DELIGNE PAIRINGS AND DISCRIMINANTS OF ALGEBRAIC VARIETIES

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Abstract

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Let V be a finite dimensional complex vector space, V^* its dual, and let $X \subset \mathbb{P}(V)$ be a smooth projective variety of dimension n and degree $d \geq 2$. For a generic $n-$tuple of hyperplanes $(H_1, ..., H_n) \in \mathbb{P}(V^*)^n$, the intersection $X \cap H_1 \cap \cdots \cap H_n$ consists of d distinct points. We define the "discriminant of X" to be the set D_X of n-tuples for which the set-theoretic intersection is not equal to d points. Then $D_X \subset \mathbb{P}(V^*)^n$ is a hypersurface and the set of defining polynomials, which is a one-dimensional vector space, is called the "discriminant line". We show that this line is canonically isomorphic to the Deligne pairing $\langle KL^n, ..., L\rangle$ where K is the canonical line bundle of X and $L \to X$ is the restriction of the hyperplane bundle. As a corollary, we obtain a generalization of Paul's formula [14] which relates the Mabuchi K-energy on the space of Bergman metrics to Δ_X, the "hyperdiscriminant of X".

To my father, without whom I would have never considered a doctoral degree, let alone one in Mathematics.

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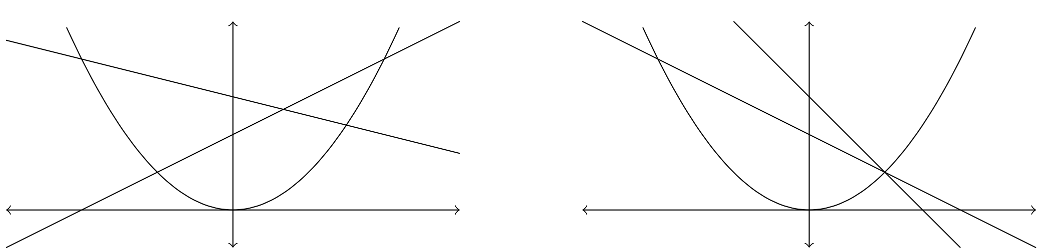
1 Introduction and Organization

Let V be a finite dimensional complex vector space, and let $\mathbb{P}(V)$ be the projectivization. Suppose further that $X \subset \mathbb{P}(V)$ is a smooth, projective manifold of dimension n and degree d. In his 1996 paper, Shouwu Zhang [22] demonstrated a canonical isomorphism between the Chow Bundle of a variety X and the Deligne Pairing [2] of n copies of the hyperplane bundle. Knudsen-Mumford [8] proved a determinant bundle expansion whose dominant term is the Chow Bundle, hence can be written as a Deligne pairing due to Zhang's result.

The work in Phong-Sturm [16] demonstrates that the subdominant term in [8] also has a Deligne Pairing interpretation, which we will call the Discriminant of X. The idea behind the Chow bundle and the Discriminant bundle can be seen via the following two toy examples:

Example 1.1. Consider $X = \{y = x^2\}$. This is a $1-$dimensional object of degree two, and it lies in a $2-$dimensional space. A hyperplane in this set up would be any equation of the form $H = \{ax + by = c\}$. For any generic choice of pairs of hyperplanes, $X \cap H_1 \cap H_2$ is empty. However, for an appropriate pair or configuration of hyperplanes, the intersection is non-empty. The set of such $n-$tuples of hyperplanes is called the Chow Variety of X. This definition will also hold for more general choices of X.

Figure 1: The intersection of X with two hyperplanes

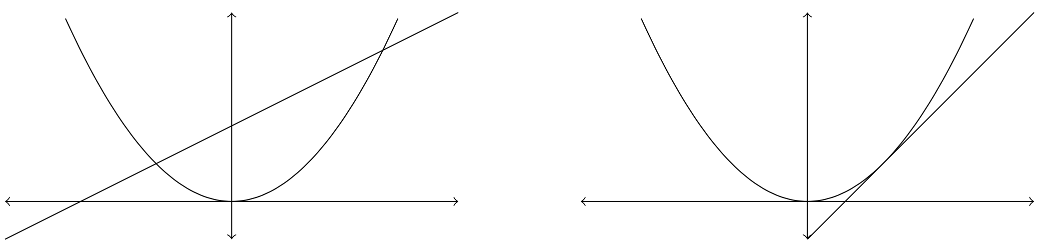


(a) Here, the hyperplanes do not have a common intersection with X

(b) Here, the hyperplanes both coincide with X precisely at the point $(1, 1)$

Example 1.2. Consider $X = \{y = x^2\}$. As before, this is a $1-$dimensional object of degree two, and it lies in a $2-$dimensional space. For any generic choice of hyperplane, $\#X \cap H = 2 = \deg(X)$. However, for an appropriate choice of hyperplane, the intersection will contain fewer than two points. In this example, this happens precisely when the hyperplane choice coincides with the tangent line of X at some point, and more generally, will happen when the set of hyperplanes nontrivially intersects with the embedded tangent space at a point. The set of these hyperplanes is what we will be calling the Discriminant of X. More generally, if V is three dimensional and $X \subseteq \mathbb{P}(V)$ is a smooth projective curves of degree at least two, let $X^\vee \subseteq \mathbb{P}(V^*)$, the dual curve, be the space of all tangent lines. In this setting, the the discriminant line is the one dimensional vector space spanned by any defining polynomial of $X^\vee$.

Figure 2: The intersection of X with one hyperplane



(a) Here, the hyperplane intersected X twice

(b) Here, the hyperplane is tangential to X

Remark 1.3. The choice to call the above set the Discriminant is motivated by the Number Theoretic definition of Discriminant (S. Lang [9]), where a generically finite to one map $\pi : X \to Y$ has a discriminant set $D \subset Y$ given by $D = \{d\pi = 0\}$ and different $\mathcal{D} \subset X$ given by $\mathcal{D}^{-1} = \{d\pi = 0\}$. To illustrate further, Let $X \subset \mathbb{P}(V)$ be a curve, and suppose $\Gamma = \{(x, H) \in X \times \mathbb{P}(V^*)\}$ with map $\pi : \Gamma \to \mathbb{P}(V^*)$. The "different" set would be the set of pairs (x, H) for which $d\pi = 0$ and the discriminant the image of the different in $\mathbb{P}(V^*)$. This set is closely related to an associated hypersurface defined in [5]. .

Example 1.4. Let $f : \mathbb{C} \to \mathbb{C}$ be the map given by $z \mapsto az^2 + bz + c$. This is generically a two to one map, and the points at which this map is not two to one correspond precisely to when $2az + b = 0$, so $\frac{b^2-4ac}{4a} = 0$.

Phong-Sturm ([16], [15]) further explored the ability of Deligne pairings to rewrite energy functionals and invariants. More recently, Biswas-Schumacher-Weng [1] obtained a Deligne Pairing representation for the Determinant bundle first cited in [8].

This thesis is organized as follows:

Section two will provide the necessary background, beginning with a very short exposition on Projective Manifolds, which can be found in [7], [10], and [6], and quickly expands to define concepts in the space of Kähler Metrics ([20], [17], [19]), Deligne Pairings ([2], [22]), the Mabuchi K-energy ([20], [11]), and the Futaki invariant ([18], [16]).

Section three will focus on a result by [22], including a proof of isometry between the Chow bundle and $\langle L, \cdots, L\rangle$. Section four provides an isomorphism between the discriminant variety and the Deligne Pairing $\langle KL^n, L, \cdots, L\rangle$. Finally, combining our result for the Discriminant of X with the work in [16], we obtain a formula for the Mabuchi K-energy using the Chow bundle and the Discriminant bundle Deligne Pairings. This result parallels a result by Sean Paul [14]. .

2 Background

2.1 Projective Manifolds

Suppose V is an $(N+1)$-dimensional complex vector space. The projectivization of V, denoted $\mathbb{P}(V)$, is the space formed by the set of one dimensional vector spaces in V. This is an N-dimensional complex manifold. Projective space is a Kähler manifold using the Fubini-Study metric, given by:

$$h_{ij} = \frac{|z|^2\delta_{ij} - \bar{z}_i z_j}{(|z|^2)^2}, \;\; \omega_{FS} = i\partial\bar{\partial}\log(|z|^2). \tag{2.1}$$

Example 2.1. Suppose $V = \mathbb{C}^{N+1}$. Then $\mathbb{P}^N$ is the set of lines through the origin in $\mathbb{C}^{N+1}$. Alternatively,

$$\mathbb{P}^N = (\mathbb{C}^{N+1} \backslash \{0\})/\sim$$

where $(z_0, \cdots, z_N) \sim (z_0^*, \cdots, z_N^*)$ if there exists $\lambda \neq 0$ so that $z = \lambda z^*$. For the coordinate maps to $\mathbb{C}^{N+1}$, we use the open cover $U_i = \{(z_0, \cdots, z_N) : z_i \neq 0\}$. Since $z_i \neq 0$ in U_i, we may divide by z_i and write U_i as the set $U_i = \{(z_0, \cdots, z_N) : z_i = 1\}$. Then $\phi_i : (z_0, \cdots, z_{i-1}, 1, z_{i+1}, \cdots, z_N) \mapsto (z_0, \cdots, z_{i-1}, z_{i+1} \cdots, z_N)$.

Example 2.2. For $\mathbb{P}^2$, the Fubini-Study metric on U_0 is:

$$h = \begin{bmatrix} 1 + |z_2|^2 & \bar{z}_1 z_2 \\ z_1 \bar{z}_2 & 1 + |z_1|^2 \end{bmatrix}$$

and the Fubini-Study form is $\omega_{FS} = i\partial\bar{\partial}\log(1 + |z_1|^2 + |z_2|)^2$.

It is convenient to write tangent bundle $T\mathbb{P}(V)$ in the following manner: for V a $\mathbb{C}-$vector space, the tangent space at any point $z \in V$ is canonically $T_z V = V$. Let I be the derivative of the canonical map $V \to \mathbb{P}(V)$. More precisely, for any point

$z \neq 0$, consider the map:

$$I_z : V = T_z V \to T_{[z]}\mathbb{P}(V), \tag{2.2}$$

where $[z]$ is the line in V containing z. Consider a path $z + tx \in T_v V$, where $t \in \mathbf{R}$. If $x = \alpha z$ for some nonzero α, then $[z] = [z + tx]$, hence the kernel of I_z is precisely $[z]$ and $I_z : V/z \to T_{[z]}\mathbb{P}(V)$ is an isomorphism. Note, $I_z(x) \neq I_{\alpha z}(x)$. Instead $I_z(x) = I_{\alpha v}(\alpha x)$ for any nonzero α, To remove this dependence on choice of $z \in [z]$, let $\lambda \in z^*$. Then $\lambda(v)I_v(x) = I_{\alpha v}(x)$. Hence, we have

$$T_v\mathbb{P}(V) = (V/v) \otimes O_x(1) \tag{2.3}$$

or, alternately,

$$T_v\mathbb{P}(V)^* = (V/v)^* \otimes O_v(-1). \tag{2.4}$$

Example 2.3. The tangent bundle $T\mathbb{P}^N$ is given by

$$T\mathbb{P}^N = \{([x], v) \in \mathbb{P}^N \times \mathbb{C}^{N+1}\}/\sim \tag{2.5}$$

where $([z], v) \sim ([z^*], v^*)$ if and only if $[z] = [z^*]$ and $v^* - \frac{z^*}{z}v \in [z]$. In this case, we are abusing notation a bit, and considering $[z]$ as both a point in $\mathbb{P}^N$ and as a line in $\mathbb{C}^{N+1}$.

We will focus on the following vector bundles on $\mathbb{P}(V)$:

1. The tautological bundle: $\mathcal{O}(-1) = \{([z], x) \in \mathbb{P}(V) \times V \,:\, x \in [z]\}$.

2. The hyperplane bundle: $\mathcal{O}(1) = \{([z], \lambda) : [z] \in \mathbb{P}(V), \lambda \in z^*\}$. Since $\lambda \in V^*$, it is a degree 1 function in $N + 1$ variables, hence defines a hyperplane in $\mathbb{P}(V)$.

3. The bundle $\mathcal{O}(-n)$ and $\mathcal{O}(n)$, which can be written as the $n-$fold tensor product of the tautological bundle and hyperplane bundle, respectively.

4. The canonical bundle $K_{\mathbb{P}} = \bigwedge^N T\mathbb{P}$.

If we have a metric h on V, then $|\cdot|_h$ is a hermitian metric on V and we may construct a hermitian metric on $\mathcal{O}(1)$ as follows: if $[x] \in \mathbb{P}(V)$, then $z \in L_{[x]}^{-1} = [x]$, so $z \in V$. For $\lambda \in \mathcal{O}_x(1)$, define $|\lambda|_h$ so that the following holds:

$$|\lambda(z)| = |\lambda|_h |z|_h. \tag{2.6}$$

For V^* with dual metric h^* and $\Lambda \in V^*$, then

$$|\Lambda|_{h^*} = \sup_{0 \neq v \in V} \frac{|\Lambda(z)|}{|z|_h} = \sup_{|z|_h = 1} |\Lambda(z)| = |\lambda(z_0)| \tag{2.7}$$

where z_0 is any vector in V such that $z_0 \perp H = \ker(\Lambda)$ and $|z_0| = 1$. Thus

$$|\Lambda(v)| \;\leq\; |\Lambda|_{h^*} |v|_h. \tag{2.8}$$

Since this metric h^* is a metric on V^*, this gives us a metric on $\mathcal{O}(1) \to \mathbb{P}(V^*)$.

We will use the following standard terminology: a projective variety X is a subspace of $\mathbb{P}(V)$ which can be realized as a zero set of a finite collection of homogeneous polynomials $\{f_1, \cdots, f_k\}$. In particular, a degree one homogeneous polynomial defines a co-dimension one subspace called a hyperplane. For a generic choice of $k \leq n$ hyperplanes $H_i \in \mathbb{P}(V)$, their intersection defines a $k-$co-dimensional linear subspace of $\mathbb{P}(V)$. An algebraic cycle of dimension n is a formal sum of $n-$dimensional closed subvarieties. The degree of X, $\deg(X)$, is defined as the number of points in the intersection of X with a generic co-dimension n hyperplane.

If $f : X \to Y$ is a smooth map of varieties, we define the relative canonical line bundle $K_{X/Y} \to X$ by

$$K_{X/Y} = K_X \otimes f^* K_Y^{-1}. \tag{2.9}$$

We shall often make use of the adjunction formula: if $X \subset Y$ is a smooth divisor of a smooth projective variety, then

$$K_X = K_Y|_X \otimes \mathcal{O}(X)|_X. \tag{2.10}$$

Finally, if $X \subseteq \mathbb{P}(V)$ is a smooth projective variety, and $\tilde{X} = \pi^{-1}X$ where π is the canonical map $\pi : V\backslash\{0\} \to \mathbb{P}(V)$, then for $\tilde{p} \in V$ and $p = \pi(\tilde{p})$, the embedded tangent space $\mathbf{E}T_pX \subseteq \mathbb{P}(V)$ is the linear subspace defined by $\pi(T_{\tilde{p}}\tilde{X})$.

2.2 Kähler Metrics

Definition 2.4. Suppose $L \to X$ is an $n-$dimensional Kähler manifold with Kähler form ω and Kähler metric h so that $\omega = \sqrt{-1}\partial\bar{\partial}\log h$. Define the space of Kähler potentials $\mathcal{H}$ by

$$\mathcal{H} = \{\phi \in C^\infty(X, \mathbf{R}) : \omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0\}. \tag{2.11}$$

For $\phi \in \mathcal{H}$, the associated metric on L is given by $h_\phi = he^{-\phi}$. In this way $\mathcal{H}$ may be identified with the space of smooth metrics on L with positive curvature.

Definition 2.5. Let $\phi \in \mathcal{H}$. The Aubin-Yau functional is given by

$$E_\omega(\phi) = \frac{1}{V}\sum_{j=0}^{n}\int_X \phi\omega_\phi^j \wedge \omega^{n-j} \tag{2.12}$$

Definition 2.6. Fix a Kähler form ω and $\phi \in \mathcal{H}$, and let ϕ_t be any path joining $\phi_0 = 0$ to ϕ. The Mabuchi K-energy is

$$\nu_\omega(\phi) = \frac{1}{V}\int_0^1 \left(\int_X \dot{\phi}_t(S - s(\omega_t))\omega_t^N\right) dt \tag{2.13}$$

where $s(\omega_t)$ is the scalar curvature of ω_t, S is the average scalar curvature, and

$\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\phi_t$.

This definition is independent of choice of path and we have the following equivalent formula for the K-energy [20] using an integration by parts argument:

$$\begin{aligned}\nu_\omega(\phi) = \frac{1}{V}\Bigg(\int_X -\log\frac{\omega_\phi^n}{\omega^n}\omega_\phi^n \\ + \sum_{i=0}^{n-1}\int_X \phi \mathrm{Ric}\,\omega\wedge\omega^i\wedge\omega^{n-1-i} - \frac{nS}{n+1}\sum_{i=0}^{n-1}\int_X \phi\omega^i\wedge\omega_\phi^{n-i}\Bigg).\end{aligned} \tag{2.14}$$

If $L \to X$ is a positive holomorphic line bundle with an associated Hermitian metric h, we may embed $X \hookrightarrow \mathbb{P}(\mathbb{C}^{N_k+1})$ using the Kodaira embedding theorem: for k sufficiently large, and an ordered basis $\underline{s} = (s_1, ..., s_{N_k+1})$ of $H^0(X, L^k)$, map $x \mapsto (s_1(x), \cdots, s_{N_k+1}(x))$ defines an imbedding $X \hookrightarrow \mathbb{P}^{N_k}$. For this embedding, the pullback of the Fubini-Study metric is given by:

$$h_{\underline{s}}^k = \frac{h^k}{\sum |s_i|_{h^k}^2}. \tag{2.15}$$

Definition 2.7. The set of metrics $\mathcal{H}_k = \{h_{\underline{s}} : \underline{s}$ is an ordered basis of $H^0(X, L^k)\}$ is the space of Bergman metrics of height k. The space of Bergman metrics is a symmetric space:

$$\mathcal{H}_k = \mathrm{GL}(N_k + 1, \mathbb{C})/U(N_k + 1, \mathbb{C}). \tag{2.16}$$

The potential $\phi_{\underline{s}}$ of $h_{\underline{s}}$ is the function for which $h_{\underline{s}} = he^{\phi_k}$, hence $\phi_{\underline{s}} = \frac{-1}{k}\log\sum|s_i|_{h^k}^2$

Remark 2.8. The space $\mathcal{H}$ is an infinitely dimensional smooth manifold, while $\mathcal{H}_k$ is a finite dimensional smooth manifold.

Remark 2.9. As $k \to \infty$, we have $\mathcal{H}_k \to \mathcal{H}$ in the following sense: For $\phi \in \mathcal{H}$, if we let $\phi_k = \phi_{\underline{s}_k}$ where $\underline{s}_k$ is an orthonormal basis of $H^0(X, L^k)$ (with respect to the natural Hilbert space inner product), then $\phi_k \to \phi$ as $k \to \infty$ in the C^∞ norm. This is a consequence of the well known Tian-Yau-Zelditch theorem [21].

Suppose X is a compact Kähler manifold, with Kähler form ω, and assume K_X^{-1} is ample. For a holomorphic vector field v on X, define the integral $F(\mathrm{v}, \omega)$ as $\int_X \mathrm{v}(f)\omega^N$, where f is the Ricci potential defined by $\sqrt{-1}\partial\bar{\partial}f = \mathrm{Ric}(\omega) - \omega$.

Theorem 2.10 (Futaki [4]). *Let X be a compact* Kähler *Fano manifold. The function $Fut(v) = F(\mathrm{v}, \omega)$ is independent of ω. Moreover*

$$\begin{aligned} Fut : hol(X) &\to \mathbb{C} \\ \mathrm{v} &\mapsto F(\mathrm{v}, \omega) \end{aligned} \tag{2.17}$$

is a Lie algebra character.

Remark 2.11. Futaki proved the invariance by the following: If ω and ω' are Kähler forms with $\omega' - \omega \in \mathcal{H}$, then $Fut_\omega(\mathrm{v}) = Fut_{\omega'}(\mathrm{v})$. This is shown by taking a path ϕ_t joining ω to ω' and noting the time derivative of Fut_{ω_t} is identically 0.

Remark 2.12. The K-energy and the Futaki invariant are related as follows:

$$\dot{\nu}_\omega(\phi_t) = Fut_{\omega_{\phi_t}}(\mathrm{v}), \tag{2.18}$$

for $\mathrm{v} = \nabla_t \dot{\phi}_t$ and f_t satisfying $\sqrt{-1}\partial\bar{\partial}f = \mathrm{Ric}(\omega_t) - \omega_t$.

2.3 Deligne Pairings

Deligne pairings provide a method to construct a line bundle over a base S from line bundles over a fiber space X. Suppose $X \to S$ is a flat morphism of schemes of relative dimension n, and suppose L_i are line bundles over the fiber space X. Then the Deligne pairing $\langle L_0, \cdots, L_n\rangle_{X/S}$ is a line bundle over S. The sections are given formally by $\langle s_0, \cdots, s_n\rangle$, where each s_i is a rational section of L_i and whose intersection of divisors is empty. The transition functions between sections are defined inductively by

$$\langle s_0, \cdots, f_j s_j, \cdots, s_n\rangle = \mathcal{N}_f[\cap_{i\neq j}\mathrm{div}(s_i)]\langle s_0, \cdots, s_j, \cdots, s_n\rangle \tag{2.19}$$

where $X_p \cap_{i\neq j} \mathrm{div}(s_i) = \sum n_k p_k$, the formal sum of zeros and poles, $\mathcal{N}_f[\cap_{i\neq j}\mathrm{div}(s_i)]$ is the product $\prod f(p_k)^{n_k}$, where the p_i are the zeros and poles in the common intersection, and n_k the multiplicities.

Example 2.13. For $\mathbb{P}^2$ and $L = \mathcal{O}(1)$, the Deligne Pairing $\langle L, L, L\rangle$ is a line whose sections are given by $\langle s_0, s_1, s_2\rangle$, where $s_i = a_i x + b_i y + c_i z$. If $s_0 = f s_0'$, then

$$\begin{aligned}\langle s_0, s_1, s_2\rangle &= \langle f s_0', s_1, s_2\rangle \\ &= \langle \frac{s_0}{s_0'} s_0', s_1, s_2\rangle \\ &= \frac{a_0(c_2b_3 - b_2c_3) + b_0(a_2c_3 - a_3c_2) + c_0(a_3b_2 - a_2b_3)}{a_0'(c_2b_3 - b_2c_3) + b_0'(a_2c_3 - a_3c_2) + c_0'(a_3b_2 - a_2b_3)}\langle s_0', s_1, s_2\rangle.\end{aligned}$$

Remark 2.14. The transition function defined in (2.19) seem to depend on choice of order, so it is not obvious that the pairing is well defined. The following example demonstrates that for curves this is not a concern (in higher dimensions the proof proceeds along similar lines).

Example 2.15. For X a Riemann surface and $L \to X$ a line bundle, the Deligne Pairing $\langle L, L\rangle$ has sections $\langle s_0, s_1\rangle$. Suppose $\langle f s_0, g s_1\rangle$ is another section. By (2.19), we see that

$$\begin{aligned}\langle f s_0, g s_1\rangle &= \prod \frac{g(\{f s_0 = 0\})}{g(\{f s_0 = \infty\})}\langle f s_0, s_1\rangle \\ &= \prod \frac{f(\{s_1 = 0\})}{f(\{s_1 = \infty\})} \prod \frac{g(\{f s_0 = 0\})}{g(\{f s_0 = \infty\})}\langle s_0, s_1\rangle,\end{aligned}$$

and that

$$\begin{aligned}\langle fs_0, gs_1\rangle &= \prod \frac{f(\{gs_1=0\})}{f(\{gs_1=\infty\})}\langle s_0, gs_1\rangle \\ &= \prod \frac{g(\{s_0=0\})}{g(\{s_0=\infty\})}\prod \frac{f(\{gs_1=0\})}{f(\{gs_1=\infty\})}\langle s_0, s_1\rangle.\end{aligned}$$

For this to be well defined, we need

$$\begin{gathered}\prod \frac{f(\{s_1=0\})}{f(\{s_1=\infty\})}\prod \frac{g(\{fs_0=0\})}{g(\{fs_0=\infty\})} = \prod \frac{g(\{s_0=0\})}{g(\{s_0=\infty\})}\prod \frac{f(\{gs_1=0\})}{f(\{gs_1=\infty\})} \\ \prod \frac{g(\{fs_0=0\})}{g(\{fs_0=\infty\})}\prod \frac{g(\{s_0=\infty\})}{g(\{s_0=0\})} = \prod \frac{f(\{gs_1=0\})}{f(\{gs_1=\infty\})}\prod \frac{f(\{s_1=\infty\})}{f(\{s_1=0\})} \\ \prod \frac{g(\{f=0\})}{g(\{f=\infty\})} = \prod \frac{f(\{g=0\})}{f(\{g=\infty\})}.\end{gathered} \tag{2.20}$$

Hence, $g(\mathrm{div}(f)) = f(\mathrm{div}(g))$. But this is just the statement of Weil's reciprocity [9], and thus the transition functions did not depend on choice of order.

Deligne pairings satisfy the following formulae:

2.3.1 Isomorphism Formulae

Suppose $\phi : X \to Y$ and $\psi : Y \to S$, are finite flat maps of integral schemes with $m = \dim(X/Y)$ and $n = \dim(Y/S)$. Suppose further $\mathcal{K}_i$ are line bundles on X and $\mathcal{L}_j$ are line bundles on Y. Then we have the following projection formulae given by pullback maps:

Definition 2.16. Projection Formula of n Pullbacks

The map given by $F : \langle k_0, ..., k_m, \phi^* l_1, ..., \phi^* l_n\rangle \mapsto \langle\langle k_0, ..., k_m\rangle, l_1, ..., l_n\rangle$ gives the following formula on n pullbacks:

$$\langle \mathcal{K}_0, ..., \mathcal{K}_m, \phi^*\mathcal{L}_1, ..., \phi^*\mathcal{L}_n\rangle_{X/S} = \langle\langle \mathcal{K}_0, ..., \mathcal{K}_m\rangle, \mathcal{L}_1, ..., \mathcal{L}_n\rangle_{Y/S}. \tag{2.21}$$

Definition 2.17. Projection Formula of $n+1$ Pullbacks

Suppose D is the number of points in $\mathrm{div}(\mathcal{K}_1)\cap\cdots\cap\mathrm{div}(\mathcal{K}_{\mathrm{m}})$ in a generic fiber. Then:

$$\langle \mathcal{K}_1, ..., \mathcal{K}_m, \phi^*\mathcal{L}_0, ..., \phi^*\mathcal{L}_n\rangle_{X/S} = \langle \mathcal{L}_0, \mathcal{L}_1, ..., \mathcal{L}_n\rangle^D_{Y/S}. \tag{2.22}$$

Definition 2.18. Projection Formula of $n+2$ Pullbacks

$$\langle \mathcal{K}_1, ..., \mathcal{K}_{m-1}, \phi^*\mathcal{L}_0, \cdots, \phi^*\mathcal{L}_{n+1}\rangle_{X/S} = \mathcal{O}_S. \tag{2.23}$$

Definition 2.19. Induction Formula

Suppose $\pi : X \to S$ and $\mathcal{L}_i$ are line bundles over X. Let l be a rational section of $\mathcal{L}_n$. Assume all components of $\mathrm{div}(l)$ are integral and flat over S. Then:

$$\langle \mathcal{L}_0, ..., \mathcal{L}_n\rangle_{X/S} \; = \; \langle \mathcal{L}_0, ..., \mathcal{L}_{n-1}\rangle_{\mathrm{div}(l)/S}. \tag{2.24}$$

To see this is true, fix a rational section l. The sections of $\langle \mathcal{L}_0, ..., \mathcal{L}_n\rangle_{X/S}$ over S are generated by the formal symbols $\langle l_0, ..., l_{n-1}, l\rangle_{X/S}$ and the isomorphism is given by the map $\langle l_0, ..., l_{n-1}, l\rangle_{X/S} \mapsto \langle l_0, ..., l_{n-1}\rangle_{\mathrm{div}(l)/S}$.

2.3.2 Metrics on Deligne Pairing

Suppose h_j is a smooth metric on $\mathcal{L}_j$. Deligne defines a metric $\langle h_0, ..., h_n\rangle$ on the line bundle $\langle \mathcal{L}_0, ..., \mathcal{L}_n\rangle$ inductively as follows:

$$\begin{aligned} \log \|\langle l_0, ..., l_n\rangle\| \; &= \; \log \|\langle l_0, ..., l_{n-1}\rangle\| \\ &+ \int_{X/S} \log |l_n| \, \omega_0 \wedge \cdots \wedge \omega_{n-1}, \end{aligned} \tag{2.25}$$

where $\omega_j = -\frac{i}{2\pi}\partial\bar\partial \log |l_j|^2$. Hence, we obtain the following isometry on (2.24):

$$\langle h_0, ..., h_n\rangle \; = \; \langle h_0, ..., h_{n-1}\rangle \exp\left(-\int_{\mathcal{X}/S} \log |l_n|\, \omega_0 \wedge \cdots \wedge \omega_{n-1}\right). \tag{2.26}$$

Remark 2.20. Suppose $\phi_0, ..., \phi_n$ are smooth functions on X. Applying formula (2.24) gives us that

$$\langle h_0 e^{-\phi_0}, \cdots, h_n e^{-\phi_n}\rangle = \langle h_0, \cdots, h_n\rangle \exp(-E(\phi_0, ..., \phi_n)) \tag{2.27}$$

where

$$E(\phi_0, ..., \phi_n) = \sum_{j=0}^{n} \int_{X/S} \phi_j \,(\bigwedge_{k<j} \omega_{\phi_k}) \wedge (\bigwedge_{k>j} \omega_k) \tag{2.28}$$

and $\omega_{\phi_k} = \omega_k + \frac{i}{2\pi}\partial\bar\partial\phi_k$. In particular, if $\mathcal{L}_0 = \cdots = \mathcal{L}_n$ and $h_0 = \cdots = h_n$, then setting $E_{\mathcal{X}}(\phi) = E(\phi, ..., \phi)$ we obtain the Aubin-Yau functional:

$$E_X(\phi) = \sum_{j=0}^{n} \int_{\mathcal{X}/S} \phi_j \, \omega_\phi^j \, \omega^{n-j}. \tag{2.29}$$

Let Y be a projective manifold of dimension m, $L \to Y$ an ample line bundle, and d a positive integer. Let $V = c_1(L)^m$ and $\mathcal{N} \to \mathbb{P}(H^0(Y, L^d))$ be the hyperplane line bundle. Let h be a positively curved hermitian metric on L with curvature $\omega = -i\partial\bar\partial \log h > 0$. Define a norm $\mathrm{D}(h)$ on the vector space $H^0(Y, L^d)$ as follows: if $0 \neq f \in H^0(Y, L^d)$ then

$$\log \|f\|^2_{D(h)} = \frac{1}{\mathrm{Vol}(Y)} \int_Y \log |f|^2_{h^d}\, \omega^m \tag{2.30}$$

In particular, $\mathrm{D}(h)$ makes $\mathcal{N}$ into a hermitian line bundle.

Remark 2.21. : The metric $\mathrm{D}(h)$ is not equal to $\mathrm{Hilb}(h)$, but $\mathrm{D}(h) = e^{\psi}\mathrm{Hilb}(h)$ for some bounded smooth function ψ on $H^0(Y, L^d)$, where $\psi(\lambda v) = \psi(v)$ for $\lambda > 0$.

Let $0 \neq f \in H^0(X, L^d)$ and assume $Z = \{f = 0\} \subseteq Y$ is smooth.

The map

$$I_f : \langle L, ..., L\rangle_Z \;\to\; \langle L, ..., L^d\rangle_Y \tag{2.31}$$

given by $\langle s_0, ..., s_{m-1}\rangle_Z \mapsto \langle s_0, ..., s_{m-1}, f\rangle_Y$ is an isomorphism. Since $I_{\alpha f} = \alpha^c I_f$,

$$\mathcal{N}_{[f]}^{-V} = \langle L, ..., L^d\rangle_Y \otimes \langle L, ..., L\rangle_Z^{-1}. \tag{2.32}$$

Equivalently, there is a canonical isomorphism

$$J_{[f]} : \mathcal{N}_{[f]}^{V} \to \langle L, \cdots, L\rangle_Z \otimes \langle L, \cdots, L\rangle_Y^{-d}. \tag{2.33}$$

Using the $D(h)$ metric on the left and the Deligne metric on the right, this is an isometry. Moreover, J is $G \subseteq GL(H^0(Y, L^d))$ equivariant, where $G = Aut(Y, L)$.

Let $L \to Y$ be a holomorphic line bundle on a projective manifold Y and h a smooth metric on L. Suppose $G \subseteq \mathrm{Aut}(Y, L)$ is a semi-simple Lie group, and write $\sigma^* h = he^{-\phi_\sigma}$ for $\sigma \in G$.

Corollary 2.22. *Let $f \in H^0(Y, L^d)$ be such that $Z = \{f = 0\}$ is a smooth submanifold. If E_Z is the Aubin-Yau functional on Z then for all $\sigma \in G$ we have*

$$E_Z(\phi^\sigma) = \frac{1}{V} \log\left(\frac{\|f^\sigma\|^2_{D(h)}}{\|f\|^2_{D(h)}}\right). \tag{2.34}$$

2.3.3 Mabuchi K-Energy

Suppose $X \to S$ is a flat morphism of schemes of relative dimension n, and suppose L_i are line bundles over the space X, and $K_{X/S}$, the relative canonical bundle, is well defined. Let h be a positively curved metric on L with curvature ω. Define h_K^{-1} as a metric on K^{-1} by $h_K^{-1} = \omega^n$. Using this set up, define the following $\mathbb{Q}$ line bundles as in [16].

1. The Mabuchi Line Bundle

$$\mathcal{M}_h = \langle K, L, \cdots, L \rangle^{\frac{1}{c_1(L)^n}} \langle L, \cdots, L \rangle^{\frac{-\mu}{c_1(L)^n}}, \tag{2.35}$$

   where $c_1(L)^n$ is computed on a generic fiber, and $\mu \in \mathbb{Q}$ is uniquely determined by requiring that the metric is scale invariant. It follows from the definitions that $nc_1(K)c_1(L)^{n-1} - \mu(n+1)c_1(L)^n = 0$ so

$$\mu = \frac{n}{n+1} \frac{c_1(K)c_1(L)^{n-1}}{c_1(L)^n}. \tag{2.36}$$

2. The Futaki Line Bundle

$$\mathcal{F}_h = \langle K^{-1}, \cdots, K^{-1} \rangle \tag{2.37}$$

   .

3. The Aubin-Yau Bundle

$$\mathcal{A}_h = \langle K, L, \cdots, L \rangle^{\frac{1}{c_1(L)^n}}. \tag{2.38}$$

Remark 2.23. Phong-Sturm [16] proved the Mabuchi energy, Futaki invariant, and Aubin-Yau functionals arise as the change of metrics of these respectively named bundles. More precisely,

$$\mathcal{M}_{he^\phi} = \mathcal{M}_h \otimes \mathcal{O}(\nu_\omega(\phi)/2), \;\; \mathcal{F}_{he^\phi} = \mathcal{F}_h \otimes \mathcal{O}(Fut_\omega(\phi)/2), \;\; \mathcal{A}_{he^\phi} = \mathcal{A}_h \otimes \mathcal{O}(E_\omega(\phi)/2) \tag{2.39}$$

3 The Chow Line

Suppose V is an $N+1$ dimensional $\mathbb{C}-$vector space, $X \subseteq \mathbb{P}(V)$ a degree d, n-dimensional subspace. Let $\mathbb{P} = \mathbb{P}(V^*)^{n+1} = \mathbb{P}_0 \times \cdots \times \mathbb{P}_n$, where $\mathbb{P}_i = \mathbb{P}(V)^*$ for $i = 0, \cdots, n$, and denote $\dim(\mathbb{P}) = N(n+1) = m$. Recall, the degree of X is given by the number of intersection points with a generic co-dimension n hyperplane. The following definitions can be found in the book of Gelfand-Kapranov-Zelevinsky [5].

Definition 3.1. The Chow variety is the set $\mathbf{G}(n, d, N+1)$ of all degree d and dimension n algebraic cycles.

Definition 3.2. The associated hypersurface of X is the set

$$\mathcal{Z}(X) = \{(H_0, \cdots, H_n) \in \mathbb{P} : H_0 \cap \cdots \cap H_n \cap X \neq \emptyset\} \tag{3.1}$$

Definition 3.3. The j^{th} associated hypersurface of X is the set

$$\mathcal{Z}_j(X) = \{(H_0, \cdots, H_{n-j} \subset \mathbb{P}(V^*)^{n-j+1} : \dim(L \cap \mathbf{E}T_x X) \geq j \text{ for some } x \in L \cap X\} \tag{3.2}$$

where $L = H_0 \cap \cdots \cap H_{n-j}$.

Remark 3.4. The 0^{th} associated hypersurface $\mathcal{Z}_0(X)$ is precisely the associated hypersurface.

Definition 3.5. The Chow form of X is the defining polynomial for $\mathcal{Z}(X)$ [5]. By abuse of notation, we identify $C(X) = \mathcal{Z}(X)$.

Example 3.6. Consider the projective variety $X = \{xy - z^2\}$ and a hyperplane $H = ax + by + cz$. Then $H \in D(X)$ if the intersection $\{xy - z^2\} \cap \{ax + by + cz\}$

contains fewer than 2 points. But,

$$\{xy - z^2\} \cap \{ax + by + cz\} \iff x\left(\frac{cz - ax}{b}\right) = -z^2$$

$$ax^2 - (cz)x - (bz^2) = 0.$$

Hence, the intersection contains one point when $c^2 - 4ab = 0$. On the other hand, $H \in \mathcal{Z}_1(X)$ if, at a point of intersection (x_0, y_0, z_0), the hypersurface H is also contained in the embedded tangent space. But the embedded tangent space at the point (x_0, y_0, z_0) is given by

$$y_0(x - x_0) + x_0(y - y_0) - 2z_0(z - z_0) = 0$$

$$y_0 x + x_0 y - 2z_0 z = 2x_0 y_0 - 2z_0^2,$$

and H intersects X at the point (x_0, y_0, z_0) when $y_0 = a$, $x_0 = b$, and $2z_0 = c$. Hence, we obtain $2x_0y_0 - 2z_0^2 = 0$, so $4x_0y_0 - 4z_0^2 = 4ab - c^2 = 0$.

Let $L \to \mathbb{P}(V)$ and $M \to \mathbb{P}(V^*)$ be the hyperplane bundles $\mathcal{O}_{\mathbb{P}(V)}(1)$ and $\mathcal{O}_{\mathbb{P}(V^*)}(1)$, respectively. The theorem of Zhang states the following:

Theorem 3.7 (Zhang)**.** *There is a canonical isomorphism*

$$\langle L, \cdots, L\rangle_{X/S} = C(X)$$

Proof of (3.7). We give a slightly simplified proof of Zhang's theorem.

3.1 Chow Bundle Isomorphism

Define a line bundle over $\mathbb{P}$ by

$$\mathcal{M} = \pi_1^* M \otimes \cdots \otimes \pi_{n+1}^* M = M_1 \otimes \cdots \otimes M_{n+1} \to \mathbb{P}$$

Figure 3: The Projection Diagram for the Chow Line Case

$$
\begin{tikzcd}
 & & X \times \mathbb{P}(\mathrm{V}^*)^{n+1} \arrow[dl, "\pi"'] \arrow[dr, "p_i"] & & \\
L \arrow[r] & X \arrow[dr] & & \mathbb{P}(\mathrm{V}^*) \arrow[dl] & M \arrow[l] \\
 & & * & &
\end{tikzcd}
$$

and let $\pi_X : X \times \mathbb{P} \to X$ and $\pi_{\mathbb{P}} : X \times \mathbb{P} \to \mathbb{P}$ be the projection maps. Consider the following Deligne Pairing:

$$\mathcal{B} = \langle \underbrace{\pi_{\mathbb{P}}^* \mathcal{M}, \cdots, \pi_{\mathbb{P}}^* \mathcal{M}}_{m}, \underbrace{\pi_X^* L \otimes \pi_{\mathbb{P}}^* M_1, \cdots, \pi_X^* L \otimes \pi_{\mathbb{P}}^* M_{n+1}}_{n+1} \rangle_{X \times \mathbb{P}/*} \tag{3.3}$$

where $*$ is a point. First, let

$$\Gamma = \{(x, H_1, \cdots., H_{n+1}) \in X \times \mathbb{P} : x \in H_1 \cap \cdots \cap H_{n+1}\} \tag{3.4}$$

and let $Z = \pi_{\mathbb{P}}(\Gamma)$. Then $Z = C(X) \subseteq \mathbb{P}$ is the Chow hypersurface of X. The line $\langle \mathcal{M}, \cdots, \mathcal{M} \rangle_{Z/*} = \langle \mathcal{M}, \cdots, \mathcal{M} \rangle_Z$ is called the Chow line. We may construct Γ using the following sections s_i of $\pi_X^* L \otimes \pi_{\mathbb{P}}^* M_i$. The section $s_i(x, H) \in x^* \otimes H^* = \mathrm{Hom}(x \otimes H, \mathbb{C})$ is the restriction of the canonical paring $\mathrm{Hom}(V \otimes V^*, \mathbb{C})$, so

$$s_i(x, H)(z, \lambda) \;=\; \lambda(z) \tag{3.5}$$

for all $z \in x$ and $\lambda \in H$. Note that $s_i(x, H) = 0$ if and only if $x \in H$. Applying (2.24) $n + 1$ times we obtain:

$$\mathcal{B} = \langle \underbrace{\mathcal{M}, \mathcal{M}, \cdots, \mathcal{M}}_{m} \rangle_{Z/*} \tag{3.6}$$

On the other hand, expanding the last $n+1$ terms on (3.3) we have $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2$ where

$$\mathcal{B}_1 = \langle \underbrace{\pi_{\mathbb{P}}^*\mathcal{M}, \cdots \pi_{\mathbb{P}}^*\mathcal{M}}_{m}, \underbrace{\pi_X^* L, \cdots, \pi_X^* L}_{n+1} \rangle_{X\times\mathbb{P}} \tag{3.7}$$

and

$$\mathcal{B}_2 = \prod_{i=1}^{n+1} \langle \underbrace{\pi_{\mathbb{P}}^*\mathcal{M}, \cdots, \pi_{\mathbb{P}}^*\mathcal{M}}_{m}, \underbrace{\pi_X^* L \otimes \pi_{\mathbb{P}}^* M_1, \cdots, \pi_{\mathbb{P}}^* M_i, \cdots \pi_X^* L \otimes \pi_{\mathbb{P}}^* M_{n+1}}_{n+1} \rangle_{X\times\mathbb{P}} \tag{3.8}$$

Using (2.22), we find that

$$\mathcal{B}_1 = \langle L, \cdots, L \rangle_X^{\deg(\mathcal{M})} \tag{3.9}$$

Expanding $\mathcal{B}_2$ in the last $n+1$ entries and applying (2.23) we find the only nontrivial terms are of the form

$$\underbrace{\langle \pi_{\mathbb{P}}^* M_i, \pi_X^* L, \cdots \pi_X^* L \rangle}_{n+1}{}_{X\times\mathbb{P}} \tag{3.10}$$

when combined with the first m terms. Applying (2.22) again, we find:

$$\begin{aligned} \mathcal{B}_2 &= \prod_{i=1}^{n+1} \langle \underbrace{\mathcal{M}, \cdots, \mathcal{M}}_{m}, M_i \rangle_{\mathbb{P}}^d \\ &= \underbrace{\langle \mathcal{M}, \cdots, \mathcal{M} \rangle}_{m+1}{}_{\mathbb{P}}^d \end{aligned} \tag{3.11}$$

where $d = c_1(L)^n$. Since $\deg(\mathcal{M}) = 1$, we have

$$\langle L, \cdots, L \rangle_X \;=\; \langle \mathcal{M}, \cdots, \mathcal{M} \rangle_{C(X)} \otimes \langle \mathcal{M}, \cdots, \mathcal{M} \rangle_{\mathbb{P}}^{-d} \tag{3.12}$$

3.2 Chow Bundle Isometry

To prove the above isomorphism is an isometry, we need to define the spaces $\Gamma_j \in X \times \mathbb{P}$ as follows:

Let $\Gamma_0 = X \times \mathbb{P}$. For $j \geq 1$, we define:

$$\Gamma'_j = \{(x, H_1, \cdots., H_j) \in X \times \mathbb{P}_1 \times \cdots \times \mathbb{P}_j : x \in H_1 \cap \cdots \cap H_j\} \tag{3.13}$$

and $\Gamma_j = \Gamma'_j \times \mathbb{P}_{j+1} \times \cdots \times \mathbb{P}_n$ by

$$\begin{aligned} \Gamma_j &= \{(x, H_1, \cdots., H_{n+1}) \in X \times \mathbb{P} : x \in H_1 \cap \cdots \cap H_j \ \} \\ &= \{z \in \Gamma_{j-1} : s_j(z) = 0\}, \end{aligned} \tag{3.14}$$

so $\dim(\Gamma_j) = m + n - j$. Using this construction, $\Gamma_{n+1} = \Gamma$.

We consider the Delinge Pairing

$$\mathcal{B}_j = \langle \underbrace{\pi_{\mathbb{P}}^*\mathcal{M}, , \cdots \pi_{\mathbb{P}}^*\mathcal{M}}_{m}, \underbrace{\pi_X^* L \otimes \pi_{\mathbb{P}}^* M_j, \cdots, \pi_X^* L \otimes \pi_{\mathbb{P}}^* M_{n+1}}_{n-j+1} \rangle_{\Gamma_j}. \tag{3.15}$$

The Deligne metric along the above pairing, combined with the induction formula (2.24) implies that the map $\mathcal{B}_{j-1} \to \mathcal{B}_j e^{-I_j}$ is an isometry where I_j is the integral

$$\begin{aligned} I_j &= \int_{\Gamma_{j-1}} \log |s_j|^2 c_1(\mathcal{M})^m \wedge c_1(L \otimes M_{j+1}) \wedge \cdots \wedge c_1(L \otimes M_{n+1}) \\ &= \int_{\Gamma'_{j-1} \times \mathbb{P}_j \times \cdots \times \mathbb{P}_{n+1}} \log |s_j(x, H_j)|^2 c_1(\mathcal{M})^m \wedge c_1(L \otimes M_{j+1}) \wedge \cdots \wedge c_1(L \otimes M_{n+1}) \end{aligned} \tag{3.16}$$

To compute I_j, we will use the following facts:

1. The above curvature of a direct product can be distributed as a sum of the

curvatures, so $c_1(\mathcal{M})^m = (c_1M_0+\cdots+c_1M_n)^m$ and $c_1(L\otimes M_i) = c_1(L)+c_1(M_i)$.

2. Suppose η_1, η_2 are volume forms on X_1, X_2, and let $f(x_1, x_2)$ be a function on $X_1 \times X_2$. Then,

$$\int_{X_1\times X_2} f(x_1,x_2)\,\pi_1^*\eta_1 \wedge \pi_2^*\eta_2 = \int_{X_2}\left(\int_{X_1} f(x_1,x_2)\eta_1(x_1)\right)\eta_2(x_2). \tag{3.17}$$

If $d = \dim X_1 \times X_2$ and $r + s = d$, then

$$\int_{X_1\times X_2} f(x_1,x_2)\pi_1^*\omega_1 \wedge \cdots \wedge \pi_1^*\omega_r \wedge \pi_2^*\theta_1 \wedge \cdots \wedge \pi_2^*\theta_s = 0. \tag{3.18}$$

In the case where $r = \dim(X_1)$ and $s = \dim(X_2)$, this can be written as

$$\int_{X_2}\left(\int_{X_1} f(x_1,x_2)\omega_1 \wedge \cdots \wedge \omega_r\right)\theta_1 \wedge \cdots \wedge \theta_s \tag{3.19}$$

3. If $\omega_1, \cdots, \omega_p$ are Kähler forms on X_1 with $p > \dim(X_1)$ then

$$\pi_1^*\omega_1 \wedge \cdots \wedge \pi_1^*\omega_p = 0 \text{ on } X_1 \times X_2 \tag{3.20}$$

Using facts (1) and (3),

$$\begin{aligned}
&\int_{\Gamma'_{j-1}\times\mathbb{P}_j\times\cdots\times\mathbb{P}_{n+1}} \log|s_j|^2 c_1(\mathcal{M})^m \wedge c_1(L\otimes M_j) \wedge \cdots \wedge c_1(L\otimes M_{n+1})\\
&= \int_{\Gamma'_{j-1}\times\mathbb{P}_j\times\cdots\times\mathbb{P}_n} \log|s_j|^2 c_1(\mathcal{M})^m \wedge c_1(L) \wedge \cdots \wedge c_1(L)\\
&= \int_{\Gamma'_{j-1}\times\mathbb{P}_j\times\cdots\times\mathbb{P}_n} \log|s_j|^2 (c_1M_1 + \cdots + c_1M_{n+1})^m \wedge c_1(L)^{n-(j-1)}\\
&= \int_{\Gamma'_{j-1}\times\mathbb{P}_j\times\cdots\times\mathbb{P}_n} \log|s_j|^2 C(c_1(M_1)^N \wedge \cdots \wedge c_1(M_{n+1})^N) \wedge c_1(L)^{n-(j-1)}
\end{aligned} \tag{3.21}$$

where the constant C is the coefficient from the expansion of $c_1(\mathcal{M})^m = (c_1M_0+$

$\cdots + c_1 M_n)^n$, hence $C = \binom{m}{N}\binom{m-N}{N}\cdots\binom{N}{N}$.

Using fact (2), we can separate the above integral as

$$\begin{aligned}
&\int_{\Gamma'_{j-1}\times\mathbb{P}_{j-1}\times\cdots\times\mathbb{P}_n} C\log|s_{j-1}|^2(c_1(M_0)^N\cdots c_1(M_n)^N)\wedge c_1(L)^{n-(j-1)} \\
&= \int_{\Gamma'_{j-1}\times\mathbb{P}_j\times\cdots\times\mathbb{P}_n}\int_{\mathbb{P}_{j-1}} C\log|s_{j-1}|^2 c_1(M_0)^N\cdots c_1(M_n)^N\wedge c_1(L)^{n-(j-1)} \\
&= \int_{\Gamma'_{j-1}\times\mathbb{P}_j\times\cdots\times\mathbb{P}_n}\left(\int_{\mathbb{P}_{j-1}} C\log|s_{j-1}|^2 c_1(M_{j-1})^N\right) \\
&\qquad\qquad (c_1(M_0)^N\cdots c_1(M_{j-2})^N\wedge c_1(M_j)^N\cdots c_1(M_n)^N)\wedge c_1(L)^{n-(j-1)} \\
&\left(= \int_{\Gamma_j\times\mathbb{P}_j} C\log|s_{j-1}|^2 c_1(L)^n\wedge c_1(M_j)^N\wedge\cdots\wedge c_1(M_n)^N\right)
\end{aligned} \tag{3.22}$$

To compute $\int_{\mathbb{P}_{j-1}} C\log|s_{j-1}|^2 c_1(M_{j-1})^N$, we define the section $s_{i-1}(x,H)\in x^*\otimes H_i^* = \mathrm{Hom}(x\otimes H_i,\mathbb{C})$ as the restriction $\mathrm{Hom}(V\otimes V^*,\mathbb{C})$, where for $z\in x$ and $\lambda_i\in H_i$,

$$s_i(x,H_i)(z,\lambda) = \lambda(z). \tag{3.23}$$

Choosing an isometry $V\to\mathbb{C}^N$ and $V^*\to\mathbb{C}^{N+1}$ the dual isometry, we may write $z\in\mathbb{C}^{N+1}$ and $\lambda\in\mathbb{C}^{N+1}$ and

$$|s_i|(x,H) = \frac{|<z,\lambda>|}{|z|\cdot|\lambda|} \tag{3.24}$$

where $<z,\lambda> = \sum_{j=0}^N z_j\bar{\lambda}_j$ is the standard euclidean inner product.

Thus

$$\int_{\mathbb{P}_{j-1}}\log|s_{j-1}|^2 c_1(M_{j-1})^N = \int_{\mathbb{P}^N}\log\left(\frac{|<z,\lambda>|}{|z|\cdot|\lambda|}\right)^2\omega_{FS}^N(\lambda), \tag{3.25}$$

where $\frac{|<z,\lambda>|}{|z|\cdot|\lambda|}$ is independent of z. Thus, the integral is independent of z.

4 Discriminant Line

Recall, V is an $(N+1)$ dimensional $\mathbb{C}-$vector space, $X \subseteq \mathbb{P}(V)$ a degree d, n dimensional subspace. Call $\mathbb{P} = \mathbb{P}_1 \times \cdots \times \mathbb{P}_n$ for $\mathbb{P}_i = \mathbb{P}(V^*)$ and denote $\dim(\mathbb{P}) = Nn = m$. As before, Let $L \to \mathbb{P}(V)$ and $M \to \mathbb{P}(V^*)$ be the hyperplane bundles $\mathcal{O}_{\mathbb{P}(V)}(1)$ and $\mathcal{O}_{\mathbb{P}(V^*)}(1)$, respectively, and let $K \to X$ be the canonical bundle.

Definition 4.1. Suppose $X \subseteq \mathbb{P}(V)$ is a degree d, n dimensional subspace. Define the Discriminant of X as the set

$$D(X) = \{(H_1, \cdots H_n) : \#(H_1, \cap \cdots, H_n \cap X) \neq d\}. \tag{4.1}$$

We claim this definition is equivalent to the first associated hypersurface $\mathcal{Z}_1(X)$. We will demonstrate $D(X) = \mathcal{Z}_1(X)$.

First, assume $(H_1, \cdots, H_n) \in D(X) \backslash \mathcal{Z}_1(X)$. Then the intersection $H_1, \cap \cdots, H_n$ is a line which is not contained in the embedded tangent space $\mathbf{E}T_x X$ for any $x \in H_1, \cap \cdots, H_n \cap X$ and hence is transversal to $\mathbf{E}T_x X$. So, each $x \in H_1, \cap \cdots, H_n \cap X$ has divisor multiplicity one, and $\#(X \cdot H_1 \cdot \cdots \cdot H_n) < d$, which is a contradiction.

Alternately, if $(H_1, \cdots, H_n) \in \mathcal{Z}_1(X) \backslash D(X)$, then $X \cap H_1 \cap \cdots \cap H_n$ contains d points. Fix x as the point for which $\dim(\mathrm{H_1 \cap \cdots \cap H_n \cap \mathbf{E}T_x X}) \geq 1$ and a small neighborhood B_x of x. Let λ_i be a section whose vanishing set is H_i and let s be any section which does not vanish at x. Hence, $\left(\frac{\lambda_i}{s}\right)(x)$ are holomorphic functions which vanish at x and they define a map $f : B_x \to \mathbb{C}^n$. This map has the following properties [3]:

1. $f^{-1}(0) = x$
2. The associated map Df has kernel $H_1 \cap \cdots \cap H_n \cap \mathbf{E}T_x X$
3. It induces a map $H^{2n}(B\backslash\{x\}, \mathbf{Z}) \to H^{2n}(\mathbb{C}^n\backslash\{0\}, \mathbf{Z})$, given by multiplication by the winding number k, where $k = \#\{\phi^{-1}(v)\}$ for any $v \neq 0 \in \mathbb{C}^n$ near the

origin. Alternately, $k = \#(B_x \cdot H_1 \cdot \cdots \cdot H_n)$.

By our assumption, $k = B_x \cdot H_1 \cdots H_n \geq 2$, so $\#(X \cdot H_1 \cdot \cdots \cdot H_n) \geq 2 + (d-1) = d+1$, which is a contradiction.

Figure 4: The Projection Diagram for the Discriminant Line Case

$$\begin{tikzcd}
 & L \arrow[d] & X \times \mathbb{P}(\mathrm{V}^*)^n \arrow[dl, "\pi"'] \arrow[dr, "p_i"] & & \\
K \arrow[r] & X \arrow[dr] & & \mathbb{P}(\mathrm{V}^*) \arrow[dl] & M \arrow[l] \\
 & & * & &
\end{tikzcd}$$

We pose the following theorem:

Original Theorem 4.2. *There is a canonical isomorphism*

$$\langle KL^n, L, \cdots, L \rangle = D(X)$$

.

Proof of (4.2). We construct the following Deligne Pairing:

$$\mathcal{B}' = \langle \underbrace{\pi_{\mathbb{P}}^* \mathcal{M}, \cdots, \pi_{\mathbb{P}}^* \mathcal{M}}_{m}, \pi_X^*(K \otimes L^n) \otimes \pi_{\mathbb{P}}^* \mathcal{M}, \underbrace{\pi_X^* L \otimes \pi_{\mathbb{P}}^* M_1, \cdots, \pi_X^* L \otimes \pi_{\mathbb{P}}^* M_n}_{n} \rangle_{X \times \mathbb{P}/*}. \tag{4.2}$$

This factors as $\mathcal{B}' = \mathcal{B}'_1 \otimes \mathcal{B}'_2$, where

$$\mathcal{B}'_1 = \langle \underbrace{\pi_{\mathbb{P}}^* \mathcal{M}, \cdots \pi_{\mathbb{P}}^* \mathcal{M}}_{m}, \pi_X^*(K \otimes L^n), \underbrace{\pi_X^* L \otimes \pi_{\mathbb{P}}^* M_1, \cdots, \pi_X^* L \otimes \pi_{\mathbb{P}}^* M_n}_{n} \rangle_{X \times \mathbb{P}/*} \tag{4.3}$$

and

$$\mathcal{B}'_2 = \langle \underbrace{\pi_{\mathbb{P}}^*\mathcal{M}, \cdots, \pi_{\mathbb{P}}^*\mathcal{M}}_{m}, \pi_{\mathbb{P}}^*\mathcal{M}, \underbrace{\pi_X^*L \otimes \pi_{\mathbb{P}}^*M_1, \cdots, \pi_X^*L \otimes \pi_{\mathbb{P}}^*M_n}_{n} \rangle_{X\times\mathbb{P}/*} \tag{4.4}$$

Applying (2.22) to each $\mathcal{B}'_1$ and $\mathcal{B}'_2$, we have

$$\mathcal{B}' = \langle \mathcal{M}, \cdots, \mathcal{M} \rangle_{\mathbb{P}}^d \otimes \langle K \otimes L^n, \underbrace{L, \cdots, L}_{n} \rangle_X, \tag{4.5}$$

where $d = (n+1)c_1(L)^n + c_1(K)c_1(L)^{n-1}$. On the other hand, define the space $\Gamma \subset X \times \mathbb{P}$ as follows:

$$\Gamma = \{(x, H_1, \cdots, H_n) \in X \times \mathbb{P} : x \in H_1 \cap \cdots \cap H_n)\}. \tag{4.6}$$

which we obtain by applying (3.5) a total of n times. Hence, we have

$$\mathcal{B}' = \langle \underbrace{\pi_{\mathbb{P}}^*\mathcal{M}, \cdots, \pi_{\mathbb{P}}^*\mathcal{M}}_{m}, \pi_X^*(K \otimes L^n) \otimes \pi_{\mathbb{P}}^*\mathcal{M} \rangle_{\Gamma/*}. \tag{4.7}$$

We need to define a section $s : \Gamma \to \pi_X^*(K \otimes L^n) \otimes \pi_{\mathbb{P}}^*\mathcal{M}$. Recall that

$$\begin{aligned} K \otimes L^n &= \wedge^n TX \otimes \mathcal{O}(n) \\ &= \wedge^n TX \otimes (\mathcal{O}(1))^{\otimes n} \\ &= \wedge^n TX \otimes ((\mathcal{O}(-1))^*)^{\otimes n} \end{aligned} \tag{4.8}$$

and that

$$\mathcal{M} = M_1 \otimes \cdots \otimes M_n. \tag{4.9}$$

Hence, at a point $(x, H_1, \cdots, H_n) \in \Gamma$, we have

$$\begin{aligned} s(x, H_1, \cdots, H_n) &\in \wedge^n T_x X \otimes ((\mathcal{O}_x(-1))^*)^{\otimes n} \otimes H_1^* \otimes \cdots \otimes H_n^* \\ &= \wedge^n T_x X \otimes ((\mathcal{O}_x(-1))^{\otimes n} H_1 \otimes \cdots \otimes H_n)^* \\ &= \wedge^n T_x X \otimes ((\mathcal{O}_x(-1)) \otimes H_1 \otimes \cdots \otimes \mathcal{O}_x(-1)) \otimes H_n)^*. \end{aligned} \tag{4.10}$$

By (2.4), we have $\mathcal{O}_x(-1)) \otimes H_i \subset T_X\mathbb{P}(V)^*$. So, for $\eta_i \in \mathcal{O}_x(-1)) \otimes H_i$, we have $\eta_i \in T_X\mathbb{P}(V^*)$ and $\eta_1 \wedge \cdots \wedge \eta_n \in \wedge^n T_X\mathbb{P}(V)^*$. Hence if $\omega \in \Lambda^n T_x X$ we have a canonical multilinear map $[\Lambda^n(T_xX)] \times [O_x(-1) \otimes H_1] \times \cdots [O_x(-1) \otimes H_n] \to \mathbb{C}$ given by $(\omega, \eta_1, \cdots, \eta_n) \mapsto (\eta_1 \wedge \cdots \wedge \eta_n)[\omega] \in \mathbb{C}$, and the condition $\{(\eta_1 \wedge \cdots \wedge \eta_n)[\omega] = 0\}$ defines the set $D(X)$. Applying (2.24) once more, we have

$$\mathcal{B}' = \langle \mathcal{M}, \cdots, \mathcal{M} \rangle_{D(X)}. \tag{4.11}$$

Hence,

$$\langle \mathcal{M}, \cdots, \mathcal{M} \rangle_{\mathbb{P}}^d \otimes \langle K \otimes L^n, L, \cdots, L \rangle = \langle \mathcal{M}, \cdots, \mathcal{M} \rangle_{D(X)} \tag{4.12}$$

or

$$\langle K \otimes L^n, L, \cdots, L \rangle = \langle \mathcal{M}, \cdots, \mathcal{M} \rangle_{\mathbb{P}}^{-d} \otimes \langle \mathcal{M}, \cdots, \mathcal{M} \rangle_{D(X)}. \tag{4.13}$$

4.1 Discrimant Variety Isometry

Original Theorem 4.3. *The above isomorphism (4.2) is an isometry.*

Proof of (4.3). To prove this, we need to define the following series of subspaces of $X \times \mathbb{P}$: Let $\Gamma_0 = X \times \mathbb{P}$. For $1 \leq j \leq n$, we define:

$$\Gamma'_j \;=\; \{(x, H_1, ..., H_j) \in X \times \mathbb{P}_1 \times \cdots \times \mathbb{P}_j \;:\; x \in H_1 \cap \cdots \cap H_j \;\} \tag{4.14}$$

and $\Gamma_j \ = \Gamma'_j \times \mathbb{P}_j \times \cdots \times \mathbb{P}_n$, so

$$\begin{aligned} \Gamma_j \ &= \ \{(x, H_1, ..., H_n) \in X \times \mathbb{P} \ : \ x \in H_1 \cap \cdots \cap H_j \ \} \\ &= \ \{z \in \Gamma_{j-1} : s_j(z) = 0\} \end{aligned} \tag{4.15}$$

Hence, we have $\dim(\Gamma_j) = m + n - j$. The subspace Γ_{n+1} is defined by a section s_{n+1} of $(K \otimes L^n) \otimes \mathcal{M}$:

$$\Gamma_{n+1} \ = \ \{(x, H_1, ..., H_n) \in \Gamma_n : [\eta_1 \wedge \cdots \wedge \eta_n](\omega) \ = \ 0\} \tag{4.16}$$

for $\eta_i \in \mathcal{O}_x(-1) \otimes H_i \subseteq T_X\mathbb{P}(V)^*$ and $\omega \in \Lambda^n T_x X$. Using this construction, $\Gamma_{N+1} \neq D$, but $\Gamma_{n+1} \to D$ is a generically d to one map. As before in (3.2), we will define the following spaces:

$$\begin{aligned} &\mathcal{B} = \\ &\langle \underbrace{\pi_{\mathbb{P}}^*\mathcal{M}, , ...\pi_{\mathbb{P}}^*\mathcal{M}}_{m}, \pi_X^*(K \otimes L^n) \otimes \pi_{\mathbb{P}}^*\mathcal{M}, \pi_X^* L \otimes \pi_{\mathbb{P}}^* M_1, ..., \pi_X^* L \otimes \pi_{\mathbb{P}}^* M_n \rangle_{X \times \mathbb{P}} \end{aligned} \tag{4.17}$$

For $1 \leq j \leq n$,

$$\begin{aligned} &\mathcal{B}_j = \\ &\langle \underbrace{\pi_{\mathbb{P}}^*\mathcal{M}, , ...\pi_{\mathbb{P}}^*\mathcal{M}}_{m}, \pi_X^*(K \otimes L^n) \otimes \pi_{\mathbb{P}}^*\mathcal{M}, \pi_X^* L \otimes \pi_{\mathbb{P}}^* M_{j+1}, ..., \pi_X^* L \otimes \pi_{\mathbb{P}}^* M_n \rangle_{\Gamma_j} \ \to \ \Gamma_j \end{aligned} \tag{4.18}$$

For $j = n + 1$,

$$\mathcal{B}_{n+1} = \langle \underbrace{\pi_{\mathbb{P}}^*\mathcal{M}, , ...\pi_{\mathbb{P}}^*\mathcal{M}}_{m} \rangle_D \ \longrightarrow \ D \tag{4.19}$$

The Deligne metric along the above pairing, combined with the induction formula implies that the map $\mathcal{B}_{j-1} \to \mathcal{B}_j e^{-I_j}$ is an isometry, where I_j is the integral:

$$I_j \;=\; \int_{\Gamma_{j-1}} \log|s_j|^2 c_1(\mathcal{M})^m \wedge c_1(K\otimes L^n \otimes \mathcal{M}) \wedge c_1(L\otimes M_{j+1}) \wedge \cdots \wedge c_1(L\otimes M_n) \quad (4.20)$$

The proof of isometry for I_j in the $j \leq n$ case follows similarly from 3.2, leaving only to compute I_j in the $j = n+1$ case,

$$I_{n+1} \;=\; \int_{\Gamma_n} \log|s_{n+1}|^2 c_1(\mathcal{M})^m \quad (4.21)$$

To compute the above integral, we need to consider the following:

1. For certain $(x, H_1, \cdots, H_n) \in \Gamma_n,\ s_{n+1} = 0$, hence $\log|s_{n+1}| = -\infty$ and the integrand is unbounded.

2. The form $c_1(\mathcal{M})$ itself is a form on the space $\mathbb{P}$, so we would like to rewrite $c_1(\mathcal{M})^m$ for the space $\Gamma_n \subset X \times \mathbb{P}$

3. The bundle $\pi_X^*(K\otimes L^n) \otimes \pi_{\mathbb{P}}^*\mathcal{M} = K_{\Gamma/\mathbb{P}}$, hence we may write $|s_{n+1}|^2$ as

$$|s_{n+1}|^2 = \frac{c_1(\mathcal{M})^m}{(c_1(\mathcal{M}) + \omega)^m} \leq 1 \quad (4.22)$$

We suppose (4.22) is true. Rewrite $c_1(\mathcal{M})^m$ as follows:

$$\begin{aligned} c_1(\mathcal{M})^m &= c_1(\mathcal{M})^m \frac{(c_1(\mathcal{M}) + \omega)^m}{(c_1(\mathcal{M}) + \omega)^m} \\ &= \frac{(c_1(\mathcal{M})^m}{(c_1(\mathcal{M}) + \omega)^m}(c_1(\mathcal{M}) + \omega)^m \\ &= |s_{n+1}|^2 (c_1(\mathcal{M}) + \omega)^m \end{aligned} \quad (4.23)$$

We may rewrite I_{n+1} as follows:

$$I_{n+1} \;=\; \int_{\Gamma_n} |s_{n+1}|^2 \log|s_{n+1}|^2 (c_1(\mathcal{M}) + \omega)^m \quad (4.24)$$

Now the form $(c_1(\mathcal{M})+\omega)^m$ is a differential form over the space Γ_n, and the integrand is of the form $x\log x$, which is bounded for $|x|\leq 1$. We have:

$$\begin{aligned} |I_{n+1}| &= \left|\int_{\Gamma_n} |s_{n+1}|^2 \log|s_{n+1}|^2 (c_1(\mathcal{M})+\omega)^m\right| \\ &\leq \int_{\Gamma_n} \left||s_{n+1}|^2\log|s_{n+1}|^2\right|(c_1(\mathcal{M})+\omega)^m \\ &\leq \int_{\Gamma_n}(c_1(\mathcal{M})+\omega)^m = \mathrm{Vol}(\Gamma_\mathrm{n}) \end{aligned} \tag{4.25}$$

So it remains to show (4.22). Call $\Gamma' = \{(x, H_1, ..., H_n) \in \mathbb{P}(V)\times\mathbb{P} : x \in H_1\cap\cdots\cap H_n\}$ and note that $\Gamma\subset\Gamma'$ with $K_{\Gamma'/\mathbb{P}(V)} = K_{\Gamma/X}$. By definition:

1. $K_{\Gamma/\mathbb{P}} = K_\Gamma \otimes K_{\mathbb{P}}^{-1}$.

2. $K_{\Gamma/X} = K_\Gamma\otimes K_X^{-1}$.

   So $K_\Gamma = K_{\Gamma/X}\otimes K_X$.

3. $K_{\Gamma'/\mathbb{P}(V)} = K_{\Gamma'}\otimes K_{\mathbb{P}(V)}^{-1}$.

4. $K_{\Gamma'} = K_{\mathbb{P}(V)\times\mathbb{P}}\otimes\mathcal{O}(\Gamma)$.

   So $K_{\Gamma'} = K_{\mathbb{P}(V)\times\mathbb{P}}\otimes(L\otimes M_1)\otimes\cdots\otimes(L\otimes M_n)$,

   or, $K_{\Gamma'} = K_{\mathbb{P}(V)\times\mathbb{P}}\otimes L^n\otimes\mathcal{M} = K_{\mathbb{P}(V)}\otimes K_{\mathbb{P}}\otimes L^n\otimes\mathcal{M}$

Applying the above sequentially, we see that

$$\begin{aligned} K_{\Gamma/\mathbb{P}} &= K_\Gamma\otimes K_{\mathbb{P}}^{-1} \\ &= K_{\Gamma/X}\otimes K_X\otimes K_{\mathbb{P}}^{-1} \\ &= K_{\Gamma'/\mathbb{P}(V)}\otimes K_X\otimes K_{\mathbb{P}}^{-1} \\ &= K_{\Gamma'}\otimes K_{\mathbb{P}(V)}^{-1}\otimes K_X\otimes K_{\mathbb{P}}^{-1} \\ &= K_{\mathbb{P}(V)}\otimes K_{\mathbb{P}}\otimes L^n\otimes\mathcal{M}\otimes K_{\mathbb{P}(V)}^{-1}\otimes K_X\otimes K_{\mathbb{P}}^{-1} \\ &= L^n\otimes\mathcal{M}\otimes K_X = K\otimes L^n\otimes\mathcal{M} \end{aligned} \tag{4.26}$$

which completes the proof.

4.2 The Mabuchi K-Energy Formula

As a consequence of 4.2, we have the following corollary.

Original Corollary 4.4. *For $X \subseteq \mathbb{P}(V)$ a smooth variety of dimension n and degree $d \geq 2$. Then*

$$\nu_\omega(\phi_\sigma) \ = \ \deg(C_X)\log\frac{\|D_X^\sigma\|^2}{\|D_X\|^2} \ - \ \deg(D_X)\log\frac{\|C_X^\sigma\|^2}{\|C_X\|^2}, \tag{4.27}$$

where ν_ω is the Mabuchi K-energy and the norm is the Deligne norm defined by (2.30).

This corollary parallels a theorem of Paul, who proves in [14] the following:

Theorem 4.5 (Paul)**.** *Let $X \hookrightarrow \mathbb{P}(C)$ be a smooth, linearly normal complex algebraic variety of degree $d \geq 2$. Let R_X denote the X-Resultant. and let $\Delta_{X\times\mathbb{P}(V)^{n-1}}$ denote the $X-$hyperdiscriminant. Then the Mabuchi energy restricted to the space of Bergman metrics is given by*

$$\nu_\omega(\phi_\sigma) = \deg(R_X)\log\frac{\|\sigma\cdot\Delta_{X\times\mathbb{P}(V)^{n-1}}\|^2}{\|\Delta_{X\times\mathbb{P}(V)^{n-1}}\|^2} - \deg(\Delta_{X\times\mathbb{P}(V)^{n-1}})\log\frac{\|\sigma\cdot R_X\|^2}{\|R_X\|^2}. \tag{4.28}$$

The proof of the corollary follows from (2.35) and (2.23), along with the results from (3.7) and (4.2).

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