RELIABILITY ANALYSIS FOR SYSTEMS SUBJECT TO DEGRADATION AND SHOCKS

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ABSTRACT OF THE DISSERTATION

Reliability Analysis for Systems Subject to Degradation and Shocks

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Engineering systems usually deteriorate due to some underlying degradation processes and possibly random shocks. Accurate modeling of the effect of these processes on the system leads to better system reliability estimation. This dissertation investigates the impact of degradation processes and random shocks on systems and studies the optimal maintenance policies for such systems.

We develop new reliability models which offer more realistic system reliability estimation. Model 1 considers the effect of system's age on shock damage magnitudes. This is due to the fact that systems become more vulnerable to shock damage magnitudes as it ages. Model 2 extends Model 1 by considering the correlation between degradation processes and accumulated shock damages. Both models assume that the shock damage magnitudes are s-independent and time-variant. For each model, the system reliability expression is developed and the associated parameter estimation method is presented. Numerical analyses and Monte Carlo simulation are conducted on different parameters of reliability expression to validate the analytical reliability expressions.

Furthermore, we develop a generalized threshold-type condition-based maintenance (CBM) policy for a system subject to multiple competing risks including degradation process and sudden failure where the maintenance is considered to be imperfect. The objective of this study is to obtain the optimal preventive maintenance threshold maximizing system's average availability. The model can also accommodate the correlation among multiple failure modes. The special case of such a system subject to two independent competing risks, degradation and sudden failure is studied where the degradation process is described according to Model 2. Finally, numerical optimization analyses and sensitivity analyses on optimum policy are conducted and presented accordingly.

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DEDICATION

To my wonderful mother Golghand Abbasi, and my wonderful father Mohammad Ghorbani, and my lovely sister Maryam Ghorbani

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Chapter 1 : Introduction

1.1. The Importance of Degradation Models and its Applications

Nowadays users expect products and services to be more reliable with longer lifetimes and higher quality. Beside these expectations, introduction of new complex technologies with their tight economic requirements as well as maintenance costs make reliability analysis of assets and estimation of their useful remaining lifetime an important issue in the engineering field.

Traditionally, reliability analysis is performed using field data or accelerated failure time data. However, with the increase in more reliable products, it is usually costly and difficult to obtain failure time data in a reasonable time horizon. In such cases reliability analysis using degradation modeling becomes a viable and important alternative. The failure mechanism of many systems can be explained via one or multiple underlying degradation processes which can be affected by randomly changing covariates such as operating conditions. Gorjian *et al.* (2009) generally define degradation of a system as the reduction in the performance or the reliability of the system over time. Lehman (2006) defines the degradation in the engineering field as the irreversible accumulation of damage in the system over its life span, which finally leads to its failure. The degradation process in an item usually is traced by measuring one index or multiple ones. The degradation measures can be visibly observed or not. For example, the crack growth in civil infrastructures can be visually recorded, while the reduction in a voltage output of an electronic device cannot be observed and could be recorded by monitoring sensors.

Examples of physical degradation measures in engineering fields generally include fatigue, corrosion, fractures in materials like concrete (Lehmann, 2006) and vibration frequencies in bearings (Gebraeel *et al.*, 2005). Degradation models have been widely applied to the reliability assessment for steel coatings (Heutink *et al.*, 2004, Nicolai *et al.*, 2007), high-speed railway tracks (Meier-Hirmer *et al.*, 2005), circuits (Elsayed, 1996), bridges and civil infrastructures (van Noortwijk and Frangopol, (2004), Pandey *et al.*, (2005)), berm breakwaters (van Noortwijk and van Gelder, 1996) and automobile brake pads (Crowder and Lawless, 2007).

1.2. Types of Degradation Models

From the reliability analysis perspective, there are three classic degradation models in the literature of degradation and reliability analysis named degradation-threshold (DT) models, degradation-shock (DS) models, and degradation-threshold-shock (DTS) models. DT models assume that the system failure is due to degradation process only. In DS models, the system failure is due to occurrence of traumatic events only, while DTS models consider that the system is subject to degradation processes and random shocks. The last two models will be discussed in detail in the next section. The classical DT models consider that the system failure happens when the degradation measure exceeds a certain critical threshold level. The time when the system reaches this level is referred to as first-passage time (Lehmann, 2010). Figure 1-1 shows an example of the critical threshold, C, and degradation path traced by crack length growth. Time t^* is the system's first-passage time.

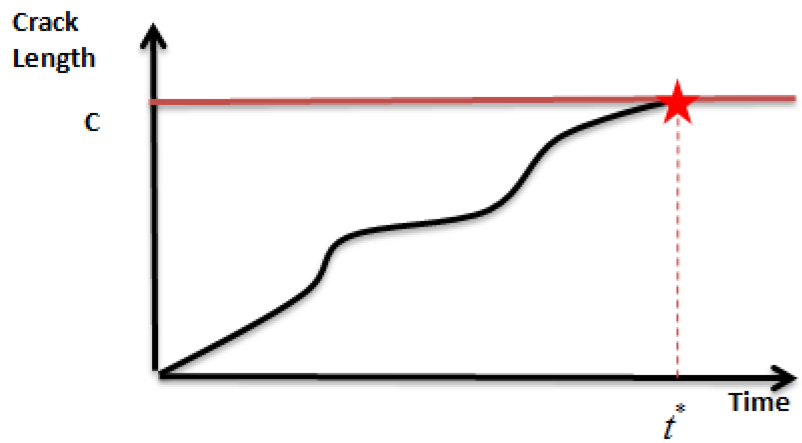

Figure 1-1: The system degradation process

From modeling perspective, the degradation models are, in general, classified into two classes named normal degradation models and accelerated degradation models/testing (ADT). The normal degradation models make an inference about the reliability metrics from degradation data collected at normal operating conditions. However, accelerated degradation models assess the reliability using degradation data obtained at accelerated stresses or environmental conditions (Gorjian *et al.*, 2009). The normal degradation models are also classified into two groups: models without stress factors and models with stress factors. The normal degradation models without stress factors make inference about reliability and degradation measures at a constant stress. General degradation paths, continuous-time stochastic processes such as gamma process, Weiner process, and Markov models fall into this category. On the other hand, the normal degradation models with stress factors model the situations when the system is subject to sporadic stresses. Shock models and stress-inference models are such models. Figure 1-2 from Gorjian *et al.* (2009) addresses the general degradation models.

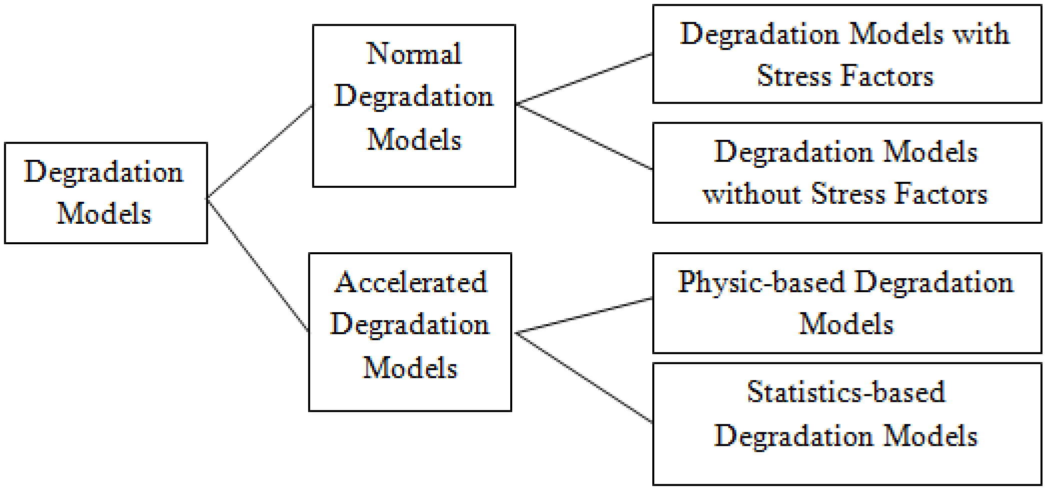

Figure 1-2: The Classification of degradation models

Among normal degradation models without stress factors, stochastic Markov models such as gamma process, Weiner process, and Markov chains have been widely applied to engineering problems. For example, Markov chain models have been employed to model degradation in nuclear pipes (Veeramany and Pandey, 2011). Gamma processes have been successfully applied to crack growth in materials (Noortwijk and Frangopol, 2004), corrosion of steel coatings (Nicolai *et al.*, 2007), civil infrastructures such as bridges (Pandey *et al.*, 2005), and water dam storages (Moron, 1956). Moreover, Weiner processes have successfully modeled degradation process in rotating element bearings (Gebraeel *et al.*, 2005), bridge beams (Wang, 2010), LED lamps (Tseng and Tang, 2003), self-regulating heat cables (Whitmore, 1997), and etc. The degradation models classified in Figure 1-2 will be discussed in detail in chapter 2, literature review.

1.3. Degradation and Random Shock Models

Most of the systems are usually subject to degradation or aging due to internal processes and some sporadic shock or stresses. According to Gorjian *et al.* (2009), the continuous-time degradation models are not appropriate models for systems subject to shocks. On the other hand, shock models existed in literature such as the ones in Gut and Husler (2005), Lam and Zhang (2004), Lam (2009), and Nakagawa (2007), usually assume that the system is absolutely reliable in absence of random shocks. Therefore, many researchers have focused on studying degradation and shock models, Li and Pham (2005), van Noortwijk *et al.* (2007), Sanchez-Silva *et al.* (2011), Jiang *et al.* (2011).

The degradation and random shock models are divided into two classes of models based on whether the shocks and degradation processes are dependent. Usually, there are two kinds of dependency between random shocks and degradations:

1) Degradation makes the system more vulnerable to random shocks. That is, the probability that a shock is fatal to the system increases by degradation. Huynh *et al.* (2011) study a system subject to degradation and non-homogeneous Poisson process shocks where the degradation increases fatal shock occurrence rates. Therefore, the system fails sooner.
2) The shocks accelerate the degradation process. The random shocks can generally have two types of impact on degradation process. They can accelerate the degradation by a sudden jump, or they increase the rate of degradation in the system. Cha and Finkelstein (2009) and Wang and Pham (2011) study a system subject to degradation and random shocks where a shock results in system failure

with probability $p(t)$ and it adds a sudden jump to degradation by probability $q(t)=1-p(t)$.

1.4. Maintenance Models

The objective of system maintenance is to ensure the system availability (Elsayed, 2012). There has been an increasing interest in developing optimal maintenance policies for degrading systems over last few decades. According to Duffuaa *et al.* (2001), maintenance strategies can be, in general, categorized into two major groups of corrective maintenance (CM) and preventive maintenance (PM). CM is referred to activities restoring the system's state to its required level of functioning after failure (Blanchard *et al.* 1995), while PM is performed prior to the system failure in order to improve the system's state (Usher *et al.*, 1998). The maintenance techniques in existing literature are classified into time-based maintenance (TBM) or condition-based maintenance (CBM). time-based maintenance analysis is based on system's age, while condition-based maintenance suggests maintenance actions based on the system's state which can be determined through inspection or continuous condition monitoring. The advantage of condition-based maintenance over time-based maintenance is providing better system's health management, lower maintenance cost over system's life span, and avoidance from catastrophic failure (Ahmad and Kamaruddin, 2012). Hence, this dissertation focuses on investigating optimal CBM policies for degrading systems due to degradation and shocks.

1.5. Overview of The Dissertation

Chapter 1 presents the motivation, basic concepts in the dissertation. Chapter 2 reviews the previous studies on degradation and shock models and their associated characteristics in detail and presents the organization and direction for the research. Chapter 3 develops a reliability model for a system subject to degradation and random shocks with time-dependent damage magnitudes to capture the effect of system's age and shock damage magnitudes. Chapter 4 extends the model in chapter 3 by incorporating some correlation between accumulated shock damage and degradation process. The associated dependency is described in a way that the random shocks affect the system by adding sudden damage increments and accelerating the underlying degradation process as well. Chapter 5 addresses a generalized optimal threshold-type CBM policy for the system described in Chapter 4 where it is subject to multiple competing risks and imperfect maintenance. The proposed optimal maintenance policy aims to maximize the system average achieved availability. Finally, chapter 6 discusses the conclusions and future research ideas.

Chapter 2 : Literature Review

In the previous chapter, the basic concepts and motivations for degradation and shock models were discussed and the objective of the dissertation was briefly given. As it was discussed in chapter one, degradation models can be divided into two major categories namely normal degradation models and accelerated degradation models. This chapter presents a comprehensive literature survey on each category of degradation models developed over the last two decades and thoroughly reviews all the recent studies on degradation and shock models. Finally, the direction and organization of research in this dissertation is discussed.

2.1. Normal Degradation Models

Many systems, subsystems, and components exhibit a degradation process. This might be attributed to aging, use, environmental conditions, and applied stresses. The degradation analysis is an alternative approach when degradation data can be obtained in lieu of failure time (Lehmann, 2009). The degradation models appropriately evaluate and predict the system's health state in order to provide a better insight on the system behavior (Jiang and Jardine, 2008). In literature and applied engineering, the degradation paths are usually considered to be a stochastic process rather than a deterministic one. The degradation models can be classified in regards with various factors. For example, the degradation models can be categorized into two basic groups named discrete and continuous models (Jiang and Jardine, 2008). A general degradation path can be defined as a linear function or nonlinear function in time (Li *et al.*, 2011). Based on Gorjian *et al.* (2009), degradation models can also be categorized into two big groups of normal and accelerated degradation models. Normal degradation models are usually used to estimate the degradation obtained at normal operating conditions. However, accelerated degradation models make inference about degradation and reliability metrics at normal conditions from data obtained under accelerated conditions. Normal degradation models can further be divided into two groups: degradation models without stress factors and degradation models with stress factor.

Degradation models without stress factor are the ones where the degradation indicator is defined at a fixed stress level. Good examples of these models are general degradation path models and linear/ nonlinear regression models. However, other degradation models such as cumulative damage or shock models are considered as degradation models with stress factors since the degradation indicator is a function of defined sporadic stresses (Gorjian *et al.*, 2009). These models are explained below.

2.1.1. Normal Degradation Models without Stress Factors

There are several methods which can be used to model the degradation data of a component when the stress is not present. Some of these models are general degradation path models, Markov models, and continuous-time stochastic processes.

2.1.1.1. General Degradation Path

The general degradation path model is a regression model with random or fixed coefficients fitted to the degradation observations. Both linear and nonlinear models are used to model degradation, while most of the degradation paths are nonlinear and not intrinsically linear (Gorjian *et al.*, 2009). A simple instance of such a model can be $D(t) = bt$ where b is a random or fixed variable. The simplicity of these models is an advantage; however, they might not be a good representative for the actual degradation path. In practice, the degradation function of most of systems cannot be described well with these models (van Noortwijk, 2009). A traditional method to model the degradation path is linear regression model in which the degradation path can be represented as $d_i(t) = \beta_0 + \beta_1 t + \varepsilon_i$ where $d_i(t)$ represent the degradation measure from unit *i* at time *t*. β_i is the fixed-effect parameter, and $\varepsilon_1, \varepsilon_2 ..., \varepsilon_n$ are assumed to be independent of time, mutually independent, and identically distributed with normal distribution with mean zero and constant variance. A major problem with linear regression model is the difficulty to estimate the matrix of correlation especially when the data is unbalanced. An extension to this model is to consider the variance of degradation as a function of mean deterioration, or use nonlinear regression to fit the degradation model (Yuan and Pandey, 2009). In general, a general degradation path model can be represented as:

$$d_i(t_j) = f(t_j, \Phi, \Theta_i) + \varepsilon_{ij} \quad and \qquad \varepsilon_{ij} \sim N(0, \sigma^2)$$

Where t_j is the time of j^{th} degradation measurement, f is a non-decreasing function addressing the actual degradation path, and d_{ij} is the degradation measure for unit *i* at the j^{th} degradation measurement. Also, Φ is a vector of fixed-effect parameters, and Θ_i is the vector of random parameters for i^{th} unit. ε_{ij} is the error term with constant variance and independent of Θ_i (Gorjian *et al.*, 2009). Lu and Meeker (1993) employ a general degradation path to find the distribution of time to failure for the system. The degradation path is described as $x_{ij} = f(t_i; \phi, \theta_j) + \varepsilon_{ij}$ where x_{ij} is the degradation measure from unit *j* at pre-specified time t_i, ϕ represents common parameters for all units, and θ_j represents multivariate normal distributed parameter for each unit j. ε_{ij} represents the error term for unit i at j^{th} measurement time. The errors can be correlated or stress dependent. The authors develop a two-stage algorithm to estimate the parameters of the degradation path. The algorithm firstly fits the degradation model to the path obtained for each unit and calculates the new parameter estimates using the estimations computed before. Bae and Kvam (2004) model the degradation path of highly reliable light display components such as plasma display panels and vacuum fluorescent as a nonlinear random-coefficient model to capture the burn-in characteristics of the component. Bae *et al.* (2007) investigate the reliability characteristics of a single component for two cases of general additive and multiplicative degradation models. The general additive degradation model is defined as $D(t,\Theta,X)=\eta(t,\Theta)+X$ where $\eta(t,\Theta)$ is the deterministic mean degradation at time t, and X represents the additive random noise around the mean degradation. Θ is also considered to be a fixed effect parameter. In the multiplicative form of degradation model, the random variable is multiplicative to the mean degradation with mathematical form of $D(t,\Theta,X)=X.\eta(t,\Theta)$ where X is a random variable. Yuan and Pandey (2009) develop an advanced nonlinear mixed effect (NLME) degradation model for degradation data obtained from nuclear piping systems. This model provides improved degradation prediction by reducing the variance associated with the degradation of each unit. Haghighi and Nikulin (2010) employ parametric and non-parametric methods to estimate the survival function and its parameters for a system with multiple conditionally independent failure modes where the degradation path is in form of linear multiplicative function, $D(t)=\frac{t}{A}$, with A as the random deterioration rate.

2.1.1.2. Markov Models

A Markov chain model is a stochastic process which is defined as a set of discrete state space and discrete time space. Markov chains with continuous-time space are referred to Markov processes (Li *et al.*, 1996). Consider that $u < t < \tau$ and $X(t)$ is a stochastic process representing the state of a system at step t. $X(t)$ is considered to be a Markov process if $X(\tau)$ given $X(t)$ does not depend on $X(u)$. In other words, the conditional distribution of a future state is independent of the past states of the process (Montoro-Carzola and Perez-Ocon, 2006). Thus, the sojourn time of each state is exponentially distributed and the transition probability to each state is independent of the process history. In degradation models, $X(n)$, addresses the state of an observable degradation measure, such as the length of a crack, at the end of n^{th} time interval $X(t)$ is usually modeled as a Markov chain, and the wear of the system at n^{th} time interval is defined as $X(n+1) - X(n)$ (Singpurwalla, 1995). Welte *et al.* (2006) model the degradation process of hydro power plant by a Markov chain with four finite states where the sojourn time of the process is modeled by gamma distribution. Markov processes have been extended to more general models such as semi-Markov processes and hidden Markov processes (See Blischke and Murthy (2000), Kim and Makis (2009), van Noortwijk, (2009), and Peng and Dong (2011)). A semi-Markov process is the integration of renewal theory and Markov chains. In such a model, the sojourn time distributions depend on the next state that the process visits (Pijnenburg, 1991). Kharoufeh (2003) and Kharoufeh and Cox (2005) estimate the residual life distribution for a single-unit system subject to Markovian environment-based degradation with finite states. They consider that the environment follows a time-homogeneous Markov chain with finite states with different degradation rate for each state of environment. Kharoufeh *et al.* (2010) extend this model by considering that the environment-based degradation is described by a semi-Markov process. Veeramany and Pandey (2011) compare homogenous Markov model and semi-Markov model to estimate the reliability of pipes in nuclear power plants. Chryssaphinou *et al.* (2011) derive the reliability indices for a system consisting of m components where the deterioration of each component is described by a time-discrete semi-Markov chain. The process is time-discrete because the sojourn time of each state follows a discrete distribution. According to Gorjian *et al.* (2009), these models are efficient and appropriate to model the incomplete set of data; however, they assume that the degradation process is a single monotonic path. This assumption is not a valid for all the engineering problems. For example, fatigue cracks sometimes may heal (Lehmann, 2009).

2.1.1.3. Continuous-Time Stochastic Processes

Continuous-time models or continuous-time Markov processes are helpful to model the continuous stochastic degradation processes. Gamma processes, compound Poisson processes, and Weiner processes, which are also called Brownian motion with drift or Gaussian processes (van Noortwijk, 2009), are such models. These models can be considered as special types of levy processes. Abdel-Hameed (2010) discusses levy processes and their extensions in detail. The literature survey shows that these models especially gamma process and Weiner process have been widely applied to many engineering problems.

2.1.1.3.1. Gamma Processes

A thorough review of literature shows that Abdel-Hameed (1975) is the first to suggest that the gamma process is a suitable model to describe the random deterioration path. Gamma process is useful when the gradual damage is monotonically increasing over the time such as fatigue, corrosion, and crack. Gamma process is a stochastic process with independent non-negative increments where each increment is distributed with gamma distribution with the same scale parameter. The non-negativity assumption for increments is valid for many degradation processes in practice. A gamma process is mathematically defined as follows:

Let $\{X(t) \geq 0, t\}$ be a gamma process representing the accumulated degradation by time *t*. Let $x = X(t+1) - X(t)$; therefore, the probability density function for x is given by

$$Gamma(x;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

Where $\alpha > 0$ is the shape parameter and $\beta \geq 0$ is the scale parameter. In general, the classic gamma process has the following properties (Lawless and Crowder, 2004):

1) With probability one, $X(0) = 0$
2) $X(t)$ has independent increments
3) $X(u) - X(v) \sim Gamma((u-v)\alpha, \beta) \quad \textit{for all } u > v \geq 0$

If the expected deterioration is linear in time, the gamma process is called stationary (van Noortwijk, 2009). The gamma process can be viewed as a compound Poisson process where the rate parameter tends to infinity and the gamma-distributed increments are so close to zero (Singpurwalla, 1995). The gamma process fits well to the degradation data of fatigue crack growth, corrosion of steel coatings, civil infrastructures such as bridges, and water dam storages (See Moron (1956), Grall *et al.* (2002), van Noortwijk and Frangopol (2004), Nicolai *et al.* (2007), van Noortwijk *et al.* (2007), Wang (2009), Baussaron *et al.*, (2010)). Lawless and Crowder (2004) develop a random effect gamma process with covariates to capture the heterogeneity of the degradation path. They consider that the scale parameter of the gamma process is defined as $z\varepsilon$ where z is a random variable and ε represents the scale parameter of the gamma distribution. Pandey *et al.*, (2005) compare gamma process with a random deterioration rate model on modeling the degradation path of civil structures. They point out that the gamma process has more versatility to capture the variations of degradation path compared to the random deterioration rate model. The conventional methods to estimate the parameters of a gamma process are maximum likelihood, method of moments, and method of Bayesian statistics (van Noortwijk, 2009). Guo and Tan (2009) update the parameter estimates of a gamma process using the Bayesian approach. Wang (2009) develops a nonparametric method namely pseudo-likelihood to estimate the unknown parameters of a non-stationary gamma process. The extensions to gamma process are multivariate gamma process, extended gamma process/weighted gamma processes, and non-stationary gamma processes. Buijs *et al.* (2005) use a bivariate gamma distribution to evaluate the reliability of a flood defense which was subject to two causes of deterioration, namely the crest level and vegetation quality. The bivariate gamma process has also been used to model the degradation path of light emitting diodes (LED) and components with two different fatigue crack positions (See Zhou *et al.* (2010) and Pan and Balakrishnan (2011)). Dykstra and Laud (1981) define the extended gamma process as $Z(t) = \int_0^t u(s)dY(s)$ where

$u(t)$ is a non-decreasing, real-valued, and right continuous function in time and $Y(t)$ is a gamma process with shape parameter of $v(t)$. Non-stationary gamma process is considered to be another extension to the classic gamma process. Non-stationary gamma process is similar to classic gamma process in definition, but it assumes that increments follow a gamma distribution with the shape parameter described as a general function in time (Gorjian *et al.*, 2009). That is,

$$X(u) - X(v) \sim Gamma((\alpha(u) - \alpha(v)), \beta) \quad for\ all\ u > v \geq 0$$

Wang *et al.* (2000) study the residual life distribution of water pumps at a soft drink manufacturing plant where the degradation level of the system is modeled by a non-stationary gamma process. Non-stationary gamma process has been applied to describe the degradation path of coating of steel and concrete creep (Cinlar (1977), Nicolai *et al.* (2004)). Cinlar (1977) also addresses how a non-stationary gamma process can be transformed into a stationary one.

2.1.1.3.2. Weiner Processes

Weiner process is also called Gaussian process or Brownian motion with drift. Brownian motion first was introduced by Robert Brown, botanist, in 1827. It was used to describe the effect of water molecules striking the pollens and distributing them. Ross (1996) defines Brownian motion or standard Weiner process, $W(t)$, as it follows:

1) $W(0) = 0$
2) $\{W(t), t \geq 0\}$ is a stationary continuous stochastic process with independent real-value increments. That is, $W(t_i) - W(t), W(t_j) - W(t)$ are independent normally-

distributed random variables for any non-overlapping time intervals of $(t, t_i), (t, t_j)$.

3) Each increment follows a normal distribution with mean zero and variance *t*.

Weiner process is a non-monotonic process because the increments are not imposed to be nonnegative. Kahle (1994) use Brownian motion with drift to define the accumulated damage level of a system which is mathematically given by:

$$X(t) = X_0 + \mu(t) + \sigma W(t)$$

Where $\mu(t)$ is the trend or drift parameter and σ is the standard deviation or diffusion parameter. $W(t)$ stands for the standard Brownian motion, and $X(t)$ represents the accumulated damage level up to time *t*. Usually, it is assumed that the trend is a linear function in time; that is, $\mu(t) = \mu_1 t$ where μ_1 is the average degradation rate. Basically, at each time $t, X(t)$ is considered to be distributed as $N(\mu t, \sigma^2 t)$. The Weiner process has additive effect on degradation path. It is also known that the time that a Weiner process reaches to a fixed level follows an inverse Gaussian distribution, $IG(\eta, \lambda)$, with the following mathematical presentation (Tang and Su, (2008)):

$$f(t) = \sqrt{\frac{\lambda}{2\pi}} t^{-\frac{3}{2}} \exp\{-\frac{\lambda(t-\mu)^2}{2\mu^2 t}\}, \quad t > 0$$

$$\eta = \frac{a}{\mu} \quad and \quad \lambda = \frac{a^2}{\sigma^2}$$

Where η and λ are the parameters of the distribution and a is the fixed level or the critical failure threshold. Weiner process can also be thought as a compound Poisson process with normal distributed increments and rate parameter of infinity. Kahle and Lehmann (2010) discuss Weiner process degradation models and their associated parameter estimation problems. Weiner process has been used to describe the degradation path for many engineering systems. Some of these systems are rotating element bearings, bridge beams, LED lamps, self-regulating heat cables, and fatigue cracks dynamics (See Gebraeel *et al.* (2005), Wang (2010), Tseng and Tang (2003), Whitmore (1997), and Ray and Tangirala (1996)). Wang and Coit (2004) study the reliability of a system subject to multiple Weiner-based degradation processes for the cases where the degradation processes can be independent or dependent. Barker and Newby (2009) develop an optimal inspection policy for a multi-component system where the degradation path of each component is independently modeled by Weiner process. Nicolai *et al.* (2007) compare the goodness-of-fit of three different stochastic models namely Brownian motion with nonlinear drift, a non-stationary gamma process with non-linear shape parameter, and a two stage hit-and-grow (TSHG) physical process in modeling the degradation level of organic coatings of steel structure. Note that the last method has been developed using the physical properties of the material. The results indicate that the non-stationary gamma process, Brownian motion, and some versions of TSHG describe the degradation path well. Wang (2010) models the degradation level of bridge beams by a Weiner process with random drift and diffusion parameters. He also uses the maximum likelihood estimates (MLE) to estimate the associated parameters. Si *et al.* (2012) assess the residual life distribution of inertial platforms which are key components in the weapon systems and space equipment. The author describes the degradation level of the system by a Weiner process where the drift parameter is updated using a recursive filter algorithm. Elsayed and Liao (2004) propose a new Brownian motion-based process namely Geometric Brownian motion. Unlike Brownian motion process, this process has nonnegative increments. Therefore, it can be used to describe the monotonically increasing degradation path of many engineering systems. In such a model, the degradation level at time *t*, $Z(t)$, is expressed as

$$Z(t) = \mu e^{bt} e^{\sigma_1 W(t)}$$

Where μ and b are the initial degradation and the drift parameter, respectively. $W(t)$ is the standard Brownian motion, and σ_1^2 is the diffusion parameter representing the impact of other factors such as the internal reactions, and variation of unknown or known stresses.

2.1.1.4. Other Models

In previous years, researchers have suggested many other approaches for modeling degradation paths. For instance, Gebraeel *et al.* (2005) suggest a Bayesian updating method which utilizes the real-time continuous signal information from a single monitored component in order to update the stochastic parameters of degradation models with multiplicative random errors. The authors develop two exponential degradation models, one with i.i.d. random error terms and the other one with error terms following a modified Brownian motion with independent and identically distributed increments. Later on, they derive the associated system residual life distribution for each model. Bordes *et al.* (2010a) define the degradation path as $D(t) = Y(t) + \tau B(t)$ where $D(t)$ describe the degradation level at time *t*, and it is defined as a combination of a gamma process, $Y(t)$, and a Brownian motion, $B(t)$. τ is a positive constant. It is also assumed that the gamma process and Brownian motion are independent. The authors, in addition, discuss the parameter estimation for this model. Later on, Bordes *et al.* (2010b) study the parameter estimation problem for the similar model with covariates. Gebraeel and Lawley (2008) propose a neural network-based degradation model for rolling contact thrust bearings where the model utilizes the condition-based sensory signals to update the failure times of partially degraded components and computes the residual life distributions of these components. The proposed neural network module predicts the initial failure times. Then, using Bayesian approach, the prior distribution is updated by using the subsequent real failure times. Li *et al.* (2011) estimate the reliability function for a system consisting of multiple components with dependent degradation paths. The dependency among degradation paths is modeled by describing each component's degradation path as the convex combination of an independent degradation mean and common random degradation factors. Zhou *et al.* (2011) propose a Baysian framework updating degradation distribution using the sensor information collected over a short period of time. This non-parametric model also can provide some inference on the predicted degradation rate. The degradation level at time t is defined as $D(t) = \mu(t) + X(t) + \sigma\varepsilon(t)$. $\mu(t)$ is the mean degradation, and $X(t)$ is the stochastic variable presenting the deviation from mean. Using the real-time degradation data, the posterior distribution of $X(t)$ is obtained and the estimation of the residual life is updated.

2.1.2. Normal Degradation Models with Stress Factor

Normal degradation models with stress factor describe the models where the system degradation level is a function of defined stresses (Gorjian *et al.*, 2009). Stress-strength inference (SSI) models and cumulative damage/shock models are the important types of such models. We will discuss these models in detail in this section as it follows.

2.1.2.1. Stress-Strength Inference Models

In SSI models, the strength is assumed to degrade over the time. The strength deterioration can be due to fatigue, corrosion, or aging. The underlying system in these models is assumed to be exposed to disperse applied loads or stresses. The system is considered failed if the realized applied load, $L(t)$, is greater than the inherent strength of the system, $S(t)$. A common assumption in these models is that the load and strength process are independent. Both load and strength can be considered deterministic or random variable (Xue and Yang, 1997). Previous research shows that the strength degradation has been commonly modeled by deterministic linear degradation model, random-coefficient degradation model, and random-increment models (See Lu and Meeker (1993), Nelson (1981), Nelson (1990)). Xue and Yang (1997) present the lower and upper bounds for the reliability of a system when both strength and load are normally distributed. Wu *et al.* (2011) estimate the reliability of a system where the strength reduction in the system is modeled by a gamma process and the load process follows a Poisson process.

2.1.2.2. Cumulative Damage/Shock Models

Shock models are usually used to study the failure and degradation mechanisms of the systems which are subject to sporadic shocks with random damage magnitudes. These models are also called random shock models if the shock process is defined by a random process. The classic shock model assumes that the system is perfectly reliable in absence of shocks. That is, no aging or degradation process is responsible for deterioration of system performance. The general set up in shock models is given by $\{(X_n, T_n), n \geq 0\}$ where n is the number of shocks, and X_n, T_n represent the magnitude of damage due to n^{th}

shock and the time interval between the (n-1)st shock and n^{th} shock, respectively (Gut and Husler, 2010). Some of common assumptions in many shock models studied in previous literature are as it follows. It is usually assumed that the shock arrival process follows a Poisson process and no shock arrives to the system at time zero. It is also assumed that the shock damage magnitudes are *s*-independent and identically distributed by a common distribution. In practice, shocks can cause sudden failure in the system. These shocks are called fatal shocks or traumatic events in literature, and the subsequent failure due to those events is referred to hard failure. Cha and Finkelstein (2009) simply model this concept by considering that each shock can be fatal or traumatic to the system with probability $p(t)$ and nonfatal with probability $q(t) = 1 - p(t)$. In general, three classic random shock models have been studied in the literature as it is listed below:

1) Cumulative damage/shock model
2) $\delta -$shock model
3) Extreme shock model

In addition, mixed shock models, run shock models, and independent damage models are other extensions of classic shock models in the past years.

2.1.2.2.1. Cumulative Shock Model

In a cumulative shock model, the system fails when the cumulative damage due to shocks reaches to a critical threshold of failure. The particular time that this event happens is referred to the system's first passage time. Consider that $S_{N(t)} = \sum_{i=1}^{N(t)} X_i$ and $Z_{N(t)} = \sum_{i=1}^{N(t)} T_i$.

Let x denote the critical threshold of failure and $\{N(t), t \geq 0\}$ represent the number of shocks arrived by time *t*. It is usually assumed that $N(t)$ follows a Poisson distribution. Therefore, the system first passage time, $\tau(t)$, in a cumulative shock model is defined as

$$\tau(t) = \min_t \{S_{N(t)} > x\}$$

Gut and Husler (2005) develop a generalized cumulative shock model where only a final recent portion of damage summands are of interest. In such a model, the time to failure, $\upsilon(t)$, is expressed as

$$v(t) = \min\{n : S_{k_n, n} = \sum_{j=n-k_n+1}^{n} X_j > t\}, \quad t \geq 0$$

Where k_n is the number of recent shocks considered to damage the system after occurrence of n shocks. Frostig and Kenzin (2009) investigate the average limiting availability of a deteriorating system subject to random shocks. The shocks are arriving to the system according to Poisson process, and the shock deterioration is modeled by cumulative shock models. Two models are developed. Model I assumes phase-type distribution for the distribution of shock damage magnitudes, and it is not affected by random environmental conditions. Model II includes the effects of random environments, which is modeled by a continuous time Markov chain, on the shock arrival process and shock damage magnitudes. Van der Weide *et al.* (2010) also study the reliability and an optimal maintenance policy for a system degrading based on cumulative shock model.

2.1.2.2.2. $\delta -$Shock Model

Let T be the system lifetime and T_i denotes the time that shock i arrives to the system. A $\delta -$shock model assumes that the system failure happens if the time interval between two consecutive shocks is less than a predetermined threshold, δ. In other words,

$\{T \le t\} \Leftrightarrow \{\min_i \{T_i - T_{i-1}\} > \delta\} \ \forall i = 1, ..., N(t)$

Lam and Zhang (2004) and Lam (2009) investigate an optimal maintenance policy for a system subject to random shocks where the system failure mechanism is modeled by a $\delta-$shock model. The shocks are also assumed to arrive according to a Poisson process. Tang and Lam (2006) develop a maintenance model for a system degrading according to $\delta-$shock model where the shocks are arriving according to Weibull or gamma distribution. Li and Kong (2007) present the analytical reliability function and its properties for two cases of $\delta-$shock models where the shocks are arriving according to homogeneous and non-homogeneous Poisson process. Also, Ma and Li (2010) study the lifetime properties of censored $\delta-$shock models. Eryilmaz (2012) studies the survival function and mean time failure for a system subject to random shocks degrading based on an extended $\delta-$shock model. This model extends the classic $\delta-$shock model by considering that the system fails when k consecutive shock inter-arrival times are less than the threshold δ. They also introduce a model with two competing failure modes described by a run shock model and the former model. Under the new model, the system failure happens when either k consecutive shock inter-arrival times are less than a threshold δ or m shock magnitudes are recorded above a pre-defined threshold.

2.1.2.2.3. Extreme Shock Model

Let y be a prefixed threshold for the shock magnitude and X_i denote the magnitude of i^{th} shock. Extreme shock model means that the system fails as soon as the magnitude of a single shock exceeds a certain threshold, which can be mathematically shown as:

$\{T \le t\} \Leftrightarrow \{\min_i \{X_i\} \le y\}$

Gut and Husler (2005) develop a generalized extreme shock model where they consider a critical window for nonfatal shock damage. They assume that the critical threshold for shock magnitudes decreases each time a fairly medium shock appears. Cirillo and Husler (2009) study generalized extreme shock model introduced by Gut and Husler (2005). They develop a triangular urn process to indirectly analyze the effect of shocks on the system’s load threshold. The urn process considers that the state of the system and shocks are equivalent to the color of balls and balls, respectively. For example, red denotes highly risky state and white denotes the safe state. Each time a ball is sampled from the urn, the system enters to a new state depending on the ball color. Cirillo and Husler (2011) employ Bayesian approach to build a new approach for the same analysis done in Cirillo and Husler (2009). Cha and Lee (2010) propose a generalized extreme shock model where the shocks can be fatal to the system and nonfatal shocks are categorized into two types. Type I shocks does not damage the system significantly, while type II shocks add a time-dependent damage increment to the system.

2.1.2.2.4. Other Shock Models

Mallor and Omey (2001) introduce the run shock model where the system fails if k shocks with magnitudes over a critical threshold occur to the system. The extreme shock model is a special case of run shock model if $k=1$. Kahle and Wendt (2004) just mention that the damage process can also be considered as $X_n = U_n e^{\delta T_n}$ where X_n and T_n are the damage magnitude and arrival time of n^{th} shock, respectively. U_n is assumed to be a continuous random variable but independent of T_n. Wang and Zhang (2005) analyze the reliability of a repairable system with two-types of independent failures due to random shocks. They combine extreme shock model and $\delta-$shock Model to model the system failure mechanisms. In other words, the system failure happens when either the shock damage magnitude is greater than a threshold or the time lag between any consecutive shocks is greater than a δ. Gut and Husler (2005) introduce the mixed shock model and develop a generalized mixed shock model. The mixed shock model is a competing risk model with two dependent modes of failure due to cumulative damage of shocks and the time lag between consecutive shocks. In such a model, the system fails whenever either the cumulative damage due to shocks reaches to a threshold or a single relatively large shock arrives to the system. The generalized mixed shock model introduced in Gut and Husler (2005) is completely similar to the former model except that a generalized extreme shock model is used to model the second failure mode. The generalized extreme shock model of interest only considers some portion of recent shock damage magnitudes in cumulative shock model, and it assumes that the system load threshold is weakened after appearance of a fairly medium shock. Nakagawa (2007) discusses other variations in shock model in detail including independent shock model, imperfect shock model, damage annealing, and so on. Finkelstein and Marais (2010) use Laplace transform to analyze the lifetime distribution of a repairable system subject to random shocks for different cases of failure modes. The shocks are assumed to arrive to the system according to homogeneous Poisson process, and each shock can lead to system failure. Case I assumes that the probability of a fatal shock event is dependent on system's state of health which is modeled by a random time-dependent function. Case II considers that the next shock is fatal if the corresponding inter-arrival time is less than a threshold. At last, Case III categorizes the shocks into two types of harmful and harmless shocks. Under case III, the system fails when the time lag between two consecutive harmful shocks is less than a threshold. Van der Weide *et al.* (2011) analyze the reliability and maintenance cost for a single-component from nuclear reactors which is subject to random shocks arriving to the system according to non-homogeneous Poisson process. The cumulative damage to the system is modeled as an increasing sequence of random variables in order to represent the nonlinear nature of damage increments. Ye *et al.* (2011) propose two models to study the system reliability for of both repairable and non-repairable items subject to both degradation process and shocks. The shocks can be destructive, and they are assumed to follow non-homogeneous Poisson process. In this model, the destruction power of a shock depends on the cumulative potential hazard of the system, a surrogate for the system's health state. Model I assumes just one mode of failure due to shock for an item. However Model II presents analysis for several traumatic failure modes from several sources of shocks. The models have been successfully applied to laser devices.

2.2. Degradation Models with Random Shocks

Many systems, in practice, are subject to sporadic shocks while they are aging due to some internal processes or the underlying environmental conditions. Stochastic continuous time models are not suitable to model the degrading systems subject to sporadic shocks (van Noortwijk *et al.* (2007) and Gorjian *et al.* (2009)). Hence, researchers, in recent years, have focused on developing various models taking both sources of damage into the consideration such as DS and DTS models. The random shocks and degradation processes can be dependent or independent. If the system is subject to both degradation and catastrophic/traumatic events or fatal shocks, the system can fail due to two competing modes of failure, soft and hard failure. Hard failure occurs when a fatal shock hit the system, while soft failure happens when the cumulative damage to the system, due to degradation and nonfatal shocks, exceeds the critical threshold of failure (Jiang et al, 2011). Kou and Wang (2003) study the first passage time distribution and the joint distribution of the first passage time and the process overshoot for doubly stochastic diffusion process. The system under study is subject to shocks and degradation where the degradation is modeled using a Brownian motion process and the shock damage magnitudes are distributed as double exponential distribution. Li and Pham (2005) study the analytical reliability expression and a condition-based maintenance framework for a system subject to three competing failure causes namely two independent degradation processes and random shocks. The shock damage to the system is modeled according to cumulative shock model. The system failure occurs when the cumulative damage level due to all sources of damage exceeds the critical threshold of failure. The degradation paths in this work are described as it follows. The first degradation path is described as $Y_1(t) = A + Bg(t)$ where A is the initial degradation in the unit and *B* is the mean rate of degradation. Function *g* is considered to be an increasing function in time. The second degradation path is described as a logistic function which models the s-shaped behavior of degradation based on component usage. van Noortwijk *et al.* (2007) combine the deterioration process and load fluctuation in order to evaluate the resistance, reliability, of sea defenses. The deterioration process is modeled as gamma process with independent increments, and the load excess is modeled by a Poisson process. The system failure happens when the resistance of the system, which has decreased due to deterioration, falls below the load threshold. Finkelstein (2007) studies the properties of mortality rate of different degradation models for the human body which is degrading due to aging and exposure to random imperfect external harmful events such as external stresses. Cha and Finkelstein (2009) combine an extreme shock model with linear deterministic degradation path. Such a model can be considered as the extension of the model in Brown-Proschan (1983) in a way that the random shocks will be fatal to the system with probability $p(t)$, or they will result in acceleration of deterioration rate by probability $q(t) = 1 - p(t)$. Jiang *et al.* (2011) investigate the reliability expression for two dependent failure modes, soft and hard failure, where the load threshold, the system's strength threshold, depends on the number of shocks. The degradation process is modeled as a linear function where the degradation rate is normally distributed with stationary parameters. The model considers that the system's strength threshold drops to a lower level when *k* shocks occur to the system. Zhu *et al.* (2010) study an optimal condition-based maintenance policy which maximizes the system availability for an underlying system which is subject to two independent competing risks namely degradation and sudden failure due to random shocks. The model describes degradation process as a linear function in time with random coefficient. The degradation is not affected by shocks, and the correlation between degradation and sudden failure is ignored. Lehmann (2009) reviews all classes of degradation and shock models and their correspondent reliability expression. He particularly studies two classes of DTS model, namely general model and model with external covariates. Both models assume that the shocks occur according to non-homogeneous Poisson process. The general model is the classic DTS model, while DTS model with covariates studies the effect of environment conditions on the cumulative damage to the system as well. Sanchez-Silva *et al.* (2011) assess the reliability and residual lifetime distribution for bridge structures which are exposed to earthquakes, corrosion, and fatigue. The earthquakes are considered as random shocks, and corrosion is referred to degradation process. The study is conducted for two scenarios of deterministic and random degradation path. The damage due to shocks is modeled by cumulative shock model. The system fails when the total cumulative damage level of the system exceeds the critical threshold of failure. The authors also study the reliability metrics for the former model when the shock inter-arrival times are considered to be stochastic and deterministic. Huynh *et al.* (2011) studies a dependent competing risks model to assess the reliability distribution of a single unit system subject to aging and fatal shocks. The aging is modeled using a gamma process, and a non-homogenous Poisson process describes the random shocks arrivals to the system. The system failure modes are: 1) deterioration due to aging and 2) fatal shocks. To model the impact of system's state on shock process the authors assume the rate parameter of Poisson process is changing based on the level of system's degradation according to a simple step function. Wang and Pham (2011) derive the system reliability expression for a single component subject to degradation and random shocks. The shocks can be deadly to the system. They study the effect of random shocks on the degradation path considering two approaches: 1) by increasing the system failure rate after each shock 2) by a sudden random change in the degradation path after the shock happens. They derive the reliability function associated with each approach when shock arrivals are fixed or follow a Poisson process. Li and Pham (2005) study the reliability model for a single-component subject to s-independent multiple competing risks which are random shocks and two degradation processes where the degradation processes are described as discrete states

Markov chain and shocks can be fatal to the system. Nonfatal shocks accelerate the degradation processes, while the fatal shocks results in system's immediate failure. The impact of shocks on the system is explained through cumulative damage model. Wang and Pham (2011) also derive the system reliability of a single component with imperfect repair where the component failure mechanism is defined by two dependent competing modes namely degradation process and fatal shocks. They also suggest an optimal age-dependent maintenance policy for such a system. In their work, the system's cumulative level is defined by a linear deterministic degradation path and damage increments due to random shocks. The shocks can be destructive, and the probability of such an event is time-variant. Liu *et al.* (2008) investigate the reliability distribution for multi-component systems, parallel and series, where the system is subject s-independent competing risks namely internal degradation and random shocks of each component. The shocks are assumed to be completely nonfatal to the system. The degradation process and shock damage magnitudes for each component are also considered to be independent. Li *et al.* (2010) study the reliability distribution of an individual component subject to multiple degradation processes and random shocks for two scenarios when 1) the degradation processes are independent 2) they are correlated. The shocks damage the system according to cumulative shock model. In addition, the dependency among shocks and degradation processes is neglected. Wang and Pham (2012) extend Li and Pham (2005)'s model in the way that they employ copula method to assess the reliability of a single-component system subject to the s-dependent competing risks namely random shocks and multiple degradation processes. The random shocks also can fatal to the system. Let $X(t)$ indicate the system degradation level by time t and $S(t)=\sum_{i=1}^{N(t)} w_i$ denote the cumulative damage level due to shocks by time t. In this work, the authors borrow the main idea of accelerate life testing to model the impact of shocks on degradation processes by function $X(te^{\phi S(t)})$ where ϕ is a constant parameter. Jiang *et al.* (2012) study the reliability expression and an optimal maintenance strategy for three systems subject to two correlated competing risks, soft and hard failures as it was defined in Jiang *et al.* (2011). The model in this work, for the first time, introduces the impact of shocks on the critical threshold of hard failure. A thorough survey on past research shows that the critical failure threshold is usually considered a fixed value which is independent of any other system's factors. The authors assume that the critical threshold of failure for hard failure is shifted to a lower value as soon as k shocks arrive to the system. The analytical reliability expressions of three distinct models are studied. These models are listed here. The first model assumes that the value of hard failure threshold drops to a lower value when the first shock with a magnitude over a certain value arrives to the system. The second model considers that the shift in the hard failure threshold happens if the inter-arrival time between two consecutive shocks is less than δ. At last, the third model assumes a shift in the hard failure threshold right after m consecutive shocks with magnitudes over a certain value arrive to the system.

2.3. Accelerated Degradation Models

The main purpose of accelerated degradation models is to make inference about reliability of systems at normal condition using the degradation data collected from situations with accelerated environmental conditions such as increased stresses (Gorjian *et al.*, 2009). Liao and Elsayed (2005) state that the accelerated degradation testing (ADT) consists of a statistical model and acceleration function representing the relationship between the degradation indicator and elevated levels of stress. One important category of models among accelerated degradation models is degradation models with covariates which express the relationship between the covariates and the system reliability. Usually, the degradation mechanism of a component is dependent on some explanatory variables such as environmental conditions and stresses. For example, the wear rate of a tire depends on the type of road, weight of the load, and weather conditions such as temperature and humidity (Bagdonavicius *et al.*, 2010). The accelerated degradation models, in general, can be classified into physic-based models and empirical statistical models (Gorjian *et al.*, 2009).The physics-based models use the related physics theory to describe the effect of elevated temperature or stresses on the degradation indicators. For example, Meeker *et al.* (1998) use linear and nonlinear degradation path to explain how the rate of chemical reactions is affected by elevated temperature. Escobar and Meeker (2006) also review many accelerated test models in more details. Pinheiro and Bates (2000) introduce the linear and nonlinear mixed-effect model with fixed time-dependent parameters and random parameters dependent on various covariates for the degradation data. Bagdonavicius and Nikulin (2001) investigate degradation modeling and reliability analysis of systems in presence of dynamic environment. They also study the lifetime distribution of a system subject to degradation process and traumatic events where the traumatic events are dependent on level of degradation and both degradation and shock processes are affected by random dynamic environment conditions. Zhao and Elsayed (2004) assess the reliability distribution of a system subject to competing risks namely catastrophic events and degradation processes under accelerated conditions where the degradation process is described as a Brownian motion and the arrival time of a catastrophic event is modeled by Weibull distribution. The model can be successfully applied to LEDs when the degradation indicator is the voltage. Liao and Elsayed (2005) propose a 3-step optimal robust methodology in order to design systems optimally and more robustly. They successfully apply the methodology to hydraulically activated disc brakes. At first step, they find the baseline ADT model. Secondly, A set of optimal ADT plans are screened to determine the optimal estimators for some critical parameters such as test termination time, assignment of test units to stress levels, and so on. At last, the reliability of each test unit is estimated using the ADT model under nominal operating conditions, and the robustness of reliability estimations are checked in presence of uncertainty in operating conditions. Kharoufeh (2003), Kharoufeh and Cox (2005), and Park and Padgett (2006) proposed a generalized discrete cumulative damage model for material strength where the stress on the subject is incremented by small amounts until the component breaks down. Due to flexibility of the model, any form of accumulated damage such as additive and multiplicative can be accommodated. The accumulated damage at increment *n+1* is expressed as $X_{n+1} = X_n + D_n h(X_n)$ where X_n represents the accumulated damage up to n^{th} stress level and D_n denotes the damage to the system at n^{th} stress increment. Basically, the model involves an initial damage and damages due to different levels of stress. The authors study the degradation models and their parameter estimations for the cases where the initial degradation follows Brownian motion, geometric Brownian motion, and gamma process. They use a general function to describe the effect of degradation on the system strength. Kharoufeh *et al.* (2010) estimate the system reliability metrics such as system residual life distribution for an aircraft engine turbine blade where the degradation of which is dependent on the random environment conditions such as temperature and engine load. Bagdonavicius *et al.* (2010) discuss reliability analysis as well as parameter estimation for degradation models with covariates. Sudret (2008) describes the degradation indicator for reinforced concrete structures with a function with general form in time and random environmental conditions. They show that the mean and variance of the degradation level does not depend on the correlation structure among input random variables. Lim and Yum (2011) designed an accelerated life testing plan which describes the system's degradation level as a Wiener process with stress-dependent drift. The test plan is designed for three distinct functions with stress-dependent drift parameter, namely exponential, Arrhenius, and power functions. It is assumed that the variance of Weiner process at each stress level is constant and independent of stress level value.

2.4. Condition-Based Maintenance Models

Jardine *et al.* (2006) review the recent research on system fault diagnostics, and its associated data processing for systems using condition-based maintenance. Ahmad and Kamaruddin (2012) also present a comprehensive review on condition-based maintenance and time-based maintenance techniques in industrial applications. Most previous studies focus on the condition-based maintenance for a system degrading due to a single failure mode, usually degradation process, where the maintenance action is perfect, meaning that the maintenance actions restore the system to the state of being as-good-as-new. Also, the models assume that the system's state is detected by inspection not by continuous monitoring. Some of such studies are as follow. Grall *et al.* (2002) suggest a condition-based maintenance model including a multi-level control-limit rule for a single-unit system degrading due to a continuous stochastic degradation path. They investigate the optimal replacement threshold and optimal inspection interval which minimize the long run system operation cost per unit of time. Castanier *et al*. (2005) extend this model to the case of two-unit system. Wang *et al.* (2009) investigate an optimal condition-based maintenance policy for a multi-unit electricity generating system which maximizes the total profit. The system's state is addressed by a discrete Markov chain, and units are assumed to be s-independent. The proposed maintenance policy sets some replacement thresholds for the system taking into account the existence of redundant units.

In real world, most systems are subject to multiple competing failure modes. Some recent studies focus on developing maintenance policies for such systems. Deloux *et al.* (2009) propose a condition-based maintenance policy combined with statistical process control (SPC) for a system subject to two competing failure risks, degradation process and stresses. The degradation process is modeled by a continuous stochastic process, and stresses affect the system according to the extreme shock model. Inspections take place to determine the system's state, and SPC is utilized to detect the failure due to intensive stress. The maintenance actions involve only replacements and no minimal repairs. The objective of this study is to investigate the optimal periodic inspection interval as well as optimal replacement threshold for the degradation process which minimize the system long run cost per unit of time. Zhu *et al.* (2010) study an optimal inspection interval for a repairable unit subject to two competing failure risks namely degradation process and sudden failure to maximize the system availability. The system is assumed to be restored to the state of being as-good-as-new after the maintenance. Neves *et al.* (2011) study the input parameters estimation for the problem discussed by Zhu *et al.* (2010) except they assume that they describe the system's state by a discrete Markov chain. Wang and Pham (2011) propose a multi-objective optimal imperfect maintenance policy based on periodic inspection for a system subject to dependent competing risks namely degradation and random shocks. Wang (2012) investigate an optimal multi-threshold condition-based maintenance for a system with two dependent failure modes, degradation process and shocks to minimize the system operating and maintenance cost. The study does not focus on obtaining the optimal maintenance policy which maximizes the system availability. Liao *et al.* (2006) develop an optimal condition-based maintenance strategy for a system subject to aging by gamma process with consideration of imperfect repair and stochastic maintenance threshold. The optimization analysis is performed in respective to system availability and short-run availability. The short-run availability indicates when the system should be replaced. The model is only limited to the systems degrading due to one failure mode.

2.5. Direction of Research

A thorough review of previous literature on degradation and random shocks reveals that many previous studies have concentrated on the systems subject to competing failure modes due to degradation and random shocks where the shocks can lead to sudden system failure or accelerate the degradation process by adding some damage increments. However, a few studies have been focused on the areas such as the impact of the system's age or state on the shock damage magnitude. Also, few studies have addressed the optimal maintenance decision making for such systems. In the scope of this dissertation, we focus on addressing those issues in the field of degradation and shock modeling. The objectives we are seeking to achieve in this dissertation can be more specifically determined as it follows:

1) To develop the system reliability model for a system subject to degradation and random shocks with time-dependent damage magnitudes to capture the effect of system's age and shock damage magnitudes.
2) To extend the model developed in 1 by considering some dependency structure between accumulated damage due to shocks and degradation process.
3) To develop a generalized optimal imperfect threshold-type CBM policy for the systems subject to multiple competing risks including degradation and sudden failure where the maintenance is imperfect.

2.6. Organization of the Dissertation

This dissertation is organized as it follows.

Chapter 1 presents the motivation of this dissertation as well as the basic concepts for degradation and shocks model. At last, an overview on this dissertation is given.

In chapter 2, a comprehensive review on degradation models, random shocks models, and associated reliability metrics is presented and the direction of this dissertation and areas for further research are addressed.

Chapter 3 develops a new reliability model for a system subject to degradation and random shocks where the shock damage magnitudes are time-dependent. Gamma process and cumulative shock model are employed to describe the impact of degradation process and random shocks on the system in this research. No other dependency structure is assumed between degradation and random shocks. This chapter extends the existing literature by incorporating time-variant shock damage magnitudes in order to address the impact of system's age on shock damage magnitudes.

Chapter 4 studies the system considered in chapter 3 where there exist some degree of correlation between the degradation process and accumulated shock damage magnitudes. The system reliability distribution for such a system is formulated, and a method to estimate model parameters is presented. To study the system reliability behavior, a numerical analysis is conducted on all the parameters involved in the mathematical expression of the system reliability distribution.

Chapter 5 develops a generalized optimal threshold-type CBM policy for the system described in chapter 4 when it is subject to multiple competing failure modes including the degradation process and sudden failure. The degradation process is described similar to the one in Chapter 4, and the maintenance is considered to be imperfect. The model can incorporate both scenarios when the competing risks are either independent or dependent. The objective of this study is to maximize the system average achieved reliability. The optimization problem for such a system is formulated, and numerical results are presented for the scenario when the system is subject only to two competing risks, namely degradation process and sudden failure.

At last, chapter 6 discusses the conclusions and the future research in the field of degradation and random shock models.

Chapter 3 : Modeling Degradation and Random Shocks with Time-Dependent Damage Magnitudes

3.1. Introduction

Traditional reliability analysis requires information on system failure time data. In many systems, collecting failure time data can be time-consuming and expensive due to being highly reliable. However, there are many situations where the potential failure of the system can be monitored by observing the system's degradation in time. In engineering, the degradation data can be generally obtained by measuring a physical indicator (Lehman, 2010) directly or indirectly. An example for a degradation indicator can be crack propagation in a bridge or wear (which is measured by contaminants in the lubrication), and bearings vibration level measured by sensors. In such systems, the system failure time is defined as the time when the degradation level exceeds a critical threshold level. Degradation of systems occurs due to many internal or external failure processes, but most of systems usually deteriorate due to aging and exposure to random shocks and environments. Each shock can result in a sudden system failure or accelerate the degradation process. A bridge made of reinforced concrete is subject to both degradation process and random shocks. The degradation process of the bridge may be due to crack grow thin the concrete or corrosion in the concrete steel rebars, and the earthquakes, tornados, or heavy snow may be considered as random shocks to the bridge. All these processes accumulate some amount of damage in the bridge, which lead to the reduction in bridge's strength that will eventually result in its failure. In general, the system failure occurs when the cumulative damage due to degradation process and random shocks exceeds a predetermined critical failure threshold.

There are many studies that address modeling degradation data obtained from systems subject to aging degradation. Some of recent models describe general degradation path, semi-Markov chains, and continuous-time stochastic processes(See Lu and Meeker (1993), Bae *et al.* (2007), Kharoufeh (2003), Kharoufeh *et al.* (2010),van Noortwijk *et al.* (2007), and Gorjian *et al.* (2009)).Among these models, continuous-time stochastic processes such as gamma process and Weiner process have been widelyapplied in modeling of continuous degradation data over the time. Some of recent research work addressing these processes include Pandey *et al.* (2005), Nicolai *et al.* (2007), Baussaron *et al.*, (2010), Elsayed and Liao (2004) and Wang (2010). Gamma process and Weiner process do not appropriately model the degradation data obtained from systems subject to random shocks (Gorjian *et al.*, 2009).The models that address random shocks are classified into three groups of models, namely cumulative damage model, $\delta-$shock model, and extreme shock model. There exists other variation of shock model like run shock model and mixed shock model described by Gut and Husler (2005). According to Gut and Husler (2005), in cumulative damage model, the system failure happens when the cumulative damage due to random shocks exceeds the critical failure threshold. The $\delta-$shock model considers that the system fails if the time interval between two consecutive shocks is less than a threshold, δ. Finally, the extreme shock model considers that the system fails as soon as a shock with a damage magnitude over a certain threshold occurs to the system. There have been many extensions to these models (See Gut and Husler (2005), Wang and Zhang (2005), Nakagawa (2007), Lam (2009), and Van der Weide *et al.* (2011)).

Usually, systems are subject to both degradation and random shocks. Finkelstein (2007) analyzes the lifetime function for human body where the body is considered as a system subject to aging and imperfect external shocks such as external stresses. The shocks are modeled by cumulative damage model, and the damages due to shocks are independently distributed with a common distribution. Jiang *et al.* (2011) study the reliability expression for a system subject to two dependent failure competing risks, namely traumatic shocks and accumulated damage, where the shock load threshold changes. The damage to the system is accumulated due to degradation process and non-fatal random shocks. Also, the system may immediately fail if a traumatic shock occurs. They assess the system reliability for this problem where the random shocks are modeled by run shock model and extreme shock model. Li and Pham (2005) assess the reliability for a system subject to two independent degradation process and random shocks. The first degradation path is described as $Y_1(t) = A + Bg(t)$ where A is the initial degradation and B is the mean degradation rate. Function g is considered to be an increasing function in time. The second degradation process is modeled by a logistic degradation path which matches the S-shaped degradation based on component usage. Cha and Finkelstein (2009) analyze the lifetime distribution of a system subject to linear degradation and random shocks which are modeled by the extreme shock model. The random shocks can be fatal to the system with probability $p(t)$, or they may lead to acceleration in the degradation process by probability $1-q(t)$. Lehmann (2009) reviews all classes of degradation and shock models and their corresponding reliability expressions. The author investigates the reliability function for two classes of degradation and shock models, the one without covariates and the other with external covariates. Both models describe a system subject to degradation and shocks where the shocks are arriving according to a non-homogenous Poisson process. However, in the second model, the degradation process and shock arrival are considered to be dependent on some external factors, such as load or environmental conditions. Wang and Pham (2011) address the reliability function for a single component where the component failure mechanism is defined by two dependent competing failure modes, damage accumulation and fatal shocks. The damage accumulation is considered to be due to a linear degradation path with stationary parameters and random shocks. Huynh *et al.* (2011) develop the reliability expression for a single unit system subject to two dependent competing risks, damage accumulated in the system due to degradation and traumatic shocks where the shock arrival rates are dependent on the degradation process. The degradation is modeled by a gamma process, and the shock arrivals follow non-homogenous Poisson process. They assume that shocks are either fatal to the system or they don't damage the system at all.

Our research differs from all recent related works as follows. In previous studies of systems subject to degradation and random shocks, the impact of system's age on shock damage magnitudes has been ignored and instead it was assumed that shock damage magnitudes are independent of the system's state or age and distributed according to a common distribution. However, as the system ages, the system gets more vulnerable to shocks (Huynh *et al.*, 2011).That is, as the system degrades in time, we can assume that the average damage of shocks tends to increase in time. An example of such a problem is the bridge made of reinforced concrete. The length of crack growth or bridge strength at any point of time can represent the degradation level of the bridge. The tornados and earthquakes can be considered as the random shock happening to the bridge. The strength of reinforced concrete will be reduced in the long term horizon, after almost 20 years of life (Nowak and Collins, 2000). Due to this fact, the intensity of damage due to a 4-Richter earthquake can be more destructive when the bridge is 25 years old compared with when the bridge is newly built. Our contribution in this study is to develop a new reliability model which extends the existing studies by incorporating the impact of system's age on shock damage magnitudes. Therefore, it is considered that shock damage magnitudes are time-dependent but s-independent. In other words, the damage magnitude of each shock depends on its arrival time. No dependency between degradation process and random shocks are also assumed.

Figures 3-1 and 3-2 highlight the difference between the model which is presented in this chapter and the existing models when the shock damage magnitudes follow a normal distribution. Figure 3-1 indicates the common assumption on shock damage magnitudes in previous studies, while Figure 3-2 shows that shock damage magnitudes intensity increases in time.

A brief overview of the current chapter is given as follows. First, we formulate the reliability expression for a one-component system subject to degradation and random shocks where the shock damage magnitudes are mutually independent but time-variant. The degradation process is described by a gamma process, and the shock damages are modeled by cumulative shock model. Furthermore, the parameter estimation procedure for the accumulated damage path is developed. A Monte Carlo simulation is developed as well to validate the mathematical results. Finally, a numerical analysis is conducted on system reliability distribution to study the behavior of system reliability function.

The remainder of the chapter is organized as follows: Section 3.2 addresses problem description and mathematical notations used in this chapter. Section 3.3 presents the reliability estimation for the system for different scenarios, and section 3.4 develops the parameter estimation method for accumulated damage path. Section 3.5 describes the numerical analysis conducted on system reliability function and the Monte Carlo simulation results. Finally, section 3.6 presents the conclusion.

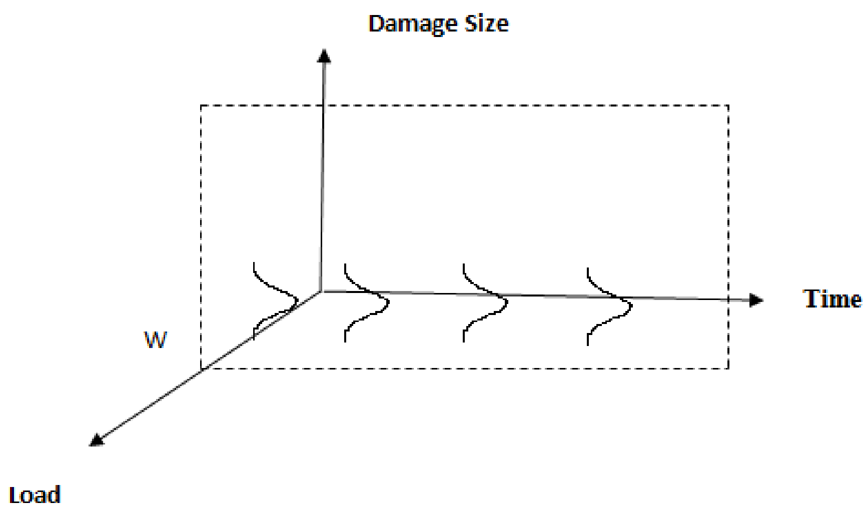

Figure 3-1: The damage magnitudes are distributed with a common distribution and independent of total damage in system

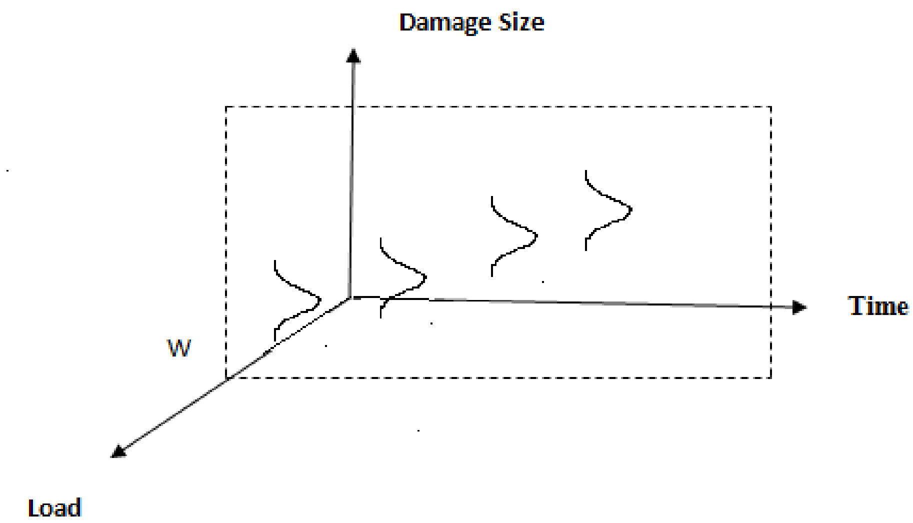

Figure 3-2: The damage magnitudes are time-variant with different but independent distributions

3.2. System Description

The system under study is a single component subject to degradation and random shocks with a stochastic critical failure threshold. In some cases such as the degradation and vibrating-damage data, the model with stochastic critical failure threshold models the system reliability more precisely compared to a model with the predetermined threshold (Chen and Chen, 2012). In this work, the system reliability for the critical threshold of failure is considered to be exponentially distributed with a parameter θ. In general, the assumptions in our model are similar to the ones in Cha and Finkelstein (2009) and Wang and Pham (2011). The details on shocks and degradation models are described as follows.

3.2.1. Shock Model

The random shocks impact the system according to a homogeneous Poisson process (HPP) with rate parameter λ. $N(t)$ denotes the number of shocks affecting the system by time *t*. The damages due to random shocks are modeled by a cumulative shock model. Gut and Husler (2005), Cha and Finkelstein (2009), and Wang and Pham (2011) assume that shock damage magnitudes are mutually independent and commonly distributed with no dependency on the state of system. The shock model in this chapter differs from the ones of Gut and Husler (2005), Cha and Finkelstein (2009), and Wang and Pham (2011) in the way that the shock damage magnitudes are considered to be s-independent but time-variant. That is, the damage magnitude of each shock depends on the time when the shock arrives. Therefore, the level of accumulated damage of the system due to shocks by time $t, Z(t)$, is defined as

$$Z(t) = \sum_{i=1}^{N(t)} w(t_i) \tag{1}$$

3.2.2. Degradation Process

The deterioration due to aging and wear in most engineering systems is a monotonic path which can be appropriately described by a gamma process. Therefore, in this work, the system degradation process, $X(t)$, is described by a gamma process with independent increments where $X(t)$ indicates the system degradation level at time t which is distributed as a gamma process with shape parameter α, scale parameter β, and independent increments. Thus, the density function for X is given by

$$Gamma(x;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \tag{2}$$

Hence, $\{X(t) \geq 0, t\}$ is a continuous gamma process with parameters $\alpha > 0$ and $\beta > 0$ with the following properties (van Noortwijk *et al.*, 2007):

4) $X(0)=0$ with probability one.

5) $X(t)$ has independent increments

6) $X(t)-X(\nu) \sim Gamma((t-\nu)\alpha, \beta) \quad for\ all\ t > \nu \geq 0$

Hence, the total damage accumulated to the system by time t is given by

$$D(t) = X(t) + Z(t) \tag{3}$$

Where $D(t)$ denotes the accumulated damage of the system by time t and $X(t)$ and $Z(t)$ are the total level of damage of the system due to degradation and random shocks, respectively. Other assumptions used in this problem are as they follow:

- $N(t)$ is independent of damage magnitudes.
- The degradation process and random shock are independent.
- The damage magnitudes of shocks are mutually independent.

3.2.3. Notation

$R(t)$: The system reliability at time t

$D(t)$: The level of accumulated damage to the system by time t

$Z(t)$: The level of accumulated damage to the system due to shocks by time t

t_i: The arrival time of the i^{th} shock to the system

$w(t_i)$: The damage magnitude due to i^{th} shock, dependent on the time when shock i arrives to the system

$X(t)$: The level of degradation of the system by time t

α : Shape parameter of the gamma distribution

β : Scale parameter of the gamma distribution

$N(t)$: Number of shocks arriving to the system by time t

λ : The shock arrival rate parameter

S: The critical failure threshold of the system

θ : The rate parameter for critical failure threshold distribution

$Y_{w(t_1),...,w(t_n)}(y_1,..,y_n)$: The joint probability distribution for damage magnitudes $w(t_1),...,w(t_n)$

M_w : Moment generation function for shock damage magnitudes

$f_T(t)$: The probability distribution function for the system time to failure

μ : The rate parameter of exponentially distributed shock damage magnitude

Ψ : The set of parameters for the shock damage magnitude probability distribution

3.3. Reliability Estimation

3.3.1. System reliability estimation in presence of both degradation process and shocks: In this case, the system is subject to both degradation and random shocks. The system fails when the accumulated damage to the system exceeds the critical threshold. Thus, the corresponding system reliability estimation can be given as:

$$
\begin{aligned}
R_s(t) &= \Pr\{D(t) < S\} \\
&= \Pr\{X(t) + \sum_{i=1}^{N(t)} w(t_i) < S\} \qquad (4) \\
&= \exp\{\alpha t \ln(\frac{\beta}{\beta+\theta}) + \int_0^t \lambda[M_{W(x)}(-\theta) - 1]dx\}
\end{aligned}
$$

Where $M_{w(t)}$ is the moment generation function for the time-dependent shock damage magnitudes.

Proof. To derive the reliability expression, we can integrate out the variables in the reliability expression. These results were inducted from theorem 1 of Cha and Finkelstein (2009). Therefore, we can start obtaining the reliability expression in the following way:

$$
\Pr(X(t) + \sum_{i=1}^{N(t)} w(t_i) < S \mid N(t) = n, X(t) = x, w(t_1), w(t_2), \ldots, w(t_n), t_1, ..., t_{N(t)}) = e^{-\theta\{x + \sum_{i=1}^{n} w(t_i)\}}
$$

Thus,

$$
\begin{aligned}
&\Pr(X(t) + \sum_{i=1}^{N(t)} w(t_i) < S \mid N(t) = n, X(t) = x, t_1, ..., t_{N(t)}) \\
&= \int_{w(t_1), w(t_2), \ldots, w(t_n)} e^{-\theta\{x + \sum_{i=1}^{n} w(t_i)\}} \Upsilon(w(t_1), w(t_2), \ldots, w(t_n)) dw(t_1) dw(t_2) \ldots dw(t_n) \\
&= e^{-\theta x} . \int_{w(t_1), w(t_2), \ldots, w(t_n)} e^{-\theta \sum_{i=1}^{n} w(t_i)} \Upsilon(w(t_1), w(t_2), \ldots, w(t_n)) dw(t_1) dw(t_2) \ldots dw(t_n) \\
&= e^{-\theta x} . E[e^{-\theta \sum_{i=1}^{n} w(t_i)}] \\
&= e^{-\theta x} . E[e^{-\theta w(t_1) - \theta w(t_2) - \cdots - \theta w(t_n)}] \\
&= e^{-\theta x} . E[e^{-\theta w(t_1)}] . E[e^{-\theta w(t_2)}] ... E[e^{-\theta w(t_n)}] \\
&= e^{-\theta x} . \prod_{i=1}^{n} M_{w(t_i)}(-\theta) \\
&= \exp\{-\theta x + \sum_{i=1}^{n} \ln M_{w(t_i)}(-\theta)\}
\end{aligned}
$$

$$\Pr\{X(t)+\sum_{i=1}^{N(t)} w(t_i) < S \mid N(t)=n, t_1,\dots,t_{N(t)})$$

$$= \int_0^\infty \mathrm{e}^{-\theta x} . e^{\sum_{i=1}^{n} \ln M_{w(t_i)}(-\theta)} \frac{\beta^{\alpha t}}{\Gamma(\alpha t)} e^{-\beta x} x^{\alpha t-1} dx$$

$$= e^{\sum_{i=1}^{n} \ln M_{w(t_i)}(-\theta)} \int_0^\infty \frac{\beta^{\alpha t}}{\Gamma(\alpha t)} e^{-(\beta+\theta)x} x^{\alpha t-1} dx$$

$$= e^{\sum_{i=1}^{n} \ln M_{w(t_i)}(-\theta)} \left(\frac{\beta}{\beta+\theta}\right)^{\alpha t}$$

Therefore,

$$\begin{aligned} R_s(t \mid t_1,\dots,t_{N(t)}) &= \Pr\{D(t) < S \mid t_1,\dots,t_{N(t)}\} \\ &= \Pr\{X(t)+\sum_{i=1}^{N(t)} w(t_i) < S \mid t_1,\dots,t_{N(t)}\} \\ &= \sum_{n=0}^{\infty} \Pr\{X(t)+\sum_{i=1}^{N(t)} w(t_i) < S \mid N(t)=n, t_1,\dots,t_{N(t)}\} \Pr\{N(t)=n\} \\ &= \sum_{n=0}^{\infty} \left(\frac{\beta}{\beta+\theta}\right)^{\alpha t} \exp\{\sum_{i=1}^{n} \ln M_{w(t_i)}(-\theta)\} \frac{e^{-\lambda t}(\lambda t)^n}{n!} \end{aligned}$$

(5)

We obtain the distribution of $t_1, t_2, \dots, t_n$ as follows. Let $m(t) \equiv E[N(t)] = \int_0^t \lambda dx$ .The assumptions used in this problem guarantee that $m^{-1}(t)$ exists (Finkelstein and Cha, 2009). We define $N^*(t) \equiv N(m^{-1}(t)), t \ge 0$ and $T_i^* \equiv m(t_i), i \ge 1$. It is known from Cinlar (1975) that $\{N^*(t), t \ge 0\}$ is a stationary Poisson process with intensity equal to 1, and $T_i^*, i \ge 1$ is the occurrence time of shocks in the new time scale. Let $s = m(t)$. Thus,

$$\exp\{\sum_{i=1}^{N(t)} \ln M_{w(t_i)}(-\theta)\}$$
$$= \exp\{\sum_{i=1}^{N^*(s)} \ln M_{w(m^{-1}(T_i^*))}(-\theta)\}$$
$$= E_{N^*(s)}[E_{T_i^*}[\exp\{\sum_{i=1}^{N^*(s)} \ln M_{w(m^{-1}(T_i^*))}(-\theta)\} \mid N^*(s) = n]]$$

The joint distribution of $(T_1^*, T_2^*, \ldots, T_n^*)$ given $N^*(s) = n$ is corresponding to the joint distribution of $(V_{(1)}, V_{(2)}, \ldots, V_{(n)})$ where $V_{(1)} \le V_{(2)} \le \cdots \le V_{(n)}$ are the order statistics of i.i.d random variables $V_1, V_2, \ldots, V_n$ which are uniformly distributed on the interval $[0, s] = [0, m(t)]$. Therefore, the last conditional expectation will be:

$$E_{T_i^*}[\exp\{\sum_{i=1}^{N^*(s)} \ln M_{w(m^{-1}(T_i^*))}(-\theta)\} \mid N^*(s) = n]$$
$$= E_{T_i^*}[\exp\{\sum_{i=1}^{n} \ln M_{w(m^{-1}(T_i^*))}(-\theta)\}]$$
$$= E_{V_{(i)}}[\exp\{\sum_{i=1}^{n} \ln M_{w(m^{-1}(V_{(i)}))}(-\theta)\}]$$
$$= E_{V_i}[\exp\{\sum_{i=1}^{n} \ln M_{w(m^{-1}(V_i))}(-\theta)\}]$$
$$= E_{V_1}[\exp\{n \ln M_{w(m^{-1}(V_1))}(-\theta)\}]$$
$$= E_U[\exp\{n \ln M_{w(m^{-1}(sU))}(-\theta)\}]$$
$$= E_U[(M_{w(m^{-1}(sU))}(-\theta))^n]$$

Where $U \equiv \dfrac{V_1}{s} = \dfrac{V_1}{m(t)}$ is a random variable uniformly distributed on the unit interval, $[0,1]$. Thus,

$$
\begin{aligned}
&E_U[(M_{w(m^{-1}(sU))}(-\theta))^n] \\
&= (E_U[M_{w(m^{-1}(sU))}(-\theta)])^n \\
&= (\int_0^1 M_{w(m^{-1}(sU))}(-\theta)du)^n \\
&= (\int_0^1 M_{w(m^{-1}(m(t)u))}(-\theta)du)^n \\
&= (\frac{\lambda}{m(t)}\int_0^t M_{w(x)}(-\theta)dx)^n
\end{aligned}
$$

The last equation is obtained using a variable change of $x = m^{-1}(m(t)u)$. Therefore, it is concluded that

$$
\begin{aligned}
R(t) &= (\frac{\beta}{\beta+\theta})^{\alpha t}\sum_{n=0}^{\infty}[\int_0^t \frac{\lambda}{m(t)} M_{w(x)}(-\theta)dx]^n \frac{s^n}{n!}e^{-s} \\
&= \exp\{\alpha t \ln(\frac{\beta}{\beta+\theta})\}e^{-s}\sum_{n=0}^{\infty}\frac{[\int_0^t \frac{\lambda s}{m(t)} M_{w(x)}(-\theta)dx]^n}{n!} \\
&= \exp\{\alpha t \ln(\frac{\beta}{\beta+\theta})\}e^{-s}\exp\{\frac{s}{m(t)}\lambda\int_0^t M_{w(x)}(-\theta)dx\} \\
&= \exp\{\alpha t \ln(\frac{\beta}{\beta+\theta})\}\exp\{-\int_0^t \lambda dx\}\exp\{\lambda\int_0^t M_{w(x)}(-\theta)dx\} \\
&= \exp\{\alpha t \ln(\frac{\beta}{\beta+\theta}) + \int_0^t \lambda[M_{w(x)}(-\theta)-1]dx\}
\end{aligned}
$$

3.3.2. System Reliability estimation in absence of shocks: One special case of the problem under study is that the system is only subject to degradation and no shocks occur to the system. This is equivalent to $\lambda = 0$. In this case, the system failure happens when the damage accumulation due to degradation process exceeds critical threshold value. Thus, the system reliability is expressed as:

$$
\begin{aligned}
R_s(t) &= \Pr\{D(t) < S\} \\
&= \Pr\{X(t) < S\} \\
&= \int_0^{\infty} \Pr\{S > X(t) \mid X(t) = x\} \frac{\beta^{\alpha t}}{\Gamma(\alpha t)} e^{-\beta x} x^{\alpha t-1} dx \\
&= \int_0^{\infty} e^{-\theta x} \frac{\beta^{\alpha t}}{\Gamma(\alpha t)} e^{-\beta x} x^{\alpha t-1} dx \\
&= (\frac{\beta}{\beta+\theta})^{\alpha t} \\
&= \exp\{\alpha t \ln(\frac{\beta}{\beta+\theta})\}
\end{aligned} \tag{6}
$$

Comparison between Equation (4) and Equation (6) shows that they are equivalent when $\lambda = 0$.

3.3.3. System Reliability estimation in absence of degradation: Another special case of the problem under study is when the system is not subject to degradation but experiences shocks. Therefore, the accumulated damage to the system is only due to shocks. We claim that the system reliability estimation for this scenario is:

$$
\begin{aligned}
R_s(t) &= \Pr(\sum_{i=1}^{N(t)} w(t_i) < S) \\
&= \exp\{\int_0^t \lambda[M_{W(x)}(-\theta) - 1] dx\}
\end{aligned}
$$

To obtain system reliability expression each conditional probability is integrated out in respective to the corresponding variables. This approach is valid since we assume that all the variables are mutually independent. Thus, we obtain:

$$\Pr(\sum_{i=1}^{N(t)} w(t_i) < S \mid N(t) = n, w(t_1), w(t_2), \ldots, w(t_n), t_1, ..., t_{N(t)}) = e^{-\theta(\sum_{i=1}^{n} w(t_i))}$$

$$\begin{aligned}\Pr(\sum_{i=1}^{N(t)} w(t_i) < S \mid N(t) = n, t_1, ..., t_{N(t)}) &= \int_{w(t_1), w(t_2), \ldots, w(t_n)} \{e^{-\theta(\sum_{i=1}^{n} w(t_i))} \\ &\quad \times f(w(t_1), w(t_2), \ldots, w(t_n))\} dw(t_1) dw(t_2) \ldots dw(t_n) \\ &= E(e^{-\theta(\sum_{i=1}^{n} w(t_i))}) \\ &= E(e^{-\theta w(t_1)}) \ldots E(e^{-\theta w(t_n)}) \\ &= M_{w(t_1)}(-\theta) \ldots M_{w(t_n)}(-\theta) \\ &= \exp\{\sum_{i=1}^{n} \ln M_{w(t_i)}(-\theta)\}\end{aligned}$$

$$\Pr(\sum_{i=1}^{N(t)} w(t_i) < S \mid t_1, ..., t_{N(t)}) = \sum_{n=0}^{\infty} \exp\{\sum_{i=1}^{n} \ln M_{w(t_i)}(-\theta)\} \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$

Using the framework presented to prove Equation (5), we eventually know that

$$\exp\{\sum_{i=1}^{N(t)} \ln M_{w(t_i)}(-\theta)\} = (\frac{\lambda}{m(t)} \int_0^t M_{w(x)}(-\theta) dx)^n$$

Thus,

$$\begin{aligned}\Pr(\sum_{i=1}^{N(t)} w(t_i) < S) &= \sum_{n=0}^{\infty} (\frac{\lambda}{m(t)} \int_0^t M_{w(x)}(-\theta) dx)^n \frac{e^{-\lambda t} (\lambda t)^n}{n!} \\ &= \exp\{\int_0^t \lambda [M_{W(x)}(-\theta) - 1] dx\}\end{aligned} \tag{7}$$

The comparison between Equation (7) and Equation (4) reveals that the expressions are corresponding if the degradation term in (4) is ignored.

3.4. Parameter Estimation

In the current section, we review a maximum likelihood method (MLE) to estimate the parameters of the system reliability model. To use MLE, we need to find the probability distribution function for the system lifetime. It is known that the probability distribution of time to failure, $f(t)$, is calculated as

$$f_T(t) = -\frac{dR_s(t)}{dt} = -\{\alpha \ln(\frac{\beta}{\beta+\theta}) + \lambda[M_{W(t)} - 1]\} \exp\{\alpha t \ln(\frac{\beta}{\beta+\theta}) + \int_0^t \lambda[M_{W(x)}(-\theta) - 1]dx\} \tag{8}$$

Suppose that n items are put to test and $\{T_1, T_2, ..., T_n\}$ represent the failure times. Therefore, the likelihood function is written as

$$\begin{aligned} L(T;\alpha,\beta,\theta,\lambda,\Psi) &= f(T_1)f(T_2)...f(T_n) \\ l(T;\alpha,\beta,\theta,\lambda,\Psi) &= \sum_{i=1}^{n} \ln(f(T_i)) \end{aligned} \tag{9}$$

Ψ is the parameters set for the shock damage magnitude probability distribution. For example,

If $w(t) \sim \exp(\frac{\mu}{t})$ then $\Psi=\{\mu\}$
If $w(t) \sim N(\mu t, \sigma^2)$ then $\Psi=\{\mu, \sigma^2\}$

In general, differentiating the likelihood function in (9) in respective to each parameter and solving the new system yields the parameter estimation. Also, we can use the following procedure to estimate the model parameters if the system the degradation indicator is monitored continuously by sensors and it is attached to a self-contained acceleration shock recorder. According to Kearns (1994), this recorder can measure any change in operating conditions or any other shock events, and it stores the time, peak acceleration, and pulse width of shocks. Using the data from shock recorder instrument, the shock arrival times can be obtained and their associated damage to the system can be discerned from the damage due to the degradation process. Therefore, we can fit appropriate distributions to shock arrival times and shock damage magnitudes. Using the degradation data, we can estimate α and β of the gamma process based on MLE or methods of moments presented by van Noortwijk (2007).

3.5. Numerical Analysis

In this section, we conduct a numerical analysis for different parameters of the system reliability function in order to study the behavior of the system reliability function. Furthermore, a Monte Carlo simulation is developed to validate the mathematical results of the system reliability. Later on, a comparison between our model and the existing studies is presented. To conduct the analysis the following parameters are assumed. First, it is assumed that the shocks arrive according to Poisson process with rate parameter $\lambda = 0.6$, and the degradation process is defined as a gamma process with parameters $\alpha = 1$ and $\beta = 3$. Second, the critical threshold is assumed to be exponentially distributed with parameter $\theta = 0.01$. In all the following numerical analyses, it is assumed that the shock damage magnitude follows an exponential distribution except where the relationship of damage and time is of interest. It is important to mention that all the plots and computations in this section have been obtained using MAPLE 14. The results of the numerical analysis are presented as it follows.

3.5.1. Numerical analysis on the relationship of damage and time

In this section, the system reliability function is compared for two scenarios when 1) the mean of damage magnitude distributions is linearly time-dependent and 2) mean of damage magnitude distributions is a quadratic function in time. Figures 3-3 to 3-6 show the system reliability function in both scenarios with different distributions for damage magnitudes namely, exponential, gamma, uniform, and normal distribution. Table 3.1 also shows the associated system reliability values over time. The results in Figures 3-3 to 3-6 and Table 3.1 show that the system reliability is always higher in scenario 1 regardless of the type of distribution function for shock damage magnitude. This is because the system is expected to accumulate less damage due to random shocks if the mean of the damage magnitude distribution is linearly time-dependent. Table 3.1 implies that the gap between the reliability function of two scenarios is increasing over time. In other words, the reliability dramatically decreases over time as the order of time-dependency in damage mean increases. Initially the reliability values are close because the gap between order one and order two of time-dependency is not large. However, as the time goes by, the system reliability function associated with the second order of time-dependency falls below the other one due to such a large gap. For example, at $t = 10$, the system reliability for the second order is approximately 60% less than the one of the first order scenario.

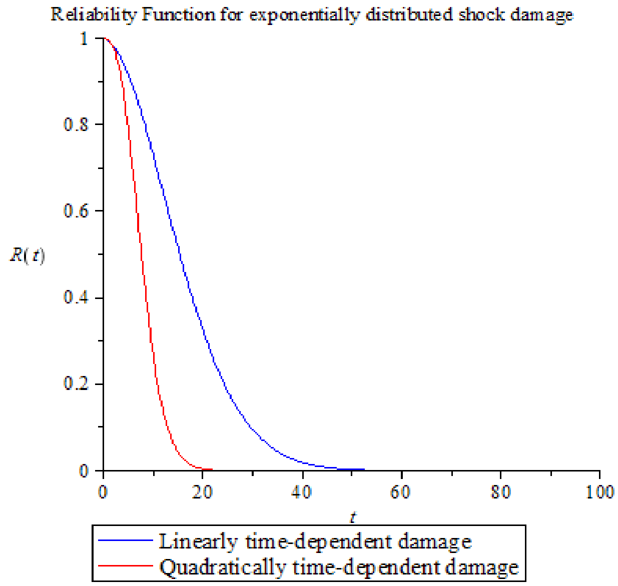

Figure 3-3: System reliability evaluation for $w(t) \sim \exp(\frac{1}{t})$ versus $w(t) \sim \exp(\frac{1}{t^2})$

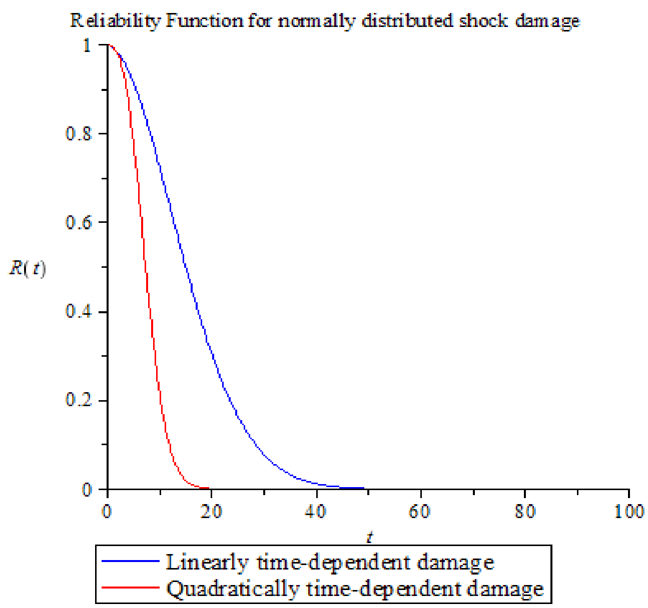

Figure 3-4: System reliability evaluation for $w(t) \sim N(t, 0.01)$ versus $w(t) \sim N(t^2, 0.01)$

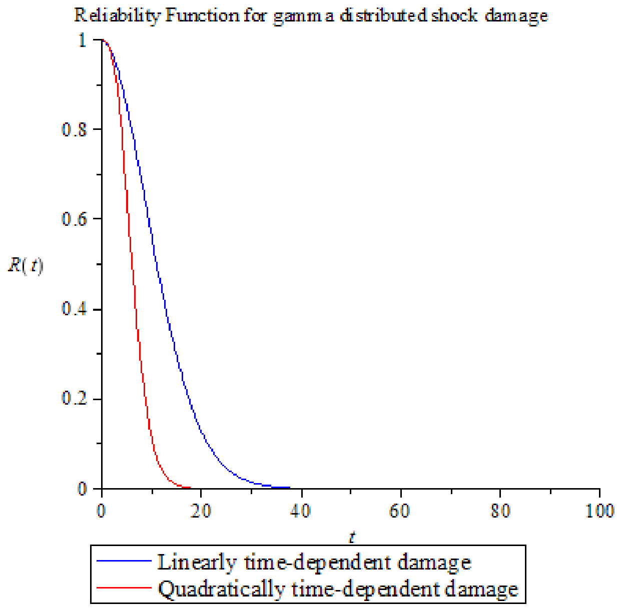

Figure 3-5: System reliability evaluation for $w(t) \sim Gamma(t, 2)$ versus $w(t) \sim Gamma(t^2, 2)$

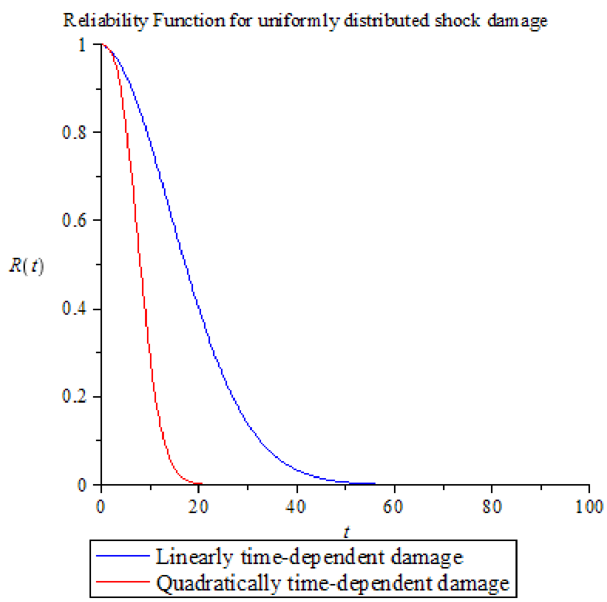

Figure 3-6: System reliability evaluation for $w(t) \sim U(0.5t, t)$ versus $w(t) \sim U(0.5t^2, t^2)$

Table 3.1: System reliability for different type of damage magnitude distribution and relationship to time

Damage Magnitude Distribution	Relation to Time	Time			
		0	10	20	30
exponential	Linear	1	0.730032	0.323919	0.094611
	Quadratic	1	0.266898	0.004411	2.48E-05
Normal	Linear	1	0.723616	0.304116	0.078172
	Quadratic	1	0.211758	0.001143	2.81E-06
gamma	Linear	1	0.560609	0.126621	0.014212
	Quadratic	1	0.113372	0.000529	1.44E-06
uniform	Linear	1	0.77681	0.397841	0.138571
	Quadratic	1	0.289151	0.002595	6.97E-06

3.5.2. Numerical analysis on other parameters

Table 3.2 shows the results of numerical analysis of $\theta, \mu, \lambda, \alpha, \beta$ for system reliability function. The impact of each parameter on system reliability function is plotted in Figures 3-7 to 3-11. Figure 3-3 plots system reliability for different critical threshold parameters. Figure 3-7 and the associated values in Table 3.2 imply that the system reliability increases as θ decreases. This is because, on average, it is expected that the mean of exponentially distributed critical threshold increases as θ decreases. This means that the system lifetime is expected to be longer. Table 3.2 shows that the reliability improved from 0.005 to 0.73 when θ changes from 0.5 to 0.01. However, such an impact is not observed when θ gets 1000 times smaller. This is because the fact that increasing the critical threshold of failure when it is already relatively large enough slightly impacts the system reliability. Figure 3-8 shows the system reliability plots for various rates of exponentially distributed damage magnitude. It can be noted from Figure 3-8 and results in Table 3.2 that the system reliability improves as the damage magnitude increases. This happens because the mean damage magnitude decreases in μ. Therefore, it is expected that the shock damage magnitudes influence the system less, which leads to longer lifetime for the system. This is equivalent to higher system reliability. Figure 3-9 addresses the system reliability plots for a variety of values of shock arrival rate. It can be noted from Figure 3-9 and Table 3.2 that the system reliability decreases as λ increases. With larger λ, the system is expected to experience more shocks for a certain time interval. Thus, on average, the system will accumulate more damage due to random shocks, and this corresponds to decrease in system reliability. Figures 3-10 and 3-11 illustrate the system reliability for different values of shape and scale parameters of gamma process. Basically, these plots show how the system reliability is affected by different parameters of degradation process. Figure 3-10 shows that the system reliability deteriorates in α. Increase in α corresponds to more degradation in the system. Therefore, the system reliability reduces in α. According to Figure 3-11 and Table 3.2, the system reliability improves with β.As β increases, the degradation reduces. That is the reason why the system reliability is improved when β increases. Table 3.2 shows that an increase in β improves the system reliability since the degradation accumulated in the system, on average, decreases. The results imply that the system reliability improves slightly for β over 6.

According to lifetime values in Table 3.2, among all parameters, the system reliability is more sensitive to the shock arrival rate parameter, λ, and is less sensitive to β.

Table 3.2: Numerical analysis on system reliability for different parameters of the mathematical model

θ /Time	0	10	20	30	40	50
0.000001	1	0.999967	0.999873	0.99972	0.999507	0.999234
0.001	1	0.967409	0.882418	0.759787	0.618225	0.475892
0.01	1	0.730032	0.323919	0.094611	0.019356	0.002913
μ /Time	0	10	20	30	40	50
1	1	0.730032	0.323919	0.094611	0.019356	0.002913
4	1	0.89848	0.699831	0.475744	0.284032	0.149799
8	1	0.931957	0.807266	0.651077	0.489763	0.344187
20	1	0.952916	0.881475	0.791759	0.690763	0.58552
λ /Time	0	10	20	30	40	50
0	1	0.96727	0.935611	0.904988	0.875367	0.846716
0.5	1	0.765085	0.386556	0.137846	0.036534	0.007498
1	1	0.605161	0.159709	0.020996	0.001525	6.64E-05
5	1	0.092718	0.000136	6.08E-09	1.40E-14	2.51E-21
α /Time	0	10	20	30	40	50
2	1	0.706138	0.303062	0.085622	0.016943	0.002467
5	1	0.639046	0.248209	0.063462	0.011365	0.001497
10	1	0.541091	0.177948	0.038524	0.005841	0.000652
20	1	0.387923	0.091463	0.014196	0.001543	0.000123
β /Time	0	10	20	30	40	50
1.5	1	0.706216	0.303129	0.08565	0.016951	0.002468
6.4	1	0.743043	0.335567	0.099761	0.020773	0.003182
10.6	1	0.747651	0.339743	0.101628	0.021293	0.003282
30	1	0.750971	0.342767	0.102988	0.021674	0.003356

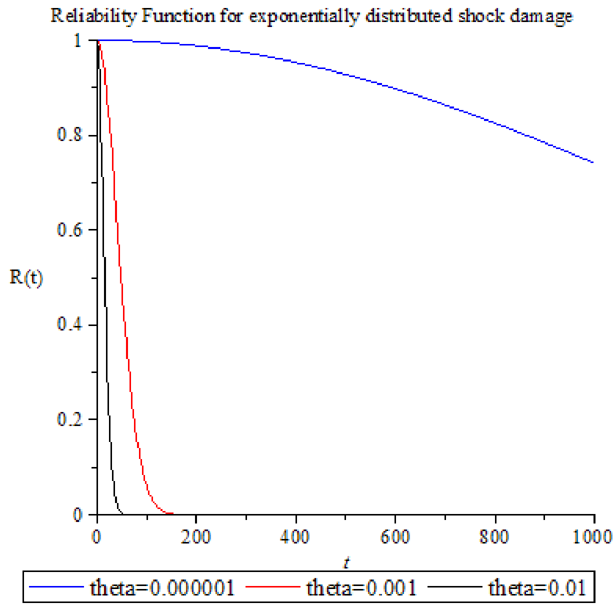

Figure 3-7: System reliability for various values of critical failure threshold parameter, θ

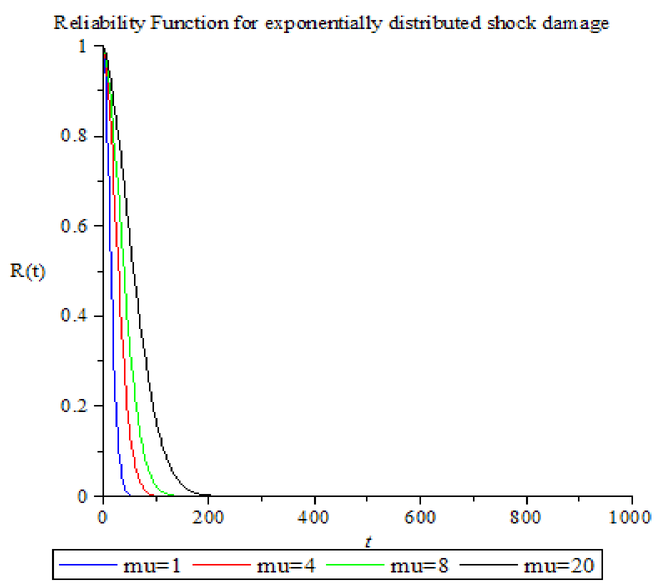

Figure 3-8: System reliability for various values of shock damage rate, μ

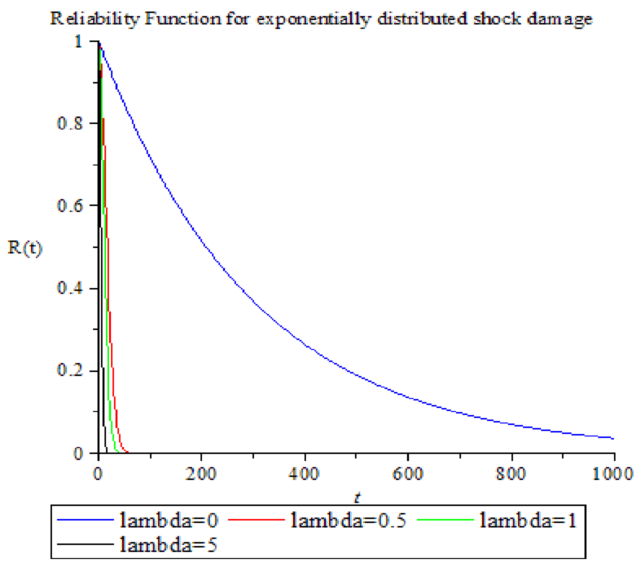

Figure 3-9: System reliability for various values of shock arrival rate, λ

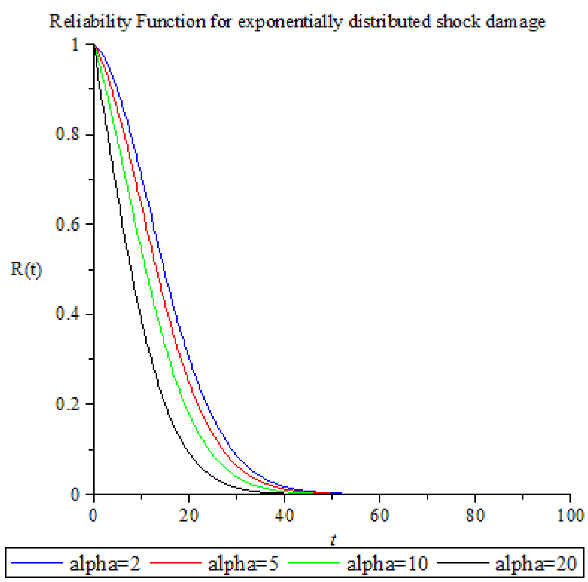

Figure 3-10: System reliability for various values of α

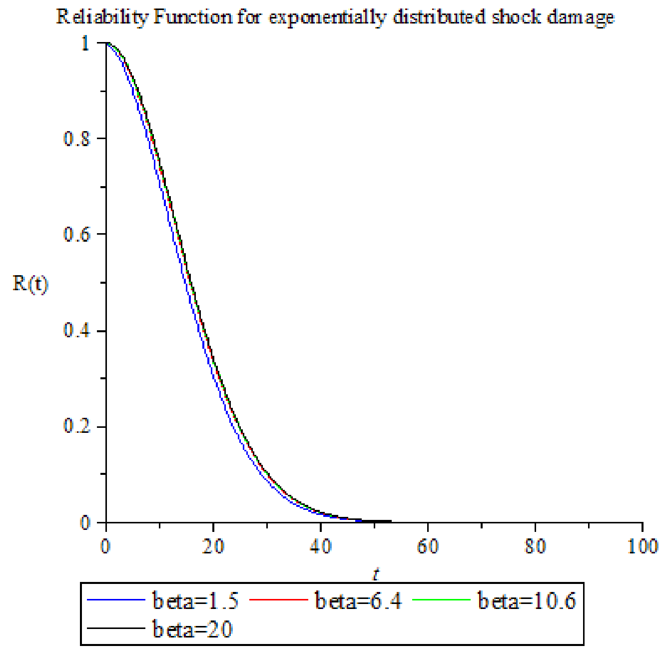

Figure 3-11: System reliability for various values of β

Let the shock damage magnitude follows $N(\mu(\mathrm{t}) = \mu_0 + \mu t, \sigma^2 = 0.01)$ where $\mu_0 = 1$. $\frac{\mu}{\mu_0}$ shows the degree to which the system's age affects the shock damage magnitudes. Thus, a sensitivity analysis of system reliability is performed on this ratio. The result of this analysis is shown in Figures 3-12. It can be indicated from Figure 3-12 that the ratio of $\frac{\mu}{\mu_0}$ impacts both $R(t)$ and $f_T(t)$ significantly. When $\frac{\mu}{\mu_0}$ increases (μ increases from 0.1 to 10 at a fixed $\mu_0 = 1$), $R(t)$ decreases and $f_T(t)$ shifts to the right which is corresponding to the smaller mean time-to-failure for the system. The results show that a small ratio of $\frac{\mu}{\mu_0}$ assures better reliability measures for the system.

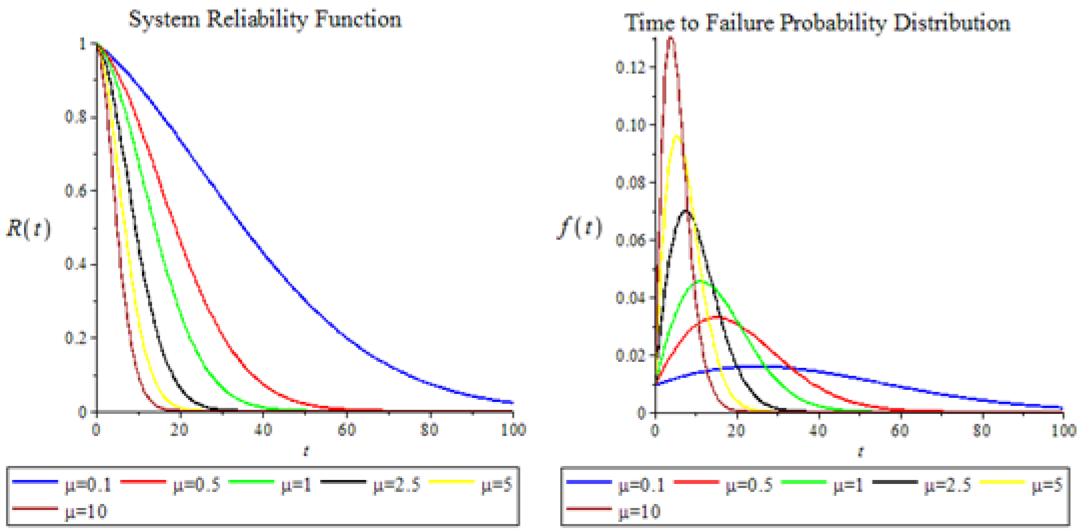

Figure 3-12: Sensitivity analysis of $R(t)$ and $f_T(t)$ on $\frac{\mu}{\mu_0}$ for Model I ($\mu_0 = 1$)

3.5.3. Comparison with Previous Studies

In this section, we aim to compare the system reliability results of the current model with the ones of a reference reliability model as it is described below:

Reference Reliability Model: In this model, the total degradation level in the system is modeled by a degradation process following a gamma process and random shock model borrowed from Cha and Finkelstein (2009), Wang and Pham (2011), and Jiang *et al.* (2012). These studies assume that the shock damage magnitudes are s-independent and follow a common probability distribution. Therefore, the level of accumulated degradation in the system is defined as $D(t) = X(t) + \sum_{i=1}^{N(t)} w_i$. Therefore, the system reliability in the reference model is stated as:

$$R(t) = \Pr(X(t) + \sum_{i=1}^{N(t)} w(t_i) < S) = \exp\{[\alpha \ln(\frac{\beta}{\beta+\theta}) + \lambda(M_W(-\theta) - 1)]\, t\} \tag{10}$$

It should be noted that the difference between our model and the reference model is in the concept of time-dependent shock magnitude. To compare the reliability distribution from each model we consider the following case study as it follows:

$\alpha=\beta=\mu=1$ and $\lambda=0.6, \theta=0.1$, and the shock damage magnitudes are exponentially distributed. Table 3.3 addresses the corresponding system reliability values, and the system reliability plots associated with each model is illustrated in Figure 3-8. Also, Figure 3-9 illustrates the model comparisons for various model parameters.

Table 3.3: System reliability values for different models

Time	4	10	20	30
Our Model	0.466541968	0.061162641	0.000666	0.0000036
Reference Reliability Model	0.549128646	0.223452516	0.049931	0.0111572

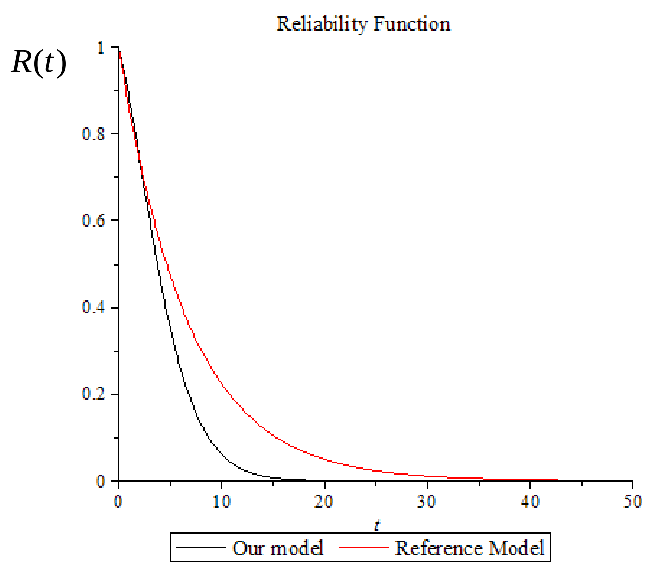

Figure 3-13: The system reliability comparison between our model and the reference model

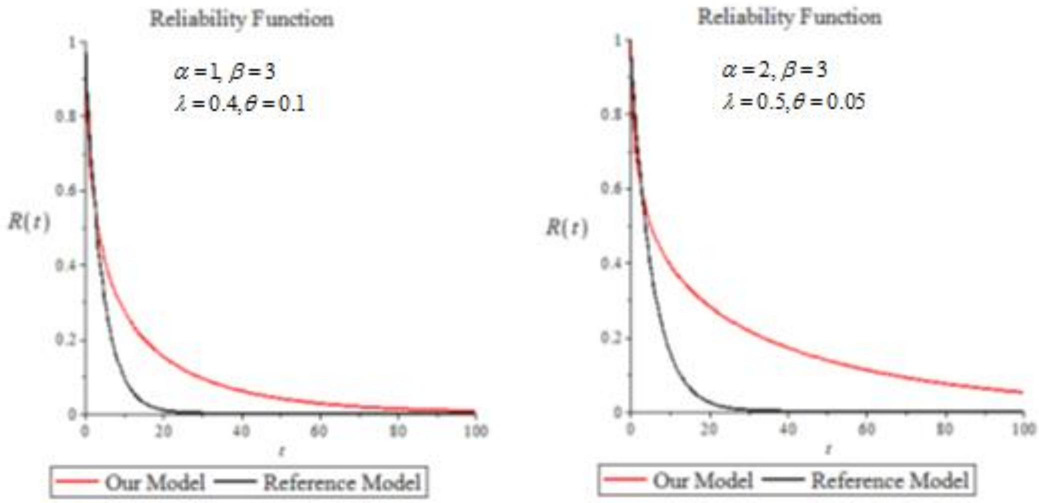

Figure 3-14: The system reliability comparison between our model and the reference model

The results from Figure 3-13 and Table 3.3 reveal that the system reliability according to our model lies below the one of the reference model over the time. This result is consistent with the real situation because the system becomes more vulnerable to the shock damages over the time. Therefore, it is expected that the shock damage magnitudes statistically increase in time. Also, the analysis shows that the system reliability values are close at the beginning since the item is relatively new yet.

3.5.4. Monte Carlo Simulation

A Monte Carlo simulation is developed using MATLAB R2010b to validate the mathematical results. The simulation procedure is described as follows:

For any time t, 10,000 damage paths are generated and the system reliability is estimated as $r = \frac{N}{10,000}$ where N denotes the number of items survived and r addresses the system reliability. Using the described procedure, the system reliability, at any time *t*, is calculated several times. This calculation helps to find the simulation standard deviation for system reliability at time *t*. Table 3.4 addresses the parameters for two exemplary systems used to compare Monte Carlo simulation and the mathematical results. Table 3.5 shows the system reliability values calculated from both methods over time. In Table 3.5, SD represents the standard deviation of system reliability values obtained from the simulation procedure and Error addresses the difference of reliability values computed by both methods. Based on Table 3.5, the absolute difference of values obtained from simulation and mathematical expression is within one standard deviation. Therefore, the difference between two results can be considered statistically insignificant. In other words, the simulation results validate the results from mathematical expressions.

Table 3.4: Case Studies

Case	θ	α	β	λ	**Distribution of** $w(t)$
Case 1	0.1	1	3	0.4	$\exp(\frac{1}{0.1t})$
Case 2	0.005	1	3	0.4	$\exp(\frac{1}{2t})$

Table 3.5: Monte Carlo simulation versus formulation

Case 1							
Time	0	10	20	30	40	50	60
Formulation	1	0.597207	0.255909	0.082983	0.021222	0.004423	0.000771
Simulation	1	0.59527	0.25535	0.08341	0.02163	0.00423	0.00088
SD	0	0.004659	0.006167	0.003014	0.002048	0.000577	0.000225
Error	0	0.001937	0.000559	0.000427	0.000408	0.000193	0.000109
Error/ SD	0	0.415787	0.09061	0.141806	0.199047	0.334615	0.482265
Case 2							
Time	0	10	20	30	40	50	60
Formulation	1	0.815262	0.476902	0.211107	0.073702	0.02097	0.004993
Simulation	1	0.8172	0.48	0.2105	0.0741	0.0215	0.005
Error	0	0.001938	0.003098	0.000607	0.000398	0.00053	7.27E-06
SD	0	0.0055	0.003	0.0027	0.0021	0.0022	0.0006
Error/ SD	0	0.352342	1.032516	0.225	0.189396	0.240882	0.01211

3.6. Conclusions

This chapter contributes to the knowledge of degradation and shock models by developing a new reliability model for a system subject to degradation and shocks by incorporating the effect of system’s age and shock damage magnitudes, which has been ignored in the previous studies. In this chapter, a system subject to both degradation and random shocks is studied where the degradation process is modeled by gamma process with independent increments and the effect of random shocks on the system is modeled by cumulative shock model with time-dependent shock damage magnitudes. The concept of time-variant shock damage magnitude addresses the effect of system’s age and shock damage magnitudes. The mathematical expression for the system reliability is formulated, and a parameter estimation procedure for accumulated damage path is developed. A numerical analysis is conducted on all parameters of system reliability model in order to study the behavior of system reliability expression. Finally, a Monte Carlo simulation is developed for some examples in order to validate the mathematical results. The comparison between the results derived from Monte Carlo simulation and the mathematical expression validates the mathematical formula.

The model developed in this chapter for the underlying system does not consider the possible correlation among degradation process and accumulated shock damage. Hence, in future research, we will include such a dependency structure in the reliability modeling of a system subject to degradation and shocks.

Chapter 4 : Dependent Degradation and Random Shocks with Time-Dependent Shock Damage

4.1. Introduction

Many engineering systems are subject to an underlying degradation process and random sporadic shocks. Some examples of such systems are bridges, bearings, circuits, and others. The degradation process is usually due to system aging and/or underlying dynamic environmental conditions In general, the random shocks either cause system's sudden failure or accelerate the degradation process. Based on Pandey *et al.* (2005) and Barker and Newby (2009), including both degradation and shock processes in the system reliability analysis provides more satisfactory results in terms of system maintenance. Because of the crucial impacts of reliability and degradation analysis on the system maintenance, many engineers have focused on degradation and shock models during recent decades.

In general, the shocks can impact the system in three ways: 1) lead to system's sudden failure, 2) add some damage increment to the system, and 3) accelerate the underlying degradation process (aging). A thorough literature review shows that cases 1-2 have been studied by many researchers (Kou and Wang (2003), Van Noortvijk *et al.* (2007), Liu *et al.* (2008), Deloux *et al.* (2009), Jiang *et al.* (2011) and (2012), Ye *et al.* (2011), Sanchez-Silva *et al.* (2011), Wang *et al.* (2011)). In these studies, the authors study the system reliability given that the shocks and underlying degradation process are s-independent.

The correlation between degradation process and shocks can be described in two ways: 1) The shocks accelerate the degradation process 2) The shock magnitudes are affected by the state of the system degradation process. A thorough literature survey indicates that few previous studies have focused on the dependency between degradation and accumulated shock damage. For example, Huynh *et al.* (2011) recently develop a condition-based maintenance policy for a system subject to degradation and random fatal shocks where the rate parameter of fatal events arrival is dependent on the system's degradation level. For the first time, they consider the dependency between degradation and shock processes by introducing degradation-based shock arrival parameter. In their work, the rate parameter of shock process, $\lambda(t)$, is formulated as

$$\lambda(X(t)) = \lambda_1(t)1_{\{X(t)\leq M\}} + \lambda_2(t)1_{\{X(t)>M\}}$$

where $\lambda_1(t)$ and $\lambda_2(t)$ are, respectively, the rate parameter of shock arrival before and after when the degradation process, $X(t)$, exceeds a threshold *M*. Wang and Pham (2012) assess the reliability distribution for a system subject to multiple degradation processes and random shocks where the degradation processes and shocks are s-dependent. The shocks add sudden damage increments to the system and accelerate the degradation process as well. A time-scaled function $g(N(t), w_1, ..., w_{N(t)}) = \alpha_1 N(t) + \alpha_2 \sum_{i=1}^{N(t)} w_i$ is introduced to model the impact of shock process on degradation process. α_1 and α_2 are fixed weighting parameters. The idea of such a function is borrowed from accelerated life testing model. Therefore, the system's degradation level at time t, is expressed as $X(te^{g(N(t), w_1, ..., w_{N(t)})})$.

For the first time, we studied the effect of system's age and shock damage magnitudes in chapter 3. In reality, there exists some dependency between degradation and accumulated shock damage. Therefore, in the current chapter, we develop a reliability model for a system subject to the degradation process and random shocks where the shock damage magnitudes are time-dependent, and the cumulative shock damage accelerates the underlying degradation process. This study extends the model presented in chapter 3 by considering correlation between degradation process and accumulated shock damage, and it presents more realistic results for the system reliability metrics. A time-scaled function inspired by Wang and Pham (2012) is employed to model the correlation between degradation process and accumulated shock damage. Also, the shock damage magnitudes are considered to be time-dependent. The reliability expression for such a system is formulated and presented later. Also, a numerical analysis to study the effect of the system's parameters on system reliability is conducted. Finally, a Monte Carlo simulation is developed in order to validate the results obtained from analytical expression.. The remainder of this chapter is organized as follows:

Section 4.2 defines the system under investigation and its associated assumptions. Section 4.3 presents the reliability expression and an approximation algorithm for computation purposes in numerical analysis. Section 4.4 also addresses a parameter estimation procedure to estimate the reliability model parameters. Section 4.5 discusses the numerical analysis results of the model. A numerical analysis is performed to insure the behavior of system reliability model, and a Monte Carlo simulation is developed to verify the mathematical model. Finally, section 4.6 presents the conclusions.

4.2. Problem Description

In this chapter, we study a single-component system subject to both degradation process and random shocks. The shocks add spike damage increments to the system, and the accumulated shock damage accelerates the degradation process. Also, the shock damage magnitudes are time dependent. Such an assumption is incorporated in the model using time-variant shock damage magnitudes. It is also assumed that the damage magnitudes tend to increase as the system ages since the system becomes more vulnerable to shock damage as it ages. Total damage accumulated in the system is assumed to be the indicator for the state of system. The rest of characteristics for degradation and random shock processes are considered to be exactly similar to the ones described in chapter 3. A brief summary of the system characteristics is given as:

A time-scaled function $Q(t;\rho, N(t), w(t_i)) = \exp\{\rho \sum_{i=1}^{N(t)} w(t_i)\}$ similar to the one introduced by Wang (2012) is employed to model the correlation between degradation process and accumulated shock damage. At any time t, $Q(.)$ describes the effect of all prior shock magnitudes on the system degradation level where t_i is the arrival time of shock i and $t_i < t$ for $\forall i = 1,..., N(t)$. The idea of this function was firstly borrowed from accelerated life testing models. Therefore, the level of total accumulated damage to the system by time t, $D(t)$, is described as

$$D(t) = X(tQ(t;\rho)) + \sum_{i=1}^{N(t)} w(t_i) = X(te^{\rho \sum_{i=1}^{N(t)} w(t_i)}) + \sum_{i=1}^{N(t)} w(t_i) \tag{1}$$

Where ρ is a constant between 0 and 1 and called shock correlation coefficient. This parameter represents to which degree the accumulated shock damage affects the degradation process. Function $Q(.)$ indicates the accumulated shock damage exponentially accelerates the system virtual age which is equivalent with higher degradation level in the system. That is, as the number of shocks and the intensity of their associated damage magnitudes increase, it is expected that the system degrades more rapidly. The second term in (1) addresses the accumulated shock damage magnitudes in the system. We use the same assumption of the system presented in chapter 3 which are summarized below.

- The system is a single-component subject to both degradation and random shocks.
- Shocks are arriving to the system according to a homogenous Poisson process (HPP) with rate parameter λ.
- The degradation process, $X(t)$, is described by a stationary gamma process with parameters α and β. The gamma process is described in detail in section 3.2.2.
- Random shock process is modeled by a cumulative damage model where the shocks are s-independent with time-dependent damage magnitudes.
- $N(t)$, the number of shocks arriving to the system until time *t*, is considered to be independent of shock damage magnitude and degradation process.
- The critical failure threshold, S, is considered to be stochastic and exponentially distributed with parameter θ.

4.2.1. Notation

$R(t)$: The system reliability at time t

$D(t)$: The total accumulated damage to the system by time t

$X(t)$: The degradation magnitude of the system by time t

α: Shape parameter of gamma distribution

β: Scale parameter of gamma distribution

$N(t)$: Number of shocks arriving to the system by time t

λ: The shock arrival rate parameter

S: The critical failure threshold of the system

θ: The rate parameter for the critical failure threshold distribution

$t_{(i)}$: The arrival time of the i^{th} shock to the system

$w(t_i)$: The damage magnitude due to i^{th} shock, dependent on the time when the shock arrives to the system

ρ: The shock correlation coefficient which is between 0 and 1

$Q(t;\rho, N(t), w(t_i))$: The time-scaled function dependent on shock process parameters which describes how shocks accelerate the degradation process

$f_T(t)$: The probability density function for the system failure time distribution

N_L : An approximated threshold for the number of shocks arriving to the system

Φ: Normal cumulative distribution function

μ : The mean value for the shock damage magnitude

σ : The standard deviation for the shock damage magnitude

4.3. Reliability Estimation

The system fails when the total accumulated damage to the system, $D(t)$, exceeds the critical failure threshold. For special cases when either degradation process or shocks are present in the system, the system reliability expression is given by equations (6) and (7) of chapter 3, respectively. For the case where the system is subject to both degradation and random shocks, the system reliability estimation is expressed as

$$R(t) = P(D(t) < S) = P(X(te^{\rho\sum_{i=1}^{N(t)} w(t_i)}) + \sum_{i=1}^{N(t)} w(t_i) < S) \tag{2}$$

To derive the reliability function it is necessary to integrate the conditional probabilities with each random variable. We consider all the random variables given except S. Therefore,

$$\begin{aligned}
&P(D(t) < S \mid X(te^{\rho\sum_{i=1}^{N(t)} w(t_i)}) = x, N(t) = n, w(t_1), w(t_2), \ldots, w(t_{N(t)}), t_1, ..., t_{N(t)}) \\
&= P(X(te^{\rho\sum_{i=1}^{N(t)} w(t_i)}) + \sum_{i=1}^{N(t)} w(t_i) < S \mid X(te^{\rho\sum_{i=1}^{N(t)} w(t_i)}) = x, N(t) = n, w(t_1), w(t_2), \ldots, w(t_{N(t)}), t_1, ..., t_{N(t)}) \\
&= P(x + \sum_{i=1}^{n} w(t_i) < S) \\
&= e^{-\theta\{x + \sum_{i=1}^{n} w(t_i)\}}
\end{aligned} \tag{2.1}$$

$$P(X(te^{\rho\sum_{i=1}^{N(t)} w(t_i)}) + \sum_{i=1}^{N(t)} w(t_i) < S \mid N(t) = n, w(t_1), w(t_2), \ldots, w(t_{N(t)}), t_1, ..., t_{N(t)})$$

$$= \int_0^\infty e^{-\theta\{x+\sum_{i=1}^{n} w(t_i)\}} . \frac{\beta^{\alpha te^{\rho\sum_{i=1}^{n} w(t_i)}}}{\Gamma(\alpha te^{\rho\sum_{i=1}^{n} w(t_i)})} \times x^{\alpha te^{\rho\sum_{i=1}^{n} w(t_i)}-1} e^{-\beta x} dx$$

$$= \int_0^\infty e^{-\theta x} . e^{-\theta\sum_{i=1}^{n} w(t_i)} . \frac{\beta^{\alpha te^{\rho\sum_{i=1}^{n} w(t_i)}}}{\Gamma(\alpha te^{\rho\sum_{i=1}^{n} w(t_i)})} . x^{\alpha te^{\rho\sum_{i=1}^{n} w(t_i)}-1} e^{-\beta x} dx$$

$$= e^{-\theta\sum_{i=1}^{n} w(t_i)} . \int_0^\infty \frac{\beta^{\alpha te^{\rho\sum_{i=1}^{n} w(t_i)}}}{\Gamma(\alpha te^{\rho\sum_{i=1}^{n} w(t_i)})} . x^{\alpha te^{\rho\sum_{i=1}^{n} w(t_i)}-1} e^{-(\beta+\theta)x} dx$$

$$= (\frac{\beta}{\beta+\theta})^{\alpha te^{\rho\sum_{i=1}^{n} w(t_i)}} . e^{-\theta\sum_{i=1}^{n} w(t_i)} \tag{2.2}$$

Let $G_{w(t)}^{(n)}(y)$ be the n-fold convolution cumulative distribution function (CDF) for the distribution of time-dependent shock damage magnitude, $w(t)$. Thus, the CDF for $Z_n = \sum_{i=1}^{n} w(t_i)$ can be obtained from the following recursive formulas as follows:

$$G_{w(t)}(y) = P(w(t_1) < y)$$

$$G_{w(t)}^{(2)}(y) = P(w(t_1) + w(t_2) < y) = \int_0^\infty G_{w(t_2)}(y-u) g_{w(t_1)}(u) du$$

$$\vdots$$

$$G_{w(t)}^{(n)}(y) = P(w(t_1) + w(t_2) + \ldots + w(t_n) < y) \tag{2.3}$$

Please note that results in Equation (2.3) are valid because the shock damage magnitudes are considered to be s-independent random variables following uncommon distributions. Hence,

$$R(t\,|\,t_1,...,t_{N(t)}) = P(X(te^{\rho\sum_{i=1}^{N(t)} w(t_i)}) + \sum_{i=1}^{N(t)} w(t_i) < S\,|\,t_1,...,t_{N(t)})$$

$$= \sum_{n=0}^{\infty}\{\int_{-\infty}^{\infty} (\frac{\beta}{\beta+\theta})^{\alpha te^{\rho y}} .e^{-\theta y}.dG_{w(t)}^{n}(y)\} \times \frac{e^{-\lambda t}(\lambda t)^n}{n!} \tag{2.4}$$

In general, it is too difficult to find convolution CDF for $Z_{N(t)} = \sum_{i=1}^{N(t)} w(t_i)$ used in (2.3); however, we can use the Central Limit Theorem to approximate that distribution. According to Ross (1996), since $w(t_i)$ s are i.i.d, we have:

$$E[e^{\delta Z_{N(t)}}] = E[e^{\delta(w(t_1)+...+w(t_{N(t)}))}] = E[e^{\delta w(t_1)}]...\mathrm{E}[e^{\delta w(t_{N(t)})}]$$

$$\approx (1-\mu t_1\delta - \frac{\sigma^2}{2N(t)}\delta^2 + \mathrm{O}(\delta^3))...(1-\mu t_{N(t)}\delta - \frac{\sigma^2}{2N(t)}\delta^2 + \mathrm{O}(\delta^3))$$

$$\approx (1-(\mu\sum_{i=1}^{N(t)} t_i)\delta - \frac{1}{2}N(t)\sigma^2\delta^2 + \mathrm{O}(\delta^3))^{N(t)} \xrightarrow{N(t)\to\infty} e^{(\mu\sum_{i=1}^{N(t)} t_i)\delta + \frac{1}{2}N(t)\sigma^2\delta^2}$$

Using Taylor series approximation, When $N(t)$ is a large number, the moment generation function for $Z_{N(t)}$ converges to the one of normal distribution. Therefore, using the Central Limit Theorem, we can assume that $Z_{N(t)}$ approximately follows a normal distribution. In other words,

$w(t_i) \sim N(\mu(t_i),\sigma^2)$ where $\mu(t_i) = \mu t_i$

Therefore,

$$Z_{N(t)} = \sum_{i=0}^{N(t)} w(t_i) \sim N(\mu\sum_{i=1}^{N(t)} t_i, N(t)\sigma^2)$$

Where $t_1,...,t_{N(t)}$ is a sequence of random s-independent variables representing the shock arrival times. Thus, the system reliability expression is obtained as

$$R(t \mid \sum_{i=1}^{n} t_i, N(t)=n) = \int_{-\infty}^{\infty} (\frac{\beta}{\beta+\theta})^{\alpha t e^{\rho y}} e^{-\theta y} d\Phi(\frac{y-\mu\sum_{i=1}^{n} t_i}{\sigma\sqrt{n}}) \tag{2.5}$$

Where Φ denotes the standard normal CDF for $Z_{N(t)}$. The random shocks follow a Poisson process. According to Finkelstein and Cha (2009), the joint distribution of $(t_1, t_2, \ldots, t_{N(t)})$ given the fact that $N(t)=n,$ is the same as the distribution of $(U_{(1)}, U_{(2)}, \ldots, U_{(n)})$ which are the order statistics of i.i.d. random variables $(U_1, U_2, \ldots, U_n)$ which are uniformly distributed on the interval of $[0,1]$. Note that t_i represents the time when shock *i* arrives, so we can write

$$\begin{aligned} R(t \mid N(t)=n) &= \int_{0}^{t}\int_{-\infty}^{\infty} (\frac{\beta}{\beta+\theta})^{\alpha t e^{\rho y}} .e^{-\theta y} d\Phi(\frac{y-\mu n t_{(1)}}{\sigma\sqrt{n}}) dt_{(1)} \\ &= \int_{0}^{1}\int_{-\infty}^{\infty} (\frac{\beta}{\beta+\theta})^{\alpha t e^{\rho y}} .e^{-\theta y} d\Phi(\frac{y-\mu n U_1}{\sigma\sqrt{n}}) dU_1 \\ &= \int_{0}^{1}\int_{-\infty}^{\infty} (\frac{\beta}{\beta+\theta})^{\alpha t e^{\rho y}} .e^{-\theta y} d\Phi(\frac{y-\mu n t u}{\sigma\sqrt{n}}) du \end{aligned} \tag{2.6}$$

Hence, the system reliability expression can be expressed as

$$R(t) = \sum_{n=0}^{\infty} \{\int_{0}^{1}\int_{-\infty}^{\infty} (\frac{\beta}{\beta+\theta})^{\alpha t e^{\rho y}} .e^{-\theta y} d\Phi(\frac{y-\mu n t u}{\sigma\sqrt{n}}) du\} \times \frac{e^{-\lambda t}(\lambda t)^n}{n!} \tag{2.7}$$

In order to approximate the system reliability from Equation (2.7), an approximation algorithm introduced by Liu *et al.* (2008) is employed. Liu *et al.* (2008) assume a threshold N_L such that $P(\lambda t - N_L \le N(t) \le \lambda t + N_L) \simeq 1$. Thus, the system reliability in (2.7) can be approximated as

$$R(t) \approx \sum_{n=\max\{0, t\lambda - N_L\}}^{t\lambda + N_L} \{\int_{0}^{1}\int_{-\infty}^{\infty} (\frac{\beta}{\beta+\theta})^{\alpha t e^{\rho y}} .e^{-\theta y} d\Phi(\frac{y-\mu n t u}{\sigma\sqrt{n}}) du\} \times \frac{e^{-\lambda t}(\lambda t)^n}{n!} \tag{2.8}$$

The closer $P(\max\{0, t\lambda - N_L\} \le N(t) \le t\lambda + N_L)$ to 1, the more precise the approximation of $R(t)$. This is because the highly probable situations of shock process have been included in the approximation results. To determine N_L $\sum_{n=\max\{0,t\lambda-N_L\}}^{t\lambda+N_L} P(N(t)=n)$ is evaluated for different values of t and minimum N_L satisfying the closeness criterion is chosen.

4.4. Parameter Estimation

We can use either MLE or method of moments to estimate the parameters of the system reliability model. Using MLE approach, the probability density function of time to failure, $f_T(t)$, can be stated as

$$
\begin{aligned}
f_T(t) = -\frac{dR(t)}{dt} = & \sum_{n=1}^{\infty}\{\int_0^1\int_{-\infty}^{\infty}(\frac{\beta}{\beta+\theta})^{\alpha te^{\rho y}}.e^{-\theta y}d\Phi(\frac{y-\mu ntu}{\sigma\sqrt{n}})du\}\times(\frac{e^{-\lambda t}(\lambda t)^{n-1}}{n!})\times(\lambda^2 t-n) \\
& -\sum_{n=0}^{\infty}\{\int_0^1\int_{-\infty}^{\infty}\alpha e^{\rho y}\ln(\frac{\beta}{\beta+\theta})(\frac{\beta}{\beta+\theta})^{\alpha te^{\rho y}}.e^{-\theta y}d\Phi(\frac{y-\mu ntu}{\sigma\sqrt{n}})du\}\times\frac{e^{-\lambda t}(\lambda t)^n}{n!} \\
& -\sum_{n=0}^{\infty}\{\int_0^1\int_{-\infty}^{\infty}(\frac{\beta}{\beta+\theta})^{\alpha te^{\rho y}}.e^{-\theta y}\ \frac{\mu u}{\sigma^2\sqrt{2\pi}}\times e^{-\frac{1}{2}(\frac{y-\mu ntu}{\sigma\sqrt{n}})^2}dydu\}\times\frac{e^{-\lambda t}(\lambda t)^n}{n!}
\end{aligned}
\tag{3}
$$

Suppose that n items are subject to test and $\{t_1, t_2, ..., t_n\}$ denote their corresponding failure times. Thus, the likelihood function is given as

$$
\begin{aligned}
L(t;\alpha,\beta,\theta,\lambda,\mu,\sigma,\rho) &= f(t_1)f(t_2)...f(t_n) \\
l(t;\alpha,\beta,\theta,\lambda,\mu,\sigma,\rho) &= \sum_{i=1}^{n}\ln(f(t_i))
\end{aligned}
\tag{4}
$$

Differentiating the likelihood function with respective to each parameter and solving the resulting equations after equating them to zeros yields the parameter estimation for the model. Also, if the system is monitored by a shock recorder device which is mentioned by Kearns (1994), the parameter estimation procedure presented in 3.4 can be utilized to estimate shock process parameters. Let $\{a_0 = 0, a_1, a_2, ..., a_n = t\}$ be an equally-spaced partition on time interval $[0,t]$ and $\{X_0 = 0, X_1, X_2, ..., X_n\}$ be the corresponding damage to the system due to only degradation process. It should be noted that we can find the damage data due to only degradation process using the argument in 3.4. Thus, we can define $\Delta t_i = \Delta t = t_i - t_{i-1}$ and $\Delta X_i = X_i - X_{i-1} \quad \forall i = 0,1,..,n$. It is known that

$$E[\Delta X \mid Z(t)] = \frac{\alpha \Delta t}{\beta} e^{\rho Z(t)}$$

$$Var[\Delta X \mid Z(t)] = \frac{\alpha \Delta t}{\beta^2} e^{\rho Z(t)} \tag{5}$$

Then, we can estimate $\alpha, \beta,$ and ρ using Least Square method (LS).

4.5. Numerical Results

4.5.1. Numerical Analysis

In this section, an example of a system under study is discussed. Numerical analysis of the effect of various parameters on the system reliability model, $\mu, \lambda, \theta, \rho, \alpha, \beta$, is performed. Let the degradation path be defined by a gamma process with parameters $\alpha = 1$ and $\beta = 3$, and the shocks follow a homogeneous Poisson process with rate $\lambda = 0.6$. The shock damage magnitudes are normally distributed with mean μt with $\mu = 1$ and standard deviation $\sigma = 0.2$. Furthermore, the critical failure threshold is considered to be exponentially distributed with rate $\theta = 0.01$, and the shock correlation coefficient, related to the effect of shocks on degradation is assumed to be equal to $\rho = 0.3$. As it is mentioned in last section, we need to calculate N_L firstly to be able to approximate the system reliability expression. Table 4.1 shows values of $P(\lambda t - N_L \le N(t) \le \lambda t + N_L)$ for various N_L for $\lambda = 1$. Since this probability is increasing in λ and all reliability analysis in this section is based on $\lambda \le 1$, we determine N_L when $\lambda = 1$ in the manner described before.

Figure 4-1 illustrates $\sum_{n=\max\{0,t\lambda - N_L\}}^{t\lambda + N_L} P(N(t)=n)$ for different values of N_L and t when $\lambda = 1$ and Table 4.1 shows those values at $t = 500$. In addition, it shows that $\sum_{n=\max\{0,t\lambda - N_L\}}^{t\lambda + N_L} P(N(t)=n)$ is decreasing in time and increasing in N_L. Table 4.1 also presents the probability of having at most N_L shocks up to $t = 500$. According to Table 4.1, the minimum value for $t \in [0, 500]$ at $N_L = 80$ is 0.999672. Therefore, $N_L = 80$ is selected as the threshold value for the approximations in the system reliability computation.

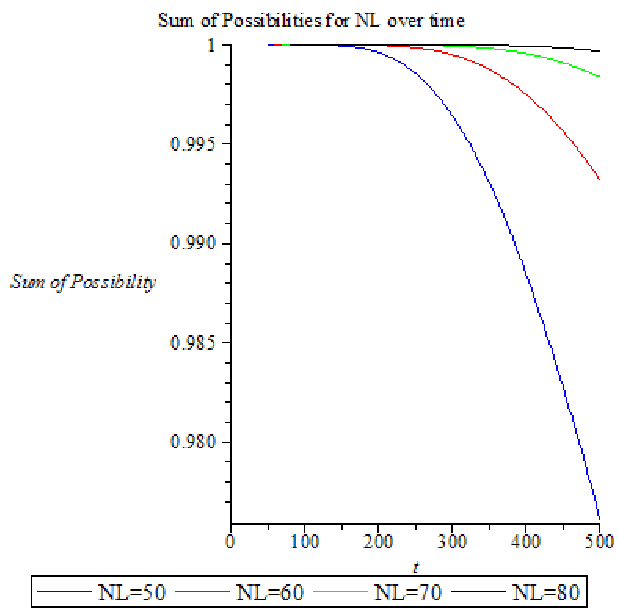

Figure 4-1: $P(\lambda t - N_L \le N(t) \le \lambda t + N_L)$ in respective to N_L over the time when $\lambda = 1$

Table 4.1: $P(\lambda t - N_L \le N(t) \le \lambda t + N_L)$ in respective to N_L at $t = 500$ and $\lambda = 1$

N_L	50	60	70	80
$P(\lambda t - N_L \le N(t) \le \lambda t + N_L)$	0.976107	0.993174	0.998368	0.999672

Table 4.2 shows the results of the numerical analysis for different parameters of system reliability model (Equation (2.8) in section 4.3).

The results of numerical analysis of mean value for the shock damage magnitude which is illustrated in Table 4.2 imply that the system reliability is decreasing in the mean value for shocks' damage magnitude. This is because the system is expected to accumulate more damage on average when each shock arrives to the system; therefore, the system lifetime tends to decrease on average. According to Table 4.2, the system reliability is decreasing with the increase in λ, ρ, α as well. However, it is increasing with the increase in the parameter β. Recall that the degradation process is defined as a gamma process, so the mean degradation value of the system for is defined as $\frac{\alpha}{\beta}$ per unit of time. Hence, an increase in α value increases the mean degradation accumulated in the system and decreases the system reliability. On the other hand, an increase in β reduces the mean degradation and improves the system mean time to failure and the reliability metrics. An increase in λ corresponds to the case that more shocks are expected to arrive to the system. Thus, the system in average will be more damaged, and the system reliability decreases accordingly.

As it is mentioned earlier, ρ is a fixed parameter which represents the impact of shock process, including damage magnitude and number of shocks, on the degradation process. Larger ρ is equivalent to more acceleration in the degradation process due to shock process, which means more degradation in the system and lower system reliability. Table 4.2 also addresses the system reliability values for various values of θ. It can be noted from results in Table 4.2 that the system reliability is strictly decreasing in θ. The critical failure threshold is exponentially distributed with rate parameter θ, so the mean critical failure threshold is reciprocal to θ. Hence, an increase in θ lowers the mean critical failure threshold, which reduces the system reliability.

Table 4.2: Numerical analysis for system reliability

μ /Time	0	10	20	30	40	50
0.5	1	0.998104	0.848616	0.294294	0.052124	0.007368
1	1	0.984699	0.434579	0.070952	0.009103	0.001151
4	1	0.52543	0.065589	0.008242	0.001062	0.000143
λ /Time	0	10	20	30	40	50
0.2	1	0.998098	0.848608	0.291813	0.051919	0.00725
0.5	1	0.9877	0.205146	0.002984	2.03E-05	1.18E-07

1	1	0.859792	0.002225	1.43E-07	4.5E-12	1.06E-16
θ /Time	0	10	20	30	40	50
0.0001	1	0.998103	0.847282	0.292221	0.051727	0.007325
0.001	1	0.981546	0.591613	0.122454	0.016507	0.00199
0.01	1	0.846084	0.236366	0.028879	0.003125	0.000367
ρ /Time	0	10	20	30	40	50
0.2	1	0.998108	0.849137	0.292715	0.05195	0.007301
0.5	1	0.963415	0.318412	0.044989	0.005565	0.000727
1	1	0.742749	0.114886	0.014536	0.001809	0.000231
α /Time	0	10	20	30	40	50
1	1	0.998106	0.849216	0.293283	0.051866	0.00718
3	1	0.995332	0.750021	0.203473	0.032239	0.004044
5	1	0.992581	0.689786	0.169063	0.024622	0.003108
β /Time	0	10	20	30	40	50
3	1	0.998101	0.848643	0.294544	0.052402	0.007238
4	1	0.998451	0.868894	0.316817	0.057769	0.008044
5	1	0.998661	0.882846	0.336783	0.06385	0.009121

4.5.2. Monte Carlo Simulation

In order to validate the analytical expression derived for the system reliability, a Monte Carlo simulation is developed using MATLAB. The simulation procedure is briefly described as follows:

At any time t, 10,000 damage paths up to time t are generated. For each path, a unique value of critical failure threshold is generated as well. Each damage path indicates the accumulated damages in an item due to degradation process and random shocks. Secondly, the number of damage paths with accumulated damage below critical failure threshold at time t is recorded as N, and the system reliability, r, is estimated as $r = \frac{N}{10000}$. Finally, the system reliability is calculated several times for any time t. The purpose of calculating multiple values for system reliability at each time is to estimate the standard deviation of the simulation procedure because the simulation procedure is a stochastic module and gives different results for each run. Figure 4-2 illustrates the simulation procedure in detail.

At time *t*
For n=1 to 10,000

1) Generate $N(t)$, number of shocks arriving to the system by time *t*, by generating a number from $Poisson(\lambda t)$
2) If $N(t) > 0$, Generate the arrival times of shocks, $t_1, ..., t_{N(t)}$. Otherwise, let $W = 0$ and go to step 4. W is the total damage due to shocks.
3) Generate the shock damage magnitudes from distribution $N(\mu t_i, \sigma^2)$ and calculate W accordingly.
4) Generate $X(t)$, the level of degradation up to time *t*, from distribution $Gamma(\alpha t e^{\rho W}, \beta)$
5) Generate the critical failure threshold, *S*, such that $S \sim \exp(\theta)$
6) Let $D(t) = W + X(t)$ and $I_n = \begin{cases} 0 & if\ D(t) > S \\ 1 & O.W. \end{cases}$

End

7) Let $N = \sum_{n=1}^{10,000} I_n$ and estimate $r = \dfrac{N}{10000}$

Figure 4-2: The pseudo code for Monte Carlo simulation

A numerical experiment is designed and used to compare the simulation and formulation results. The system characteristics in this experiment are shown in Table 4.3. The parameters are plugged into Monte Carlo simulation and the mathematical formula and the results are calculated accordingly. Table 4.4 tabulates the results obtained from both simulation procedure and mathematical expression. Moreover, it shows the error, absolute difference between values obtained from both methods, standard deviation of simulated reliability (SD), and the ratio of error over SD. As it can be seen in Table 4.4, at any time t, the error is within one standard variation. Therefore, based on the results in Table 4.4, the results from both simulation procedure and mathematical expressions correspond, meaning that the Monte Carlo simulation validates the system reliability mathematical expression presented in section 4.3. It should be noted that all the calculations based on mathematical expressions have been made in MAPLE.

Table 4.3: The specification of designed system for comparison between simulation and formula

Characteristics of System	θ	α	β	λ	ρ	Distribution of $w(t)$
Values	0.001	1	3	0.2	0.3	$N(t, 0.04)$

Table 4.4: The system reliability values obtained from Monte Carlo simulation and mathematical formulation

Time	0	10	20	30	40	50	60
Simulation	1	0.7861	0.1351	0.0157	0.0016	0	0
Formulation	1	0.78581	0.13562	0.01621	0.00169	0.0002	0.00002
SD	0	0.001883	0.002718	0.001519	0.000314	0.000125	4.22E-05
Error	0	0.00029	0.00052	0.00051	9E-05	0.0002	0.00002
Error/ SD	0	0.154015	0.191334	0.335725	0.28636	1.603567	0.474342

Figure 4-3 indicates a sensitivity analysis of system reliability on shock effect coefficient, ρ. Model I is the model in chapter 3 where $\rho = 0$. Figure 4-3 also shows that the system reliability is decreasing as ρ slightly increases. Thus, the correlation between accumulated shock damage and degradation should be carefully estimated, and its ignorance leads to a large error in system reliability estimation.

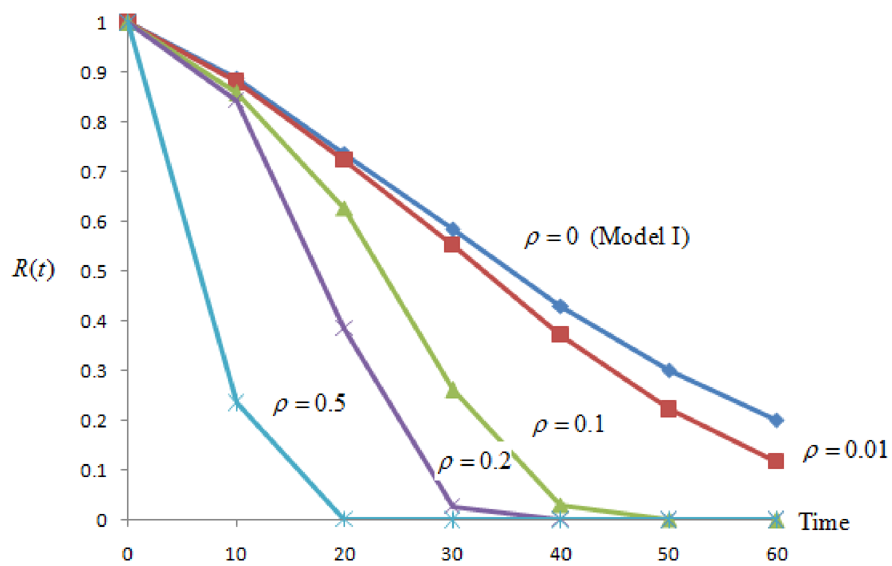

Figure 4-3: Sensitivity analysis of $R(t)$ on ρ

4.6. Conclusion

This chapter provides an extension to the knowledge of the degradation and random shocks modeling by developing a reliability model which considers some dependency between degradation process and accumulated shock damage where the shock damage magnitudes are dependent on system's age. The problem studied in this chapter differs from the one in chapter 3 only in taking the dependency structure between degradation process and accumulated shock damage into account. A time-scaled transformation function is employed to describe the effect of accumulated shock damage on the degradation process. The system reliability expression for such a problem is formulated, and a numerical analysis is conducted on the system reliability model parameters in order to study the behavior of system reliability function. The results show that the correlation between accumulated shock damage and the degradation process is highly important and should be estimated precisely, otherwise, it will lead to a large error in system reliability estimation. Furthermore, an approximation method from Liu *et al.* (2008) is employed to approximate the system reliability values. At last, a Monte Carlo simulation is developed in order to validate the analytical results.

Chapter 5 : Optimal Condition-Based Imperfect Maintenance Policy for Systems Subject to Multiple Competing Risks

5.1. Introduction

With the increase in system's complexity and rising customer expectations for product availability, the concept of system durability and its maintenance is further highlighted. The main objective of maintenance is to restore the system to a state at which it can fulfill its required functions. In general, maintenance actions depend on many factors such as the complexity of system, its degree of deterioration, maintenance costs, and others. Traditionally, maintenance policies are time-based. That is, the system is inspected for maintenance at some pre-determined periods of time based on the analysis of historical data. Nowadays, with the introduction of advanced and sophisticated measurement devices such as sensors, it is possible to monitor the system condition continuously. Therefore, many engineers direct their focus to the development of condition-based maintenance policy (CBM) for various engineering systems. Such policy is highly applicable in important systems like aircraft engines and braking system in elevators (Zhu *et al.*, 2010).

There has been extensive research work, in the literature, on developing CBM policies for systems subject to a single failure mode (Marseguerra *et al.* (2002), Grall *et al.* (2002), Nakagawa (2007), Wang *et al.* (2009), Wang *et al.* (2012)). However, most engineering systems are subject to multiple competing risks such as degradation processes and sudden or catastrophic events. From another point of view, CBM policies can be divided into threshold-type or periodic-inspection-type policies. The former proposes an optimal preventive maintenance threshold, and the latter suggests an optimal periodic inspection. Zhu *et al.* (2010) study the optimal preventive maintenance period maximizing the average system availability when the system is subject to independent failure modes, degradation process and sudden failure events. They assume perfect maintenance after each failure. Tai and Chan (2010) develop an optimal CBM policy for a system subject to continuous degradation without assuming a certain form of degradation function. The proposed policy can handle both perfect and imperfect maintenance assumption and investigates the optimal preventive maintenance periods which maximize the system availability. Chen *et al.* (2011) study an optimal inspection period for a system subject to dependent degradation process and shocks in order to minimize the long-run maintenance cost. The proposed maintenance policy only considers perfect maintenance. Neves *et al.* (2011) also investigate an optimal periodic inspection policy with perfect repair for a system subject to degradation and sudden failure which minimizes the expected cost of maintenance where the input information can be imperfect. Wang and Pham (2011) investigate an optimal threshold-type CBM policy for a system subject to two dependent, linear, and deterministic degradation processes in order to minimize the expected maintenance cost. The policy considers imperfect maintenance which is handled by means of an improvement factor method. In the model, no maintenance is carried out as long as the levels of both of degradations are below their associated threshold, L_i. If the degradation level of either of degradation processes exceeds the relevant threshold, N imperfect maintenance actions with a period of T is performed, and the system is totally replaced at the $(N+1)$ maintenance action.

The objective of this model is to find optimum N^* and L_i^*. Liao *et al.* (2006) study an optimal preventive maintenance threshold-type policy for a system subject to only degradation where the maintenance is imperfect and the mean time to repair is increasing with the number of maintenance actions.

The literature review shows that most studies assume perfect maintenance for the system. However, in practice, the system is not restored to the state of as-good-as-new after maintenance. Sometimes, a minimal repair or imperfect maintenance restores the state of system to a state somewhere between as-good-as-new and as-bad-as-old which can be enough for the system to function according to its requirements. Besides, such a policy offers lower maintenance cost per system's life cycle without compromising the system functionality. Moreover, most studies in the literature set goals to minimize the maintenance cost. According to Liao *et al.* (2006), system availability is a better alternative criterion for an objective function since the cost is sensitive to uncertainty in estimations, while we can accurately estimate the system uptimes and downtimes. Therefore, it is preferable and more realistic to investigate an optimal CBM policy where the model simultaneously includes multiple failure modes for the system, imperfect maintenance and its change with number of maintenance, and system availability as the objective function. Such a problem has not been investigated.

Therefore, we propose an optimal threshold-type CBM policy for a system subject to multiple dependent failure modes, which maximizes the system average availability. This policy considers imperfect maintenance for the system and assumes that the mean repair time is positively correlated with the number of maintenance actions. The following research also generalizes the model proposed by Liao *et al.* (2006) by including multiple failure competing risks for the system. The remainder of this chapter is organized as follows:

Section 5.2 describes the problem assumptions, notations, and the maintenance processes. Section 5.3 presents the mathematical formulation of the maintenance optimization. Section 5.4 studies an especial case of the generalized maintenance problem where the system is subject to only two competing risks, degradation process and sudden failure. Section 5.5 presents a numerical application and multiple sensitivity analyses for this problem. Finally, Section 5.6 presents the conclusions of this chapter.

5.2. Maintenance Model Description and Assumptions

In this chapter, we study a single-component system subject to n competing risks. The competing risks can be either dependent or independent. It is assumed that the system is subject to at least one degradation-type failure mode. The underlying system is continuously monitored, and the monitoring sensors are assumed to be error-free, i.e. the monitoring is perfect. Maintenance actions for the system include preventive and corrective actions. The replacement is referred as the corrective maintenance or perfect maintenance; however, the preventive maintenance is assumed to be imperfect i.e. the system state is not restored to the as-good-as-new state after this type of maintenance. We aim to investigate the optimal condition-based preventive maintenance threshold for the system considering a number of assumptions.

Notation

T_i : The i^{th} inter-maintenance time

A_1 : The achieved average availability in a cycle

$A_2(i)$: The average short-run availability after i^{th} maintenance

$A_{2\min}$: The minimum requirement for average short-run availability

T_{total} : The lower bound on the system total operating time before replacement

$M_i^{(j)}$: The time required to perform i^{th} maintenance for the j^{th} failure mode

$M_i^{(s)}$: The time required to perform i^{th} maintenance if the system failed due to sudden failure

$M_i^{(d)}$: The time required to perform i^{th} maintenance if the degradation reaches to maintenance threshold

S_{PM} : The maintenance threshold for the system

S_n : The critical failure threshold of the system due to the n^{th} failure mode

$XR_i^{(j)}$: The state of system immediately after i^{th} maintenance if the failure was due to j^{th} failure mode

$f_{XR_i^{(j)}}(x)$: The probability distribution for the state of system immediately after i^{th} maintenance for j^{th} failure mode

$D(t)$: The level of degradation process in the system at time t

I_j : Indicator function $= j$ if the system is maintained for j^{th} failure mode

$f_j(t)$: The probability distribution for the system inter-maintenance time due to j^{th} failure mode

$T^{(j)}$: The system inter-maintenance time due to j^{th} failure mode

$F_{T^{(1)},T^{(2)},...,T^{(n)}}(t_1,t_2,...,t_n)$: The joint CDF for inter-maintenance times corresponding to n competing failure modes

p : The probability that a sudden failure event arrives to the system

λ : The rate parameter at which shocks arrive to the system

Φ : The cumulative distribution function for standard normal distribution

5.2.1. Renewal Process

We define a cycle as a period of time between two consecutive replacements. The maintenance is performed when either of the indicator level of some failure modes exceeds the preventive maintenance threshold, S_{PM} or the system fails due to sudden failure. The system is set aside for maintenance immediately after either of scenarios occurs. To consider the effect of system's age on the inter-maintenance times and repair times of each failure mode, it is assumed that the inter-maintenance times in expectation are negatively correlated with the number of maintenance actions. Besides, the mean maintenance times in expectation are considered to be increasing in the number of maintenance actions over the time to incorporate the system's age impact. Figure 5-1 shows an example of a cycle for the system subject to n degradation processes and sudden failure including preventive maintenance and replacement actions. T_1 indicates the first inter-maintenance time where the system is taken to maintenance because the level of first degradation process exceeds S_{PM}. However, the system is repaired due to sudden failure in the second maintenance because none of degradation processes have not exceeded S_{PM} . $XR_1^{(1)}$ is the residual damage level in the system after the first maintenance which is performed since the indicator of first failure mode exceeds S_{PM} . As it can be noted from Figure 5-1, the state of system after maintenance is stochastically increasing in the number of maintenance, and the inter-maintenance times are stochastically decreasing. Consequently, the system's availability after each maintenance is decreasing. Thus, we continue to maintain the system until when the system's availability falls below a minimum required threshold. T^* represents such a time in Figure 5-1. Therefore, the system is not worthy of maintaining any more, and the system is replaced.

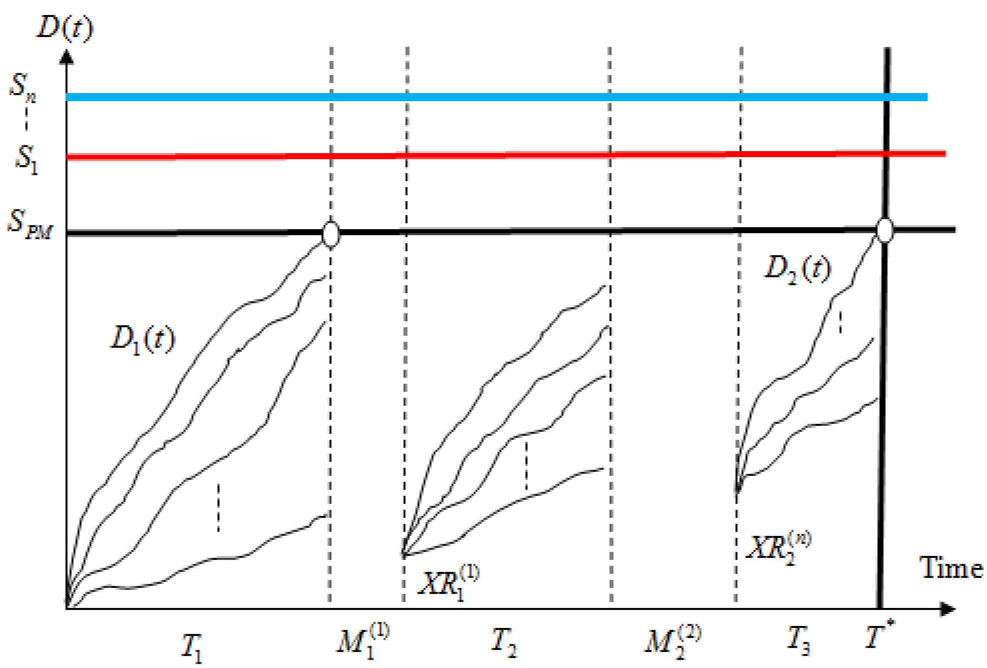

Figure 5-1: An instance of a cycle for the system including maintenance and replacement

5.2.2. Inter-Maintenance Times

The system is subject to *n* competing risks. As it was stated earlier, the failure modes can be either dependent or independent. We assume that the failure modes include sudden failure and several degradation processes. The maintenance is carried out either when the system fails due to sudden failure or the level of some degradation processes reaches the maintenance threshold. Therefore, the system inter-maintenance time is in general defined as $T = \min\{T^{(1)}, T^{(2)}, ..., T^{(n)}\}$. Therefore, we have:

$$\begin{aligned} P(T > t) &= P(\min\{T^{(1)}, T^{(2)}, ..., T^{(n)}\} > t) \\ &= P(T^{(1)} > t, T^{(2)} > t, ..., T^{(n)} > t) \\ &= 1 - F_{T^{(1)}, T^{(2)}, ..., T^{(n)}}(t, t, ..., t) \end{aligned} \tag{1.1}$$

Where $F_{T^{(1)}, T^{(2)}, ..., T^{(n)}}(t, t, ..., t)$ is the joint CDF of inter-maintenance times due to each failure mode at time *t*. If the failure modes are considered to be s-independent, we can rewrite (1.1) as:

$$\begin{aligned} P(T > t) &= P(T^{(1)} > t) P(T^{(2)} > t) ... P(T^{(n)} > t) \\ &= \prod_{j=1}^{n} [1 - F_j(t)] \end{aligned} \tag{1.2}$$

Where $F_j(t)$ is the CDF for inter-maintenance time due to failure mode *j*. Please note that Equation (1.2) is based on the assumption that all failure modes are independent. Therefore, in general, the mean inter-maintenance time is calculated as

$$E[T] = \int_0^{\infty} \{1 - F_{T^{(1)}, T^{(2)}, ..., T^{(n)}}(t, t, ..., t)\} \, dt \tag{2}$$

5.2.3. Imperfect Maintenance Impact on Inter-Maintenance Times

In practice, the maintenance actions are performed imperfectly. Therefore, some residual damage remains in the system after performing the maintenance. We develop a maintenance model which considers residual damages. The mean inter-maintenance times will be decreasing over time. In general,

$$E[T_i] = E[E[T_i \mid XR_{i-1}^{(j)}]] = \int_0^{S_{PM}} \sum_{j=1}^{n} E[T_i \mid XR_{i-1}^{(j)}] \, f_{XR_{i-1}^{(j)}}(x) 1\{\mathrm{I}_{i-1} = j\} \, dx \tag{3}$$

Where $f_{XR_{i-1}^{(j)}}(x)$ denotes the probability density function for the state of system immediately after $(i-1)^{th}$ maintenance for j^{th} failure mode, and $E[T_i]$ is the i^{th} mean inter-maintenance time in the cycle. Equation (3) implies the impact of imperfect maintenance on the mean inter-maintenance times. In order to model the effect of residual damage in the system after maintenance, we use the relevant assumptions in *Liao et al.* (2006). That is, we assume that $XR_i^{(j)}$ falls randomly in the interval of $[0, S_{PM}]$. We can utilize several probability density functions including beta distribution and truncated distributions to describe the probability distribution for the level of residual damage after maintenance. In this work, we specifically utilize beta distribution as follows:

$$f_{XR_i^{(j)}}(x) = \frac{1}{S_{PM}} \frac{\Gamma(\varphi_{ij}+\varepsilon_{ij})}{\Gamma(\varphi_{ij})\Gamma(\varepsilon_{ij})} \left(\frac{x}{S_{PM}}\right)^{\varphi_{ij}-1} \left(1-\frac{x}{S_{PM}}\right)^{\varepsilon_{ij}-1} \tag{4}$$

Where $\varphi_{ij} > 0$ and $\varepsilon_{ij} > 0$ are the parameters of beta distribution for j^{th} failure mode, respectively, and $\Gamma(.)$ is the gamma function. We assume that the mean of residual damage after maintenance due to each type of failure is increasing in the number of maintenance actions. This assumption is expressed as follows:

$$\begin{aligned} E\left[\frac{XR_i^{(j)}}{S_{PM}}\right] &= \frac{\varphi_{ij}}{\varphi_{ij}+\varepsilon_{ij}} = 1-e^{-i\mu_j} \\ Var\left[\frac{XR_i^{(j)}}{S_{PM}}\right] &= \frac{\varphi_{ij}\varepsilon_{ij}}{(\varphi_{ij}+\varepsilon_{ij})(1+\varphi_{ij}+\varepsilon_{ij})} = \sigma^2 \end{aligned} \tag{5}$$

Where $XR_i^{(j)}$ denotes the residual damage in the system immediately after maintenance i for j^{th} failure mode and μ_j and σ^2 are positive constants which are referred to rectification effort and rectification variance for j^{th} failure mode, respectively.

The repair time is stochastic and usually follows a probability distribution function. Also, in general, it is reasonable to assume that the average length of maintenance is positively correlated with the initial state of the system after maintenance. This is because the system ages and as a result of that the mean residual damage of the system right after maintenance is increasing with the number of maintenance actions. Let M_i denotes the length of the i^{th} maintenance action and $M_i^{(j)}$ defines the conditional length of i^{th} maintenance action for failure mode j. In general, we assume that $M_i^{(j)}$ is exponentially distributed with mean $E[M_i^{(j)}] = \psi_j \exp(i\eta_j S_{PM})$ where ψ_j and η_j are constants and independent of S_{PM}. Please note that $\psi_j > 0$ and $\eta_j \geq 0$. Therefore, the average maintenance time is described as

$$E(M_i) = \sum_{j=1}^{n} E[M_i^{(j)} \mid I_i = j] P(I_i = j) \tag{6}$$

Where $P(I_i = j)$ is the probability that the system is under maintenance due to the j^{th} failure mode at the i^{th} maintenance action.

5.3. Maintenance Policy Formulation

In this section, we develop the maintenance policy characteristics for a system subject to multiple competing risks and formulate an optimization model in order to obtain the optimal preventive maintenance threshold for such a system.

5.3.1. Maintenance Policy Metrics

As it stated earlier, a system's cycle is defined as a period between two consecutive replacements. We consider two measures of system availability firstly defined by *Liao et al.* (2006). These measures are as follows:

$$A_1 = \frac{\text{(expected total inter-maintenance time)/cycle}}{\text{(expected total inter-maintenance time + downtime+replacement time)/cycle}}$$

$$A_2(i) = \frac{\text{expected inter-maintenance time after i}^{th}\text{maintenance}}{\text{expected inter-maintenance time + downtime after i}^{th}\text{maintenance}} \tag{7}$$

Where A_1 refers to the average system availability achieved in a cycle and $A_2(i)$ presents the average short-run availability after i^{th} maintenance. The maintenance policy in this section is described as follows:

The system is considered unavailable when it undergoes repair due to either of failure modes and it becomes readily available after repair. However, some residual damage remains in the system after the maintenance. The mean of the residual damage increases as the number of maintenance actions increases. The maintenance continues in this manner until when the system $A_2(i)$ is lower than a pre-determined minimum requirement for average short-run availability. Then, the system is replaced.

5.3.2. Problem Formulation

As described earlier, the objective is to find the optimal maintenance threshold that maximizes the achieved availability for a continuously monitored system. Therefore, we formulate the problem as follows.

$$
\begin{aligned}
&Max \quad A_1(S_{PM}) \\
&s.t. \\
&\qquad 0 \le S_{PM} \le \min\{S_1, S_2, ..., S_n\} && (8.1) \\
&\qquad A_2(i) > A_{2\min} \qquad i \in [0, N-1] && (8.2) \\
&\qquad A_2(\mathrm{N}) \le A_{2\min} && (8.3) \\
&\qquad E[\sum_{i=1}^{N+1} T_i] \ge T_{total} && (8.4)
\end{aligned}
$$

The objective function, the achieved average system availability, is a function of preventive maintenance threshold. Constraints 8.2 and 8.3 address the fact that the system is repaired as long as it meets the minimum requirement for the average short-run availability. Constraint 8.4 states the lower bound for total operating times before a replacement. N is described as $N = \inf_{i \in Z^+}\{\mathrm{i} : |\frac{E[T_{i+1}]}{E[T_{i+1}] + E[M_i]} \le A_{2\min}\}$. That is, the average short-run availability no longer meets the minimum requirement at the N^{th} maintenance; thus, the system is replaced.

5.4. Case study

In this section, we investigate the optimal threshold maintenance policy for a system subject to two independent competing risks namely shocks and degradation. The shocks can be fatal to the system. The system degrades due to aging process and non-fatal shock damage accumulation. The system fails when either the level of the degradation level or the total damage to the system exceeds the degradation failure threshold or a sudden failure event or fatal shock arrives to the system. Figure 5-2 shows an example of a cycle for such a system. T_1 is the first inter-maintenance time due to the fact that the level of degradation indicator exceeds S_{PM}. However, the second inter-maintenance time is due to sudden failure. In general, XR_1 shows the state of system after the first maintenance. The maintenance process continues until T^*, system's replacement time.

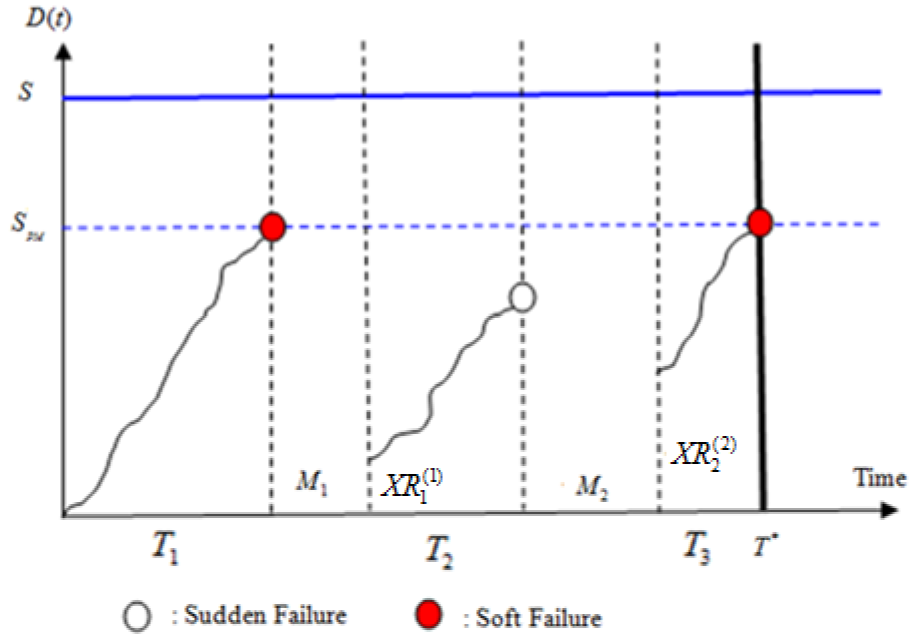

Figure 5-2: An example of a cycle in a system subject to degradation and sudden failure

5.4.1. Inter-Maintenance Times

We assume that the random shocks arrive to the system according to a Homogeneous Poisson process with rate parameter λ, and each shock can cause a sudden failure to the system with probability p. If the shock is not fatal, it adds some damage increment to the system. The system is maintained either when the total degradation level in system, $D(t)$, exceeds the maintenance threshold, S_{PM}, or the system fails due to fatal shocks. In this case, it is assumed that the degradation process and shock process remain the same as the ones in Chapter 4 except that the system experiences sudden failure and the critical failure threshold is assumed to be predetermined and fixed. Therefore, the level of degradation in the system or the total damage to the system is defined as follows:

$$D(t) = X(te^{\rho \sum_{i=1}^{N(t)} w(t_i)}) + \sum_{i=1}^{N(t)} w(t_i) \tag{9}$$

Where $X(t)$ denotes the aging process according to a gamma process with shape and scale parameters α and β, respectively. Also, the second term, $\sum_{i=1}^{N(t)} w(t_i)$, refers to accumulated damage due to non-fatal shocks. The system operates until when either the system is taken down for maintenance or a fatal shock arrives to the system. Therefore, the system inter-maintenance time is in general defined as $T = \min\{T_d, T_s\}$. Therefore, we have the following expression:

$$\begin{aligned}
P(T > t) &= P(\min\{T_d, T_s\} > t) \\
&= P(T_d > t, T_s > t) \\
&= \sum_{n=0}^{\infty} P(T_d > t, T_s > t \mid N(t) = n) P(N(t) = n) \\
&= \sum_{n=0}^{\infty} P(T_d > t \mid T_s > t, N(t) = n) P(T_s > t \mid N(t) = n) P(N(t) = n) \\
&= \sum_{n=0}^{\infty} P(T_d > t \mid N(t) = n) P(T_s > t \mid N(t) = n) P(N(t) = n)
\end{aligned} \tag{10}$$

Where T_d and T_s are the system's inter-maintenance time due to degradation process and sudden failure (fatal shock arrival), respectively. It is assumed that shocks arrive to the system based on Homogeneous Poisson process with rate parameter λ, and they can be fatal to the system with probability p; thus, we can conclude that $P(T_s > t \mid N(t) = n) = (1-p)^n$. Also, based on Lehmann (2009), we know that

$$P(T_d > t \mid N(t) = n) = P(D(t) < \mathrm{S}_{PM} \mid N(t) = n) = P(X(te^{\rho \sum_{i=1}^{n} w(t_i)}) + \sum_{i=1}^{n} w(t_i) < S_{PM}) \tag{11}$$

According to Chapter 4, it is difficult to obtain the convolution CDF for $Z(t) = \sum_{i=1}^{N(t)} w(t_i)$.

However, using the Central Limit Theorem, we can approximately assume that the accumulated shock damage magnitudes follow a normal distribution. In other words, the density function of $\mathrm{Z}(t)$ can be estimated by a normal distribution with the mean and variance as follows:

$$\mu_{Z(t)} = E[\sum_{i=1}^{N(t)} w(t_i)] = (\lambda + \frac{\mu}{2})t \quad (12.1)$$

$$\sigma^2_{Z(t)} = Var[\sum_{i=1}^{N(t)} w(t_i)] = \lambda t(\sigma^2 + \lambda t + \frac{\mu t}{2}) \quad (12.2)$$

Therefore, the expression in (10) can be written as

$$\begin{aligned} P(T_d > t \mid N(t) = n) = P(T_d > t \mid N(t) = n) &= P(D(t) < S_{PM} \mid N(t) = n) \\ &= \int_{-\infty}^{\infty} P(X(te^{\rho\sum_{i=1}^{n} w(t_i)}) + \sum_{i=1}^{n} w(t_i) < S_{PM} \mid N(t) = n, \sum_{i=1}^{n} w(t_i) = z) P(\sum_{i=1}^{n} w(t_i) = z) \\ &= \int_{-\infty}^{\infty} P(X(te^{\rho y}) < S_{PM} - y) \times d\Phi(\frac{y - \mu_{Z(t)}}{\sigma_{Z(t)}}) \\ &= \int_{-\infty}^{\infty} G_X(S_{PM} - y) \times d\Phi(\frac{y - \mu_{Z(t)}}{\sigma_{Z(t)}}) \end{aligned} \quad (13)$$

Where $G_X(.)$ is the gamma CDF function. Thus, by substituting Equation (13) in Equation (10), the system's mean time-to-maintenance is calculated as

$$P(T>t)=\sum_{n=0}^{\infty}(1-p)^n\{\int_{-\infty}^{\infty}P(X(te^{\rho y})<S_{PM}-y)\times d\Phi(\frac{y-\mu_{Z(t)}}{\sigma_{Z(t)}})\}\frac{e^{-\lambda t}(\lambda t)^n}{n!}$$
$$=\sum_{n=0}^{\infty}\{\int_{-\infty}^{\infty}G_X(S_{PM}-y)\times d\Phi(\frac{y-\mu_{Z(t)}}{\sigma_{Z(t)}})\}\frac{e^{-\lambda t}[\lambda t(1-p)]^n}{n!}$$

and (14)

$$\mathrm{E[T]}=\int_0^{\infty}P(T>t)dt$$

Since the maintenance is imperfect and the residual damage after maintenance action is stochastically increasing, the average inter-maintenance time decreases in number of maintenance actions. Therefore,

$$E[T_i]=E[E[T_i\mid XR_{i-1}]]=\int_0^{S_{PM}}E[T_i\mid XR_{i-1}]\,f_{XR_{i-1}}(x)dx$$

where (15)

$$f_{XR_{i-1}}(x)=P(I_{i-1}=1)\,f_{XR_{i-1}^{(1)}}(x)+P(I_{i-1}=2)\,f_{XR_{i-1}^{(2)}}(x)$$

Where $I_{i-1}=1$ refers to the event that the system is maintained at $(i-1)^{th}$ maintenance because the level of degradation exceeds S_{PM}. Also, $I_{i-1}=2$ states that the system is repaired due to sudden failure at $(i-1)^{th}$ maintenance. Taking into consideration all the above assumptions and Equation (15), the mean inter-maintenance time after $(i-1)^{th}$ maintenance is expressed as:

$$E[T_i\mid XR_{i-1}]=\int_{t=0}^{\infty}P(T_i>t\mid XR_{i-1})dt=\int_{t=0}^{\infty}e^{-p\lambda t}\times P(D(t)+XR_{i-1}<S_{PM})dt$$

$$E[T_i]=E[E[T_i\mid XR_{i-1}]]=\int_0^{S_{PM}}E[T_i\mid XR_{i-1}]\,f_{XR_{i-1}}(x)dx \quad (16.1)$$
$$=\int_{x=0}^{S_{PM}}\{\int_{t=0}^{\infty}P(T_i>t\mid XR_{i-1})dt\}\times f_{XR_{i-1}}(x)dx$$

Also,

$$
\begin{aligned}
P(T_i > t \mid XR_{i-1} = r) &= \sum_{n=0}^{\infty} R_{T_{di}}(t \mid N(t) = n, XR_{i-1} = r) R_{T_{si}}(t \mid N(t) = n, XR_{i-1} = r) P(N(t) = n) \\
&= \sum_{n=0}^{\infty} P(X(te^{\rho \sum_{i=1}^{n} w(t_i)}) + \sum_{i=1}^{n} w(t_i) + r < S_{PM})(1-p)^n \frac{e^{-\lambda t}(\lambda t)^n}{n!} \\
&= \sum_{n=0}^{\infty} \{ \int_{-\infty}^{\infty} G_X(S_{PM} - y - r) \times d\Phi(\frac{y - \mu_{Z(t)}}{\sigma_{Z(t\times u)}}) dy \} \frac{e^{-\lambda t}[\lambda t(1-p)]^n}{n!}
\end{aligned}
\tag{16.2}
$$

Where Φ is the standard normal cumulative distribution.

5.4.2. Repair Times

$M_i^{(2)}$ and $M_i^{(1)}$ refer to the system's i^{th} maintenance time exactly after sudden failure or when the degradation level exceeds S_{PM}, respectively. Therefore, the average maintenance time is described as

$$
\begin{aligned}
E(M_i) &= P(I_i = 1)E[M_i \mid I_i = 1] + P(I_i = 2)E[M_i \mid I_i = 2] \\
&= P(T_{di} < T_{si})\,\mathrm{E}[M_i^{(1)}] + P(T_{si} < T_{di})\,\mathrm{E}[M_i^{(2)}]
\end{aligned}
\tag{17.1}
$$

$$
\begin{aligned}
P(T_{di} < T_{si}) &= \int_0^{\infty} P(T_{si} > t) f_{T_{di}}(t) dt = \int_0^{\infty} P(T_{si} > t) dR_{T_{di}}(t) \\
&= \int_0^{\infty} e^{-p\lambda t} \times dP(D(t) + XR_{i-1} < S_{PM}) \\
&= \int_0^{\infty} e^{-p\lambda t} \times d\{ \int_0^1 \int_{-\infty}^{\infty} P(X(te^{\rho z}) < S_{PM} - y - r) \times d\Phi(\frac{y - \mu_{Z(t)}}{\sigma_{Z(t)}}) f_{XR_i}(r) dr \} \\
&= \int_0^{\infty} e^{-p\lambda t} \times d\{ \int_0^1 \int_{-\infty}^{\infty} G_X(S_{PM} - y - r) \times d\Phi(\frac{y - x - \mu_{Z(t\times u)}}{\sigma_{Z(t\times u)}\sqrt{n}}) f_{XR_i}(r) dr \}
\end{aligned}
\tag{17.2}
$$

Where $P(I_i = 1) = P(T_{di} < T_{si})$ addresses the probability of an event that i^{th} maintenance is performed given that the total degradation level reaches to the maintenance threshold before a fatal shock arrives to the system. As noted from equation (17.2), the probability of maintenance due to either of degradation or hard failure are dependent on several factors including the number of maintenance actions, residual damage level after maintenance, and the maintenance threshold. That is, these probabilities change over the system life span.

Now that the mathematical expressions for inter-maintenance and repair times have been presented, we use the optimization formulation presented in 5.3.2 as presented below:

$$Max \quad A_1(S_{PM})$$
$$s.t.$$
$$0 \le S_{PM} \le S \tag{18.1}$$
$$A_2(i) > A_{2\min} \qquad i \in [0, N-1] \tag{18.2}$$
$$A_2(\mathrm{N}) \le A_{2\min} \tag{18.3}$$
$$E[\sum_{i=1}^{N+1} T_i] \ge T_{total} \tag{18.4}$$

5.4.3. Optimization Procedure

In order to obtain the optimal maintenance threshold we utilize the following optimization algorithm which was firstly proposed by *Liao et al.* (2006).

1) Set S_{PM} to a small value within $[0, S]$.

While $S_{PM} < S$ **do**

2) Calculate all mean operating times, $E[T_i]$, and mean downtimes, $E[M_{i-1}]$.

3) Calculate the average short-run availability, $A_2(i)$ for all maintenance actions i.

4) Find the minimum number of maintenance, N, such that it violates $A_2(N) > A_{2\min}$.

5) Calculate the total operating times, $\sum_{i=1}^{N+1} E[T_i]$; If $\sum_{i=1}^{N+1} E[T_i] \ge T_{total}$, do steps 6 and 7, otherwise go to step 8.

6) Calculate the objective function, the system achieved average availability, $A_1(S_{PM})$.

7) Record the associated value to S_{PM}.

8) Increase S_{PM} by a small increment.

End

9) Choose optimal S^*_{PM} such that it maximizes $A_1(S_{PM})$.

5.5. Numerical Analysis

Consider that the degradation process follows a stationary gamma process with parameters $\alpha = 1$ and $\beta = 3$. The shocks are arriving to the system according to a Poisson process with rate parameter $\lambda = 0.1$. Each shock causes sudden failure with probability $p = 0.2$, and the shock effect coefficient is considered to be $\rho = 0.1$. The shock damage magnitudes are assumed to be independent, time-dependent, and distributed according to $N(0.5t, 0.05^2)$. The system critical failure threshold due to degradation process is $S = 15$, and we require that the average short-run availability of the system after each maintenance not to be less than $A_{2\min} = 0.9$. The replacement time is equal to $\varepsilon = 1$, and the repair time parameters when the system due to degradation are $\gamma_0 = 0.02$ and $\eta_0 = 0.02$. Also, those parameters are set to $\gamma_1 = 0.04$ and $\eta_1 = 0.03$ when the system fails due to hard failure. It is assumed that $\mu_1 = 0.5, \mu_2 = 0.7,$ and $\sigma^2 = 0.005$. The minimum total operating time for this system is defined to be $T_{total} = 35$. We aim to investigate the optimal preventive maintenance threshold for this system which maximizes the system average achieved availability, A_1, and meets constraints 18.2-4.

Figure 5-3 plots the system achieved average availability versus S_{PM}. The increments in S_{PM} is by 1 unit until it reaches to $S = 15$. Note that the achieved availability for $S_{PM} < 4$ is not shown in Figure 5-3 because the total operating time in a cycle does not exceed $T_{total} = 35$ for those values of S_{PM}. As it can be noted from Figure 5-3, the optimal preventive maintenance threshold is 9, i.e. $S_{PM}^* = 9$, and the maximum achieved availability for the system is 0.9712. That is $A_1^* = 0.9712$.

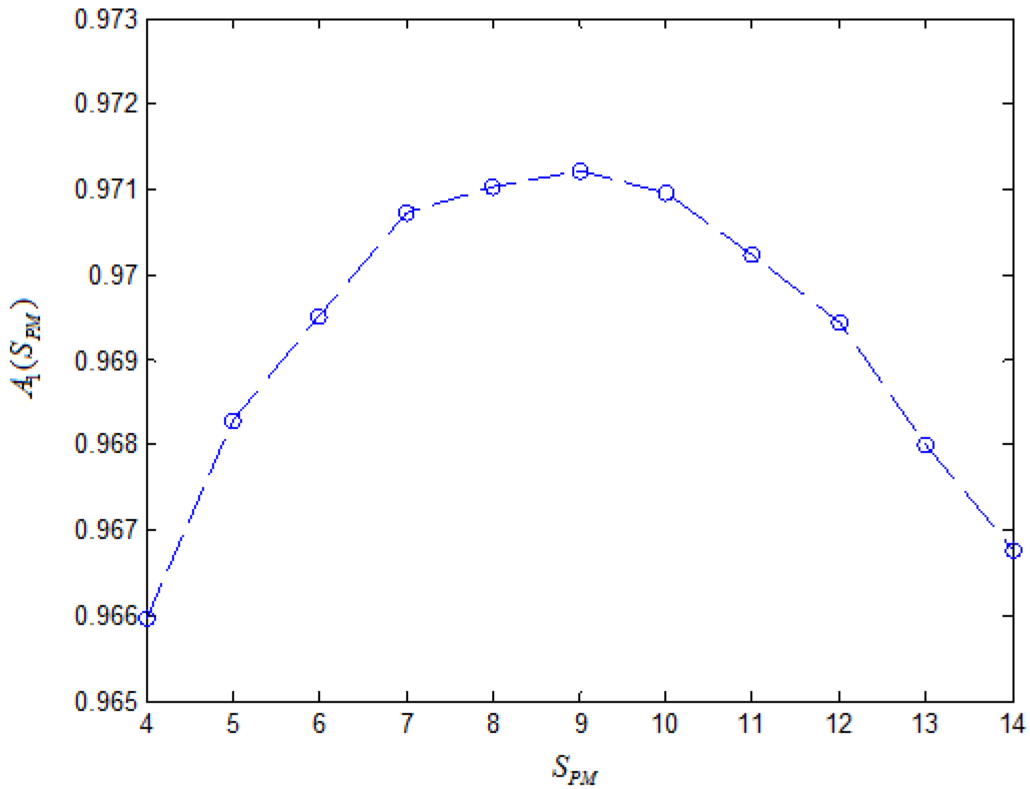

Figure 5-3: The system achieved availability versus S_{PM}

Table 5.1 shows the average short-run availability for up to maintenance number 6 versus S_{PM} and the system total operating time. As it can be noted from Table 5.1, the total operating time is increasing in S_{PM}.

Table 5.1: The average short-run availability versus maintenance number and S_{PM}

Maintenance Action Number							
S_{PM}	1	2	3	4	5	6	Total Operating Time
4	0.991928	0.991405	0.991138	0.990963	0.990773	0.990579	36.96
5	0.993282	0.991807	0.990713	0.990237	0.989776	0.989348	41.005

6	0.994792	0.992605	0.990467	0.989628	0.988808	0.987985	44.2125
7	0.995749	0.993429	0.991135	0.988745	0.987476	0.98661	47.8975
8	0.996152	0.994234	0.990996	0.98855	0.986364	0.985016	50.595
9	0.996476	0.99452	0.991504	0.987571	0.985848	0.983293	53.4675
10	0.996715	0.994805	0.991629	0.987892	0.984231	0.981365	55.9475
11	0.996805	0.99501	0.991671	0.987181	0.982625	0.979574	57.9175
12	0.996923	0.995109	0.991642	0.986889	0.980952	0.976409	60.2975
13	0.996948	0.995235	0.991433	0.985993	0.979477	0.973897	61.9575
14	0.996889	0.99521	0.991362	0.98487	0.977647	0.970623	64.7025

Table 5.2 addresses the effect of the rate parameter of shock arrivals, λ on sensitivity the optimal maintenance policy . Note that Table 5.2 shows the results for when constraint (18.4) is relaxed. Otherwise, there is no feasible optimal maintenance policy for $\lambda \geq 0.5$. It can also be seen that the optimal maintenance threshold and optimal achieved average availability are decreasing in λ. This is because the likelihood of hard failure increases as λ increases. Such an event deteriorates the mean inter-maintenance time, average short-run availability, and system's average achieved availability. Also, incrementing S_{PM} , at the same time, leads to reduction in the probability of maintenance action due to degradation reaching the preventive maintenance threshold and to increase in the probability of sudden failure. Therefore, the effect of S_{PM} on the overall system mean inter-maintenance time, $E[T]$, is considerably reduced.

Table 5.2: Sensitivity analysis of optimal maintenance policy on λ

λ	0	0.05	0.1	0.2	0.5	0.8	1	2
S_{PM}^*	11	10	9	6	4	1	1	1
A_1^*	0.9817	0.9761	0.9715	0.9640	0.9528	0.9518	0.9513	0.9497

Figure 5-4 plots $P(T_{di} < T_{si})$ versus S_{PM} in respective with maintenance numbers. Note that $P(T_{di} < T_{si})$ refers to the probability that the system is repaired given that the degradation level exceeds S_{PM}. As noted from Figure 5-4, $P(T_{di} < T_{si})$ decreases in S_{PM}; however, it increases in the maintenance actions number. As S_{PM} increases, the probability that the degradation exceeds S_{PM} reduces, which means that the probability of sudden failure in the system increases.

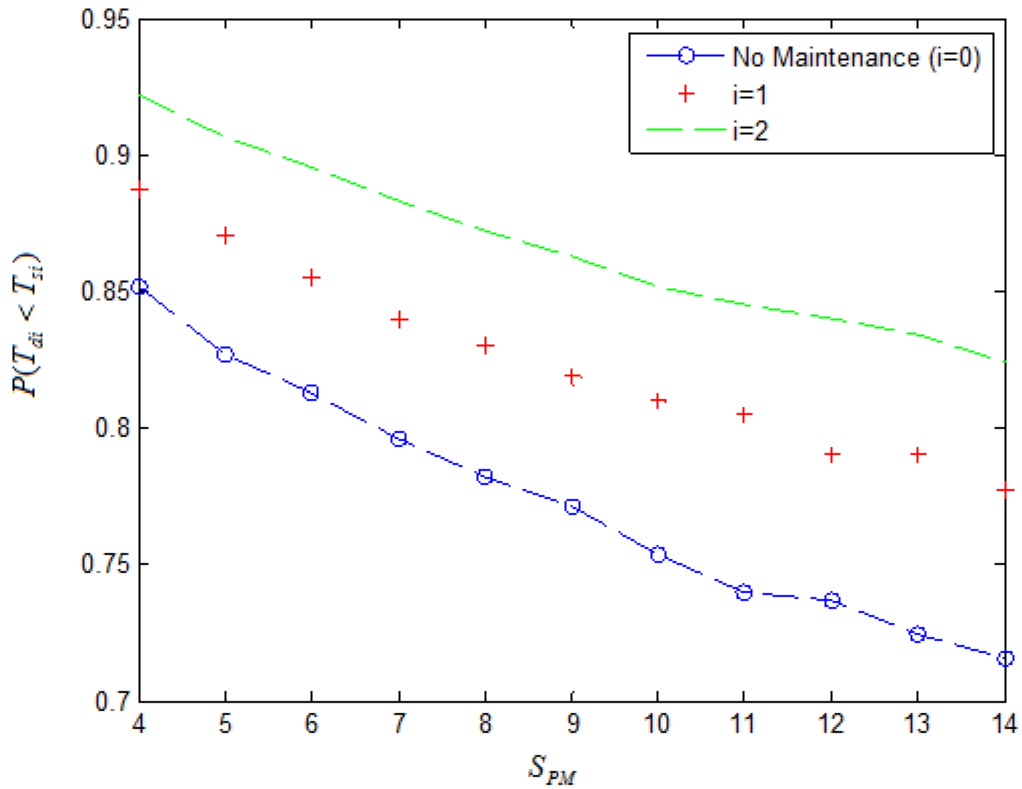

Figure 5-4: Probability of failure due to degradation versus S_{PM} for N maintenance actions

Table 5.3 shows a sensitivity analysis of optimal maintenance policy on replacement time and repair parameters due to degradation process. The analyses for repair parameters due to sudden failure are symmetric. Thus, they are not shown here. The objective of this analysis is to investigate how the deviation from actual parameters affects the optimal maintenance policy. Table 5.3 shows that S_{PM}^{*} is robust to the deviations in replacement time and η_0. However, A_1^{*} is overestimated if all parameters are 10% underestimated.

Table 5.3: Sensitivity analysis of optimal maintenance policy on $\varepsilon, \gamma_0, \eta_0$

	γ_0		η_0		Replacement Time (ε)	
	-10%	+10%	-10%	+10%	-10%	+10%
S_{PM}^{*}	10	9	9	9	9	9
A_1^{*}	0.9719	0.9702	0.9721	0.9703	0.9730	0.9695

5.6. Conclusion

This chapter contributes to the knowledge of degradation and shock modeling by developing a generalized optimal threshold-based CBM policy for a system subject to competing risks including degradation and sudden failure where imperfect maintenance and system availability as the objective function are simultaneously included. The model can accommodate the dependency or independency structure among competing risks. The maintenance is performed either when the system fails due to some kind of sudden failure or the level of one of degradation type of failure modes exceeds the maintenance threshold. To incorporate the effect of imperfect maintenance it is assumed that the state of system after maintenance does not restore it to as-good-as-new, and some residual damage remains in the system. Thus, the inter-maintenance times are decreasing in maintenance actions number. Also, it is assumed that the maintenance times are positively correlated with preventive maintenance threshold and number of maintenance actions. The corresponding mathematical problem is formulated. Sensitivity analyses on optimal preventive threshold are conducted, and the results are presented. The results show that the optimal policy is sensitive to the rate of shock arrival to the system, and the optimal preventive maintenance threshold is in general robust to ten percent deviation in replacement time.

Chapter 6 : Conclusions and Future Work

We have investigated three problems in reliability modeling of systems subject to degradation and random shocks, they are: (a) modeling of the system reliability considering the effect of system's age and shock damage magnitudes, (b) modeling of the system reliability when the accumulated shock damages accelerate the underlying aging process, and (3) developing an optimal threshold-type CBM policy for systems subject to multiple competing risks with imperfect maintenance. In this chapter, we present the summary and conclusions of this dissertation and describe future research related to this dissertation.

6.1. Summary and Conclusions

6.1.1. Modeling Degradation and Random Shocks with Time-Dependent Damage

In chapter 3, we study a system subject to both degradation and random shocks where the degradation process follows a gamma process with independent increments and the random shocks affect the system according to cumulative shock model. We assume that the shock damage magnitudes are time-dependent in order to model the effect of the system's age and shock damage magnitudes. The mathematical formulation for system reliability and a parameter estimation procedure for accumulated damage path are developed. Numerical analysis and Monte Carlo simulation of the system are conducted in order to validate the developed models. Results show that the proposed model are effective in estimating system reliability and indeed, the use of gamma process to describe the degradation of the system and the cumulative damage model to model the effect of the shocks is a realistic approach.

6.1.2. Modeling Dependent Degradation and Random Shocks with Time-Dependent Damage Magnitudes

In chapter 4, we extend the models presented in chapter 3 by incorporating a dependency between degradation and random shocks. We formulate the reliability expression for a system subject to degradation and random shocks where the shock damage magnitudes are time-dependent and accumulated shock damage accelerate the underlying degradation process (aging). In this model, a time-scaled transformation function similar to what was originally introduced by Wang (2011) is used to describe the effect of accumulated shock damage magnitudes on the degradation process. We formulate the system reliability expression for the problem. We also use an approximation approach based on Liu *et al.* (2008) to obtain system reliability expression. Similar to Chapter 3, we develop numerical analysis and simulation model to validate the analytical derivations of the system's reliability. The results show that the developed model is effective and realistic.

6.1.3. Optimal Condition-Based Imperfect Maintenance Policy for Systems Subject to Multiple Competing Risks

Chapter 5 studies a generalized optimal threshold-based CBM policy for a system subject to multiple competing risks including degradation and sudden failure where maintenance is imperfect. This study extends the existing literature by developing an optimal CBM policy which simultaneously incorporates three important and practical concepts in maintenance namely: multiple competing risks, imperfect maintenance, and maximizing system availability instead of minimizing the long term system cost. The model also generalizes the other models by accommodating the dependency or independency structure among competing risks. Maintenance is carried out either when failure happens or when the level of degradation indicator corresponding to one of failure modes reaches the preventive maintenance threshold. The imperfect maintenance is incorporated in the system by considering a residual damage in the system after maintenance. Therefore, the system inter-maintenance times are expected to be decreasing with the maintenance number. The maintenance is carried out until the average short-run availability immediately after maintenance is less than the minimum required threshold; Next, a special case of such a problem is studied where the underlying system is subject to only two independent competing risks named degradation process and sudden failure. The results conclude that the optimal policy is robust to deviations in replacement time and it is decreasing with the rate of shock arrival to the system.

6.2. Future Research

Future research of this dissertation focuses on the following problems:

Problem 1: This dissertation focuses on reliability estimation for systems subject to degradation and shock where the system is only composing of single component. This can be extended by studying the reliability modeling of multi-unit systems subject to degradation and shocks where the correlation among unit's degradation is taken into consideration. This is applicable to different system's configurations such as parallel, series-parallel and parallel-series configurations.

Problem 2: we studied the effect of system's age on shock damage magnitude. This can be extended by incorporating the effect of system's state, i.e. total damage, on shock damage magnitudes. The systems are more vulnerable to damage as time goes by. The system degradation is not only due to aging but also it can be due to cumulative damage from non-fatal shocks. Thus, it is reasonable to consider a correlation between the level of total degradation and the shock damage magnitudes.

Problem 3: The proposed optimal threshold-type CBM can be extended by incorporating multi-objective optimization of long term maintenance cost and system's average achieved availability when the system has multiple components with standby units. Budgeting and costs always play a key role in any engineering decision making framework. At higher cost, the system can be replaced more often in order to achieve higher system availability. Thus, it is valuable to evaluate the presented maintenance policy optimization problem when both availability and cost are taken into consideration. Our optimization problem can be studies for a case when standby components are available to operate whenever the system is under repair. Such a policy can improve the system's overall availability; however, this strategy costs more due to the maintenance costs of the standby components. Therefore, studying such a problem with the consideration of multi-objective framework can be a valuable future work.

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