examining a possible ground

(3) Examining a conflicting conjecture

FIGURE 4

TESTING OF A CONJECTURE

As for the origination phase of mathematical inquiry, it is not contended that the patterns listed here for the testing phase of the logic of discovery in mathematics are complete. They are only suggested by Polya's work based on a Fallibilist orientation to the growth of mathematical knowledge.

III. THE PROVING OF MATHEMATICAL CONJECTURES

The proving phase of mathematical inquiry follows after some mathematical conjecture has been corroborated. Corroboration of a conjecture <u>suggests</u> that it might be possible to prove the conjecture. The process of corroboration can give clues and ideas as to the scope of the conjecture, to its field of application, and in doing so perhaps provide insights as to how it might be possible to prove the conjecture.

The following discussion of the proving phase of the logic of discovery in mathematics draws on the work of Lakatos, especially his series of articles "Proofs and Refutations".[39] In this essay, Lakatos considers both the problem of how a mathematical conjecture is obtained, and how, once a conjecture is produced, a proof of the conjecture is constructed. Moreover, in analysing proof strategies Lakatos identifies three methods or techniques of 'saving' a conjecture; that is to say, techniques which are utilized when a conjecture is threatened by refutation due to the discovery of a counterexample. Furthermore, the nature or types of counterexamples are delineated with respect to the decomposition of the conjecture under consideration.

As a consequence, three general areas are discussed in this section. The first subsection discusses the nature of counterexamples and their relation to the conjecture being studied. Second, conjecture saving techniques are explicated with consideration being given to their advantages and disadvantages. Finally, two patterns of conjecture origination are delineated and their relationship to the general Fallibilist strategy of conjecture and refutation is explained.

"Proofs and Refutations" is written as a dialogue which takes place in an imaginary classroom. The problem which the class is interested in is that of whether there is a relationship among the number of faces, edges and vertices of polyhedra analogous to that which exists for the number of edges and vertices of a polygon; namely that the number of

---

39 Imre Lakatos, "Proofs and Refutations," The British Journal for the Philosophy of Science, Vol. 14, 1963, pp. 1-25, 120-139, 221-245, 296-342.

edges equals the number of vertices: V = E. The origination of the conjecture that the relationship is V-E+F = 2 is discussed by Polya.[40] Lakatos' articles proceed from the point at which the conjecture has been originated and corroborated in a few instances. The problem is then to prove the conjecture.

Conjectures and Counterexamples

In the process of looking for a proof of a conjecture it is sometimes advantageous to break the conjecture down into a series of lemmas; that is, the conjecture is decomposed so that its proof becomes dependent on the proof of a number of subconjectures. The advantage of this procedure is that ". . . it opens new vistas for testing. The decomposition deploys the conjecture on a wider front, so that our criticism has more targets."[41] Hence, instead of having only the original conjecture for which to find counterexamples, each lemma becomes susceptible to refutation by a counterexample. The conjecture thus becomes testable over a wider range--it becomes riskier and better testable, that is, more highly falsifiable.

In this view the concept of proof takes on a new meaning. Proof can now be thought of as the "decomposition of the original conjecture into subconjectures."[42] This is in contradistinction to a conception of

---

[40]Polya, op. cit., MPR, Vol. 1, pp. 35-41.

[41]Lakatos, op. cit., p. 11.

[42]Ibid., p. 15.

proof as being a guarantee of the certain truth of the original conjecture. Using the former conception of proof it becomes possible to distinguish between two types of counterexamples.

The first type of counterexample is one which is local; that is, this type of counterexample refutes a subconjecture or lemma but not necessarily the main conjecture. On the other hand, a global counterexample refutes the main conjecture. Hence, "A local, but not global, counterexample is a criticism of the proof, but not of the conjecture."[43] The importance of this distinction and that of the distinction between the two meanings of proof is that when counterexamples are found they can be classified. If the counterexample is local, then the proof-analysis, the system of subconjectures, has to be reconsidered, but it does not mean that the original conjecture has been refuted. Furthermore, if the counterexample is global but <u>not</u> local, then the original conjecture has been refuted but not the proof. It then becomes important to determine what it is the proof actually proves if it is not the original conjecture. With this orientation, global counterexamples do not end the discussion of a proof, but rather act as a spur to pursue the proof in order to determine its meaning. Hence the growth of mathematical knowledge is fostered by refutation rather than stopped by refutation. Lakatos summarizes this Fallibilistic position when he has Alpha state:

> So according to your philosophy--while a counterexample (if it is not global at the same time) is a criticism of the proof, but not of the conjecture--a global counterexample is a criticism of

---

[43] <u>Ibid</u>., p. 12.

the conjecture, but not necessarily of the proof. You agree to surrender as regards the conjecture, but you defend the proof.[44]

A third situation may arise, however; that is the case of a counterexample which is both global and local. But far from refuting the conjecture, this situation actually corroborates the conjecture. As Lakatos points out, this corresponds to the paradox of confirmation. As a result, the second type of counterexample, that which is global but not local, causes the greatest difficulty. Lakatos identifies three strategies which may be utilized in such a situation.

Conjecture Saving Techniques

Monsterbarring. Two general responses are possible when a global but not local counterexample is discovered. The first response is to accept the counterexample as valid which in turn necessitates the rejection of the original conjecture. This represents total surrender. The second response is not to accept the counterexample. This means the rejection of the counterexample as not REALLY being a counterexample which, of course, allows one to retain the original conjecture. But on what grounds could a global counterexample be rejected? This technique usually involves the redefinition or clarification of the terms used in stating the conjecture. Lakatos' classroom teacher (who is obviously a Fallibilist) states that: ". . . refutation by counterexample depends on the meaning of the terms in question. If a counterexample is to be an objective criticism, we have to agree on the meaning of our terms. . . .

[44] Ibid., p. 15.

DEFINITIONS ARE FREQUENTLY PROPOSED AND ARGUED ABOUT WHEN COUNTEREXAMPLES EMERGE."[45]

This technique of surreptitious redefinition of terms is designed to save a conjecture from monsters. Lakatos calls this the monster-barring technique. He writes:

> I think we should refuse to accept Delta's strategy for dealing with global counterexamples, although we should congratulate him on his skilful execution of it. We could aptly label his method THE METHOD OF MONSTERBARRING. Using this method one can eliminate any counterexample to the original conjecture by a sometimes deft but always AD HOC redefinition . . . of defining terms. . . . We should somehow treat counterexamples with more respect, and not stubbornly exorcise them by dubbing them monsters.[46]

The main criticism of this technique is its ad hocness; that is, it is not designed to create knowledge but to preserve and conserve the original conjecture. However, it should be noted that monster-barring techniques can serve as a goad to the creation of clarified definitions and the uncovering of hidden assumptions and lemmas. If monsterbarring is not allowed to save a conjecture simply for the sake of saving it, then its function of clarification can be useful and potentially fruitful.

Exception-barring. At least that is the point of view expressed by exception-barrers. This technique means that one accepts:

> . . . the method of monsterbarring in so far as it serves for finding THE DOMAIN OF VALIDITY OF THE ORIGINAL CONJECTURE; I REJECT it in so far as it functions as a linguistic trick for rescuing

---

[45]Ibid., p. 18.

[46]Ibid., p. 25.

> 'nice' theorems by restrictive concepts. These two functions . . . should be kept separate. I should like to baptise MY method which is characterized by the first of these functions only, 'THE EXCEPTION-BARRING METHOD'.[47]

This technique represents a third type of response to a counterexample; it does not take a conjecture or leave it, but tries to improve the conjecture. It is preferable to both monsterbarring and outright surrender. However, in this form it fails because it is impossible to KNOW if all the exceptions have been noted. Scepticism enters for it is impossible to know if all the exceptions have been enunciated in a way analogous to the sceptic criticism of the Euclidean Programme of finding ultimate trivially true axioms.

A modification of this technique is that of rather than restricting the conjecture bit-by-bit with each new counterexample which is discovered, the conjecture is withdrawn to a 'safe' domain.

> In this new, modified version of the exception-barring method, . . . piecemeal withdrawal has been replaced by a strategic retreat into a domain hoped to be a stronghold of the conjecture. You are playing for safety.[48]

The question then is whether such a withdrawal is too radical? Such a withdrawal could so restrict a conjecture that consequences outside its domain could be valid. Moreover, such a restriction of the conjecture may not be restrictive enough; it could still be an overstatement. The point is that it is impossible to know.

Exception-barring does improve a conjecture, but it cannot claim

---

[47] Ibid., p. 122.

[48] Ibid., p. 125.

to have perfected the conjecture for the reasons given above. Hence, exception-barring is an improved response to counterexamples over monsterbarring and surrender, but it is not without its limitations.

Monster-adjustment. A fourth response to the emergence of a counterexample is that which Lakatos labels 'monster-adjustment'.[49] Monster-adjusters criticize the techniques of monsterbarring and exception-barring as not taking the counterexamples seriously. This method attempts to get around the counterexample, to explain the counterexample, by claiming that counterexamples are not really monsters. To see this, monster-adjusters claim, one needs only to have his vision clarified--to purge his mind from error. In describing this method Lakatos has Rho state that:

> One has to purge one's mind from perverted illusions, one has to learn how to see and how to define correctly what one sees. My method is therapeutic: where you--erroneously--'see' a counterexample, I teach you how to recognise--correctly--an example. I adjust your monstrous vision.[50]

This method depends obviously on the light of 'pure' vision analogous to the Euclidean light of 'pure' reason. Both forms of illumination are in the head of the beholder. Monster-adjustment is a dogmatic response to counterexamples; the theory of perverted vision is employed to explain why some individuals cannot see what is manifestly true.

Hence, three monster treatment methods have been identified. Two

---

[49] Ibid., p. 127.

[50] Ibid., p. 128.

of them, monsterbarring and monster-adjustment, reject counterexamples; the first by refusing to consider them, and the second by claiming that they are not really counterexamples but the product of perverted vision. The third method, that of exception-barring, restricts the domain of the conjecture and the proof in order to place the counterexamples outside this domain. But whether the area of this domain is too large or too small is unanswerable and constitutes a weakness of this method. Nevertheless, exception-barring does serve to improve a conjecture if the above caution concerning the area of the domain is kept in mind.

Two Patterns of Guessing

On the basis of Lakatos' alternate definition of the nature of proof--proof as being a decomposition of the original conjecture into subconjectures--it is possible, according to Lakatos, to identify two patterns of guessing. At least one of these patterns presents a different view of origination as derived from Polya's work.

The first pattern is the one which starts with a problem followed by a conjecture which is a proposed solution to the problem. The conjecture is then subjected to tests and finally a proof is developed. This pattern Lakatos has labelled as naive guessing.

Naive Guessing.

Naive guessing is a pattern of origination of mathematical conjectures which is dependent on the strategy of conjecture and refutation. Naive guessing is not an inductive process, but rather it is a guess followed by the testing of the guess. The conjecture and refutation process stretches the original conjecture; it refines and redefines the original guess SPURRED by the discovery of counterexamples, by being refuted. Hence, refutations, rather than destroying a conjecture, encourage the growth of knowledge. How this can be so is denoted by Lakatos as the strategem of lemma-incorporation.

Lemma-incorporation is really a fifth response to the discovery of a counterexample to a conjecture. Instead of surrendering, barring the counterexample, making an exception of the counterexample, or treating the counterexample as not really being a refutation, lemma-incorporation takes the counterexample seriously and accepts it as a refutation of the original conjecture, but not as a refutation of the proof of the conjecture. It then becomes necessary to modify the conjecture, to improve the conjecture, by incorporating into the conjecture a condition which eliminates the counterexample. The function of proof then becomes not a process of establishing the conjecture with certainty, but that of IMPROVING the conjecture. In this view, proofs do not necessarily prove the original conjecture, but IMPROVE this conjecture. The proof does not prove the original conjecture but it can be a proof of the new modified conjecture. In summarizing this method, Lakatos states:

> I hope that now all of you see that proofs, even though they may not PROVE, certainly do help to IMPROVE our conjecture.[1] THE EXCEPTION-BARRERS IMPROVED IT TOO, BUT IMPROVING WAS INDEPENDENT OF PROVING. OUR METHOD IMPROVES BY PROVING. THIS INTRINSIC UNITY BETWEEN THE 'LOGIC OF DISCOVERY' AND THE 'LOGIC OF JUSTIFICATION' IS THE MOST IMPORTANT ASPECT OF THE METHOD OF LEMMA-INCORPORATION.[51]

---

[51] Ibid., p. 134. Lakatos' footnote states: "Hardy, Littlewood, Wilder and Polya seem to have missed this point (see footnote I, p. 125.)."

This is of course the Fallibilist viewpoint, a viewpoint which Lakatos admits is not widely accepted. Most people, including mathematicians, cannot see how it is possible to at one time prove AND refute a conjecture:

> Most mathematicians, because of ingrained heuristical dogmas, are incapable of setting out simultaneously to prove AND refute a conjecture. They would EITHER prove it OR refute it. Moreover, they are particularly incapable of improving conjectures by refuting them if the conjectures happen to be their own. THEY WANT TO IMPROVE THEIR CONJECTURES WITHOUT REFUTATIONS; NEVER BY REDUCING FALSEHOOD BUT BY THE MONOTONOUS INCREASE OF TRUTH; THUS THEY PURGE THE GROWTH OF KNOWLEDGE FROM THE HORROR OF COUNTEREXAMPLES. This is perhaps the background to the approach of the best sort of exceptionbarrers: they START by 'playing for safety', by devising a proof for the 'safe' domain and CONTINUE by submitting it to a thorough critical investigation, testing whether they have made use of each of the imposed conditions. If not, they 'sharpen' or 'generalise' the first modest version of their theorem, i.e., specify the lemmas on which the proof hinges, and incorporate them.[52]

It seems obvious that Lakatos' lemma-incorporation is an improved and refined method of exception-barring. The method of lemma-incorporation, or proof and refutation as Lakatos later calls it, is predicated on the principle of the retransmission of falsity.[53] Lakatos explains this principle in his paper "Infinite Regress and Foundations of Mathematics."

> The basic definitional characteristic of a . . . deductive system is the PRINCIPLE OF RETRANSMISSION OF FALSITY from the 'bottom' to the 'top', from the conclusions to the premises: a counterexample to a conclusion will be a counterexample to at least one of the premises.[54]

With respect to mathematical conjectures and their proofs, the principle

---

[52] Ibid.

[53] Ibid., p. 229.

[54] Lakatos, "IRFM", p. 158.

of retransmission of falsity:

> demands the global counterexample be also local: falsehood should be transmitted from the naive conjecture to the lemmas, from the consequent of the theorem to its antecedent. If a global but not local counterexample violates this principle, we restore it by adding a suitable lemma to the proof-analysis. The Principle of Retransmission of Falsity is therefore a REGULATIVE PRINCIPLE for proof-analysis in statu nascendi, and a global but not local counterexample is a fermenting agent in the growth of proof-analysis.[55]

The implication of this statement is that refutation and not corroboration leads to the growth of knowledge. "Refutation makes us learn," according to Lakatos, while "corroboration makes us forget."[56] Corroboration leads to uncritical acceptance of conjectures, of theorems, and of theories. Refutation, on the other hand, sustains suspicion and focuses attention on seemingly self-evident truths.[57]

Refutation, then, stretches and expands concepts. In opposition to this, monsterbarring keeps concepts invariant; monsterbarring cannot foster the growth of knowledge. But it does function to show how refutation expands concepts. From this point of view, monsterbarring is important since it makes it possible to see how and why concepts grow.

In summary of this pattern of conjecture origination and expansion Lakatos writes:

> The impact of proofs and refutations on naive concepts is. . . (that) they ERASE the crucial naive concepts completely and REPLACE them by proof-generated concepts.[58]

---

[55]Lakatos, op. cit., p. 226.

[56]Lakatos, "IRFM", p. 161.

[57]Lakatos illustrates this result with respect to Euclid's geometry and Newton's mechanics and theory of gravitation in a footnote to "Proofs and Refutations", p. 228.

[58]Lakatos, P & R, p. 320.

OR

NAIVE CONJECTURES ARE SUPERCEDED BY IMPROVED CONJECTURES (THEOREMS) AND CONCEPTS (PROOF-GENERATED OR THEORETICAL CONCEPTS) GROWING OUT OF THE METHOD OF PROOFS AND REFUTATIONS.[59]

But what of the other pattern of conjecture origination? How else can knowledge grow if not by naive guessing followed by refutation spurring redefinition and growth of naive conjectures? Lakatos contends there is another pattern which he illustrates again with respect to the problem of the relation among the vertices, edges and faces of polyhedra.

Deductive Guessing. In this pattern, the starting point is an IDEA, not data or facts. In the case of the polyhedra problem, the idea is that for polygon, $V = E$. But in the case of polyhedra, $V \neq E$. The problem then is to determine where the relationship $V = E$ broke down in switching from polygons to polyhedra.

From the basic idea which is taken as given, a series of deductive steps follow which end eventually in a conjecture. But there is an important difference with this conjecture, for this conjecture has a proof built into its origination; that is, since the conjecture is originated by deductive means the conjecture is proven while it is being originated. This does not mean that the proof is final for counterexamples still may exist or hidden lemmas may be found. Therefore, even though the conjecture was originated deductively, it still is necessary

[59]Ibid., 322.

to subject the conjecture to tests, to attempted refutations. Indeed, the deductively generated conjecture may only be a creature of history, for it may represent only the finished product of the naive process of proof and refutation generated conjectures. However, Lakatos like Polya states that:

> We certainly have to learn BOTH heuristic patterns: DEDUCTIVE GUESSING is best, but NAIVE GUESSING is better than no guessing at all. But NAIVE GUESSING IS NOT INDUCTION: THERE ARE NO SUCH THINGS AS INDUCTIVE CONJECTURES.[60]

Polya supports this conclusion in his case history of the polyhedra problem when he enumerates at least four conjectures that were made before achieving success in the fourth attempt: (1) F increases with V; (2) E increases with F; (3) E increases with V. Each of these conjectures being refuted, the fourth conjecture was made: F + V increases with E. This proved to be the fruitful conjecture.[61] This sequence is certainly not inductive, but yet Polya concludes that it is inductive. This conclusion does not seem to be justified by the evidence Polya himself presents.

It is of utmost importance to recognize both of the patterns of conjecture origination. Attention should not be focused on the one to the detriment of the other. Polya's books represent a plea for the teaching of naive guessing. This plea is understandable in light of the emphasis on deductive guessing which has pervaded mathematics

---

[60]Ibid., p. 303.

[61]Polya, MPR, Vol. 1, pp. 35-37.

teaching for at least the past half century. That Polya may seem to overstate his case may be justified by the hope that in doing so the pendulum of emphasis would swing back from deductive guessing not to the opposite extreme of naive guessing, but to some position between these extremes.

Polya's contribution is important for another reason; that is his stress on the similarities between mathematical heuristic and scientific heuristic. These heuristics are the same in that both are based on conjectures, proofs and refutations (or, in the case of science, on conjectures, explanations, and refutations). As has been pointed out earlier (see page 79) the difference between these two fields lies primarily in the nature of their potential falsifiers, the counterexamples in the case of mathematics, and the refuting intersubjectively testable experiments in the case of science. Polya's only weakness, according to Lakatos, is that of concluding that mathematics is inductive, a conclusion based on his view of science as being inductive. This point has been noted by Lakatos who states of Polya:

> . . . he (Polya) never questioned that science is inductive, and because of his correct vision of deep analogy between scientific and mathematical heuristic he was led to think that mathematics is also inductive.[62]

Mathematical heuristic, based on a Fallibilistic philosophical basis, then, can follow two basic patterns: naive guessing and deductive

---

[62] Ibid., p. 304. This quote is contained in a footnote on the page noted. Incidently, Lakatos' footnotes form an integral part of the body of the text in this series of articles. Indeed, in many instances they give the historical background and justification of some of his conclusions.

guessing. Naive guessing proceeds from a problem to a conjecture and then to proving and refutation phases. Deductive guessing begins also from a problem proceeds to a deductive proof and then a conjecture followed by testing and refutation. In both patterns, refutation encourages and fosters growth. Corroboration leads only to over-confidence and a dimming of the light of criticism. Refutation increases the voltage of the current of criticism thereby focusing concentrated light on the weaknesses of present knowledge. The illumination of such weaknesses promotes revision, expansion, and modification of the existent state of knowledge, and, as a result, spurs the growth of mathematical knowledge.

It is only when something goes wrong, when a theory cannot account for some occurrence, that the basic assumptions of the system are called into question. As a consequence, refutation encourages re-examination of basic concepts and consequently concept redefinition and expansion. It can be argued that in mathematics this need not be the only case; growth can be achieved by simply considering alternate axiomatic structures. Nevertheless, refutation would spur the consideration of such alternatives whereas corroboration tends to diminish the necessity for the active search for alternate structures. If corroboration does spur growth, it may be as the result of boredom with the existing structures.

IV. THE MODE OF INQUIRY OF MATHEMATICS

The mode of inquiry of mathematics from a Fallibilistic viewpoint may be seen now to consist of three distinct though interrelated phases.

The origination phase is concerned with the generation of mathematical conjectures. The proving phase of mathematical inquiry is designed to decompose the original conjecture into subconjectures or lemmas in order to improve the possibilities of testing the conjecture. Here proof is not taken as a means of obtaining an absolutely certain conjecture, but rather as a means of exposing, expanding and defining the conjecture and lemmas so that they are more susceptible to criticism. A third phase of mathematical inquiry is that of the testing of a conjecture. This phase is designed to expose the weaknesses and limitations of the original conjecture and its proof-analysis or system of subconjectures. The goal here is to attempt to refute either the original conjecture or its proof; the goal is not the corroboration of the conjecture, for as was mentioned previously, Popper and Lakatos contend that only refutation can lead to a growth of knowledge, mathematical or otherwise, whereas corroboration leads to stagnation.

The goal of this section is to develop an over-all model of the mode of inquiry of mathematics. Consequently, the purpose is to describe the relationship among the three previously identified phases of mathematical inquiry. Furthermore, not only is it desirable to describe the interrelationships among these phases, but if an order of precedence--patterns of how and why one phase of inquiry precedes another phase--could be identified, then this model would have implications for how mathematical learning activities should be structured. The work of Polya and Lakatos described earlier is utilized in this section to describe the operation of a <u>particular</u> phase of mathematical inquiry. Indeed, Polya and Lakatos do not attempt to describe an overall pattern of mathematical inquiry. In this section, the investigator goes beyond the description of a particular phase of mathematical inquiry in order to make explicit the relationship among the phases of inquiry in mathematics and to describe some orders of precedence for these phases. Accordingly, the interrelationship between Polya's patterns of origination and those of Lakatos are discussed as are similar relationships in the testing phase of mathematical inquiry. In addition, the mode of inquiry developed depicts these relationships and the order of precedence existent among these phases. The description of the relationships and the orders of precedence have not been dealt with previously by either Polya or Lakatos.

Because two patterns of guessing can be and have been identified, two general patterns of mathematical heuristic--two orders of precedence--may be identified, namely, a naive heuristic and a deductive heuristic.

Origination Phase

Two basic patterns of origination have been identified. These two patterns should not be conceived of as being separate entities. There is a constant switching from the one to the other depending on the applicability of a particular pattern in a particular situation.

The first pattern of origination has been called naive origination; the second pattern may be termed deductive origination. Both types of conjecturing begin with a problem.

Naive Origination. Starting from a problem, naive origination proceeds directly to a conjecture. In such a process many conjectures may be put forth and quickly refuted before a satisfactory conjecture is obtained (for example, Polya's four conjectures concerning polyhedra). Conjecturing in this situation is a guessing and testing phase until a conjecture is discovered which appears to be a satisfactory solution of the problem, at least temporarily satisfactory.

In attempting to generate a conjecture by naive processes, the two strategies of analogy and specialization identified by Polya may be utilized. Analogy requires that similar problems and solutions to the one under consideration be studied with the desire of possibly using such similarities as guide posts to the origination of a new conjecture. Specialization, on the other hand, requires a limiting of the problem under consideration in an attempt to solve the limited problem, but from whose solution conjectures with respect to the main problem could be generated. These two strategies cannot guarantee that a fruitful hypothesis will be generated. These two strategies are heuristic patterns which may be helpful and fruitful, but they are certainly not guaranteed to be successful. As naive origination moves from a problem to a conjecture, this movement might possibly be aided by analogy and/or specialization. Regardless of whether this movement is aided by these strategies, naive origination remains a guessing process. Analogy and specialization can only serve to perhaps aid one in obtaining conjectures which might be somewhat more rational as opposed to simply wild guessing. However, a guess is usually called wild only if it fails; if it succeeds, the guess is usually called a daring one.

Not all conjectures are the product of naive origination. Some conjectures are obtained deductively.

Deductive Origination. The distinguishing characteristic of a deductively originated conjecture is that the conjecture comes with a built-in proof-analysis. In this process, according to Lakatos, the starting point is again a problem, but rather than moving directly to a conjecture, the next step or series of steps is a deductive sequence of conclusions which finally terminates in a conjecture. The deductive sequence of conclusions constitutes the proof-analysis of the deductively obtained conjecture. In order to make this process work, the originator must after identifying the problem begin with an idea (for example, the idea that $V = E$ for polygons). From this idea the mathematician proceeds deductively utilizing deductive patterns such as modus ponens to eventually arrive at a fully developed conjecture. As an example of this deductive process, the deductive sequence of steps enumerated by Lakatos for the polyhedra problem is given below. The speaker is Alpha, one of the students in Lakatos' hypothetical class:

> I have a different point. We started from
> (1) one vertex is one vertex.
> We deduced from this
> (2) $V = E$ for all perfect polygons.
> We deduced from this
> (3) $V - E + F = 1$ for all normal open polygonal systems.
> From this
> (4) $V - E + F = 2$ for all normal closed polygonal systems, i. e., polyhedra.

From this again in turn

(5) $V - E + F = 2 - 2(n - 1)$ for normal n-spheroid polyhedra.

(6) $V - E + F = 2 - 2(n - 1) + \sum_{k=1}^{F} e_k$ for normal n-spheroid polyhedra with multiply-connected faces.

(7) $V - E + F = \sum_{j=1}^{K} \left\{ 2 - 2(n - 1) + \sum_{n=1}^{F} e_k \right\}$ for normal n-spheroid polyhedra with multiply-connected faces and with cavities.[63]

The above development provides an illustration of how deductively originated conjectures generates an ever-increasing or expanding concept to polyhedra. It must be noted, however, that this development may have first been preceded by a naive generation process because at each stage from step four on, the new expansion of the concept of polyhedra was necessitated by a refutation, a counterexample, to the previous generalization. Furthermore, a counterexample may still exist to this final conjecture or hidden lemmas and assumptions may yet be uncovered in one of the steps. Hence the need of testing is not diminished in this process, but is probably heightened due to the fact that the derivation looks too perfect. The example also illustrates the general mathematical strategy of reducing a problem to a simpler version and then expanding a proposed solution to the larger problem.

The two patterns of origination just described may be depicted as shown below in Figure 5. Both patterns begin with a problem and end with a conjecture. The intermediate steps in naive origination are the strategies of analogy and specialization identified by Polya. In this

[63] Ibid., p. 311.

pattern, the origination of the conjecture is characterized by a guess

Problem	Problem
↓ (1) Analogy	↓ (1) Idea
(2) Specialization	(2) Proof-analysis
Conjecture	Conjecture
Naive Origination	Deductive Origination

FIGURE 5

TWO PATTERNS OF CONJECTURE ORIGINATION

and test process--by plausible reasoning. The intermediate steps in deductive origination are the conception of a beginning idea followed by the construction of a proof-analysis, a process described by Lakatos. Deductive origination as its name implies is characterized by deductive patterns of reasoning, principally modus ponens.

Testing Phase

The second identifiable phase of the logic of mathematical discovery may be called the testing of proposed conjectures. The testing phase follows the origination phase regardless of whether the conjecture is naively generated or deductively generated. However, it is possible in the case of a naively originated conjecture that the testing phase may be preceded by a proof-analysis; that is, by a decomposition of the original conjecture into a system of subconjectures. This does not alter the fact, however, that testing always follows origination though perhaps it is not the next immediate step.

From a plausible reasoning point of view, Polya's viewpoint, the testing phase is designed to determine if the conjecture is one worthy of further study and consideration. Three of Polya's patterns of plausible inference have been described and illustrated. Each of these, examining a consequence, examining a possible ground, and the examination of a conflicting conjecture, are propogated as means of determining the credibility of the conjecture. The Fallibilistic restrictions on the third plausible reasoning pattern have been noted; a conflicting conjecture only increases its credibility if it has been subjected to severe tests.

Even though Polya identifies these three strategies as plausible reasoning patterns they only become such if the test of the conjecture fails; that is to say, they are patterns which attempt to describe the change in the credibility of a conjecture after the conjecture has been tested and the test has failed. Hence, these patterns of plausible inference arise as the <u>result</u> of a conjecture and attempted refutation procedure. In this sense they are 'after-the-fact' type of patterns, patterns which only come into play if an honest attempt to refute a conjecture is not successful. Even then, they can only indicate the direction of evidence and not the strength of evidence with respect to the conjecture under consideration.

As a consequence, the deductive testing of a conjecture, which functions via the Principle of Retransmission of Falsity, and the plausible point of view are not in competition. If a test of some conjecture is successful, that is, the conjecture is refuted, then the retransmission of falsity principle applies. If the test is not successful, then Polya's patterns come into operation. But as has been noted earlier (see page 115) plausible reasoning patterns which are corroborative in nature do not foster the growth of knowledge. Only refutation spurs the growth of knowledge, for refutation demands a renewed attack on the problem--corroboration does not.

Therefore, attempted refutation precedes corroboration in order of precedence. Corroboration only follows if attempted refutations fail. The corroboration of a conjecture only serves to indicate that it is worthy of <u>further</u> testing.

In the case of a deductively generated conjecture the testing phase serves an additional purpose. Testing in this situation is designed to uncover weaknesses in the proof-analysis, to discover hidden lemmas and to foster the emergence of counterexamples. The greatest danger for the growth of mathematical knowledge in the case of a deductively obtained conjecture is that the proof-analysis will not be subjected to severe criticism. If this happens then mathematical knowledge stagnates for it is assumed that the problem for which the conjecture is a solution does <u>in fact</u> solve the problem. This may not be the case.

Figure 6 diagrammatically describes the testing phase of mathematical inquiry. The primary goal of the testing phase of mathematical inquiry is refutation. Failing that, plausible reasoning patterns are identified which indicate the credibility of the conjecture being examined.

Testing Phase

Conjecture--------------------------------Corroboration

(1) Attempted refutation

(2) Examining a consequence

(3) Examining a possible ground

(4) Examining a conflicting conjecture

FIGURE 6

TESTING PHASE OF MATHEMATICAL INQUIRY

Proving Phase

The third identifiable phase of mathematical inquiry is that of the construction of a proof of some conjecture previously obtained. The placement of the proving phase in an order of precedence, an order that indicates which phase is preceded by what other phase, depends on the particular pattern of origination that is utilized in obtaining the conjecture. If deductive origination is utilized to obtain the conjecture, then the proving phase becomes part of the origination phase. On the other hand, if the conjecture is obtained by naive processes, then the proving phase follows origination although not necessarily immediately because the testing phase may intervene. The testing phase would intervene if the mathematician is not able to immediately construct a proof-analysis. Hence it is not possible to establish an order of precedence for the three phases of mathematical inquiry unless the overall heuristic pattern being utilized is identified.

It seems somewhat contradictory to contend that the proving phase is concerned with both the proving AND refuting of a conjecture. Nevertheless, this is indeed the case if the two definitions of proof given earlier (see page 114ff.) are kept in mind, especially the definition of proof as the decomposition of the conjecture into subconjectures. The goal of this decomposition is to provide a larger target, so to speak, for criticism and refutation. From this orientation, proof becomes a concept expansion process--a growth process. In such a process, concepts are proof-generated; that is, concepts are expanded, redefined and modified under the influence of proof generation which encounters counterexamples. Hence, counterexamples rather than hindering progress encourage growth and expansion of knowledge.

Three techniques drawn from Lakatos' work of dealing with counterexamples have been described: monsterbarring, exception-barring, and lemma-incorporation. A fourth technique, monster-adjustment, is dismissed as being a dogmatic defense mechanism which does not foster the growth of knowledge. The usefulness of the monsterbarring technique is that it identifies why and how concepts grow. Consequently, it has mainly a historical function. This function, however, should not be underestimated in the teaching and learning of mathematics. Moreover, even in purely mathematical research, the classification of counterexamples as monsters can serve to sharpen the mathematicians awareness of and possible extension of his original conjecture.

The exception-barring and lemma-incorporation techniques serve basically the same purpose although in different ways. Both procedures attempt to expand the original conjecture and the concepts with which the conjecture deals. Exception-barring does this without a consideration of the proof-analysis whereas lemma-incorporation does not. As a result, lemma-incorporation is a refinement of, an improvement of, the exception-barring technique. Both methods of dealing with counterexamples treat the counterexamples seriously. Exception-barring responds by <u>trying</u> to retreat to safe ground, while lemma-incorporation includes the offending counterexample in the proof-analysis and hence renders the conjecture even more risky--better testable. Rather than trying to increase the logical probability of the conjecture, lemma-incorporation attempts to decrease the logical probability of the conjecture thereby increasing its testability and degree of corroboration.

In Figure 7 below, the proving phase of mathematical inquiry is shown with note being taken of the techniques utilized in dealing with counterexamples.

Proving Phase

Conjecture--Proof-analysis

(1) Monsterbarring

(2) Exception-barring

(3) Lemma-incorporation

FIGURE 7

PROVING PHASE OF MATHEMATICAL INQUIRY

The proving phase of the mode of inquiry of mathematics is concerned with the construction of a proof-analysis, or, to put it another way, with the decomposition of the original conjecture into a series of subconjectures. The latter two techniques of dealing with counterexamples are fermenting agents for the growth of mathematical knowledge, whereas the first technique serves a historical function in the logic of discovery in mathematics.

A Model

It is now possible to bring these three phases together in order to construct a model of the mode of inquiry of mathematics. In doing so it becomes clear that there are basically two heuristic patterns in mathematical inquiry. These two patterns may be called the NAIVE HEURISTIC and the DEDUCTIVE HEURISTIC.

The naive heuristical pattern is characterized by a flow chart which begins with a problem, moves to a naively generated conjecture, followed by a testing phase which in turn leads to the proving phase. The proving phase leads to proof-generated concepts, expanded concepts, and thence to further testing. It is possible in this heuristic pattern that the testing phase may not follow immediately after the origination phase. In this case, origination is followed by the proving phase and concept expansion and then the testing phase. This testing phase may itself generate new problems and the whole cycle is repeated.

The deductive heuristical pattern is characterized by a flow which begins with a problem, proceeds to an idea and the proving phase and only then to a conjecture. The testing phase follows leading to concept expansion and more problems. Both patterns then are seen to begin and end with a problem.

Naive heuristic and deductive heuristic are not isolated patterns. The working mathematician may and probably does utilize both patterns in creating mathematical knowledge. It seems reasonable to conclude that the naive pattern may be utilized MORE in the early stages of the investigation of some mathematical problems whereas the deductive pattern may be utilized in later stages when the mathematician is polishing his results. However, this conclusion is obviously not valid in all circumstances.

If the origination phase is denoted by 'O', the testing phase by 'T', and the proving phase by 'P', then six possible permutations of the processes can be obtained. They are the following:

(1) O - T - P

(2) O - P - T

(3) P - O - T

(4) P - T - O

(5) T - O - P

(6) T - P - O

Of these six cycles, the first two comprise the naive heuristic as described above. The third cycle is the deductive heuristic. The last three cycles are not individually feasible, although they may occur as parts of longer patterns of heuristical sequences. For example, if pattern or cycle one above was followed by cycle two so that the flow of cycles was the following:

$$(O - T - P) - (O - P - T)$$

then a rearrangement of the parentheses would yield cycle six as shown:

$$O - (T - P - O) - P - T.$$

However, even in this case the initial testing phase is preceded by an origination phase. The point is that cycles four to six above could not be the initial cycle in a long pattern of heuristical cycles.

An additional point needs to be made with respect to the levels of universality obtained by heuristic cycles being strung together. It is not contended that the mathematician solves a problem simply by working through one cycle. Rather he probably proceeds from one cycle to another, sometimes using a naive cycle and sometimes a deductive cycle. In this process, it is likely that he is proceeding to ever higher levels of universality in order to obtain results with greater generalizability, but which are riskier because they apply to a larger area of mathematics with the result that it becomes possible to test the generalization over a wider range of application. Consequently, the model of the mode of inquiry depicted in Figure 8 on page 143 has a third dimension in that one cycle follows another and may develop in a spiral fashion in an attempt to attain a higher level of universality. This is a concept expansion process in that a higher level of universality--generalizations with greater scope--is the goal. Furthermore, where long patterns of cycles are concerned it may be the case that each of the various phases act as stimuli for the generation of higher level and riskier hypotheses.

Consequently, three distinct orders of precedence are identifiable. The first two orders (O-T-P and O-P-T) have been combined and designated as the naive heuristic of mathematical inquiry. The third order (P-O-T) has been called the deductive heuristic. Because of the varying orders in which the three phases may occur, it is not possible to describe only one pattern of mathematical inquiry in the fashion Lee has for an order of precedence in science.[64] Nevertheless, an order of precedence can be established if the general heuristical pattern of mathematical is known.

The generation of the two heuristical patterns is based on the identification of the three phases of mathematical inquiry. These phases were derived from the work of Polya and Lakatos. However, the present study goes beyond the mere identification of phases to a point where two orders of precedence have been identified.

Figure 8 shows the two heuristical patterns. The fact that the two circles meet at a point is meant to symbolize the interplay between the two patterns. The dashed line in the naive heuristical pattern is designed to convey the fact that in this pattern the first testing phase may be eliminated with the mathematician moving directly to proof-analysis. The dashed line serves to identify the distinct orders of precedence which exist in the naive heuristic. Only the main phases of inquiry have been identified in Figure 8 as the details of each of these phases are provided in previous sections of the present chapter.

---

[64]Donald S. Lee. Unpublished materials obtained from Dr. Lee in 1963. The materials deal with the depiction of science as an order of precedence.

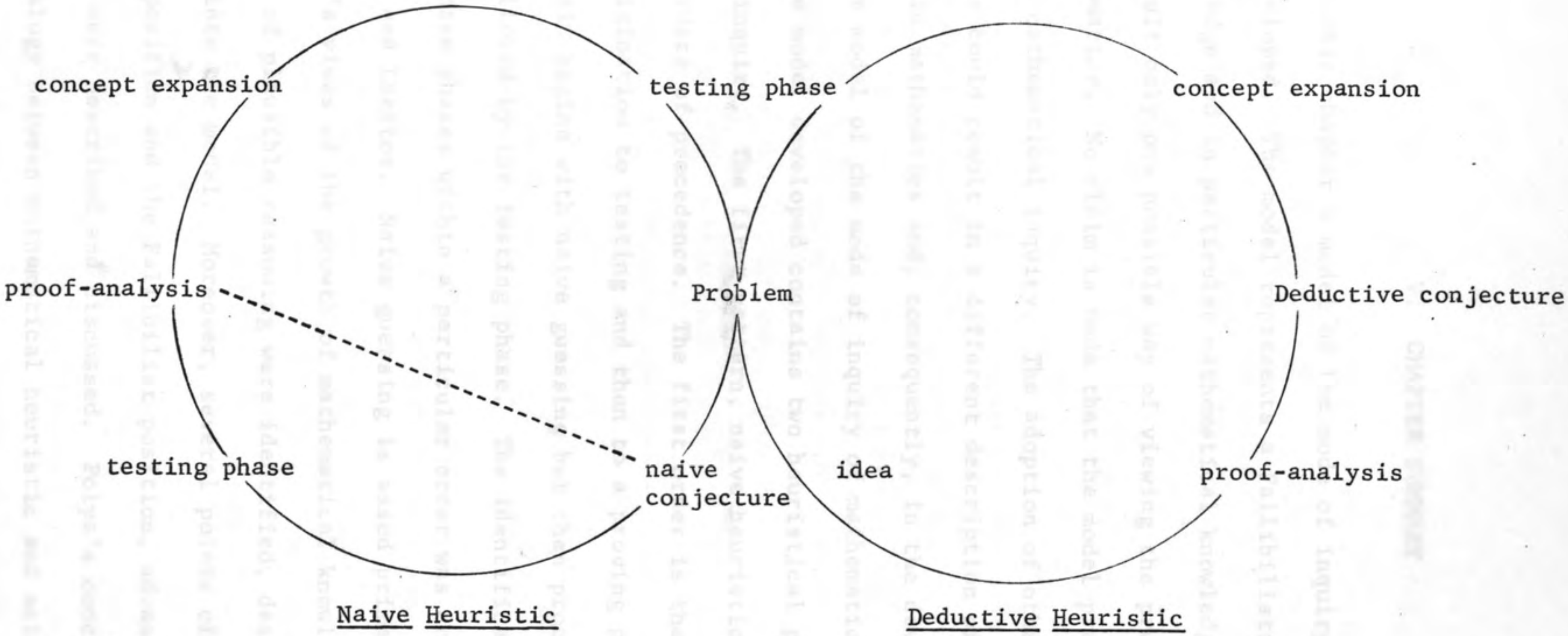

FIGURE 8

THE MODE OF INQUIRY OF MATHEMATICS

V. CHAPTER SUMMARY

In this chapter a model of the mode of inquiry of mathematics has been developed. The model represents a Fallibilistic view of the growth of knowledge and in particular mathematical knowledge. The model is as a result only one possible way of viewing the processes of inquiry in mathematics. No claim is made that the model presented here is <u>the</u> model of mathematical inquiry. The adoption of other philosophical positions could result in a different description of the processes of inquiry in mathematics and, consequently, in the development of a different model of the mode of inquiry of mathematics.

The model developed contains two heuristical patterns of mathematical inquiry. The first pattern, naive heuristic, is characterized by two orders of precedence. The first order is that of moving from naive origination to testing and then to a proving phase. The second order again begins with naive guessing but then proceeds to the proving phase followed by the testing phase. The identification and description of the three phases within a particular order was drawn from the works of Polya and Lakatos. Naive guessing is based primarily on an analysis of Polya's views of the growth of mathematical knowledge. Polya's patterns of plausible reasoning were identified, described and incorporated into the model. Moreover, several points of conflict between Polya's position and the Fallibilist position, advanced by Popper and Lakatos, were described and discussed. Polya's conclusions as to the close analogy between mathematical heuristic and scientific heuristic was noted with the qualification that Polya failed, according to Lakatos, to question the inductive nature of scientific research. This failure provided the basis of the difference between Polya's position and the position held by the Fallibilists.

In the origination phase, the disagreement concerning what is and what is not inductive seems to degenerate to a disagreement over the names attached to ideas. Such a disagreement is of no concern to mathematics educators who are primarily interested in the general patterns of origination and not the names attached to these patterns. In the area of the confirmation of a conjecture, however, the disagreement over induction seems somewhat more serious since it has been argued that Polya's position is contradictory. This does not, however, establish the correctness of the view taken by the Fallibilist. It does raise a possible weakness in Polya's position; a weakness the Fallibilists avoid by simply dismissing induction.

The second pattern of mathematical inquiry is characterized by an order of precedence which begins with an idea followed by a proof-analysis which results in a conjecture. This phase of origination and proof is followed by the testing phase. The phase of deductive origination is based on Popper's and derivatively Lakatos' views of the growth of mathematical knowledge. In their view all knowledge grows by means of proofs and refutations. Moreover, the proving phase is a concept generating and expanding process. Even the origination phase of the naive heuristic is characterized by conjecture and refutation procedures. The difference between the two heuristical patterns depends on the order in which the phases follow one another. Consequently, both patterns incorporate the same phases, but in different orders. The three phases are those of the origination, testing, and proving of a mathematical conjecture.

Consequently, taking the philosophical position of Critical Fallibilism yields a description of mathematical inquiry which characterizes the growth of mathematics as being a conjecture and refutation process. Further, when refutations do appear in the form of counterexamples strategies such as monsterbarring, exception-barring, and lemma-incorporation function as part of the 'situational logic' utilized by mathematicians in dealing with such refutations. If refutations do not appear, and in this view they should be actively sought, then the corroboration patterns identified by Polya serve as part of the 'situational logic'. Even then the original conjecture is actively and aggressively tested in order to identify its weaknesses and possible counterexamples to it.

Finally, the orders of precedence identified and the interrelationships of the phases can now serve as the basis for the construction of a <u>model of instruction</u> for mathematics teaching. The model of inquiry developed in the present chapter has identified phases, orders of precedence, and heuristical patterns which a corresponding Fallibilistic model of instruction should incorporate. The construction, description and illustration of such a model of instruction is undertaken in the next chapter.

CHAPTER V

THE MADISON PROJECT AS A FALLIBILISTIC APPROACH TO THE TEACHING OF MATHEMATICS

I. INTRODUCTION

In the present chapter, the paradigm previously developed is taken as the basis for the derivation of a Fallibilistically oriented model of instruction. In this situation the model of the mode of inquiry of mathematics is taken as given. It is utilized as a guide for the generation of a model of instruction _for_ the teaching of mathematics. Each of the strategies identified in the model of instruction are illustrated by means of hypothetical classroom situations. These illustrations serve the dual function of explaining a particular teaching stratagem or phase of a stratagem as well as illustrating how teachers and students could come to operate Fallibilistically. Consequently, the second section of the chapter develops, explains and illustrates a model of instruction derived from a Fallibilistic viewpoint.

Since the illustrations of the Fallibilistic approach are hypothetical and as a consequence have not been explicitly utilized in actual classrooms, a currently active curriculum project is examined and appraised as being an example of a Fallibilistic approach to the teaching of mathematics. The Madison Project is chosen for this purpose. The question may be posed as to why the materials of the

CHAPTER V

THE MADISON PROJECT AS A FALLIBILISTIC APPROACH TO THE TEACHING OF MATHEMATICS

I. INTRODUCTION

In the present chapter, the paradigm previously developed is taken as the basis for the derivation of a Fallibilistically oriented model of instruction. In this situation the model of the mode of inquiry of mathematics is taken as given. It is utilized as a guide for the generation of a model of instruction <u>for</u> the teaching of mathematics. Each of the strategies identified in the model of instruction are illustrated by means of hypothetical classroom situations. These illustrations serve the dual function of explaining a particular teaching stratagem or phase of a stratagem as well as illustrating how teachers and students could come to operate Fallibilistically. Consequently, the second section of the chapter develops, explains and illustrates a model of instruction derived from a Fallibilistic viewpoint.

Since the illustrations of the Fallibilistic approach are hypothetical and as a consequence have not been explicitly utilized in actual classrooms, a currently active curriculum project is examined and appraised as being an example of a Fallibilistic approach to the teaching of mathematics. The Madison Project is chosen for this purpose. The question may be posed as to why the materials of the

Madison Project are chosen for examination? Why not the materials produced by some of the other curriculum projects in mathematics such as the School Mathematics Study Group (SMSG), the University of Illinois Arithmetic Project (UIAP), or the University of Illinois Committee on School Mathematics (UICSM)? The answer to these questions stems from two considerations: (1) whether the project focused on teaching strategies, and (2) whether the project materials encompassed the K to 9 or 10 grade range.

The SMSG project, for example, states its objectives as the fostering of ". . . research and development in the teaching of school mathematics."[1] The primary instructional procedure followed by this project is "normal classroom procedures."[2] Consequently, this project is not concerned with instructional procedures which could be classified as Fallibilistic in nature. The UIAP program, on the other hand, is concerned with instructional strategies: ". . . the project seeks novel ways of doing old mathematics . . . ."[3] The difficulty with this project from the point of view of the present study is that the UIAP does not have a systematic approach to teaching strategies and the fact that their materials only cover the range from kindergarten to grade six.[4] In contrast to this project, the UICSM program focuses on the senior

---

[1] J. David Lockard, editor, <u>Sixth Report of the International Clearinghouse on Science and Mathematics Curricular Developments</u>, 1968. A Joint Project of the American Association for the Advancement of Science and the Science Teaching Center, University of Maryland. P. 321.

[2] <u>Ibid</u>.

[3] <u>Ibid</u>., p. 374.

[4] <u>Ibid</u>., p. 375.

high school mathematics curriculum. Although this project is concerned with teaching strategies--the discovery approach--their published materials only cover the grade nine to twelve range.[5] Consequently, the SMSG project, the UIAP program and the UICSM program were not selected since they did not meet the requirements deemed essential for the purposes of the present study. The Madison Project was chosen because it could meet these requirements. An examination of the Madison Project objectives and an appraisal of the teaching strategies developed by the Madison Project is given in section three of the present chapter. Suffice it here to say that the Madison Project is concerned with instructional strategies and the materials published by the Project cover the range from kindergarten to at least grade nine thereby encompassing both the elementary and secondary school mathematics curriculum.

Section three of the chapter is divided into two subsections. The first subsection examines the stated goals and objectives of the Madison Project in order to determine in what ways these objectives exhibit Fallibilistic tendencies. Both goals for the curriculum and goals for students are examined and analysed. The second subsection describes and appraises the instructional strategies developed by the Madison Project. In doing so, the model of instruction developed in section two of the chapter is utilized in an attempt to determine in which ways the Madison Project adopts a Fallibilistic approach to the teaching of mathematics.

---

[5] Ibid., p. 378.

II. A FALLIBILISTIC MODEL OF INSTRUCTION

The purpose of this section is to develop, explain, and provide illustrations of the use of a Fallibilistic model of instruction--a Fallibilistic approach to the teaching of mathematics. The components of the model to be developed are derived from the model of inquiry presented in the previous chapter. Accordingly, the model of instruction should exhibit the two heuristic patterns and the three orders of precedence identified earlier. Moreover, the model should explain the relationship of the phases within a particular order of precedence and, in addition, provide illustrations of how strategies of teaching within particular phases could become an operational reality in the classroom. In this way it would be hoped that the model of instruction to be developed would aid teachers in functioning Fallibilistically as well as assisting them in creating Fallibilistic learning situations in the mathematics classrooms, situations in which the children could also operate Fallibilistically. The illustrations of the use of a Fallibilistic approach to the teaching of mathematics are hypothetical classroom situations created by the investigator. As such they are suggestive of how students and teachers might create a Fallibilistic climate in the classroom. Before these illustrations may be considered, however, it is necessary to have a frame of reference, a model, for the Fallibilistic approach to the teaching of mathematics. Such a model is developed in the next subsection.

A Model for Instruction

The model of the Fallibilistic mode of inquiry of mathematics developed in the previous chapter is characterized by two heuristical patterns within which there are three orders of precedence. It would seem to follow then that a model for instruction based on Fallibilism should encompass the same components in terms not only of heuristical patterns and orders of precedence, but also in terms of the phases within a particular pattern and order. This would imply that the model for instruction should be composed of two instructional patterns, three stratagems of teaching, and the three phases within each of these stratagems. Because three orders of precedence have been found to exist in a Fallibilistic orientation to the description of the mode of inquiry of mathematics, there should be corresponding to each of these orders a strategy of teaching, a pattern of approach, to the creation and implementation of mathematics learning situations in the classroom. In other words, from a Fallibilistic point of view the mathematics teacher has basically three ways of approaching the teaching of some mathematical topic. Two of these strategies are derived from the naive heuristic; the third strategy is derived from the deductive heuristic. The former strategy is called, for ease of reference, the naive stratagem of teaching while the latter is called the deductive stratagem of teaching, denoted by DED. Within the naive stratagem, two distinct strategies are extant; the testing-proving strategy, denoted by TP, and the proving-testing strategy, denoted by PT.

The TP strategy is characterized by an order of precedence in which the naive guessing phase is followed by a testing phase and subsequently the proving phase. The difference between this strategy and the PT strategy is in the order of the occurrence of the testing and proving phases. In the PT strategy, the proving phase preceded the testing phase and, as a consequence, the testing phase serves to uncover hidden lemmas and counterexamples to the proof-analysis created in the proving phase. The DED strategy is characterized by an order of precedence in which the proving and origination phases proceed simultaneously. The testing phase follows these two phases. Again, the testing phase serves to critically examine the proof-analysis created by means of deductive origination.

The distinctions among these three strategies of teaching or models for instruction stems from the order in which particular phases of inquiry are utilized. The distinction does not arise from basic difference among the phases themselves. One exception does exist, however, for there is a difference between origination in the naive stratagem and origination in the deductive stratagem. In the former stratagem, hypothesis generation is a guessing and testing procedure whereas in the latter stratagem hypothesis generation is a deductive procedure. This distinction provides the basis for the identification of two separate heuristical patterns in mathematical inquiry and, derivatively, the two teaching stratagems in the model for instruction.

Figure 9 depicts a Fallibilistic model for instruction in skeleton form identifying the two stratagems of teaching, naive and deductive, and the two strategies within the naive stratagem, namely

the TP strategy and the PT strategy.

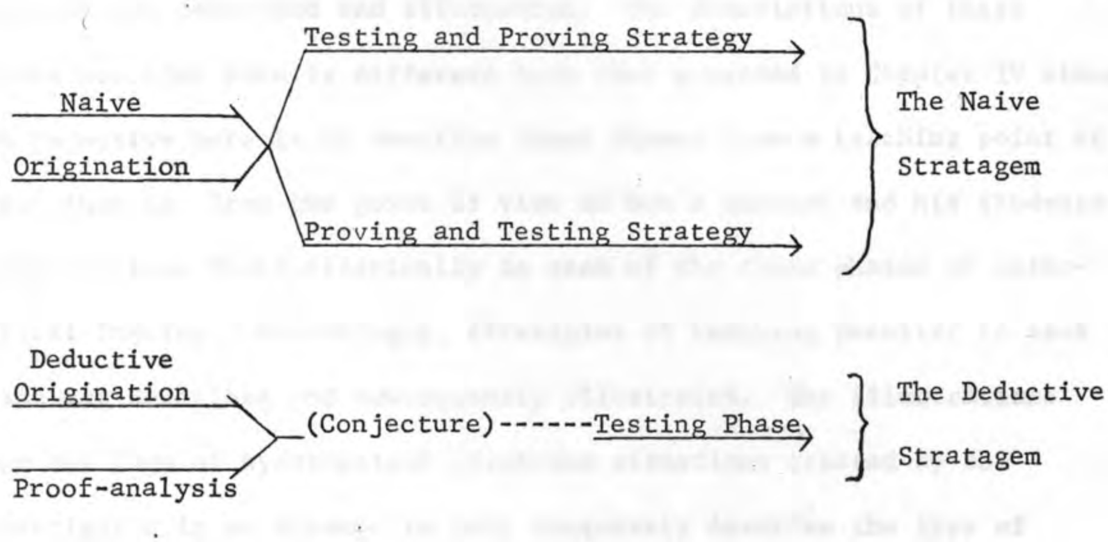

FIGURE 9

A FALLIBILISTIC MODEL FOR INSTRUCTION

Figure 9 is a paradigm of the three strategies of teaching derivable from the model of the mode of inquiry of mathematics given in Chapter IV. As such it depicts the over-all strategies of a Fallibilistic model for instruction. However, only the outlines of these strategies are present in the model. The next subsection provides the details for each phase of the three strategies along with illustrations of how such strategies for the teaching of mathematics might be made operational in the classroom.

The Phases of Fallibilistic Teaching Strategies

This subsection is designed to provide the details of the Fallibilistic strategies of teaching described in outline form above. Consequently, each of the phases within the particular strategies (TP, PT, and DED) are described and illustrated. The descriptions of these phases provided here is different from that provided in Chapter IV since the objective here is to describe these phases from a teaching point of view; that is, from the point of view of how a teacher and his students would function Fallibilistically in each of the three phases of mathematical inquiry. Accordingly, strategies of teaching peculiar to each phase are described and subsequently illustrated. The illustrations take the form of hypothetical classroom situations created by the investigator in an attempt to more adequately describe the type of classroom climate necessary for teachers and students to function Fallibilistically. The situations are designed to exemplify the characteristics of the particular phase of mathematical inquiry being discussed. Despite the hypothetical nature of these illustrations, they do have some basis in reality as they represent variants of situations the researcher has seen in use or has constructed in actual classrooms with children at the grade levels specified.

Origination Strategies. Two stratagems of teaching mathematics based on Fallibilism have been identified: the naive stratagem and the deductive stratagem. The first phase of the naive teaching stratagem is that of naive guessing. This phase precedes the testing and proving phases regardless of whether the TP or the PT strategy is to be utilized. As has been noted earlier (see page 120 ff.), naive guessing is a conjecture process--a guessing process--followed by the testing of this conjecture; that is, within the origination phase itself guessing and testing procedures are evident. This guessing and testing procedure has as its objective the creation of a plausible conjecture which can then be subjected to more severe testing and possibly proof. If within this origination process a counterexample is produced to a conjecture, then attempts are made to improve the conjecture in order to account for the counterexample. This is the strategy of lemma-incorporation. Once a viable conjecture is produced, a conjecture for which no obvious counterexamples exist, then the stage is set for proceeding to either the TP strategy or the PT strategy. If the next phase is one of attempting to provide a proof-analysis for the conjecture, then the PT strategy is being utilized. If, however, the conjecture is then subjected to severe tests, then the TP strategy is operative.

A teaching strategy designed to provide opportunities for students and teachers to operate using naive guessing as a pattern should establish a situation which first presents the student with a problem, a problem about which the student can make a conjecture as to its solution. Second, such a strategy must also provide the means by which each conjecture may be tested, preferably independently of an authority figure such as the teacher. The students must be allowed to guess the solution, attempt to discover a pattern, and to evaluate their proposed solutions in a climate which does not penalize what may seem to be wild guesses.

It may be contended that to allow students to guess wildly in this way would be disruptive of the classroom situation making classroom management difficult. However, this need not be the case, <u>if</u> the students are provided with means to test their guesses. The students themselves may provide the testing procedures or these procedures may be provided by the teacher. In either case, if such procedures are available to the student, he is not as likely to guess in a vacuum, as it were, but he will make his guesses in a way which seems appropriate to the testing procedures. For example, if the student was attempting to find the roots for the quadratic equation $(\square \times \square) - (5 \times \square) - 6 = 0$ it is not likely he would guess 157 as one of the roots if he realizes he has to test this guess on his own by substitution. Accordingly, the testing procedures can function as a control mechanism, a mechanism which does not involve the teacher in a disciplinary role.

If the teacher <u>is</u> actively involved in the learning situation, his role is one of providing guidance devoid of value judgments as to the fruitfulness of a student's guess or conjecture. Students should be encouraged to be critical of their own and their classmates conjectures. Such a classroom climate, that of rational criticism, is designed to foster the growth of mathematical knowledge in a Fallibilistic fashion, an approach which contends that knowledge grows by a conjecture and refutation process.

For example, a student's first acquaintance with quadratic equations may be so structured that opportunities are provided for students to guess the coefficient rules; that is, that the product of the two roots is the coefficient of the x^o term, and the sum of the two roots is the additive inverse of the absolute value of the coefficient of the x^1 term.

These opportunities could be provided in such a way that students are able to test their guesses independently of the teacher.[6] In this situation, the students confront the mathematical concepts directly and not through the teacher. A Fallibilistic approach on the part of students requires that as far as is possible the student should interact with mathematical ideas and his peer group rather than the teacher. The teacher's role in this situation is to set the problem, provide the means for students to test their guesses independently of the teacher, and to guide the discussion in a 'light-handed' fashion--the teacher uses the light touch in Davis' terminology.[7]

As an illustration of naive origination consider the following hypothetical classroom discussion which illustrates the strategy of naive guessing a solution to a problem. The problem is one of discovering the pattern utilized to generate the following sequence of numbers: 4, 16, 37, ... The class is a hypothetical group of grade ten students who would perhaps be classified as advanced in their creative abilities in mathematics. The teacher opens the discussion by posing the problem to be solved:

Teacher: I would like you to try to determine the rule I am using to generate the following sequence of numbers: 4, 16, 37, ... Can anyone guess what the next number in the sequence would be and hence

---

[6]Robert B. Davis, Explorations in Mathematics (Addison-Wesley Publishing Company, Don Mills, Ontario, 1967), pp. 112-114.

[7]Ibid., p. 4

how the sequence is obtained?

Doug: I think the next number is 1369.

Scott: That can't work, because 16 squared isn't 37. (Scott realizes that Doug looked only at the first two numbers in the sequence, guessed the squaring hypothesis, and then squared 37 to obtain 1369.)

Scott: Is the next number 49? (Scott seems to have focused on the difference between 16 and 4, namely 12, and the difference between 37 and 16, namely 21. He is guessing that perhaps these differences alternate.)

Teacher: No, the number is not 49. I'll tell you the next number in the sequence. It is 58. Does that help? (The sequence is now 4, 16, 37, 58, ...)

Susie: Well, the differences now are 12, 21, and 21. Is the next number either--let's see--70 or 79? (Susie is guessing naively that there may be a pattern of differences which is either 12, 21, 21, 12 or 12, 21, 21, 21.)

Teacher: No, I'm sorry but neither of those are the next number in the sequence. I'll give you the next number in the sequence--it is 89. (General puzzlement follows. The sequence is now 4, 16, 37, 58, and 89. The teacher by giving additional numbers is attempting to provide a wider basis on which the students can test their guesses. As a result of the addition of this last number, Susie's conjecture seems to be refuted.)

Jeff: Does it (the pattern) have anything to do with the squaring of the numbers? (Jeff is looking for patterns not just numbers. The actual numbers serve only to <u>test</u> the pattern and the pattern is the real conjecture.)

Teacher: Perhaps. (He's not too helpful.)

Susie: Is the next number 120 or 102? (Susie has many conjectures, but she remains focused on 'differences' between numbers in the sequence. In the first case, she guesses the sequence of differences to be 12, 21, 21, 31, 31, and in the second case that it might be 12, 21, 21, 31, 13.)

Teacher: No neither of those is the next number.[8]

Several things should be noted in this illustration. First, the conjectures put forth by the students are naive; that is, they are guesses which are not deductively obtained, but rather which seem plausible on the basis of the sequence of numbers. Second, the students are able to test their conjecture by using their proposed 'rule' to determine the next number in the sequence. They then ask the teacher for refutation or corroboration. Note also that this refutation is devoid of personal criticism of the merit of the student's guess. Moreover, there is no inductive leap here, but rather a series of guesses and tests.

Note also that in order to solve this problem, the students would be required to proceed to a higher level of complexity; that is, there is not a linear relationship between the elements of the sequence, but rather a higher level relationship. It is true that the next number in

---

[8]The next number in the sequence, for those readers who have not guessed it themselves, is obtained by summing the squares of the digits of the last given number in the sequence.

the sequence is obtained by addition, but only after the digits have been squared. Hence, this requires a higher level of generalization than a strictly linear relationship.

The role of the teacher is to set the problem and then to inform the students if their guesses as to the next number in the sequence is correct or not. This latter function is performed by the teacher without penalty or praise in order that the students may interact with the mathematical problem without attempting to conform to some preconceived behavior patterns established by the teacher.

The other major stratagem identified was the deductive stratagem (DED). In this stratagem, deductive guessing is utilized as a means of generating conjectures. Deductive guessing begins with an idea from which there flows a series of deductive steps culminating with a conjecture. The difference between the conjecture obtained in this instance and the conjecture arrived at by naive guessing techniques is that the former type of origination comes with a built-in proof-analysis. In other words, the phases of proof and origination proceed simultaneously in this stratagem with the proof-analysis which is created giving rise to a deductively generated hypothesis.

A model or pattern of teaching based on deductive guessing should include, first, a basic idea from which a solution can be derived to the problem under consideration. Second, the proof-analysis must be developed which eventuates in a conjecture, a conjecture which is 'proven' in the course of its origination. Finally, the conjecture and proof-analysis must be subjected to severe tests in order to uncover hidden lemmas or assumptions, faulty deductive sequences, and limiting cases to the conjecture or its proof. The proof in this situation is, of course, a proof-analysis, a system of lemmas or sub-conjectures as described in Chapter IV.

The particular deductive patterns used in creating the proof-analysis in this phase are described elsewhere.[9] Among these patterns are the strategies of modus ponens and proof by contradiction both of which are or should be well known to the secondary school teacher. The particular strategy of proof utilized within the deductive guessing phase is not crucial, although students must be acquainted with all of these methods of attacking the proof of some conjectures. It should be noted, however, that in the origination phase where a conjecture is being generated the strategy of proof by contradiction is not appropriate since the utilization of this strategy depends on the existence of a conjecture to be proved. This strategy can only be used when the mathematician or student has a conjecture he wishes to prove. Consequently, this strategy is appropriate to the proving phase of mathematical inquiry but not the deductive guessing phase. Modus ponens remains the main deductive pattern to be used in attempting to generate a conjecture deductively.

As a result, deductive origination flows in a direct fashion from some basic idea to a conjecture. A hypothetical classroom situation

---

[9]See, for example, The Growth of Mathematical Ideas (Twenty-fourth Yearbook of the National Council of Teachers of Mathematics, New York, 1959), chapters five and nine.

which illustrates this strategy of origination is presented next. In this illustration, the conjecture ultimately obtained and proven is that the sum of two odd numbers is an even number. The class is an average group of grade-six students. The teacher opens the discussion with the following statement:

Teacher: We have been discussing even and odd numbers during the past few days. Today we want to try to find out something more about even and odd numbers. Can anyone show us something about these numbers which we don't already know?

Debbie: Well, we know that any odd number can be written as $2n + 1$ where n is a natural number. (This was probably a naively obtained result corroborated by Polya's pattern of examining a consequence.)

Wade: Yes, and any even number can be expressed as $2n$ where n is a natural number. (A result probably obtained in a fashion similar to Debbie's.)

Barry: We also found out that if we double any number we get an even number. (Again, a naively obtained result.)

Teacher: Yes, we discovered all of these things. But can you use these results and ideas to obtain a new conjecture?

Debbie: Let's see now. What if we add two odd numbers. We would have two numbers of the form $2k + 1$ and $2n + 1$, say. If we add these we get $2k + 2n + 2$ or $2(k + n + 1)$. But Barry pointed out that we know that the double of any number is an even number so we now have <u>proven</u> that the sum of two odd numbers is always an even number. (Debbie's conjecture is obtained by deducing results from previously known ideas;

that is, it is deductively generated.)

Wade: This seems correct, but remember we haven't really proved that the double of any number is always an even number. If that result is false, then your proof breaks down even though your conjecture may be correct. This weakness leaves the proof-analysis in doubt. (Wade has identified the stratagem of setting out to simultaneously prove and refute a conjecture. Moreover, he seems to have identified in a very unsophisticated fashion the distinction between local and global counter-examples.)

It should be noted that the children had already obtained some naively generated results which served as a basis for the proof-analysis which Debbie developed. Accordingly, it may be the case that in the actual pursuit of mathematical knowledge naive origination always precedes deductive origination. Furthermore, as Wade's response indicates, Debbie's proof-analysis may not actually prove her result, but it does broaden the field for possible refutations and tests of her conjecture precisely as Lakatos argues proof-analysis should. Wade's response to Debbie's proof characterizes him as a true Fallibilist who attempts to use the stratagem of lemma-incorporation in order to strengthen a conjecture.

The above description of origination strategies dealt with naive guessing and deductive guessing. Naive guessing is the means of conjecture generation utilized in the naive stratagem of teaching for both the TP and the PT strategies. Deductive guessing is the one means of generating conjectures in the DED strategy of teaching. The two strategies of naive and deductive guessing are derived from the model of mathematical inquiry given in Chapter IV. Consequently, they represent a Fallibilistic approach to the creation of mathematical conjectures in the classroom situation.

Each of the two strategies of origination was illustrated by means of hypothetical classroom situations. These illustrations showed the relatively passive role of the teacher and the active role of the student in Fallibilistic learning situations. The central focus of these situations was the guessing procedures conducted by the students. The teacher's role was to set the problem, assist the students in creating testing procedures, and then guiding and focusing the classroom discussion. The students were allowed to pursue the discussion on their own governed principally by the criticism of their classmates.

In terms of the instructional model given on Page 153, the origination phases in both the naive and the deductive stratagems of teaching have been described and illustrated. The next step is that of describing and illustrating the testing and proving phases. Since the testing phase is the next phase in both the TP and DED strategies, this phase is considered next.

Testing Strategies. The testing phase of the Fallibilistic mode of inquiry was seen to be composed of four techniques. These four elements or techniques were those of attempted refutation, examining a consequence, examining a possible ground, and examining a conflicting conjecture. It would seem appropriate then that strategies of teaching be developed corresponding to each of these elements of the testing phase. Teachers of mathematics must be aware of these techniques within the testing phase in order that they may acquaint students with the strengths and weaknesses of them.

A cautionary note must be sounded with regard to the latter three techniques. Because these techniques are only plausible reasoning patterns great care must be exercised in teaching students to use these techniques. The reason for this of course is that children may come to the conclusion that plausible reasoning patterns are <u>sufficient</u> to establish a mathematical conjecture when in fact they are not. Hence, students can be made aware of such techniques, but care must be taken to prevent these patterns of plausible reasoning from obscurring the basic pattern of refutation.

Given this cautionary note, the goal of this subsection is to describe and illustrate the techniques of testing a conjecture. The primary function of the testing phase from a Fallibilistic viewpoint is the attempted refutation of the conjecture in the case of a naively generated conjecture and the identification of hidden lemmas and weaknesses in the proof-analysis of a deductively generated conjecture. Consequently, the primary function of the testing phase is the refutation of a conjecture or its proof-analysis. Only if such attempted refutations fail do the patterns of plausible reasoning become operative.

The techniques of attempted refutation is actually the development of a critical, rational attitude on the part of students. It is designed to foster the student's ability to criticize all possible solutions to a problem, to not accept any solution at its face value. In order to develop such skills and attitudes in children the teacher could and should pursue two plans of action.

The first method would be to encourage students to question their classmates suggestions and proposed conjectures. The teacher can do this by actively criticizing student suggestions in the hope that other students will follow the teacher's example. However, because of the teacher being an authority figure in the classroom, the student might respond negatively to the teacher functioning in this way. Consequently, a second plan would be for the teacher to put forth conjectures which are designed so that they may be refuted. In this way the students could be brought to an understanding that criticism and attempted refutation is not a personal attack on individuals but an attack rather on ideas. The teacher in this situation should gladly and constructively accept the criticism and refutation of his conjectures so that children may copy the teachers behavior patterns.

The following hypothetical classroom situation gives some idea of how a teacher could foster attitudes of criticism and rational thought on the part of students. The class would probably be classified as an average grade nine class.

Teacher: We have been studying various geometical shapes and the sum of the angles contained by these shapes. I would like you to consider the following statement regarding triangles: The sum of the angles of a triangle is greater than 180°. Would you agree with this statement?

Doug: No, I don't agree.

Teacher: Why not? Can you support your denial of the statement?

Doug: Well, let's draw some triangles and measure the angles. (This is done.) Now in all these cases we see that within certain measurement limitations that the sum of the angles of these triangles is 180^{o}. So I have refuted your statement.

Teacher: But what about your measurement errors? Could not all of your errors been such that you erred by getting too small a sum?

Scott: That might be true. The counting of instances doesn't prove anything, but if we can show one instance where your statement is wrong, then that would destroy your guess.

Brian: Can we create just one such instance? Can we provide a deductive refutation of this statement?

Doug: Can we prove that the sum of the angles of a triangle is 180^{o}?

Jeff: I think so. (Jeff provides the usual Euclidean Proof.) There now that refutes your statement.

Teacher: It would seem to, but consider this example. Suppose we take the world to be a perfect sphere. Consider two distinct lines of longitude which meet at the north pole and cut the equator. These two lines and the equator would form a triangular shape. Would you agree?

Class: Yes. (This is said with scepticism.)

Teacher: Alright. Now what is the angle between the equator and any line of longitude?

Jeff: Ninety degrees.

Teacher: Good. But in our triangular figure we have two distinct lines of longitude and consequently the angle between them at the pole is greater than zero degrees. Would you all agree?

Doug: I suspect a red herring, but yes I would agree.

Teacher: If you agree to that, then we have a triangle which contains two right angles and a third angle which is greater than zero. Hence I have shown you a triangle in which the sum of the angles IS greater than 180°. Hence, I have refuted your proof Jeff.

Brian: But that is not really a triangle.

This discussion could be pursued further. However, what is important to note here is how first the students and then the teacher actively sought to refute each others conjectures. Moreover, the students provided a deductive refutation of the teacher's conjecture by providing a proof for a rival conjecture. The teacher responded by providing a physical refutation of the student's proof-analysis. Further, in providing this counterexample the teacher is actually leading the students to a discussion of non-Euclidean geometry. He is attempting to expand the class conception of triangles and introduce the students to spherical geometry. The students would no doubt attempt to refute the teacher's example by arguing that his figure is not <u>really</u> a triangle.

This example also indicates how the growth of mathematical knowledge is characterized by an never-ending pursuit to seek results with greater generalizability. The teacher in this example is actually forcing the students to jump to a higher level of universality by identifying limiting cases. In doing so, the teacher is trying to expand the students' knowledge by looking to the empirical world in order to refute a deductively proven conjecture. Accordingly, the principle of the retransmission of falsity is operating here since a deductively obtained result is being refuted by an empirical test.

The entire tenor of this illustration is that teacher and students actively attempt to refute each other's conjectures. This is a criticism of ideas, a rational discussion of proposals, devoid of affective overtones. It is designed to illustrate how attitudes of critical and rational thought could be fostered in the mathematics classroom.

But what if the attempted refutations fail? Then the three patterns of plausible inference become operative. All of these patterns serve to increase or decrease the credibility of some conjecture. It will be recalled that plausible patterns can only suggest the direction of evidence but not its strength. The first of these patterns of plausible inference is that of examining a consequence. The strategy followed with this pattern is to examine the consequences of some conjecture which has been previously obtained.

In the following example, an imaginary classroom situation is presented in which children are using the pattern of examining a consequence to provide plausible evidence in support of a conjecture. The teacher sets the stage for the discussion:

Teacher: We decided yesterday that any even number could be written as 2n where n is a natural number, and any odd number could be written as 2n + 1 where n is a natural number. Can anyone guess what the double of an even or odd number would look like? (This is that grade six class again. This sequence is designed to indicate how they might have corroborated one of their naive conjectures.)

Wade: Well, if you double 2 you get 4 and these are both even numbers.

Debbie: Yes, if we double 4, 6, 8, and 10 we get 8, 12, 16, and 20 respectively, which are also even numbers.

Barry: But, what if we double 1 and 3? Then we get 2 and 6 which are even numbers whereas 1 and 3 are odd numbers. Maybe if we double any number we get an even number. (A naive conjecture.)

Wade: Let's make a chart and see. In one row we'll put the numbers from 1 to 20 and in the next row the doubles of these numbers. Here I have it:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2n	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40

Barry: Yes, it does seem that the double of any number is an even number.

Wade: At least as far as we have tested it anyway.

In this illustration, a naively obtained conjecture was tested by looking at consequences of the conjecture. The examination of such consequences gives support to Barry's conjecture in a positive fashion, but of course the strength of such support is impossible to assess. And as Wade notes, the conjecture is not really proven, but remains plausible only.

The connection between this illustration and the previous example where the student "proved" that the sum of two odd numbers is an even number can now be made clear. In point of time, a class would be guided through the above illustration prior to attempting to prove the summation problem. This would seem reasonable both from logical and from pedagogical grounds. However, the essential point in this ordering of these learning situations is that the two examples taken together illustrate how one moves from one level of universality to a higher level. In the testing sequence above, the children are at a low level of universality not far removed from the empirical world. In the former example, however, they have progressed to a higher level generalization by considering the universe of all whole numbers. Consequently, their proof of the sum problem is a generalization on a higher plane of universality, but again one which can be referred back to an empirical base.

The testing patterns of the examination of a conflicting conjecture and the examination of a possible ground are illustrated when the proving phase is described and illustrated. As a consequence, these patterns are not dealt with here since such a discussion would be repetitious. Suffice it to say that the important point with all these techniques from a pedagogical viewpoint is that they are only suggestive of conjectures which may be worthy of further consideration. The only true test of a conjecture is a severe and honest attempt to refute it. Knowledge grows with refutation even for the student of mathematics.

If the deductive stratagem of teaching is being utilized then the testing phase takes on an additional aspect. In the deductive strategy, the conjecture being considered is obtained by deductive guessing which means that the conjecture comes with a proof-analysis. In this situation, the testing phase is designed to uncover hidden lemmas and assumptions, faulty deductive steps, and local or global counterexamples.

The earlier hypothetical teaching situation concerned with the sum of the angles of a triangle provides an example of the testing phase serving this function. Recall that the teacher presented a counterexample to a conjecture which Jeff had proven deductively; that is, Jeff provided the traditional Euclidean proof of the conjecture dealing with the sum of the angles of a triangle. The teacher in giving his physical counterexample is identifying a hidden assumption (two dimensional figures) in this proof. Thus the testing taking place here is designed not to destroy the conjecture, but to improve the conjecture and its proof-analysis. This is in line with Lakatos' contention that refutation can be a concept-expansion process and that proof can serve to prove and refute simultaneously some conjecture.

The third phase of the two teaching stratagems is the proof of a conjecture. In the naive stratagem, proof precedes the testing phase in the PT strategy and follows the testing phase in the TP strategy. The proving phase forms part of the origination phase in the DED strategy.

<u>Proving Strategies</u>. The purpose of this subsection is to describe and illustrate the strategies of teaching appropriate for the proving phase of mathematical inquiry. The Fallibilistic view of proof is two-fold in that the creation of a proof-analysis is designed to prove <u>and</u> refute the conjecture. Proof for a Fallibilist is the creation of a system of subconjectures or lemmas which have the effect of making the original conjecture more susceptible to criticism. It gives rise to the idea of local and global counterexamples in that a counterexample may,

in the first case, refute a subconjecture but not the original conjecture, and in the second instance, a counterexample may refute the main conjecture but not its system of lemmas. Consequently, proof in Fallibilistic terms is not designed to provide final 'truth'. Various techniques have been identified as part of the proving phase. The application of these techniques to teaching and learning situations forms the basis of the remainder of this section.

The three techniques identified as being part of the proving phase of mathematical inquiry were those of monsterbarring, exception-barring, and lemma-incorporation. Teaching strategies corresponding to each of these techniques would seem most appropriate for senior high school students due to the sophisticated nature of these techniques. This is probably not the case for teaching strategies discussed previously. Indeed, as will be shown in the next major section of this chapter, Davis has successfully used the preceding instructional models with children as young as grade five students.

Teaching strategies designed to illustrate the techniques of monsterbarring, exception-barring, and lemma-incorporation could be designed for very elementary problems. For example, the problem of finding a general formula for the solution of quadratic equations could be approached by a conjecturing process. Beginning with quadratic equations whose roots are unequal primes might lead to the conjecture of the coefficient rules. A more complicated quadratic equation such as the following $2x^2 - 5x - 3 = 0$ might be ruled out as a monster or barred as an exception. Lemma-incorporation would demand, however, that this case not be ruled out, but that a better conjecture be sought. In order to specifically identify the various techniques, the teacher could purposely torpedo student conjectures, thus forcing a rejection of the conjecture or the proof-analysis.[10]

A teaching strategy designed to provide opportunities for students and teachers to develop proof-analyses and to utilize Fallibilistic techniques for dealing with counterexamples must first have a problem to be solved. A conjecture may or may not be available as a possible solution to the problem. If a conjecture is available, then the proving phase is part of the naive stratagem. If a conjecture is not available, then one must be generated either by naive or by deductive processes. The teacher should encourage students to try to prove <u>and</u> refute the proposed solution however it may be obtained. In addition, the teacher should purposely provide counterexamples to the proposed solution so that the scope of the conjecture may be broadened. The students should of course function in a similar way in order to develop the skills and attitudes of rational criticism.

It is not necessary for the <u>names</u> of the various ways of dealing with counterexamples be used. Indeed, it would probably force premature closure of a developing concept. The concepts themselves, the techniques and responses, should be utilized in order that children

[10]The process of 'torpedoing', a term used by Davis, is discussed in the next section when the Madison Project materials are analyzed.

become aware of the defense strategies available for the rescuing of a conjecture from counterexamples.

The hypothetical classroom situation given below attempts to illustrate how a teacher and an advanced group of grade ten students might attempt to generate the general formula for the solution of quadratic equations using Fallibilistic techniques. The various ways of dealing with counterexamples are illustrated. Some of the patterns of plausible reasoning identified by Polya are also utilized in order to illustrate how these techniques may serve to increase the credibility of some conjecture. It is assumed that the children have had some initial experience with very simple quadratic equations such as $x^2 - 5x + 6 = 0$. The teacher opens the discussion by stating:

Teacher: Let us continue our discussion of quadratic equations. Remember that eventually we want to derive a formula for the general solution of <u>all</u> quadratic equations. Can someone tell me two numbers which would satisfy the equation $x^2 - 8x + 15 = 0$?

Jeff: The numbers are 3 and 5. (A naive conjecture.)

Teacher: Do you all agree? Can you test Jeff's guess to see if he is correct? (This is done.) (This is the testing of a naive conjecture, the plausible reasoning pattern of examining a consequence.)

Teacher: Good! These two numbers do seem to satisfy the equation. Has any one found a secret way of determining the answer to such problems? (Several children raise their hands. The teacher is attempting to have the children expand their conjecture beyond just <u>specific</u> instances.) How many secrets do you have Susie?

Susie: Well! I have one.

Scott: I have one also.

Teacher: Fine. Try your secrets out on this equation: $x^2 - 20x + 96 = 0$. (The teacher is torpedoing some of the student's conjectures. There is a pause while the children attempt to determine the answer to this particular problem. Recall though that the goal of the lesson is to develop a general solution for all quadratic equations.)

Teacher: Do you have an answer Scott?

Scott: Yes, I think the numbers are 2 and 48. (Scott is obviously using the product rule as his secret way of getting the answer. More obviously it is a naive conjecture in need of severe testing.)

Susie: No, the numbers are 4 and 24. (A rival conjecture.)

Jeff: You are both wrong. The numbers are 8 and 12. (Another rival.)

Teacher: Who is right or are all of you correct? Test your guesses. Have you determined who is correct?

Mark: Jeff is right because his numbers work, but Scott's and Susie's do not satisfy the equation. (The destruction of a rival conjecture, in this case Scott's and Susie's, can only lead to the strengthening of the other conjecture, in this case Jeff's secret conjecture of the coefficient rules. Note, however, that Jeff's conjecture is only strengthened if it is tested. It is not strengthened simply because Scott and Susie's conjectures failed. Furthermore, Scott's and Susie's conjectures represented possible grounds for the solution of the problem, possible grounds which did not work. These are Polya's other two patterns of plausible inference.)

Teacher: What happened to your secret Susie--Scott? (The teacher forces the students to analyse their errors so that they may learn from them.)

Susie: It does not seem to work, but yet it does. I just don't know. (Susie's secret works for 8 and 12 since $8 \times 12 = 96$, but with only one secret at her disposal 4 and 24 are ruled out since $4 + 24 \neq 20$. Hence, a refutation forces her, and Scott as well, to revise their conjecture--to expand the scope of their secret.)

Teacher: How many secrets have you got Jeff?

Jeff: I have two secrets.

Teacher: Yes--there are two secrets. Do you think they will work for every quadratic equation Jeff? (A rhetorical question.) Try your secrets on this equation: $2x^2 - 5x - 3 = 0$.

Mark: That's not a quadratic. It has a 2 as a coefficient of the x^2 term. It's a monster and should be thrown out as such.

Brian: Why should we? It satisfies the condition for quadratics, namely, that the highest power in the equation is 2. Can we solve it using Jeff's coefficient rules? (Jeff has revealed his secrets to the class.)

Mark: No, it is a monster and should be barred. It is not within the conditions of the problem. You are redefining the situation which was set up to start with. No one said anything about having coefficients of the x^2 term other than one. (Monsterbarring often gives rise to the redefinition of the conditions of the problem. "Good" monsterbarring leads to exception-barring.)

Scott: So what! If we can find two numbers which satisfy the equation, then we have accomplished what we set out to do. Anyway, we want to find a solution for all quadratics and this is very definitely a quadratic equation.

Susie: I have a solution. The numbers are 3 and $-\frac{1}{2}$. Try them--they both work. (Susie seems to have successively expanded her conjecture to include something more than the coefficient rules.)

Mark: But they are not both integers and one of them is even a negative number! (Monsterbarrers do not give up easily.)

Scott (aside to Jeff): I guess he thinks only positive integers are 'good' numbers.

Susie: But look! If we divide both sides of the equation by 2, then we get $x^2 - \frac{5}{2}x - \frac{3}{2} = 0$ and Jeff's rules still work since $- \frac{1}{2} \times 3 = - \frac{3}{2}$ and $- \frac{1}{2} + 3 = - \frac{5}{2}$, so why should be bar this equation as a monster. Just because you couldn't solve it doesn't mean that it is a monster and we have to dismiss it from our discussion. (Susie is beginning to switch to a deductive heuristic pattern.)

Teacher: I agree. In its original form the equation did not appear as though it could be solved using Jeff's rules, but Susie has shown they will work. So there is no reason to treat this equation as a monster.

Scott: I have an equation for which Jeff's rules don't work. It is $x^2 - 6x + 4 = 0$. (The identification of a counterexample which is global.)

Teacher: Indeed. What do you say to that Jeff?

Jeff: I can't guess any two numbers that will work, but of course that doesn't mean they don't exist. (Jeff seems to still be functioning naively, but at the same time he realizes the tentativeness of such an approach.)

Mark: Ha! Another monster. How are you going to get around this example. Maybe my monsterbarring technique is not so bad after all.

Brian: You never give up do you? Instead of barring this example as a monster let's treat it as an exception and stipulate that Jeff's rules work except for examples of this type. (An exception-barrer.)

Susie: But how do you know that there are not more exceptions, or how do you know that Jeff's rules will not work? You are retreating too quickly and you may retreat too far so that we have nothing left to work with but very simple equations. We want to get a general solution to all quadratics and hence we don't want exceptions if we can help it. Anyway, I can show that Jeff's rules still work. (Susie has identified the weakness of the strategy of exception-barring.)

Teacher: Please do.

Susie: The equation is $x^2 - 6x + 4 = 0$. If we transform the equation by the addition of 5 to both sides of the equation, we obtain the result $x^2 - 6x + 9 = 5$.

Brian: But now you do not have zero on the right side of the equation.

Jeff: Another exception!!! (Jeff is very skeptical about monster-barrers and exception-barrers.)

Susie: Let me finish. If we factor the quadratic we get the following equation: $(x - 3)^2 = 5$ and as a consequence $x = 3 \pm \sqrt{5}$. Now, using Jeff's rules we have $(3 + \sqrt{5}) + (3 - \sqrt{5}) = 6$, and $(3 + \sqrt{5})\ (3 - \sqrt{5}) = 4$, and so his rules still work. (Susie is completely in the deductive heuristic now. Jeff's rules still work, but they are not very useful in actually finding the solution to these equations.)

Scott: Beautiful!

Doug: Hence, we shouldn't make an exception of this case, nor should we bar it as a monster. Susie's proof-analysis is very neat. It shows that we must incorporate this example and her proof-analysis in our derivation of the general formula. (Doug had identified the strategy of lemma-incorporation.)

Brian: But now we have irrational roots!

Mark: Now who is the devil's advocate? (Good point, but Mark and Brian both serve a useful function in that they attempt to identify weaknesses in the proof-analysis.)

Jeff: What's wrong with irrational roots? We accepted rationals, both positive and negative, so why not irrational roots. My rules work so let's incorporate quadratic equations with irrational roots into our proof-analysis as well. (Jeff is somewhat of a pragmatist.)

Teacher: I agree. But let me summarize. We are trying to find a general solution for all quadratic equations. We started with equations of the form $x^2 + bx + c = 0$ where the roots were unequal primes. Then we dealt with equations whose roots are composite numbers. Susie expanded our conception of the quadratic equation with the example of $2x^2 - 5x - 3 = 0$ so that we now have the general form of the quadratic as being $ax^2 + bx + c = 0$ in which case the roots might be irrational numbers. Can anyone now provide a proof-analysis which would yield a general formula?

Jeff and Susie: We have a proof-analysis--at least we think we do. (After progressing through a period of utilizing the naive heuristic pattern, Jeff and Susie are now ready to provide a deductively obtained proof-analysis.)

Teacher: Show it to us please.

Jeff and Susie: Starting with the general equation

$$ax^2 + bx + c = 0$$

we transform this equation to give the following equation:

$$ax^2 + bx = -c.$$

Dividing both sides of this equation by 'a' gives the result that

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

If we follow Susie's pattern, we can add the term $(\frac{b}{2a})^2$ to both sides of the equation giving the following equation:

$$x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 = (\frac{b}{2a})^2 - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}.$$

Factoring the left side of this equation gives the result

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}.$$

As a consequence of taking the square root of both sides of this equation, we obtain

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

and finally we get the general formula given below:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hence, it is obvious that the roots could be irrational. (Jeff and Susie have provided the required proof-analysis.)

Mark: Yes, but what if $b^2 < 4ac$?

Jeff: Then we should have to invent a new number system.

The imaginary situation created above serves to indicate the techniques of monsterbarring, exception-barring, and lemma-incorporation on a very elementary level. In addition, it illustrates how counterexamples serve to expand the focus of a problem--to foster the growth and understanding of a concept. The progressive development from prime unequal roots to irrational roots was spurred by examples which did not seem to fit the pattern. Indeed, this concept expansion led to the observation that number systems <u>are</u> created, a significant finding in and of itself. The fact that new number systems had to be utilized to solve the general problem also allowed the class to progress to higher levels of universality. They moved from the level of dealing with only prime unequal roots to the quite sophisticated level of complex numbers. This represents a jump in the level of universality of the problem, and, indeed, its ultimate solution. This is further illustrated later when an example is provided which deals specifically with the progression from one number system to another and the corresponding rise in the level of universality. Note also how Jeff and Susie incorporated patterns (factoring, etc.) developed earlier in the lesson into their final proof-analysis. They were expanding their conception of the problem and at the same time developing techniques which ultimately became part of their proof-analysis.

The illustration also serves to indicate how a conjecture and refutation process gives rise to a deductive sequence of steps which form the basis for the proof-analysis; that is, it may be that the construction of a proof-analysis always comes after a naive origination process. As pointed out previously, the naive and deductive heuristics constantly interact with the result that it is doubtful if one heuristical pattern is ever pursued independently of the other. As Figure 8 on Page 143 indicates, the naive and deductive heuristics can and do intersect and interact in that the student or mathematician may pass around either circle of naive heuristic or deductive heuristic and progress to the other circle. To conclude, however, that the naive heuristic always precedes the deductive heuristic is not justifiable on the basis of the evidence available.

The students in this illustration were the prime movers in the development of the proof-analysis. They utilized the techniques of dealing with counterexamples which resulted finally in an expanded conception of quadratic equations and the generation of a general solution for all quadratics. Although the final solution was obtained deductively, the naive processes of origination were used initially. This phase of the development of the general formula probably helps the students to grasp the problem under study, to become familiar with its dimensions, and to tentatively see patterns which gave rise to the final solution. Consequently the naive stratagem and the naive heuristic may serve the very important function of being explorers of unknown terrain. This stratagem and heuristic probably constitute the first halting steps in moving from an unknown situation to a known situation.

The teacher's role in this illustration was to set the problem and to guide the students in their explorations. The teacher's job was not to give a complete map of the unknown terrain (unknown to the students), but rather to get the students started, to keep them from wandering too far off track (although some wandering is desirable), and to encourage the students to broaden their conception of the terrain by providing counterexamples which force the expansion of naive conjectures.

The broad curricular and instructional viewpoint of a Fallibilistically oriented teacher is one which displays at least the following characteristics. The Fallibilistic teacher is one who is concerned with students learning how to learn rather than with students learning specific materials. The end result of any mathematics course should not be, according to the Fallibilistic view, the acquisition by students of the ability to memorize and produce on demand a great number of assumptions, definitions, and theorems. Rather the student should develop the skills and attitudes for attacking problems in a Fallibilistic fashion. The Fallibilistic teacher bases his philosophy on the fact that there are no immutable truths. Consequently, his students should learn how to investigate unknown terrain in mathematics not with the goal of finding truth but with the desire of obtaining an ever improved map of the terrain.

In attempting to chart the unknown terrain, students are guided by the teacher, but they are not given the complete map. Indeed, from a Fallibilistic orientation such a complete map is not available, not even to the teacher. The teacher must allow students to create, revise and expand their own map. The teacher aids the students by guiding them over the terrain, by putting forth conjectures and counterexamples which focus their attention on specific points of their map.

In a year of working with children at any grade level, the Fallibilistic teacher would attempt to develop a critical and rational attitude in his students whatever the particular mathematical content might be. His goal would be to have students increase their origination, testing and proving skills. What is originated, tested and proven is not of central importance. What is of importance is that students develop these skills and expand their knowledge of whatever terrain they happen to be investigating.

The Fallibilistic view of the growth of mathematical knowledge and the subsequent attitude of the classroom teacher who is oriented to a Fallibilistic approach to the teaching of mathematics is summarized by Hull in the following passage in which he discusses the history, growth and teaching of mathematics:

> When we look at the history of mathematics, we see a kind of life-like elemental rhythm. There are periods of exuberant untidy growth, when exciting, vital structures rise upon untried assumptions, and loose ends lie about all over the place. Logic and precision are not unduly honoured; because restlessness, enthusiasm, daring, and ability to tolerate a measure of confusion, are the appropriate qualities of mind at these times. Such periods are followed by pauses for consolidation, when the analysts and systematisers get to work: material is logically ordered, gaps are filled, loose ends are neatly tied up, and rigorous proofs supplied. Solemn commentators sit in judgment upon great innovators. . . . Work of this kind, at its best, is also creative: new ideas grow from the critical examination of old, and the cycle is renewed. Periods of these two kinds may overlap;

or a growth period in one field may coincide with a period of consolidation in another: but the fundamental alternation would seem to subsist pretty generally. And one thing about which there can be no doubt is that the analyst and systematiser must play second fiddle to the innovator. They depend on him for their job, because they have no function until there is something significant to be criticised or ordered.

This has some obvious implications for individual learning. The early stages should be exploratory. Experiment, intuition, and informal inference should all be involved; the approach should be through particular problems and situations which excite curiosity; general principles should be only gradually evolved; while formal rigour will be out of place. There should then be a stage of systematisation, when regions of knowledge begin to show an increasing logical articulation, and proofs (when they are really necessary) are subject to a gradually more rigorous examination. Areas of organization will grow, and merge into one another; precise definitions and sets of axioms may eventually appear. But what we must beware of doing is to begin with the deductive development from definitions and axioms. New mathematics does not arise in this way, and existing mathematics should not be so presented.[11]

In this section a Fallibilistic model of instruction has been developed. The model was based on the description of the Fallibilistic model of the mode of inquiry in mathematics given in the previous chapter. As a result, two basic stratagems were identified, namely, the naive stratagem and the deductive stratagem. The naive stratagem was seen as being composed of two strategies which were designated as the TP strategy (in which the testing phase is followed by the proving phase), and the PT strategy (in which the proving phase is followed by the testing phase). The deductive stratagem encompassed the DED strategy. The distinguishing characteristic of the DED strategy is that origination in this strategy is of a deductive nature. In the naive stratagem,

[11] L. W. H. Hull, "The Superstition of Educated Men," Mathematics Teaching, Number Forty-three, Summer, 1968. Pp. 29 - 30.

the origination processes were of the guess-and-test variety; that is, naive origination.

Each of the three phases within the two stratagems were discussed at length. The point of reference for this discussion was how students and teachers could utilize Fallibilistic strategies in creating teaching and learning situations. Origination strategies for both the naive and deductive stratagems were discussed and illustrated. The testing strategies for teaching focused on the attempted refutation of the proposed solutions. However, the utilization of patterns of plausible reasoning were also demonstrated as being part of the testing strategies. They were further illustrated in the hypothetical situation created when proving strategies were discussed. Finally, an extended illustration of the utilization of proving strategies was given.

A Global Illustration

The following illustration is designed to serve several purposes. First, all of the previous illustrations deal basically with isolated problems and do not give a more or less global view of how a Fallibilistic approach could be utilized over a relatively large amount of work. Second, although levels of universality have been discussed with respect to some of the previous examples, none of them were designed to exemplify the growth of the levels of universality with respect to mathematical systems. The next example has this as a primary goal. Third, the empirical basis of mathematical systems (or quasi-empirical in Lakatos' view) has not been illustrated to any great degree in the previous illustrations. It is hoped that the following illustration will be more clear in this respect. Finally, the illustration is designed to show how a unit of work <u>might</u> be organized in an instructional sense from a Fallibilistic point of view.

The material chosen for this illustration is that of the development of the various number systems beginning with the positive whole numbers and expanding progressively to the complex number system. The illustration provides only one possible way of organizing the instructional sequence of this material--a Fallibilistic approach. Many other ways are, of course, possible. The general focus of the entire class discussion centers around the solution of equations and the mathematical universe for which such solutions are possible. The class would probably be classified as an advanced group of grade ten students although some evidence is available that at least portions of this material can be dealt with by much younger children. The teacher opens the discussion by setting the learning situation.

Teacher: We have been considering and working with the beam balance during the past few days. You found that you could solve mathematical sentences such as these:

$$\square + 5 = 7$$

$$\square - 7 = 10$$

$$2\square + 3 = 27 \quad \text{and so on.}$$

Today I would like you to attempt to solve some more open sentences using the balance, if possible. (At this point, the children are working with the positive whole numbers only. Consequently, the scope or range of open sentences which they can solve is rather limited. They are at a low level of universality with respect to number systems. Because they are working with a balance, it is possible for the students to either test proposed solutions in an empirical fashion, or, indeed, to obtain a solution simply by empirical-naive guessing.) Would you try to find the solution set for the following open sentence:

$$\square + 5 = 3$$

Doug: You can't solve that equation--not with a positive whole number anyway.

Scott: That's right! There is no positive whole number which will make a true statement out of that sentence.

Brian: Not only that! We can't use the balance on that question so we'll have to throw it away as a means of finding answers.

Jeff: Maybe, maybe not. What we need to solve this question is a negative whole number. So we should expand our universe of discourse to the set of all integers rather than remaining with just the positive whole numbers. If we do that, then we could use the balance if we think of the addition of negative numbers as taking away that quality.

Teacher: Yes, we could interpret it in that way. But more important is your suggestion to expand the universe of discourse. Could we explore that idea a bit further. (Physical tests of mathematical systems are enlightening but not necessary. The teacher is guiding the students to deeper mathematical concepts rather than have them focus on the physical representation of such concepts.)

Susie: Well, for positive whole numbers we found that the following rules seem to hold. We could think of them as the basis for the system of positive whole numbers. If 'a' and 'b' are positive whole numbers, then we have that:

(1) $a + b = c$ and $ab = c$, where c is a positive whole number.

(2) $a + b = b + a$ and $ab = ba$ (Commutative Property)

(3) $a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$ (Associative Property)

(4) $a(b + c) = ab + ac$ (Distributive Property)

(5) $a \times 1 = a$ (Multiplicative Identity)

(The class probably arrived at these basic axioms by a process of naive guessing and perhaps deductive guessing. In either case, they have been accepted without proof.) Now, if we expand the universe to include all integers instead of just the positive whole numbers, we get two additional basic statements, namely

(6) $a + 0 = a$ (Additive Identity)

(7) $a + a^{-1} = 0$ where a^{-1} is called the additive inverse of 'a' and vice versa.

Since we now have negative numbers, we can solve the open sentence you gave, namely, $\square + 5 = 3$ by substituting $^{-}2$ into the frame.

Teacher: Very good Susie. We could indeed consider the above seven statements as the axioms for the integers since the integers do satisfy these axioms. (In expanding the number system to include all integers, Susie has raised the level of universality of the system since an additional set of numbers has been added to the system and, moreover, made it possible to solve an expanded set of equations. Her set of new axioms are more encompassing. Consequently, they risk the chance of being falsified over a wider range of application.)

Jeff: Yes, and as Susie shows, we can now solve an entirely new set of open sentences so we have greater power to solve equations.

Teacher: Good. Are there still equations we couldn't solve though?

Brian: Sure, that is easy. We couldn't solve this equation:

$$2\square + 3 = 8.$$

Teacher: Why not?

Doug: Because there is not an integer which will make that statement a true sentence. We can see that by approximation. For example, if we substitute 2, we have $(2 \times 2) + 3 = 7$ which is too small. But if we substitute 3, we have $(2 \times 3) + 3 = 9$ which is too large. The solution must be a number somewhere between 2 and 3 and hence couldn't be an integer. (Doug has reverted to a quasi-empirical basis to test his statement of the non-existence of a suitable integer. He has corroborated his statement. This is an instance where the corroboration of a statement asserting the non-existence of a solution spurs the growth of knowledge. However, in Popper's view, Doug has actually refuted the existential statement, "There exists an integer which will satisfy the equation," and, hence, refutation spurs the expansion of knowledge.)

Scott: I would agree. What we have to do now is to expand our number system to include fractional numbers. We need such numbers in order to solve this equation since the number that will make a true statement out of this sentence is $2\frac{1}{2}$.

Susie: But then what we really need is a new axiom, namely,

(8) $a \times a^{-1} = 1$, where a^{-1} is called the multiplicative inverse of 'a' and vice versa. Isn't there another name for fractional numbers

which mathematicians use?

Teacher: Yes, they are called rational numbers. That is very good Susie. We now have eight statements which are satisfied by the rational number system. We have expanded our number system considerably just by looking for the solution to equations. (The addition of axioms generates a system which has a higher level of universality. The progression thus far has moved the focus from the positive whole numbers to the class of all integers--positive and negative--and now to the rational numbers. This progression has enabled the students to solve an ever increasing number of open sentences. Furthermore, as Doug has shown, it is possible to revert to a quasi-empirical basis in order to test a conjecture.)

Susie: I guess we can solve any equation you give us now eh!

Jeff: Don't be too sure. For example, I bet you can't solve this equation using rational numbers:

$$\square \times \square = 2.$$

Susie: That should be easy. It is probably about $1\frac{1}{2}$. (That is a pretty shrewd guess, naive though it is.) Let's see now. $1\frac{1}{2} \times 1\frac{1}{2}$ would be $\frac{9}{4}$ or $2\frac{1}{4}$. I guess $1\frac{1}{2}$ is too large. (Susie is following Doug's pattern of successive approximations--a quasi-empirical strategy.) How about $1\ \frac{1}{3}$ --no, that would be $1\ \frac{7}{9}$ which is too small, but it is closer to 2. Anyway, the answer we're looking for must be between $1\ \frac{1}{3}$ and $1\frac{1}{2}$. (The class then proceeded to test various conjectures attempting to get closer to the answer. They refined each guess on the basis of the previous refutation gradually coming to the conclusion that $1\ \frac{4}{10}$ was pretty close to the right answer.)

Teacher: Do you think you will ever get an exact answer--a rational number?

Doug: I don't know, but at the rate we're proceeding it might take us a while.

Teacher: How could you express the general form of a rational number? (The teacher is guiding the class here in an attempt to have them move to the deductive heuristic. The goal is to get the class to prove the non-existence of a solution.)

Susie: Let's see. We could write any rational number as $\frac{a}{b}$ where 'a' and 'b' are positive or negative integers, and b is not zero.

Jeff: Yes, we could do that. That would mean that the number we are looking for, let's call it $\frac{c}{d}$, is such that

$$\frac{c}{d} \times \frac{c}{d} = 2$$

$$\text{or } \frac{c^2}{d^2} = 2$$

Brian: Is $\frac{c}{d}$ reduced to lowest terms?

Jeff: Let's assume that it is.

Susie: O. K. Then we have that $c^2 = 2d^2$ and as a result c^2 must be an even number.

Doug: But 'c' could be even or odd.

Jeff: No, it couldn't. It must be even.

Teacher: Why do you say that Jeff?

Jeff: Well, let's assume that 'c' is odd. Then it could be written in the form $2k + 1$. Hence, $c^2 = 4k^2 + 4k + 1$ which is odd. But we know that c^2 is even. Therefore, c must be an even number and have the form $c = 2k$.

Susie: Yes, that is right. Moreover, since $c^2 = 2d^2$, we obtain the result that $c^2 = 4k^2 = 2d^2$ or $d^2 = 2k^2$, which means that 'd' is an even number also.

Doug: But that is a contradiction. We now have the 'c' as an even number and 'd' is also an even number. Therefore, they must have a common factor of at least 2, which contradicts our assumption that $\frac{c}{d}$ was reduced to lowest terms.

Teacher: What have you found then?

Jeff: That we can't solve $\square \times \square = 2$ by using a rational number.

Susie: And that's about as irrational as you can get!

Teacher: That's a very sick pun Susie. Nevertheless, if we are to solve this equation, we need to expand our number system again.

Brian: Yes, and we might as well call these new numbers of ours irrationals since they don't make much sense to me either.

Teacher: Do we need to add any new axioms to our system in order to incorporate these new numbers? (After much discussion and considerable work with irrational numbers, the class decided this wasn't necessary. The eight axioms they now have with the addition of the Euclidean axioms are sufficient to generate the real number system. They are not necessary, however, as even this set of axioms can be derived from Peano's axioms. The expansion to the irrational numbers was fostered by the inability to find a solution to the equation being studied. The class conjectured that there was a solution in the rational numbers, and subsequently refuted this conjecture. A consequence derived from a basic assumption has shown to be contradictory, and by the principle of the retransmission of falsity, the initial assumption was shown to be false. This led to the expansion to the irrational numbers.)

Jeff: I know another equation which we still can't solve even with the real numbers as the universe. (The class is now using the term 'real number system' to denote the number system formed by combining the rationals and irrationals.) Remember when we did that quadratic problem we ended with the general solution being the following:

$$x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

Mark asked what would happen if $b^2 < 4ac$, and I laughed and said that we would have to make a new number system. Well, what would happen if $b^2 < 4ac$? How do we find the square root of a negative number? What is the square root of $^-1$ for example?

Teacher: That's a very good question. Does anyone have any ideas? (None of the students have had any contact with complex numbers. The teacher will have to create a learning situation in which the students could generate this new number system.)

Jeff: I can't even imagine what such a number would look like.

The hypothetical classroom situation described above illustrated in a global fashion how a Fallibilistic approach could be utilized over a larger unit of work. It presented a global view of the operation of a Fallibilistic approach in the classroom. The general approach was one of dealing with equations which could not be solved in terms of the number system being utilized at that point in the illustration. This necessitated an expansion of the universe of discourse--the number system being used--with the result that new axioms were introduced into the mathematical system. As a consequence of the introduction of these new axioms, the class was forced to work with number systems which were on a higher level of universality than those previously used. Furthermore, the illustration contained sequences which enabled the class to use the quasi-empirical basis of mathematics in order to test their conjectures. Finally, the situation created in this illustration indicates how a unit of work might be organized Fallibilistically in the classroom.

III. THE MADISON PROJECT

This section of the chapter is designed to examine and appraise the Madison Project as an example of the Fallibilistic approach to the teaching of mathematics. Since the Madison Project was not developed with Fallibilism as a foundation, it is necessary to first provide both a description of this project, including its goals, and to document the Fallibilistic tendencies of the Madison Project. The first subsection examines and analyses the goals and objectives of the Madison Project from a Fallibilistic point of view, whereas the second subsection describes and assesses the instructional strategies developed by the Madison Project utilizing the model of instruction developed in the previous section.

The Madison Project: A Fallibilistic Interpretation

The basis for the interpretation which follows derives from an examination of the objectivesof the Madison Project, both objectives for the curriculum and objectives for the student. These objectives are given and subsequently analysed to determine what, if any, characteristics of Fallibilism are extant among them. In order to complete this analysis the model of instruction developed in the previous section and the model of the mode of inquiry created in Chapter IV are taken as the basis for identifying the inclination of the Madison Project towards a Fallibilistic approach to the teaching of mathematics.

The Madison Project was initiated in 1957 at Syracuse University under the direction of Robert B. Davis. The materials available dealing with this project include papers and manuscripts prepared by Davis and his associates, two sets of teacher and student discussion guides which are suitable for grades four to eight, some physical apparatus, an experimental course report for kindergarten, an experimental course report for grade nine, an in-service teacher training course, and a series of films--videotape kinescopes--of groups of students using the Project's materials and being instructed by Project techniques. The Project in its present day form is concerned with mathematics as a process rather than the finished products of a mathematician's activities. Davis has recently argued that ". . . mathematics is the process."[12] This is in contrast to the view of mathematics as being an end-product, a view which Polya rejects, as does Davis: "The end result is not mathematics;

---

[12]Robert B. Davis "Mathematics Teaching--with Special Reference to Epistemological Problems," Journal of Research and Development in Education, Monograph Number one, Fall 1967, p. 32.

the process was the mathematics."[13]

The overriding or general purpose of the Madison Project as enunciated by Davis is that task of seeking "the best experience with mathematics which can be provided for children at the pre-college level."[14] This general objective may be broken into at least two other areas; goals for the curriculum, and goals for the student. With respect to the curriculum, the Madison Project desires to broaden the curriculum, to expand the curriculum, in order to make it more representative of the mathematics needed in today's world, evidence of the Project's concern with the substantive structure of mathematics.[15] Second, the Project attempts to inject a more creative flavor into the curriculum. This means, at least in part, listening to the children, and realizing that children can and do discover and invent mathematics. As an example of the fostering of this objective, consider Davis' description of what he calls 'Kye's Arithmetic'.

> A third-grade teacher was introducing subtraction, with "borrowing" and "carrying":
>
> $$\begin{array}{r} 64 \\ \underline{-28} \end{array}$$
>
> She said: "You can't subtract 8 from 4, so you take 10 from 60 . . ." A third-grade boy named Kye interrupted: "Oh, yes you can!"

---

[13] Ibid., p. 29.

[14] Robert B. Davis, "The Madison Project's Approach to a Theory of Instruction," Journal of Research in Science Teaching, Vol. 2, 1964. p. 146.

[15] See, for example, Robert B. Davis, Matrices, Functions and other Topics. Elementary Mathematics Series, Madison Project, 1963.

$$4 - 8 = {}^{-}4$$

$$\begin{array}{r} 64 \\ \underline{-28} \\ {}^{-}4 \end{array}$$

$$\text{and } 60 - 20 = 40$$

$$\begin{array}{r} 64 \\ \underline{-28} \\ {}^{-}4 \\ 40 \end{array}$$

$$\text{and } 40 - 4 = 36$$

$$\begin{array}{r} 64 \\ \underline{-28} \\ {}^{-}4 \\ \underline{40} \\ 36 \end{array}$$

> The teacher did nothing here to SOLICIT originality, but when she was confronted with it, she LISTENED to the student, tried to UNDERSTAND, and WELCOMED and APPRECIATED his contribution.[16]

Here can be seen an instance of a student conjecturing a 'different' solution to a traditional problem in mathematics. Moreover, he provides confirming evidence of his conjecture, evidence which is plausible only, but which could provide the basis for the construction of a proof-analysis at a later point in time. The student does however exhibit characteristics of a highly creative orientation to the solution of mathematical problems. Note also that the creative response offered by the student dealt with the process for the solution of the problem, not with the final answer. Indeed, Westcott and Smith argue that mathematical creativity is both a

---

[16] Robert B. Davis, "Some Remarks on 'Learning by Discovery'," The Madison Project, July, 1966, pp. 4-5.

product and a process.[17] They contend that in a predominant number of instances where creativity manifests itself in the elementary and secondary school classroom, this creativity manifests itself in new methods of solving problems rather than in finding new answers; that is, students exhibit behavior which is mathematically creative in the methods of solving problems. In Davis' view, mathematics is the process and consequently it is there that creativity on the part of students can become evident.

Third, the Project attempts to provide a greater variety of experiences for children, and greatly increased student participation. In this respect, the Madison Project tends towards an activity curriculum, an activist approach to the teaching and learning of mathematics.[18] In doing so, the Madison Project is focusing on instructional models with the goal of obtaining a variety of such approaches--approaches in which the child is actively participating.

Although the Fallibilistic approach would not demand any specific curriculum content and consequently does not imply whether the curriculum should be expanded or reduced, the Fallibilistic orientation does imply allowing students to explore unknown terrain which might be beyond the scope of a prescribed curriculum. With respect to the second and third objectives for the curriculum maintained by the Madison Project,

---

[17] Alvin M. Westcott and James A. Smith, Creative Teaching of Mathematics. Allyn and Bacon, Inc., 1967, pp. 2-3.

[18] Robert B. Davis, "The Madison Project--A Brief Introduction to Materials and Activities," The Madison Project, Revised September, 1965, pp. 2-4.

they are definitely in agreement with a Fallibilistic philosophy for the teaching of mathematics. As evidenced by the example of Kye's Arithmetic, the Madison Project's idea of creativity is allowing students to make guesses, to test these guesses, and in general listening sympathetically to a student's conjecture. This is a Fallibilistic approach. Furthermore, the provision of greater opportunities for students to participate in the mathematics learning situation is axiomatic to a Fallibilistic approach: the growth of knowledge is based on an individual's conjecturing and refutation processes or activities.

With respect to the objectives for the student, the present goals of the Project, as stated by Davis, are the following:

(i) the ability to discover pattern in abstract situations;

(ii) the ability (or propensity) to use independent creative explorations to extend 'open-ended' mathematical situations;

(iii) the possession of a suitable set of mental symbols that serve to picture mathematical situations in a psuedo-geometical, psuedo-isomorphic fashion, somewhat as described by the psychologist Tolman and the mathematician George Polya;

(iv) a good understanding of basic mathematical concepts (such as variable, function, isomorphism, linearity, etc.) and of their inter-relations;

(v) a reasonable mastery of important techniques;

(vi) knowledge of mathematical facts.[19]

From the Fallibilistic viewpoint, the first two objectives can be seen as stressing the conjecture-testing-refutation process; that is,

---

[19]Robert B. Davis, "The Madison Project's Approach to a Theory of Instruction," Journal of Research in Science Teaching, Vol. 2, 1964, pp. 158-162.

students are encouraged to 'discover patterns', to conjecture as to the relationship in abstract situations, in order to foster the growth of their knowledge. Moreover, in endeavoring to expand their knowledge the goal is not to obtain 'final' truths, but rather to extend and expand their conception of mathematical situations.

The third and fourth objectives can be seen as goals which are required so that students can function in a way compatible with the first two objectives. Nevertheless, the acquisition of 'mental symbols', the understanding of 'basic mathematical concepts', the 'mastery of important techniques', and a 'knowledge of mathematical facts', are end-products which according to the Madison Project view should be acquired by processes which stress the conjecture and testing process proposed by the Fallibilists.

The above objectives are the so-called 'cognitive' or 'mathematical' objectives. Below are listed the more general student objectives:

(1) a <u>belief</u> that <u>mathematics</u> <u>is</u> <u>discoverable</u>;

(ii) a realistic assessment of one's own ability to discovery mathematics;

(iii) an "emotional" recognition (or "acceptance") of the open-endedness of mathematics;

(iv) honest personal self-critical ability;

(v) a personal commitment to the value of abstract rational analysis;

(vi) recognition of the valuable role of "educated intuition";

(vii) a feeling that mathematics is "fun" or "exciting" or "challenging" or "rewarding" or "worthwhile".

Actually, there is another important objective. We want the child to know who he is in relation to the human cultural past. By developing mathematics through discovery and through student initiative, we have brought history right into the classroom![20]

Several Fallibilistic tendencies may be observed in this set of objectives. Objectives one and three stress the tentativeness and discoverability of knowledge. For mathematics to be discoverable, it is necessary for someone to propose conjectures which can be tested and analysed. The testing and analysing processes are never-ending. Consequently, the discipline of mathematics must remain open-ended and tentative. This is the Fallibilistic contention: ultimate truths will never be obtained and all knowledge is tentative and subject to refutation. The emphasis in the second, fourth, and fifth objectives is on the critical rationalistic approach to the growth of knowledge advocated by Popper. It may be recalled that Fallibilism is characterized by a critical attitude towards problems--an attitude concerned with the rational analysis of problems and a critical attitude towards proposed solutions to these problems. The three above mentioned objectives would seem to be encouraging the development of such attitudes in students. In speaking of "educated intuition" in objective six, Davis is recognizing the role of guessing in the growth of mathematical knowledge. As will be seen later, Davis encourages students to guess solutions to problems and to subject these guesses to severe tests with the goal of making better and better conjectures. This is of course the Fallibilistic orientation. Finally, objective seven while not peculiar to a Fallibilistic approach to the

---

[20] *Ibid*.

teaching of mathematics does emphasize the fact that learning mathematics need not be drudgery. It can be "fun" if it is approached with a critical attitude which recognizes the tentativeness of knowledge and the challenge offered in being able to 'create' mathematics.

Clearly, all of these objectives have psychological overtones as does the entire Fallibilistic approach to the learning and teaching of mathematics. However, the consideration of these overtones and their possible compatibility with modern learning theory, Piaget's or Dienes' approach for example, would require an additional study based on psychological considerations alone. As a consequence, these overtones are not considered here, but the researcher is cognizant of their importance and note is made of this in Chapter VI when the implications for further research are given.

The Project is based on two seemingly contradictory hypothesis. It is assumed, on the one hand, that children need to have informal exploratory exercises in order to grasp intuitively the mathematical concepts they are dealing with and learning about. These informal exploratory exercises may be actual physical experiments or psuedo-geometrical experiences. On the other hand, the Project insists upon a formal axiomatic foundation as the only correct basis for mathematics. These two hypotheses are not contradictory, however, but rather constitute the distinction between the naive and the deductive stratagems of teaching.

When speaking of the Madison Project's materials dealing with fractions, Davis states that:

In fact, Madison Project materials for fractions cover three stages

> of the child's growth: first, informal relatively unstructured explorations; second, discovery of patterns and generalizations; third, proof (from axioms) of the relevant theorems.[21]

The first stage utilizes the naive stratagem as does the second stage. The third stage utilizes the deductive stratagem.

That the Project believes that learning exhibits a conjecture and refutation process is evidenced by the following statement which Davis makes when discussing the child's 'natural modes of learning':

> The learner starts with a relevant, but 'wrong' cognitive structure--a vastly over-simplified 'map' of home or city or English linguistics. As a result of new experiences, the learner modifies this structure, either mildly (which Piaget calls 'assimilation'), or else drastically (which Piaget calls 'accommodation'). In this way the experience has been coded by the learner in a highly personal form, not coded absolutely and independently of all else, but used to modify the learner's personal mental imagery and the dynamics thereof.[22]

Moreover, Davis argues that people do not see reality; rather "they map the world into their cognitive schema."[23] Hence, Davis views the learning of mathematics as a process--a conjecture and refutation process--which fosters a student's growth of knowledge of mathematics by means of a revision of the student's cognitive map of the discipline of mathematics. As with the Fallibilistic approach, the Madison Project views knowledge as being tentative and inconclusive and subject to continual modification and improvement.

---

[21] Robert B. Davis, "The Madison Project--A Brief Introduction to Materials and Activities," The Madison Project, Revised September, 1965, p. 10.

[22] Robert B. Davis, "Goals for School Mathematics: The Madison Project View," Journal of Research in Science Teaching, Vol. 2, 1964, p. 313.

[23] Davis, op. cit., "Mathematics Teaching . . . ," p. 11.

From a Fallibilistic point of view, what Davis seems to be arguing for, as evidenced by the general student's objectives, is that mathematics is a process which is open-ended; that is, a process which is not conclusive or final, but which is characterized by successes and failures, by corroboration and refutation. The analysis of these objectives would seem to support the claim that the Madison Project emphasizes a critical and rational approach to the discovery of mathematical knowledge. Furthermore, in viewing mathematics as a process rather than an end-product, the Madison Project focuses on the mode of inquiry of mathematics. Such a view means that the creative portion of mathematics is its processes of creation and not the finished results found in the theorems of textbooks.

Moreover, Davis' view of the three stages of discovery closely parallel the Fallibilist model of naive and deductive stratagems. Even the Piagetian leanings of the Madison Project are not entirely foreign to the Fallibilist for the successive modification and extension of cognitive schema is similar to the origination, testing, proving and concept extension pattern of the Fallibilistic model. An in-depth investigation of this relationship between Critical Fallibilism and Piagetian psychology could prove to be a potentially fruitful piece of research.

The Madison Project is sympathetic to Polya's views of the plausible reasoning aspect of mathematics. Indeed, the Madison Project's emphasis on mathematics as a process can be seen as an application of Polya's appeal for the teaching of mathematical 'know-how'. It was shown earlier (see Chapter IV, section II) that the focusing on mathematical know-how, on plausible reasoning, constitutes a basis for the naive heuristic and consequently the naive stratagem of teaching.

In discussing the role and nature of theories, Davis has recently produced the following description of the utility of theories:

(A'_1) Theories are derived from theories by EXTENSION.

(A'_2) Theories shape facts.

(B') Theories often determine the acceptance or rejection of facts.[24]

Popper's description of Fallibilism is based on similar conclusions when he argues that the falsification of theories gives rise to new and better theories; when he contends that facts are indeed theoretical terms; and when he conjectures that theories provide new interpretations of so-called immutable facts.

Hence, it seems that the Madison Project and the philosophy of its director, Robert B. Davis, without specifically attempting to do so tends towards a Fallibilistic orientation to the growth of mathematical knowledge. The conclusion that the Madison Project exhibits all the characteristics of a Fallibilistic approach to the teaching of mathematics cannot be supported however. The Project does not identify the three phases of mathematical inquiry which the Fallibilistic approach demands. Nor does the Project identify the two stratagems of teaching, although it seems reasonable to conclude on the basis of the analysis of objectives given earlier that the Project is concerned with the naive generation of hypotheses and the testing of such hypotheses. But these are indications of tendencies only--tendencies towards a Fallibilistic approach to the

---

[24] Ibid., p. 7.

teaching of mathematics. Any stronger conclusion than this is simply not justifiable on the basis of the evidence which is available.

How far the Madison Project has progressed along the path of developing instructional models compatible with the Fallibilistic viewpoint is the topic of the next subsection. Any deficiencies which may exist in the Madison Project as a Fallibilistic approach to the teaching of mathematics should not be construed as criticism of the Madison Project. Such deficiencies do suggest, however, future areas of research and development in the area of mathematics education.

The Madison Project: A Fallibilistic Appraisal

In this subsection, instructional strategies developed by the Madison Project are described and assessed in terms of the Fallibilistic model of instruction derived in the second section of the present chapter. This appraisal takes the form of determining to what degree these models of instruction illustrate Fallibilistic strategies of teaching. Points of agreement and discrepancy between the two approaches--Fallibilistic and Madison Project--are noted and discussed.

It was seen in the previous subsection that the Madison Project exhibits in its statement of objectives certain Fallibilistic tendencies. As a consequence, it is not surprising that the instructional models of the Madison Project also exhibit such tendencies. Hence, the appraisal which follows assesses the degree to which Madison Project instructional models illustrate Fallibilistic strategies of teaching. It should also be obvious that such an appraisal cannot be done on a quantative basis, but must be done qualitatively. The assessment indicates the points of contact between the Fallibilistic strategies of teaching and the Madison Project instructional models. Moreover, the instructional models discussed are not exhaustive of all the models developed by the Madison Project, but they are representative of each of the three phases of mathematical inquiry and consequently the three stratagems identified earlier. The models used are illustrative of the Madison Project's focus on mathematics as a process--a conjecture and refutation process--which is open-ended and which views the acquisition of knowledge as a modification of cognitive schema. These cognitive schema are in no way final, but, as with all knowledge, are tentative and inconclusive. However, as with theories, each cognitive schema gives rise to other schema, schema which are hopefully more inclusive and more powerful than those previously available.

Madison Project Origination Strategies. In the model of instruction developed earlier, two basic stratagems of teaching were derived and described. The distinction between the two stratagems--naive and deductive--stemmed from the form of the origination procedures utilized in each of the stratagems. The origination process in the naive stratagem was described as being a guessing and testing procedure which has as its objective the creation of a plausible conjecture which can then be subjected to more severe testing and possibly proof. The conjecture and attempted refutation procedure is of course the Fallibilistic approach to the creation of knowledge. The Madison Project incorporates this instructional strategy of naive origination into their materials and classroom lessons. Indeed, in filmed sequences of Madison Project classes, Davis (or other Project personnel) encourage students to guess at solutions and encourages other students to challenge these proposed solutions. The following excerpt taken from the Madison Project film "Guessing Functions" exhibits the Fallibilistic characteristics of naive guessing.[25]

The situation this class was presented with is the following: three students are asked to make up an open sentence of the form $mx + b = y$; that is, a linear equation. The remainder of the class is charged with the task of 'guessing' what function these three students are using in creating a table of values for the function. The class suggests values for 'x' and the three students using their 'rule', as it is called, determine the corresponding value of 'y'. In the Madison Project form of equation writing, the sentence under discussion was $(\square \times 11) - 4 = \triangle$. A table is kept of the suggested numbers and the panel's responses to these suggestions. For the sentence given above, the table might begin with one student suggesting (as he did in the film) the numeral seven:

□	△
7	

The panel applying their 'rule' would respond with seventy-three.

□	△
7	73

[25] Kinescope of a videotape produced at Saint Louis in 1964. This film is available from the Madison Project. The children used were seventh-graders and the teacher was Robert B. Davis.

Another student might suggest nine receiving the response of ninety-five from the panel. The table would then appear as follows:

□	△
7	73
9	95

At this point in the film sequence, a student named Will conjectured that the rule is (□ x 10) + 3 = △. Davis in responding to this guess suggested that Will should test his guess. Obviously, the conjecture is made on the basis of the first pair of values only. The conjecture is immediately refuted when nine is substituted into the proposed solution. What is of importance here is that Davis encourages Will to try to improve his guess--to study the errors in his guess with a view to coming up with an improved conjecture. The rule is then applied to the numerals five, two, and one obtaining respectively the responses fifty-one, eighteen and seven so that the table now appears as follows:

□	△
7	73
9	95
5	51
2	18
1	7

Another student guesses the rule to be (□ x 7) + 14 = △. This is also refuted. The process through the sequence this far is one of conjecture and refutation. No attempt is made to suggest patterns of improving these conjectures. The guessing is naive. However, later in the filmed sequence, the following dialogue takes place (the speaker is Robert Davis).

> Let's see now. Seven gave us seventy-three, and nine produced ninety-five. If the panel were to use their rule on eight, what do you suppose they would tell us? Eighty-four! That would be my guess, but don't bet on it. What does the panel say? Eighty-four!

A student named Lori then guesses that the rule is $(\square \times 11) - 4 = \triangle$. What Davis has done of course is to focus the student's attention on the difference between the numbers in the right-hand column of the table. The students make the additional guess that the value corresponding to eight is halfway between seventy-three and ninety-five: hence eighty-four. The difference between two consecutive values is eleven. It could then be argued that the rule is $\square \times 11$. In the case of eight, this rule would yield eighty-eight, whereas the desired response is eighty-four. A simple adjustment yields the conjecture that the rule is the one made by Lori.

The final arrival at this conjecture is a guess and test process. The student makes a guess, tests it, and revises it if it is refuted. In such a classroom situation, the climate is one which encourages guessing, one in which no stigma is attached to wrong guesses. There is nothing wrong with being wrong. Indeed, as Davis points out, everything one knows is to some extent wrong:

> Every idea of every one of us on every subject is wrong--partly wrong, that is. We learn by successive approximations, and there IS no final and absolutely perfect "ultimate version" in any of our minds. We are wrong, but we can learn; having learned, we shall still be wrong, but less so; and, after that, we can still develop a yet more accurate cognitive representation, within our minds, for the various structures that exist independently of our minds.[26]

---

[26] Davis, op. cit., "The Madison Project's Approach . . . ," p. 153.

The successive approximations which Davis talks about is the Fallibilistic pattern of origination and testing. This instructional strategy would seem to exhibit the characteristics of the Fallibilistic pattern of naive guessing.

The guessing pattern of naive origination is also utilized by the Madison Project in other instructional settings such as their treatment of the topic of quadratic equations.[27] Consequently, this instructional pattern is representative of a mode of teaching utilized by the Madison Project which is Fallibilistic in nature.

It was noted earlier that the Madison Project insists on a formal axiomatic foundation for mathematics. As a means of developing this foundation, the Madison Project also utilizes deductive guessing as an origination procedure. Deductive guessing as may be recalled begins with an idea from which there flows a series of deductive steps culminating with a conjecture. The difference between the conjecture obtained in this manner and the naively generated conjecture is that the former type of origination comes with a built-in proof-analysis.

The Madison Project in stressing the deductive aspects of mathematics have developed or adopted instructional strategies which foster the development of the use of deductive guessing as a means of conjecture generation. This deductive strategy is illustrated in the next example which is taken from Explorations in Mathematics, a book written by Davis,

---

[27] Davis, op. cit., Explorations in Mathematics.

and shows how in a game situation the processess of deductive guessing may be utilized.[28] The game is entitled 'Clues' and is an adaptation of the game 'Hidden Numbers' developed by David Page of Educational Services, Incorporated. The description of the game and an illustration of it are given below:

The rules for the game of clues are as follows:

One team (or one person) has a secret. Let's call this team TWS, for "team with secret". The other team seeks to discover this secret. Let's call this team DISC, for discovery.

1. TWS writes some numbers on a piece of paper which is then sealed in an envelope, or otherwise put where it cannot be read. (For example, someone can fold the paper and sit on it.)

2. DISC seeks to force TWS to disclose the "secret" numbers, and to let everyone read the paper.

3. Only positive integers are allowed. Repetitions are allowed; for example, the secret numbers might be:

1,3,5,7,7,7,7.

4. In guessing the secret numbers, DISC does <u>not</u> have to guess the <u>order</u> in which they are written; for example,

7,3,5,7,1,7,7,

would count as the same list as the one given in the rule preceding.

5. TWS writes <u>clues</u> on the board, labeling the clues a, b, c, . . . and so on (it is desirable to omit "F" and "T" as labels, since we have a different use for them).

6. The clues may be TRUE or they may be FALSE.

7. Anytime that DISC believes there is a <u>contradiction</u> in a certain set of clues, DISC <u>lists</u> the clues in question and tries to show that there <u>is</u> a contradiction in these clues.

---

[28] <u>Ibid</u>., pp. 156-160.

8. DISC is right about the contradiction if the clues they list do contain a contradiction, and if no proper subset of the clues on the list contains a contradiction.

9. DISC is wrong about the contradiction if the clues they list do not contain a contradiction or if a proper subset of the clues does contain a contradiction.

10. At the start of the game, DISC has 5 points.

11. Anytime DISC is wrong about a contradiction, it loses one point.

12. Anytime DISC is right about a contradiction, TWS must mark T (for true) or F (for false) beside each clue that is involved in the contradiction. TWS must be correct in marking T's and F's (even though TWS is allowed to make some of the clues themselves false.)

13. The game ends in one of two ways: If DISC loses all 5 points, then TWS tears up the secret paper and never allows it to be read (DISC has "lost"). If, on the other hand, DISC is able to force disclosure of the paper, then everyone on the DISC team is allowed to read it, and DISC has "won".

14. The procedure by which DISC may be able to force disclosure of the secret is this: whenever it believes it is in a position to do so, DISC can list the numbers that it believes must be written on the paper, and can bet TWS that no other collection of numbers would satisfy all the known truth values of the clues. (That is, no other collection of numbers would make true statements of all the clues labeled T and false statements of all the statements labeled F.) If TWS can find any other collection of numbers that will be consistent with the T's and F's, then DISC loses the bet, and DISC's points are reduced to zero. (Which, of course, means the secret paper is torn up and the numbers never disclosed.)

If TWS cannot find any other collection of numbers that will be consistent with the indicated T's and F's, then DISC wins the bet, and TWS is forced to disclose the secret.

In order to make the game interesting, TWS must provide a growing collection of interesting clues.

Here is a sample game:

DISC begins, of course, with 5 points.

TWS begins by listing these clues.

a. 5 numbers on paper.
b. All odd numbers.
c. Their sum is 26.
d. The largest number is 7.
e. The smallest number is 8.

DISC says there is a contradiction in clues a, b, and c, because an odd number of odd numbers cannot add up to an even total. (This is of course a conjecture obtained deductively from previously known results.)

Since DISC is right about {a, b, c}, it is necessary for TWS to label a, b, and c as either T or F; TWS does this as follows:

F a. 5 numbers on paper.
T b. All odd numbers.
F c. Their sum is 26.
d. The largest number is 7.
e. The smallest number is 8.

TWS changes the clues to look like this:

a. 7 numbers on paper.
T b. All odd numbers.
c. Their sum is 12.
d. The largest number is 7.
e. The smallest number is 8.

DISC says that {a, b, c} still contains a contradiction; an odd number of odd numbers cannot add up to an even sum. (Again, a deductively produced conjecture. Both of these conjectures are not tested but accepted as being valid.)

Since DISC is right about this contradiction, TWS must label a, b and c as T or F. They do this as follows:

T a. 7 numbers on paper.
T b. All odd numbers.
F c. Their sum is 12.
d. The largest number is 7.
e. The smallest number is 8.

DISC says that { d, e} contains a contradiction, because the largest number cannot be smaller than the smallest number. (An obvious result.)

Since DISC is right about this, TWS must mark T's and F's on {d, e}. They do this as follows:

T a. 7 numbers on paper.

T b. All odd numbers.
F c. Their sum is 12.
T d. The largest number is 7.
F e. The smallest number is 8.

TWS changes the clues to read like this:

T a. 7 numbers on paper.
T b. All odd numbers.
c. Their sum is 13.
T d. The largest number is 7.
F e. The smallest number is 8.

Although they are not forced to do so, TWS labels clue c as T, in order to make the game move along faster. The clues now look like this:

T a. 7 numbers on paper.
T b. All odd numbers.
T c. Their sum is 13.
T d. The largest number is 7.
F e. The smallest number is 8.[29]

The Madison Project goals for this lesson are ". . . to give children experience with such mathematical ideas as IMPLICATION, CONTRADICTION, and UNIQUENESS."[30]

But how did the pattern of deductive guessing apply in this situation? The conjecture which was finally made was that the set of numbers is 7,1,1,1,1,1,1. This conjecture was based on the final set of clues given above. These clues were arrived at by seeking contradictions in the first set of clues given. Hence, this was a process of attempting to uncover contradictions in order to improve one's conjecture as to the final set of numbers. Consequently, refutation is seen as a conjecture improving process. Moreover, this instructional model gives children an

[29] Ibid., pp. 156-159.

[30] Ibid., p. 156.

opportunity to develop their critical abilities. They test clues provided in order to uncover hidden contradictions, and they test them in a deductive fashion.

Once this final set of clues is obtained, the emergence of a conjecture is a deductive process. Davis reconstructs the deductive sequence as follows:

> Since we have only odd numbers, the largest of which is 7, we know that there is at least one 7 on the paper and that the other numerals, if any, are 1, 3, and 5.
>
> Now, since there are 7 numerals on the paper, they cannot be too large, or the sum will exceed 13. Let's see, is 7,1,1,1,1,1,1 possible? Yes, since 7 + 1 + 1 + 1 + 1 + 1 + 1 = 13.
>
> But . . . . if we increase any numeral on the list the sum will be too large! Hence, 7,1,1,1,1,1,1 is the only possible answer.[31]

This is a deductive process which yields a conjecture--a conjecture with a built-in proof-analysis. It is difficult to identify some basic idea from which this proof-analysis developed. Nevertheless, it does illustrate the creation of a conjecture by deductive means. Moreover, if the conjecture happened to be refuted, this refutation would force a reexamination of the proof-analysis to perhaps uncover contradictions not yet identified in the set of clues. Hence, again refutation is utilized in this deductive stratagem to improve a conjecture. The goal of setting out simultaneously to prove and refute a conjecture is illustrated by the above example. The children attempt to refute the clues as being consistent in an attempt to finally provide a valid proof-analysis.

Moreover, the conjecture which was finally obtained was produced

---

[31] Ibid., p. 159.

only after a series of deductively derived conjectures had identified contradictions in the sets of clues provided. It is indeed doubtful if a problem is ever solved by the immediate production of a complete proof-analysis, at least not problems which can be classified as non-trivial. The preliminary steps in the development of a proof-analysis may be deductive or they may in fact follow naive patterns of origination.

Madison Project Testing Strategies. As indicated earlier when a model of the mode of inquiry and the model of instruction were developed, the testing phase of mathematical inquiry and derivatively mathematics teaching strategies is composed of the process of attempted refutation and some plausible reasoning techniques. The primary function of the testing phase from a Fallibilistic orientation is the attempted refutation of a conjecture if this conjecture was naively generated, and the identification of weaknesses in the proof-analysis of a deductively generated conjecture. The testing phase is designed to refute a conjecture or its proof-analysis. Only if such attempted refutations fail do the patterns of plausible reasoning become operative. It was noted however that plausible reasoning techniques cannot provide final answers, but are only suggestive of potentially fruitful conjectures.

From a pedagogical point of view, the testing phase should develop a learner's ability and skill in criticizing in a rational manner proposed solutions to a problem. It is designed to foster the learner's ability to criticize all possible solutions to a problem and to encourage the learner not to accept any proposed solution at its face value.

The Madison Project's instructional strategies encourage learners to adopt a critical attitude towards solutions. Furthermore, the Project has developed a technique, called 'torpedoing', which is designed to make students aware of the tentativeness of conjectures which are supported by plausible reasoning techniques only. The following example not only illustrates the Madison Project's testing strategy, but it also illustrates the way in which the Madison Project views mathematics as discoverable and open-ended.

> In the present instance, remember, we wanted the children to get some experience using VARIABLES, and working with the ARITHMETIC of signed numbers. We consequently gave them quadratic equations to solve, beginning with
>
> $$(\square \times \square) - (5 \times \square) + 6 = 0$$
>
> and gradually progressing to harder problems, such as
>
> $$(\square \times \square) - (20 \times \square) + 96 = 0.$$
>
> Now, at first the only method available to the student was, of course, trial and error. If he makes no discoveries, the student continues with this method, and gets full benefit from the "basic" part of the lesson; that is, he gets a great deal of experience using variables and signed numbers, and in a situation where he does <u>not</u> regard this as "drill".
>
> But--if the student discovers the so-called "coefficient rules" for quadratic equations, his use of trial-and-error can be guided to maximum efficiency. He has discovered a "secret"--and one which his classmates don't know. They may <u>never</u> know![32]

In presenting this situation to children, the Madison Project usually tells the children there is a 'secret' way of solving these problems. In doing so, the Madison Project is attempting to have children take a constructive view of errors--a view which focuses on the critical

---

[32]Davis, <u>op</u>. <u>cit</u>., "Some Remarks . . . ," p. 2.

analysis of proposed solutions to problems with the goal of identifying errors and weaknesses--and, indeed, this active searching for errors and the criticizing of proposed solutions represents the difference between 'raw' trial-and-error and the conjecture and refutation process.

Davis has the following to say on the technique of 'torpedoing' a student's naive conjecture which the student has now supported by corroborating instances:

> In a similar way, with the quadratic equations we begin by using only unequal prime roots, so that one of the two "coefficient rules" (the product rule) is extremely obvious. Using it alone leads to easy solutions of the equations, such as
>
> $$(\square \times \square) - (5 \times \square) + 6 = 0$$
>
> $$(\square \times \square) - (12 \times \square) + 35 = 0 \; . \; . \; .$$
>
> and so on.
>
> Here also, once the student is really pleased with his "discovery" and with the new power it has given him, we confront him--unobstrusively and unexpectedly--with a variant problem which will tend to confound his theory.
>
> In this instance, we slip in a problem having composite roots, instead of the prime roots the student previously dealt with.
>
> The product rule now seems to indicate more than two roots; for example, with
>
> $$(\square \times \square) - (9 \times \square) + 20 = 0$$
>
> many students will say the roots are $\{2, 10, 4, 5\}$.
>
> Trial by substitution shows that this is wrong. Again, by persevering, the student finds that there is a broader theory, of which he has found only a narrower part.[33]

By torpedoing Davis means the presentation of a refuting instance--

---

[33] Ibid., p. 3.

a counterexample--which destroys a student's naively generated and plausibly tested conjecture. The example illustrates how a student makes a conjecture, the product rule in the above example, which is confirmed or corroborated in several instances. Moreover, it is easy to corroborate because only unequal prime roots are used initially. The student's confidence in his conjecture is increased--the credibility of the conjecture increases. This is the plausible reasoning pattern which Polya calls the examining of a consequence and which is part of the testing phase of mathematical inquiry. However, the conjecture is scuttled when quadratic equations with composite roots are introduced. Obviously caution must be exercised when using plausible reasoning techniques. Accordingly, the example shows how knowledge grows by refutation, for it is the fact that the primitive conjecture is torpedoed which forces the student to revise and expand his conjecture.

The Madison Project then would seem to utilize testing procedures in its instructional models. Indeed, the Project seems to have successfully developed teaching strategies which not only make use of plausible patterns of inference, but also one which adopts the Fallibilistic approach of refutation, or torpedoing in the Project's terminology.

Madison Project Proving Strategies. The Madison Project has as a stated orientation the development of mathematics on a firm foundation which is deductive in nature; that is, a basis which views the end-product of a mathematician's activities as being a deductive sequence of statements. The Madison Project attempts to have students become aware of this basis by dealing with the elements of symbolic logic, the nature of axioms, definitions and theorems. The Madison Project book Explorations in Mathematics written by Davis provides several excellent examples of this orientation. The particular example chosen for discussion below is concerned with the Project's treatment of mathematical identities.[34]

In this sequence of activities, the Madison Project has children create their own list of identities. These identities are initially called "open sentences that will become true for every legal substitution."[35] An example of an identity which children seem to have no difficulty in creating is the following:

$$\square \times 0 = 0$$

The process of creating such a list of identities is accomplished on an inituitive, trial-and-error basis. After the initial list has been developed, the Madison Project approach does encourage children to become more explicit and systematic in the creation of more identities.

The next step in the sequence of activities dealing with identities is that of taking the list which has been developed and shortening it. Here, the Project is attempting to intuitively define the nature, role, and function of axioms and theorems.[36] Once a primitive set of axioms has been obtained (and this may not be the most primitive set possible, if such a list is possible to achieve), the children are asked to reverse

---

[34]Davis, op. cit., Explorations . . . , pp. 169-200.

[35]Ibid., p. 171.

[36]Ibid., p. 178.

the process in order to develop theorems from their axioms.[37]

Though the Madison Project does not explicitly deal with levels of universality in this example, it does seem that implicitly the Project is attempting to have the children identify the different levels of generalizability in a mathematical system. The identification of a set of basic axioms and the subsequent proof of other statements as theorems seems to be an attempt to have the children place the various statements they have generated by naive guessing on different levels of universality.

In developing or creating a proof-analysis of some theorem from the accepted list of axioms, the Project encourages learners to be critical and to attempt to find 'shorter' or 'better' proofs than the ones previously offered. The children are encouraged to challenge the proof-analysis put forth by other students. However, the Madison Project does not attempt to identify the monster treatment techniques discussed earlier. The children using the Project's materials and approach do attack proofs with a critical and rational attitude, but they are not instructed in the various techniques of saving a conjecture or proof-analysis.

The Madison Project does not deal with the techniques of treating counterexamples in a way which would be characteristic of a Fallibilistic approach to the teaching of mathematics. Although the Project does utilize the refutation approach to the teaching of mathematics, it does not explicitly or implicitly identify the monsterbarring, exception-

---

[37] Ibid., pp. 180-184.

barring, or lemma-incorporation techniques. This is not surprising however for a number of reasons. First, the Madison Project though tending towards Fallibilism is not based on Fallibilism. Second, the techniques of treating counterexamples were not explicitly identified, to the investigator's knowledge, until after the Madison Project materials were published. Finally, it is possible that the Madison Project does not consider these techniques, if they are aware of them, appropriate to the age range of children for which their materials are designed.

This is not to say that the Madison Project does not concern itself with the proving aspects of mathematics and the teaching strategies appropriate to this phase of mathematical inquiry. Even a cursory glance at the Madison Project's materials will substantiate this view. However, the Project has not and does not consider the techniques of treating counterexamples which Lakatos has identified.

IV. CHAPTER SUMMARY

The present chapter set out to accomplish two aims. First, the chapter was designed to develop an instructional model or teaching strategies based on the Fallibilistic model of the mode of inquiry of mathematics created in Chapter IV. Second, the instructional approaches utilized by the Madison Project were discussed in terms of the Fallibilistic model of instruction.

The model of instruction was seen to be composed of two basic stratagems, the naive stratagem and the deductive stratagem. The naive stratagem was composed of two substrategies, so to speak, which were designated as the TP strategy (denoting the fact that the testing phase preceded the proving phase), and the PT strategy (in which the proving phase precedes the testing phase). The deductive stratagem differed from the naive stratagem in the area of conjecture origination. The deductive stratagem utilized deductive guessing and teaching strategies pertaining thereto, whereas the origination strategy characteristic of the naive stratagem was that of naive guessing. The various phases of each of these stratagems were discussed at length and examples were given which illustrated how a teacher and students could come to function Fallibilistically.

The broad curricular and instructional viewpoint of a Fallibilistically oriented teacher was described. It was noted that the teacher's role was to set the learning situation, aid the students in developing a means of testing their conjectures, and to guide the students unobtrusively and without moralistic assessment towards a growth of their mathematical knowledge. The teacher's overall objective was to have students develop their skills in originating, testing, and proving mathematical conjectures. The teacher designed the learning situation which enabled the students to explore unknown terrain on their own with a minimum of guidance from the teacher. The teacher accomplished this task by adopting a rational and critical attitude towards proposed conjectures, an attitude which avoids making affective decisions relative to the student who made the conjecture.

Once the students had become involved in the learning situation, the teacher was viewed as assuming a relatively passive role. His task attempt to solve the problem situation set by the teacher. (It should be pointed out that it is not always the teacher who sets the problem. The students could do this initially, or they may identify other problems as they are working in a learning situation.) Within this general approach, the teacher and student utilized the TP and the PT strategies of the naive stratagem and the DED strategy of the deductive stratagem. In each of these strategies, the techniques of monster treatment were utilized. Furthermore, the growth of mathematical knowledge was fostered by the constant progression to ever higher levels of universality. However, this did not mean that the student cannot or did not revert to the quasi-empirical basis of mathematics in order to test their conjectures. The illustrations provided gave instances of students reverting to an empirical test of their, and the teacher's, conjectures.

These examples and illustrations were hypothetical only and as a result their fruitfulness or usefulness in actual classroom situations is undetermined. However, the fact that philosophically based strategies of teaching can be derived (and which can then be empirically tested) speaks in support of the approach taken in the study as being fruitful.

It was concluded on the basis of an analysis of the objectives of the Madison Project that this project does indeed have Fallibilistic tendencies in its orientation to the teaching of mathematics. The Project was seen as encouraging the use of a conjecture and refutation process for the growth of a student's mathematical knowledge. In addition, such knowledge was concluded to be open-ended or tentative thus agreeing with the Fallibilistic viewpoint concerning the status of knowledge (and, it may be noted, Schwab's as well). The general student objectives gave rise to the conclusion that the Madison Project has as a primary goal the development of a critical, rational attitude to the growth of knowledge. Accordingly, the growth of knowledge was seen as being characterized by more than a trial-and-error procedure, a point made earlier by Popper. Finally, the Madison Project seems more concerned with the processes of mathematical inquiry than with the substantive structure of this discipline.

A psychological overtone was identified in the statement of the Madison Project's objectives. Indeed, the Piagetian leanings of the Project is evidence of the influence modern psychological theory is having on the Project. However, the psychological base of the Project was not considered here for it is beyond the scope of the study. The study takes a philosophical approach rather than a psychological one. As a consequence, the psychology of learning has not been considered and this fact must be noted as a limitation of the study.

In assessing the instructional models developed by the Project, it was observed that the Fallibilistic strategies of teaching were adequately dealt with by the Project's instructional models in the areas of origination and testing. Some deficiencies in the Madison Project instructional models were identified in the area of proving strategies. The techniques of monsterbarring, exception-barring, and lemma-incorporation have not been treated by the Madison Project. These deficiencies should be taken as a spur for further research and empirical testing of the proposed strategies of teaching.

Additionally, as evidenced by the successes of the Madison Project not only by increasing student interest and involvement in the creative processes of mathematics, but also by the simplified yet fundamental means of treating advanced topics at earlier grade levels, the Fallibilistic approach to the teaching of mathematics provides ever increasing grounds for potentially fruitful research.

It would be hoped that if teachers are made aware of the particular Fallibilistic approach to the teaching of mathematics developed in this study, then they will begin to attempt to deepen children's understanding of the processes of mathematical inquiry with the result that the now existing superficial distinction between inductive and deductive patterns of mathematical inquiry will disappear.[38]

---

[38]For an assessment of the successes and failures of the Madison Project see Robert B. Davis, _A Modern Mathematics Program as it Pertains to the Interrelationship of Mathematical Content, Teaching Methods and Classroom Atmosphere_. _The Madison Project_. (Final Report submitted to the U.S. Department of Health, Education, and Welfare, 1967.)

CHAPTER VI

CONCLUSIONS AND IMPLICATIONS

I. INTRODUCTION

The concluding chapter of a study is designed to provide a summary of and major conclusions of the report. In addition, the implications the study may have for further research should be outlined. Consequently, in order to relate the conclusions and findings to the study as a whole, a restatement of the objectives of the study is given first. This is provided so that the relationship between the objectives and the conclusions of the study are available for easier comparison. After the statement of the purposes of the study, a summary of and the general conclusions of the study are presented. These are followed by a statement of the limitations both of the study and of the range of generalizability of the conclusions. The final two sections of the present chapter deal with the implications of the study: implications, first, for mathematics education, and, second, implications for further research.

The goals of the present chapter then are to provide (1) a restatement of the objectives and summary of the study; (2) a statement of the conclusions and limitations of the report; and (3) a statement of implications and indications of areas in need of further research.

II. THE PURPOSE OF THE STUDY

The study was divided into two phases in order to deal with the basic objective of the study which was to describe the practical applications of a philosophically based model of mathematical inquiry by an appraisal of a modern curriculum and instruction project. The first phase of the study was to develop a model of mathematical inquiry utilizing Critical Fallibilism as a philosophical basis. Since this philosophical position was adopted, the study included a description of Critical Fallibilism and its application to mathematics. The emphasis in that section of the report was on describing Critical Fallibilism and not on evaluating the philosophical acceptability of it. Hence, Fallibilism and its application to mathematics was taken as given and the model was deduced from this orientation.

The second phase of the study had several purposes. A Fallibilistic model of instruction was derived from the previously obtained model of a mode of inquiry in mathematics. This instructional model was then used to develop teaching stratagems and the resultant substrategies of these larger stratagems. Finally, the instructional paradigm was used to assess a sample of the instructional models created by the Madison Project.

The specific goals of the study were to (1) develop a model of a mode of inquiry of mathematics using Critical Fallibilism as a philosophical basis, (2) derive a model of instruction and develop teaching stratagems from this model, and (3) assess in which ways the Madison Project exhibits Fallibilistic tendencies in its instructional strategies.

III. SUMMARY AND CONCLUSIONS

This section is divided into three subsections. These subsections deal first with a summary and conclusions concerning Critical Fallibilism as a philosophical position and its application to the growth of mathematical knowledge. The second subdivision summarizes and draws conclusions with respect to Polya's results dealing with plausible reasoning aspects of mathematics as well as Lakatos' application of Fallibilism to a description of the growth of mathematical knowledge. As well, it includes the findings embodied in the model of mathematical inquiry with special emphasis being given to the naive and deductive heuristic. Finally, the third subdivision presents the results of the derived strategies of teaching.

Fallibilism and Mathematics

Critical Fallibilism is a particular philosophical orientation developed by Karl Popper to the problem of the growth of knowledge. Five basic characteristics of this position were described; that is, Popper's results concerning the problem of induction, conjectures and refutations, his falsifiability thesis, degrees of testability, and degrees of corroboration. Of these five areas, the problem of induction, and degrees of testability and corroboration are in need of further research and study as to their applicability to mathematics. As a result, only the major conclusions dealing with the areas of conjectures and refutations and Popper's falsifiability thesis are dealt with here. Furthermore, conclusions concerning the implications of Fallibilism for mathematics are given. The conclusions that may be drawn on the basis of the description of Fallibilism and its application to mathematics provided in the study are discussed below.

The growth of knowledge is characterized by a process of conjecture and refutation; that is, in attempting to solve some problem, a conjecture--a guess--is put forth as a possible solution to this problem. This conjecture is then subjected to severe criticism: this criticism is designed to refute the conjecture. It must be noted, however, that the Fallibilist also realizes that for knowledge to grow not only must conjectures be refuted, but also positive successes must be obtained. The deciding criterion of whether or not a hypothesis is a potentially satisfactory one is whether or not the conjecture is testable.

Popper's falsifiability thesis is designed to provide a line of demarcation between statements which may be classified as scientific and those which must be relegated to the class of metaphysical statements. His criterion is that of testability or falsifiability; that is, the hypothesis which is the better testable or stands the better chance of being falsified is to be preferred. The reason for the preference of the riskier hypothesis seems to be the fact that if it is indeed false it will be refuted more quickly. A less risky conjecture may survive longer, but at the same time it would be less informative and ultimately false. As a consequence, the proposing of a less risky hypothesis would slow the growth of knowledge: at least, this is the Fallibilistic view as put forth by Popper.

Fallibilism, then, sees the growth of knowledge as being spurred by refutation rather than stifled by refutation. Positive successes must be obtained, but if only positive successes are met, knowledge would not grow and expand. If a conjecture is always corroborated, if not even minor refutations to it are discovered, then the conjecture attains the position of not being tested--not revised--and as a result, the growth of knowledge slows and is concerned only with the attempt to find confirming instances of the conjecture. This the Fallibilist cannot accept.

The reason the Fallibilist cannot accept this situation is that he views all knowledge as tentative. No ultimate and final foundations of knowledge can be achieved, according to the Fallibilist, although knowledge can approach ultimate truth. In order to do so, however, the Fallibilist was seen as arguing that the process of approaching ultimate truth and knowledge can be fostered only by a constant criticism, revision, and expansion of existing knowledge. Hence, although ultimate truth may exist, this truth cannot be achieved according to the Fallibilistic view, but it can by a process of conjecture and refutation, by concept revision and expansion, by rational and critical analysis, be constantly approaching this situation.

As applied to mathematics, the Fallibilist position, as interpreted by Lakatos, views the growth of mathematical knowledge as a conjecture, proving and refutation process. A conjecture is offered in an attempt to solve a mathematical problem. A proof of this conjecture is constructed. Then both the conjecture and the proof-analysis are subjected to severe criticism--criticism in the form of the attempted discovery of counterexamples to either the conjecture or to the proof-analysis. The acquisition of mathematical knowledge on this basis would be a three stage process--origination, testing, and proving.

Moreover, from a Fallibilistic viewpoint ultimate foundations of mathematics can not be obtained. This conclusion, which is drawn by Lakatos, is based on the skeptics criticism of the Euclidean Programme, the Empiricist Programme, the Inductivist Programme, and the Hilbertian Programme. The question of whether a primitive set of axioms exists which could provide a foundation for mathematics was answered in the negative.

These then are the major conclusions which may be drawn from the description of Critical Fallibilism given in the study. They represent a Fallibilistic view of the growth of knowledge, and, specifically, the growth of mathematical knowledge. They were taken as a basis for the description of the methodology, the heuristics, of mathematical inquiry as well as for the development of the model of the mode of inquiry of mathematics.

Methodology of Mathematics

The results outlined in this section are those dealing with the heuristics, the situational logic, of mathematics. As such, the section includes Polya's and Lakatos' results as well as the conclusions to be derived from the description of a model of mathematical inquiry.

The application of Fallibilism to a description of the growth of mathematical knowledge resulted in a characterization of the processes of inquiry in mathematics as consisting of three phases. These three phases were called the origination phase, the testing phase, and the proving phase of mathematical inquiry. It was found that an order of precedence could not be established for these three phases other than that the origination phase must precede the testing phase. Two basic heuristic patterns were also identified; that is, the naive heuristic in which origination precedes the testing and proving phase, and the deductive heuristic in which case the proving phase precedes the origination and testing phases.

Two patterns of flow were identified in association with the naive heuristic. The first pattern was one of origination followed by the testing and proving phases respectively. The second pattern began again with origination, but in this case the proving phase preceded the testing phase. On the other hand, the deductive heuristic had only one pattern associated with it. In this heuristical pattern, the proving phase is followed by the origination phase and subsequently the testing phase. Indeed, it could be argued that with the deductive heuristic pattern the proving and origination phases progress simultaneously, as it were, since the immediate result of the construction of the proof-analysis is a mathematical conjecture. As a consequence, the conjecture comes with a built-in proof-analysis which can then be subjected to severe tests.

Polya's first contribution to a description of mathematical inquiry came in his description of the origination phase. The role of analogy and specialization in the origination of mathematical conjectures was described. These were seen as means of POSSIBLY obtaining a conjecture--a mathematical guess--which are not guaranteed to produce a potentially satisfactory hypothesis, but which are only helpful in perhaps delimiting the problem under consideration or in identifying analogous problems. As such these techniques were seen as serving a suggestive function; suggestive in the sense that they may be useful in obtaining a conjecture, but which do not guarantee success. Lakatos called this pattern of obtaining a conjecture naive guessing. He classifies it as being non-inductive, a conclusion derived from his Fallibilistic orientation to the growth of mathematical knowledge. Whether it is inductive or not is not of central importance. What is of importance is that there is an identifiable pattern of origination which is characterized by a conjecturing process--a guessing process. Moreover, this pattern of origination is distinct from the pattern of origination called deductive guessing. With this latter pattern, Lakatos contends that the proving phase precedes or at least progresses along with the origination phase. Furthermore, deductive guessing was seen as progressing from some simple idea and proceeding deductively to a conjecture. Hence, the conjecture is the result of the construction of a proof-analysis.

The testing phase of mathematical inquiry has as its function the refutation, attempted refutation, of a conjecture and/or proof-analysis. Polya's three patterns of plausible inference were seen to apply only when a test of a conjecture had failed; that is, the conjecture was not refuted. Hence, these patterns of plausible inference--examining a consequence, examining a possible ground, and the examination of a conflicting conjecture--came into operation AFTER a test of the conjecture had been made. Moreover, Polya concluded that even in this situation, these patterns could only indicate the direction of support provided by the test, and not the strength of evidence with respect to the conjecture under consideration.

From the Fallibilistic viewpoint, namely Lakatos' interpretation of the position, the testing phase is governed by the Principle of the Retransmission of Falsity; that is, if a counterexample to a conclusion is obtained, then this counterexample will also be a counterexample to either the conjecture or to the proof-analysis constructed for the conjecture. It was concluded, however, that this principle explicated by Lakatos and the patterns of plausible inference propogated by Polya were not in competition. If the test of a conjecture is successful, then the conjecture is refuted by virtue of the principle of the retransmission of falsity. If the test is unsuccessful, then the conjecture is corroborated and Polya's patterns apply. In addition, for a deductively originated conjecture the testing phase of mathematical inquiry was seen as fulfilling another purpose. This purpose was the identification, the exposure, of weaknesses in the proof-analysis, to discover hidden lemmas and to foster the emergence of counterexamples.

The third phase of mathematical inquiry was the proving phase--the construction of a proof-analysis. In the case of a naively generated hypothesis, the proving phase comes second or third in order, whereas in the case of a deductively generated conjecture, the proving phase precedes all other phases.

The proving phase is designed to construct a proof-analysis of some conjecture. In this Fallibilistic interpretation, proof becomes the decomposition of the conjecture into subconjectures in order to provide a larger target, so to speak, for criticism and possible refutation. Lakatos identified several techniques which form part of the situational logic of the creative mathematician: the techniques of monsterbarring, exception-barring, and lemma-incorporation. While monsterbarring was seen as serving mainly a historical function, exception-barring and lemma-incorporation were both viewed as procedures which attempt to expand the original conjecture and the concepts with which the conjecture deals.

A Fallibilistic Model of Instruction

A model of instruction was derived from the Fallibilistic paradigm of mathematical inquiry. This model was seen as being composed of two basic teaching stratagems. These instructional approaches were called the naive stratagem and the deductive stratagem. The latter stratagem differed from the former in the area of conjecture origination. The deductive stratagem utilized deductive guessing and teaching techniques pertaining thereto, whereas the origination strategy characteristic of the naive stratagem was that of naive guessing.

The naive stratagem was composed of two substrategies which were designated as the TP strategy and the PT strategy. The distinguishing characteristic of the TP strategy was that in the order of precedence of the phases of instruction the origination phase preceded the testing phase which was followed by the proving phase. On the other hand, the PT strategy was distinguished by the fact that the proving phase preceded the testing phase. The deductive stratagem (DED) was characterized by an order of precedence in which the proving and origination phases proceeded simultaneously. The testing phase follows these two phases.

With respect to the paradigm of mathematical inquiry from which the model of instruction was derived, it was seen that the naive stratagem of teaching derives from the naive heuristic, a heuristic in which two patterns of discovery were identified. The deductive stratagem of teaching was likewise derived from the deductive heuristic, a heuristic pattern in which the origination and proof of a mathematical conjecture were accomplished at the same time. Consequently, the two basic stratagems of the Fallibilistic model of instruction, the PT and the TP strategies, and the phases within each of these strategies were derived from and correspond to the heuristics, phases and orders of precedence extant in the Fallibilistic model of the mode of inquiry of mathematics.

The various phases of each of the stratagems were discussed at length and examples were given which illustrated how a teacher and students could come to function Fallibilistically. These examples and illustrations were hypothetical only. As a result, their fruitfulness or usefulness in actual classroom situations is undetermined. They did provide however indications of the roles to be assumed by teachers and students if they are to function Fallibilistically.

The teacher's role was one of setting the initial problem and learning situation, aiding the children in developing effective testing procedures, and as part of the classroom discussion to provide counterexamples and suggestions designed to force the students to expand their mathematical knowledge. The teacher is a guide whose role is to aid students in mapping an unknown mathematical terrain. The teacher must allow students to create, revise and expand their own map, a map which will never be complete in every detail. It follows that the student's role is to create, analyse, revise and expand his map of the mathematical wilderness which he is exploring. In doing so, the student, like the teacher, adopts a rational and critical attitude towards proposed maps. The student has a very active role in this learning situation. The primary goal for the student is not to memorize the map and to reproduce it on demand. Rather, the student was seen as learning how to create the map, and, consequently, how to develop the skills and attitudes for attacking problems Fallibilistically.

It was concluded on the basis of an analysis of the objectives of the Madison Project that this project does indeed have Fallibilistic tendencies in its orientation to the teaching of mathematics. The Project was seen as encouraging the use of a conjecture and refutation process for the growth of a student's mathematical knowledge. The general student objective gave rise to the conclusion that the Madison Project has as a primary goal the development of a critical and rational attitude to the growth of mathematical knowledge. Accordingly, the growth of knowledge was seen as being characterized by more than a trial-and-error procedure, a point made earlier by Popper. Finally, the Madison Project seems more concerned with the processes of mathematical inquiry than with the substantive structure of this discipline.

In assessing the instructional models developed by the Project, it was observed that the Fallibilistic strategies of teaching were adequately dealt with by the Project's instructional models in the areas of origination and testing. Some deficiencies in the Madison Project instructional models were identified in the area of proving strategies. The techniques of monsterbarring, exception-barring, and lemma-incorporation have not been treated by the Madison Project. These deficiencies should be taken as a spur for further research and empirical testing of the proposed Fallibilistic strategies of teaching.

Finally, a philosophical approach to problems in the teaching and learning of mathematics seems to be a fruitful path for research. The identification of areas of weakness in the Madison Project provides evidence in support of this conclusion. If philosophically based models of instruction can sharpen the distinctions in teaching strategies, then the fruitfulness of such an approach seems to be corroborated. Moreover, if this approach can lead to the decline of the superficial treatment of strategies of teaching denoted by the terms 'inductive' and 'deductive' then such an approach is worthwhile.

These conclusions carry with them, however, certain limitations. These are dealt with in the next section. Furthermore, these conclusions and their limitations to be discussed presently, foster implications for mathematics education and for areas in need of further research. The implications which may be derived in these two areas are considered after the limitations of the study have been noted.

IV. LIMITATIONS

The scope and generalizability of the study is limited by the choice of only one philosophical position as the basis of the model of the mode of inquiry of mathematics. Due to the selection of Critical Fallibilism as a basis for the model, a bias is introduced into the study; that is, a bias in the sense that the model developed represents only one possible way of interpreting the growth of mathematical knowledge. Moreover, it cannot be contended that this is the only Fallibilistic model of inquiry possible for even it is determined to some extent by the writer's interpretation of Fallibilism, although an accurate and complete description of Fallibilism as espoused by Popper, and its application to mathematics as described by Lakatos, has been attempted.

These restrictions limit the generalizability of the characterization of mathematical inquiry given here. However, the philosophically based model developed is taken as a point of reference. This being the case, then the strategies of teaching described would seem to follow with the stipulation, of course, that such strategies should exemplify the processes of mathematical inquiry as described in the model. Furthermore, these strategies of teaching are in need of empirical investigation in the classroom with children being taught mathematics. It is only once this is done that the points of view presented in the study can move from the realm of the metaphysical to the area of scientific research. The appraisal of the Madison Project provides only *prima facie* evidence as to the fruitfulness of the model of inquiry and strategies of teaching developed. As such, this appraisal provides plausible evidence that the model and strategies are at least potentially useful. However, such evidence is just as subject to the restrictions of plausible inference as have been identified in this study for the use of this type of reasoning relative to mathematical hypotheses.

A further limitation must be noted with respect to the Fallibilistic model of instruction and the stratagems and strategies of teaching encompassed by this model. It is not feasible to derive from such a paradigm how the total mathematics curriculum (over a one year period, for example) should be organized. An instructional model cannot answer the question as to what mathematical material should be selected for inclusion in the curriculum. In other words, an instructional model like the one developed in the present study cannot be utilized to make decisions regarding the scope and sequence of the mathematics curriculum. These decisions must be made on a much broader basis, a basis which includes at least a study of the nature of the child, psychological theories of learning, the nature of society, and the nature of the community in which the curriculum is to be implemented. Consequently, it has not been possible in the present study to even outline how a year's work in mathematics might be organized from a Fallibilistic point of view. An instructional model regardless of its basis can contribute to the making of such decisions, but it would be folly to assume that it could serve as the only determinant of the mathematics curriculum. However, once the decision has been made as to what the particular mathematical content is to be studied, then the instructional model can help in making decisions as to the teaching strategies to be utilized in presenting this material. Even in this case, however, an instructional model cannot be the only consideration in making these decisions. Note should be made of this limitation of the role which can be played by instructional models in curriculum decisions.

As a consequence, the results and conclusions of the study reported in the previous section are tentative and subject to the restrictions noted above. Despite this fact, these conclusions do tend to support the continued exploration and investigation of both a model of mathematical inquiry and derived strategies of teaching as being potentially fruitful areas of research.

V. IMPLICATIONS FOR MATHEMATICS EDUCATION

Subject to the restrictions given in the preceding section several implications for mathematics education may be presented. In the interest of conciseness and since more detailed discussions in support of these recommendations can be found throughout the study, these implications are presented here in point form.

1. An adequate description of the mode of inquiry of mathematics is a necessity if the two-fold nature of the discipline of mathematics is to be reflected in the curriculum of the schools. Mathematics instruction should provide for the development of a student's awareness of both the mode of inquiry and the substantive structure of mathematics.

2. In developing the student's awareness and understanding of the processes of mathematical inquiry attention should be directed to the dual nature of mathematical heuristic--the naive and deductuve heuristics. Moreover, the students understanding of and experience in utilizing the processes of mathematical investigations should include learning situations exemplifying the origination, testing, and proving phases of mathematical inquiry.

3. Students should be encouraged to approach each new problem situation with a guessing and testing orientation to the solution of the problem. If students are encouraged to approach problems situations in this way, perhaps student errors can then be treated cognitively rather than affectively. Errors and refutations are part of the processes of the growth of knowledge and the growth of student knowledge is no exception.

4. Mathematics instruction should foster the students ability to uncover and identify errors and counterexamples. In doing so, it would be hoped that a greater awareness on the part of students would be developed relevant to the role of refutation in mathematics. From a Fallibilist view, refutation is a growth engendering process--not a destructive process--and consequently students should come to view refutation as a desirable state of affairs. A positive attitude towards refutation is more likely to encourage students to accept errors cognitively rather than affectively.

5. Experimental investigation of teaching strategies based on philosophical models should be undertaken in order to facilitate both

an improved and multi-perspective point of view towards the mode of inquiry in mathematics, and a more adequate description of a theory of instruction for mathematics. In addition, such empirical investigations are necessary in order to transpose what is now metaphysical speculation into scientifically testable hypotheses.

6. The description of mathematical inquiry reported in this study implies a releasing of students from a situation in which rote memorization of definitions, generalization, and algorithms is required, to situations in which students develop and criticize mathematical systems, definitions, generalizations, and algorithms on their own. It would be hoped that such a situation would enable students to develop a positive attitude towards mathematics and to operate confidently in mathematical situations.

7. Because of the variety of techniques identified within each phase of mathematical inquiry, it would seem desirable that the traditional, stagnant, and superficial distinction between inductive and deductive techniques in mathematics curricula should be eliminated. Students should be made aware of and develop a much deeper understanding of the processes of mathematical inquiry than is conveyed by the deductive-inductive distinction.

8. The phases of inquiry developed and the derived strategies of mathematical inquiry deserve to be taught in their own right. They are techniques which students should master and teachers should instruct students in their use. Perhaps if this situation were to develop then the mode of inquiry of mathematics would occupy a position comparable to that

of the substantive structure in the mathematics curriculum of the schools.

VI. IMPLICATIONS FOR FURTHER RESEARCH

In an exploratory study such as this one many more problems are generated than are solved. As a result, some of the areas of research and problems in need of investigation are outlined below.

In terms of the original flow of the study, the study was seen as flowing from a theoretical base to the area of instructional models and finally to the area of classroom activities. This progression can be depicted in the following way:

$$\boxed{\text{Theoretical Base}} \longrightarrow \boxed{\text{Instructional Models}} \longrightarrow \boxed{\text{Classroom Activities}}$$

The theoretical base for the present study is a philosophical one, that is, Critical Fallibilism. However, many other theoretical bases could be adopted. For example, one might choose another philosophical orientation such as Rationalism, or one might opt for a psychological base such as Piagetian theory. From these alternate starting points one could then develop instructional models and derivatively classroom activities characteristic of the theoretical starting point chosen. If such studies were to be initiated, then it would be hoped that alternate views of the mode of inquiry in mathematics would be developed. The goal in all these studies would be of course an ever improving descripttion of the methods of inquiry in mathematics.

Future studies need not just flow from the theoretical area through the instructional model area to the realm of classroom activities. Indeed, some studies and projects should be designed to proceed in the opposite direction. The Madison Project seems to have adopted this strategy. In this latter type of research, actual classroom activities give rise to functional instructional models which, in turn, can be linked to a theoretical foundation. It is the writer's contention that both of the above types of research design are necessary--necessary for an improved description of fruitful instructional models for the teaching of mathematics.

An additional avenue of further research which might be fruitful is that of combining a philosophical orientation and a psychological orientation into a more encompassing theoretical base. Superficially at least, Critical Fallibilism and Piagetian learning theory seem to have many commonalities, commonalities which should be investigated. This investigation could very well give rise to a more pervasive theoretical foundation and subsequently to instructional models with more scope and power for the generation of classroom activities.

Specific areas of further research which the present study suggests are numerous. Lakatos' initial results in depicting mathematics as quasi-empirical are in need of further study. His concept of "heuristic falsifier" requires further exploration and examination. Similarly, Polya's concept of credibility and the psychological basis implied by this idea should be studied further. The effect corroborating evidence has on an individual's psychological acceptance of a conjecture could be part of the study of the concept of credibility.

BIBLIOGRAPHY

BIBLIOGRAPHY

A. BOOKS

Barker, S. F. Philosophy of Mathematics. Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1966.

Benacerraf, P., and H. Putnam, editors. Philosophy of Mathematics. Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1964.

Bruner, Jerome S. The Process of Education. Harvard University Press, Cambridge, Massachusetts, 1960.

______. On Knowing: Essays for the left-hand. Belnap Press of Harvard University Press, Cambridge, Massachusetts, 1962.

______. Toward a Theory of Instruction. Harvard University Press, Cambridge, Massachusetts, 1966.

Bunge, M., editor. The Critical Approach to Science and Philosophy. The Free Press of Glencoe, Collier-MacMillan Limited, London, 1964.

Davis, R. B. (1962a). A Modern Mathematics Program As It Pertains to the Interrelationship of Mathematical Content, Teaching Methods and Classroom Atmosphere. The Madison Project. Report submitted to the Commissioner of Education, U. S. Department of Health, Education and Welfare, 1962.

______. (1962b). Axioms for Arithmetic and Algebra. The Madison Project, 1962.

______. (1962c). Notes on the Film: A Lesson with Second Graders. The Madison Project, 1962.

______. (1962d). Notes on the Film: First Lesson. The Madison Project, 1962.

______. (1963a). A Modern Mathematics Program As It Pertains to the Interrelationship of Mathematical Content, Teaching Methods, and Classroom Atmosphere. The Madison Project. Report submitted to the Commissioner of Education, U. S. Department of Health, Education and Welfare, 1963.

______. (1963b). Matrices, Functions, and other Topics: Student Discussion Guide. The Madison Project, 1963.

_______. (1963c). Matrices, Functions, and other Topics: A Text for Teachers. The Madison Project, 1963.

_______. (1964a). Discovery in Mathematics: Student Discussion Guide. Addison-Wesley Publishing Company, Don Mills, Ontario, 1964.

_______. (1964b). Discovery in Mathematics: A Text for Teachers. Addison-Wesley Publishing Company, Don Mills, Ontario, 1964.

_______. (1965a). In-Service Course I. The Madison Project, 1965.

_______. (1965b). The Madison Project: A Brief Introduction to Materials and Activities. The Madison Project, 1965.

_______. (1966a). Explorations in Mathematics: Student Discussion Guide. Addison-Wesley Publishing Company, Don Mills, Ontario, 1966.

_______. (1966b). Explorations in Mathematics: A Text for Teachers. Addison-Wesley Publishing Company, Don Mills, Ontario, 1966.

_______. (1966c). Some Remarks on 'Learning by Discovery'. The Madison Project, 1966.

Dienes, Z. P. Building Up Mathematics. Hutchinson Educational Press, London, 1960.

_______. Mathematics in Primary Education. International Study Group for Mathematics Learning, Hamburg, 1966.

_______. The Power of Mathematics. Hutchinson Educational Press, London, 1964.

_______, and E. W. Golding. Exploration of Space and Practical Measurement. Herder and Herder, New York, 1966.

_______. Learning Logic, Logical Games. Herder and Herder, New York, 1966.

_______. Modern Mathematics for Young Children. Herder and Herder, New York, 1966.

_______. Sets, Numbers and Powers. Herder and Herder, New York, 1966.

Downey, L. W. The Secondary Phase of Education. Blaisdell Press, Toronto, 1965.

Educational Services Incorporated. Goals for School Mathematics. The Report of the Cambridge Conference on School Mathematics. Houghton Mifflin Company, Boston, 1963.

Fawcett, H. P., editor. The Nature of Proof. The Thirteenth Yearbook of the National Council of Teachers of Mathematics, New York, 1938.

Goodlad, J. I. School Curriculum Reform in the United States. The Fund for the Advancement of Education, New York, 1964.

Hanson, N. R. Patterns of Discovery. Cambridge University Press, 1958.

Hesse, M. B. Models and Analogies in Science. University of Notre Dame Press, 1966.

Kaplan, A. The Conduct of Inquiry. Chandler Publishing Company, San Francisco, California, 1964.

King, A. R. and J. A. Brownell. The Curriculum and the Discipline of Knowledge. John Wiley and Sons, Inc., New York, 1966.

Luchins, A. S. and E. H. Luchins. Logical Foundations of Mathematics for Behavioral Scientists. Holt, Rinehart and Winston Inc., Toronto, 1965.

Polyani, M. Personal Knowledge. University of Chicago Press, 1958.

Polya, G. Mathematics and Plausible Reasoning, two volumes. Princeton University Press, Princeton, New Jersey, 1954.

_______. How To Solve It. Doubleday Anchor Books, Garden City, New York, 1957.

_______. Mathematical Discovery: On understanding, learning and teaching problem solving, two volumes. John Wiley and Sons Inc., New York, 1962.

Popper, K. R. The Logic of Scientific Discovery. Basic Books Inc., New York, 1959.

_______. Conjectures and Refutations. Basic Books Inc., New York, 1962.

_______. Of Clouds and Clocks: An Approach to the Problem of Rationality and the Freedom of Man. The Arthur Holly Compton Memorial Lecture, presented at Washington University, April, 1965. Washington University, Saint Louis, Missouri.

Reeve, W. D., editor. The Teaching of Arithmetic. The Tenth Yearbook of the National Council of Teachers of Mathematics, New York, 1935.

_______. Arithmetic in General Education. The Sixteenth Yearbook of the National Council of Teachers of Mathematics, New York, 1941.

Russell, B. My Philosophical Development. George Allen and Unwin Limited, London, 1959.

Scheffler, I. Conditions of Knowledge. Scott, Foresman and Company, Chicago, 1965.

Schwab, J. J. and P. Brandwein. The Teaching of Science. Harvard University Press, Cambridge, 1964.

Shumsky, A. In Search of Teaching Style. Appleton-Century-Crofts, New York, 1965.

Scott, L. Trends in Elementary School Mathematics. Rand McNally and Company, Chicago, 1966.

Westcott, A. M. and J. A. Smith. Creative Teaching of Mathematics in the Elementary School. Allyn and Bacon, Inc., Boston, 1968.

B. ARTICLES IN COLLECTIONS

Brodbeck, M. "Models, Meaning, and Theories," Symposium on Sociological Theory, L. Gross, editor. Harper and Row Publishers, New York, 1959. Pp. 373-406.

Carnap, R. "Truth and Confirmation," Readings in Philosophical Analysis, H. Feigl and W. Sellars, editors. Appleton-Century-Crofts Inc., New York, 1949. Pp. 119-127.

_______. "Testability and Meaning," Readings in the Philosophy of Science, H. Feigl and M. Brodbeck, editors. Appleton-Century-Crofts Inc., New York, 1953. Pp. 47-92.

d'Abro, A. "The Controversies on the Nature of Mathematics," The Growth of Mathematics, R. W. Marks, editor. Bantam Matrix Edition, 1964. Pp. 45-74.

Davis, R. B. "Some Remarks on a Theory of Instruction," Piaget Rediscovered, V. N. Rockcastle and R. E. Ribble, editors. A report of the Conference on Cognitive Studies and Curriculum Development, School of Education, Cornell University, March, 1964. Pp. 134-138.

_______. "Mathematics," New Curriculum Developments, G. Unruh, editor. Association for Supervision and Curriculum Development, 1965. Pp. 45-56.

Easley, J. R. "Logic and Heuristic in Mathematics Curriculum Reform," Problems in the Philosophy of Mathematics, I. Lakatos, editor. North-Holland Publishing Company, Amsterdam, 1967. Pp. 208-230.

Feigl, H. "The Logical Character of the Principle of Induction," Readings in Philosophical Analysis, H. Feigl and W. Sellars, editors. Appleton-Century-Crofts Inc., New York, 1949. Pp. 297-304.

Hanson, N. R. "Is There a Logic of Discovery," Current Issues in the Philosophy of Science, H. Feigl and G. Maxwell, editors. Holt, Rinehart and Winston Inc., New York, 1961. Pp. 20-34.

Hildebrandt, E. H. C. "Mathematical Modes of Thought," The Growth of Mathematical Ideas. The Twenty-fourth Yearbook of the National Council of Teachers of Mathematics, New York, 1959. Pp. 370-404.

Kalmar, L. "Foundations of Mathematics--Whither Now?" Problems in the Philosophy of Mathematics, I. Lakatos, editor. North-Holland Publishing Company, Amsterdam, 1967. Pp. 187-194.

Lakatos, I. "A renaissance of empiricism in the recent philosophy of mathematics," Problems in the Philosophy of Mathematics, I. Lakatos, editor. North-Holland Publishing Company, Amsterdam, 1967. Pp. 199-202.

Popper, K. R. "On the Theory of Deduction: Part I, Derivation and its Generalization," Proceedings of the Kon. Nederlandsche Akademie van Wetenschappen, Volume 51, Number 2, 1948. P. 173ff.

_______. "On the Theory of Deduction: Part II, The Definitions of Classical and Inutitionist Negation," Proceeding of the Kon. Nederlandsche Akademie van Wetenschappen, Volume 51, Number 3, P. 322ff.

_______. "Some Comments on Truth and the Growth of Knowledge," Proceedings of the 1960 International Congress for Logic, Methodology, and Philosophy of Science, E. Nagel, P. Suppes, and A. Tarski, editors. Stanford University Press, 1962. Pp. 285-292.

Schwab, J. J. "Problems, Topics, and Issues," Education and the Structure of Knowledge, S. Elam, editor. Rand McNally and Company, Chicago, 1964. Pp. 4-43.

_______. "Structuring of the Disciplines: Meaning and Significance," The Structure of Knowledge and the Curriculum, G. W. Ford, and L. Pugno, editors. Rand McNally and Company, Chicago, 1964. Pp. 1-30.

_______. "The Teaching of Science as Enquiry," The Teaching of Science, J. J. Schwab and P. Brandwein, editors. Harvard University Press, Cambridge, 1964. Pp. 1-104.

Smith, E. R. and K. B. Henderson. "Proof," The Growth of Mathematical Ideas. The Twenty-fourth Yearbook of the National Council of Teachers of Mathematics, New York, 1959. Pp. 111-181.

C. ARTICLES IN PERIODICALS

Davis, R. B. "The Syracuse University 'Madison Project'," American Mathematical Monthly, 67:178-180, February, 1960.

______. "The Evolution of School Mathematics," Journal of Research in Science Teaching, 1:260-264.

______. "Math Takes a New Path," The PTA Magazine, 57:8-11, February, 1963.

______. "The Madison Project's Approach to a Theory of Instruction," Journal of Research in Science Teaching, 2:146-162.

______. "Report on the Syracuse-Webster College Madison Project," American Mathematical Monthly, 71:306-308, March, 1964.

______. "Goals for School Mathematics: The Madison Project View," Journal of Research in Science Teaching, 2:309-315, December, 1964.

______. "The Next Few Years," The Arithmetic Teacher, 13:355-362, May, 1966.

______. "Mathematics Teaching--with Special Reference to Epistemological Problems," Journal of Research and Development in Education, Monograph Number 1, Fall, 1967.

______. "The Range of Rhetorics, Scale and Other Variables," Journal of Research and Development in Education, 1:51-74, Fall, 1967.

Hull, L. W. H. "The Superstition of Educated Men," Mathematics Teaching, 43:26-33, Summer, 1968.

Lakatos, I. "Infinite Regress and Foundations of Mathematics," The Aristotelian Society, Supplementary Volume, 34:155-184, 1962.

______. "Proofs and Refutations," The British Journal for the Philosophy of Science, Volume 14, 1963. Pp. 1-25, 120-139, 221-245, and 296-342.

Polya, G. "Mathematics as a subject for learning plausible reasoning," The Mathematics Teacher, 52:7-9, January, 1959.

Popper, K. R. "New Foundations for Logic," Mind, 56:193-236.

______. "What Can Logic Do For Philosophy," Aristotelian Society, Supplementary Volume, 22:141-154.

_______. "Two Autonomous Systems for the Calculus of Probabilities," The British Journal for the Philosophy of Science, 6:51-57.

_______. "Probabalistic Independence and Corroboration by Empirical Tests," The British Journal for the Philosophy of Science, 10:315-318.

D. UNPUBLISHED MATERIALS

Harrison, D. B. "Reflective Intelligence and Mathematics Learning," Unpublished Doctoral Dissertation, The University of Alberta, Edmonton, 1967.

Lakatos, I. "A renaissance of empiricism in the recent philosophy of mathematics," (Mimeographed, Limited Distribution).

Lakatos, I. "Essays in the Logic of Mathematical Discovery," Unpublished Doctoral Dissertation, Cambridge University, London, 1961.

Lee, D. S. Unpublished materials obtained from Dr. Lee at Washington University, Saint Louis, Missouri. These materials deal with the description of the scientific method as an order of precedence. Dr. Lee is presently located at Tulane University, New Orleans.

E. FILMS

The Madison Project. A Lesson with Second Graders.

_______. Creative Learning Experiences.

_______. First Lesson.

_______. Guessing Functions.

_______. Matrices.

_______. Postman Stories.

_______. Second Lesson.