ON THE BIRATIONALITY OF TORIC DOUBLE MIRRORS

by

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A dissertation submitted to the

Graduate School—New Brunswick

Rutgers, The State University of New Jersey

In partial fulfillment of the requirements

For the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Lev A. Borisov

And approved by

New Brunswick, New Jersey

May, 2014

ABSTRACT OF THE DISSERTATION

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We prove that generic complete intersections associated to double mirror nef-partitions are all birational. This result solves a conjecture of Batyrev and Nill in [6] under some mild assumptions. This dissertation is based on my paper [18].

Acknowledgements

I am deeply indebted to my advisor, Professor Lev A. Borisov, for his encouragement, immense knowledge, and constant guidance: in our weekly meetings, he never shows impatience when I stick to trivial questions, instead, he explains to me every details on blackboard or even writes down every tiny things on piece of papers for my record; he is never too busy to talk with; and he is always willing to reply my long emails consisting of math questions. I always appreciate his generosity of sharing ideas and even proofs with me. Without his supervision, this dissertation would have never been written.

I would like to thank Professor Anders S. Buch who taught me commutative algebra and first course of algebraic geometry. It is his encouragement and kindness that make algebraic geometry less formidable.

I would like to thank Professor Xiaojun Huang who has been helping me from my first day at Rutgers. He is guiding me on academics as well as on my life.

To Professor Chris Woodward, it is always my pleasure to observe him together with Lev shooting questions to seminar speakers; To Professor Jerrold B. Tunnell for his crystal clear lectures and enlightening seminars; To Professor Simon Thomas for helps on many occasions during my early years. I would like to thank the faculty and staff of Mathematics Department at Rutgers, especially Professor Stephen Miller, Professor Michael Saks, Professor Zhengchao Han, Ms. Lynn Braun, Ms. Demetria Carpenter and Ms. Katie Guarino, for their patience, help and support.

I would like to thank Rutgers University for providing excellent academic environment and financial support; to the organizers of the conferences, workshops and seminars I spoke on and attended.

To Howard Nuer for many helpful discussions and constant help.

Many thanks go to my friends: Dr. Jin Wang, Zhuo Feng, Simão Herdade, Edward Chien, Ming Xiao, John Miller, Mo Wang, Hanlong Fang, Liming Sun, Dr. Knight Fu, Sjuvon Chung, Dr. Tian Yang, Xu Dong, Dr. Jingang Xiong, Dr. Jinwei Yang, Professor Yuan Yuan, Yuxiang Qin, Fanxing Hu, Hongbin Wang, Dr. Kaige Zhu... And to many many people I don't know their names, it is their kindness that survives me from those difficult days.

To my girlfriend Bai Yixiu for her patient waiting...

To my relatives, especially my grandmother Ms. Fu Zhe for her unwavering confidence in me.

Finally, I would give all my gratitude to my parents for raising me up, supporting me, understanding me and trusting me. Without them, any math is meaningless.

Dedication

To My Parents.

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Chapter 1

Introduction

Mirror symmetry was first discovered in string theory as a duality between families of 3-dimensional Calabi-Yau manifolds. Since its discovery more than twenty years ago, it has drawn much attention from physicists and mathematicians. Batyrev [2] used Δ-regular hypersurfaces in toric varieties associated to reflexive polytopes as a way to construct a large set of mirror pairs. In this case, the mirror pair consists of the family of Δ-regular hypersurfaces associated to a reflexive polytope and the family of Δ-regular hypersurfaces associated to its dual polytope. Borisov [7] generalized Batyrev's construction by considering nef-partitions of reflexive polytopes. A nef-partition of a reflexive polytope corresponds to a decomposition of the boundary divisor into nef divisors. In this case, the mirror pairs are constructed as the family of complete intersections associated to a nef-partition and the family of complete intersections associated to its dual nef-partition. These complete intersections are Calabi-Yau varieties, and their string-theoretic Hodge numbers behave as predicted by mirror symmetry [4].

Compared to hypersurfaces, complete intersections associated to nef-partitions are more complicated. In particular, they may exhibit nontrivial double mirror phenomenon, i.e. two Calabi-Yau varieties $X, \tilde{X}$ may have the same mirror Y. If this is the case, the homological mirror symmetry conjecture [17] implies that the derived categories of coherent sheaves on $X, \tilde{X}$ are equivalent. Indeed, according to the conjecture, the derived categories of $X, \tilde{X}$ are expected to be equivalent to the Fukaya categories of their mirrors, which in this case are the same because $X, \tilde{X}$ are double mirrors.

Instead of derived equivalence, Batyrev and Nill [6] asked whether toric double mirrors are birational. We give an affirmative answer to this question in Theorem 4.3.1 under some mild assumptions:

Theorem. *Let $X, \tilde{X}$ be toric double mirrors, then there exists a variety D, called the determinantal variety, with morphism*

$$\begin{tikzcd} X \arrow[dr] & & \tilde{X} \arrow[dl] \\ & D & \end{tikzcd}$$

such that if $X, \tilde{X}$ and D are all irreducible with $\dim D = \dim X = \dim \tilde{X}$, then $X, \tilde{X}$ are birational.

Now, we describe briefly the content of each chapter:

In Chapter 2, we fix the notations used throughout the paper. We give relevant background information on reflexive Gorenstein cones and nef-partitions. At the end of the chapter, we prove Proposition 2.2.1 which connects the notions of nef-partitions and reflexive Gorenstein cones. This will be used to reformulate Batyrev and Nill's original question in the language of Gorenstein cones. We also give a constructive proof of the converse of Proposition 2.2.1 in Proposition 2.2.2. In Chapter 3, we reformulate the question of Batyrev and Nill using reflexive Gorenstein cones. We also discuss the motivation of this question and give an example which motivates our proof. In Chapter 4, we give a proof for the main result Theorem 4.3.1. We also discuss the necessity of its assumptions. In Chapter 5, we present some open questions related to the subject. In the appendix, we give the definition of Δ-regularity and discuss its properties. We show that the singularities of Δ-regular intersections are inherited from the ambient toric variety. In particular, the complete intersections considered in the paper are Calabi-Yau varieties with canonical, Gorenstein singularities. This fact is used in the proof of the main theorem.

Chapter 2

Background

2.1 Gorenstein cones and Nef-partitions

We fix the following notations throughout the paper. Let $M \cong \mathbb{Z}^d$ be a lattice of rank d, and $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ be its dual lattice with pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$. Let $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ be the $\mathbb{R}$-linear extensions. The pairing between M, N can be extended to $\langle \cdot, \cdot \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$. Let $\overline{M} = \mathbb{Z}^s \oplus M$ be the lattice extended from M, and $\overline{N} = \mathbb{Z}^s \oplus N$ be its dual lattice with pairing:

$$\overline{M} \times \overline{N} \to \mathbb{Z}$$
$$(a_1, \cdots, a_s; m) \times (b_1, \cdots, b_s; n) \mapsto \sum_{i=1}^{s} a_i b_i + \langle m, n \rangle \ ,$$

where the integer s should be obvious from the context.

The purpose of introducing notations $\overline{M}, \overline{N}$ will become clear in a moment: if a nef-partition lives in M (or N), then the corresponding reflexive Gorenstein cone will live in $\overline{M}$ (or $\overline{N}$). Sometimes we also use lattice M_1 and its dual lattice N_1. The convention is as follows: we always use M (or N) to denote the lattice where *polytopes* live, if the *cones* come from nef-partitions, we use $\overline{M}$ (or $\overline{N}$) to denote the lattice where they live. However, when talking about general *cones* which do not come from nef-partitions, we use M_1 (or N_1) to denote the lattice where they live.

Let $S \subset M_{\mathbb{R}}$ be a set, we use $\mathrm{Conv}(S)$ to denote its convex hull.

If $\Delta \subset M_{\mathbb{R}}$ with the origin 0 in the interior is a lattice polytope (i.e. the convex hull of a finite set of lattice points), then $\Delta^{\vee} := \{y \in N_{\mathbb{R}} \mid \langle x, y \rangle \geq -1, \forall\ x \in \Delta\}$ is its dual polytope. We use $\mathrm{Vert}(\Delta)$ to denote the set of *vertices* of a lattice polytope Δ, and $l(\Delta)$ to denote the set of its *lattice* points, i.e. $l(\Delta) = \Delta \cap M$.

Definition 2.1.1. Let Δ be a lattice polytope with the origin $0 \in \Delta$ as an interior point. If the dual polytope $\Delta^\vee$ is also a lattice polytope, then Δ is called reflexive polytope.

Definition 2.1.2. (See [5]) A d-dimensional rational polyhedral cone $K \subset (M_1)_\mathbb{R}$ is called a Gorenstein cone, if it is generated by lattice points which are contained in an affine hyperplane $\{x \in (M_1)_\mathbb{R} \mid \langle x, n \rangle = 1\}$ for some $n \in N_1$.

This n is uniquely determined if $\dim K = \operatorname{rank} M_1$, and this is the only case considered in the paper. We denote this unique element by $\deg^\vee$, and call it the degree element. By definition, $\deg^\vee$ must live in $K^\vee \cap N_1$, where $K^\vee := \{y \in (N_1)_\mathbb{R} \mid \langle x, y \rangle \geq 0, \forall\, x \in K\}$ is the dual cone of K.

In general, K is a Gorenstein cone does not imply $K^\vee$ is a Gorenstein cone. However, if this is the case, we arrive at the notion of reflexive Gorenstein cone.

Definition 2.1.3. (See [5]) A Gorenstein cone K is called reflexive Gorenstein cone if $K^\vee$ is also a Gorenstein cone. Let $\deg \in K, \deg^\vee \in K^\vee$ be the degree elements in $K, K^\vee$ respectively, then $\langle \deg, \deg^\vee \rangle$ is called the index of this pair of dual reflexive Gorenstein cones.

We will see in a moment how reflexive Gorenstein cones relate to nef-partitions. Before doing this we should briefly recall the notion of nef-partition. In the projective toric variety defined by a reflexive polytope, a nef-partition is equivalent to a decomposition of the boundary divisor into a summation of nef divisors. On the other hand, there exists a purely combinatorial definition of nef-partition without invoking toric variety constructions. For simplicity, we use this combinatorial definition here. The readers can find its equivalent form and its motivation in Borisov's original paper [7].

Definition 2.1.4. If the Minkowski sum of s lattice polytopes $\sum_{i=1}^{s} \Delta_i$ is a reflexive polytope, and the origin $0 \in \Delta_i$ (0 may not be an interior point) for each i, then $\{\Delta_i \mid i = 1, \ldots, s\}$ is called a length s nef-partition of the convex hull $\mathrm{Conv}(\cup_{i=1}^{s} \Delta_i)$.

Nef-partitions arise in pairs [7]: if we fixed a nef-partition $\{\Delta_i \mid i = 1, \ldots, s\}$ with $\Delta_i \subset M_\mathbb{R}$, then there exists a dual nef-partition $\{\nabla_i \mid i = 1, \ldots, s\}$ with $\nabla_i \subset N_\mathbb{R}$. The relations between them are

$$(\sum_{i=1}^{s} \Delta_i)^\vee = \mathrm{Conv}(\cup_{i=1}^{s} \nabla_i)$$

$$(\sum_{i=1}^{s} \nabla_i)^\vee = \mathrm{Conv}(\cup_{i=1}^{s} \Delta_i).$$

Furthermore, they satisfy the property

$$\min\langle \Delta_i, \nabla_j \rangle \geq -\delta_{ij},$$

and $\forall\ w_j \in \mathrm{Vert}(\nabla_j) - \{0\}$, the minimum value can be achieved, that is

$$\min_{x \in \Delta_i} \langle x, w_j \rangle = -\delta_{ij}.$$

2.2 Relationship between nef-partitions and reflexive Gorenstein cones

From a nef-partition, one can construct a reflexive Gorenstein cone [5]. On the other hand, from a reflexive Gorenstein cone associated to a nef-partition, if we have a decomposition of the degree element $\deg^\vee$, we can construct another nef-partition. Now we will give a precise statement of the above relations, which appeared in a slightly different form in [6]. In fact, we will prove a general result.

Let $K, K^\vee$ be full dimensional reflexive Gorenstein cones in $(M_1)_\mathbb{R}$, $(N_1)_\mathbb{R}$, with degree elements $\deg, \deg^\vee$ in $K, K^\vee$ respectively. Suppose the index is $\langle \deg, \deg^\vee \rangle = s$ and

$$\deg^\vee = \sum_{i=1}^{s} e_i,$$

with $e_i \in N_1 \cap K^\vee$, $e_i \neq 0$.

Let

$$S = \{x \in K \mid \langle x, \deg^\vee \rangle = 1\}$$

$$S_i = \{x \in K \mid \langle x, e_i \rangle = 1, \langle x, e_j \rangle = 0, j \neq i\}$$

$$T = \{y \in K^\vee \mid \langle \deg, y \rangle = 1\}.$$

Because K is a Gorenstein cone, any vertex v of S is a lattice point. Thus $\langle v, e_i \rangle$ are nonnegative integers which add up to 1. Hence, there exists precisely one e_i such that

$\langle v, e_i \rangle = 1$. On the other hand, for any e_j, because $e_j \neq 0$ and K is a full dimensional cone, there exists at least one vertex w of S such that $\langle w, e_j \rangle = 1$. Using these facts, one can show that $\{e_1, \ldots, e_s\}$ must be part of a $\mathbb{Z}$-basis of N_1.

Let

$$\operatorname{Ann}(e_1, \ldots, e_s) := \{m \in M_1 \mid \langle m, e_i \rangle = 0, \forall\, i, 1 \leq i \leq s\}$$

be a sublattice of M_1 (we also use $\operatorname{Ann}(e)$ for simplicity if no confusion arises), and

$$\operatorname{Span}_{\mathbb{Z}}\{e_1, \ldots, e_s\} := \sum_{i=1}^{s} \mathbb{Z} e_i$$

be a sublattice of N_1. From the fact that $\{e_1, \ldots, e_s\}$ is part of a $\mathbb{Z}-$basis, it follows that the pairing between M and N induces a pairing

$$\operatorname{Ann}(e_1, \ldots, e_s) \;\times\; (N_1/\operatorname{Span}_{\mathbb{Z}}\{e_1, \ldots, e_s\}) \to \mathbb{Z},$$

which identifies $\operatorname{Ann}(e_1, \ldots, e_s)$ and $N_1/\operatorname{Span}_{\mathbb{Z}}\{e_1, \ldots, e_s\}$.

Proposition 2.2.1. *Under the above notations, the lattice polytope*

$$\sum_{i=1}^{s} S_i - \deg \subset \operatorname{Ann}(e_1, \ldots, e_s)_{\mathbb{R}}$$

is a reflexive polytope.

Proof. We will show that the dual polytope of $\sum_{i=1}^{s} S_i - \deg$ is exactly

$$\overline{T} \subset (N_1/\operatorname{Span}_{\mathbb{Z}}\{e_1, \ldots, e_s\})_{\mathbb{R}},$$

where $\overline{T}$ is the image of T under the projection $(N_1)_{\mathbb{R}} \to (N_1/\operatorname{Span}_{\mathbb{Z}}\{e_1, \ldots, e_s\})_{\mathbb{R}}$.

First, we show that $\overline{T}$ has 0 as an interior point. Because $\deg^\vee$ is in the interior of $K^\vee$, $\frac{1}{s}\deg^\vee$ is also in the interior of $K^\vee$, and thus in the interior of T. This property is kept under the projection map $T \to \overline{T}$. The image of $\frac{1}{s}\deg^\vee$ is 0 in $(N_1/\operatorname{Span}_{\mathbb{Z}}\{e_1, \ldots, e_s\})_{\mathbb{R}}$, and thus 0 is in the interior of $\overline{T}$.

Second, we show $0 \in \sum_{i=1}^{s} S_i - \deg$ is an interior point. Let $\deg = \sum_{i \in I} \lambda_i v_i$, where $\lambda_i \in \mathbb{R}$ and $v_i \in \operatorname{Vert}(S)$ be vertices of S. We have

$$1 = \langle \deg, e_1 \rangle = \langle \sum_{i \in I} \lambda_i v_i, e_1 \rangle = \sum_{i \in I_1} \lambda_i,$$

where $I_1 = \{i \in I \mid \langle v_i, e_1\rangle = 1\}$. This implies $\sum_{i\in I_1} \lambda_i v_i \in S_1$. By continuing this procedure, one can show $\deg \in \sum_{i=1}^s S_i$. If $w \in \mathrm{Vert}(T)$ is a vertex of T, then $\langle \sum_{i=1}^s S_i, w\rangle$ cannot always be zero. Indeed otherwise, all the S_i would be contained in a facet of K, which is impossible. Thus, for any $w \in \mathrm{Vert}(T)$, there exists $v \in \sum_{i=1}^s S_i$ such that $\langle v - \deg, w\rangle \geq 0$. If $\sum_{i=1}^s S_i - \deg$ did not have 0 as an interior point, then $\mathbb{R}_{\geq 0}(\sum_{i=1}^s S_i - \deg) \neq (M_1)_{\mathbb{R}}$. We have already showed that $\overline{T}$ had 0 as an interior point, so $\mathbb{R}_{\geq 0}\overline{T} = (N_1/\operatorname{Span}_{\mathbb{Z}}(e_1, \dots, e_s))_{\mathbb{R}}$. In particular, there exists a vertex $\bar{w}$ of $\overline{T}$, and thus a vertex $w \in T$, such that $\langle \sum_{i=1}^s S_i - \deg, w\rangle < 0$, a contradiction.

Next, we show that $\sum_{i=1}^s S_i - \deg \subset \mathrm{Ann}(e_1, \dots, e_s)_{\mathbb{R}}$ is a reflexive polytope with dual $\overline{T} \subset (N_1/\operatorname{Span}_{\mathbb{Z}}(e_1, \dots, e_s))_{\mathbb{R}}$. Because

$$\begin{aligned} \min\langle \sum_{i=1}^s S_i - \deg, \overline{T}\rangle &= \min\langle \sum_{i=1}^s S_i - \deg, T\rangle \\ = \min\langle \sum_{i=1}^s S_i, T\rangle &- \langle \deg, T\rangle \geq 0 - 1 = -1, \end{aligned}$$

we have $\overline{T} \subseteq (\sum_{i=1}^s S_i - \deg)^\vee$.

We only need to show the other inclusion $\overline{T} \supseteq (\sum_{i=1}^s S_i - \deg)^\vee$. Let $y \in (\sum_{i=1}^s S_i - \deg)^\vee$ such that there exists $x \in \sum_i^s S_i - \deg$ with $\langle x, y\rangle = -1$ (this y corresponding to some boundary point of the dual polytope of $\sum_i^s S_i - \deg$). We will show for this y, $y \in \overline{T}$. Then it follows for arbitrary $y \in (\sum_{i=1}^s S_i - \deg)^\vee$, $y \in \overline{T}$.

Let $\theta_i = \min_{x\in S_i}\langle x, y\rangle$ and set $y' = y - \sum_{i=1}^s \theta_i e_i$. We claim $y' \in K^\vee$. Indeed, $K = \sum_{i=1}^s t_i S_i$ with $t_i \geq 0$, and we have

$$\begin{aligned} \min\langle K, y'\rangle &= \min\langle \sum_{i=1}^s t_i S_i, y - \sum_{i=1}^s \theta_i e_i\rangle = \sum_{i=1}^s \min\langle t_i S_i, y - \sum_{i=1}^s \theta_i e_i\rangle \\ &= \sum_{i=1}^s \min\left(t_i(\langle S_i, y\rangle) - \sum_{j=1}^s \langle S_i, \theta_j e_j\rangle \right) = \sum_{i=1}^s (t_i(\min\langle S_i, y\rangle - \theta_i)) \geq 0. \end{aligned}$$

Finally, we will show $y' \in T$ and this will imply $y \in \overline{T}$. By the assumption on y, we have $\min\langle \sum_{i=1}^s S_i - \deg, y\rangle \geq -1$, and there exists $x \in \sum_{i=1}^s S_i - \deg$, such that $\langle x, y\rangle = -1$. Let $x = \sum_{i=1}^s x_i - \deg$ with $x_i \in S_i$, then we must have $\langle x_i, y\rangle = \theta_i$. Indeed, otherwise there exists k such that $\langle x_k, y\rangle > \theta_k$, and all the others satisfy $\langle x_i, y\rangle \geq \theta_i$.

Thus

$$-1 = \min\langle\sum_{i=1}^{s} S_i - \deg, y\rangle = \left(\sum_{i=1}^{s} \min\langle S_i, y\rangle\right) - \langle\deg, y\rangle$$
$$= \left(\sum_{i=1}^{s} \theta_i\right) - \langle\deg, y\rangle = \left(\sum_{i=1}^{s} \langle x_i, y\rangle\right) - \langle\deg, y\rangle,$$

a contradiction.

We have

$$\langle\deg, y'\rangle = \langle\deg, y - \sum_{i=1}^{s} \theta_i e_i\rangle = \langle\deg, y\rangle - \sum_{i=1}^{s} \theta_i = 1$$

and this implies $y' \in T$.

□

The converse is proved in [6] Theorem 2.6. We will give a direct proof by constructing the dual cone $K^\vee$ explicitly.

Proposition 2.2.2. *Let $\Delta_1, \dots, \Delta_s \subset M_\mathbb{R}$ be lattice polytopes such that the Minkowski sum $\sum_{i=1}^s \Delta_i$ has dimension $\dim(M_\mathbb{R})$ and $\sum_{i=1}^s \Delta_i - m$ be a reflexive polytope for some $m \in M$. Let $\overline{M} = \mathbb{Z}^s \oplus M$, then the associated cone in $\overline{M}_\mathbb{R}$*

$$K = \{(a_1, \dots, a_s; \sum_{i=1}^{s} a_i \Delta_i) \mid a_i \geq 0\}$$

is a reflexive Gorenstein cone of index $\langle\deg, \deg^\vee\rangle = s$.

Proof. Let $\nabla = (\sum_{i=1}^s \Delta_i - m)^\vee$, and for any vertex $w_j \in \mathrm{Vert}(\nabla)$, we set $m_{ij} = -\min_{x \in \Delta_i} \langle x, w_j\rangle$. Then we claim $K^\vee \subset \overline{N}_\mathbb{R}$ is generated by the lattice points

$$\{(m_{1j}, m_{2j}, \dots, m_{sj}; w_j) \in \overline{N} \mid w_j \in \mathrm{Vert}(\nabla)\}$$
$$\cup\{\underbrace{(0, \dots, 1, \dots 0}_{\text{1 at the i-th position}}; 0) \in \overline{N} \mid 1 \leq i \leq s\}\ .$$

Suppose these lattice points generate a cone C, then it is straightforward to check $C \subseteq K^\vee$. The difficult part is to show $K^\vee \subseteq C$.

Let $(a_1, \dots, a_r; t) \in K^\vee$, then we must have $a_i + \min\langle\Delta_i, t\rangle \geq 0$ for all i. Subtracting a non-negative combination of the $(0, \dots, 1, \dots, 0; 0)$ if necessary, we have $a_i' + \min\langle\Delta_i, t\rangle = 0$ for all i. In this case, if one can show $(a_1', \dots, a_s'; t) \in C$, then adding back those non-negative combination of the $(0,\ldots,1,\ldots,0;0)$, we have $(a_1,\ldots,a_s;t)\in C$. By the above argument, we can assume without loss of generality, $a_i+\min\langle\Delta_i,t\rangle=0$ for all i. Moreover, if $t=0$, then we are done. If $t\neq 0$, then one can multiply t by a positive real number λ, such that λt lands on the boundary of ∇. In this case, we still have $\lambda a_i+\min\langle\Delta_i,\lambda t\rangle=0$, and if one can show $(\lambda a_1,\ldots,\lambda a_s;\lambda t)\in C$, then certainly $(a_1,\ldots,a_s;t)\in C$. Thus, we can reduce to the case when t is on the boundary of ∇, particularly, it is on some facet $F_v\subset\nabla$. Here, v is a vertex of $\sum_{i=1}^s\Delta_i-m$ such that $\langle v,F_v\rangle=-1$, where we have used the $1-1$ correspondence between vertices and facets in dual reflexive polytopes. Let $v=\sum_{i=1}^s v_i-m$, with v_i a vertex of Δ_i, and $t=\sum_j\lambda_jt_j, \lambda_j\geq 0$, $\sum_j\lambda_j=1$, with t_j vertices of F_v. Then because

$$-1+\langle m,t\rangle=\min\langle\sum_{i=1}^s\Delta_i,t\rangle=\sum_{i=1}^s\min\langle\Delta_i,t\rangle=\sum_{i=1}^s\langle v_i,t\rangle,$$

we have

$$-\min\langle\Delta_i,t\rangle=-\langle v_i,t\rangle=-\sum_j\lambda_j\langle v_i,t_j\rangle=-\sum_j\lambda_j\min\langle\Delta_i,t_j\rangle.$$

The last equation uses the fact that $t_j\in F_v$, and because

$$\begin{aligned}-1=\min\langle\sum_{i=1}^s\Delta_i-m,t_j\rangle&=\left(\min\sum_{i=1}^s\langle\Delta_i,t_j\rangle\right)-\langle m-t_j\rangle\\ =\langle\sum_{i=1}^s v_i-m,t_j\rangle&=\left(\sum_{i=1}^s\langle v_i,t_j\rangle\right)-\langle m,t_j\rangle,\end{aligned}$$

we must have $\langle\Delta_i,t_j\rangle=\langle v_i,t_j\rangle$. Putting everything together, we have

$$(-\min\langle\Delta_1,t\rangle,\ldots,-\min\langle\Delta_s,t\rangle,t)$$

$$=\sum_j\lambda_i(-\min\langle\Delta_1,t_j\rangle,\ldots,-\min\langle\Delta_1,t_j\rangle;t_j).$$

This proves the claim $K^\vee\subseteq C$.

In order to show $K^\vee$ is also a Gorenstein cone, let $\deg=(1,1,\ldots,1;m)$. By using the property $\min\langle\sum_{i=1}^s\Delta_i-m,v_j\rangle=-1$, it is straightforward to show that for the vertex v_j of ∇,

$$\langle\deg,(m_{1j},m_{2j},\ldots,m_{rj};v_j)\rangle=1$$

and

$$\langle \deg, (\underbrace{0,\ldots,1,\ldots,0}_{\text{1 at the i-th position}};0)\rangle = 1.$$

Thus we finish the argument K is a reflexive Gorenstein cone. Because $\deg^\vee = (1,1,\ldots,1;0)$, the index is $\langle \deg, \deg^\vee\rangle = s$.

$\square$

The above theorem can be applied to the case of nef-partitions, where $\sum_{i=1}^s \Delta_i$ itself is a reflexive polytope with dual polytope $(\sum_{i=1}^s \Delta_i)^\vee = \mathrm{Conv}(\cup_{i=1}^s \nabla_i)$. Because $0 \in \nabla_i$, and $\min\langle \Delta_i, \nabla_j\rangle = -\delta_{ij}$, we can write the reflexive Gorenstein cones associated to this pair of nef-partitions in a symmetric way

$$K = \{(a_1,\ldots,a_s;\sum_{i=1}^s a_i\Delta_i) \subset (\overline{M})_{\mathbb{R}} \mid a_i \geq 0\}$$

$$K^\vee = \{(b_1,\ldots,b_s;\sum_{i=1}^s b_i\nabla_i) \subset (\overline{N})_{\mathbb{R}} \mid b_i \geq 0\}.$$

This result can also be proved directly as in [5].

The following is our key construction which is used to reformulate the question from polytopes to reflexive cones.

Now we start off with a nef-partition $\{\Delta_i \mid 1 \leq i \leq s\}$, and let K be the reflexive Gorenstein cone associated to it as above with the degree element $\deg^\vee \in K^\vee$. If $\deg^\vee = \sum_{i=1}^s \tilde{e}_i$, with $\tilde{e}_i \neq 0, \tilde{e}_i \in K^\vee \cap \overline{N}$, then we can similarly define $\tilde{S}_i$ as in Proposition 2.2.1. In this case, $\left(\sum_{i=1}^s \tilde{S}_i - \deg\right)$ is a reflexive polytope in $\mathrm{Ann}(\tilde{e}_1,\ldots,\tilde{e}_s)$. Without loss of generality, we can assume

$$\tilde{e}_i = (\underbrace{0,\ldots,1,\ldots,0}_{\text{1 at the i-th position}};p_i) \in \mathbb{Z}^s \oplus N.$$

We claim that there exists a lattice isomorphism

$$\phi : \mathrm{Ann}(\tilde{e}_1,\ldots,\tilde{e}_s) \to M$$

defined by restricting to the projection $p : \mathbb{Z}^s \oplus M \to M$. In fact, if $\phi(x) = 0$, then $x = (a_1,\ldots,a_s;0)$, but $x \in \mathrm{Ann}(\tilde{e}_1,\ldots,\tilde{e}_s)$ implies that $\forall\ i, a_i = 0$, thus ϕ is injective.

The surjectivity comes from the fact that for $m \in M$, if we let $a_i = -\langle m, p_i \rangle$, then $(a_1, \ldots, a_s; m) \in \mathrm{Ann}(\tilde{e}_1, \ldots, \tilde{e}_s)$ maps to m under ϕ.

Under this isomorphism, we can identify $\mathrm{Ann}(\tilde{e}_1, \ldots, \tilde{e}_s)$ with M. Let $\tilde{\Delta}_i = p(\tilde{S}_i)$, one can verify directly that

$$\mathrm{Conv}(\bigcup_{i=1}^{s} \tilde{\Delta}_i) = \mathrm{Conv}(\bigcup_{i=1}^{s} \Delta_i).$$

Moreover, since $\phi(\deg) = 0$, and by Proposition 2.2.1, $(\sum_{i=1}^{s} \tilde{S}_i - \deg)$ is a reflexive polytope in $\mathrm{Ann}(e_1, \ldots, e_s)_{\mathbb{R}}$. Hence, $(\sum_{i=1}^{s} \tilde{\Delta}_i)$ is a reflexive polytope in M. Because $(\underbrace{0, \ldots, 1, \ldots, 0}_{\text{1 at the i-th position}}; 0) \in \tilde{S}_i$, we have $0 \in \tilde{\Delta}_i$, and this implies $\{\tilde{\Delta}_i \mid 1 \leq i \leq s\}$ is another nef-partition of $\mathrm{Conv}(\cup_{i=1}^{s} \Delta_i)$ (see Definition 2.1.1).

Remark 2.2.3. One cannot exhaust *all* the nef-partitions of length s of $\mathrm{Conv}(\cup_{i=1}^{s} \Delta_i)$ using the above construction (i.e. first construct reflexive Gorenstein cone $K, K^{\vee}$, then decompose $\deg^{\vee} = \sum_{i=1}^{s} \tilde{e}_i$, and finally construct $\tilde{\Delta}_i$). For example, any subsets of the vertices of an octahedron will give a nef-partition, but some subsets cannot be obtained from the above construction. However, the above process will give exactly the combinatorial data for toric double mirrors (details see Theorem 3.1.3).

Next, we give the geometry meaning of this construction.

Let $X(\Sigma)$ be the toric variety defined by the fan

$$\Sigma := \{0\} \cup \{\mathbb{R}_{\geq 0}\theta \mid \theta \subset \mathrm{Conv}(\cup_i \Delta_i) \text{ is a face}\},$$

and

$$\mathcal{L}_i = \sum_{\rho \in \mathrm{Vert}(\Delta_i) \setminus \{0\}} D_\rho, \qquad \tilde{\mathcal{L}}_i = \sum_{\rho \in \mathrm{Vert}(\tilde{\Delta}_i) \setminus \{0\}} D_\rho$$

be the nef divisors corresponding to $\{\Delta_i\}, \{\tilde{\Delta}_i\}$ respectively, where D_ρ is the torus invariant divisor associated to the primitive element ρ. The following result gives a characterization of the nef-partitions obtained from reflexive Gorenstein cones as above.

Proposition 2.2.4. *The nef-partition* $\{\tilde{\Delta}_i \mid 1 \leq i \leq s\}$ *of* $\mathrm{Conv}\,(\cup_{i=1}^{s} \Delta_i)$ *is obtained from the same reflexive Gorenstein cone if and only if the corresponding divisors* $\{\tilde{\mathcal{L}}_i \mid 1 \leq i \leq s\}$ *and* $\{\mathcal{L}_i \mid 1 \leq i \leq s\}$ *are pairwise linearly equivalent.*

Proof. Suppose $\deg^\vee = \sum_{i=1}^s \tilde{e}_i = \sum_{i=1}^s e_i$. Without loss of generality, we can assume $\tilde{e}_i - e_i = p_i \in N$. Then one can check that $\tilde{\mathcal{L}}_i - \mathcal{L}_i$ is exactly the principle divisor (X^{p_i}) on $X(\Sigma)$.

On the other hand, suppose $\tilde{\mathcal{L}}_i, \mathcal{L}_i$ are linearly equivalent divisors for each i, then there exists $p_i \in N$ such that $\tilde{\mathcal{L}}_i - \mathcal{L}_i = (X^{p_i})$. one can check that $\tilde{e}_i = e_i + (0; p_i)$ satisfies the requirement. $\square$

One may ask what is the relation between associated Gorenstein cones of these double mirror nef-partitions. As one can imagine, they are all isomorphic.

In fact, let $\tilde{e}_i \in \mathbb{Z}^s \oplus M$ as before, then $\eta_i = (\underbrace{0, \ldots, 1, \ldots, 0}_{\text{1 at the i-th position}}; 0) \in \mathbb{Z}^s \oplus M$ must be in S_i by definition. Let $\tilde{S}_i = S_i - \eta_i$, then we have (1) $0 \in \tilde{S}_i \subset \mathrm{Ann}(\tilde{e}_1, \cdots, \tilde{e}_s)$; (2) $\sum_{i=1}^s \tilde{S}_i$ is reflexive; (3) $\tilde{S}_i$ has the same image as S_i under the aforementioned projection. Also, because $\mathrm{Ann}(\tilde{e}_1, \cdots, \tilde{e}_s)$ is isomorphic to M under the same projection, we can identify $\{\tilde{S}_i\}$ with the nef-partition $\tilde{\Delta}_i$. Hence we only need to show the claim for $\{\tilde{S}_i\}$ in lattice $\mathrm{Ann}(\tilde{e}_1, \cdots, \tilde{e}_s)$.

This is straightforward to check, because

$$\sum_{i=1}^s \mathbb{Z}\eta_i + \mathrm{Ann}(\tilde{e}_1, \cdots, \tilde{e}_s) = \mathbb{Z}^s \oplus M,$$

and the reflexive Gorenstein cone associated to $\tilde{S}_i$ is

$$\tilde{K} = \sum_{i=1}^s r_i(\eta_i + \tilde{S}_i) = \sum_{i=1}^s r_i S_i = K, \quad r_i \geq 0.$$

We will prove the birationality for the Δ-regular complete intersections associated to nef-partitions (i.e. double mirror nef-partition) which are obtained from above.

Chapter 3

The main question

3.1 The main question and its motivation

After establishing the relation between reflexive Gorenstein cones and nef-partitions, we are ready to state the question asked in [6] more explicitly.

Let us repeat the construction in the last part of Section 2 in order to extract the main ingredients. Let $\Delta \subset M$ be a reflexive polytope, $\mathrm{Conv}(\cup_{i=1}^{s}\Delta_i) = \Delta$, and $\{\Delta_i \mid i = 1, \ldots, s\}$ be a nef-partition of Δ. Let $\overline{M} = \mathbb{Z}^s \oplus M$, $\overline{N} = \mathbb{Z}^s \oplus N$, and $K \subset \overline{M}_{\mathbb{R}}$ be the reflexive Gorenstein cone associated to this nef-partition. The dual cone of K is $K^\vee \subset \overline{N}_{\mathbb{R}}$ and $\deg^\vee \in K^\vee$ is the degree element. Then $\deg^\vee = \sum_{i=1}^{s} e_i$ with $e_i = (\underbrace{0, \ldots, 1, \ldots, 0}_{\text{1 at the i-th position}}; 0)$ gives back the original nef-partition $\{\Delta_i\}$. If there exists another decomposition $\deg^\vee = \sum_{i=1}^{s} \tilde{e}_i$ with $\tilde{e}_i \neq 0, \tilde{e}_i \in K^\vee \cap \overline{N}$, then we can associate to it $\{\tilde{\Delta}_i\}$ which gives another nef-partition of Δ.

Whenever one has a polytope, there is a family of Laurent polynomials associated to it. Let $l(\Delta_i)$ be the set of lattice points in Δ_i, then the family of Laurent polynomials associated to Δ_i is

$$f_i = \sum_{v \in l(\Delta_i)} c_v X^v \in \mathbb{C}[M],$$

where c_v is a complex coefficient only depends on the vertex v. Here we abuse notations, using v to represent the lattice point as well as its coordinate in M. For example, if $v = (a_1, \ldots, a_n) \in M$, then $X^v = x_1^{a_1} \cdots x_n^{a_n} \in \mathbb{C}[M]$. In the same fashion, $\tilde{\Delta}_j$ produces a family of Laurent polynomials

$$\tilde{f}_j = \sum_{v \in l(\tilde{\Delta}_j)} c_v X^v \in \mathbb{C}[M].$$

Remark 3.1.1. We should emphasize that for the same vertex $v, v \neq 0$, the coefficient c_v is the same in all Laurent polynomials. However, the coefficient of the origin, c_0 (i.e. the constant term) might be different in different Laurent polynomials. We abuse notations to avoid writing $c_{0,j}$ in place of c_0.

We can take the zero locus of all f_i in $(\mathbb{C}^*)^d = \operatorname{Spec}(\mathbb{C}[M])$, and denote this variety by $X_{(\Delta_i)}$. To be precise $X_{(\Delta_i)} \subset (\mathbb{C}^*)^d$ is defined by:

$$X_{(\Delta_i)}: \quad f_1 = f_2 = \cdots = f_s = 0,$$

and similarly, $X_{(\tilde{\Delta}_i)} \subset (\mathbb{C}^*)^d$ is defined by

$$X_{(\tilde{\Delta}_i)}: \quad \tilde{f}_1 = \tilde{f}_2 = \cdots = \tilde{f}_s = 0.$$

Remark 3.1.2. From toric variety point of view, this construction can be stated as follows. Let $X := X(\Sigma(\nabla))$ be the projective toric variety associated to the polytope $\sum_{i=1}^{s} \Delta_i$, $T \subset X$ be the big torus. Let $\mathcal{L}_i$ be the line bundle associated to the dual nef-partition $\{\nabla_i \mid 1 \leq i \leq s\}$. Generic global sections in $H^0(X, \mathcal{L}_i)$ can be identified with Laurent polynomials with Newton polytopes Δ_i. In particular, for $1 \leq i \leq s$, $f_i = \sum_{v \in l(\Delta_i)} c_v X^v \in H^0(X, \mathcal{L}_i)$. Let $(f_i)_0$ be the zero locus of f_i, then

$$X_{(\Delta_i)} = T \cap (f_1)_0 \cap \cdots \cap (f_s)_0.$$

We will return to this point of view in the appendix.

The following question was asked by Batyrev and Nill in [6] Question 5.2:

(**Nef-partition version**)

> Are the Calabi-Yau complete intersections $X_{(\Delta_i)}$ and $X_{(\tilde{\Delta}_i)}$ birational to each other?

We can reformulate this question in terms of reflexive Gorenstein cones as follows.

Let $\tilde{S} = \{x \in K \mid \langle x, \deg^\vee \rangle = 1\}$, $\tilde{S}_i = \{v \in K \mid \langle v, \deg^\vee \rangle = \langle v, \tilde{e}_i \rangle = 1\}$. Because $\deg^\vee = \sum_{i=1}^{s} \tilde{e}_i$, for each lattice point v in $\tilde{S}$, that is $v \in l(\tilde{S})$, there exists a unique

i, such that $\langle v, \tilde{e}_i \rangle = 1$. We have a disjoint union $l(\tilde{S}) = \coprod_{i=1}^{s} l(\tilde{S}_i)$. One can define a Laurent polynomial in $\mathbb{C}[\overline{M}]$ by setting:

$$\tilde{g}_i = \sum_{v \in l(\tilde{S}_i)} c_v X^v.$$

For any lattice point w_i such that $\langle w_i, \tilde{e}_i \rangle = 1, \langle w_i, \tilde{e}_j \rangle = 0, i \neq j$, $X^{-w_i} \cdot \tilde{g}_i$ is a Laurent polynomial in $\mathbb{C}[\mathrm{Ann}(\tilde{e}_1, \ldots, \tilde{e}_r)]$. We can similarly define an intersection $X_{(\tilde{e}_i)} \subset (\mathbb{C}^*)^d = \mathrm{Spec}(\mathbb{C}[\mathrm{Ann}(\tilde{e}_1, \ldots, \tilde{e}_s)])$ by

$$X_{(\tilde{e}_i)} \ : \ X^{-w_1} \cdot \tilde{g}_1 = X^{-w_2} \cdot \tilde{g}_2 = \cdots = X^{-w_s} \cdot \tilde{g}_s = 0$$

This intersection does not depend on the choice of w_i, because any other choice will differ by a factor $X^w, w \in \mathbb{C}[\mathrm{Ann}(\tilde{e}_1, \ldots, \tilde{e}_s)]$ and this will not affect the zero loci defined in $(\mathbb{C}^*)^d$.

Similarly, we can construct S_i and g_i associated to the decomposition $\deg^\vee = \sum_{i=1}^{r} e_i$, and an intersection $X_{(e_i)} \subset (\mathbb{C}^*)^d = \mathrm{Spec}(\mathbb{C}[\mathrm{Ann}(e_1, \ldots, e_s)])$ by

$$X_{(e_i)} \ : \ X^{-w'_1} \cdot g_1 = X^{-w'_2} \cdot g_2 = \cdots = X^{-w'_s} \cdot g_s = 0.$$

We can compare the equations defined by these intersections with the equations defined the intersections above by nef-partitions. Because the lattice isomorphism

$$\phi : \mathrm{Ann}(\tilde{e}_1, \ldots, \tilde{e}_r) \to M$$

sends $\tilde{S}_i - \deg$ to $\tilde{\Delta}_i$, we can identify $\tilde{g}_i \in \mathbb{C}[\mathrm{Ann}(\tilde{e}_1, \ldots, \tilde{e}_r)]$ with $f_i \in \mathbb{C}[M]$ up to a factor $X^{v_i}, v_i \in \mathbb{C}[M]$. Hence, $X_{(\tilde{e}_i)}$ and $X_{(\tilde{\Delta}_i)}$ are isomorphic varieties. The same thing is true for $X_{(e_i)}$ and $X_{(\Delta_i)}$ as well.

The importance of the above construction is explained in the following theorem.

Theorem 3.1.3. *The complete intersections $X_{(\tilde{e}_i)}$ and $X_{(e_i)}$ are toric double mirror in the sense that they both mirror to the same family.*

We abuse the notations: $X_{(\tilde{e}_i)}$ here means the *family* parameterized by the coefficients c_v, and the same for $X_{(e_i)}$.

Proof. By the toric mirror construction in [2] [3], the mirror of $X_{(\tilde{e}_i)}$ is a family of generic complete intersections defined by divisors $\{\tilde{\mathcal{L}}_i | 1 \le i \le s\}$ in the toric variety $X(\Sigma)$ (see the notations above Proposition 2.2.4). Likewise, the mirror of $X_{(e_i)}$ is a family of generic complete intersections defined by divisors $\{\mathcal{L}_i | 1 \le i \le s\}$ in $X(\Sigma)$. By Proposition 2.2.4, $\{\tilde{\mathcal{L}}_i | 1 \le i \le s\}, \{\mathcal{L}_i | 1 \le i \le s\}$ consist of pairwise linearly equivalent divisors and hence they defined the same family of complete intersections which is the mirror of both $X_{(\tilde{e}_i)}$ and $X_{(e_i)}$.

□

Viewing the original question from this perspective, we can ask:

(**Reflexive Gorenstein cone version**)

Are the toric double mirror $X_{(e_i)}, X_{(\tilde{e}_i)}$ birational?

We give an affirmative answer to this question in Theorem 4.3.1 under some technical assumptions.

3.2 Example

In this section, we will illustrate the basic idea of the proof by an explicit example.

Let $\{u_1, \ldots, u_{15}\}$ be a basis of $\mathbb{Z}^{15}$, and we consider a sublattice $M \subset \mathbb{Z}^{15}$ which is defined by

$$M := \{\sum_{i=1}^{15} l_i u_i \in \mathbb{Z}^{15} \mid \sum_{i=1}^{5} l_i = \sum_{i=6}^{10} l_i = \sum_{i=11}^{15} l_i\}.$$

The rank of M is 13, it contains a cone $K = \mathbb{Z}^{15}_{\ge 0} \cap M$ which is defined by nonnegativity of all l_i. The 125 generators of rays of K are given by $u_{i_1} + u_{i_2} + u_{i_3}$ with $5j - 4 \le i_j \le 5j$, and let $c_{ijk} \in \mathbb{C}$ denote coefficients. Suppose $\{v_1, \ldots, v_{15}\}$ is the dual basis of $\{u_1, \ldots, u_{15}\}$, then the dual lattice $M^\vee$ is the quotient of $\mathbb{Z}^{15}$:

$$M^\vee = \mathbb{Z}^{15} / \operatorname{Span}_{\mathbb{Z}}\{\sum_{i=1}^{5} v_i - \sum_{i=6}^{10} v_i, \sum_{i=1}^{5} v_i - \sum_{i=11}^{15} v_i\}.$$

The dual cone $K^\vee$ is the image of $\mathbb{Z}^{15}_{\ge 0}$ in $M^\vee$, and its rays are generated by $v_i, 1 \le i \le 15$. The degree elements $\deg, \deg^\vee$ are given by $\sum_{i=1}^{15} u_i$ and $\sum_{i=1}^{5} v_i$ respectively.

There are three different ways of decomposing $\deg^\vee$ as a summation of lattice points in $K^\vee$:

$$\deg^\vee = \sum_{i=1}^{5} v_i, \quad \deg^\vee = \sum_{i=6}^{10} v_i, \quad \deg^\vee = \sum_{i=11}^{15} v_i.$$

This gives three different complete intersections in $\mathbb{P}^4 \times \mathbb{P}^4$.

For $\deg^\vee = \sum_{i=1}^{5} v_i$, the equations of this decomposition can be expressed as

$$\begin{aligned}
\sum_{1\le j,k\le 5} c_{1jk}\ x_1 y_j z_k &= 0\\
\sum_{1\le j,k\le 5} c_{2jk}\ x_2 y_j z_k &= 0\\
&\vdots\\
\sum_{1\le j,k\le 5} c_{5jk}\ x_5 y_j z_k &= 0 \quad .
\end{aligned}$$

Here $[x_1, \cdots, x_5]$ are homogenous coordinates of $\mathbb{P}^4$, and similarly for y_j, z_k.

As explained before, we can multiply each equation a factor in order to make it well defined in $M \cap \mathrm{Ann}(v_1, \ldots, v_5)$. Hence, let

$$f_i(y,z) = x_i^{-1} \sum_{1\le j,k\le 5} c_{ijk}\ x_i y_j z_k = \sum_{1\le j,k\le 5} c_{ijk}\ y_j z_k = 0, \quad 1 \le i \le 5.$$

This can be viewed as five bidegree $(1,1)$ equations in $\mathbb{P}^4 \times \mathbb{P}^4$. Similarly, for $\deg^\vee = \sum_{i=6}^{10} v_i$ and $\deg^\vee = \sum_{i=11}^{15} v_i$ we have defining equations:

$$\begin{aligned}
g_j(x,z) &= \sum_{1\le i,k\le 5} c_{ijk}\ x_i z_k = 0, \quad 1 \le j \le 5,\\
h_k(x,y) &= \sum_{1\le i,j\le 5} c_{ijk}\ x_i y_j = 0, \quad 1 \le k \le 5.
\end{aligned}$$

Our question thus becomes whether these three complete intersections are birational for generic choice of c_{ijk}.

Let X_1 be the variety defined by $f_i = 0, 1 \le i \le 5$. Let $A_1(z)$ be 5×5 matrix

$$A_1(z) = \left(\sum_{k=1}^{5} c_{ijk} z_k \right)_{ij} \quad , \quad 1 \le i,j \le 5,$$

then $f_i = 0, 1 \leq i \leq 5$ can be written as a matrix equation

$$A_1(z) \begin{pmatrix} y_1 \\ \vdots \\ y_5 \end{pmatrix} = 0.$$

Notice that $([y_1, \dots, y_5], [z_1, \dots, z_5]) \in \mathbb{P}^4 \times \mathbb{P}^4$ satisfy $f_i = 0, 1 \leq i \leq 5$ if and only if $\det(A_1(z)) = 0$ in $\mathbb{P}^4$. Let D_1 denote the variety defined by $\det(A_1(z)) = 0$. For generic coefficients, one can show X_1 and D_1 are birational.

Similarly, the variety X_2 defined by $g_j = 0, 1 \leq j \leq 5$ can be written as

$$(x_1, \cdots, x_5)\, A_2(z) = 0$$

where

$$A_2(z) = \left(\sum_{k=1}^{5} c_{ijk} z_k \right)_{ij}, \quad 1 \leq i, j \leq 5.$$

Let D_2 be the variety defined be $\det(A_2(z)) = 0$. The same argument as above shows that X_2 is birational to D_2. On the other hand, D_1 and D_2 are the same varieties, and hence X_1, X_2 are birational. We notice that despite drastically different defining equations, the three complete intersections are all birational.

This example suggests us to look at the determinantal variety defined by a "common" matrix of different nef-partitions. However, it is not very clear how to construct this "common" matrix at present stage. Besides that, there are following more pressing issues: (1) the dimension of $\mathrm{Span}_{\mathbb{R}}\{\tilde{e}_1 - e_1, \dots, \tilde{e}_s - e_s\}$ might be smaller than $s - 1$ which leads to considering the intersection of several determinantal varieties; (2) "non-saturatedness" might occur, which forces us to work in auxiliary lattices; (3) in order to show the birationality, we have to take into account of the singularities of the complete intersection. This leads us to consider Δ-regular intersections.

Chapter 4

The main theorem

4.1 Results on the decomposition of lattices

Let Δ be a reflexive polytope, $\{\Delta_i \mid 1 \leq i \leq s\}$ be a nef-partition of Δ, and $\{\nabla_i \mid 1 \leq i \leq s\}$ be its dual nef-partition. In the following, we assume $\dim \Delta = \dim M_{\mathbb{R}}$. Because

$$\Delta \subset \sum_{i=1}^{s} \Delta_i \ ,$$

we have $\dim(\sum_{i=1}^{s} \Delta_i) = \dim M_{\mathbb{R}}$. We use $\mathrm{Span}_{\mathbb{R}}\{p_1, \ldots, p_s\}$ to denote the vector space spanned by $p_i \in N_{\mathbb{R}}, 1 \leq i \leq s$. The following lemma is crucial for our argument.

Lemma 4.1.1. *Let $p_i \in \nabla_i$. If $\sum_{i=1}^{s} p_i = 0$, and*

$$\dim(\mathrm{Span}_{\mathbb{R}}\{p_1, \ldots, p_s\}) = s - r,$$

then there exist disjoint sets $I_k \subset \{1, \ldots, s\}, 1 \leq k \leq r$, such that $\coprod_{k=1}^{r} I_k = \{1, \ldots, s\}$ and for each k, we have $\sum_{i \in I_k} p_i = 0$.

Proof. Suppose l is the maximum number such that there exist l nonempty disjoint sets $I_j, 1 \leq j \leq l$ satisfying

$$I_1 \coprod \cdots \coprod I_l = \{1, \cdots, s\}$$

and $\forall\ j, \sum_{i \in I_j} p_i = 0$.

Because these l equations are linearly independent, we have

$$s - r = \dim(\mathrm{Span}_{\mathbb{R}}\{p_1, \ldots, p_s\}) \leq s - l,$$

and hence $l \leq r$. All we need to show is $l = r$.

Otherwise, suppose $l < r$, then there must exist at least one equation $\sum_{1 \leq i \leq s} a_i p_i = 0$, which is not a linear combination of $\sum_{i \in I_j} p_i = 0$. Hence, there must exist an index

j, such that for $i \in I_j, a_i$ are not identically the same. Suppose a_m is a minimal element in $\{a_i \mid i \in I_j\}$. After reindexing the set, we can assume $j = 1$ and $m = 1$. Let C be a sufficiently large number, then

$$0 = \sum_{1 \leq i \leq s} a_i p_i - a_1 \sum_{i \in I_1} p_i + C \cdot \sum_{i \in I_2 \sqcup \cdots \sqcup I_l} p_i = \sum_{2 \leq i \leq s} b_i p_i$$

satisfies $b_i > 0$ when $i \in I_2 \sqcup \cdots \sqcup I_l$, and $b_i \geq 0$ when $i \in I_1$. Moreover, there exists at least one element $t \in I_1$ such that $b_t > 0$ (because a_i are not identically the same for $i \in I_1$). Let $S = \{i \mid b_i \neq 0\}$ be the index set corresponding to nonzero coefficients.

Set $P = \sum_{i \in S} p_i = \sum_{i \in S} (1 - cb_i) p_i$ with c sufficiently big such that $\forall\, i, (1 - cb_i) < 0$. When $k \notin S$, we have

$$\langle \Delta_k, \sum_{i \in S} p_i \rangle \geq 0$$

$$\langle \Delta_k, \sum_{i \in S} (1 - cb_i) p_i \rangle \leq 0.$$

Hence $\langle \Delta_k, P \rangle = 0$ for $k \notin S$.

In the following, we will show $P = 0$. Otherwise, there exists $v \in M_{\mathbb{R}}$ such that $\langle v, P \rangle > 0$. Because $M_{\mathbb{R}} = \sum_{i=1}^{s} \mathbb{R}_{\geq 0} \Delta_i$, we can chose $v = \sum_{1 \leq i \leq s} v_i$ with $v_i \in \Delta_i$. Then we have

$$\langle v, P \rangle = \langle \sum_{i \in S} v_i + \sum_{i \notin S} v_i, P \rangle = \sum_{i \in S} \langle v_i, -\sum_{j \notin S} p_j \rangle + \sum_{i \notin S} \langle v_i, P \rangle.$$

We use the assumption $\sum_{j=1}^{s} p_j = 0$, and thus $P = -\sum_{j \notin S} p_j$ in the second equation. However, $\sum_{i \in S} \langle v_i, -\sum_{j \notin S} p_j \rangle \leq 0$, and $\sum_{i \notin S} \langle v_i, P \rangle = 0$ because $\langle \Delta_k, P \rangle = 0$ for $k \notin S$. This contradiction implies $P = \sum_{i \in S} p_i = 0$.

Because $I_1 \cap S \neq \emptyset$ and $I_1 \not\subseteq S$, the index set $I_1' := I_1 \cap S$ must satisfy $\emptyset \subsetneqq I_1' \subsetneqq I_1$. Since $I_2 \sqcup \cdots \sqcup I_l \subset S$, we have

$$\sum_{j \in I_1'} p_j = P - \sum_{i \in I_2 \sqcup \cdots \sqcup I_l} p_i = 0.$$

But this implies

$$\sum_{j \in I_1'} p_j = \sum_{j \in I_1 \setminus I_1'} p_j = 0$$

which gives a further decomposition of I_1. This is a contradiction to the maximality of l.

$\square$

Remark 4.1.2. Under the notation of Lemma 4.1.1, we observe that for each k, $\dim(\mathrm{Span}_{\mathbb{R}}\{p_i \mid i \in I_k\}) = \#(I_k) - 1$.

Let $\overline{M} = \mathbb{Z}^s \oplus M$, and $K \subset \overline{M}_{\mathbb{R}}$ be the reflexive Gorenstein cone associated to a nef-partition $\{\Delta_1, \ldots, \Delta_s\}$ in $M_{\mathbb{R}}$ as it is in Proposition 2.2.2. This nef-partition corresponds to $\deg^\vee = \sum_{i=1}^s e_i \in K^\vee$, where $e_i = (\underbrace{0, \ldots, 1, \ldots, 0}_{\text{1 at the i-th position}};0)$. If $\deg^\vee = \sum_{i=1}^s \tilde{e}_i$ with $\tilde{e}_i \neq 0, \tilde{e}_i \in \overline{N} \cap K^\vee$, then we can assume without loss of generality that

$$\tilde{e}_i = (\underbrace{0, \ldots, 1, \ldots, 0}_{\text{1 at the i-th position}};p_i), \quad p_i \in N \cap \nabla_i.$$

Note that

$$\dim(\mathrm{Span}_{\mathbb{R}}\{e_1, \ldots, e_s, \tilde{e}_1, \ldots, \tilde{e}_s\}) = s + \dim(\mathrm{Span}_{\mathbb{R}}\{p_1, \ldots, p_s\}),$$

hence, if $\dim(\mathrm{Span}_{\mathbb{R}}\{p_1, \ldots, p_s\}) = s - r$, by Lemma 4.1.1 there exists disjoint index sets $I_k, 1 \leq k \leq r$, such that $\coprod_{k=1}^r I_k = \{1, \ldots, s\}$. For each k, we have $\sum_{i \in I_k} p_i = 0$, with $\dim(\mathrm{Span}_{\mathbb{R}}\{p_i \mid i \in I_k\}) = \#(I_k) - 1$.

Let $n_k = \#(I_k)$ from now on, and let

$$\mathrm{Ann}(e) := \mathrm{Ann}(e_1, \ldots, e_s) = \{m \in \overline{M} \mid \langle m, e_i \rangle = 0, \forall\ 1 \leq i \leq s\}.$$

If $\operatorname{rank} M = d$, then $\mathrm{Ann}(e)$ is a sublattice of $\overline{M}$ with rank d, and

$$\begin{aligned} \mathrm{Ann}(e, \tilde{e}) :&= \mathrm{Ann}(e_1, \ldots, e_s, \tilde{e}_1, \ldots, \tilde{e}_s) \\ &= \{m \in \overline{M} \mid \langle m, e_i \rangle = \langle m, \tilde{e}_i \rangle = 0, \forall\ 1 \leq i \leq s\} \end{aligned}$$

a sublattice of $\overline{M}$ with rank $d + r - s$.

For our convenience, we use $\{(k1), (k2), \ldots, (kn_k)\}$ as the index set of I_k, and reindex the corresponding elements. For example

$$\sum_{i \in I_k} p_i = 0$$

becomes
$$\sum_{i=1}^{n_k} p_{ki} = 0$$
under the new indexing.

Because $\dim(\mathrm{Span}_{\mathbb{R}}\{p_{k1}, \ldots, p_{kn_k}\}) = n_k - 1$, we can choose
$$\{p_{12}, \ldots, p_{1n_1}, \ldots, p_{r2}, \ldots, p_{rn_r}\}$$
as a $\mathbb{R}$-linearly independent set.

Another important fact of $\{p_i \mid 1 \leq i \leq s\}$ is that they form a saturated sublattice in N, that is, the abelian group $N/\left(\sum_{k=1}^{s} \mathbb{Z}p_i\right)$ is torsion free. The following combinatorial proof is due to Borisov.

Lemma 4.1.3. *The sublattice $\sum_{i=1}^{s} \mathbb{Z}p_i \subset N$ is saturated.*

Proof. Suppose otherwise, there exists $n = \sum_{i=1}^{s} a_i p_i$ with $a_i \in \mathbb{Q}$ such that $n \in N$ but $n \notin \sum_{i=1}^{s} \mathbb{Z}p_i$. Furthermore, we can assume $\forall\, i, 0 \leq a_i < 1$.

Recall that $\forall\, i, p_i \in \Delta_i$, hence $a_i p_i \in \Delta_i$. By the property of nef-partition, we have
$$n \in \sum_{i}^{s} \Delta_i = (\bigcup_{i}^{s} \nabla_i)^{\vee}.$$
If $n \neq 0$, then there exists a lattice $m \in \cup_i^s \nabla_i$ such that $-1 \leq \langle n, m \rangle < 0$. Because n is a lattice, we have $\langle n, m \rangle = -1$.

On the other hand, the set $\{m \in \cup_i^s \nabla_i \mid \langle n, m \rangle = -1\}$ must contain some vertices of $\cup_i^s \nabla_i$ and hence some vertices of ∇_i due to nef-partition, without loss of generality, we can assume $m \in \nabla_k$. Then use the property that $\min\langle \Delta_i, \nabla_j \rangle \geq -\delta_{ij}$ (see the discussion after Definition 2.1.4), we have
$$-1 = \langle n, m \rangle = \sum_{i=1}^{s} a_i \langle p_i, m \rangle \geq -a_k > -1,$$
this is a contradiction. Thus $n = 0$, but this contradicts our initial assumption on $n \notin \sum_{i=1}^{s} \mathbb{Z}p_i$.

$\square$

Using the above two lemmas, we can decompose the lattice to fulfil our purpose.

Lemma 4.1.4. *The lattice* $\mathrm{Ann}(e) \subset \overline{M}$ *can be decomposed as follows:*

$$\begin{aligned}\mathrm{Ann}(e) = &\mathrm{Ann}(e, \tilde{e}) \\ &\oplus \mathbb{Z}[w_{12}] \oplus \cdots \oplus \mathbb{Z}[w_{1n_1}] \\ &\oplus \cdots \\ &\oplus \mathbb{Z}[w_{r2}] \oplus \cdots \oplus \mathbb{Z}[w_{rn_r}].\end{aligned}$$

Where $w_{ki} \in \overline{M}$ *satisfies the following requirements (where by our indexing,* w_{ki} *starts from* w_{k2}*):*

1. $\langle w_{ki}, \tilde{e}_{k1} \rangle = -1,\ \langle w_{ki}, \tilde{e}_{ki} \rangle = 1$ *for* $i \geq 2$.

2. $\langle w_{ki}, \tilde{e}_{lj} \rangle = 0$ *for all* $\tilde{e}_{lj} \neq \tilde{e}_{k1}, \tilde{e}_{ki}$.

3. $\langle w_{ki}, e_{lj} \rangle = 0$ *for all* e_{lj}.

Proof. First, if we already have w_{ki} satisfying the given properties, then by definition, we have

$$\mathrm{Ann}(e, \tilde{e}) \oplus \mathbb{Z}[w_{12}] \oplus \cdots \oplus \mathbb{Z}[w_{1n_1}] \oplus \cdots \oplus \mathbb{Z}[w_{r2}] \oplus \cdots \oplus \mathbb{Z}[w_{rn_r}] \subset \mathrm{Ann}(e)$$

as a sublattice. On the other hand, $\forall\ m \in \mathrm{Ann}(e)$, we set

$$m - \sum_{k} \sum_{\substack{ki \\ i \geq 2}} \langle m, \tilde{e}_{ki} \rangle w_{ki},$$

then by definition, one can check

$$\begin{aligned}&m - \sum_{k} \sum_{\substack{ki \\ i \geq 2}} \langle m, \tilde{e}_{ki} \rangle w_{ki} \\ &\in \mathrm{Ann}(e) \cap \mathrm{Ann}(\tilde{e}_{12}, \cdots, \tilde{e}_{1n_1}, \cdots, \tilde{e}_{r2}, \cdots, \tilde{e}_{rn_r}).\end{aligned}$$

Using the fact that $\forall\ k, \sum_{t \in I_k} e_t = \sum_{t \in I_k} \tilde{e}_t$, we have

$$m - \sum_{k} \sum_{\substack{ki \\ i \geq 2}} \langle m, \tilde{e}_{ki} \rangle w_{ki} \in \mathrm{Ann}(e, \tilde{e}).$$

Thus, we only need to show the existence of w_{ki}. Let lattice map

$$\theta : M \to \mathbb{Z}^{s-r}$$

be defined by

$$m \mapsto (\langle m, p_{12}\rangle, \ldots, \langle m, p_{1n_1}\rangle, \ldots, \langle m, p_{r2}\rangle, \ldots, \langle m, p_{rn_r}\rangle).$$

We claim that θ is a surjective lattice map. Because of the saturatedness (Lemma 4.1.3),

$$\{p_{12}, \ldots, p_{1n_1}, \ldots, p_{r2}, \ldots, p_{rn_r}\}$$

forms part of $\mathbb{Z}$-basis of N. It follows that θ is surjective.

We can choose m such that $\langle m, p_{ij}\rangle = 0\ \forall j \geq 2$ except $\langle m, p_{ki}\rangle = 1$, and set

$$w_{ki} = (0, 0, \ldots, 0; m) \in \overline{M},$$

then w_{ki} satisfies the required properties.

$\square$

Now let

$$L = \mathrm{Span}_{\mathbb{Z}}\{w_{12}, \cdots, w_{1n_1}, \cdots, w_{r2}, \cdots, w_{rn_r}\} \subset \overline{M}$$

we have

$$\mathrm{Ann}(e) = \mathrm{Ann}(e, \tilde{e}) \oplus L\ .$$

Because of the above decomposition of lattices, we have a corresponding decomposition of toric varieties:

$$\mathrm{Spec}(\mathbb{C}[\mathrm{Ann}(e)]) = \mathrm{Spec}(\mathbb{C}[\mathrm{Ann}(e, \tilde{e})]) \times \mathrm{Spec}(\mathbb{C}[L])\ .$$

For any closed point in $\mathrm{Spec}(\mathbb{C}[\mathrm{Ann}(e)])$ with coordinate x , we will write $x = (y, \omega)$ with $y \in \mathrm{Spec}(\mathbb{C}[\mathrm{Ann}(e, \tilde{e})]),\ \omega \in \mathrm{Spec}(\mathbb{C}[L])$ respectively.

4.2 Construction of the determinantal variety

The main ingredient in the proof of Theorem 4.3.1 is a determinantal variety D which serves as a bridge to connect two complete intersections. We will show how to construct this variety in $\mathrm{Spec}(\mathbb{C}[\mathrm{Ann}(e, \tilde{e})])$ which heavily relies on Lemma 4.1.4.

Now, let

$$S_{i,j} = \{v \in K \mid \langle v, \deg^{\vee}\rangle = 1, \langle v, e_i\rangle = \langle v, \tilde{e}_j\rangle = 1\}$$

be a polytope, and

$$g_{i,j} = \sum_{v \in l(S_{i,j})} c_v X^v$$

be the Laurent polynomial associated to $S_{i,j}$ with coefficients $c_v \in \mathbb{C}$. Let $u_{ki} \in \overline{M}$ satisfy:

1. $\langle u_{ki}, e_{ki} \rangle = \langle u_{ki}, \tilde{e}_{k1} \rangle = 1$
2. $\langle u_{ki}, e_{lj} \rangle = 0$ for all $e_{lj} \neq e_{ki}$
3. $\langle u_{ki}, \tilde{e}_{lj} \rangle = 0$ for all $\tilde{e}_{lj} \neq \tilde{e}_{k1}$.

We point out that unlike those w_{ki} constructed before, u_{ki} starts from u_{k1} for each k. The existence of u_{ki} follows from the similarly reason as in Lemma 4.1.4, and we do not repeat it here.

Next, we proceed to the construction of the determinantal variety D.

Let $A_k(y)$ be the $n_k \times n_k$ matrix with entries in $\mathbb{C}[M]$,

$$\begin{aligned} & A_k(y) \\ = & \begin{pmatrix} X^{-u_{k1}} g_{k1,k1} & X^{-u_{k1}-w_{k2}} g_{k1,k2} & \cdots & X^{-u_{k1}-w_{kn_k}} g_{k1,kn_k} \\ X^{-u_{k2}} g_{k2,k1} & X^{-u_{k2}-w_{k2}} g_{k2,k2} & \cdots & X^{-u_{k2}-w_{kn_k}} g_{k2,kn_k} \\ \vdots & \vdots & & \vdots \\ X^{-u_{kn_k}} g_{kn_k,k1} & X^{-u_{kn_k}-w_{k2}} g_{kn_k,k2} & \cdots & X^{-u_{kn_k}-w_{kn_k}} g_{kn_k,kn_k} \end{pmatrix} \end{aligned}$$

Notice that the first column is not constructed identically as the rest. The reason for writing the matrix $A_k(y)$ as a function of y is that every entry of this matrix is in $\mathbb{C}[\mathrm{Ann}(e, \tilde{e})]$, as one can verify. Thus, according to the above decomposition $\mathrm{Spec}(\mathbb{C}[\mathrm{Ann}(e)]) = \mathrm{Spec}(\mathbb{C}[\mathrm{Ann}(e, \tilde{e})]) \times \mathrm{Spec}(\mathbb{C}[L])$, we use y to represent the corresponding coordinates in $\mathrm{Spec}(\mathbb{C}[\mathrm{Ann}(e, \tilde{e})])$.

Next, we define $n_k \times 1$ matrix

$$\mathbf{w}_k = (1, X^{w_{k2}}, \cdots, X^{w_{kn_k}})^t,$$

where t means the transpose of a matrix. And also define the $1 \times n_k$ matrix

$$\mathbf{u}_k = (X^{u_{k1}}, X^{u_{k2}}, \ldots, X^{u_{kn_k}}).$$

We claim that the condition

$$A_k(y) \cdot \mathbf{w}_k = 0$$

is exactly the same as

$$\begin{pmatrix} X^{-u_{k1}} g_{k1} \\ \vdots \\ X^{-u_{kn_k}} g_{kn_k} \end{pmatrix} = 0.$$

Indeed, recall (Section 3) by definition, we have

$$g_{ki} = \sum_{v \in l(S_{ki})} c_v X^v$$

where

$$S_{ki} = \{v \in K \mid \langle v, \deg^\vee \rangle = 1, \langle v, e_{ki} \rangle = 1\}$$

(Notice: this is not the same as $S_{k,i}$ defined before).

Because of the relation $\sum_{i \in I_k} e_i = \sum_{i \in I_k} \tilde{e}_i$, for any $v \in l(S_{ki})$, $\langle v, \sum_{i \in I_k} e_i \rangle = 1$ implies $\langle v, \sum_{i \in I_k} \tilde{e}_i \rangle = 1$, thus there exists kj, such that $v \in l(\tilde{S}_{kj})$, where $\tilde{S}_{kj} = \{v \in K \mid \langle v, \deg^\vee \rangle = 1, \langle v, \tilde{e}_{kj} \rangle = 1\}$. This means $v \in l(S_{ki,kj})$, and in particular, we have a disjoint union

$$l(S_{ki}) = \coprod_{kj \in I_k} l(S_{ki,kj}).$$

Hence, $g_{ki} = \sum_{kj \in I_k} g_{ki,kj}$, and this justifies the claim.

On the other hand,

$$\mathbf{u}_k \cdot A_k(y) = 0$$

is exactly the same as

$$\left(X^{-w_{k1}} \tilde{g}_{k1}, \cdots, X^{-w_{kn_k}} \tilde{g}_{kn_k} \right) = 0,$$

where $\tilde{g}_{kj} = \sum_{v \in l(\tilde{S}_{kj})} c_v X^v = \sum_{ki \in I_k} g_{ki,kj}$ because of the disjoint union

$$l(\tilde{S}_{ki}) = \coprod_{ki \in I_k} l(\tilde{S}_{ki,kj}) \ .$$

Let

$$D_k := \{\det A_k(y) = 0\}$$

in Spec($\mathbb{C}[\mathrm{Ann}(e,\tilde{e})]$). Let

$$D = \bigcap_{k=1}^{r} D_k$$

with its reduced induced subscheme structure. This D will serve as a bridge to prove the birationality of two complete intersections.

Remark 4.2.1. We have $\det A_k(y) \not\equiv 0$ for generic coefficients because we always have

$$(\underbrace{0,\ldots,1,\ldots,0}_{\text{1 at the i-th position}};0) \in S_{i,i}.$$

Thus by the definition of determinant, after choosing generic coefficients, these elements will give a nonzero summand in $\det A_k(y)$, hence, D_k is a hypersurface in Spec($\mathbb{C}[\mathrm{Ann}(e,\tilde{e})]$), and $\dim D_k = d + r - s - 1$, with $d = \operatorname{rank} M$.

We need the following lemma about the degree of a morphism and points in the generic fibres. This proof below is indebted from the discussion with Professor Qing Liu.

Lemma 4.2.2. *Let $f : X \to Y$ be a dominant morphism of varieties over $\mathbb{C}$. Suppose $[K(X) : K(Y)] = n$. Then there exists a dense open subset U of Y such that $f^{-1}(y)$ consists of n (distinct) points for all $y \in U$.*

In particular, if f is a dominant, injective morphism, then $[K(X) : K(Y)] = 1$, so X, Y are birational.

Proof. First we can reduce it to the case when f is a finite morphism. In fact, the problem is local in Y, hence we can assume $Y = \mathrm{Spec}(A)$ to be affine. Let $X' \subset X$ be a non empty open affine subset, it suffice to prove the result for $X' \to Y$. This is because $[K(X) : K(Y)] < \infty$, then $Y \setminus \overline{f(X \backslash X')} \neq \emptyset$, hence the open set $f^{-1}(Y \setminus \overline{f(X \backslash X')}) \subset X'$. Now we assume $X = \mathrm{Spec}(B)$.

The dominant morphism f corresponds to an injective homomorphism $A \to B$. Write $k(B) = k(A)[t]$ where $k(B), k(A)$ are quotient fields of B, A and t annihilates a polynomial $P(T) \in k(A)[T]$ of degree n (theorem of primitive element). Replacing A by a localization A_a with $a \in A$ such that the element t becomes integral over A (also localizing B correspondingly). As B is a finitely generated algebra over A, localizing further A, we can suppose $A \subset B \subset A[t]$ (because each element of B belong to some $A_a[t]$, it is enough to inverse a common denominator for a system of generators of B over A). As B and $A[t]$ have the same field of fractions and B is finite over A, localizing A again, we have $B = A[t] = A[T]/(P(T))$. The discriminant Δ of $P(T)$ belongs to A (we may need to localize A for this) and is non-zero because $P(T)$ is separable in $k(A)[T]$. Let U be the principal open subset $D(\Delta) \subset Y$. Then for any $y \in Y$, the fiber $f^{-1}(y)$ is given by the algebra $k(y)[T]/(\bar{P}(T))$ where $k(y) = \mathbb{C}$ denotes the residue field at y and $\bar{P}(T) \in k(y)[T]$ is the canonical image of $P(T)$. Its discriminant is $\Delta(y) \neq 0$, so it has n (distinct) roots.

$\square$

4.3 Proof of the main theorem

In this section, we will show that $X_{(e_i)}$ and $X_{(\tilde{e}_i)}$ are both birational to the determinantal variety D. In fact, we will show that the morphism $X_{(e_i)}$ to D induced by the projection from $\mathrm{Spec}(\mathbb{C}[\mathrm{Ann}(e)])$ to $\mathrm{Spec}(\mathbb{C}[\mathrm{Ann}(e, \tilde{e})])$ gives the birational morphism, and similarly for $X_{(\tilde{e}_i)}$ to D. We recall our setup first:

Let M, N be rank d lattices, and let $K, K^\vee$ be reflexive Gorenstein cones associated to a length s nef-partition. Let $\deg^\vee = \sum_{i=1}^s e_i = \sum_{i=1}^s \tilde{e}_i$ where $e_i, \tilde{e}_i \in K^\vee \cap \overline{N}, e_i, \tilde{e}_i \neq 0$ as before. Again, without loss of generality, we assume $\forall\ 1 \leq i \leq s$,

$$e_i = (\underbrace{0, \ldots, 1, \ldots, 0}_{\text{1 at the i-th position}}; 0), \qquad \tilde{e}_i = (\underbrace{0, \ldots, 1, \ldots, 0}_{\text{1 at the i-th position}}; p_i) \in \overline{N}\ .$$

By Lemma 4.1.1, we have a decomposition of $\{p_1, \ldots, p_s\}$ into subsets $I_k = \{p_{k1}, \ldots, p_{kn_k}\}$, for each $1 \leq k \leq r$. We define the intersections $X_{(e_i)}, X_{(\tilde{e}_i)}$ as in Section 3. With this notation, we have the following birationality result:

Theorem 4.3.1. *For generic coefficients, if $X_{(e_i)}, X_{(\tilde{e}_i)}, D$ are irreducible with $\dim D = \dim X_{(e_i)} = \dim X_{(\tilde{e}_i)}$, then the complete intersections $X_{(e_i)}$ and $X_{(\tilde{e}_i)}$ are birational.*

Proof. When $s = 1$, then $X_{(e_i)} = X_{(\tilde{e}_i)}$, so nothing needs to be proved. Now we assume $s \geq 2$.

From the discussion after Lemma 4.1.4, we have

$$\mathrm{Ann}(e) = \mathrm{Ann}(e, \tilde{e}) \oplus L \subset M.$$

For any closed point $x \in X_{(e_i)}$, we can write $x = (y, \omega) \in \mathrm{Spec}(\mathbb{C}[\mathrm{Ann}(e)])$ with $y \in \mathrm{Spec}(\mathbb{C}[\mathrm{Ann}(e, \tilde{e})])$, and $\omega \in \mathrm{Spec}(\mathbb{C}[L])$ respectively. We claim that there exists a morphism π:

$$\pi : X_{(e_i)} \to D$$

defined by $x \mapsto y$.

Indeed, by Lemma 4.1.4, we have a lattice decomposition of $\mathrm{Ann}(e)$ in M

$$\begin{aligned} &\mathrm{Ann}(e) \\ =\, &\mathrm{Ann}(e, \tilde{e}) \oplus \mathbb{Z}[w_{12}] \oplus \cdots \oplus \mathbb{Z}[w_{1n_1}] \oplus \cdots \oplus \mathbb{Z}[w_{r2}] \oplus \cdots \oplus \mathbb{Z}[w_{rn_r}]. \end{aligned}$$

By the construction of $A_k(y)$, the following matrix equation

$$\begin{pmatrix} A_1(y) & & & \\ & A_2(y) & & \\ & & \ddots & \\ & & & A_r(y) \end{pmatrix} \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \vdots \\ \mathbf{w}_r \end{pmatrix} = 0$$

gives the variety $X_{(e_i)}$, where $\mathbf{w}_k = (1, X^{w_{k2}}, \cdots, X^{w_{kn_k}})^t$.

Hence, for a closed point $(y, \omega) \in X_{(e_i)}$ with $y \in \mathrm{Spec}(\mathbb{C}[\mathrm{Ann}(e, \tilde{e})])$ and $\omega \in \mathrm{Spec}(\mathbb{C}[L])$, we have for all k, $A_k(y)\mathbf{w}_k = 0$. Because $\mathbf{w}_k \not\equiv 0$, we must have $\det(A_k(y)) = 0$. Hence, for all k, y lives in D_k, and thus $y \in D = \cap_{k=1}^r D_k$. This shows that the natural projection $\mathrm{Spec}(\mathbb{C}[\mathrm{Ann}(e)]) \to \mathrm{Spec}(\mathbb{C}[\mathrm{Ann}(e, \tilde{e})])$ maps $X_{(e_i)}$ to D. We denote this morphism by π.

Next, we show that π is generically injective, that is, π is injective on a nonempty open subset of $X_{(e_i)}$. Roughly speaking, the proof rests on the fact that a Calabi-Yau variety cannot be uniruled. We show that if π is not generically injective, then $X_{(e_i)}$ is a uniruled variety. However, we can construct a compactification $\overline{X_{(e_i)}}$ of $X_{(e_i)}$ which is a projective, Calabi-Yau variety with canonical, Gorenstein singularities. Put these facts together, and we get a contradiction. The details are given in the follows:

Suppose π is not generically injective. By a theorem of Chevalley ([10] Chapter II Ex.3.22(e)), there exists a nonempty open set $V \subset \pi(X_{(e_i)})$ such that over V, the fibres have the same dimension h. Let $y \in V$, and let $(X_{(e_i)})_y$ be the fibre over y. We claim that there exists a birational morphism

$$\theta_y : (X_{(e_i)})_y \to \mathbb{P}^h.$$

In fact, let

$$\Omega_y = \{\omega \in \mathrm{Spec}(\mathbb{C}[L]) \mid (y, \omega) \in (X_{(e_i)})_y\},$$

and let

$$W_y = \{ \sum_{i \in I, \#(I) < \infty} \lambda_i \omega_i \mid \omega_i \in \Omega_y, \lambda_i \in \mathbb{C}\}$$

be the affine subspace of $\mathrm{Spec}(\mathbb{C}[L])$ generated by Ω_y, where I is some finite index set. We claim that $\Omega_y \subset W_y$ is a dense open subvariety. To see this, notice by the matrix equation

$$\begin{pmatrix} A_1(y) & & & \\ & A_2(y) & & \\ & & \ddots & \\ & & & A_r(y) \end{pmatrix} \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \vdots \\ \mathbf{w}_r \end{pmatrix} = 0,$$

if

$$\sum_{i \in I, \#(I) < \infty} \lambda_i \neq 0 \quad \text{and} \quad \sum_{i \in I, \#(I) < \infty} \lambda_i \omega_i \in \mathrm{Spec}(\mathbb{C}[L]),$$

then we have

$$(y, \sum_{i \in I, \#(I) < \infty} \lambda_i \omega_i) \in (X_{(e_i)})_y.$$

However, the closed points in W_y which satisfy

$$\sum_{i \in I, \#(I) < \infty} \lambda_i \neq 0 \quad \text{and} \quad \sum_{i \in I, \#(I) < \infty} \lambda_i \omega_i \in (\mathbb{C}^*)^{s-r}$$

form an open variety, and this justifies the claim.

Now, $\dim \Omega_y = h$ implies $\dim W_y = h$, and the natural morphism

$$\theta_y : (X_{(e_i)})_y \hookrightarrow W_y \hookrightarrow \mathbb{P}^h$$

is a birational morphism. Moreover, if π is not generically injective, then $h \geq 1$, i.e. the general fibres of π are positive dimensional (one might need to pass to some small open subvariety of V in order to make the fibre contains distinct closed points). Then we can construct a birational morphism

$$\begin{aligned} \pi^{-1}(V) &\to V \times \mathbb{P}^h \\ (y, \omega) &\mapsto (y, \theta_y(\omega)). \end{aligned}$$

This shows that $X_{(e_i)}$ is a ruled variety and, in particular, a uniruled variety.

In the appendix (cf. Remark 6.0.18, Proposition 6.0.17), we construct a compactification $\overline{X_{(e_i)}}$ of $X_{(e_i)}$, such that $\overline{X_{(e_i)}}$ is a projective, Calabi-Yau variety with canonical, Gorenstein singularities. Let $\widetilde{X_{(e_i)}}$ be a desingularization of $\overline{X_{(e_i)}}$. It is also a uniruled variety. Because $\overline{X_{(e_i)}}$ is a Calabi-Yau variety with canonical singularities, the canonical divisor $K_{(e_i)}$ of $\widetilde{X_{(e_i)}}$ is

$$K_{(e_i)} = \sum c_j E_j, \quad c_j \geq 0,$$

where E_j are the exceptional divisors. Hence, $H^0(\widetilde{X_{(e_i)}}, \mathcal{O}(K_{(e_i)})) \neq 0$. However, because $\widetilde{X_{(e_i)}}$ is a smooth, proper uniruled variety over $\mathbb{C}$, we have $H^0(\widetilde{X_{(e_i)}}, \mathcal{O}(K_{(e_i)})) = 0$ ([15] IV Corollary 1.11). This is a contradiction, and hence π is generically injective.

Let $U \subset X_{(e_i)}$ be an open set where $\pi|_U$ is injective. Because for generic coefficients, $X_{(e_i)}$ is smooth of dimension $d - s$ (Proposition 6.0.14), $\pi(U)$ is a constructible subset of D with dimension $d - s$. This is the same dimension as D by assumption. Thus π is dominant as well. By Lemma 4.2.2, $X_{(e_i)}$ is birational to D.

A similar argument can be used to show that $X_{(\tilde{e}_i)}$ is birational to D as well. We sketch the argument below:

First, by the proof of Lemma 4.1.4, one has a decomposition of lattices

$$\begin{aligned}\mathrm{Ann}(\tilde{e}) = &\mathrm{Ann}(e,\tilde{e})\\ &\oplus \mathbb{Z}[u_{12}-u_{11}] \oplus \cdots \oplus \mathbb{Z}[u_{1n_1}-u_{11}]\\ &\oplus \cdots\\ &\oplus \mathbb{Z}[u_{r2}-u_{r1}] \oplus \cdots \oplus \mathbb{Z}[u_{rn_r}-u_{r1}].\end{aligned}$$

where u_{ki} is as defined in Section 4. We can view $u_{ki}-u_{k1}$ as w_{ki} when $i \geq 2$ because it satisfies the required relation of Lemma 4.1.4 (with $e_{ki}, \tilde{e}_{ki}$ switched), and this is enough for the existence of the decomposition. Correspondingly, we have a decomposition of the torus:

$$\mathrm{Spec}(\mathbb{C}[\mathrm{Ann}(\tilde{e})]) = \mathrm{Spec}(\mathbb{C}[\mathrm{Ann}(e,\tilde{e})]) \times (\mathbb{C}^*)^{s-r}.$$

We can similarly define $X_{(\tilde{e}_i)} \to D$ as before, and for the same reason, this is a birational morphism.

Hence $X_{(e_i)}$ and $X_{(\tilde{e}_i)}$ are both birational to D, and this completes the proof.

$\square$

Remark 4.3.2. It is necessary in our argument for D to be irreducible. When we consider the case $s = 2$, with $S_{2,1} = \emptyset$, we see that D is a union of zero loci of $g_{1,1}$ and $g_{2,2}$, where $g_{i,i} = \sum_{v \in l(S_{i,i})} c_v X^v, i = 1, 2$. By the proof of the theorem, we see that $X_{(e_i)}$ is birational to the zero locus of $g_{2,2}$, but $X_{(\tilde{e}_i)}$ is birational to the zero locus of $g_{1,1}$. A priori, one cannot expect that the two loci be birational.

There is a result due to Batyrev and Borisov ([3] Theorem 3.3) which asserts that $X_{(e_i)}$ is irreducible if the nef-partition is 2-independent. This means there exists *no* integer $n > 0$ nor any subset of the nef-partition $\{\Delta_{k_1}, \ldots, \Delta_{k_n}\} \subset \{\Delta_1, \ldots, \Delta_s\}$ such that $\dim(\Delta_{k_1} + \cdots + \Delta_{k_n}) \leq n$.

Remark 4.3.3. It is reasonable to require that $\dim D = \dim X_{(e_i)} = d - s$. Indeed $D = \cap_{i=1}^r D_i$ is a variety in $\mathrm{Spec}(\mathbb{C}[\mathrm{Ann}(e,\tilde{e})]) \cong (\mathbb{C}^*)^{d-(s-r)}$ defined by the intersection of r hypersurfaces. Thus D is expected to have dimension $d - s$ for generic choice of coefficients.

Chapter 5

Open questions: D-equivalence and K-equivalence

Let $D^b(Coh(X))$ be the derived category of bounded complexes of coherent sheaves on X. For smooth varieties X, Y, if $D^b(Coh(X))$ is equivalent to $D^b(Coh(Y))$ as derived categories, then X, Y are called D-equivalent.

Let K_X, K_Y be canonical divisors of X and Y respectively. If there exists a birational correspondence

$$X \xleftarrow{\pi_X} Z \xrightarrow{\pi_Y} Y$$

such that $\pi_X^* K_X \cong \pi_Y^* K_Y$, then X, Y are called K-equivalent. There is a surprising relation between D-equivalence and K-equivalence [11]. A theorem of Kawamata [13] says: if X, Y are projective smooth varieties of general type over an algebraically closed field, then X, Y are D-equivalent implies they are K-equivalent. We have the following conjecture of Kawamata [13]

Conjecture 5.0.4. If X, Y are smooth projective varieties, then X, Y are K-equivalent implies they are D-equivalent.

This conjecture has been settled for smooth Calabi-Yau threefolds [8] and toroidal varieties [14].

Back to the case considered in this paper. We have proved that $X_{(e_i)}, X_{(\tilde{e}_i)}$ are birational Calabi-Yau varieties, and their compactifications $\overline{X_{(e_i)}}$, $\overline{X_{(\tilde{e}_i)}}$ are automatically K-equivalent. According to the conjecture, we expect to have D-equivalence $D^b(Coh(\overline{X_{(e_i)}})) \cong D^b(Coh(\overline{X_{(\tilde{e}_i)}}))$.

Conjecture 5.0.5 ([6] Conjecture 5.3)**.** There exists an equivalence (of Fourier-Mukai type) between the derived category of coherent sheaves on the two Calabi-Yau complete intersections $\overline{X_{(e_i)}}$ and $\overline{X_{(\tilde{e}_i)}}$.

One might consider sheaves on smooth DM-stacks associated to $\overline{X_{(e_i)}}$, $\overline{X_{(\tilde{e}_i)}}$ because of the possible singularities. Moreover, when we consider the homological mirror symmetry conjecture, it is plausible to have such D-equivalence.

Chapter 6

Appendix: Δ-regularity, singularities and Calabi-Yau varieties

Roughly speaking, Δ-regularity is a condition on the smoothness of stratifications with correct dimension. In this appendix, we generalize the concept of Δ-regularity [1] [2] of a hypersurface to an intersection of several hypersurfaces in toric varieties. We will show that for general coefficients (meaning for a nonempty open set of the parameter space of coefficients), the complete intersections defined by a nef-partition are Δ-regular, and thus form a large family of intersections associated to a nef-partition. Under the Δ-regular assumption, the singularities of the complete intersection are inherited from the ambient toric variety. Using these results, we will show that an irreducible Δ-regular complete intersection associated to a nef-partition is a Calabi-Yau variety with canonical, Gorenstein singularities. This fact is used in the proof of Theorem 4.3.1 by showing that the morphism π is generically injective.

Let $\Sigma \subset N_{\mathbb{R}}$ be a fan, and $X(\Sigma)$ be the toric variety defined by Σ. If $\sigma \in \Sigma$ is a cone, let T_σ be the torus corresponding to σ. Then we have the following stratification:

$$X(\Sigma) = \bigcup_{\sigma \in \Sigma} T_\sigma \ .$$

Definition 6.0.6. Let $V_i, 1 \leq i \leq s$, be hypersurfaces of $X(\Sigma)$, and let $V = \bigcap_i^s V_i$ be the scheme-theoretic intersection. Then V is called Δ-regular if and only if V is equidimensional and $\forall\ \sigma \in \Sigma,\ T_\sigma \cap V$ is either empty or smooth of codimension s in T_σ.

Remark 6.0.7. The Δ-regular condition requires the linear independence of the cotangent spaces at a common intersection point. This takes care both of smoothness and of codimension.

We use the name Δ-regularity following Batyrev [1] [2], where Δ is a polytope, and the regularity is about a hypersurface defined by a Laurent polynomial with Newton polytope inside Δ.

One can consider the family of Δ-regular complete intersections associated to a nef-partition. In fact, let $\Delta \subset M_{\mathbb{R}}$ be a reflexive polytope with nef-partition $\{\Delta_i \mid 1 \leq i \leq s\}$, in particular, we have $\mathrm{Conv}(\cup_{i=1}^{s}\Delta_i) = \Delta$. Let $\{\nabla_i \mid 1 \leq i \leq s\}$ be the dual nef-partition, then

$$\nabla = \mathrm{Conv}(\bigcup_{i=1}^{s} \nabla_i) = (\sum_{i=1}^{s} \Delta_i)^{\vee}.$$

Let

$$\Sigma(\nabla) = \{0\} \cup \{\mathbb{R}_{\geq 0}\theta \mid \theta \text{ is a face of } \nabla\}$$

be a fan, and $X(\Sigma(\nabla))$ be the toric variety defined by fan $\Sigma(\nabla)$. One can show that $X(\Sigma(\nabla))$ is the same as the projective toric variety associated to the polytope $(\sum_{i=1}^{s} \Delta_i)$.

By the construction of a nef-partition, we have a nef torus invariant (Cartier) divisor $\mathcal{L}_i$:

$$\mathcal{L}_i = \sum_{\rho \in \mathrm{Vert}(\nabla_i)\setminus\{0\}} D_\rho$$

where D_ρ is the torus invariant divisor associated to the primitive element ρ.

One can identify the global sections of $\mathcal{L}_i$ with Laurent polynomials associated to Δ_i [9]:

$$H^0(X(\Sigma(\nabla)), \mathcal{L}_i) \cong \{\sum_{v \in l(\Delta_i)} c_v X^v \mid c_v \in \mathbb{C}\}.$$

Let $g_i = \sum_{v \in l(\Delta_i)} c_v X^v$, and let $V_i = (g_i)_0$ be the zero locus of g_i on $X(\Sigma(\nabla))$. Then

$$\{V = \bigcap_{i=1}^{s} V_i \mid V_i = (g_i)_0, g_i \in H^0(X(\Sigma(\nabla)), \mathcal{L}_i)\}$$

is a family of subschemes of $X(\Sigma(\nabla))$ parameterized by the coefficients of $g_i, 1 \leq i \leq s$. To show the general elements is Δ-regular, we first show that the general elements satisfy the requirement on the codimension for each T_σ.

Proposition 6.0.8. *For general coefficients $c_v \in \mathbb{C}$ of $g_i = \sum_{v \in l(\Delta_i)} c_v X^v$, $1 \le i \le s$, the scheme $T_\sigma \bigcap V = T_\sigma \bigcap (\cap_{i=1}^s V_i)$ is either empty or smooth of codimension s for every T_σ.*

Proof. Using the same notation as before. Because nefness and basepoint freeness are equivalent on toric varieties, the linear system $|\mathcal{L}_i|$ is basepoint free.

Next, we generalize Bertini's theorem ([10] III Corollary 10.9 and Remark 10.9.2) to show that for general coefficients, either $T_\sigma \cap V$ is empty or smooth of codimension s, where $\sigma \in \Sigma$. If the dimension of the linear system $|\mathcal{L}_i|$ is n_i, then together they define a morphism

$$f : T_\sigma \hookrightarrow X(\Sigma(\nabla)) \to \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}.$$

Let $\mathbb{P} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$, and we consider it as a homogeneous space under the action of $G := \mathrm{PGL}(n_1) \times \cdots \times \mathrm{PGL}(n_s)$. Let $H_i \to \mathbb{P}^{n_i}$ be the inclusion of a hyperplane $H_i \cong \mathbb{P}^{n_i - 1}$, and

$$g : H_1 \times \cdots \times H_s \to \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$$

be the product of these inclusions.

Next, we set $H := H_1 \times \cdots \times H_s$, and for $\tau \in G$, let H^τ be H with the morphism $\tau \circ g$ to $\mathbb{P}$. We can apply Kleiman's theorem ([10] III Theorem 10.8) to g and conclude that there exists a nonempty open set $W \subset G$, such that $\forall\, \tau \in W$, $T_\sigma \times_{\mathbb{P}} H^\tau$ is nonsingular and either empty or of codimension s. However, one can show that $f^{-1}(H^\tau)$ is exactly the scheme theoretic intersection $T_\sigma \cap V$ defined by the linear systems $|\mathcal{L}_i|, 1 \le i \le s$. This completes the proof.

□

Remark 6.0.9. It worth while to point out that not only the complete linear system $|\mathcal{L}_I|$ is basepoint free, but also the linear system $\{\sum_{v \in \mathrm{Vert}(\Delta_i)} c_v X^v \mid c_v \in \mathbb{C}\}$ is basepoint free, where $\mathrm{Vert}(\Delta_i)$ denotes the set of vertices of Δ_i.

Proposition 6.0.10. *For general coefficients, $V = \bigcap_{i=1}^s V_i$ is a reduced scheme.*

Proof. We follow the idea in the proof of [12] Theorem 6.3(3). For $1 \leq i \leq s$, let

$$n_i + 1 = \dim(H^0(X(\Sigma(\Delta)), \mathcal{L}_i)),$$

then we can choose a basis $f_0^{(i)}, \cdots, f_{n_i}^{(i)}$ of $H^0(X(\Sigma(\Delta)), \mathcal{L}_i)$. Let

$$Z := \{(x; u_0^{(0)}, \cdots, u_{n_0}^{(0)}; \cdots; u_0^{(s)}, \cdots, u_{n_s}^{(s)}) \mid \sum_{j=1}^{s} \sum_{k=0}^{n_j} u_k^{(j)} f_k^{(j)}(x) = 0\}$$

be a subscheme of $X \times \mathbb{C}^{n_0+1} \times \cdots \times \mathbb{C}^{n_s+1}$, and π be the natural projection:

$$\pi : Z \to \mathbb{C}^{n_0+1} \times \cdots \times \mathbb{C}^{n_s+1}.$$

Then Z is an integral scheme. Indeed, at any open affine variety $\operatorname{Spec} B \subset X(\Sigma(\Delta))$ where $f_k^{(j)}$ trivializes to $b_k^{(j)} \in B$, the inverse image under the projection $Z \to X$ is

$$\operatorname{Spec}\left(B[u_k^{(j)}]/(\sum_{j=1}^{s} \sum_{k=0}^{n_j} u_k^{(j)} b_k^{(j)}) \right) := \operatorname{Spec}(B').$$

Then, because $X(\Sigma(\Delta)$ is an integral variety, B is an integral domain, and hence B' is also an integral domain.

Let $Y := \overline{\pi(Z)}$ be the closure of scheme-theoretic image. By using the fact that $\operatorname{char}(\mathbb{C}) = 0$ and Z is integral, one can show that the fibre of $\pi : Z \to Y$ over the generic point η of Y is geometric reduced over the field $K(\eta)$, where $K(\eta)$ is the local ring at η. Because the set of points over which the fibres are geometric reduced is a constructible set in Y (see [12] Theorem 4.10), for general elements $\xi \in Y$, $\pi^{-1}(\xi)$ is geometric reduced, and in particular reduced. On the other hand, $\pi^{-1}(\xi) \cong \bigcap_{i=1}^{s} V_i$ where $V_i := \{\sum_{k=0}^{n_i} u_k^{(i)} f_k^{(i)}(x) = 0\}$. This shows that for general coefficients, V is a reduced scheme.

□

Next, we will show that if V have the property that $T_\sigma \bigcap V = T_\sigma \bigcap (\cap_{i=1}^{s} V_i)$ is either empty or smooth of codimension s, then the singularities of V is inherited from the singularities of the ambient toric variety.

First, recall that toric Gorenstein, canonical and terminal singularities are characterized by the combinatoric properties of cones [20] (See also [1]):

Proposition 6.0.11. *Let* $n_1, \ldots, n_r \in N$ *be primitive integral generators of all* 1-*dimensional faces of a cone* $\sigma \subset N_{\mathbb{R}}$.

1. U_σ *has Gorenstein singularity if and only if* $n_1, \ldots, n_r$ *are contained in an affine hyperplane*

$$H_\sigma := \{y \in N_{\mathbb{R}} \mid \langle k_\sigma, y \rangle = 1\},$$

for some $k_\sigma \in M$.

2. *Assume* U_σ *has Gorenstein singularity, then it has canonical singularity if and only if*

$$N \cap \sigma \cap \{y \in N_{\mathbb{R}} \mid \langle k_\sigma, y \rangle < 1\} = \{0\}.$$

3. *Assume* U_σ *has Gorenstein singularity, then it has terminal singularity if and only if*

$$N \cap \sigma \cap \{y \in N_{\mathbb{R}} \mid \langle k_\sigma, y \rangle \leq 1\} = \{0, n_1, \ldots, n_r\}.$$

Theorem 6.0.12. *Let* $X(\Sigma)$ *be the toric variety defined by a fan* Σ *and* $V := \bigcap_{i=1}^{s} V_i$ *be the intersection defined by nef-partitions. Suppose* $X(\Sigma)$ *has Gorenstein, canonical (resp. terminal) singularities, and* $T_\sigma \bigcap V$ *is either empty or smooth of codimension* s *for any* $\sigma \in \Sigma$, *then* V *also has Gorenstein, canonical (resp. terminal) singularities.*

Proof. For $\sigma \in \Sigma$, let T_σ be the torus corresponding to σ. Let $U_{\sigma,N}$ be the toric variety associated to the cone σ in the lattice N, and $N(\sigma)$ be the lattice $N \cap \mathbb{R}\sigma$. Then we have

$$U_{\sigma,N} \cong U_{\sigma,N(\sigma)} \times (\mathbb{C}^*)^{d-l}.$$

with $\operatorname{rank} N = d, \operatorname{rank} N(\sigma) = l$.

Under this identification, $T_\sigma \cong p_\sigma \times (\mathbb{C}^*)^{d-l}$, where $p_\sigma \in U_{\sigma,N(\sigma)}$ is the unique torus invariant point with coordinate $(0, \ldots, 0)$. Let $f_1, \ldots, f_s$ be the restriction of Δ-regular Laurent polynomials on $U_{\sigma,N}$. This should be understood as follows: since $U_{\sigma,N(\sigma)} = \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M(\sigma)]), (\mathbb{C}^*)^{d-l} = \operatorname{Spec}(\mathbb{C}[t_1^{\pm}, \ldots, t_{n-l}^{\pm}])$, $f_1, \ldots, f_s$ should be viewed as elements in $\mathbb{C}[x_1, \ldots, x_l; t_1^{\pm}, \ldots, t_{n-l}^{\pm}]$. By the Δ-regular assumption, if V_{f_i}

denotes the zero locus of f_i, for any $(0, \dots, 0; a_1, \dots, a_{n-l}) \in T_\sigma \cap V_{f_1} \cap \dots \cap V_{f_s}$, the Jacobian matrix

$$\left(\frac{\partial f_i}{\partial t_j}(0, \dots, 0; a_1, \dots, a_{n-l}) \right)_{ij}, \quad 1 \le i \le s, 1 \le j \le n-l$$

has rank $s = \dim(T_\sigma) - \dim(T_\sigma \cap V_{f_1} \cap \dots \cap V_{f_s})$. By continuity, in an analytic neighborhood of $(0, \dots, 0; a_1, \dots, a_{n-l}) \in \mathbb{C}^l \times (\mathbb{C}^*)^{n-l}$, the matrix

$$\left(\frac{\partial f_i}{\partial t_j}(x_1, \dots, x_l; a_1, \dots, a_{n-l}) \right)_{ij}, \quad 1 \le i \le s, 1 \le j \le n-l$$

has rank s. Without loss of generality, we can assume the $s \times s$ minor with $1 \le i \le s, l+1 \le j \le l+s$ is nonvanishing. Thus, we can apply the implicit function theorem to $f_1, \dots, f_s$. It shows that there are s analytic functions $u_1, \dots, u_s$ defined on an open neighborhood of $(0, \dots, 0; a_{s+1}, \dots, a_{n-l}) \in \mathbb{C}^l \times (\mathbb{C}^*)^{n-l-s}$ such that for points satisfying $f_1 = \dots = f_s = 0$ on $\mathbb{C}^l \times (\mathbb{C}^*)^{n-l}$, we have

$$f_i = f_i(x_1, \dots, x_l; u_1, \dots, u_s, t_{n-l-s+1}, \dots, t_{n-l}), \quad 1 \le i \le s.$$

Thus, when we restrict to a neighborhood of

$$(0, \dots, 0; a_1, \dots, a_{n-l}) \in U_{\sigma, N(\sigma)} \times (\mathbb{C}^*)^{n-l} \subset \mathbb{C}^l \times (\mathbb{C}^*)^{n-l},$$

it is locally, analytically isomorphic to a product of a neighborhood of $p_\sigma = (0, \dots, 0)$ in $U_{\sigma, N(\sigma)}$ with a neighborhood of $(a_{s+1}, \dots, a_{n-l})$ in $(\mathbb{C}^*)^{n-l-s}$. Moreover, Gorenstein singularity is a locally analytic property. This is because the completion of the local ring of a variety is the same as the completion of the local ring of the analytic space associated to that variety, and a local ring is Gorenstein if and only if its completion is Gorenstein. Likewise, canonical and terminal singularities are both local analytic property ([19] Proposition 4-4-4). Hence, we have proved the claim.

$\square$

Remark 6.0.13. The same argument also shows that the intersection V is normal, because $X(\Sigma)$ is normal, and normality is preserved under analytic isomorphism.

Next we show that a large family of intersections associated to nef-partition are Δ-regular intersections.

Theorem 6.0.14. *For general coefficients, $V = \bigcap_{i=1}^{s} V_i$ is a Δ-regular intersection.*

Proof. From Proposition 6.0.8, we know that for general coefficients, $V \cap T_\sigma$ is either empty or smooth of codimension s. Because V is also normal (see Remark 6.0.13), the irreducible components of V cannot intersect. By the proof of Theorem 6.0.12, we know that each component has dimension $n-s$ (because for any point on an irreducible component, we show that a neighborhood of that point is locally analytically isomorphism to an open neighborhood of dimension $n-s$). Finally, it is proved in Theorem 6.0.12 that the singularities of each component are canonical and Gorenstein . $\square$

Remark 6.0.15. From the above argument, one can show further that for general coefficients, and for any subset $I \subset \{1, 2, \cdots, s\}$, the scheme-theoretic intersection $\bigcap_{i \in I} V_i$ is Δ-regular.

In the last part of this section, we apply the adjunction formula to a Δ-regular complete intersection of a nef-partition to show that it has trivial canonical divisor (i.e. Calabi-Yau). As Proposition 6.0.14, we assume V to be a Δ-regular intersection associated to a nef-partition. First recall following proposition about the adjunction formula on a Cohen-Macaulay scheme ([16]Proposition 5.73).

Proposition 6.0.16. *Let P be a projective Cohen-Macaulay scheme of pure dimension n over a field k, and $D \subset P$ an effective Cartier divisor. Then $\omega_D \cong \omega_P(D) \otimes \mathcal{O}_D$. Here ω_D, ω_P are dualizing sheaves of D, P respectively.*

Applying this result and combining with Theorem 6.0.12, we have the following proposition.

Proposition 6.0.17. *If an irreducible variety V is an intersection of general elements of $|\mathcal{L}_i|, 1 \leq i \leq s$, then V is a Calabi-Yau variety (i.e. the canonical divisor $K_V = 0$).*

Proof. Let $V_i \in |\mathcal{L}_i|$ be the general element which is a Weil divisor and associates to the effective Cartier divisor $\mathcal{L}_i$. By definition (see Proposition 6.0.14), we have $V = \bigcap_i^s V_i$.

A Gorenstein ring is naturally Cohen-Macaulay, so $X := X(\Sigma(\nabla))$ is a Cohen-Macaulay scheme, and we can apply Proposition 6.0.16 to get

$$\begin{aligned}
&\omega_{V_1} \cong \omega_X(V_1) \otimes \mathcal{O}_{V_1} \\
&\omega_{V_1 \cap V_2} \cong \omega_X(V_1 + V_2) \otimes \mathcal{O}_{V_1 \cap V_2} \\
&\qquad \vdots \\
&\omega_V \cong \omega_X(V_1 + V_2 + \cdots + V_s) \otimes \mathcal{O}_V.
\end{aligned}$$

Because of the nef-partition, we have

$$-K_X \cong \sum_{i=1}^{s} V_s \quad .$$

We have

$$\omega_X(V_1 + V_2 + \cdots + V_s) \cong \mathcal{O}_X(-K_X + V_1 + V_2 + \cdots + V_s) \cong \mathcal{O}_X,$$

hence $\omega_V \cong \mathcal{O}_V$. On a normal variety, the dualizing sheaf is equivalent to the canonical sheaf ([16]Proposition 5.77). Using the fact that V is a normal variety, we have $K_V = 0$. This shows that V is a Calabi-Yau variety.

$\square$

In summary, we have proved that for general coefficients, the variety V associated to a nef-partition is a Δ-regular Calabi-Yau variety with canonical, Gorenstein singularities.

Remark 6.0.18. If the nef-partition $\{\Delta_i \mid 1 \leq i \leq s\}$ comes from $\deg^\vee = \sum_{i=1}^{s} e_i$ as in Section 3, then $V \cap (\mathbb{C}^*)^d = X_{(e_i)}$. In other words, V is a projective compactification of $X_{(e_i)}$, and we denote it by $\overline{X_{(e_i)}}$ in Theorem 4.3.1.

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