ON ROTA-BAXTER NIJENHUIS TD ALGEBRA

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Abstract

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There was a long standing problem of G. C. Rota regarding the classification of all linear operators on associative algebras that satisfy algebraic identities. Initially, only very few of such operators were known, for example, the derivative operator, average operator, difference operator and Rota-Baxter operator. Recently, in a paper by L. Guo, W. Sit and R. Zhang, the authors revisited Rota's problem by concentrating on two classes of operators; differential type operators and Rota-Baxter type operators. One of the Rota-Baxter type operators they found is the Rota-Baxter Nijenhuis TD (RBNTD) operator which puts together the terms of the well-known Rota-Baxter operator, Nijenhuis operator and Leroux' TD operator. In this dissertation, we initiate a systematic study of the RBNTD operator, extending the previous works on the Rota-Baxter, Nijenhuis and TD operators. After giving basic properties and examples, we construct free commutative and then free (non-commutative) RBNTD algebras. We then use free RBNTD algebras to obtain an extension of the renowned dendriform algebra with five binary operations.

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Contents

1 Introduction

The subject of this thesis is motivated by several well-known operators from mathematics and physics, including the Rota-Baxter operator, Nijenhuis operator and TD operator.

A Rota-Baxter algebra is an associative algebra equipped with a linear operator, called the Rota-Baxter operator(RBO), that generalizes the integral operator in analysis. The Rota-Baxter operator was introduced in 1960 [5] by G. Baxter to study the theory of fluctuations in probability. Later, other well-known mathematicians such as Atkinson, Cartier, and especially G. C. Rota have shown keen interest in Baxter algebras. Their fundamental papers brought the subject into the domains of algebra and combinatorics. The study of Baxter algebras continued through the 1960s and 1970s [7, 32, 33] and recently has led to remarkable results with applications to renormalization in quantum field theory [8,9,14,15], multiple zeta values in number theory [11, 25], umbral calculus in combinatorics [20], and also in Loday's work on dendriform algebras [27] and Hopf algebras [3].

The concept of a Nijenhuis operator of a Lie algebra is a generalization of the Nijenhuis tensor, which was introduced in the 1950s by Nijenhuis [31] in his study of pseudo-complex manifolds. It is related to the well-known Schouten-Nijenhuis bracket [17]. The Nijenhuis operator on an associative algebra was introduced by Carinena *et al.* [6] to study quantum bi-Hamiltonian systems. Nijenhuis operators are also constructed by analogy with Poisson-Nijenhuis geometry, from the relative Rota-Baxter algebras [34].

The TD operator, which was introduced by Leroux [26], and therefore was also known as the Leroux' TD operator. The operator gives a tridendriform algebra and is a particular case of the Rota-Baxter operator of generalized weight.

In 1995, Rota asked for a classification of all linear operators on an associative algebra. Guo, Sit and Zhang [24] recently formulated Rota's problem, in which they worked on Rota's problem in the framework of free operated algebras by viewing an associative algebra with a linear operator as one which satisfies a certain operated polynomial identity. They have also used rewriting systems, Gröbner-Shirshov bases and the help of computer algebra. In their research work, the authors have obtained a possibly complete list of 14 Rota-Baxter type operators and some other differential type operators as a partial solution to Rota's problem.

Given the phenomenal interest in the Rota-Baxter, Nijenhuis and TD operators, further research should be carried out around these newly discovered operators. One of these operators is the combines the Rota-Baxter, Nijenhuis, and TD operators to form the Rota-Baxter Nijenhuis TD operator (RBNTD), which is the subject of the present research work. Our approach to study this operator is based on algebraic constructions. We first construct free RBNTD commutative algebras from algebraic structures of combinatorial objects, such as generalized shuffles extending the work of L. Guo and W. Keigher [22] and of E. Fard and P. Leroux [16]. Later, we construct free non-commutative RBNTD algebras from the algebraic structures of bracketed words and rooted trees. This is an extension of previous works of E. Fard and L. Guo [12, 13], P. Lei and L. Guo [23] and C. Zhou [35]. Finally, our work on constructing RBNTD-dendriform algebras gets and the motivation comes from operads, and is an extension of Loday's work on dendriform algebra which has two binary operations. In our case, there are five binary operations.

2 Organization

The organization of this work is as follows. All standard material is drawn from [21]. In Section 3, we review the necessary definitions and provide examples of RBNTD operators using Mathematica. We begin Section 4 with the basic definitions of shuffle products and then proceed with an explicit construction of free commutative RBNTD algebras. We will also observe some interesting results about the relationship between the RBNTD shuffle product and the Nijenhuis shuffle product. In Section 5, we start with the concept of rooted trees and some previous results of Rota-Baxter, Nijenhuis and TD algebras. Our goal is to give an explicit construction of free non-commutative algebras. In Section 6, we recall the definitions of dendriform and tridendriform algebras and review some previous results and background on operads. We obtain the RBNTD-dendriform algebra that satisfies binary, quadratic, and non-symmetric identities, which are compatible with those of RBNTD algebras.

3 Definitions and examples

We will use $\mathbb{N}_{>0}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively to denote the set of positive integers, non-negative integers, integers, rational numbers, real numbers and complex numbers.

To fix the notations and to be self-contained, we briefly recall definitions. Refer to [21].

3.1 Definitions

In the following, by a ring we always mean a unitary ring, that is, a set A with binary operations $+$ and $\cdot$ (which will often be suppressed) such that $(A, +)$ is an abelian group, $(A, \cdot)$ is a monoid and $\cdot$ is distributive over $+$. The unit of the monoid is called the identity element of A, denoted by $\mathbf{1}_A$ or simply 1. A ring homomorphism is assumed to preserve the unit. We use $\mathbf{k}$ to denote a commutative ring with identity element denoted by $\mathbf{1}$ or simply 1.

Let A be a ring. A **(left)** A**-module** M is an abelian group M together with a **scalar multiplication** $A \times M \to M$ such that

$$a(x + y) = ax + ay,\ (a + b)x = ax + bx,\ (a\,b)x = a(bx),\ \forall a, b \in A,\ x, y \in M.$$

Definition 3.1. Let $\mathbf{k}$ be a commutative ring. A $\mathbf{k}$**-algebra** is a ring A together with a unitary $\mathbf{k}$-module structure on the underlying abelian group of A such that

$$k(ab) = (ka)b = a(kb),\ \forall\, k \in \mathbf{k},\ a, b \in A.$$

All $\mathbf{k}$-algebras are taken to be unitary.

A Rota-Baxter algebra of weight zero (or simply, a Rota Baxter algebra) is thus an associative algebra equipped with a linear operator that generalizes the integral operator in analysis. Rota-Baxter algebras (initially known as Baxter algebras) originated in 1960 [5] from the probability study by G. Baxter to understand the Spitzer's identity in fluctuation theory. This concept drew the attention of many well-known mathematicians such as Atkinson, Cartier, and especially G. C. Rota, whose fundamental papers brought the subject into the areas of algebra and combinatorics around 1970. In 1980s, Lie algebras were studied independently by mathematical physicists C.N. Yang and R. Baxter under the name (operator form) of the classical Yang Baxter Equation (CYBE). In 2000, Aguiar discovered that the Rota-Baxter algebra of weight zero and the associative analog of CYBE are related [2]. He also showed that the Rota-Baxter algebra of weight zero naturally carries the structure of a dendriform algebra which was introduced by Loday in his study of K-theory [27]. Also in 2000, Guo and Keigher showed that the free Rota-Baxter algebras can be constructed via generalization of the shuffle algebra [22], called the mixable shuffle algebra.

Definition 3.2. Let λ be a given element of $\mathbf{k}$. A **Rota-Baxter k-algebra of weight** λ, or simply an **RBA of weight** λ, is a pair (R, P) consisting of a $\mathbf{k}$-algebra R and a linear operator $P : R \to R$ that satisfies the **Rota-Baxter identity**

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \quad \forall x, y \in R. \tag{3.1}$$

Then P is called a **Rota-Baxter operator (RBO) of weight** λ.

Definition 3.3. A **Nijenhuis algebra** is an associative algebra R with a linear endomorphism P satisfying the **Nijenhuis identity**:

$$P(x)P(y) = P(P(x)y) + P(xP(y)) - P^2(xy), \quad \forall x, y \in R. \tag{3.2}$$

The associative analog of the Nijenhuis relation may be regarded as the "homogeneous" version of the Rota-Baxter relation. Some of its algebraic aspects with regard to the notion of quantum bi-Hamiltonian systems were studied by Carinena [6]. The Lie algebraic version of the associative Nijenhuis relation was studied in the context of the classical Yang-Baxter equation which is closely related to the Lie algebraic version of Rota-Baxter relation [18, 19].

The TD operator was introduced by Leroux [26], and therefore is also known as Leroux' TD operator. The operator gives a tridendriform algebra and is a particular case of the Rota-Baxter operator of generalized weight [4].

Definition 3.4. A **TD k-algebra** is an associative **k**-algebra R with a **k**-linear endomorphism $P : R \to R$ satisfying the **TD identity:**

$$P(x)P(y) = P(P(x)y) + P(xP(y)) - P(xP(1)y), \quad \forall x, y \in R. \tag{3.3}$$

In 1995, Rota posed a question about finding all linear operators that satisfy an algebraic identities on an associative algebra. More precisely, Rota's question involved an associative **k**-algebra R with a **k**-linear unary operator P. The operations: addition, multiplication, scalar multiplication, and P, already are required to satisfy certain identities such as the commutative law of addition, the associative laws, the distributive law, and **k**-linearity for P. Rota wanted to find "all possible polynomial identities that could be satisfied by P on an algebra" and to "classify all such identities". He also wanted to find "a complete list of such identities."

Taking this work forward, Guo, Sit and Zhang recently published a paper [24], in which they worked on Rota's problem and put together the framework of free operated algebras by viewing associative algebra with a linear operator, as one that is compatible with a certain operated polynomial identity. They have also used rewriting systems, Gröbner-Shirshov bases and the help of computer algebra. In that research work, the authors have obtained a possibly complete list of 14 Rota-Baxter type operators and some other differential type operators as a partial solution to Rota's problem.

The completeness of the list of Rota-Baxter type identities that Guo, Sit and Zhang found is still a conjecture and further work should be done. One of the identities in their framework is a combination of (3.1),(3.2) and (3.3). This new identity, which gives rise to a new class of associative **k**-algebras known as **RBNTD k-algebras** is the **RBNTD identity:**

$$P(x)P(y) = P(P(x)y) + P(xP(y)) + \lambda P(xy) - P^2(xy) - P(xP(1)y), \quad \forall x, y \in R. \tag{3.4}$$

3.2 Examples

Consider RBNTD operators on the two-dimensional algebra $k \times k$. We use Mathematica to do computations.

Proposition 3.5. Consider the **k**-algebra $R = \mathbf{k} \times \mathbf{k}$, where the operators are defined componentwise. The matrices of RBNTD operators of weight zero on R w.r.t. the basis $e_1 = (1,0)$ and $e_2 = (0,1)$ is

$$M = \begin{pmatrix} 0 & 0 \\ c & -c \end{pmatrix},\ c \in \mathbf{k}$$

Consider RBNTD operators on the three-dimensional algebra $k \times k \times k$.

Proposition 3.6. Consider the $\mathbf{k}$-algebra $R = \mathbf{k} \times \mathbf{k} \times \mathbf{k}$, where the operators are defined componentwise. The matrices of RBNTD operators of weight zero on R w.r.t. the basis $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ are

$$M_1 = \begin{pmatrix} 0 & P_{12} & -P_{12} \\ 0 & 0 & 0 \\ 0 & P_{12} & -P_{12} \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ P_{31} & 0 & -P_{31} \end{pmatrix}, M_3 = \begin{pmatrix} -P_{12} & P_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M_4 = \begin{pmatrix} -P_{12} & P_{12} & 0 \\ 0 & 0 & 0 \\ -P_{12} & P_{12} & 0 \end{pmatrix}, M_5 = \begin{pmatrix} -P_{13} & 0 & P_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, M_6 = \begin{pmatrix} 0 & 0 & 0 \\ P_{21} & -P_{21} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Remark 3.7. We verify that they are indeed RBNTD operators.

4 Free Commutative RBNTD algebras and shuffle products

The shuffle product is an important concept in algebra, combinatorics and topology. Several generalizations that have been found in recent years that are applied to algebra, combinatorics and number theory.

4.1 The shuffle product

The shuffle product can be defined in two ways, namely recursively or explicitly. Let V be a $\mathbf{k}$-module. Consider the $\mathbf{k}$-module

$$T(V) := \bigoplus_{n\geq 0} V^{\otimes n} = \mathbf{k} \oplus V \oplus V^{\otimes 2} \oplus \cdots .$$

Here the tensor products are taken over $\mathbf{k}$ and we take $V^{\otimes 0} = \mathbf{k}$.

Let two pure tensors $\mathfrak{a} = a_1 \otimes \cdots \otimes a_m \in V^{\otimes m}$ and $\mathfrak{b} = b_1 \otimes \cdots \otimes b_n \in V^{\otimes n}$ be given. To describe the shuffle product of the two tensors, think of them as two decks of cards. A shuffle of $\mathfrak{a}$ and $\mathfrak{b}$ is just a shuffle of the two decks of cards — a tensor list from the factors of $\mathfrak{a}$ and $\mathfrak{b}$ in which the natural orders of the a_i's and b_j's are preserved. To form the shuffle product of $\mathfrak{a}$ and $\mathfrak{b}$, one sums together all possible shuffles of the two pure tensors.

Example 4.1. Let $\mathfrak{a} = a_1$ and $\mathfrak{b} = b_1 \otimes b_2$. Then, their shuffles are

$$a_1 \otimes b_1 \otimes b_2, b_1 \otimes a_1 \otimes b_2, b_1 \otimes b_2 \otimes a_1. \tag{4.1}$$

Hence, their shuffle product is $a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1$.

Recall that the vector space $T(V)$ is naturally equipped with a grading, where $|u| = n$ for $u \in V^{\otimes n}$. The shuffle product on $T(V)$ starts with the shuffles of permutations. For $m,\ n \in \mathbb{N}_+$, define the set of (m,n)**-shuffles** by

$$S(m,n) = \left\{ \sigma \in S_{m+n} \;\middle|\; \begin{array}{l} \sigma^{-1}(1) < \sigma^{-1}(2) < \ldots < \sigma^{-1}(m), \\ \sigma^{-1}(m+1) < \sigma^{-1}(m+2) < \ldots < \sigma^{-1}(m+n) \end{array} \right\}.$$

Here S_{m+n} is the symmetric group on the set $\{1, 2, \cdots, m+n\}$.

Example 4.2. Consider the case of $m = 1$ and $n = 2$. Then we have

$$\begin{aligned} S(1,2) &= \left(\sigma \in S_3 \mid \sigma^{-1}(2) < \sigma^{-1}(3)\right\} \\ &= \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \right\}. \end{aligned} \tag{4.2}$$

For a $\sigma \in S_n,\ n \geq 1$, the map

$$f_\sigma : V^n \to V^{\otimes n}, f_\sigma(a_1, \ldots, a_n) = a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}, (a_1, \cdots, a_n) \in V^n,$$

is multi-linear. Hence it induces a linear map

$$\sigma : V^{\otimes n} \to V^{\otimes n}, \sigma(a_1 \otimes \cdots \otimes a_n) = a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}.$$

For $\mathfrak{a} = a_1 \otimes \ldots \otimes a_m \in V^{\otimes m}$, $\mathfrak{b} = b_1 \otimes \ldots \otimes b_n \in V^{\otimes n}$ and $\sigma \in S(m,n)$, the element

$$\sigma(\mathfrak{a} \otimes \mathfrak{b}) = u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \ldots \otimes u_{\sigma(m+n)} \in V^{\otimes(m+n)},$$

where

$$u_k = \begin{cases} a_k, & 1 \leq k \leq m, \\ b_{k-m}, & m+1 \leq k \leq m+n. \end{cases}$$

is called the **shuffle of** $\mathfrak{a}$ **and** $\mathfrak{b}$ **via** σ . The **shuffle product** $\mathfrak{a} ш \mathfrak{b}$ is the sum over all shuffles of $\mathfrak{a}$ and $\mathfrak{b}$.

$$\mathfrak{a} ш \mathfrak{b} := \sum_{\sigma \in S(m,n)} \sigma(\mathfrak{a} \otimes \mathfrak{b}) \tag{4.3}$$

Also, by convention, $\mathfrak{a} ш \mathfrak{b}$ is the scalar product if either $m = 0$ or $n = 0$, that is, if either $\mathfrak{a} \in \mathbf{k}$ or $\mathfrak{b} \in \mathbf{k}$. The operation ш extends to a commutative and associative binary operation on $T(V)$, making $T(V)$ into a commutative algebra with identity, called the **shuffle product algebra** generated by V.

Example 4.3. Consider the case when $m = 1$ and $n = 2$ as in Example 4.2. Then $\mathfrak{a} = a_1$ and $\mathfrak{b} = b_1 \otimes b_2$. From Eq. (4.1) we obtain

$$a_1 ш (b_1 \otimes b_2) = a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1.$$

The shuffle product on $T(V)$ can also be recursively defined as follows. First define the product with $V^{\otimes 0} = \mathbf{k}$ to be the scalar product. In particular, $\mathbf{1_k}$ is the identity of $T(V)$. Next for any $m, n \geq 1$, $\mathfrak{a} := a_1 \otimes \cdots \otimes a_m \in V^{\otimes m}$ and $\mathfrak{b} := b_1 \otimes \cdots \otimes b_n \in V^{\otimes n}$, define $\mathfrak{a} ш \mathfrak{b}$ by induction on the sum $m + n \geq 2$. When $m + n = 2$, we have $\mathfrak{a}, \mathfrak{b} \in V$ and define

$$\mathfrak{a} ш \mathfrak{b} = \mathfrak{a} \otimes \mathfrak{b} + \mathfrak{b} \otimes \mathfrak{a}.$$

Assume that $\mathfrak{a} ш \mathfrak{b}$ has been defined for $m + n \leq k$ with $k \geq 2$. Consider $\mathfrak{a}$ and $\mathfrak{b}$ with $m + n = k + 1$. Then $m + n \geq 3$ and so at least one of m and n is greater than 1. We then define

$$
\begin{aligned}
&\mathfrak{a}ш\mathfrak{b}\\
&= \begin{cases}
a_1 \otimes b_1 \otimes \cdots \otimes b_n + b_1 \otimes (a_1 ш (b_2 \otimes \cdots \otimes b_n)), m = 1, n \geq 2, \\
a_1 \otimes ((a_2 \otimes \cdots \otimes a_m) ш b_1) + b_1 \otimes a_1 \otimes \cdots \otimes a_m, m \geq 2, n = 1, \\
a_1 \otimes ((a_2 \otimes \cdots \otimes a_m) ш (b_1 \otimes \cdots \otimes b_n)) \\
\qquad +b_1 \otimes ((a_1 \otimes \cdots \otimes a_m) ш (b_2 \otimes \cdots \otimes b_n)), m, n \geq 2.
\end{cases}
\end{aligned}
\tag{4.4}
$$

Here the products by ш on the right hand side of the equation are well-defined by the induction hypothesis. In short, writing $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$ where $\mathfrak{a}' = a_2 \otimes \cdots \otimes a_m$ if $m \geq 2$, and $\mathfrak{a} = a_1$ if $m = 1$, in which case putting $\mathfrak{a}'$=$\mathbf{1}_{\mathrm{k}}$=1 and identifying $\mathfrak{a} = a_1$ with $a_1 \otimes \mathbf{1}_{\mathrm{k}} \in V \otimes \mathbf{k}$, we have

$$
\mathfrak{a}ш\mathfrak{b} = a_1 \otimes (\mathfrak{a}' ш \mathfrak{b}) + b_1 \otimes (\mathfrak{a} ш \mathfrak{b}'). \tag{4.5}
$$

Example 4.4. Consider the case when $m = 1$ and $n = 2$ again. Then $\mathfrak{a} = a_1$ and $\mathfrak{b} = b_1 \otimes b_2$. Then we have

$$
\begin{aligned}
a_1 ш (b_1 \otimes b_2) &= a_1 \otimes (1 ш (b_1 \otimes b_2)) + b_1 \otimes (a_1 ш b_2) \\
&= a_1 \otimes b_1 \otimes b_2 + b_1 \otimes (a_1 \otimes b_2 + b_2 \otimes a_1) \\
&= a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1.
\end{aligned}
$$

This agrees with Example 4.3.

4.2 Generalized shuffle products

We now recall the tensor algebra, $T(A)$, over an associative unitary **k-**algebra A equipped with three generalized shuffle products, namely the mixable shuffle product defined in [22] and the left and right shuffle product defined in [16]. First, we will give explicit formulae for these products and then their recursive definitions.

Given an (m,n)-shuffle $\sigma \in S(m,n)$, a pair of indices $(k, k+1)$, $1 \leq k < m+n$, is called an **admissible pair** for σ if $\sigma(k) \leq m < \sigma(k+1)$. Let $\mathcal{T}^{\sigma}$ be the set of admissible pairs for σ. For a subset T of $\mathcal{T}^{\sigma}$, call the pair (σ, T) a **mixable** (m,n)**-shuffle**. Let $|T|$ be the cardinality of T. By convention, if $T = \emptyset$, then $(\sigma, T) = \sigma$ is just a shuffle. If $T \neq \emptyset$, then call $(\sigma; T)$ a **proper mixable** (m,n)**-shuffle**. Let

$$\bar{S}(m,n) = \{(\sigma, T) \mid \sigma \in S(m,n),\ T \subset \mathcal{T}^{\sigma}\} \tag{4.6}$$

be the set of mixable (m,n)-shuffles.

As remarked in § 4.1, an (m,n)-shuffle is a permutation σ of $\{1, \ldots, m,\ m+1, \ldots, m+n\}$ such that the natural orders of $\{1, \ldots, m\}$ and $\{m+1, \ldots, m+n\}$ are preserved in

$$\{\sigma(1), \ldots, \sigma(m), \sigma(m+1), \ldots, \sigma(m+n)\}.$$

Further, a mixable (m,n)-shuffle (σ, T) is a (m,n)-shuffle σ in which pairs of indices from T represent positions where "merging" will occur.

Example 4.5. By Example 4.2, the $(1,2)$-shuffles are

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

For σ_1, the pair $(1,2)$ is an admissible pair. For σ_2, the pair $(2,3)$ is an admissible pair. There are no admissible pairs for σ_3.

4.2.1 Mixable shuffles of tensors

Let A be a commutative unitary $\mathbf{k}$-algebra. For $\mathfrak{a} = a_1 \otimes \ldots \otimes a_m \in A^{\otimes m}$, $\mathfrak{b} = b_1 \otimes \ldots \otimes b_n \in A^{\otimes n}$ and $(\sigma, T) \in \bar{S}(m,n)$, define

$$\sigma(\mathfrak{a} \otimes \mathfrak{b}; T) = u_{\sigma(1)} \hat{\diamond} u_{\sigma(2)} \hat{\diamond} \ldots \hat{\diamond} u_{\sigma(m+n)} \in A^{\otimes(m+n-|T|)},$$

where for each pair $(k, k+1)$, $1 \leq k < m+n$,

$$u_{\sigma(k)} \hat{\diamond} u_{\sigma(k+1)} = \begin{cases} u_{\sigma(k)} u_{\sigma(k+1)}, & (k, k+1) \in T \\ u_{\sigma(k)} \otimes u_{\sigma(k+1)}, & (k, k+1) \notin T. \end{cases}$$

$\sigma(\mathfrak{a} \otimes \mathfrak{b}; T)$ is called a **mixable shuffle** of the pure tensors $\mathfrak{a}$ and $\mathfrak{b}$ (with respect to T). When $T = \emptyset$, a mixable shuffle is just a shuffle and when $T \neq \emptyset$, we call $\sigma(\mathfrak{a} \otimes \mathfrak{b}; T)$ a proper mixable shuffle of $\mathfrak{a}$ and $\mathfrak{b}$.

Example 4.6. $\sigma_1(\mathfrak{a} \otimes \mathfrak{b}; (1,2)) = a_1 b_1 \otimes b_2, \sigma_2(\mathfrak{a} \otimes \mathfrak{b}; (2,3)) = b_1 \otimes a_1 b_2$, recovering the two proper mixable shuffles found there.

Now fix $\lambda \in \mathbf{k}$. Define, for $\mathfrak{a}$ and $\mathfrak{b}$ as above, the **mixable shuffle product**

$$\mathfrak{a} Ш_\lambda \mathfrak{b} := \sum_{(\sigma,T) \in \bar{S}(m,n)} \lambda^{|T|} \sigma(\mathfrak{a} \otimes \mathfrak{b}; T) \in \bigoplus_{1 \leq k \leq m+n} A^{\otimes k}. \tag{4.7}$$

We then extend the operation $Ш_\lambda$ to a binary operation on

$$\bigoplus_{k \geq 1} A^{\otimes k} = A \oplus A^{\otimes 2} \oplus \ldots$$

by biadditivity. Further extend $Ш_\lambda$ to

$$Ш^+(A) := Ш^+_{\mathbf{k},\lambda}(A) := \bigoplus_{k \in \mathbb{N}} A^{\otimes k} = \mathbf{k} \oplus A \oplus A^{\otimes 2} \oplus \ldots. \tag{4.8}$$

by letting $\mathbf{1} \in \mathbf{k}$ multiply as the identity.

Example 4.7. Continuing with Example 4.6, we have

$$
\begin{aligned}
& a_1 Ш_\lambda (b_1 \otimes b_2) && (4.9)\\
= & \, a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 \ \text{(improper mixable shuffles)}\\
& + \lambda a_1 b_1 \otimes b_2 + \lambda b_1 \otimes a_1 b_2 \ \text{(proper mixable shuffles).}
\end{aligned}
$$

Theorem 4.8. [22] The **k**-module $Ш^+(A)$ with the mixable shuffle product $Ш_\lambda$ is a commutative unitary **k**-algebra. Further, the tensor product algebra $Ш(A) := A \otimes Ш^+(A)$ with the linear operator

$$P_A : Ш(A) \to Ш(A); \quad P_A(\mathfrak{a}) = \mathbf{1}_A \otimes \mathfrak{a},$$

is the free commutative Rota-Baxter algebra on A.

4.2.2 Left and right shift shuffles

Let A be a commutative unitary **k**-algebra. For $\mathfrak{a} = a_1 \otimes \ldots \otimes a_m \in A^{\otimes m}$, $\mathfrak{b} = b_1 \otimes \ldots \otimes b_n \in A^{\otimes n}$ and $(\sigma, T) \in \bar{S}(m, n)$, define

$$\sigma_q(\mathfrak{a} \otimes \mathfrak{b}; T) = u_{\sigma(1)} \hat{\diamond}_q u_{\sigma(2)} \hat{\diamond}_q \ldots \hat{\diamond}_q u_{\sigma(m+n)} \in A^{\otimes(m+n-|T|)},$$

where for each pair $(k, k+1)$, $1 \leq k < m+n$,

$$u_{\sigma(k)} \hat{\diamond}_q u_{\sigma(k+1)} = \begin{cases} -\mathbf{1}_A \otimes (u_{\sigma(k)} u_{\sigma(k+1)}), & (k, k+1) \in T \text{ and } q = r, \\ -(u_{\sigma(k)} u_{\sigma(k+1)}) \otimes \mathbf{1}_A, & (k, k+1) \in T \text{ and } q = l, \\ u_{\sigma(k)} \otimes u_{\sigma(k+1)}, & (k, k+1) \notin T. \end{cases}$$

$\sigma_q(\mathfrak{a} \otimes \mathfrak{b}; T)$ is called the **left-shift shuffle product** (when q=l) and the **right-shift shuffle product** (when q=r) of the pure tensors $\mathfrak{a}$ and $\mathfrak{b}$ (with respect to T).

The **left-shift shuffle product** and **right-shift shuffle product** respectively can also be defined with the convention $x \otimes \mathbf{1}_k = x$, by

$$\mathfrak{a} \hat{\diamond}_l \mathfrak{b} = a_1 \otimes (\mathfrak{a}' \hat{\diamond}_l \mathfrak{b}) + b_1 \otimes (\mathfrak{a} \hat{\diamond}_l \mathfrak{b}') - a_1 b_1 \otimes \mathbf{1}_A \otimes (\mathfrak{a}' \hat{\diamond}_l \mathfrak{b}')$$
$$\mathfrak{a} \hat{\diamond}_r \mathfrak{b} = a_1 \otimes (\mathfrak{a}' \hat{\diamond}_r \mathfrak{b}) + b_1 \otimes (\mathfrak{a} \hat{\diamond}_r \mathfrak{b}') - \mathbf{1}_A \otimes a_1 b_1 \otimes (\mathfrak{a}' \hat{\diamond}_r \mathfrak{b}')$$

where $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$ and $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$ as before.

Example 4.9. We have

$$\begin{aligned} & a_1 \hat{\diamond}_l (b_1 \otimes b_2) \\ = & a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 \quad \text{(shuffles)} \\ & - (a_1 b_1) \otimes \mathbf{1}_A \otimes b_2 - b_1 \otimes (a_1 b_2) \otimes \mathbf{1}_A \quad \text{(left-shift shuffles).} \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} & a_1 \hat{\diamond}_r (b_1 \otimes b_2) \\ = & a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 \quad \text{(shuffles)} \\ & - \mathbf{1}_A \otimes (a_1 b_1) \otimes b_2 - b_1 \otimes \mathbf{1}_A \otimes (a_1 b_2) \quad \text{(right-shift shuffles).} \end{aligned} \tag{4.11}$$

Theorem 4.10. [16] The $\mathbf{k}$-module $Ш^+(A)$ with the right-shift (resp. left-shift) shuffle product $\hat{\diamond}_r$ (resp. $\hat{\diamond}_l$) is a commutative unitary $\mathbf{k}$-algebra. Further, the tensor product algebra $\overrightarrow{Ш}(A) := A \otimes Ш^+(A)$ (resp. $\overleftarrow{Ш}(A) := A \otimes Ш^+(A)$) with the linear operators

$$P_A : \overrightarrow{Ш}(A) \to \overrightarrow{Ш}(A); \quad P_A(\mathfrak{a}) = \mathbf{1}_A \otimes \mathfrak{a},$$
$$Q_A : \overleftarrow{Ш}(A) \to \overleftarrow{Ш}(A)); \quad Q_A(\mathfrak{a}) = \mathfrak{a} \otimes \mathbf{1}_A,$$

is the free commutative Nijenhuis (resp. TD) algebra on A.

4.3 RBNTD shuffle product

Now, let us define our new RBNTD-shuffle product, denoted by $\circledast_\lambda$, on

$$Ш^+(A) := Ш^+_{\mathbf{k},\lambda}(A) := \bigoplus_{k\in\mathbb{N}} A^{\otimes k} = \mathbf{k} \oplus A \oplus A^{\otimes 2} \oplus \cdots$$

For this, we only need to define the product of two pure tensors and then to extend by bilinearity. Any tensor $\mathfrak{a}$ in $A^{\otimes k}$ is said to have length k, denoted by $|\mathfrak{a}|$. For two pure tensors $\mathfrak{a} = a_1 \otimes \ldots \otimes a_m \in A^{\otimes m}$ and $\mathfrak{b} = b_1 \otimes \ldots \otimes b_n \in A^{\otimes n}$, we define $\mathfrak{a} \circledast_\lambda \mathfrak{b}$ by induction on $m+n$. If $m = n = 0$, then $\mathfrak{a}, \mathfrak{b} \in \mathbf{k}$ and we just define $\mathfrak{a} \circledast_\lambda \mathfrak{b} = ab \in \mathbf{k}$. Assume the product has been defined for a, b with $m+n \leq k$, and let $\mathfrak{a}, \mathfrak{b}$ be pure tensors with $m+n = k+1$. If either m or n is zero, then we define $\mathfrak{a} \circledast_\lambda \mathfrak{b}$ by the scalar product. If neither m nor n is zero, then we have $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$, $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$ with $a_1, b_1 \in A$ and $\mathfrak{a}' \in A^{\otimes(m-1)}$ and $\mathfrak{b}' \in A^{\otimes(n-1)}$. We then define

$$\begin{aligned} \mathfrak{a} \circledast_\lambda \mathfrak{b} = {} & a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}) + b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}') + \lambda(a_1 b_1) \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}') \\ & - \mathbf{1}_A \otimes (a_1 b_1) \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}') - a_1 b_1 \otimes ((\mathfrak{a}' \circledast_\lambda \mathbf{1}_A) \circledast_\lambda \mathfrak{b}') \end{aligned} \tag{4.12}$$

Here the terms on the right hand side are well-defined by the induction hypothesis, and $a_1 b_1$ is the product in A.

Before we do any further computations, we need another result.

Lemma 4.11. For $|\mathfrak{a}|>0$, $\mathbf{1}_A \circledast_\lambda \mathfrak{a} = \lambda\mathfrak{a} = \mathfrak{a} \circledast_\lambda \mathbf{1}_A$, where $|\mathfrak{a}|$=length of $\mathfrak{a}$.

Proof. We will divide the proof into two cases:

Case I: $|\mathfrak{a}|=1$. Then,

$$\mathbf{1}_A \circledast_\lambda \mathfrak{a} = \mathbf{1}_A \otimes \mathfrak{a} + \mathfrak{a} \otimes \mathbf{1}_A + \lambda\mathfrak{a} - \mathbf{1}_A \otimes \mathfrak{a} - \mathfrak{a} \otimes ((\mathbf{1}_{\mathbf{k}} \circledast_\lambda \mathbf{1}_A) \circledast_\lambda \mathbf{1}_{\mathbf{k}}) = \lambda\mathfrak{a}.$$

Therefore, $\mathbf{1}_A \circledast_\lambda \mathfrak{a} = \lambda\mathfrak{a}$ for $|\mathfrak{a}|=1$.

Case II: $|\mathfrak{a}|>1$, i.e. $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$

$$\begin{aligned}
\mathbf{1}_A \circledast_\lambda a &= \mathbf{1}_A \otimes (\mathbf{1}_\mathrm{k} \circledast_\lambda \mathfrak{a}) + a_1 \otimes (\mathbf{1}_A \circledast_\lambda \mathfrak{a}') + \lambda a_1 \otimes (\mathbf{1}_\mathrm{k} \circledast_\lambda \mathfrak{a}') \\
&\quad -\mathbf{1}_A \otimes a_1 \otimes (\mathbf{1}_\mathrm{k} \circledast_\lambda \mathfrak{a}') - a_1 \otimes ((\mathbf{1}_A \circledast_\lambda \mathbf{1}_\mathrm{k}) \circledast_\lambda \mathfrak{a}') \\
&= \mathbf{1}_A \otimes \mathfrak{a} + a_1 \otimes (\mathbf{1}_A \circledast_\lambda \mathfrak{a}') + \lambda a_1 \otimes \mathfrak{a}' - \mathbf{1}_A \otimes a_1 \otimes \mathfrak{a}' - a_1 \otimes (\mathbf{1}_A \circledast_\lambda \mathfrak{a}') \\
&= \lambda a_1 \otimes \mathfrak{a}' \\
&= \lambda \mathfrak{a}.
\end{aligned}$$

By a similar computation, we get

$$\mathfrak{a} \circledast_\lambda \mathbf{1}_A = \lambda\mathfrak{a}.$$

This completes the proof of the lemma. □

Example 4.12. Let $\mathfrak{a} = a_1,\ \mathfrak{b} = b_1 \otimes b_2$. Then,

$$\begin{aligned}
& a_1 \circledast_\lambda (b_1 \otimes b_2) \\
&= a_1 \otimes (\mathbf{1}_\mathrm{k} \circledast_\lambda (b_1 \otimes b_2)) + b_1 \otimes (a_1 \circledast_\lambda b_2) + \lambda(a_1 b_1) \otimes (\mathbf{1}_\mathrm{k} \circledast_\lambda b_2) \\
&\quad - \mathbf{1}_A \otimes (a_1 b_1) \otimes (\mathbf{1}_\mathrm{k} \circledast_\lambda b_2) - (a_1 b_1) \otimes (\mathbf{1}_\mathrm{k} \circledast_\lambda \mathbf{1}_A) \circledast_\lambda b_2)) \\
&= a_1 \otimes (b_1 \otimes b_2) + b_1 \otimes (a_1 \circledast_\lambda b_2) + \lambda(a_1 b_1) \otimes b_2 - \mathbf{1}_A \otimes (a_1 b_1) \otimes b_2 - a_1 b_1 \otimes (\mathbf{1}_A \circledast_\lambda b_2) \\
&= a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 + b_1 \otimes \lambda(a_1 b_2) - b_1 \otimes \mathbf{1}_A \otimes a_1 b_2 \otimes \mathbf{1}_\mathrm{k} \\
&\quad - b_1 \otimes a_1 b_2 \otimes ((\mathbf{1}_\mathrm{k} \circledast_\lambda \mathbf{1}_A) \circledast_\lambda \mathbf{1}_\mathrm{k}) + \lambda(a_1 b_1) \otimes b_2 - \mathbf{1}_A \otimes (a_1 b_1) \otimes b_2 - a_1 b_1 \otimes (\mathbf{1}_A \circledast_\lambda b_2). \\
&= a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 + b_1 \otimes \lambda(a_1 b_2) - b_1 \otimes \mathbf{1}_A \otimes a_1 b_2 \\
&\quad - b_1 \otimes a_1 b_2 \otimes \mathbf{1}_A + \lambda(a_1 b_1) \otimes b_2 - \mathbf{1}_A \otimes (a_1 b_1) \otimes b_2 - a_1 b_1 \otimes (\mathbf{1}_A \circledast_\lambda b_2).
\end{aligned}$$

By applying above lemma, we will get

$$
\begin{aligned}
& a_1 \circledast_\lambda (b_1 \otimes b_2) \\
& = a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 + b_1 \otimes \lambda(a_1 b_2) - b_1 \otimes \mathbf{1}_A \otimes a_1 b_2 \\
& \quad - b_1 \otimes a_1 b_2 \otimes \mathbf{1}_A + \lambda(a_1 b_1) \otimes b_2 - \mathbf{1}_A \otimes (a_1 b_1) \otimes b_2 - a_1 b_1 \otimes (\lambda b_2) \\
& = a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 + b_1 \otimes \lambda(a_1 b_2) - b_1 \otimes \mathbf{1}_A \otimes a_1 b_2 \\
& \quad - b_1 \otimes a_1 b_2 \otimes \mathbf{1}_A - \mathbf{1}_A \otimes (a_1 b_1) \otimes b_2.
\end{aligned}
$$

Now, we prove the associativity of the RBNTD shuffle product.

Theorem 4.13. The RBNTD shuffle product is associative. i.e.

$$(\mathfrak{a} \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c} = \mathfrak{a} \circledast_\lambda (\mathfrak{b} \circledast_\lambda \mathfrak{c}).$$

Proof. We will prove the associativity in five cases:

Case I: If $|\mathfrak{a}|= 0,\ |\mathfrak{b}|= 0,\ |\mathfrak{c}|= 0$, then all are scalars and the associativity holds.

Case II: If one of $|\mathfrak{a}|,\ |\mathfrak{b}|= 0, |\mathfrak{c}|$ arbitrary, then also the associativity holds.

Case III: If $|\mathfrak{a}|= 1,\ |\mathfrak{b}|= 1,\ |\mathfrak{c}|= 1$, then to check

$$(\mathfrak{a} \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c} = \mathfrak{a} \circledast_\lambda (\mathfrak{b} \circledast_\lambda \mathfrak{c}).$$

We will start with computing the left hand side which is:

$$
\begin{aligned}
(\mathfrak{a} \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c} &= (\mathfrak{a} \otimes \mathfrak{b} + \mathfrak{b} \otimes \mathfrak{a} + \lambda \mathfrak{a}\mathfrak{b} - \mathbf{1}_A \otimes \mathfrak{a}\mathfrak{b} - \mathfrak{a}\mathfrak{b} \otimes \mathbf{1}_A) \circledast_\lambda \mathfrak{c} \\
&= (\mathfrak{a} \otimes \mathfrak{b}) \circledast_\lambda \mathfrak{c} + (\mathfrak{b} \otimes \mathfrak{a}) \circledast_\lambda \mathfrak{c} + (\lambda \mathfrak{a}\mathfrak{b}) \circledast_\lambda \mathfrak{c} \\
&\quad -(\mathbf{1}_A \otimes \mathfrak{a}\mathfrak{b}) \circledast_\lambda \mathfrak{c} - (\mathfrak{a}\mathfrak{b} \otimes \mathbf{1}_A) \circledast_\lambda \mathfrak{c}.
\end{aligned}
$$

Expanding further each term:

$$(\mathfrak{a} \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c} = \mathfrak{a} \otimes (\mathfrak{b} \circledast_\lambda \mathfrak{c}) + \mathfrak{c} \otimes (\mathfrak{a} \otimes \mathfrak{b}) + \lambda \mathfrak{a}\mathfrak{c} \otimes \mathfrak{b} - \mathbf{1}_A \otimes \mathfrak{a}\mathfrak{c} \otimes \mathfrak{b}$$

$$-\mathfrak{ac} \otimes (\mathfrak{b} \circledast_\lambda \mathbf{1}_A) + \mathfrak{b} \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{c}) + \mathfrak{c} \otimes (\mathfrak{b} \otimes \mathfrak{a}) + \lambda \mathfrak{bc} \otimes \mathfrak{a}$$
$$-\mathbf{1}_A \otimes \mathfrak{bc} \otimes \mathfrak{a} - \mathfrak{bc} \otimes (\mathfrak{a} \circledast_\lambda \mathbf{1}_A) + \lambda \mathfrak{ab} \otimes \mathfrak{c} + \mathfrak{c} \otimes \lambda \mathfrak{ab}$$
$$+\lambda^2 \mathfrak{abc} - \mathbf{1}_A \otimes \lambda \mathfrak{abc} - \lambda \mathfrak{abc} \otimes \mathbf{1}_A - \mathbf{1}_A \otimes (\mathfrak{ab} \circledast_\lambda \mathfrak{c})$$
$$-\mathfrak{c} \otimes (\mathbf{1}_A \otimes \mathfrak{ab}) - \lambda \mathfrak{c} \otimes \mathfrak{ab} + \mathbf{1}_A \otimes \mathfrak{c} \otimes \mathfrak{ab} + \mathfrak{c} \otimes (\mathfrak{ab} \circledast_\lambda \mathbf{1}_A)$$
$$-\mathfrak{ab} \otimes (\mathbf{1}_A \circledast_\lambda \mathfrak{c}) - \mathfrak{c} \otimes (\mathfrak{ab} \otimes \mathbf{1}_A) - \lambda \mathfrak{abc} \otimes \mathbf{1}_A$$
$$+\mathbf{1}_A \otimes \mathfrak{abc} \otimes \mathbf{1}_A + \mathfrak{abc} \otimes (\mathbf{1}_A \circledast_\lambda \mathbf{1}_A).$$

Further expanding $\circledast_\lambda$ product, we get:

$$\begin{aligned}
(\mathfrak{a} \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c} \quad = \quad & \mathfrak{a} \otimes (\mathfrak{b} \otimes \mathfrak{c}) + \mathfrak{a} \otimes (\mathfrak{c} \otimes \mathfrak{b}) + \mathfrak{a} \otimes \lambda \mathfrak{bc} - \mathfrak{a} \otimes (\mathbf{1}_A \otimes \mathfrak{bc}) \\
& -\mathfrak{a} \otimes (\mathfrak{bc} \otimes \mathbf{1}_A) + \mathfrak{c} \otimes (\mathfrak{a} \otimes \mathfrak{b}) + \lambda \mathfrak{ac} \otimes \mathfrak{b} - \mathbf{1}_A \otimes \mathfrak{ac} \otimes \mathfrak{b} \\
& -\mathfrak{ac} \otimes \lambda \mathfrak{b} + \mathfrak{b} \otimes (\mathfrak{a} \otimes \mathfrak{c}) + \mathfrak{b} \otimes (\mathfrak{c} \otimes \mathfrak{a}) + \mathfrak{b} \otimes \lambda \mathfrak{ac} \\
& -\mathfrak{b} \otimes \lambda \mathfrak{ac} - \mathfrak{b} \otimes (\mathbf{1}_A \otimes \mathfrak{ac}) + \mathfrak{c} \otimes (\mathfrak{b} \otimes \mathfrak{a}) + \lambda \mathfrak{bc} \otimes \mathfrak{a} \\
& -\mathbf{1}_A \otimes \mathfrak{bc} \otimes \mathfrak{a} - \mathfrak{bc} \otimes (\mathfrak{a} \circledast_\lambda \mathbf{1}_A) + \lambda \mathfrak{ab} \otimes \mathfrak{c} + \mathfrak{c} \otimes \lambda \mathfrak{ab} \\
& +\lambda^2 \mathfrak{abc} - \mathbf{1}_A \otimes \lambda \mathfrak{abc} - \lambda \mathfrak{abc} \otimes \mathbf{1}_A - \mathbf{1}_A \otimes (\mathfrak{ab} \otimes \mathfrak{c}) \\
& -\mathbf{1}_A \otimes (\mathfrak{c} \otimes \mathfrak{ab}) - \mathbf{1}_A \otimes \lambda \mathfrak{abc} + \mathbf{1}_A \otimes \mathbf{1}_A \otimes \mathfrak{abc} + \mathbf{1}_A \otimes \mathfrak{abc} \otimes \mathbf{1}_A \\
& -\mathfrak{c} \otimes (\mathbf{1}_A \otimes \mathfrak{ab}) - \lambda \mathfrak{c} \otimes \mathfrak{ab} + \mathbf{1}_A \otimes \mathfrak{c} \otimes \mathfrak{ab} + \mathfrak{c} \otimes \lambda(\mathfrak{ab}) \\
& -\mathfrak{ab} \otimes (\lambda \mathfrak{c}) - \mathfrak{c} \otimes (\mathfrak{ab} \otimes \mathbf{1}_A) - \lambda \mathfrak{abc} \otimes \mathbf{1}_A \\
& +\mathbf{1}_A \otimes \mathfrak{abc} \otimes \mathbf{1}_A + \mathfrak{abc} \otimes (\lambda \mathbf{1}_A).
\end{aligned}$$

Simplifying:

$$\begin{aligned}
(\mathfrak{a} \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c} \quad = \quad & \underbrace{\mathfrak{a} \otimes (\mathfrak{b} \otimes \mathfrak{c})}_{L_1} + \underbrace{\mathfrak{a} \otimes (\mathfrak{c} \otimes \mathfrak{b})}_{L_2} + \underbrace{\mathfrak{a} \otimes \lambda \mathfrak{bc}}_{L_3} - \underbrace{\mathfrak{a} \otimes (\mathbf{1}_A \otimes \mathfrak{bc})}_{L_4} \\
& - \underbrace{\mathfrak{a} \otimes (\mathfrak{bc} \otimes \mathbf{1}_A)}_{L_5} + \underbrace{\mathfrak{c} \otimes (\mathfrak{a} \otimes \mathfrak{b})}_{L_6} - \underbrace{\mathbf{1}_A \otimes \mathfrak{ac} \otimes \mathfrak{b}}_{L_7} + \underbrace{\mathfrak{b} \otimes (\mathfrak{a} \otimes \mathfrak{c})}_{L_8}
\end{aligned}$$

$$+\underbrace{\mathfrak{b}\otimes(\mathfrak{c}\otimes\mathfrak{a})}_{L_9}+\underbrace{\mathfrak{b}\otimes\lambda\mathfrak{a}\mathfrak{c}}_{L_{10}}-\underbrace{\mathfrak{b}\otimes(\mathbf{1}_A\otimes\mathfrak{a}\mathfrak{c})}_{L_{11}}-\underbrace{\mathfrak{b}\otimes(\mathfrak{a}\mathfrak{c}\otimes\mathbf{1}_A)}_{L_{12}}$$
$$+\underbrace{\mathfrak{c}\otimes(\mathfrak{b}\otimes\mathfrak{a})}_{L_{13}}-\underbrace{\mathbf{1}_A\otimes\mathfrak{b}\mathfrak{c}\otimes\mathfrak{a}}_{L_{14}}+\underbrace{\mathfrak{c}\otimes\lambda\mathfrak{a}\mathfrak{b}}_{L_{15}}+\underbrace{\lambda^2\mathfrak{a}\mathfrak{b}\mathfrak{c}}_{L_{16}}$$
$$-\underbrace{\mathbf{1}_A\otimes\lambda\mathfrak{a}\mathfrak{b}\mathfrak{c}}_{L_{17}}-\underbrace{\mathbf{1}_A\otimes(\mathfrak{a}\mathfrak{b}\otimes\mathfrak{c})}_{L_{18}}-\underbrace{\mathbf{1}_A\otimes\lambda\mathfrak{a}\mathfrak{b}\mathfrak{c}}_{L_{19}}+\underbrace{\mathbf{1}_A\otimes\mathbf{1}_A\otimes\mathfrak{a}\mathfrak{b}\mathfrak{c}}_{L_{20}}$$
$$+\underbrace{\mathbf{1}_A\otimes\mathfrak{a}\mathfrak{b}\mathfrak{c}\otimes\mathbf{1}_A}_{L_{21}}-\underbrace{\mathfrak{c}\otimes(\mathbf{1}_A\otimes\mathfrak{a}\mathfrak{b})}_{L_{22}}-\underbrace{\mathfrak{c}\otimes(\mathfrak{a}\mathfrak{b}\otimes\mathbf{1}_A)}_{L_{23}}-\underbrace{\lambda\mathfrak{a}\mathfrak{b}\mathfrak{c}\otimes\mathbf{1}_A}_{L_{24}}$$
$$+\underbrace{\mathbf{1}_A\otimes\mathfrak{a}\mathfrak{b}\mathfrak{c}\otimes\mathbf{1}_A}_{L_{25}}.$$

Now, computing the right hand side which is $\mathfrak{a}\circledast_\lambda(\mathfrak{b}\circledast_\lambda\mathfrak{c})$.

$$\begin{aligned}
\mathfrak{a}\circledast_\lambda(\mathfrak{b}\circledast_\lambda\mathfrak{c}) &= \mathfrak{a}\circledast_\lambda(\mathfrak{b}\otimes\mathfrak{c}+\mathfrak{c}\otimes\mathfrak{b}+\lambda\mathfrak{b}\mathfrak{c}-\mathbf{1}_A\otimes\mathfrak{b}\mathfrak{c}-\mathfrak{b}\mathfrak{c}\otimes\mathbf{1}_A)\\
&= \mathfrak{a}\circledast_\lambda(\mathfrak{b}\otimes\mathfrak{c})+\mathfrak{a}\circledast_\lambda(\mathfrak{c}\otimes\mathfrak{b})+\mathfrak{a}\circledast_\lambda(\lambda\mathfrak{b}\mathfrak{c})\\
&\quad -\mathfrak{a}\circledast_\lambda(\mathbf{1}_A\otimes\mathfrak{b}\mathfrak{c})-\mathfrak{a}\circledast_\lambda(\mathfrak{b}\mathfrak{c}\otimes\mathbf{1}_A).
\end{aligned}$$

By a similar computation as for the left hand side, we get

$$\mathfrak{a}\circledast_\lambda(\mathfrak{b}\circledast_\lambda\mathfrak{c}) = \underbrace{\mathfrak{a}\otimes(\mathfrak{b}\otimes\mathfrak{c})}_{R_1}+\underbrace{\mathfrak{b}\otimes(\mathfrak{a}\otimes\mathfrak{c})}_{R_2}+\underbrace{\mathfrak{b}\otimes(\mathfrak{c}\otimes\mathfrak{a})}_{R_3}+\underbrace{\mathfrak{b}\otimes(\lambda\mathfrak{a}\mathfrak{c})}_{R_4}$$
$$-\underbrace{\mathfrak{b}\otimes\mathbf{1}_A\otimes\mathfrak{a}\mathfrak{c}}_{R_5}-\underbrace{\mathfrak{b}\otimes\mathfrak{a}\mathfrak{c}\otimes\mathbf{1}_A}_{R_6}-\underbrace{\mathbf{1}_A\otimes\mathfrak{a}\mathfrak{b}\otimes\mathfrak{c}}_{R_7}+\underbrace{\mathfrak{a}\otimes(\mathfrak{c}\otimes\mathfrak{b})}_{R_8}$$
$$+\underbrace{\mathfrak{c}\otimes(\mathfrak{a}\otimes\mathfrak{b})}_{R_9}+\underbrace{\mathfrak{c}\otimes(\mathfrak{b}\otimes\mathfrak{a})}_{R_{10}}+\underbrace{\mathfrak{c}\otimes\lambda\mathfrak{a}\mathfrak{b}}_{R_{11}}-\underbrace{\mathfrak{c}\otimes\mathbf{1}_A\otimes\mathfrak{a}\mathfrak{b}}_{R_{12}}$$
$$-\underbrace{\mathfrak{c}\otimes\mathfrak{a}\mathfrak{b}\otimes\mathbf{1}_A}_{R_{13}}-\underbrace{\mathbf{1}_A\otimes\mathfrak{a}\mathfrak{c}\otimes\mathfrak{b}}_{R_{14}}+\underbrace{\mathfrak{a}\otimes\lambda\mathfrak{b}\mathfrak{c}}_{R_{15}}+\underbrace{\lambda^2\mathfrak{a}\mathfrak{b}\mathfrak{c}}_{R_{16}}$$
$$-\underbrace{\mathbf{1}_A\otimes\lambda\mathfrak{a}\mathfrak{b}\mathfrak{c}}_{R_{17}}-\underbrace{\lambda\mathfrak{a}\mathfrak{b}\mathfrak{c}\otimes\mathbf{1}_A}_{R_{18}}-\underbrace{\mathfrak{a}\otimes\lambda\mathfrak{b}\mathfrak{c}}_{R_{19}}-\underbrace{\mathbf{1}_A\otimes(\mathfrak{b}\mathfrak{c}\otimes\mathfrak{a})}_{R_{20}}$$
$$-\underbrace{\mathbf{1}_A\otimes\lambda\mathfrak{a}\mathfrak{b}\mathfrak{c}}_{R_{21}}+\underbrace{\mathbf{1}_A\otimes\mathbf{1}_A\otimes\mathfrak{a}\mathfrak{b}\mathfrak{c}}_{R_{22}}+\underbrace{\mathbf{1}_A\otimes\mathfrak{a}\mathfrak{b}\mathfrak{c}\otimes\mathbf{1}_A}_{R_{23}}-\underbrace{\mathfrak{a}\otimes(\mathfrak{b}\mathfrak{c}\otimes\mathbf{1}_A)}_{R_{24}}$$
$$+\underbrace{\mathbf{1}_A\otimes\mathfrak{a}\mathfrak{b}\mathfrak{c}\otimes\mathbf{1}_A}_{R_{25}}.$$

For $1\le i\le 25$, let L_i (resp R_i) be the i^{th} term in the left hand side (resp. in the right hand side). Then by the induction hypothesis, $L_i = R_{\sigma(i)}$. Here the permutation $\sigma \in \Sigma_{25}$ is given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 1 & 8 & 15 & 19 & 24 & 9 & 14 & 2 & 3 & 4 & 5 & 6 & 10 \end{pmatrix.}$$

$$\begin{matrix} 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 \\ 20 & 11 & 16 & 17 & 7 & 21 & 22 & 25 & 12 & 13 & 18 & 23 \end{pmatrix}.$$

Case IV: If $|\mathfrak{a}|= 1,\ |\mathfrak{b}|= 1,\ |\mathfrak{c}|> 1$, then similar to above case.

Case V: If $|\mathfrak{a}|\geq 2,\ |\mathfrak{b}|\geq 2$ and $|\mathfrak{c}|$ arbitrary. Then, let $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$, $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$ and $\mathfrak{c} = c_1 \otimes \mathfrak{c}'$.

We will prove this result using induction on $|\mathfrak{a}| + |\mathfrak{b}| + |\mathfrak{c}|$.

If $|\mathfrak{a}|= 0, |\mathfrak{b}|= 0, |\mathfrak{c}|= 0$, holds by Case I.

Assume that the associativity holds for $|\mathfrak{a}| + |\mathfrak{b}| + |\mathfrak{c}|\leq$ k.

To check for $|\mathfrak{a}| + |\mathfrak{b}| + |\mathfrak{c}|$=k+1.

Now, first we will work on the left hand side $(\mathfrak{a} \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c}$.

$$\begin{aligned}(\mathfrak{a} \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c} \ &= \ \big[a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}) + b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}') + \lambda a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}') \\ &\quad -\mathbf{1}_A \otimes a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}') - a_1 b_1 \otimes ((\mathfrak{a}' \circledast_\lambda \mathbf{1}_A) \circledast_\lambda \mathfrak{b}')\big] \circledast_\lambda \mathfrak{c}.\end{aligned}$$

Since $|\mathfrak{a}'|\geq 1(|\mathfrak{a}|\geq 2)$, apply Lemma 4.11 on the last term,

$$\begin{aligned}(\mathfrak{a} \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c} \ &= \ \big[a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}) + b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}') + \lambda a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}') \\ &\quad -\mathbf{1}_A \otimes a_1 b_1 \otimes (a' \circledast_\lambda b') - a_1 b_1 \otimes ((\lambda \mathfrak{a}' \circledast_\lambda \mathfrak{b}')\big] \circledast_\lambda \mathfrak{c}.\end{aligned}$$

Cancelling the 3rd and the 5th term, we get:

$$(\mathfrak{a} \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c} = \left[a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}) + b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}') - \mathbf{1}_A \otimes a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')\right] \circledast_\lambda \mathfrak{c}.$$

Expanding further, we get

$$\begin{aligned}
(\mathfrak{a} \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c} &= a_1 \otimes [(\mathfrak{a}' \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c}] + c_1 \otimes [(a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b})) \circledast_\lambda \mathfrak{c}'] \\
&\quad + \lambda a_1 c_1 \otimes [(\mathfrak{a}' \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c}'] - \mathbf{1}_A \otimes a_1 c_1 \otimes [(\mathfrak{a}' \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c}'] \\
&\quad - a_1 c_1 \otimes [((\mathfrak{a}' \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathbf{1}_A) \circledast_\lambda \mathfrak{c}'] + b_1 \otimes [(\mathfrak{a} \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c}] \\
&\quad + c_1 \otimes [(b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}'] + \lambda b_1 c_1 \otimes [(\mathfrak{a} \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c}'] \\
&\quad - \mathbf{1}_A \otimes b_1 c_1 \otimes [(\mathfrak{a} \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c}'] - b_1 c_1 \otimes [((\mathfrak{a} \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathbf{1}_A) \circledast_\lambda \mathfrak{c}'] \\
&\quad - \mathbf{1}_A \otimes [(a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}] - c_1 \otimes [(\mathbf{1}_A \otimes a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}'] \\
&\quad - \lambda c_1 \otimes [(a_1 b_1 \otimes (a' \circledast_\lambda b')) \circledast_\lambda c'] + \mathbf{1}_A \otimes c_1 \otimes [(a_1 b_1 \otimes (a' \circledast_\lambda b')) \circledast_\lambda c'] \\
&\quad + c_1 \otimes [((a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathbf{1}_A) \circledast_\lambda \mathfrak{c}'].
\end{aligned}$$

Applying Lemma 4.11 on the 5th term, 10th term and 15th term, we get:

$$\begin{aligned}
(\mathfrak{a} \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c} &= a_1 \otimes [(\mathfrak{a}' \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c}] + c_1 \otimes [(a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b})) \circledast_\lambda \mathfrak{c}'] \\
&\quad + \lambda a_1 c_1 \otimes [(\mathfrak{a}' \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c}'] - \mathbf{1}_A \otimes a_1 c_1 \otimes [(\mathfrak{a}' \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c}'] \\
&\quad - a_1 c_1 \otimes [(\lambda \mathfrak{a}' \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c}'] + b_1 \otimes [(\mathfrak{a} \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c}] \\
&\quad + c_1 \otimes [(b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}'] + \lambda b_1 c_1 \otimes [(\mathfrak{a} \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c}'] \\
&\quad - \mathbf{1}_A \otimes b_1 c_1 \otimes [(\mathfrak{a} \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c}'] - b_1 c_1 \otimes [\lambda(\mathfrak{a} \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c}'] \\
&\quad - \mathbf{1}_A \otimes [(a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}] - c_1 \otimes [(\mathbf{1}_A \otimes a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}'] \\
&\quad - \lambda c_1 \otimes [(a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}'] + \mathbf{1}_A \otimes c_1 \otimes [(a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}'] \\
&\quad + c_1 \otimes [(\lambda a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}'].
\end{aligned}$$

Simplifying:

$$
\begin{aligned}
(\mathfrak{a} \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c} \;=\;& a_1 \otimes [(\mathfrak{a}' \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c}] + c_1 \otimes [(a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b})) \circledast_\lambda \mathfrak{c}'] \\
& -\mathbf{1}_A \otimes a_1 c_1 \otimes [(\mathfrak{a}' \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c}'] + b_1 \otimes [(\mathfrak{a} \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c}] \\
& +c_1 \otimes [(b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}'] - \mathbf{1}_A \otimes b_1 c_1 \otimes [(\mathfrak{a} \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c}'] \\
& -\mathbf{1}_A \otimes [(a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}] - c_1 \otimes [(\mathbf{1}_A \otimes a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}'] \\
& +\mathbf{1}_A \otimes c_1 \otimes [(a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}'].
\end{aligned}
$$

Now using induction on the 1st term, we get:

$$
\begin{aligned}
a_1 \otimes [(\mathfrak{a}' \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c}] &= a_1 \otimes [\mathfrak{a}' \circledast_\lambda (\mathfrak{b} \circledast_\lambda \mathfrak{c})] \\
&= a_1 \otimes [\mathfrak{a}' \circledast_\lambda \big(b_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}) + c_1 \otimes (\mathfrak{b} \circledast_\lambda \mathfrak{c}') + \lambda b_1 c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}') \\
&\quad -\mathbf{1}_A \otimes b_1 c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}') - b_1 c_1 \otimes ((\mathfrak{b}' \circledast_\lambda 1) \circledast_\lambda \mathfrak{c}')\big)].
\end{aligned}
$$

Apply Lemma 4.11 on the last term, and cancel with the 3rd term, we get:

$$
\begin{aligned}
a_1 \otimes [(\mathfrak{a}' \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c}] &= a_1 \otimes [\mathfrak{a}' \circledast_\lambda \big(b_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}) + c_1 \otimes (\mathfrak{b} \circledast_\lambda \mathfrak{c}') \\
&\quad -\mathbf{1}_A \otimes b_1 c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}')\big)]. \\
&= a_1 \otimes [\mathfrak{a}' \circledast_\lambda (b_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c})] + a_1 \otimes [\mathfrak{a}' \circledast_\lambda c_1 \otimes (\mathfrak{b} \circledast_\lambda \mathfrak{c}')] \\
&\quad -a_1 \otimes [\mathfrak{a}' \circledast_\lambda (\mathbf{1}_A \otimes b_1 c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}'))].
\end{aligned}
$$

Similarly, expanding the 7th term and apply Lemma 4.11, we get:

$$
\begin{aligned}
\mathbf{1}_A \otimes [(a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}] &= \mathbf{1}_A \otimes [a_1 b_1 \otimes \big((\mathfrak{a}' \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c}\big)] \\
&\quad +\mathbf{1}_A \otimes [c_1 \otimes \big((a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}'\big)] \\
&\quad -\mathbf{1}_A \otimes [\mathbf{1}_A \otimes a_1 b_1 c_1 \otimes ((\mathfrak{a}' \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c}')].
\end{aligned}
$$

Substituting, we will get:

$$
\begin{aligned}
(\mathfrak{a} \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c} \quad = \quad & \underbrace{a_1 \otimes [\mathfrak{a}' \circledast_\lambda (b_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c})]}_{L_1} + \underbrace{a_1 \otimes [\mathfrak{a}' \circledast_\lambda c_1 \otimes (\mathfrak{b} \circledast_\lambda \mathfrak{c}')]}_{L_2} \\
& - \underbrace{a_1 \otimes [\mathfrak{a}' \circledast_\lambda (\mathbf{1}_A \otimes b_1 c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}'))]}_{L_3} + \underbrace{c_1 \otimes [(a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b})) \circledast_\lambda \mathfrak{c}']}_{L_4} \\
& - \underbrace{\mathbf{1}_A \otimes a_1 c_1 \otimes [(\mathfrak{a}' \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c}']}_{L_5} + \underbrace{b_1 \otimes [(\mathfrak{a} \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c}]}_{L_6} \\
& + \underbrace{c_1 \otimes [(b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}']}_{L_7} - \underbrace{\mathbf{1}_A \otimes b_1 c_1 \otimes [(\mathfrak{a} \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c}']}_{L_8} \\
& - \underbrace{\mathbf{1}_A \otimes [a_1 b_1 \otimes \big((\mathfrak{a}' \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c})]}_{L_9} - \underbrace{\mathbf{1}_A \otimes [c_1 \otimes \big((a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}')]}_{L_{10}} \\
& + \underbrace{\mathbf{1}_A \otimes [\mathbf{1}_A \otimes a_1 b_1 c_1 \otimes ((\mathfrak{a}' \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c}')]}_{L_{11}} - \underbrace{c_1 \otimes [(\mathbf{1}_A \otimes a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}']}_{L_{12}} \\
& + \underbrace{\mathbf{1}_A \otimes c_1 \otimes [(a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}']}_{L_{13}}.
\end{aligned}
$$

L_{10} and L_{13} will cancel each other, so we get:

$$
\begin{aligned}
(\mathfrak{a} \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c} \quad = \quad & \underbrace{a_1 \otimes [\mathfrak{a}' \circledast_\lambda (b_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c})]}_{L_1} + \underbrace{a_1 \otimes [\mathfrak{a}' \circledast_\lambda c_1 \otimes (\mathfrak{b} \circledast_\lambda \mathfrak{c}')]}_{L_2} \\
& - \underbrace{a_1 \otimes [\mathfrak{a}' \circledast_\lambda (\mathbf{1}_A \otimes b_1 c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}'))]}_{L_3} + \underbrace{c_1 \otimes [(a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b})) \circledast_\lambda \mathfrak{c}']}_{L_4} \\
& - \underbrace{\mathbf{1}_A \otimes a_1 c_1 \otimes [(\mathfrak{a}' \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c}']}_{L_5} + \underbrace{b_1 \otimes [(\mathfrak{a} \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c}]}_{L_6} \\
& + \underbrace{c_1 \otimes [(b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}']}_{L_7} - \underbrace{\mathbf{1}_A \otimes b_1 c_1 \otimes [(\mathfrak{a} \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c}']}_{L_8} \\
& - \underbrace{\mathbf{1}_A \otimes [a_1 b_1 \otimes \big((\mathfrak{a}' \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c})]}_{L_9} + \underbrace{\mathbf{1}_A \otimes [\mathbf{1}_A \otimes a_1 b_1 c_1 \otimes ((\mathfrak{a}' \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c}')]}_{L_{10}} \\
& - \underbrace{c_1 \otimes [(\mathbf{1}_A \otimes a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}']}_{L_{11}}.
\end{aligned}
$$

Similarly, for the right hand side,

$$
\mathfrak{a} \circledast_\lambda (\mathfrak{b} \circledast_\lambda \mathfrak{c}) \quad = \quad \mathfrak{a} \circledast_\lambda [b_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}) + c_1 \otimes (\mathfrak{b} \circledast_\lambda \mathfrak{c}') + \lambda b_1 c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}')
$$

$$-\mathbf{1}_A \otimes b_1c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}') - b_1c_1 \otimes \big((\mathfrak{b}' \circledast_\lambda \mathbf{1}_{\mathrm{k}}) \circledast_\lambda \mathfrak{c}')].$$

Since $|\mathfrak{b}'| \geq 1 (|\mathfrak{b}| \geq 2)$, apply Lemma 4.11 on the last term,

$$\begin{aligned}\mathfrak{a} \circledast_\lambda (\mathfrak{b} \circledast_\lambda \mathfrak{c}) &= \mathfrak{a} \circledast_\lambda [b_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}) + c_1 \otimes (\mathfrak{b} \circledast_\lambda \mathfrak{c}') + \lambda b_1c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}') \\ &\quad -\mathbf{1}_A \otimes b_1c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}') - b_1c_1 \otimes \lambda(\mathfrak{b}' \circledast_\lambda \mathfrak{c}')].\end{aligned}$$

Cancelling the 3rd and the 5th term, we get:

$$\mathfrak{a} \circledast_\lambda (\mathfrak{b} \circledast_\lambda \mathfrak{c}) = \mathfrak{a} \circledast_\lambda [b_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}) + c_1 \otimes (\mathfrak{b} \circledast_\lambda \mathfrak{c}') - \mathbf{1}_A \otimes b_1c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}')].$$

Expanding further,

$$\begin{aligned}\mathfrak{a} \circledast_\lambda (\mathfrak{b} \circledast_\lambda \mathfrak{c}) &= a_1 \otimes [\mathfrak{a}' \circledast_\lambda (b_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}))] + b_1 \otimes [\mathfrak{a} \circledast_\lambda (\mathfrak{b}' \circledast_\lambda \mathfrak{c})] \\ &\quad \lambda a_1b_1 \otimes [\mathfrak{a}' \circledast_\lambda (\mathfrak{b}' \circledast_\lambda \mathfrak{c})] - \mathbf{1}_A \otimes a_1b_1 \otimes [\mathfrak{a}' \circledast_\lambda (\mathfrak{b}' \circledast_\lambda \mathfrak{c})] \\ &\quad a_1b_1 \otimes [(\mathfrak{a}' \circledast_\lambda \mathbf{1}_{\mathrm{k}}) \circledast_\lambda (\mathfrak{b}' \circledast_\lambda \mathfrak{c})] + a_1 \otimes [\mathfrak{a}' \circledast_\lambda (c_1 \otimes (\mathfrak{b} \circledast_\lambda \mathfrak{c}'))] \\ &\quad +c_1 \otimes [\mathfrak{a} \circledast_\lambda (\mathfrak{b} \circledast_\lambda \mathfrak{c}')] + \lambda a_1c_1 \otimes [\mathfrak{a}' \circledast_\lambda (\mathfrak{b} \circledast_\lambda \mathfrak{c}')] \\ &\quad -\mathbf{1}_A \otimes a_1c_1 \otimes [\mathfrak{a}' \circledast_\lambda (\mathfrak{b} \circledast_\lambda \mathfrak{c}')] - a_1c_1 \otimes [(\mathfrak{a}' \circledast_\lambda \mathbf{1}_{\mathrm{k}}) \circledast_\lambda (\mathfrak{b} \circledast_\lambda \mathfrak{c}')] \\ &\quad -a_1 \otimes [\mathfrak{a}' \circledast_\lambda (\mathbf{1}_A \otimes b_1c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}'))] - \mathbf{1}_A \otimes [\mathfrak{a} \circledast_\lambda (b_1c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}'))] \\ &\quad -\lambda a_1 \otimes [\mathfrak{a}' \circledast_\lambda (b_1c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}'))] + \mathbf{1}_A \otimes a_1 \otimes [\mathfrak{a}' \circledast_\lambda (b_1c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}'))] \\ &\quad -a_1 \otimes [(\mathfrak{a}' \circledast_\lambda \mathbf{1}_{\mathrm{k}}) \circledast_\lambda (b_1c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}'))].\end{aligned}$$

Apply lemma 4.11 on the 5th, 10th and 15th term, and then simplify:

$$\begin{aligned}\mathfrak{a} \circledast_\lambda (\mathfrak{b} \circledast_\lambda \mathfrak{c}) &= a_1 \otimes [\mathfrak{a}' \circledast_\lambda (b_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}))] + b_1 \otimes [\mathfrak{a} \circledast_\lambda (\mathfrak{b}' \circledast_\lambda \mathfrak{c})] \\ &\quad -\mathbf{1}_A \otimes a_1b_1 \otimes [\mathfrak{a}' \circledast_\lambda (\mathfrak{b}' \circledast_\lambda \mathfrak{c})] + a_1 \otimes [\mathfrak{a}' \circledast_\lambda (c_1 \otimes (\mathfrak{b} \circledast_\lambda \mathfrak{c}'))] \\ &\quad +c_1 \otimes [\mathfrak{a} \circledast_\lambda (\mathfrak{b} \circledast_\lambda \mathfrak{c}')] - \mathbf{1}_A \otimes a_1c_1 \otimes [\mathfrak{a}' \circledast_\lambda (\mathfrak{b} \circledast_\lambda \mathfrak{c}')]\end{aligned}$$

$$-a_1 \otimes [\mathfrak{a}' \circledast_\lambda (\mathbf{1}_A \otimes b_1c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}'))] - \mathbf{1}_A \otimes [\mathfrak{a} \circledast_\lambda (b_1c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}'))]$$
$$+\mathbf{1}_A \otimes a_1 \otimes [\mathfrak{a}' \circledast_\lambda (b_1c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}'))].$$

Now using the induction hypothesis on the 5th term, we get:

$$\begin{aligned}
c_1 \otimes [\mathfrak{a} \circledast_\lambda (\mathfrak{b} \circledast_\lambda \mathfrak{c}')] &= c_1 \otimes [(\mathfrak{a} \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c}'] \\
&= c_1 \otimes \big[(a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}) + b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}') + \lambda a_1b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}') \\
&\quad -\mathbf{1}_A \otimes a_1b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}') - a_1b_1 \otimes (((\mathfrak{a}' \circledast_\lambda \mathbf{1}_{\mathrm{k}})\mathfrak{b}')) \circledast_\lambda c'\big] \\
&= c_1 \otimes \big[(a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}) + b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}') + \lambda a_1b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}') \\
&\quad -\mathbf{1}_A \otimes a_1b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}') - a_1b_1 \otimes ((\lambda\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}'\big] \\
&= c_1 \otimes \big[(a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}) + b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}') \\
&\quad -\mathbf{1}_A \otimes a_1b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}'\big] \\
&= c_1 \otimes \big[(a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}) \circledast_\lambda \mathfrak{c}'\big] + c_1 \otimes \big[b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c}'\big] \\
&\quad -c_1 \otimes \big[\mathbf{1}_A \otimes a_1b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}'\big].
\end{aligned}$$

Similarly expanding the 8th term and applying Lemma 4.11, we get

$$\begin{aligned}
\mathbf{1}_A \otimes [\mathfrak{a} \circledast_\lambda (b_1c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}'))] &= \mathbf{1}_A \otimes [a_1 \otimes (\mathfrak{a}' \circledast_\lambda (b_1c_1 \otimes (b' \circledast_\lambda c')))] \\
&\quad +\mathbf{1}_A \otimes [b_1c_1 \otimes (\mathfrak{a} \circledast_\lambda (\mathfrak{b}' \circledast_\lambda \mathfrak{c}'))] \\
&\quad -\mathbf{1}_A \otimes [\mathbf{1}_A \otimes a_1b_1c_1 \otimes (\mathfrak{a}' \circledast_\lambda (\mathfrak{b}' \circledast_\lambda \mathfrak{c}'))].
\end{aligned}$$

Substituting, we obtain

$$\begin{aligned}
\mathfrak{a} \circledast_\lambda (\mathfrak{b} \circledast_\lambda \mathfrak{c}) &= \underbrace{a_1 \otimes [\mathfrak{a}' \circledast_\lambda (b_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}))]}_{R_1} + \underbrace{b_1 \otimes [\mathfrak{a} \circledast_\lambda (\mathfrak{b}' \circledast_\lambda \mathfrak{c})]}_{R_2} \\
&\quad - \underbrace{\mathbf{1}_A \otimes a_1b_1 \otimes [\mathfrak{a}' \circledast_\lambda (\mathfrak{b}' \circledast_\lambda c)]}_{R_3} + \underbrace{a_1 \otimes [\mathfrak{a}' \circledast_\lambda (c_1 \otimes (\mathfrak{b} \circledast_\lambda \mathfrak{c}'))]}_{R_4}
\end{aligned}$$

$$
\begin{aligned}
&+\underbrace{c_1 \otimes \left[(a_1 \otimes (\mathfrak{a}' \circledast_\lambda b) \circledast_\lambda \mathfrak{c}'\right]}_{R_5} + \underbrace{c_1 \otimes \left[b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}') \circledast_\lambda \mathfrak{c}'\right]}_{R_6} \\
&- \underbrace{c_1 \otimes \left[\mathbf{1}_A \otimes a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}'\right]}_{R_7} - \underbrace{\mathbf{1}_A \otimes a_1 c_1 \otimes [\mathfrak{a}' \circledast_\lambda (\mathfrak{b} \circledast_\lambda \mathfrak{c}')]}_{R_8} \\
&- \underbrace{a_1 \otimes [\mathfrak{a}' \circledast_\lambda (\mathbf{1}_A \otimes b_1 c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}'))]}_{R_9} - \underbrace{\mathbf{1}_A \otimes [a_1 \otimes \left(\mathfrak{a}' \circledast_\lambda (b_1 c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}'))\right)]}_{R_{10}} \\
&- \underbrace{\mathbf{1}_A \otimes [b_1 c_1 \otimes (\mathfrak{a} \circledast_\lambda (\mathfrak{b}' \circledast_\lambda \mathfrak{c}'))]}_{R_{11}} + \underbrace{\mathbf{1}_A \otimes [\mathbf{1}_A \otimes a_1 b_1 c_1 \otimes \left(\mathfrak{a}' \circledast_\lambda (\mathfrak{b}' \circledast_\lambda \mathfrak{c}')\right)]}_{R_{12}} \\
&+ \underbrace{\mathbf{1}_A \otimes a_1 \otimes [\mathfrak{a}' \circledast_\lambda (b_1 c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}'))]}_{R_{13}}.
\end{aligned}
$$

R_{10} and R_{13} will cancel with each other. Hence, we have

$$
\begin{aligned}
\mathfrak{a} \circledast_\lambda (\mathfrak{b} \circledast_\lambda \mathfrak{c}) \quad = \quad & \underbrace{a_1 \otimes [\mathfrak{a}' \circledast_\lambda (b_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}))]}_{R_1} + \underbrace{b_1 \otimes [\mathfrak{a} \circledast_\lambda (\mathfrak{b}' \circledast_\lambda \mathfrak{c})]}_{R_2} \\
&- \underbrace{\mathbf{1}_A \otimes a_1 b_1 \otimes [\mathfrak{a}' \circledast_\lambda (\mathfrak{b}' \circledast_\lambda c)]}_{R_3} + \underbrace{a_1 \otimes [\mathfrak{a}' \circledast_\lambda (c_1 \otimes (b \circledast_\lambda \mathfrak{c}'))]}_{R_4} \\
&+ \underbrace{c_1 \otimes \left[(a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b})) \circledast_\lambda \mathfrak{c}'\right]}_{R_5} + \underbrace{c_1 \otimes \left[(b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}'\right]}_{R_6} \\
&- \underbrace{c_1 \otimes \left[\mathbf{1}_A \otimes a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \circledast_\lambda \mathfrak{c}'\right]}_{R_7} - \underbrace{\mathbf{1}_A \otimes a_1 c_1 \otimes [\mathfrak{a}' \circledast_\lambda (\mathfrak{b} \circledast_\lambda \mathfrak{c}')]}_{R_8} \\
&- \underbrace{a_1 \otimes [\mathfrak{a}' \circledast_\lambda (\mathbf{1}_A \otimes b_1 c_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{c}'))]}_{R_9} - \underbrace{\mathbf{1}_A \otimes [b_1 c_1 \otimes (a \circledast_\lambda (\mathfrak{b}' \circledast_\lambda \mathfrak{c}'))]}_{R_{10}} \\
&+ \underbrace{\mathbf{1}_A \otimes [\mathbf{1}_A \otimes a_1 b_1 c_1 \otimes \left(\mathfrak{a}' \circledast_\lambda (\mathfrak{b}' \circledast_\lambda \mathfrak{c}')\right)]}_{R_{11}}.
\end{aligned}
$$

For $1 \leq i \leq 11$, let L_i (resp R_i) be the i^{th} term in the left hand side (resp. in the right hand side). Then by the induction hypothesis, $L_i = R_{\sigma(i)}$. Here the permutation $\sigma \in \Sigma_{11}$ is given by

$$
\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 4 & 9 & 5 & 8 & 2 & 6 & 10 & 3 & 11 & 7 \end{pmatrix}
$$

Therefore, the associtivity is proved. $\square$

Now, we also prove the commutativity of the RBNTD shuffle product.

Theorem 4.14. The RBNTD shuffle product $\circledast_\lambda$ is commutative. More precisely,

$$\mathfrak{a} \circledast_\lambda \mathfrak{b} = \mathfrak{b} \circledast_\lambda \mathfrak{a}.$$

Proof. For two pure tensors a and b, $\mathfrak{a} := a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}$ and $\mathfrak{b} := b_1 \otimes \cdots \otimes b_n \in A^{\otimes n}$, let $\mathfrak{a} = a_1 \otimes \mathfrak{a}',\ \mathfrak{a}' \in A^{\otimes m-1}$ and $\mathfrak{b} = b_1 \otimes \mathfrak{b}',\ b' \in A^{\otimes n-1}$. We will prove the result by induction on m+n.

For m+n=0, a and b are scalars, so the result holds . Assume that the commutativity holds for $m + n \leq k$. Now, we check for m+n=k+1.

$$\begin{aligned} \mathfrak{a} \circledast_\lambda \mathfrak{b} &= a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}) + b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}') + \lambda a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}') \\ &\quad -\mathbf{1}_A \otimes a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}') - a_1 b_1 \otimes ((\mathfrak{a}' \circledast_\lambda \mathbf{1}_A) \circledast_\lambda \mathfrak{b}'). \end{aligned}$$

For $\mid \mathfrak{a}' \mid \neq 0$, we get $\mathfrak{a}' \circledast_\lambda 1 = \lambda \mathfrak{a}'$ which gives us

$$\begin{aligned} \mathfrak{a} \circledast_\lambda \mathfrak{b} &= a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}) + b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}') \\ &\quad -\mathbf{1}_A \otimes a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}'). \end{aligned}$$

Applying the induction hypothesis on all the terms because the sum of lengths $\leq k$, we get

$$\begin{aligned} \mathfrak{a} \circledast_\lambda \mathfrak{b} &= a_1 \otimes (\mathfrak{b} \circledast_\lambda \mathfrak{a}') + b_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{a}) \\ &\quad -\mathbf{1}_A \otimes a_1 b_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{a}'). \end{aligned}$$

Similarly, we compute $\mathfrak{b} \circledast_\lambda \mathfrak{a}$.

$$\mathfrak{b} \circledast_\lambda \mathfrak{a} = b_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{a}) + a_1 \otimes (\mathfrak{b} \circledast_\lambda \mathfrak{a}') + \lambda b_1 a_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{a}')$$

$$-\mathbf{1}_A \otimes b_1 a_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{a}') - b_1 a_1 \otimes ((\mathfrak{b}' \circledast_\lambda \mathbf{1}_A) \circledast_\lambda \mathfrak{a}').$$

For $|\ \mathfrak{b}'\ | \neq 0$, we get $\mathfrak{b}' \circledast_\lambda 1 = \lambda \mathfrak{b}'$, which gives us

$$\begin{aligned} \mathfrak{b} \circledast_\lambda \mathfrak{a} &= b_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{a}) + a_1 \otimes (\mathfrak{b} \circledast_\lambda \mathfrak{a}') \\ &\quad -\mathbf{1}_A \otimes b_1 a_1 \otimes (\mathfrak{b}' \circledast_\lambda \mathfrak{a}'). \end{aligned}$$

So, they both agree since A is commutative. □

Now we define the free commutative RBNTD algebras.

Definition 4.15. Let A be a commutative $\mathbf{k}$-algebra. A free commutative RBNTD algebra over A is a commutative RBNTD $\mathbf{k}$-algebra $F_{\mathrm{T}}(A)$ with a RBNTD operator P_A and a commutative $\mathbf{k}$-algebra homomorphism $j_A : A \to F_{\mathrm{T}}(A)$ such that, for any commutative RBNTD algebra (T, P) and any algebra homomorphism $f : A \to T$, there is a unique RBNTD $\mathbf{k}$-algebra homomorphism $\bar{f} : F_{\mathrm{T}}(A) \to T$ such that $\bar{f} \circ j_A = f$:

$$\begin{tikzcd} A \arrow[r, "j_A"] \arrow[dr, "f"'] & F_{\mathrm{T}}(A) \arrow[d, "\bar{f}"] \\ & T \end{tikzcd}$$

Let A be a commutative $\mathbf{k}$-algebra A. Let

$$Ш_{\mathrm{T}}^{+}(A) = \bigoplus_{k \geq 0} A^{\otimes k}$$

be the algebra equipped with the RBNTD shuffle product $\circledast_\lambda$ defined earlier .

Define

$$Ш_{\mathrm{T}}(A) := Ш_{\mathrm{T},\lambda}(A) := A \otimes Ш_{\mathrm{T}}^{+}(A) = \oplus_{k \geq 1} A^{\otimes k}. \tag{4.13}$$

to be the tensor product algebra whose product is denoted by $\diamond$ and is called the **augmented total shuffle product of weight** λ. More precisely, the product $\diamond := \diamond_\lambda$ on $Ш(A)_{\mathrm{T}}$ is defined by:

$$(a_0 \otimes \mathfrak{a}) \diamond (b_0 \otimes \mathfrak{b}) := (a_0 b_0) \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}),\ a_0,\ b_0 \in A,\ \mathfrak{a},\ \mathfrak{b} \in Ш_{\mathrm{T}}^{+}(A). \tag{4.14}$$

Thus we have the algebra isomorphism (embedding to the second tensor factor)

$$\alpha : (Ш_{\mathrm{T}}^{+}(A), \circledast_\lambda) \to (\mathbf{1}_A \otimes Ш_{\mathrm{T}}^{+}(A), \diamond). \tag{4.15}$$

Define the **k**-linear endomorphism P_A on $Ш_{\mathrm{T}}(A)$ by assigning

$$\begin{aligned} P_A(a_0 \otimes \mathfrak{a}) &= \mathbf{1}_A \otimes a_0 \otimes \mathfrak{a},\ \mathfrak{a} \in A^{\otimes n},\ n \geq 1, \\ P_A(a_0 \otimes c) &= \mathbf{1}_A \otimes c a_0,\ c \in A^{\otimes 0} = \mathbf{k} \end{aligned}$$

and extending by additivity. Let $j_A : A \to Ш_{\mathrm{T}}(A)$ be the canonical inclusion map. Call $(Ш_{\mathrm{T}}(A), P_A)$ the (**total**) **shuffle RBNTD k-algebra of weight** λ **on** A. The term is justified by the following theorem.

Theorem 4.16. The shuffle RBNTD algebra $(Ш_{\mathrm{T}}(A), P_A)$, together with the natural embedding j_A, is a free commutative RBNTD algebra on A with weight λ. More precisely, for any commutative RBNTD **k**-algebra (T, P) of weight λ and algebra homomorphism $f : A \to T$, there is a unique RBNTD **k**-algebra homomorphism $\bar{f} : (Ш_{\mathrm{T}}(A), P_A) \to (T, P)$ such that $f = \bar{f} \circ j_A$.

4.4 The proof of Theorem 4.16

We first prove that $(Ш_{\mathrm{T}}(A), P_A)$ is a commutative RBNTD algebra. For pure tensors $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$ and $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$,

$$\begin{aligned}
P_A(a) \diamond P_A(b) &= (\mathbf{1}_A \otimes \mathfrak{a}) \diamond (\mathbf{1}_A \otimes \mathfrak{b}) \\
&= \mathbf{1}_A \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}) \\
&= \mathbf{1}_A \otimes (a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b})) + \mathbf{1}_A \otimes (b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}')) \\
&\quad + \mathbf{1}_A \otimes ((\lambda a_1 b_1) \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) - \mathbf{1}_A \otimes (\mathbf{1}_A \otimes a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')) \\
&\quad - \mathbf{1}_A \otimes (a_1 b_1 \otimes ((\mathfrak{a}' \circledast_\lambda \mathbf{1}_A) \circledast_\lambda \mathfrak{b}')) \\
&= P_A(\mathfrak{a} \diamond P_A(\mathfrak{b})) + P_A(\mathfrak{b} \diamond P_A(\mathfrak{a})) + \lambda P_A(\mathfrak{a} \diamond \mathfrak{b}) - P_A^2(\mathfrak{a} \diamond \mathfrak{b}) \\
&\quad - P_A((\mathfrak{a} \diamond P_A(\mathbf{1}_A)) \diamond \mathfrak{b}).
\end{aligned}$$

To verify the universal property of $(Ш_{\mathrm{T}}(A), P_A)$, let (T, P) be a commutative RBNTD algebra and let $f : A \to T$ be an algebra homomorphism. To define $\bar{f} : Ш_{\mathrm{T}}(A) \to T$, we need to define $\bar{f}(\mathfrak{a})$ for any pure tensor $\mathfrak{a} = a_1 \otimes \cdots \otimes a_n \in A^{\otimes n}$. For this, we use induction on n. When $n = 1$, we have $A = j_A(A)$. So, we must have $\bar{f}(\mathfrak{a}) = f(\mathfrak{a})$. Assume that $\bar{f}(\mathfrak{a})$ has been defined for $n \leq k$ and let $\mathfrak{a} \in A^{\otimes(k+1)}$. Then $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$ with $\mathfrak{a}' \in A^{\otimes k}$. Note that $\mathfrak{a} = a_1 \diamond P_A(\mathfrak{a}')$. Since $\bar{f}$ is to be a RBNTD algebra homomorphism, we must have

$$\bar{f}(\mathfrak{a}) = \bar{f}(a_1)\bar{f}(P_A(\mathfrak{a}')) = f(a_1)P(\bar{f}(\mathfrak{a}')). \tag{4.16}$$

In fact, if $\mathfrak{a} = a_1 \otimes a_2 \otimes \cdots \otimes a_{k+1}$, then we must have

$$\bar{f}(\mathfrak{a}) = f(a_1)P(f(a_2)P(f(a_3) \cdots f(a_{k+1}))).$$

So the uniqueness of $\bar{f}$ is proved.

Now, we just need to prove that the $\bar{f}$ obtained this way, i.e., by Eq. (4.16), is a RBNTD algebra homomorphism. From the construction we obviously have $\bar{f}\circ P_A = P\circ\bar{f}$. So, it remains to prove the multiplicativity. Let $\mathfrak{a}\in A^{\otimes m}$ and $\mathfrak{b}\in A^{\otimes n}$, we will verify

$$\bar{f}(\mathfrak{a}\diamond\mathfrak{b}) = \bar{f}(\mathfrak{a})\bar{f}(\mathfrak{b}) \tag{4.17}$$

by induction on $m+n$. Then $m+n\geq 2$. When $m+n=2$, then $m=n=1$ and so $a, b\in A$. Then $a\diamond b = ab$, so Eq.(4.17) follows since f is an algebra homomorphism.

Assume that the equation is proved for $m+n\leq k$ and let $\mathfrak{a} = a_1\otimes\mathfrak{a}' = a_1\diamond P_A(\mathfrak{a}')\in A^{\otimes m}$, $\mathfrak{b} = b_1\otimes\mathfrak{b}' = b_1\diamond P_A(\mathfrak{b}')\in A^{\otimes n}$ with $m+n = k+1$. Then by Eq. (4.16) and the induction hypothesis, we have

$$\begin{aligned}
\bar{f}(\mathfrak{a}\diamond\mathfrak{b}) &= \bar{f}((a_1b_1)\diamond P_A(\mathfrak{a}')\diamond P_A(\mathfrak{b}'))\\
&= \bar{f}\big((a_1b_1)\diamond P_A\big(\mathfrak{a}'\diamond P_A(\mathfrak{b}') + P_A(\mathfrak{a}')\diamond\mathfrak{b}' + \lambda\mathfrak{a}'\diamond\mathfrak{b}'\\
&\quad -P_A(\mathfrak{a}'\diamond\mathfrak{b}') - (\mathfrak{a}'\diamond P_A(1))\diamond\mathfrak{b}'\big)\big).\\
&= \bar{f}(a_1b_1)(\bar{f}\circ P_A)\big(\mathfrak{a}'\diamond(1\otimes\mathfrak{b}') + (1\otimes\mathfrak{a}')\diamond\mathfrak{b}' + \lambda(\mathfrak{a}'\diamond\mathfrak{b}')\\
&\quad -P_A(\mathfrak{a}'\diamond\mathfrak{b}') - (\mathfrak{a}'\diamond P_A(1))\diamond\mathfrak{b}'\big)\big).\\
&= f(a_1b_1)(P\circ\bar{f})\big(\mathfrak{a}'\diamond(1\otimes\mathfrak{b}') + (1\otimes\mathfrak{a}')\diamond\mathfrak{b}' + \lambda(\mathfrak{a}'\diamond\mathfrak{b}')\\
&\quad -P_A(\mathfrak{a}'\diamond\mathfrak{b}') - (\mathfrak{a}'\diamond P_A(1))\diamond\mathfrak{b}'\big)\big).\\
&= f(a_1b_1)P(\bar{f}(\mathfrak{a}')\bar{f}(1\otimes\mathfrak{b}') + \bar{f}(1\otimes\mathfrak{a}')\bar{f}(\mathfrak{b}') + \lambda\bar{f}(\mathfrak{a}')\bar{f}(\mathfrak{b}')\\
&\quad -P(\bar{f}(\mathfrak{a}')\bar{f}(\mathfrak{b}')) - \bar{f}(\mathfrak{a}')\bar{f}(P_A(1))\bar{f}(\mathfrak{b}')).\\
&= f(a_1)f(b_1)P(\bar{f}(\mathfrak{a}')P(\bar{f}(\mathfrak{b}')) + P(\bar{f}(\mathfrak{a}'))\bar{f}(\mathfrak{b}') + \lambda\bar{f}(\mathfrak{a}')\bar{f}(\mathfrak{b}')\\
&\quad -P(\bar{f}(\mathfrak{a}')\bar{f}(\mathfrak{b}')) - \bar{f}(\mathfrak{a}')P(\bar{f}(1))\bar{f}(\mathfrak{b}')).
\end{aligned}$$

$$= \quad f(a_1)f(b_1)P(\bar{f}(\mathfrak{a}'))P(\bar{f}(\mathfrak{b}'))$$

$$= \quad f(a_1)P(\bar{f}(\mathfrak{a}'))\, f(b_1)P(\bar{f}(\mathfrak{b}'))$$

$$= \quad \bar{f}(\mathfrak{a})\bar{f}(\mathfrak{b}).$$

This completes the induction and the proof of Theorem 4.16.

4.5 An observation

Recall how **Nijenhuis shuffle product** (right-shift product in §4.2.2) denoted by $\hat{\diamond}_r$ is defined. Once again, for two pure tensors $\mathfrak{a} = a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}$ and $\mathfrak{b} = b_1 \otimes \cdots \otimes b_n \in A^{\otimes n}$, where $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$, $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$, $\mathfrak{a}' \in A^{\otimes(m-1)}$, $\mathfrak{b}' \in A^{\otimes(n-1)}$, we define

$$\mathfrak{a}\hat{\diamond}_r\mathfrak{b} = a_1 \otimes (\mathfrak{a}'\hat{\diamond}_r\mathfrak{b}) + b_1 \otimes (\mathfrak{a}\hat{\diamond}_r\mathfrak{b}') - \mathbf{1}_A \otimes a_1b_1 \otimes (\mathfrak{a}'\hat{\diamond}_r\mathfrak{b}')$$

Proposition 4.17. For $|\,\mathfrak{a}\,|,\ |\,\mathfrak{b}\,| \geq 1$, if $|\,\mathfrak{a}\,| > 1$ or $|\,\mathfrak{b}\,| > 1$, then the RBNTD shuffle product $\circledast_\lambda$ satisfies the same recursion as the Nijenhuis product $\hat{\diamond}_r$.

$$\mathfrak{a} \circledast_\lambda \mathfrak{b} = a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}) + b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}') - \mathbf{1}_A \otimes a_1b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}')$$

Remark. It is interesting to see that, even though the RBNTD product and Nijenhuis product satisfy the same recursion, they are not the same because of their different initial conditions.

Proof. We only need to consider the case when $|\,\mathfrak{a}\,| > 1$ because the product is commutative (Theorem 4.14).

For $|\,\mathfrak{a}\,| > 1$, there are two cases to consider: either $|\,\mathfrak{b}\,| = 1$ or $|\,\mathfrak{b}\,| > 1$.

Case I: $|\mathfrak{a}| > 1, |\mathfrak{b}| = 1$ i.e. $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$, $\mathfrak{b} = b_1$ where $|\mathfrak{a}'| > 0$. Then,

$$\begin{aligned}\mathfrak{a} \circledast_\lambda \mathfrak{b} = {} & a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}) + b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathbf{1}_{\mathrm{k}}) + \lambda a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathbf{1}_{\mathrm{k}}) \\ & -\mathbf{1}_A \otimes a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathbf{1}_{\mathrm{k}}) - a_1 b_1 \otimes ((\mathfrak{a}' \circledast_\lambda \mathbf{1}_A) \circledast_\lambda \mathbf{1}_{\mathrm{k}}).\end{aligned}$$

Using Lemma 4.11 on the last term and simplifying other terms, we will get

$$\begin{aligned}\mathfrak{a} \circledast_\lambda \mathfrak{b} = {} & a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}) + b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathbf{1}_{\mathrm{k}}) + \lambda a_1 b_1 \otimes \mathfrak{a}' \\ & -\mathbf{1}_A \otimes a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathbf{1}_{\mathrm{k}}) - a_1 b_1 \otimes \lambda \mathfrak{a}'.\end{aligned}$$

Now, the 3rd and the 5th term will cancel with each other, so we will get

$$\mathfrak{a} \circledast_\lambda \mathfrak{b} = a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}) + b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathbf{1}_{\mathrm{k}}) - \mathbf{1}_A \otimes a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathbf{1}_{\mathrm{k}}),$$

which is the same as the recursion of $\mathfrak{a} \hat{\diamond}_r \mathfrak{b}$.

Case II: If $|\mathfrak{a}|>1, |\mathfrak{b}|>1$, i.e. $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$, $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$ where $|\mathfrak{a}'| > 0$, $|\mathfrak{b}'| > 0$, then

$$\begin{aligned}\mathfrak{a} \circledast_\lambda \mathfrak{b} = {} & a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}) + b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}') + \lambda a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}') \\ & -\mathbf{1}_A \otimes a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}') - a_1 b_1 \otimes ((\mathfrak{a}' \circledast_\lambda \mathbf{1}_A) \circledast_\lambda \mathfrak{b}').\end{aligned}$$

Using Lemma 4.11 on the last term and simplifying other terms, we will get

$$\begin{aligned}\mathfrak{a} \circledast_\lambda \mathfrak{b} = {} & a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}) + b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}') + \lambda a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}') \\ & -\mathbf{1}_A \otimes a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}') - a_1 b_1 \otimes (\lambda \mathfrak{a}' \circledast_\lambda \mathfrak{b}').\end{aligned}$$

Now, the 3rd and the 5th term will cancel with each other, so we will get

$$\mathfrak{a} \circledast_\lambda \mathfrak{b} = a_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}) + b_1 \otimes (\mathfrak{a} \circledast_\lambda \mathfrak{b}') - \mathbf{1}_A \otimes a_1 b_1 \otimes (\mathfrak{a}' \circledast_\lambda \mathfrak{b}').$$

which is the same as the recursion of $\mathfrak{a} \hat{\diamond}_r \mathfrak{b}$.

□

4.6 A special case of the free commutative RBNTD algebra

As a particular example, we consider $Ш_T(\mathbf{k})$, the free commutative RBNTD $\mathbf{k}$-algebra on $\mathbf{k}$.

If we choose $A = \mathbf{k}$ in the construction of the free commutative RBNTD $\mathbf{k}$-algebra, then we get

$$Ш_T(\mathbf{k}) = \bigoplus_{n=0}^{\infty} \mathbf{k}^{\otimes(n+1)}.$$

Since the tensor product is over $\mathbf{k}$, we have $\mathbf{k}^{\otimes(n+1)} = \mathbf{k}\mathbf{1}^{\otimes(n+1)}$, where $\mathbf{1}^{\otimes(n+1)} = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(n+1)-\text{factors}}$. In particular, $\mathbf{1}^{\otimes 1} = \mathbf{1}_{\mathbf{k}}$ is the unit of $\mathbf{k}$. Thus $Ш_T(\mathbf{k})$ is a free $\mathbf{k}$-module on the basis $\mathbf{1}^{\otimes n},\ n \geq 1$. The following result gives the multiplication table of $Ш_T(\mathbf{k})$. As a special case of Lemma 4.11, we have

Lemma 4.18. For any $m \in \mathbb{N}_{\geq 0}$,

$$\mathbf{1}_A \circledast_\lambda \mathbf{1}_A^{\otimes m} = \lambda \mathbf{1}_A^{\otimes m}.$$

Theorem 4.19. For any $m, n \in \mathbb{N}$,

$$\mathbf{1}_A^{\otimes(m+1)} \circledast_\lambda \mathbf{1}_A^{\otimes(n+1)} = \lambda \mathbf{1}_A^{\otimes(m+n+1)}.$$

Proof. We will prove this using the induction hypothesis on the sum $m + n$.

For $m + n = 0$, the only option we have is $m = n = 0$,

$$\mathbf{1}_A \circledast_\lambda \mathbf{1}_A = \lambda \mathbf{1}_A \text{ (by the previous lemma).}$$

For $m+n=2$, if either of m and n is zero, then we will get the previous lemma. So our result still holds.

If $m=n=1$, then we have

$$\begin{aligned}\mathbf{1}_A^{\otimes 2} \circledast_\lambda \mathbf{1}_A^{\otimes 2} &= \mathbf{1}_A \otimes (\mathbf{1}_A \circledast_\lambda \mathbf{1}_A^{\otimes 2}) + \mathbf{1}_A(\mathbf{1}_A^{\otimes 2} \circledast_\lambda \mathbf{1}_A) + \lambda \mathbf{1}_A \cdot \mathbf{1}_A \otimes (\mathbf{1}_A \circledast_\lambda \mathbf{1}_A) \\ &\quad - \mathbf{1}_A \otimes \mathbf{1}_A \otimes (\mathbf{1}_A \circledast_\lambda \mathbf{1}_A) - \mathbf{1}_A \otimes ((\mathbf{1}_A \circledast_\lambda \mathbf{1}_A) \circledast_\lambda \mathbf{1}_A).\end{aligned}$$

By using previous lemma on each term and twice on the last term, we obtain

$$\begin{aligned}\mathbf{1}_A^{\otimes 2} \circledast_\lambda \mathbf{1}_A^{\otimes 2} &= \mathbf{1}_A \otimes \lambda \mathbf{1}_A^{\otimes 2} + \mathbf{1}_A \otimes \lambda \mathbf{1}_A^{\otimes 2} + \lambda^2 \mathbf{1}_A \otimes \mathbf{1}_A \\ &\quad - \mathbf{1}_A \otimes \lambda \mathbf{1}_A^{\otimes 2} - \mathbf{1}_A \otimes \lambda^2 \mathbf{1}_A \\ &= \lambda \mathbf{1}_A^{\otimes 3}.\end{aligned}$$

Assume that the result holds for $m+n \leq$ k,which gives

$$\mathbf{1}_A^{\otimes(m+1)} \circledast_\lambda \mathbf{1}_A^{\otimes(n+1)} = \lambda \mathbf{1}_A^{\otimes(m+n+1)}.$$

Now using the induction hypothesis, our goal is to check

$$\mathbf{1}_A^{\otimes(m+1)} \circledast_\lambda \mathbf{1}_A^{\otimes(n+1)} = \lambda \mathbf{1}_A^{\otimes(k+2)}, \text{where m+n= k+1.}$$

In fact we have

$$\begin{aligned}\mathbf{1}^{\otimes(m+1)} \circledast_\lambda \mathbf{1}^{\otimes(n+1)} &= \mathbf{1}_A \otimes (\mathbf{1}_A^{\otimes m} \circledast_\lambda \mathbf{1}_A^{\otimes(n+1)}) + \mathbf{1}_A \otimes (\mathbf{1}_A^{\otimes(m+1)} \circledast_\lambda \mathbf{1}_A^{\otimes n}) \\ &\quad + \lambda \mathbf{1}_A \cdot \mathbf{1}_A \otimes (\mathbf{1}_A^{\otimes m} \circledast_\lambda \mathbf{1}_A^{\otimes n)}) - \mathbf{1}_A \otimes (\mathbf{1}_A \cdot \mathbf{1}_A) \otimes (\mathbf{1}_A^{\otimes m} \circledast_\lambda \mathbf{1}_A^{\otimes n}) \\ &\quad - \mathbf{1}_A \otimes ((\mathbf{1}_A^{\otimes m} \circledast_\lambda \mathbf{1}_A) \circledast_\lambda \mathbf{1}_A^{\otimes n}).\end{aligned}$$

By applying the previous lemma to the last term , we obtain

$$\begin{aligned}\mathbf{1}^{\otimes(m+1)} \circledast_\lambda \mathbf{1}^{\otimes(n+1)} &= \mathbf{1}_A \otimes (\mathbf{1}_A^{\otimes m} \circledast_\lambda \mathbf{1}_A^{\otimes(n+1)}) + \mathbf{1}_A \otimes (\mathbf{1}_A^{\otimes(m+1)} \circledast_\lambda \mathbf{1}_A^{\otimes n}) \\ &\quad +\lambda\mathbf{1}_A \cdot \mathbf{1}_A \otimes (\mathbf{1}_A^{\otimes m} \circledast_\lambda \mathbf{1}_A^{\otimes n)}) - \mathbf{1}_A \otimes (\mathbf{1}_A \cdot \mathbf{1}_A) \otimes (\mathbf{1}_A^{\otimes m} \circledast_\lambda \mathbf{1}_A^{\otimes n}) \\ &\quad -\mathbf{1}_A \otimes ((\lambda\mathbf{1}_A^{\otimes m}) \circledast_\lambda \mathbf{1}_A^{\otimes n}).\end{aligned}$$

Now applying the induction hypothesis to all the terms, we have

$$\begin{aligned}\mathbf{1}^{\otimes(m+1)} \circledast_\lambda \mathbf{1}^{\otimes(n+1)} &= \mathbf{1}_A \otimes (\lambda\mathbf{1}_A^{\otimes(m+n)}) + \mathbf{1}_A \otimes (\lambda\mathbf{1}_A^{\otimes(m+n)}) \\ &\quad +\lambda\mathbf{1}_A^{\otimes 1} \otimes \lambda\mathbf{1}_A^{\otimes(m+n-1)} - \mathbf{1}_A^{\otimes 2} \otimes (\lambda\mathbf{1}_A^{\otimes(m+n-1)}) \\ &\quad -\mathbf{1}_A \otimes (\lambda(\lambda\mathbf{1}_A^{\otimes(m+n-1)})) \\ &= \lambda\mathbf{1}_A^{\otimes(m+n+1)} + \lambda\mathbf{1}_A^{\otimes(m+n+1)} + \lambda^2\mathbf{1}_A^{\otimes(m+n)} - \lambda\mathbf{1}_A^{\otimes(m+n+1)} \\ &\quad -\lambda^2\mathbf{1}_A^{\otimes(m+n)}.\end{aligned}$$

By cancelling the 3rd and the 5th term, we get

$$\begin{aligned}\mathbf{1}^{\otimes(m+1)} \circledast_\lambda \mathbf{1}^{\otimes(n+1)} &= \lambda\mathbf{1}_A^{\otimes(m+n+1)} \\ &= \lambda\mathbf{1}_A^{\otimes(k+2)}.\end{aligned}$$

This completes the proof of the theorem. $\square$

5 Free non-commutative RBNTD algebras and bracketed words

We start with the definition of free RBNTD algebras.

Definition 5.1. Let A be a $\mathbf{k}$-algebra. A free Rota-Baxter Nijenhuis TD algebra over A is a RBNTD algebra $F^{NC}_{\mathrm{T}}(A)$ with a RBNTD operator P_A and an algebra homomorphism $j_A : A \to F^{NC}_{\mathrm{T}}(A)$ such that, for any RBNTD algebra T and any algebra homomorphism $f : A \to T$, there is a unique RBNTD algebra homomorphism $\bar{f} : F^{NC}_{\mathrm{T}}(A) \to T$ such that $\bar{f} \circ j_A = f$:

$$\begin{tikzcd} A \arrow[r, "j_A"] \arrow[dr, "f"'] & F^{NC}_{\mathrm{T}}(A) \arrow[d, "\bar{f}"] \\ & T \end{tikzcd}$$

For the construction of free Rota- Baxter Nijenhuis TD algebras, we follow the construction of free Rota-Baxter algebras [13, 21] by bracketed words. Alternatively, one can follow [12] to give the construction by rooted trees that is more in the spirit of operads [30]. We first display a $\mathbf{k}$-basis of the free Rota-Baxter Nijenhuis TD algebra in terms of bracketed words in § 5.1. The product on the free RBNTD algebra is given in § 5.2 and the universal property of the free RBNTD algebra is proved in § 5.3.

5.1 A basis of the free RBNTD algebra

For any set X, let $S(X)$ and $M(X)$ denote the free semigroup and free monoid respectively generated by X. For the rest of this chapter , A is taken to be the free

k-algebra on X : $A= \mathrm{k}\langle X\rangle$. We first display a **k**-basis $\tilde{\mathfrak{X}}_\infty$ of $F^{NC}_{\mathrm{T}}(A)$ in terms of bracketed words from the alphabet set X. Define

$$\mathfrak{X}_0 = S(X), \quad \tilde{\mathfrak{X}}_0 = M(X).$$

Let $\lfloor$ and $\rfloor$ be symbols, called brackets, and let $\tilde{\mathfrak{X}}'_0 = \tilde{\mathfrak{X}}_0 \cup \{\lfloor, \rfloor\}$. Let $M(X')$ denote the free monoid generated by X'.

Definition 5.2. ([13, 21]) Let Y, Z be two subsets of $M(X')$. Define the **alternating product** of Y and Z to be

$$\begin{aligned}\tilde{\Lambda}_X(Y,Z) \;\; = \;\; & \Big(\bigcup_{r\geq 1}(Y\lfloor Z\rfloor)^r\Big)\bigcup\Big(\bigcup_{r\geq 0}(Y\lfloor Z\rfloor)^r Y\Big) \\ & \bigcup\Big(\bigcup_{r\geq 1}(\lfloor Z\rfloor Y)^r\Big)\bigcup\Big(\bigcup_{r\geq 0}(\lfloor Z\rfloor Y)^r \lfloor Z\rfloor\Big)\bigcup\{1\}.\end{aligned}$$

We construct a sequence $\tilde{\mathfrak{X}}_n,\ n \geq 1$ of subsets of $M(X')$ by the following recursion. Let $\mathfrak{X}_0 = S(X),\ \tilde{\mathfrak{X}}_0 = M(X)$ and, for $n \geq 0$, define

$$\tilde{\mathfrak{X}}_{n+1} = \tilde{\Lambda}_X(\mathfrak{X}_0, \tilde{\mathfrak{X}}_n).$$

For example,

$$\tilde{\mathfrak{X}}_1 = \tilde{\Lambda}_X(\mathfrak{X}_0, \tilde{\mathfrak{X}}_0).$$

Further, define

$$\tilde{\mathfrak{X}}_\infty \;\; = \;\; \bigcup_{n\geq 0}\tilde{\mathfrak{X}}_n = \varinjlim \tilde{\mathfrak{X}}_n. \tag{5.1}$$

Here the second equation in Eq.(5.1) follows since

$$\tilde{\mathfrak{X}}_1 \supseteq \tilde{\mathfrak{X}}_0$$

and, assuming

$$\tilde{\mathfrak{X}}_n \supseteq \tilde{\mathfrak{X}}_{n-1},$$

we have

$$\tilde{\mathfrak{X}}_{n+1} = \tilde{\Lambda}_X(\mathfrak{X}_0, \tilde{\mathfrak{X}}_n) \supseteq \tilde{\Lambda}_X(\mathfrak{X}_0, \tilde{\mathfrak{X}}_{n-1}) = \tilde{\mathfrak{X}}_n.$$

By [13, 21] we have the disjoint union

$$\begin{aligned} \tilde{\mathfrak{X}}_\infty &= \Big(\bigsqcup_{r\geq 1} (\mathfrak{X}_0 \lfloor \tilde{\mathfrak{X}}_\infty \rfloor)^r\Big) \bigsqcup \Big(\bigsqcup_{r\geq 0} (\mathfrak{X}_0 \lfloor \tilde{\mathfrak{X}}_\infty \rfloor)^r \mathfrak{X}_0\Big) \\ & \bigsqcup \Big(\bigsqcup_{r\geq 1} (\lfloor \tilde{\mathfrak{X}}_\infty \rfloor \mathfrak{X}_0)^r\Big) \bigsqcup \Big(\bigsqcup_{r\geq 0} (\lfloor \tilde{\mathfrak{X}}_\infty \rfloor \mathfrak{X}_0)^r \lfloor \tilde{\mathfrak{X}}_\infty \rfloor\Big) \bigcup \{1\}. \end{aligned} \tag{5.2}$$

Further, every $\mathbf{x} \in \tilde{\mathfrak{X}}_\infty$ has a unique decomposition

$$\mathbf{x} = \mathbf{x}_1 \cdots \mathbf{x}_b, \tag{5.3}$$

where $\mathbf{x}_i$, $1 \leq i \leq b$, is alternatively in $\mathfrak{X}_0$ or in $\lfloor \tilde{\mathfrak{X}}_\infty \rfloor$. This decomposition will be called the **standard decomposition** of $\mathbf{x}$. For $\mathbf{x}$ in $\tilde{\mathfrak{X}}_\infty$ with standard decomposition $\mathbf{x}_1 \cdots \mathbf{x}_b$, we define b to be the **breadth** $b(\mathbf{x})$ of $\mathbf{x}$, we define the **head index** $h(\mathbf{x})$ of $\mathbf{x}$ to be 0 (resp. 1) if $\mathbf{x}_1$ is in X (resp. in $\lfloor \tilde{\mathfrak{X}}_\infty \rfloor$). Similarly define the **tail index** $t(\mathbf{x})$ of $\mathbf{x}$ to be 0 (resp. 1) if $\mathbf{x}_b$ is in $\tilde{\mathfrak{X}}_0$ (resp. in $\lfloor \tilde{\mathfrak{X}}_\infty \rfloor$). The **depth** $d(\mathbf{x})$ of $\mathbf{x}$ in $\tilde{\mathfrak{X}}_\infty$, is the smallest $n \geq 0$ such that $\mathbf{x} \in \mathfrak{X}_n$.

Elements in $\tilde{\mathfrak{X}}_\infty$ are called **RBNTD bracketed words**.

5.2 The product in a free RBNTD algebra

Let

$$F_{\mathrm{T}}^{NC}(A) = \bigoplus_{\mathbf{x}\in\tilde{\mathfrak{X}}_\infty} \mathbf{k}\mathbf{x}.$$

We now define a product $\diamond$ on $F_{\mathrm{T}}^{NC}(A)$ by defining $\mathbf{x} \diamond \mathbf{x}' \in F_{\mathrm{T}}^{NC}(A)$ for $\mathbf{x}, \mathbf{x}' \in \tilde{\mathfrak{X}}_\infty$ and then extending bilinearly. Roughly speaking, the product of $\mathbf{x}$ and $\mathbf{x}'$ is defined to be the concatenation whenever $t(\mathbf{x}) \neq h(\mathbf{x}')$. When $t(\mathbf{x}) = h(\mathbf{x}')$, the product is defined by the product in A or by the RBNTD relation in Eq. (3.4).

To be precise, we use induction on the sum $n := d(\mathbf{x}) + d(\mathbf{x}')$ of the depths of $\mathbf{x}$ and $\mathbf{x}'$. Then $n \geq 0$. If $n = 0$, then $\mathbf{x}, \mathbf{x}'$ are in $\mathfrak{X}_0$ and so are in A and we define

$$\mathbf{x} \diamond \mathbf{x}' = \mathbf{x} \cdot \mathbf{x}' \in A \subseteq F_{\mathrm{T}}^{NC}(A).$$

Here $\cdot$ is the product in A. Suppose $\mathbf{x} \diamond \mathbf{x}'$ have been defined for all $\mathbf{x}, \mathbf{x}' \in \tilde{\mathfrak{X}}_\infty$ with $0 \leq n \leq k$, and let $\mathbf{x}, \mathbf{x}' \in \tilde{\mathfrak{X}}_\infty$ with $n = k + 1$. First assume that the breadth $b(\mathbf{x}) = b(\mathbf{x}') = 1$. Then $\mathbf{x}$ and $\mathbf{x}'$ are in $\mathfrak{X}_0$ or $\lfloor \tilde{\mathfrak{X}}_\infty \rfloor$. Since $n = k + 1$ is at least one, $\mathbf{x}$ and $\mathbf{x}'$ cannot be both in $\mathfrak{X}_0$. We accordingly define

$$\mathbf{x} \diamond \mathbf{x}' = \begin{cases} \mathbf{x}\mathbf{x}', & \text{if } \mathbf{x} \in \mathfrak{X}_0, \mathbf{x}' \in \lfloor \tilde{\mathfrak{X}}_\infty \rfloor, \\ \mathbf{x}\mathbf{x}', & \text{if } \mathbf{x} \in \lfloor \tilde{\mathfrak{X}}_\infty \rfloor, \mathbf{x}' \in \mathfrak{X}_0, \\ \lfloor \lfloor \overline{\mathbf{x}} \rfloor \diamond \overline{\mathbf{x}}' \rfloor + \lfloor \overline{\mathbf{x}} \diamond \lfloor \overline{\mathbf{x}}' \rfloor \rfloor & \text{if } \mathbf{x} = \lfloor \overline{\mathbf{x}} \rfloor, \mathbf{x}' = \lfloor \overline{\mathbf{x}}' \rfloor, \overline{\mathbf{x}} = \overline{\mathbf{x}}_1 \cdots \overline{\mathbf{x}}_b, \\ +\lambda \lfloor \overline{\mathbf{x}} \diamond \overline{\mathbf{x}}' \rfloor - \lfloor \lfloor \overline{\mathbf{x}} \diamond \overline{\mathbf{x}}' \rfloor \rfloor & \overline{\mathbf{x}}' = \overline{\mathbf{x}}'_1 \cdots \overline{\mathbf{x}}'_c, \overline{\mathbf{x}}_b \in \mathfrak{X}_0 \text{ and } \overline{\mathbf{x}}'_1 \in \mathfrak{X}_0, \\ -\lfloor \overline{\mathbf{x}} \lfloor 1 \rfloor \overline{\mathbf{x}}' \rfloor, & \\ \lfloor \lfloor \overline{\mathbf{x}} \rfloor \diamond \overline{\mathbf{x}}' \rfloor + \lfloor \overline{\mathbf{x}} \diamond \lfloor \overline{\mathbf{x}}' \rfloor \rfloor & \text{if } \mathbf{x} = \lfloor \overline{\mathbf{x}} \rfloor, \mathbf{x}' = \lfloor \overline{\mathbf{x}}' \rfloor, \overline{\mathbf{x}} = \overline{\mathbf{x}}_1 \cdots \overline{\mathbf{x}}_b, \\ -\lfloor \lfloor \overline{\mathbf{x}} \diamond \overline{\mathbf{x}}' \rfloor \rfloor, & \overline{\mathbf{x}}' = \overline{\mathbf{x}}'_1 \cdots \overline{\mathbf{x}}'_c, \overline{\mathbf{x}}_b \in \lfloor \tilde{\mathfrak{X}}_\infty \rfloor \text{ or } \overline{\mathbf{x}}'_1 \in \lfloor \tilde{\mathfrak{X}}_\infty \rfloor. \end{cases} \tag{5.4}$$

Here, the product in case I and II is by concatenation, whereas, in case III and IV it is by the induction hypothesis for the four and three products on the right-hand side respectively, since we have

$$\begin{aligned} d(\lfloor\overline{\mathbf{x}}\rfloor) + d(\overline{\mathbf{x}}') &= d(\lfloor\overline{\mathbf{x}}\rfloor) + d(\lfloor\overline{\mathbf{x}}'\rfloor) - 1 = d(\mathbf{x}) + d(\mathbf{x}') - 1, \\ d(\overline{\mathbf{x}}) + d(\lfloor\overline{\mathbf{x}}'\rfloor) &= d(\lfloor\overline{\mathbf{x}}\rfloor) + d(\lfloor\overline{\mathbf{x}}'\rfloor) - 1 = d(\mathbf{x}) + d(\mathbf{x}') - 1, \\ d(\overline{\mathbf{x}}) + d(\overline{\mathbf{x}}') &= d(\lfloor\overline{\mathbf{x}}\rfloor) - 1 + d(\lfloor\overline{\mathbf{x}}'\rfloor) - 1 = d(\mathbf{x}) + d(\mathbf{x}') - 2 \end{aligned}$$

which are all less than or equal to k. Now assume $b(\mathbf{x}) > 1$ or $\mathrm{b}(\mathbf{x}') > 1$. Let $\mathbf{x} = \mathbf{x}_1 \cdots \mathbf{x}_b$ and $\mathbf{x}' = \mathbf{x}'_1 \cdots \mathbf{x}'_c$ be the standard decompositions from Eq.(5.3). We then define

$$\mathbf{x} \diamond \mathbf{x}' = \mathbf{x}_1 \cdots \mathbf{x}_{b-1} (\mathbf{x}_b \diamond \mathbf{x}'_1)\, \mathbf{x}'_2 \cdots \mathbf{x}'_c \tag{5.5}$$

where $\mathbf{x}_b \diamond \mathbf{x}'_1$ is defined by Eq. (5.4) and the rest is given by concatenation. The concatenation is well-defined since by Eq. (5.4), we have $h(\mathbf{x}_b) = h(\mathbf{x}_b \diamond \mathbf{x}'_1)$ and $t(\mathbf{x}'_1) = t(\mathbf{x}_b \diamond \mathbf{x}'_1)$. Therefore, $t(\mathbf{x}_{b-1}) \neq h(\mathbf{x}_b \diamond \mathbf{x}'_1)$ and $h(\mathbf{x}'_2) \neq t(\mathbf{x}_b \diamond \mathbf{x}'_1)$.

We have the following simple properties of $\diamond$.

Lemma 5.3. Let $\mathbf{x}, \mathbf{x}' \in \tilde{\mathfrak{X}}_\infty$. We have the following statements.

1. $h(\mathbf{x}) = h(\mathbf{x} \diamond \mathbf{x}')$ and $t(\mathbf{x}') = t(\mathbf{x} \diamond \mathbf{x}')$.

2. If $t(\mathbf{x}) \neq h(\mathbf{x}')$, then $\mathbf{x} \diamond \mathbf{x}' = \mathbf{x}\mathbf{x}'$ (concatenation).

3. If $t(\mathbf{x}) \neq h(\mathbf{x}')$, then for any $\mathbf{x}'' \in \tilde{\mathfrak{X}}_\infty$,

$$(\mathbf{x}\mathbf{x}') \diamond \mathbf{x}'' = \mathbf{x}(\mathbf{x}' \diamond \mathbf{x}''), \quad \mathbf{x}'' \diamond (\mathbf{x}\mathbf{x}') = (\mathbf{x}'' \diamond \mathbf{x})\mathbf{x}'.$$

We will define another product $\tilde{\diamond}$ on $F^{NC}_{\mathrm{T}}(A)$ by defining $\tilde{\diamond}$ in the same way as $\diamond$ except replacing Eq. (5.4) by the following equation.

$$
\mathbf{x}\tilde{\diamond}\mathbf{x}' = \begin{cases} \mathbf{x}\mathbf{x}', & \text{if } \mathbf{x} \in X, \mathbf{x}' \in \lfloor \mathfrak{X}_\infty \rfloor, \\ \mathbf{x}\mathbf{x}', & \text{if } \mathbf{x} \in \lfloor \mathfrak{X}_\infty \rfloor, \mathbf{x}' \in X, \\ \lfloor \lfloor \overline{\mathbf{x}} \rfloor \tilde{\diamond} \overline{\mathbf{x}}' \rfloor + \lfloor \overline{\mathbf{x}} \tilde{\diamond} \lfloor \overline{\mathbf{x}}' \rfloor \rfloor + \lambda \lfloor \overline{\mathbf{x}} \tilde{\diamond} \overline{\mathbf{x}}' \rfloor & \\ -\lfloor \lfloor \overline{\mathbf{x}} \tilde{\diamond} \overline{\mathbf{x}}' \rfloor \rfloor - \lfloor (\overline{\mathbf{x}} \tilde{\diamond} \lfloor 1 \rfloor) \tilde{\diamond} \overline{\mathbf{x}}' \rfloor, & \text{if } \mathbf{x} = \lfloor \overline{\mathbf{x}} \rfloor, \mathbf{x}' = \lfloor \overline{\mathbf{x}}' \rfloor \in \lfloor \mathfrak{X}_\infty \rfloor. \end{cases} \tag{5.6}
$$

This will be the product that we will use later in place of $\diamond$. To prove that the two products agree and hence the second product is well-defined, we first prove a lemma.

Lemma 5.4. We have the following simple properties of $\tilde{\diamond}$.

1. For $\mathbf{x} \in \lfloor \tilde{\mathfrak{X}}_\infty \rfloor$, we have $\mathbf{x}\tilde{\diamond}\lfloor 1 \rfloor = \lfloor 1 \rfloor \tilde{\diamond} \mathbf{x} = \lambda \mathbf{x}$.

2. For $\mathbf{x}, \mathbf{x}' \in \tilde{\mathfrak{X}}$, we have

$$
\mathbf{x}\tilde{\diamond}\lfloor 1 \rfloor = \begin{cases} \lambda \mathbf{x}, & \text{if } t(\mathbf{x}) = 1, \\ \mathbf{x}\lfloor 1 \rfloor, & \text{if } t(\mathbf{x}) = 0. \end{cases}
$$

$$
\lfloor 1 \rfloor \tilde{\diamond} \mathbf{x}' = \begin{cases} \lambda \mathbf{x}', & \text{if } h(\mathbf{x}') = 1, \\ \lfloor 1 \rfloor \mathbf{x}', & \text{if } h(\mathbf{x}') = 0. \end{cases}
$$

3. For $\mathbf{x} = \lfloor \overline{\mathbf{x}} \rfloor, \mathbf{x}' = \lfloor \overline{\mathbf{x}}' \rfloor$, we have

$$
\lfloor (\overline{\mathbf{x}} \tilde{\diamond} \lfloor 1 \rfloor) \tilde{\diamond} \overline{\mathbf{x}}' \rfloor = \begin{cases} \lfloor \overline{\mathbf{x}} \lfloor 1 \rfloor \overline{\mathbf{x}}' \rfloor, & \text{if} t(\overline{\mathbf{x}}) = h(\overline{\mathbf{x}}') = 0, \\ \lambda \lfloor \overline{\mathbf{x}} \tilde{\diamond} \overline{\mathbf{x}}' \rfloor, & \text{if} t(\overline{\mathbf{x}}) = 1 \text{ or } h(\overline{\mathbf{x}}') = 1. \end{cases}
$$

Proof. (1) Let $\mathbf{x} = \lfloor \overline{\mathbf{x}} \rfloor$. Then we have

$$\begin{aligned}
\mathbf{x}\tilde{\diamond}\lfloor 1 \rfloor &= \lfloor \overline{\mathbf{x}}\tilde{\diamond}\lfloor 1 \rfloor \rfloor + \lfloor \mathbf{x}\tilde{\diamond}1 \rfloor + \lambda\lfloor \overline{\mathbf{x}}\tilde{\diamond}1 \rfloor - \lfloor\lfloor \overline{\mathbf{x}}\tilde{\diamond}1 \rfloor\rfloor - \lfloor(\overline{\mathbf{x}}\tilde{\diamond}\lfloor 1 \rfloor)\tilde{\diamond}1 \rfloor \\
&= \lfloor \overline{\mathbf{x}}\tilde{\diamond}\lfloor 1 \rfloor \rfloor + \lfloor \mathbf{x} \rfloor + \lambda\lfloor \overline{\mathbf{x}} \rfloor - \lfloor\lfloor \overline{\mathbf{x}} \rfloor\rfloor - \lfloor \overline{\mathbf{x}}\tilde{\diamond}\lfloor 1 \rfloor \rfloor \\
&= \lambda \mathbf{x}.
\end{aligned}$$

$$\begin{aligned}
\lfloor 1 \rfloor\tilde{\diamond}\mathbf{x} &= \lfloor 1\tilde{\diamond}\mathbf{x} \rfloor + \lfloor\lfloor 1 \rfloor\tilde{\diamond}\overline{\mathbf{x}} \rfloor + \lambda\lfloor 1\tilde{\diamond}\overline{\mathbf{x}} \rfloor - \lfloor\lfloor 1\tilde{\diamond}\overline{\mathbf{x}} \rfloor\rfloor - \lfloor\lfloor(1\tilde{\diamond}\lfloor 1 \rfloor)\tilde{\diamond}\overline{\mathbf{x}} \rfloor \\
&= \lfloor \mathbf{x} \rfloor + \lfloor\lfloor 1 \rfloor\tilde{\diamond}\overline{\mathbf{x}} \rfloor + \lambda\lfloor \overline{\mathbf{x}} \rfloor - \lfloor\lfloor \overline{\mathbf{x}} \rfloor\rfloor - \lfloor\lfloor 1 \rfloor\tilde{\diamond}\overline{\mathbf{x}} \rfloor \\
&= \lambda \mathbf{x}.
\end{aligned}$$

(2) Let $\mathbf{x} = \mathbf{x}_1 \cdots \mathbf{x}_b$ be the standard decomposition of $\mathbf{x}$. If $t(\mathbf{x}) = 1$ which gives $\mathbf{x}_b \in \lfloor \tilde{\mathfrak{X}}_\infty \rfloor$, then $\mathbf{x}\tilde{\diamond}\lfloor 1 \rfloor = \lambda\mathbf{x}$ (from part 1 above). If $t(\mathbf{x}) = 0$ which gives $\mathbf{x}_b \in X$, then the product is defined by concatenation. Therefore, we get $\mathbf{x}\tilde{\diamond}\lfloor 1 \rfloor = \mathbf{x}\lfloor 1 \rfloor$.

Similarly, let $\mathbf{x}' = \mathbf{x}'_1 \cdots \mathbf{x}'_c$ be the standard decomposition of $\mathbf{x}'$. For $h(\mathbf{x}') = 1$, $\mathbf{x}'_1 \in \lfloor \tilde{\mathfrak{X}}_\infty \rfloor$. So $\lfloor 1 \rfloor\tilde{\diamond}\mathbf{x}' = \lambda\mathbf{x}'$ (from part 1 above). For $h(\mathbf{x}') = 0$, $\mathbf{x}'_1 \in X$, then the product is defined by concatenation. Therefore, we get $\lfloor 1 \rfloor\tilde{\diamond}\mathbf{x}' = \lfloor 1 \rfloor\mathbf{x}'$.

(3) For $\lfloor(\overline{\mathbf{x}}\tilde{\diamond}\lfloor 1 \rfloor)\tilde{\diamond}\overline{\mathbf{x}}' \rfloor$, we need to take three cases:

Case I: When $t(\overline{\mathbf{x}}) = h(\overline{\mathbf{x}}') = 0$, then the product is by concatenation. Therefore, we get $\lfloor(\overline{\mathbf{x}}\tilde{\diamond}\lfloor 1 \rfloor)\tilde{\diamond}\overline{\mathbf{x}}' \rfloor = \lfloor \overline{\mathbf{x}}\lfloor 1 \rfloor\overline{\mathbf{x}}' \rfloor$.

Case II: When $t(\overline{\mathbf{x}}) = 1$, then by part 2, we get $\lfloor(\overline{\mathbf{x}}\tilde{\diamond}\lfloor 1 \rfloor)\tilde{\diamond}\overline{\mathbf{x}}' \rfloor = \lambda\lfloor \overline{\mathbf{x}}\tilde{\diamond}\overline{\mathbf{x}}' \rfloor$.

Case III: When $h(\overline{\mathbf{x}}') = 1$, then also by part 2, we get $\lfloor(\overline{\mathbf{x}}\tilde{\diamond}\lfloor 1 \rfloor)\tilde{\diamond}\overline{\mathbf{x}}' \rfloor = \lambda\lfloor \overline{\mathbf{x}}\tilde{\diamond}\overline{\mathbf{x}}' \rfloor$. □

Proposition 5.5. The product $\diamond$ in Eq. (5.4) equals product $\tilde{\diamond}$.

Proof. Since $\diamond$ and $\tilde{\diamond}$ are defined by the same recursion except their recursive formulas in Eq. (5.6) and Eq. (5.6), we just need to verify that the product $\tilde{\diamond}$ satisfies the same recursion as for $\diamond$ in Eq. (5.4). Applying Lemma 5.4 to the expression $(\overline{\mathbf{x}} \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}'$ in Eq. (5.6), we obtain

$$\mathbf{x}\tilde{\diamond}\mathbf{x}' = \begin{cases} \mathbf{x}\mathbf{x}', & \text{if } \mathbf{x} \in X, \mathbf{x}' \in \lfloor \mathfrak{X}_\infty \rfloor, \\ \mathbf{x}\mathbf{x}', & \text{if } \mathbf{x} \in \lfloor \mathfrak{X}_\infty \rfloor, \mathbf{x}' \in X, \\ \lfloor \lfloor \overline{\mathbf{x}} \rfloor \tilde{\diamond} \overline{\mathbf{x}}' \rfloor + \lfloor \overline{\mathbf{x}} \tilde{\diamond} \lfloor \overline{\mathbf{x}}' \rfloor \rfloor + \lambda \lfloor \overline{\mathbf{x}} \tilde{\diamond} \overline{\mathbf{x}}' \rfloor & \\ -\lfloor \lfloor \overline{\mathbf{x}} \tilde{\diamond} \overline{\mathbf{x}}' \rfloor \rfloor - \lfloor \overline{\mathbf{x}} \lfloor 1 \rfloor \overline{\mathbf{x}}' \rfloor, & \text{if } t(\overline{\mathbf{x}}) = h(\overline{\mathbf{x}}') = 0, \\ \lfloor \lfloor \overline{\mathbf{x}} \rfloor \tilde{\diamond} \overline{\mathbf{x}}' \rfloor + \lfloor \overline{\mathbf{x}} \tilde{\diamond} \lfloor \overline{\mathbf{x}}' \rfloor \rfloor - \lfloor \lfloor \overline{\mathbf{x}} \tilde{\diamond} \overline{\mathbf{x}}' \rfloor \rfloor, & \text{if } t(\overline{\mathbf{x}}) = 1 \text{ or } h(\mathbf{x}') = 1. \end{cases}$$

This is the same recursion as Eq. (5.4). Therefore, $\diamond = \tilde{\diamond}$. □

Extending $\diamond$ bilinearly, we obtain a binary operation

$$F_{\mathrm{T}}^{NC}(A) \otimes F_{\mathrm{T}}^{NC}(A) \to F_{\mathrm{T}}^{NC}(A).$$

For $\mathbf{x} \in \tilde{\mathfrak{X}}_\infty$, define

$$T_A(\mathbf{x}) = \lfloor \mathbf{x} \rfloor. \tag{5.7}$$

Obviously $\lfloor \mathbf{x} \rfloor$ is again in $\tilde{\mathfrak{X}}_\infty$. Thus T_A extends to a linear operator T_A on $F_{\mathrm{T}}^{NC}(A)$. Let

$$j_X : X \to \tilde{\mathfrak{X}}_\infty \to F_{\mathrm{T}}^{NC}(A)$$

be the natural injection which extends to an algebra injection

$$j_A : A \to F_{\mathrm{T}}^{NC}(A). \tag{5.8}$$

The following is our main result which will be proved in the next subsection.

Theorem 5.6. Let A be a $\mathbf{k}$-algebra with a $\mathbf{k}$-basis X.

1. The pair $(F_{\mathrm{T}}^{NC}(A), \diamond)$ is an **k**-algebra.

2. The triple $(F_{\mathrm{T}}^{NC}(A), \diamond, T_A)$ is a RBNTD **k**-algebra.

3. The quadruple $(F_{\mathrm{T}}^{NC}(A), \diamond, T_A, j_A)$ is the free RBNTD algebra on the algebra A.

The immediate corollary of above theorem will be used in later section.

Corollary 5.7. Let M be a **k**-module and let $T(M) = \bigoplus_{n\geq 0} M^{\otimes n}$ be the tensor algebra over M. Then $F_{\mathrm{T}}^{NC}(T(M))$, together with the natural injection

$$i_M : M \to T(M) \xrightarrow{j_{T(M)}} F_{\mathrm{T}}^{NC}(T(M)),$$

is a free RBNTD algebra over M, in the sense that, for any RBNTD algebra N and **k**-module map $f : M \to N$ there is a unique RBNTD algebra homomorphism $\hat{f} : F_{\mathrm{T}}^{NC}(T(M)) \to N$ such that $\hat{f} \circ i_M = f$.

Proof. This follows immediately from Theorem 5.6 and the fact that the construction of the free algebra on a module (resp. free RBNTD algebra on an algebra, resp. free RBNTD on a module) is the left adjoint functor of the forgetful functor from algebras to modules (resp. from RBNTD algebras to algebras, resp. from RBNTD algebras to modules), and the fact that the composition of two left adjoint functors is the left adjoint functor of the composition. □

5.3 The proof of Theorem 5.6

1. We just need to verify the associativity. For this, we only need to verify

$$(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}''' = \mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''') \tag{5.9}$$

for $\mathbf{x}', \mathbf{x}'', \mathbf{x}''' \in \tilde{\mathfrak{X}}_\infty$. We will do this by induction on the sum of the depths $n := d(\mathbf{x}') + d(\mathbf{x}'') + d(\mathbf{x}''')$. If $n = 0$, then all of $\mathbf{x}', \mathbf{x}'', \mathbf{x}'''$ have depth zero and so are in X. In this case the product $\diamond$ is given by the product $\cdot$ in A and so is associative. Assume the associativity holds for $n \leq k$ and assume that $\mathbf{x}', \mathbf{x}'', \mathbf{x}''' \in \tilde{\mathfrak{X}}_\infty$ have $n = d(\mathbf{x}') + d(\mathbf{x}'') + d(\mathbf{x}''') = k + 1$. If $t(\mathbf{x}') \neq h(\mathbf{x}'')$, then by Lemma 5.3,

$$(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}''' = (\mathbf{x}'\mathbf{x}'') \diamond \mathbf{x}''' = \mathbf{x}'(\mathbf{x}'' \diamond \mathbf{x}''') = \mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''').$$

A similar argument holds when $t(\mathbf{x}'') \neq h(\mathbf{x}''')$. Thus, we only need to verify the associativity when $t(\mathbf{x}') = h(\mathbf{x}'')$ and $t(\mathbf{x}'') = h(\mathbf{x}''')$. We will next reduce the breadths of the words.

Lemma 5.8. If the associativity

$$(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}''' = \mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''')$$

holds for all $\mathbf{x}', \mathbf{x}''$ and $\mathbf{x}'''$ in $\tilde{\mathfrak{X}}_\infty$ of breadth one, then it holds for all $\mathbf{x}', \mathbf{x}''$ and $\mathbf{x}'''$ in $\tilde{\mathfrak{X}}_\infty$.

Proof. We use induction on the sum of breadths $m := b(\mathbf{x}') + b(\mathbf{x}'') + b(\mathbf{x}''')$. Then $m \geq 3$. The case when $m = 3$ is the assumption of the lemma. Assume that the associativity holds for $3 \leq m \leq j$ and take $\mathbf{x}', \mathbf{x}'', \mathbf{x}''' \in \tilde{\mathfrak{X}}_\infty$ with $m = j + 1$. Then $j + 1 \geq 4$. So at least one of $\mathbf{x}', \mathbf{x}'', \mathbf{x}'''$ have breadth greater than or equal to 2. First assume $b(\mathbf{x}') \geq 2$. Then $\mathbf{x}' = \mathbf{x}'_1\mathbf{x}'_2$ with $\mathbf{x}'_1, \mathbf{x}'_2 \in \tilde{\mathfrak{X}}_\infty$ and $t(\mathbf{x}'_1) \neq h(\mathbf{x}'_2)$. Thus by Lemma 5.3, we obtain

$$(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}''' = ((\mathbf{x}'_1\mathbf{x}'_2) \diamond \mathbf{x}'') \diamond \mathbf{x}''' = (\mathbf{x}'_1(\mathbf{x}'_2 \diamond \mathbf{x}'')) \diamond \mathbf{x}''' = \mathbf{x}'_1((\mathbf{x}'_2 \diamond \mathbf{x}'') \diamond \mathbf{x}''').$$

Similarly,

$$\mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''') = (\mathbf{x}'_1\mathbf{x}'_2) \diamond (\mathbf{x}'' \diamond \mathbf{x}''') = \mathbf{x}'_1(\mathbf{x}'_2 \diamond (\mathbf{x}'' \diamond \mathbf{x}''')).$$

Thus $(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}''' = \mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''')$ whenever $(\mathbf{x}'_2 \diamond \mathbf{x}'') \diamond \mathbf{x}''' = \mathbf{x}'_2 \diamond (\mathbf{x}'' \diamond \mathbf{x}''')$. The latter follows from the induction hypothesis. A similar proof works if $b(\mathbf{x}''') \geq 2$. Finally if $b(\mathbf{x}'') \geq 2$, then $\mathbf{x}'' = \mathbf{x}''_1\mathbf{x}''_2$ with $\mathbf{x}''_1, \mathbf{x}''_2 \in \tilde{\mathfrak{X}}_\infty$ and $t(\mathbf{x}''_1) \neq h(\mathbf{x}''_2)$. By applying Lemma 5.3 repeatedly, we obtain

$$(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}''' = (\mathbf{x}' \diamond (\mathbf{x}''_1\mathbf{x}''_2)) \diamond \mathbf{x}''' = ((\mathbf{x}' \diamond \mathbf{x}''_1)\mathbf{x}''_2) \diamond \mathbf{x}''' = (\mathbf{x}' \diamond \mathbf{x}''_1)(\mathbf{x}''_2 \diamond \mathbf{x}''').$$

In the same way, we have

$$(\mathbf{x}' \diamond \mathbf{x}''_1)(\mathbf{x}''_2 \diamond \mathbf{x}''') = \mathbf{x}' \diamond (\mathbf{x}'' \diamond \mathbf{x}''').$$

This again proves the associativity. □

To summarize, our proof of the associativity has been reduced to the special case when $\mathbf{x}', \mathbf{x}'', \mathbf{x}''' \in \tilde{\mathfrak{X}}_\infty$ are chosen so that

1. $n := d(\mathbf{x}') + d(\mathbf{x}'') + d(\mathbf{x}''') = k + 1 \geq 1$ with the assumption that the associativity holds when $n \leq k$.
2. the elements have breadth one and
3. $t(\mathbf{x}') = h(\mathbf{x}'')$ and $t(\mathbf{x}'') = h(\mathbf{x}''')$.

By 2, the head and tail of each of the elements are the same. Therefore by 3, either all the three elements are in X or they are all in $\lfloor \tilde{\mathfrak{X}}_\infty \rfloor$. If all of $\mathbf{x}', \mathbf{x}'', \mathbf{x}'''$ are in X, then as already shown, the associativity follows from the associativity in A. Now only case to check is when $\mathbf{x}', \mathbf{x}'', \mathbf{x}'''$ are all in $\lfloor \tilde{\mathfrak{X}}_\infty \rfloor$. Then

$$\mathbf{x}' = \lfloor \overline{\mathbf{x}}' \rfloor, \mathbf{x}'' = \lfloor \overline{\mathbf{x}}'' \rfloor, \mathbf{x}''' = \lfloor \overline{\mathbf{x}}''' \rfloor$$

with $\overline{\mathbf{x}}', \overline{\mathbf{x}}'', \overline{\mathbf{x}}''' \in \tilde{\mathfrak{X}}_\infty$.

For $(\mathbf{x}' \diamond \mathbf{x}'') \diamond \mathbf{x}'''$, using Eq. (5.6) and bilinearity of the product $\diamond$, we will get

$$\begin{aligned}
(\lfloor \overline{\mathbf{x}}' \rfloor \diamond \lfloor \overline{\mathbf{x}}'' \rfloor) \diamond \lfloor \overline{\mathbf{x}}''' \rfloor &= (\lfloor \lfloor \overline{\mathbf{x}}' \rfloor \diamond \overline{\mathbf{x}}'' \rfloor + \lfloor \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \rfloor \rfloor + \lambda \lfloor \overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'' \rfloor \\
&\quad - \lfloor \lfloor \overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'' \rfloor \rfloor - \lfloor (\overline{\mathbf{x}}' \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}'' \rfloor) \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \\
&= (\lfloor \lfloor \overline{\mathbf{x}}' \rfloor \diamond \overline{\mathbf{x}}'' \rfloor) \diamond \lfloor \overline{\mathbf{x}}''' \rfloor + (\lfloor \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \rfloor \rfloor) \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \\
&\quad + (\lambda \lfloor \overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'' \rfloor) \diamond \lfloor \overline{\mathbf{x}}''' \rfloor - (\lfloor \lfloor \overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'' \rfloor \rfloor) \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \\
&\quad - (\lfloor (\overline{\mathbf{x}}' \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}'' \rfloor) \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \\
&= \lfloor (\lfloor \lfloor \overline{\mathbf{x}}' \rfloor \diamond \overline{\mathbf{x}}'' \rfloor) \diamond \overline{\mathbf{x}}''' \rfloor + \lfloor (\lfloor \overline{\mathbf{x}}' \rfloor \diamond \overline{\mathbf{x}}'') \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \rfloor \\
&\quad + \lambda \lfloor (\lfloor \overline{\mathbf{x}}' \rfloor \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}''' \rfloor - \lfloor \lfloor (\lfloor \overline{\mathbf{x}}' \rfloor \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}''' \rfloor \rfloor \\
&\quad - \lfloor ((\lfloor \overline{\mathbf{x}}' \rfloor \diamond \overline{\mathbf{x}}'') \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}''' \rfloor + \lfloor \lfloor \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \rfloor \rfloor \diamond \overline{\mathbf{x}}''' \rfloor \\
&\quad + \lfloor (\overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \rfloor) \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \rfloor + \lambda \lfloor (\overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \rfloor) \diamond \overline{\mathbf{x}}''' \rfloor \\
&\quad - \lfloor \lfloor (\overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \rfloor) \diamond \overline{\mathbf{x}}''' \rfloor \rfloor - \lfloor ((\overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \rfloor) \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}''' \rfloor \\
&\quad + \lambda \lfloor \lfloor \overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''' \rfloor + \lambda \lfloor (\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \rfloor \\
&\quad + \lambda^2 \lfloor (\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}''' \rfloor - \lambda \lfloor \lfloor (\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}''' \rfloor \rfloor \\
&\quad - \lambda \lfloor ((\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}''' \rfloor - \lfloor \lfloor \lfloor \overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'' \rfloor \rfloor \diamond \overline{\mathbf{x}}''' \rfloor \\
&\quad - \lfloor \lfloor \overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'' \rfloor \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \rfloor - \lambda \lfloor \lfloor \overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''' \rfloor \\
&\quad + \lfloor \lfloor \lfloor \overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''' \rfloor \rfloor + \lfloor (\lfloor \overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'' \rfloor \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}''' \rfloor \\
&\quad - \lfloor \lfloor (\overline{\mathbf{x}}' \diamond \lfloor 1 \rfloor) \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''' \rfloor - \lfloor ((\overline{\mathbf{x}}' \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}'') \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \rfloor \\
&\quad - \lambda \lfloor ((\overline{\mathbf{x}}' \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}''' \rfloor + \lfloor \lfloor ((\overline{\mathbf{x}}' \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}''' \rfloor \rfloor
\end{aligned}$$

$$+\lfloor(((\overline{\mathbf{x}}' \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}'') \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor.$$

Applying the induction hypothesis in n for the 7th and then expanding the 7th and the 17th term using Eq. (5.6), we obtain

$$\begin{aligned}
(\lfloor\overline{\mathbf{x}}'\rfloor \diamond \lfloor\overline{\mathbf{x}}''\rfloor) \diamond \lfloor\overline{\mathbf{x}}'''\rfloor \quad = \quad & \lfloor\lfloor\lfloor\overline{\mathbf{x}}'\rfloor \diamond \overline{\mathbf{x}}''\rfloor \diamond \overline{\mathbf{x}}'''\rfloor + \lfloor(\lfloor\overline{\mathbf{x}}'\rfloor \diamond \overline{\mathbf{x}}'') \diamond \lfloor\overline{\mathbf{x}}'''\rfloor\rfloor \\
& +\lambda\lfloor(\lfloor\overline{\mathbf{x}}'\rfloor \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}'''\rfloor - \lfloor\lfloor(\lfloor\overline{\mathbf{x}}'\rfloor \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}'''\rfloor\rfloor \\
& -\lfloor((\lfloor\overline{\mathbf{x}}'\rfloor \diamond \overline{\mathbf{x}}'') \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor + \lfloor\lfloor\overline{\mathbf{x}}' \diamond \lfloor\overline{\mathbf{x}}''\rfloor\rfloor \diamond \overline{\mathbf{x}}'''\rfloor \\
& +\lfloor\overline{\mathbf{x}}' \diamond (\lfloor\lfloor\overline{\mathbf{x}}''\rfloor \diamond \overline{\mathbf{x}}'''\rfloor)\rfloor + \lfloor\overline{\mathbf{x}}' \diamond (\lfloor\overline{\mathbf{x}}'' \diamond \lfloor\overline{\mathbf{x}}'''\rfloor)\rfloor \\
& +\lambda\lfloor\overline{\mathbf{x}}' \diamond (\lfloor\overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}'''\rfloor)\rfloor - \lfloor\overline{\mathbf{x}}' \diamond (\lfloor\lfloor\overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}'''\rfloor\rfloor)\rfloor \\
& -\lfloor\overline{\mathbf{x}}' \diamond (\lfloor(\overline{\mathbf{x}}'' \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor)\rfloor + \lambda\lfloor(\overline{\mathbf{x}}' \diamond \lfloor\overline{\mathbf{x}}''\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor \\
& -\lfloor\lfloor(\overline{\mathbf{x}}' \diamond \lfloor\overline{\mathbf{x}}''\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor\rfloor - \lfloor((\overline{\mathbf{x}}' \diamond \lfloor\overline{\mathbf{x}}''\rfloor) \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor \\
& +\lambda\lfloor\lfloor\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}''\rfloor \diamond \overline{\mathbf{x}}'''\rfloor + \lambda\lfloor(\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \lfloor\overline{\mathbf{x}}'''\rfloor\rfloor \\
& +\lambda^2\lfloor(\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}'''\rfloor - \lambda\lfloor\lfloor(\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}'''\rfloor\rfloor \\
& -\lambda\lfloor((\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor - \lfloor\lfloor\lfloor\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}''\rfloor\rfloor \diamond \overline{\mathbf{x}}'''\rfloor \\
& -\lfloor\lfloor\lfloor\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}''\rfloor \diamond \overline{\mathbf{x}}'''\rfloor\rfloor - \lfloor\lfloor(\overline{\mathbf{x}} \diamond \overline{\mathbf{x}}') \diamond \lfloor\overline{\mathbf{x}}''\rfloor\rfloor\rfloor \\
& -\lambda\lfloor\lfloor(\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}'''\rfloor\rfloor + \lfloor\lfloor\lfloor(\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}'''\rfloor\rfloor\rfloor \\
& +\lfloor\lfloor((\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor\rfloor - \lambda\lfloor\lfloor\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}''\rfloor \diamond \overline{\mathbf{x}}'''\rfloor \\
& +\lfloor\lfloor\lfloor\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}''\rfloor \diamond \overline{\mathbf{x}}'''\rfloor\rfloor + \lfloor(\lfloor\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}''\rfloor \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor \\
& -\lfloor\lfloor(\overline{\mathbf{x}}' \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}''\rfloor \diamond \overline{\mathbf{x}}'''\rfloor - \lfloor((\overline{\mathbf{x}}' \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}'') \diamond \lfloor\overline{\mathbf{x}}'''\rfloor\rfloor \\
& -\lambda\lfloor((\overline{\mathbf{x}}' \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}'''\rfloor + \lfloor\lfloor((\overline{\mathbf{x}}' \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}'''\rfloor\rfloor \\
& +\lfloor(((\overline{\mathbf{x}}' \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}'') \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}''\rfloor.
\end{aligned}$$

Applying the induction hypothesis in n, using Eq. (5.6) to the 14th and the 28th term, we obtain

$$\begin{aligned}
\lfloor((\overline{\mathbf{x}}' \diamond \lfloor\overline{\mathbf{x}}''\rfloor) \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor &= \lfloor(\overline{\mathbf{x}}' \diamond (\lfloor\overline{\mathbf{x}}''\rfloor \diamond \lfloor 1\rfloor)) \diamond \overline{\mathbf{x}}'''\rfloor \\
&= \lfloor(\overline{\mathbf{x}}' \diamond (\lfloor\lfloor\overline{\mathbf{x}}''\rfloor \diamond 1\rfloor)) \diamond \overline{\mathbf{x}}'''\rfloor + \lfloor(\overline{\mathbf{x}}' \diamond (\lfloor\overline{\mathbf{x}}'' \diamond \lfloor 1\rfloor\rfloor))\rfloor \diamond \overline{\mathbf{x}}'''\rfloor \\
&\quad +\lambda\lfloor(\overline{\mathbf{x}}' \diamond (\lfloor\overline{\mathbf{x}}'' \diamond 1\rfloor)) \diamond \overline{\mathbf{x}}'''\rfloor - \lfloor(\overline{\mathbf{x}}' \diamond (\lfloor\lfloor\overline{\mathbf{x}}''\rfloor\rfloor)) \diamond \overline{\mathbf{x}}'''\rfloor \\
&\quad -\lfloor(\overline{\mathbf{x}}' \diamond (\lfloor(\overline{\mathbf{x}}'' \diamond \lfloor 1\rfloor) \diamond 1\rfloor)) \diamond \overline{\mathbf{x}}'''\rfloor
\end{aligned}$$

$$\begin{aligned}
\lfloor(\lfloor\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}''\rfloor \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor &= \lfloor(\lfloor\lfloor\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}''\rfloor \diamond 1\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor + \lfloor(\lfloor(\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \lfloor 1\rfloor\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor \\
&\quad +\lambda\lfloor(\lfloor(\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond 1\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor - \lfloor(\lfloor\lfloor(\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond 1\rfloor\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor \\
&\quad -\lfloor(\lfloor((\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \lfloor 1\rfloor) \diamond 1\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor.
\end{aligned}$$

Substituting these into the original equation, we will get

$$\begin{aligned}
(\lfloor\overline{\mathbf{x}}'\rfloor \diamond \lfloor\overline{\mathbf{x}}''\rfloor) \diamond \lfloor\overline{\mathbf{x}}'''\rfloor &= \lfloor\lfloor\lfloor\overline{\mathbf{x}}'\rfloor \diamond \overline{\mathbf{x}}''\rfloor \diamond \overline{\mathbf{x}}'''\rfloor + \lfloor(\lfloor\overline{\mathbf{x}}'\rfloor \diamond \overline{\mathbf{x}}'') \diamond \lfloor\overline{\mathbf{x}}'''\rfloor\rfloor \\
&\quad +\lambda\lfloor(\lfloor\overline{\mathbf{x}}'\rfloor \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}'''\rfloor - \lfloor\lfloor(\lfloor\overline{\mathbf{x}}'\rfloor \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}'''\rfloor\rfloor \\
&\quad -\lfloor((\lfloor\overline{\mathbf{x}}'\rfloor \diamond \overline{\mathbf{x}}'') \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor + \lfloor\lfloor\overline{\mathbf{x}}' \diamond \lfloor\overline{\mathbf{x}}''\rfloor\rfloor \diamond \overline{\mathbf{x}}'''\rfloor \\
&\quad +\lfloor\overline{\mathbf{x}}' \diamond (\lfloor\lfloor\overline{\mathbf{x}}''\rfloor \diamond \overline{\mathbf{x}}'''\rfloor)\rfloor + \lfloor\overline{\mathbf{x}}' \diamond (\lfloor\overline{\mathbf{x}}'' \diamond \lfloor\overline{\mathbf{x}}'''\rfloor)\rfloor \\
&\quad +\lambda\lfloor\overline{\mathbf{x}}' \diamond (\lfloor\overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}'''\rfloor)\rfloor - \lfloor\overline{\mathbf{x}}' \diamond (\lfloor\lfloor\overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}'''\rfloor\rfloor)\rfloor \\
&\quad -\lfloor\overline{\mathbf{x}}' \diamond (\lfloor(\overline{\mathbf{x}}'' \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor)\rfloor + \lambda\lfloor(\overline{\mathbf{x}}' \diamond \lfloor\overline{\mathbf{x}}''\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor \\
&\quad -\lfloor\lfloor(\overline{\mathbf{x}}' \diamond \lfloor\overline{\mathbf{x}}''\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor\rfloor - \lfloor(\overline{\mathbf{x}}' \diamond (\lfloor\lfloor\overline{\mathbf{x}}''\rfloor \diamond 1\rfloor)) \diamond \overline{\mathbf{x}}'''\rfloor \\
&\quad -\lfloor(\overline{\mathbf{x}}' \diamond (\lfloor\overline{\mathbf{x}}'' \diamond \lfloor 1\rfloor\rfloor))\rfloor \diamond \overline{\mathbf{x}}'''\rfloor - \lambda\lfloor(\overline{\mathbf{x}}' \diamond (\lfloor\overline{\mathbf{x}}'' \diamond 1\rfloor)) \diamond \overline{\mathbf{x}}'''\rfloor \\
&\quad +\lfloor(\overline{\mathbf{x}}' \diamond (\lfloor\lfloor\overline{\mathbf{x}}''\rfloor\rfloor)) \diamond \overline{\mathbf{x}}'''\rfloor + \lfloor(\overline{\mathbf{x}}' \diamond (\lfloor(\overline{\mathbf{x}}'' \diamond \lfloor 1\rfloor) \diamond 1\rfloor)) \diamond \overline{\mathbf{x}}'''\rfloor \\
&\quad +\lambda\lfloor(\lfloor\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}''\rfloor) \diamond \overline{\mathbf{x}}'''\rfloor + \lambda\lfloor(\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \lfloor\overline{\mathbf{x}}'''\rfloor\rfloor
\end{aligned}$$

$$
\begin{aligned}
&+\lambda^2\lfloor(\overline{\mathbf{x}}'\diamond\overline{\mathbf{x}}'')\diamond\overline{\mathbf{x}}'''\rfloor-\lambda\lfloor\lfloor(\overline{\mathbf{x}}'\diamond\overline{\mathbf{x}}'')\diamond\overline{\mathbf{x}}'''\rfloor\rfloor\\
&-\lambda\lfloor((\overline{\mathbf{x}}'\diamond\overline{\mathbf{x}}'')\lfloor 1\rfloor)\diamond\overline{\mathbf{x}}'''\rfloor-\lfloor\lfloor\lfloor\overline{\mathbf{x}}'\diamond\overline{\mathbf{x}}''\rfloor\rfloor\diamond\overline{\mathbf{x}}'''\rfloor\\
&-\lfloor\lfloor\lfloor\overline{\mathbf{x}}'\diamond\overline{\mathbf{x}}''\rfloor\diamond\overline{\mathbf{x}}'''\rfloor\rfloor-\lfloor\lfloor(\overline{\mathbf{x}}'\diamond\overline{\mathbf{x}}'')\diamond\lfloor\overline{\mathbf{x}}'''\rfloor\rfloor\rfloor\\
&-\lambda\lfloor\lfloor(\overline{\mathbf{x}}'\diamond\overline{\mathbf{x}}'')\diamond\overline{\mathbf{x}}'''\rfloor\rfloor+\lfloor\lfloor\lfloor(\overline{\mathbf{x}}'\diamond\overline{\mathbf{x}}'')\diamond\overline{\mathbf{x}}'''\rfloor\rfloor\rfloor\\
&+\lfloor\lfloor((\overline{\mathbf{x}}'\diamond\overline{\mathbf{x}}'')\diamond\lfloor 1\rfloor)\diamond\overline{\mathbf{x}}'''\rfloor\rfloor-\lambda\lfloor\lfloor\overline{\mathbf{x}}'\diamond\overline{\mathbf{x}}''\rfloor\diamond\overline{\mathbf{x}}'''\rfloor\\
&+\lfloor\lfloor\lfloor\overline{\mathbf{x}}'\diamond\overline{\mathbf{x}}''\rfloor\diamond\overline{\mathbf{x}}'''\rfloor\rfloor+\lfloor(\lfloor\lfloor\overline{\mathbf{x}}'\diamond\overline{\mathbf{x}}''\rfloor\diamond 1\rfloor)\diamond\overline{\mathbf{x}}'''\rfloor\\
&+\lfloor(\lfloor(\overline{\mathbf{x}}'\diamond\overline{\mathbf{x}}'')\diamond\lfloor 1\rfloor\rfloor)\diamond\overline{\mathbf{x}}'''\rfloor+\lambda\lfloor(\lfloor(\overline{\mathbf{x}}'\diamond\overline{\mathbf{x}}'')\diamond 1\rfloor)\diamond\overline{\mathbf{x}}'''\rfloor\\
&-\lfloor(\lfloor\lfloor(\overline{\mathbf{x}}'\diamond\overline{\mathbf{x}}'')\diamond 1\rfloor\rfloor)\diamond\overline{\mathbf{x}}'''\rfloor-\lfloor(\lfloor((\overline{\mathbf{x}}'\diamond\overline{\mathbf{x}}'')\diamond\lfloor 1\rfloor)\diamond 1\rfloor)\diamond\overline{\mathbf{x}}'''\rfloor\\
&-\lfloor\lfloor(\overline{\mathbf{x}}'\diamond\lfloor 1\rfloor)\diamond\overline{\mathbf{x}}''\rfloor\diamond\overline{\mathbf{x}}'''\rfloor-\lfloor((\overline{\mathbf{x}}'\diamond\lfloor 1\rfloor)\diamond\overline{\mathbf{x}}'')\diamond\lfloor\overline{\mathbf{x}}'''\rfloor\rfloor\\
&-\lambda\lfloor((\overline{\mathbf{x}}'\diamond\lfloor 1\rfloor)\diamond\overline{\mathbf{x}}'')\diamond\overline{\mathbf{x}}'''\rfloor+\lfloor\lfloor((\overline{\mathbf{x}}'\diamond\lfloor 1\rfloor)\diamond\overline{\mathbf{x}}'')\diamond\overline{\mathbf{x}}'''\rfloor\rfloor\\
&+\lfloor(((\overline{\mathbf{x}}'\diamond\lfloor 1\rfloor)\diamond\overline{\mathbf{x}}'')\diamond\lfloor 1\rfloor)\diamond\overline{\mathbf{x}}''\rfloor.
\end{aligned}
$$

By a similar computation on $\mathbf{x}'\diamond(\mathbf{x}''\diamond\mathbf{x}''')$, we obtain

$$
\begin{aligned}
\lfloor\overline{\mathbf{x}}'\rfloor\diamond(\lfloor\overline{\mathbf{x}}''\rfloor\diamond\lfloor\overline{\mathbf{x}}'''\rfloor) \quad = \quad &\lfloor\lfloor\lfloor\overline{\mathbf{x}}'\rfloor\diamond\overline{\mathbf{x}}''\rfloor\diamond\overline{\mathbf{x}}'''\rfloor+\lfloor\lfloor\overline{\mathbf{x}}'\diamond\lfloor\overline{\mathbf{x}}''\rfloor\rfloor\diamond\overline{\mathbf{x}}'''\rfloor\\
&+\lambda\lfloor\lfloor\overline{\mathbf{x}}'\diamond\overline{\mathbf{x}}''\rfloor\diamond\overline{\mathbf{x}}'''\rfloor-\lfloor\lfloor\lfloor\overline{\mathbf{x}}'\diamond\overline{\mathbf{x}}''\rfloor\rfloor\diamond\overline{\mathbf{x}}'''\rfloor\\
&-\lfloor\lfloor(\overline{\mathbf{x}}'\diamond\lfloor 1\rfloor)\diamond\overline{\mathbf{x}}''\rfloor\diamond\overline{\mathbf{x}}'''\rfloor+\lfloor\overline{\mathbf{x}}'\diamond(\lfloor\lfloor\overline{\mathbf{x}}''\rfloor\diamond\overline{\mathbf{x}}'''\rfloor)\rfloor\\
&+\lambda\lfloor\overline{\mathbf{x}}'\diamond(\lfloor\overline{\mathbf{x}}''\rfloor\diamond\overline{\mathbf{x}}''')\rfloor-\lfloor\lfloor\overline{\mathbf{x}}'\diamond(\lfloor\overline{\mathbf{x}}''\rfloor\diamond\overline{\mathbf{x}}''')\rfloor\rfloor\\
&-\lfloor\overline{\mathbf{x}}'\diamond\lfloor\lfloor 1\rfloor\diamond\overline{\mathbf{x}}''\rfloor\diamond\overline{\mathbf{x}}'''\rfloor-\lfloor\overline{\mathbf{x}}'\diamond(\lfloor\lfloor\overline{\mathbf{x}}''\rfloor\rfloor\diamond\overline{\mathbf{x}}''')\rfloor\\
&-\lambda\lfloor\overline{\mathbf{x}}'\diamond(\lfloor\overline{\mathbf{x}}''\rfloor\diamond\overline{\mathbf{x}}''')\rfloor+\lfloor\overline{\mathbf{x}}'\diamond(\lfloor\lfloor\overline{\mathbf{x}}''\rfloor\rfloor\diamond\overline{\mathbf{x}}''')\rfloor\\
&+\lfloor\overline{\mathbf{x}}'\diamond\lfloor\lfloor 1\rfloor\diamond(\overline{\mathbf{x}}''\rfloor\diamond\overline{\mathbf{x}}''')\rfloor+\lfloor\lfloor\overline{\mathbf{x}}'\rfloor\diamond(\overline{\mathbf{x}}''\diamond\lfloor\overline{\mathbf{x}}'''\rfloor)\rfloor\\
&+\lfloor\overline{\mathbf{x}}'\diamond(\lfloor\overline{\mathbf{x}}''\diamond\lfloor\overline{\mathbf{x}}'''\rfloor\rfloor)\rfloor+\lambda\lfloor\overline{\mathbf{x}}'\diamond(\overline{\mathbf{x}}''\diamond\lfloor\overline{\mathbf{x}}'''\rfloor)\rfloor\\
&-\lfloor\lfloor\overline{\mathbf{x}}'\diamond(\overline{\mathbf{x}}''\diamond\lfloor\overline{\mathbf{x}}'''\rfloor)\rfloor\rfloor-\lfloor(\overline{\mathbf{x}}'\diamond\lfloor 1\rfloor)\diamond(\overline{\mathbf{x}}''\diamond\lfloor\overline{\mathbf{x}}'''\rfloor)\rfloor
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda\lfloor\lfloor\overline{\mathbf{x}}'\rfloor\diamond(\overline{\mathbf{x}}''\diamond\overline{\mathbf{x}}''')\rfloor+\lambda\lfloor\overline{\mathbf{x}}'\diamond\lfloor\overline{\mathbf{x}}''\diamond\overline{\mathbf{x}}'''\rfloor\rfloor\\
&+\lambda^2\lfloor\overline{\mathbf{x}}'\diamond(\overline{\mathbf{x}}''\diamond\overline{\mathbf{x}}''')\rfloor-\lambda\lfloor\lfloor\overline{\mathbf{x}}'\diamond(\overline{\mathbf{x}}''\diamond\overline{\mathbf{x}}''')\rfloor\rfloor\\
&-\lambda\lfloor(\overline{\mathbf{x}}'\diamond\lfloor 1\rfloor)\diamond(\overline{\mathbf{x}}''\diamond\overline{\mathbf{x}}''')\rfloor-\lfloor\lfloor\lfloor\overline{\mathbf{x}}'\rfloor\diamond(\overline{\mathbf{x}}''\diamond\overline{\mathbf{x}}''')\rfloor\rfloor\\
&-\lfloor\lfloor\overline{\mathbf{x}}'\diamond(\lfloor\overline{\mathbf{x}}''\diamond\overline{\mathbf{x}}'''\rfloor\rfloor)\rfloor-\lambda\lfloor\lfloor\overline{\mathbf{x}}'\diamond(\overline{\mathbf{x}}''\diamond\overline{\mathbf{x}}''')\rfloor\rfloor\\
&+\lfloor\lfloor\lfloor\overline{\mathbf{x}}'\diamond(\overline{\mathbf{x}}''\diamond\overline{\mathbf{x}}''')\rfloor\rfloor\rfloor+\lfloor\lfloor(\overline{\mathbf{x}}'\diamond\lfloor 1\rfloor)\diamond(\overline{\mathbf{x}}''\diamond\overline{\mathbf{x}}''')\rfloor\rfloor\\
&-\lfloor\overline{\mathbf{x}}'\diamond\lfloor\lfloor\overline{\mathbf{x}}''\diamond\overline{\mathbf{x}}'''\rfloor\rfloor\rfloor-\lambda\lfloor\overline{\mathbf{x}}'\diamond\lfloor\overline{\mathbf{x}}''\diamond\overline{\mathbf{x}}'''\rfloor\rfloor\\
&+\lfloor\lfloor\overline{\mathbf{x}}'\diamond\lfloor\overline{\mathbf{x}}''\diamond\overline{\mathbf{x}}'''\rfloor\rfloor\rfloor+\lfloor\overline{\mathbf{x}}'\diamond\lfloor\lfloor 1\rfloor\diamond(\overline{\mathbf{x}}''\diamond\overline{\mathbf{x}}''')\rfloor\rfloor\\
&+\lfloor\overline{\mathbf{x}}'\diamond\lfloor\lfloor\overline{\mathbf{x}}''\diamond\overline{\mathbf{x}}'''\rfloor\rfloor\rfloor+\lambda\lfloor\overline{\mathbf{x}}'\diamond\lfloor\overline{\mathbf{x}}''\diamond\overline{\mathbf{x}}'''\rfloor\rfloor\\
&-\lfloor\overline{\mathbf{x}}'\diamond\lfloor\lfloor\overline{\mathbf{x}}''\diamond\overline{\mathbf{x}}'''\rfloor\rfloor\rfloor-\lfloor\overline{\mathbf{x}}'\lfloor\lfloor 1\rfloor\diamond(\overline{\mathbf{x}}''\diamond\overline{\mathbf{x}}''')\rfloor\rfloor\\
&-\lfloor\lfloor\overline{\mathbf{x}}'\rfloor\diamond((\overline{\mathbf{x}}''\diamond\lfloor 1\rfloor)\diamond\overline{\mathbf{x}}''')\rfloor-\lfloor\overline{\mathbf{x}}'\diamond\lfloor(\overline{\mathbf{x}}''\diamond\lfloor 1\rfloor)\rfloor\diamond\overline{\mathbf{x}}'''\rfloor\\
&-\lambda\lfloor\overline{\mathbf{x}}'\diamond((\overline{\mathbf{x}}''\diamond\lfloor 1\rfloor)\diamond\overline{\mathbf{x}}''')\rfloor+\lfloor\lfloor\overline{\mathbf{x}}'\diamond((\overline{\mathbf{x}}''\diamond\lfloor 1\rfloor)\diamond\overline{\mathbf{x}}''')\rfloor\rfloor\\
&+\lfloor(\overline{\mathbf{x}}'\diamond\lfloor 1\rfloor)\diamond((\overline{\mathbf{x}}''\diamond\lfloor 1\rfloor)\diamond\overline{\mathbf{x}}''')\rfloor.
\end{aligned}
$$

Some of the terms on the left hand side cancel among themselves. Those terms are: 15th and 18th, 25th and 31st, 30th and 34th, 32nd and 35th, 33rd and 36th. After simplifying, we will get

$$
\begin{aligned}
(\lfloor\overline{\mathbf{x}}'\rfloor\diamond\lfloor\overline{\mathbf{x}}''\rfloor)\diamond\lfloor\overline{\mathbf{x}}'''\rfloor \;=\;& \underbrace{\lfloor\lfloor\lfloor\overline{\mathbf{x}}'\rfloor\diamond\overline{\mathbf{x}}''\rfloor\diamond\overline{\mathbf{x}}'''\rfloor}_{L_1}+\underbrace{\lfloor(\lfloor\overline{\mathbf{x}}'\rfloor\diamond\overline{\mathbf{x}}'')\diamond\lfloor\overline{\mathbf{x}}'''\rfloor\rfloor}_{L_2}\\
&+\underbrace{\lambda\lfloor(\lfloor\overline{\mathbf{x}}'\rfloor\diamond\overline{\mathbf{x}}'')\diamond\overline{\mathbf{x}}'''\rfloor}_{L_3}-\underbrace{\lfloor\lfloor(\lfloor\overline{\mathbf{x}}'\rfloor\diamond\overline{\mathbf{x}}'')\diamond\overline{\mathbf{x}}'''\rfloor\rfloor}_{L_4}\\
&-\underbrace{\lfloor((\lfloor\overline{\mathbf{x}}'\rfloor\diamond\overline{\mathbf{x}}'')\diamond\lfloor 1\rfloor)\diamond\overline{\mathbf{x}}'''\rfloor}_{L_5}+\underbrace{\lfloor(\lfloor\overline{\mathbf{x}}'\diamond\lfloor\overline{\mathbf{x}}''\rfloor\rfloor)\diamond\overline{\mathbf{x}}'''\rfloor}_{L_6}\\
&+\underbrace{\lfloor\overline{\mathbf{x}}'\diamond(\lfloor\lfloor\overline{\mathbf{x}}'\rfloor\diamond\overline{\mathbf{x}}''\rfloor)\rfloor}_{L_7}+\underbrace{\lfloor\overline{\mathbf{x}}'\diamond(\lfloor\overline{\mathbf{x}}''\diamond\lfloor\overline{\mathbf{x}}'''\rfloor\rfloor)\rfloor}_{L_8}\\
&+\underbrace{\lambda\lfloor\overline{\mathbf{x}}'\diamond(\lfloor\overline{\mathbf{x}}''\diamond\overline{\mathbf{x}}'''\rfloor)\rfloor}_{L_9}-\underbrace{\lfloor\overline{\mathbf{x}}'\diamond(\lfloor\lfloor\overline{\mathbf{x}}''\diamond\overline{\mathbf{x}}'''\rfloor\rfloor)\rfloor}_{L_{10}}
\end{aligned}
$$

$$
\begin{aligned}
&-\underbrace{\lfloor \overline{\mathbf{x}}' \diamond (\lfloor (\overline{\mathbf{x}}'' \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}''' \rfloor) \rfloor}_{L_{11}} + \underbrace{\lambda \lfloor (\overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \rfloor) \diamond \overline{\mathbf{x}}''' \rfloor}_{L_{12}} \\
&-\underbrace{\lfloor \lfloor (\overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \rfloor) \diamond \overline{\mathbf{x}}''' \rfloor \rfloor}_{L_{13}} - \underbrace{\lfloor (\overline{\mathbf{x}}' \diamond (\lfloor \lfloor \overline{\mathbf{x}}'' \rfloor \diamond 1 \rfloor)) \diamond \overline{\mathbf{x}}''' \rfloor}_{L_{14}} \\
&-\underbrace{\lambda \lfloor (\overline{\mathbf{x}}' \diamond (\lfloor \overline{\mathbf{x}}'' \diamond 1 \rfloor)) \diamond \overline{\mathbf{x}}''' \rfloor}_{L_{15}} + \underbrace{\lfloor (\overline{\mathbf{x}}' \diamond (\lfloor \lfloor \overline{\mathbf{x}}'' \rfloor \rfloor)) \diamond \overline{\mathbf{x}}''' \rfloor}_{L_{16}} \\
&+\underbrace{\lambda \lfloor (\lfloor \overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'' \rfloor) \diamond \overline{\mathbf{x}}''' \rfloor}_{L_{17}} + \underbrace{\lambda \lfloor (\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \rfloor}_{L_{18}} \\
&+\underbrace{\lambda^2 \lfloor (\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}''' \rfloor}_{L_{19}} - \underbrace{\lambda \lfloor \lfloor (\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}''' \rfloor \rfloor}_{L_{20}} \\
&-\underbrace{\lambda \lfloor ((\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}''' \rfloor}_{L_{21}} - \underbrace{\lfloor (\lfloor \lfloor \overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'' \rfloor \rfloor) \diamond \overline{\mathbf{x}}''' \rfloor}_{L_{22}} \\
&-\underbrace{\lfloor \lfloor (\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \rfloor \rfloor}_{L_{23}} - \underbrace{\lambda \lfloor \lfloor (\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}''' \rfloor \rfloor}_{L_{24}} \\
&+\underbrace{\lfloor \lfloor \lfloor (\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}''' \rfloor \rfloor \rfloor}_{L_{25}} + \underbrace{\lfloor \lfloor ((\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}''' \rfloor \rfloor}_{L_{26}} \\
&-\underbrace{\lfloor \lfloor ((\overline{\mathbf{x}}' \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}'' \rfloor) \diamond \overline{\mathbf{x}}''' \rfloor}_{L_{27}} - \underbrace{\lfloor ((\overline{\mathbf{x}}' \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}'') \diamond \lfloor \overline{\mathbf{x}}''' \rfloor \rfloor}_{L_{28}} \\
&-\underbrace{\lambda \lfloor ((\overline{\mathbf{x}}' \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}''' \rfloor}_{L_{29}} + \underbrace{\lfloor \lfloor ((\overline{\mathbf{x}}' \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}'') \diamond \overline{\mathbf{x}}''' \rfloor \rfloor}_{L_{30}} \\
&+\underbrace{\lfloor (((\overline{\mathbf{x}}' \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}'') \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}'' \rfloor}_{L_{31}}. \qquad (5.10)
\end{aligned}
$$

Similarly, on the right hand side, there are also some terms which cancel among themselves: 9th and 13th, 25th and 31st, 29th and 33rd, 30th and 34th, 32nd and 36th.

$$
\begin{aligned}
\lfloor \overline{\mathbf{x}}' \rfloor \diamond (\lfloor \overline{\mathbf{x}}'' \rfloor \diamond \lfloor \overline{\mathbf{x}}''' \rfloor) \quad = \quad &\underbrace{\lfloor \lfloor \lfloor \overline{\mathbf{x}}' \rfloor \diamond \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''' \rfloor}_{R_1} + \underbrace{\lfloor \lfloor \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \rfloor \rfloor \diamond \overline{\mathbf{x}}''' \rfloor}_{R_2} \\
&+\underbrace{\lambda \lfloor \lfloor \overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''' \rfloor}_{R_3} - \underbrace{\lfloor \lfloor \lfloor \overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'' \rfloor \rfloor \diamond \overline{\mathbf{x}}''' \rfloor}_{R_4} \\
&-\underbrace{\lfloor \lfloor (\overline{\mathbf{x}}' \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''' \rfloor}_{R_5} + \underbrace{\lfloor \overline{\mathbf{x}}' \diamond (\lfloor \lfloor \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''' \rfloor) \rfloor}_{R_6} \\
&+\underbrace{\lambda \lfloor \overline{\mathbf{x}}' \diamond (\lfloor \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''') \rfloor}_{R_7} - \underbrace{\lfloor \lfloor \overline{\mathbf{x}}' \diamond (\lfloor \overline{\mathbf{x}}'' \rfloor \diamond \overline{\mathbf{x}}''') \rfloor \rfloor}_{R_8}
\end{aligned}
$$

$$
\begin{aligned}
&-\underbrace{\lfloor \overline{\mathbf{x}}' \diamond (\lfloor\lfloor \overline{\mathbf{x}}''\rfloor\rfloor \diamond \overline{\mathbf{x}}''')\rfloor}_{R_9} - \underbrace{\lambda\lfloor \overline{\mathbf{x}}' \diamond (\lfloor \overline{\mathbf{x}}''\rfloor \diamond \overline{\mathbf{x}}''')\rfloor}_{R_{10}} \\
&+\underbrace{\lfloor \overline{\mathbf{x}}' \diamond (\lfloor\lfloor \overline{\mathbf{x}}''\rfloor\rfloor \diamond \overline{\mathbf{x}}''')\rfloor}_{R_{11}} + \underbrace{\lfloor\lfloor \overline{\mathbf{x}}'\rfloor \diamond (\overline{\mathbf{x}}'' \diamond \lfloor \overline{\mathbf{x}}'''\rfloor)\rfloor}_{R_{12}} \\
&+\underbrace{\lfloor \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \diamond \lfloor \overline{\mathbf{x}}'''\rfloor\rfloor\rfloor}_{R_{13}} + \underbrace{\lambda\lfloor \overline{\mathbf{x}}' \diamond (\overline{\mathbf{x}}'' \diamond \lfloor \overline{\mathbf{x}}'''\rfloor)\rfloor}_{R_{14}} \\
&-\underbrace{\lfloor\lfloor \overline{\mathbf{x}}' \diamond (\overline{\mathbf{x}}'' \diamond \lfloor \overline{\mathbf{x}}'''\rfloor)\rfloor\rfloor}_{R_{15}} - \underbrace{\lfloor (\overline{\mathbf{x}}' \diamond \lfloor 1\rfloor) \diamond (\overline{\mathbf{x}}'' \diamond \lfloor \overline{\mathbf{x}}'''\rfloor)\rfloor}_{R_{16}} \\
&+\underbrace{\lambda\lfloor\lfloor \overline{\mathbf{x}}'\rfloor \diamond (\overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}''')\rfloor}_{R_{17}} + \underbrace{\lambda\lfloor \overline{\mathbf{x}}' \diamond \lfloor \overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}'''\rfloor\rfloor}_{R_{18}} \\
&+\underbrace{\lambda^2\lfloor \overline{\mathbf{x}}' \diamond (\overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}''')\rfloor}_{R_{19}} - \underbrace{\lambda\lfloor\lfloor \overline{\mathbf{x}}' \diamond (\overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}''')\rfloor\rfloor}_{R_{20}} \\
&-\underbrace{\lambda\lfloor (\overline{\mathbf{x}}' \diamond \lfloor 1\rfloor) \diamond (\overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}''')\rfloor}_{R_{21}} - \underbrace{\lfloor\lfloor\lfloor \overline{\mathbf{x}}'\rfloor \diamond (\overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}''')\rfloor\rfloor}_{R_{22}} \\
&-\underbrace{\lambda\lfloor\lfloor \overline{\mathbf{x}}' \diamond (\overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}''')\rfloor\rfloor}_{R_{23}} + \underbrace{\lfloor\lfloor\lfloor \overline{\mathbf{x}}' \diamond (\overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}''')\rfloor\rfloor\rfloor}_{R_{24}} \\
&+\underbrace{\lfloor\lfloor (\overline{\mathbf{x}}' \diamond \lfloor 1\rfloor) \diamond (\overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}''')\rfloor\rfloor}_{R_{25}} - \underbrace{\lfloor \overline{\mathbf{x}}' \diamond \lfloor\lfloor \overline{\mathbf{x}}'' \diamond \overline{\mathbf{x}}'''\rfloor\rfloor\rfloor}_{R_{26}} \\
&-\underbrace{\lfloor\lfloor \overline{\mathbf{x}}'\rfloor \diamond ((\overline{\mathbf{x}}'' \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}''')\rfloor}_{R_{27}} - \underbrace{\lfloor \overline{\mathbf{x}}' \diamond \lfloor (\overline{\mathbf{x}}'' \diamond \lfloor 1\rfloor)\rfloor \diamond \overline{\mathbf{x}}'''\rfloor}_{R_{28}} \\
&-\underbrace{\lambda\lfloor \overline{\mathbf{x}}' \diamond ((\overline{\mathbf{x}}'' \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}''')\rfloor}_{R_{29}} + \underbrace{\lfloor\lfloor \overline{\mathbf{x}}' \diamond ((\overline{\mathbf{x}}'' \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}''')\rfloor\rfloor}_{R_{30}} \\
&+\underbrace{\lfloor (\overline{\mathbf{x}}' \diamond \lfloor 1\rfloor) \diamond ((\overline{\mathbf{x}}'' \diamond \lfloor 1\rfloor) \diamond \overline{\mathbf{x}}''')\rfloor}_{R_{31}}. \qquad (5.11)
\end{aligned}
$$

Lemma 5.9. For $1 \leq i \leq 31$, let L_i (resp R_i) be the i^{th} term in Eq. (5.10) (resp. in Eq. (5.11)). Then $L_i = R_{\sigma(i)}$. Here the permutation $\sigma \in \Sigma_{31}$ is given by

$$
\sigma = \left(\begin{array}{ccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 1 & 12 & 17 & 22 & 27 & 2 & 6 & 13 & 18 & 26 & 28 & 7 & 8 & 9 & 10 \end{array} \right. \qquad (5.12)
$$

$$
\left. \begin{array}{cccccccccccccccc} 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 \\ 11 & 3 & 14 & 19 & 20 & 29 & 4 & 15 & 23 & 24 & 30 & 5 & 16 & 21 & 25 & 31 \end{array} \right).
$$

This permuation is too long to fit in one line, so is split into two.

Proof. We divide the proof of this lemma by type of terms into three cases:

Case I: Those terms on the left hand side i.e. in Eq. (5.10) that exactly match with certain terms on the right hand side i.e. Eq. (5.11) . They are:

$L_1 = R_1$, $L_6 = R_2$, $L_7 = R_6$, $L_8 = R_{13}$, $L_9 = R_{18}$, $L_{10} = R_{26}$, $L_{17} = R_3$, $L_{22} = R_4$

Case II: Those terms on the left hand side that match with the right hand side using induction on sum of depths of $\overline{\mathbf{x}}', \overline{\mathbf{x}}'', \overline{\mathbf{x}}'''$ i.e. $d(\overline{\mathbf{x}}') + d(\overline{\mathbf{x}}'') + d(\overline{\mathbf{x}}''') \leq n + 1$. They are:

$L_2 = R_{12}, L_3 = R_{17}, L_4 = R_{22}, L_{12} = R_7, L_{13} = R_8, L_{14} = R_9, L_{15} = R_{10}, L_{16} = R_{11}$, $L_{18} = R_{14}, L_{19} = R_{19}, L_{20} = R_{20}, L_{23} = R_{15}, L_{24} = R_{23}, L_{25} = R_{24}$

Case III: Those remaining terms that we want match are the special terms in which $\lfloor 1 \rfloor$ is involved. They are :

$L_5 = R_{27}, L_{11} = R_{28}, L_{21} = R_{29}, L_{26} = R_{30}, L_{27} = R_5, L_{28} = R_{16}, L_{29} = R_{21}, L_{30} = R_{25}$, $L_{31} = R_{31}$.

We only need to verify Case III. Before we check that these terms match, we need two claims.

Lemma 5.10. For all $\mathbf{x}', \mathbf{x}'' \in M(X)$, $(\mathbf{x}' \diamond \mathbf{x}'') \diamond \lfloor 1 \rfloor = \mathbf{x}' \diamond (\mathbf{x}'' \diamond \lfloor 1 \rfloor)$.

Proof. We will prove the result by induction on n: $n = d(\mathbf{x}') + d(\mathbf{x}'') \geq 0$.

For n=0, $\mathbf{x}', \mathbf{x}'' \in M(X)$, $\mathbf{x}' = \mathbf{x}_1 \mathbf{x}_2 \cdots \mathbf{x}_k$,

$$(\mathbf{x}' \diamond \mathbf{x}'') \diamond \lfloor 1 \rfloor = (\mathbf{x}_1 \cdots \mathbf{x}_k \mathbf{x}'') \diamond \lfloor 1 \rfloor = \mathbf{x}_1 \cdots \mathbf{x}_k (\mathbf{x}'' \diamond \lfloor 1 \rfloor) = \mathbf{x}' \diamond (\mathbf{x}'' \diamond \lfloor 1 \rfloor).$$

Assume the result holds for n$\leq$ k. Consider $n = d(\mathbf{x}') + d(\mathbf{x}'') = k + 1$.

If either $\mathbf{x}'$ or $\mathbf{x}''$ has depth 0, then the result holds from n=0 case. Otherwise,

$\mathbf{x}' = \lfloor \overline{\mathbf{x}}' \rfloor$, then the using induction hypothesis, we have

$$(\mathbf{x}' \diamond \mathbf{x}'') \diamond \lfloor 1 \rfloor = (\lfloor \overline{\mathbf{x}}' \rfloor \diamond x'') \diamond \lfloor 1 \rfloor = \lfloor \overline{\mathbf{x}}' \rfloor \diamond (\mathbf{x}'' \diamond \lfloor 1 \rfloor) = \mathbf{x}' \diamond (\mathbf{x}'' \diamond \lfloor 1 \rfloor).$$

□

Claim 5.11. For $\mathbf{x} \in \mathfrak{X}_\infty$, $d(\mathbf{x} \diamond \lfloor 1 \rfloor) \leq d(\mathbf{x}) + 1$, where $d(\mathbf{x})$=depth of $\mathbf{x}$.

Proof. Let $\mathbf{x} = \mathbf{x}_1 \cdots \mathbf{x}_k, \mathbf{x}_i \in X \bigsqcup \lfloor M(X) \rfloor$. If $\mathbf{x}_k = \mathbf{x} \in$ X, then $\mathbf{x} \diamond \lfloor 1 \rfloor = \mathbf{x}_1 \cdots \mathbf{x}_k \lfloor 1 \rfloor$. This gives us : d($\mathbf{x} \diamond \lfloor 1 \rfloor$)=max$\{d(\mathbf{x}),\ d(\lfloor 1 \rfloor)\} \leq d(\mathbf{x}) + 1$.

When $\mathbf{x}_k = \lfloor \overline{\mathbf{x}}_k \rfloor$, then $d(\mathbf{x}_k \diamond \lfloor 1 \rfloor) \leq d(\mathbf{x}_k$) +1 (by definition of $\diamond$). Now, d($\mathbf{x} \diamond \lfloor 1 \rfloor$) = d(($\mathbf{x}_1 \cdots \mathbf{x}_k) \diamond \lfloor 1 \rfloor$).

Now by lemma 5.3, d(($\mathbf{x}_1 \cdots \mathbf{x}_k) \diamond \lfloor 1 \rfloor$) =d($\mathbf{x}_1 \cdots \mathbf{x}_{k-1}(\mathbf{x}_k \diamond \lfloor 1 \rfloor)$), hence we get

$$\begin{aligned}
d(\mathbf{x} \diamond \lfloor 1 \rfloor) &= d(\mathbf{x}_1 \cdots \mathbf{x}_{k-1}(\mathbf{x}_k \diamond \lfloor 1 \rfloor)) \\
&= max\{d(\mathbf{x}_1), \cdots, d(\mathbf{x}_k \diamond \lfloor 1 \rfloor)\} \\
&\leq max\{d(\mathbf{x}_1), \cdots, d(\mathbf{x}_k) + 1\} \\
&\leq max\{d(\mathbf{x}_1) + 1, \cdots, d(\mathbf{x}_k) + 1\} \\
&= max\{d(\mathbf{x}_1), \cdots, d(\mathbf{x}_k)\} + 1 \\
&= d(\mathbf{x}) + 1.
\end{aligned}$$

This proves the claim. □

Now, coming back to our lemma, we will check some of them since the proofs are similar:

$L_5 = \lfloor ((\lfloor \overline{\mathbf{x}}' \rfloor \diamond \overline{\mathbf{x}}'') \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}''' \rfloor$

$= \lfloor (\lfloor \overline{\mathbf{x}}' \rfloor \diamond (\overline{\mathbf{x}}'' \diamond \lfloor 1 \rfloor)) \diamond \overline{\mathbf{x}}''' \rfloor$ (By claim (5.10) and the induction hypothesis)

$= \lfloor \lfloor \overline{\mathbf{x}}' \rfloor \diamond ((\overline{\mathbf{x}}'' \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}''') \rfloor$ (By claim (5.11) and the induction hypothesis)

$= R_{27}.$

$$
\begin{aligned}
L_{21} &= \lfloor ((\overline{\mathbf{x}}' \diamond \overline{\mathbf{x}}'') \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}''' \rfloor \\
&= \lfloor (\overline{\mathbf{x}}' \diamond (\overline{\mathbf{x}}'' \diamond \lfloor 1 \rfloor)) \diamond \overline{\mathbf{x}}''' \rfloor \quad \text{(By claim (5.10) and the induction hypothesis)} \\
&= \lfloor \overline{\mathbf{x}}' \diamond ((\overline{\mathbf{x}}'' \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}''') \rfloor \quad \text{(By claim (5.11) and the induction hypothesis)} \\
&= R_{29}.
\end{aligned}
$$

$$
\begin{aligned}
L_{31} &= \lfloor (((\overline{\mathbf{x}}' \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}'') \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}'' \rfloor \quad \text{(By claim (5.10) and the induction hypothesis)} \\
&= \lfloor ((\overline{\mathbf{x}}' \diamond \lfloor 1 \rfloor) \diamond (\overline{\mathbf{x}}'' \diamond \lfloor 1 \rfloor)) \diamond \overline{\mathbf{x}}''' \rfloor \quad \text{(By claim (5.10) and the induction hypothesis)} \\
&= \lfloor (\overline{\mathbf{x}}' \diamond \lfloor 1 \rfloor) \diamond ((\overline{\mathbf{x}}'' \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}''') \rfloor \\
&= R_{31}.
\end{aligned}
$$

□

This completes the proof of Theorem 5.6.1.

2. The proof follows from the definition $T_A(\mathbf{x}) = \lfloor \mathbf{x} \rfloor$ and Eq. (5.6).

3. Let $(T, *, P)$ be a RBNTD algebra with multiplication $*$. Let $f : A \to T$ be a $\mathbf{k}$-algebra homomorphism. We will construct a $\mathbf{k}$-linear map $\bar{f} : F^{NC}_{\mathrm{T}}(A) \to T$ by defining $\bar{f}(\mathbf{x})$ for $\mathbf{x} \in \tilde{\mathfrak{X}}_\infty$. We achieve this by defining $\bar{f}(\mathbf{x})$ for $\mathbf{x} \in \mathfrak{X}_n$, $n \geq 0$, inductively on n. For $\mathbf{x} \in \mathfrak{X}_0 := X$, define $\bar{f}(\mathbf{x}) = f(\mathbf{x})$. Suppose $\bar{f}(\mathbf{x})$ has been defined for $\mathbf{x} \in \mathfrak{X}_n$ and consider $\mathbf{x}$ in $\mathfrak{X}_{n+1}$ which is, by definition and Eq. (5.2),

$$
\begin{aligned}
\tilde{\Lambda}_X(X, \mathfrak{X}_n) = & \Big(\bigsqcup_{r \geq 1} (X \lfloor \mathfrak{X}_n \rfloor)^r \Big) \bigsqcup \Big(\bigsqcup_{r \geq 0} (X \lfloor \mathfrak{X}_n \rfloor)^r X \Big) \\
& \bigsqcup \Big(\bigsqcup_{r \geq 0} \lfloor \mathfrak{X}_n \rfloor (X \lfloor \mathfrak{X}_n \rfloor)^r \Big) \bigsqcup \Big(\bigsqcup_{r \geq 0} \lfloor \mathfrak{X}_n \rfloor (X \lfloor \mathfrak{X}_n \rfloor)^r X \Big).
\end{aligned}
$$

Let $\mathbf{x}$ be in the first union component $\bigsqcup_{r \geq 1} (X \lfloor \mathfrak{X}_n \rfloor)^r$ above. Then

$$
\mathbf{x} = \prod_{i=1}^{r} (\mathbf{x}_{2i-1} \lfloor \mathbf{x}_{2i} \rfloor)
$$

for $\mathbf{x}_{2i-1} \in X$ and $\mathbf{x}_{2i} \in \mathfrak{X}_n$, $1 \leq i \leq r$. By the construction of the multiplication $\diamond$

and the RBNTD operator T_A, we have

$$\mathbf{x} = \diamond_{i=1}^{r}(\mathbf{x}_{2i-1} \diamond \lfloor \mathbf{x}_{2i} \rfloor) = \diamond_{i=1}^{r}(\mathbf{x}_{2i-1} \diamond T_A(\mathbf{x}_{2i})).$$

Define

$$\bar{f}(\mathbf{x}) = *_{i=1}^{r}(\bar{f}(\mathbf{x}_{2i-1}) * T(\bar{f}(\mathbf{x}_{2i}))). \tag{5.13}$$

where the right hand side is well-defined by the induction hypothesis. Similarly define $\bar{f}(\mathbf{x})$ if $\mathbf{x}$ is in the other union components. For any $\mathbf{x} \in \tilde{\mathfrak{X}}_\infty$, we have $T_A(\mathbf{x}) = \lfloor \mathbf{x} \rfloor \in \tilde{\mathfrak{X}}_\infty$, and by the definition of $\bar{f}$ in Eq.(5.13), we have

$$\bar{f}(\lfloor \mathbf{x} \rfloor) = P(\bar{f}(\mathbf{x})). \tag{5.14}$$

So $\bar{f}$ commutes with the RBNTD operators. Combining this equation with Eq. (5.13) we see that if $\mathbf{x} = \mathbf{x}_1 \cdots \mathbf{x}_b$ is the standard decomposition of $\mathbf{x}$, then

$$\bar{f}(\mathbf{x}) = \bar{f}(\mathbf{x}_1) * \cdots * \bar{f}(\mathbf{x}_b). \tag{5.15}$$

Note that this is the only possible way to define $\bar{f}(\mathbf{x})$ in order for $\bar{f}$ to be a RBNTD algebra homomorphism extending f.

It remains to prove that the map $\bar{f}$ defined in Eq. (5.13) is indeed an algebra homomorphism. For this we only need to check the multiplicity

$$\bar{f}(\mathbf{x} \diamond \mathbf{x}') = \bar{f}(\mathbf{x}) * \bar{f}(\mathbf{x}') \tag{5.16}$$

for all $\mathbf{x}, \mathbf{x}' \in \tilde{\mathfrak{X}}_\infty$. For this, we use induction on the sum of depths $n := d(\mathbf{x}) + d(\mathbf{x}')$. Then $n \geq 0$. When $n = 0$, we have $\mathbf{x}, \mathbf{x}' \in X$. Then Eq. (5.16) follows from the multiplicity of f. Assume the multiplicity holds for $\mathbf{x}, \mathbf{x}' \in \tilde{\mathfrak{X}}_\infty$ with $n \geq k$ and take

$\mathbf{x}, \mathbf{x}' \in \tilde{\mathfrak{X}}_\infty$ with $n = k+1$. Let $\mathbf{x} = \mathbf{x}_1 \cdots \mathbf{x}_b$ and $\mathbf{x}' = \mathbf{x}'_1 \cdots \mathbf{x}'_c$ be the standard decompositions. Since $n = k+1 \geq 1$, at least one of $\mathbf{x}_b$ and $\mathbf{x}'_c$ is in $\lfloor \tilde{\mathfrak{X}}_\infty \rfloor$. Then by Eq. (5.6) we have,

$$\bar{f}(\mathbf{x}_b \diamond \mathbf{x}'_1) = \begin{cases} \bar{f}(\mathbf{x}_b \mathbf{x}'_1), & \text{if } \mathbf{x}_b \in X, \mathbf{x}'_1 \in \lfloor \tilde{\mathfrak{X}}_\infty \rfloor, \\ \bar{f}(\mathbf{x}_b \mathbf{x}'_1), & \text{if } \mathbf{x}_b \in \lfloor \tilde{\mathfrak{X}}_\infty \rfloor, \mathbf{x}'_1 \in X, \\ \bar{f}(\lfloor \lfloor \overline{\mathbf{x}}_b \rfloor \diamond \overline{\mathbf{x}}'_1 \rfloor + \lfloor \overline{\mathbf{x}}_b \diamond \lfloor \overline{\mathbf{x}}'_1 \rfloor \rfloor + \lambda \lfloor \overline{\mathbf{x}}_b \diamond \overline{\mathbf{x}}'_1 \rfloor & \\ -\lfloor \lfloor \overline{\mathbf{x}}_b \diamond \overline{\mathbf{x}}'_1 \rfloor \rfloor - \lfloor (\overline{\mathbf{x}}_b \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}'_1 \rfloor), & \text{if } \mathbf{x}_b = \lfloor \overline{\mathbf{x}}_b \rfloor, \mathbf{x}'_1 = \lfloor \overline{\mathbf{x}}'_1 \rfloor \in \lfloor \tilde{\mathfrak{X}}_\infty \rfloor. \end{cases}$$

In the first two cases, the right hand side is $\bar{f}(\mathbf{x}_b) * \bar{f}(\mathbf{x}'_1)$ by the definition of $\bar{f}$. In the third case, we have, by Eq. (5.14), the induction hypothesis and the RBNTD relation of the operator P on T,

$$\begin{aligned}
&\bar{f}(\lfloor \lfloor \overline{\mathbf{x}}_b \rfloor \diamond \overline{\mathbf{x}}'_1 \rfloor + \lfloor \overline{\mathbf{x}}_b \diamond \lfloor \overline{\mathbf{x}}'_1 \rfloor \rfloor + \lambda \lfloor \overline{\mathbf{x}}_b \diamond \overline{\mathbf{x}}'_1 \rfloor - \lfloor \lfloor \overline{\mathbf{x}}_b \diamond \overline{\mathbf{x}}'_1 \rfloor \rfloor - \lfloor (\overline{\mathbf{x}}_b \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}'_1 \rfloor) \\
=&\bar{f}(\lfloor \lfloor \overline{\mathbf{x}}_b \rfloor \diamond \overline{\mathbf{x}}'_1 \rfloor) + \bar{f}(\lfloor \overline{\mathbf{x}}_b \diamond \lfloor \overline{\mathbf{x}}'_1 \rfloor \rfloor) + \lambda \bar{f}(\lfloor \overline{\mathbf{x}}_b \diamond \overline{\mathbf{x}}'_1 \rfloor) \\
&- \bar{f}(\lfloor \lfloor \overline{\mathbf{x}}_b \diamond \overline{\mathbf{x}}'_1 \rfloor \rfloor) - \bar{f}(\lfloor (\overline{\mathbf{x}}_b \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}'_1 \rfloor) \\
=&P(\bar{f}(\lfloor \overline{\mathbf{x}}_b \rfloor \diamond \overline{\mathbf{x}}'_1)) + P(\bar{f}(\overline{\mathbf{x}}_b \diamond \lfloor \overline{\mathbf{x}}'_1 \rfloor)) + \lambda P(\bar{f}(\overline{\mathbf{x}}_b \diamond \overline{\mathbf{x}}'_1 \rfloor) \\
&- P(\bar{f}(\lfloor \overline{\mathbf{x}}_b \diamond \overline{\mathbf{x}}'_1 \rfloor)) - P(\bar{f}((\overline{\mathbf{x}}_b \diamond \lfloor 1 \rfloor) \diamond \overline{\mathbf{x}}'_1) \\
=&P(\bar{f}(\lfloor \overline{\mathbf{x}}_b \rfloor) * \bar{f}(\overline{\mathbf{x}}'_1)) + P(\bar{f}(\overline{\mathbf{x}}_b) * \bar{f}(\lfloor \overline{\mathbf{x}}'_1 \rfloor)) \\
&+ \lambda P(\bar{f}(\overline{\mathbf{x}}_b) * P((\overline{\mathbf{x}}'_1)) - P(P(\bar{f}(\overline{\mathbf{x}}_b) * \bar{f}(\overline{\mathbf{x}}'_1))) - P(\bar{f}(\overline{\mathbf{x}}_b \diamond \lfloor 1 \rfloor) * P(\bar{f}\overline{\mathbf{x}}'_1) \\
=&P(P(\bar{f}(\overline{\mathbf{x}}_b)) * \bar{f}(\overline{\mathbf{x}}'_1)) + P(\bar{f}(\overline{\mathbf{x}}_b) * P(\bar{f}(\overline{\mathbf{x}}'_1))) \\
&+ \lambda P(\bar{f}(\overline{\mathbf{x}}_b) * P((\overline{\mathbf{x}}'_1)) - P(P((\bar{f}(\overline{\mathbf{x}}_b) * \bar{f}(\overline{\mathbf{x}}'_1))) - P(\bar{f}(\overline{\mathbf{x}}_b \diamond \lfloor 1 \rfloor) * P(\bar{f}\overline{\mathbf{x}}'_1)) \\
=&P(\bar{f}(\overline{\mathbf{x}}_b)) * P(\bar{f}(\overline{\mathbf{x}}'_1)) \\
=&\bar{f}(\lfloor \overline{\mathbf{x}}_b \rfloor) * \bar{f}(\lfloor \overline{\mathbf{x}}'_1 \rfloor) \\
=&\bar{f}(\mathbf{x}_b) * \bar{f}(\mathbf{x}'_1).
\end{aligned}$$

Therefore $\bar{f}(\mathbf{x}_b \diamond \mathbf{x}'_1) = \bar{f}(\mathbf{x}_b) * \bar{f}(\mathbf{x}'_1)$. Then

$$\begin{aligned}
\bar{f}(\mathbf{x} \diamond \mathbf{x}') &= \bar{f}(\mathbf{x}_1 \cdots \mathbf{x}_{b-1}(\mathbf{x}_b \diamond \mathbf{x}'_1)\mathbf{x}'_2 \cdots \mathbf{x}'_c) \\
&= \bar{f}(\mathbf{x}_1) * \cdots * \bar{f}(\mathbf{x}_{b-1}) * \bar{f}(\mathbf{x}_b \diamond \mathbf{x}'_1) * \bar{f}(\mathbf{x}'_2) \cdots \bar{f}(\mathbf{x}'_c) \\
&= \bar{f}(\mathbf{x}_1) * \cdots * \bar{f}(\mathbf{x}_{b-1}) * \bar{f}(\mathbf{x}_b) * \bar{f}(\mathbf{x}'_1) * \bar{f}(\mathbf{x}'_2) \cdots \bar{f}(\mathbf{x}'_c) \\
&= \bar{f}(\mathbf{x}) * \bar{f}(\mathbf{x}').
\end{aligned}$$

This proves our result.

6 RBNTD algebras and RBNTD-dendriform algebras

We first introduce the concepts of a dendriform algebra and a tridendriform algebra and recall some results from the links between a Rota-Baxter algebra and the dendriform and the tridendriform algebra [21]. We then determine the binary quadratic nonsymmetric algebra called the RBNTD-dendriform algebra that is compatible with the RBNTD-algebra.

In this section, **k** is a commutative unitary ring.

6.1 Dendriform algebras and tridendriform algebras

The concept of a dendriform algebra was introduced by Loday [28] in 1995 with motivation from algebraic K-theory.

Definition 6.1. ([28]) A **dendriform k-algebra** (previously also called a dendriform dialgebra) is a **k**-module D with two binary operations $\prec$ and $\succ$ that satisfy the following relations.

$$\begin{aligned}
(x \prec y) \prec z &= x \prec (y \prec z + y \succ z), \\
(x \succ y) \prec z &= x \succ (y \prec z), \quad x, y, z \in D, \\
(x \prec y + x \succ y) \succ z &= x \succ (y \succ z).
\end{aligned} \tag{6.1}$$

Dendriform algebras have been further studied with connections to several areas in mathematics and physics, including operads, homological algebra, Hopf algebra, Lie and Leibniz algebra, combinatorics, arithmetic and quantum field theory.

A few years later, Loday and Ronco defined the tridendriform algebra in their study [29] of polytopes and Koszul duality.

Definition 6.2. ([29]) A **tridendriform k-algebra** (previously also called a dendriform trialgebra) is a **k**-module T equipped with three binary operations $\prec, \succ$ and $\cdot$ that satisfy the following relations.

$$\begin{aligned}
(x \prec y) \prec z &= x \prec (y \star z), \\
(x \succ y) \prec z &= x \succ (y \prec z), \\
(x \star y) \succ z &= x \succ (y \succ z), \\
(x \succ y) \cdot z &= x \succ (y \cdot z), \quad x, y, z \in T, \\
(x \prec y) \cdot z &= x \cdot (y \succ z), \\
(x \cdot y) \prec z &= x \cdot (y \prec z), \\
(x \cdot y) \cdot z &= x \cdot (y \cdot z).
\end{aligned} \tag{6.2}$$

Here we have used the notation

$$x \star y := x \prec y + x \succ y + x \cdot y \tag{6.3}$$

Proposition 6.3. 1. Let $(D, \prec, \succ)$ be a dendriform algebra. The operation $\star := \star_D$ on D defined by

$$x \star y := x \prec y + x \succ y, \quad x, y \in D, \tag{6.4}$$

is associative.

2. Let $(T, \prec, \succ, \cdot)$ be a tridendriform algebra. The operation $\star := \star_T$ on T de-

fined by

$$x \star y := x \prec y + x \succ y + x \cdot y, \quad x, y \in T, \tag{6.5}$$

is associative.

Thus, a dendriform algebra and tridendriform algebra share the property that the sum of the binary operations $\star := \prec + \succ$ for a dendriform algebra or $\star := \prec + \succ + \cdot$ for a tridendriform algebra is associative. Such a property is called a "splitting the associativity".

Proof. We just prove Item 1. The proof of Item 2 is similar.

Adding the left hand sides of Eqs. (6.1) , we obtain

$$\begin{aligned}
&(x \prec y) \prec z + (x \succ y) \prec z + (x \prec y + x \succ y) \succ z \\
=&(x \prec y + x \succ y) \prec z + (x \prec y + x \succ y) \succ z \\
=&(x \star y) \prec z + (x \star y) \succ z \\
=&(x \star y) \star z.
\end{aligned}$$

Similarly, adding the right hand sides of these equations, we obtain $x \star (y \star z)$. Thus we have proved the associativity of $\star$. □

There is a link between Rota-Baxter algebras and dendriform algebras stated in the following theorem.

Theorem 6.4. 1. [1] A Rota-Baxter algebra (R, P) of weight zero defines a dendriform algebra $(R, \prec_P, \succ_P)$, where

$$x \prec_P y = xP(y), \; x \succ_P y = P(x)y, \; \forall x, y \in R. \tag{6.6}$$

2. [10] A Rota-Baxter algebra (R, P) of weight λ defines a tridendriform algebra $(R, \prec_P, \succ_P, \cdot_P)$, where

$$x \prec_P y = xP(y),\ \ x \succ_P y = P(x)y,\ \ x \cdot_P y = \lambda xy,\ \ \forall x, y \in R. \tag{6.7}$$

6.2 RBNTD-dendriform algebras

Suppose (T, P) is a RBNTD algebra and define binary operations [21, 23]:

$$x \prec_P y = xP(y), x \succ_P y = P(x)y, x \cdot_P y = \lambda xy, x \bullet_P y = -P(xy), x *_P y = -xP(1)y \tag{6.8}$$

Our question is, what quadratic nonsymmetric relations could $(T, \prec_P, \succ_P, \cdot_P, \bullet_P, *_P)$ satisfy? We recall some background on binary quadratic nonsymmetric operads in order to make the question precise. We then determine all the quadratic nonsymmetric relations that are consistent with the RBNTD operator. For details on binary quadratic nonsymmetric operads, see [21, 30].

Definition 6.5. Let $\mathbf{k}$ be a field.

1. A **graded vector space** is a sequence $\mathcal{P} := \bigoplus\limits_{n\geq 0} \mathcal{P}_n$ of $\mathbf{k}$-vector spaces $\mathcal{P}_n,\ \ n \geq 0$.

2. A **nonsymmetric (ns) operad** is a graded vector space $\mathcal{P} = \bigoplus\limits_{n\geq 0} \mathcal{P}_n$ equipped with **partial compositions**:

$$\circ_i := \circ_{m,n,i} : \mathcal{P}_m \otimes \mathcal{P}_n \longrightarrow \mathcal{P}_{m+n-1}, \quad 1 \leq i \leq m, \tag{6.9}$$

such that, for $\lambda \in \mathcal{P}_\ell, \mu \in \mathcal{P}_m$ and $\nu \in \mathcal{P}_n$, the following relations hold.

(i) (Sequential composition)

$$(\lambda \circ_i \mu) \circ_{i-1+j} \nu = \lambda \circ_i (\mu \circ_j \nu), \quad 1 \leq i \leq \ell, 1 \leq j \leq m.$$

(ii) (Parallel composition)

$$(\lambda \circ_i \mu) \circ_{k-1+m} \nu = (\lambda \circ_k \nu) \circ_i \mu, \quad 1 \leq i < k \leq \ell.$$

(iii) (Identity) There is an element $\mathrm{id} \in \mathcal{P}_1$ such that $\mathrm{id} \circ \mu = \mu$ and $\mu \circ \mathrm{id} = \mu$ for $\mu \in \mathcal{P}_n, n \geq 0$.

3. Let $\mathcal{P} = \bigoplus \mathcal{P}_k$ be a ns operad. A graded vector space $Q = \bigoplus_{k\geq 1} Q_k$ is called a **graded subspace** of $\mathcal{P}$ if $Q_k \leq \mathcal{P}_k$ (Q_k is a subspace of $\mathcal{P}_k$).

4. A graded subspace $Q = \bigoplus Q_k$ is called a **suboperad** $\mathcal{P} = \bigoplus \mathcal{P}_k$ if Q is closed under compositions. (Equivalently, $Q = \bigoplus Q_k$ with the restrictions of $\circ_i$ is also an operad.)

5. Let $\mathcal{B} \subseteq \mathcal{P}$. The suboperad $Op(\mathcal{B}) = Op_{\mathcal{P}}(\mathcal{B})$ of $\mathcal{P}$ generated by $\mathcal{B}$ is the smallest suboperad of $\mathcal{P}$ containing $\mathcal{B}$.

6. A suboperad Q of $\mathcal{P}$ is called an **operad ideal** of $\mathcal{P}$ if

$$Q_m \circ_i \mathcal{P}_n \subseteq Q_{m+n-1}$$
$$\mathcal{P}_m \circ_i Q_n \subseteq Q_{m+n-1}$$

(Recall: A subring H of R is an ideal if HR⊆ H, RH ⊆H)

7. An **operad ideal** Q of $\mathcal{P}$ is called to be generated by $\mathcal{B} \subseteq \mathcal{P}$ if Q is the smallest operad ideal of $\mathcal{P}$ containing $\mathcal{B}$, denoted by $Op\ Id(\mathcal{B})$.

8. $\mathcal{P} = \bigoplus \mathcal{P}_k$ is called **binary** if $\mathcal{P} = Op_{\mathcal{P}}(\mathcal{P}_2)$. ($\mathcal{P}$ is generated by $\mathcal{P}_2$), i.e. if $\mathcal{P}_1 = \mathbf{k}.\mathrm{id}$ and $\mathcal{P}_n,\ n \geq 3$ are induced from $\mathcal{P}_2$ by composition.

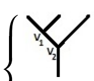
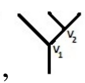
9. Let V be a vector space. A binary operad $\mathcal{F}(V) = \bigoplus\limits_{k\geq 1} \mathcal{F}_k(V)$ is called a **free (binary) operad** on V (with i: V $\to \mathcal{F}_2$) if for any binary operad $Q = \bigoplus\limits_{k\geq 1} Q_k$ and any linear map f:V $\to Q_2$, there exists a unique morphism $\bar{f} : \mathcal{F}(V) \to Q$

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "f"'] & \mathcal{F}(V) \arrow[d, "\bar{f}"] \\ & Q \end{tikzcd}$$

Then in particular, for the free binary operad, we have

$$\mathcal{F}_3 = (\mathcal{F}_2 \circ_1 \mathcal{F}_2) \oplus (\mathcal{F}_2 \circ_2 \mathcal{F}_2), \tag{6.10}$$

which can be identified with $\mathcal{F}_2^{\otimes 2} \oplus \mathcal{F}_2^{\otimes 2}$. The free binary operad $\mathcal{F}(V)$ can be constructed by planar binary trees with vertices decorated by the elements in V. Here, $\mathcal{F}_k(V)$ consists of such trees with k leaves.

For example, $\mathcal{F}_3(V) = K \Bigg\{$  ,  $\Bigg| V_1, V_2 \in V \Bigg\}$.

10. A binary ns operad $\mathcal{P}$ is called **quadratic** if all relations among the binary operations in $\mathcal{F}_2$ are derived from $\mathcal{F}_3$. Thus a binary, quadratic, ns operad is determined by a pair (V, R) where $V = \mathcal{F}_2$, called the **space of generators**, and R is a subspace of $V^{\otimes 2} \oplus V^{\otimes 2}$, called the **space of relations.** So we can denote $\mathcal{P} = \mathcal{F}(V)/(R)$. In other words, a binary operad $\mathcal{P} = \mathcal{F}(V)/(R)$ is called quadratic if $R \subseteq \mathcal{F}_3(V)$.

So a typical element of $V^{\otimes 2}$ is of the form $\sum\limits_{i=1}^{k} \odot_i^{(1)} \otimes \odot_i^{(2)}$ with $\odot_i^{(1)}, \odot_i^{(2)} \in V, 1 \leq$

$i \leq k$.

Thus a typical element of $V^{\otimes 2} \oplus V^{\otimes 2}$ is of the form

$$\left(\sum_{i=1}^{k} \odot_i^{(1)} \otimes \odot_i^{(2)}, \sum_{j=1}^{m} \odot_j^{(3)} \otimes \odot_j^{(4)}\right), \quad \odot_i^{(1)}, \odot_i^{(2)}, \odot_j^{(3)}, \odot_j^{(4)} \in V, 1 \leq i \leq k, 1 \leq j \leq m, \ k, m \geq 1.$$

For a given binary quadratic ns operad $\mathcal{P} = \mathcal{P}(V)/(R)$, a **k**-vector space A is called a **$\mathcal{P}$-algebra** if A has binary operations (indexed by) V and if, for

$$(\sum_{i=1}^{k} \odot_i^{(1)} \otimes \odot_i^{(2)}, \sum_{j=1}^{m} \odot_j^{(3)} \otimes \odot_j^{(4)}) \in R \subseteq V^{\otimes 2} \oplus V^{\otimes 2}$$

with $\odot_i^{(1)}, \odot_i^{(2)}, \odot_j^{(3)}, \odot_j^{(4)} \in V$, $1 \leq i \leq k$, $1 \leq j \leq m$, we have

$$\sum_{i=1}^{k} (x \odot_i^{(1)} y) \odot_i^{(2)} z = \sum_{j=1}^{m} x \odot_j^{(3)} (y \odot_j^{(4)} z), \quad \forall\ x, y, z \in A. \tag{6.11}$$

Theorem 6.6. Let $V = \mathbf{k}\{\prec, \succ, \cdot, \bullet, *\}$ be the vector space with basis $\{\prec, \succ, , \cdot, \bullet, *\}$ and let $\mathcal{P} = \mathcal{P}(V)/(R)$ be a binary quadratic ns operad. The following statements are equivalent.

1. For every RBNTD algebra (T, P), the hextuple $(T, \prec_P, \succ_P, \cdot_P, \bullet_P, *_P)$ is a $\mathcal{P}$-algebra.

2. The relation space R of $\mathcal{P}$ is contained in the subspace of $V^{\otimes 2} \oplus V^{\otimes 2}$ spanned by

$$(\prec \otimes \prec, \prec \otimes \star), \quad (\succ \otimes \star, \succ \otimes \succ), \quad (\prec \otimes \cdot, \cdot \otimes \succ), \tag{6.12}$$

$$(\prec \otimes \bullet, \bullet \otimes \succ), \quad (\prec \otimes *, * \otimes \succ), \quad (\succ \otimes \prec, \succ \otimes \prec),$$

$$(\succ \otimes \cdot, \succ \otimes \cdot), \quad (\succ \otimes *, \succ \otimes *), \quad (\cdot \otimes \prec, \cdot \otimes \prec),$$

$$(\cdot \otimes \cdot, \cdot \otimes \cdot), \quad (\cdot \otimes *, \cdot \otimes *), \quad (\bullet \otimes *, \bullet \otimes *),$$

$$(*\otimes \prec, *\otimes \prec), \quad (* \otimes \cdot, * \otimes \cdot), \quad (* \otimes *, * \otimes *), \quad (* \otimes \cdot, - \cdot \otimes \bullet),$$

$$(\succ \otimes \bullet + \bullet \otimes \prec + \bullet \otimes \bullet - \cdot \otimes \bullet + * \otimes \bullet, \succ \otimes \bullet + \bullet \otimes \prec + \bullet \otimes \bullet - \bullet \otimes \cdot + \bullet \otimes *)$$

where $\star = \prec + \succ + \cdot + \bullet + *$. More precisely, any $\mathcal{P}$-algebra A satisfies the relations

$$(x \prec y) \prec z = x \prec (y \star z), \quad (x \star y) \succ z = x \succ (y \succ z), \quad (x \prec y) \cdot z = x \cdot (y \succ z),$$

$$(x \prec y) \bullet z = x \bullet (y \succ z), \quad (x \prec y) * z = x * (y \succ z), \quad (x \succ y) \prec z = x \succ (y \prec z),$$

$$(x \succ y) \cdot z = x \succ (y \cdot z), \quad (x \succ y) * z = x \succ (y * z), \quad (x \cdot y) \prec z = x \cdot (y \prec z),$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \quad (x \cdot y) * z = x \cdot (y * z), \quad (x \bullet y) * z = (x \bullet y) \cdot z,$$

$$(x * y) \prec z = x *_(y \prec z), \quad (x * y) \cdot z = x * (y \cdot z), \quad (x * y) * z = x * (y * z),$$

$$x * (y \cdot z) = -x \cdot (y \bullet z), \quad \forall x, y, z \in A \tag{6.13}$$

$$(x \succ y) \bullet z + (x \bullet y) \prec z + (x \bullet y) \bullet z - (x \cdot y) \bullet z + (x * y) \bullet z$$

$$= x \succ (y \bullet z) + x \bullet (y \prec z) + x \bullet (y \bullet z) - x \bullet (y \cdot z) + x \bullet (y * z)$$

Note that the relations of the dendriform in Eq. (6.1) and the tridendriform in Eq. (6.2) is contained in the space spanned by the relations in Eq. (6.12). We call $\mathcal{P}$ defined by the relations in Eq. 6.12 the **RBNTD-dendriform operad** and call a hextuple $(T, \prec, \succ, \cdot, \bullet, *)$ satisfying Eq. (6.13) an **RBNTD-dendriform algebra**. Let **RBNTDD** denote the category of RBNTD-dendriform algebras and **RBNTDA** be the category of RBNTD algebras.

Then we have the following immediate corollary of Theorem 6.6.

Corollary 6.7. 1. There is a natural functor

$$\mathcal{F} : \mathbf{RBNTDA} \to \mathbf{RBNTDD}, \quad (T, P) \mapsto (T, \prec_P, \succ_P, \cdot_P, \bullet_P, *_P). \qquad (6.14)$$

2. In a RBNTD-dendriform algebra $(T, \prec_P, \succ_P, \cdot_P, \bullet_P, *_P)$, the operation

$$\star := \prec + \succ + \cdot + \bullet + *$$

is associative.

Proof. The proof of 2 is the same as the for Prop. 6.3.

Thus, a RBNTD-dendriform algebra gives a five part splitting of the associativity.

□

6.3 The proof of Theorem 6.6

With $V = \mathbf{k}\{\prec, \succ, \cdot, \bullet, *\}$, we have

$$V^{\otimes 2} \oplus V^{\otimes 2} = \bigoplus_{\odot_1, \odot_2, \odot_3, \odot_4 \in \{\prec, \succ, \cdot, \bullet, *\}} \mathbf{k}(\odot_1 \otimes \odot_2, \odot_3 \otimes \odot_4).$$

Thus any element r of $V^{\otimes 2} \oplus V^{\otimes 2}$ is of the form

$$\begin{aligned} r \ := \ & a_1(\prec \otimes \prec, 0) + a_2(\prec \otimes \succ, 0) + a_3(\prec \otimes \cdot, 0) + a_4(\prec \otimes \bullet, 0) + a_5(\prec \otimes *, 0) \\ & + b_1(\succ \otimes \prec, 0) + b_2(\succ \otimes \succ, 0) + b_3(\succ \otimes \cdot, 0) + b_4(\succ \otimes \bullet, 0) + b_5(\succ \otimes *, 0) \\ & + c_1(\cdot \otimes \prec, 0) + c_2(\cdot \otimes \succ, 0) + c_3(\cdot \otimes \cdot, 0) + c_4(\cdot \otimes \bullet, 0) + c_5(\cdot \otimes *, 0) \\ & + d_1(\bullet \otimes \prec, 0) + d_2(\bullet \otimes \succ, 0) + d_3(\bullet \otimes \cdot, 0) + d_4(\bullet \otimes \bullet, 0) + d_5(\bullet \otimes *, 0) \\ & + e_1(* \otimes \prec, 0) + e_2(* \otimes \succ, 0) + e_3(* \otimes \cdot, 0) + e_4(* \otimes \bullet, 0) + e_5(* \otimes *, 0) \end{aligned}$$

$$+f_1(0, \prec \otimes \prec) + f_2(0, \prec \otimes \succ) + f_3(0, \prec \otimes \cdot) + f_4(0, \prec \otimes \bullet) + f_5(0, \prec \otimes *)$$

$$+g_1(0, \succ \otimes \prec) + g_3(0, \succ \otimes \succ) + g_3(0, \succ \otimes \cdot) + g_4(0, \succ \otimes \bullet) + g_5(0, \succ \otimes *)$$

$$+h_1(0, \cdot \otimes \prec) + h_2(0, \cdot \otimes \succ) + h_3(0, \cdot \otimes \cdot) + h_4(0, \cdot \otimes \bullet) + h_5(0, \cdot \otimes *)$$

$$+i_1(0, \bullet \otimes \prec) + i_2(0, \bullet \otimes \succ) + i_3(0, \bullet \otimes \cdot) + i_4(0, \bullet \otimes \bullet) + i_5(0, \bullet \otimes *)$$

$$+j_1(0, * \otimes \prec) + j_2(0, * \otimes \succ) + j_3(0, * \otimes \cdot) + j_4(0, * \otimes \bullet) + j_5(0, * \otimes *)$$

where the coefficients are in **k**.

(1 $\Rightarrow$ 2) Let $\mathcal{P} = \mathcal{P}(V)/(R)$ be an operad satisfying the condition in Item 1. Let r be in R expressed in the above form. Then for any RBNTD algebra (T, P), the hextuple $(T, \prec_P, \succ_P, \cdot_P, \bullet_P, *_P)$ is a $\mathcal{P}$-algebra. Thus

$$a_1(x \prec_P y) \prec_P z + a_2(x \prec_P y) \succ_P z + a_3(x \prec_P y) \cdot_P z + a_4(x \prec_P y) \bullet_P z + a_5(x \prec_P y) *_P z$$

$$+b_1(x \succ_P y) \prec_P z + b_2(x \succ_P y) \succ_P z + b_3(x \succ_P y) \cdot_P z + b_4(x \succ_P y) \bullet_P z + b_5(x \succ_P y) *_P z$$

$$+c_1(x \cdot_P y) \prec_P z + c_2(x \cdot_P y) \succ_P z + c_3(x \cdot_P y) \bullet_P z + c_4(x \cdot_P y) \bullet_P z + c_5(x \cdot_P y) *_P z$$

$$+d_1(x \bullet_P y) \prec_P z + d_2(x \bullet_P y) \succ_P z + d_3(x \bullet_P y) \cdot_P z + d_4(x \bullet_P y) \bullet_P z + d_5(x \bullet_P y) *_P z$$

$$+e_1(x *_P y) \prec_P z + e_2(x *_P y) \succ_P z + e_3(x *_P y) \cdot_P z + e_4(x *_P y) \bullet_P z + e_5(x *_P y) *_P z$$

$$+f_1 x \prec_P (y \prec_P z) + f_2 x \prec_P (y \succ_P z) + f_3 x \prec_P (y \cdot_P z) + f_4 x \prec_P (y \bullet_P z) + f_5 x \prec_P (y *_P z)$$

$$+g_1 x \succ_P (y \prec_P z) + g_2 x \succ_P (y \succ_P z) + g_3 x \succ_P (y \cdot_P z) + g_4 x \succ_P (y \bullet_P z) + g_5 x \succ_P (y *_P z)$$

$$+h_1 x \cdot_P (y \prec_P z) + h_2 x \cdot_P (y \succ_P z) + h_3 x \cdot_P (y \cdot_P z) + h_4 x \cdot_P (y \bullet_P z) + h_5 x \cdot_P (y *_P z)$$

$$+i_1 x \bullet_P (y \prec_P z) + i_2 x \bullet_P (y \succ_P z) + i_3 x \bullet_P (y \cdot_P z) + i_4 x \bullet_P (y \bullet_P z) + i_5 x \bullet_P (y *_P z)$$

$$+j_1 x *_P (y \prec_P z) + j_2 x *_P (y \succ_P z) + j_3 x *_P (y \cdot_P z) + j_4 x *_P (y \bullet_P z) + j_5 x *_P (y *_P z) = 0,$$

$\forall x, y, z \in N$.

Substituting the definitions of $\prec_P, \succ_P, \cdot_P, \bullet_P, *_P$ from Eq. (6.8),

$$x \prec_P y = xP(y), x \succ_P y = P(x)y, x \cdot_P y = \lambda xy, x \bullet_P y = -P(xy), x *_P y = -xP(1)y$$

we have

$$\begin{aligned}
&a_1 xP(y)P(z) + a_2 P(xP(y))z + \lambda xP(y)z - a_4 P(xP(y)z) - a_5 xP(y)P(1)z \\
&+b_1 P(x)yP(z) + b_2 P(P(x)y)z + \lambda b_3 (P(x)y)z - b_4 P(P(x)yz) - b_5 P(x)yP(1)z \\
&+\lambda c_1 xyP(z) + \lambda c_2 P(xy)z + \lambda^2 c_3 xyz - \lambda c_4 P(xyz) - \lambda c_5 xyP(1)z \\
&-d_1 P(xy)P(z) - d_2 P(P(xy))z - \lambda d_3 P(xy)z + d_4 P(P(xy)z) - d_5 P(xy)P(1)z \\
&+e_1 (xP(1)y)P(z) - e_2 P(xP(1)y)z - \lambda e_3 xP(1)yz + e_4 P(xP(1)yz) + e_5 (xP(1)y)P(1)z \\
&+f_1 xP(yP(z)) + f_2 xP(P(y)z) + \lambda f_3 xP(yz) - f_4 xP(P(yz)) - f_5 xP(yP(1)z) \\
&+g_1 P(x)yP(z) + g_2 P(x)P(y)z + \lambda g_3 P(x)yz - g_4 P(x)P(yz) - g_5 P(x)(yP(1)z) \\
&+\lambda h_1 xyP(z) + \lambda h_2 xP(y)z + \lambda^2 h_3 xyz - \lambda h_4 xP(yz) - \lambda h_5 x(yP(1)z) \\
&-i_1 P(xyP(z)) - i_2 P(xP(y)z) - \lambda i_3 P(xyz) + i_4 P(xP(yz)) + i_5 P(x(yP(1)z)) \\
&-j_1 xP(1)yP(z) - j_2 xP(1)P(y)z - \lambda j_3 xP(1)yz + j_4 xP(1)P(yz) + j_5 xP(1)(yP(1)z) = 0.
\end{aligned}$$

Since P is a RBNTD operator:

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy) - P(P(xy)) - P(xP(1)y).$$

We futher have,

$$\begin{aligned}
&a_1 xP(yP(z)) + a_1 xP(P(y)z) + \lambda a_1 xP(yz) - a_1 xP^2(yz) - a_1 xP(yP(1)z) \\
&+a_2 P(xP(y))z + \lambda a_3 xP(y)z - a_4 P(xP(y)z) - \lambda a_5 xP(y)z \\
&+b_1 P(x)yP(z) + b_2 P(P(x)y)z + \lambda b_3 (P(x)y)z - b_4 P(P(x)yz) - b_5 P(x)yP(1)z
\end{aligned}$$

$$+\lambda c_1 xyP(z) + \lambda c_2 P(xy)z + \lambda^2 c_3 xyz - \lambda c_4 P(xyz) - \lambda c_5 xyP(1)z$$

$$-d_1 P(xyP(z)) - d_1 P(P(xy)z) - \lambda d_1 P(xyz) + d_1 P^2(xyz) + d_1 P(xyP(1)z)$$

$$-d_2 P(P(xy))z - \lambda d_3 P(xy)z + d_4 P(P(xy)z) - \lambda d_5 P(xy)z$$

$$+e_1(xP(1)y)P(z) - e_2 P(xP(1)y)z - \lambda e_3 xP(1)yz + e_4 P(xP(1)yz) + e_5(xP(1)y)P(1)z$$

$$+f_1 xP(yP(z)) + f_2 xP(P(y)z) + \lambda f_3 xP(yz) - f_4 xP(P(yz)) - f_5 xP(yP(1)z)$$

$$+g_1 P(x)yP(z) + g_2 P(xP(y))z + g_2 P(P(x)y)z + \lambda g_2 P(xy)z - g_2 P^2(xy)z$$

$$-g_2 P(xP(1)y)z + \lambda g_3 P(x)yz - g_4 P(xP(yz)) - g_4 P(P(x)yz) - \lambda g_4 P(xyz)$$

$$+g_4 P^2(xyz) + g_4 P(xP(1)yz) - g_5 P(x)(yP(1)z)$$

$$+\lambda h_1 xyP(z) + \lambda h_2 xP(y)z + \lambda^2 h_3 xyz - \lambda h_4 xP(yz) - \lambda h_5 x(yP(1)z)$$

$$-i_1 P(xyP(z)) - i_2 P(xP(y)z) - \lambda i_3 P(xyz) + i_4 P(xP(yz)) + i_5 P(x(yP(1)z))$$

$$-j_1 xP(1)yP(z) - \lambda j_2 xP(y)z - \lambda j_3 xP(1)yz + \lambda j_4 xP(yz) + j_5 xP(1)(yP(1)z) = 0.$$

Collecting similar terms, we obtain

$$(a_1 + f_1)xP(yP(z)) + (a_1 + f_2)xP(P(y)z) + (a_1 + f_3)\lambda xP(yz) - (a_1 + f_4)xP(P(yz))$$

$$-(a_1 + f_5)xP(yP(1)z) + (a_2 + g_2)P(xP(y))z + (a_3 + h_2)\lambda xP(y)z - (a_4 + i_2)P(xP(y)z)$$

$$-(a_5 + j_2)\lambda xP(y)z + (b_1 + g_1)P(x)yP(z)$$

$$+(b_2 + g_2)P(P(x)y)z + (b_3 + g_3)\lambda P(x)yz - (b_4 + g_4)P(P(x)yz) - (b_5 + g_5)P(x)yP(1)z$$

$$-(c_1 + h_1)\lambda xyP(z) + (c_2 + g_2)\lambda P(xy)z + (c_3 + h_3)\lambda^2(xyz)$$

$$-(c_4 + d_1)\lambda P(xyz) - (c_5 + h_5)\lambda xyP(1)z - (d_1 + i_1)P(xyP(z)) - (h_4 - j_4)\lambda xP(yz)$$

$$-(d_1 + d_4)P(P(xy)z) + (d_1 + g_4)P^2(xyz) + (d_1 + i_5)P(xyP(1)z) - (d_2 + g_2)P^2(xy)z$$

$$-(d_3 + d_5)\lambda P(xy)z - (e_1 + j_1)xP(1)yP(z) - (e_2 + g_2)P(xP(1)y)z - (e_3 + j_3)\lambda xP(1)yz$$

$$+(e_4 + g_4)P(xP(1)yz) + (e_5 + j_5)xP(1)yP(1)z - (g_4 - i_4)P(xP(yz)) - (g_4 + i_3)\lambda P(xyz) = 0.$$

Now we take the special case when (T, P) is the free RBNTD algebra $(F_{\mathrm{T}}^{NC}(T(M)), P_{T(M)})$ defined in Corollary 5.7 for our choice of $M = \mathbf{k}\{x, y, z\}$ and $P_{T(M)}(u) = \lfloor u \rfloor$. Then the above equation is just

$$
\begin{aligned}
&(a_1 + f_1)x\lfloor y\lfloor z\rfloor\rfloor + (a_1 + f_2)x\lfloor\lfloor y\rfloor z\rfloor + (a_1 + f_3)\lambda x\lfloor yz\rfloor - (a_1 + f_4)x\lfloor\lfloor yz\rfloor\rfloor \\
&-(a_1 + f_5)x\lfloor y\lfloor 1\rfloor z\rfloor + (a_2 + g_2)\lfloor x\lfloor y\rfloor\rfloor z + (a_3 + h_2)\lambda x\lfloor y\rfloor z - (a_4 + i_2)\lfloor x\lfloor y\rfloor z\rfloor \\
&-(a_5 + j_2)\lambda x\lfloor y\rfloor z + (b_1 + g_1)\lfloor x\rfloor y\lfloor z\rfloor \\
&+(b_2 + g_2)\lfloor\lfloor x\rfloor y\rfloor z + (b_3 + g_3)\lambda\lfloor x\rfloor yz - (b_4 + g_4)\lfloor\lfloor x\rfloor yz\rfloor - (b_5 + g_5)\lfloor x\rfloor y\lfloor 1\rfloor z \\
&-(c_1 + h_1)\lambda xy\lfloor z\rfloor + (c_2 + g_2)\lambda\lfloor xy\rfloor z + (c_3 + h_3)\lambda^2(xyz) \\
&-(c_4 + d_1)\lambda\lfloor xyz\rfloor - (c_5 + h_5)\lambda xy\lfloor 1\rfloor z - (d_1 + i_1)\lfloor xy\lfloor z\rfloor\rfloor - (h_4 - j_4)\lambda x\lfloor yz\rfloor \\
&-(d_1 + d_4)\lfloor\lfloor xy\rfloor z\rfloor + (d_1 + g_4)\lfloor\lfloor xyz\rfloor\rfloor + (d_1 + i_5)\lfloor xy\lfloor 1\rfloor z\rfloor - (d_2 + g_2)\lfloor\lfloor xy\rfloor\rfloor z \\
&-(d_3 + d_5)\lambda\lfloor xy\rfloor z - (e_1 + j_1)x\lfloor 1\rfloor y\lfloor z\rfloor - (e_2 + g_2)\lfloor x\lfloor 1\rfloor y\rfloor z - (e_3 + j_3)\lambda x\lfloor 1\rfloor yz \\
&+(e_4 + g_4)\lfloor x\lfloor 1\rfloor yz\rfloor + (e_5 + j_5)x\lfloor 1\rfloor y\lfloor 1\rfloor z - (g_4 - i_4)\lfloor x\lfloor yz\rfloor\rfloor - (g_4 + i_3)\lambda\lfloor xyz\rfloor = 0.
\end{aligned}
$$

Note that the set of elements

$$x\lfloor y\lfloor z\rfloor\rfloor, x\lfloor\lfloor y\rfloor z\rfloor, x\lfloor yz\rfloor, x\lfloor\lfloor yz\rfloor\rfloor, x\lfloor y\lfloor 1\rfloor z\rfloor, \lfloor x\lfloor y\rfloor\rfloor z, x\lfloor y\rfloor z, \lfloor x\lfloor y\rfloor z\rfloor, \lfloor x\rfloor y\lfloor z\rfloor$$

$\lfloor\lfloor x\rfloor y\rfloor z, \lfloor x\rfloor yz, \lfloor\lfloor x\rfloor yz\rfloor, \lfloor x\rfloor y\lfloor 1\rfloor z, xy\lfloor z\rfloor, \lfloor xy\rfloor z, xyz, \lfloor xyz\rfloor, xy\lfloor 1\rfloor z, \lfloor xy\lfloor z\rfloor\rfloor, x\lfloor yz\rfloor, \lfloor\lfloor xy\rfloor z\rfloor,$
$\lfloor\lfloor xyz\rfloor\rfloor, \lfloor xy\lfloor 1\rfloor z\rfloor, x\lfloor 1\rfloor y\lfloor z\rfloor, \lfloor\lfloor xy\rfloor\rfloor z, \lfloor x\lfloor 1\rfloor y\rfloor z, x\lfloor 1\rfloor yz, \lfloor x\lfloor 1\rfloor yz\rfloor, x\lfloor 1\rfloor y\lfloor 1\rfloor z, \lfloor x\lfloor yz\rfloor\rfloor$

is a subset of the basis $\tilde{\mathfrak{X}}_\infty$ of the free RBNTD algebra $F_{\mathrm{T}}^{NC}(T(M))$ and hence is linearly independent. Thus the coefficients must be zero, that is,

$$a_1 = -f_1 = -f_2 = -f_3 = -f_4 = -f_5,$$

$$
\begin{aligned}
&a_2 = b_2 = c_2 = d_2 = e_2 = -g_2,\\
&a_3 = -h_2,\\
&a_4 = -i_2,\\
&a_5 = -j_2,\\
&b_1 = -g_1,\\
&b_3 = -g_3,\\
&b_5 = -g_5,\\
&c_1 = -h_1,\\
&c_3 = -h_3,\\
&c_5 = -h_5,\\
&d_3 = -d_5,\\
&e_1 = -j_1,\\
&e_3 = -j_3,\\
&e_5 = -j_5,\\
&h_4 = j_4,\\
&b_4 = -c_4 = d_1 = d_4 = e_4 = -g_4 = -i_1 = i_3 = -i_4 = -i_5.
\end{aligned}
$$

Substituting these equations into the general relation r, we find that the any relation r that can be satisfied by $\prec_P, \succ_P, \cdot_P, \bullet_P, *_P$ for all RBNTD algebras (T, P) is of the form

$$
\begin{aligned}
r \;\; = \;\; & a_1\Big((x \prec y) \prec z - x \prec (y \prec z) - x \prec (y \succ z) - x \prec (y \cdot z) - x \prec (y \bullet z) - x \prec (y * z)\Big)\\
& + a_2\Big((x \prec y) \succ z + (x \succ y) \succ z + (x \bullet y) \succ z - (x \cdot y) \succ z + (x * y) \succ z - x \succ (y \succ z)\Big)
\end{aligned}
$$

$$+a_3\big((x \prec y) \cdot z - x \cdot (y \succ z)\big)$$
$$+a_4\big((x \prec y) \bullet z - x \bullet (y \prec z)\big)$$
$$+a_5\big((x \prec y) * z - x * (y \succ z)\big)$$
$$+b_1\big((x \succ y) \prec z - x \succ (y \prec z)\big)$$
$$+b_3\big((x \succ y) \cdot z - x \succ (y \cdot z)\big)$$
$$+b_5\big((x \succ y) * z - x \succ (y * z)\big)$$
$$+c_1\big((x \cdot y) \prec z - x \cdot (y \prec z)\big)$$
$$+c_3\big((x \cdot y) \cdot z - x \cdot (y \cdot z)\big)$$
$$+c_5\big((x \cdot y) * z - x \cdot (y * z)\big)$$
$$+d_3\big((x \bullet y) \cdot z - (x \bullet y) * z\big)$$
$$+e_1\big((x * y) \prec z - x * (y \prec z)\big)$$
$$+e_3\big((x * y) \cdot z - x * (y \cdot z)\big)$$
$$+e_5\big((x * y) * z - x * (y * z)\big)$$
$$+h_4\big(x \cdot (y \bullet z) + x * (y \bullet z)\big)$$
$$+b_4\big((x \succ y) \bullet z - (x \cdot y) \bullet z + (x \bullet y) \prec z + (x \bullet y) \bullet z + (x * y) \bullet z$$
$$-x \succ (y \bullet z) - x \bullet (y \prec z) + x \bullet (y \cdot z) - x \bullet (y \bullet z) - x \bullet (y * z)\big),$$

where $a_1, a_2, a_3, a_4, a_5, b_1, b_3, b_4, b_5, c_1, c_3, c_5, d_3, e_1, e_3, e_5, h_4 \in \mathbf{k}$ can be arbitrary. Thus r is in the subspace prescribed in Item 2, as needed.

$(2 \Rightarrow 1)$ We check directly that all the relations in Eq. (6.13) are satisfied by $(T, \prec_P, \succ_P, \cdot_P, \bullet_P, *_P)$ for every RBNTD algebra (T, P).

1. To check the first relation in Eq. (6.12), we have

$$\begin{aligned}(x \prec_P y) \prec_P z &= xP(y)P(z)\\ &= xP(yP(z)) + xP(P(y)z) + \lambda xP(yz)\\ &\quad -xP^2(yz) - xyP(1)z\\ &= x \prec_P (y \prec_P z) + x \prec_P (y \succ_P z) + x \prec_P (y \cdot_P z)\\ &\quad +x \prec_P (y \bullet_P z) + x \prec_P (y *_P z),\end{aligned}$$

2. For the second relation in Eq. (6.12), we similarly have

$$\begin{aligned}x \succ_P (y \succ_P z) &= P(x)P(y)z\\ &= P(xP(y))z + P(P(x)y)z + \lambda P(xy)z\\ &\quad -P^2(xy)z - xP(1)yz\\ &= (x \prec_P y) \succ_P z + (x \succ_P y) \succ_P z + (x \cdot_P y) \succ_P z\\ &\quad +(x \bullet_P y) \succ_P z + (x *_P y) \succ_P z,\end{aligned}$$

The rest of the relations except the last one are easy to verify:

3. $(x \prec_P y) \cdot_P z = \lambda xP(y)z = x \cdot_P (y \succ_P z)$

4. $(x \prec_P y) \bullet_P z = -P((xP(y))z) = -P(x(P(y)z)) = x \bullet_P (y \succ_P z)$.

5. $(x \prec_P y) *_P z = -xP(y)P(1)z = x *_P (y \succ_P z)$

6. $(x \succ_P y) \prec_P z = P(x)yP(z) = x \succ_P (y \prec_P z)$

7. $(x \succ_P y) \cdot_P z = -\lambda P(x)yz = x \succ_P (y \cdot_P z)$

8. $(x \succ_P y) *_P z = -P(x)yP(1)z = -P(x)(yP(1)z) = x \succ_P (y *_P z)$

9. $(x \cdot_P y) \prec_P z = \lambda xyP(z) = x \cdot_P (y \prec_P z)$

10. $(x \cdot_P y) \cdot_P z = \lambda(\lambda(xy)z) = \lambda(\lambda x(yz)) = x \cdot_P (y \cdot_P z)$

11. $(x \cdot_P y) *_P z = -\lambda xyP(1)z = x \cdot_P (y *_P z)$

12. $(x \bullet_P y) *_P z = -P(xy)P(1)z = -\lambda P(xy)z = (x \bullet_P y) \cdot_P z$

13. $(x *_P y) \prec_P z = xP(1)yP(z) = x *_P (y \prec_P z)$

14. $(x *_P y) \cdot_P z = \lambda(xP(1)yz) = x *_P (y \cdot_P z)$

15. $(x *_P y) *_P z = (xP(1)y)P(1)z = xP(1)(yP(1)z) = x *_P (y *_P z)$

16. $x *_P (y \cdot_P z) = xP(1)P(yz) = \lambda xP(yz) = -x \cdot_P (y \bullet_P z)$

17. For the last relation in Eq. (6.12), we check that the left hand side is

$$
\begin{aligned}
& (x \succ_P y) \bullet_P z + (x \bullet_P y) \prec_P z + (x \bullet_P y) \bullet_P z - (x \cdot_P y) \bullet_P z + (x *_P y) \bullet_P z \\
= \; & -P((P(x)y)z) - P(xy)P(z) + P(P(xy)z) + \lambda P(xyz) + P(xP(1)yz) \\
= \; & -P(P(x)yz) - P(xyP(z)) - P(P(xy)z) + P^2(xyz) - \lambda P(xyz) + P(xyP(1)z) \\
& +P(P(xy)z) + \lambda P(xyz) + P(xP(1)yz) \\
= \; & -P(P(x)yz) - P(xyP(z)) + P^2(xyz) + P(xyP(1)z) + P(xP(1)yz),
\end{aligned}
$$

which agrees with the right hand side

$$
\begin{aligned}
& x \succ_P (y \bullet_P z) + x \bullet_P (y \prec_P z) + x \bullet_P (y \bullet_P z) - x \bullet_P (y \cdot_P z) + x \bullet_P (y *_P z) \\
= \; & -P(x)P(yz) - P(x(yP(z))) + P(xP(yz)) + P(x(\lambda yz)) + P(x(yP(1)z)) \\
= \; & -P(xP(yz)) - P(P(x)yz) + P^2(xyz) - \lambda P(xyz) + P(xP(1)yz) - P(x(yP(z))) \\
& +P(xP(yz)) + \lambda P(xyz) + P(x(yP(1)z)) \\
= \; & -P(P(x)yz) + P^2(xyz) - P(xyP(z)) + P(xP(1)yz) + P(x(yP(1)z)).
\end{aligned}
$$

Thus, if the relation space R of an operad $\mathcal{P} = \mathcal{P}(V)/(R)$ is contained in the subspace spanned by the vectors in Eq. (6.12), then the corresponding relations are linear combinations of the equations in Eq. (6.13) and hence are satisfied by $(T, \prec_P, \succ_P, \cdot_P, \bullet_P, *_P)$ for each RBNTD algebra (T, P). Therefore $(T, \prec_P, \succ_P, \cdot_P, \bullet_P, *_P)$ is a $\mathcal{P}$-algebra. This completes the proof of Theorem 6.6

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