Dynamic Revenue and Inventory Management Models

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A dissertation submitted to the

Graduate School-Newark

Rutgers, The State University of New Jersey

in partial fulfillment of requirements

for the degree of

Doctor of Philosophy

Graduate Program in Management

Operations Research Major

Written under the direction of

Dr. Jian Yang

and approved by

Newark, New Jersey

May, 2014

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ABSTRACT OF THE DISSERTATION

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Effective pricing and inventory controls are very important for the success of a company, especially in an environment with many uncertainties such as random demand and fluctuating cost.

In this work, we first consider pure dynamic pricing. Indeed, we consider three cases: markup in which price can only go up, markdown in which price can only go down, and reversible pricing in which price can go either direction.

We also consider a joint pricing and inventory control model in which the raw material price evolves as a Markov process. For this model, we suppose production is make-to-order, so that the conversion from raw material to finished product is carried out only when demand arrives.

For the pure pricing model, we establish the optimality of threshold-like policies. We also develop efficient and numerically stable algorithms. For the make-to-order joint inventory-pricing model, we demonstrate the optimality of a base-stock-list-price policy. In addition, we identify conditions under which policy parameters would exhibit monotone trends. Moreover, we showed the significant benefit of adopting cost-dependent base-stock list-price policy.

ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to my dissertation advisor, Dr. Jian Yang, for his tremendous support, enthusiastic encouragement, and patient guidance. During the past six years, he has been not only my mentor but also my friend. His dedication to work has impacted me profoundly and will continue to inspire me in my future career. It is my great honor to be one of his Ph.D. students.

I would also like to offer my special thanks to Department Chair, Dr. Michael N. Katehakis, for his support and valuable advice. I am also very grateful to Dr. Hui Xiong, Dr. Andrzej P. Ruszczyński, and Dr. Eugene A. Feinberg for their valuable and constructive suggestions.

My grateful thanks are extended to my friends, Yi Yang, Xiangqian Yu, Lina, Lu, Jicheng Wang, Yan Wang, Yipeng Cao, Zhenqing Zheng, Shengbin Wang, and Qiang Wu for their help.

Finally, I would particularly like to thank my parents for their love, support and understanding.

Contents

List of Tables

List of Figures

CHAPTER 1

Introduction

Dynamic pricing is also called time-based pricing which is a price strategy based on the time when the service or commodity is provided. It is playing a more and more important role in the success of business. As time goes by, the situation that a firm faces, like the quality, attractiveness, substitute, or the inventory of a product as well as the appetite of customers, will change. For example, flight ticket becomes more attractive when the time closes to the departing date; on the other hand, there are probably only a few flight tickets remaining available at that time. Demand is also fluctuating with higher rate during peak season like in the Christmas holiday and lower rate in normal business time. The firm may miss the chance to obtain more revenue, or lose demand on the other side, by charging a same price when environment changes. Therefore, it is necessary for firms to adopt dynamic pricing to capture the evolvement of market to maximize their revenues. According to Davis (1994), airlines increase at least $500 millions annually through dynamic pricing.

However, dynamic pricing is not the only issue to be concerned by companies, especially for manufacturers. Due to demand uncertainty, firms need to make effective production planning to hedge supply and demand risk. All the demand can be satisfied if they produce large enough inventory which, unfortunately, will introduce huge inventory holding cost that includes cost of capital, material handling, storage, damage, insurance, and tax. Should they produce less, they will pay penalty for the demand which can not be satisfied immediately due to lost good will, promotion to hold demand, or even lost sales. On the upstream of the supply chain, fluctuating raw material price is also an important factor to affect the production decision. It is intuitive that a firm should purchase more to save purchasing cost when the raw material is cheap. However, it is really hard to determine when the raw material price has touched the bottom and how much should be ordered. Therefore, effective production planning is also very critical to building a successful company.

Almost every company faces demand uncertainty. As the mentioned airline industry, people may prefer to travel during holiday seasons which brings high book rates for the flight tickets. Even though we can imagine the demand will be higher in the holidays than normal business days, we still can not be sure what is the exact number of demand on a special day due to diverse preference of travelers. Besides seasonal factor, demand for a product or a service is also affected by many other reasons such as available substitutes, quality, popularity, necessity and so on. Those factors will lead to fluctuating demand as time goes by. Another important factor to influence the demand is the sales price. Due to different income, customers' ability to afford a good or a service ranges widely. Expensive products like luxury cars are designed for a few rich people and cheap products like shampoos are suitable for everyone. Thus, it is straightforward to see, even for the same product, high price will incur less demand than lower price. Simply speaking, we can think demand is time-fluctuating, price-sensitive, and random.

Because of regulation, wars, technology, and many other elements, raw material price is fluctuating as well. Take crude oil as an example to illustrate. It increases gradually from around \$60 in Jan 2006 to \$145 in Jul 2008, and suddenly drops sharply to around \$60 in Oct 2008. Then, it rebounds from \$40 in Dec 2008 to more than \$100 in May 2011, and decreases again after that. Even during the long trend of increasing or decreasing, the crude oil price vibrates frequently and goes to the opposite direction very often. Some experts predict the crude oil price will soar to \$200 before it goes to \$145 in Jul 2008. However, the prediction never comes true. Besides crude oil, the prices of steel, copper, latex, and others also fluctuate wildly. Thus, it is very hard to predict the price of raw material. To purchase raw material earlier, firms may overleap the chance to get a even lower price and have to pay a holding cost in the meantime; to procure later, they may come across a higher price and lose customer demand. Taking uncertain raw material price into account is therefore a big concern for production planning.

To satisfy customer demand, firms can either build raw material inventory and make production when order comes, or build both raw material and finished product inventories before order comes. The former case is called the make-to-order and the latter is make-to-stock. The make-to-order case fits the situation where it is very expensive to hold finished product inventory, the firm has very good relationships with customers, and customers are willing to wait. For example, producers of automobiles, computer servers, and aircrafts can adopt this production strategies. Make-to-stock can be suitable for almost every manufacturer. Normally, the holding cost is more expensive for finished product than raw material. As we can imagine, holding an aircraft would be less economical than holding its parts. To avoid high holding cost, firms may prefer to store raw material instead of finished product. Due to price uncertainty of raw material, building raw material inventory can also hedge input price risk.

In this work, we will consider a pure dynamic pricing model and a make-to-order inventory control model with pricing. The pure dynamic pricing model contains three cases: the markup case where the concerned firm can only charge consecutively increasing prices, the markdown case where the price can be changed in a decreasing direction, and the reversible pricing case where any price can be adopted at any time.

In all these cases, the concerned firm aims to maximize revenue through dynamic pricing where price can be switched to another one from a given finite set at any time of the sales season. Demand follows a poisson process with rates in the form of a price-dependent term multiplying a time-sensitive term. Product inventory is built in advance and can not be replenished. In the make-to-order inventory control model, the concerned firm needs to periodically make decisions on raw material purchase and product pricing to maximize its profit. Raw material price evolves as a Markov process with finite supports. The random demand takes a general non-stationary price-elastic form. The firm makes immediate production with available raw material inventory when demand comes and unsatisfied demand is backlogged.

For applications, we can consider the following stories. An airline company usually starts to sell tickets one year ahead of departing time. In the beginning, all the seats are available and customers face too many uncertainties for traveling one year later. The airline company usually offers very cheap price for the seats to capture some cash flow. When it closes to the departing date, the airline company carrying only a few empty seats will charge a much higher even full price since it now targets at the customers who are urgent to fly and care less about the price. The markup case is designed for such companies as the airlines or hotels who consecutively charge an increasing sequence of prices along the time. Retailers who sell fashion goods like the down jacket or new technologies like the iPhone4 usually adopt the markdown pricing strategy in which they charge prices in a decreasing direction. Fashion goods are usually very attractive or good to use when launched, but are less popular as time goes by because newer products will come to compete for the same customer base. As a result, those retailers will tend to charge higher prices in the beginning and gradually decrease the price as time goes by. In many other cases, firms can freely charge prices. The reversible case is appropriate for this situation.

When inventory replenishment and production are involved, firms need to be aware of the fluctuating raw material price. Consider a die making shop which purchases metals, stores them, and carries out production when order arrives. At the beginning of each month, the company needs to make joint decisions on how much raw material should be procured and how much it should charge for the die. In case that the on-hand raw material is not enough to satisfy all the demand, the unsatisfied demand is backlogged to the next month with some rebates. The time for shipping raw material is negligible and production can be finished in less than one month to the extent that lead time and production time can be assumed to be zero. Then, this die making shop can apply our make-to-order model to its practice.

We have found optimal pricing strategies for all three cases of the pure dynamic models and for the make-to-order model as well as an optimal purchasing strategy for the make-to-order model. Threshold-like pricing policy are optimal for all the three dynamic pricing cases. For the make-to-order model, we establish the optimality of base-stock-list-price policy. Such policy says the firm should increase its inventory level up to a base stock level and charge a list price when its inventory level is lower than the base stock level; when starting with a higher inventory level, the firm should charge a even lower price. In addition, we identify monotone trends for the various policies. In particular, we find the threshold levels of the markup and markdown cases hold decreasing trend in inventory, while that of the reversible case is decreasing in both inventory and price. In the make-to-order model, the base stock level is decreasing in the raw material price. List price may be expected to decrease as well. However, this turns out to be false. Instead, it is the average inventory of the next period that is decreasing in the raw material price. Moreover, we develop algorithms to solve the threshold policies and offer guidance to efficiently find the base-stock-list-price policy. We conduct numerical experiments which show that our threshold policies, considering non-stationary demand, generate much more revenue than that in the literature, considering stationary demand, does and our cost-adjusted base-stock-list policy outperforms the cost-independent policy in the literature significantly as well. Our numerical study also confirms, for the pure pricing model, that threshold policies may not be optimal under general demand forms, and for the make-to-order model, monotone trends may not be retained when certain conditions are violated.

This work is organized as in the following. In chapter 2, we review the related literature; in chapter 3, we concentrate on the pure dynamic pricing model; in chapter 4, we analyze the make-to-order model; in chapter 5, we provide numerical examples that supplement theoretical results; finally, we make conclusion in chapter 6.

CHAPTER 2

Literature Review

Research on dynamic pricing/revenue management (RM) has been flourishing for many decades. One earlier work Rothstein (1971) studied an overbooking model for the airline industry. Glover et al. (1982) studied a network model with different fare classes, different flight segment. Weatherford and Bodily (1992) provided a detailed review on the literature of revenue management by that time. Gallego and van Ryzin (1994) started treating dynamic pricing from the perspective of control theory. They showed the monotonicity of pricing policies with respect to the inventory and remaining time. In addition, they proposed a fixed-price heuristic method to reach the optimal pricing policy asymptotically. Feng and Gallego (1995) established the optimality of threshold policies for both the markup and markdown case that evolve only one price change. Feng and Xiao (2000b) made generalization to cases involving multiple price switch.

When demand is time-varing, Bitran and Mondschein (1997) characterized an optimal pricing policy which has time-monotonicity and inventory-monotonicity when the customers' valuation distribution is homogeneous. Zhao and Zheng (2000) extended their work to nonhomogeneous valuation distribution and showed that the inventory-monotonicity of the optimal threshold pricing policy is still preserved. As for time-monotonicity, they found that it is preserved under the condition that the willingness of a customer to pay a premium is not increasing over time.

One of the earliest works on inventory management is Harris (1913) in which the classical economic order quantity model was studied. Another important basic model in inventory is the newsvendor model which can date back to Arrow et al. (1951). Based on these two basic model, numerous inventory control models were developed; see Axsater (2006). In terms of the times of inventory monitoring, the literature on inventory control can be divided into continuous review, in which inventory is monitored continuously, and periodic review, in which inventory is only reviewed after certain time. Papers considering continuous review include Moinzadeh and Nahmias (1988), Qi et al. (2009), Yang and Xia (2009), and so on; papers dealing with periodic review include Tsitsiklis (1984), Cheng and Sethi (1999), Jian et al. (2006) Zhao and Katehakis (2006), Katehakis and Sonin (2014), and so on. However, as mentioned in the introduction, they are the same in some sense.

Quite a few works have dealt with inventory management under fluctuating raw material prices. Kalymon (1971) considered a Markov raw material price and showed that a price-dependent (s, S) policy can minimize the total discounted expected cost. Golabi (1985) studied a periodic-review inventory control problem with stochastic input cost but deterministic demand. Li and Kouvelis (1999) compared two type of supply contracts that can be used to hedge the risk evolved in uncertain sourcing cost. With random demand and purchasing costs, Ozekici and Parlar (1999) established the optimality of a base-stock policy when the ordering cost is linear and that of an (s, S) policy when there is a fixed cost. However, these papers mentioned above did not take into account pricing, neither did they show policy trend with respect to fluctuating raw material prices. Yang and Xia (2009) treated a continuous-time version of our joint control problem without pricing. They identified time-continuity and mean-reversion as sufficient conditions for the acquisition base-stock level to be monotone in the raw material price. Secomandi (2010) also took advantage of the time-continuity property in his study of a raw commodity speculation problem.

Many authors have worked on pricing problems involving procurement as well. Whitin (1955) considering a single-period deterministic problem. Thomas (1974) proposed a heuristic method to study a multi-period joint pricing and production problem with random demand. For a detailed review, readers are refered to Eliashberg and Steinberg (1993). On this subject, recent major breakthroughs were accomplished by Federgruen and Heching (1999) and Chen and Simchi-Levi (2004). Federgruen and Heching considered a linear ordering cost. They characterized one optimal policy as base-stock-list-price for both finite and infinite horizons. Chen and Simchi-Levi extended the above work to the case with fixed costs and introduced the concept of symmetric-k-convexity to cope with the added difficulty. Under additive demand form, they identified an (s, S, p) policy which says the inventory policy follows (s, S) policy and the price should be charged based on the inventory level.

Next, we turn to the Markovian assumption of the raw material price process. There is ample literature in support of the Markov-process modeling of commodity prices. On the validation of treating actual price processes as Markov processes, we have Ryan (1973) and Fielitz and Bhargava (1973); on estimating for transition matrices of Markov processes, there are Anderson and Goodman (1957) and Ryan (1973); also, quite some operations management papers base their studies on the premise that their commodity price process follows a Markov process; see, e.g., Fabian et al. (1959), Andersen (2010), and Secomandi (2010). Some times, one can obtain the transition matrix of a Markov chain relatively easily when the chain is the discrete-state counterpart of a well-behaving continuous-state Markov process; see Yang and Xia (2009).

As mentioned, the linear infinite-location model used to motivate our demand function is intimately related to Hotelling (1929)'s linear two-location model and Salop (1979)'s circular multi-location model. Hotelling (1929) studied a two-player competition problem and found an equilibrium price. In this model, two sellers, locating at two ends of a line, and competes customers which are evenly distributed on the line and buy products based on their evaluation on the prices of and their distances to the sellers. Salop (1979) extended the above work and considered multiple suppliers and customers that are located on a circle. He introduced customers' preference over the supplies/brand and showed the existence and properties of a symmetric zero-profit Nash-equilibrium. Many more works examined the effects of input costs and competition on demand; see, e.g., Dixit and Stiglitz (1977) and Perloff and Salop (1985).

The part of irreversible-pricing cases, including the markup and the markdown cases, is a generalization of Feng and Xiao (2000b) which considered stationary demand. Similarly, we have reached the threshold-like optimal pricing policy. However, we have identified a minor error in their markup case. In addition, to characterize the optimal policy, we use quite different approaches such as an ordinary differential equation and its solution, instead of advanced tools like Karlin's (1968) total positivity results. Moreover, we demonstrate the difference between the markup and the markdown cases. In particular, the markup case possesses a complementary property between price flexibility and inventory, while is not enjoyed by the markdown case. Besides, our construction of threshold policies and value functions lead to efficient and numerically stable algorithms to solve for optimal policies.

Works most relevant to our make-to-order are Federgruen and Heching (1999) on joint pricing-procurement control and Yang and Xia (2009) on acquisition management with fluctuating raw material prices. Comparing to Federgruen and Heching (1999), we have identified the monotone trend of the optimal base stock level and the expected next-period inventory with the raw material price. In particular, when the input cost goes up, the targeted base stock level should be lowered, while the sales price may not necessarily increase; and even when it does, it should be charged to aim at attracting an expected demand that will lead to the expected next-period inventory position being reduced. Contrasting against Yang and Xia (2009), we have added the pricing component and demonstrated the monotone trends for the corresponding pricing policy. In addition, Chen et al. (2014) studied a joint pricing and inventory control model with both on-site sales market and long-distance market. They showed that the long-distance market has advantage because of delivery flexibility when inventory level is low. Yang (2014) introduced time-consistent coherent and Markov risk measure in the literature of joint inventory and pricing activities.

Besides the above most relevant literature, we note that some authors took into account the strategic behavior of buyers. Strategic customers will monitor prices and decide when and which price to buy a product. Su (2007) studied a revenue management problem evolving strategic customers with different valuations and different patient level. He identified dynamic pricing policies under different compositions of strategic customers. Aviv and Pazgal (2008) proposed a model to study the impact of customers' strategic behavior on retailer's pricing strategies. They considered two pricing schemes and identified a subgame-perfect equilibrium for both cases. Liu and van Ryzin (2008) investigated how rationing would affect both monopoly market and oligopoly market in presence of strategic customers. In addition, due to the development of the E-commerce, auction becomes a popular procurement strategy. Katehakis and Puranama (2012a) addressed an auction problem in which a firm, aiming to maximize the revenue, procures products through sequencial bidding actions and resells them to customers. In Katehakis and Puranama (2012b), a firm adopts two procurement strategies, buy-it-now and sequencial auctions, to meet fixed demand with minimum cost. Monotone properties for the optimal value function and bidding strategies are identified. Yang et al. (2005) and Yang and Qi (2010) considered the availability of outsoucing when the firm has to make both production and inventory management decisions. Yin et al. (2009) explored a game-theoretical model between a retailer who has two choices of displaying its inventory and customers who behave strategically. There is also a sizable body of literature involving competition. Perakis and Sood (2006) used the robust optimization approach to study dynamic pricing under competition. Xu and Hopp (2006) found a weakly perfect Bayesian equilibrium for an oligopolistic pricing model. Lin and Sibdari (2008) established a Nash equilibrium for retailers with complete information of competitors' inventory levels. Meanwhile, Araman and Caldentey (2009), Besbes and Zeevi (2009), and Farias and van Roy (2010) brought different approaches to bear on the task of fusing together dynamic pricing decisions and real-time learning of demand information. Also recently, Xu and Hopp (2009) gave conditions under which optimal price paths would exhibit monotone trends in a stochastic sense.

CHAPTER 3

Pure Dynamic Pricing Model

We will consider three cases: markup case, markdown case, and reversible case to emulate real practice. For airlines, markup is more probable; for retailers, markdown is more prevalent; for most other cases, reversible pricing is more commonly seen.

3.1 Problem Setup

We consider a concerned firm that wants to sell N items of its product during the sales season $[0, T]$. After the end time T, all the products unsold will become useless and have salvage value 0. Therefore, the firm will try to maximize the revenue it can get by selling its inventory through effective pricing. The prices the firm can choose are given, denoted by $\{\bar{p}^k, k = 0, 1, ..., K\}$, with $p^{k_1} \leq p^{k_2}$ for $0 \leq k_1 \leq k_2 \leq K$. In the markup case, the firm can only choose ever increasing prices as time goes by. Similarly, in the markdown case, the firm can only charge decreasing prices as time goes by. The reversible case doesn't have any restriction on which direction the prices can go.

Demand follows a Poisson process with rate $\bar{\alpha}^k\beta(t)$ when the firm charges price $\bar{p}^k$ at time t. We assume $\beta(\cdot)$ is continuous on time t. The Poisson process means that the demand coming during $[s, t]$ at price $\bar{p}^k$, $N^k(t) - N^k(s)$, follows a Poisson distribution with rate $\int_s^t \bar{\alpha}^k\beta(\tau)d\tau$. In addition, $N^k(s_2) - N^k(s_1)$ is independent with

$N^k(t_2) - N^k(t_1)$ for $0 \leq s_1 \leq s_2 \leq t_1 \leq t_2 \leq T$. For $n = 0, 1, ...$, we have

$$P[N^k(t) - N^k(s) = n] = \exp(-\bar{\alpha}^k \cdot \hat{\beta}(s,t)) \cdot \frac{(\bar{\alpha}^k \cdot \hat{\beta}(s,t))^n}{n!}. \tag{3.1}$$

The Poisson arrival process is commonly assumed in literature; see Gallego and van Ryzin (1994) and Feng and Xiao (2000b). The term $\bar{\alpha}^k$ reflects the elasticity of demand to the price $\bar{p}^k$ and the term $\beta(t)$ implies the sensitivity of demand to the time t. To understand a time-varying $\beta(t)$, we may take the example of Christmas tree. Demand is very high one month before Christmas, while it will all but disappear after Christmas. Another example is that demand for hotel rooms follows very clear seasonal patterns. Furthermore, for high-tech products like iPhone 4S, demand usually peaks right after release time and then it starts to decline. It can be said that we use the $\beta(t)$ to reflect factors like seasonality, new product release, and so on so forth. We make the following assumptions:

(S1) for the revenue rates $\bar{p}^k \bar{\alpha}^k \beta(t)$, we have

$$\bar{p}^0 \bar{\alpha}^0 > \bar{p}^1 \bar{\alpha}^1 > \cdots > \bar{p}^K \bar{\alpha}^K,$$

which results in the weaker condition:

$(S1^{'})$ for the demand arrival rates $\bar{\alpha}^k \beta(t)$, we have

$$\bar{\alpha}^0 > \bar{\alpha}^1 > \cdots > \bar{\alpha}^K.$$

Assumption (S1) is reasonable because if it were not true, people would always choose the high price.Now, define $\overline{\lambda}$, so that

$$\overline{\lambda} = \bar{\alpha}^0 \cdot \sup_{t \in [0,T]} \beta(t). \tag{3.2}$$

By the continuity of $\beta(\cdot)$ and the compactness of $[0, T]$, we know that $\overline{\lambda}$ is a strictly positive and finite constant. By (S1'), we know that $\overline{\lambda}$ is the highest instantaneous arrival rate that can ever be achieved.

Let's define a threshold policy. Simply put, we denote it by $\tau = (\tau_n^k \mid k = 1, 2, ..., K,\ n = 1, 2, ..., N) \in (\Delta_N)^K$, where $\Delta_N \subset [0, T]^N$ is defined through

$$\Delta_N = \{(\tau_1, \tau_2, ..., \tau_N) \mid 0 \leq \tau_N \leq \tau_{N-1} \leq \cdots \tau_1 \leq T\}. \tag{3.3}$$

The presence of (3.3) implies the inventory monotonicity property. In the markup case, a firm adopting policy τ should increase its price to $\bar{p}^k$ when the current price is lower than $\bar{p}^k$ and the inventory level drops to n before the threshold point τ_n^k. For the markdown case, a firm adopting policy τ should decrease its price from $\bar{p}^k$ to $\bar{p}^{k-1}$ when the inventory is n and time passes the threshold point τ_n^k. For the reversible pricing case, a firm adopting policy τ should charge price $\bar{p}^k$ when the current time t satisfies $\tau_n^{k+1} \leq t \leq \tau_n^k$ and the inventory is n.

We define the value function as the maximum value or revenue the firm can make during the rest of sales season when it charges a given price with a certain inventory level. For irreversible cases, including markup case and markdown case, let us define $v_n^k(t)$ as the maximum revenue the firm can make when it starts at time t with price $\bar{p}^k$ and inventory level n. For the reversible pricing case, let us define $v_n(t)$ as the maximum revenue the firm can make when it starts at time t with inventory n. When the firm has zero inventory, it will have nothing to sell to generate revenue. Therefore, it holds that $v_0^k(t) = v_0(t) = 0$. When the sales season ends, we assume there is no salvage value which means $v_n^k(T) = v_n(T) = 0$.

3.2 The Markup Case

Since the highest price $\bar{p}^K$ is the last price for the firm to charge, we have

$$v_n^K(t) = \bar{p}^K \cdot E[(N^K(T) - N^K(t)) \wedge n]. \tag{3.4}$$

For $k = K-1, K-2, ..., 0$, the firm still has the chance to increase its price to $\bar{p}^{k+1}$ when it is currently charging $\bar{p}^k$. Hence,

$$v_n^k(t) = \sup_{\tau \in \mathcal{T}} E[\bar{p}^k \cdot \{(N^k(\tau) - N^k(t)) \wedge n\} + v^{k+1}_{(n-N^k(\tau)+N^k(t))^+}(\tau)]. \tag{3.5}$$

Here, it is obvious to see that stopping time τ is the moment to switch the price from p^k to p^{k+1}. However, the problem is how we can find these stopping times which also compose the threshold policy defined above. In the following, we will provide the procedure to find this threshold policy and prove its optimality. In addition, we will show how to obtain the value functions.

3.2.1 A Constructing Procedure

Before deriving the value functions and threshold policy, we define the infinitesimal generator $\mathcal{G}_n^k(t)$ corresponding to price $\bar{p}^k$, inventory level n, and time t. When this operator applies to a well-defined function vector $u = (u_n(t) \mid n = 0, 1, ..., N, \ t \in [0, T])$, it will follow that

$$\mathcal{G}_n^k(t) \circ u = d_t u_n(t) + \bar{\alpha}^k \cdot \beta(t) \cdot (u_{n-1}(t) - u_n(t)). \tag{3.6}$$

Note that it will be an abuse of notation to call the left-hand side $\mathcal{G}^k u_n(t)$, as knowing $u_n(t)$ at the particular n and t alone will not help one get to the right-hand side. If we consider the sequence u as a sequence of value function, then the first term of this generator stands for the change of time value of the value function and the second term represents the change of the value function when demand occurs. The time value can be understood as the longer the sales season is, the more chance to sell more inventory the firm have. Also note that if we set $\mathcal{G}_n^k(t) \circ u + \bar{\alpha}^k \beta(t) \cdot \bar{p}^k$ equal to zero, then it can be simplified into the following form:

$$d_s f(s) = b(s) - a(s) \cdot f(s), \qquad \forall s \in (0, t). \tag{3.7}$$

where, $f(s) = u_n(s)$, $a(s) = -\bar{\alpha}^k \cdot \beta(s)$, and $b(s) = -\bar{\alpha}^k \cdot \beta(s) \cdot (u_{n-1}(s) + \bar{p}^k)$.

For this ordinary differential equation, we know from Carrier and Pearson (1991) that it has a unique solution $f(\cdot)$, so that for any $s \in [0, t]$,

$$\begin{aligned} f(s) &= f(0) \cdot \exp(-\textstyle\int_0^s a(u)du) + \int_0^s b(u) \cdot \exp(-\int_u^s a(v)dv) \cdot du \\ &= f(t) \cdot \exp(\textstyle\int_s^t a(u)du) - \int_s^t b(u) \cdot \exp(\int_s^u a(v)dv) \cdot du. \end{aligned} \tag{3.8}$$

Late, we will see (3.7) and (3.8) will play critical role in our derivation.

For the case where the time multiplier $\beta(\cdot)$ is stationary, Theorem 1 of Feng and Xiao (2000b) offers sufficient conditions for a vector of functions to be the value functions $v_n^k(t)$. But this result can be easily generalized to the case where $\beta(\cdot)$ is time-variant. In the same spirit of this theorem, we have the following.

Proposition 3.1. *For any $k = K-1, K-2, ..., 0$, a function vector $u \equiv (u_n(t) \mid n = 0, 1, ..., N,\ t \in [0, T])$ that is uniformly bounded and absolutely continuous in t for every n will be $v^k \equiv (v_n^k(t) \mid n = 0, 1, ..., N,\ t \in [0, T])$, if it satisfies the following:*

(i) $u_n(t) \geq v_n^{k+1}(t)$ for every $n = 0, 1..., N$ and $t \in [0, T]$,

(ii) $u_n(T) = 0$ for every $n = 0, 1, ..., N$ and $u_0(t) = 0$ for every $t \in [0, T]$,

(iii) for $n = 1, 2, ..., N$ and $t \in [0, T]$, $u_n(t) = v_n^{k+1}(t)$ implies $\mathcal{G}_n^k(t) \circ u + \bar{p}^k \bar{\alpha}^k \cdot \beta(t) \leq 0$,

(iv) for $n = 1, 2, ..., N$ *and* $t \in [0, T]$, $u_n(t) > v_n^{k+1}(t)$ *implies* $\mathcal{G}_n^k(t) \circ u + \bar{p}^k \bar{\alpha}^k \cdot \beta(t) = 0$.

Proposition 3.1 provides a hint as to how the value functions $v_n^k(t)$ and threshold levels τ_n^k can be constructed. First, due to (3.4), we can show $\mathcal{G}_n^K(t) \circ v^K(t) + \bar{p}^K \bar{\alpha}^K \beta(t) = 0$. Hence, we can establish all the $v_n^K(t)$ values in an n-loop by using the equation (3.7) and its solution (3.8) as well as the fact that $v_0^K(t) = 0$,

Then, for any $k = K-1, K-2, ..., 0$, suppose $v_n^{k+1}(t)$ is known for all n and t. We can then go through an n-loop to find all the $v_n^k(t)$'s. First, let $v_0^k(t) = 0$ as suggested by (ii) of the proposition. Second, suppose $v_{n-1}^k(t)$ is known for all t and some $n = 1, 2, ..., N$. Then, we have $v_n^k(T) = 0$ due to (ii) of the proposition. Next, we can obtain $v_n^k(t)$ for ever smaller t values by solving the differential equation $\mathcal{G}_n^k(t) \circ v^k + \bar{p}^k \bar{\alpha}^k \cdot \beta(t) = 0$ as indicated by (iv) of the proposition. We stop at the t when $v_n^k(t)$ is to sink below $v_n^{k+1}(t)$, which is not allowed by (i) of the proposition. For time t' earlier than this t, which is marked as τ_n^{k+1}, we let $v_n^k(t')$ be $v^{k+1}(t')$.

According to (iii) of the proposition, we still need $\mathcal{G}_n^k(t') \circ v^k + \bar{p}^k \bar{\alpha}^k \cdot \beta(t') \leq 0$ for $t' < \tau_n^{k+1}$ for the thus constructed $v_n^k(\cdot)$ to be the true value function. Nevertheless, let us go ahead with the construction procedure thus outlined. Not knowing whether what shall be constructed are the true value functions, we call them u's instead of v's. Formally, here is the iterative procedure for constructing function vector $u = (u_n^k(t) \mid k = 0, 1, ..., K,\ n = 0, 1, ..., N,\ t \in [0, T])$ and point vector $\tau = (\tau_n^k \mid k = 1, 2, ..., K,\ n = 1, 2, ..., N)$.

First, let

$$u_0^k(t) = 0, \qquad \forall k = K, K-1, ..., 0,\ t \in [0, T]. \tag{3.9}$$

Then, for $n = 1, 2, ..., N$ and $t \in [0, T]$, let

$$u_n^K(t) = \bar{\alpha}^K \cdot \int_t^T \beta(s) \cdot (\bar{p}^K + u_{n-1}^K(s)) \cdot \exp(-\bar{\alpha}^K \cdot \hat{\beta}(t,s)) \cdot ds. \tag{3.10}$$

Next, we go over an outer loop on $k = K-1, K-2, ..., 0$ and an inner loop on $n = 1, 2, ..., N$. At each k and n, first let

$$u_n^k(t) = \bar{\alpha}^k \cdot \int_t^T \beta(s) \cdot (\bar{p}^k + u_{n-1}^k(s)) \cdot \exp(-\bar{\alpha}^k \cdot \hat{\beta}(t,s)) \cdot ds, \qquad \forall t \in [0, T]. \tag{3.11}$$

Then, let

$$\tau_n^{k+1} = \inf\{t \in [0, T] \mid u_n^k(t) > u_n^{k+1}(t)\}, \tag{3.12}$$

with the understanding that $\tau_n^k = 0$ when the concerned inequality is always true and $\tau_n^k = T$ when it is never true. Finally, let

$$u_n^k(t) = u_n^{k+1}(t), \qquad \forall t \in [0, \tau_n^{k+1}]. \tag{3.13}$$

3.2.2 Optimality and Characteristics

First, let's introduce a few concepts such as concavity, supermodularity, and increasing differences.

Definition 3.1. For a set $S \subseteq R^n$, we say it is convex, if for any $\lambda \in [0, 1]$ and $x, y \in S$, $\lambda x + (1-\lambda)y \in S$.

Definition 3.2. Given a continuous function $f(s)$ defined on a convex subset $S \subseteq R^n$, it is concave on S, if for any $\lambda \in [0, 1]$ and $x, y \in S$, $\lambda f(x) + (1-\lambda)f(y) \leq f(\lambda x + (1-\lambda)y)$. For a discrete function $f(s)$ on a convex subset S of all integers Z, it is concave, if for any $s \in S$, $(f(s-1) + f(s+1))/2 \leq f(s)$. A function $f(s)$ is convex on S if $-f(s)$ is concave.

As shown in Howe (1982), a continuous concave or convex function is differentiable almost everywhere. The first-order derivative of a continuous concave(convex) function is decreasing(increasing).

Definition 3.3. Given a set $S \subseteq R^n$, it is called a lattice, if for any $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in S$, $x \wedge y = (\min\{x_1, y_1\}, \min\{x_2, y_2\}, ..., \min\{x_n, y_n\}) \in S$ and $x \vee y = (\max\{x_1, y_1\}, \max\{x_2, y_2\}, ..., \max\{x_n, y_n\}) \in S$.

Definition 3.4. Given a real-valued function $f(s)$ defined on a lattice $S \subseteq R^n$, it is supermodular on S if $f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$. If $-f(s)$ is supermodular on S, then $f(s)$ is submodular on S.

Definition 3.5. For a function $f(x, y)$ defined on $X \times Y$ where X and Y are both interval subsets of the integer set Z, we say $f(x, y)$ has increasing differences in (x, y), if $f(x_2, y_2) - f(x_1, y_2) \geq f(x_2, y_1) - f(x_1, y_1)$ for any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ satisfying $x_1 < x_2$ and $y_1 < y_2$. If $-f(x, y)$ has increasing differences, then $f(x, y)$ has decreasing differences.

Now we show that the construction process in the previous subsection will lead to an optimal threshold policy and the true value function. Let us first prove two properties of the value function $v_n^k(t)$ using induction and sample-path arguments that were first used by Zhao and Zheng (2000) in the dynamic pricing context.

Proposition 3.2. *For any fixed $k = 0, 1, ..., K$ and $t \in [0, T]$, the value function $v_n^k(t)$ is concave in n.*

The above shows that decreasing marginal value of inventory.

Proof: We can use a sample-path argument to prove that

$$v_n^k(t) \geq \frac{v_{n-1}^k(t) + v_{n+1}^k(t)}{2}. \tag{3.14}$$

We allow four pools of inventories, termed 1, 2, $\bar{1}$, and $\bar{2}$, to start at time t with the same price $\bar{p}^k$ and experience the same sample path over the interval $[t, T]$. These pools have different starting inventory levels and may exert different price controls, though. Pools 1 and 2 start with $n+1$ and $n-1$ initial items and pools $\bar{1}$ and $\bar{2}$ with n items.

Besides applying optimal time-increasing stopping-time pricing policies to pools 1 and 2, we apply the higher of the two prices for pools 1 and 2 to pool $\bar{1}$ and the lower of the two prices to pool $\bar{2}$, until the first moment, say s, when pool 1 is to have generated one more demand arrival than pool $\bar{1}$. After s, we let pool $\bar{1}$ follow pool 1's decisions and pool $\bar{2}$ follow pool 2's decisions. As both the minimum and maximum of two decreasing functions are decreasing functions themselves, pools $\bar{1}$ and $\bar{2}$ can be considered as adopting time-increasing stopping-time pricing policies as well.

Suppose the moment ever occurred, i.e., $s \in [t, T)$. Then, it has already been shown by Zhao and Zheng that the total sales revenue made by pools $\bar{1}$ and $\bar{2}$ amounts to the same as that by pools 1 and 2. Suppose the moment never occurred, i.e., $s = T$. Then, it has been shown by Zhao and Zheng that pools $\bar{1}$ and $\bar{2}$ can generate as high a total sales revenue as pools 1 and 2. So regardless, on every sample path, pools $\bar{1}$ and $\bar{2}$ can generate as high a total revenue as pools 1 and 2. Thus, we have proved (3.14). □

Proposition 3.3. *For any fixed $t \in [0, T]$, the value function $v_n^k(t)$ has decreasing differences between k and n.*

This proposition means $v_{n+1}^k(t) - v_n^k(t) \geq v_{n+1}^{k+1}(t) - v_n^{k+1}(t)$. It reflects that the marginal value of inventory is decreasing in price. Note that a lower price stands for more price choices in this markup case. This relationship therefore means the complementarity between price flexibility and inventory.

Proof: We use a sample-path approach to show the following: for any $n = 0, 1, ..., N-$

1,

$$v_{n+1}^{k}(t) - v_{n}^{k}(t) \geq v_{n+1}^{k+1}(t) - v_{n}^{k+1}(t), \qquad \forall k = 0, 1, ..., K-1,\ t \in [0, T]. \tag{3.15}$$

We prove by induction on the inventory level n. Let us first prove

$$v_{1}^{k}(t) - v_{0}^{k}(t) \geq v_{1}^{k+1}(t) - v_{0}^{k+1}(t), \qquad \forall k = 0, 1, ..., K-1,\ t \in [0, T]. \tag{3.16}$$

For every possible k and t, we introduce four pools of inventories, 1, 2, $\bar{1}$, and $\bar{2}$, that experience identical sample paths. At time t, pools 1 and $\bar{1}$ are with price index $k+1$, and pools 2 and $\bar{2}$ are with price index k. Also, pools 1 and $\bar{2}$ have one item, while pools $\bar{1}$ and 2 are out of stock. Apparently, pools $\bar{1}$ and 2 will continue to hold zero inventory. On the other hand, pool $\bar{2}$ can immediately raise its price to $\bar{p}^{k+1}$ and then match actions taken by pool 1. Hence, (3.16) is true.

Now, for some $n = 1, 2, ...N-1$, suppose it is true that, for any $m = 0, 1, ..., n-1$,

$$v_{m+1}^{k}(t) - v_{m}^{k}(t) \geq v_{m+1}^{k+1}(t) - v_{m}^{k+1}(t), \qquad \forall k = 0, 1, ..., K-1,\ t \in [0, T]. \tag{3.17}$$

We now prove that

$$v_{n+1}^{k}(t) - v_{n}^{k}(t) \geq v_{n+1}^{k+1}(t) - v_{n}^{k+1}(t), \qquad \forall k = 0, 1, ..., K-1,\ t \in [0, T]. \tag{3.18}$$

For any possible k and t, we rely on pools 1, 2, $\bar{1}$, and $\bar{2}$ that experience identical sample paths. At time t, pools 1 and $\bar{1}$ are with price index $k+1$, and pools 2 and $\bar{2}$ are with price index k. Also, pools 1 and $\bar{2}$ both have $n+1$ items, while pools $\bar{1}$ and 2 both have n items. For $s \in [t, T]$, let us use $N_i(s)$ to denote the inventory level of pool i at time s. For instance, we have $N_{\bar{1}}(t) = n$ and $N_{\bar{2}}(t) = n+1$. We let pools 1 and 2 execute their respective optimal decisions. For a certain period, we let pool $\bar{1}$ follow pool 1's decisions and pool $\bar{2}$ follow pool 2's decisions. Note that prices adopted by all pools are increasing over time.

We let this certain period end at the first moment, say $s \in [t, T)$, when (a) pools 1 and 2 have reached the same price, (b) pools 2 and $\bar{2}$ have just experienced one more arrival than pools 1 and $\bar{1}$, or (c) pools 1 and $\bar{2}$ both have just one item left while pools $\bar{1}$ and 2 have no item left. We may denote the case where none of the above occurs by $s = T$. This is the case when by time T, at any moment the price taken by pool 2 has always been strictly lower than that taken by pool 1, yet all pools have admitted the same demand arrivals, and none of the pools have run out of stock. We caution that the opposite to (b) will not occur, since before its price "catches up" with that of pool 1, pool 2 always charges a strictly lower price than pool 1 and hence, by (S1'), has more chance to realize sales.

For all cases, pools $\bar{1}$ and $\bar{2}$ will have together generated the same revenue as pools 1 and 2 by time s. This also means that we are already done when $s = T$.

When (a) ever occurs, we may let pools $\bar{1}$ and $\bar{2}$ both execute optimal decisions from time s on. Then, within the time interval $[s, T]$, pool $\bar{1}$ will behave exactly the same as pool 2, and pool $\bar{2}$ will behave exactly the same as pool 1. So, pools $\bar{1}$ and $\bar{2}$ will continue to together produce the same revenue as pools 1 and 2 do.

When (b) ever occurs, denote the price taken by pool 1 at time s by k_1 and the price taken by pool 2 at time s by k_2. We have $k_1 > k_2$, and

$$n + 1 = N_1(t) \geq N_1(s) = N_{\bar{1}}(s) + 1 = N_{\bar{2}}(s) + 1 = N_2(s) + 2. \tag{3.19}$$

Hence, from the induction hypothesis (3.17), we have

$$v^{k_2}_{N_{\bar{2}}(s)}(s) - v^{k_2}_{N_2(s)}(s) \geq v^{k_1}_{N_{\bar{2}}(s)}(s) - v^{k_1}_{N_2(s)}(s). \tag{3.20}$$

But from Proposition 3.2, we also have

$$v^{k_1}_{N_{\bar{2}}(s)}(s) - v^{k_1}_{N_2(s)}(s) \geq v^{k_1}_{N_1(s)}(s) - v^{k_1}_{N_{\bar{1}}(s)}(s). \tag{3.21}$$

Combining (3.20) and (3.21), we obtain

$$v^{k_2}_{N_{\bar{2}}(s)}(s) - v^{k_2}_{N_2(s)}(s) \geq v^{k_1}_{N_1(s)}(s) - v^{k_1}_{N_{\bar{1}}(s)}(s). \tag{3.22}$$

In view of the memorylessness property of the Poisson process, (3.22) means that, conditioned on the same sample path up to the provocation of (b), pools $\bar{1}$ and $\bar{2}$ will on average together earn more than pools 1 and 2 in the time interval $[s, T]$.

When (c) ever occurs, denote the price taken by pool 1 at time s by k_1 and the price taken by pool 2 at time s by k_2. We have $k_1 > k_2$, and

$$N_1(s) = N_{\bar{1}}(s) + 1 = N_{\bar{2}}(s) = N_2(s) + 1 = 1. \tag{3.23}$$

From the induction hypothesis (3.17), we have

$$v^{k_2}_{N_{\bar{2}}(s)}(s) - v^{k_2}_{N_2(s)}(s) \geq v^{k_1}_{N_1(s)}(s) - v^{k_1}_{N_{\bar{1}}(s)}(s). \tag{3.24}$$

In view of the memorylessness property of the Poisson process, (3.24) means that, conditioned on the same sample path up to the provocation of (c), pools $\bar{1}$ and $\bar{2}$ will on average together earn more than pools 1 and 2 in the time interval $[s, T]$.

In view of all the above, we see that (3.18) is true. We have hence completed the induction process. Therefore, (3.15) is true. $\square$

We now demonstrate the optimality of threshold policy obtained from the constructive procedure. For convenience, we let the yet undefined $\tau_n^{K+1} = 0$ for $n = 1, 2, ..., N$ and $v_n^{K+1}(t) = u_n^K(t)$ for $n = 0, 1, ..., N$ and $t \in [0, T]$.

Theorem 3.1. *The u and τ as constructed from (3.9) to (3.13) satisfy the following for $k = K, K-1, ..., 0$:*

(a[k]) for any $n = 1, 2, ..., N$, $(\mathcal{G}_n^k)^+(\tau_n^{k+1}) \circ v^{k+1} + \bar{p}^k \bar{\alpha}^k \cdot \beta(\tau_n^{k+1}) \leq 0$;

(b[k]) $\mathcal{G}_n^k(t) \circ v^{k+1}/\beta(t)$ is increasing in t for $n = 1, 2, ..., N$ and $t \in (0, T)$;

(c[k]) $u_n^k(t) = v_n^k(t)$ for any $n = 0, 1, ..., N$ and $t \in [0, T]$;

(d[k]) for any $n = 1, 2, ..., N$, we have $v_n^k(t) = v_n^{k+1}(t)$ and $\mathcal{G}_n^k(t) \circ v^k + \bar{p}^k \bar{\alpha}^k \cdot \beta(t) \leq 0$ for $t \in (0, \tau_n^{k+1})$, and $\mathcal{G}_n^k(t) \circ v^k + \bar{p}^k \bar{\alpha}^k \cdot \beta(t) = 0$ for $t \in (\tau_n^{k+1}, T)$;

(e[k]) τ_n^{k+1} is decreasing in n;

(f[k]) on top of $v_n^k(t)$ having decreasing differences between n and t, it is further true that $d_t v_n^k(t)/\beta(t)$ is decreasing in t for $n = 1, 2, ..., N$ and $t \in (0, T)$.

As a consequence, τ provides an optimal policy for the firm; under this policy, the firm should switch to price $\bar{p}^{k+1}$ when its inventory level drops to n before time τ_n^{k+1} while its price level is $\bar{p}^k$.

In this theorem, $(\mathcal{G}_n^k)^+$ is just $\mathcal{G}_n^k$ with d_t being replaced by d_t^+. c[k] declares that the constructed function $u_n^k(t)$ is just the true value function. The first half of d[k] states that the firm should charge $\bar{p}^{k+1}$ when it has inventory n before the threshold time τ_n^{k+1}, and the second half implies that the firm should choose $\bar{p}^k$ when it has inventory n and the time passes τ_n^{k+1}. The equation of the second half is the optimal condition, i.e. the change rate of value function is equal to the revenue rate at price $\bar{p}^k$, for choosing $\bar{p}^k$ between the time interval (τ_n^{k+1}, T). e[k] shows the inventory monotonicity of the threshold policy, implying that the more inventory the firm has, the earlier it should switch price. As we will see later, inventory monotonicity of the threshold policy holds for both irreversible and reversible cases, the time monotonicity only exists in the reversible case. f[k] says that the ability of an additional unit to capture revenue will fade away as time passes. The last sentence of this theorem tells how the concerned firm should apply this optimal pricing strategy. Once another demand comes, check τ_{n-1}^i for $i = 1, 2, ..., K$ to see if price can be raised. We provide the proof of this theorem in the following. The proof uses Proposition 3.3.

Proof: The proof has use of the following lemma, which was originated in Karlin (1968) and also used in Feng and Xiao (2000b).

Lemma 3.1. *Let $k > 0$ and $\phi(t) = \int_t^{+\infty} \rho(s) \cdot exp(-k \cdot (s-t)) \cdot ds$. Then, $\phi(t)$ will be decreasing in $t \geq 0$ if $\rho(s)$ is decreasing in $s \geq 0$.*

We prove by induction on k. Let us focus on proving (a[K]) to (f[K]) first. Take $n = 1, 2, ..., N$. From (3.4), we have

$$v_n^K(t) = \bar{p}^K \cdot E[(N^K(T) - N^K(t)) \wedge n] = \bar{p}^K \cdot (n - \sum_{m=0}^{n-1} (n-m) \cdot P[N^K(t,T) = m]), \quad (3.25)$$

which, by (3.1), amounts to

$$v_n^K(t) = \bar{p}^K \cdot n - \bar{p}^K \cdot \exp(-\bar{\alpha}^K \cdot \hat{\beta}(t,T)) \cdot \sum_{m=0}^{n-1} \frac{(n-m) \cdot (\bar{\alpha}^K \cdot \hat{\beta}(t,T))^m}{m!}. \quad (3.26)$$

Taking derivative over t, we find that, for $t \in (0, T)$,

$$d_t v_n^K(t) = -\bar{p}^K \bar{\alpha}^K \cdot \beta(t) \cdot \exp(-\bar{\alpha}^K \cdot \hat{\beta}(t,T)) \cdot \sum_{m=0}^{n-1} \frac{(\bar{\alpha}^K \cdot \hat{\beta}(t,T))^m}{m!}. \quad (3.27)$$

Taking differences over n, we find that

$$v_n^K(t) - v_{n-1}^K(t) = \bar{p}^K \cdot (1 - \exp(-\bar{\alpha}^K \cdot \hat{\beta}(t,T)) \cdot \sum_{m=0}^{n-1} \frac{(\bar{\alpha}^K \cdot \hat{\beta}(t,T))^m}{m!}). \quad (3.28)$$

From (3.27) and (3.28), it can be checked that

$$\mathcal{G}_n^K(t) \circ v^K(t) + \bar{p}^K \bar{\alpha}^K \cdot \beta(t) = 0, \qquad \forall t \in (0, T). \quad (3.29)$$

Consulting (3.7) and (3.8), we obtain

$$v_n^K(t) = \bar{\alpha}^K \cdot \int_t^T \beta(s) \cdot (\bar{p}^K + v_{n-1}^K(s)) \cdot \exp(-\bar{\alpha}^K \cdot \hat{\beta}(t,s)) \cdot ds. \tag{3.30}$$

In view of the construction (3.10), we may see (c[K]), that

$$u^K = (u_n^K(t) \mid n = 0,1,...,N,\ t \in [0,T]) = v^K = (v_n^K(t) \mid n = 0,1,...,N,\ t \in [0,T]). \tag{3.31}$$

From (3.29), we may confirm (d[K]) with the understanding that $\tau_n^{K+1} = 0$ for $n = 1,2,...,N$. The convention for the τ_n^{K+1}'s also leads directly (e[K]). By (3.29) and the convention on τ_n^{K+1} and $v_n^{K+1}(t)$, we may see that (a[K]) and (b[K]) are both true. To verify (f[K]), we can use the same method on (f[k]) for $k = K-1, K-2, ..., 0$, which is presented near the end of this proof.

Suppose for some $k = K-1, K-2, ..., 0$, we have (a[$k+1$]), that

$$(\mathcal{G}_n^{k+1})^+(\tau_n^{k+2}) \circ v^{k+2} + \bar{p}^{k+1}\bar{\alpha}^{k+1} \cdot \beta(\tau_n^{k+2}) \leq 0, \qquad \forall n = 1,2,...,N, \tag{3.32}$$

(b[$k+1$]), that $\mathcal{G}_n^{k+1}(t) \circ v^{k+2}/\beta(t)$ is increasing in t for $n = 1,2,...,N$ and $t \in (0,T)$, (c[$k+1$]), that

$$u_n^{k+1}(t) = v_n^{k+1}(t), \qquad \forall n = 0,1,...,N,\ t \in [0,T], \tag{3.33}$$

(d[$k+1$]), that, for $n = 1,2,...,N$,

$$v_n^{k+1}(t) = v_n^{k+2}(t) \text{ and } \mathcal{G}_n^{k+1}(t) \circ v^{k+1} + \bar{p}^{k+1}\bar{\alpha}^{k+1} \cdot \beta(t) \leq 0, \qquad \forall t \in (0, \tau_n^{k+2}), \tag{3.34}$$

and

$$\mathcal{G}_n^{k+1}(t) \circ v^{k+1} + \bar{p}^{k+1}\bar{\alpha}^{k+1} \cdot \beta(t) = 0, \qquad \forall t \in (\tau_n^{k+2}, T). \tag{3.35}$$

(e[$k+1$]), that τ_n^{k+2} is decreasing in n, and (f[$k+1$]), that $v_n^{k+1}(t)$ has decreasing differences between n and t.

Now we embark on showing (a[k]) to (f[k]). For $n = 1, 2, ..., N$, by the definition of τ_n^{k+1} through (3.12), we know that

$$u_n^k(t) > v_n^{k+1}(t), \qquad \forall t \in (\tau_n^{k+1}, T), \tag{3.36}$$

$$u_n^k(\tau_n^{k+1}) = v_n^{k+1}(\tau_n^{k+1}), \tag{3.37}$$

and

$$d_t^+ u_n^k(\tau_n^{k+1}) \geq d_t^+ v_n^{k+1}(\tau_n^{k+1}). \tag{3.38}$$

By (3.7) and (3.8), we may see that the construction (3.11) renders

$$\mathcal{G}_n^k(t) \circ u^k + \bar{p}^k \bar{\alpha}^k \cdot \beta(t) = 0, \qquad \forall t \in (\tau_n^{k+1}, T). \tag{3.39}$$

Our construction through (3.11) to (3.13) also guarantees that

$$u_{n-1}^k(\tau_n^{k+1}) \geq v_{n-1}^{k+1}(\tau_n^{k+1}). \tag{3.40}$$

Combining (3.37), (3.38), (3.39), and (3.40), we obtain

$$(\mathcal{G}_n^k)^+(\tau_n^{k+1}) \circ v^{k+1} + \bar{p}^k \bar{\alpha}^k \cdot \beta(\tau_n^{k+1}) \leq 0. \tag{3.41}$$

Thus we have (a[k]).

Note that (a[$k+1$]) means (3.32). This, (b[$k+1$]), and (d[$k+1$]) together lead to the fact that $\mathcal{G}_n^{k+1}(t) \circ v^{k+2}/\beta(t)$ is increasing in t and below $-\bar{p}^{k+1}\bar{\alpha}^{k+1}$. From the definition of $u_n^{k+1}(t)$ for $t \in [0, \tau_n^{k+2}]$ through (3.13), (c[$k+1$]), and (e[$k+1$]), we may see that

$$\frac{\mathcal{G}_n^{k+1}(t)\circ v^{k+1}}{\beta(t)} = \frac{\mathcal{G}_n^{k+1}(t)\circ v^{k+2}}{\beta(t)}, \qquad \forall t\in(0,\tau_n^{k+2}). \tag{3.42}$$

Hence, in view of the above and (b$[k+1]$) again, we may see that $\mathcal{G}_n^{k+1}(t)\circ v^{k+1}/\beta(t)$ is increasing in t and below $-\bar{p}^{k+1}\bar{\alpha}^{k+1}$ when $t\in(0,\tau_n^{k+2})$, and is flat at $-\bar{p}^{k+1}\bar{\alpha}^{k+1}$ for $t\in(\tau_n^{k+2},T)$. Now, note that

$$\frac{\mathcal{G}_n^{k}(t)\circ v^{k+1}-\mathcal{G}_n^{k+1}(t)\circ v^{k+1}}{\beta(t)} = (\bar{\alpha}^k-\bar{\alpha}^{k+1})\cdot(v_{n-1}^{k+1}(t)-v_n^{k+1}(t)), \tag{3.43}$$

which, by (S1') and (f$[k+1]$), is increasing in t. This and the just proved result together lead to the increase of $\mathcal{G}_n^k(t)\circ v^{k+1}/\beta(t)$ in t. Hence, we have (b$[k]$).

From (a$[k]$) and (b$[k]$), we obtain, for $n=1,2,...,N$,

$$\mathcal{G}_n^k(t)\circ v^{k+1}+\bar{p}^k\bar{\alpha}^k\cdot\beta(t)\le 0, \qquad \forall t\in(0,\tau_n^{k+1}). \tag{3.44}$$

Our construction (3.13) and (c$[k+1]$) dictate that

$$u_n^k(t)=v_n^{k+1}(t), \qquad \forall t\in[0,\tau_n^{k+1}]. \tag{3.45}$$

By its construction, $u_n^k(t)$ is uniformly bounded by $N\overline{\lambda}T$; it is also Lipschitz continuous in t with coefficient $N\overline{\lambda}$, and hence absolutely continuous in t. By (3.36), (3.39), (3.44), and (3.45), and as well as the fact that $v_n^{k+1}(T)=0$ for every n, we may see that $u^k\equiv(u_n^k(t)\mid n=0,1,...,N,\ t\in[0,T])$ satisfies the sufficient conditions (i) to (iv) stipulated in Proposition 3.1. Hence, we have shown (c$[k]$), that $u_n^k(t)=v_n^k(t)$ for every $n=0,1,...,N$ and $t\in[0,T]$.

From (3.39), (3.44), (3.45), and (c$[k]$), we easily have (d$[k]$).

For $n = 1, 2, ..., N-1$, we have, from (3.12), (c[$k+1$]), and (c[k]),

$$v_n^k(t) - v_n^{k+1}(t) > 0, \qquad \forall t \in (\tau_n^{k+1}, T). \tag{3.46}$$

By Proposition 3.3, however, we have

$$v_{n+1}^k(t) - v_{n+1}^{k+1}(t) \geq v_n^k(t) - v_n^{k+1}(t). \tag{3.47}$$

Combining (3.46) and (3.47), we obtain

$$v_{n+1}^k(t) - v_{n+1}^{k+1}(t) > 0, \qquad \forall t \in (\tau_n^{k+1}, T). \tag{3.48}$$

But in view of (3.12), (c[$k+1$]), and (c[k]), this leads to $\tau_{n+1}^{k+1} \leq \tau_n^{k+1}$. Therefore, we have (e[$k$]).

Let us now turn to the proof of (f[k]). For convenience, we denote $d_t v_n^k(t)/\beta(t)$ by $w_n^k(t)$. When $t \in (0, \tau_n^{k+1})$, which is $\emptyset$ when $k = K$, we have $w_n^k(t) = w_n^{k+1}(t)$ from (d[k]). Hence, following (f[$k+1$]), we know that $w_n^k(t)$ is decreasing in t. Let $t \in (\tau_n^{k+1}, T)$ then. By (d[k]), we know

$$v_n^k(t) - v_{n-1}^k(t) = \bar{p}^k + \frac{w_n^k(t)}{\bar{\alpha}^k}. \tag{3.49}$$

By the boundary conditions $v_n^k(T) = v_{n-1}^k(T) = 0$, we therefore have

$$w_n^k(T^-) = -\bar{p}^k \bar{\alpha}^k. \tag{3.50}$$

Taking derivative on (3.49), it follows that

$$\mathcal{G}_n^k(t) \circ w^k = 0, \qquad \forall t \in (\tau_n^{k+1}, T). \tag{3.51}$$

In view of (3.7), (3.8), and (3.50), we can solve (3.51) to obtain, for $t \in (\tau_n^{k+1}, T)$,

$$w_n^k(t) = -\bar{p}^k \bar{\alpha}^k \cdot \exp(-\bar{\alpha}^k \cdot \hat{\beta}(t,T)) + \bar{\alpha}^k \cdot \int_t^T \beta(s) \cdot w_{n-1}^k(s) \cdot \exp(-\bar{\alpha}^k \cdot \hat{\beta}(t,s)) \cdot ds, \quad (3.52)$$

which, by the identity

$$\bar{\alpha}^k \cdot \int_t^T \beta(s) \cdot \exp(-\bar{\alpha}^k \cdot \hat{\beta}(t,s)) \cdot ds = 1 - \exp(-\bar{\alpha}^k \cdot \hat{\beta}(t,T)), \quad (3.53)$$

results in

$$w_n^k(t) = -\bar{p}^k \bar{\alpha}^k + \bar{\alpha}^k \cdot \int_t^T \beta(s) \cdot (\bar{p}^k \bar{\alpha}^k + w_{n-1}^k(s)) \cdot \exp(-\bar{\alpha}^k \cdot \hat{\beta}(t,s)) \cdot ds. \quad (3.54)$$

Since $\hat{\beta}(0,\cdot)$ is a strictly increasing function on $[0,T]$, we can define strictly increasing function $\theta(\cdot)$ on $[0, \hat{\beta}(0,T)]$, so that

$$\hat{\beta}(t, \theta(y)) = \hat{\beta}(0, \theta(y)) - \hat{\beta}(0,t) = y - \hat{\beta}(0,t), \qquad \forall y \in [0, \hat{\beta}(0,T)]. \quad (3.55)$$

This then leads to

$$d_y \theta(y) = \frac{1}{d_s \hat{\beta}(0,s)} \mid_{s=\theta(y)} = \frac{1}{\beta(\theta(y))}. \quad (3.56)$$

In view of the above, we can bring a change of variables to the integral involved in (3.54), so that the latter becomes

$$w_n^k(t) = -\bar{p}^k \bar{\alpha}^k + \bar{\alpha}^k \cdot \int_{\hat{\beta}(0,t)}^{\hat{\beta}(0,T)} (\bar{p}^k \bar{\alpha}^k + w_{n-1}^k(\theta(y))) \cdot \exp(-\bar{\alpha}^k \cdot (y - \hat{\beta}(0,t))) \cdot dy. \quad (3.57)$$

Suppose $w_{n-1}^k(t)$ is decreasing in t for $t \in (0,T)$, then since $\theta(\cdot)$ is increasing, we know $w_{n-1}^k(\theta(y))$ is decreasing in y for $y \in (0, \hat{\beta}(0,T))$. From (3.50) which also applies to $w_{n-1}^k(T^-)$, we may see that

$$\bar{p}^k \bar{\alpha}^k + w_{n-1}^k(\theta(y)) \geq 0, \qquad \forall y \in (0, \hat{\beta}(0,T)), \quad (3.58)$$

and

$$\bar{p}^k\bar{\alpha}^k + w^k_{n-1}(\theta((\hat{\beta}(0,T))^-)) = 0. \tag{3.59}$$

Hence, (3.57) can be rewritten as

$$w^k_n(t) = -\bar{p}^k\bar{\alpha}^k + \bar{\alpha}^k \cdot \int_{\hat{\beta}(0,t)}^{+\infty} (\bar{p}^k\bar{\alpha}^k + w^k_{n-1}(\theta(y)))^+ \cdot \exp(-\bar{\alpha}^k \cdot (y - \hat{\beta}(0,t))) \cdot dy. \tag{3.60}$$

By the decrease of $(\bar{p}^k\bar{\alpha}^k + w^k_{n-1}(\theta(y)))^+$ in $y \geq 0$, the increase of $\hat{\beta}(0,t)$ in t, and Lemma 3.1, we can get the decrease of $w^k_n(t)$ in t on (τ^{k+1}_n, T). But combining with the earlier result on the other half interval, we may get the decrease of $w^k_n(t)$ in t on the entire $(0,T)$ from that of $w^k_{n-1}(t)$. As $w^k_0(t) = 0$ for all $t \in (0,T)$, we can therefore use induction on n to prove the decrease of $w^k_n(t)$ in $t \in (0,T)$ for all $n = 0, 1, ..., N$.

For $t \in (0, \tau^{k+1}_n)$, which is $\emptyset$ when $k = K$ and a subset of $(0, \tau^{k+1}_{n-1})$ by (e[k]), we have, by (d[k]),

$$v^k_n(t) - v^k_{n-1}(t) = v^{k+1}_n(t) - v^{k+1}_{n-1}(t). \tag{3.61}$$

Hence, $v^k_n(t) - v^k_{n-1}(t)$ is decreasing in t by (f[$k+1$]). For $t \in (\tau^{k+1}_n, T)$, we can achieve the same property by (3.49) and the just proved decrease of $w^k_n(t)$ in t. Thus, we have shown (f[k]). Note that, when $k = K$, the proof has no involvement of any [$k+1$]-property.

We have now completed the induction process. Therefore, (a[k]) to (f[k]) are all true for $k = K, K-1, ..., 0$. From these, we see that, for any $k = K-1, K-2, ..., 0$ and $n = 1, 2, ..., N$,

$$v^k_n(t) \begin{cases} = v^{k+1}_n(t), & \forall t \in [0, \tau^{k+1}_n], \\ > v^{k+1}_n(t), & \forall t \in (\tau^{k+1}_n, T). \end{cases} \tag{3.62}$$

Hence, we may see that each τ^{k+1}_n offers an optimal time by which the firm is to raise its price from $\bar{p}^k$ to $\bar{p}^{k+1}$ when it has n remaining items. $\square$

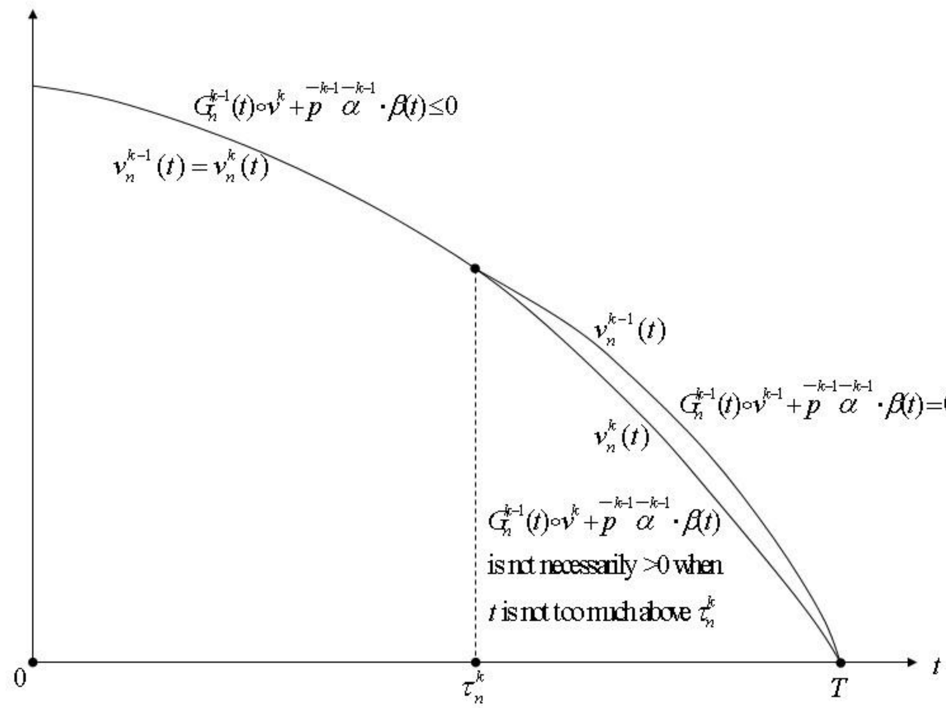

Figure 3.1: Illustration for the Markup Case

While the threshold point τ_n^k is a zero-crossing point for $v_n^{k-1}(t)-v_n^k(t)$ for the markup case, it is not necessarily one for $\mathcal{G}_n^{k-1}(t)\circ v^k+\bar{p}^{k-1}\bar{\alpha}^{k-1}\cdot\beta(t)$: Though the term is below 0 when $t \in (0, \tau_n^k)$, we can verify through computation that the term is not necessarily above 0 when $t \in (\tau_n^k, T)$ is not too much above τ_n^k. This marks a huge contrast with the markdown case, for which we can learn from chapter 3.3 that the threshold point τ_n^k is the zero-crossing point for both $\mathcal{G}_n^k(t) \circ v^{k-1} + \bar{p}^k\bar{\alpha}^k \cdot \beta(t)$ and $v_n^k(t) - v_n^{k-1}(t)$; see Figures 3.1 and 3.2 for a demonstration of this discrepancy. This previously unnoticed point determines that, in order to obtain the n-monotone pattern of the τ_n^k points for the markup case, we have to deal with the dependence of $v_n^{k-1}(t) - v_n^k(t)$ on n rather than that of $\mathcal{G}_n^{k-1}(t) \circ v^k$.

Feng and Xiao's (2000b) treatment of the stationary-demand markup case relied on properties of $\mathcal{G}_n^{k-1}(t) \circ v^k$ rather than those of $v_n^{k-1}(t) - v_n^k(t)$. Temporarily, let us consider the stationary-demand case where $\beta(t) = 1$ for all $t \in [0, T]$. Their Lemma 1 claimed that the increase of $\mathcal{G}_n^{k-1}(t) \circ v^k$ in t and n alone, without other benefits

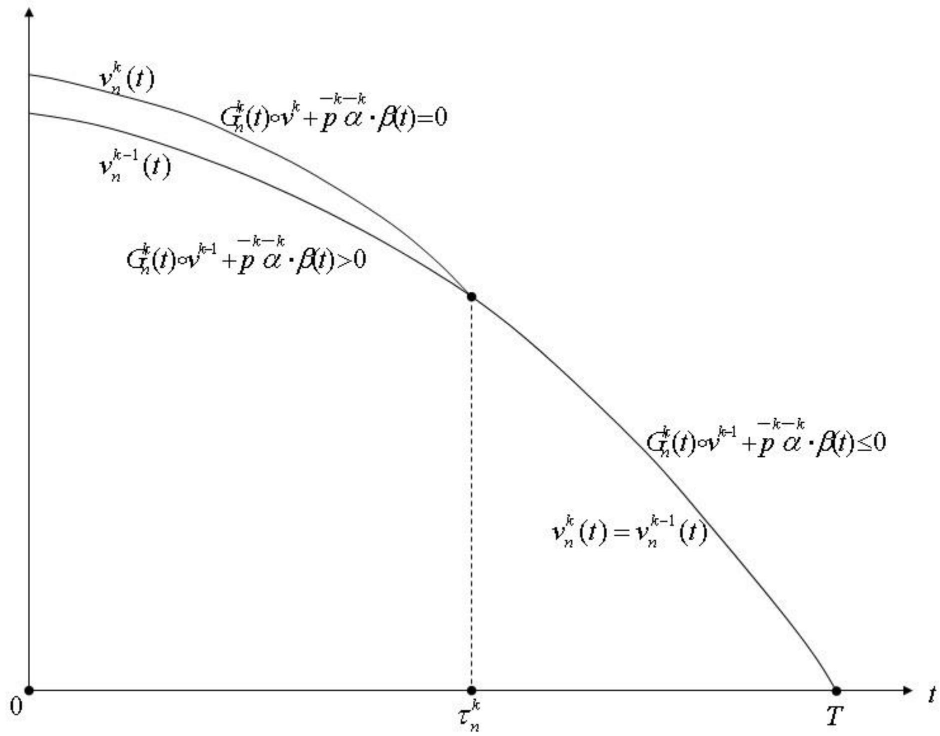

Figure 3.2: Illustration for the Markdown Case

that might come from the $v_n^k(\cdot)$'s being truly value functions of a markup problem, would lead to optimal threshold levels τ_n^k that necessarily satisfy $0 \leq \tau_N^k \leq \tau_{N-1}^k \leq \cdots \tau_1^k \leq T$. While its counterpart for the markdown case is correct, we have a counter example in the following to the current claim.

Consider an example with $T = 1$, $K = 1$, $N = 2$, and $\bar{\alpha}^0 = 2$. Let $v_0^1(t) = 0$, $v_1^1(t) = -2t + 2$, and $v_2^1(t) = t^2 - 4t + 3$. We can check that $d_t v_1^1(t) = -2$ and $d_t v_2^1(t) = 2t - 4$. These lead to $\mathcal{G}_1^0(t) \circ v^1 = 4t - 6$ and $\mathcal{G}_2^0(t) \circ v^1 = -2t^2 + 6t - 6$. Hence, $\mathcal{G}_1^0(t) \circ v^1$ and $\mathcal{G}_2^0(t) \circ v^1$ are both increasing in $t \in [0, 1]$. In addition, $\mathcal{G}_2^0(t) \circ v^1 - \mathcal{G}_1^0(t) \circ v^1 = -2t^2 + 2t$, which is positive for $t \in [0, 1]$. That is, $\mathcal{G}_n^0(t) \circ v^1$ is increasing in n too. On the other hand, we may let $v_1^0(t) = -t^2 - t + 2$ and $v_2^0(t) = v_2^1(t) = t^2 - 4t + 3$. Now, $v_1^0(t) - v_1^1(t) = -t^2 + t$, which is strictly positive for $t \in (0, 1)$, and $v_2^0(t) - v_2^1(t) = 0$. Thus, it is not true that $v_1^0(t) - v_1^1(t) \leq v_2^0(t) - v_2^1(t)$ on $t \in [0, 1]$. Among other violations, the last point consists of a violation of Proposition 3.3. So the possibility that these $v_n^k(\cdot)$'s form actual value functions for a markup problem has been ruled out.

When we pretend that these $v_n^k(t)$'s form value functions for a markup problem, however, we should have τ_1^1 as the smallest t such that $v_1^0(t) - v_1^1(t) > 0$ and τ_2^1 the smallest t such that $v_2^0(t) - v_2^1(t) > 0$; also, each threshold level should be set at 0 when the corresponding strict positivity is always true and set at $T = 1$ when the corresponding strict positivity is never true. Therefore, we should have $\tau_1^1 = 0$ and $\tau_2^1 = 1$ for this example. It is therefore not true that $\tau_1^1 \geq \tau_2^1$. But the latter is required for a threshold policy.

The above should suffice as a main justification for our new approach for the markup case. In this approach, Proposition 3.3 has been set aside for the sole purpose of illustrating the complementarity between price flexibility and inventory, a property that is both previously unknown and not possessed by the seemingly symmetric markdown case. In addition, property (a[k]) in Theorem 3.1, concerning right derivatives of the value functions, has no counterpart in earlier literature. Moreover, Feng and Xiao (2000b) used the increase of $\mathcal{G}_n^{k-1}(t) \circ v_n^k$ in n (their Lemmas 1 and 2) that we have no use of. Our computational studies confirmed that this is not necessarily true.

From Theorem 3.1, we know that the threshold policy holds the inventory monotonicity which is $\tau_n^k \leq \tau_{n-1}^k$. It may be straightforward to think that it also possesses time monotonicity or k monotonicity, i.e. $\tau_n^{k+1} \leq \tau_n^k$. Time monotonicity says the firm should switch to a higher price at an earlier time for a given inventory. However, it is not always true, as we can see from the following result which is directly from the definition (3.12) and Theorem 3.1.

Proposition 3.4. *Suppose $\tau_n^k < T$ for some $k = 1, 2, ..., K-1$ and $n = 1, 2, ..., N$. Then, we have $\tau_n^{k+1} \leq \tau_n^k$ if and only if $v_n^{k-1}(t) - v_n^k(t) > 0$ will lead to $v_n^k(t) - v_n^{k+1}(t) > 0$.*

As confirmed by one of our computational studies, the threshold level τ_n^k for the markup case is not necessarily decreasing in k. Hence, this case may have its "leapfrog" phenomenon: When it is time to switch to the price level $\bar{p}^{k+1}$, it may also be the time to switch to the next higher price $\bar{p}^{k+2}$, and so on and so forth. Therefore, when it is time to make the price swith from $\bar{p}^k$, the ultimate target should be some $\bar{p}^{\tilde{k}^+(k,n)}$, where $\tilde{k}^+(k,n)$ is not necessarily $k+1$. For each $n = 1, 2, ..., N$, we can use the following iterative procedure to find $(\tilde{k}^+(k,n) \mid k = 0, 1, ..., K-1)$:

for $k = K-1$ down to 0
  let $l = k+1$;
  while $l \leq K-1$ and $\tau_n^{l+1} \geq \tau_n^{k+1}$ **do**
    let $l = \tilde{k}^+(l,n)$;
  let $\tilde{k}^+(k,n) = l$.

3.2.3 Algorithm

We now can conclude the constructed threshold policy is optimal and the constructed value function is true. Based on the construction procedure, we can establish an efficient and numerical stable algorithm to calculate the optimal threshold policy and the value function. The time interval $[0,T]$ is discretized by a grid $0, \Delta T, 2\Delta T, ..., Q \cdot \Delta T$, where Q is a large positive integer and $\Delta T = T/Q$. For $k = 0, 1, ..., K$ and $q = 0, 1, ..., Q$, let λ_q^k be $\lambda^k(q \cdot \Delta T)$ and v_{nq}^k be $v_n^k(q \cdot \Delta T)$. We note that for $0 \leq t_1 \leq t_2 \leq T$ and certain k values, (3.10) and (3.11) will lead to

$$u_n^k(t_1) = \int_{t_1}^{t_2} \lambda^k(s) \cdot (\bar{p}^k + u_{n-1}^k(s)) \cdot \exp(-\hat{\lambda}^k(t_1, s)) \cdot ds + \exp(-\hat{\lambda}^k(t_1, t_2)) \cdot u_n^k(t_2), \quad (3.63)$$

where $\hat{\lambda}^k(s,t)$ means $\int_s^t \lambda^k(u) \cdot du$. When $t_2 = t_1 + \Delta T$ and ΔT is very small, (3.63) can be approximated by

$$u_n^k(t_1) = \lambda^k(t_1) \cdot (\bar{p}^k + u_{n-1}^k(t_1)) \cdot \Delta T + \exp(-\lambda^k(t_1) \cdot \Delta t)) \cdot u_n^k(t_2). \quad (3.64)$$

Then, we can follow the recursive procedure described from (3.9) to (3.13) to the following algorithm *Markup1*.

for $k = 0$ to K

  for $q = 0$ to Q

    let $v^k_{0q} = 0$;

for $n = 1$ to N

  let $v^K_{nQ} = 0$;

  for $q = Q - 1$ down to 0

    let $v^K_{nq} = \lambda^K_q \cdot \Delta T \cdot (\bar{p}^K + v^K_{n-1,q}) + \exp(-\lambda^K_q \cdot \Delta T) \cdot v^K_{n,q+1}$;

for $k = K - 1$ down to 0

  for $n = 1$ to N

    let $q = Q$ and $v^k_{nq} = 0$;

    while $q = Q$, or $q \geq 0$ and $v^k_{nq} > v^{k+1}_{nq}$ **do**

      let $q = q - 1$;

      if $q \geq 0$

        let $v^k_{nq} = \lambda^k_q \cdot \Delta T \cdot (\bar{p}^k + v^k_{n-1,q}) + \exp(-\lambda^k_q \cdot \Delta T) \cdot v^k_{n,q+1}$;

    let $\tau^{k+1}_n = (q + 1) \cdot \Delta T$;

    for $r = q$ down to 0

      let $v^k_{nr} = v^{k+1}_{nr}$.

The algorithm's time complexity is apparently $O(KNQ)$.

3.3 The Markdown Case

In the markdown case, the firm has to consecutively charge a decreasing sequence of prices $\bar{p}^K, \bar{p}^{K-1}, ..., \bar{p}^0$. For example, a fashion product such as a down jacket will lose attractiveness as spring comes. The retailers would try to deplete its inventory by providing discounts or rebates when the time closes to the end of winter. Again, we establish a threshold policy $\tau = (\tau_n^k \mid k = 1, 2, ..., K,\ n = 1, 2, ..., N) \in (\Delta_N)^K$ and the value function $v_n^k(t)$ for $k = 0, 1, ..., K,\ n = 0, 1, ..., N$, and $t \in [0, T]$. As addressed earlier, under such threshold policy, the firm should switch its price to $\bar{p}^{k-1}$ when it charges price $\bar{p}^k$ with inventory n and the time has passed τ_n^k. Also, $v_n^k(t)$ is the maximum revenue the firm can make during the time interval $[t, T]$ when it starts at time t with price $\bar{p}^k$ and inventory level n.

We know the last price a firm can charge, before running out of stock, is the lowest prie $\bar{p}^0$. Then, we have

$$v_n^0(t) = \bar{p}^0 \cdot E[(N^0(T) - N^0(t)) \wedge n]. \tag{3.65}$$

When the firm is charging any other price $\bar{p}^k$ for $k = 1, 2, ..., K$, it has yet to dynamically decide the time to switch to the next price $\bar{p}^{k-1}$. Hence, we have

$$v_n^k(t) = \sup_{\tau \in \mathcal{T}} E[\bar{p}^k \cdot \{(N^k(\tau) - N^k(t)) \wedge n\} + v^{k-1}_{(n - N^k(\tau) + N^k(t))^+}(\tau)]. \tag{3.66}$$

Here, stopping time τ is the moment that the firm will switch price from p^k to p^{k-1}, $k = 1, 2, ..., K$.

3.3.1 A Constructing Procedure

Similar to the markup case, we have the following proposition to offer the hint of how to construct the threshold policy and value function.

Proposition 3.5. *For any $k = 1, 2, ..., K$, a function vector $u \equiv (u_n(t) \mid n = 0, 1, ..., N,\ t \in [0, T])$ that is uniformly bounded and absolutely continuous in t for every n will be the value-function sub-vector $v^k \equiv (v_n^k(t) \mid n = 0, 1, ..., N,\ t \in [0, T])$, if it satisfies the following:*

(i) $u_n(t) \geq v_n^{k-1}(t)$ for every $n = 0, 1..., N$ and $t \in [0, T]$,

(ii) $u_n(T) = 0$ *for every* $n = 0, 1, ..., N$ *and* $u_0(t) = 0$ *for every* $t \in [0, T]$,

(iii) for $n = 1, 2, ..., N$ *and* $t \in [0, T]$, $u_n(t) = v_n^{k-1}(t)$ *implies* $\mathcal{G}_n^k(t) \circ u + \bar{p}^k \bar{\alpha}^k \cdot \beta(t) \leq 0$,

(iv) for $n = 1, 2, ..., N$ *and* $t \in [0, T]$, $u_n(t) > v_n^{k-1}(t)$ *implies* $\mathcal{G}_n^k(t) \circ u + \bar{p}^k \bar{\alpha}^k \cdot \beta(t) = 0$.

Here is the idea of establishing the value functions $v_n^k(t)$ and threshold levels τ_n^k. First, due to (3.65), $v_n^0(t)$ can be constructed in an n-loop as $v_n^K(t)$ was in the markup case.

Then, for any $k = 1, 2, ..., K$, suppose $v_n^{k-1}(t)$ is known. We can next go through an n-loop to find all the $v_n^k(t)$'s. First, let $v_0^k(t) = 0$ as suggested by (ii) of the proposition. Second, suppose $v_{n-1}^k(t)$ is known for some $n = 1, 2, ..., N$, we can equate $v_n^k(t)$ to $v_n^{k-1}(t)$ for bigger t values, starting with $t = T$, until it is to occur that $\mathcal{G}_n^k(t) \circ v^k + \bar{p}^k \bar{\alpha}^k \cdot \beta(t) > 0$. The latter is not allowed by (iii) and (iv) of the proposition. For time t' earlier than this t, which we mark as τ_n^k, we let $v_n^k(t')$ be the solution to $\mathcal{G}_n^k(t') \circ v^k + \bar{p}^k \bar{\alpha}^k \cdot \beta(t') = 0$.

According to (i) of the proposition, we still need $v_n^k(t') \geq v_n^{k-1}(t')$ for $t' < \tau_n^k$ for the thus constructed $v_n^k(\cdot)$ to be the true value function. Nevertheless, let us go ahead with the construction procedure thus outlined. Not knowing whether what shall be constructed are the true value functions, we call them u's instead of v's. Formally, here is the iterative procedure.

First, let

$$u_0^k(t) = 0, \qquad \forall k = 0, 1, ..., K, \; t \in [0, T]. \tag{3.67}$$

Then, for $n = 1, 2, ..., N$ and $t \in [0, T]$, let

$$u_n^0(t) = \bar{\alpha}^0 \cdot \int_t^T \beta(s) \cdot (\bar{p}^0 + u_{n-1}^0(s)) \cdot \exp(-\bar{\alpha}^0 \cdot \hat{\beta}(t, s)) \cdot ds. \tag{3.68}$$

Next, we go over an outer loop on $k = 1, 2, ..., K$ and an inner loop on $n = 1, 2, ..., N$. At each k and n, first let

$$\tau_n^k = \inf\{t \in [0, T] \mid \mathcal{G}_n^k(t) \circ u^{k-1} + \bar{p}^k \bar{\alpha}^k \cdot \beta(t) \leq 0\}, \tag{3.69}$$

with the understanding that $\tau_n^k = 0$ when the concerned inequality is always true and $\tau_n^k = T$ when it is never true. Then, let

$$u_n^k(t) = u_n^{k-1}(t), \qquad \forall t \in [\tau_n^k, T], \tag{3.70}$$

and when $t \in [0, \tau_n^k)$,

$$u_n^k(t) = u_n^{k-1}(\tau_n^k) \cdot \exp(-\bar{\alpha}^k \cdot \hat{\beta}(t, \tau_n^k)) + \bar{\alpha}^k \cdot \int_t^{\tau_n^k} \beta(s) \cdot (\bar{p}^k + u_{n-1}^k(s)) \cdot \exp(-\bar{\alpha}^k \cdot \hat{\beta}(t, s)) \cdot ds. \tag{3.71}$$

3.3.2 Optimality and Characteristics

To prove the optimality of the threshold policy and value function, we also need the concavity of the value function.

Proposition 3.6. *For any fixed $k = 0, 1, ..., K$ and $t \in [0, T]$, the value function $v_n^k(t)$ is concave in n.*

This proposition implies the same meaning as Proposition 3.2. The marginal revenue of unit product is decreasing in inventory. The more inventory the firm has, the less marginal revenue it captures. The decreasing margin of revenue always appears in the literature of economics. Since the proof is similar to Proposition 3.2, we omit it here.

Now, we introduce the main result of markdown case in Theorem 3.2. For convenience, we have let the yet undefined $\tau_n^0 = T$ for $n = 1, 2, ..., N$ and $v_n^{-1}(t) = u_n^0(t)$ for

$n = 1, 2, ..., N$ and $t \in [0, T]$.

Theorem 3.2. *The u and τ as constructed from (3.67) to (3.71) satisfy the following for $k = 0, 1, ..., K$:*

(a[k]) $\mathcal{G}_n^k(t) \circ v^{k-1}/\beta(t)$ is decreasing in t for $n = 1, 2, ..., N$ and $t \in (0, T)$;

(b[k]) $u_n^k(t) = v_n^k(t)$ for any $n = 0, 1, ..., N$ and $t \in [0, T]$;

(c[k]) for any $n = 1, 2, ..., N$, we have $\mathcal{G}_n^k(t) \circ v^k + \bar{p}^k \bar{\alpha}^k \cdot \beta(t) = 0$ for $t \in (0, \tau_n^k)$ and $\mathcal{G}_n^k(t) \circ v^k + \bar{p}^k \bar{\alpha}^k \cdot \beta(t) \leq 0$ for $t \in (\tau_n^k, T)$;

(d[k]) τ_n^k is decreasing in n;

(e[k]) more than having decreasing differences between n and t, $v_n^k(t)$ satisfies the following for every $t \in (0, T)$:

$$d_t v_1^k(t) \leq 0,$$

and for $n = 1, 2, ..., N - 1$,

$$d_t v_{n+1}^k(t) - d_t v_n^k(t) \leq \bar{\alpha}^k \cdot \beta(t) \cdot (v_{n-1}^k(t) - 2v_n^k(t) + v_{n+1}^k(t)),$$

which is negative due to Proposition 3.6.

As a consequence, τ provides an optimal policy for the firm; under this policy, the firm should switch to price $\bar{p}^{k-1}$ when time t hits τ_n^k while its price level is $\bar{p}^k$ and inventory level is n.

In Theorem 3.2, b[k] confirms that the constructed function $u_n^k(t)$ is the true value function. c[k] shows the structural characteristic of the threshold points. Its first half tells the firm to charge price $\bar{p}^k$ before time τ_n^k when it has n items; its second half tells the firm to charge price $\bar{p}^{k-1}$ when the time has passed τ_n^k and it has n items. d[k] states that the threshold policy in the markdown case is also inventory monotone. e[k] says that the marginal revenue of a unit product under a fixed price is decreasing in time. In addition, the decreasing rate of the marginal revenue in time is bounded. The last sentence clarifies how the firm should apply this pricing strategy.

Proof: We prove by induction on k. Let us first focus on proving (a[0]) to (e[0]). Take $n = 1, 2, ..., N$. Following the same logic in Theorem 3.1 from (3.25) to (3.30), we obtain

$$\mathcal{G}_n^0(t) \circ v^0(t) + \bar{p}^0\bar{\alpha}^0 \cdot \beta(t) = 0, \qquad \forall t \in (0, T). \tag{3.72}$$

and (b[0]), that

$$u^0 = (u_n^0(t) \mid n = 0, 1, ..., N,\ t \in [0, T]) = v^0 = (v_n^0(t) \mid n = 0, 1, ..., N,\ t \in [0, T]). \tag{3.73}$$

From (3.72), we may confirm (c[0]) with the understanding that $\tau_n^0 = T$ for $n = 1, 2, ..., N$. The convention for the τ_n^0's also leads directly (d[0]). From (3.65), we have

$$v_1^0(t) = \bar{p}^0 \cdot E[N^0(t, T) \wedge 1] = \bar{p}^0 \cdot P[N^0(t, T) \geq 1] < \bar{p}^0. \tag{3.74}$$

As $v_0^0(t) = 0$ for any $t \in [0, T]$, we can apply (3.72) at $n = 1$ to get

$$d_t v_1^0(t) = \bar{\alpha}^0 \cdot \beta(t) \cdot (v_1^0(t) - \bar{p}^0), \qquad \forall t \in (0, T). \tag{3.75}$$

which is negative by (3.74). For $n = 1, 2, ..., N - 1$, we can apply (3.72) at both n and $n + 1$ to obtain

$$d_t v_{n+1}^0 - d_t v_n^0(t) = \bar{\alpha}^0 \cdot \beta(t) \cdot (v_{n-1}^0(t) - 2v_n^0(t) + v_{n+1}^0(t)), \qquad \forall t \in (0, T). \tag{3.76}$$

So (e[0]) is satisfied as well. Finally, by our default definition of $v_n^{-1}(t)$ and (3.72), we know that (a[0]) is true.

Suppose for some $k = 1, 2, ..., K$, we have (a[$k - 1$]), that $\mathcal{G}_n^{k-1}(t) \circ v^{k-2}/\beta(t)$ is decreasing in t for $n = 1, 2, ..., N$ and $t \in (0, T)$, (b[$k - 1$]), that

$$u_n^{k-1}(t) = v_n^{k-1}(t), \qquad \forall n = 0, 1, ..., N,\ t \in [0, T], \tag{3.77}$$

(c[$k-1$]), that, for $n = 1, 2, ..., N$,

$$\mathcal{G}_n^{k-1}(t) \circ v^{k-1} + \bar{p}^{k-1}\bar{\alpha}^{k-1} \cdot \beta(t) = 0, \qquad \forall t \in (0, \tau_n^{k-1}), \tag{3.78}$$

and

$$\mathcal{G}_n^{k-1}(t) \circ v^{k-1} + \bar{p}^{k-1}\bar{\alpha}^{k-1} \cdot \beta(t) \leq 0, \qquad \forall t \in (\tau_n^{k-1}, T), \tag{3.79}$$

(d[$k-1$]), that τ_n^{k-1} is decreasing in n, and (e[$k-1$]), that, for $t \in (0, T)$,

$$d_t v_1^{k-1}(t) \leq 0, \qquad \forall t \in (0, T), \tag{3.80}$$

and for $n = 1, 2, ..., N-1$,

$$d_t v_{n+1}^{k-1} - d_t v_n^{k-1}(t) \leq \bar{\alpha}^{k-1} \cdot \beta(t) \cdot (v_{n-1}^{k-1}(t) - 2v_n^{k-1}(t) + v_{n+1}^{k-1}(t)), \qquad \forall t \in (0, T). \tag{3.81}$$

Now we embark on showing (a[k]) to (e[k]). Take $n = 1, 2, ..., N$. From the definition of τ_n^{k-1} through (3.69), (b[$k-1$]), and (c[$k-1$]), we may see that

$$\mathcal{G}_n^{k-1}(t) \circ v^{k-1} + \bar{p}^{k-1}\bar{\alpha}^{k-1} \cdot \beta(t) = 0 < \mathcal{G}_n^{k-1}(t) \circ v^{k-2} + \bar{p}^{k-1}\bar{\alpha}^{k-1} \cdot \beta(t), \tag{3.82}$$

for $t \in (0, \tau_n^{k-1})$, and

$$\mathcal{G}_n^{k-1}(t) \circ v^{k-1} + \bar{p}^{k-1}\bar{\alpha}^{k-1} \cdot \beta(t) = \mathcal{G}_n^{k-1}(t) \circ v^{k-2} + \bar{p}^{k-1}\bar{\alpha}^{k-1} \cdot \beta(t) \leq 0, \tag{3.83}$$

for $t \in (\tau_n^{k-1}, T)$. Therefore,

$$\mathcal{G}_n^{k-1}(t) \circ v^{k-1} + \bar{p}^{k-1}\bar{\alpha}^{k-1} \cdot \beta(t) = [\mathcal{G}_n^{k-1}(t) \circ v^{k-2} + \bar{p}^{k-1}\bar{\alpha}^{k-1} \cdot \beta(t)] \wedge 0, \tag{3.84}$$

and hence by the strict positivity of $\beta(\cdot)$,

$$\frac{\mathcal{G}_n^{k-1}(t)\circ v^{k-1}}{\beta(t)} = [\frac{\mathcal{G}_n^{k-1}(t)\circ v^{k-2}}{\beta(t)} + \bar{p}^{k-1}\bar{\alpha}^{k-1}] \wedge 0 - \bar{p}^{k-1}\bar{\alpha}^{k-1}. \tag{3.85}$$

Note that the above is even true for $k=1$ due to the default definitions. Combining (a$[k-1]$) and (3.85), we obtain the decrease of $\mathcal{G}_n^{k-1}(t)\circ v^{k-1}/\beta(t)$ in t. Now, note that

$$\frac{\mathcal{G}_n^{k}(t)\circ v^{k-1} - \mathcal{G}_n^{k-1}(t)\circ v^{k-1}}{\beta(t)} = (\bar{\alpha}^{k-1} - \bar{\alpha}^{k})\cdot(v_n^{k-1}(t) - v_{n-1}^{k-1}(t)), \tag{3.86}$$

which, by (S1') and (e$[k-1]$), is decreasing in t. This and the just proved result together lead to the decrease of $\mathcal{G}_n^{k}(t)\circ v^{k-1}/\beta(t)$ in t. Hence, we have (a$[k]$).

By the definition of τ_n^k through (3.69) and (b$[k-1]$), we have

$$\mathcal{G}_n^{k}(t)\circ v^{k-1} + \bar{p}^{k}\bar{\alpha}^{k}\cdot\beta(t) > 0, \qquad \forall t\in(0,\tau_n^k). \tag{3.87}$$

Due to the strict positivity of $\beta(\cdot)$ and (b$[k-1]$), the threshold τ_n^k also satisfies

$$\tau_n^k = \inf\{t\in[0,T] \mid \frac{\mathcal{G}_n^{k}(t)\circ v^{k-1}}{\beta(t)} + \bar{p}^{k}\bar{\alpha}^{k} \le 0\}. \tag{3.88}$$

But this and (a$[k]$) will lead to

$$\frac{\mathcal{G}_n^{k}(t)\circ v^{k-1}}{\beta(t)} + \bar{p}^{k}\bar{\alpha}^{k} \le 0, \qquad \forall t\in(\tau_n^k,T). \tag{3.89}$$

Combining (3.87), (3.89), and the strict positivity of $\beta(\cdot)$, we arrive to

$$\mathcal{G}_n^{k}(t)\circ v^{k-1} + \bar{p}^{k}\bar{\alpha}^{k}\cdot\beta(t) \begin{cases} > 0, & \forall t\in(0,\tau_n^k), \\ \le 0, & \forall t\in(\tau_n^k,T). \end{cases} \tag{3.90}$$

Our construction (3.70) and (b$[k-1]$) dictate that

$$u_n^k(t) = v_n^{k-1}(t), \qquad \forall t \in [\tau_n^k, T]. \tag{3.91}$$

Also, by (3.7) and (3.8), we may see that the construction (3.71) renders

$$\mathcal{G}_n^k(t) \circ u^k + \bar{p}^k \bar{\alpha}^k \cdot \beta(t) = 0, \qquad \forall t \in (0, \tau_n^k), \tag{3.92}$$

From the first half of (3.90), we have, for any $n = 1, 2, ..., N$ and $t \in [0, \tau_n^k)$,

$$v_n^{k-1}(t) < v_n^{k-1}(\tau_n^k) \cdot \exp(-\bar{\alpha}^k \cdot \hat{\beta}(t, \tau_n^k)) + \bar{\alpha}^k \cdot \int_t^{\tau_n^k} \beta(s) \cdot (\bar{p}^k + v_{n-1}^{k-1}(s)) \cdot \exp(-\bar{\alpha}^k \cdot \hat{\beta}(t, s)) \cdot ds. \tag{3.93}$$

Using (3.71) and (3.93), as well as the fact that $u_0^k(t) = v_0^{k-1}(t) = 0$ for any $t \in [0, T]$, we can use induction over n to prove that

$$u_n^k(t) > v_n^{k-1}(t), \qquad \forall n = 1, 2, ..., N \text{ and } t \in [0, \tau_n^k). \tag{3.94}$$

By its construction, $u_n^k(t)$ is uniformly bounded by $N\overline{\lambda}T$; it is also Lipschitz continuous in t with coefficient $N\overline{\lambda}$, and hence absolutely continuous in t. By (3.92), (3.90), (3.91), and (3.94), as well as the fact that $v_n^{k-1}(T) = 0$ for every n, we may see that $u^k \equiv (u_n^k(t) \mid n = 0, 1, ..., N, \ t \in [0, T])$ satisfies the sufficient conditions (i) to (iv) stipulated in Proposition 3.5. Hence, we have shown (b$[k]$), that $u_n^k(t) = v_n^k(t)$ for every $n = 0, 1, ..., N$ and $t \in [0, T]$.

From (3.92), the second half of (3.90), as well as (3.91), and (b$[k]$), we easily have (c$[k]$).

Now take $n = 1, 2, ..., N-1$. By (S1'), (e$[k-1]$), and Proposition 3.6, we have

$$\begin{aligned} d_t v_{n+1}^{k-1}(t) - d_t v_n^{k-1}(t) &\leq \bar{\alpha}^{k-1} \cdot \beta(t) \cdot (v_{n-1}^{k-1}(t) - 2v_n^{k-1}(t) + v_{n+1}^{k-1}(t)) \\ &\leq \bar{\alpha}^{k} \cdot \beta(t) \cdot (v_{n-1}^{k-1}(t) - 2v_n^{k-1}(t) + v_{n+1}^{k-1}(t)) \leq 0. \end{aligned} \tag{3.95}$$

We therefore have the negativity of

$$\begin{aligned} &\mathcal{G}_{n+1}^k(t) \circ v^{k-1} - \mathcal{G}_n^k(t) \circ v^{k-1} \\ &\quad = d_t v_{n+1}^{k-1}(t) - d_t v_n^{k-1}(t) + \bar{\alpha}^k \cdot \beta(t) \cdot (2v_n^{k-1}(t) - v_{n-1}^{k-1}(t) - v_{n+1}^{k-1}(t)). \end{aligned} \tag{3.96}$$

But by the definition of τ_n^k and τ_{n+1}^k, this leads to $\tau_{n+1}^k \leq \tau_n^k$. Therefore, we have (d$[k]$).

Let us turn to the proof of (e$[k]$). When $t \in (\tau_1^k, T)$, we have

$$d_t v_1^k(t) = d_t v_1^{k-1}(t), \tag{3.97}$$

which is negative by (3.80). When $t \in (0, \tau_1^k)$, we have

$$\begin{aligned} d_t v_1^k(t) &= \bar{\alpha}^k \cdot \beta(t) \cdot (v_1^k(t) - \bar{p}^k) \\ &= \bar{\alpha}^k \cdot \beta(t) \cdot [v_1^{k-1}(\tau_1^k) \cdot \exp(-\bar{\alpha}^k \cdot \hat{\beta}(t, \tau_1^k)) \\ &\qquad + \bar{p}^k \bar{\alpha}^k \cdot \textstyle\int_t^{\tau_1^k} \beta(s) \cdot \exp(-\bar{\alpha}^k \cdot \hat{\beta}(t, s)) \cdot ds - \bar{p}^k] \\ &= \bar{\alpha}^k \cdot \beta(t) \cdot \exp(-\bar{\alpha}^k \cdot \hat{\beta}(t, \tau_1^k)) \cdot (v_1^{k-1}(\tau_1^k) - \bar{p}^k), \end{aligned} \tag{3.98}$$

where the first equality is due to (c$[k]$), the second equality is from (b$[k]$) and (3.71), and the last equality is by the following result from integration by parts:

$$\bar{\alpha}^k \cdot \int_t^{\tau_1^k} \beta(s) \cdot \exp(-\bar{\alpha}^k \cdot \hat{\beta}(t, s)) \cdot ds = 1 - \exp(-\bar{\alpha}^k \cdot \hat{\beta}(t, \tau_1^k)). \tag{3.99}$$

But from (3.65), (3.66), and the fact that $\bar{p}^k > \bar{p}^{k-1} > \cdots > \bar{p}^0$, it is obvious that $v_1^{k-1}(\tau_1^k) \leq \bar{p}^{k-1} < \bar{p}^k$. So we know that $d_t v_1^k(t) \leq 0$ for $t \in (0, \tau_1^k)$ as well.

Now let $n = 1, 2, ..., N-1$. For $t \in (\tau_n^k, T)$, we have $t \in (\tau_{n+1}^k, T)$ as well due to (d[k]). Thus, we have

$$v_n^k(t) = v_n^{k-1}(t) \quad \text{and} \quad v_{n+1}^k(t) = v_{n+1}^{k-1}(t), \tag{3.100}$$

and hence

$$\begin{aligned} d_t v_{n+1}^k(t) - d_t v_n^k(t) &= d_t v_{n+1}^{k-1}(t) - d_t v_n^{k-1}(t) \\ &\leq \bar{\alpha}^{k-1} \cdot \beta(t) \cdot (v_{n-1}^{k-1}(t) - 2v_n^{k-1}(t) + v_{n+1}^{k-1}(t)) \\ &\leq \bar{\alpha}^k \cdot \beta(t) \cdot (v_{n-1}^{k-1}(t) - 2v_n^{k-1}(t) + v_{n+1}^{k-1}(t)) \\ &\leq \bar{\alpha}^k \cdot \beta(t) \cdot (v_{n-1}^k(t) - 2v_n^k(t) + v_{n+1}^k(t)), \end{aligned} \tag{3.101}$$

where the first inequality is by (e[$k-1$]), the second inequality is by (S1') and Proposition 3.6, and the last inequality is from (3.100) and the fact that $v_{n-1}^k(t) \geq v_{n-1}^{k-1}(t)$. For $t \in (0, \tau_n^k)$, we have, by (c[k]),

$$\mathcal{G}_{n+1}^k(t) \circ v^k + \bar{p}^k \bar{\alpha}^k \cdot \beta(t) \leq 0 = \mathcal{G}_n^k(t) \circ v^k + \bar{p}^k \bar{\alpha}^k \cdot \beta(t). \tag{3.102}$$

This leads to

$$d_t v_{n+1}^k(t) - d_t v_n^k(t) \leq \bar{\alpha}^k \cdot \beta(t) \cdot (v_{n-1}^k(t) - 2v_n^k(t) + v_{n+1}^k(t)). \tag{3.103}$$

Therefore, we have (e[k]).

We have thus completed the induction process. Therefore, (a[k]) to (e[k]) are all true for $k = 0, 1, ..., K$. From these, we see that, for any $k = 1, 2, ..., K$ and $n = 1, 2, ..., N$,

$$v_n^k(t) \begin{cases} > v_n^{k-1}(t), & \forall t \in [0, \tau_n^k), \\ = v_n^{k-1}(t), & \forall t \in [\tau_n^k, T]. \end{cases} \tag{3.104}$$

Hence, we may see that each τ_n^k offers an optimal time beyond which the firm is to drop its price from $\bar{p}^k$ to $\bar{p}^{k-1}$ when it has n remaining items. $\square$

For the thus constructed threshold levels, it may be tempting to conjecture that τ_n^k is decreasing in k as well. We have the following relevant result.

Proposition 3.7. *Suppose $\tau_n^k > 0$ for some $k = 1, 2, ..., K-1$ and $n = 1, 2, ..., N$. Then, we have $\tau_n^{k+1} \leq \tau_n^k$ if and only if*

$$v_n^k(\tau_n^k) - v_{n-1}^k(\tau_n^k) \leq \frac{\bar{p}^k \bar{\alpha}^k - \bar{p}^{k+1}\bar{\alpha}^{k+1}}{\bar{\alpha}^k - \bar{\alpha}^{k+1}}.$$

Proof: If $\tau_n^k = T$, we have both $\tau_n^{k+1} \leq T = \tau_n^k$ and, due to (S1),

$$v_n^k(T) - v_{n-1}^k(T) = 0 \leq \frac{\bar{p}^k \bar{\alpha}^k - \bar{p}^{k+1}\bar{\alpha}^{k+1}}{\bar{\alpha}^k - \bar{\alpha}^{k+1}}. \tag{3.105}$$

Now suppose $\tau_n^k \in (0, T)$. From Theorem 3.2, we know that

$$\mathcal{G}_n^k(t) \circ v^k + \bar{p}^k \bar{\alpha}^k \cdot \beta(t) = 0, \qquad \forall t \in (0, \tau_n^k]; \tag{3.106}$$

we have $\tau_n^{k+1} \leq \tau_n^k$ if and only if

$$\mathcal{G}_n^{k+1}(\tau_n^k) \circ v^k + \bar{p}^{k+1}\bar{\alpha}^{k+1} \cdot \beta(\tau_n^k) \leq 0. \tag{3.107}$$

Due to (S1') and (3.106), the above (3.107) is equivalent to

$$v_n^k(\tau_n^k) - v_{n-1}^k(\tau_n^k) \leq \frac{\bar{p}^k \bar{\alpha}^k - \bar{p}^{k+1}\bar{\alpha}^{k+1}}{\bar{\alpha}^k - \bar{\alpha}^{k+1}}. \tag{3.108}$$

This completes our proof. $\square$

In the last inequality in Proposition 3.7, the left-hand side is independent of $\bar{p}^{k+1}$ or $\bar{\alpha}^{k+1}$; yet, the right-hand side is dependent on both, and can be arbitrarily small. Hence, we may see that τ_n^k is not necessarily decreasing in k. Thence, there exists a possibility for the following "leapfrog" phenomenon: Right after the time has gone past τ_n^k, it has also passed the time τ_n^{k-1}, and so on and so forth. Therefore, when the time passes beyond τ_n^k and it is currently charging $\bar{p}^k$ while with n items, the firm should ultimately switch to some price $\bar{p}^{\tilde{k}^-(k,n)}$, where $\tilde{k}^-(k,n)$ is not necessarily $k-1$. For each $n = 1, 2, ..., N$, we can use the following iterative procedure to find $(\tilde{k}^-(k,n) \mid k = 1, 2, ..., K)$:

for $k = 1$ to K

  let $l = k - 1$;

  while $l \geq 1$ and $\tau_n^l \leq \tau_n^k$ **do**

    let $l = \tilde{k}^-(l, n)$;

  let $\tilde{k}^-(k,n) = l$.

One of our computational studies shall confirm that a threshold policy for the markdown case is not necessarily k-monotone.

3.3.3 Algorithm

According to subsection 3.3.1, we can also establish an efficient and numerically stable algorithm to compute the optimal threshold policy and value function for the markdown case. Similar to the markup case, we denote $\lambda^k(q \cdot \Delta t)$ by λ_q^k and $v_n^k(q \cdot \Delta t)$ by v_{nq}^k. For $0 \leq t_1 \leq t_2 \leq T$, (3.68) leads to (3.63) for certain k values; through integration by parts, (3.71) for $0 \leq t_1 \leq t_2 \leq \tau_n^k$ leads to

$$\begin{aligned} u_n^k(t_1) &= u_n^{k-1}(\tau_n^k) + \int_{t_1}^{\tau_n^k} \lambda^k(s) \cdot (\bar{p}^k + u_{n-1}^k(s) - u_n^{k-1}(\tau_n^k)) \cdot \exp(-\hat{\lambda}^k(t_1, s)) \cdot ds \\ &= u_n^{k-1}(\tau_n^k) + \int_{t_1}^{t_2} \lambda^k(s) \cdot (\bar{p}^k + u_{n-1}^k(s) - u_n^{k-1}(\tau_n^k)) \cdot \exp(-\hat{\lambda}^k(t_1, s)) \cdot ds \\ &\qquad + \exp(-\lambda^k(t_1, t_2)) \cdot (u_n^k(t_2) - u_n^{k-1}(\tau_n^k)). \end{aligned} \tag{3.109}$$

The same approach can be applied to (3.119). In this algorithm, we will use (3.63) and (3.109) when $t_2 - t_1$ is as small as the single-step size ΔT.

We can adapt the recursive procedure described from (3.67) to (3.71) into the following algorithm *Markdown1*.

for $k = 0$ to K

  for $q = 0$ to Q

    let $v^k_{0q} = 0$;

for $n = 1$ to N

  let $v^0_{nQ} = 0$;

  for $q = Q - 1$ down to 0

    let $v^0_{nq} = \lambda^0_q \cdot \Delta T \cdot (\bar{p}^0 + v^0_{n-1,q}) + \exp(-\lambda^0_q \cdot \Delta T) \cdot v^0_{n,q+1}$;

for $k = 1$ to K

  for $n = 1$ to N

    let $q = Q$;

    do

      let $G = v^{k-1}_{nq} - v^{k-1}_{n,q-1} + \lambda^k_q \cdot \Delta T \cdot (v^{k-1}_{n-1,q} - v^{k-1}_{nq}) + \bar{p}^k \lambda^k_q \cdot \Delta T$;

      if $G \leq 0$

        let $q = q - 1$;

    while $G \leq 0$ and $q \geq 1$;

    let $\tau^k_n = q \cdot \Delta T$;

    for $r = Q$ down to q

      let $v^k_{nr} = v^{k-1}_{nr}$;

    for $r = q - 1$ down to 0

      let $v^k_{nr} = v^{k-1}_{nq} + \lambda^k_r \cdot \Delta T \cdot (\bar{p}^k + v^k_{n-1,r} - v^{k-1}_{nq}) + \exp(-\lambda^k_r \cdot \Delta T) \cdot (v^k_{n,r+1} - v^{k-1}_{nq})$.

The algorithm's time complexity is apparently $O(KNQ)$.

3.4 The Reversible Case

In this section, we consider the reversible pricing case in which the firm can choose any price from a given set of price choices to maximize its profit. For instance, HP touchpad 32G was sold using this reversible pricing strategy. It was priced at \$499.99 when just released. The price declined by \$100 one month later. To spur demand, management sold the product even cheaper at \$149.99 later. When customers reacted enthusiastically, the price was brought back up to \$280.

3.4.1 Threshold Policy and Value Function

Once again, the concerned firm can choose any price from $\bar{p}^0, \bar{p}^1, ..., \bar{p}^K$ at any time to suit its needs. It concerns the threshold policy $\tau = (\tau_n^k \mid k = 1, 2, ..., K,\ n = 1, 2, ..., N) \in (\Delta_N)^K$, under which, the firm should charge price $\bar{p}^k$ when its inventory level is n at a time $t \in [\tau_n^{k+1}, \tau_n^k)$, with the understanding that the latter interval is $\emptyset$ when $\tau_n^{k+1} = \tau_n^k$, and that $\tau_n^{K+1} = 0$ and $\tau_n^0 = T$ for $n = 1, 2, ..., N$.

Note that $v_n(t)$ is the firm's optimal remaining value function when it has n remaining items at time t. Because the firm can choose any price at any time now, the value function is not related with the price $\bar{p}^k$. The value functions satisfy the following Hamilton-Jacobi-Bellman (HJB) equations:

$$\max_{k=0}^{K}[\mathcal{G}_n^k(t) \circ v + \bar{p}^k \bar{\alpha}^k \cdot \beta(t)] = 0, \qquad \forall n = 1, 2, ..., N,\ t \in (0, T). \tag{3.110}$$

In addition, boundary and terminal conditions are that $v_0(t) = 0$ for every $t \in [0, T]$

and $v_n(T) = 0$ for every $n = 1, 2, ..., N$. Equation (3.110) can be rewritten as

$$\begin{aligned} &d_t v_n(t) + \beta(t) \cdot \max\nolimits_{k=0}^{K} [\bar{p}^k \bar{\alpha}^k - \bar{\alpha}^k \cdot (v_n(t) - v_{n-1}(t))] = 0, \\ &\qquad \forall n = 1, 2, ..., N,\ t \in (0, T). \end{aligned} \tag{3.111}$$

To solve (3.111), we can consider the following convex function $f(\cdot)$ on $[0, +\infty)$ through

$$f(x) = \max_{k=0}^{K} f^k(x), \qquad \text{where for } k = 0, 1, ..., K, \quad f^k(x) = \bar{p}^k \bar{\alpha}^k - \bar{\alpha}^k \cdot x. \tag{3.112}$$

In fact, some prices may not be selected, depending on the shape of $f(x)$. To find $f(x)$ rigorously, we define an increasing sequence $(\bar{x}^k \mid k = 0, 1, ..., K, K+1)$ with $\bar{x}^0 = 0$ and $\bar{x}^{K+1} = +\infty$, so that $\bar{U} = \{k = 0, 1, ..., K \mid \bar{x}^k < \bar{x}^{k+1}\}$ forms the set of un-dominated price indices. As for the sequence itself, we rely on the following procedure for its generation.

$$\bar{x}^k = \max_{j=0}^{k-1} \frac{\bar{p}^j \bar{\alpha}^j - \bar{p}^k \bar{\alpha}^k}{\bar{\alpha}^j - \bar{\alpha}^k}, \tag{3.113}$$

which is allowed by (S1'). Let l be the smallest index among $0, 1, ..., k-1$ such that

$$f^l(\bar{x}^k) = f^k(\bar{x}^k). \tag{3.114}$$

Now, let $\bar{x}^{l+1} = \bar{x}^{l+2} = \cdots = \bar{x}^{k-1} = \bar{x}^k$. Then, let $k = l$ and go back to the step involving (3.113) unless $k = 0$ already. By (S1), this procedure will guarantee that

$$\bar{x}^0 \equiv 0 < \bar{x}^1 \leq \bar{x}^2 \leq \cdots \leq \bar{x}^K < +\infty \equiv \bar{x}^{K+1}. \tag{3.115}$$

From (3.113), one may see that the sequence is linked with the maximum concave envelope of Feng and Xiao (2000a).

With (3.113) to (3.115), we will have the following proposition which tells the value of $f(x)$ for each x.

Proposition 3.8. *For $k = 0, 1, ..., K$, we have $f^k(x) = \max_{j=0}^{K} f^j(x)$ for $x \in (\bar{x}^k, \bar{x}^{k+1}]$, with the understanding that the latter set is $\emptyset$ when $\bar{x}^k = \bar{x}^{k+1}$.*

Proof: For $j, k = 0, 1, ..., K$ with $j < k$, define $\bar{y}^{jk}$ so that

$$\bar{y}^{jk} = \frac{\bar{p}^j \bar{\alpha}^j - \bar{p}^k \bar{\alpha}^k}{\bar{\alpha}^j - \bar{\alpha}^k}. \tag{3.116}$$

Using (3.112), (S1), and (S1'), it is easy to see that $f^j(x) \le f^k(x)$ if and only if $x \ge \bar{y}^{jk}$. By (3.113), we see that $\bar{x}^k = \max_{j=0}^{k-1} \bar{y}^{jk}$, and hence

$$f^k(x) \ge \max_{j=0}^{k-1} f^j(x), \quad \text{if and only if} \quad x \ge \bar{x}^k. \tag{3.117}$$

By our construction at $k = K+1$ and K, we may see the veracity of our claim at $k = K$. That is, $f^K(x) = \max_{k=0}^{K-1} f^k(x)$ when $x \in (\bar{x}^K, \bar{x}^{K+1}]$. Suppose the claim is true for $K, K-1, ..., k+1$ for some $k = 0, 1, ..., K-1$. Let $j \ge k+1$ be the index such that $\bar{x}^{k+1} = \bar{x}^{k+2} = \cdots = \bar{x}^j < \bar{x}^{j+1}$. That is, $\bar{x}^j$ is the last point that is constructed through (3.113). There are two possibilities: $f^k(\bar{x}^j) < f^j(\bar{x}^j)$ or $f^k(\bar{x}^j) = f^j(\bar{x}^j)$. When the first possibility is true, we have $\bar{x}^k = \bar{x}^{k+1} = \cdots = \bar{x}^j$ by (3.114), and hence the claim is always true by virtue of its condition being void.

Depending on whether there is any $l \le k-1$ satisfying $f^l(\bar{x}^j) = f^j(\bar{x}^j)$, we have two sub-cases under the second possibility. When there is such an l, we have $\bar{x}^{l+1} = \cdots = \bar{x}^k = \bar{x}^{k+1} = \cdots = \bar{x}^j$ by (3.114), and hence the claim is again true. Otherwise, we see that $\bar{x}^k$ will be constructed through (3.113). By (3.117), this will guarantee that $f^k(x) \ge \max_{u=0}^{k-1} f^u(x)$ when $x > \bar{x}^k$. On the other hand, note the following three facts: $f^k(\bar{x}^{k+1}) = f^k(\bar{x}^j) = f^j(\bar{x}^j)$, the dominance of $f^j(\bar{x}^j)$ over all $f^u(\bar{x}^j)$ by the induction hypothesis along with the continuity of the $f^u(\cdot)$ functions, and

(S1'). These together will guarantee that $f^k(x) \geq \max_{u=k+1}^K f^u(x)$ when $x \leq \bar{x}^{k+1}$. Therefore, we will have $f^k(x) = \max_{u=0}^K f^u(x)$ when $x \in (\bar{x}^k, \bar{x}^{k+1}]$. □

Equipped with the sequence $(\bar{x}^k \mid k = 0, 1, ..., K, K+1)$, we can construct function vector $u = (u_n(t) \mid n = 0, 1, ..., N,\ t \in [0, T])$ and point vector $\tau = (\tau_n^k \mid k = 1, 2, ..., K,\ n = 1, 2, ..., N)$ through an iterative procedure. First, let

$$u_0(t) = 0, \qquad \forall t \in [0, T]. \tag{3.118}$$

Then, we go over an outer loop on $n = 1, 2, ..., N$. At each n, we first let $u_n(T) = 0$ and $\tau_n^0 = T$, and then go over an inner loop on $k = 0, 1, ..., K-1$. At each k, we let

$$u_n(t) = u_n(\tau_n^k)\cdot\exp(-\bar{\alpha}^k\cdot\hat{\beta}(t,\tau_n^k))+\bar{\alpha}^k\cdot\int_t^{\tau_n^k}\beta(s)\cdot(\bar{p}^k+u_{n-1}(s))\cdot\exp(-\bar{\alpha}^k\cdot\hat{\beta}(t,s))\cdot ds, \tag{3.119}$$

for $t \in [\tau_n^{k+1}, \tau_n^k]$, where τ_n^{k+1} is defined through

$$\tau_n^{k+1} = \inf\{t \in [0, \tau_n^k] \mid u_n(t) - u_{n-1}(t) \leq \bar{x}^{k+1}\}, \tag{3.120}$$

with the understanding that $\tau_n^{k+1} = 0$ when the concerned inequality is always true and $\tau_n^{k+1} = \tau_n^k$ when it is never true.

The following theorem demonstrates that the above constructed threshold policy is optimal and $u_n(t)$ is the true value function. Its proof relies on some results of Zhao and Zheng (2000).

Theorem 3.3. *The u and τ as constructed from (3.118) to (3.120) satisfy the following:*

(a) $u_n(t) = v_n(t)$ for any $n = 0, 1, ..., N$ and $t \in [0, T]$;

(b) τ_n^k is decreasing in both n and k.

As a consequence, τ provides an optimal policy for the firm. Indeed, only prices in $\bar{U} = \{k = 0, 1, ..., K \mid \bar{x}^k < \bar{x}^{k+1}\}$ will be charged.

The information implied by this theorem is straightforward. (a) verifies the constructed function is just the value function. (b) shows the threshold policy has both time monotonicity and inventory-monotonicity, which means that the price should decrease along the time given a fixed inventory and that higher inventory leads to lower price at a fixed time. The last sentence confirms the constructed threshold policy is optimal. Under such a policy, the firm with inventory n should choose price $\bar{p}^k$ when the time is between τ_n^{k+1} and τ_n^k. The last sentence also conveys the information that some prices may not be chosen in the whole sales season. To go further, if a company wants to decide what price should be selected as a potential one, instead of choosing from a given set, it can check whether the price is in $\bar{U} = \{k = 0, 1, ..., K \mid \bar{x}^k < \bar{x}^{k+1}\}$. Since our product-form arrival pattern satisfies Zhao and Zheng's (2000) key assumption that $\lambda^k(t)$ has log-decreasing differences between k and t, time monotonicity has been predicted by the earlier paper. Later analysis actually takes much advantage of the earlier paper, to the extent that it appears simpler than the derivation employed in Feng and Xiao (2000a) for the stationary-demand case.

Proof: Let $p_n(t)$ be the lowest price that achieves the supremum in (3.110). We have the following:

Fact 1, that $v_n(t)$ is increasing in n, as having more items will not hurt the revenue; and,

Fact 2, that $v_n(t)$ is decreasing in t, as having less time to sell will not boost the revenue.

Zhao and Zheng (2000) also proved the following:

Fact 3, that $v_n(t)$ is concave in n (their Theorem 1);

Fact 4, that $v_n(t)$ has decreasing differences between n and t (their Theorem 2; note the different t-definitions); and,

Fact 5, that the optimal pricing policy $p = (p_n(t) \mid n = 1, 2, ..., N,\ t \in [0, T))$ is in existence and for each $t \in [0, T)$, $p_n(t)$ is decreasing in n (their Theorem 3).

By fact 4, we know that $v_n(t) - v_{n-1}(t)$ is decreasing in t. Hence, we may define $\tilde{\tau}_n^k$ for $k = 1, 2, ..., K$ and $n = 1, 2, ..., N$, such that

$$\tilde{\tau}_n^k = \inf\{t \in [0, T] \mid v_n(t) - v_{n-1}(t) \leq \bar{x}^k\}, \tag{3.121}$$

with the understanding that $\tilde{\tau}_n^k = 0$ when the concerned inequality is always true and $\tilde{\tau}_n^k = T$ when it is never true. By the terminal condition $v_n(T) = 0$, it follows that

$$\tilde{\tau}_n^{K+1} \equiv 0 \leq \tilde{\tau}_n^K \leq \tilde{\tau}_n^{K-1} \leq \cdots \leq \tilde{\tau}_n^1 < T \equiv \tilde{\tau}_n^0; \tag{3.122}$$

also, we will have $\tilde{\tau}_n^{k+1} < \tilde{\tau}_n^k$ when both $k \in \bar{U}$ and $\tilde{\tau}_n^k > 0$, and $\tilde{\tau}_n^{k+1} = \tilde{\tau}_n^k$ otherwise. From (3.111) and Proposition 3.8, we may see that

$$p_n(t) = \bar{p}^k, \qquad \forall t \in [\tilde{\tau}_n^{k+1}, \tilde{\tau}_n^k). \tag{3.123}$$

Note that $[\tilde{\tau}_n^{k+1}, \tilde{\tau}_n^k)$ will be nonempty if and only if $k \in \bar{U}$ and $\tilde{\tau}_n^k > 0$, with the latter condition amounting to $\bar{x}^k < v_n(0) - v_{n-1}(0)$. Therefore, only prices in some lower end of $\bar{U}$ will be chosen. Thus, we have

Fact 6, that $p_n(t)$ is decreasing in t.

Indeed, we can also obtain fact 5 from fact 3. The latter says that $v_n(t) - v_{n-1}(t)$ is decreasing in n. This and fact 4 lead us to conclude that $\tilde{\tau}_n^k$ as defined by (3.121) is decreasing in n.

From (3.111) and (3.123), we have, for $t \in (\tilde{\tau}_{n+1}^k, \tilde{\tau}_n^k)$,

$$d_t v_n(t) - \bar{\alpha}^k \cdot \beta(t) \cdot v_n(t) + \bar{\alpha}^k \cdot \beta(t) \cdot (\bar{p}^k + v_{n-1}(t)) = 0. \tag{3.124}$$

By (3.7) and (3.8), this leads to

$$v_n(t) = v_n(\tilde{\tau}_n^k)\cdot\exp(-\bar{\alpha}^k\cdot\hat{\beta}(t,\tilde{\tau}_n^k))+\bar{\alpha}^k\cdot\int_t^{\tilde{\tau}_n^k}\beta(s)\cdot(\bar{p}^k+v_{n-1}(s))\cdot\exp(-\bar{\alpha}^k\cdot\hat{\beta}(t,s))\cdot ds, \tag{3.125}$$

for $t\in[\tilde{\tau}_n^{k+1},\tilde{\tau}_n^k]$. Comparing the way in which we obtain $u_n(t)$ and τ_n^k through (3.118) to (3.120) with the current $v_n(t)$ and $\tilde{\tau}_n^k$, we may see that $u_n(t)$ is merely $v_n(t)$, while τ_n^k merely $\tilde{\tau}_n^k$. $\square$

3.4.2 Algorithm

We can adapt the recursive procedure described from (3.113) to (3.114) as well as from (3.118) to (3.120) to the following algorithm *Reversible1*.

let $k = K$;
while $k \geq 1$
  let $l = k-1$ and $\bar{x}^k = (\bar{p}^l\bar{\alpha}^l - \bar{p}^k\bar{\alpha}^k)/(\bar{\alpha}^l - \bar{\alpha}^k)$;
  for $j = k-2$ down to 0
    let $x = (\bar{p}^j\bar{\alpha}^j - \bar{p}^k\bar{\alpha}^k)/(\bar{\alpha}^j - \bar{\alpha}^k)$;
    if $x \geq \bar{x}^k$
      let $l = j$ and $\bar{x}^k = x$;
  for $j = l+1$ to $k-1$
    let $\bar{x}^j = \bar{x}^k$;
  let $k = l$;
for $q = 0$ to Q
  let $v_{0q} = 0$;
for $n = 1$ to N
  let $q = Q$, $v_{nq} = 0$, and $k = 0$;
  for $r = Q-1$ down to 0

let $v_{nr} = v_{nq} + \lambda_r^k \cdot \Delta T \cdot (\bar{p}^k + v_{n-1,r} - v_{nq}) + \exp(-\lambda_r^k \cdot \Delta T) \cdot (v_{n,r+1} - v_{nq})$;

if $v_{nr} - v_{n-1,r} > \bar{x}^{k+1}$

let $r = r + 1$, $q = r$, $k = k + 1$, and $\tau_n^k = q \cdot \Delta T$;

while $k \leq K - 1$ and $\bar{x}^{k+1} = \bar{x}^k$

let $k = k + 1$ and $\tau_n^k = q \cdot \Delta T$.

In *Reversible1*, the first part up to the end of the **while** loop is devoted to an analysis of the convex function $f(\cdot)$ defined in (3.112). Here, we have utilized the fact that for $j \leq k - 1$, one will have $f^j(\bar{x}^k) < f^k(\bar{x}^k)$ when $(\bar{p}^j \bar{\alpha}^j - \bar{p}^k \bar{\alpha}^k)/(\bar{\alpha}^j - \bar{\alpha}^k) < \bar{x}^k$. The algorithm's time complexity is $O(N \cdot (K + Q))$.

CHAPTER 4

Make-to-order Inventory Control with Pricing

Now we add the production component into revenue management. The concerned firm can purchase the raw material from an external market and convert the material into finished product. The firm takes control of its raw material acquisition activities and the pricing of the finished product. At the same time, the raw material price is assumed to follow a Markov process.

4.1 Problem Setup

We name periods in a backward fashion and let period 0 be the terminal period. Each period t can be understood as the time interval $[t, t-1)$. There is also a per-period discount factor $\alpha \in [0, 1)$. All exogenous model features are taken as stationary over time. This is mostly to ensure simplicity of presentation only; our finite-horizon results can be easily extended to the case with time-varying exogenous features. We suppose that the unit raw material cost evolves as a homogeneous Markov process with states in $[\underline{\pi}, \overline{\pi}]$ for some strictly positive constants $\underline{\pi}$ and $\overline{\pi}$. We use $(\Pi' \mid \pi)$ to denote, generically, the next-period random cost Π' conditioned on the present-period cost π. Also, we use $s(\pi)$ to denote the discounted version of the expected next-period cost given current period's π:

$$s(\pi) = \alpha \cdot E[\Pi' \mid \pi]. \tag{4.1}$$

We use a stylized model in the spirit of Hotelling (1929) and Salop (1979) to further motivate the impact of a commonly felt input cost on the demand curve of a firm under price competition. In it, firms are located at points $0, \pm 1, \pm 2, \ldots$. At cost $\pi \geq 0$ and demand level w, a firm can earn $(p - \pi) \cdot w$ in profit when it charges a unit sales price p. Customers are evenly distributed on the entire real line, and the potential demand generated by customers in any interval $[a, b]$ is $b - a$. Customers have very high valuations for the product sold by firms, and each one of them goes to the firm with the lowest $p + d$ value, where p is the unit price charged by the firm and d is the distance between the customer and the firm. Through the analysis in section 4.1.1, we can show that the only symmetric equilibrium arrangement is for all firms to charge the same π-dependent unit sales price

$$p^*(\pi) = \pi + 1. \tag{4.2}$$

Then, when one particular firm charges a unit sales price $p \in [\pi, \pi + 2]$ while all other firms abide by the equilibrium price given in (4.2), the former firm would be able to attract demand

$$w(\pi, p) = \pi - p + 2. \tag{4.3}$$

Naturally, the demand level $w(\pi, p)$ is decreasing in the firm's sales price p. That $w(\pi, p)$ is increasing in the input cost π is a more noteworthy phenomenon. It can be explained by the tendency for a raised input cost to push competitors' prices higher, thus making it easier for the current firm to attract demand at any particular price level.

Now, we model the interplay, in a given period t, between the realized raw material cost, the sales price, and demand through the use of a positive function $w(\cdot, \cdot)$ as well as a positive random variable Θ_t. When the raw material cost is π and the firm charges a unit sales price p, demand Δ_t for this period will follow

$$\Delta_t = w(\pi, p) + \Theta_t. \tag{4.4}$$

For instance, $w(\pi, p)$ can take the form in (4.3). The demand Δ_t can thus be understood as resulting from the current firm's competition with other firms, all under the sway of the common raw material cost. The extra term Θ_t emphasizes uncontrollable factors in the model, stating that the firm can control demand up to an additive random error. This additive-demand form was adopted by many researchers; see, e.g., Mills (1959) on a study of the price-demand relationship and Petruzzi and Dada (1999) on an investigation of a newsvendor who faces price-sensitive demands. Note the positivity requirement on Θ_t is used to ensure the positivity of Δ_t; it is not required for later derivation.

We suppose $w(\pi, \cdot)$ is continuous and strictly decreasing, to the effect that it has a continuous and strictly decreasing inverse $p(\pi, \cdot)$ on the positive real line $\Re^+$. We let $p(\pi, w)$ be continuous in π too. We now take the view that the firm uses the lever $w = w(\pi, p)$ to influence its demand–when it decides on a w, it will charge $p(\pi, w)$ for every unit of the finished product, and its demand will satisfy

$$\Delta_t = w + \Theta_t. \tag{4.5}$$

We suppose the Θ_t's are independent across different periods. Also, the process $(\Theta_t \mid t = 1, 2, ...)$ is independent of the raw material cost process $(\Pi_t \mid t = 0, 1, 2, ...)$. In addition, each Θ_t is distributed as a generic random variable Θ with $E[\Theta] = \theta \geq$

0.

Let $h(\cdot)$ be the firm's holding-backlogging cost per period. We suppose that $h(x) \geq 0$ for $x \in \Re$, $h(0) = 0$, and that $h(\cdot)$ is convex. A concave or convex function defined on the real line $\Re$ is continuous; it is differentiable almost everywhere as well; see, e.g., Howe (1982). In the sequel, we will apply derivative to a concave or convex function, even though left- and right-derivatives may not agree on a measure-zero set of points. For the just defined convex function $h(\cdot)$, we assume that $b_\infty \equiv -\lim_{x \to -\infty} dh(x)/dx$ and $h_\infty \equiv \lim_{x \to +\infty} dh(x)/dx$ are both finite values. That is, upper bounds exist for unit backlogging and holding costs.

In every period t, the firm first observes the current raw material cost π and its own inventory level x. It then decides the raw material replenishment quantity z, which, upon immediate delivery, would bring the firm's post-procurement inventory level up to $y = x + z$. Next, the firm decides on the demand lever w which would translate into a retail price $p(\pi, w)$ and imply a random demand Δ_t in the form of (4.5). When the realized demand is δ_t, the firm's inventory level would be reduced to $y - \delta_t$. In the ensuing period $t - 1$, the random raw material cost the firm expects to face is $(\Pi' \mid \pi)$.

We suppose that the firm's choice w is within the range $[0, w^U]$ for some finite w^U. As w^U can be made arbitrarily large, its finiteness would not render our results less practicable. For $\pi \in [\underline{\pi}, \overline{\pi}]$ and $w \in [0, w^U]$, define the revenue function

$$r(\pi, w) = E[p(\pi, w) \cdot (w + \Theta)] = p(\pi, w) \cdot (w + \theta). \tag{4.6}$$

It stands for the average revenue the firm can make under raw material cost π and demand lever w. Recall that unsatisfied orders will still earn the firm revenue in the current period, though they will cost it delay penalties in ensuing periods.

Let $\tilde{f}_t(\pi, x)$ be the maximum total discounted average profit that the firm can earn from period t onward till the terminal period 0, when the raw material cost is π and the firm's inventory level is x. For any period $t = 1, 2, ...$, we have the following recursive relationship:

$$\begin{aligned} \tilde{f}_t(\pi, x) \quad &= \sup_{w \in [0, w^U], z \in \Re^+} E[p(\pi, w) \cdot (w + \Theta) - \pi z - h(x + z - w - \Theta) \\ &\quad + \alpha \cdot \tilde{f}_{t-1}(\Pi', x + z - w - \Theta) | \pi]. \end{aligned} \tag{4.7}$$

In (4.7), the expectation is conditioned on the period-t raw material cost π. Inside this expectation, the first term is the revenue the firm can earn from the demand, the second term is the firm's raw material acquisition cost, the third term is its inventory holding-backlogging cost, and the last term is the total discounted profit from period $t - 1$ onward that the firm can earn. For the terminal period, we assume

$$\tilde{f}_0(\pi, x) = \pi x. \tag{4.8}$$

That is, the firm can sell its remaining inventory or fill up its shortage using the raw material market. For convenience, we make the transformation

$$f_t(\pi, x) = \tilde{f}_t(\pi, x) - \pi x. \tag{4.9}$$

The newly defined $f_t(\pi, x)$ is the maximum total discounted average profit the firm can make minus its present net inventory worth. Now, (4.8) will result in

$$f_0(\pi, x) = 0. \tag{4.10}$$

Meanwhile, (4.7) will lead to

$$f_t(\pi, x) = \sup_{w \in [0, w^U], y \in [x, +\infty)} g_t(\pi, w, y), \qquad \forall t = 1, 2, ..., \tag{4.11}$$

where

$$\begin{aligned} g_t(\pi, w, y) &= -\theta \cdot s(\pi) - s(\pi) \cdot w + r(\pi, w) - (\pi - s(\pi)) \cdot y \\ &\quad - E[h(y - w - \Theta)] + \alpha \cdot E[f_{t-1}(\Pi', y - w - \Theta) \mid \pi]. \end{aligned} \tag{4.12}$$

In (4.12), the post-procurement inventory level y has replaced the acquisition level z as one of the firm's main decision variables.

4.1.1 Derivation of the Stylized Model

Let us find a price $p^*(\pi)$ that can be adopted by all firms to reach equilibrium when all of them experience the same unit cost π. Under the same cost π, suppose all firms at locations $\pm 1, \pm 2, \ldots$ have adopted a common sales price p. Let us use $\tilde{p}^*(\pi, p)$ to denote the largest best response that the firm at location 0 can muster. When $\tilde{p}^*(\pi, \cdot)$ is obtainable, we can just let $p^*(\pi)$ be a fixed point for the function, given that such a point exists.

It turns out that $\tilde{p}^*(\pi, p)$ for an arbitrary (π, p)-pair is difficult to compute. Instead, we shall first deal with the "lesser" entity $\tilde{p}^0(\pi, p)$, merely location 0's largest best response to the (π, p)-pair under the extra restriction of it being between $p - 1$ and $p + 1$. Under this restriction, the best response turns out to be solvable and unique. We shall then identity a unique fixed point $p^0(\pi)$ for the function $\tilde{p}^0(\pi, \cdot)$.

Since $p^*(\pi)$ must also be a fixed point for $\tilde{p}^0(\pi, \cdot)$, it either does not exist or is exactly $p^0(\pi)$. The latter is actually true, as we can show that $p^0(\pi)$ is a fixed point for not only $\tilde{p}^0(\pi, \cdot)$, but also $\tilde{p}^*(\pi, \cdot)$. Indeed, $p^0(\pi)$ is the unique best response to the $(\pi, p^0(\pi))$-pair, without the requirement that the best response be in $[p^0(\pi) - 1, p^0(\pi) + 1]$.

In summary, we can avoid analyzing $\tilde{p}^*(\pi, p)$ for every (π, p)-pair in order to ultimately reach $p^*(\pi)$. Rather, a thorough understanding of the "lesser" $\tilde{p}^0(\pi, p)$ is more attainable, and based on it, we can obtain a unique"tied-hand equilibrium" price $p^0(\pi)$. But we have $p^0(\pi) = \tilde{p}^*(\pi, p^0(\pi))$ as well. Thus, there is a unique $p^*(\pi)$ in the form of $p^0(\pi)$.

Here comes our detailed derivation. Under a given π, suppose all firms at locations $\pm 1, \pm 2, ...$ have adopted a common sales price p. Now let the firm at location 0 use some price $p' \in [p-1, p+1]$. Then the customers attracted to this firm would lie in the interval $[-a^0(p', p), +a^0(p', p)]$, where the boundary point $a^0(p', p) \in [0, 1]$ satisfies

$$p' + a^0(p', p) = p + (1 - a^0(p', p)). \tag{4.13}$$

This guarantees that customers in the interval $[-a^0(p', p), +a^0(p', p)]$ would prefer the firm at 0, while those in $[+a^0(p', p), +1]$ would prefer the firm at $+1$ and those in $[-1, -a^0(p', p)]$ the firm at -1. We can solve (4.13) to obtain $a^0(p', p) = (p - p' + 1)/2$. Therefore, the firm at 0 can attract demand

$$w^0(p', p) = 2 \cdot a^0(p', p) = p - p' + 1, \tag{4.14}$$

and earn profit $q^0(p', \pi, p) = (p' - \pi) \cdot w^0(p', p) = (p' - \pi) \cdot (p - p' + 1)$. Re-express this and we can get

$$q^0(p', \pi, p) = -(p' - \frac{\pi + p + 1}{2})^2 + \frac{(\pi + p + 1)^2}{4} - \pi \cdot p - \pi. \tag{4.15}$$

By our earlier definition, we reckon that

$$\tilde{p}^0(\pi, p) = \max \ \mathrm{argmax}_{p-1 \leq p' \leq p+1} q^0(p', \pi, p). \tag{4.16}$$

From the quadratic form exhibited in (4.15), we can get

$$\tilde{p}^0(\pi, p) = \begin{cases} p - 1, & \text{when } p > \pi + 3, \\ (\pi + p + 1)/2, & \text{when } \pi - 1 \leq p \leq \pi + 3, \\ p + 1, & \text{when } p < \pi - 1. \end{cases} \tag{4.17}$$

At each (π, p)-pair, the thus obtained $\tilde{p}^0(\pi, p)$ is actually the unique maximizer for $q^0(\cdot, \pi, p)$ at $p^0 \in [p - 1, p + 1]$ since the objective function, as shown in (4.15), is strictly concave. Due to (4.17), any fixed point $p^0(\pi)$ for $\tilde{p}^0(\pi, \cdot)$ must be in the range $[\pi - 1, \pi + 3]$. Within this range, we can solve $p^0(\pi) = (\pi + p^0(\pi) + 1)/2$ to obtain the unique solution

$$p^0(\pi) = \pi + 1, \tag{4.18}$$

which is indeed within the range $[\pi - 1, \pi + 3]$.

Any fixed point for $\tilde{p}^*(\pi, \cdot)$ must first be a fixed point for $\tilde{p}^0(\pi, p)$, so the fixed point $p^*(\pi)$ is either nonexistent or exactly equal to $p^0(\pi)$. We shall verify the latter.

Under an input cost π, suppose firms at locations $\pm 1, \pm 2, ...$ all charge the same price $p^0(\pi) = \pi + 1$ and the firm at location 0 charges some price p'. When $p' < \pi$, the firm at location 0 would surely not make any profit. When $p' > \pi + 2$, it would not attract any demand when even customers at 0 are siphoned away by firms at ± 1. So to make any profit, the firm must choose $p' \in [\pi, \pi + 2]$ even though it is allowed a free reign over $[0, +\infty)$. That is, the firm would voluntarily use $p' \in [p^0(\pi) - 1, p^0(\pi) + 1]$. But within this range, we know the best choice is $p^0(\pi)$ itself. Indeed, by charging $p^0(\pi) = \pi + 1$, the firm would earn $q^0(\pi + 1, \pi, \pi + 1) = 1$, a strictly positive sum.

By our earlier definition, this line of reasoning amounts to

$$p^0(\pi) = \tilde{p}^*(\pi, p^0(\pi)). \tag{4.19}$$

That is, $p^0(\pi)$ is a fixed point for $\tilde{p}^*(\pi,\cdot)$. Thus, it follows that

$$\text{the unique } p^*(\pi) = p^0(\pi) = \pi + 1. \tag{4.20}$$

We have thus obtained a unique symmetric equilibrium pricing policy $p^*(\cdot)$ in the form of $\pi + 1$. □

4.2 Properties under a Changing π

Now we introduce the following assumptions:

(MO1) For every $\pi \in [\underline{\pi}, \overline{\pi}]$, the revenue function $r(\pi,\cdot)$ is concave on $[0, w^U]$.

(MO2) $b_\infty \geq [(1-\alpha)\cdot\overline{\pi}] \vee [\partial^+ r(\overline{\pi}, 0)/\partial w - s(\overline{\pi})]$.

(MO3) $h_\infty \geq s(\overline{\pi}) \vee (\alpha\overline{\pi} - \underline{\pi})^+ \vee [s(\overline{\pi}) - \partial^- r(\pi, w^U)/\partial w]$.

(MO4) For every $w \in (0, w^U)$, the term $\partial r(\pi, w)/\partial w - s(\pi)$ is increasing in π.

(MO5) For every $w \in (0, w^U)$, the term $\partial r(\pi, w)/\partial w - \pi$ is decreasing in π.

(MO6) The random next-period raw material cost $(\Pi' \mid \pi)$ is stochastically increasing in the current-period raw material cost $\pi \in [\underline{\pi}, \overline{\pi}]$; namely, for any increasing function $u(\cdot)$, the conditional average $E[u(\Pi') \mid \pi]$ is increasing in π.

Assumption (MO1) is the concavity requirement used for our derivation. When not concerning randomness, $r(\pi, w)$ can be understood as $p(\pi, w)\cdot w$, where $p(\pi,\cdot)$ is the inverse of the demand function $w(\pi,\cdot)$ mentioned earlier. This concavity requirement implies the price elasticity of demand, and is widely used in the literature; see, e.g., Feichtinger and Hartl (1985), Gallego and van Ryzin (1994), and Ziya, Ayhan, and Foley (2004). Actually, we can obtain from (4.3) the relationship $p(\pi, w) = \pi - w + 2$. This would result in

$$r(\pi, w) = (\pi - w + 2)\cdot w = (\frac{\pi}{2} + 1)^2 - (w - (\frac{\pi}{2} + 1))^2. \tag{4.21}$$

The latter is certainly concave in w. To see the price elasticity of demand implied by (MO1), let us suppose that $p(\pi,\cdot)$ is twice differentiable. From (4.6), we see that (MO1) is equivalent to

$$\frac{\partial^2 r(\pi,w)}{\partial w^2} = \frac{\partial^2 p(\pi,w)}{\partial w^2}\cdot(w+\theta) + 2\cdot\frac{\partial p(\pi,w)}{\partial w} \le 0, \qquad \forall w \in [0, w^U]. \tag{4.22}$$

By the inverse relationship between $w(\pi,\cdot)$ and $p(\pi,\cdot)$, we may derive that

$$\begin{cases} \partial p(\pi,w)/\partial w = 1/(\partial w(\pi,p)/\partial p)\mid_{p=p(\pi,w)}, \\ \partial^2 p(\pi,w)/\partial w^2 = -\partial p(\pi,w)/\partial w\cdot \partial^2 w(\pi,p)/\partial p^2 \mid_{p=p(\pi,w)} /(\partial w(\pi,p)/\partial p\mid_{p=p(\pi,w)})^2. \end{cases} \tag{4.23}$$

Plugging (4.23) into (4.22) and noting the negativity of $\partial p(\pi,\cdot)/\partial w$, we obtain

$$\frac{\partial^2 w(\pi,p)}{\partial p^2} \le \frac{2\cdot(\partial w(\pi,p)/\partial p)^2}{w(\pi,p)+\theta}. \tag{4.24}$$

The inequality (4.24) and hence (MO1), means that, the pace at which demand decreases with the sales price does not slow down much as the sales price increases.

By (4.6),

$$\frac{\partial r(\pi,w)}{\partial w} = \frac{\partial p(\pi,w)}{\partial w}\cdot(w+\theta) + p(\pi,w). \tag{4.25}$$

So, by considering (4.23), we must have

$$-\frac{\partial w(\pi,p)}{\partial p} \ge \frac{w(\pi,p)+\theta}{p}. \tag{4.26}$$

This means that the rate at which demand decreases with the sales price is greater than the ratio between demand and price. Therefore, imposing (MO1) confines us to products with elastic demands.

Conditions (MO2) and (MO3) are introduced to ensure the finiteness of certain optimal solutions. By bloating up unit backlogging and holding costs at rarely-encountered extreme inventory levels, they can always be realized.

Assumptions (MO4) and (MO5) indicate that the growth rate of the revenue with respect to the expected sales is increasing with the current-period cost at a rate sandwiched between the unit rate of one and the increasing rate of the discounted average next-period cost. As the current raw material cost increases, (MO4) says that it will be more likely for the former to justify the deferment of acquisition to the next period, and (MO5) says that it will be less likely for the marginal revenue per unit demand to justify any immediate acquisition. If we follow (4.3) and hence (4.21), we will have

$$\frac{\partial r(\pi, w)}{\partial w} = \pi - 2w. \tag{4.27}$$

With (4.27), we see that (MO4) would be true if $s(\pi) - \pi$ is decreasing in π; also, (MO5) would always be true.

By (MO6), we have the positive growth of the discounted average next-period cost $s(\pi) = \alpha \cdot E[\Pi' \mid \pi]$ in π. Note that an equivalent expression for (MO6) is that of $P[\Pi' \geq \pi' \mid \pi]$ being increasing in π for any given π'. So, (MO6) also means that a higher present-period cost is more likely to lead to a higher next-period cost. Therefore, we may view (MO6) as a time-continuity requirement, stipulating that cost is not expected to change drastically in a very short period of time. Taken together, (MO4) to (MO6) would imply that $s(\pi)$ is increasing in π at a mild pace, or when $s(\cdot)$ is differentiable,

$$0 \leq \frac{ds(\pi)}{d\pi} \leq 1. \tag{4.28}$$

The second inequality in (4.28) means the following mean-reversion tendency: as the raw material cost increases, it will become more difficult for the next-period cost to have the same degree of increase.

To see that time-continuity and mean-reversion are reasonable assumptions, note that the continuous-time iron ore spot price process $(\Pi_s \mid s \in \Re^+)$ may roughly be treated as an Ornstein-Uhlenbeck (OU) process. Under this setup, one has

$$d\Pi_s = \rho \cdot (\pi_0 - \Pi_s) \cdot ds + \sigma \cdot dW_s, \tag{4.29}$$

where $(W_s \mid s \in \Re^+)$ is the Wiener process, and ρ, π_0, and σ are positive constants. Meanwhile, Yang and Xia (2009) have shown that a discrete-state approximation of the OU process meets the mean-reversion requirement exactly and the time-continuity requirement asymptotically.

Let's give an example to illustrate all the assumptions (MO1)-(MO6) can be satisfied. In this example, there are five constants A, B, C, D, and γ with $A \in [0, 1)$, $B, C \in R^+$, $D \in [0, \bar{\alpha} \wedge A]$, and $\gamma \in (0, 1)$, as well as, a random variable Π^0 with its support on $[\underline{\pi}, \overline{\pi}]$ and mean at a certain $\pi^0 \equiv E[\Pi^0] \in [\underline{\pi}, \overline{\pi}]$. For the inverse-demand function, we let

$$p(\pi, w) = A\pi + \frac{B}{(w + \theta + C)^\gamma}, \tag{4.30}$$

while for the raw material price process, we let

$$(\Pi'|\pi) = \frac{D}{\alpha}\pi + (1 - \frac{D}{\alpha})\Pi^0. \tag{4.31}$$

From (4.6), we obtain that

$$r(\pi, w) = (A\pi + \frac{B}{(w + \theta + C)^\gamma}) \cdot (w + \theta), \tag{4.32}$$

which can be easily checked that $r(\pi, w)$ is concave. In addition, we get

$$\frac{\partial r(\pi, w)}{\partial w} = A\pi + \frac{B}{(w + \theta + C)^\gamma} - \frac{\gamma B \cdot (w + \theta)}{(w + \theta + C)^{1+\gamma}}. \tag{4.33}$$

Thus, we can verified that assumptions (MO1) and (MO3)-(MO6) can be all satisfied. With large enough b_∞ and h_∞, (MO2) and (MO3) can be satisfied as well.

Theorem 4.1. *For $t = 0, 1, 2, ...$, $f_t(\pi, x)$ is concave in x; and, for $t = 1, 2, ...$, $g_t(\pi, w, y)$ is both jointly concave and supermodular in (w, y).*

The first concavity on $f_t(\pi, \cdot)$ means that the marginal value of any additional inventory decreases with the present inventory level; the joint concavity of $g_t(\pi, \cdot, \cdot)$ makes the optimization problem (4.11) relatively easy to solve; also, the last supermodularity asserts the often-seen complementarity between supply (y) and demand (w).

Proof: We prove by induction. First, by (4.10), $f_0(\pi, x) = 0$ is clearly concave in $x \in \Re$. Then, suppose that $f_{t-1}(\pi, x)$ is concave in $x \in \Re$ for some $t = 1, 2, ...$

We first prove that $g_t(\pi, w, y)$ is jointly concave in $(w, y) \in \Re^2$. Let $(w^0, y^0), (w^1, y^1) \in \Re^2$ and $\lambda \in [0, 1]$. By (4.12), we have

$$g_t(\pi, (1-\lambda)w^0+\lambda w^1, (1-\lambda)y^0+\lambda y^1)-(1-\lambda)\cdot g_t(\pi, w^0, y^0)-\lambda\cdot g_t(\pi, w^1, y^1) = T_1+T_2+T_3, \tag{4.34}$$

where

$$\begin{cases} T_1 & = r(\pi, (1-\lambda)w^0 + \lambda w^1) - (1-\lambda) \cdot r(\pi, w^0) - \lambda \cdot r(\pi, w^1), \\ T_2 & = E[(1-\lambda) \cdot h((y^0 - w^0) - \Theta) + \lambda \cdot h((y^1 - w^1) - \Theta) \\ & \qquad -h((1-\lambda)(y^0 - w^0) + \lambda(y^1 - w^1) - \Theta)], \\ T_3 & = \alpha \cdot E[f_{t-1}(\Pi', (1-\lambda)(y^0 - w^0) + \lambda(y^1 - w^1) - \Theta) \\ & \qquad -(1-\lambda) \cdot f_{t-1}(\Pi', (y^0 - w^0) - \Theta) - \lambda \cdot f_{t-1}(\Pi', (y^1 - w^1) - \Theta) \mid \pi]. \end{cases} \tag{4.35}$$

We know that $T_1 \geq 0$ because of (MO1); we know that $T_2 \geq 0$ due to the convexity of $h(\cdot)$; and, by the induction hypothesis that $f_{t-1}(\pi, x)$ is concave in x, we know that $T_3 \geq 0$. Therefore, the left-hand side of (4.34) is positive.

We then prove that $g_t(\pi, w, y)$ is supermodular in $(w, y) \in \Re^2$. Let $(w, y) \in \Re^2$ and $(\delta w, \delta y) \in (\Re^+)^2$. By (4.12), we have

$$g_t(\pi, w+\delta w, y+\delta y)+g_t(\pi, w, y)-g_t(\pi, w+\delta w, y)-g_t(\pi, w, y+\delta y) = T_4+T_5, \quad (4.36)$$

where

$$\begin{cases} T_4 &= E[h(y-w-\Theta-\delta w)+h(y-w-\Theta+\delta y) \\ & \qquad -h(y-w-\Theta+\delta y-\delta w)-h(y-w-\Theta)], \\ T_5 &= \alpha \cdot E[f_{t-1}(\Pi', y-w-\Theta+\delta y-\delta w)+f_{t-1}(\Pi', y-w-\Theta) \\ & \qquad -f_{t-1}(\Pi', y-w-\Theta-\delta w)-f_{t-1}(\Pi', y-w-\Theta+\delta y) \mid \pi]. \end{cases} \quad (4.37)$$

We know that $T_4 \geq 0$ due to the convexity of $h(\cdot)$; and, by the induction hypothesis that $f_{t-1}(\pi, x)$ is concave in x, we know that $T_5 \geq 0$. Therefore, the left-hand side of (4.36) is positive.

Lastly, we prove that $f_t(\pi, x)$ is concave in $x \in \Re$. Let $x^0, x^1 \in \Re$ and $\lambda \in [0, 1]$. Let $(w^{0(1)}, y^{0(1)})$ be the optimal (w, y)-pair for (4.11) corresponding to $f_t(\pi, x^{0(1)})$. Hence, we have $0 \leq w^{0(1)} \leq w^U$, $y^{0(1)} \geq x^{0(1)}$, and

$$f_t(\pi, x^{0(1)}) = g_t(\pi, w^{0(1)}, y^{0(1)}). \quad (4.38)$$

Therefore, we have

$$\begin{cases} 0 \leq (1-\lambda)w^0 + \lambda w^1 \leq w^U, \\ (1-\lambda)y^0 + \lambda y^1 \geq (1-\lambda)x^0 + \lambda x^1. \end{cases} \quad (4.39)$$

Thus, we may obtain

$$\begin{aligned} f_t(\pi,(1-\lambda)x^0+\lambda x^1) &\geq g_t(\pi,(1-\lambda)w^0+\lambda w^1,(1-\lambda)y^0+\lambda y^1) \\ &\geq (1-\lambda)\cdot g_t(\pi,w^0,y^0)+\lambda\cdot g_t(\pi,w^1,y^1)=(1-\lambda)\cdot f_t(\pi,x^0)+\lambda\cdot f_t(\pi,x^1), \end{aligned} \tag{4.40}$$

where the first inequality is from (4.11) and (4.39), the second inequality is due to the joint concavity of $g_t(\pi,w,y)$ in (w,y), and the last equality is apparently due to (4.38).

We have thus completed the induction procedure. □

To go any further, we find it necessary to carry out a re-formulation of the problem. In it, we change decision variables from (w,y) to (y,v), where $v=y-w$. Note that $v_t-\theta=y_t-(w_t+\theta)$ is the average period-$(t-1)$ starting inventory level. Also, we define $j_t(\pi,y,v)$ so that

$$j_t(\pi,y,v)=g_t(\pi,y-v,y), \quad \text{and equivalently, } g_t(\pi,w,y)=j_t(\pi,y,y-w). \tag{4.41}$$

The earlier optimization problem involving (4.11) and (4.12) will now become

$$f_t(\pi,x)=\sup_{y\in[x,+\infty),v\in[y-w^U,y]} j_t(\pi,y,v), \tag{4.42}$$

where

$$\begin{aligned} j_t(\pi,y,v) &= -\theta\cdot s(\pi)-\pi y+s(\pi)\cdot v+r(\pi,y-v)-E[h(v-\Theta)] \\ &\quad +\alpha\cdot E[f_{t-1}(\Pi',v-\Theta)\mid\pi]. \end{aligned} \tag{4.43}$$

At this particular t, we can obtain some useful properties for $j_t(\pi,y,v)$. Taking derivatives on (4.12), we obtain

$$(\frac{\partial}{\partial w}+\frac{\partial}{\partial y})g_t(\pi,w,y)=\frac{\partial r(\pi,w)}{\partial w}-\pi. \tag{4.44}$$

From Theorem 4.1, we already know that

(f1) $\partial f_t(\pi,x)/\partial x$ is decreasing in x;

(g1) $\partial g_t(\pi,w,y)/\partial w$ is decreasing in w;

(g2) $\partial g_t(\pi,w,y)/\partial y$ is decreasing in y; and,

(g3) $\partial g_t(\pi,w,y)/\partial w$ is increasing in y, which is equivalent to the increase of $\partial g_t(\pi,w,y)/\partial y$ in w.

Also, we see from (MO1) and (4.44) that

(g4) $(\partial/\partial w+\partial/\partial y)g_t(\pi,w,y)$ is decreasing in w; and,

(g5) $(\partial/\partial w+\partial/\partial y)g_t(\pi,w,y)$ is invariant in y.

Recall that the re-formulation is about changing decision variables from (w,y) to (y,v) where $v=y-w$. From (4.41), we see that

$$\begin{cases}\partial j_t(\pi,y,v)/\partial y=\partial g_t(\pi,y-v,y)/\partial y=(\partial g_t(\pi,w,y)/\partial w+\partial g_t(\pi,w,y)/\partial y)\mid_{w=y-v},\\ \partial j_t(\pi,y,v)/\partial v=\partial g_t(\pi,y-v,y)/\partial v=-\partial g_t(\pi,w,y)/\partial w\mid_{w=y-v},\\ (\partial j_t(\pi,y,v)/\partial y+\partial j_t(\pi,y,v)/\partial v)=(\partial g_t(\pi,y-v,y)/\partial y+\partial g_t(\pi,y-v,y)/\partial v)\\ \qquad\qquad =\partial g_t(\pi,w,y)/\partial y\mid_{w=y-v}.\end{cases} \tag{4.45}$$

Before translating the properties of function $g_t(\pi,w,y)$ into those of $j_t(\pi,y,v)$, let us define diagonal dominance first:

Definition 4.1. An $n\times n$ matrix A is called diagonal dominant if $|a_{ii}|\geq\sum_{i\neq j}^{n}|a_{ij}|$, where a_{ij} is the element in the i^{th} row and j^{th} column. A real valued function, $f(s)$, defined on $s\in R^n$ is called diagonal dominant if its hessian matrix, $\bigtriangledown^2 f(s)$, is diagonal dominant.

With this definition, we know a joint concave function $f(x,y)$ is diagonal dominant if it has the property that $(\partial^2/\partial x^2+\partial^2/\partial x\partial y)f(x,y) \leq 0$ and $(\partial^2/\partial y^2+\partial^2/\partial x\partial y)f(x,y) \leq 0$. Thsi means $(\partial/\partial x + \partial/\partial y)f(x,y)$ is decreasing in x and y.

Now, (g1) to (g5) translate into the following, respectively:

(j1) $\partial j_t(\pi,y,v)/\partial v$ is decreasing in v, meaning that $j_t(\pi,y,v)$ is concave in v;

(j-) $(\partial/\partial y + \partial/\partial v)j_t(\pi,y,v)\mid_{y=y_0+u,v=v_0+u}$ is decreasing in u;

(j2) $\partial j_t(\pi,y,v)/\partial v\mid_{y=y_0+u,v=v_0+u}$ is decreasing in u, or equivalently, $(\partial/\partial y + \partial/\partial v)$ $j_t(\pi,y,v)$ is decreasing in v, meaning that $j_t(\pi,y,v)$ has the (v,y)-diagonal dominance property;

(j3) $\partial j_t(\pi,y,v)/\partial y$ is increasing in v, or equivalently, $\partial j_t(\pi,y,v)/\partial v$ is increasing in y, meaning that $j_t(\pi,y,v)$ is supermodular in (y,v); and,

(j4) $\partial j_t(\pi,y,v)/\partial y\mid_{y=y_0+u,v=v_0+u}$ is invariant in u, or equivalently, $(\partial/\partial y + \partial/\partial v)$ $j_t(\pi,y,v)$ is invariant in y, meaning that $j_t(\pi,y,v)$ has the (y,v)-diagonal balance property.

Note that (j3) and (j4) will together lead to

(j5) $\partial j_t(\pi,y,v)/\partial y$ is decreasing in y, meaning that $j_t(\pi,y,v)$ is concave in y.

At the same time, (j1), (j2), (j4), and (j5) shall guarantee that $j_t(\pi,y,v)$ is jointly concave in (y,v).

Let $w^0(\pi)$ be the largest $w \in [0,w^U]$ that maximizes the function $\{r(\pi,w)-\pi w\}$, i.e.,

$$w^0(\pi) = \sup \operatorname{argmax}_{w\in[0,w^U]}\{r(\pi,w)-\pi w\}. \tag{4.46}$$

By (MO1), the following must be true:

$$w^0(\pi)\begin{cases} = w^U, & \text{if } \partial^- r(\pi,w^U)/\partial w \geq \pi, \\ \in (0,w^U), & \text{else, if } \partial r(\pi,w^0(\pi))/\partial w = \pi, \\ = 0, & \text{otherwise.} \end{cases} \tag{4.47}$$

Here as well as later, $du(x)/dx = 0$ for a concave function means that $d^- u(x)/dx \geq 0 > d^+ u(x)/dx$.

Before going to the next theorem, let's first propose the following lemma:

Lemma 4.1. *Let $\underline{t}(\cdot)$ and $\bar{t}(\cdot)$ be two mildly increasing functions satisfying $\underline{t}(s) \leq \bar{t}(s)$ for every $s \in \Re$. Consider function $u(\cdot,\cdot)$ defined on the lattice $\{(s,t) \mid s \in \Re, t \in [\underline{t}(s), \bar{t}(s)]\}$. Suppose $u(s,\cdot)$ is concave on $[\underline{t}(s), \bar{t}(s)]$ for every $s \in \Re$, and $\hat{t}(s)$ is the largest solution for $\sup_{t\in[\underline{t}(s),\bar{t}(s)]} u(s,t)$. Then, $\hat{t}(\cdot)$ will be an increasing function when $u(\cdot,\cdot)$ is supermodular; furthermore, the increasing trend will be mild, i.e., with rate between 0 and 1, when $\partial u(s,t)/\partial t \mid_{s=s'+\Delta s, t=t'+\Delta s}$ is decreasing in Δs for every (s',t').*

Proof: As $u(s,\cdot)$ is concave on $[\underline{t}(s), \bar{t}(s)]$, the solution $\hat{t}(s)$ will satisfy the following:

$$\hat{t}(s) \begin{cases} = \bar{t}(s), & \text{if } \partial^- u(s, \bar{t}(s))/\partial t \geq 0, \\ \in (\underline{t}(s), \bar{t}(s)), & \text{else, if } \partial u(s, \hat{t}(s))/\partial t = 0, \\ = \underline{t}(s), & \text{otherwise.} \end{cases} \tag{4.48}$$

For the increasing part, let $\Delta s \geq 0$. When $\hat{t}(s) \leq \underline{t}(s+\Delta s)$, we are done since

$$\hat{t}(s+\Delta s) \geq \underline{t}(s+\Delta s) \geq \hat{t}(s). \tag{4.49}$$

Otherwise, $\hat{t}(s) > \underline{t}(s+\Delta s)$, and by the increasing property of $\bar{t}(\cdot)$,

$$\hat{t}(s) \leq \bar{t}(s) \leq \bar{t}(s+\Delta s). \tag{4.50}$$

Therefore, $\hat{t}(s)$ is a feasible candidate for $\hat{t}(s+\Delta s)$. But by the supermodularity of $u(\cdot,\cdot)$ and (4.48),

$$\frac{\partial^- u(s+\Delta s, \hat{t}(s))}{\partial t} \geq \frac{\partial^- u(s, \hat{t}(s))}{\partial t} \geq 0. \tag{4.51}$$

This, the concavity of $u(s,\cdot)$, and (4.48) will together lead to

$$\hat{t}(s+\Delta s) \geq \hat{t}(s). \tag{4.52}$$

Thus, we are done with the increasing part.

For the mildness part, let again $\Delta s \geq 0$. When $\hat{t}(s)+\Delta s \geq \bar{t}(s+\Delta s)$, we are done since

$$\hat{t}(s+\Delta s) \leq \bar{t}(s+\Delta s) \leq \hat{t}(s)+\Delta s. \tag{4.53}$$

Otherwise, $\hat{t}(s)+\Delta s < \bar{t}(s+\Delta s)$, and by the mild increasing of $\underline{t}(\cdot)$,

$$\hat{t}(s)+\Delta s \geq \underline{t}(s)+\Delta s \geq \underline{t}(s+\Delta s). \tag{4.54}$$

Therefore, $\hat{t}(s)+\Delta s$ is a feasible candidate for $\hat{t}(s+\Delta s)$. But by the last hypothesis on $u(\cdot,\cdot)$ and (4.48),

$$\frac{\partial^+ u(s+\Delta s, \hat{t}(s)+\Delta s)}{\partial t} \leq \frac{\partial^+ u(s,\hat{t}(s))}{\partial t} < 0. \tag{4.55}$$

This, the concavity of $u(s,\cdot)$, and (4.48) will together lead to

$$\hat{t}(s+\Delta s) \leq \hat{t}(s)+\Delta s. \tag{4.56}$$

Thus, we are done with the mildness part as well. □

The following now partially characterizes one optimal policy in terms of $(y_t^*(\pi,x), v_t^*(\pi,x) \mid t=1,2,...,\pi \in [\underline{\pi},\overline{\pi}], x \in \Re)$.

Theorem 4.2. *For $t=0,1,..,$ we have $\lim_{x\to-\infty} \partial f_t(\pi,x)/\partial x = 0$ and $\lim_{x\to+\infty} \partial f_t(\pi,x)/\partial x \leq 0$. For $t=1,2,...,$ let $\pi \in [\underline{\pi},\overline{\pi}]$ be given. There is in the (y,v)-plane a mildly increasing curve $v=\tilde{v}_t(\pi,y)$, i.e., $\tilde{v}_t(\pi,y) \leq \tilde{v}_t(\pi,y+\Delta y) \leq \tilde{v}_t(\pi,y)+\Delta y$,* *such that*

$$\tilde{v}_t(\pi, y) = \sup \operatorname{argmax}_{v \in [y-w^U, y]} j_t(\pi, y, v).$$

This curve intersects the curve $y = v + w^0(\pi)$ *at a point* $(y_t^0(\pi), y_t^0(\pi) - w^0(\pi))$ *and the curve* $y = v + w^U$ *at a point* $(y_t^U(\pi), y_t^U(\pi) - w^U)$*. It turns out that* $y_t^U(\pi) - y_t^0(\pi) \geq w^U - w^0(\pi) \geq 0$*, and one optimal solution pair* $(y_t^*(\pi, x), v_t^*(\pi, x))$ *for (4.42) satisfies the following:*

$$(y_t^*(\pi, x), v_t^*(\pi, x)) = \begin{cases} (y_t^0(\pi), y_t^0(\pi) - w^0(\pi)), & \text{when } x \leq y_t^0(\pi), \\ (x, \tilde{v}_t(\pi, x)), & \text{when } y_t^0(\pi) < x \leq y_t^U(\pi), \\ (x, x - w^U), & \text{when } x > y_t^U(\pi). \end{cases}$$

The policy identified by the theorem can be interpreted as a base-stock-list-price policy with the extra π dependence, and hence the raw-cost-dependent version of Federgruen and Heching's (1999) main result. We can think of $y_t^0(\pi)$ as a base-stock point for the firm's procurement activity. It should make raw material acquisition to bring its inventory level up to this point when its starting inventory level x is below the level, and should do nothing otherwise. At the same time, $p(\pi, w^0(\pi))$ serves as the firm's list price, which is to be charged when the firm's starting inventory level is below $y_t^0(\pi)$. When its starting as well as post-procurement inventory level x grows beyond the base-stock point, the firm should charge the price $p(\pi, \tilde{v}_t(\pi, x))$ which is decreasing in x. When x reaches the exorbitantly high level $y_t^U(\pi)$, the firm should start to charge its lowest allowed price $p(\pi, w^U)$. Without discussing the range of x, we can concisely express the firm's sales price as

$$p_t^*(\pi, x) = p(\pi, y_t^*(\pi, x) - v_t^*(\pi, x)) = p(\pi, \tilde{w}_t(\pi, (x \vee y_t^0(\pi)) \wedge y_t^U(\pi))), \tag{4.57}$$

where $\tilde{w}_t(\pi, y) = y - \tilde{v}_t(\pi, y)$ is mildly increasing in y as well.

The above policy dictates that the (π, x)-plane $[\underline{\pi}, \overline{\pi}] \times \Re$ be divided into three regions R_t^1 to R_t^3, so that

$$f_t(\pi, x) = \begin{cases} j_t(\pi, y_t^0(\pi), y_t^0(\pi) - w^0(\pi)), & \\ \qquad\qquad\qquad\qquad \text{when } (\pi, x) \in R_t^1, \text{ where } x \le y_t^0(\pi), & \\ j_t(\pi, x, \tilde{v}_t(\pi, x)), \qquad \text{when } (\pi, x) \in R_t^2, \text{ where } y_t^0(\pi) < x \le y_t^U(\pi), & \\ j_t(\pi, x, x - w^U), \qquad \text{when } (\pi, x) \in R_t^3, \text{ where } x > y_t^U(\pi). & \end{cases} \tag{4.58}$$

Remark 4.1. A comparable but more complex result than Theorem 4.2 can be obtained when demand follows a more general form than (4.4):

$$\Delta_t = \Omega_t \cdot w(\pi, p) + \Theta_t, \tag{4.59}$$

where Ω_t is a random multiplicative factor. Unfortunately, our forthcoming main result, Theorem 4.3, can only be reached for the additive-demand case.

Proof: We prove the theorem through induction. For a convex or concave function $g(\cdot)$, limits on $dg(x)/dx$ exist when $x \to \pm\infty$. We shall use $dg(\pm\infty)/dx$ to denote $\lim_{x\to\pm\infty} dg(x)/dx$.. By (4.10), we know

$$\frac{\partial f_0(\pi, -\infty)}{\partial x} = \frac{\partial f_0(\pi, +\infty)}{\partial x} = 0. \tag{4.60}$$

Suppose for some $t = 1, 2, ...$, we have

$$\frac{\partial f_{t-1}(\pi, -\infty)}{\partial x} \ge 0, \qquad \text{and} \qquad \frac{\partial f_{t-1}(\pi, +\infty)}{\partial x} \le 0. \tag{4.61}$$

For each $\pi \in [\underline{\pi}, \overline{\pi}]$, let $\tilde{y}_t(\pi, v)$ be the largest $y \in [v, v + w^U]$ that maximizes $j_t(\pi, y, v)$ for every $v \in \Re$, and let $\tilde{v}_t(\pi, y)$ be the largest $v \in [y - w^U, y]$ that maximizes $j_t(\pi, y, v)$

for every $y \in \Re$. On the other hand, note that (4.43) leads to

$$\begin{cases} \partial j_t(\pi, y, v)/\partial y &= -\pi + \partial r(\pi, y - v)/\partial w, \\ \partial j_t(\pi, y, v)/\partial v &= s(\pi) - \partial r(\pi, y - v)/\partial w - E[dh(v - \Theta)/dx] \\ & \qquad +\alpha \cdot E[\partial f_{t-1}(\Pi', v - \Theta)/\partial x \mid \pi]. \end{cases} \tag{4.62}$$

This and (j5) would together result in

$$\tilde{y}_t(\pi, v) \begin{cases} = v + w^U, & \text{if } \partial^- r(\pi, w^U)/\partial w \geq \pi, \\ \in (v, v + w^U), & \text{else, if } \partial r(\pi, \tilde{y}_t(\pi, v) - v)/\partial w = \pi, \\ = v, & \text{otherwise.} \end{cases} \tag{4.63}$$

Comparing this with (4.47), we see that $\tilde{y}_t(\pi, v) = v + w^0(\pi)$. Note that $v = y - w^U$ and $v = y$ are both mildly increasing curves in the (y, v)-plane. Hence, by (j2), (j3), and Lemma 4.1, we know that $\tilde{v}_t(\pi, y)$ is mildly increasing in y.

From (4.62), we have

$$\lim_{y \to -\infty} \frac{\partial^- j_t(\pi, y, y)}{\partial v} = s(\pi) - \frac{\partial^+ r(\pi, 0)}{\partial w} + b_\infty + \alpha \cdot E[\frac{\partial f_{t-1}(\Pi', -\infty)}{\partial x} \mid \pi], \tag{4.64}$$

which is positive by (MO2), (MO4), and the induction hypothesis (4.61). By (j1), this means that $\tilde{v}_t(\pi, y) = y$ when y is small enough. From (4.62) again,

$$\lim_{y \to +\infty} \frac{\partial^+ j_t(\pi, y, y - w^U)}{\partial v} = s(\pi) - \frac{\partial^- r(\pi, w^U)}{\partial w} - h_\infty + \alpha \cdot E[\frac{\partial f_{t-1}(\Pi', +\infty)}{\partial x} \mid \pi], \tag{4.65}$$

which is strictly negative by (MO1), (MO3), and the induction hypothesis (4.61). By (j1), this means that $\tilde{v}_t(\pi, y) = y - w^U$ when y is large enough.

Let $y_t^0(\pi)$ be the smallest y that satisfies $y - w^0(\pi) = \tilde{v}_t(\pi, y)$ when $w^0(\pi) = w^U$ and the largest such y otherwise. Let $y_t^U(\pi)$ be the smallest y that satisfies $y - w^U = \tilde{v}_t(\pi, y)$. From the conclusions of (4.64) and (4.65), we know that both $y_t^0(\pi)$ and $y_t^U(\pi)$ are finite. As $\tilde{v}_t(\pi,\cdot)$ is mildly increasing, we have $y_t^U(\pi) - w^U \geq y_t^0(\pi) - w^0(\pi)$; that is, $y_t^U(\pi) - y_t^0(\pi) \geq w^U - w^0(\pi) \geq 0$.

By the Karush-Kuhn-Tucker condition, $(y_t^*(\pi,x), v_t^*(\pi,x))$ will be an optimal solution pair for (4.42) if and only if the following are true: for $y_t^*(\pi,x)$,

$$\begin{cases} \text{either } y_t^*(\pi,x) = v_t^*(\pi,x) + w^U = x, \\ \quad \text{or } y_t^*(\pi,x) = x < v_t^*(\pi,x) + w^U \text{ and } \partial^+ j_t(\pi,x,v_t^*(\pi,x))/\partial y \leq 0, \\ \quad \text{or } y_t^*(\pi,x) = v_t^*(\pi,x) > x \text{ and } \partial^+ j_t(\pi,y_t^*(\pi,x),v_t^*(\pi,x))/\partial y \leq 0, \\ \quad \text{or } y_t^*(\pi,x) = v_t^*(\pi,x) + w^U > x \text{ and } \partial^- j_t(\pi,y_t^*(\pi,x),v_t^*(\pi,x))/\partial y \geq 0, \\ \quad \text{or } y_t^*(\pi,x) > x,\ v_t^*(\pi,x) < y_t^*(\pi,x) < v_t^*(\pi,x) + w^U, \\ \qquad\qquad \text{and } \partial j_t(\pi,y_t^*(\pi,x),v_t^*(\pi,x))/\partial y = 0; \end{cases} \tag{4.66}$$

for $v_t^*(\pi,x)$,

$$\begin{cases} \text{either } v_t^*(\pi,x) = y_t^*(\pi,x) - w^U \text{ and } \partial^+ j_t(\pi,y_t^*(\pi,x),v_t^*(\pi,x))/\partial v \leq 0, \\ \quad \text{or } v_t^*(\pi,x) = y_t^*(\pi,x) \text{ and } \partial^- j_t(\pi,y_t^*(\pi,x),v_t^*(\pi,x))/\partial v \geq 0, \\ \quad \text{or } y_t^*(\pi,x) - w^U < v_t^*(\pi,x) < y_t^*(\pi,x) \text{ and } \partial j_t(\pi,y_t^*(\pi,x),v_t^*(\pi,x))/\partial v = 0. \end{cases} \tag{4.67}$$

We can check that the chosen solution pair has satisfied the above requirements.

Finally, we derive limiting conditions for $\partial f_t(\pi,x)/\partial x$. According to the policy chosen, we have

$$\frac{\partial f_t(\pi,-\infty)}{\partial x} = 0, \tag{4.68}$$

whereas

$$\frac{\partial f_t(\pi,+\infty)}{\partial x} = \lim_{x\to+\infty}[\frac{\partial j_t(\pi,x,x-w^U)}{\partial y} + \frac{\partial j_t(\pi,x,x-w^U)}{\partial v}]. \tag{4.69}$$

By (4.62), the latter results in

$$\frac{\partial f_t(\pi,+\infty)}{\partial x} = -\pi + s(\pi) - h_\infty + \alpha \cdot E[\frac{\partial f_{t-1}(\Pi',+\infty)}{\partial x} \mid \pi]. \tag{4.70}$$

Using (MO3) to (MO5), as well as the induction hypothesis (4.61), we can confirm that the above left-hand side is strictly below 0. We are thus done with the induction process. $\square$

Now we show that, with a higher raw material cost π, each additional unit of initial inventory brings less marginal profit to the firm; consequently, all π-dependent points and curves that help define the firm's policy as delineated in Theorem 4.2 are decreasing in π.

Theorem 4.3. *For $t = 0, 1, 2, ...$, the function $f_t(\pi, x)$ is submodular in (π, x). Also, for $t = 1, 2, ...$, the function $j_t(\pi, y, v)$ is submodular in both (π, y) and (π, v). Moreover, for $t = 1, 2, ...$, $\tilde{v}_t(\pi, y)$, $y_t^0(\pi)$, $y_t^U(\pi)$, and $v_t^0(\pi) \equiv y_t^0(\pi) - w^0(\pi)$ are all decreasing in π; as a consequence, $y_t^*(\pi, x)$ is decreasing in π, and $v_t^*(\pi, x)$ is decreasing in π.*

In Theorem 4.3, the submodularity of $f_t(\pi, x)$ in (π, x) is expected. Under a higher raw material cost, the firm will be able to extract less profit from each additional unit in inventory. Similarly, the submodularity of $j_t(\pi, y, v)$ in (π, y) and (π, v) reflects the decreasing marginal value of inventory with respect to the raw material cost. The theorem also says that boundaries of the three-region description of $f_t(\pi, x)$ given in (4.58) have declining trends in π. Figure 4.1 is a depiction of $f_t(\pi, x)$.

From the theorem's decreasing trend of $y_t^*(\pi, x)$ in π, we know that the firm should lower its acquisition target level when there is an increase in the raw material cost. This is the same message conveyed by the main result of Yang and Xia (2009), which is not concerned with pricing. The lowered acquisition target level helps the firm's profitability in the present period.

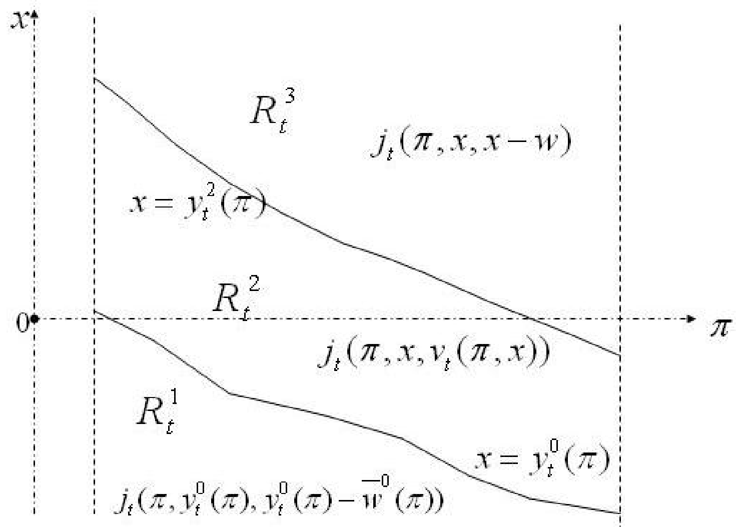

Figure 4.1: Depiction of $f_t(\pi, x)$ in the (π, x)-plane

When the raw material inventory level is lower than the base stock level, the firm will charge list price $p(\pi, w^0(\pi))$. Assume $p(\pi, w)$ is increasing the cost π which is not used in our model. Then, $p(\pi, w^0(\pi))$ is increasing in π because $w^0(\pi)$ is decreasing in π and $p(\pi, w)$ increases in w. However, when the raw material inventory level is too high, the firm will charge sales price $p(\pi, x - \tilde{v}_t(\pi, x))$ which may increase or decrease in π. As verifiable by numerical tests, no definitive conclusion can be reached on the direction in which the firm should adjust its sales price or its targeted sales volume in the face of a changing raw material cost. Meanwhile, the decreasing trend of $v_t^*(\pi, x)$ in π indicates that the pricing policy should be carried out to lower the expected next-period inventory level when the raw material cost increases.

Note (a) the relation

$$\begin{aligned} \text{expected next-period inventory level} &= \text{post-procurement inventory level} \\ &\quad - \text{expected sales volume,} \end{aligned}$$

(b) the decreasing trend of the post-procurement inventory level over the raw material cost, and (c) the fact that the sales volume is negatively correlated with the sales price charged. Hence, the trend is about "moderation in pricing". In the face of higher raw material costs, the firm should not charge too high a sales price to raise the expected next-period inventory level. A lowered expected next-period inventory level is actually reasonable, as it prepares the firm for inflated raw material costs to come in the near future.

Before providing the proof of Theorem 4.3, we first present the following lemma which is from Lemma 2.8.1 of Topkis (1998).

Lemma 4.2. *Let $\underline{t}$ and $\bar{t}$ be two constants satisfying $\underline{t} \leq \bar{t}$. Consider function $u(\cdot,\cdot)$ defined on $\Re \times [\underline{t}, \bar{t}]$. Suppose $u(s,\cdot)$ is concave on $[\underline{t}, \bar{t}]$ for every $s \in \Re$, and $\hat{t}(s)$ is the largest solution for $\sup_{t \in [\underline{t}, \bar{t}]} u(s,t)$. Then, $\hat{t}(\cdot)$ will be a decreasing function when $u(\cdot,\cdot)$ is submodular.*

We now prove the Theorem 4.3 through induction.

Proof: First, by (4.10), $f_0(\pi, x) = 0$ is clearly submodular in (π, x). Then, for some $t = 1, 2, ...,$ suppose that $f_{t-1}(\pi, x)$ is submodular in (π, x).

Now, we prove that $j_t(\pi, y, v)$ is submodular in (π, y). Recall (4.62) says

$$\frac{\partial j_t(\pi, y, v)}{\partial y} = -\pi + \frac{\partial r(\pi, y - v)}{\partial w}. \tag{4.71}$$

From (MO5), we know that $\partial j_t(\pi, y, v)/\partial y$ is decreasing in π.

Then, we prove that $j_t(\pi, y, v)$ is submodular in (π, v). Recall (4.62) says

$$\frac{\partial j_t(\pi, y, v)}{\partial v} = s(\pi) - \frac{\partial r(\pi, y - v)}{\partial w} - E[\frac{dh(v - \Theta)}{dx}] + \alpha \cdot E[\frac{\partial f_{t-1}(\Pi', v - \Theta)}{\partial x} \mid \pi]. \tag{4.72}$$

By (MO4), we know that $s(\pi) - \partial r(\pi, y - v)/\partial w$ is decreasing in π. By the induction hypothesis, we know that $E[\partial f_{t-1}(\pi, v - \Theta)/\partial x]$ is decreasing in π. This and (MO6) lead to that $E[\partial f_{t-1}(\Pi', v - \Theta)/\partial x \mid \pi]$ is decreasing in π. Therefore, $\partial j_t(\pi, y, v)/\partial v$ is decreasing in π.

Lastly, we prove that $f_t(\pi, x)$ is submodular in (π, x) and along the way, the π-monotonicity of the policy-defining constants and curves. Because we are to use Theorem 4.2 extensively, we will call upon its results without further mentioning the theorem. From (MO1) and (MO5), we may know that $w^0(\pi)$ is decreasing in π. By $j_t(\pi, y, v)$'s submodularity in (π, v) and its concavity in v, we may use Lemma 4.2 to establish that $\tilde{v}_t(\pi, y)$ is decreasing in π at any fixed y.

Note that $(y_t^0(\pi), y_t^0(\pi) - w^0(\pi))$ is the intersection of the two curves $y = v + w^0(\pi)$ and $v = \tilde{v}_t(\pi, y)$. For $\delta\pi \geq 0$, suppose curves $y = v + w^0(\pi)$ and $v = \tilde{v}_t(\pi + \delta\pi, y)$ intersect at (y', v'). Then, since $\tilde{v}_t(\cdot, y)$ is decreasing, we have that $v' \leq y_t^0(\pi) - w^0(\pi)$. But $\tilde{v}_t(\pi + \delta\pi, \cdot)$ is itself increasing, so $y' \leq y_t^0(\pi)$. Because $w^0(\cdot)$ is decreasing, the y-component of the intersection between $y = v + w^0(\pi + \delta\pi)$ and $v = \tilde{v}_t(\pi + \delta\pi, y)$ must be below that of the intersection between $y = v + w^0(\pi)$ and $v = \tilde{v}_t(\pi + \delta\pi, y)$. That is, $y_t^0(\pi + \delta\pi) \leq y'$. Therefore, we have

$$y_t^0(\pi + \delta\pi) \leq y' \leq y_t^0(\pi). \tag{4.73}$$

Hence, $y_t^0(\pi)$ is decreasing in π. This, together with that $\tilde{v}_t(\pi, y)$ is decreasing in π and increasing in y, lead to $v_t^0(\pi) = y_t^0(\pi) - w^0(\pi) = \tilde{v}_t(\pi, y_t^0(\pi))$ being decreasing in π. Meanwhile, $(y_t^U(\pi), y_t^U(\pi) - w^U)$ is the intersection of the two curves $y = v + w^U$ and $v = \tilde{v}_t(\pi, y)$. Since $\tilde{v}_t(\pi, y)$ is decreasing in π, $y_t^U(\pi)$ is decreasing in π.

Let $x' = -x$, $y' = -y$, and $v' = -v$. Then $j_t(\pi, y', v') = j_t(\pi, -y, -v)$ is supermodular in (π, y') and (π, v'). Due to (j3), $j_t(\pi, y', v')$ is also supermodular in (y', v'). As a result, $j_t(\pi, y', v')$ is supermodular in (π, y', v'). To prove that $f(\pi, x)$ is submodular in (π, x), we can equivalently show that $f(\pi, x') = f(\pi, -x)$ is supermodular in (π, x'). It is easy to verify that $S' = \{(\pi, x') | \pi \in [\underline{\pi}, \overline{\pi}], x \in \Re\}$ and $D' = \{(\pi, y', v') | y' \leq x', y' \leq v' \leq y' + w^U\}$ are lattices. Let $\pi_1 < \pi_2$, $x'_1 < x'_2$, (y'_1, v'_1) and (y'_2, v'_2) be the optimal solution for $f(\pi_1, x'_1)$ and $f(\pi_2, x'_2)$, respectively. Then, we have

$$\begin{aligned} f_t(\pi_1, x'_1) + f_t(\pi_2, x'_2) &= j_t(\pi_1, y'_1, v'_1) + j_t(\pi_2, y'_2, v'_2) \\ &\leq j_t(\pi_1 \vee \pi_2, y'_1 \vee y'_2, v'_1 \vee v'_2) + j_t(\pi_1 \wedge \pi_2, y'_1 \wedge y'_2, v'_1 \wedge v'_2) \\ &\leq f_t(\pi_1 \vee \pi_2, x'_1 \vee x'_2) + f_t(\pi_1 \wedge \pi_2, x'_1 \wedge x'_2). \end{aligned} \tag{4.74}$$

Therefore, $f_t(\pi, x')$ is supermodular in (π, x') and $f_t(\pi, x)$ is submodular in (π, x). We have thus completed the induction procedure.

For the final trends at a particular t, let us suppose π increases while $(\pi, x) \in R_t^1$. Then, $y_t^*(\pi, x)$ will be $y_t^0(\pi)$, which is decreasing in π; meanwhile, $v_t^*(\pi, x)$ will be $v_t^0(\pi)$, which is decreasing in π as well. Suppose π increases while $(\pi, x) \in R_t^2$. Then, $y_t^*(\pi, x)$ will be fixed at x, while $v_t^*(\pi, x)$ will be $\tilde{v}_t(\pi, x)$, which is decreasing in π. Suppose π increases while $(\pi, x) \in R_t^3$. Then, $y_t^*(\pi, x)$ will be fixed at x, while $v_t^*(\pi, x)$ will be fixed at $x - w^U$.

At the boundary between R_t^1 and R_t^2, the policy is continuous, as $x = y_t^0(\pi) = \tilde{v}_t(\pi, x) + w^0(\pi)$. At the boundary between R_t^2 and R_t^3, the policy is also continuous, as $x = y_t^U(\pi) = \tilde{v}_t(\pi, x) + w^U$.

Therefore, we have that, on any trajectory in the (π, x)-plane with a varying π, both $y_t^*(\pi, x)$ and $v_t^*(\pi, x)$ will decrease with π. □

4.3 The Infinite Horizon Case

We can extend earlier structural results on value functions and policies to the case where the firm faces statistically the same environment in an infinite number of periods. Let $\tilde{f}(\pi, x)$ be the maximum total discounted expected profit the firm can earn when starting with state (π, x) and given an infinite number of periods. For the terminal period, we now assume the following instead of (4.8):

$$\tilde{f}_0(\pi, x) = -\frac{h(x)}{1-\alpha} - \frac{b_\infty \theta}{(1-\alpha)^2}. \tag{4.75}$$

By (4.9), this would lead to

$$f_0(\pi, x) = -\pi x - \frac{h(x)}{1-\alpha} - \frac{b_\infty \theta}{(1-\alpha)^2}. \tag{4.76}$$

For the current infinite-horizon case, the presence of infinite discountings renders the choice of the terminal-period value function immaterial. But our current choice (4.75) serves to simplify analysis, as can be seen from the following monotone result.

Proposition 4.1. *The sequence* $(f_t(\pi, x) \mid t = 0, 1, 2, ...)$ *is ascending. That is, for any* $\pi \in [\underline{\pi}, \overline{\pi}]$ *and* $x \in \Re$, *we have*

$$f_0(\pi, x) \le f_1(\pi, x) \le f_2(\pi, x) \le \cdots .$$

Proof: Suppose the firm starts period t with state (π, x). In the first $t-1$ periods, it may execute an optimal policy suitable for the situation where time 1, instead of time 0, is the end of the planning horizon; in the last period, period 1, the firm may opt to acquire nothing and charge the demand-minimizing price $p(\Pi_1, 0)$.

Note that

$$\frac{1}{(1-\alpha)^2} = 1 + 2\alpha + 3\alpha^2 + 4\alpha^3 + \cdots , \tag{4.77}$$

while b_∞ is the maximum penalty cost to the firm for any one unit loss of its inventory. So $\tilde{f}_0(\pi, x)$ as defined by (4.75) is a lower bound to the firm's infinite-horizon total discounted average profit when it starts with inventory level x. This bound is achieved when the firm receives no revenue, makes no acquisition, and keeps on charging demand-minimizing prices. Thus, the firm's total payoff will be above $\tilde{f}_{t-1}(\pi, x)$.

As the firm has the potential to do better with a different policy, we have

$$\tilde{f}_t(\pi, x) \geq \tilde{f}_{t-1}(\pi, x). \tag{4.78}$$

This, through (4.9), translates into

$$f_t(\pi, x) \geq f_{t-1}(\pi, x). \tag{4.79}$$

Therefore, the sequence $(f_t(\pi, x) \mid t = 0, 1, 2, ...)$ is ascending, and hence we are done proving the proposition. $\square$

Also, we can establish loose bounds for post-procurement inventory levels.

Proposition 4.2. *For $x \geq 0$, it is true that $\partial f_t(\pi, x)/\partial x \leq 0$. For the optimal post-procurement inventory level $y_t^*(\pi, x)$ provided by Theorem 4.2, it is true that*

$$y_t^*(\pi, x) \leq x \vee y^0,$$

for some positive constant y^0.

Proof: When $t \geq 2$, Theorem 4.2 states that $\partial f_{t-1}(\pi, -\infty)/\partial x = 0$, which, in view of $f_{t-1}(\pi, \cdot)$'s concavity, leads to $\partial f_{t-1}(\pi, x)/\partial x \leq 0$ for $x \in \Re$. Combining this with (4.76), we may see that, for any $t = 1, 2, ...,$

$$\frac{\partial f_{t-1}(\pi, x)}{\partial x} \leq 0, \tag{4.80}$$

when $x \in \Re^+$. We now consider $y \in \Re^+$. Due to (4.12) and (4.80), we can establish that

$$\frac{\partial g_t(\pi, w, y)}{\partial y} \leq -\pi + s(\pi) - E[\frac{dh(y - w - \Theta)}{dx}]. \tag{4.81}$$

Due to the bounds of π, this will lead to

$$\frac{\partial g_t(\pi, w, y)}{\partial y} \leq \alpha\overline{\pi} - \underline{\pi} - E[\frac{dh(y - w - \Theta)}{dx}]. \tag{4.82}$$

By the convexity of $h(\cdot)$ and (MO3), we know that there exists some x^0, so that $dh(x)/dx \geq (h_\infty + (\alpha\overline{\pi} - \underline{\pi})^+)/2$ when $x \geq x^0$. Due again to the convexity of $h(\cdot)$, we may derive from (4.82) that

$$\frac{\partial g_t(\pi, w, y)}{\partial y} \leq \alpha\overline{\pi} - \underline{\pi} + b_\infty \cdot P[y < x^0 + w + \Theta)] - \frac{h_\infty + (\alpha\overline{\pi} - \underline{\pi})^+}{2} \cdot P[y \geq x^0 + w + \Theta]. \tag{4.83}$$

Define constant β so that

$$\beta = \frac{\alpha\overline{\pi} - \underline{\pi} + b_\infty}{(h_\infty + (\alpha\overline{\pi} - \underline{\pi})^+)/2 + b_\infty}, \tag{4.84}$$

which is within $[0, 1)$ by (MO2) and (MO3). Now let θ^0 be such that

$$P[\Theta \leq \theta^0] > \beta. \tag{4.85}$$

For $y \geq x^0 + w + \theta^0$, we have

$$P[y \geq x^0 + w + \Theta] = P[\Theta \leq y - x^0 - w] \geq P[\Theta \leq \theta^0] > \beta, \tag{4.86}$$

which, by (4.83), implies

$$\frac{\partial g_t(\pi, w, y)}{\partial y} < \alpha\overline{\pi} - \underline{\pi} + b_\infty \cdot (1 - \beta) - \frac{h_\infty + (\alpha\overline{\pi} - \underline{\pi})^+}{2} \cdot \beta = 0. \tag{4.87}$$

Due to (4.11), we must have

$$y_t^*(\pi, x) \leq x \vee (x^0 + w^U + \theta^0). \tag{4.88}$$

So, the claim of Proposition 4.2 is true with $y^0 = x^0 + w^U + \theta^0$. $\square$

The function $r(\cdot,\cdot)$ as defined in (4.6) is continuous on the compact set $[\underline{\pi},\overline{\pi}] \times [0, w^U]$ as $p(\cdot,\cdot)$ is. We may let $r^M \equiv \sup_{\pi\in[\underline{\pi},\overline{\pi}], w\in[0,w^U]} r(\pi, w)$, which is a finite number. Thus, even given an infinite number of periods, the firm's total discounted average profit will not exceed $r^M/(1-\alpha)$.

Using (4.10), (4.11), and (4.12), one can establish the loose bound

$$f_t(\pi, x) \leq -\pi x + \frac{\bar{r}^M}{1-\bar{\alpha}}. \tag{4.89}$$

This says that the sequence $(f_t(\pi, x) \mid t = 0, 1, 2, ...)$ is bounded from above. (4.89), Proposition 4.1, and Proposition 4.2 together entail the pointwise convergence of the sequence $(f_t(\pi, x) \mid t = 0, 1, 2, ...)$ to some $f(\pi, x)$. We now suppose that the random factors Θ are independent of each other. Let $\tilde{f}(\pi, x)$ be the maximum total discounted expected profit the firm can earn when starting with state (π, x) and given an infinite number of periods. The following reveals that the limit point $f(\pi, x)$ is indeed the profit function for the infinite-horizon problem, much like what $f_t(\pi, x)$ is for the t-period problem. Also, there is more to the aforementioned pointwise convergence.

The following shows that the limit of finite-horizon value functions serves as the value function for the limiting, infinite-horizon, case.

Theorem 4.4. *The sequence $(f_t(\pi, x) \mid t = 0, 1, 2, ...)$ converges to some $f(\pi, x)$ uniformly in any (π, x)-region with a bounded x-range. For every $\pi \in [\underline{\pi},\overline{\pi}]$ and $x \in \Re$, we have, much like (4.9),*

$$f(\pi, x) = \tilde{f}(\pi, x) - \pi x.$$

Proof: When given an infinite number of periods, say periods $t, t-1, ..., 1, 0, -1, -2, ...$, a firm may execute a t-period optimal policy for the first t periods. Then, starting from period 0, the firm may keep on charging demand-minimizing prices, $p(\Pi_0, 0)$, $p(\Pi_{-1}, 0), ...$, while carrying out no acquisition. In view of (4.75), the firm will earn more than $\tilde{f}_t(\pi, x)$ on average. Therefore, we have

$$\tilde{f}(\pi, x) \geq \tilde{f}_t(\pi, x). \tag{4.90}$$

On the other hand, when given t periods $t, t-1, ..., 1$, the firm may execute an optimal infinite-horizon policy till the end of the horizon. If it were given the chance to continue executing the policy from period 0 onwards, the firm would have earned at most an extra $\alpha^t \cdot r^M/(1-\alpha)$ in period-t money. Now, with time 0 as the terminal point, the firm will be charged with the inventory cost of $\alpha^t \cdot h(X_0)/(1-\alpha)$ in period-t money, where X_0 is the firm's inventory level at time 0 after it has started with (π, x) and experienced random shocks $\Theta_t, ..., \Theta_1$, $\Pi_{t-1}, ..., \Pi_1$, as well as the optimal infinite-horizon policy. By Proposition 4.2, we know that

$$X_0 \leq \mid x \mid + y^0, \tag{4.91}$$

where y^0 is a positive constant defined in the proposition. On the other hand, due to w's bounded range, we have

$$X_0 \geq - \mid x \mid - w^U \cdot t - (\Theta_t + \cdots + \Theta_1). \tag{4.92}$$

Therefore, we have

$$E[h(X_0)] \leq h_\infty \cdot (\mid x \mid + y^0) + b_\infty \cdot (\mid x \mid + (w^U + \theta) \cdot t). \tag{4.93}$$

Combining the above, we obtain

$$\tilde{f}(\pi,x)-\tilde{f}_t(\pi,x)\leq\frac{\alpha^t}{1-\alpha}\cdot(r^M+E[h(X_0)])\leq\frac{\alpha^t}{1-\alpha}\cdot(a^0\cdot\mid x\mid+b^0\cdot t+c^0),\quad(4.94)$$

where we have let

$$a^0=h_\infty+b_\infty,\qquad b^0=b_\infty\cdot(w^U+\theta),\qquad c^0=h_\infty\cdot y^0+r^M.\tag{4.95}$$

Note that

$$\lim_{t\to+\infty}\alpha^t=\lim_{t\to+\infty}t\cdot\alpha^t=0.\tag{4.96}$$

Therefore, we may know from (4.90) and (4.94) that the sequence $(\tilde{f}_t(\pi,x)\mid t=0,1,2,...)$ converges to $\tilde{f}(\pi,x)$. From (4.9) and the pointwise convergence of $(f_t(\pi,x)\mid t=0,1,2,...)$ to $f(\pi,x)$, we have

$$f(\pi,x)=\tilde{f}(\pi,x)-\pi x.\tag{4.97}$$

By (4.9) and (4.97), we will derive from (4.94) that

$$f(\pi,x)-f_t(\pi,x)\leq\frac{\alpha^t}{1-\alpha}\cdot(a^0\cdot\mid x\mid+b^0\cdot t+c^0).\tag{4.98}$$

This dictates that the aforementioned convergence is uniform in any (π,x)-region with a bounded x-range. □

Now we can follow (4.43) to define $j(\pi,y,v)$, so that

$$j(\pi,y,v)=-\theta\cdot s(\pi)-\pi y+s(\pi)\cdot v+r(\pi,y-v)-E[h(v-\Theta)]+\alpha\cdot E[f(\Pi',v-\Theta)\mid\pi].\tag{4.99}$$

The next result reaches three points. It shows that (i) $j(\pi,w,y)$ is the limit point of its finite-horizon counterparts, (ii) the limiting version of the finite-horizon relationship (4.42) is valid between the infinite-horizon entities $f(\pi, x)$ and $j(\pi, y, v)$, and (iii) one of the limiting control problem's optimal policies enjoys the same characteristics as those described in Theorems 4.2 and 4.3 for finite-horizon policies.

Theorem 4.5. *For every $\pi \in [\underline{\pi}, \overline{\pi}]$ and $(y, v) \in \Re^2$, the sequence $(j_t(\pi, y, v) \mid t = 1, 2, ...)$ converges to $j(\pi, y, v)$. Like in (4.42), we have*

$$f(\pi, x) = \sup_{y \in [x, +\infty), v \in [y - w^U, y]} j(\pi, y, v).$$

Consequently, one optimal policy for the above problem has the same characteristics as the one presented in Theorem 4.2; furthermore, the policy $(y^*(\pi, x), v^*(\pi, x))$ just identified possesses properties described for its finite-horizon counterparts in Theorem 4.3: $y^*(\pi, x)$ is decreasing in π, and $v^*(\pi, x)$ is decreasing in π unless x is high enough to elicit the lowest possible sales price $p(\pi, w^U)$.

Theorem 4.5 asserts that policy structures known for the finite-horizon case is still true for the infinite-horizon case. When remaining operational indefinitely, the firm should still use a raw-cost-dependent base-stock-list-price policy to carry out its procurement and sales activities; also, the base-stock level $y_t^0(\pi)$ for acquisition should decrease in the face of higher raw material costs π.

The theorem's last and most important claim is that the expected next-period starting inventory level $v^*(\pi, x) + \theta$ should decrease with the material cost π. It conveys the following message to practitioners: when raw material becomes more expensive, acquisition should be scaled back; sales may or may not have to be reduced; regardless, there should be "moderation in pricing," in that the reduction in sales should not exceed that in procurement.

Proof: From (4.76) and Lemma 4.1, we know that $f_t(\pi, x)$ has a lower bound:

$$f_t(\pi, x) \geq -\pi x - \frac{h(x)}{1-\alpha} - \frac{b_\infty \theta}{(1-\alpha)^2}. \tag{4.100}$$

On the other hand, using (4.76), (4.11), and (4.12), one can establish the loose bound

$$f_t(\pi, x) \leq -\pi x + \frac{r^M}{1-\alpha}. \tag{4.101}$$

where r^M is defined in the proof of Theorem 4.4.Hence, loosely speaking,

$$\mid f_t(\pi, x) \mid \leq \pi \cdot \mid x \mid + \frac{r^M + h(x)}{1-\alpha} + \frac{b_\infty \theta}{(1-\alpha)^2}. \tag{4.102}$$

So, using Theorem 4.4 and the dominated convergence theorem, we can show that the sequence $(g_t(\pi, w, y) \mid t = 1, 2, ...)$, defined through (4.12), converges to some $g(\pi, w, y)$; as for the latter, it is linked to $j(\pi, y, v)$ already defined in (4.99) through

$$j(\pi, y, v) = g(\pi, y - v, y), \quad \text{and equivalently, } g(\pi, w, y) = j(\pi, y, y - w). \tag{4.103}$$

Due to Theorem 4.4, we also know that the convergence is uniform in any (π, w, y)-region with a bounded (w, y)-range.

As concavity and modularity properties are preserved under even pointwise convergence, we know from Theorems 4.1, 4.3, and 4.4, as well as the above, that $f(\pi, x)$ is concave in x and submodular in (π, x), $g(\pi, w, y)$ is jointly concave and supermodular in (w, y), and $j(\pi, y, v)$ is submodular in both (π, y) and (π, v). From Theorems 4.2 and 4.4, we also know

$$\frac{\partial f(\pi, -\infty)}{\partial x} = 0, \qquad \text{and} \qquad \frac{\partial f(\pi, +\infty)}{\partial x} \leq 0. \tag{4.104}$$

With these, we can use the same reasoning as used in the proof of Theorems 4.2

and 4.3 to establish that one optimal policy $(y*(\pi,x), v^*(\pi,x))$ for the optimization problem $\sup_{y\in[x,+\infty),v\in[y-w^U,y]} j(\pi,y,v)$ enjoys all properties possessed by its finite-t counterparts.

Let y^0 be as defined in Lemma 4.2. As $[0,w^U]\times[x,x\vee y^0]$ is a bounded region in the (w,y)-plane, we may use the earlier convergence to show that the sequence $(\sup_{w\in[0,w^U],y\in[x,x\vee y^0]} g_t(\pi,w,y) \mid t=1,2,...)$ converges to $\sup_{w\in[0,w^U],y\in[x,x\vee y^0]} g(\pi,w,y)$ for every (π,x)-pair. On the other hand, Lemma 4.2 says that $\partial f_t(\pi,x)/\partial x\le 0$ for $x\ge 0$. This along with Theorem 4.4 leads to $\partial f(\pi,x)/\partial x\le 0$ for $x\ge 0$. Using the same argument as used in the proof of Lemma 4.2, we can then establish that $y^*(\pi,x)\le x\vee y^0$. Hence,

$$\sup_{w\in[0,w^U],y\in[x,x\vee y^0]} g(\pi,w,y)=\sup_{w\in[0,w^U],y\in[x,+\infty)} g(\pi,w,y). \tag{4.105}$$

We already know the finite-t counterpart of (4.105) through Lemma 4.2. Therefore, we have the convergence of $(\sup_{w\in[0,w^U],y\in[x,+\infty)} g_t(\pi,w,y) \mid t=1,2,...)$ to $\sup_{w\in[0,w^U],y\in[x,+\infty)} g(\pi,w,y)$ for every (π,x)-pair. In view of (4.12) and Theorem 4.4, however, the only possibility is that

$$f(\pi,x)=\sup_{w\in[0,w^U],y\in[x,+\infty)} g(\pi,w,y). \tag{4.106}$$

But by (4.103), this translates into

$$f(\pi,x)=\sup_{y\in[x,+\infty),v\in[y-w^U,y]} j(\pi,y,v). \tag{4.107}$$

We have thus proved the theorem. $\square$

CHAPTER 5

Numerical Study

In this chapter, we conduct numerical study to explore the results we haven't found from the previous models. Through out this chapter, we focus on the following questions:

(a) whether the markdown case does not have complementarity between price flexibility and inventory, a property that is essential for the markup case;

(b) whether earlier treatment of the markup case needs minor corrections;

(c) whether k-monotonicity is in general not true for the threshold policies of the irreversible-pricing cases;

(d) whether heeding the time-variability of $\beta(t)$ helps reap huge benefits;

(e) whether optimal policies for arrival patterns more general than the current product form in dynamic pricing models are not necessarily threshold-like;

(f) whether there is emperical evidence supporting for the assumption of Markovian raw material cost;

(g) whether considering cost-dependent policy provides significant profit benefit;

(h) whether the increment of the cost can be fully passed to the customer; and,

(i) whether violation of assumption (M4)-(MO6) will lead to the break of π-monotone properties of the optimal policy in the make-to-order model.

5.1 Parameters Setup

For answering question (a)-(e) which are raised for the dynamic pricing cases, we take the following setup: the horizon length $T = 1$, $K = 4$ so that there are five different price levels, the number of initial stock level $N = 20$, the price-level vector $(\bar{p}^k \mid k = 0, 1, ..., K) = (1, 2, 3, 4, 5)$, the time-multiplier vector $(\bar{\alpha}^k \mid k = 0, 1, ..., K) = (4N, 1.7N, 1.0N, 0.7N, 0.54N)$ unless otherwise specified, and the number of discrete time intervals $Q = 1,000,000$. Unless otherwise specified, we define the time multiplier $\beta(\cdot)$ to be used in the product form by

$$\beta(t) = \frac{W}{1 - e^{-W}} \cdot e^{W\cdot(t-T)/T}, \qquad \forall t \in [0, T], \tag{5.1}$$

where parameter $W \in \{1, 2, \cdots, 10\}$. It is easy to check that the time average $\int_0^T \beta(t) \cdot dt/T = 1$ at all W values, and that W reflects the degree of time-variability of the arrival pattern $\beta(\cdot)$. We set the default value of W at 5.

In addition, we want to introduce brute-force algorithms for the markup, markdown, and reversible cases, which we call *Markup2*, *Markdown2*, and *Reversible2*, respectively. The brute-force algorithms will be used to test whether the optimal policy is still threshold-like when demand is not product form. Also, *Markup3*, *Markdown3*, and *Reversible3* are proposed for checking the benefit brought by considering fluctuating $\beta(t)$.

We first describe our brute-force algorithm *Markup2*.

for $k = 0$ to K

  for $n = 0$ to N

    let $v_{nQ}^k = 0$;

  for $q = 0$ to $Q - 1$

let $v^k_{0q} = 0$;

for $q = Q-1$ down to 0

for $n = 1$ to N

let $v^K_{nq} = \lambda^K_q \cdot \Delta T \cdot (\bar{p}^K + v^K_{n-1,q+1}) + (1 - \lambda^K_q \cdot \Delta T) \cdot v^K_{n,q+1}$;

for $k = K-1$ down to 0

for $q = Q-1$ down to 0

for $n = 1$ to N

let $v^k_{nq} = \lambda^k_q \cdot \Delta T \cdot (\bar{p}^k + v^k_{n-1,q+1}) + (1 - \lambda^k_q \cdot \Delta T) \cdot v^k_{n,q+1},\ p^k_{nq} = \bar{p}^k$,

and $u = \lambda^{k+1}_q \cdot \Delta T \cdot (\bar{p}^{k+1} + v^{k+1}_{n-1,q+1}) + (1 - \lambda^{k+1}_q \cdot \Delta T) \cdot v^{k+1}_{n,q+1}$;

if $u > v^k_{nq}$

let $v^k_{nq} = u$ and $p^k_{nq} = \bar{p}^{k+1}$.

In this algorithm, each p^k_{nq} indicates the new price taken by the firm when it has n items and is charging price $\bar{p}^k$ right before time $q \cdot \Delta T$. The following is algorithm *Markdown2*.

for $k = 0$ to K

for $n = 0$ to N

let $v^k_{nQ} = 0$;

for $q = 0$ to $Q-1$

let $v^k_{0q} = 0$;

for $q = Q-1$ down to 0

for $n = 1$ to N

let $v^0_{nq} = \lambda^0_q \cdot \Delta T \cdot (\bar{p}^0 + v^0_{n-1,q+1}) + (1 - \lambda^0_q \cdot \Delta T) \cdot v^0_{n,q+1}$;

for $k = 1$ to K

for $q = Q-1$ down to 0

for $n = 1$ to N

let $v^k_{nq} = \lambda^k_q \cdot \Delta T \cdot (\bar{p}^k + v^k_{n-1,q+1}) + (1 - \lambda^k_q \cdot \Delta T) \cdot v^k_{n,q+1},\ p^k_{nq} = \bar{p}^k$,

and $u = \lambda^{k-1}_q \cdot \Delta T \cdot (\bar{p}^{k-1} + v^{k-1}_{n-1,q+1}) + (1 - \lambda^{k-1}_q \cdot \Delta T) \cdot v^{k-1}_{n,q+1}$;

if $u > v^k_{nq}$

let $v^k_{nq} = u$ and $p^k_{nq} = \bar{p}^{k-1}$.

The time complexity for either of the two brute-force algorithms is also $O(KNQ)$.

The brute-force algorithm *Reversible2* is as follows.

for $n = 0$ to N

let $v_{nQ} = 0$;

for $q = 0$ to $Q - 1$

let $v_{0q} = 0$;

for $n = 1$ to N

for $q = Q - 1$ down to 0

let $v_{nq} = \lambda^K_q \cdot \Delta T \cdot (\bar{p}^K + v_{n-1,q+1}) + (1 - \lambda^K_q \cdot \Delta T) \cdot v_{n,q+1}$ and $p_{nq} = \bar{p}^K$;

for $k = K - 1$ down to 0

let $u = \lambda^k_q \cdot \Delta T \cdot (\bar{p}^k + v_{n-1,q+1}) + (1 - \lambda^k_q \cdot \Delta T) \cdot v_{n,q+1}$;

if $u \geq v_{nq}$

let $v_{nq} = u$ and $p_{nq} = \bar{p}^k$.

In the algorithm, each p_{nq} stores the price to be taken at time $q \cdot \Delta T$ when the firm has n items at that time. The complexity of *Reversible2* is $O(KNQ)$.

What follows is algorithm *Markup3*.

for $k = 0$ to K

for $n = 0$ to N

let $v^{+\prime k}_{nQ} = 0$;

for $q = 0$ to $Q - 1$

let $v^{+\prime k}_{0q} = 0$;

for $q = Q - 1$ down to 0

for $n = 1$ to N

let $v^{+\prime K}_{nq} = \lambda^K_q \cdot \Delta T \cdot (\bar{p}^K + v^{+\prime K}_{n-1,q+1}) + (1 - \lambda^K_q \cdot \Delta T) \cdot v^{+\prime K}_{n,q+1}$;

for $k = K - 1$ down to 0

  for $q = Q - 1$ down to 0

    for $n = 1$ to N

      if $q < \tau_n^{+\prime,k+1} \cdot Q$

        let $v_{nq}^{+\prime k} = \lambda_q^{k+1} \cdot \Delta T \cdot (\bar{p}^{k+1} + v_{n-1,q+1}^{+\prime,k+1}) + (1 - \lambda_q^{k+1} \cdot \Delta T) \cdot v_{n,q+1}^{+\prime,k+1}$;

      else

        let $v_{nq}^{+\prime k} = \lambda_q^{k} \cdot \Delta T \cdot (\bar{p}^{k} + v_{n-1,q+1}^{+\prime k}) + (1 - \lambda_q^{k} \cdot \Delta T) \cdot v_{n,q+1}^{+\prime k}$.

Now comes algorithm *Markdown3*.

for $k = 0$ to K

  for $n = 0$ to N

    let $v_{nQ}^{-\prime k} = 0$;

  for $q = 0$ to $Q - 1$

    let $v_{0q}^{-\prime k} = 0$;

for $q = Q - 1$ down to 0

  for $n = 1$ to N

    let $v_{nq}^{-\prime 0} = \lambda_q^{0} \cdot \Delta T \cdot (\bar{p}^{0} + v_{n-1,q+1}^{-\prime 0}) + (1 - \lambda_q^{0} \cdot \Delta T) \cdot v_{n,q+1}^{-\prime 0}$;

for $k = 1$ to K

  for $q = Q - 1$ down to 0

    for $n = 1$ to N

      if $q \geq \tau_n^{-\prime k} \cdot Q$

        let $v_{nq}^{-\prime k} = \lambda_q^{k-1} \cdot \Delta T \cdot (\bar{p}^{k-1} + v_{n-1,q+1}^{-\prime,k-1}) + (1 - \lambda_q^{k-1} \cdot \Delta T) \cdot v_{n,q+1}^{-\prime,k-1}$;

      else

        let $v_{nq}^{-\prime k} = \lambda_q^{k} \cdot \Delta T \cdot (\bar{p}^{k} + v_{n-1,q+1}^{-\prime k}) + (1 - \lambda_q^{k} \cdot \Delta T) \cdot v_{n,q+1}^{-\prime k}$.

Both *Markup3* and *Markdown3* are of complexity $O(KNQ)$, and in both, we have only allowed switches between neighboring prices, and hence have excluded "leapfrogging". But as $K \ll Q$, consecutive price changes in this discrete-time setting can approximate the continuous-time phenomenon reasonably well. The following is algorithm *Reversible3*.

for $n = 0$ to N

let $v'_{nQ} = 0$;

for $q = 0$ to $Q - 1$

let $v'_{0q} = 0$;

for $n = 1$ to N

let $k = 0$;

for $q = Q - 1$ down to 0

let $v'_{nq} = \lambda_q^k \cdot \Delta T \cdot (\bar{p}^k + v'_{n-1,q+1}) + (1 - \lambda_q^k \cdot \Delta T) \cdot v'_{n,q+1}$;

while $k \leq K - 1$ and $\tau_n'^{k+1} \cdot Q > q$

let $k = k + 1$.

The complexity of *Reversible3* is $O(N \cdot (K + Q))$.

To address questions (f)-(i) which are raised for the make-to-order inventory control with pricing model, we focus on the infinite horizon case. Both the state space, i.e., the (π, x)-plane, and the control space, i.e., the (w, y)- or (y, v)-planare discretized. We can get the same policy structures as predicted in Theorem 4.5, as long as we replace derivatives used in earlier continuous-state derivations with differences. To facilitate later definitions, we define an operator INT$(\cdot)$ that converts a random variable with a finite support into a "nearby" random variable with an integer-valued support. Suppose, for some n constants $\phi_1, ..., \phi_n$ and n positive constants $p_1, ..., p_n$ satisfying $p_1 + \cdots + p_n = 1$, we have that Φ is a random variable satisfying $P[\Phi = \phi_i] = p_i$ for $i = 1, 2, ..., n$. Then, INT(Φ) is a random variable that satisfies, for every integer k,

$$P[\mathrm{INT}(\Phi) = k] = \sum_{i=1}^{n} p_i \cdot (\mathbf{1}(\phi_i \in (k-1, k]) \cdot (\phi_i - k + 1) + \mathbf{1}(\phi_i \in (k, k+1)) \cdot (k + 1 - \phi_i)), \tag{5.2}$$

where $\mathbf{1}(\cdot)$ is the indicator function. Basically, INT(Φ) assigns weights to integer values based on densities of the ϕ_i points surrounding them.

We let Θ be an integer-valued random variable uniformly distributed in an interval $[\theta - S, \theta + S]$ for some $S \in [0, \theta]$. For the inventory holding-backlogging cost, we let

$$h(x) = H \cdot (x^+ \wedge X) + L \cdot (x^- \wedge X) + M \cdot (x - X)^+ + M \cdot (-x - X)^+, \qquad (5.3)$$

for positive constants X and M, as well as constants $H, L \in [0, M]$. Here, H serves as an ordinary holding cost rate and L an ordinary backlogging cost rate. We shall let X be a large "rarely exceeded" absolute value of the inventory level, and let M be large enough to render the satisfaction of (MO2) and (MO3) concerning b_∞ and h_∞ a foregone conclusion.

We let the inverse-demand function in Section 4.1 be

$$p(\pi, w) = A\pi + \frac{B}{(w + \theta)^C}, \qquad (5.4)$$

where A, B, C and D with $A \in [0, 1)$, $B \in \Re^+$, and $C \in (0, 1)$.

For the firm to enjoy any premium p' over A times the raw material cost π, (5.4) suggests that competition will force the firm to accept the expected demand size $(B/p')^{1/C}$, which dwindles as p' rises. From (5.4), we can check that $p(\pi, 0) = A\pi + B/\theta^C$ is finite and $p(\pi, \cdot)$ is continuous and strictly decreasing. We may obtain from (4.6) and (5.4) that

$$r(\pi, w) = A\pi \cdot (w + \theta) + B \cdot (w + \theta)^{1-C}, \qquad (5.5)$$

and hence

$$\frac{\partial r(\pi, w)}{\partial w} = A\pi + \frac{B \cdot (1 - C)}{(w + \theta)^C}. \qquad (5.6)$$

It can be easily checked that $r(\pi, w)$ is concave in w.

Under any given scenario, we conduct value iteration on a variant of the formulation involving (4.10), (4.11), and (4.12). This way, we may obtain the infinite-horizon profit values $f(\pi, x)$ along with the corresponding optimal controls $w^*(\pi, x)$ and $y^*(\pi, x)$, for $\pi \in [\underline{\pi}, \overline{\pi}]$ and $x \in [-X, X]$. As for convergence criterion, we adopt

$$|| f_t - f_{t-1} || \equiv \max_{\underline{\pi} \leq \pi \leq \overline{\pi}, -X \leq x \leq X} | f_t(\pi, x) - f_{t-1}(\pi, x) | < 0.01. \tag{5.7}$$

5.2 Computational Result

In test (a), we use v_{nq}^{-k} to denote the maximum value achieved by *Markdown1* in the markdown case when the firm charges price $\bar{p}^k$ at time $q\Delta t$ and remaining inventory n. To check whether the markdown case possesses complementarity between price choice and inventory, we define the following ratio μ^-:

$$\mu^- = \frac{\sum_{q=0}^{Q} \sum_{n=0}^{N-1} \sum_{k=0}^{K-1} (v_{n+1,q}^{-k} + v_{nq}^{-,k+1} - v_{nq}^{-k} - v_{n+1,q}^{-,k+1})^+}{\sum_{q=0}^{Q} \sum_{n=0}^{N-1} \sum_{k=0}^{K-1} | v_{n+1,q}^{-k} + v_{nq}^{-,k+1} - v_{nq}^{-k} - v_{n+1,q}^{-,k+1} |}. \tag{5.8}$$

Note that μ^- is always between 0 and 1; it will be 0 if and only if $v_{nq}^{-k} + v_{n+1,q}^{-,k+1} \geq v_{n+1,q}^{-k} + v_{nq}^{-,k+1}$ for any (k, n, q), that is, if and only if v_{nq}^{-k} has increasing differences between k and n. For the markdown case, a higher k means more price choices in the future. Thus, μ^- measures the degree to which complementarity between price flexibility and inventory is violated for the markdown case. When $W = 5$, we find $\mu^- \approx 25.0\%$. Figure 5.1 further shows the different μ^- values at different W levels. From this figure, we can see that the complementarity property doesn't exist in the markdown case, which confirms the distinction between the markdown case and markup case.

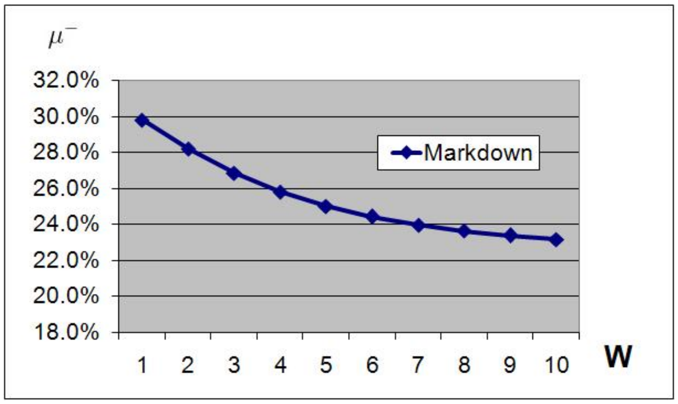

Figure 5.1: Violations of Complementarity between Price-flexibility and Inventory at Different W's

In test (b), we only have to use $\beta(t) = 1$ for all t's. Concerning the sign of $\mathcal{G}_n^{k-1}(t) \circ v^k + \bar{p}^{k-1}\bar{\alpha}^{k-1} \cdot \beta(t)$ within (τ_n^k, T), we find that $\mathcal{G}_6^3(t) \circ v^4 + \bar{p}^3\bar{\alpha}^3 \cdot 1 < -1.7$ when $t \in [0.6065, 0.6165]$, where $\tau_6^4 \simeq 0.6065$ is the threshold level for $k = 4$ and $n = 6$. Here, any $d_t u(t)$ is approximated by $[u(t + \Delta t) - u(t)]/\Delta t$. Therefore, $\mathcal{G}_n^{k-1}(t) \circ v^k + \bar{p}^{k-1}\bar{\alpha}^{k-1} \cdot \beta(t)$ may be negative when $t > \tau_n^k$. Concerning the monotonicity of $\mathcal{G}_n^{k-1}(t) \circ v^k$ in n, we find that $\mathcal{G}_{17}^0(t) \circ v^1 - \mathcal{G}_{16}^0(t) \circ v^1 \simeq -0.340$ when $t \simeq 0.2622$. That is, we do not necessarily have the increase of $\mathcal{G}_n^{k-1}(t) \circ v^k$ in n.

In test (c), we use $\tau_n^{\pm k}$ to denote threshold levels found by applying *Markup(down)1* to the markup(down) problem. To check whether the irreversible cases enjoy the k-monotonicity, we define the $\delta^\pm$ ratios:

$$\delta^\pm = \frac{\sum_{n=1}^N \sum_{k=0}^{K-1} (\tau_n^{\pm,k+1} - \tau_n^{\pm k})^+}{\sum_{n=1}^N \sum_{k=0}^{K-1} \mid \tau_n^{\pm,k+1} - \tau_n^{\pm k} \mid}. \tag{5.9}$$

Note that $\delta^\pm$ is always between 0 and 1; it will be 0 if and only if $\tau_n^{\pm k+1} \leq \tau_n^{\pm k}$ at every possible (k, n). Thus, it measures the degree to which k-monotonicity has been violated. For $W = 5$ and time-multiplier vector $(\bar{\alpha}^k \mid k = 0, 1, ..., K) = (2.0N, 0.76N, 0.5N, 0.25N, 0.1N)$, we have $\delta^+ \approx 11.2\%$ for the markup case. For the markdown case, we have to use a different time-multiplier vector $(\bar{\alpha}^k \mid k = 0, 1, ..., K) = (3N, N, 0.5N, 0.35N, 0.27N)$ to achieve unequivocal results. Under this

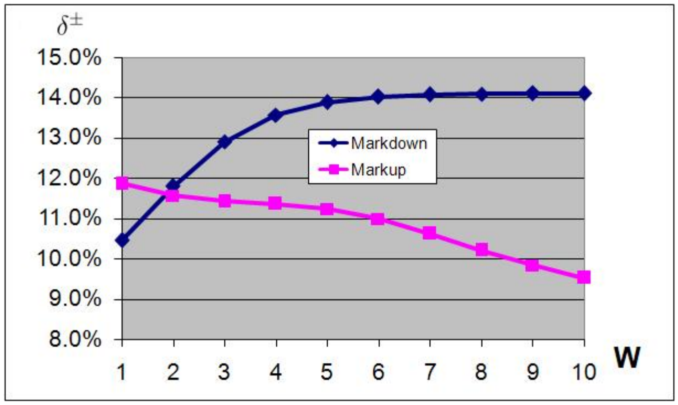

Figure 5.2: Violations of k-monotonicity at Different W's

and $W = 5$, we find $\delta^- \approx 13.9\%$. We can also plot the $\delta^{\pm}$ values when W changes in Figure 5.2. This figure confirms that the irreversible cases doesn't have the k-monotonicity which distinguish them from the reversible case.

In test (d), let $\beta^{'}(t) = 1$ when the firm is not aware of the time-fluctuating demand. Then, we denote v_{nq}^{+k} and τ_n^{+k} as the values and threshold points resulting from applying *Markup1* to the markup problem defined by $\beta(t)$. Also, we define $\tau_n^{+\prime k}$ be the threshold levels resulting from applying *Markup1* to the corresponding stationary problem when $\beta'(t) = 1$. To find the values $v_{nq}^{+\prime k}$ by using the sub-optimal policy $\tau_n^{+\prime k}$ in the time-fluctuating situation with $\beta(t)$, we adopt algorithm *Markup3*.

Corresponding to *Markup3*, we have algorithms *Markdown3* and *Reversible3* for the markdown and reversible-pricing cases, respectively. For the markdown case, the relevant values will be denoted by v_{nq}^{-k} and $v_{nq}^{-\prime k}$, while for the reversible-pricing case, these values will be denoted by v_{nq}^{0} and $v_{nq}^{0\prime}$.

To measure the losses due to neglecting the time-variability of $\beta(\cdot)$, we may define $\eta^{\pm(0)}$ for the markup(down) and reversible-pricing cases:

$$\eta^{\pm} = \frac{\sum_{k=0}^{K}(v_{N0}^{\pm k} - v_{N0}^{\pm\prime k})}{\sum_{k=0}^{K} v_{N0}^{\pm k}}, \qquad \eta^{0} = \frac{v_{N0}^{0} - v_{N0}^{0\prime}}{v_{N0}^{0}}. \tag{5.10}$$

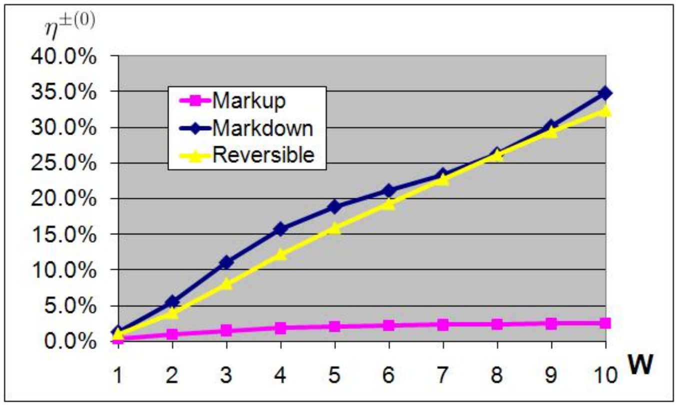

Figure 5.3: Benefits of Heeding Demand's Time-variability at Different W's

When $W = 5$, we have $\eta^+ \approx 2.0\%$, $\eta^- \approx 18.9\%$, and $\eta^0 \approx 15.8\%$. Hence, the benefit of heeding demand's time-variability in each of the three cases is substantial. When W varies, we show the $\eta^{\pm(0)}$ values in Figure 5.3. It is quite clear from the figure that the benefit increases with the degree of the fluctuation.

In test (e), we choose the following form:

$$\lambda^k(t) = \bar{\alpha}^k \cdot [1 + 0.8 \cdot \sin(2\pi \cdot (\frac{k}{0.3} + t))]. \tag{5.11}$$

Note that (5.11) is not product-form demand rate. Since our algorithm *Markup1*, *Markdown1*, and Reversible1 is achieved under product-form demand rate, we use the brute-force algorithms *Markup2*, *Markdown2*, and Reversible2 to solve the optimal pricing policy here. In fact, the results differ widely between the threshold policy and brute-force policy in the case of (5.11). We show, in figure 5.4, the pricing decisions obtained by *Markup2*, *Markdown2*, and Reversible2.

Since none of the above pricing decisions is decreasing in t, we simply can not define τ_n^k for any of these cases. Therefore, threshold policies for those three cases stop at the product-form case for the time being. Note that $\lambda^{k+1}(t)/\lambda^k(t)$ under (5.11) is not decreasing in t. Therefore, Zhao and Zheng's (2000) Assumption 1 is violated, and our counter example for the reversible-pricing case is not in contradiction with

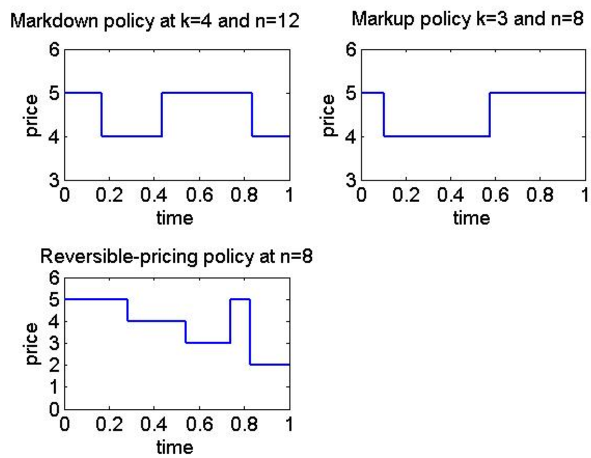

Figure 5.4: Non-threshold Pricing Decisions when Arrival is not of Product Form

Zhao and Zheng's prediction for time-monotone policies when arrival patterns are well behaved.

Then, we study the lost opportunity involved in the markup and markdown practices. In this study, we let $\beta(\cdot)$ take the form

$$\beta(t) = [1 + X \cdot \sin(\frac{2\pi Y}{T} \cdot t)] \times [\frac{2Z}{T} \cdot t + 1 - Z], \qquad \forall t \in [0, T], \tag{5.12}$$

where X is a constant in $[0, 1)$, Y a constant in $\{1, 2, \cdots, 10\}$, and Z a constant in $(-1, 1)$. For the arrival pattern, X controls its oscillation volume, Y its oscillation frequency, and Z its incremental trend. Under each (X, Y, Z) tuple, we compute the optimal value v_{N0}^{+0} at $k = 0$, $n = N$, and $t = 0$ for the markup case using *Markup1*, the optimal value v_{N0}^{-K} at $k = K$, $n = N$, and $t = 0$ for the markdown case using *Markdown1*, and the optimal value v_{N0}^{0} at $n = N$ and $t = 0$ for the reversible-pricing case using *Reversible1*. Define $\epsilon^{\pm}$ by

$$\epsilon^{+} = \frac{v_{N0}^{0} - v_{N0}^{+0}}{v_{N0}^{0}}, \qquad \epsilon^{-} = \frac{v_{N0}^{0} - v_{N0}^{-K}}{v_{N0}^{0}}. \tag{5.13}$$

Note that ϵ^{+} reflects the loss due to the obligation of always marking up, and ϵ^{-}

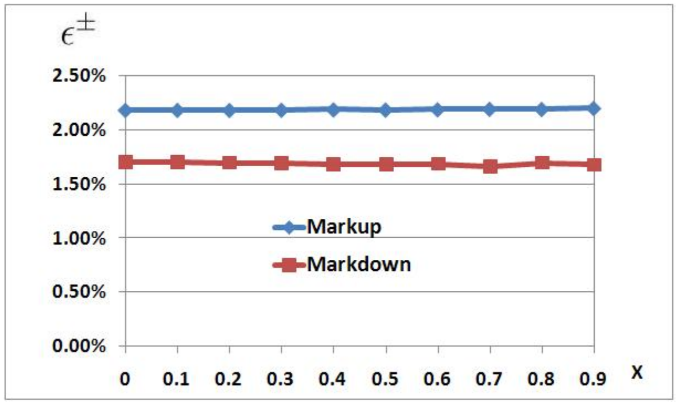

Figure 5.5: Average $\epsilon^{\pm}$ Values when X Varies

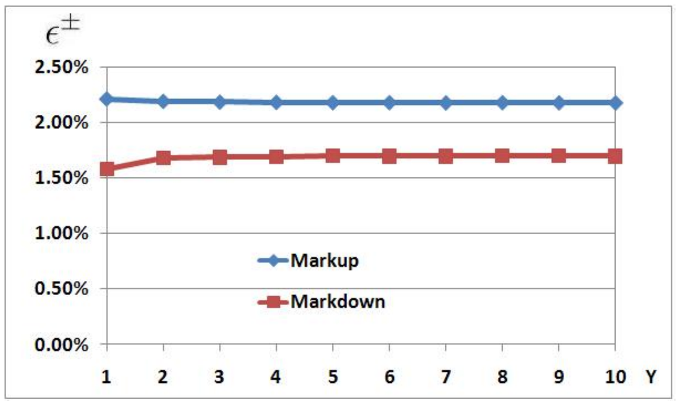

Figure 5.6: Average $\epsilon^{\pm}$ Values when Y Varies

reflects the loss due to the obligation of always marking down.

In Figures 5.5 to 5.7, we show average $\epsilon^{\pm}$ values when one of X, Y, and Z varies, respectively, while at each such fixed parameter, the other two parameters go over 100 random samples generated from uniform distributions on their respective ranges. These figures always show $\epsilon^{+} < 2.5\%$ and $\epsilon^{-} < 2\%$. Also, the losses do not alter much when the parameters vary. The only notable trend is that ϵ^{-} will increase when Z increases. This is to be expected–when demand is on the rise, it is unwise to keep on making price concessions. Our overall conclusion from this part of the computational study may be that, at least for product-form demand, irreversible pricing practices are viable alternatives to the less consumer-friendly practice of reversible pricing.

In test (f), we have gathered weekly LME Copper Cash Price, US dollar/tonne, from

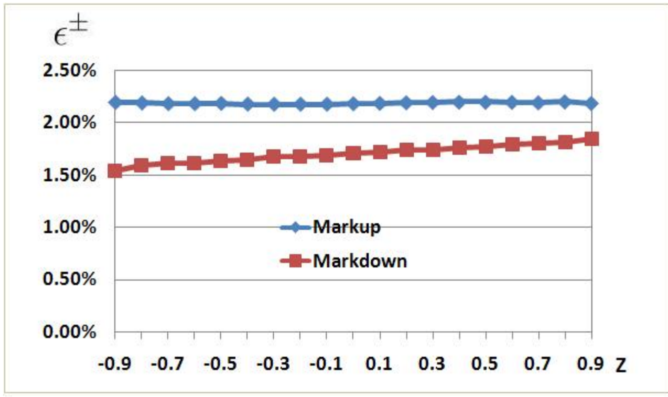

Figure 5.7: Average $\epsilon^{\pm}$ Values when Z Varies

Jan/2/2004 to Dec/27/2013 and monthly China Iron Ore Spot Price Shandong/Zibo, CNY/tonne, from Jan/31/2008 to Dec/31/2013 from Bloomberg Terminal. Every two-year period of copper prices makes up a time series, and every three-year period of iron ore prices makes up a time series as well. We have five time series on cooper prices and two time series on iron ore prices.

For each time series (X_t), we start off with the following Autoregressive(AR) model

$$X_t = \mu(1 - \sum_{i=1}^{k} r_i) + \sum_{i=1}^{k} r_i X_{t-i} + \epsilon_t, \tag{5.14}$$

where k is the order of the model reflecting the depth at which each X_t is dependent on the past, μ is the mean of X_t, and ϵ_t represents noise. Each r_i for $i = 1, 2, ..., k$ reflects the correlation between X_t and X_{t-i}. Within the statistics software R, we can call the "ar" function to get the order k and the "arima" function using the maximum likelihood method to obtain other parameters.

Our tests show that all cases have order $k = 1$ except for two. For copper prices from January 2012 to December 2013 and that from January 2008 to December 2009, we find k to be 2 and 4, respectively. To be specific, the AR model for copper prices from January 2012 to December 2013 is

$$X_t = 325.23 + 0.895X_{t-1} + 0.065X_{t-2} + \epsilon_t. \quad (5.15)$$

However, the p-value for r_2 is 0.42 indicating insignificance of the dependence. For copper prices from January 2008 to December 2009, we have

$$X_t = 139.20 + 0.993X_{t-1} + 0.181X_{t-2} - 0.041X_{t-3} - 0.156X_{t-4} + \epsilon_t. \quad (5.16)$$

Here, p-values for r_2, r_3, and r_4 are, respectively, 0.187, 0.763, and 0.112. Again, the higher-order dependencies are not significant. We can rerun the AR models after excluding the insignificant terms.

In the following, we present all the AR(1) parameters for the seven time series.

Table 5.1: AR(1) Parameters

Date, Period	Item	r_1	μ	p-value of r_1	p-value of μ
01/2004-12/2005, weekly	copper	0.928	2856.86	$\ll 0.001$	$\ll 0.001$
01/2006-12/2007, weekly	copper	0.986	4929.19	$\ll 0.001$	0.001
01/2008-12/2009, weekly	copper	0.986	5388.00	$\ll 0.001$	0.002
01/2010-12/2011, weekly	copper	0.996	6387.00	$\ll 0.001$	0.013
01/2012-12/2013, weekly	copper	0.956	8116.30	$\ll 0.001$	$\ll 0.001$
01/2008-12/2010, monthly	iron ore	0.902	1244.28	$\ll 0.001$	$\ll 0.001$
01/2011-12/2013, monthly	iron ore	0.899	1259.02	$\ll 0.001$	$\ll 0.001$

Note "$\ll 0.001$" in Table 5.1 means that the concerned p-value is much less than 0.001. From the low p-values, we can see that the copper and iron ore price processes are all accurately describable by AR(1) models. In other words, they are Markovian in the sense that the present-period price can be predicted by the previous-period one, but not anything more in the past.

In test (g), to assess the benefit of adopting cost-dependent control, we compare the total profit generated by our cost-dependent policy to that the best that a costblind policy can achieve when both face randomly evolving raw material costs. For the latter alternative, we suppose the policy is tailored to the case where the raw material cost stays the flat at $\hat{\pi} = E[\hat{\Pi}]$. Once the infinite-horizon π-independent policy $(\hat{y}(x), \hat{v}(x))$ is found, we plug it back to the π-varying environment to obtain infinite-horizon profit function $\hat{f}(\pi, x)$. To be clear, we briefly describe their model in (5.17), (5.18), and (5.19). The raw material price for all the period is $\hat{\pi} = E[\Pi]$. Let $\tilde{f}_t(x)$ be the maximum total discounted expected profit that the firm can earn from period t onward till the terminal period 0, when the firm's inventory level is x. For any period $t = 1, 2, ...$, the recursive relationship in (4.7) now becomes:

$$\begin{aligned} \tilde{f}_t(x) &= \sup_{w\in[0,w^U],z\in\Re^+} E[p(\hat{\pi}, w) \cdot (w + \Theta) - \hat{\pi} z - h(x + z - w - \Theta) \\ &\quad +\alpha \cdot \tilde{f}_{t-1}(x + z - w - \Theta)]. \end{aligned} \tag{5.17}$$

Let $f_t(x) = \tilde{f}_t(x) - \hat{\pi} x$, we have the counterparts of (4.42) and (4.43) as the following:

$$f_t(x) = \sup_{y\in[x,+\infty),v\in[y-w^U,y]} j_t(y, v), \tag{5.18}$$

where

$$\begin{aligned} j_t(y, v) &= -\alpha\theta\hat{\pi} - \hat{\pi} y + \alpha\hat{\pi} v + r(\hat{\pi}, y - v) - E[h(v - \Theta)] \\ &\quad +\alpha \cdot E[f_{t-1}(v - \Theta)]. \end{aligned} \tag{5.19}$$

Suppose the profit is $f(\pi, x)$ when our truly π-dependent optimal policy is applied to the π-varying environment. We then define ρ to capture the relative loss of profit due to the firm's blindness to raw material cost changes:

$$\rho = \frac{E[f(\hat{\Pi}, 0) - \hat{f}(\hat{\Pi}, 0)]}{E[f(\hat{\Pi}, 0)]} = \frac{\sum_{\pi=\underline{\pi}}^{\overline{\pi}} q(\pi) \cdot (f(\pi, 0) - \hat{f}(\pi, 0))}{\sum_{\pi=\underline{\pi}}^{\overline{\pi}} q(\pi) \cdot f(\pi, 0)}. \tag{5.20}$$

Basically, we compute the relative loss in average profits when the raw material cost is in steady state and the firm starts with no product in inventory.

We let the raw material cost process be governed by the probability transition matrix

$$M = \begin{pmatrix} m(\underline{\pi},\underline{\pi}) & m(\underline{\pi},\underline{\pi}+1) & ... & m(\underline{\pi},\overline{\pi}) \\ m(\underline{\pi}+1,\underline{\pi}) & m(\underline{\pi}+1,\underline{\pi}+1) & ... & m(\underline{\pi}+1,\overline{\pi}) \\ & & ... & \\ m(\overline{\pi},\underline{\pi}) & m(\overline{\pi},\underline{\pi}+1) & ... & m(\overline{\pi},\overline{\pi}) \end{pmatrix},$$

where each $m(\pi,\pi')$ is the probability of moving from state π to π' in one period. Let vector $Q = (q(\underline{\pi}), q(\underline{\pi}+1), ..., q(\overline{\pi}))$ be the invariable probability distribution of the raw material cost, which satisfies both $QM = Q$ and $\sum_{q=\underline{\pi}}^{\overline{\pi}} q(\pi) = 1$. Let $\hat{\Pi}$ be the generic random raw material cost in steady state.

At every π, we let $Var[\Pi'|\pi] = E[(\Pi' - E[\Pi'|\pi])^2|\pi]$ be the variance of the next-period cost Π' conditioned on the present-period cost π, where for any function $f(\cdot)$, we have $E[f(\Pi')|\pi] = \sum_{\pi=\underline{\pi}}^{\overline{\pi}} f(\pi') \cdot m(\pi,\pi')$. We define $\gamma = \sum_{\pi=\underline{\pi}}^{\overline{\pi}} q(\pi) \cdot Var[\Pi'|\pi]$ to capture the randomness of the raw material cost's move from one period to the next.

We compute (γ, ρ) for 1,000 randomly generated scenarios. With regard to how the cost-transition matrix M is constructed, these scenarios can be divided into ten 100-strong batches. For each of the 100 scenarios within the b-th batch, where $b = 1, 2, ..., 10$, we let each or each pair of the middle elements in every row of M share $0.1b - 0.05$. The remaining entries in every row of M are randomly generated so that they occupy the entire $1.05 - 0.1b$ in a uniform fashion. All 1,000 scenarios share the same other parameters; e.g., discount factor $\alpha = 0.95$, Θ-defining parameters $\theta = 3$ and $S = 2$, demand-lever upper bound $w^U = 70$, cost bounds $\underline{\pi} = 1$ and $\overline{\pi} = 20$, $r(\pi, w)$-defining constants $A = 0.6$, $B = 100$, and $C = 0.6$, and inventory-related parameters $X = 50$ and $M = 100$. As for the two inventory-related constants, we maintain the relationship $L = 2H = 1$.

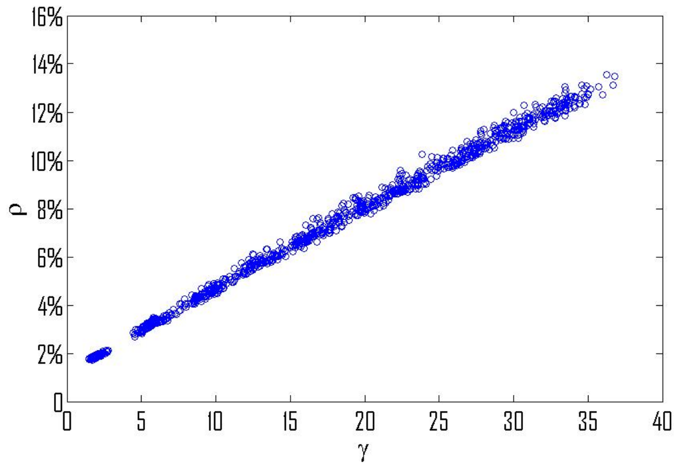

Figure 5.8: Profit vs. Price Variance

In figure 5.8, we plot the (γ, ρ)-points for the 1,000 scenarios.

In the range where γ increases from 0 to 40, the relative loss ρ steadily grows from 2% to 14%. So not only is the profit improvement from cost-sensitive control significant, it will also grow in importance as the cost changes more randomly and faster over time.

In test (h), we let

$$\Delta p(\pi, x) = p^*(\pi + 1, x) - p^*(\pi, x), \tag{5.21}$$

where $p^*(\pi, x)$ is the optimal sales price under cost π and inventory level x. So $\Delta p(\pi, x) \geq 1$ means that cost increment is fully passed over to customers, $0 \leq \Delta p(\pi, x) < 1$ means that cost increment is only partially passed over, and $\Delta p(\pi, x) < 0$ means that price is dropped in the face of cost increase.

From now on, we let the per-period raw material cost evolution be guided by

$$(\Pi'|\pi) = \frac{D}{\alpha}\pi + (1 - \frac{D}{\alpha})\Pi^0, \tag{5.22}$$

where $D \in (0, A\alpha)$ and Π^0 is a random variable uniformly distributed on $[\underline{\pi}, \overline{\pi}]$. Thus, the next-period cost is (D/α)-portion the current cost and $(1 - D/\alpha)$-portion the random update Π^0. This leads to $ds(\pi)/d\pi = D$. We also use real data to calibrate the involved parameters. To track the monthly China Iron Ore Spot Price Shandong/Zibo from January 31, 2010 to December 31, 2013 which range from 955 to 1,520 CNY/tonne and follow an AR(1) model with autocorrelation coefficient 0.899, we let $\underline{\pi} = 95$, $\overline{\pi} = 152$, and $D/\alpha = 0.9$, so that per unit increase of π corresponds to 10 CNY/tonne. For other parameters, we let $\alpha = 0.95$, $\theta = 3$, $S = 2$, $w^U = 70$, $A = 0.6$, $B = 300$, $C = 0.5$, $X = 50$, $M = 100$, and $L = 2H = 20$.

In Figure 5.9, we use different colors to reflect different $\Delta p(\pi, x)$ levels at various (π, x)-points.

We can see from Figure 5.9 that the base stock level decreases in the raw material cost with the maximum level $y^0(\underline{\pi}) = 16$ and the minimum level $y^0(\overline{\pi}) = 5$. In most cases, the firm passes cost increases to customers in various degrees. However, there is no clear monotone trend of this behavior with respect to inventory-level changes. There are even occasions where $\Delta p(\pi, x) < 0$, indicating price drops in the face of cost increases.

In test (i), we do not insist that $Q = 0$. Thus, Ω is not necessarily equal to 1 almost surely. For some parameter $Q \in [0, 1]$, we let Ω take values $1 - Q$, 1, $1 + Q$, and $1 + 2Q$ with, respectively, probabilities $4P$, $1 - 7P$, $2P$, and P for some $P \in [0, 0.125]$. Demand will remain additive when $P = 0$ or $Q = 0$. The current arrangement ensures that $E[\Omega] = 1$ regardless. When Ω is involved, the expression $(y - \Omega w - \Theta)$ is changed to $\mathrm{INT}(y - w - \Theta)$. We let the Ω-defining constant $P = 0.11$

For the inverse-demand function $p(\cdot, \cdot)$ in (5.4) and raw material cost process $(\Pi' \mid \cdot)$ in (5.22), we shall make sure that (MO1) on the concavity of $r(\pi, \cdot)$ is always satisfied. Yet, we allow (MO4) to (MO6) to be turned on and off through adjustments of certain parameters. In particular, $A \in [0, 2)$, $B \in \Re^+$, $C \in (0, 1)$, and $D \in [-\alpha, 2\alpha]$.

When $A \in [0, 1)$ and $D \in [0, \alpha \wedge A]$, we may check that (MO4) to (MO6) will be

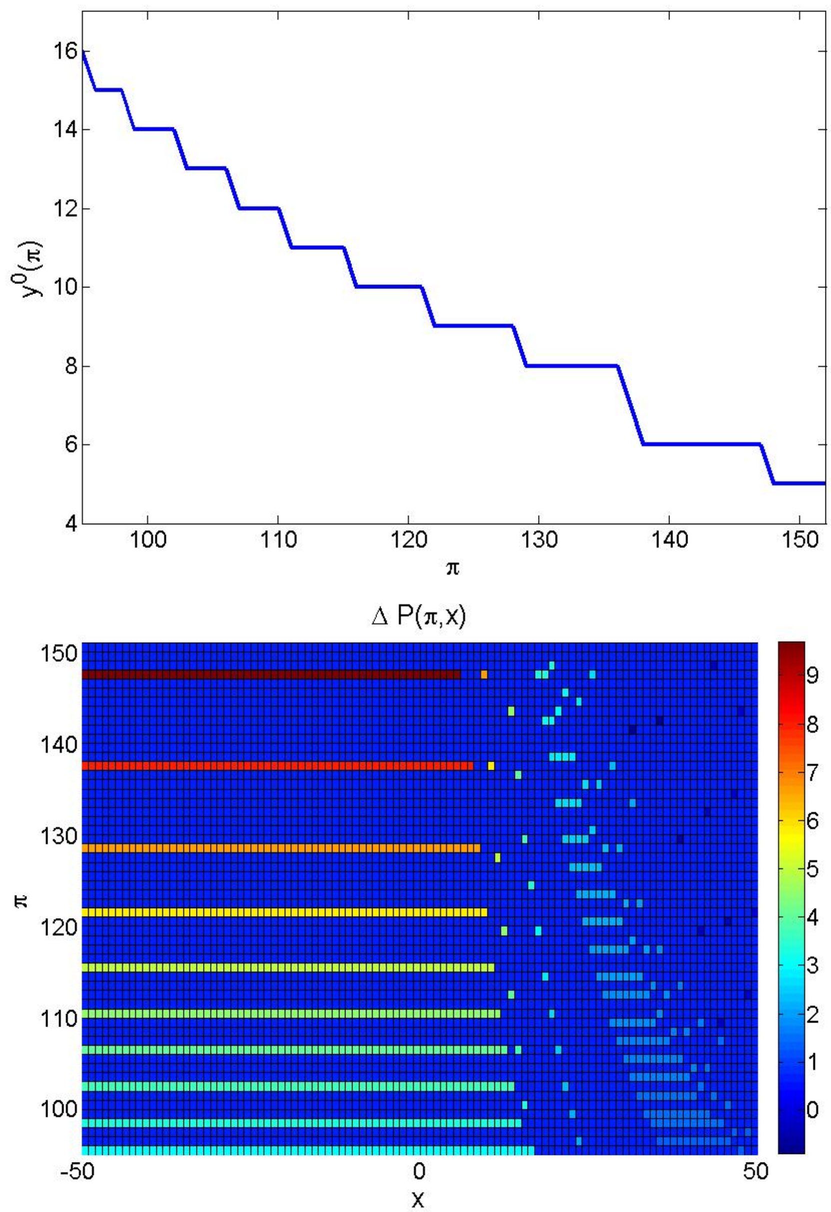

Figure 5.9: Cost Increases Passed on to Pricing

satisfied. But when $A \in [1,2)$ or $D \in [-\alpha, 0) \cup (\alpha \wedge A, 2\alpha]$, some of them may be violated.

In all scenarios, we fix the discount factor $\alpha = 0.95$, Θ-defining parameters $\theta = 3$ and $S = 2$, the demand-lever upper bound $w^U = 70$, the cost bounds $\underline{\pi} = 1$ and $\overline{\pi} = 20$, the $r(\pi, w)$-defining constant $B = 200$ and $C = 0.8$, and the inventory-related parameters $X = 50$ and $M = 100$. As for the two inventory-related constants, we maintain the relationship $L = 2H = 1$. As default values, we let the Ω-defining parameter Q $= 0$, the $p(\pi, w)$-defining parameters $A = 0.8$, and $(\Pi'|\pi)$-defining parameter $D = 0.5\alpha = 0.475$.

In our study, we let Q, A, and D vary around their default parameters. We may define ratios of total variations R_4 to R_6 that measure, respectively, degrees to which assumptions (MO4) to (MO6) are violated. More particularly, we let

$$R_i = \frac{V_i^+}{V_i^+ + V_i^-}, \qquad \forall i = 4, 5, 6, \tag{5.23}$$

where we take the convention that $0/0 = 0$ and assume the following:

$$\begin{cases} V_4^{\pm} & = \sum_{\pi=\underline{\pi}}^{\overline{\pi}-1} \sum_{w=0}^{w=w^U} (\pm(\delta_w r(\pi, w) - s(\pi) - \delta_w r(\pi+1, w) + s(\pi+1)))^+, \\ V_5^{\pm} & = \sum_{\pi=\underline{\pi}}^{\overline{\pi}-1} \sum_{w=0}^{w=w^U} (\pm(\delta_w r(\pi+1, w) - (\pi+1) - \delta_w r(\pi, w) + \pi))^+, \\ V_6^{\pm} & = \sum_{\pi=\underline{\pi}}^{\overline{\pi}-1} (\pm(s(\pi) - s(\pi+1)))^+, \end{cases} \tag{5.24}$$

where $\delta_w r(\pi, w) = r(\pi, w+1) - r(\pi, w)$ is a proxy for $\partial r(\pi, w)/\partial w$. For $i = 4, 5, 6$, each R_i is between 0 and 1. It will be 0 when assumption (MOi) is totally satisfied. For example, if (MO4) is satisfied, then $\delta_w r(\pi, w) - s(\pi) \leq \delta_w r(\pi+1, w) - s(\pi+1)$ for any $w \in \{0, 1, ...w^M\}$ and $\pi \in \{\underline{\pi}, \underline{\pi}+1, ..., \overline{\pi}\}$. Therefore, $(\delta_w r(\pi, w) - \delta_w r(\pi+1, w) + s(\pi+1) - s(\pi))^+ = 0$ which leads to $V_4^+ = 0$ and $R_4 = 0$. Basically, $R_i, i = 4, 5, 6$ offers the degree at which the assumption (MOi) has been violated.

We may also define ratios of total variations R_w, R_p, R_y, and R_v that measure, respectively, degrees to which π-monotonicity properties have been violated by the $w^*(\cdot,x)$, $p(\cdot,w^*(\cdot,x))$, $y^*(\cdot,x)$, and $(y^*(\cdot,x)-w^*(\cdot,x))$ curves. More particularly, we let

$$R_x = \frac{V_x^+}{V_x^+ + V_x^-}, \qquad \forall x = w, p, y, z, \tag{5.25}$$

where we take the convention that $0/0 = 0$ and assume the following:

$$\begin{cases} V_w^{\pm} &= \sum_{\pi=\underline{\pi}}^{\overline{\pi}-1} \sum_{x=-X}^{X} (\pm(w^*(\pi+1,x) - w^*(\pi,x)))^+, \\ V_p^{\pm} &= \sum_{\pi=\underline{\pi}}^{\overline{\pi}-1} \sum_{x=-X}^{X} (\pm(p(\pi,w^*(\pi,x)) - p(\pi+1,w^*(\pi+1,x))))^+, \\ V_y^{\pm} &= \sum_{\pi=\underline{\pi}}^{\overline{\pi}-1} \sum_{x=-X}^{X} (\pm(y^*(\pi+1,x) - y^*(\pi,x)))^+, \\ V_v^{\pm} &= \sum_{\pi=\underline{\pi}}^{\overline{\pi}-1} \sum_{x=-X}^{X} (\pm(y^*(\pi+1,x) - w^*(\pi+1,x) - y^*(\pi,x) + w^*(\pi,x)))^+. \end{cases} \tag{5.26}$$

Interpretations of the $R_{w(p,y,v)}$'s are very similar to those of the R_i's. When the raw material cost π is increased, $R_w = 0$ means the sales volume will always be reduced, $R_p = 0$ means the sales price will always be raised, $R_y = 0$ means the post-procurement inventory level will always be reduced, and $R_v = 0$ means the expected next-period inventory level will always be lowered.

When $Q = 0$ and hence $\Omega = 1$, and A and D are at their default values, we can computationally confirm that $R_4 = R_5 = R_6 = 0$, and more importantly, that $R_w \approx 43.7\% > 0$, $R_p \approx 2.7\% > 0$, and $R_y = R_v = 0$. The strict positivity of R_w confirms that the firm does not always have to scale back on its sales volume when the raw material cost rises; meanwhile, $R_p \approx 2.7\%$ means that the firm does not always have to raise its sales price when the same thing happens, though it should under majority of circumstances. The above understanding has in a sense absolved us from not producing any monotone results on either the sales volume or the sales price. That $R_y = R_v = 0$ is consistent with our theoretical monotone results, that both post-procurement and expected next-period inventory levels should decrease in the

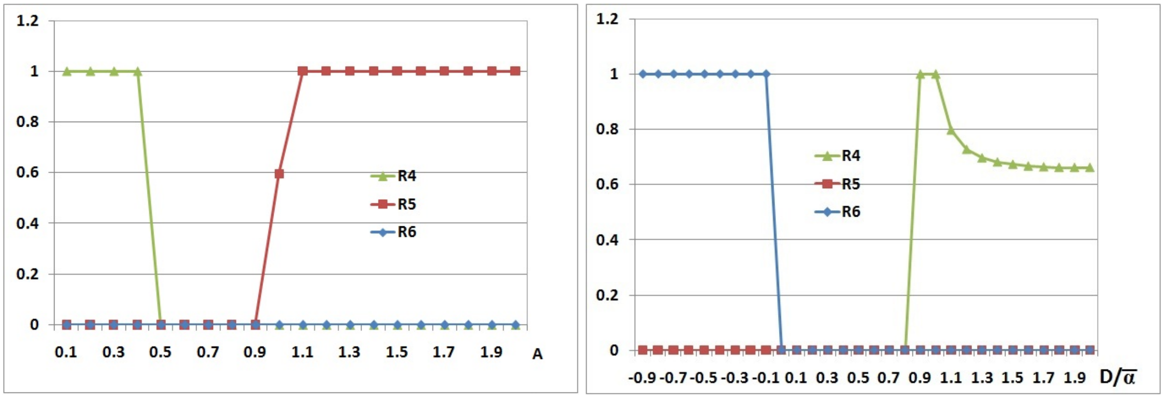

Figure 5.10: Depictions of R_4 to R_6 when A and D Change

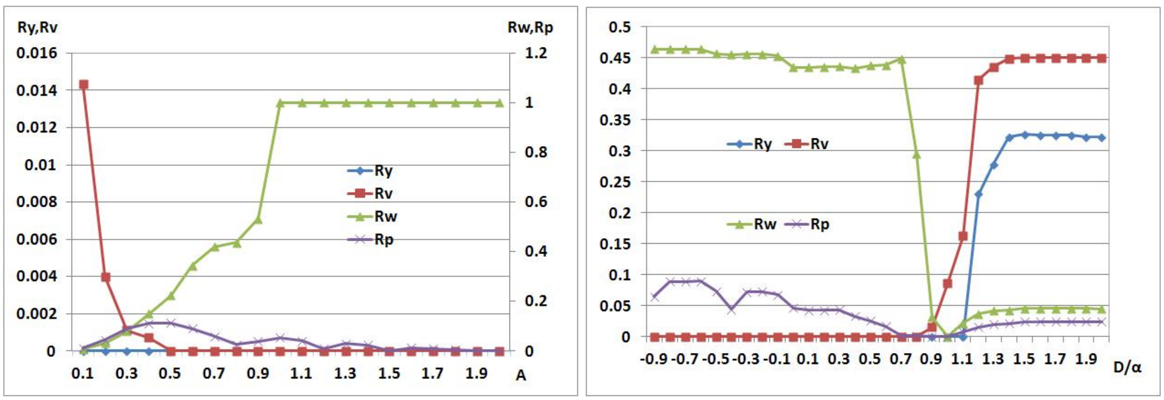

Figure 5.11: Depictions of R_w, R_p, R_y, and R_v when A and D Change

raw material cost.

Next, we conduct two experiments in which A and D vary, respectively. When one of the two parameters varies, we present values R_4 to R_6 in Figure 5.10 and values R_w, R_p, R_y, and R_v in Figure 5.11.

From Figures 5.10 and 5.11, we see that when $A \in [0.5, 1.0]$ and D is at its default value, there are many occasions when $R_4 = R_5 = R_6 = 0$ and yet $R_w, R_p > 0$. This again suggests the firm need not always try to pass on its cost burdens.

Furthermore, when $A \in [0.1, 0.4]$ and D is at its default value, we have $R_4 > 0$ and $R_v > 0$; and, when A is at its default value and $D/\alpha \in [1.1, 2.0]$, we have $R_4 > 0$, $R_y > 0$, and $R_v > 0$. Hence, it can be said that (MO4) is essential for π-monotonicity properties of the optimal policy. On the other hand, we may see

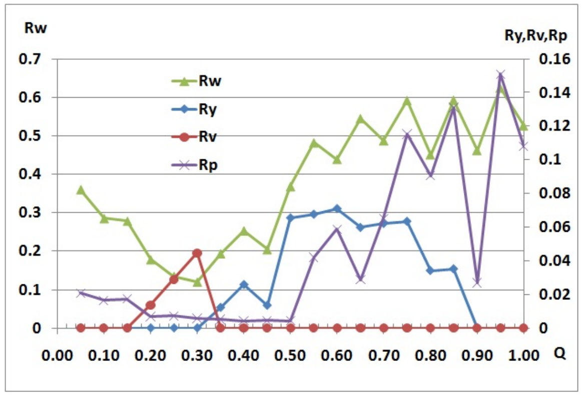

Figure 5.12: Depictions of R_w, R_p, R_y, and R_v when Q Changes

that, when $A \in [1.1, 2.0)$ and D is at its default value, we have $R_5 > 0$, and yet $R_y = R_v = 0$; also, when A is at its default value and $D/\alpha \in [-0.9, -0.1]$, we have $R_6 > 0$, and yet $R_y = R_v = 0$; also, So thus far the indispensability of either (MO5) or (MO6) has not been revealed.

Indeed, with $Q = 0$ and hence Ω being at 1 almost surely, we have not found cases to confirm the necessity of (MO5) or (MO6). But counter examples abound as long as Q deviates from 0. For instance, when $Q = 0.35$, we have found an example where the violation of (MO5) is the culprit: $A = 1.3$, $D = 0.5\alpha$, and $R_y \approx 7.7\%$; in addition, we have found another example where the violation of (MO6) is the culprit for the break-down of the π-monotonicity properties: $A = 0.8$, $D = -0.5\alpha$, and $R_y \approx 14.9\%$.

On the other hand, we may still observe monotone trends when $Q > 0$ as long as A and D are kept at their default values. Figure 5.12 plots R_w, R_p, R_y, and R_v values at various Q points.

From Figure 5.12, we see that $R_y = R_v = 0$ as long as Q is below 0.15. That is, the monotonicity result of Theorem 4.5 is robust within a small neighborhood of the additive-demand setting. However, monotone trends will start to crumble when Ω deviates further from 1, as we may observe $R_y > 0$ or $R_v > 0$ when $Q > 0.15$. This, on the other hand, means that our main result can not be fully extended to the case with a more general demand form such as (4.59). Throughout, we have the strict positivity of both R_w and R_p, indicating again that there is no guaranteed trend on either the sales volume or the sales price.

CHAPTER 6

Conclusion

In summary, we considered three dynamic pricing cases including the markup case, the markdown case, and the reversible case for the firms with perishable products under product-form demand. We established an optimal threshold policy for each case and developed efficient and numerically stable algorithms to solve the corresponding optimal policies. More importantly, we identified the distinction between the markup case and the markdown case as well as the difference between the irreversible cases and the reversible case. Further, we studied a make-to-order inventory control with pricing model for the firms facing fluctuating raw material price. The optimal policy was found to be a base-stock-list-price policy and the base stock level is decreasing in the raw material price. When the raw material cost increases, the base-stock level decreases. More interestingly, the trend of pricing is not necessarily increasing, instead, the average next period inventory level is decreasing in the raw material price. In addition, we found that the benefit of considering cost-dependent policy is very significant.

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