CANONICAL KÄHLER METRICS WITH CONE SINGULARITIES

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A dissertation submitted to the

Graduate School—New Brunswick

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Jian Song

and approved by

New Brunswick, New Jersey

May, 2014

ABSTRACT OF THE DISSERTATION

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This dissertation consists of some results on the existence and regularity of canonical Kähler metrics with cone singularities. First, a much shorter proof is provided for a result of H. Guenancia and M. Păun, that solutions to some complex Monge-Ampère equations with conical singularities along effective simple normal crossing divisors are uniformly equivalent to a conical metric along that divisor. It is also shown that such metrics can always be approximated, in the Gromov-Hausdorff topology, by smooth metrics with a uniform Ricci lower bound and uniform diameter bound. As an application, it is proved that the regular set of these metrics is convex.

Next, the existence of conical Kähler-Einstein metrics and conical Kähler-Ricci solitons on toric manifolds is studied in relation to the greatest lower bounds for the Ricci and the Bakry-Emery Ricci curvatures. It is also shown that any two toric manifolds of the same dimension can be connected by a continuous path of toric manifolds with conical Kähler-Einstein metrics in the Gromov-Hausdorff topology. In the final chapter, the greatest lower bound for the Bakry-Emery Ricci curvature is studied on Fano manifolds. In particular, it is related to the solvability of some soliton-type complex Monge-Ampère equations and the properness of a twisted Mabuchi energy, extending previous work of Székelyhidi on the greatest lower bound for Ricci curvature.

Acknowledgements

First and foremost, I would like thank my advisor Jian Song, for his guidance and support, without which this dissertation would not have been possible. I am deeply indebted to him for all that he has taught me, for supplying a continuous stream of new problems and directions, and above all, for his generosity in sharing his ideas and vision. Most of the complex geometry I know was learnt through the informal seminars organized by him, with carefully chosen topics and an emphasis on detailed presentations. But more importantly, his energy and enthusiasm for the subject were very infectious, always driving me to work harder. In addition to all this, I will also remember his occasional ramblings on mathematics, life, and other random topics, with great fondness.

I would like to thank Sagun Chanillo, Xiaojun Huang, and Xiaowei Wang for agreeing to serve on my thesis committee. I have benefitted greatly from some of the courses they have offered, and even more from innumerable discussions on various mathematical topics. A special thanks to Sagun Chanillo for introducing me to the beautiful subject of harmonic analysis in my first year, and for being a great mentor of sorts. I will always be indebted to him for pointing me towards complex geometry, and my eventual advisor.

I would also like to thank Duong H. Phong, Jacob Sturm, Gabor Székelyhidi, Valentino Tosatti, Zhenlei Zhang, Bin Guo and Yuan Yuan for many fruitful and enlightening conversations. I would also like to express my gratitude towards my teachers R. Sridharan, V. Balaji, S. Kesavan, M.A. Prasad, and Z.Ahmed for having had a great influence on me during my formative years. A very special thanks to my friends here at Rutgers, for making my stay particularly enjoyable, and for all the memories that I will cherish for years to come.

Finally, I am immensely grateful to my wonderful family. My father, for passing on his love and appreciation of the sciences in general, and mathematics in particular, my mother for her unyielding support, and my sister for always making me laugh when I most needed to. At Rutgers, I also met my closest friend, confidante and loving wife. Her support, especially in these last few months, has been invaluable. Thanks Aishwarya, for teaching me to ease up a bit, and for providing me with a shoulder to lean on.

Dedication

this thesis is dedicated to

my parents

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Chapter 1

Introduction

The existence of Kähler-Einstein metrics has been a central problem in Kähler geometry since the solution to the Calabi conjecture by Yau in 1976. A Kähler manifold is specified by the data (X, J, ω), where X is an n-dimensional complex manifold with complex structure J, and ω is a smooth, closed, positive definite, real $(1,1)$ form called the Kähler form. Along with the complex structure, the Kähler form naturally induces a Riemannian metric g by the formula $g(v,w) = \omega(v, Jw)$. The Riemannian structure is compatible with the complex structure, in the sense that J is parallel with respect to the Levi-Civita connection of g. In local holomorphic coordinates, the Kähler form can be expressed as

$$\omega = \frac{\sqrt{-1}}{2} \sum_{\alpha,\beta} g_{\alpha\bar{\beta}}\ dz^{\alpha} \wedge d\bar{z}^{\beta}$$

where $\{g_{\alpha\bar{\beta}}\}$ is a positive definite hermitian matrix, and the Ricci form, which is related to the Riemannian Ricci curvature in the same way that ω is related to g, is given by

$$Ric(\omega) = -\frac{\sqrt{-1}}{2} \frac{\partial^2 \log\det(g_{\alpha\bar{\beta}})}{\partial z^{\lambda} \partial \bar{z}^{\nu}}\ dz^{\lambda} \wedge d\bar{z}^{\nu}$$

The triple (X, J, ω) is said to be *Kähler-Einstein*, and ω is said to be a Kähler-Einstein metric, if the Ricci form is a multiple of the Kähler form.

In the 1950s, Calabi asked whether there exist Kähler-Einstein metrics on any Kähler manifold. Since the Ricci form represents the first Chern class $c_1(X)$, an obvious obstruction is that $c_1(X)$ must have a sign, and indeed the problem splits naturally into three cases depending on the first Chern class being negative, zero or positive. In the first two cases, the answer to Calabi's question is affirmative, as was proved by Yau [83] when $c_1 = 0$, and independently by Aubin [3] and Yau [83] when $c_1 < 0$. The uniqueness of the Kähler-Einstein metrics in both these cases was proven by Calabi himself. When

$c_1 > 0$, i.e when X is a Fano manifold, there are other well known obstructions due to Matsushima [58] and Futaki [42]. It was shown by Tian [69] that for complex surfaces these were the only obstructions. The obstructions of Matsushima and Futaki, are both related to the existence of holomophic vector fields. Somewhat surprisingly, in 1997 Tian constructed Fano three-folds without any non-trivial holomorphic vector fields, which had no Kähler-Einstein metrics.

For general Fano manifolds, Yau conjectured that existence of Kähler-Einstein metrics was related to algebro-geometric stability, though the correct formulation of stability in this context was not clear. In 1997, Tian introduced the notion of K-stability which was further extended to more algebraic settings and to the case of constant scalar curvature metrics by Donaldson [34]. The precise conjecture then takes the form

Conjecture 1.0.1 (Yau-Tian-Donaldson)**.** *A Kähler manifold with positive Chern class admits a Kähler-Einstein metric if and only if it is K-stable.*

While, the necessity of K-stability was demonstrated by Tian in [72], the sufficiency was a long standing open problem in the field that was proved only last year, independently by Chen-Donaldson-Sun [22, 23, 24] and Tian [74] following a program laid out by Donaldson in [36]. Donaldson proposed to study the existence problem for Kähler-Einstein metrics by deforming Kähler metrics with cone singularities to smooth Kähler-Einstein metrics. Interest in conical Kähler metrics goes back, at the very least, to the works of McOwen [60] and Luo-Tian [55] on Riemann surfaces with marked points. In higher dimensions, applications of such metrics have been proposed and applied to obtain various Chern number inequalities [71, 66].

Recall that a model flat cone metric on $\mathbb{C}$ is given in polar coordinates by $ds^2 = dr^2 + \beta^2 r^2 d\theta^2$. Geometrically one can think of a cone obtained by gluing together the radial edges of an infinite sector of angle $2\pi\beta$ for some $\beta \in (0, 1]$. The corresponding Kähler form in complex coordinates is $\sqrt{-1}\partial\bar{\partial}|z|^{2\beta}$. It is easy to check that the Ricci form of this metric is the Dirac measure at the origin. In general, one can consider Kähler metrics with cone angles along *effective simple normal crossing klt* divisors. Such metrics are Kähler currents, that are smooth on the complement of the divisor, and in a neighborhood of the divisor are modeled on the following edge metric in $\mathbb{C}^n$,

$$\omega_e = \frac{\sqrt{-1}}{2} \sum_{j=1}^{k} \beta_j^2 \, \frac{dz_j \wedge d\bar{z}_j}{|z_j|^{2(1-\beta_j)}} + \frac{\sqrt{-1}}{2} \sum_{j=k+1}^{n} dz_j \wedge d\bar{z}_j.$$

where $\beta_j \in (0,1)$, and the divisor is locally given by $\sum_{j=1}^{k}(1-\beta_j)[z_j = 0]$. A Kähler current is called a *conical Kähler-Einstein* metric if in addition it satisfies

$$Ric(\omega) = \alpha\omega + [D] \tag{1.0.1}$$

for some $\alpha \in \mathbb{R}$. In the special case, when $\alpha \in (0,1)$ and $D = (1-\alpha)\tilde{D}$ where $\tilde{D}$ is a smooth anti-canonical divisor, this is precisely the continuity method introduced by Donaldson in [36] in relation to the Yau-Tian-Donaldson conjecture. The linear theory for such equations was completely worked out in [36]. Since then, the existence and regularity of solutions to Donaldson's continuity equation, and more generally to equation (1.0.1), have been very active areas of research.

When α is zero or negative, existence of weak solutions, goes back to Yau's seminal paper on the Calabi conjecture [83]. When $\alpha > 0$, it was shown very recently by Berman [6] and Li-Sun [54], that solutions always exist provided D has one component, and α is small. Moreover, in the case that D has only one component, very precise regularity results have been obtained, by Brendle when $\alpha < 1/2$, and more generally for any $\alpha \in (0,1)$ by Jeffres-Mazzeo-Rubinstein [46] and Chen-Donaldson-Sun [23]. In the present dissertation, we address questions of existence and regularity for Kähler-Einstein metrics with cone singularities along normal crossing divisors, and study some geometric consequences. The organization of this thesis is as follows.

In Chapter 2, we provide a quick introduction to Kähler geometry, toric manifolds and Gromov-Hausdorff distance. Most of the material in this chapter can be found in standard texts, and hence there are very few proofs. The section on blow-ups, and especially the description of toric blow-ups, and the adjunction formula, will be important for some of the calculations in Chapter 5. Chapter 3 deals with conical Kähler-Einstein metrics with cone angles along normal crossing divisors. The regularity theory in this case is not as developed as in the case of divisors with only one component. Following Donaldson [36], we first give a precise definition of a cone metric along a simple normal crossing kit divisor. Next, we provide a short proof of a result of Guenancia -Păun [39] that any Kähler current satisfying (1.0.1) is quasi-isometric to a cone metric. We also prove that such metrics can be approximated by smooth metrics with a uniform diameter bound and a uniform lower bound on the Ricci curvature, thereby extending a result of Chen-Donaldson-Sun [22] and Tian [74] on Fano manifolds. As an application, we prove that the regular set $X\backslash D$ is in fact convex, and that the theorems of Myers and Bishop-Gromov extend to this singular setting. This will be very useful in studying degenerations of conical Kähler-Einstein metrics on toric manifolds in Chapter 5.

Next, in Chapter 4, we study the existence of conical Kähler-Einstein metrics, and more generally conical Kähler-Ricci solitons, on toric manifolds. We introduce three invariants, and provide a complete classification of toric conical Kähler-Einstein metrics and Kähler-Ricci solitons in relation to these invariants. This generalizes the work of Wang-Zhu [80] on existence of Kähler-Ricci solitons on Fano manifolds. We also explicitly compute the three invariants in terms of the polytope associated to the toric manifold. In Chapter 5, we study degenerations of toric conical Kähler-Einstein metrics in the Gromov-Hausdorff topology. In particular, we show that any two toric manifolds of the same dimension can be connected by a continuous path of conical Kähler-Einstein metrics. Finally, in Chapter 6 we study the greatest lower bound on Bakry-Emery Ricci curvature on Fano manifolds. We characterize this invariant in terms of the solvability of some soliton-type complex Monge-Ampère equations and the properness of a twisted Mabuchi energy, extending results of Székelyhidi on the greatest lower bound for the Ricci curvature.

Chapter 2

Preliminaries.

In this chapter, we collect some standard facts from Kähler, Riemannian and toric geometry that will be needed in the sequel. Most theorems will be stated without proofs since they are rather well known. For more details, interested readers can refer to [44, 47, 73] for the first two sections, [41, 35] for the third section, and [61] for the final section.

2.1 An introduction to Kähler geometry.

Let X be a smooth manifold with $\dim_{\mathbb{R}} X = 2n$. An *almost complex structure* on X is a smooth map $J \in End(TX)$ such that $J^2 = -I$. If such a map exists, then X is called an almost complex manifold. The eigenvalues of J are $\pm\sqrt{-1}$, and the complexified tangent bundle $T_{\mathbb{C}}X = TX \otimes_{\mathbb{R}} \mathbb{C}$ accordingly splits into

$$T_{\mathbb{C}}X = T^{(1,0)}X \oplus T^{(0,1)}X$$

where $T^{(1,0)}X = \{\xi \in T_{\mathbb{C}}X \mid J\xi = i\xi\}$. J also induces a map on the dual space T^*X and one has the corresponding splitting of $T^*_{\mathbb{C}}X$ into $T^{*(1,0)}X$ and $T^{*(0,1)}X$.

Definition 2.1.1. *We say that the pair (X, J) defines a complex structure, and that X is a complex n-dimensional manifold, if in the neighborhood of every point one can choose coordinates $(x^1, \cdots, x^n, y^1, \cdots, y^n)$ such that*

$$J(\frac{\partial}{\partial x^j}) = \frac{\partial}{\partial y^j} \ ; \ J(\frac{\partial}{\partial y^j}) = -\frac{\partial}{\partial x^j} \tag{2.1.1}$$

In such a case the complex coordinates, $z^j = x^j + \sqrt{-1}y^j$ define a holomorphic coordinate system, in that the transition functions from one coordinate system to another are bi-holomorphic. This is because (2.1.1) is equivalent to the Cauchy Riemann equations for the transition functions. It is most convenient to abandon the "real" coordinates and work almost exclusively with these "complex" coordinates. Our convention is to use greek letters to denote complex coordinates. The differential forms $dz^\alpha = dx^\alpha + \sqrt{-1}dy^\alpha$ and $d\bar{z}^\beta = dx^\beta - \sqrt{-1}dy^\beta$ give a local basis for $T^{*(1,0)}X$ and $T^{*(0,1)}X$ respectively. Their duals are given by

$$\frac{\partial}{\partial z^\alpha} = \frac{1}{2}\Big(\frac{\partial}{\partial x^\alpha} - \sqrt{-1}\frac{\partial}{\partial y^\alpha}\Big) \; ; \; \frac{\partial}{\partial \bar{z}^\beta} = \frac{1}{2}\Big(\frac{\partial}{\partial x^\beta} + \sqrt{-1}\frac{\partial}{\partial y^\beta}\Big)$$

respectively. The following celebrated theorem of Newlander and Nirenberg gives sufficient conditions under which an almost complex structure induces a complex structure.

Theorem 2.1.1 (Newlander-Nirenberg)**.** *An almost complex structure J on X defines a complex structure if and only if*

$$[\xi, \eta] \in T^{(1,0)}X$$

for any $\xi, \eta \in T^{(1,0)}X$. Here $[\ ,\ ]$ denotes the Lie bracket.

On a complex manifold, the k-forms split as

$$A^k(X) = \oplus_{p+q=k} A^{p,q}(X)$$

where $A^{p,q}(X)$ denotes the forms of type (p, q) spanned by $\{dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_q}\}$. Then the exterior derivative also splits naturally as $d = \partial + \bar{\partial}$.

A Riemannian metric g on TX is said to be Hermitian, if $g(Jv, Jw) = g(v, w)$ for all $v, w \in TX$. g can be extended bi-linearly to all of $T_{\mathbb{C}}X$, and one can define a skew-symmetric $(1, 1)$ form by setting

$$\omega(\xi, \eta) = g(J\xi, \eta)$$

for $\xi, \eta \in T_{\mathbb{C}}X$. Locally ω is given by

$$\omega = \frac{\sqrt{-1}}{2}\sum_{\alpha,\bar{\beta}} g_{\alpha\bar{\beta}}\; dz^\alpha \wedge d\bar{z}^\beta$$

The fact that ω is $(1, 1)$, is equivalent to $g(\xi, \eta)$ being zero whenever ξ and η are of the same type. The matrix for g in the real coordinates has the block form

$$\begin{pmatrix} Re(g_{\alpha\bar{\beta}}) & Im(g_{\alpha\bar{\beta}}) \\ -Im(g_{\alpha\bar{\beta}}) & Re(g_{\alpha\bar{\beta}}) \end{pmatrix}$$

Since g is symmetric and positive definite, we immediately see that the matrix $(g_{\alpha\bar{\beta}})$ is a hermitian and positive definite matrix.

Definition 2.1.2. *The form ω is said to be Kähler, and the triple (X, J, ω) a Kähler manifold, if*

$$d\omega = 0$$

We use the words Kähler metric and Kähler form interchangeably. Next, if ∇ is the Levi-Civita connection of the Riemannian metric g, a fundamental fact of Kähler geometry is that ∇ is compatible with J and ω if and only if (M, J, ω) is a Kähler manifold. More precisely,

Theorem 2.1.2. *The following are equivalent to $d\omega = 0$*

1. $\nabla J = 0$

2. $\nabla \omega = 0$

3. *There exists holomorphic coordinates $\{z^\alpha\}$, centered at $p \in X$ such that*

$$g_{\alpha\bar{\beta}}(p) = \delta_{\alpha,\beta} \ ; \ g_{\alpha,\bar{\beta},\lambda}(p) = g_{\alpha\bar{\beta},\bar{\nu}}(p) = 0$$

Remark 2.1.1. *Setting $h(\xi, \eta) = g(\xi, \bar{\eta})$ defines a hermitian product on the holomorphic vector bundle $T^{(1,0)}X$. It is a general fact that there exists a unique connection, called the Chern connection, compatible with this hermitian metric and the holomorphic structure. The above equivalences imply that the Levi-Civita connection restricted to $T^{(1,0)}X$ and the Chern connection coincide in the special case of Kähler manifolds.*

A particularly nice feature of Kähler manifolds is that the formulas for Christoffel symbols and curvatures are much nicer as compared to usual Riemannian geometry. For instance, the only non zero Christoffel symbols are given by

$$\Gamma^{\tau}_{\lambda\alpha} = g_{\alpha\bar{\beta},\lambda} g^{\bar{\beta}\tau} \ ; \ \Gamma^{\bar{\mu}}_{\bar{\nu}\bar{\beta}} = \overline{\Gamma^{\mu}_{\nu\beta}}$$

Recall that the Riemannian curvature endomorphism is defined by $R(u, v)z = \nabla_u \nabla_v z - \nabla_v \nabla_u z - \nabla_{[u,v]} z$, and the curvature 4-tensor is given by $Rm(u, v, w, z) = g(R(w, z)v, u)$.

Since J is parallel with respect to ∇, it is easy that $Rm(u,v,w,z) = Rm(u,v,Jw,Jz)$. Consequently, Rm is non-zero if and only if w and z are of different types, and so are u and v. If locally, $R = R^{\tau}_{\alpha\lambda\bar{\nu}}\, dz^{\lambda}\wedge dz^{\bar{\nu}}\otimes dz^{\alpha}\otimes\frac{\partial}{\partial z^{\tau}}$ and $Rm = R_{\alpha\bar{\beta}\lambda\bar{\nu}}\, dz^{\alpha}\wedge dz^{\bar{\beta}}\otimes dz^{\lambda}\wedge dz^{\bar{\nu}}$ respectively, then

$$R^{\tau}_{\alpha\lambda\bar{\nu}} = -\partial_{\bar{\nu}}\Gamma^{\tau}_{\lambda\alpha},$$

$$R_{\alpha\bar{\beta}\lambda\bar{\nu}} = R^{\tau}_{\alpha\lambda\bar{\nu}}\; g_{\tau\bar{\beta}} = -g_{\alpha\bar{\beta},\lambda\bar{\nu}} + g^{\bar{\mu}\tau} g_{\alpha\bar{\mu},\lambda} g_{\tau\bar{\beta},\bar{\nu}},$$

and we have the fundamental curvature identity

$$[\nabla_{\lambda}, \nabla_{\bar{\nu}}]\; \xi^{\tau} = R^{\tau}_{\alpha\lambda\bar{\nu}}\; \xi^{\alpha}$$

The Riemannian Ricci curvature, defined as the trace of the endomorphism R, is also compatible with the complex structure. That is $Rc(Jv,Jw) = Rc(v,w)$. Then, analogous to the definition of the Kähler form, one can define the Ricci form.

Definition 2.1.3. *The Ricci form of the Kähler form ω is defined by*

$$Ric(\omega)(v,w) = Rc(Jv,w)$$

As with the metric and the Käbler form, it is clear that the Ricci curvature and the Ricci form determine each other. A simple calculation shows that the Ricci form can be very simply expressed in terms of the matrix $\{g_{\alpha\bar{\beta}}\}$.

Lemma 2.1.1. *If ω is a Kähler metric, then the Ricci form is given by*

$$Ric(\omega) = \frac{\sqrt{-1}}{2}\; R_{\alpha\bar{\beta}}\; dz^{\alpha}\wedge d\bar{z}^{\beta} = -\frac{\sqrt{-1}}{2}\; \partial\bar{\partial}\log\det(g_{\alpha\bar{\beta}}) \tag{2.1.2}$$

The lemma follows from the elementary observation that $\partial_{\lambda}\partial_{\bar{\nu}}\log\det(g_{\alpha\bar{\beta}}) = \partial_{\bar{\nu}} g^{\bar{\beta}\alpha} g_{\alpha\bar{\beta},\lambda}$. We end with another fundamental fact about Kähler manifolds, which is a simple consequence of the Hodge decomposition theorem.

Lemma 2.1.2 ($\partial\bar{\partial}$-Lemma)**.** *If η_1 and η_2 are two smooth cohomologous d-closed* $(1,1)$ *forms, then there exists a function $\varphi \in C^{\infty}(X)$ such that*

$$\eta_2 = \eta_1 + \sqrt{-1}\partial\bar{\partial}\varphi$$

The above lemma is very useful in turning many tensor equations involving curvatures into scalar equations, thereby making them more tractable in the Kähler setting.

2.2 Line bundles and divisors.

Let X be a compact complex manifold.

Definition 2.2.1. *A holomorphic line bundle L is a topological space, with a projection $\pi : L \to X$ such that*

1. *Each $L_x = \pi^{-1}(x)$ is a one dimensional complex vector space.*
2. *There exists open cover $\{U_\alpha\}$ of X and trivializations $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$, such that for every $x \in U_\alpha$, the restriction $\phi_\alpha\Big|_{\pi^{-1}\{x\}}$ is a complex linear automorphism from $\pi^{-1}(x)$ to $\{x\} \times \mathbb{C}$.*
3. *The transition functions $f_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}(.,.) : U_\alpha \cap U_\beta \to GL(1, \mathbb{C}) \approx \mathbb{C}^*$, are holomorphic maps.*

Using the transition maps, L can be endowed with the structure of a complex manifold.

Given a line bundle L, one can define its dual in an obvious way. Similarly one can also define tensor products of two line bundles to obtain another line bundle. A *section* of the line bundle is a smooth map $s : X \to L$ such that $\pi \circ s = id$. The space of holomorphic sections is denoted by $\Gamma(X, L)$.

Example 2.2.1.
1. *The product $\mathcal{O} = X \times \mathbb{C}$ with projection onto the first factor is a line bundle, called the trivial line bundle.*
2. *An important example is obtained by taking the top exterior product of the holomorphic co-tangent bundle, $\Lambda^n T^{1,0}X^*$ with standard projection map. On a coordinate neighborhood U_α with coordinates $z = (z^1, \cdots, z^n)$, the trivializations defined by $\phi_\alpha^{-1}(z, \lambda) = \lambda dz^1 \wedge \cdots \wedge dz^n$, give it a structure of a holomorphic line bundle, called the canonical line bundle and denoted by K_X. The transition functions in this case are precisely the determinants of the Jacobian matrix and hence holomorphic. The dual of this line bundle is called the anti-canonical line bundle.*

The equivalence class of all holomorphic line bundles naturally forms an Abelian group under the operation of tensor products, called the Picard group and denoted by

$Pic(X)$. The trivial line bundle acts as the identity element of this group, while the dual acts as the inverse. Note that the transition functions satisfy the cocyle relation :

$$f_{\alpha\beta}\ f_{\beta\gamma}\ f_{\gamma\alpha} = 1;\ f_{\alpha\alpha} = 1$$

and hence define a homomorphism $\mathcal{J} : Pic(X) \to H^1(X, \mathcal{O}^*)$, where $\mathcal{O}^*$ denotes the sheaf of germs of non-vanishing holomorphic functions on X, and $H^1(X, \mathcal{O}^*)$ is the first Čech cohomology group of this sheaf. Conversely, given transition functions $\{f_{\alpha\beta}\} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ satisfying the cocyle relations, one can construct a holomorphic line bundle by setting $L = \coprod_\alpha U_\alpha \times \mathbb{C}/\sim$, where $(x, \lambda) \sim (y, \mu)$ for $x \in U_\alpha, y \in U_\beta$, if and only if $x = y$ and $\lambda = f_{\alpha\beta}(x)\mu$. This proves that $\mathcal{J}$ is surjective. By tracing the above construction, it can be shown that the kernel of the map is trivial, and hence $Pic(X)$ is isomorphic to $H^1(X, \mathcal{O}^*)$. This identification allows one to define the first Chern class of a line bundle. The short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2\pi\sqrt{-1}} \mathcal{O} \xrightarrow{exp} \mathcal{O}^* \to 0$$

induces a long exact sequence at the cohomology level, and in particular a boundary map $H^1(X, \mathcal{O}^*) \xrightarrow{c_1} H^2(X, \mathbb{Z})$. The image $c_1(L)$ of a line bundle under this map is called the *first Chern class.*

A hermitian metric is a collection of smooth positive functions $h = \{h_\alpha\}$ such that on $U\alpha \cap U_\beta$, $h_\alpha = |f_{\alpha\beta}|^2 h_\beta$. If $\{e_\alpha\}$ is a local unitary basis element and $s = s_\alpha e_\alpha$ a section, then the norm is defined by $\langle s, s\rangle = |s_\alpha|^2 h_\alpha$. A unitary connection compatible with the holomorphic structure is a map $D_h = \nabla_h + \overline{\nabla}_h : \Gamma(X, L) \to \Lambda^1 T^* X \otimes \Gamma(X, L)$ satisfying

$$d\langle s_1, s_2\rangle = \langle D_h s_1, s_2, \rangle + \langle s_1, D_h s_2\rangle$$

$$\overline{\nabla}_h(s_\alpha e_\alpha) = \overline{\partial} s_\alpha \otimes e_\alpha$$

Locally such a connection is given by a connection 1-form $\theta_\alpha = \partial \log h_\alpha$, where $\nabla e_\alpha = \theta_\alpha \otimes e_\alpha$. The curvature is then defined to be $F_h = D_h \circ D_h$, and is locally given by a $(1, 1)$-form Θ_α on U_α. By the Cartan structure equation

$$\Theta_\alpha = d\theta_\alpha = -\sqrt{-1}\partial\overline{\partial} \log h_\alpha \tag{2.2.3}$$

Clearly $\Theta_\alpha = \Theta_\beta$ on $U_\alpha \cap U_\beta$, and $d\Theta_\alpha = 0$. So one obtains a global closed curvature $(1,1)$-form Θ. A fundamental fact, which follows from the de Rham isomorphism theorem, is that $c_1(L) = (i/2\pi)[\Theta]$.

Example 2.2.2. *A Kähler metric with components $\{g_{\alpha\bar{\beta}}\}$ induces a Hermitian metric on the anti-canonical bundle given by the* $\det(g_{\alpha\bar{\beta}})$*. The curvature $-\sqrt{-1}\partial\overline{\partial}\log\det(g_{\alpha\bar{\beta}})$ is precisely the Ricci form* (2.1.2)*. The first Chern class of the manifold $c_1(X) := c_1(-K_X)$, is therefore the cohomology class of the Ricci form.*

Next we define another Abelian group, called the group of *divisors*, and denoted by $Div(X)$. Recall that a hyperfurface $V \subset X$ is locally the zero set of a single non trivial holomorphic function, called the defining function. It's *multiplicity* is defined as the order of vanishing of this defining function. It is said to be *irreducible* if it cannot be written as the sum of two hyper-surfaces $V_1, V_2 \neq V$. Equivalently V is irreducible if and only if for any $p \in V$, the local defining function is an irreducible element in the ring $\mathcal{O}_p$ of holomorphic functions near p. An irreducible hyper-surface is also called a *prime divisor*. Now let V be a prime divisor with defining function s. For any local holomorphic function h on X, one can define its order along V at $p \in V$ as the largest integer a such that $h = s^a g$ in $\mathcal{O}_p$. It turns out that the order is locally constant. More generally for a meromorphic function that is locally given by $f = g/h$, the order is defined as $ord_V(f) = ord_V(g) - ord_V(h)$. Then the divisor associated to f is defined by

$$(f) = \sum_V ord_V(f) \cdot V$$

It is clear that this sum will in fact be a finite one. More generally we have the following definition.

Definition 2.2.2. *A divisor D is a locally finite linear combination*

$$D = \sum_{j=1}^{N} a_j V_j$$

of irreducible hyper-surfaces V_j with $a_j \in \mathbb{Z}$. The set of divisors form an Abelian group $Div(X)$ under addition. It is said to be effective if $a_j \geq 0$ for all j. We say D_1 is *linearly equivalent to* D_2*, and write* $D_1 \sim D_2$*, if there exists a meromorphic function* f *such that* $D_2 - D_1 = (f)$*. It is said to be an* $\mathbb{R}$*-divisor if* a_j *can be taken to be in* $\mathbb{R}$*.*

One can define a map $\partial : Div(X) \to Pic(X)$ in the following way. Take a covering $\{U_\alpha\}$ and suppose $V|_{U_\alpha} = (f_{\alpha j} = 0)$. Then define the transition functions by $f_{\alpha\beta} = s_\alpha / s_\beta$ where

$$s_\alpha = \prod_{j=1}^{N} f_{\alpha j}^{a_j}$$

and let $\partial(D) = L_D$ be the line bundle associated to these transition functions. Note that the collection $\{s_\alpha\}$ defines a meromorphic section for L_D. It is holomorphic if and only if D is effective. Conversely, for any line bundle L that has a meromorphic section s, there exists a divisor D, such that $L = L_D$. One can also understand this correspondence as a map between Čech cohomology groups. Let $\mathcal{M}^*$ denote the multiplicative sheaf of meromorphic functions that are not identically zero, and recall that $\mathcal{O}^*$ is the subsheaf of non vanishing holomorphic functions.. Then divisors can be identified with the global sections of the quotient sheaf $\mathcal{M}^* \backslash \mathcal{O}^*$ i.e $Div(X) \approx H^0(X, \mathcal{M}^* \backslash \mathcal{O}^*)$. One has the canonical exact sequence

$$0 \to \mathcal{O}^* \xrightarrow{i} \mathcal{M}^* \xrightarrow{j} \mathcal{M}^* \backslash \mathcal{O}^* \to 0$$

This induces a long exact sequence at the level of cohomology. Then the above correspondence between divisors and line bundles is precisely the boundary map

$$\cdots \xrightarrow{i_*} H^0(X, \mathcal{M}^*) \xrightarrow{j_*} H^0(X, \mathcal{M}^* \backslash \mathcal{O}^*) \xrightarrow{\partial} H^1(X, \mathcal{O}^*) \xrightarrow{i_*} \cdots$$

where $Pic(X)$ can be identified with $H^1(X, \mathcal{O}^*)$. Note that $H^0(X, \mathcal{M}^*)$ is the set of all divisors linearly equivalent to the trivial divisor. So the pre-image of the trivial line bundle are precisely those divisors that are linearly equivalent to the trivial divisor. A divisor $D = \sum a_j V_j$, represents a class in $H_2(X, \mathbb{Z})$ and defines a current of integration on $A^{(n-1,n-1)}(X)$,

$$T_D(\psi) = \sum_j a_j \int_{V_j} \psi = \int_X [D] \wedge \psi$$

where $[D]$ is given explicitly by the Poincaré-Lelong formula.

Lemma 2.2.1 (Poincaré-Lelong equation)**.** *For any prime divisor D, if s is the defining section, and $[D]$ denotes the current of integration of $(n-1, n-1)$ forms along D, then*

$$[D] = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|s|^2 \tag{2.2.4}$$

This equation can be extended linearly to define the current of integration along any divisor. We next discuss the notion of ampleness.

Definition 2.2.3. *A line bundle L is said to be ample if there exists a smooth hermitian metric h on it whose curvature form Θ_h is a positive form. A divisor is said to be ample if the corresponding line bundle L_D is ample. A $(1,1)$ class in $H^2(X,\mathbb{R})$ is said to be ample if there exists a positive representative in that class.*

In this thesis, we will be concerned with Kähler currents $\omega = \hat{\omega} + \sqrt{-1}\partial\overline{\partial}\varphi$ that are solutions to

$$Ric(\omega) = \lambda\omega + [D]$$

where $\lambda \in \mathbb{R}^+$, $D = \sum a_j V_j$ is an effective $\mathbb{R}$-divisor with $a_j \in (0,1)$, and $\hat{\omega}$ is a smooth Kähler metric. Such currents are called *Kähler-Einstein* currents. At the level of co-homologies this forces $-K_X - D$ to be an ample class. This motivates the next definition.

Definition 2.2.4. *Let X be a smooth projective variety. The pair (X, D) is called a log pair if one can write $D = \sum a_J V_j$ where $a_j \in [0,1)$. The pair is furthermore said to be log Fano if in addition, $-(K_X + D)$ is an ample class. X is said to be log Fano if there exists such a divisor D.*

So, by the above discussion, a necessary condition that there exist Kähler-Einstein currents is that X is log Fano. In the next section, we will show that all toric manifolds are indeed log Fano.

2.3 Brief survey of toric geometry.

In this section, we collect some well-known facts of projective toric manifolds. For the rest of the thesis, we will stick to lower indices to denote coordinates, unless specified otherwise.

Definition 2.3.1. *A Kähler manifold (X^n, J, ω) is called a toric manifold if it admits an effective Hamiltonian action of the standard torus $\mathbb{T}^n = (S^1)^n$, which extends to holomorphic action of the complexified torus $(\mathbb{C}^*)^n$ with a free open dense orbit $X_0 \approx (\mathbb{C}^*)^n$.*

Since the action is Hamiltonian, there exists a moment map $\mu : X \to \mathbb{R}^n$, where $\mathbb{R}^n$ is identified to the dual of the Lie algebra of $\mathbb{T}^n$. By the Atiyah-Guillemin-Sternberg convexity theorem, $\mu(X)$ is the closure of a convex polytope $P \subset \mathbb{R}$ which is uniquely determined, up to translations, by the cohomology class of the Kähler metric ω. In the converse direction, Delzant classified all polytopes that can arise as moment polytopes of toric manifolds.

Definition 2.3.2. *A convex polytope $P \subset \mathbb{R}^n$ is called a* Delzant polytope *if a neighborhood of any vertex $p \in P$ is $SL(n, \mathbb{Z})$ equivalent to $\{x_j \geq 0, j = 1, ..., n\} \subset \mathbb{R}^n$. P is called* an integral Delzant *polytope if each vertex $p \in P$ is a lattice point in $\mathbb{Z}^n \subset \mathbb{R}^n$.*

Let P be a Delzant polytope (not necessarily integral) in $\mathbb{R}^n$ defined by

$$P = \{x \in \mathbb{R}^n \mid l_j(x) > 0, j = 1, ..., N\}, \tag{2.3.5}$$

where

$$l_j(x) = v_j \cdot x + \lambda_j$$

and v_i is a primitive integral vector in $\mathbb{Z}^n$ and $\lambda_j \in \mathbb{Z}_+$ for all $j = 1, ..., N$. It was shown by Delzant that X can be constructed globally as a symplectic quotient of $\mathbb{C}^N$ by a compact Lie group. More explicitly, one can use the polytope data to construct complex coordinate charts for X in the following way : Recall that there is a standard torus $X_0 \approx (\mathbb{C}^*)^n \subset X$ with coordinates $(t_1, \cdots, t_n)$. Then, to each pair $(p, \{v_{p,j}\}_{j=1}^n)$, of a vertex p with the normals to the n faces intersecting at p, we associate a coordinate neighborhood $U_p = \mathbb{C}^n$ with coordinates $(z_1, \cdots, z_n)$. By the definition of a Delzant polytope, there exists a unique $\sigma_p = (a_{ij}) \in SL(n, \mathbb{Z})$ mapping $\{v_{p,j}\}$ to the standard unit vectors $\{e_j\}$. We then define the coordinates by

$$z_i = \prod_{j=1}^{n} t_j^{a_{ij}} \tag{2.3.6}$$

Now, for two vertices p, p', if we let $\sigma_{p,p'} = \sigma_{p'}^{-1} \circ \sigma_p = (b_{ij}) \in SL(n, \mathbb{Z})$, then

$$\sigma_{p,p'} v_{p,j} = v_{p',j}$$

If the corresponding coordinates are given by $z = (z_1, \cdots, z_n)$ and $z' = (z'_1, \cdots, z'_n)$, then the coordinate change is given by

$$z'_i = \prod_{j=1}^{n} z_j^{b_{ij}} \tag{2.3.7}$$

The inverse images of the faces $[l_j = 0]$ under the moment maps correspond to all the irreducible toric divisors D_j on X. In the above coordinates, these are precisely the closures of the sub-varieties $\{z_j = 0\}$. Note that the transition functions, and hence the construction of X, depend only on the normals to the faces, and not the constants λ_j. These constants, in fact, determine the cohomology class that the Kähler metric ω sits inside, via the following formula

$$[\omega] = \sum_{j=1}^{N} \lambda_j [D_j]$$

where $[D_j]$ is the Poincaré dual of $D_j \in H_{2n-2}(X, \mathbb{Z})$. Moreover, any $\mathbb{R}$-divisor D cohomologous to ω is given by

$$D = \sum_{j=1}^{N} l_j(\tau) D_j$$

for some $\tau \in \mathbb{R}^n$. If the Delzant polytope is integral, then $[\omega]$ represents an integral class, and hence $[\omega] = c_1(L)$ for some ample line bundle L. Then the space of global sections of L is given by

$$H^0(X, \mathcal{O}(L)) = \bigoplus_{\alpha \in P \cap \mathbb{Z}^n} \mathbb{C} \cdot t^{\alpha}, \tag{2.3.8}$$

where $t = (t_1, \cdots, t_n)$ is the standard coordinate on $(\mathbb{C}^*)^n$. The anti-canonical divisor on toric manifolds can also be explicitly expressed in terms of the toric divisors.

Lemma 2.3.1.

$$-K_X = \sum_{j=1}^{N} D_j$$

Proof. This follows from the fact that, if $(t_1, \cdots, t_n)$ are the standard coordinates on $(\mathbb{C}^*)^n$, then

$$\Omega = \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}$$

extends to global meromorphic $(n, 0)$ form. In fact if $(z_1, \cdots, z_n)$ are the coordinates on U_p, where p is a vertex of P, then a simple calculation shows that

$$\frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} = \pm \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$$

and clearly the right hand side has poles of order -1 on $\{z_j = 0\}$. Hence $K_X = -\sum_{j=1}^N D_j$ and the lemma is proved by taking duals. $\square$

Recall the definition 2.2.4 of a log Fano variety from the previous section. We can now prove that all toric manifolds are log Fano.

Lemma 2.3.2. *Let X be any projective toric manifold. Then X is log Fano.*

Proof. Since X is toric and projective, there exists a toric, effective and ample divisor $A = \sum_{j=1}^N a_j D_j$. Let $D = -K_X - \varepsilon A$. Then for $0 < \varepsilon \ll 1$, $0 \leq (1 - \varepsilon a_j) < 1$ and so, (X, D) is a log pair. Moreover $-(K_X + D) = \varepsilon A$, is ample. So (X, D) is a log Fano pair and X is log Fano. $\square$

We next describe Kähler metrics on toric manifolds. If ω is a $\mathbb{T}^n$ invariant Kähler metric on X, then by the Poincaré lemma, $\omega\Big|_{(\mathbb{C}^*)^n} = \sqrt{-1}\partial\bar{\partial}\varphi$ for some potential function φ. Working with logarithmic-angular coordinates, $\rho_j = \log |t_j|^2$ and θ_j, by the torus invariance φ is a function of only $(\rho_1, \cdots, \rho_n)$. In fact the metric can be explicitly written down as

$$\omega\Big|_{(\mathbb{C}^*)^n} = \frac{1}{4} \frac{\partial^2 \varphi}{\partial \rho_j \partial \rho_k}\, d\rho_j \wedge d\theta_k$$

and so, $\omega > 0$ is equivalent to the convexity of $\varphi = \varphi(\rho_1, \cdots, \rho_n)$. The moment map now, up to a constant, is just given by $\mu = \nabla\varphi$. Recall that given a Delzant polytope, the toric manifold can be realized as the symplectic quotient of $\mathbb{C}^N$ by a compact Lie group. The standard flat Euclidean metric on $\mathbb{C}^N$ induces a canonical Kähler metric on X whose image under the moment map is the polytope P. The potential for this metric is most conveniently expressed in terms of "symplectic" coordinates or momentum-angular coordinates, which we now introduce. Using the moment map, $X_0 \approx (\mathbb{C}^*)^n$ can be identified to $P \times \mathbb{T}^n$ with coordinates $(x^1, \cdots, x^n, \theta_1, \cdots, \theta_n)$ where

$\vec{x} = \nabla\varphi(\rho_1, \cdots, \rho_n)$. In these coordinates the Kähler metric and the Riemannian metric are given by

$$\omega = \sum_{j=1}^{n} dx^j \wedge d\theta_j \tag{2.3.9}$$

$$ds^2 = \sum_{j=1}^{n} u_{jk} dx^j dx^k + u^{jk} d\theta_j d\theta_k \tag{2.3.10}$$

where $u \in C^\infty(P) \cap C^0(\overline{P})$ is the Legendre transform of φ,

$$u(x) = x \cdot \rho - \varphi(\rho) \ ; \ \rho = \nabla u(x)$$

and is called the *symplectic potential* associated to the metric ω. u_{jk} and u^{jk} denote the Hessian of u and its inverse respectively. The following theorem, proved by Guillemin [45], classifies all possible symplectic potentials such that the corresponding metric ds^2 on $P \times \mathbb{T}^n$ extends to a global metric on X.

Proposition 2.3.1. *1. The symplectic potential for the canonical Kähler metric induced from the standard flat metric on $\mathbb{C}^N$ is given by*

$$\hat{u} = \sum_{j=1}^{N} l_j(x) \log l_j(x) - l_\infty(x) \tag{2.3.11}$$

where

$$l_\infty(x) = \sum_{j=1}^{N} v_j \cdot x \tag{2.3.12}$$

2. Moreover a potential $u \in C^\infty(P) \cap C^0(\overline{P})$ corresponds to a global Riemannian metric on X via the formula (2.3.10) *if and only if*

$$u - \hat{u} \in C^\infty(\overline{P}) \tag{2.3.13}$$

In the sequel, such a potential (or its generalization to the conical case) will be said to be *admissible*. Equation (4.2.13) specifies the asymptotics of any admissible potential near the boundary, and is referred to as the Guillemin boundary condition. We will state a generalization of this result to the case of conical toric metrics in section 4.2.

2.4 Blowing up along sub-varieties.

We wish to discuss the general procedure of blowing up, the adjunction formula and more specifically the description of toric blow-ups in terms of the polytope data. Consider the unit polydisc $\Delta \subset \mathbb{C}^n$ with standard coordinates $(z_1, \cdots, z_n)$. The blow up of Δ along the sub-variety $V = (z_{k+1} = 0, \cdots, z_n = 0)$ is defined the be the smooth variety

$$\tilde{\Delta} = \{(z, \xi) \in \Delta \times \mathbb{C}P^{n-k-1} \mid z_j \xi_l = z_l \xi_j,\ k+1 \leq j, l \leq n\}$$

together with the map $\pi : \tilde{\Delta} \to \Delta$. The pre-image $E = \pi^{-1}(V)$ is called the exceptional divisor and is a $\mathbb{C}P^{n-k-1}$ bundle over V. We can cover $\tilde{\Delta}$ by coordinate neighborhoods $\tilde{U_l} = \pi^{-1}(U_l)$, where $U_l = (\xi_l \neq 0)$ are the standard coordinate neighborhoods for $\mathbb{C}P^{n-k-1}$, and define coordinates

$$\begin{cases} z^l_j = z_j \ ; \ j = 1, \cdots k \\ z^l_j = \frac{\xi_j}{\xi_l} = \frac{z_j}{z_l} \ ; \ j = k+1, \cdots, \hat{l}, \cdots, n \\ z^l_l = z_l \end{cases} \tag{2.4.14}$$

The exceptional divisor is then given by $E = [z^l_l = 0]$.

Lemma 2.4.1 (Kodaira's Lemma). *Let $L \to X$ be an ample line bundle, and $\pi^* : \tilde{X} \to X$ be the blow-up along a k-dimensional sub-variety V. Then there exists an $\varepsilon_0 > 0$ such that $L_\varepsilon = \pi^* L - \varepsilon[E]$ is ample for all $\varepsilon \in (0, \varepsilon_0)$.*

Proof. We only outline a proof. Let h be a smooth metric on L so that the curvature form Ω is positive on all of X. Then $\pi^*\Omega$ is a curvature form for $\pi^* L$ wither expect to the metric $\pi^* h$. Clearly $\pi^*\Omega \geq 0$ everywhere on X. Moreover, $\pi^*\Omega(v, \bar{v})$ is zero for some $v \in T^{(1,0}\tilde{X}$, if and only if $\pi_*(v) = 0$. Recall that we have a fibration $\pi : E \to V$ where for all $x \in V$, the fibre $E_x \approx \mathbb{C}P^{n-k-1}$. Then $\pi_*(v) = 0$ if and only if v is tangent to $E_{\pi(x)}$ at some $x \in E$. Summarizing

$$\pi^*\Omega \;=\; \begin{cases} \geq 0 \text{ everywhere} \\ > 0 \text{ on } \tilde{X}\backslash E \\ > 0 \text{ on } T_x^{(1,0)}\tilde{X}/T_x^{(1,0)}E_{\pi(x)} \text{ with } x \in E \\ = 0 \text{ on } T_x^{(1,0)}E_{\pi(x)} \text{ with } x \in E \end{cases}$$

On the other hand, from the coordinates above, it is clear that for $l \neq m$, $f_{lm} = z_l/z_m = \xi_l/\xi_m$ are the transition functions for the line bundle generated by E. So $[E]\Big|_E$ is simply the pullback of $\mathcal{O}(-1)$ on $\mathbb{C}P^{n-k-1}$ and hence is negative precisely on $T_x^{(1,0)}E_{\pi(x)}$, and lower bounded everywhere else on $\tilde{X}$. A simple partition of unity argument now completes the proof. □

Lemma 2.4.2 (Adjunction formula)**.** *Let $\pi : \tilde{X} \to X$ be the blow-up along a k-dimensional sub-variety V. Then the respective canonical bundles $K_{\tilde{X}}$ and K_X are related by*

$$K_{\tilde{X}} = \pi^* K_X + (n-k-1)E$$

Proof. We prove this in the special case that K_X has a global meromorphic section Ω. In general, one needs to show that the transition functions transform correctly, and the local computations are very similar to the ones given below. Choose local coordinates $(z_1, \cdots, z_n)$ on X so that $V = [z_{k+1} = 0, \cdots, z_n = 0]$, and let

$$\Omega = f(z)dz_1 \wedge \cdots \wedge dz_n$$

By the above construction, these induce local coordinates on $\tilde{X}$. The map π is given by

$$\begin{cases} z_j = z_j^l \;;\; j = 1, \cdots k \\ z_j = z_j^l \cdot z_l^l \;;\; j = k+1, \cdots, \hat{l}, \cdots, n \\ z_l = z_l^l \end{cases}$$

So, $\pi^*(dz_1 \wedge \cdots \wedge dz_n) = (z_l^l)^{n-k-1} dz_1^l \wedge \cdots \wedge dz_j^l$ and

$$\pi^*\Omega = \pi^*(f)(z_l^l)^{n-k-1} dz_1^l \wedge \cdots \wedge dz_j^l$$

But $\pi^*\Omega$ is a meromorphic volume form on $\tilde{X}$ and hence is a section on $K_{\tilde{X}}$, while $\pi^*(f)$ and z_l^l are sections of $\pi^* K_X$ and E respectively. This proves the theorem. □

Recall the correspondence between polarized toric manifolds and polytopes from last section. Toric blow-ups also have a particularly nice description in terms of the polytope picture.

Theorem 2.4.1 (Toric blow-ups)**.** *Let X be an n-dimensional toric manifold, α an ample class it, and*

$$P = \{x \in \mathbb{R}^n \mid l_j(x) = v_j \cdot x + \lambda_j > 0,\ j = 1, \cdots, N-1\}$$

the corresponding polytope with moment map $\mu : X \to P$. If $V = \mu^{-1}\Big(\cap_{j=1}^{n-k}[l_j = 0]\Big)$, is a toric k-dimensional sub-variety, and $\pi : \tilde{X} \to X$ is the blow-up along V with exceptional divisor E, then for small ε, the corresponding polytope for $\tilde{X}$ associated to the Kähler class $\alpha_\varepsilon = \pi^*\alpha - \varepsilon E$ is given by

$$\tilde{P} = P \cap \{x \mid l_N(x) = v_N \cdot x + \lambda_N^\varepsilon > 0\}$$

where

$$v_N = \sum_{j=1}^{n-k} v_j \ ; \ \lambda_N^\varepsilon = \sum_{j=1}^{n-k} \lambda_j - \varepsilon$$

Proof. For clarity of exposition, we provide an outline of a proof in the special case of blowing up a point. The general case follows with the obvious modifications. Since the argument is essentially local, it also suffices to consider the model case

$$P = \{x \in \mathbb{R}^n \mid x_j > 0\}$$
$$V = \mu^{-1}\big([x = 0]\big)$$

corresponding to $\mathbb{C}^n$ with standard coordinates $(t_0, \cdots, t_n)$. Then

$$\tilde{P} = P \cap \{x \mid \sum_{j=1}^{n} x_j > \varepsilon\}$$

with vertices $p_l = (0, \cdots, \varepsilon, \cdots, 0)$ with ε at the l^{th} place and corresponding normals $(e_1, \cdots, e_{l-1}, v, e_{l+1}, \cdots, e_n)$ where e_l are the standard basis for $\mathbb{R}^n$ and $v = (1, \cdots, 1)$.

Then the corresponding $SL(n,\mathbb{Z})$ element in the definition of a Delzant polytope is given by $\sigma^l = (a_{jk})$ where

$$a_{jk} = \begin{cases} \delta_{jk} \ ; \ k \neq l \\ -1 \ ; \ k = l, j \neq l \\ 1 \ ; \ k = j = l \end{cases}$$

Each vertex p_l corresponds to a coordinate neighborhood U_l for the toric manifold corresponding to $\tilde{P}$ with coordinates $(z_1^l, \cdots, z_n^l)$. Then by equation (2.3.6) and the description of σ^l above, it is clear that

$$\begin{cases} z_j^l = \frac{t_j}{t_l} \ ; \ j \neq l \\ z_l^l = t_l \end{cases}$$

But these are precisely the coordinates of blow-up at a point (2.4.14), and hence $\tilde{P}$ corresponds to blow-up of $\mathbb{C}^n$ at 0. $\square$

We end this section with a weak factorization theorem of toric manifolds first proved in [81] (cf. also [1]).

Theorem 2.4.2 (Weak factorization theorem)**.** *Let $f : X \dashrightarrow Y$ be a toric birational map between two complete nonsingular toric varieties X and Y over $\mathbb{C}$, and let $U \subset X$ be an open set where f is an isomorphism. Then f can be factored into a sequence of blow-ups and blow-downs with nonsingular irreducible toric centers disjoint from U, namely, there is a sequence of birational maps between complete nonsingular toric varities*

$$X = X_0 \overset{f_1}{\dashrightarrow} X_1 \overset{f_2}{\dashrightarrow} \cdots \overset{f_i}{\dashrightarrow} X_i \overset{f_{i+1}}{\dashrightarrow} \cdots \overset{f_n}{\dashrightarrow} X_n = Y,$$

where

1. *$f = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$,*

2. *f_i is an isomorphism on U, and*

3. *either $f_i : X_{i-1} \dashrightarrow X_i$ or $f_i^{-1} : X_i \dashrightarrow X_{i-1}$ is a morphism obtained by blowing up a nonsingular irreducible toric center disjoint from U.*

This theorem essentially say that any two toric manifolds of the same dimension can be connected algebraically by a sequence of toric blow-ups and blow-downs. We use this theorem in Chapter 5 to reduce the proof of Theorem 5.1.1 to the case of one blow-up. Indeed our results in Chapter 5 can be viewed as differential geometric analogues of the above theorem.

2.5 Some metric geometry.

In this final section, we review some basic comparison geometry, and the notions of Gromov-Hausdorff distance and convergence. We fix (M, g) to be a complete m-dimensional Riemannian manifold inducing a distance function d. We start with the Myer's theorem and the Bishop-Gromov volume comparison theorems.

Theorem 2.5.1 (Myer's theorem)**.** *If $Ric(g) > (m-1)\lambda g$ for some $\lambda > 0$, then*

$$diam(M, g) < \frac{\pi}{\sqrt{\lambda}}$$

Theorem 2.5.2 (Bishop-Gromov volume comparison)**.** *If $Ric(\omega) > (m-1)\lambda g$ for some $\lambda \in \mathbb{R}$ and $\tilde{M}$ is the m-dimensional space form with constant sectional curvature λ. Then*

1. *If $K \subset M$ is any star convex set centered at x, then for $0 < r_1 < r_2 (< \pi/\sqrt{\lambda}$ if $\lambda > 0)$,*

$$\frac{Vol(B_d(x, r_2) \cap K) - Vol(B_d(x, r_1) \cap K)}{\tilde{V}(r_2) - \tilde{V}(r_1)} \leq \frac{Vol(\partial B_d(x, r_1) \cap K)}{Vol(\partial \tilde{B}(r_1))}$$

   where $\tilde{B}(r)$ is a ball of radius r in $\tilde{M}$ and $\tilde{V}(r) = Vol(\tilde{B}(r))$.

2. *For all $x \in M$, the volume ratio*

$$\frac{Vol(B_d(x, r))}{\tilde{V}(r)}$$

   is non-increasing in r.

Using the above theorem, one can prove a relative comparison lemma of Gromov's [38], which will be crucial in studying Gromov-Hausdorff convergence in Chapters 3 and 5.

Lemma 2.5.1. *Let (M, g), be a Riemannian manifold of dimension m and $T \subset M$ be any compact set with a smooth boundary, such that*

- $$Ric(g) > -Ag \ ; \ diam(M, g) < \Lambda$$
- *For some points $p_1, p_2 \in M$ with $B(p_j, \varepsilon) \cap T = \emptyset$ for $j = 1, 2$, every minimal geodesic from p_1 to points in $B(p_2, \varepsilon)$ intersects T.*

Then, there exists a constant $c = c(n, \varepsilon, A, \Lambda)$ such that

$$Vol(\partial T, g) > cVol(B(p_2, \varepsilon), g)$$

Next, we recall the definition of Gromov-Hausdorff distance. Let (Y, d_Y) and (Z, d_Z) be two compact metric spaces.

Definition 2.5.1. *A map $f : (Y, d_Y) \to (Z, d_Z)$ is said to be an ε-distortion if*

1. *$f(Y)$ is ε-dense in (Z, d_Z).*
2. *For all $y_1, y_2 \in Y$,*
$$|d_Z(f(y_1), f(y_2)) - d_Y(y_1, y_2)| < \varepsilon$$

Definition 2.5.2. *We define the Gromov-Hausdorff distance between (Y, d_Y) and (Z, d_Z) by*

$$d_{GH}((Y, d_Y), (Z, d_Z)) = \inf \{\varepsilon \mid \textit{there exist } \varepsilon\textit{-distortions } f : Y \to Z \textit{ and } g : Z \to Y\}$$

One can check that this defines a distance function on the set of all metric spaces, once isometric spaces are identified. One can then define convergence of metric spaces and it is a fact that the set of metric spaces is actually complete with respect to the Gromov-Hausdorff distance. For Riemannian manifolds, the fundamental compactness theorem is the following result of Gromov.

Theorem 2.5.3 (Gromov's compactness theorem)**.** *The space of compact Riemannian manifolds (M, g) of the same dimension, with $Ric(g) > -Ag$ and $diam(M, g) < \Lambda$, is pre-compact in the Gromov-Hausdorf topology.*

Chapter 3

Metrics with cone singularities along simple normal crossing divisors

3.1 Introduction.

Let $(X, \hat{\omega})$ be an n-dimensional Kähler manifold with a smooth Kähler metric $\hat{\omega}$. A model edge metric on $\mathbb{C}^n$ with cone angles $2\pi\beta_j \in (0, 2\pi]$ along $[z_j = 0]$ is given by

$$\omega_\beta = \frac{\sqrt{-1}}{2} \sum_{j=1}^{k} \beta_j^2 \, \frac{dz_j \wedge d\bar{z}_j}{|z_j|^{2(1-\beta_j)}} + \frac{\sqrt{-1}}{2} \sum_{j=k+1}^{n} dz_j \wedge d\bar{z}_j. \tag{3.1.1}$$

On X, we fix a divisor

$$D = \sum_{j=1}^{N} (1 - \beta_j) D_j \tag{3.1.2}$$

where $\beta_j \in (0, 1)$ and D_j's are irreducible smooth divisors. We further assume that D is a *simple normal crossing divisor* i.e for any $p \in Supp(D)$ lying in the intersection of exactly k divisors $D_1, \cdots, D_k$, there exists a coordinate chart $(U_p, \{z_j\})$ containing p, such that $D_j|_{U_p} = \{z_j = 0\}$ for $j = 1, \cdots, k$. We set $z_j = \rho_j e^{i\theta_j}$, $r_j = |z_j|^{\beta_j}$ for $j = 1, \cdots, k$ and $r_j = |z_j|$ otherwise, and consider the following $(1, 0)$ forms

$$\epsilon_j = \begin{cases} e^{\sqrt{-1}\theta_j}(dr_j + \sqrt{-1}\beta_j r_j d\theta_j) & ; \; j = 1, \cdots, k \\ e^{\sqrt{-1}\theta_j}(dr_j + \sqrt{-1} r_j d\theta_j) & ; \; j = k+1, \cdots, n \end{cases} \tag{3.1.3}$$

Then $\{\epsilon_i \wedge \bar{\epsilon}_j\}_{i,j=1}^n$ is a local orthogonal (with respect to ω_β) basis for the $(1, 1)$ forms.

Definition 3.1.1. *For $\gamma < \min_j\{\beta_j^{-1} - 1\}$, a $(1, 1)$ form η is said to be in $\mathcal{C}^{,\gamma,D}$ if*

1. *$\eta \in C^{,\gamma}$ on $X \backslash D$ with respect to any smooth metric.*

2. *It's restriction to U_p can be written as*

$$\eta\Big|_{U_p} = \sum_{i,j=1}^{n} \eta_{i\bar{j}} \; \epsilon_i \wedge \bar{\epsilon}_j$$

where, $\eta_{i\bar{j}} \in \mathcal{C}^{,\gamma}$ *with respect to the metric* ω_β *for each* i, j*, and* $\eta_{i\bar{j}} \to 0$ *as* $z_i \to 0$*, if* $i = 1, \cdots, k$*.*

A function φ *is said to be in* $\mathcal{C}^{2,\gamma,D}$ *if* $\sqrt{-1}\partial\overline{\partial}\varphi \in \mathcal{C}^{,\gamma,D}$*. If the divisor has only one component* $(1-\beta)D$ *we use the notation* $\mathcal{C}^{,\gamma,\beta}$ *and* $\mathcal{C}^{2,\gamma,\beta}$ *instead.*

With the restriction that $\gamma < \beta_j^{-1} - 1$, $C^\infty \subset C^{2,\gamma,D}$. Next, we adapt the following precise definition of conical Kähler metrics from [36] to the case of *snc* divisors.

Definition 3.1.2. *A Kähler current* ω *is said to be a cone metric with angles* $2\pi\beta_j$ *along* D_j *if*

1. ω *is a smooth Kähler metric in* $X \backslash D$
2. *Locally at a* $p \in D$*,* ω *is equivalent to the model edge metric i.e*
$$C^{-1}\omega_\beta < \omega|_{U_p} < C\omega_\beta$$
3. $\omega = \hat{\omega} + \sqrt{-1}\partial\overline{\partial}\varphi$ *for some* $\varphi \in \mathcal{C}^{2,\gamma,D}$*.*

If only the first two conditions are satisfied, it is said to be quasi isometric to a conical metric with angles $2\pi\beta_j$ *along* D_j*.*

In this chapter we will be concerned with the following singular complex Monge-Ampère equation

$$\begin{cases} (\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi)^n = \dfrac{e^{-\lambda\varphi}\Omega}{\prod_{j=1}^N |s_j|_{h_j}^{2(1-\beta_j)}} \\ \omega = \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \end{cases} \tag{3.1.4}$$

where $\lambda \in \mathbb{R}$, Ω is a smooth volume form, s_j is the defining function of D_j and h_j is a metric on the line bundle generated by D_j. By rescaling one can always assume that $\lambda = 1, 0, -1$. In the case that $\lambda = 0$, we impose an additional normalization that $\sup_M \varphi = 0$. The above equation arises when one considers the problem of prescribing the Ricci curvature of a conical metric. The Ricci curvature of ω satisfies

$$Ric(\omega) = \lambda\omega + \chi + [D] \tag{3.1.5}$$

where Ω and χ are related by

$$\sqrt{-1}\partial\overline{\partial}\log\Omega + \lambda\hat{\omega} + \chi = \sum_{j=1}^{N}(1-\beta_j)\sqrt{-1}\partial\overline{\partial}\log h_j \tag{3.1.6}$$

A detailed study of such equations was carried out by Yau in his seminal paper on the Calabi conjecture [82]. He proved that, when $\lambda = -1, 0$, there always exist unique, globally bounded solutions to the above equation. Moreover, any bounded solution to (3.1.4) (even when $\lambda = 1$) is smooth away from the divisor D. Since then, there has been an effort to understand the behavior of solutions close to the divisor.

More recently, for *smooth divisors* i.e when there is only one smooth component $(1-\beta)D$, very precise regularity results for solutions to the equation were obtained by Brendle [9] in the case when $\beta < 1/2$ and by Donaldson [36], Jeffres-Mazzeo-Rubinstein [46], and Chen-Donaldson-Sun [22, 23] for all $\beta \in (0,1]$. Unfortunately, many linear systems do not contain smooth divisors. So, for geometric applications, it is important to address the questions of regularity for non-smooth divisors. The following theorem, proved independently by Guenancia -Păun [39] and the author, with Jian Song [30], is the first step in this direction.

Theorem 3.1.1. *[30] Any solution ω to* (3.1.4) *is quasi-isometric (cf. definition 3.1.2) to a cone metric with angles $2\pi\beta_j$ along D_j.*

We next show that $X\backslash D$ is convex with respect to the metric ω. Since ω is smooth on $X\backslash D$, it defines a length functional $\mathcal{L}_\omega$ and in turn, a distance function

$$d_\omega(p,q) = \inf\{\mathcal{L}_\omega(\gamma) \mid \gamma : [0,1] \to X\backslash D \text{ piecewise smooth },\ \gamma(0) = p, \gamma(1) = q\}$$

By the above theorem, ω is locally equivalent to a model edge metric, and hence it is easily seen that the metric completion of $X\backslash D$ under this distance function is homeomorphic to X itself, and we set

$$(X,d) = \overline{(X\backslash D, d_\omega)}$$

We first prove an approximation theorem for ω, extending results of [22, 74] for conical Kähler-Einstein metrics in the Fano case.

Proposition 3.1.1. *[31] Let* $\omega = \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi$ *be a solution to* (3.1.4) *with* $\varphi \in PSH(X, \hat{\omega}) \cap L^\infty(X)$*. Then there exist uniform constants* $A, \Lambda \gg 1$*, and a sequence* $\omega_\eta \in [\omega]$ *of smooth Kähler metrics such that*

1. $Ric(\omega_\eta) > -A\omega_\eta$ *;* $diam(X, \omega_\eta) < \Lambda$

2. *As* $\eta \to 0$*,*

$$(X, \omega_\eta) \xrightarrow{d_{GH}} (X, d)$$

where (X, d) *is the metric completion of* $(X \backslash D, \omega|_{X \backslash D})$*.*

It should be noted that in the Fano case with $\chi = 0$ and a smooth pluri-canonical divisor D, Chen-Donaldson-Sun [22] and Tian [74], prove a much stronger result, namely one can approximate with the *same* Ricci lower bound as the conical metric. For such a result, it is of course necessary that X is Fano.

Next, recall that a unit-speed path $\gamma : [0, l] \to X$ joining p, q is said to be a *minimal geodesic* if $d(p, q) = l$. It is said to be a *limiting geodesic* if there exists a sub-sequence $\{\eta_j\}$ with ω_{η_j}-geodesics $\gamma^{\eta_j} : [0, l_j] \to X$ such that $l_j \to j$ and $\gamma^{\eta_j} \to \gamma$ point wise. Limiting geodesics can usually be found in abundance. We then have,

Theorem 3.1.2. *[31]* $X \backslash D \subset (X, d)$ *is geodesically convex, in the following sense: if any interior point of a limiting minimal geodesic lies in* $X \backslash D$*, then all the interior points must lie in* $X \backslash D$*.*

The theorem is proved by combining the above smoothening with the results of Colding-Naber [28] on the Hölder continuity of tangent cones for limit spaces. It must be noted that the theorem does not rule out the possibility of *some* geodesic connecting $p, q \in X \backslash D$ passing through D, though it is expected that such a scenario will not occur. A nice consequence of the above theorem, as can be seen from Corollary 3.4.2, is that between any two points of $X \backslash D$, there always exists a smooth minimal geodesic contained in $X \backslash D$.

Notation. Distances with respect to ω_η and ω are denoted by d_η, d respectively. Paths connecting points p, q are denoted by γ_{pq}. Minimal geodesics are denoted with superscripts to specify the reference metric. For example an d_η-minimal and d-minimal geodesics are denoted by γ^η_{pq} and γ^d_{pq} respectively.

3.2 Preliminaries.

Here we collect some facts about conical metrics along divisor with only one smooth component that will be needed in the rest of the chapter. Fix a divisor $(1-\beta)D$ on X, where $D = [s = 0]$ is a smooth divisor, and let h be a smooth Hermitian metric on the line bundle generated by D. The model edge metric in this case on $\mathbb{C}^n$ with angle $2\pi\beta$ along $[z_1 = 0]$ is given by

$$\omega_\beta = \frac{\sqrt{-1}}{2}\beta^2 \frac{dz_j \wedge d\bar{z}_j}{|z_j|^{2(1-\beta)}} + \frac{\sqrt{-1}}{2}\sum_{j=2}^{N} dz_j \wedge d\bar{z}_j. \tag{3.2.7}$$

Now, set

$$\mathcal{H}_{\gamma,\beta} = \{\varphi \in C^{2,\gamma,\beta} \mid \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}$$

Then $\omega_\varphi = \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi$ is a cone metric with angle $2\pi\beta$ along D for any $\varphi \in \mathcal{H}_{\gamma,\beta}$ and we denote this space of cone metrics by $\mathcal{K}_{\gamma,\beta}$.

Lemma 3.2.1. *[46] For any ϵ small enough,*

$$\eta = \hat{\omega} + \epsilon\sqrt{-1}\partial\bar{\partial}|s|^{2\beta}_h$$

belongs to $\mathcal{K}_{\gamma,\beta}$ for all $\gamma < \beta^{-1} - 1$. Moreover the bi-sectional curvature of η is bounded above, i.e there exists a constant $C > 0$ such that at any $p \in X\backslash D$ and any unit vectors $\nu, \xi \in T^{1,0}_p X$,

$$Rm_\eta(\nu, \bar{\nu}, \xi, \bar{\xi}) < C$$

The first part of the above lemma follows from a simple computation. For the second part, the reader should refer to the appendix in [46]. Next, consider the Monge-Ampère equation

$$\begin{cases} (\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi})^n = \frac{e^{-\lambda\tilde{\varphi}}\tilde{\Omega}}{|s|^{2(1-\beta)}_h} \\ \tilde{\omega} := \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi} > 0 \end{cases} \tag{3.2.8}$$

where $\tilde{\Omega}$ is a smooth volume form. Then we have the following regularity result.

Theorem 3.2.1. *[9, 46, 23] For all $\gamma < \beta^{-1} - 1$, any bounded solution to* (3.2.8) *is in* $C^{2,\gamma,\beta}$.

In fact, one can also prove that in suitable holomorphic coordinates around any $p \in D$, the metric $\tilde{\omega}$ is asymptotic to ω_β.

Proposition 3.2.1. *[23, Prop. 26] For any $\zeta > 0$, there exists $\bar{r} = \bar{r}(\zeta)$ and holomorphic coordinates such that*

$$(1-\zeta)\omega_\beta < \tilde{\omega} < (1+\zeta)\omega_\beta$$

on $B_{\tilde{\omega}}(p, \bar{r})$.

A remark on the proof is in order. In [23] this proposition is proved for conical Kähler-Einstein metrics. A key technical point is the observation that the conical re-scalings of $\tilde{\omega}$ defined by $\tilde{\omega}_\epsilon = \epsilon^{-2}T_\epsilon^*\tilde{\omega}$, where $T_\epsilon(z_1, \cdots, z_n) = (\epsilon^{1/\beta}z_1, \epsilon z_2, \cdots, \epsilon z_n)$, converge to a metric cone on $\mathbb{C}^n$. In the present context, by Theorem 3.1.1 one can approximate $\tilde{\omega}$ by smooth metrics with uniform Ricci lower bound. Then the convergence of the re-scalings to a metric cone is a consequence of general results of Cheeger-Colding [17]. We end this section with an estimate on the volume density of $\tilde{\omega}$ at a point on the divisor D. This is needed in the final section in the proof of geodesic convexity. The volume density for any Riemannian metric g on X is defined to be

$$V_g(p,r) := \frac{Vol_g(B_g(p,r))}{r^{2n}}$$

We start off with an elementary lemma.

Lemma 3.2.2. *Let ω_β denote the model edge metric on $\mathbb{C}^n$ with cone angle $2\pi\beta$ along $[z_1 = 0]$ given by* (3.2.7)*. Then for any $r > 0$,*

$$V_{\omega_\beta}(0,r) = \alpha(n)\beta$$

Proof. The ω_β-minimal geodesic connecting the origin to any $(z_1, z') := (z_1, z_2, \cdots, z_n) \in \mathbb{C}^n$ is given by $\gamma(t) = (t^{1/\beta}z_1, tz_2 \cdots, tz_n)$, and it is easily seen that $\mathcal{L}_{\omega_\beta}(\gamma) = |z_1|^{2\beta} +$

$|z'|^2$. So $B_{\omega_\beta}(0,r) = \{z \in \mathbb{C}^n \mid |z_1|^{2\beta} + |z'|^2 < r\}$. But then, using polar coordinates $z_j = \rho_j e^{i\theta_j}$, and the change of variables $u = \rho^{2\beta}$ in the third line,

$$
\begin{aligned}
r^{2n} V_{\omega_\beta}(0,r) &= \int_{B_{\omega_\beta}(0,r)} \frac{\omega_\beta^n}{n!} \\
&= \beta^2 (2\pi)^n \int_{\rho_1^{2\beta}+\cdots+\rho_n^2<r} \frac{(\rho_1 d\rho_1)(\rho_2 d\rho_2)\cdots(\rho_n d\rho_n)}{\rho_1^{2(1-\beta)}} \\
&= \beta (2\pi)^n \int_{u^2+\cdots+\rho_n^2<r} (udu)(\rho_2 d\rho_2)\cdots(\rho_n d\rho_n) \\
&= \beta \int_0^{2\pi} \cdots \int_0^{2\pi} \int_{u^2+\cdots+\rho_n^2<r} (udud\theta_1)(\rho_2 d\rho_2 d\theta_2)\cdots(\rho_n d\rho_n d\theta_n) \\
&= \beta \alpha(n) r^{2n}
\end{aligned}
$$

□

Finally, we prove that the volume density of $\tilde{\omega}$ at any point on the divisor is strictly bounded away from the Euclidean volume density. More precisely,

Lemma 3.2.3. *For any $p \in D$,*

$$\lim_{r\to 0} V_{\tilde{\omega}}(p,r) = \beta\alpha(n)$$

Proof. By Prop. 3.2.1, for all $\zeta > 0$, there exists an $\bar{r} > 0$ such that in some holomorphic coordinates centered at $p \in D$,

$$(1-\zeta)\omega_\beta < \tilde{\omega} < (1+\zeta)\omega_\beta$$

on $B_{\tilde{\omega}}(p,\bar{r})$. But then it is easy to see that

$$\Big(\frac{1-\zeta}{1+\zeta}\Big)^n V_\beta(0, \frac{r}{\sqrt{1+\zeta}}) < V_{\tilde{\omega}}(p,r) < \Big(\frac{1+\zeta}{1-\zeta}\Big)^n V_\beta(0, \frac{r}{\sqrt{1-\zeta}}); \qquad \forall r < \bar{r}$$

First letting $r \to 0$, then $\zeta \to 0$ and using Lemma 3.2.2 we complete the proof. □

3.3 C^2 estimates for cone metrics along simple normal crossing divisors.

In this section, we prove theorem 3.1.1. Recall that $\omega = \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi$ for some $\varphi \in PSH(X,\hat{\omega}) \cap L^\infty(X) \cap C^\infty(X\backslash D)$. The proof is essentially equivalent to obtaining certain second order estimates on the potential φ. Note that form the previous section, If s_j is the defining section of D_j and h_j is any smooth metric on the line bundle induced by $[D_j]$, then for sufficiently small $\epsilon_j > 0$, $\theta_j = \hat{\omega} + \epsilon_j\sqrt{-1}\partial\bar{\partial}|s_j|^{2\beta_j}_{h_j}$ gives a Kähler metric on $X\backslash D_j$ with cone angle $2\pi\beta_j$ along D_j. Now, set

$$\theta = \sum_{j=1}^{N}\theta_j \tag{3.3.9}$$

The same proof as that of Lemma 3.2.1 shows that θ is a conical metric with cone angles $2\pi\beta_j$ along D_j. Then, to prove Theorem 3.1.1 it suffices to prove the following

Proposition 3.3.1. *[39, 30] If φ is any bounded solution to* (3.1.4)*, then there exists a constant $C > 0$ such that*

$$C^{-1}\theta \leq \omega_\varphi \leq C\theta \tag{3.3.10}$$

on $X\backslash Supp(D)$.

Proof. We present the proof in [30]. The key idea is to smoothen out all but one divisor, and use the regularity results in the case of one smooth divisorial component.

Step-1: There exists a constant $a > 0$ such that, for any $j \in \{1, \cdots, N\}$

$$\omega \geq a\theta_j \tag{3.3.11}$$

on $X\backslash Supp(D)$.

We first assume that $\lambda = 1$. The proof in the other two cases is similar (cf. Remark 3.3.1). We set $f_j = \log\left(\prod_{i=1,\ldots,N, i\neq j}|s_i|^{2(1-\beta_i)}_{h_i}\right)$. Then for some constant $A >> 1$, $\sqrt{-1}\partial\bar{\partial}f_j > -A\omega$ as currents. By Demailly's regularization theorem [33], there exist functions $F_{j,k}, \psi_k \in C^\infty(X)$ such that $F_{j,k} \searrow f_j$, $\psi_k \searrow \varphi$ and $F_{j,k}, \psi_k \in PSH(X, A\hat{\omega})$. Now, consider the following family of Monge-Ampère equations

$$\begin{cases} (\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi_{j,k})^n = \frac{e^{(-\psi_k - F_{j,k} + c_{j,k})}\Omega}{|s_j|^{2(1-\beta_j)}_{h_j}} \\ \omega_{j,k} = \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi_{j,k} > 0 \end{cases} \tag{3.3.12}$$

It is well known [36, 8, 46, 23] that there always exists a solution $\varphi_{j,k}$ in $C^{2,\gamma,\beta_j}(X)$ for some $\alpha \in (0,1)$. Note that by integrating both sides of the equation, it is easy to see that the constants $c_{j,k}$ are uniformly bounded and converge to zero as $k \to \infty$.

Since $\beta_j \in (0,1)$, from (3.3.12) it is clear that $\omega_{j,k}^n/\Omega \in L^{1+\epsilon}(X,\Omega)$ for some $\epsilon > 0$ with uniform control over the $L^{1+\epsilon}$ norm. So, by Kolodziej's theorem [49], the solutions $\varphi_{j,k}$ are uniformly bounded in the C^0 norm. In fact, since from the equation $\omega_{j,k}^n/\Omega \to \omega_\varphi^n/\Omega$ in $L^1(X,\Omega)$, by the stability of solutions of complex Monge-Ampère equations [50], $|\varphi_{j,k} - \varphi|_{C^0(X)} \to 0$, as $k \to \infty$.

To obtain second order estimates, we note that $tr_{\omega_{j,k}}\theta_j$ is bounded since $\varphi_{j,k} \in C^{2,\gamma,\beta_j}(X)$, and so for any $\delta > 0$ and $B > 0$, the quantity

$$Q = \log\left(|s_j|_{h_j}^{2\delta} tr_{\omega_{j,k}}\theta_j\right) - B\left(\varphi_{j,k} - \epsilon_j |s_j|_{h_j}^{2\beta_j}\right) \tag{3.3.13}$$

attains its maximum value at some $p_{max} \in X \backslash Supp(D_j)$. Without loss of generality, we can assume that $|s_j|_{h_j} \leq 1$ on X. First, it follows from (3.3.12) and the fact that $\omega \leq c\theta$ for some $c > 0$, that there exists a uniform $C > 0$ such that $Ric(\omega_{j,k}) > -C\theta_j$. Next, the bisectional curvature of θ_j is bounded above [46]. Hence by the Chern-Lu inequality [26, 56, 82], there exist constants $B, C > 0$ independent of j, k and δ such that

$$\Delta_{\omega_{j,k}} Q \geq tr_{\omega_{j,k}}\theta_j - C.$$

By the maximum principle and the uniform C^0 estimates, there exists an $a > 0$ such that $tr_{\omega_{j,k}}\theta_j \leq a^{-1}/|s_j|_{h_j}^{2\delta}$ on X. Letting $\delta \to 0$,

$$\omega_{j,k} \geq a\theta_j.$$

Taking limit as $k \to \infty$ we prove that $\omega \geq a\theta_j$ as currents. But since ω is in fact smooth away from D, the inequality must be point-wise and this completes the first step.

Step-2: By adding the lower bounds from (3.3.11) for $j = 1, \cdots, N$, there exists $C > 0$ such that

$$\omega \geq C^{-1}\theta.$$

Since θ is locally equivalent to a cone metric with angle $2\pi\beta_j$ along D_j, it is easy to check that

$$\theta^n = \frac{\Omega'}{\prod_{j=1}^N |s_j|_{h_j}^{2(1-\beta_j)}}$$

for some continuous nowhere vanishing volume form Ω', i.e., θ^n and ω^n are uniformly equivalent on $X\backslash D$. Together with the lower bound on the metric, it directly gives the required upper bound on the metric.

$\square$

Remark 3.3.1. *The proof with* $\lambda = -1, 0$ *can be carried out exactly as above, and in fact is even simpler. By the work of Yau [83] and Aubin-Yau [3, 83] one does not need to approximate* φ *by* ψ_k *on the right hand side of* (3.3.12). *When* $\lambda = 0$, *as noted earlier, we need to impose additional normalization that* $\sup_X \varphi_{j,k} = 0$.

3.4 Smoothening and geodesic convextiy.

In this section, we construct an approximating sequence and prove Proposition 3.1.1. Once again by Demailly's regularization theorem [33], there exists a sequence $\psi_\eta \in C^\infty(X) \cap PSH(X, \hat{\omega})$ such that $\psi_\eta \searrow \varphi$ point wise as $\eta \to 0$. Note that all the ψ_η's are uniformly bounded in the L^∞ norm. The metrics ω_η are then constructed as the solutions to the following perturbation of (3.1.4)

$$(\hat{\omega} + \sqrt{-1}\partial\overline{\partial}\varphi_\eta)^n = \frac{e^{-\lambda\psi_\eta + c_\eta}\Omega}{\prod_{j=1}^N (|s_j|^2_{h_j} + \eta)^{(1-\beta_j)}} \tag{3.4.14}$$

where c_η is a constant such that the integrals on both sides match-up. By Yau's work on the Calabi conjecture [83], there always exists a smooth solution to the above equation for $\eta > 0$. It is easy to see that $|c_\eta|$ is uniformly bounded, and in fact tends to zero as $\eta \to 0$.

Lemma 3.4.1. *If* $\omega_\eta = \hat{\omega} + \sqrt{-1}\partial\overline{\partial}\varphi_\eta$ *is a solution to* (3.4.14), *then there exists an* $A \gg 1$ *such that*

$$Ric(\omega_\eta) > -A\hat{\omega}$$

Proof. We follow the computation in [22]. First observe that for any smooth $f > 0$

$$\sqrt{-1}\partial\overline{\partial}\log(f+\eta) \geq \frac{f}{f+\eta}\sqrt{-1}\partial\overline{\partial}\log f$$

So, on $X\backslash D$

$$Ric(\omega_\eta) = \lambda\sqrt{-1}\partial\overline{\partial}\psi_\eta - \sqrt{-1}\partial\overline{\partial}\log\Omega + \sum_{j=1}^N (1-\beta_j)\sqrt{-1}\partial\overline{\partial}\log(|s_j|^2_{h_j} + \eta)$$

$$\geq -\lambda\hat{\omega} - \sqrt{-1}\partial\overline{\partial}\log\Omega + \sum_{j=1}^{N}(1-\beta_j)\frac{|s_j|^2_{h_j}}{(|s_j|^2_{h_j}+\eta)}\sqrt{-1}\partial\overline{\partial}\log|s_j|^2_{h_j}$$

$$= -\sum_{j=1}^{N}(1-\beta_j)\frac{\eta}{(|s_j|^2_{h_j}+\eta)}\sqrt{-1}\partial\overline{\partial}\log h_j + \chi \qquad \text{(by equation 3.1.6)}$$

$$\geq -A\hat{\omega}$$

if for instance $-\sqrt{-1}\partial\overline{\partial}\log h_j > -A\hat{\omega}/2N$, and $\chi > -A\hat{\omega}/2$. We also use the fact that $1-\beta_j \geq 0\ \forall j$ in the second line. So this argument will not work if the divisor is not effective. □

Next, we obtain uniform C^0 and C^2 estimates on φ_η.

Proposition 3.4.1. *There exists a constant* $C = C(n, A, ||\omega^n/\Omega||_{L^{1+\delta}(X,\Omega)}, ||Rm(\hat{\omega})||) \gg 1$ *independent of* η*, such that*

1.

$$||\varphi_\eta||_{C^0(X)} < C$$

2.

$$C^{-1}\hat{\omega} < \omega_\eta < \frac{C\hat{\omega}}{\prod_{j=1}^{N}(|s_j|^2_{h_j}+\eta)^{(1-\beta_j)}}$$

Proof. The proof is standard. The right hand side of equation (3.4.14) is uniformly in $L^{1+\varepsilon}(X,\omega)$ for some $\varepsilon > 0$, since all the β_j's are strictly positive, ψ_η, $|c_\eta|$ are uniformly bounded, and D is a simple normal crossing divisor. The C^0 estimate now follows directly from the work of Kolodziej [49, 50]. For the C^2 estimate, we consider the following quantity :

$$Q = \log \mathrm{tr}_{\omega_\eta}\hat{\omega} - B\varphi_\eta \qquad (3.4.15)$$

By Lemma 3.4.1, $Ric(\omega_\eta) > -A\hat{\omega}$ for some $A \gg 1$. Then by the Chern-Lu inequality, there exist constants $B, C \gg 1$ depending on A, the dimension n, and an upper bound for the bisectional curvature of $\hat{\omega}$, such that

$$\Delta_\eta Q \geq \mathrm{tr}_{\omega_\eta}\hat{\omega} - C$$

By maximum principle and uniform C^0 estimates,

$$\mathrm{tr}_{\omega_\eta}\hat{\omega} \leq C$$

But then using the equation (3.4.14), and an elementary arithmetic-geometric mean inequality

$$\begin{aligned}\mathrm{tr}_{\hat{\omega}}\omega_\eta &\leq (\mathrm{tr}_{\omega_\eta}\hat{\omega})^{n-1}\frac{\omega_\eta^n}{\hat{\omega}^n} \\ &\leq \frac{C}{\prod_{j=1}^N (|s_j|_{h_j}^2+\eta)^{(1-\beta_j)}}\end{aligned}$$

□

As a straightforward corollary

Corollary 3.4.1. *If the ω_η are solutions to* (3.4.14) *then there exists a constant $A, \Lambda > 0$ such that*

1. *$Ric(\omega_\eta) > -A\omega_\eta$; $diam(X, \omega_\eta) < \Lambda$.*
2. *$\omega_\eta \xrightarrow{C^\infty_{loc}(X\backslash D)} \omega$ on $X\backslash D$ as $\eta \to 0$.*
3. *For all open sets $U \subset X$,*

$$Vol(U, \omega_\eta) \xrightarrow{\eta\to 0} Vol(U, \omega)$$

The proof of the Gromov-Hausdorff convergence follows along the lines of [86, 74, 29]. The main technical ingredient is the following relative comparison lemma of Gromov [38].

The next lemma proves that for almost all points, $X\backslash D$ is geodesic convex. This was proved by Cheeger-Colding [18] for Gromov-Hausdorff limits of Riemannian manifolds with a Ricci lower bound. In our case, we haven't yet identified the Gromov-Haudorff limit with X, and so we give an elementary proof using the above comparison lemma and smooth convergence on $X\backslash D$.

Lemma 3.4.2. *Let $K \subset\subset X\backslash D$, and $d(\partial K, D) > 4\varepsilon$. Then there exists a $\delta = \delta(n, \varepsilon, A, L)$, such that if T is a neighborhood of D in $X\backslash K$ with $d(p, \partial T) > 2\varepsilon\ \forall\ p \in K$, and $Vol_d(\partial T) < \delta$, then, for all $p, q \in K$, there exists a $q' \in B_d(q, \varepsilon)$ and a minimal d-geodesic $\gamma^d_{pq'} : [0, 1] \to X\backslash T$ connecting p to q'.*

Proof. ***Claim 1:*** For η small,

$$B_{d_\eta}(q, \varepsilon/2) \subset B_d(q, \varepsilon)$$

.

Suppose not, then for arbitrarily small η, there exists a $x \in X$ such that $d_\eta(q, x) < \varepsilon/2$, but $d(q, x) > \varepsilon$. The minimal η-geodesic γ^η_{qx} has a first point of contact $\tilde{x} \in \partial B_d(q, \varepsilon)$. Then $\mathcal{L}_d(\gamma^\eta_{q\tilde{x}}) \geq \varepsilon$, and hence $d_\eta(q, \tilde{x}) = \mathcal{L}_\eta(\gamma_{q,\tilde{x}}) > 3\varepsilon/4$ if η is sufficiently small, by uniform smooth convergence on $X \backslash T$ and the fact that γ^η_{qx} is minimal. This is a contradiction and the claim is proved.

Claim-2: For η small, and any $p, q \in K$, there exists $q_\eta \in B_{d_\eta}(q, \varepsilon/2)$ and a minimal d_η-geodesic $\gamma^\eta_{pq_\eta} : [0, 1] \to X \backslash T$.

If not, then by Lemma 2.5.1, uniform non-local collapsing and volume convergence (cf. Lemma 3.4.1),

$$c\kappa\varepsilon^{2n} \leq cVol(B_{d_\eta}(q, \varepsilon/2)) \leq Vol_{\omega_\eta}(\partial T) \leq 2Vol_\omega(\partial T) \leq 2\delta$$

Pick $\delta = c\kappa\varepsilon^{2n}/4$ to get a contradiction.

So there is a sequence of points $q_\eta \in B_{d_\eta}(q, \varepsilon/2) \subset B_d(q, \varepsilon)$ and η-minimal geodesics $\gamma^\eta_{pq_\eta} \subset X \backslash T$. By compactness and smooth convergence away from D, there exists a $q' \in B_\beta(q, \varepsilon)$ and a limiting d-geodesic $\gamma_{pq'} : [0, 1] \to X \backslash T$ from p to q'.

Claim-3: $\gamma_{pq'}$ is d-minimal. i.e

$$\mathcal{L}_d(\gamma_{pq'}) = d(p, q')$$

.

If not, then by definition of d, there exists a path $\tilde{\gamma}_{pq'} : [0, 1] \to X \backslash D$ such that $\mathcal{L}_d(\tilde{\gamma}_{pq'}) < \mathcal{L}_d(\gamma_{pq'}) - \zeta$, for some $\zeta > 0$. For η small, $d(q', q_\eta) < \zeta/8$. The minimal d-geodesic $\gamma^d_{q'q_\eta}$ doesn't hit ∂T. So once again by smooth convergence $\mathcal{L}_{d_\eta}(\gamma^d_{q'q_\eta}) < \zeta/4$. On the other hand, for η small,

$$\mathcal{L}_{d_\eta}(\tilde{\gamma}_{pq'}) < \mathcal{L}_d(\tilde{\gamma}_{pq'}) + \zeta/8 < \mathcal{L}_d(\gamma_{pq'}) - 7\zeta/8 < \mathcal{L}_{d_\eta}(\gamma^\eta_{pq_\eta}) - 6\zeta/8$$

So the concatenation $\tilde{\gamma}_{pq'} \cdot \gamma^d_{q'q_\eta}$ is a path from p to q_η with length $\mathcal{L}_{d_\eta}(\tilde{\gamma}_{pq'} \cdot \gamma^d_{q'q_\eta}) < \mathcal{L}_{d_\eta}(\gamma^\eta_{pq_\eta}) - 6\zeta/8 + \zeta/4 = \mathcal{L}_{d_\eta}(\gamma^\eta_{pq_\eta}) - \zeta/2$, contradicting the minimality of $\gamma^\eta_{pq_\eta}$. Hence $\mathcal{L}_d(\gamma_{pq'}) = d(p, q')$. □

Proof of Proposition 3.1.1. Fix a small $\varepsilon > 0$, and choose a tubular neighborhood E of D such that $K = X \backslash E$ is ε-dense with respect to the distance d and $Vol(E, \omega) < \varepsilon^{4n}$. We denote the distances with respect to ω and ω_η by d and d_η respectively. The proof of the Gromov-Hausdorff convergence is completed in two steps:

Claim-1: There exists a $\eta_0 = \eta_0(\varepsilon) > 0$ such that $\forall \eta < \eta_0$, K is ε-dense with respect to d_η.

Proof. If not, then there exists a sequence $p_\eta \in E$ such that $B_{d_\eta}(p_\eta, \varepsilon) \subset E$. Using volume comparison, diameter bound and the fact that volumes of balls converge, for some uniform $\kappa > 0$ and η small,

$$\kappa\varepsilon^{2n} < Vol_{\omega_\eta}(B_{d_\eta}(p_\eta, \varepsilon)) < Vol_{\omega_\eta}(E) < 2Vol_\omega(E) < 2\varepsilon^{4n}$$

which is a contradiction if ε is small.

Claim-2: There exists a $\eta_0 = \eta_0(\varepsilon) > 0$ such that $\forall \eta < \eta_0$ and for all $p, q \in K$,

$$|d_\eta(p, q) - d(p, q)| < \varepsilon$$

Proof. Let $\tilde{\varepsilon} = d_\beta(\partial K, D)/4$, so that in particular $\tilde{\varepsilon} < \varepsilon/4$. We first claim that a neighborhood T of D can be chosen with $Vol(\partial T, \omega)$ arbitrarily small. For a unit polydisc in $\mathbb{C}^n$ with a model edge metric with cone angle $2\pi\beta_j$ along $[z_j = 0]$, such a neighborhood can be constructed explicitly. One can then glue together these local neighborhoods to obtain a neighborhood of D in X with the volume of the boundary arbitrarily small. In particular one can construct a T such that $d(\partial T, K) > \tilde{\varepsilon}/2$ and $Vol(\partial T, \omega) < \delta$ where $\delta = \delta(n, \varepsilon, A, L)$ is the constant in Lemma 3.4.2.

Next, by Lemma 3.4.2, for all $p, q \in K$ there exists $q' \in B_d(q, \tilde{\varepsilon})$ and a minimal d-geodesic $\gamma^d_{pq'} \subset X \backslash T$. Like in the argument for the proof of Lemma 3.4.2, for η small,

$d_\eta(q,q') < 2\tilde{\varepsilon}$. Then by uniform smooth convergence on $X\backslash T$, there exists $\eta_0 > 0$ such that for $\eta < \eta_0$ and all $p, q \in K$,

$$d_\eta(p,q) < \mathcal{L}_\eta(\gamma^d_{pq'}) + d_\eta(q,q') < \mathcal{L}_d(\gamma^d_{pq'}) + 3\tilde{\varepsilon} = d(p,q') + 3\tilde{\varepsilon} < d(p,q) + 4\tilde{\varepsilon} < d(p,q) + \varepsilon$$

On the other hand, recall that $\gamma^d_{pq'}$ is constructed as the limit of η-minimal geodesics $\gamma^\eta_{pq_\eta}$ with $q_\eta \in B_{d_\eta}(q, \tilde{\varepsilon}/2) \subset B_d(q, \tilde{\varepsilon})$, and $q_\eta \to q'$. So,

$$d(p,q) < d(p,q') + \tilde{\varepsilon} < \mathcal{L}_{d_\eta}(\gamma^\eta_{pq_\eta}) + 2\tilde{\varepsilon} = d_\eta(p,q_\eta) + 2\tilde{\varepsilon} < d_\eta(p,q) + 5\tilde{\varepsilon}/2 < d_\eta(p,q) + \varepsilon$$

We can now complete the proof of the theorem. For small $\eta > 0$,

$$\begin{aligned} & d_{GH}((X,d_\eta),(X,d)) \\ \leq\ & d_{GH}((X,d_\eta),(K,d_\eta)) + d_{GH}((K,d_\eta),(K,d)) + d_{GH}((K,d),(X,d)) \\ <\ & 3\varepsilon, \end{aligned}$$

□

Theorem 3.1.2 follows from a theorem of Colding-Naber [28]. By Proposition 3.1.1, (X,d) is the Gromov-Hausdorff limit of smooth Riemannian metrics. The crucial point in proving convexity is that the regular set in the sense of Cheeger-Colding [17] coincides with $X\backslash D$, and hence is open. To prove this, we need to show that the volume density of balls in (X,d) centered on the divisor is strictly less than the Euclidean volume density. We do this by reducing to the case of a smooth divisor (i.e when $N = 1$), and using Lemma 3.2.3.

Proposition 3.4.2. *There exists an $\zeta > 0$ and $r(\zeta) > 0$, such that for any $r < r(\zeta)$, and any $p \in D$,*

$$V_d(p,r) := \frac{Vol(B_d(p,r))}{r^{2n}} < (1-\zeta)\alpha(n)$$

where $\alpha(n) = \pi^n/n!$ is the volume of the unit Euclidean ball in $\mathbb{C}^n$.

Proof. We first smoothen out all but one divisor. Without loss of generality, let $p \in D_1$, and consider the equation

$$\begin{cases} \omega_\epsilon^n = \frac{e^{-\lambda\psi_\epsilon - f_\epsilon + c_\epsilon}\Omega}{|s_1|_{h_1}^{2(1-\beta_1)}} \\ \omega_\epsilon = \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi_\epsilon > 0 \end{cases}$$

where ψ_ϵ is the sequence approximating φ from section 2, $f_\epsilon = \log\left(\prod_{j=2}^N (|s_j|_{h_j}^2 + \epsilon)^{(1-\beta_j)}\right)$ and c_ϵ is a constant such that the integrals match up. By Prop. 3.1.1 there exists a sequence $\omega_{\epsilon,\eta}$ of smooth Kähler metrics and constants A and L such that

$$Ric(\omega_{\epsilon,\eta}) > -A\omega_{\epsilon,\eta} \ ; \ diam(X, \omega_{\epsilon,\eta}) < L$$
$$\omega_{\epsilon,\eta} \xrightarrow{C^\infty_{loc}(X\backslash D_1)} \omega_\epsilon$$
$$(X, \omega_{\epsilon,\eta}) \xrightarrow{d_{GH}} (X, \omega_\epsilon)$$

By the Bishop-Gromov comparison theorem for the metrics $\omega_{\epsilon,\eta}$ and Colding's convergence theorem [27], for $r' < r$

$$\frac{V_{\omega_\epsilon}(p,r)}{V_{-A}(\tilde{p},r)} \leq \frac{V_{\omega_\epsilon}(p,r')}{V_{-A}(\tilde{p},r')}$$

where $V_{-A}(\tilde{p},r)$ is the volume ratio for the space form of constant sectional curvature $-A/(2n-1)$. Taking $r' \to 0$, by Lemma 3.2.3

$$\begin{aligned} V_{\omega_\epsilon}(p,r) &\leq \beta_1 V_{-A}(\tilde{p},r) \\ &\leq \frac{1+\beta_1}{2}\alpha(n) \end{aligned}$$

if $r < \bar{r} = \bar{r}(A)$. Moreover, since the Ricci lower bounds for $\omega_{\epsilon,\eta}$ are uniform, by an elementary diagonalization argument, $\omega_\epsilon \xrightarrow{d_{GH}} \omega$ as $\omega_\epsilon \to 0$. Then once again by Coldings theorem on volume convergence

$$V_\omega(p,r) < \frac{1+\beta_1}{2}\alpha(n)$$

This proves the proposition with $\zeta = \frac{1}{2}max((1-\beta_1), \cdots, (1-\beta_N))$ and $r(\zeta) = \bar{r}$. $\square$

Since (X, d) is the Gromov-Hausdorff limit of (X, ω_η), one can talk about the regular set, in the sense of Cheeger-Colding. It is defined as

$$\mathcal{R} = \{p \in X \mid (X, r_j^{-2}d, p) \xrightarrow{d_{GH}} (\mathbb{R}^n, d_{euc}, 0) \text{ for any sequence } r_j \to 0\}$$

Lemma 3.4.3. *$\mathcal{R}$ is open and dense in (X, d).*

Proof. Let $\mathcal{R}$ be the regular set in Gromov-Hausdorff limit (X, d), in the sense of Cheeger-Colding i.e for all $p \in X$, $(X, r_j^{-2}d, p) \xrightarrow{d_{GH}} (\mathbb{C}^n, d_{eucl}, 0)$. By smooth convergence away from D, it is clear that $X \backslash D \subset \mathcal{R}$. On the other hand, suppose $p \in \mathcal{R}$, then by Colding's volume convergence, there exist small $r > 0$ such that the volume density $V_d(p, r)$ is arbitrarily close to the $\alpha(n)$, but then p cannot belong to D since this would contradict with Proposition 3.4.2. Hence $\mathcal{R} = X \backslash D$, and is consequently open. The denseness follows from the fact that $X \backslash D$ has full measure. □

Proof of Theorem 3.1.2. We follow the line of argument in [28]. By Colding and Naber's result on the Hölder continuity of the tangent cones of limiting spaces of sequences with a Ricci lower bound [28, Cor. 1.5], the set of regular points in the interior of a limiting geodesic is closed. On the other hand, by the above lemma, this set is also open. Therefore, as soon as one interior point lies in $X \backslash D$, all must, and the theorem is proved. □

Corollary 3.4.2. *Let $p, q \in X \backslash D$ with $l = d(p, q)$. Then there exists a smooth unit speed geodesic $\gamma : [0, l] \to X \backslash D$ with $\gamma(0) = p$ and $\gamma(l) = q$.*

Proof. For every $\eta > 0$, there exists a unit speed η-minimal geodesic $\gamma^\eta : [0, l_\eta] \to X$ connecting p and q with $l_\eta \to l$. By the Ascoli-Arzela theorem for Gromov-Haudorff limits, there exists a continuous limiting geodesic $\gamma : [0, l] \to X$ connecting p and q. By Theorem 3.1.2, γ stays away from D. For any $\gamma(t_0)$ with, there exists a small ball $B_d(\gamma(t_0), \epsilon) \subset X \backslash D$. By the argument of Claim-1 in the proof of Lemma 3.4.2 $B_{d_\eta}(\gamma(t_0), \epsilon/2) \subset B_d(\gamma(t_0), \epsilon)$ for η small enough. By convergence of geodesics and the fact that the geodesics are of unit speed, there exists a δ such that, for η small enough $\gamma^\eta(t) \in B_d(\gamma(t_0), \epsilon)$ for all $|t - t_0| < \delta$. By smooth convergence, it is easily seen that $\gamma|_{(t_0 - \delta, t_0 + \delta)}$ must be smooth, and hence the entire γ must be smooth. □

3.5 Comparison theorems for Kähler metrics with cone singularities.

For this section we fix D to be a simple normal crossing divisor given by (3.1.2), and ω to be a conical Kähler metric along D inducing the metric d on X. The aim of this section is to present extensions of some classical comparison theorems to this singular setting. The crucial point is that the cut locus has measure zero. This is already proved in [40, 28]. For the convenience of the reader, we offer an elementary proof in the conical case exploiting smooth convergence away from the divisor.

For a point $p \in X \backslash D$, let

$$\mathcal{E}_p = \{v \in T_pX \mid \exists \text{ geodesic } \gamma : [0,1] \to X\backslash D \text{ with } \gamma(0) = p \ , \ \gamma'(0) = v\}$$

The exponential map is well defined and smooth on $\mathcal{E}$. The following lemma follows directly from Theorem 3.1.2.

Lemma 3.5.1. *$exp_p : \mathcal{E}_p \to X\backslash D$ is surjective.*

We define the cut locus and conjugate locus int he usual way.

Definition 3.5.1. *1. For a $p \in X\backslash D$, the cut locus is defined by*

$$\mathcal{C}_p = \{x \in X \mid d(p,x) + d(x,z) > d(p,z) \ \forall z \in X\}$$

2. The conjugate locus is defined by

$$Conj(p) = \{x \in X\backslash D \mid \exists \ v \in exp_p^{-1}(x) \text{ such that } exp_p \text{ is singular at } v\}$$

Furthermore, $x = \gamma(t_0)$ is said to be conjugate to p along a unit speed geodesic $\gamma : [0,l] \to X\backslash D$ if exp_p is singular at $v = t_0\gamma'(0)$.

The following useful characterization of the cut locus from standard Riemannian geometry [15] also extends to this setting.

Lemma 3.5.2. *Let $\gamma : [0,l] \to X\backslash D$ be a smooth unit-speed geodesic emanating from p. Then $x = \gamma(t_0) \in \mathcal{C}_p$ if and only if one of the following holds at $t = t_0$ and neither holds for any smaller value of t :*

1. *x is conjugate to p along γ.*

2. *There exists a unit speed minimal limiting geodesic $\sigma \neq \gamma$ connecting p and x.*

Proof. Suppose $\gamma(t_0) \in \mathcal{C}_p$, $\epsilon_j \to 0$ and $\sigma_j : [0, l_j] \to X \backslash D$ be a unit speed smooth limiting minimal geodesic connecting p to $x_j = \gamma(t_0 + \epsilon_j)$. By continuity of the distance function, $l_j \to t_0$. By the same argument as in the proof of Cor. 3.4.2, one can show that there exists a $\sigma : [0, t_0] \to X \backslash D$ connecting p and x such that $\sigma_j \to \sigma$ smoothly. If $\sigma \neq \gamma$, criteria (2) is satisfied. If not, then arbitrarily small neighborhoods of $t_0\gamma'(0)$ in $\mathcal{E}_p$ have two distinct vectors, namely $l_j\sigma_j'(0)$ and $(t_0 + \epsilon_j)\gamma'(0)$, mapped to the same point x_j under the exponential map. By the inverse function theorem, $t_0\gamma'(0)$ is a singular point of $\exp_p$ or equivalently x is a conjugate point along γ. $\square$

As an immediate corollary we have

Corollary 3.5.1. *$\mathcal{C}_p$ has measure zero with respect to ω.*

Proof. From the previous Lemma $\mathcal{C}_p \subset \{$singular values of $exp_p\} \cup \{r$ is not differentiable$\}$. The first one has measure zero by Sard's theorem, while the second one has measure zero because r is Lipshitz. $\square$

Definition 3.5.2. *We say that*

$$Ric(\omega) > -A\omega$$

if there exists a smooth positive closed (1,1) form χ such that

$$Ric(\omega) + A\omega = \chi + [D]$$

We now present some classical comparison theorems. We also recall the proofs to emphasize that geodesic convexity, even of the slightly weaker kind proved in this note, is all that is needed for the extensions to the conical setting.

Theorem 3.5.1 (Laplacian comparison)**.** *Suppose $Ric(\omega) > (2n-1)\lambda\omega$ for some $\lambda \in \mathbb{R}$, and $\tilde{X}$ is the $2n$-dimensional space form with constant sectional curvature λ. Let $r(x)$*

and $\tilde{r}(\tilde{x})$ be distance functions to some fixed points in X and $\tilde{X}$ respectively. Then for any $x \in X\backslash D$ where r is smooth, and any $\tilde{x} \in \tilde{X}$ where $\tilde{r}$ is smooth with $r(x) = \tilde{r}(\tilde{x})$,

$$\Delta r(x) \leq \tilde{\Delta}\tilde{r}(\tilde{x})$$

Proof. By Bochner formula,

$$\begin{aligned} 0 &= |\nabla^2 r|^2 + \frac{\partial(\Delta r)}{\partial r} + Ric(\nabla r, \nabla r) \\ &\geq (\Delta r)^2 + \frac{\partial(\Delta r)}{\partial r} + (2n-1)\lambda \end{aligned}$$

Note that equality holds in the case of $\tilde{X}$. So, if $\gamma \subset X\backslash D$ and $\tilde{\gamma}$ are unit speed minimal geodesics joining the reference points to x and $\tilde{x}$ respectively, then $u(t) = \Delta r(\gamma(t)) - \tilde{\Delta}\tilde{r}(\tilde{\gamma}(t))$ satisfies the differential inequality

$$\dot{u} + gu \leq 0$$

where $g = \Delta r(\gamma(t)) + \tilde{\Delta}\tilde{r}(\tilde{\gamma}(t))$. Moreover

$$\lim_{t\to 0} |\Delta r(\gamma(t)) - \Big(\frac{2n-1}{t}\Big)| = \lim_{t\to 0} |\tilde{\Delta}\tilde{r}(\tilde{\gamma}(t)) - \Big(\frac{2n-1}{t}\Big)| = 0$$

i.e $u(0) = 0$. By the method of integrating factors for first order ODEs, it is easily seen that $u(t) \leq 0 \ \forall \ t$.

□

Theorem 3.5.2 (Myer's theorem)**.** *With D as above, suppose ω is a conical Kähler metric along D satisfying $Ric(\omega) > (2n-1)\lambda\omega$ for some $\lambda > 0$. Then*

$$diam(X, d) < \frac{\pi}{\sqrt{\lambda}}$$

Proof. By explicit calculation, if $\lambda > 0$, and $\tilde{X}$ is the space form with sectional curvature λ, then along a unit speed minimal geodesic $\tilde{\gamma}$,

$$\tilde{\Delta}\tilde{r}(\tilde{\gamma}(t)) = (2n-1)\sqrt{\lambda}\frac{\cos(\sqrt{\lambda}t)}{\sin(\sqrt{\lambda}t)}$$

Fix a point $p \in X\backslash D$. For any other point $x \in X\backslash D$, if γ is the minimal unit speed geodesic joining them, then

$$\Delta r(\gamma(t)) \leq (2n-1)\sqrt{\lambda}\frac{\cos(\sqrt{\lambda}t)}{\sin(\sqrt{\lambda}t)}$$

Since right hand side goes to $-\infty$ as $t \to \pi/\sqrt{\lambda}$, t, and hence the length of γ, can be at most $\pi/\sqrt{\lambda}$. $\square$

Next, the exponential map is a diffeomorphism from an open subset of $\mathcal{E}_p$ onto $X \backslash (D \cup C_p)$. Moreover, since $C_p \cup D$ has measure zero, standard arguments as in [**?**] can be used to prove the Bishop-Gromov volume comparison.

Theorem 3.5.3 (Bishop-Gromov volume comparison)**.** *If $Ric(\omega) > (2n-1)\lambda\omega$ for some $\lambda \in \mathbb{R}$ and $\tilde{X}$ is the $2n$-dimensional space form with constant sectional curvature λ. Then*

1. *If $K \subset X \backslash D$ is any star convex set centered at x, then for $0 < r_1 < r_2 (< \pi/\sqrt{\lambda}$ if $\lambda > 0)$,*

$$\frac{Vol(B_d(x,r_2) \cap K) - Vol(B_d(x,r_1) \cap K)}{\tilde{V}(r_2) - \tilde{V}(r_1)} \leq \frac{Vol(\partial B_d(x,r_1) \cap K)}{Vol(\partial \tilde{B}(r_1))}$$

   where $\tilde{B}(r)$ is a ball of radius r in $\tilde{X}$ and $\tilde{V}(r) = Vol(\tilde{B}(r))$.

2. *For all $x \in X$, the volume ratio*

$$V(x,r;\lambda) := \frac{Vol(B_d(x,r))}{\tilde{V}(r)}$$

   is non-increasing in r.

Remark 3.5.1. *As a corollary to Theorem 3.5.3 above, Lemma 2.5.1 generalizes to conical metrics satisfying equation* (3.1.5)*, and in particular to conical Kähler-Einstein metrics. This will be very useful in Chapter 5 to study degenerations of toric conical KE metrics.*

Chapter 4

Conical Soliton Metrics on Toric Manifolds

4.1 Introduction.

After the generalities of the previous chapter, we now turn to the study of conical Kähler-Einstein metrics and conical Kähler-Ricci solitons on toric manifolds. Recall that when X is a Fano manifold, Donaldson [36] proposed to study the conical Kähler-Einstein equation

$$Ric(\omega) = \beta\omega + (1-\beta)[D], \tag{4.1.1}$$

where D is smooth simple divisor in the anticanonical class $[-K_X]$ and $\beta \in (0,1)$.

The solvability of equation (4.1.1) is closely related to the following holomorphic invariant for Fano manifolds which is known as the greatest Ricci lower bound first introduced by Tian in [70].

Definition 4.1.1. *Let X be a Fano manifold. The greatest Ricci lower bound $R(X)$ is defined by*

$$R(X) = \sup\{\beta \mid Ric(\omega) \geq \beta\omega,\ \textit{for some } \omega \in c_1(X) \cap \mathcal{K}(X)\}, \tag{4.1.2}$$

where $\mathcal{K}(X)$ is the space of all Kähler metrics on X.

It is proved by Szekelyhidi in [67] that $[0, R(X))$ is the maximal interval for the continuity method to solve the Kähler-Einstein equation on a Fano manifold X. In particular, it is independent of the choice for the initial Kähler metric when applying the continuity method. The invariant $R(X)$ is explicitly calculated for $\mathbb{P}^2$ blown up at one point by Szekelyhidi [67], and for all toric Fano manifolds by Li [52]. Recent results [53] show that $R(X) = 1$ if and only if X is K semi-stable, and such a Fano manifold satisfies the Chern-Miyaoka inequality [66]. It is shown in [66, 54] that (4.1.1) cannot be solved for $\beta > R(X)$, answering a question of Donaldson [36] while it can always be solved for $\beta \in (0, R(X))$ if one replace D by a smooth divisor in the pluri-anticanonical system of X. In this paper, we will give various generalizations of the greatest Ricci lower bound.

The Bakry-Emery Ricci curvature on a Riemannian manifold (M, g) is defined by

$$Ric_f(g) = Ric(g) + Hess f$$

for a smooth real valued function f on M [4]. If (M, g, f) satisfies the equaiton $Ric_f(g) = \lambda g$ for some $\lambda \in \mathbb{R}$, it is called a gradient Ricci soliton with the gradient vector field $V = \nabla f$. We can define the greatest Bakry-Emery-Ricci lower bound on Fano manifolds as an analogue of the greatest Ricci lower bound.

Definition 4.1.2. *Let X be a Fano manifold. The greatest Bakry-Emery-Ricci lower bound $R_{BE}(X)$ is defined by*

$$R_{BE}(X) = \sup\{\beta \mid Ric(\omega) \geq \beta\omega + \mathcal{L}_{Re\xi}\omega, \textit{for some } \omega \in c_1(X) \cap \mathcal{K}(X) \textit{ and } \xi \in H^0(X, TX)\}.$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ.

Since X is Fano, it is simply connected and $\mathcal{L}_{Re\xi}\omega = -\sqrt{-1}\partial\overline{\partial} f_\xi$ for some real-valued smooth function f_ξ with $\nabla_{z_i}\nabla_{z_j} f_\xi = 0$ in holomorphic coordinates. This implies that $Ric(\omega) \geq \beta\omega + \mathcal{L}_{Re\xi}\omega$ is equivalent to

$$R_{ij} + \nabla_i\nabla_j f_\xi \geq \beta g_{ij}$$

in real coordinates. Hence

$$R_{BE}(X) = \sup\{\beta \mid Ric(\omega) + \sqrt{-1}\partial\overline{\partial} f \geq \beta\omega,\ \omega \in c_1(X) \cap \mathcal{K}(X),\ f \in C^\infty(X), \uparrow \overline{\partial} f \text{ is holomorphic } \}.$$

We will see in the next chapter, that one can relate $R_{BE}(X)$ to the continuity method for solving the Kähler-Ricci soliton equation on X as introduced in [76] as analogue of $R(X)$ and explicitly calculate the value of $R_{BE}(X)$ for toric Fano manifolds. In fact, we conjecture that $R_{BE}(X) = 1$ for any Fano manifold X. However, in this chapter, we are more interested in generalizing $R(X)$ and $R_{BE}(X)$ for log Fano manifolds and more specifically, toric conical metrics on toric manifolds.

We start with a few definitions. Let X be an n-dimensional toric manifold and L a Kähler class (or equivalently, an ample $\mathbb{R}$ divisor) on X. In [36, 66], smooth toric conical Kähler metrics are defined and studied in detail and a brief review is given in the next section. We let $\mathcal{K}_c(X)$ be the set of all smooth toric conical Kähler metrics with each cone angle in $(0, 2\pi]$ (cf. Definition 4.2.1).

Definition 4.1.3. *Let X be a toric manifold. Let $\omega \in \mathcal{K}_c(X)$ be a smooth toric conical Kähler metric on X. We say*

$$Ric(\omega) > \alpha\omega$$

if there exists $\eta \in \mathcal{K}_c(X)$ and an effective toric divisor D such that

$$Ric(\omega) = \alpha\omega + \eta + [D].$$

In fact, ω and η have the same cone angles and the divisor D can be explicitly calculated in terms of the cone angles of ω.

Definition 4.1.4. *A smooth toric conical Kähler metric $\omega \in \mathcal{K}_c(X)$ is called a conical Kähler-Ricci soliton metric if it satsifies*

$$Ric(\omega) = \alpha\omega + \mathcal{L}_\xi\omega + [D]$$

for some holomorphic vector field ξ and effective toric divisor D. If $\xi = 0$, the metric is a smooth toric conical Kähler-Einstein metric.

Associated to any toric Kähler class, we define the following geometric invariants $\mathcal{R}(X, L)$, $\mathcal{R}_{BE}(X, L)$ and $\mathcal{S}(X, L)$.

Definition 4.1.5. *Let X be a toric manifold and L be a Kähler class on X. Let $\{D_j\}_{j=1}^N$ be the set of all prime toric divisors on X. Then we define*

1. $\mathcal{R}(X, L) = \sup\{\alpha \mid Ric(\omega) > \alpha\omega$ *for some* $\omega \in c_1(L) \cap \mathcal{K}_c(X)\}$,
2. $\mathcal{R}_{BE}(X, L) = \sup\{\alpha \mid Ric(\omega) + \mathcal{L}_\xi\omega > \alpha\omega$ *for a* $\omega \in c_1(L) \cap \mathcal{K}_c(X)$ *and a toric* $\xi \in H^0(X, TX)\}$,
3. $\mathcal{S}(X, L) = \sup\left\{\alpha \mid \text{there exists } D = \sum_{j=1}^N a_j D_j \sim -K_X - \alpha L \text{ with } a_j \in [0, 1)\right\}$.

$\mathcal{R}(X,L)$ and $\mathcal{R}_{BE}(X,L)$ are natural generalizations of $R(X)$ and $R_{BE}(X)$ for log Fano manifolds with polarization L. $\mathcal{S}(X,L)$ characterizes when (X,D) is log Fano as by definition K_X+D is klt and negative. In the special case that X is toric Fano and $L=-K_X$, $\mathcal{R}(X,-K_X)$ is the usual greatest Ricci lower bound studied in [67] and $\mathcal{S}(X,-K_X)=1$. In fact, for any toric pair (X,L), $\mathcal{R}(X,L)$ and $\mathcal{S}(X,L)$ are both positive. In general, one can define $\mathcal{R}(X,L)$ and $\mathcal{R}_{BE}(X,L)$ for any log Fano pair (X,L) by requiring $Ric(\omega)-\alpha\omega\geq 0$ and $Ric(\omega)+\mathcal{L}_\xi\omega-\alpha\omega\geq 0$ in the current sense.

Any toric manifold X is induced by a Delzant polytope P and P determines a Kähler class on X. Without loss of generality, we let

$$P=\{x\in\mathbb{R}^n \mid l_j(x)>0,\ j=1,...,N\}, \tag{4.1.3}$$

where $l_j(x)=v_j\cdot x+\lambda_j$, v_j is a prime integral integral vector in $\mathbb{Z}^n$ and $\lambda_j\in\mathbb{R}$ for all $j=1,...,N$. As a special case, when X is Fano, one can choose $\lambda_j=1$ for all j and the polytope gives the anti-canonical polarization of X. The existence of smooth toric Kähler-Einstein and Kähler-Ricci soliton metrics on toric Fano manifolds is completely settled by Wang-Zhu [80]. In collaboration with Bin Guo, Jian Song, and Xiaowei Wang, we were able to generalize their results to toric conical Kähler-Einstein and Kähler-Ricci soliton metrics on any toric manifold.

Theorem 4.1.1. *[29] Let X be an n-dimensional toric Kähler manifold and L be the Kähler class on X induced by the Delzant polytope P. Then*

1. *$\mathcal{R}_{BE}(X,L)=\mathcal{S}(X,L)>0$ and*

$$\mathcal{R}_{BE}(X,L)=\sup\{\alpha \mid \text{there exists } \tau\in P \text{ with } 1-\alpha l_j(\tau)>0, j=1,...,N\}, \tag{4.1.4}$$

2. *For any $\alpha\in(0,\mathcal{S}(X,L))$ and $\tau\in P$ satisfying $1-\alpha l_j(\tau)\geq 0$ for all j, there exists a unique $\omega\in L\cap\mathcal{K}_c(X)$ solving the Kähler-Ricci soliton equation*

$$Ric(\omega)=\alpha\omega+\mathcal{L}_\xi\omega+[D]. \tag{4.1.5}$$

Moreover the divisor D and the vector field ξ are given by

$$D=\sum_{j=1}^N(1-\alpha l_j(\tau))D_j,\quad \xi=\sum_{i=1}^n c_i t_i\frac{\partial}{\partial t_i}, \tag{4.1.6}$$

where t_i's are the standard coordinates on $(\mathbb{C}^*)^n$ and $c \in \mathbb{R}^n$ is uniquely given by

$$\tau = \frac{\int_P x e^{c\cdot x}\, dx}{\int_P e^{c\cdot x}\, dx}. \tag{4.1.7}$$

3. *There does not exist a toric conical Kähler-Ricci soliton metric $\omega \in L \cap \mathcal{K}_c(X)$ solving the soliton equation (4.1.5) for any $\alpha > \mathcal{R}_{BE}(X, L)$.*

The existence of toric conical Kähler-Ricci soliton metrics on log Fano toric varieties is derived in [7] and for general toric manifolds by allowing the cone angle in $(0, \infty)$ [51]. Our result gives a complete classification for the existence of toric conical Kähler-Einstein and Kähler-Ricci soliton metrics using the invariants $\mathcal{R}(X, L)$ and $\mathcal{R}_{BE}(X, L)$ for any Kähler class. We are only interested in the toric conical Kähler metrics with cone angle in $(0, 2\pi)$ since the smooth part is geodesic convex and various Riemannian geometric properties can be applied. As a special case, we obtain an existence result for conical Kähler-Einstein metrics on toric manifolds and apply it to characterize the invariant $\mathcal{R}(X, L)$ in terms of the polytope data.

Corollary 4.1.1. *[29]Let X be an n-dimensional toric Kähler manifold and L be the Kähler class on X induced by the Delzant polytope P. Let P_C be the barycenter of P. Then*

1. *$\mathcal{R}(X, L) > 0$ and*

$$\mathcal{R}(X, L) = \sup\{\alpha \mid 1 - \alpha l_j(P_C) > 0,\ j = 1, ..., N\}. \tag{4.1.8}$$

2. *For all $\alpha \in (0, \mathcal{R}(X, L)]$, there exists a unique toric conical Kähler-Einstein metric $\omega \in L \cap \mathcal{K}_c(X)$ solving*

$$Ric(\omega) = \alpha\omega + [D]. \tag{4.1.9}$$

Moreover the divisor D is given by

$$D = \sum_{j=1}^{N} (1 - \alpha l_j(P_c)) D_j. \tag{4.1.10}$$

3. *There does not exist a toric conical Kähler-Einstein metric $\omega \in L \cap \mathcal{K}_c(X)$ solving the equation (4.1.9) for any $\alpha > \mathcal{R}(X, L)$.*

In the special case when X is Fano and $L = -K_X$, $l_j(0) = 1$ and so $1 - \alpha l_j(P_C) = (1-\alpha) l_j(\frac{-\alpha P_C}{1-\alpha})$. By the theorem, $\mathcal{R}(X, L)$ is the maximum of all α such that $\frac{-\alpha P_C}{1-\alpha}$ remains inside the polytope, generalizing the results in the smooth case in [67] and [52]. There is a subtle difference between Theorem 4.1.1 and Theorem 4.1.1 that in Theorem 4.1.1, α can be taken to be $\mathcal{R}(X, L)$ while in Theorem 4.1.1, α has to be strictly less than $\mathcal{R}_{BE}(X, L)$. Such a phenomena will be explained in Example 4.5.1.

4.2 Weighted function spaces and conical metrics on toric manifolds.

In this section, we define certain weighted function spaces and extend the Guillemin boundary conditions to conical toric metrics. Fix a toric manifold X, an ample class L and an associated polytope P given by

$$P = \{x \in \mathbb{R}^n \mid l_j(x) = v_j \cdot x + \lambda_j > 0, j = 1, \cdots, N\}$$

Recall that each vertex p of the polytope P corresponds to a coordinate chart $\{U_p, (z_1, \cdots, z_n)\}$ where U_p is a copy of $\mathbb{C}^n$, and the transition functions are determined by the normals v_j to the n faces intersecting at p. The closure of $[z_j = 0]$ gives a smooth toric divisor of X. For any function $f(z)$ invariant under the $(S^1)^n$-action, and any multi index $\beta = (\beta_1, \cdots, \beta_n)$, we can lift it to a function

$$\tilde{f}(w) = f(z)$$

by letting

$$|w_i| = |z_i|^{\beta_i}, \ w = (w_1, ..., w_n) \in \mathbb{C}^n,$$

and clearly $\tilde{f}(w)$ is also $(S^1)^n$-invariant. $w \in \mathbb{C}^n$ can be regarded a β-covering of $z \in \mathbb{C}^n$. Now we introduce consider the $(S^1)^n$-invariant function space for $k \in \mathbb{Z}_+$ and $\gamma \in [0, 1]$

$$C^{k,\gamma}_{\beta,p} = \{f(z) = f(|z_1|, ..., |z_n|) \mid \tilde{f}(w) \in C^{k,\gamma}(\mathbb{C}^n)\}.$$

This in turn defines the weighted function space

$$C^{k,\gamma}_{\beta}(X), \beta = (\beta_1, ..., \beta_N) \in (\mathbb{R}_+)^N$$

whose restriction on each chart belongs to $C^{k,\gamma}_{\beta}$ with respect to the weight β and β_j corresponding to the divisor induced by $l_j(x) = 0$.

Definition 4.2.1. *A Kähler current $\omega \in L$ is said to be a smooth β-conical toric metric if for each vertex p of the polytope P,*

$$\omega|_{U_p} = \sqrt{-1}\partial\overline{\partial}\varphi_p$$

for some $\varphi_p \in C^{\infty}_{\beta,p}$. Such a metric naturally has a cone angle of $2\pi\beta_j$ along the divisor D_j.

Theorem 4.2.1 can now be generalized to the conical case.

Proposition 4.2.1. *1. The symplectic potential for the canonical smooth β-conical toric metric induced from the standard edge metric on $\mathbb{C}^N$ is given by*

$$\hat{u} = \sum_{j=1}^{N} \left(\frac{l_j(x)}{\beta_j}\right) \log\left(\frac{l_j(x)}{\beta_j}\right) - l^{\beta}_{\infty}(x) \tag{4.2.11}$$

where

$$l^{\beta}_{\infty}(x) = \sum_{j=1}^{N} \frac{v_j \cdot x}{\beta_j} \tag{4.2.12}$$

2. Moreover a potential $u \in C^{\infty}(P) \cap C^0(\overline{P})$ corresponds to a global smooth β-conical toric metric on X if and only if

$$u - \hat{u} \in C^{\infty}(\overline{P}) \tag{4.2.13}$$

The main advantage of dealing with conical metrics on toric manifolds is that one has all the curvature bounds. The reader should contrast this with Lemma 3.2.1.

Lemma 4.2.1. *[66] Let g be a smooth toric conical metric on a toric manifold X. Let D be a toric divisor consist of all toric prime divisors and let Rm denote the full curvature tensor of g. Then for any $k \geq 0$ there exists a constant C_k such that for all $p \in X \backslash D$,*

$$|\nabla^k_g \mathrm{Rm}|^2_g(p) \leq C_k. \tag{4.2.14}$$

4.3 The set-up and the continuity method.

We once and for all fix the reference metric to be the β-conical Guillemin metric, by setting $\hat{\omega} = \sqrt{-1}\partial\overline{\partial}\hat{\varphi}$ on $(\mathbb{C}^*)^n$ where,

$$\hat{u} = \sum_{j=1}^{N} \Big(\frac{l_j(x)}{\beta_j}\Big) \log\Big(\frac{l_j(x)}{\beta_j}\Big) - l_\infty^\beta(x) \tag{4.3.15}$$

and $\hat{\varphi}$ is the Legendre transform of $\hat{u}$. Then, by the discussion in the last section, $\hat{\omega}$ is a global toric smooth conical metric with angles $2\pi\beta_j$ along the divisor D_j.

Our aim in this section is to solve the following conical soliton equation

$$Ric(\omega) = \alpha\omega + \mathcal{L}_\xi\omega + [D], \tag{*}$$

where $\alpha > 0$, ω is a smooth toric conical Kähler metric, ξ is a holomorphic toric vector field on X and D is an effective toric $\mathbb{R}$-divisor. This is a generalization of Wang-Zhu [80] in the case of smooth Fano manifolds. We will prove our estimates in the framework of [80] combined with some techniques of [36]. On the open part $(\mathbb{C}^*)^n$, we can write $\omega = \sqrt{-1}\partial\overline{\partial}\varphi$. ξ being holomorphic then implies that $\mathcal{L}_\xi\omega = \partial\bar{\partial}\xi(\varphi)$. Since ξ is also toric , it is generated by the standard vector fields $\{t_i\partial/\partial t_i\}$. Consequently, there exists a vector $\vec{c} \in \mathbb{R}^n$ such that

$$\xi(\phi) = \sum_{i=1}^{n} c_i \frac{\partial\varphi}{\partial\rho_i}.$$

Since on the open part one does not see the divisor, the soliton equation can be re-written as a real Monge-Ampere equation -

$$\det(\nabla^2\varphi) = e^{-\alpha\varphi - c\cdot\nabla\varphi + \alpha\tau\cdot\rho} \tag{4.3.16}$$

for some $\tau \in \mathbb{R}^n$. Here the linear part shows up when one gets rid of the $\partial\bar{\partial}$ and in some sense corresponds to the divisor and controls the blow up of the metric as is seen below.

Lemma 4.3.1. *If there exists a solution of* (4.3.16) *then* τ *and the vector* $\vec{c}$ *must satisfy*

$$\tau = \frac{\int_P x e^{c\cdot x}\,dx}{\int_P e^{c\cdot x}\,dx}. \tag{4.3.17}$$

Moreover, the divisor D in (*) is given by

$$D = \sum_j (1 - \alpha l_j(\tau)) D_j$$

and so the cone angle along each D_j is $2\pi\beta_j$ where $\beta_j = \alpha l_j(\tau)$.

Proof. Since $\det(\nabla^2\varphi) \to 0$ as $|\rho| \to \infty$, for $\alpha > 0$,

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \nabla(e^{-\alpha\varphi + \alpha\tau\cdot\rho})\,d\rho \\ &= \alpha\tau \int_P e^{c\cdot x}\,dx - \alpha \int_P x e^{c\cdot x}\,dx. \end{aligned}$$

To compute the cone angles we consider the asymptotics at infinity. The equation for the symplectic potential u, the Legendre transform of φ, is given by

$$\det(\nabla^2 u) = e^{-\alpha u + \alpha(x-\tau)\cdot\nabla u - c\cdot x}. \tag{4.3.18}$$

By the conic Gullemein boundary conditions,

$$u = \hat{u} + f(x) = \sum_{j=1}^{N} \Big(\frac{l_j(x)}{\beta_j}\Big) \log\Big(\frac{l_j(x)}{\beta_j}\Big) - l_\infty^\beta(x) + f(x)$$

for some $f \in C^\infty(\bar{P})$. By direct computation it can be seen that

$$\det(\nabla^2 u) = \frac{G(x)}{l_1(x)\dots l_N(x)}$$

for some non vanishing $G \in C^\infty(\bar{P})$. On the other hand, once again using the formula for u, the order of l_j on the right hand side of equation (4.3.18) can be seen to be $\alpha l_j(\tau)/\beta_j$. Comparing the orders of $l_j(x)$ on both sides of the equation we conclude that $\beta_j = \alpha l_j(\tau)$. □

Hence D is effective if and only if $1 - \alpha l_j(\tau) > 0$ for all j and in this case Lemma 4.3.1 implies that $\tau \in P$. Conversely, we have the following Lemma due to Wang-Zhu and Donaldson [80], [36]

Lemma 4.3.2. *For each $\tau \in P$, there exists a unique vector $\vec{c} \in \mathbb{R}^n$ satisfying*

$$\tau = \frac{\int_P xe^{c\cdot x}\,dx}{\int_P e^{c\cdot x}\,dx}. \tag{4.3.19}$$

Proof. By translating the polytope by τ, we can assume without loss of generality that $\tau = 0$. Consider the function

$$F(\vec{c}) = \int_P e^{c\cdot x}\,dx.$$

Clearly this function is strictly convex as can be seen by differentiating it twice. It is also proper. This follows from 0 being an interior point. Hence the function has a unique minimum $\vec{c}$. But then $\nabla F(\vec{c}) = 0$ which is precisely what we need.

□

By the Cartan formula, for any Kähler metric ω, $\mathcal{L}_\xi \omega = di_\xi \omega$. Since ξ is holomorphic, clearly $\bar{\partial} i_\xi \omega = 0$. Now, all toric manifolds are simply connected i.e $H^{0,1}(X, \mathbb{C}) = 0$. So there exists a potential function θ_ξ such that $\xi = \nabla \theta_\xi$. Of course the function also depends on the metric. The Lie derivative is now given by

$$\mathcal{L}_\xi \omega = \sqrt{-1}\partial\bar{\partial}\theta_\xi. \tag{4.3.20}$$

From now on, we fix $\tau \in P$ with $1 - \alpha l_j(\tau) > 0$ for all j and ξ is the unique holomorphic vector field determined by τ as in Lemma 4.3.1.

For the continuity method, we need to set up the Monge-Ampere equation. For that we need an analogue of the $\partial\bar{\partial}$-lemma in this conical setting. We also set $\beta(\alpha) = (\alpha l_1(\tau), \ldots, \alpha l_N(\tau))$. By lifting the smooth conical Kähler metric $\hat{\omega}$ to each uniformization covering, we can obtain the following lemma.

Lemma 4.3.3. *There exists a unique (up to constants) function $h \in C^\infty_{\beta(\alpha)}(X)$ satisfying*

$$Ric(\hat{\omega}) - \alpha\hat{\omega} - [D] - \mathcal{L}_\xi\hat{\omega} = \sqrt{-1}\partial\bar{\partial}h. \tag{4.3.21}$$

We now write the Monge-Ampere equation for the conical soliton. Set $\omega = \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi$. Then the equation for the conical soliton is

$$\begin{cases} (\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{-\alpha\psi - \xi(\psi) + h}\hat{\omega}^n \\ \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi > 0 \\ \psi \in C^{\infty}_{\beta(\alpha)}(X), \end{cases} \tag{$**$}$$

where h is from the above lemma.

To solve this equation, like usual, we introduce a parameter $s \in [0, \alpha]$ and look at the following family of equations.

$$\begin{cases} (\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi_s)^n = e^{-s\psi_s - \xi(\psi_s) + h}\hat{\omega}^n \\ \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi_s > 0 \\ \psi_s \in C^{\infty}_{\beta(\alpha)}(X) \end{cases} \tag{$(**)_s$}$$

or equivalently

$$Ric(\omega_s) = s\omega_s + (\alpha - s)\hat{\omega} + \mathcal{L}_{\xi}\omega_s + D. \tag{$(**)_s$}$$

The corresponding linearized operator is given by

$$L_s(\psi) = L(\psi) + s\psi = \Delta\psi + \xi(\psi) + s\psi.$$

Recall that we are only looking at the space of functions invariant under the toric action. One can define an inner product by

$$(\psi_1, \psi_2) = \int_X \psi_1\bar{\psi}_2 e^{\theta_\xi}\omega^n$$

and denote the corresponding Hilbert space of square integrable functions by $L^2(e^{\theta_\xi})$. Then L restricted to $C^{\infty}_{\beta}(X)$ is self adjoint and hence can be thought of as an operator from $L^2(e^{\theta_\xi})$ to itself. Also, by virtue of being self adjoint, L only has real eigenvalues. The linear theory for the spaces $C^{k,\gamma}_{\beta}(X)$ is summarized below.

Lemma 4.3.4. *Let ω be a β-conical metric, Δ be the corresponding Laplacian and L be defined as above. Then*

(1) For $k \geq 2$, $\Delta : C^{k,\gamma}_{\beta}(X) \to C^{k-2,\gamma}_{\beta}(X)$ is an invertible operator, modulo constants

(2) The Fredholm alternative holds for L.

(3) All nonzero eigenvalues of $-L$ are positive. Moreover, if $Ric(\omega) > t\omega + \mathcal{L}_\xi \omega$ and $-L\psi = \lambda\psi$, then $\lambda > t$.

The lemma that follows is essentially an observation of Zhu [87], adapted to the conical setting and is required in all the subsequent estimates. The proof in the toric case is in fact much easier.

Lemma 4.3.5. *There exists a uniform constant C depending only on $\hat{\omega}$ and ξ such that, for any function $\psi \in C^\infty_\beta(X) \cap PSH(X, \hat{\omega})$*

$$|\xi(\psi)| \leq C.$$

Proof. In the toric situation, the proof is almost trivial. Locally on $(\mathbb{C}^*)^n$, $\hat{\omega} = \partial\bar{\partial}\hat{\varphi}$ and for any $\varphi = \hat{\varphi} + \psi$, since ψ is globally bounded and plurisubharmonic, it is easy to see that $\nabla\varphi(\mathbb{R}^n) = \nabla\hat{\varphi}(\mathbb{R}^n) = P$ and $\partial\bar{\partial}\varphi$ extends to a global Kähler metric. So there exists a uniform constant C such that

$$|\nabla\psi| \leq C.$$

But then, since ξ is given by

$$\xi = \sum_{i=1}^{n} c_i t_i \frac{\partial}{\partial t_i},$$

we have that

$$\xi(\psi) = c \cdot \nabla\psi.$$

This gives us the required bound.

$\square$

The following proposition shows that there exists a solution to equation $(**)_s$ at $s = 0$

Proposition 4.3.1. *For any $h \in C^{\infty}_{\beta(\alpha)}(X)$ there exists a unique function $\psi \in C^{\infty}_{\beta(\alpha)}(X)$ satisfying*

$$\begin{cases} (\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{h-\xi(\psi)}\hat{\omega}^n \\ \sup \psi = 0 \\ \omega = \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi > 0. \end{cases}$$

Proof. We proceed by the continuity method. Consider the family of equations

$$\begin{cases} (\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi_s)^n = e^{sh-\xi(\psi_s)}\hat{\omega}^n \\ \sup \psi_s = 0 \\ \omega = \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi_s > 0 \end{cases} \tag{4.3.22}$$

and set $S = \{s \in [0,1]|$ equation (4.3.22) has a solution $\psi_s \in C^{3,\gamma}_{\beta(\alpha)}(X)$ at $s\}$. The set S is clearly nonempty, since $0 \in S$. In what follows we suppress the index s for convinience.

Openness. This follows straight from part(a) of Lemma 4.3.4 and the implicit function theorem on the space $C^{\infty}_{\beta(\alpha)}(X)$, since the linearized operator is just L.

C^0 *estimates.* By Lemma 6.2.2, the right hand side of the equation is uniformly bounded in s and hence in particular there exists a uniform L^p bound for any $p > 1$. Now, Kolodziej's results and their generalizations [49, 37, 85] give a uniform C^0 bound.

Second order estimates. Consider the quantity

$$H_s = \log tr_{\hat{\omega}}\omega_s - A\psi,$$

where A is some large number to be chosen later. Since both $\hat{\omega}$ and ω_s have poles of same order, the quantity is bounded. Let $\sup_X H_s = H_s(q)$. We lift all the local calculations to the $(S^1)^n$ invariant β- covering space. The second order estimates easily follow from [82] and [76].

C^3 *and higher order estimates.* Calabi's method for third order estimates can again be carried out by lifting the calculations to the β-cover. The reader should refer to [62] for the simplified computations. Higher order derivatives can be obtained by a standard bootstrapping argument. Closedness now follows from Ascoli-Arzela. Hence $1 \in S$.

$\square$

4.4 The main C^0 estimate.

For later applications, we need to get the precise dependence of the C^0 estimate on the polytope. So we introduce some notation. Recall that

$$P = \{x \in \mathbb{R}^n \mid l_j(x) = v_j \cdot x + \lambda_j > 0, j = 1, \cdots, N\}.$$

We let ν and σ be two constants such that

$$\nu^{-1} < Vol(P) < \nu,$$
$$(\sigma)^{-1} < diam(P) < \sigma.$$

On $(\mathbb{C}^*)^n$ we write $\hat{\omega} = \sqrt{-1}\partial\overline{\partial}\hat{\varphi}$ and $\omega_s = \sqrt{-1}\partial\overline{\partial}\varphi_s$. Using the standard logarithmic coordinates like before one can rewrite equation $(**)_s$ as a real Monge-Ampere on $\mathbb{R}^n$

$$\begin{cases} \det(\nabla^2\varphi_s) = e^{-s(\varphi_s - \tau\cdot\rho)-(\alpha-s)(\hat{\varphi}-\tau\cdot\rho)-c\cdot\nabla\varphi_s}, \\ u_s = \mathcal{L}\varphi_s = \sum_{j=1}^N (\beta_j)^{-1} l_j(x)\log l_j(x) + f(x), \qquad f \in C^\infty(\overline{P}) \\ \nabla^2\varphi_s > 0. \end{cases} \tag{4.4.23}$$

Proposition 4.4.1. *For any $s_0 \in (0,\alpha)$ there exists a constant $C = C(n, s_0, \nu, \sigma, \Lambda, \sup_{j,P}|l_j|)$ such that*

$$|\varphi_s - \hat{\varphi}| \leq C$$

for all $s \in [s_0, \alpha)$. Here Λ is the constant from Lemma 4.4.2 below.

We use the arguments in [80] with inputs and simplifications from [36], most notably the last step which helps us avoid the Harnack inequality. We first need two technical lemmas.

Lemma 4.4.1. *Suppose $v \geq 0$ is a strictly convex function on $\mathbb{R}^n$ such that $v(0) = 0$ and $\det(\nabla^2 v) \geq \lambda$ on $v \leq 1$. Then there exists $C > 0$ such that*

$$Vol(v \leq 1) \leq C\lambda^{-1/2}. \tag{4.4.24}$$

The proof is a standard barrier function argument and so we skip it.

Lemma 4.4.2. *If $\hat{\varphi}$ is the Legendre transform of $\hat{u}$ and we define the function*

$$g_j(\rho) = \log(l_j(\nabla\hat{\varphi}(\rho))). \tag{4.4.25}$$

Then, there exists a constant Λ such that

$$\sup_{\mathbb{R}^n} |\nabla g_j| \leq \Lambda. \tag{4.4.26}$$

Here, Λ depends only on β_j, N, n and the normal vectors v_j.

Proof. Recall that the polytope P is given by faces $l_j(x) = v_j \cdot x + \lambda_j$ and let $\hat{u}$ be the usual conical symplectic potential given by

$$\hat{u} = \sum_{j=1}^{N} \Big(\frac{l_j(x)}{\beta_j}\Big) \log\Big(\frac{l_j(x)}{\beta_j}\Big) - l_\infty^\beta(x) \tag{4.4.27}$$

We then set,

$$V = \{v_{j\gamma}\}, \qquad A = \{\frac{v_{j\gamma}}{\sqrt{\beta_j l_j}}\} := \{a_{j\gamma}\},$$

where $v_j = \{v_{j\gamma}\}$ is the vector normal to the face l_j. For any $J = \{j_1, \dots j_n\} \subset \{1, \dots N\}$ in some order, we let M_J be the corresponding $n \times n$ minor of V and $C_J = det(M_J)$. Let $\tilde{M}_J$ and $\tilde{C}_J$ be the corresponding quantities for A. Then, in our notation

$$\nabla^2\hat{u} := \{\hat{u}_{\gamma\mu}\} = (A)^t A. \tag{4.4.28}$$

Claim 1

$$\det(\nabla^2\hat{u}) = \sum_{j_1<\dots<j_n} \tilde{C}^2_{j_1,\dots j_n} = \sum_{j_1<\dots<j_n} \frac{C^2_{j_1,\dots j_n}}{(\beta_{j_1} l_{j_1})\dots(\beta_{j_n} l_{j_n})}$$

.

This is known as the Cauchy-Binet formula in literature. The proof is of course just a simple exercise in undergraduate linear algebra and so we skip it.

Now, for $J = \{j_1, \ldots, j_{n-1}\}$ let $M_{J;\gamma}$ be the minor obtained by deleting the γ^{th} column from the matrix of row vectors v_{j_i}. We once again set $C_{J;\gamma} = det(M_{J;\gamma})$

Claim 2

$$\det(\nabla^2 \hat{u}) \sum_{\mu=1}^{n} a_{j\mu} \hat{u}^{\mu\gamma} = (-1)^{\gamma+1} \sum_{\substack{j_1 < \ldots < j_{n-1} \\ j_i \neq j}} \tilde{C}_{j,j_1,\ldots j_{n-1}} \tilde{C}_{j_1,\ldots j_{n-1};\gamma} \tag{4.4.29}$$

$$= \frac{(-1)^{\gamma+1}}{\sqrt{\beta_j l_j}} \sum_{\substack{j_1 < \ldots < j_{n-1} \\ j_i \neq j}} \frac{C_{j,j_1,\ldots j_{n-1}} C_{j_1,\ldots j_{n-1};\gamma}}{(\beta_{j_1} l_{j_1}) \ldots (\beta_{j_{n-1}} l_{j_{n-1}})}, \tag{4.4.30}$$

where $\{u^{\mu\gamma}\}$ denotes the inverse matrix of $\nabla^2 \hat{u}$.

Proof. The proof proceeds along the lines of the proof for Cauchy-Binnet formula, only it requires more book keeping and is as follows - Denoting by $\chi_{\gamma\mu}$, the co-factor matrix of $\nabla^2 u$ and employing Cramer's rule,

$$\begin{aligned}
\det(\nabla^2 \hat{u}) \sum_{\mu=1}^{n} a_{j\mu} \hat{u}^{\mu\gamma} &= \sum_{\mu=1}^{n} a_{j\mu} \chi_{\gamma\mu} \\
&= \sum_{\mu=1}^{n} a_{j\mu} \sum_{\substack{\sigma:\{1,\ldots,\hat{\mu},\ldots,n\} \\ \to\{1,\ldots,\hat{\gamma},\ldots n\}}} (-1)^{\gamma+\mu} sgn(\sigma) u_{1\sigma(1)} \ldots u_{l-1\sigma(l-1)} u_{l+1\sigma(l+1)} \ldots u_{n\sigma(n)} \\
&= \sum_{\mu=1}^{n} a_{j\mu} \sum_{j_1=1}^{N} \cdots \sum_{j_{n-1}=1}^{N} \sum_{\substack{\sigma:\{1,\ldots,\hat{\mu},\ldots,n\} \\ \to\{1,\ldots,\hat{\gamma},\ldots n\}}} (-1)^{\gamma+\mu} sgn(\sigma) a_{j_1 1} a_{j_1\sigma(1)} \ldots a_{j_{n-1}n} a_{j_{n-1}\sigma(n)},
\end{aligned}$$

where the product in the above summation includes exactly two entries from all columns except the μ^{th} and the γ^{th} ones which have one entry each. Clearly, the innermost summation, as σ runs over all permutations, is some determinant. More precisely,

$$\det(\nabla^2 \hat{u}) \sum_{\mu=1}^{n} a_{j\mu} \hat{u}^{\mu\gamma} = \sum_{\mu=1}^{n} a_{j\mu} \sum_{j_1=1}^{N} \cdots \sum_{j_{n-1}=1}^{N} (-1)^{\gamma+\mu} a_{j_1 1} \ldots a_{j_{n-1}n} \tilde{C}_{j_1,\ldots j_{n-1};\gamma}.$$

Again, like in the proof of the first claim, $\tilde{C}_{j_1,\ldots j_{n-1};\gamma} = 0$ unless the j_i's are distinct. Also, if $j_1 < \ldots j_{n-1}$ and τ any permutation of this indices, then $\tilde{C}_{\tau(j_1),\ldots \tau(j_{n-1});\gamma} = \tilde{s}gn(\tau) C_{j_1,\ldots j_{n-1};\gamma}$. Thus,

$$\begin{aligned}
det(\nabla^2\hat{u})\sum_{\mu=1}^{n} a_{j\mu}\hat{u}^{\mu\gamma} &= \sum_{\mu=1}^{n} a_{j\gamma}\sum_{j_1<\ldots<j_{n-1}}\sum_{\tau}(-1)^{\gamma+\mu}sgn(\tau)a_{\tau(j_1)1}\ldots a_{\tau(j_{n-1})n}\tilde{C}_{j_1,\ldots j_{n-1};\gamma}\\
&= \sum_{\mu=1}^{n} a_{j\mu}\sum_{j_1<\ldots<j_{n-1}}\sum_{\substack{\tau:\{1,\ldots,\hat{\mu},\ldots n\}\\ \to\{1,\ldots,\hat{\mu},\ldots n\}}}(-1)^{\gamma+\mu}sgn(\tau)a_{j_1\tau(1)}\ldots a_{j_{n-1}\tau(n)}\tilde{C}_{j_1,\ldots j_{n-1};\gamma}\\
&= \sum_{j_1<\ldots<j_{n-1}}\sum_{\mu=1}^{n}(-1)^{\gamma+\mu}a_{j\mu}\tilde{C}_{j_1,\ldots j_{n-1};\mu}\tilde{C}_{j_1,\ldots j_{n-1};\gamma}\\
&= (-1)^{\gamma+1}\sum_{j_1<\ldots<j_{n-1}}\tilde{C}_{j,j_1,\ldots j_{n-1}}\tilde{C}_{j_1,\ldots j_{n-1};\gamma}.
\end{aligned}$$

□

Proof of Lemma 4.4.2 We compute the derivative of g_j using the correspondence $\nabla^2\hat{\varphi} = (\nabla^2\hat{u})^{-1}$ and $x = \nabla\hat{\varphi}$.

$$\begin{aligned}
\frac{\partial g_j}{\partial\rho_\gamma} &= \frac{l_j(\nabla(\varphi_\gamma))}{l_j(\nabla\varphi)} = \frac{1}{l_j(\nabla\varphi)}\sum_{\mu=1}^{n} v_{j\mu}\hat{u}^{\mu\gamma} = \frac{\sqrt{\beta_j}}{\sqrt{l_j}}\sum_{\mu=1}^{n} a_{j\mu}\hat{u}^{\mu\gamma}\\
&= \frac{1}{det(\nabla^2\hat{u})}\sum_{\substack{j_1<\ldots<j_{n-1}\\ j_i\neq j}}\frac{1}{l_j}\frac{C_{j,j_1,\ldots j_{n-1}}C_{j_1,\ldots j_{n-1};\gamma}}{(\beta_{j_1}l_{j_1})\ldots(\beta_{j_{n-1}}l_{j_{n-1}})}\\
&= \frac{\sum_{\substack{j_1<\ldots<j_{n-1}\\ j_i\neq j}}\frac{1}{l_j}\frac{C_{j,j_1,\ldots j_{n-1}}C_{j_1,\ldots j_{n-1};\gamma}}{(\beta_{j_1}l_{j_1})\ldots(\beta_{j_{n-1}}l_{j_{n-1}})}}{\sum_{j_1<\ldots<j_n}\frac{C^2_{j_1,\ldots j_n}}{(\beta_{j_1}l_{j_1})\ldots(\beta_{j_n}l_{j_n})}}\\
&\leq \frac{\sum_{\substack{j_1<\ldots<j_{n-1}\\ j_i\neq j}}\frac{1}{l_j}\frac{C_{j,j_1,\ldots j_{n-1}}C_{j_1,\ldots j_{n-1};\gamma}}{(\beta_{j_1}l_{j_1})\ldots(\beta_{j_{n-1}}l_{j_{n-1}})}}{\sum_{j_1<\ldots<j_{n-1}}\frac{C^2_{j,j_1,\ldots j_{n-1}}}{(\beta_j l_j)(\beta_{j_1}l_{j_1})\ldots(\beta_{j_{n-1}}l_{j_{n-1}})}}.
\end{aligned}$$

The summation can be taken to be only over all $J = \{j_1,\ldots j_{n-1}\}$ such that $C_{jJ}\neq 0$. For such terms, one has the trivial bound

$$\left|\frac{C_{j_1,\ldots j_{n-1};\gamma}}{C_{j,j_1,\ldots j_{n-1}}}\right| \leq M',$$

where M' depends only on upper bounds on $|v_j|$ and β_j and a positive lower bound on $|C_J|$ as $J\subset\{1,\ldots N\}$ varies over all subsets with $C_J\neq 0$. Together with the above computation, we get

$$\left|\frac{\partial g_j}{\partial\rho_\gamma}\right| \leq \beta_j\Lambda'.$$

This proves the Lemma with $\Lambda = \beta_j\Lambda'$.

□

Proof of Proposition 4.4.1 There are several steps following [80] and [36] combined with Lemma 4.4.2. Let $\phi_s = \varphi_s - \tau \cdot \rho$, $\hat{\phi} = \hat{\varphi} - \tau \cdot \rho$ and define

$$w_s = s\phi_s + (\alpha - s)\hat{\phi}. \tag{4.4.31}$$

Set

$$m_s := \inf_{\mathbb{R}^n} w_s = w_s(\rho_s)$$

Step 1. We claim that there exist $C, \zeta > 0$ independent of s such that for all $s \in [s_0, \alpha]$,

$$(a)\ |m_s| \leq C \tag{4.4.32}$$

$$(b)\ w_s \geq \zeta|\rho - \rho_s| - C. \tag{4.4.33}$$

It follows from the definition of w_s that $\det(\nabla^2 w_s) \geq s^n \det(\nabla^2 \phi_s) \geq s_0^n \det(\nabla^2 \phi_s)$. Set $K = \{m_s \leq w_s \leq m_s + 1\}$, $K_\mu = \{m_s \leq w_s \leq m_s + \mu\}$ and $V_\mu = Vol(K_\mu)$. From the equation, $\det(\nabla^2 \phi_s) = e^{-w_s - c \cdot \nabla \varphi_s}$ and so on K,

$$\begin{aligned}
\det(\nabla^2 w_s) &\geq s_0^n \det(\nabla^2 \phi_s) \\
&= s_0^n e^{-w_s - c \cdot \nabla \varphi_s} \\
&\geq C e^{-m_s},
\end{aligned}$$

where C only depends on s_0 and σ which is an upper bound for $|\nabla \varphi_s|$. So, Lemma 4.4.1 applied to $v = w_s - m_s$ implies that $Vol(K) \leq Ce^{m_s/2}$. But $K_\mu \subseteq \mu K$, where by μK, we mean dilation with center ρ_s. So we have the volume estimate

$$V_\mu \leq C\mu^n e^{m_s/2}.$$

Now,

$$\begin{aligned}
\nu^{-1} \leq Vol(X) &= \int_{\mathbb{R}^n} \det(\nabla^2 \phi_s)\, d\rho \\
&= \int_{\mathbb{R}^n} e^{-w_s - c \cdot \nabla \varphi_s}\, d\rho \\
&\leq Ce^{-m_s} \int_0^\infty e^{-\mu} V_\mu\, d\mu \\
&\leq Ce^{-m_s/2}
\end{aligned}$$

and so $m_s \leq C(n, s_0, \nu, \sigma)$. For the lower bound, notice that $\nabla w_s(\mathbb{R}^n) = P - \tau$ and so $|\nabla w_s| \leq 2\sigma$. This implies that K contains a ball of radius $1/2\sigma$. But the volume of K is bounded above by $Ce^{m_s/2}$ and so we immediately have a lower bound for m_s. Hence (a) is proved with $C = C(n, s_0, \nu, \sigma)$.

Suppose now there exists a point $\rho \in K$ such that $|\rho - \rho_s| = R$. Because $B = B(\rho_s, 1/(2\sigma)) \subseteq K$, by convexity, the entire cone κ with vertex at ρ and base as B lies inside K. So, $Vol(K) \geq CR$, where C depends only on dimension and σ. But $Vol(K) \leq Ce^{m_s/2}$ and so is less than some fixed constant C by part (a). Hence R is uniformly bounded. That is, there exists a uniform R such that $K \subseteq B(\rho_s, R)$. But then, convexity implies that $K_\mu \subseteq B(\rho_s, \mu R)$. From this and the lower bound on m_s, it easily follows that

$$w_s \geq \frac{1}{R}|\rho - \rho_s| - C.$$

This proves (b) with $\zeta = 1/R$.

Step 2. We now claim that, there exists uniform constant C such that

$$|\rho_s| \leq C. \tag{4.4.34}$$

We first observe,

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \nabla(e^{-w_s}) = -\int_{\mathbb{R}^n} [s(\nabla\varphi_s - \tau) + (\alpha - s)(\nabla\hat{\varphi} - \tau)]e^{-w_s}\, d\rho \\ &= -\int_{\mathbb{R}^n} (\alpha - s)(\nabla\hat{\varphi} - \tau)]e^{-w_s}, \end{aligned}$$

where we use the change of coordinates $x = \nabla\varphi_s(\rho)$ along with the equation $e^{-w_s} = \det(\nabla^2\varphi_s)e^{c\cdot\nabla\varphi_s}$ and the fact that $\vec{c}$ and τ are compatible to conclude that the first term is zero. This computation gives us the crucial identity

$$\int_{\mathbb{R}^n} (\nabla\hat{\varphi} - \tau)e^{-w_s}\, d\rho = 0,$$

or equivalently,

$$\frac{1}{\tilde{V}_s}\int_{\mathbb{R}^n} (\nabla\hat{\varphi})e^{-w_s}\, d\rho = \tau, \tag{4.4.35}$$

where $\tilde{V}_s$ is the weighted volume given by

$$\tilde{V}_s = \int_{\mathbb{R}^n} e^{-w_s}\, d\rho.$$

Note that when the Futaki invariant vanishes, this is precisely the identity in the paper of Wang and Zhu since in that case τ is the barycenter which is zero.

Suppose the claim is false i.e for all $M > 0$ there exists a pair (s, ρ_s) with $|\rho_s| > M$. Applying l_j to both sides of the identity (4.4.35),

$$\frac{1}{\tilde{V}_s} \int_{\mathbb{R}^n} l_j(\nabla\hat{\varphi}) e^{-w_s}\, d\rho = l_j(\tau) > \delta \tag{4.4.36}$$

for some j and some $\delta > 0$. Fix an $\epsilon > 0$. From the estimates in the previous step there exists an $R_\epsilon >> 1$ such that

$$\int_{\mathbb{R}^n \setminus B(\rho_s, R_\epsilon)} e^{-w_s}\, d\rho \leq \epsilon. \tag{4.4.37}$$

Recall that as ρ goes to infinity, the image under $\nabla\hat{\varphi}$ goes to the boundary of P. So, by hypothesis, on can choose a big $M >> 1$ such that $|\rho_s| > M$ and $\log\left(l_j(\nabla\hat{\varphi}(\rho_s))\right) < -M$ for some s and some face l_j. By the gradient estimate in Lemma 4.4.2 there exists a constant Λ (which does not depend on s) such that on $B = B(\rho_s, R_\epsilon)$

$$\log\left(l_j(\nabla\hat{\varphi}(\rho))\right) < -M + \Lambda R_\epsilon < \frac{-M}{2} < \log\epsilon \tag{4.4.38}$$

for M sufficiently big. So combining 4.4.37 and 4.4.38 we estimate the integral in 4.4.36,

$$\begin{aligned}\frac{1}{\tilde{V}} \int_{\mathbb{R}^n} l_j(\nabla\hat{\varphi}) e^{-w_s} &= \frac{1}{\tilde{V}} \int_B l_j(\nabla\hat{\varphi}) e^{-w_s} + \frac{1}{\tilde{V}} \int_{\mathbb{R}^n \setminus B} l_j(\nabla\hat{\varphi}) e^{-w_s} \\ &\leq \epsilon + C\epsilon,\end{aligned}$$

where C only depends on an upper bound for the image of P under l_j and a lower bound for the total volume of X. Now choose ϵ small enough so that $\epsilon + C\epsilon < \delta/2$. But then, this contradicts (4.4.36), completing Step 2.

Step 3. We first observe the elementary identity from convex analysis

$$\sup_{\mathbb{R}^n} |\varphi_s - \hat{\varphi}| = \sup_P |u_s - \hat{u}|. \tag{4.4.39}$$

So, to complete the proof, one only needs to control the C^0 norm of u_s since from the definition it is easy to see that the bound for $\hat{u}$ only depends on β_j and an upper bound on l_j. From (4.4.33) and (4.4.34),

$$w_s(\rho) \geq \zeta|\rho| - C. \tag{4.4.40}$$

Let u_s be the Legendre transform of φ_s, then for any $p > n$,

$$\begin{aligned}
\int_P |\nabla u_s|^p \, dx &= \int_{\mathbb{R}^n} |\rho|^p e^{-w_s - c\cdot\nabla\varphi_s} \, d\rho \\
&\leq C \int_{\mathbb{R}^n} |\rho|^p e^{-\zeta|\rho|} \, d\rho \\
&\leq C(p).
\end{aligned}$$

By Morrey's inequality $osc_{\bar{P}} u_s < C$ for some C independent of s. Now, if we set $x_s = \nabla\varphi_s(\rho_s)$, then,

$$u_s(x_s) = \rho_s \cdot x_s - \varphi_s(\rho_s).$$

The first term is clearly bounded from Step-2. Moreover by Step-1, $w_s(\rho_s)$ is bounded. Since ρ_s stays bounded, there is a uniform bound on $\hat{\varphi}(\rho_s)$, which in turn gives a uniform bound on $\varphi_s(\rho_s)$. This shows that $|u_s(x_s)|$ is uniformly bounded. Hence the oscillation bound implies

$$|u_s|_{C^0(P)} \leq C.$$

This completes the proof of the proposition.

$\square$

4.5 Proof of Theorem 4.1.1 and Corollary 4.1.1.

We are now in a position to prove Theorem 4.1.1 and Corollary 4.1.1.

Proof of Theorem 4.1.1.

Step 1. We first characterize the invariant $\mathcal{S}(X, L)$ in terms of the polytope data as follows - The polytope for the linear system $|-K_X - \alpha L|$ can be taken to be $P^\alpha = \{x \in \mathbb{R}^n \mid l_j^\alpha = v_j \cdot x + 1 - \alpha\lambda_j\}$. For any $\alpha > 0$ and any j,

$$\tau \in P \text{ with } 1 - \alpha l_j(\tau) \geq 0$$

$$\Leftrightarrow 0 \le 1 - \alpha l_j(\tau) \le 1$$

$$\Leftrightarrow 0 \le v_j \cdot (-\alpha\tau) + 1 - \alpha\lambda_j \le 1$$

$$\Leftrightarrow 0 \le l_j^{\alpha}(-\alpha\tau) \le 1.$$

But then divisor $D = \sum l_j^{\alpha}(-\alpha\tau) D_j$ is an effective divisor in $|-K_X - \alpha L|$ with coefficients less than 1.

Step 2. We next outline a proof of the existence of solutions to the soliton equation. Let $S = \{s \in [0,\alpha] | \exists$ a solution $\psi \in C^{3,\gamma}_{\beta(\alpha)}$ to eqn. $(**)_s$ $\}$. By proposition 4.3.1, $0 \in S$ and hence S is nonempty. We now need to show that S is both open and closed.

Openness- The linearized operator for equation $(**)_s$ is $L_s = \Delta_s + \xi + sI$. Since $[D] \ge 0$, $Ric(\omega_s) > s\omega_s + \mathcal{L}_\xi \omega_s$. By lemma 4.3.4 all eigenvalues of $-L_s$ are strictly positive and hence the Fredholm alternative implies that L_s is invertible. Implicit function theorem then implies that S is open.

C^0 *estimates*- Since there is a solution at $s = 0$ by openness there exists an s_0 such that there is a solution on $[0, s_0]$. With this choice of s_0, by proposition 4.4.1 there exists a constant C independent of s such that

$$|\psi_s| = |\varphi_s - \hat{\varphi}| \le C.$$

C^2 *and higher order estimates* - Once the uniform bound is obtained, the argument for the second and higher order estimates is the same as that in the proof of proposition 4.3.1. Hence the upshot is that S is nonempty, open and closed. Hence $S = [0, \alpha]$ and in particular $\alpha \in S$. This completes the proof of the second part of the theorem.

Step 3. Finally, to complete the proof of Theorem 4.1.1, we now prove that $\mathcal{R}_{BE}(X, L) = \mathcal{S}(X, L)$. From the existence part of the theorem, it is easy to see that $\mathcal{R}_{BE}(X, L) \ge \mathcal{S}(X, L)$. In order to prove the reverse inequality, let $\alpha \in (0, \mathcal{R}_{BE})$. Then by definition, there exist smooth toric β-conical metrics $\omega = \sqrt{-1}\partial\bar{\partial}\varphi$ and $\eta = \sqrt{-1}\partial\bar{\partial}\psi$, and a holomorphic vector field ξ vector $\tau \in \mathbb{R}^n$, such that

$$Ric(\omega) = \alpha\omega + \mathcal{L}_\xi \omega + \eta + [D]$$

for some smooth conical Kähler metric η and some effective divisor D. Note that the volume form can be expressed as

$$\omega^n = \frac{\Omega}{\prod_{j=1}^N |s_j|_{h_j}^{2(1-\beta_j)}}$$

for some global volume form Ω with $\log \Omega$ bounded. From this, it is clear that the divisor is given by

$$D = \sum_{j=1}^N (1-\beta_j) D_j$$

and consequently one can take the polytope for η to be

$$P^\eta = \{x \in \mathbb{R}^n | l_j^\eta = v_j \cdot x + \beta_j - \alpha\lambda_j > 0 \; j = 1, \ldots, N\}.$$

Locally on $(\mathbb{C}^*)^n$, $\omega = \sqrt{-1}\partial\overline{\partial}\varphi$ and $\eta = \sqrt{-1}\partial\overline{\partial}\psi$, and the corresponding real Monge-Ampere equation reads

$$\det \nabla^2 \varphi = e^{-\alpha\varphi - \psi - c\cdot\nabla\varphi - \tau\cdot\rho}$$

for some $\tau \in \mathbb{R}^n$. As before, we take φ so that $\nabla\varphi(\mathbb{R}^n) = P$. Furthermore we normalize ψ so that $\nabla\psi(\mathbb{R}^n) = P^\eta$. With this normalization, we claim that $\tau = 0$.

Since $\nabla\varphi$ is bounded, It suffices to prove that

$$|\log \det \nabla^2 \varphi + \alpha\varphi + \psi| \tag{4.5.41}$$

is bounded. Let $\overline{\varphi} = \alpha\varphi + \psi$ be the potential for the smooth β-conical metric $\overline{\omega} = \alpha\omega + \eta$. Then $\overline{\omega}^n/\omega^n$ is a global bounded function. This is because both the metrics have the same poles at the divisors. Consequently it is enough to show that

$$|\log \det \nabla^2 \overline{\varphi} + \overline{\varphi}|$$

is bounded. But, as in the proof of Lemma 4.3.1,

$$|\log \det \nabla^2 \overline{\varphi} + \overline{\varphi}| \leq |\sum_{j=1}^N \left(1 + \frac{x \cdot v_j}{\beta_j}\right) \log \bar{l}_j| + C$$

$$\leq |\sum_{j=1}^{N}\Big(1-\frac{\bar{l}_j(0)}{\beta_j}\Big)\log \bar{l}_j| + C$$
$$\leq C,$$

where $\bar{l}_j(x) = v_j \cdot x + \beta_j$ and we used the fact that $\bar{l}_j \log \bar{l}_j$ is a bounded function in the second line. Note that the polytope for $\overline{\varphi}$ is given precisely by the intersection of $\bar{l}_j > 0$ for $j = 1, \ldots, N$. This completes the proof of (4.5.41) and hence proves the claim that $\tau = 0$. But then using the integration by parts trick from the proof of Lemma 4.3.1

$$0 = \int_{\mathbb{R}^n} \nabla(e^{-\alpha\varphi-\psi})d\rho = -\alpha \int_P xe^{c\cdot x}dx - \int_P \nabla\psi e^{c\cdot x}dx.$$

So, if we set

$$\bar{\tau} = \frac{\int_P xe^{c\cdot x}dx}{\int_P e^{c\cdot x}dx} = \frac{-\int_P \nabla\psi e^{c\cdot x}dx}{\alpha\int_P e^{c\cdot x}dx}.$$

Obviously, $\bar{\tau} \in P$ and applying l_j , we have

$$1-\alpha l_j(\bar{\tau}) = \frac{\int_P(1+v_j\cdot\nabla\psi-\alpha\lambda_j)e^{c\cdot x}dx}{\int_P e^{c\cdot x}dx} \geq 1-\beta_j \geq 0.$$

where we used the definition of P^η for the first inequality and $D \geq 0$ for the second inequality. Hence $\alpha < \mathcal{S}(X.L)$ and this completes the proof of the theorem. □

Proof of Theorem 4.1.1. This theorem follows directly from Theorem 4.1.1 by taking $\tau = P_C$. For uniqueness we refer to results of Berndtsson [8]. The only slightly subtle point is the existence of conical Kähler-Einstein metrics for $\alpha = \mathcal{R}(X.L)$. This follows from the fact that barycenter always stays in the interior of the polytope and hence Proposition 4.4.1 also holds for this choice of α (Contrast this, for instance, with the case when $\alpha = \mathcal{S}(X, L)$ as discussed in Example 4.5.1 below). All the higher order estimates then follow from the C^0 bound exactly as in the proof above.

The following example illustrates that as $\gamma \to \mathcal{S}(X, \alpha)$ a complete end might develop along one of the divisors, and the soliton equation might not have any solution at $\gamma = \mathcal{S}(X, \alpha)$.

Example 4.5.1. *Let $X = \mathbb{P}^2\#\overline{\mathbb{P}^2}$, i.e., $\mathbb{P}^2$ blow-up at one point, given by a polytope P defined by $l_0 = x + y + \varepsilon > 0$, $l_1 = y + 1 > 0$, $l_2 = x + 1 > 0$, and $l_\infty = -x - y + \varepsilon > 0$, where $\varepsilon > 0$ is a small constant. Let α be the Kähler class induced by P. Note that the* *class being ample is equivalent to* $\varepsilon \in (0,2)$. *By Theorem 1.1,* $\frac{1}{\mathcal{S}(X,\alpha)}$ *can be characterized as*

$$\inf_{(x,y)\in P} \max\{x+1, y+1, x+y+\varepsilon, -x-y+\varepsilon\}. \tag{4.5.42}$$

By the symmetry of $x, y \in P$, *the extremal point must be at the line* $y - x = 0$, *hence the aimed function is reduced to*

$$\inf_{x\in(-\varepsilon/2,\varepsilon/2)} \max\{x+1, 2x+\varepsilon, -2x+\varepsilon\}. \tag{4.5.43}$$

We have three cases: $\varepsilon \in (0, \frac{2}{5}], (\frac{2}{5}, 1), [1, 2)$.

1. *When* $\varepsilon \in (0, \frac{2}{5}]$, *the unique extremal point of* (4.5.43) *is at* $x = -\frac{\varepsilon}{2}$, *or, that of* (4.5.42) *is at* $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, *which is at the boundary of* P, *hence the conical Kähler-Ricci soliton equation cannot be solved at* $\mathcal{S}(X,\alpha)$ *for this case.*

2. *When* $\varepsilon \in (\frac{2}{5}, 1)$, *the unique extremal point of* (4.5.43) *is at* $x = -\frac{1-\varepsilon}{3}$, *and the point* $(-\frac{1-\varepsilon}{3}, -\frac{1-\varepsilon}{3})$ *is in the interior of* P, *hence by our proof above, the conical Kähler-Ricci soliton equation can be solved at* $\mathcal{S}(X,\alpha)$ *in this case.*

3. *When* $\varepsilon \in [1, 2)$, *the unique extremal point of* (4.5.43) *is at* $x = 0$, *and the origin* $(0,0)$ *is always in* P, *so in this case, the conical Kähler-Ricci soliton equation can also be solved up to* $\mathcal{S}(X,\alpha)$.

Chapter 5

Connecting toric manifolds by conical Kähler-Einstein metrics

5.1 Introduction.

It is a well known theorem of Bando and Mabuchi [5], that smooth Kähler-Einstein metrics on a Fano manifold are unique modulo the identity component of the automorphism group. In particular, the moduli space of n-dimensional toric manifolds with smooth Kähler-Einstein metrics consists of only isolated points. Although the corresponding moduli space for n-dimensional toric manifolds with conical Kähler-Einstein metrics is much bigger, in collaboration with Bin Guo, Jian Song and Xiaowei Wang, we were able to prove that it is connected in the Gromov-Hausdorff topology.

Theorem 5.1.1. *[29] Let X_0 and X_1 be two n-dimensional toric manifolds. Suppose $\omega_0 \in \mathcal{K}_c(X_0)$ and $\omega_1 \in \mathcal{K}_c(X_1)$ are two smooth toric conical Kähler-Einstein metrics on X_0 and X_1 respectively. Then, there exist a family $\{(X_t, \omega_t)\}_{t\in[0,1]}$ of n-dimensional toric manifolds X_t with smooth toric conical Kähler-Einstein metrics $\omega_t \in \mathcal{K}_c(X_t)$ for $t \in [0,1]$, such that*

1. *(X_t, ω_t) is a continuous path in Gromov-Hausdorff topology for $t \in [0,1]$,*
2. *$K \subset\subset (\mathbb{C}^*)^n$ and for $t \to t_0$,*

$$\omega_t \xrightarrow{C^\infty(K)} \omega_{t_0}$$

Theorem 5.1.1 can be considered to be an analytic analogue of the weak factorization theorem 2.4.2 for toric varieties in algebraic geometry. Combined with Theorem 4.1.1, it implies that any two toric manifolds of same dimension can be joined by a continuous path of conical Kähler-Einstein spaces in Gromov-Hausdorff topology. More generally, it is a natural question to ask if for any two birationally equivalent log Fano manifolds, there exists a continuous path connecting them by log Fano varieties coupled with conical Kähler-Einstein metrics, in Gromov-Hausdorff topology. This is related to the connectedness of moduli space of log Fano varieties coupled with conical Kähler-Einstein metrics.

5.2 Reducing to the case of one blow-up.

Let us fix a toric manifold Y with an ample toric line bundle $\mathcal{L}$ and corresponding polytope P. Let X be the blow-up of Y along a k-dimensional smooth toric variety V with $\pi : X \to Y$ as the blow-down map. Set $L_t = \pi^*\mathcal{L} + tA$ for some ample toric line bundle A on X and for $t \in [0, 1]$. Recalling the definition of the invariant $\mathcal{R}$, we make the following simple observation

Lemma 5.2.1. *Let (X_t, L_t) be a family of toric manifolds with ample $\mathbb{R}$-line bundles L_t for $t \in (0, 1]$. Then as long as the corresponding polytopes P_t stay bounded, we have*

$$\inf_t \mathcal{R}(X_t, L_t) > 0. \tag{5.2.1}$$

Proof. By Corollary 4.1.1

$$\begin{aligned}\mathcal{R}(X, L) &= \sup\{\alpha | 1 - \alpha l_j(P_C) > 0,\ j = 1, \dots, N\} \\ &= \inf_j \{\frac{1}{l_j(P_C)}\},\end{aligned}$$

which stays bounded away from zero if the polytopes stay bounded. □

For $t \in [0, t_0]$ we can now choose a continuous path α_t such that

$$0 < \alpha_t < \min(R(X, L_t), R(Y, \mathcal{L}))$$

and let ω_t be the unique toric conical Kähler-Einstein metrics in $c_1(L_t) \cap \mathcal{K}_c(X)$ with Einstein constant α_t. We also let $\omega_Y \in c_1(\mathcal{L}) \cap \mathcal{K}_c(Y)$ be the the toric conical Kähler-Einstein metric on Y with Einstein constant α_0. Denoting the corresponding Riemannian metrics by g_t and g_Y, we have the following proposition.

Proposition 5.2.1. (X, g_t) *is a continuous path in the Gromov-Hausdorff topology and*

$$(X, g_t) \xrightarrow{t \to 0} (Y, g_Y).$$

By restricting α_t to be less than $R(X, L_t)$ we ensure that g_t are geodesically convex, thus facilitating the application of tools from comparison geometry. In particular, we will make use of lemma 2.5.1, or rather its generalization to the conical setting (cf. Remark 3.5.1). Taking the above proposition for granted, we now prove Theorem 5.1.1

Proof of Theorem 5.1.1. We fix (X_j, ω_j) for $j = 0, 1$ as in the statement of the theorem and we let L_j be the Kähler class of ω_j and $\alpha_j < R(X, L_j)$, and we call $(L_j, \omega_j, \alpha_j)$ compatible triples on X_j. By the factorization theorem 2.4.2, there exist a sequence $0 = t_0 < t_1 < \ldots < t_k = 1$ and pairs (X_{t_i}, f_{t_i}) such that

$$X = X_0 \overset{f_{t_1}}{\dashrightarrow} X_{t_1} \overset{f_{t_2}}{\dashrightarrow} \cdots \overset{f_{t_i}}{\dashrightarrow} X_{t_i} \overset{f_{t_{i+1}}}{\dashrightarrow} \cdots \overset{f_{t_k}}{\dashrightarrow} X_{t_k} = X_1.$$

We start from the left and construct the family of metrics inductively. Suppose we are at stage t_i i.e we have already constructed $(X_{t_i}, L_{t_i}, \alpha_{t_i}, \omega_{t_i})$. Then there are two cases

Case-1 - $f_{t_{i+1}}$ is a blow-down map.

On $X_{t_{i+1}}$, we take an arbitrary choice of compatible triples $(L_{t_{i+1}}, \alpha_{t_{i+1}}, \omega_{t_{i+1}})$. Then we connect this to $(X_{t_i}, L_{t_i}, \alpha_{t_i}, \omega_{t_i})$ in two steps. Fix a $\mu \in (t_i, t_{i+1})$ and ample line bundle A on X_{t_i}.

Step-1 For $t \in [\mu, t_{i+1}]$, set

$$\begin{aligned}
&X_t = X_{t_i} \\
&L_t = f^*_{t_{i+1}} L_{t_{i+1}} + (t_{i+1} - t)A \\
&\alpha_t < \min(R(X_t, L_t), R(X_{t_{i+1}}, L_{t_{i+1}}))
\end{aligned}$$

where we choose α_t to be continuous. We now let ω_t be the α_t - conical Kähler-Eisntein metric in L_t. Then by Proposition 5.2.1, (X_t, ω_t) is continuous for $t \in [\mu, t_{i+1})$ and converges to $(X_{t_{i+1}}, \omega_{t_{i+1}})$ in the Gromov-Hausdorff topology.

Step-2 - For $t \in [t_i, \mu]$ set

$$
\begin{aligned}
X_t &= X_{t_i} \\
L_t &= \frac{\mu - t}{\mu - t_i} L_{t_i} + \frac{t - t_i}{\mu - t_i} L_\mu \\
\alpha_t &< R(X_t, L_t)
\end{aligned}
$$

Again, let ω_t be the corresponding α_t - conical Kähler-Einstein metrics. Since in this case, L_t is uniformly Kähler all the estimates of Proposition 5.2.1 go through and we in fact get that (X_{t_i}, ω_t) is continuous in t in the smooth topology on X_{t_i}.

Case 2. $f_{t_{i+1}}$ ***is a blow-up map.*** This can be treated by the same argument as in Case 1 by moving t backward from t_{i+2} to t_{i+1}.

The smoothness of g_t on the complex torus $(\mathbb{C}^*)^n$ follows if we take α_t to be a smooth path in t.

□

5.3 Uniform estimates and proof of Proposition 5.2.1.

In this section we prove Proposition 5.2.1, thereby completing the proof of Theorem 5.1.1. Throughout the section we fix an $\alpha > 0$, such that $\alpha \in (0, \min(R(X, L_t), R(Y, \mathcal{L})))$.

Without loss of generality, we can assume that the polytope P that induces the toric manifold Y is given by $(N-1)$ defining functions $l_j(x) = v_j \cdot x + \lambda_j \geq 0$, $j = 1, \ldots, N-1$. Let P^A be the poytope corresponding to the ample line bundle A on X with N defining functions $l_j^A(x) = v_j \cdot x + \lambda_j^A$ for $j = 1, \ldots, N$. The blow-up process corresponds to the $(n-1)-$dimensional face given by l_N contracting to a k-dimensional face given by the intersection of $(n-k)$ co-dimension one faces, say $l_1, \ldots, l_{n-k}$. We denote the divisor corresponding to l_j on X by D_j with defining section s_j, while on Y we denote the divisor corresponding to l_j by $\tilde{D}_j$ and the corresponding section by $\tilde{s}_j$. Then it follows from the definition of blow-ups that

$$
\begin{cases}
\pi^* \tilde{D}_j = D_j + D_N, & j = 1, \ldots, n-k, \\
\pi^* \tilde{D}_j = D_j, & j = n-k+1, \ldots, N-1,
\end{cases}
\tag{5.3.2}
$$

where as before D_j denotes the divisor corresponding to l_j and π^* is the total transform. Using this fact, one can explicitly write down the polytope P^t for L_t by defining

$$\begin{cases} l_j^t(x) = v_j \cdot x + \lambda_j + t\lambda_j^A, & j = 1, \dots, N-1, \\ l_N^t(x) = v_N \cdot x + (\sum_{j=1}^{n-k} \lambda_j) + t\lambda_N^A, \end{cases}$$

where $v_N = \sum_{j=1}^{n-k} v_j$.

We denote the barycenters of the evolving polytopes by P_C^t and the corresponding angles by $\beta_j^t = \alpha l_j^t(P_C^t)$. We then set l_j^0, P_C^0 and β_j^0 to be the limit of the respective quantities as t goes to zero. We first prove an important identity that will be very useful, among other things, in proving that the limiting Monge-Ampere equation descends to the Einstein equation on Y.

Lemma 5.3.1.

$$(1 - \beta_N^0) - \sum_{j=1}^{n-k} (1 - \beta_j^0) = -(n - k - 1). \tag{5.3.3}$$

Proof. Since D_N is obtained by blowing up the intersection of $D_1, \dots, D_{n-k}$, it is well known that

$$v_N = \sum_{j=1}^{n-k} v_j.$$

Now at $t = 0$, there are $(n - k + 1)$ affine linear functions $l_1, \dots, l_{n-k}$ and l_N vanishing on a k- dimensional face. So they must be linearly related i.e there exist real numbers a_j so that

$$l_N^0 = \sum_{j=1}^{n-k} a_j l_j^0.$$

But then, since v_j's are linearly independent, the two equations together force the a_j's to be one i.e

$$l_N^0 = \sum_{j=1}^{n-k} l_j^0.$$

The lemma now follows.

□

Example 5.3.1. *Let $X = \mathbb{P}^2 \# \overline{\mathbb{P}^2}$ and $Y = \mathbb{P}^2$. On Y we take $\mathcal{L}$ to be the anti-canonical bundle and the corresponding $P \subset \mathbb{R}^2$ to be defined by $\{x+1 \geq 0, y+1 \geq 0, 1-x-y \geq 0\}$. One can view X as a projective bundle over $\mathbb{P}^1$ with a zero section D_0 and a section D_∞ at infinity. We take A to be $2[D_\infty] - [D_0]$, with the polytope P^A given by $\{x \geq 0, y \geq 0, 2-x-y \geq 0, -1+x+y \geq 0\}$. It follows from the Nakai criteria that A is ample. Then the polytope for L_t is given by the inequalities $\{x+1 \geq 0, y+1 \geq 0, 1+2t-x-y \geq 0, 2-t+x+y \geq 0\}$. Computing the β_j^0 for this example we see that $\beta_1^0 = 1, \beta_2^0 = 1, \beta_3^0 = 1, \beta_4^0 = 2$. One can now easily verify Lemma 5.3.1 for this simple example.*

Now let ω_t and ω_Y be the unique toric conical Kähler-Einstein metrics with Einstein constant α on X and Y in the class $c_1(L_t)$ and $c_1(\mathcal{L})$ respectively. In section 3, we worked with conical reference metrics coming from the symplectic potential. However, for dealing with convergence issues as the Kähler class degenerates, it is more convenient to work with smooth reference forms. So we pick a Kähler form $\tilde{\omega}_Y \in c_1(\mathcal{L})$ and a Kähler form $\chi \in c_1(A)$. More explicitly, by taking the embedding of Y into a big projective space via the sections coming from the lattice points of P, we can set $\tilde{\omega}_Y$ to be the pull-back of the Fubini-Study metric. One can make a similar choice for χ. We then set $\tilde{\omega}_t = \pi^*\tilde{\omega}_Y + t\chi$. Clearly, there exist locally bounded functions ψ_t such that $\omega_t = \tilde{\omega}_t + \sqrt{-1}\partial\bar{\partial}\psi_t$. We similarly have a potential ψ_Y for ω_Y.

Lemma 5.3.2. *There exists a uniform constant C independent of t such that*

$$\sup_X \psi_t - \inf_X \psi_t \leq C. \tag{5.3.4}$$

Proof. We fix a volume form on X say $\Omega = \chi^n$. Recall from section 3, that ω_t is given locally on $(\mathbb{C}^*)^n$ by $\omega_t = \sqrt{-1}\partial\overline{\partial}\varphi_t$, where φ_t is a function only of $\rho \in \mathbb{R}^n$ and satisfies the real Monge-Ampere

$$\det(\nabla^2\varphi_t) = e^{-\alpha(\varphi_t - P_C^t \cdot \rho)} = e^{-w_t}$$

The volume for ω_t^n is given by

$$\omega_t^n = (\det \nabla^2\varphi_t) d\rho_1 \wedge \ldots \wedge \rho_n \wedge d\theta_1 \ldots \wedge d\theta_n$$

All the estimates in the proof of Proposition 4.4.1 remain uniform under small perturbations of the polytope. In particular, the estimate (4.4.40) holds with constants ζ and C independent of t. That is

$$\det(\nabla^2\varphi_t) < Ce^{-\zeta|\rho|}.$$

Similarly on $(\mathbb{C}^*)^n$, $\chi = \sqrt{-1}\partial\overline{\partial}\phi$. Since χ is the pull back of the Fubini-Study metric, one can take

$$\phi = \log\big(\sum_{\nu\in P^A\cap\mathbb{Z}^n} e^{\nu\cdot\rho}\big).$$

By an elementary calculation, there exist constant B_1, B_2, B_3, B_4 depending only on P^A such that

$$B_3e^{-B_1|\rho|} < \det(\nabla^2\phi) < B_4e^{-B_2|\rho|}.$$

Now, consider the trivial identity

$$(\tilde{\omega}_t + \sqrt{-1}\partial\overline{\partial}\psi_t)^n = \omega_t^n = \frac{\omega_t^n}{\Omega}\Omega. \tag{5.3.5}$$

We claim that ω_t^n/Ω is in $L^{1+\epsilon}(X,\Omega)$ for some $\epsilon > 0$. This is because

$$\begin{aligned}
\int_X \left(\frac{\omega_t^n}{\Omega}\right)^{1+\epsilon}\Omega &= \int_{(S^1)^n}\int_{\mathbb{R}^n}\left(\frac{\det(\nabla^2\varphi_t)}{\det(\nabla^2\phi)}\right)^{1+\epsilon}\det(\nabla^2\phi)\,d\rho d\theta \\
&\leq C\int_{\mathbb{R}^n}\left(\frac{e^{-\zeta|\rho|}}{e^{-B_1|\rho|}}\right)^{\epsilon} e^{-B_2|\rho|}\,d\rho \\
&\leq C\int_{\mathbb{R}^n} e^{-(B_2+\epsilon(B_1-\zeta))|\rho|}\,d\rho \\
&\leq C
\end{aligned}$$

if ϵ is small enough. Then in view of (5.3.5), since we have a uniform $L^{1+\epsilon}(X,\Omega)$ bound on ω_t^n/Ω, applying [49, 37, 85] we directly obtain a uniform bound on the oscillation of ψ_t.

□

We now spend some time in deriving a complex Monge-Ampere equation satisfied by ψ_t. Let Ω and Ω_Y be two fixed volume forms on X and Y respectively and let ξ_Y, ξ_A be metrics on $\mathcal{L}$ and A such that $\omega_Y = -\sqrt{-1}\partial\overline{\partial}\log\xi_Y$ and $\chi = -\sqrt{-1}\partial\overline{\partial}\log\xi_A$.

One can also view Ω and Ω_Y as metrics on $-K_X$ and $-K_Y$. We recall the adjunction formula

$$K_X = \pi^* K_Y + (n-k-1)[D_N].$$

By the $\partial\bar{\partial}$-lemma there exists a metric h_N on $[D_N]$ such that

$$\Omega = \frac{\pi^*\Omega_Y}{|s_N|_{h_N}^{2(n-k-1)}}. \tag{5.3.6}$$

Next, since $-K_Y = \alpha\mathcal{L} + \tilde{D}$, one can choose smooth hermitian metrics $\tilde{h}_1, \ldots, \tilde{h}_{N-1}$ on $\tilde{D}_1, \ldots, \tilde{D}_{N-1}$ such that

$$\prod_{j=1}^{N-1} \tilde{h}_j^{(1-\beta_j^Y)} = \pi^*\left(\frac{\Omega_Y}{(\xi_Y)^\alpha}\right). \tag{5.3.7}$$

Using 5.3.2, we then define smooth metrics on D_j for $j < N$ by setting

$$\begin{cases} h_j = \pi^*\tilde{h}_j / h_N & j = 1, \ldots, n-k \\ h_j = \pi^*\tilde{h}_j & j = n-k+1, \ldots, N-1. \end{cases} \tag{5.3.8}$$

Finally we define a family of metrics on $[D_N]$ by

$$h_N(t) = \left(\frac{\Omega\xi_A^{-t\alpha}\pi^*(\xi_Y^{-\alpha})}{\prod_{j=1}^{N-1} h_j^{(1-\beta_j^t)}}\right)^{\frac{1}{1-\beta_N^t}}. \tag{5.3.9}$$

We claim

Lemma 5.3.3. *At all points of X,*

$$\lim_{t\to 0}\frac{h_N(t)}{h_N} = 1. \tag{5.3.10}$$

Proof. If we consider Ω as a metric on $-K_X$ and $\pi^*\Omega_Y$ as a pull back metric on $-\pi^*K_Y$, then by equation (5.3.6), $\Omega = \pi^*\Omega_Y h_N^{-(n-k-1)}$.

$$\begin{aligned}
\lim_{t\to 0}\frac{h_N(t)}{h_N} &= \left(\frac{\Omega\pi^*(\xi_Y)^{-\alpha}}{h_N^{(1-\beta_N^0)}\prod_{j=1}^{N-1} h_j^{(1-\beta_j^0)}}\right)^{\frac{1}{1-\beta_N^0}} \\
&= \left(\frac{\pi^*(\Omega_Y\xi_Y^{-\alpha})h_N^{-(n-k-1)}}{h_N^{(1-\beta_N^0)}\prod_{j=1}^{N-1} h_j^{(1-\beta_j^0)}}\right)^{\frac{1}{1-\beta_N^0}} \\
&= \left(\frac{\pi^*(\Omega_Y\xi_Y^{-\alpha})}{\prod_{j=1}^{N-1}\pi^*\tilde{h}_j^{(1-\beta_j^Y)}}\right)^{\frac{1}{1-\beta_N^0}}
\end{aligned}$$

$$= 1,$$

where we used lemma 5.3.1, equation (5.3.8) in line three and equation (5.3.7) in line four.

□

By applying logarithm and taking $\partial\bar{\partial}$ we see that $h_1, \ldots, h_{N-1}$ and $h_N(t)$ satisfy

$$-\partial\bar{\partial}\log\Omega = \alpha\tilde{\omega}_t - \sum_{j=1}^{N-1}(1-\beta_j^t)\partial\bar{\partial}\log h_j - (1-\beta_N^t)\partial\bar{\partial}\log h_N(t).$$

The purpose of the above constructions is that ψ_t and ψ_Y now satisfy, possibly after modification by some constant, the following Monge-Ampere equations

$$(\tilde{\omega}_t + \sqrt{-1}\partial\overline{\partial}\psi_t)^n = \frac{e^{-\alpha\psi_t}\Omega}{|s_N|_{h_N(t)}^{2(1-\beta_N^t)}\prod_{j=1}^{N-1}|s_j|_{h_j}^{2(1-\beta_j^t)}} \qquad (*)_t$$

$$(\tilde{\omega}_Y + \sqrt{-1}\partial\overline{\partial}\psi_Y)^n = \frac{e^{-\alpha\psi_Y}\Omega_Y}{\prod_{j=1}^{N-1}|\tilde{s}_j|_{\tilde{h}_j}^{2(1-\beta_j^Y)}} \qquad (*)_Y$$

We remark that since modification by a constant doesn't change the oscillation, the estimate of lemma 5.3.4 holds for this modified ψ_t. Immediately, we have the following corollary from Lemma 5.3.2 because the total volume of (X, ω_t) is $[L_t]^n$ and is uniformly bounded.

Corollary 5.3.1. *There exists a unique constant $C > 0$ such that for all $t \in (0, 1]$,*

$$||\psi_t||_{L^\infty(X)} \le C. \qquad (5.3.11)$$

We next prove uniform estimates away from D on all derivatives of ψ_t

Proposition 5.3.1. *For all $l \ge 0$ and $K \subset\subset X\backslash D$, there exist constants $C_{K,l}$ independent of t such that*

$$||\psi_t||_{C^l(K)} < C_{K,l}. \qquad (5.3.12)$$

Here the norm is with respect to some fixed reference metric.

Proof. Since the usual arguments for $C^{2,\gamma}$ estimates, using the theory of Evans, Kryllov and Safanov, are local in nature they can be used in the present context. Hence, to prove the proposition, we only need uniform C^2 estimates.

We follow the argument in [65] using Tsuji's trick [79]. By Kodaira's lemma 2.4.1, $L_\epsilon = \pi^*\tilde{\omega}_y - \epsilon[D_N] > 0$ for small $\epsilon > 0$. So, we pick a new smooth hermitian metric ξ_N on $[D_N]$ such that $\eta = \pi^*\tilde{\omega}_Y + \epsilon\partial\bar{\partial}\log\xi_N > 0$ and consider

$$Q_t = \log\left(|s_N|^{2(1+A\epsilon)}_{\xi_N}\prod_{j=1}^{N-1}|s_j|^2_{h_j}tr_\eta\omega_t\right) - A\psi_t$$

for some big constant A to be chosen later. We note that Q goes to negative infinity near D. This is because the order of poles of ω_t near each D_j is $2(1-\beta_j^t)$ which is strictly less than two. So for each t, the maximum is attained in $X\backslash D$. Following Yau, we compute $\Delta_t Q_t$ where Δ_t is the Laplacian with respect to ω_t. Since on $X\backslash D$, $Ric(\omega_t) = \alpha\omega_t$, standard calculations show that there exists a constant C depending only on the dimension and curvature of η such that

$$\Delta_t Q_t \geq -Ctr_{\omega_t}\eta + (1+A\epsilon)\Delta_t\log\xi_N + \sum_{j=1}^{N-1}\Delta_t\log h_j + Atr_{\omega_t}\tilde{\omega}_t - C.$$

Also, there exists a constant C' independent of t such that

$$\Delta_t\log\xi_N > -C'tr_{\omega_t}\eta,$$

$$\Delta_t\log h_j > -C'tr_{\omega_t}\eta.$$

Combining this with the above estimate

$$\begin{aligned}\Delta_t Q_t &\geq -Ctr_{\omega_t}\eta + Atr_{\omega_t}(\tilde{\omega}_t + \epsilon\partial\bar{\partial}\log\xi_N) - C\\ &= -Ctr_{\omega_t}\eta + Atr_{\omega_t}(\eta + t\chi) - C\\ &> tr_{\omega_t}\eta - C,\end{aligned}$$

where we choose $A = C+1$. So, at the maximum point of Q_t, $tr_{\omega_t}\eta(p_t) < C$. Now, standard arguments show

$$\left(|s_N|^{2(1+A\epsilon)}_{\xi_N}\prod_{j=1}^{N-1}|s_j|^2_{h_j}\right)tr_\eta\omega_t < Ce^{\sup\psi_t - \inf\psi_t}\left(|s_N|^{2(1+A\epsilon)}_{\xi_N}\prod_{j=1}^{N-1}|s_j|^2_{h_j}\frac{\omega_t^n}{\eta^n}\right)(p_t).$$

Using the equation and the oscillation bound on ψ_t and the fact that $\beta_j^t < 1$,

$$tr_\eta \omega_t < \frac{C}{\left(|s_N|_{\xi_N}^{2(1+A\epsilon)} \prod_{j=1}^{N-1} |s_j|_{h_j}^2\right)}.$$

Hence, we have uniform second order estimates away from D and this completes the proof of the proposition.

□

With the above uniform local estimates and the uniqueness of ω_Y, we have the following local uniform convergence away from the divisors.

Proposition 5.3.2. *For any compact subset $K \subset\subset X \setminus D$, we have the following uniform convergence*

$$\omega_t \xrightarrow{C^\infty(K)} \omega_Y.$$

Using Moser iteration, one can in fact show that ψ_t converges to $\pi^*\psi_Y$ globally in L^∞. We now have to prove the global convergence, in Gromov-Hausdorff topology, of (X, ω_t) to (Y, ω_Y).

Proof of Proposition 5.2.1. We let $t \to 0$. Fix an $\epsilon > 0$. Let E be a tubular neighborhood of $D \subset Y$ such that $A = Y \backslash E$ is ϵ-dense in Y with respect to the metric g_Y. Note, that since X and Y are bi-holomorphic away from D, A can be identified as a subset of X. We also pick E close enough to D so that $Vol_{g_Y}(E) < \epsilon^{4n}$ and we set $\tilde{E} = \pi^*(E)$. Finally, we denote the distances with respect to g_t and g_Y, by d_t and d_Y respectively.

Claim 1. For t small enough, $A = Y \backslash E = X \backslash \tilde{E}$ is ϵ - dense in (X, g_t).

Proof. If not, then there exists a sequence $t_k \to 0$ and points $x_k \in \tilde{E}$ such that $B_{g_{t_k}}(x_k, \epsilon) \subset \tilde{E}$. Using volume comparison, uniform diameter bounds and the fact that the volumes converge, for small t_k

$$\kappa \epsilon^{2n} < Vol_{g_k}(B_{g_k}(x_k, \epsilon)) < Vol_{g_k}(\tilde{E}) < 2Vol_{g_Y}(E) < 2\epsilon^{4n}$$

for some constant κ if k is sufficiently large. But if ϵ is small enough, this is a contradiction.

Claim 2. There exists a $t(\epsilon)$ such that for all $0 < t < t(\epsilon)$ and for all $p, q \in A$,

$$d_t(p,q) < d_Y(p,q) + \epsilon$$

Proof. By the geodesic convexity of $Y \backslash D$, one can choose a small tubular neighborhood, $T \subset E$ of D in Y such that any two points in A can be connected by a g_Y - minimal geodesic in $Y \backslash T$. Set $\tilde{T} = \pi^{-1}(T)$. Let γ be a g_Y - minimal geodesic connecting p and q lying in $Y \backslash T$. Since the metrics converge uniformly on compact sets of $X \setminus \tilde{E}$, for t sufficiently small,

$$d_t(p,q) < \mathcal{L}_t(\gamma) < \mathcal{L}_Y(\gamma) + \epsilon = d_Y(p,q) + \epsilon,$$

where $\mathcal{L}$ denotes the length functional.

Claim 3. There exists a $t(\epsilon)$ such that for all $0 < t < t(\epsilon)$ and for all $p, q \in A$,

$$d_t(p,q) > d_Y(p,q) - \epsilon$$

Proof. The proof of this claim relies on the generalization of Lemma 2.5.1 to the conical setting (cf. Remark 3.5.1). We once again choose a tubular neighborhood T of D contained in E with smooth boundary such that for all $q \in A$,

$$B_{g_Y}(q, \epsilon/2) \subset Y \backslash T$$

$$Vol_{g_Y}(\partial T) < \delta/2$$

and set $\tilde{T} = \pi^*(T)$. Again, as in the proof of Prop. 3.1.1, one can choose δ to be arbitrarily small. Since, away from D, the metric converges uniformly, we can assume that $Vol_{g_t}(\partial T) < \delta$ by choosing t sufficiently small. Furthermore, since $d_{g_Y}(q, \partial T) > \epsilon/2$, once again by the uniform convergence of the metric on $X \backslash \tilde{T}$, for small t, $d_{g_t}(q, \partial \tilde{T}) > \epsilon/4$, i.e., $B_{g_t}(q, \epsilon/4) \subset X \backslash \tilde{T}$.

We claim that there exists at least one minimal geodesic from p to a point in $B_{g_t}(q, \epsilon/4)$ that lies entirely in $X \backslash \tilde{T}$. If not, then by Gromov's lemma there exists a constant c uniform in t (but depending on ϵ) such that

$$\kappa\epsilon^{2n} < Vol_{g_t}(B_{g_t}(q, \epsilon/4)) < cVol_{g_t}(\partial\tilde{T}) < c\delta.$$

Letting δ go to zero, we get a contradiction.

So there exists at least one g_t - minimal geodesic γ_t connecting p to a point $\tilde{q}_t \in B_{g_t}(q, \epsilon/4)$. By compactness, there exists a $\tilde{q} \in B_{g_Y}(q, \epsilon/2)$ such that $\tilde{q}_t \to \tilde{q}$ and moreover the geodesics γ_t converge to a curve, denoted by γ, joining p to $\tilde{q}$.

$$\begin{aligned} d_{g_t}(p,q) &> \mathcal{L}_{g_t}(\gamma_t) - \epsilon/4 \\ &> \mathcal{L}_{g_Y}(\gamma) - \epsilon/2 \\ &> d_{g_Y}(p,\tilde{q}) - \epsilon/2 \\ &> d_{g_Y}(p,q) - \epsilon \end{aligned}$$

and this proves Claim 3.

Now we complete the proof of the proposition. For sufficiently small t,

$$\begin{aligned} & d_{GH}((X,d_t),(Y,d_Y)) \\ \leq\ & d_{GH}((X,d_t),(A,d_t)) + d_{GH}((A,d_t),(A,d_Y)) + d_{GH}((A,d_Y),(Y,d_Y)) \\ <\ & 3\epsilon \end{aligned}$$

where we use Claim 1 to bound the first term, Claim 2 and Claim 3 to bound the second term, while the last term is bound by ϵ from the choice of A. Now, letting ϵ go to zero, we see that (X, g_t) converges in Gromov-Hausdorff distance to (Y, g_Y). $\square$

Chapter 6

Greatest Bakry-Emery Ricci lower bound on Fano manifolds

6.1 Introduction.

A Kähler metric $\omega \in c_1(X)$ on a Fano manifold X is called a Kähler-Ricci soliton if there exists a holomorphic vector field $\xi \in \mathfrak{h}(X)$ such that

$$Ric(\omega) - \omega = \mathcal{L}_\xi(\omega) \tag{6.1.1}$$

There are well known obstructions to the construction of Kähler-Einstein metrics on Fano manifolds. Kähler-Ricci solitons provide natural generalizations of Kähler-Einstein metrics, and as such, are suitable candidates for canonical metrics on these manifolds. They are also closely related to the limiting behavior of the normalized Kähler-Ricci flow.

A detailed study of equation (6.1.1), in complete analogy to earlier work of Yau and Aubin on the Kähler-Einstein equation, was initiated in [87]. Building on this, Tian and Zhu [76, 77] generalized the classical results of Bando-Mabuchi to prove uniqueness theorems for Kähler-Ricci solitons. In particular, they showed that the holomorphic vector field ξ is also uniquely determined up to a choice of a maximal compact subgroup of $Aut(X)$. They also introduced the generalized Futaki invariant as an obstruction to the existence of Kähler-Ricci solitons. In the special case of toric manifolds, as was remarked earlier, Kähler-Ricci solitons invariant under the $(S^1)^n$ action are completely classified by the work of Wang and Zhu [80], and the theory takes on a strikingly simple form.

We now fix a holomorphic vector field ξ and a reference metric $\eta \in c_1(X)$ such that $\mathcal{L}_{Im(\xi)}\eta = 0$. A classical approach to study the existence of solutions to (6.1.1) is to consider the following continuity method

$$Ric(\omega_t) = t\omega_t + (1-t)\eta + \mathcal{L}_\xi \omega_t \tag{6.1.2}$$

It is well known [87] that a solution to (6.1.2) at $t = 0$ always exists. But by the remarks above, it is clear that for most vector fields ξ, one cannot solve all the way up to $t = 1$. Our aim in this chapter is to characterize the obstructions to solving equation (6.1.2), alternatively in terms of the Bakry-Emery Ricci curvature, and the properness of a twisted Mabuchi energy. In the context of Kähler-Einstein metric and the Aubin-Yau continuity method, this was done by Székelyhidi [67], and indeed our methods are an adaptation of his work.

Since, any solution ω_t to (6.1.2) is invariant under the flow of $Im(\xi)$, we restrict attention to

$$K_\xi(X) = \{\omega \in c_1(X) \mid \omega \text{ is a smooth Kähler metric with } \mathcal{L}_{Im(\xi)}\omega = 0\}$$

By our choice, the reference metric η belongs to this space. Similar to Chapter 4, we can define the greatest Bakry-Emery Ricci curvature on Fano manifolds with respect to a *fixed* vector field by setting

$$\mathcal{R}_{BE}(X,\xi) = \sup\{t \mid \text{there exists } \omega \in \mathcal{K}_\xi(X) \text{ such that } Ric(\omega) > t\omega + \mathcal{L}_\xi\omega\} \tag{6.1.3}$$

By Zhu's theorem [87], and an openness argument from [76], it is clear that this invariant is strictly positive. On the other hand, by the maximum principle, $\mathcal{R}_{BE}(X,\xi) \leq 1$ for any ξ. Next, we define the twisted Mabuchi energy on $\mathcal{K}_\xi(X)$ by

$$\mathcal{E}^\tau_{\eta,\xi} = \mu_{\eta,\xi} + (1-\tau) i_{\eta,\xi} \tag{6.1.4}$$

where $\mu_{\eta,\xi}$ and $i_{\eta,\xi}$ are the generalized Mabuchi energy and the generalized Aubin-Yau functional respectively. These were introduced by Tian-Zhu [76, 77], and the reader can find the relevant definitions in the next section.

Definition 6.1.1. *We say that $\mathcal{E}^\tau_{\eta,\xi}$ is proper on the space of metrics $\mathcal{K}_\xi(X)$ if there exist constants $\varepsilon, C_\varepsilon > 0$ such that*

$$\mathcal{E}^\tau_{\eta,\xi}(\omega) > \varepsilon i_{\eta,\xi}(\omega) - C_\varepsilon$$

for all $\omega \in \mathcal{K}_\xi(X)$.

We are now ready to state our main theorem.

Theorem 6.1.1. *[32] Let (X, η) be a Fano manifold with $\eta \in c_1(M)$, and $\xi \in \mathfrak{h}(X)$ be a holomorphic vector field such that $\mathcal{L}_{Im(\xi)}\eta = 0$. Then for any $0 \leq \tau < 1$, the following are equivalent :*

1. $0 \leq \tau < \mathcal{R}_{BE}(M, \xi)$.
2. *There exists a $\tau' > \tau$, and $C > 0$ such that*
$$\mathcal{E}^{\tau'}_{\eta,\xi}(\omega) > -C \; ; \; \forall \omega \in \mathcal{K}_\xi(X)$$
3. $\mathcal{E}^{\tau}_{\eta,\xi}$ *is proper on* $\mathcal{K}_\xi(X)$.
4. *There exists a solution $\omega_t \in \mathcal{K}_\xi(X)$ to (6.1.2), for all $t \in [0, \tau]$.*

6.2 The generalized Mabuchi and Aubin-Yau functionals of Tian-Zhu.

We define, and collect some basic facts about generalizations of the Mabuchi and the Aubin-Yau functionals introduced by Tian-Zhu in [76, 77]. By the $\partial\bar{\partial}$-lemma 2.1.2, for any Kähler metric $\omega \in c_1(X)$ there exists a unique function $h_\omega \in C^\infty(X)$ such that

$$\begin{cases} Ric(\omega) - \omega = \sqrt{-1}\partial\bar{\partial}h_\omega \\ \int_X e^{h_\omega}\omega^n = \int_X \omega^n \end{cases} \tag{6.2.5}$$

The function h_ω is called the *Ricci potential.* If ξ is a holomorphic vector field, then it is clear that $i_\xi\omega$ is a $\bar{\partial}$-closed $(0,1)$ form. But any Fano manifold is simply connected, and hence there exists a unique potential $\theta_\xi(\omega)$ such that

$$\begin{cases} i_\xi\omega = \bar{\partial}\theta_\xi(\omega) \\ \int_X e^{\theta_\xi(\omega)}\omega^n = \int_X \omega^n \end{cases} \tag{6.2.6}$$

It is then easy to check the following.

Lemma 6.2.1. *If $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ is another Kähler metric in $c_1(X)$, then*

$$\theta_\xi(\omega_\varphi) = \theta_\xi(\omega) + \xi(\varphi)$$

Recall that $K_\xi(X)$ is the space of Kähler metrics in $c_1(X)$ which are invariant under the flow of $Im(\xi)$. This space can be identified with the space of potentials

$$\mathcal{H}_\xi(X,\eta) = \{\varphi \in C^\infty(X) \mid \eta_\varphi = \eta + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}$$

We next state an estimate of Zhu [87], that was proved in the toric context in section 4.3 (cf. Lemma 4.3.5).

Lemma 6.2.2. *There exists a uniform constant C such that for any $\varphi \in \mathcal{H}_\xi(X,\eta)$,*

$$|\xi(\varphi)| < C$$

Definition 6.2.1. *For any $\omega = \eta + \partial\bar{\partial}\varphi \in \mathcal{K}_\xi(X)$, define the generalized Mabuchi, and Aubin-Yau functionals as*

$$\mu_{\eta,\xi}(\omega) = -\int_0^1 \fint_X \dot{\varphi}_t[\sigma_t - n - \Delta\theta_t + \xi(h_t - \theta_t)]e^{\theta_t}\omega_t^n \, dt \tag{6.2.7}$$

$$i_{\eta,\xi}(\omega) = (I_{\eta,\xi} - J_{\eta,\xi})(\omega) = \fint_X \varphi(e^{\theta_\xi(\eta)}\eta^n - e^{\theta_\xi(\omega)}\omega^n) + \int_0^1 \fint_X \dot{\varphi}_t(e^{\theta_\xi(\eta)}\eta^n - e^{\theta_t}\omega_t^n) \tag{6.2.8}$$

where $\omega_t = \eta + \partial\bar{\partial}\varphi_t \in \mathcal{K}_\xi(X)$ is a path joining η to ω and $\theta_t = \theta_\xi(\omega_t) = \theta_\xi(\eta) + \xi(\varphi_t)$.

Note that $i_{\eta,\xi}(\eta) = \mu_{\eta,\xi}(\eta) = 0$. It is easy to see that the first variations of the functionals $\mu_{\eta,\xi}$ and $i_{\eta,\xi}$ are given by

$$\delta\mu_{\eta,\xi}(\omega) = -\fint_X (\delta\varphi)[\sigma_\omega - n - \Delta_\omega\theta_\xi(\omega) + \xi(h_\omega - \theta_\xi(\omega))]e^{\theta_\xi(\omega)}\omega^n \tag{6.2.9}$$

$$\delta i_{\eta,\xi}(\omega) = -\fint_X (\delta\varphi)[\Delta_\omega\varphi + \xi(\varphi)]e^{\theta_\xi(\omega)}\omega^n \tag{6.2.10}$$

where $\omega = \eta + \sqrt{-1}\partial\bar{\partial}\varphi$ and $\theta_\xi(\omega) = \theta_\xi(\eta) + \xi(\varphi)$.

A basic fact is that the functional $i_{\eta,\xi}$ is always positive. This follows from the following estimate. For the convenience of the readers, we include their proof.

Lemma 6.2.3. *[76, Lemma 3.3] There exists a uniform constant $\delta > 0$ such that for any $\omega = \eta + \sqrt{-1}\partial\bar{\partial}\varphi \in \mathcal{K}_\xi(X)$, we have that*

$$i_{\eta,\xi}(\omega) \geq \delta \fint_X \varphi(\eta^n - \omega^n)$$

In particular, $i_{\eta,\xi}$ is a positive functional on $\mathcal{K}_\xi(X)$.

Proof. Let $\omega_t = \eta + t\sqrt{-1}\partial\overline{\partial}\varphi$, $\theta_t = \theta_\xi(\omega_t)$ and $\Delta_t = \Delta_{\omega_t}$. By Lemma 6.2.2, there exists a uniform constant C independent of φ such that $|\theta_t| < C$. By equation (6.2.10) above,

$$\begin{aligned}\frac{d}{dt} i_{\eta,\xi}(\omega_t) &= -\fint_X t\varphi[\Delta_t\varphi + \xi(\varphi)]e^{\theta_t}\omega_t^n \\ &= \fint_X t|\nabla_t\varphi|^2 e^{\theta_t}\omega_t^n \\ &\geq tne^{-C}\fint_X i\partial\varphi \wedge \bar{\partial}\varphi \wedge \omega_t^n\end{aligned}$$

Integrating this while setting $\delta_0 = ne^{-C}$, we get

$$\begin{aligned}i_{\eta,\xi}(\omega) &\geq \delta_0 \fint_X i\partial\varphi \wedge \bar{\partial}\varphi \wedge \sum_{j=0}^n \binom{n}{j}\Big(\int_0^1 t^{j+1}(1-t)^{n-j}dt\Big)\omega^j \wedge \eta^{n-j} \\ &= \delta_0 \fint_X i\partial\varphi \wedge \bar{\partial}\varphi \wedge \sum_{j=0}^n \frac{j+1}{(n+1)(n+2)}\omega^j \wedge \eta^{n-j} \\ &\geq \frac{\delta_0}{(n+1)(n+2)}\fint_X \sum_{j=0}^n (\eta - \omega) \wedge \omega^j \wedge \eta^{n-j} \\ &\geq \delta \fint_X \varphi(\eta^n - \omega^n)\end{aligned}$$

where we can take $\delta = \delta_0/(n+1)(n+2)$. □

Recall that the operator

$$L_\omega = \Delta_\omega + \xi$$

appearing in the second formula above was introduced in section 4.3, and plays a role similar to that of the Laplacian in the study of Kähler-Einstein metrics. Indeed L is the linearized operator for the Kähler-Ricci soliton equation (6.1.1), and is self-adjoint with respect to the following inner product

$$\langle f, g\rangle = \int_X f\bar{g}\ e^{\theta_\xi(\omega)}\omega^n$$

We denote the corresponding L^2 space by $L^2(e^{\theta_\xi(\omega)})$. We then have the following eigenvalue estimate.

Lemma 6.2.4. *If $Ric(\omega) > t\omega + \mathcal{L}_\xi\omega$, the any eigenvalue λ of $-L_\omega$ satisfies $\lambda > t$.*

We end this section with an elementary lemma regarding the effect on the functionals on change of reference metrics. This is required in the proof of Proposition 6.3.1 below.

Lemma 6.2.5. *Let $\omega = \eta + \partial\bar{\partial}\phi \in \mathcal{K}_\xi$, then*

$$\mu_{\eta,\xi}(\omega) + \mu_{\hat{\omega},\xi}(\eta) + \mu_{\omega,\xi}(\hat{\omega}) = \mu_{\eta,\xi}(\omega) - \mu_{\eta,\xi}(\hat{\omega}) + \mu_{\omega,\xi}(\hat{\omega}) = 0 \tag{6.2.11}$$

$$i_{\eta,\xi}(\hat{\omega}) - i_{\omega,\xi}(\hat{\omega}) = i_{\eta,\xi}(\omega) - \fint_M \varphi(e^{\theta_\xi(\hat{\omega})}\hat{\omega}^n - e^{\theta_\xi(\omega)}\omega^n) \tag{6.2.12}$$

$$i_{\omega,\xi}(\eta) = -i_{\eta,\xi}(\omega) + I_{\eta,\xi}(\omega) \tag{6.2.13}$$

Proof. The first line is just the usual co-cycle property of the Mabuchi energy. The last line follows from the first by choosing $\hat{\omega} = \eta$. So we only need to prove the second equality. Let $\hat{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\psi$ and $\omega_t = \omega + t\sqrt{-1}\partial\bar{\partial}\psi$. Then

$$\begin{aligned}
\frac{d}{dt}(i_{\eta,\xi}(\omega_t) - i_{\omega,\xi}(\omega_t)) &= -\fint_M \psi[\Delta_t\varphi + \xi(\varphi)]e^{\theta_\xi(\omega_t)}\omega_t^n \\
&= -\fint_M \varphi[\Delta_t\psi + \xi(\psi)]e^{\theta_\xi(\omega_t)}\omega_t^n \\
&= -\frac{d}{dt}\Big(\fint_M \varphi e^{\theta_\xi(\omega_t)}\omega_t^n\Big)
\end{aligned}$$

Integrating this from $t = 0$ to $t = 1$, completes the proof of (6.2.12).

□

6.3 Proof of the main theorem.

The key technical ingredient in the proof of Theorem 6.1.1 is

Proposition 6.3.1. *Suppose $\omega_\tau \in \mathcal{K}_\xi(X)$ solves*

$$Ric(\omega_\tau) = \tau\omega_\tau + (1-\tau)\eta + \mathcal{L}_\xi\omega_\tau \tag{6.3.14}$$

Then, for any $\omega \in \mathcal{K}_\xi(X)$

$$\mathcal{E}^\tau_{\eta,\xi}(\omega) \geq \mathcal{E}^\tau_{\eta,\xi}(\omega_\tau) \tag{6.3.15}$$

with equality if and only if $\omega = \omega_\tau$.

In [67], this is proved for $\xi = 0$. The proof there uses a variational argument based on the work of Chen-Tian [21]. In the present piece of work, we give a more direct argument based on some standard monotonicity formulas and the continuity method.

We first recall the following extension, by Tian-Zhu, of the backward continuity method of Bando-Mabuchi [5] to the setting of solitons. The readers should refer to [76] for a proof.

Lemma 6.3.1 (Backward continuity)**.** *Suppose there exists a unique solution to* (6.1.2) *at $t = \tau$. Then for any $\omega = \eta + \sqrt{-1}\partial\overline{\partial}\varphi \in \mathcal{K}_\xi(X)$, the family of equations*

$$\begin{cases} (\omega + \sqrt{-1}\partial\overline{\partial}\nu_t)^n = e^{-t\nu_t + h_\omega + (1-\tau)\varphi - \theta_\xi(\omega) - \xi(\nu_t)}\omega^n \\ \omega_t = \omega + \sqrt{-1}\partial\overline{\partial}\nu_t > 0 \end{cases} \tag{$*_t$}$$

has a unique solution ν_t for $t \in [0, \tau]$.

Proof of Proposition 6.3.1

Step-1 We first claim, that if $\omega = \eta + \sqrt{-1}\partial\overline{\partial}\varphi$

$$\mathcal{E}^\tau_{\omega,\xi}(\omega_\tau) - (1-\tau)\fint_M \varphi(e^{\theta_\xi(\omega_\tau)}\omega_\tau^n - e^{\theta_\xi(\omega)}\omega^n) \tag{6.3.16}$$

$$= \fint_M \psi e^{\theta_\xi(\omega)}\omega^n - \log\Big(\fint_M e^\psi e^{\theta_\xi(\omega)}\omega^n\Big) - \int_0^\tau i_{\omega,\xi}(\omega_t)\,dt \tag{6.3.17}$$

$$\leq 0 \tag{6.3.18}$$

where $\psi = h_\omega + (1-\tau)\varphi - \theta_\xi(\omega)$.

This identity is similar in spirit to Lemma 5.1 in [77]. Once the identity is established, the proof of the negativity follows simply from Jensen's inequality and the positivity of the functional $i_{\omega,\xi}$. The proof of the identity goes back to the original argument of Bando-Mabuchi to prove lower boundedness of Mabuchi energy on Kähler-Einstein manifolds. It is purely computational, and is carried out in two stages. The metric ω_τ is first connected to the metric ω_0 by solving ($*_t$), and then connected to ω along another continuity method of the type used by Yau in the proof of the Calabi conjecture. The twisted Mabuchi energy is computed by integrating along these paths. For $t \leq \tau$, we first consider the following functional

$$\mathcal{F}(t) = \mu_{\omega,\xi}(\omega_t) + (1-t)i_{\omega,\xi}(\omega_t) - (1-\tau)\fint_M \varphi(e^{\theta_\xi(\omega_t)}{\omega_t}^n - e^{\theta_\xi(\omega)}\omega^n) \tag{6.3.19}$$

Then, using the shorthand of θ_t for $\theta_\xi(\omega_t)$ and h_t for the Ricci potential of ω_t,

$$\begin{aligned}\mathcal{F}'(t) &= -\fint_M \dot{\nu}_t[\sigma_t - n - \Delta\theta_t + (1-t)\Delta\nu_t + (1-\tau)\Delta\varphi]e^{\theta_t}\omega_t^n \\ &\quad - \fint_M \dot{\nu}_t[\xi(h_t - \theta_t + (1-t)\nu_t + (1-\tau)\varphi)]e^{\theta_t}\omega_t^n - i_\omega(\omega_t) \\ &= -i_\omega(\omega_t)\end{aligned}$$

where we used the equation $(*_t)$ to conclude that the first two terms are zero. Integrating this equation in t,

$$\mathcal{F}(\tau) = \mathcal{F}(0) - \int_0^\tau i_\omega(\omega_t)\,dt \tag{6.3.20}$$

Next, to compute $\mathcal{F}(0)$ we consider the following extension, introduced by Zhu [87], of Yau's original continutiy method to solve the Calabi conjecture :

$$\begin{cases}(\omega + \partial\bar{\partial}u_s)^n = e^{s\psi - \xi(u_s) + c_s}\omega^n \\ \tilde{\omega}_s = \omega + \partial\bar{\partial}u_s > 0\end{cases} \tag{$*_s$}$$

where $\psi = h_\omega + (1-\tau)\varphi - \theta_\xi(\omega)$ is a fixed function, and c_s are constants given by

$$c_s = -\log\Big(\fint_M e^{s\psi + \theta_\xi(\omega)}\omega^n\Big) \tag{6.3.21}$$

Note that $c_0 = 0$. It is proved in [87], that there always exists a solution to this family for all $t \in [0,1]$. Now, set

$$\mathcal{G}(s) = \mu_{\omega,\xi}(\tilde{\omega}_s) + i_{\omega,\xi}(\tilde{\omega}_s) - (1-\tau)\fint_M \varphi(e^{\theta_\xi(\tilde{\omega}_s)}\tilde{\omega}_s - e^{\theta_\xi(\omega)}\omega^n) \tag{6.3.22}$$

Clearly $\mathcal{G}(1) = \mathcal{F}(0)$ and $\mathcal{G}(0) = 0$. Like before, we differentiate $\mathcal{G}$ and obtain

$$\begin{aligned}\mathcal{G}'(s) &= -\fint_M \dot{u}_s[\sigma_s - n - \Delta\theta_s + \Delta u_s + (1-\tau)\Delta\varphi + \xi(h_s - \theta_s + u_s + (1-\tau)\varphi)]e^{\theta_s}\tilde{\omega}_s^n \\ &= -(1-s)\fint_M \psi L_s(\dot{u}_s)e^{\theta_s}\tilde{\omega}_s^n \\ &= -\frac{d}{ds}\Big(\fint_M (1-s)\psi e^{\theta_s}\tilde{\omega}_s^n\Big) - \fint_M \psi e^{\theta_s}\tilde{\omega}_s^n \\ &= \frac{d}{ds}\Big(c_s - \fint_M (1-s)\psi e^{\theta_s}\tilde{\omega}_s^n\Big)\end{aligned}$$

where we use equation $(*_s)$ and the self adjointness of L in the second line. Integrating this

$$\begin{aligned}\mathcal{G}(1) &= c_1 + \fint_M \psi e^{\theta_\xi(\omega)}\omega^n \\ &= -\log\Big(\fint_M e^{\psi} e^{\theta_\xi(\omega)}\omega^n\Big) + \fint_M \psi e^{\theta_\xi(\omega)}\omega^n\end{aligned}$$

Putting this together with (6.3.20) we prove our claim.

Step-2 We now complete the proof of Prop. 6.3.1 by changing the reference metric from ω to η, and using Lemma 6.2.5 (in the second and third lines)

$$\begin{aligned}0 &\geq \mathcal{E}^{\tau}_{\omega,\xi}(\omega_\tau) - (1-\tau)\fint_M \varphi(e^{\theta_\xi(\omega_\tau)}\omega_\tau^n - e^{\theta_\xi(\omega)}\omega^n) \\ &= \mu_{\omega,\xi}(\omega_\tau) + (1-\tau)i_{\omega,\xi}(\omega_\tau) + (1-\tau)i_{\eta,\xi}(\omega_\tau) - (1-\tau)i_{\omega,\xi}(\omega_\tau) - (1-\tau)i_{\eta,\xi}(\omega) \\ &= \mu_{\eta,\xi}(\omega_\tau) - \mu_{\eta,\xi}(\omega) + (1-\tau)i_{\eta,\xi}(\omega_\tau) - (1-\tau)i_{\eta,\xi}(\omega)\end{aligned}$$

And so, using the definition of $\mathcal{E}^{\tau}_{\eta,\xi}$, we prove

$$\mathcal{E}^{\tau}_{\eta,\xi}(\omega) \geq \mathcal{E}^{\tau}_{\eta,\xi}(\omega_\tau)$$

□

Proof of Theorem 6.1.1 - Clearly, if there exists a metric ω_τ such that $Ric(\omega_\tau) = \tau\omega_\tau + (1-\tau)\eta + \mathcal{L}_\xi\omega_\tau$, then $Ric(\omega_\tau) > \tau\omega + \mathcal{L}_\xi\omega_\tau$ and $\tau < \mathcal{R}_{BE}(M,\xi)$. The strict inequality is due to the fact the the linearized operator $L_{\omega_\tau} + \tau$ is invertible. Hence, 4) $\implies$ 1). To complete the proof of the equivalence, we need to establish the remaining three implications. For any t, we set $L_t = L_{\omega_t}$.

Step-1 : 1) $\implies$ 2)

Proof : By hypothesis, there exist metrics $\alpha, \omega_\tau \in \mathcal{K}_\xi(X)$, such that $Ric(\omega_\tau) = \tau\omega_\tau + (1-\tau)\alpha + \mathcal{L}_\xi\omega_\tau$ with the corresponding Monge-Ampère equation. Since $Ric(\omega_\tau) > \tau\omega_\tau$, by Lemma 6.2.4 the linearized operator of this equation, $L_\tau + \tau$, is invertible and an application of the implicit function theorem shows that there exists a $\tau' > \tau$ such that

$$Ric(\omega) = \tau'\omega + (1-\tau')\alpha + \mathcal{L}_\xi\omega$$

has a solution $\omega \in \mathcal{K}_\xi(X)$. Then by Proposition 6.3.1, $\mathcal{E}^{\tau'}_{\alpha,\xi}$ is lower bounded on $\mathcal{K}_\xi(X)$. By Lemma 6.2.5, for any $\omega \in \mathcal{K}_\xi(X)$

$$\mathcal{E}^{\tau'}_{\eta,\xi}(\omega) = \mathcal{E}^{\tau'}_{\alpha,\xi}(\omega) + \mathcal{E}^{\tau'}_{\eta,\xi}(\alpha)$$

And so, $\mathcal{E}^{\tau'}_{\eta,\xi}$ is also bounded below on $\mathcal{K}_\xi(X)$.

Step-2 : 2) $\implies$ 3)

Proof : The properness follows easily from the observation that

$$\mathcal{E}^{\tau}_{\eta,\xi} = (\tau' - \tau) i_{\eta,\xi} + \mathcal{E}^{\tau'}_{\eta,\xi}$$

Step-3 : 3) $\implies$ 4)

Proof : By now this is pretty standard, and is essentially proved (at least when $\tau = 1$) in [77, 13]. Solving (6.1.2) is equivalent to finding a solution, $\omega_t = \eta + \sqrt{-1}\partial\overline{\partial}\varphi_t \in K_\xi$, to the following family of equations, at $t = \tau$

$$(\eta + \sqrt{-1}\partial\overline{\partial}\varphi_t)^n = e^{-t\varphi - \xi(\varphi_t) + h_\eta - \theta_\xi(\eta)}\eta^n \tag{1.1$_t$}$$

It is well known [83, 76], that the only obstruction to existence are obtaining uniform *a priori* bounds on $\|\varphi_t\|_{C^0(M)}$. The deduction of such an estimate from the properness was observed by Tian [72]. Firstly, by [13], there exists a uniform constant C such that

$$\|\varphi_t\|_{C^0(M)} \leq C(i_{\eta,\xi}(\omega_t) + 1)$$

Hence it is enough to get a uniform upper bound on $i_{\eta,\xi}(\omega_t)$. Differentiating the equation we obtain $L_t\varphi_t = -\varphi_t - t\dot{\varphi}_t$. Now, proceeding as in [67], for $t < \tau$ we compute

$$\begin{aligned}
\frac{d}{dt}(\mu_{\eta,\xi}(\omega_t) + (1-\tau)i_{\eta,\xi}(\omega_t)) &= -\fint_M \dot{\varphi}_t[\sigma_t - n - \Delta\theta_t + \xi(h_t - \theta_t) + (1-\tau)L_t\varphi_t]e^{\theta_t}\omega_t^n \\
&= (\tau - t)\fint_M \dot{\varphi}_t L_t\varphi_t e^{\theta_t}\omega_t^n \\
&= -(\tau - t)\fint_M (t\dot{\varphi}_t + \varphi_t)\varphi_t e^{\theta_t}\omega_t^n \\
&= -(\tau - t)\Big\{\fint_M \varphi_t^2 e^{\theta_t}\omega_t^n - t\fint_M (L_t\dot{\varphi}_t + t\dot{\varphi}_t)\dot{\varphi}_t e^{\theta_t}\omega_t^n\Big\}
\end{aligned}$$

$$\leq 0$$

The last inequality follows from Lemma 6.2.4, since ω_t satisfies $Ric(\omega_t) > t\omega_t + \mathcal{L}_\xi \omega_t$, and hence $L_t + t$ is a negative operator. Combining this with properness,

$$\begin{aligned} \epsilon i_{\eta,\xi}(\omega_t) - C_\epsilon &\leq \mu_{\eta,\xi}(\omega_t) + (1-\tau) i_{\eta,\xi}(\omega_t) \\ &\leq \mu_{\eta,\xi}(\omega_0) + (1-\tau) i_{\eta,\xi}(\omega_0) \\ &\leq C \end{aligned}$$

and we get a uniform upper bound on $i_{\eta,\xi}(\omega_t)$. $\square$

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