INTEGRAL FORMS FOR CERTAIN CLASSES OF VERTEX OPERATOR ALGEBRAS AND THEIR MODULES

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ABSTRACT OF THE DISSERTATION

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We study integral forms in vertex operator algebras over $\mathbb{C}$. We prove general results on when a multiple of the standard conformal vector ω can be added to an integral form of a vertex operator algebra and when intertwining operators among modules for a vertex operator algebra respect integral forms in the modules. We also show when the $\mathbb{Z}$-dual of an integral form in a module for a vertex operator algebra is an integral form in the contragredient module. As examples, we consider vertex operator algebras based on affine Lie algebras and even lattices, and tensor powers of the Virasoro vertex operator algebra $L(\frac{1}{2}, 0)$. In particular, we demonstrate vertex algebraic generators of integral forms for standard modules for affine Lie algebras that were first constructed in work of Garland; we reprove Borcherds' construction of integral forms in lattice conformal vertex algebras using generators; and we find generating sets that include ω for integral forms in tensor powers $L(\frac{1}{2}, 0)^{\otimes n}$ when $n \in 4\mathbb{Z}$. We also construct integral forms in modules for these vertex operator algebras using generating sets, and we show when intertwining operators among modules for these vertex operator algebras respect integral forms in the modules.

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Dedication

To my family in heaven, on earth, and yet to come

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Chapter 1
Introduction

The purpose of this dissertation is to develop the theory of integral forms in vertex operator algebras. Integral forms in lattice vertex algebras as well as related structures have appeared in [B], [P], [BR1], [DG], and [GL]; here, we continue the study of integral forms in lattice vertex algebras, using an approach based on generating sets. We also construct integral forms in vertex operator algebras based on affine Lie algebras; in the case that the affine Lie algebra is an untwisted affine Kac-Moody algebra, the integral forms we construct are identical to the ones constructed in [G], but here we show that they have vertex algebraic structure and have natural generating sets. As a third class of examples, we construct integral forms in tensor powers of the Virasoro vertex operator algebra $L(\frac{1}{2}, 0)$; in fact, for $n \in 4\mathbb{Z}$ we show that $L(\frac{1}{2}, 0)^{\otimes n}$ has integral forms which contain the standard conformal vector ω. These tensor power vertex operator algebras are significant because the moonshine module vertex operator algebra $V^\natural$ contains a subalgebra isomorphic to $L(\frac{1}{2}, 0)^{\otimes 48}$ ([DMZ]).

In addition to constructing integral forms in vertex operator algebras, we construct integral forms in modules for all three classes of vertex operator algebras studied in this dissertation. We also consider when intertwining operators among modules for a vertex operator algebra map integral forms into integral forms. These problems are important for understanding the representation theory of integral vertex algebras, as well as for constructing integral forms in vertex operator algebras which contain a subalgebra having an integral form. In this chapter, we recall the development of vertex operator algebra theory and previous work on integral forms in vertex operator algebras, as well as review the results of this dissertation.

1.1 Motivation for vertex (operator) algebras

Vertex operators first appeared in mathematics in the concrete construction of representations of affine Kac-Moody algebras, starting with [LW1], which led to a vertex-operator-theoretic proof of the Rogers-Ramanujan partition function identities in [LW2]. The algebraic notion of vertex algebra was first formulated by Borcherds in [B], motivated by the vertex-operator construction [FLM1] of the "moonshine module" $V^\natural$, a graded vector space

$$V^\natural = \coprod_{n \geq 0} V^\natural_{(n)}$$

acted on by the Monster group, the largest of the sporadic finite simple groups. Remarkably, the graded dimension

$$\dim_* V^\natural = q^{-1} \sum_{n \geq 0} \dim V^\natural_{(n)}\, q^n$$

of the moonshine module, with $q = e^{2\pi i \tau}$, is the Fourier series expansion of the J-function, a fundamental modular form defined on the upper half plane. The moonshine module is an example of a vertex algebra, and in fact it satisfies the axioms of the more refined notion of vertex operator algebra; the Monster group, which was first constructed by Griess in [Gr], was naturally constructed as the group of vertex operator algebra automorphisms of $V^\natural$ in [FLM1], [FLM2].

A vertex operator algebra is a $\mathbb{Z}$-graded vector space equipped with infinitely many (nonassociative) products $(u, v) \mapsto u_n v$ which are most naturally written in generating function form

$$Y(u, x)v = \sum_{n \in \mathbb{Z}} u_n v\, x^{-n-1}$$

with x a formal variable; the *vertex operator* $Y(\cdot, x)\cdot$ satisfies several axioms, most notably one called the Jacobi identity due to its similarity to the Lie algebra Jacobi identity (see [FLM2]). Additionally, a vertex operator algebra admits a representation of the Virasoro Lie algebra. Among the important examples of vertex operator algebras, besides the moonshine module, are those given by representations of the Virasoro and affine Lie algebras ([FZ]); also, any even positive-definite lattice L, that is, any discrete subgroup of Euclidean space with square lengths equal to even integers, induces a vertex operator algebra V_L ([B], [FLM2]). In fact, one of the early starting points for constructing the moonshine module was the vertex operator algebra based on the Leech lattice, which gives optimal lattice sphere-packing in 24 dimensions. There is also a notion of module for a vertex operator algebra, that is, a space on which a vertex operator algebra acts naturally.

A further motivation for vertex operator algebras comes from string theory in physics, where strings moving in spacetime sweep out Riemann surfaces. In a simple case of string theory, a vertex operator algebra gives the state space of a particle represented by a string, and the vertex operator $Y(u, z)v$, with the formal variable x replaced by a non-zero complex number z, describes how a string in state u interacts with a string in state v to form a third string. This interaction is represented by the sphere $\mathbb{C} \cup \{\infty\}$ with tubes coming in at 0 and z and going out at ∞. The connection between the geometry of string theory and the theory of vertex operator algebras is made precise in [H1], in particular, the analysis of local coordinates associated with the tubes. This connection also motivates a number of further deep results in vertex operator algebra theory, such as modular invariance in vertex operators associated to tori ([Z], [H2]).

Vertex operator algebras have continued to play a deep role in several areas of mathematics. Among many examples, it has been shown recently that certain categories of modules for vertex operator algebras have the structure of modular braided tensor categories (see the review article [HL2]). This is significant partly because modular tensor categories give rise to knot invariants. As another interesting application of vertex operator algebras, [L1], [L2] develop a connection between vertex operator algebras and certain ζ-function values, essentially Bernoulli numbers, developed further in [Mi] and [DLM]. As a third example, [CLM1] and [CLM2] used vertex operator algebras associated to the Lie algebra $\widehat{\mathfrak{sl}(2,\mathbb{C})}$ to understand how the sum sides of the Rogers-Ramanujan and Gordon-Andrews partition function identities arise from the graded dimensions of certain graded vector spaces.

1.2 Integral forms in vertex (operator) algebras

While vertex algebras are ordinarily assumed to be vector spaces over $\mathbb{C}$, or over any field of characteristic zero, the axioms, in particular the Jacobi identity, make sense for any commutative ring, and so it is natural to consider vertex algebras over $\mathbb{Z}$. In particular, it is natural to look for $\mathbb{Z}$-forms of vertex algebras over $\mathbb{C}$, by analogy with the construction of Lie algebras over $\mathbb{Z}$ using Chevalley bases.

One motivation for studying integral forms in vertex algebras is constructing vertex algebras over fields of prime characteristic p. Given an integral form $V_{\mathbb{Z}}$ in a vertex algebra V, one obtains a form $V_{\mathbb{F}_p}$ over the p-element field $\mathbb{F}_p$ by taking a base for $V_{\mathbb{Z}}$ as a basis for $V_{\mathbb{F}_p}$, and then reducing all the vertex operator structure constants for the basis elements mod p. Then if $\mathbb{K}$ is any field of characteristic p, V has a $\mathbb{K}$-form $V_{\mathbb{K}} = \mathbb{K} \otimes_{\mathbb{F}_p} V_{\mathbb{F}_p}$. Several papers have considered vertex algebras over fields of prime characteristic. The modular moonshine program of Borcherds and Ryba ([R], [BR1], [BR2]) used forms of the moonshine module over fields of odd prime characteristic (see also the recent paper [GL] in which integral forms of vertex algebras are used to prove a conjecture from [BR1]). The recent papers [DR1] and [DR2] study the representation theory of vertex operator algebras over arbitary fields, in particular the Virasoro vertex operator algebra $L(\frac{1}{2},0)$ over fields of odd prime characteristic; however, these papers do not use integral forms in $L(\frac{1}{2},0)$ to construct forms over fields of prime characteristic. Also, the paper [GL] uses integral forms in vertex operator algebras based on level 1 affine Lie algebras to obtain forms over finite fields, and studies the automorphism groups of such forms.

Another motivation for studying integral forms for vertex operator algebras is the moonshine module $V^\natural$, which, as mentioned above, has the Monster group as its automorphism group. A Monster-invariant $\mathbb{Z}$-form of $V^\natural$ was constructed in [DG] using the finiteness of the Monster. However, if a Monster-invariant $\mathbb{Z}$-form $V^\natural_\mathbb{Z}$ could be constructed without using the finiteness of the Monster, this would provide a new proof of the finiteness of the Monster. This is because each weight space of $V^\natural_\mathbb{Z}$ would be a finite-rank, Monster-invariant lattice. The work in Chapters 4 and 5 of these dissertation may prove useful in constructing such an integral form of $V^\natural$.

Integral forms in vertex algebras V_L based on even lattices were introduced in [B], where they were used to construct integral forms of the universal enveloping algebras of affine Lie algebras. The details of Borcherds' work in [B], including the proof that the integral forms in V_L actually are integral forms, were verified in [P]. More recently, integral forms in lattice vertex algebras were studied in [DG], where they were used to construct integral forms in related vertex operator algebras, in particular $V^\natural$, which are invariant under finite groups. The paper [GL], which constructs integral forms in level 1 affine Lie algebra vertex operator algebras, appeared after most of the work in Chapter 3 of this dissertation, on integral forms in arbitrary positive integral level affine Lie algebra modules, was completed. Thus there is some intersection between the work in [GL] and Chapter 3 of this dissertation, but the methods used here are different and apply to all positive integral levels, as well as to modules and intertwining operators.

In this dissertation, we continue the study of integral forms for vertex algebras, revisiting some known results with new methods and proving new results as well. In particular we construct integral forms in vertex (operator) algebras based on affine Lie algebras and lattices, and in tensor powers of the Virasoro vertex operator algebra $L(\frac{1}{2}, 0)$. Our approach is based on finding generators for integral forms of vertex algebras rather than on finding bases. One advantage of this new method is that it allows the construction of an integral form containing desired elements without knowledge of the full structure in advance. This is particularly useful in constructing integral forms in vertex algebras based on affine Lie algebras and in constructing integral forms containing the standard conformal vector ω generating the Virasoro algebra, where it may be difficult to find an explicit basis. Further, defining an integral form $V_{\mathbb{Z}}$ for a vertex algebra V to be the vertex subalgebra over $\mathbb{Z}$ generated by certain elements essentially reduces the problem of proving that $V_{\mathbb{Z}}$ is in fact an integral form of V to the problem of showing that it is an integral form of V as a vector space. In the case of vertex algebras based on lattices, this is easier than proving that an integral form of V as a vector space is also a vertex subalgebra, which is the method of proof used in [P] and [DG].

In this dissertation, we also construct integral forms in modules for vertex algebras which are invariant under the action of an integral form in the algebra, using generators as in the algebra case. In particular, we construct integral forms in modules for vertex (operator) algebras based on affine Lie algebras and lattices and for tensor powers of the Virasoro vertex operator algebra $L(\frac{1}{2}, 0)$. We also determine when intertwining operators among a triple of irreducible modules for these vertex operator algebras respect integral forms in the modules.

It does not seem that much work has been done previously on integral forms in modules and intertwining operators for vertex operator algebras, in spite of the fact that modules and intertwining operators are essential in vertex operator algebra theory. As one of many examples of the importance of modules and intertwining operators, in the connection of vertex operator algebras to string theory, the irreducible modules for a vertex operator algebra correspond to the state spaces of different kinds of strings in the theory, and intertwining operators describe how different kinds of strings interact with each other. Modules are also important because of the modular braided tensor category structure of certain categories of modules for vertex operator algebras ([HL2]), and intertwining operators play a crucial role in the construction of these tensor categories.

Here we remark on an interesting subtlety that we encounter in our consideration of intertwining operators that appears in all three classes of examples that we study in this dissertation. Suppose V is one of our example vertex operator algebras with integral form $V_\mathbb{Z}$ and $W^{(i)}$ for $i = 1, 2, 3$ are V-modules with integral forms $W^{(i)}_\mathbb{Z}$. Let $V^{W^{(3)}}_{W^{(1)}W^{(2)}}$ denote the vector space of intertwining operators

$$\mathcal{Y} : W^{(1)} \otimes W^{(2)} \to W^{(3)}\{x\}.$$

We find that if the $W^{(i)}$ are irreducible modules and the $W^{(i)}_\mathbb{Z}$ are generated by the action of $V_\mathbb{Z}$ on a vector of lowest conformal weight, then it is possible to find a lattice of intertwining operators $\mathcal{Y}$ contained in $V^{W^{(3)}}_{W^{(1)}W^{(2)}}$ which satisfy

$$\mathcal{Y} : W^{(1)}_\mathbb{Z} \otimes W^{(2)}_\mathbb{Z} \to ((W^{(3)})'_\mathbb{Z})'\{x\}.$$

Here $(W^{(3)})'_\mathbb{Z}$ is the integral form in the contragredient of $W^{(3)}$ generated by the action of $V_\mathbb{Z}$ on a vector of lowest conformal weight, and $((W^{(3)})'_\mathbb{Z})'$ refers to the graded $\mathbb{Z}$-dual of $(W^{(3)})'_\mathbb{Z}$, an integral form in $W^{(3)}$ which is generally larger than $W^{(3)}_\mathbb{Z}$. That is, we are not generally able to find non-zero intertwining operators which satisfy

$$\mathcal{Y} : W^{(1)}_\mathbb{Z} \otimes W^{(2)}_\mathbb{Z} \to W^{(3)}_\mathbb{Z}\{x\};$$

rather, we find intertwining operators that satisfy

$$\langle w'_{(3)}, \mathcal{Y}(w_{(1)}, x)w_{(2)}\rangle \in \mathbb{Z}\{x\}$$

for $w_{(1)} \in W^{(1)}_\mathbb{Z}$, $w_{(2)} \in W^{(2)}_\mathbb{Z}$, and $w'_{(3)} \in (W^{(3)})'_\mathbb{Z}$.

1.3 Summary of results

We now describe the contents of this dissertation in more detail. Chapter 2 recalls basic definitions and establishes general results on integral forms in vertex (operator) algebras and modules, while Chapters 3, 4, and 5 are devoted to the study of integral forms in specific examples of vertex (operator) algebras and modules.

We start Chapter 2 by recalling the definitions of vertex (operator) algebra and module for a vertex (operator) algebra. We also recall the notions of contragredient module from [FHL] and of strongly A-graded conformal vertex algebra, where A is an abelian group, from [HLZ1]. In Section 2.2 we define the notion of integral form in a vertex (operator) algebra, a strongly A-graded conformal vertex algebra, and their modules. We also include a general result on generators for vertex algebras over $\mathbb{Z}$ and their modules that will be heavily used in this dissertation for the construction of integral forms in examples of vertex (operator) algebras and their modules.

In Section 2.3, we prove results concerning when the standard conformal vector ω generating the action of the Virasoro algebra may be added to an integral form of a vertex operator algebra. Most significantly, we prove that if V is a vertex operator algebra with an integral form $V_{\mathbb{Z}}$ generated by homogeneous lowest weight vectors for the Virasoro algebra and if $k \in \mathbb{Z}$ is such that $k^2 c \in 2\mathbb{Z}$ (where c is the central charge of V) and $k\omega$ is in the $\mathbb{Q}$-span of $V_{\mathbb{Z}}$, then $V_{\mathbb{Z}}$ can be extended to an integral form of V containing $k\omega$.

In Section 2.4 we recall the definition of intertwining operator among three modules for a vertex operator algebra and define the notion of an intertwining operator which is integral with respect to integral forms in the three modules, that is, an intertwining operator which respects integral forms in the modules. We also prove an important result showing that to check whether or not an intertwining operator is integral, it is sufficient to check whether or not it is integral on generators. In Section 2.5 we give conditions showing when the graded $\mathbb{Z}$-dual of an integral form in a module for a vertex operator algebra is an integral form in the contragredient module. These results generalize some results in [DG], where the module is taken to be the vertex operator algebra itself. We also recall the notion of an invariant bilinear pairing between two modules and prove a result that will be used in Chapter 4 on identifying two modules as a contragredient pair via intertwining operators and invariant pairings.

Chapter 3 is devoted to the study of integral forms in vertex (operator) algebras and modules based on an affine Lie algebra $\widehat{\mathfrak{g}}$. In Section 3.1 we recall the notion of affine Lie algebra and recall how to construct vertex algebras and modules from $\widehat{\mathfrak{g}}$. We also recall the characterization of intertwining operators among irreducible modules for an affine Lie algebra vertex operator algebra in the case that $\mathfrak{g}$ is a finite-dimensional simple Lie algebra. In Section 3.2 we construct integral forms in integral level affine Lie algebra vertex (operator) algebras and their modules using integral forms in the universal enveloping algebra $U(\widehat{\mathfrak{g}})$. We take $\mathfrak{g}$ to be finite-dimensional simple in Section 3.3 and use the integral form of $U(\widehat{\mathfrak{g}})$ from [G] (see also [Mit] and [P]); we also exhibit natural sets of generators for the resulting integral forms in the vertex operator algebras and modules. In Section 3.4 we apply the results from Sections 2.3, 2.4, and 2.5 to affine Lie algebra vertex operator algebras and modules where $\mathfrak{g}$ is finite-dimensional simple. In particular, we characterize which intertwining operators among three modules are integral with respect to integral forms in the modules.

In Chapter 4 we study integral forms in vertex algebras V_L based on an even lattice L. In his research announcement [B], Borcherds defined an integral form for such a vertex algebra and exhibited a $\mathbb{Z}$-basis for this form. It has been proved in [P] and [DG] that this structure is in fact an integral form, essentially by showing that the vertex algebra product of any two members of Borcherds' $\mathbb{Z}$-basis is a $\mathbb{Z}$-linear combination of basis elements. In Section 4.2, we provide an alternate proof by defining the integral form to be the vertex subalgebra over $\mathbb{Z}$ generated by a natural generating set and then proving that the resulting structure is an integral form of the vector space V_L. We also show that our definition of the integral form is equivalent to the definition in [B]. Further, we construct integral forms in modules for lattice vertex algebras. This problem is slightly more subtle than the problem of constructing an integral form in the algebra due to the nature of a central extension of the dual lattice L° that is needed to construct modules for V_L.

In Section 4.3 we apply the results of Sections 2.3, 2.4, and 2.5 to the integral forms in lattice vertex algebras and their modules. We show that the standard integral form of a lattice vertex algebra V_L constructed in Section 4.2 contains ω if and only the lattice L is self-dual (the "if" direction appeared in [BR1]). More generally, the standard integral form can be extended to a larger integral form containing ω if the rank of L is even and containing 2ω if the rank of L is odd; this result was observed in [B]. We also exhibit $\mathbb{Z}$-bases for the integral forms of lattice vertex algebras, their modules, and their contragredients, generalizing to the module setting a result in [DG]. We also recall the intertwining operators among V_L-modules constructed in [DL] and show when they are integral with respect to integral forms in the V_L-modules.

In Chapter 5, we consider integral forms in tensor powers of the Virasoro vertex operator algebra $L(\frac{1}{2},0)$ and their modules. In Section 5.1 we recall the definition of the tensor product of two vertex operator algebras from [FHL] and prove a result on generators for the tensor product algebra and its modules. In Section 5.2 we recall the construction of vertex operator algebras based on the Virasoro algebra and their modules, in particular $L(\frac{1}{2},0)$ and its modules. We also show the existence of $\mathbb{Q}$-forms in irreducible Virasoro vertex operator algebras and modules. In Section 5.3 we first construct integral forms in arbitrary tensor powers $L(\frac{1}{2},0)^{\otimes n}$ using generators. In the case that $n \in 4\mathbb{Z}$, we construct different integral forms in $L(\frac{1}{2},0)^{\otimes n}$ containing ω with generators indexed by elements of an appropriate binary linear code on an n-element set. In Section 5.4, we construct integral forms in irreducible modules for $L(\frac{1}{2},0)^{\otimes n}$, and in Section 5.5 we characterize which intertwining operators among irreducible $L(\frac{1}{2},0)^{\otimes n}$-modules are integral with respect to integral forms in the modules. We conclude Section 5.5 by suggesting an approach to constructing interesting integral forms in vertex operator algebras which contain a vertex operator subalgebra isomorphic to $L(\frac{1}{2},0)^{\otimes n}$, such as the lattice vertex operator algebra V_{E_8} based on the E_8 root lattice and the moonshine module $V^\natural$.

Chapter 2

Definitions and general results

In this chapter we recall the definitions of vertex (operator) algebra and module for a vertex (operator) algebra and present our definition of an integral form in a vertex (operator) algebra or module. We also derive some general results on integral forms in vertex (operator) algebras and modules. In particular, we show when the standard conformal vector ω can be added to an integral form of a vertex operator algebra, when intertwining operators among modules for a vertex operator algebra respect integral forms in the modules, and when the contragredient of a module with an integral form has an integral form.

2.1 Vertex operator algebras and modules

In this section we state the definitions of vertex algebra, vertex operator algebra, and module for a vertex (operator) algebra and recall some basic properties. Given a vector space V, we will use the space of formal series

$$V[[x, x^{-1}]] = \left\{ \sum_{n\in\mathbb{Z}} v_n x^n,\ v_n \in V \right\},$$

as well as analogous spaces in several variables. We will commonly use the formal delta function series

$$\delta(x) = \sum_{n\in\mathbb{Z}} x^n$$

as well as three-variable analogues such as

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) = \sum_{n\in\mathbb{Z}} x_0^{-n-1}(x_1 - x_2)^n = \sum_{n\in\mathbb{Z}} \sum_{i\geq 0} \binom{n}{i} x_0^{-n-1} x_1^{n-i}(-x_2)^i.$$

Note that we always use the binomial expansion convention that expressions such as $(x_1 - x_2)^n$ (whenever n is not a non-negative integer) are expanded in non-negative integral powers of the second variable.

We now recall the definition of vertex algebra from [LL] (see also the original, equivalent, definition in [B]):

Definition 2.1.1. A *vertex algebra* $(V, Y, \mathbf{1})$ consists of a vector space V, a *vertex operator* map

$$\begin{array}{rccl} Y: & V & \rightarrow & (\mathrm{End}\, V)[[x, x^{-1}]] \\ & v & \mapsto & Y(v,x) = \sum_{n\in\mathbb{Z}} v_n\, x^{-n-1} \end{array}$$

and a distinguished vector $\mathbf{1} \in V$, called the *vacuum*. The data satisfy the following axioms:

1. *Lower truncation*: for any $u, v \in V$, $u_n v = 0$ for n sufficiently positive, that is, $Y(u,x)v$ has finitely many negative powers of x.

2. The *vacuum property*:
$$Y(\mathbf{1}, x) = 1_V.$$

3. The *creation property*: for any $v \in V$, $Y(v,x)\mathbf{1}$ has no negative powers of x and its constant coefficient is v.

4. The *Jacobi identity*: for $u, v \in V$,
$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u,x_1)Y(v,x_2) - x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) Y(v,x_2)Y(u,x_1)$$
$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u,x_0)v, x_2).$$
Note that we need lower truncation for each term in this expression to be well defined.

If we take the coefficient of x_0^{-1} in the Jacobi identity, we get the *commutator formula*

$$[Y(u,x_1), Y(v,x_2)] = \mathrm{Res}_{x_0}\, x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u,x_0)v, x_2), \qquad (2.1)$$

where the notation Res_{x_0} means take the coefficient of x_0^{-1}, the formal residue. If we take the coefficient of x_1^{-1} in the Jacobi identity, we get the *iterate formula*

$$\begin{aligned} Y(Y(u,x_0)v,x_2) = \mathrm{Res}_{x_1} \bigg(& x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) Y(u,x_1)Y(v,x_2) \\ & - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right) Y(v,x_2)Y(u,x_1) \bigg). \end{aligned} \tag{2.2}$$

If in the iterate formula we further take the coefficient of x_0^{-n-1}, we get

$$Y(u_n v, x_2) = \mathrm{Res}_{x_1}\left((x_1-x_2)^n Y(u,x_1)Y(v,x_2) - (-x_2+x_1)^n Y(v,x_2)Y(u,x_1)\right).$$

Together, the commutator and iterate formulas are equivalent to the Jacobi identity (see for instance [LL]).

A vertex operator algebra has two similar properties which are also together equivalent to the Jacobi identity. The first is *weak commutativity*: for any $u, v \in V$ and for any positive integer k such that $u_n v = 0$ for $n \geq k$,

$$(x_1-x_2)^k Y(u,x_1)Y(v,x_2) = (x_1-x_2)^k Y(v,x_2)Y(u,x_1). \tag{2.3}$$

The second property is *weak associativity*: for any $u, v, w \in V$ and for any positive integer l such that $u_n w = 0$ for $n \geq l$,

$$(x_0+x_2)^l Y(Y(u,x_0)v,x_2)w = (x_0+x_2)^l Y(u,x_0+x_2)Y(v,x_2)w. \tag{2.4}$$

See [LL] for proofs of these properties; we will prove weak commutativity in Section 2.4, where we will need a generalization for intertwining operators among V-modules.

Now we state the definition of module for a vertex algebra V, which is a vector space on which V acts in such a way that all the axioms for an algebra that make sense hold:

Definition 2.1.2. A V-*module* (W, Y_W) is a vector space W equipped with a *vertex operator* map

$$\begin{aligned} Y_W :\ & V \to (\mathrm{End}\, W)[[x, x^{-1}]] \\ & v \mapsto Y_W(v,x) = \textstyle\sum_{n\in\mathbb{Z}} v_n\, x^{-n-1} \end{aligned}$$

satisfying the following axioms:

1. *Lower truncation*: for any $v \in V$, $w \in W$, $v_n w = 0$ for n sufficiently positive, that is, $Y_W(v,x)w$ has finitely many negative powers of x.

2. The *vacuum property*:
$$Y_W(\mathbf{1},x) = 1_W.$$

3. The *Jacobi identity*: for $u, v \in V$,
$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_W(u,x_1)Y_W(v,x_2) - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)Y_W(v,x_2)Y_W(u,x_1)$$
$$= x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_W(Y(u,x_0)v,x_2). \tag{2.5}$$

As in the algebra case, we have commutator and iterate formulas for the V-module W, as well as weak commutativity and weak associativity.

Most of the vertex algebras we shall study satisfy the axioms of the more refined notion of vertex operator algebra. Here we recall the definition of vertex operator algebra from [FLM2]:

Definition 2.1.3. A *vertex operator algebra* $(V, Y, \mathbf{1}, \omega)$ is a vertex algebra $(V, Y, \mathbf{1})$ where V is a $\mathbb{Z}$-graded vector space
$$V = \coprod_{n\in\mathbb{Z}} V_{(n)}$$
equipped with a *conformal vector* $\omega \in V_{(2)}$. In addition to the vertex algebra axioms, V satisfies the following additional properties:

1. The *grading restrictions*: $V_{(n)} = 0$ for n sufficiently negative and $\dim V_{(n)} < \infty$ for all n.

2. The *Virasoro algebra relations*: if we write $Y(\omega,x) = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}$, then for any $m, n \in \mathbb{Z}$,
$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}c,$$

where $c \in \mathbb{C}$ is the *central charge* of V. Moreover, for $v \in V_{(n)}$, $L(0)v = nv$ (n is called the *conformal weight* of v).

3. The $L(-1)$-*derivative property*: for any $v \in V$,

$$Y(L(-1)v, x) = \frac{d}{dx}Y(v, x).$$

We say that a vertex operator algebra is graded by conformal weights, and if $v \in V_{(n)}$, we say that wt $v = n$.

We have a notion of module for a vertex operator algebra V:

Definition 2.1.4. A V-*module* is a module for V as vertex algebra such that W is graded by $L_W(0)$-eigenvalues:

$$W = \coprod_{h \in \mathbb{C}} W_{(h)}$$

where $L_W(0)w = hw$ for $w \in W_{(h)}$, and such that the *grading restrictions* hold: for any $h \in \mathbb{C}$, $W_{(h+n)} = 0$ for $n \in \mathbb{Z}$ sufficiently negative, and $\dim W_{(h)} < \infty$ for any $h \in \mathbb{C}$.

As in the algebra case, we say that the conformal weight of a vector $w \in W_{(h)}$ is h. The Virasoro algebra relations

$$[L_W(m), L_W(n)] = (m-n)L_W(m+n) + \frac{m^3 - m}{12}\delta_{m+n,0}c$$

for $m, n \in \mathbb{Z}$ and the $L(-1)$-derivative property

$$Y_W(L(-1)v, x) = \frac{d}{dx}Y_W(v, x)$$

for any $v \in V$ are consequences of the Virasoro algebra relations and the $L(-1)$-derivative property for algebras, as well as the Jacobi identity for modules (see [LL]).

Remark 2.1.5. If the context is clear, we shall generally drop the W subscript from Y_W and $L_W(n)$.

Remark 2.1.6. The Jacobi identity (2.5) for a V-module W with u taken to be ω and $v \in V$, $w \in W$ both homogeneous implies that

$$\mathrm{wt}\, v_n w = \mathrm{wt}\, v + \mathrm{wt}\, w - n - 1 \tag{2.6}$$

for any $n \in \mathbb{Z}$. Note that this relation holds in particular when $W = V$.

If V is any vertex operator algebra and W is any V-module, the *contragredient* of W is the V-module

$$W' = \coprod_{h \in \mathbb{C}} W_{(h)}^*$$

with vertex operator map given by

$$\langle Y_{W'}(v, x)w', w\rangle = \langle w', Y_W^o(v, x)w\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between a vector space and its dual, and the *opposite vertex operator* $Y_W^o(v, x)$ is given by

$$Y_W^o(v, x) = Y_W(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}). \tag{2.7}$$

Note that $Y_W^o(v, x)w$ has finitely many positive powers of x for any $v \in V$, $w \in W$. See [FHL] for the proof that this gives a V-module structure on W'. Moreover, for any V-module W, $W \cong (W')'$, so that it makes sense to refer to contragrdient pairs of V-modules.

We can weaken the notion of vertex operator algebra by dropping the grading restriction conditions; such a structure is called a *conformal vertex algebra* in [HLZ1]. The specific conformal vertex algebras and their modules that we will study in this thesis have an additional grading by an abelian group, in addition to the weight grading:

Definition 2.1.7. ([HLZ1]) A *strongly A-graded conformal vertex algebra*, where A is an abelian group, is a conformal vertex algebra V with an A-grading

$$V = \coprod_{\alpha \in A} V^\alpha$$

that satisfies the following axioms:

1. Compatibility with the weight grading: for any $\alpha \in A$,

$$V^{\alpha} = \coprod_{n \in \mathbb{Z}} V^{\alpha}_{(n)},$$

where $V^{\alpha}_{(n)} = V^{\alpha} \cap V_{(n)}$ for $n \in \mathbb{Z}$.

2. The grading restriction conditions: for any fixed $\alpha \in A$, $V^{\alpha}_{(n)} = 0$ for n sufficiently negative and $\dim V^{\alpha}_{(n)} < \infty$ for any n.

3. $\mathbf{1} \in V^{0}_{(0)}$ and $\omega \in V^{0}_{(2)}$.

4. For any $v \in V^{\alpha}$ and $\beta \in A$,

$$Y(v, x)V^{\beta} \subseteq V^{\alpha+\beta}[[x, x^{-1}]]. \tag{2.8}$$

Definition 2.1.8. ([HLZ1]) A *module* for a strongly A-graded conformal vertex algebra V is a module W for V as conformal vertex algebra with a B-grading

$$W = \coprod_{\beta \in B} W^{\beta},$$

where B is an abelian group containing A, satisfying the following axioms:

1. Compatibility with the weight grading: for any $\beta \in B$,

$$W^{\beta} = \coprod_{h \in \mathbb{C}} W^{\beta}_{(h)},$$

where $W^{\beta}_{(h)} = W^{\beta} \cap W_{(h)}$ for $h \in \mathbb{C}$.

2. The grading restriction conditions: for any fixed $\beta \in B$, $W^{\beta}_{(h)} = 0$ for any $h \in \mathbb{C}$ and n sufficiently negative, and $\dim W^{\beta}_{(h+n)} < \infty$ for any h.

3. For any $v \in V^{\alpha}$ and $\beta \in B$,

$$Y_W(v, x)W^{\beta} \subseteq W^{\alpha+\beta}[[x, x^{-1}]]. \tag{2.9}$$

Remark 2.1.9. Note that a vertex operator algebra V is a strongly A-graded conformal vertex algebra with $A = 0$, and a V-module is a module for V as strongly A-graded conformal vertex algebra with $B = 0$.

2.2 Basic properties of vertex rings

The notions of vertex algebra and module for a vertex algebra over $\mathbb{Z}$ make sense because all numerical coefficients in the formal delta functions appearing in the Jacobi identity are integers. For convenience, we will call vertex algebras over $\mathbb{Z}$ *vertex rings*, and we will call $\mathbb{Z}$-subalgebras of vertex algebras *vertex subrings*. Recall that an integral form of a vector space V is a free abelian group $V_{\mathbb{Z}}$ such that the canonical map

$$\mathbb{C} \otimes_{\mathbb{Z}} V_{\mathbb{Z}} \to V,$$

given by

$$c \otimes_{\mathbb{Z}} v \mapsto cv$$

for $c \in \mathbb{C}$ and $v \in V$, is an isomorphism. That is, $V_{\mathbb{Z}}$ is the $\mathbb{Z}$-span of a basis of V. We can now define an integral form in a vertex algebra over $\mathbb{C}$ (or over any field of characteristic zero), and the definition of an integral form in a module for a vertex algebra is analogous:

Definition 2.2.1. If V is a vertex algebra, an *integral form* of V is a vertex subring $V_{\mathbb{Z}}$ of V which is an integral form of V as a vector space. If V is a vertex algebra with integral form $V_{\mathbb{Z}}$ and W is a V-module, an *integral form* of W is a $V_{\mathbb{Z}}$-submodule of W which is an integral form of W as a vector space.

Remark 2.2.2. An integral form $V_{\mathbb{Z}}$ of a vertex algebra V is the $\mathbb{Z}$-span of a basis for V, it contains $\mathbf{1}$, and it is closed under vertex algebra products. Likewise, an integral form $W_{\mathbb{Z}}$ of a V-module W is the $\mathbb{Z}$-span of a basis for W and it is preserved by vertex operators from $V_{\mathbb{Z}}$. Note that the notion of an integral form in W depends on the precise integral form $V_{\mathbb{Z}}$ used for V.

If V is also a vertex operator algebra or conformal vertex algebra, and so has a conformal element ω, we do not require an integral form of V to contain ω. Such a requirement would disallow many interesting integral forms. However, in this dissertation, we will require an integral form $V_{\mathbb{Z}}$ of V to be compatible with the weight grading:

$$V_{\mathbb{Z}} = \coprod_{n\in\mathbb{Z}} V_{(n)} \cap V_{\mathbb{Z}}, \tag{2.10}$$

where $V_{(n)}$ is the weight space with $L(0)$-eigenvalue n. Moreover, if V is a strongly A-graded conformal vertex algebra, we require $V_{\mathbb{Z}}$ to be compatible with the $A \times \mathbb{C}$-gradation:

$$V_{\mathbb{Z}} = \coprod_{\alpha\in A,\, n\in\mathbb{Z}} V_{(n)}^{\alpha} \cap V_{\mathbb{Z}}. \tag{2.11}$$

We require the analogues of these compatibility conditions for modules for a conformal vertex algebra or strongly A-graded conformal vertex algebra. In particular, if V is a strongly A-graded conformal vertex algebra with integral form $V_{\mathbb{Z}}$ and W is a V-module, an integral form $W_{\mathbb{Z}}$ of W should satisfy

$$W_{\mathbb{Z}} = \coprod_{\beta\in B,\, h\in\mathbb{C}} W_{(h)}^{\beta} \cap W_{\mathbb{Z}}. \tag{2.12}$$

Note the special case $B = 0$ for when V is a vertex operator algebra.

Remark 2.2.3. It is possible to assume different requirements on the relation of ω to $V_{\mathbb{Z}}$; for instance, in [DG] it is required that some integer multiple of ω be in $V_{\mathbb{Z}}$.

We include two useful general results on vertex rings in this section. The proof of the algebra part of the next proposition is exactly the same as the proof of Proposition 3.9.3 in [LL], but we include it here because the result is essential for the construction of integral forms in vertex (operator) algebras and modules from generating sets:

Proposition 2.2.4. *Suppose V is a vertex algebra; for a subset S of V, denote by $\langle S\rangle_{\mathbb{Z}}$ the vertex subring generated by S. Then $\langle S\rangle_{\mathbb{Z}}$ is the $\mathbb{Z}$-span of coefficients of products of the form*

$$Y(u_1, x_1) \dots Y(u_k, x_k)\mathbf{1} \tag{2.13}$$

where $u_1, \ldots u_k \in S$. Moreover, if W is a V-module, the $\langle S\rangle_{\mathbb{Z}}$-submodule generated by a subset T of W is the $\mathbb{Z}$-span of coefficients of products of the form

$$Y(u_1, x_1)\ldots Y(u_k, x_k)w \tag{2.14}$$

where $u_1, \ldots u_k \in S$ and $w \in T$.

Proof. Let K denote the $\mathbb{Z}$-span of coefficients of products of the form (2.13). Since any vertex subring must contain $\mathbf{1}$, it is clear that $K \subseteq \langle S\rangle_{\mathbb{Z}}$. To prove the opposite inclusion, K contains $\mathbf{1}$ and it contains S by the creation property, so it suffices to show that K is a vertex subring of V. Let K' denote the set of vectors $u \in K$ such that

$$Y(u, x)v \in K[[x, x^{-1}]]$$

for all $v \in K$. To show that $K \subseteq K'$, we see from the definition of K that it is enough to show that $\mathbf{1} \in K'$ and that if $v \in K'$, so are the coefficients of $Y(u, x)v$ for any $u \in S$. In fact, $\mathbf{1} \in K'$ because $Y(\mathbf{1}, x) = 1_V$ and if $u \in S$, $v \in K'$, the iterate formula (2.2) implies that

$$Y(Y(u, x_0)v, x_2)w \in K[[x_0, x_0^{-1}, x_2, x_2^{-1}]]$$

for any $w \in K$, since all numerical coefficients in the δ-function expressions are integers. This implies that $Y(u, x_0)v \in K'[[x_0, x_0^{-1}]]$, as desired. Thus $K = K'$ and $K = \langle S\rangle_{\mathbb{Z}}$.

The proof of the second assertion is similar. Let L denote the $\mathbb{Z}$-span of coefficients of products as in (2.14). Then L is contained in the $\langle S\rangle_{\mathbb{Z}}$-module generated by T, and since $T \subseteq L$, it suffices to show that L is an $\langle S\rangle_{\mathbb{Z}}$-submodule of W. Let $L' \subseteq \langle S\rangle_{\mathbb{Z}}$ denote the set of vectors v such that

$$Y(v, x)w \subseteq L[[x, x^{-1}]]$$

for all $w \in L$. Then $\mathbf{1} \in L'$ and the iterate formula for modules implies that if $v \in L'$, so are the coefficients of $Y(u, x)v$ for $u \in S$. By the first assertion of the proposition, this implies $L' = \langle S\rangle_{\mathbb{Z}}$, showing that L is the $\langle S\rangle_{\mathbb{Z}}$-submodule of V generated by T. $\square$

Remark 2.2.5. Proposition 2.2.4 applies even if $\langle S\rangle_{\mathbb{Z}}$ and the $\langle S\rangle_{\mathbb{Z}}$-submodule generated by T are not integral forms of their respective vector spaces.

Proposition 2.2.6. *If V is a vertex algebra with integral form $V_{\mathbb{Z}}$, then $V_{\mathbb{Z}}\cap\mathbb{C}\mathbf{1} = \mathbb{Z}\mathbf{1}$.*

Proof. Since $\mathbf{1} \in V_{\mathbb{Z}}$, it is clear that $\mathbb{Z}\mathbf{1} \subseteq V_{\mathbb{Z}}\cap\mathbb{C}\mathbf{1}$. On the other hand, since $V_{\mathbb{Z}}$ is the $\mathbb{Z}$-span of a basis $\{v_i\}$ for V, $\mathbf{1} = \sum_i n_i v_i$ where $n_i \in \mathbb{Z}$. If $c\mathbf{1} = \sum_i cn_i v_i \in V_{\mathbb{Z}} \cap \mathbb{C}\mathbf{1}$ for $c \in \mathbb{C}$, then $cn_i \in \mathbb{Z}$ for each i, so $c \in \mathbb{Q}$. If $c \notin \mathbb{Z}$, by subtracting off an integer multiple of $\mathbf{1}$ from $c\mathbf{1}$ if necessary, we may assume $0 < c < 1$. Since $V_{\mathbb{Z}}$ is closed under vertex operators and $Y(\mathbf{1}, x) = 1_V$, we see that $c^n\mathbf{1} \in V_{\mathbb{Z}}$ for all $n \geq 0$, contradicting the assumption that $V_{\mathbb{Z}}$ is an integral form of V. Thus $c \in \mathbb{Z}$. $\square$

2.3 The conformal vector in an integral form

As was remarked in the previous section, we do not require an integral form of a vertex operator algebra to contain the conformal vector ω. In this section we prove results showing when an integral form can contain ω and when ω can be added to an existing integral form to give a larger one.

Suppose A is an abelian group and V is a strongly A-graded conformal vertex algebra with conformal vector ω and central charge $c \in \mathbb{C}$. Then we have the following result on when an integral form $V_{\mathbb{Z}}$ of V can contain ω:

Proposition 2.3.1. *If $V_{\mathbb{Z}}$ contains $k\omega$ where $k \in \mathbb{C}$, then $k^2c \in 2\mathbb{Z}$.*

Proof. If $k\omega \in V_{\mathbb{Z}}$, then $V_{\mathbb{Z}}$ must also contain

$$(kL(2))(k(L(-2))\mathbf{1} = k^2L(-2)L(2)\mathbf{1} + 4k^2L(0)\mathbf{1} + k^2\frac{c(2^3-2)}{12}\mathbf{1} = \frac{k^2c}{2}\mathbf{1}.$$

By Proposition 2.2.6, we must have $k^2c \in 2\mathbb{Z}$. $\square$

In particular, the central charge of V must be an even integer if ω is in any integral form of V. Now we prove a partial converse to Proposition 2.3.1. Recall that $v \in V$ is called a *lowest weight vector* for the Virasoro algebra if it is an $L(0)$-eigenvector and $L(n)v = 0$ for $n > 0$.

Theorem 2.3.2. *Suppose $V_{\mathbb{Z}}$ is an integral form of V generated by doubly homogeneous lowest weight vectors $\{v^{(j)}\}$ for the Virasoro algebra. If $k \in \mathbb{Z}$ is such that $k^2c \in 2\mathbb{Z}$ and $k\omega$ is in the $\mathbb{Q}$-span of $V_{\mathbb{Z}}$, then $V_{\mathbb{Z}}$ can be extended to an integral form of V containing $k\omega$.*

Proof. We shall show that the vertex subring $V_{\mathbb{Z}}^*$ of V generated by $V_{\mathbb{Z}}$ and $k\omega$ is an integral form of V. By Proposition 2.2.4, $V_{\mathbb{Z}}^*$ is spanned over $\mathbb{Z}$ by coefficients of products as in (2.13) where the u_i are either $v^{(j)}$ or $k\omega$. Since the $v^{(j)}$ and $k\omega$ are homogeneous in the $A \times \mathbb{Z}$-gradation of V, (2.6) and (2.8) imply that

$$V_{\mathbb{Z}}^* = \coprod_{\alpha \in A,\, n \in \mathbb{Z}} V_{\mathbb{Z}}^* \cap V_{(n)}^{\alpha}.$$

To show that $V_{\mathbb{Z}}^*$ is an integral form of the vector space V, it is enough to show that for any $\alpha \in A$ and $n \in \mathbb{Z}$, $V_{\mathbb{Z}}^* \cap V_{(n)}^{\alpha}$ is a lattice in $V_{(n)}^{\alpha}$ whose rank is the dimension of $V_{(n)}^{\alpha}$. In fact, if $V_{\mathbb{Z}}^* \cap V_{(n)}^{\alpha}$ is a lattice, then its rank is at least the dimension of $V_{(n)}^{\alpha}$ since $V_{\mathbb{Z}}^*$ contains $V_{\mathbb{Z}}$, which already spans V over $\mathbb{C}$. In addition, since $k\omega \in V_{\mathbb{Q}}$, $V_{\mathbb{Z}}^* \subseteq V_{\mathbb{Q}}$ as well. This means that if $V_{\mathbb{Z}}^* \cap V_{(n)}^{\alpha}$ is a lattice, its rank is at most the dimension of $V_{(n)}^{\alpha}$: any vectors in $V_{\mathbb{Z}}^* \cap V_{(n)}^{\alpha}$ which are linearly independent over $\mathbb{Z}$ are also linearly independent over $\mathbb{Q}$ since a dependence relation over $\mathbb{Q}$ reduces to a dependence relation over $\mathbb{Z}$ by clearing denominators. Thus we are reduced to showing that $V_{\mathbb{Z}}^* \cap V_{(n)}^{\alpha}$ is a lattice in $V_{(n)}^{\alpha}$ for any $\alpha \in A$ and $n \in \mathbb{Z}$.

Lemma 2.3.3. *For any $m, n \in \mathbb{Z}$ and lowest weight vector v, $[L(m), v_n]$ is an integral linear combination of operators v_k for $k \in \mathbb{Z}$.*

Proof. By the commutator formula (2.1),

$$\begin{aligned}[Y(\omega,x_1),Y(v,x_2)] &= \mathrm{Res}_{x_0}\, x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(\omega,x_0)v,x_2)\\ &= \mathrm{Res}_{x_0}\, e^{-x_0\,\partial/\partial x_1}\left(x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\right)\sum_{n\in\mathbb{Z}}Y(L(n)v,x_2)x_0^{-n-2}\\ &= \sum_{n\geq -1}(-1)^{n+1}\left(\frac{\partial}{\partial x_1}\right)^{n+1}\left(x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\right)Y(L(n)v,x_2).\end{aligned}$$

Since v is a lowest weight vector, and using the $L(-1)$-derivative property,

$$[Y(\omega,x_1),Y(v,x_2)] = x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\frac{d}{dx_2}Y(v,x_2)-(\mathrm{wt}\ v)\frac{\partial}{\partial x_1}\left(x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\right)Y(v,x_2)$$

Since wt $v\in\mathbb{Z}$, all coefficients in the delta function expressions are integers, and all powers of x_1 and x_2 are integers, we see that $[L(m),v_n]$ is an integral combination of operators v_k. □

Corollary 2.3.4. *For $m\geq -1$, $L(m)$ leaves $V_{\mathbb{Z}}$ invariant.*

Proof. By Lemma 2.3.3, an expression $L(m)v_{n_1}^{(j_1)}\cdots v_{n_k}^{(j_k)}\mathbf{1}$ equals $v_{n_1}^{(j_1)}\cdots v_{n_k}^{(j_k)}L(m)\mathbf{1}$ plus an integral linear combination of terms of the form $v_{m_1}^{(j_1)}\cdots v_{m_k}^{(j_k)}\mathbf{1}$. But since $m\geq -1$, $L(m)\mathbf{1}=0$, so we see that $L(m)V_{\mathbb{Z}}\subseteq V_{\mathbb{Z}}$. □

Continuing with the proof of Theorem 2.3.2, we can use Lemma 2.3.3 to rewrite any product of operators of the form $u_{m_1}^{(1)}\cdots u_{m_k}^{(k)}$, where the $u^{(i)}$ equal either $v^{(j)}$ or $k\omega$, as an integral combination of such products in which all operators of the form $kL(m)$ appear on the left. That is, $V_{\mathbb{Z}}^*$ is the integral span of products of the form

$$(kL(m_1))\cdots(kL(m_j))v_{n_1}^{(j_1)}\cdots v_{n_k}^{(j_k)}\mathbf{1},$$

where $m_i,n_i\in\mathbb{Z}$. We can now use the Virasoro algebra relations,

$$[kL(m),kL(n)] = k(m-n)(kL(m+n))+\frac{k^2c(m^3-m)}{12}1_V, \tag{2.15}$$

to rewrite $(kL(m_1))\cdots(kL(m_j))$ as an integral combination of such products for which $m_1\leq\ldots\leq m_j$. Note that since by assumption $k^2c\in 2\mathbb{Z}$ and $\frac{m^3-m}{6}=\binom{m+1}{3}$, $\frac{k^2c(m^3-m)}{12}$ is always an integer.

Thus, using Corollary 2.3.4, we see that $V_{\mathbb{Z}}^{*}$ is the integral span of products of the form

$$(kL(-m_1))\cdots(kL(-m_j))v$$

where $m_1 \geq \ldots \geq m_j \geq 2$ and $v \in V_{\mathbb{Z}}$. To see that $V_{\mathbb{Z}}^{*} \cap V_{(n)}^{\alpha}$ is a lattice, that is, the $\mathbb{Z}$-span of finitely many vectors, suppose $v_1, \ldots, v_l \in V_{\mathbb{Z}}$ span $V_{\mathbb{Z}} \cap \left(\coprod_{i=N_\alpha}^{n} V_{(i)}^{\alpha}\right)$, where $V_{(m)}^{\alpha} = 0$ for $m < N_\alpha$. Then $V_{\mathbb{Z}}^{*} \cap V_{(n)}^{\alpha}$ is spanned by some of the vectors of the form $(kL(-m_1))\cdots(kL(-m_j))v_k$ where $1 \leq k \leq l$, $m_i \geq 2$, and $m_1 + \ldots + m_j \leq n - N_\alpha$. Since there are finitely many such vectors, $V_{\mathbb{Z}}^{*} \cap V_{(n)}^{\alpha}$ is a lattice. $\square$

Using the commutation relations (2.15), we can use an argument similar to but simpler than the proof of Theorem 2.3.2 to prove:

Proposition 2.3.5. *If V is a vertex operator algebra generated by the conformal vector ω, and ω is contained in a $\mathbb{Q}$-form of V, then V has an integral form generated by $k\omega$ if $k \in \mathbb{Z}$ and $k^2 c \in 2\mathbb{Z}$. In particular, ω generates an integral form of V if and only if $c \in 2\mathbb{Z}$.*

Proof. The vertex subring $V_{\mathbb{Z}}$ generated by $k\omega$ is compatible with the weight grading in the sense of (2.10) by (2.6) since ω is homogeneous of weight 2. Moreover, for any $n \in \mathbb{Z}$, $V_{(n)} \cap V_{\mathbb{Z}}$ spans $V_{(n)}$ over $\mathbb{C}$ since ω generates V as a vertex operator algebra. Since $k\omega$ is contained in a rational form of V, it suffices to show that $V_{(n)} \cap V_{\mathbb{Z}}$ is a lattice in $V_{(n)}$ for any n, just as in the proof of Theorem 2.3.2.

By (2.13), $V_{\mathbb{Z}}$ is the $\mathbb{Z}$-span of vectors of the form

$$(kL(m_1))\cdots(kL(m_j))\mathbf{1}$$

where $m_i \in \mathbb{Z}$, but in fact we can use the commutation relations (2.15) to straighten such products, implying that $V_{\mathbb{Z}}$ is the $\mathbb{Z}$-span of vectors of the form

$$(kL(-m_1))\cdots(kL(-m_j))\mathbf{1}$$

where now $m_1 \geq \ldots \geq m_j \geq 2$. It is now clear that $V_{(n)} \cap V_{\mathbb{Z}}$ is the $\mathbb{Z}$-span of finitely many vectors for any $n \in \mathbb{Z}$, proving the proposition. $\square$

2.4 Intertwining operators

For any vector space or $\mathbb{Z}$-module V, let $V\{x\}$ denote the space of formal series with complex powers and coefficients in V. We recall the definition of intertwining operator among a triple of modules for a vertex operator algebra (see for instance [FHL]):

Definition 2.4.1. Suppose V is a vertex operator algebra and $W^{(1)}$, $W^{(2)}$ and $W^{(3)}$ are V-modules. An *intertwining operator* of type $\binom{W^{(3)}}{W^{(1)}\,W^{(2)}}$ is a linear map

$$\begin{aligned} \mathcal{Y}: W^{(1)} \otimes W^{(2)} &\to W^{(3)}\{x\}, \\ w_{(1)} \otimes w_{(2)} &\mapsto \mathcal{Y}(w_{(1)}, x)w_{(2)} = \sum_{n\in\mathbb{C}} (w_{(1)})_n w_{(2)} x^{-n-1} \in W^{(3)}\{x\} \end{aligned}$$

satisfying the following conditions:

1. *Lower truncation*: For any $w_{(1)} \in W^{(1)}$, $w_{(2)} \in W^{(2)}$ and $n \in \mathbb{C}$,

$$(w_{(1)})_{(n+m)} w_{(2)} = 0 \quad \text{for} \;\; m \in \mathbb{N} \;\; \text{sufficiently large.} \tag{2.16}$$

2. The *Jacobi identity*:

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_{W^{(3)}}(v, x_1)\mathcal{Y}(w_{(1)}, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) \mathcal{Y}(w_{(1)}, x_2) Y_{W^{(2)}}(v, x_1) \\ = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) \mathcal{Y}(Y_{W^{(1)}}(v, x_0)w_{(1)}, x_2) \end{aligned} \tag{2.17}$$

   for $v \in V$ and $w_{(1)} \in W^{(1)}$.

3. The $L(-1)$-*derivative property:* for any $w_{(1)} \in W^{(1)}$,

$$\mathcal{Y}(L(-1)w_{(1)}, x) = \frac{d}{dx}\mathcal{Y}(w_{(1)}, x). \tag{2.18}$$

Remark 2.4.2. We use $V^{W^{(3)}}_{W^{(1)}W^{(2)}}$ to denote the vector space of intertwining operators of type $\binom{W^{(3)}}{W^{(1)}\,W^{(2)}}$, and the dimension of $V^{W^{(3)}}_{W^{(1)}W^{(2)}}$ is the *fusion rule* $N^{W^{(3)}}_{W^{(1)}W^{(2)}}$

Remark 2.4.3. Note that if W is a V-module, the vertex operator Y_W is an intertwining operator of type $\binom{W}{V\,W}$, in particular in the special case $V = W$. As in the algebra or module vertex operator case, the Jacobi identity (2.17) with $v = \omega$ implies that for $w_{(1)} \in W^{(1)}$ and $w_{(2)} \in W^{(2)}$ both homogeneous,

$$\mathrm{wt}\,(w_{(1)})_n w_{(2)} = \mathrm{wt}\,w_{(1)} + \mathrm{wt}\,w_{(2)} - n - 1 \tag{2.19}$$

for any $n \in \mathbb{C}$.

Remark 2.4.4. If $W^{(1)}$, $W^{(2)}$ and $W^{(3)}$ are irreducible V-modules, then there are complex numbers h_i for $i = 1, 2, 3$ such that the conformal weights of $W^{(i)}$ are contained in $h_i + \mathbb{N}$ for each i. If $\mathcal{Y}$ is an intertwining operator of type $\binom{W^{(3)}}{W^{(1)}\,W^{(2)}}$ and we set

$$h = h_1 + h_2 - h_3,$$

then (2.19) implies that we can write

$$\mathcal{Y}(w_{(1)}, x)w_{(2)} = \sum_{n\in\mathbb{Z}} w_{(1)}(n)w_{(2)}\, x^{-n-h} \tag{2.20}$$

for any $w_{(1)} \in W^{(1)}$ and $w_{(2)} \in W^{(2)}$, where $w_{(1)}(n) = (w_{(1)})_{n+h-1}$ for $n \in \mathbb{Z}$. In particular, for $w_{(1)} \in W^{(1)}_{(h_1)}$ and $W^{(2)}_{(h_2)}$, note that $w_{(1)}(0)w_{(2)} \in W^{(3)}_{(h_3)}$.

The Jacobi identity for intertwining operators implies commutator and iterate formulas, as in the algebra and module vertex operator case. We will in particular need the iterate formula: for any $v \in V$, $w_{(1)} \in W^{(1)}$, and $n \in \mathbb{Z}$,

$$\begin{aligned}\mathcal{Y}(v_n w_{(1)}, x_2) =& \mathrm{Res}_{x_1}(x_1 - x_2)^n Y_{W^{(3)}}(v, x_1)\mathcal{Y}(w_{(1)}, x_2)\\ &- \mathrm{Res}_{x_1}(-x_2 + x_1)^n \mathcal{Y}(w_{(1)}, x_2)Y_{W^{(2)}}(v, x_1).\end{aligned} \tag{2.21}$$

We will also need weak commutativity for intertwining operators, whose proof is exactly the same as in the algebra or module case:

Proposition 2.4.5. *Suppose $W^{(1)}$, $W^{(2)}$ and $W^{(3)}$ are V-modules and $\mathcal{Y} \in V^{W^{(3)}}_{W^{(1)}W^{(2)}}$. Then for any positive integer k such that $v_n w_{(1)} = 0$ for $n \geq k$,*

$$(x_1 - x_2)^k Y_{W^{(3)}}(v, x_1)\mathcal{Y}(w_{(1)}, x_2) = (x_1 - x_2)^k \mathcal{Y}(w_{(1)}, x_2)Y_{W^{(2)}}(v, x_1).$$

Proof. Multiply the Jacobi identity (2.17) by x_0^k and extract the coefficient of x_0^{-1}, obtaining

$$\begin{aligned}(x_1-x_2)^k Y_{W^{(3)}}(v,x_1)\mathcal{Y}(w_{(1)},x_2) - (x_1-x_2)^k\mathcal{Y}(w_{(1)},x_2)Y_{W^{(2)}}(v,x_1)\\ = \mathrm{Res}_{x_0}\, x_0^k x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}(Y_{W^{(1)}}(v,x_0)w_{(1)},x_2).\end{aligned}$$

Since $v_n w_{(1)} = 0$ for $n \geq k$, the right side contains no negative powers of x_0, so the residue is 0. □

We will need the following proposition, which is similar to Proposition 4.5.8 in [LL] and uses essentially the same proof:

Proposition 2.4.6. *Suppose $W^{(1)}$, $W^{(2)}$, and $W^{(3)}$ are V-modules and $\mathcal{Y} \in V^{W^{(3)}}_{W^{(1)}W^{(2)}}$. Then for any $v \in V$, $w_{(1)} \in W^{(1)}$, and $w_{(2)} \in W^{(2)}$, and for any $p \in \mathbb{C}$ and $q \in \mathbb{Z}$, $(w_{(1)})_p v_q w_{(2)}$ is an integral linear combination of terms of the form $v_r(w_{(1)})_s w_{(2)}$. Specifically, let k and m be nonnegative integers such that $v_n w_{(1)} = 0$ for $n \geq k$ and $v_n w_{(2)} = 0$ for $n > m+q$. Then*

$$(w_{(1)})_p v_q w_{(2)} = \sum_{i=0}^{m}\sum_{j=0}^{k}(-1)^{i+j}\binom{-k}{i}\binom{k}{j} v_{q+i+j}(w_{(1)})_{p-i-j}w_{(2)}.$$

Proof. From weak commutativity, we have

$$\begin{aligned}(w_{(1)})_p v_q w_{(2)} &= \mathrm{Res}_{x_1}\mathrm{Res}_{x_2} x_1^q x_2^p \mathcal{Y}(w_{(1)},x_2)Y_{W^{(2)}}(v,x_1)w_{(2)}\\ &= \mathrm{Res}_{x_1}\mathrm{Res}_{x_2} x_1^q x_2^p(-x_2+x_1)^{-k}[(x_1-x_2)^k\mathcal{Y}(w_{(1)},x_2)Y_{W^{(2)}}(v,x_1)w_{(2)}]\\ &= \mathrm{Res}_{x_1}\mathrm{Res}_{x_2} x_1^q x_2^p(-x_2+x_1)^{-k}[(x_1-x_2)^k Y_{W^{(3)}}(v,x_1)\mathcal{Y}(w_{(1)},x_2)w_{(2)}].\end{aligned} \tag{2.22}$$

These formal expressions are well defined because $Y_{W^{(2)}}(v,x_1)w_{(2)}$ is lower-truncated, but we cannot remove the brackets from the last expression in (2.22). We observe that the term in brackets in (2.22) may be written explicitly as

$$\sum_{j\in\mathbb{N},\, m\in\mathbb{Z},\, n\in\mathbb{C}}(-1)^j\binom{k}{j} v_m(w_{(1)})_n w_{(2)} x_1^{k-j-m-1}x_2^{j-n-1}.$$

Meanwhile, only a finite truncation of $x_1^q x_2^p(-x_2+x_1)^{-k}$ contributes to the residue since $Y_{W^{(3)}}(v,x_1)w_{(2)}$ is lower-truncated. In particular, the lowest possible integral power in $x_1^q Y_{W^{(2)}}(v,x_1)w_{(2)}$ with a non-zero coefficient is x_2^{-m-1} by definition of m. Thus we can take

$$\sum_{i=0}^{m}(-1)^{k+i}\binom{-k}{i}x_1^{q+i}x_2^{p-k-i}$$

as our truncation of $x_1^q x_2^p(-x_2+x_1)^{-k}$. Then

$$\begin{aligned}
&(w_{(1)})_p v_q w_{(2)} = \\
&\mathrm{Res}_{x_1,x_2}\sum_{i,j,m,n}(-1)^{k+i+j}\binom{-k}{i}\binom{k}{j}v_m(w_{(1)})_n w_{(2)}x_1^{q+k+i-j-m-1}x_2^{p-k-i+j-n-1} = \\
&\sum_{i=0}^{m}\sum_{j=0}^{k}(-1)^{k+i+j}\binom{-k}{i}\binom{k}{j}v_{q+k+i-j}(w_{(1)})_{p-k-i+j}w_{(2)} = \\
&\sum_{i=0}^{m}\sum_{j=0}^{k}(-1)^{i+j}\binom{-k}{i}\binom{k}{j}v_{q+i+j}(w_{(1)})_{p-i-j}w_{(2)},
\end{aligned}$$

where we have changed j to $k-j$ in the last equality. □

The Jacobi identity for intertwining operators makes sense in the vertex ring context for the same reasons that the Jacobi identity for algebras and modules does. However, the $L(-1)$-derivative property may not make sense. If $V_{\mathbb{Z}}$ is a vertex ring without a conformal vector ω, then there may be no $L(-1)$ operator on $V_{\mathbb{Z}}$-modules. In addition, since intertwining operators involve complex powers of x, the coefficients of the derivative of an intertwining operator may not make sense as maps from a $V_{\mathbb{Z}}$-module into a $V_{\mathbb{Z}}$-module.

However, suppose V is a vertex operator algebra with integral form $V_{\mathbb{Z}}$ and $W^{(1)}$, $W^{(2)}$, and $W^{(3)}$ are V-modules with integral forms $W_{\mathbb{Z}}^{(1)}$, $W_{\mathbb{Z}}^{(2)}$, and $W_{\mathbb{Z}}^{(3)}$, respectively:

Definition 2.4.7. An intertwining operator $\mathcal{Y}\in V_{W^{(1)}W^{(2)}}^{W^{(3)}}$ is *integral* with respect to $W_{\mathbb{Z}}^{(1)}$, $W_{\mathbb{Z}}^{(2)}$, and $W_{\mathbb{Z}}^{(3)}$ if for any $w_{(1)}\in W_{\mathbb{Z}}^{(1)}$ and $w_{(2)}\in W_{\mathbb{Z}}^{(2)}$,

$$\mathcal{Y}(w_{(1)},x)w_{(2)}\in W_{\mathbb{Z}}^{(3)}\{x\}.$$

Remark 2.4.8. Note that whether an intertwining operator is integral or not will generally depend on the integral forms used for the three V-modules.

The main result of this section is the following theorem which reduces the problem of showing that an intertwining operator is integral to the problem of showing that it is integral when restricted to generators of $W^{(1)}$ and $W^{(2)}$:

Theorem 2.4.9. *Suppose V is a vertex operator algebra with integral form $V_{\mathbb{Z}}$ and $W^{(1)}$, $W^{(2)}$, and $W^{(3)}$ are V-modules with integral forms $W_{\mathbb{Z}}^{(1)}$, $W_{\mathbb{Z}}^{(2)}$, and $W_{\mathbb{Z}}^{(3)}$, respectively. Moreover, suppose $T^{(1)}$ and $T^{(2)}$ are generating sets for $W_{\mathbb{Z}}^{(1)}$ and $W_{\mathbb{Z}}^{(2)}$, respectively. If $\mathcal{Y} \in V_{W^{(1)}W^{(2)}}^{W^{(3)}}$ satisfies*

$$\mathcal{Y}(t_{(1)}, x)t_{(2)} \in W_{\mathbb{Z}}^{(3)}\{x\}$$

for all $t_{(1)} \in T^{(1)}$, $t_{(2)} \in T^{(2)}$, then $\mathcal{Y}$ is integral with respect to $W_{\mathbb{Z}}^{(1)}$, $W_{\mathbb{Z}}^{(2)}$, and $W_{\mathbb{Z}}^{(3)}$.

Proof. First let $U_{\mathbb{Z}}^{(2)}$ be the sublattice of $W_{\mathbb{Z}}^{(2)}$ consisting of vectors $u_{(2)}$ such that

$$\mathcal{Y}(t_{(1)}, x)u_{(2)} \in W_{\mathbb{Z}}^{(3)}\{x\}$$

for all $t_{(1)} \in T^{(1)}$. Note that if $u_{(2)} \in U_{\mathbb{Z}}^{(2)}$, $t_{(1)} \in T^{(1)}$, and $v \in V_{\mathbb{Z}}$, Proposition 2.4.6 implies that for any $p \in \mathbb{C}$ and $q \in \mathbb{Z}$, $(t_{(1)})_p v_q u_{(2)}$ is an integral linear combination of terms of the form $v_r (t_{(1)})_s u_{(2)}$, and is thus in $W_{\mathbb{Z}}^{(3)}$. This means that

$$Y_{W^{(2)}}(v, x)u_{(2)} \in U_{\mathbb{Z}}^{(2)}[[x, x^{-1}]],$$

that is, $U_{\mathbb{Z}}^{(2)}$ is a $V_{\mathbb{Z}}$-module. Since by hypothesis $U_{\mathbb{Z}}^{(2)}$ contains $T^{(2)}$ which generates $W_{\mathbb{Z}}^{(2)}$, $U_{\mathbb{Z}}^{(2)} = W_{\mathbb{Z}}^{(2)}$ and so

$$\mathcal{Y}(t_{(1)}, x)w_{(2)} \in W_{\mathbb{Z}}^{(3)}\{x\} \tag{2.23}$$

for any $t_{(1)} \in T^{(1)}$ and $w_{(2)} \in W_{\mathbb{Z}}^{(2)}$.

Now we define $U_{\mathbb{Z}}^{(1)}$ as the sublattice of $W_{\mathbb{Z}}^{(1)}$ consisting of all vectors $u_{(1)} \in W_{\mathbb{Z}}^{(1)}$ such that

$$\mathcal{Y}(u_{(1)}, x)w_{(2)} \in W_{\mathbb{Z}}^{(3)}\{x\}.$$

for all $w_{(2)} \in W_{\mathbb{Z}}^{(2)}$. In this case, $U_{\mathbb{Z}}^{(1)}$ is a $V_{\mathbb{Z}}$-submodule of $W_{\mathbb{Z}}^{(1)}$ by the iterate formula for intertwining operators, (2.21). Since $U_{\mathbb{Z}}^{(1)}$ contains $T^{(1)}$ by (2.23), $U_{\mathbb{Z}}^{(1)} = W_{\mathbb{Z}}^{(1)}$, and we have shown that

$$\mathcal{Y}(w_{(1)}, x)w_{(2)} \in W_{\mathbb{Z}}^{(3)}\{x\}$$

for any $w_{(1)} \in W_{\mathbb{Z}}^{(1)}$ and $w_{(2)} \in W_{\mathbb{Z}}^{(2)}$, that is, $\mathcal{Y}$ is integral with respect to $W_{\mathbb{Z}}^{(1)}$, $W_{\mathbb{Z}}^{(2)}$, and $W_{\mathbb{Z}}^{(3)}$. $\square$

We conclude this section by recalling the symmetries among spaces of intertwining operators from ([FHL]), ([HL1]), and ([HLZ2]), which we will need in the next section. First, if $W^{(1)}$, $W^{(2)}$ and $W^{(3)}$ are V-modules and $\mathcal{Y}$ is an intertwining operator of type $\binom{W^{(3)}}{W^{(1)}\,W^{(2)}}$, then for any $r \in \mathbb{Z}$, there is an intertwining operator $\Omega_r(\mathcal{Y})$ of type $\binom{W^{(3)}}{W^{(2)}\,W^{(1)}}$ defined by

$$\Omega_r(\mathcal{Y})(w_{(2)}, x)w_{(1)} = e^{xL(-1)}\mathcal{Y}(w_{(1)}, e^{(2r+1)\pi i}x)w_{(2)} \tag{2.24}$$

for $w_{(1)} \in W^{(1)}$ and $w_{(2)} \in W^{(2)}$. Moreover, for any $r \in \mathbb{Z}$ there is an intertwining operator $A_r(\mathcal{Y})$ of type $\binom{(W^{(2)})'}{W^{(1)}\,(W^{(3)})'}$ defined by

$$\langle A_r(\mathcal{Y})(w_{(1)}, x)w'_{(3)}, w_{(2)}\rangle_{W^{(2)}} = \langle w'_{(3)}, \mathcal{Y}_r^o(w_{(1)}, x)w_{(2)}\rangle_{W^{(3)}} \tag{2.25}$$

for $w_{(1)} \in W^{(1)}$, $w_{(2)} \in W^{(2)}$, and $w'_{(3)} \in (W^{(3)})'$, where

$$\mathcal{Y}_r^o(w_{(1)}, x)w_{(2)} = \mathcal{Y}(e^{xL(1)}e^{(2r+1)\pi i L(0)}(x^{-L(0)})^2 w_{(1)}, x^{-1})w_{(2)}.$$

Then we have (see for example Propositions 3.44 and 3.46) in [HLZ2]):

Proposition 2.4.10. *For any $r \in \mathbb{Z}$, the map $\Omega_r : V_{W^{(1)}W^{(2)}}^{W^{(3)}} \to V_{W^{(2)}W^{(1)}}^{W^{(3)}}$ is a linear isomorphism with inverse Ω_{-r-1}. Moreover, the map $A_r : V_{W^{(1)}W^{(2)}}^{W^{(3)}} \to V_{W^{(1)}(W^{(3)})'}^{(W^{(2)})'}$ is a linear isomorphism with inverse A_{-r-1} for any $r \in \mathbb{Z}$.*

2.5 Contragredient modules and integral forms

The next three propositions plus Remark 2.5.3 below generalize Lemma 6.1, Lemma 6.2, and Remark 6.3 in [DG] to the context of contragredient pairs of modules for a vertex operator algebra V. Suppose V has an integral form $V_{\mathbb{Z}}$ and W is a V-module with integral form $W_{\mathbb{Z}}$. Then there is an integral form $W'_{\mathbb{Z}}$ for the contragredient module W' as a vector space given by

$$W'_{\mathbb{Z}} = \{w' \in W' \mid \langle w', w\rangle \in \mathbb{Z} \text{ for } w \in W_{\mathbb{Z}}\}.$$

We would like $W'_{\mathbb{Z}}$ to be a module for $V_{\mathbb{Z}}$.

Proposition 2.5.1. *Suppose $V_{\mathbb{Z}}$ is preserved by $\frac{L(1)^n}{n!}$ for $n \geq 0$. Then $W'_{\mathbb{Z}}$ is a $V_{\mathbb{Z}}$-module.*

Proof. By the definition of the module action on W', if $v \in V_{\mathbb{Z}}$, $w \in W_{\mathbb{Z}}$ and $w' \in W'_{\mathbb{Z}}$,

$$\langle Y(v,x)w', w\rangle = \langle w', Y^o(v,x)w\rangle = \left\langle w', Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w\right\rangle$$

Since $\frac{L(1)^n}{n!}$ preserves $V_{\mathbb{Z}}$ and $V_{\mathbb{Z}}$ satisfies (2.10), $e^{xL(1)}(-x^{-2})^{L(0)}v \in V_{\mathbb{Z}}[[x,x^{-1}]]$. Since also $V_{\mathbb{Z}}$ preserves $W_{\mathbb{Z}}$, we have $\langle Y(v,x)w', w\rangle \in \mathbb{Z}[[x,x^{-1}]]$ for any w and so $Y(v,x)w' \in W'_{\mathbb{Z}}[[x,x^{-1}]]$ as desired. □

Proposition 2.5.2. *If $V_{\mathbb{Z}}$ is generated by vectors v such that $L(1)v = 0$, then $V_{\mathbb{Z}}$ is preserved by $\frac{L(1)^n}{n!}$ for $n \geq 0$.*

Proof. We will use the $L(1)$-conjugation formula proved in [FHL]:

$$e^{yL(1)}Y(v,x)e^{-yL(1)} = Y\left(e^{y(1-xy)L(1)}(1-xy)^{-2L(0)}v, \frac{x}{1-xy}\right).$$

If $L(1)v = 0$, this formula simplifies to

$$e^{yL(1)}Y(v,x)e^{-yL(1)} = Y\left((1-xy)^{-2L(0)}v, \frac{x}{1-xy}\right).$$

Thus if a vector $v \in V_{\mathbb{Z}}$ is a coefficient of a monomial in

$$Y(v_1,x_1)\cdots Y(v_k,x_k)\mathbf{1} \tag{2.26}$$

where $L(1)v_i = 0$, then $\frac{L(1)^n}{n!}v$ is a coefficient of a monomial in

$$e^{yL(1)}Y(v_1,x_1)\cdots Y(v_k,x_k)\mathbf{1} =$$
$$Y\left((1-x_1y)^{-2L(0)}v_1, \frac{x_1}{1-x_1y}\right)\cdots Y\left((1-x_ky)^{-2L(0)}v_k, \frac{x_k}{1-x_ky}\right)\mathbf{1}.$$

Since the expansion of $(1-xy)^m$ for any integer m has integer coefficients, all coefficients of monomials on the right side lie in $V_{\mathbb{Z}}$. Hence $\frac{L(1)^n}{n!}v \in V_{\mathbb{Z}}$. Since $V_{\mathbb{Z}}$ is spanned by coefficients of monomials in products of the form (2.26) by Proposition 2.2.4, $\frac{L(1)^n}{n!}$ preserves $V_{\mathbb{Z}}$ for any $n \geq 0$. □

Remark 2.5.3. If an integral form $V_{\mathbb{Z}}$ of V is generated by lowest weight vectors for the Virasoro algebra, then contragredients of $V_{\mathbb{Z}}$-modules are $V_{\mathbb{Z}}$-modules.

Suppose V is equivalent as V-module to its contragredient V'. This is the case if and only if there is a nondegenerate bilinear form $(\cdot,\cdot)$ on V that is invariant in the sense that

$$(Y(u,x)v,w) = (v, Y^o(u,x)w)$$

for $u,v,w \in V$ (see [FHL]). Invariant forms on V are in one-to-one correspondence with linear functionals on $V_{(0)}/L(1)V_{(1)}$ ([Li1]). Given a choice of nondegenerate invariant bilinear form and an integral form $V_{\mathbb{Z}}$ of a vertex operator algebra V, the contragredient module $V'_{\mathbb{Z}}$ may be identified with another lattice spanning V that is invariant under the action of $V_{\mathbb{Z}}$. However, $V'_{\mathbb{Z}}$ need not be an integral form of V as a vertex algebra, because it may not be closed under vertex algebra products.

Proposition 2.5.4. *Suppose V is equivalent to V' as V-module and V has an integral form $V_{\mathbb{Z}}$ preserved by $\frac{L(1)^n}{n!}$ for $n \geq 0$; also assume $V_{(0)} = \mathbb{C}\mathbf{1}$. Identify $V'_{\mathbb{Z}}$ with a lattice in V using a non-degenerate invariant form $(\cdot,\cdot)$ such that $(\mathbf{1},\mathbf{1}) \in \mathbb{Z}\setminus\{0\}$. Then $V_{\mathbb{Z}} \subseteq V'_{\mathbb{Z}}$.*

Proof. The integral form $V'_{\mathbb{Z}} \subseteq V$ is the sublattice

$$V'_{\mathbb{Z}} = \{v' \in V \mid (v',v) \in \mathbb{Z} \text{ for } v \in V_{\mathbb{Z}}\}.$$

Thus we need to show that if $u,v \in V_{\mathbb{Z}}$, then $(u,v) \in \mathbb{Z}$. We have:

$$\begin{aligned}(u,v) &= \mathrm{Res}_x\, x^{-1}(Y(u,x)\mathbf{1},v)\\ &= \mathrm{Res}_x\, x^{-1}(\mathbf{1}, Y(e^{xL(1)}(-x^{-2})^{L(0)}u, x^{-1})v)\\ &= (\mathbf{1}, c\mathbf{1}) = c\,(\mathbf{1},\mathbf{1})\end{aligned}$$

where $c\mathbf{1} = \mathrm{Res}_x\, x^{-1}Y(e^{xL(1)}(-x^{-2})^{L(0)}u, x^{-1})v$ (since the residue is indeed in $V_{(0)}$). But all coefficients of $Y(e^{xL(1)}(-x^{-2})^{L(0)}u, x^{-1})v$ are in $V_{\mathbb{Z}}$ because $V_{\mathbb{Z}}$ is closed under vertex operators and invariant under $\frac{L(1)^n}{n!}$ for $n \geq 0$. Thus $c\mathbf{1} \in V_{(0)} \cap V_{\mathbb{Z}} = \mathbb{Z}\mathbf{1}$ by Proposition 2.2.6 and so $c\,(\mathbf{1}, \mathbf{1})$ is an integer. $\square$

In Chapter 4, we will use intertwining operators to identify contragredient pairs of modules for conformal vertex algebras based on even lattices. We say that a bilinear pairing $(\cdot,\cdot)$ between V-modules $W^{(1)}$ and $W^{(2)}$ is *invariant* if

$$(Y_{W^{(1)}}(v,x)w_{(1)}, w_{(2)}) = (w_{(1)}, Y^o_{W^{(2)}}(v,x)w_{(2)})$$

for $v \in V$, $w_{(1)} \in W^{(1)}$, and $w_{(2)} \in W^{(2)}$. It is clear that if there is a nondegenerate invariant bilinear pairing between $W^{(1)}$ and $W^{(2)}$, then $W^{(1)}$ and $W^{(2)}$ form a contragredient pair. We will need the following proposition, which is essentially a generalization of Remark 2.9 in [Li1]:

Proposition 2.5.5. *Suppose V is a vertex operator algebra equipped with a nondegenerate invariant bilinear pairing $(\cdot,\cdot)_V$, and suppose $W^{(1)}$ and $W^{(2)}$ are V-modules. If $\mathcal{Y}$ is an intertwining operator of type $\binom{V}{W^{(1)}\,W^{(2)}}$, then the bilinear pairing $(\cdot,\cdot)$ between $W^{(1)}$ and $W^{(2)}$ given by*

$$(w_{(1)}, w_{(2)}) = \mathrm{Res}_x(\mathbf{1}, \mathcal{Y}^o_0(w_{(1)}, e^{\pi i}x)e^{xL(1)}w_{(2)})_V \tag{2.27}$$

for $w_{(1)} \in W^{(1)}$ and $w_{(2)} \in W^{(2)}$ is invariant. Moreover, if $W^{(1)}$ and $W^{(2)}$ are irreducible and $\mathcal{Y}$ is non-zero, then the pairing is nondegenerate and $W^{(1)}$ and $W^{(2)}$ form a contragredient pair.

Proof. We consider the intertwining operator $\mathcal{Y}' = \Omega_0(A_0(\mathcal{Y}))$ of type $\binom{(W^{(2)})'}{V\cong V'\,W^{(1)}}$. The coefficient of x_0^{-1} in the Jacobi identity (2.17) implies that

$$Y_{(W^{(2)})'}(v,x_1)\mathcal{Y}'(\mathbf{1},x_2) - \mathcal{Y}'(\mathbf{1},x_2)Y_{W^{(1)}}(v,x_1) = 0$$

since $Y(v,x_0)\mathbf{1}$ has no negative powers of x_0. Moreover, the $L(-1)$-derivative property (2.18) implies that $\mathcal{Y}'(\mathbf{1},x)$ equals its constant term $\mathbf{1}_{-1}$, which means that $\mathbf{1}_{-1} = \varphi_{\mathcal{Y}}$

is a V-homomorphism from $W^{(1)}$ to $(W^{(2)})'$. Thus we obtain a bilinear pairing $(\cdot,\cdot)$ between $W^{(1)}$ and $W^{(2)}$ given by

$$(w_{(1)}, w_{(2)}) = \langle \varphi_{\mathcal{Y}}(w_{(1)}), w_{(2)}\rangle$$

for $w_{(1)} \in W^{(1)}$, $w_{(2)} \in W^{(2)}$, which is invariant because $\varphi_{\mathcal{Y}}$ is a homomorphism.

To show that the invariant pairing $(\cdot,\cdot)$ is also given by (2.27), we calculate using the definitions of Ω_0 and A_0 from (2.24) and (2.25), and identifying $V \cong V'$ via $(\cdot,\cdot)_V$. For $w_{(1)} \in W^{(1)}$ and $w_{(2)} \in W^{(2)}$,

$$\begin{aligned}
(w_{(1)}, w_{(2)}) = \langle \varphi_{\mathcal{Y}}(w_{(1)}), w_{(2)}\rangle &= \mathrm{Res}_x\, x^{-1}\langle \Omega_0(A_0(\mathcal{Y}))(\mathbf{1}, x)w_{(1)}, w_{(2)}\rangle\\
&= \mathrm{Res}_x\, x^{-1}\langle e^{xL(-1)}A_0(\mathcal{Y})(w_{(1)}, e^{\pi i}x)\mathbf{1}, w_{(2)}\rangle\\
&= \mathrm{Res}_x\, x^{-1}\langle A_0(\mathcal{Y})(w_{(1)}, e^{\pi i}x)\mathbf{1}, e^{xL(1)}w_{(2)}\rangle\\
&= \mathrm{Res}_x\, x^{-1}(\mathbf{1}, \mathcal{Y}_0^o(w_{(1)}, e^{\pi i}x)e^{xL(1)}w_{(2)})_V.
\end{aligned}$$

This proves the first assertion of the proposition.

To prove the nondegeneracy of $(\cdot,\cdot)$, it is enough to prove that $\varphi_{\mathcal{Y}} : W^{(1)} \to (W^{(2)})'$ is an isomorphism. Since $W^{(1)}$ and $W^{(2)}$ are irreducible, it suffices to prove that $\varphi_{\mathcal{Y}}$ is non-zero, that is, $\mathcal{Y}'(\mathbf{1}, x) \neq 0$. But if $\mathcal{Y}'(\mathbf{1}, x) = 0$, then the creation property and the iterate formula (2.21) imply that for any $v \in V$,

$$\mathcal{Y}'(v, x) = \mathcal{Y}'(v_{-1}\mathbf{1}, x) = 0.$$

This is a contradiction because by Proposition 2.4.10, $\mathcal{Y}'$ is non-zero if $\mathcal{Y}$ is. $\square$

Chapter 3

Integral forms in vertex (operator) algebras and modules based on affine Lie algebras

In this chapter we study integral forms in vertex (operator) algebras and modules based on affine Lie algebras $\widehat{\mathfrak{g}}$. Although some results in this chapter will hold for a general finite-dimensional Lie algebra $\mathfrak{g}$ equipped with a symmetric invariant bilinear form, we will especially concentrate on the case where $\mathfrak{g}$ is a finite-dimensional simple Lie algebra. We will construct natural integral forms in vertex (operator) algebras and their irreducible modules based on a finite-dimensional simple Lie algebra $\mathfrak{g}$, using the integral form $U_{\mathbb{Z}}(\widehat{g})$ of the universal enveloping algebra of $\widehat{\mathfrak{g}}$ first constructed in [G] (see also [Mit], [P]), and we will find natural generating sets for these integral forms. We will also determine when intertwining operators among irreducible modules for these vertex operator algebras respect integral forms in the modules.

3.1 Vertex (operator) algebras based on affine Lie algebras

We first recall the construction of vertex algebras based on affine Lie algebras (see [FZ] and [LL] for more details). Suppose that $\mathfrak{g}$ is a finite dimensional Lie algebra, with symmetric invariant form $\langle\cdot,\cdot\rangle$, that is,

$$\langle [a,b],c\rangle = \langle a,[b,c]\rangle$$

for any $a,b,c \in \mathfrak{g}$. Then we can form the affine Lie algebra

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}\mathbf{k}$$

where $\mathbf{k}$ is central and all other brackets are determined by

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m\langle a, b\rangle \delta_{m+n,0}\mathbf{k}, \tag{3.1}$$

where $a, b \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. The affine Lie algebra has the decomposition

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ \oplus \widehat{\mathfrak{g}}_0 \oplus \widehat{\mathfrak{g}}_-$$

where

$$\widehat{\mathfrak{g}}_\pm = \coprod_{n \in \pm\mathbb{Z}_+} \mathfrak{g} \otimes t^n, \quad \widehat{\mathfrak{g}}_0 = \mathfrak{g} \otimes t^0 \oplus \mathbb{C}\mathbf{k}.$$

If U is a finite dimensional $\mathfrak{g}$-module, then U becomes a $\widehat{\mathfrak{g}}_+ \oplus \widehat{\mathfrak{g}}_0$-module on which $\widehat{\mathfrak{g}}_+$ acts trivially, $\mathfrak{g} \otimes t^0$ acts as $\mathfrak{g}$, and $\mathbf{k}$ acts as some scalar $\ell \in \mathbb{C}$. Then we can form the generalized Verma module

$$V_{\widehat{\mathfrak{g}}}(\ell, U) = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{g}}_+ \oplus \widehat{\mathfrak{g}}_0)} U,$$

which is a $\widehat{\mathfrak{g}}$-module; we say that the *level* of $V_{\mathfrak{g}}(\ell, U)$ is ℓ. It has a unique maximal submodule and thus a unique irreducible quotient, $L_{\widehat{\mathfrak{g}}}(\ell, U)$. We use $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ to denote $V_{\widehat{\mathfrak{g}}}(\ell, U)$ where U is the trivial one-dimensional $\mathfrak{g}$-module, and $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ to denote its irreducible quotient. If we write the trivial $\mathfrak{g}$-module with $\mathbf{k}$ acting as ℓ as $\mathbb{C}1_\ell$, and then write $\mathbf{1} = 1 \otimes 1_\ell \in V_{\widehat{\mathfrak{g}}}(\ell, 0)$, then we can write

$$V_{\widehat{\mathfrak{g}}}(\ell, 0) = U(\widehat{\mathfrak{g}}_-)\mathbf{1},$$

so $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ is spanned by elements of the form

$$a_1(-n_1) \cdots a_k(-n_k)\mathbf{1} \tag{3.2}$$

where $a_i \in \mathfrak{g}$, $n_i \in \mathbb{Z}_+$, and we use the notation $a(n)$ to denote the action of $a \otimes t^n$ on $\widehat{\mathfrak{g}}$-modules.

The generalized Verma module $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ is a vertex algebra, as are all its quotients, including $L_{\widehat{\mathfrak{g}}}(\ell, 0)$, which is a simple vertex algebra. The vertex algebra structure is determined by

$$Y(a(-1)\mathbf{1}, x) = \sum_{n \in \mathbb{Z}} a(n) x^{-n-1}. \tag{3.3}$$

(The elements $a(-1)\mathbf{1}$ generate $V_{\widehat{\mathfrak{g}}}(\ell,0)$, and thus also its quotients, as vertex algebras.) As long as $\ell \neq -h$, where h is the dual Coxeter number of $\mathfrak{g}$, $V_{\widehat{\mathfrak{g}}}(\ell,0)$ is also a vertex operator algebra, with conformal vector given by

$$\omega = \frac{1}{2(\ell+h)} \sum_{i=1}^{\dim \mathfrak{g}} u_i(-1)u_i'(-1)\mathbf{1},$$

where $\{u_i\}$ is any basis of $\mathfrak{g}$ and $\{u_i'\}$ is the corresponding dual basis with respect to the form $\langle \cdot,\cdot\rangle$. The generalized Verma modules $V_{\widehat{\mathfrak{g}}}(\ell,U)$ for finite-dimensional $\mathfrak{g}$-modules U are $V_{\widehat{\mathfrak{g}}}(\ell,0)$-modules, and so are all quotients of $V_{\widehat{\mathfrak{g}}}(\ell,U)$; the irreducible $V_{\widehat{\mathfrak{g}}}(\ell,0)$-modules consist precisely of the $\widehat{\mathfrak{g}}$-modules $L_{\widehat{\mathfrak{g}}}(\ell,U)$ where U is a finite-dimensional irreducible $\mathfrak{g}$-module.

Although some of the results in this chapter will apply to vertex operator algebras based on arbitrary affine Lie algebras, we shall particularly concentrate on the case where $\mathfrak{g}$ is a finite-dimensional simple complex Lie algebra. In this case, let $\mathfrak{h}$ denote a Cartan subalgebra of $\mathfrak{g}$. The invariant bilinear form $\langle \cdot,\cdot\rangle$ on $\mathfrak{g}$ is nondegenerate on $\mathfrak{h}$ and induces a bilinear form on $\mathfrak{h}^*$. We shall normalize $\langle \cdot,\cdot\rangle$ so that

$$\langle \alpha,\alpha\rangle = 2$$

for long roots $\alpha \in \mathfrak{h}^*$. Irreducible finite-dimensional $\mathfrak{g}$-modules are in one-to-one correspondence with dominant integral weights $\lambda \in \mathfrak{h}^*$, where $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra.

Let L_λ denote the irreducible $\mathfrak{g}$-module with highest weight λ. Then for any level $\ell \in \mathbb{C}$, the irreducible $V_{\widehat{\mathfrak{g}}}(\ell,0)$-modules consist of $L(\ell,L_\lambda)$ for λ a dominant integral weight. Moreover, suppose θ is the highest root of $\mathfrak{g}$; if ℓ is a non-negative integer, the irreducible $L_{\widehat{\mathfrak{g}}}(\ell,0)$-modules are given by $L(\ell,L_\lambda)$ where λ is a dominant integral weight satisfying $\langle \lambda,\theta\rangle \leq \ell$ ([FZ]; see also [LL]).

The dual of an irreducible $\mathfrak{g}$-module L_λ is also an irreducible $\mathfrak{g}$-module with the action given by

$$\langle x\cdot v',v\rangle = -\langle v',x\cdot v\rangle$$

for $x \in \mathfrak{g}$, $v \in L_\lambda$, and $v' \in L_\lambda^*$. So we have $L_\lambda^* \cong L_{\lambda^*}$ for some dominant integral weight λ^*. In fact, $\lambda^* = -w(\lambda)$ where w is the element in the Weyl group of $\mathfrak{g}$ of maximal length ([Hu]). Then the contragredient of the $V_{\widehat{\mathfrak{g}}}(\ell, 0)$-module $L_{\widehat{\mathfrak{g}}}(\ell, L_\lambda)$ is isomorphic to $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda^*})$ ([FZ]).

Now suppose ℓ is a non-negative integer and $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})$, $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})$, and $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3})$ are irreducible $L_{\widehat{\mathfrak{g}}}(\ell, 0)$-modules, so that $\langle \lambda_i, \theta \rangle \leq \ell$ for $i = 1, 2, 3$. We recall from [FZ] (see also [Li2]) the classification of intertwining operators of type $\binom{L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3})}{L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})\, L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})}$. First, let $A(L_{\widehat{\mathfrak{g}}}(\ell, 0))$ be the Zhu's algebra of $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ (see [Z] for the definition); from [FZ], as an associative algebra,

$$A(L_{\widehat{\mathfrak{g}}}(\ell, 0)) \cong U(\mathfrak{g})/\langle x_\theta^{\ell+1} \rangle,$$

where x_θ is a root vector for the longest root θ of $\mathfrak{g}$. We also need the $A(L_{\widehat{\mathfrak{g}}}(\ell, 0))$-bimodule $A(L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1}))$, which from [FZ] is given by

$$A(L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})) \cong (L_{\lambda_1} \otimes U(\mathfrak{g}))/\langle v_{\lambda_1} \otimes x_\theta^{\ell - \langle \lambda_1, \theta \rangle + 1} \rangle,$$

where v_{λ_1} is a highest weight vector of L_{λ_1} and $\langle v_{\lambda_1} \otimes x_\theta^{\ell - \langle \lambda_1, \theta \rangle + 1} \rangle$ indicates the subbimodule generated by the indicated element. The $A(L_{\widehat{\mathfrak{g}}}(\ell, 0)) \cong U(\mathfrak{g})/\langle x_\theta^{\ell+1} \rangle$-bimodule structure on $(L_{\lambda_1} \otimes U(\mathfrak{g}))/\langle v_{\lambda_1} \otimes x_\theta^{\ell - \langle \lambda_1, \theta \rangle + 1} \rangle$ is induced by the following $U(\mathfrak{g})$-bimodule structure on $L_{\lambda_1} \otimes U(\mathfrak{g})$:

$$x \cdot (v \otimes y) = (x \cdot v) \otimes y + v \otimes xy$$

for $x, y \in U(\mathfrak{g})$, $v \in L_{\lambda_1}$, and

$$(v \otimes y) \cdot x = v \otimes yx.$$

We also recall from [FZ] that the lowest weight spaces L_{λ_2} and L_{λ_3} of $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})$ and $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3})$, respectively, are (left) $A(L_{\widehat{\mathfrak{g}}}(\ell, 0))$-modules in the natural way.

The description of the space of intertwining operators of type $\binom{L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3})}{L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})\, L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})}$ from [Li2] (see also [FZ]) is as follows:

$$V_{L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})\, L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})}^{L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3})} \cong \mathrm{Hom}_{A(L_{\widehat{\mathfrak{g}}}(\ell, 0))}(A(L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})) \otimes_{A(L_{\widehat{\mathfrak{g}}}(\ell, 0))} L_{\lambda_2}, L_{\lambda_3}).$$

This space of $A(L_{\widehat{\mathfrak{g}}}(\ell, 0))$-homomorphisms can be described more usefully using the following lemma:

Lemma 3.1.1. *As modules for $A(L_{\widehat{\mathfrak{g}}}(\ell, 0)) \cong U(\mathfrak{g})/\langle x_\theta^{\ell+1}\rangle$,*

$$A(L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})) \otimes_{A(L_{\widehat{\mathfrak{g}}}(\ell,0))} L_{\lambda_2} \cong (L_{\lambda_1} \otimes L_{\lambda_2})/W,$$

with W the $U(\mathfrak{g})/\langle x_\theta^{\ell+1}\rangle$-module generated by all vectors of the form $v_{\lambda_1} \otimes x_\theta^{\ell-\langle\lambda_1,\theta\rangle+1} \cdot w$ for $w \in L_{\lambda_2}$.

Proof. We know that $A(L_{\widehat{\mathfrak{g}}}(\ell, \lambda_1)) \cong (L_{\lambda_1} \otimes U(\mathfrak{g}))/W'$, where W' is the subbimodule generated by $v_{\lambda_1} \otimes x_\theta^{\ell-\langle\lambda_1,\theta\rangle+1}$. We first define a map

$$\Phi : (L_{\lambda_1} \otimes U(\mathfrak{g})) \otimes_{U(\mathfrak{g})} L_{\lambda_2} \to (L_{\lambda_1} \otimes L_{\lambda_2})/W$$

as follows: for $u = v \otimes x$ where $v \in L_{\lambda_1}$ and $x \in U(\mathfrak{g})$, and for $w \in L_{\lambda_2}$, we define

$$\Phi(u \otimes w) = v \otimes x \cdot w + W.$$

By the left $U(\mathfrak{g})$-module structure on $(L_{\lambda_1} \otimes U(\mathfrak{g})) \otimes_{U(\mathfrak{g})} L_{\lambda_2}$ and the tensor product $\mathfrak{g}$-module structure on $(L_{\lambda_1} \otimes L_{\lambda_2})/W$, it is easy to see that Φ is a $U(\mathfrak{g})$-homomorphism. Moreover, Φ induces a $U(\mathfrak{g})/\langle x_\theta^{\ell+1}\rangle$-module homomorphism

$$\varphi : (L_{\lambda_1} \otimes U(\mathfrak{g}))/W' \otimes_{U(\mathfrak{g})/\langle x_\theta^{\ell+1}\rangle} L_{\lambda_2}$$

because for $y, y' \in U(\mathfrak{g})$ and $w \in L_{\lambda_2}$,

$$\begin{aligned}
&\Phi(y \cdot (v_{\lambda_1} \otimes x_\theta^{\ell-\langle\lambda_1,\theta\rangle+1}) \cdot y' \otimes w) = \\
&\Phi((y \cdot v_{\lambda_1}) \otimes x_\theta^{\ell-\langle\lambda_1,\theta\rangle+1} y' \otimes w + v_{\lambda_1} \otimes y x_\theta^{\ell-\langle\lambda_1,\theta\rangle+1} y') \otimes w = \\
&(y \cdot v_{\lambda_1}) \otimes x_\theta^{\ell-\langle\lambda_1,\theta\rangle+1} y') \cdot w + W + v_{\lambda_1} \otimes (y x_\theta^{\ell-\langle\lambda_1,\theta\rangle+1} y') \cdot w + W = \\
&y \cdot (v_{\lambda_1} \otimes (x_\theta^{\ell-\langle\lambda_1,\theta\rangle+1} y') \cdot w) + W = 0.
\end{aligned}$$

To obtain an inverse homomorphism, we define a $U(\mathfrak{g})$-homomorphism

$$\Psi : L_{\lambda_1} \otimes L_{\lambda_2} \to (L_{\lambda_1} \otimes U(\mathfrak{g}))/W' \otimes_{U(\mathfrak{g})/\langle x_\theta^{\ell+1}\rangle} L_{\lambda_2}$$

by defining for $u \in V_{\lambda_1}$ and $w \in V_{\lambda_2}$:

$$\Psi(u \otimes w + W) = (u \otimes 1 + W') \otimes w.$$

The map Ψ induces a $U(\mathfrak{g})/\langle x_\theta^{\ell+1}\rangle$-homomorphism

$$\psi : (L_{\lambda_1} \otimes L_{\lambda_2})/W \to (L_{\lambda_1} \otimes U(\mathfrak{g}))/W' \otimes_{U(\mathfrak{g})/\langle x_\theta^{\ell+1}\rangle} L_{\lambda_2}$$

because for $x \in U(\mathfrak{g})$ and $w \in L_{\lambda_2}$,

$$\begin{aligned}\Psi(x \cdot (v_{\lambda_1} \otimes x_\theta^{\ell-\langle\lambda_1,\theta\rangle+1} \cdot w)) &= \Psi((x \cdot v_{\lambda_1}) \otimes x_\theta^{\ell-\langle\lambda_1,\theta\rangle+1} \cdot w + v_{\lambda_1} \otimes (x x_\theta^{\ell-\langle\lambda_1,\theta\rangle+1}) \cdot w)\\ &= ((x \cdot v_{\lambda_1}) \otimes 1 + W') \otimes x_\theta^{\ell-\langle\lambda_1,\theta\rangle+1} \cdot w + (v_{\lambda_1} \otimes 1 + W') \otimes x x_\theta^{\ell-\langle\lambda_1,\theta\rangle+1} \cdot w\\ &= ((x \cdot v_{\lambda_1}) \otimes x_\theta^{\ell-\langle\lambda_1,\theta\rangle+1} + W') \otimes w + (v_{\lambda_1} \otimes x x_\theta^{\ell-\langle\lambda_1,\theta\rangle+1} + W') \otimes w\\ &= (x \cdot (v_{\lambda_1} \otimes x_\theta^{\ell-\langle\lambda_1,\theta\rangle+1}) + W') \otimes w = 0.\end{aligned}$$

From the definitions, it is easy to see that φ and ψ are inverses of each other, so they give $A(L_{\widehat{\mathfrak{g}}}(\ell,0)) \cong U(\mathfrak{g})/\langle x_\theta^{\ell+1}\rangle$-module isomorphisms. $\square$

From this lemma, the space of intertwining operators of type $\binom{L_{\widehat{\mathfrak{g}}}(\ell,L_{\lambda_3})}{L_{\widehat{\mathfrak{g}}}(\ell,L_{\lambda_1})\, L_{\widehat{\mathfrak{g}}}(\ell,L_{\lambda_2})}$ is linearly isomorphic to

$$\mathrm{Hom}_{\mathfrak{g}}((L_{\lambda_1} \otimes L_{\lambda_2})/W, L_{\lambda_3}).$$

From [Li2] and [FZ], we can describe the isomorphism as follows. An intertwining operator $\mathcal{Y}$ of type $\binom{L_{\widehat{\mathfrak{g}}}(\ell,L_{\lambda_3})}{L_{\widehat{\mathfrak{g}}}(\ell,L_{\lambda_1})\, L_{\widehat{\mathfrak{g}}}(\ell,L_{\lambda_2})}$ induces a $\mathfrak{g}$-homomorphism

$$\pi(\mathcal{Y}) : L_{\lambda_1} \otimes L_{\lambda_2} \to L_{\lambda_3}$$

given by

$$\pi(\mathcal{Y})(w_{(1)} \otimes w_{(2)}) = w_{(1)}(0)w_{(2)} \tag{3.4}$$

using the notation of (2.20), where $w_{(1)} \in L_{\lambda_1} \subseteq L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})$ and $w_{(2)} \in L_{\lambda_1} \subseteq L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})$. The homomorphism $\pi(\mathcal{Y})$ will equal 0 on W, so it induces a homomorphism from $(L_{\lambda_1} \otimes L_{\lambda_2})/W$ to L_{λ_3}.

3.2 Construction of integral forms for general $\mathfrak{g}$

In this section we construct integral forms in integral level vertex (operator) algebras and there modules based on an affine Lie algebra $\widehat{\mathfrak{g}}$. We will need to use an appropriate integral form of the universal enveloping algebra $U(\widehat{\mathfrak{g}})$. In particular, we need a subring $U_{\mathbb{Z}}(\widehat{\mathfrak{g}}) \subseteq U(\widehat{\mathfrak{g}})$ (containing 1) which is an integral form of $U(\widehat{\mathfrak{g}})$ as a vector space.

For any finite-dimensional Lie algebra $\mathfrak{g}$ we will need an integral form $U_{\mathbb{Z}}(\widehat{\mathfrak{g}})$ of the universal enveloping algebra $U(\widehat{\mathfrak{g}})$ such that

$$U_{\mathbb{Z}}(\widehat{\mathfrak{g}}) = U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_-)U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_0)U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_+), \tag{3.5}$$

where $U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_\pm)$ and $U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_0)$ are integral forms of $U(\widehat{\mathfrak{g}}_\pm)$ and $U(\widehat{\mathfrak{g}}_0)$, respectively. We also assume that $U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_\pm)$ are graded in the sense that

$$U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_\pm) = \coprod_{n \in \pm\mathbb{N}} U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_\pm) \cap U(\widehat{\mathfrak{g}}_\pm)_n \tag{3.6}$$

where $U(\widehat{\mathfrak{g}}_\pm)_n$ is the space spanned by monomials of the form

$$(g_1 \otimes t^{n_1}) \cdots (g_k \otimes t^{n_k})$$

where $g_1, \ldots, g_k \in \mathfrak{g}$ and $n_1 + \ldots + n_k = n$.

We will show a general method for obtaining an integral form of $U(\widehat{\mathfrak{g}})$ satisfying (3.5) and (3.6), assuming that $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ has an integral form. That is, we assume that $\mathfrak{g}$ has a basis whose $\mathbb{Z}$-span $\mathfrak{g}_{\mathbb{Z}}$ is closed under brackets and such that the form $\langle \cdot, \cdot \rangle$ is $\mathbb{Z}$-valued on $\mathfrak{g}_{\mathbb{Z}}$. Then $\widehat{\mathfrak{g}}$ also has the integral form

$$\widehat{\mathfrak{g}}_{\mathbb{Z}} = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}] \oplus \mathbb{Z}\mathbf{k}.$$

Example 3.2.1. If $\mathfrak{h}$ is a finite-dimensional abelian Lie algebra with a symmetric non-degenerate form, any full-rank integral lattice $L \subseteq \mathfrak{h}$ is an integral form of $\mathfrak{h}$.

Proposition 3.2.2. *Assume $\{u_i\}$ is a $\mathbb{Z}$-basis for $\mathfrak{g}_{\mathbb{Z}}$. Then the $\mathbb{Z}$-span $U_{\mathbb{Z}}(\widehat{\mathfrak{g}})$ of monomials in the elements $u_i \otimes t^n$ for $n \in \mathbb{Z}$ and $\mathbf{k}$ is an integral form of $U(\widehat{\mathfrak{g}})$ satisfying (3.5) and (3.6).*

Proof. By definition, $U_{\mathbb{Z}}(\widehat{\mathfrak{g}})$ is a subring of $U(\widehat{\mathfrak{g}})$ which contains 1. The integral form $\widehat{\mathfrak{g}}_{\mathbb{Z}}$ has the $\mathbb{Z}$-basis $\{u_i \otimes t^n, \mathbf{k}\}$, which we order in such a way that

$$u_i \otimes t^m < u_j \otimes t^n$$

if $m < n$ and

$$u_i \otimes t^{-m} < \mathbf{k} < u_j \otimes t^n$$

if $m, n > 0$. Then the Poincaré-Birkhoff-Witt Theorem implies that ordered monomials in the elements $u_i \otimes t^n$ and $\mathbf{k}$ form a basis for $U(\widehat{\mathfrak{g}})$, and hence the $\mathbb{Z}$-span of ordered monomials is an integral form of $U(\widehat{\mathfrak{g}})$ as a vector space. But in fact $U_{\mathbb{Z}}(\widehat{\mathfrak{g}})$ equals the $\mathbb{Z}$-span of ordered monomials because any unordered monomial can be straightened: in any monomial, the order of any two basis elements of $\widehat{\mathfrak{g}}_{\mathbb{Z}}$ can be switched at the cost of a commutator term which is an integral linear combination of monomials of lower degree.

Moreover, taking $U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_{\pm})$ to be the $\mathbb{Z}$-span of monomials in $u_i \otimes t^{\pm n}$ where $n > 0$ and taking $U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_0)$ to be the $\mathbb{Z}$-span of monomials in $u_i \otimes t^0$ and $\mathbf{k}$, the conditions on the order of the basis elements of $\widehat{\mathfrak{g}}_{\mathbb{Z}}$ imply the decomposition (3.5). Also, since $U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_{\pm})$ are spanned by monomials in the elements $u_i \otimes t^n$, (3.6) holds as well. $\square$

We will now use an integral form $U_{\mathbb{Z}}(\widehat{\mathfrak{g}})$ satisfying (3.5) and (3.6) to obtain integral forms in generalized Verma modules $V_{\widehat{\mathfrak{g}}}(\ell, U)$ and their irreducible quotients. By an integral form $U_{\mathbb{Z}}$ of a $\widehat{\mathfrak{g}}_0$-module U, we will mean an integral form of U as a vector space which is invariant under $U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_0)$, where $U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_0)$ is as in (3.5). For example, if $U_{\mathbb{Z}}(\widehat{\mathfrak{g}})$ is constructed as in Proposition 3.2.2 and U as a vector space has an integral form $U_{\mathbb{Z}}$ which is a $\mathfrak{g}_{\mathbb{Z}}$-module, and $\mathbf{k}$ acts on U as an integer, then $U_{\mathbb{Z}}$ is an integral form of U.

Theorem 3.2.3. *Suppose $U_{\mathbb{Z}}$ is an integral form of a finite-dimensional $\widehat{\mathfrak{g}}_0$-module U on which $\mathbf{k}$ acts as ℓ, and suppose W is $V_{\widehat{\mathfrak{g}}}(\ell, U)$ or its irreducible quotient $L_{\widehat{\mathfrak{g}}}(\ell, U)$. Then $W_{\mathbb{Z}} = U_{\mathbb{Z}}(\widehat{\mathfrak{g}})U_{\mathbb{Z}}$ is an integral form of W as a vector space, and $W_{\mathbb{Z}}$ is compatible with the conformal weight grading of W. Moreover, $W_{\mathbb{Z}}$ is invariant under $U_{\mathbb{Z}}(\widehat{\mathfrak{g}})$.*

Proof. By definition, $W_{\mathbb{Z}}$ is invariant under $U_{\mathbb{Z}}(\widehat{\mathfrak{g}})$. Also, since $\widehat{\mathfrak{g}}_+ \cdot U = 0$ and since $U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_0)U_{\mathbb{Z}} \subseteq U_{\mathbb{Z}}$,

$$U_{\mathbb{Z}}(\widehat{\mathfrak{g}})U_{\mathbb{Z}} = U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_-)U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_0)U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_+)U_{\mathbb{Z}} = U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_-)U_{\mathbb{Z}}.$$

Since we assume $U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_-)$ is graded as in (3.6), $W_{\mathbb{Z}}$ is graded by conformal weight and the intersection of $W_{\mathbb{Z}}$ with each weight space is spanned by finitely many vectors. In fact, an upper bound on the number of vectors required to span the weight space of weight n higher than the lowest weight space is $\dim U(\widehat{\mathfrak{g}}_-)_n \cdot \dim U < \infty$. Moreover, since $W = U(\widehat{\mathfrak{g}}_-)U$, it follows that $W_{\mathbb{Z}}$ spans W as a vector space. To prove the theorem, that is, to prove that W is linearly isomorphic to $\mathbb{C} \otimes_{\mathbb{Z}} W_{\mathbb{Z}}$, we need to show that if any vectors in $W_{\mathbb{Z}}$ are linearly independent over $\mathbb{Z}$, then they are also linearly independent over $\mathbb{C}$.

In the case that $W = V_{\widehat{\mathfrak{g}}}(\ell, U)$, then $W_{\mathbb{Z}}$ is isomorphic as a $\mathbb{Z}$-module to $U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_-) \otimes_{\mathbb{Z}} U_{\mathbb{Z}}$, which is a free graded $\mathbb{Z}$-module whose weight spaces have rank equal to the dimension of the weight spaces of $W \cong U(\widehat{\mathfrak{g}}_-) \otimes_{\mathbb{C}} U$ since $U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_-)$ is a graded integral form of $U(\widehat{\mathfrak{g}}_-)$ and $U_{\mathbb{Z}}$ is an integral form of U. This proves the theorem in this case.

In the case that $W = V_{\widehat{\mathfrak{g}}}(\ell, U)$, we use essentially the same proof as in Theorem 11.3 in [G], which is an analogue of Lemma 12 in [S]; see also Theorem 27.1 in [Hu]. We first observe that $W_{\mathbb{Z}} \cap U = U_{\mathbb{Z}}$, and we use $\{u_j\}$ to denote a $\mathbb{Z}$-base of $U_{\mathbb{Z}}$. For any $w \in W_{\mathbb{Z}}$, we use $(w)_{U,j}$ for any j to denote the component of w in $\mathbb{Z}u_j$.

Now we prove that $W_{\mathbb{Z}}$ is an integral form of W by contradiction. Suppose that $\{w_i\}_{i=1}^k$ is a minimal set of (non-zero) vectors contained in $W_{\mathbb{Z}}$ that is linearly independent over $\mathbb{Z}$ but not over $\mathbb{C}$; note that for this to happen, $k \geq 2$. Then

$$\sum_{i=1}^{k} c_i w_i = 0$$

for $c_i \in \mathbb{C}^{\times}$. There is some $y \in U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_+)$ such that the component of $y \cdot w_1$ in $U_{\mathbb{Z}}$ is non-zero, because otherwise w would generate a proper $\widehat{\mathfrak{g}}$-submodule in W, which is impossible since $W = L_{\widehat{\mathfrak{g}}}(\ell, U)$ is irreducible. In particular for some j, $(y \cdot w_1)_{U,j} =$

m_1u_j with $m_1 \neq 0$. For $2 \leq i \leq k$, set $(y \cdot w_i)_{U,j} = m_iu_j$ where $m_i \in \mathbb{Z}$. Thus, since

$$0 = \left(y \cdot \sum_{i=1}^{k} c_iw_i\right)_{U,j} = \sum_{i=1}^{k} c_i(y \cdot w_i)_{U,j} = \left(\sum_{i=1}^{k} c_im_i\right)u_j,$$

we obtain $\sum_{i=1}^{k} c_im_i = 0$. Thus

$$0 = m_1\left(\sum_{i=1}^{k} c_iw_i\right) - \left(\sum_{i=1}^{k} c_im_i\right)w_1 = \sum_{i=1}^{k}(m_1c_iw_i - m_ic_iw_1) = \sum_{i=2}^{k} c_i(m_1w_i - m_iw_1).$$

The vectors $m_1w_i - m_iw_1$ for $2 \leq i \leq k$ are thus in $W_{\mathbb{Z}}$, linearly independent over $\mathbb{Z}$ (since $m_1 \neq 0$), and linearly dependent over $\mathbb{C}$, which contradicts the minimality of $\{w_i\}_{i=1}^{k}$. $\square$

We can now prove a general result on vertex algebraic integral forms that applies to any finite-dimensional Lie algebra $\mathfrak{g}$ having an integral form.

Proposition 3.2.4. *Suppose $\mathfrak{g}_{\mathbb{Z}}$ is an integral form of $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, $U_{\mathbb{Z}}(\widehat{\mathfrak{g}})$ is constructed as in Proposition 3.2.2, and $\ell \in \mathbb{Z}$. Then the integral form $V_{\widehat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$ given by Theorem 3.2.3 is the vertex subring generated by the vectors $a(-1)\mathbf{1}$ for $a \in \mathfrak{g}_{\mathbb{Z}}$. Moreover, if U is a finite-dimensional $\mathfrak{g}$-module with integral form $U_{\mathbb{Z}}$, $V_{\widehat{\mathfrak{g}}}(\ell, U)_{\mathbb{Z}}$ and $L_{\widehat{\mathfrak{g}}}(\ell, U)_{\mathbb{Z}}$ are the $V_{\widehat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$-modules generated by $U_{\mathbb{Z}}$.*

Proof. Let W be the module $V_{\widehat{\mathfrak{g}}}(\ell, U)$ or $L_{\widehat{\mathfrak{g}}}(\ell, U)$, where U is possibly $\mathbb{C}1_\ell$. From the construction of $U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_-)$ in the proof of Proposition 3.2.2,

$$W_{\mathbb{Z}} = U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_-)U_{\mathbb{Z}}$$

is the integral span of vectors of the form

$$a_1(-n_1)\cdots a_k(-n_k)u \tag{3.7}$$

where $a_i \in \mathfrak{g}_{\mathbb{Z}}$, $n_i > 0$, and $u \in U_{\mathbb{Z}}$. On the other hand, since

$$Y(a(-1)\mathbf{1}, x) = \sum_{n\in\mathbb{Z}} a(n)x^{-n-1}$$

for $a \in \mathfrak{g}$, Proposition 2.2.4 implies that the vertex subring in $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ generated by the $a(-1)\mathbf{1}$ is the integral span of vectors of the form

$$a_1(n_1) \cdots a_k(n_k)\mathbf{1} \tag{3.8}$$

for $a_i \in \mathfrak{g}_{\mathbb{Z}}$ and $n_i \in \mathbb{Z}$. This is the same as (3.7) in the case that $U = \mathbb{C}1_\ell$ because $a(n)\mathbf{1} = 0$ if $n \geq 0$ and any $a_i(n_i)$ occurring in (3.8) with $n_i > 0$ can be moved to the right using the commutation relations (3.1). This proves the first assertion of the proposition, and the second follows similarly using the second assertion of Proposition 2.2.4. □

Corollary 3.2.5. *In the setting of Proposition 3.2.4, $L_{\widehat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$ is the vertex subring of $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ generated by vectors of the form $a(-1)\mathbf{1}$ for $a \in \mathfrak{g}_{\mathbb{Z}}$.*

Proof. This follows because the vertex subring generated by vectors of the form $a(-1)\mathbf{1}$ for $a \in \mathfrak{g}_{\mathbb{Z}}$ is the integral span of vectors of the form (3.8), which is also equal to $L_{\widehat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$, just as in the proof of Proposition 3.2.4. □

3.3 Construction of integral forms for finite-dimensional simple $\mathfrak{g}$

In this section, we construct integral forms in integral level vertex (operator) algebras and modules based on an affine Lie algebra $\widehat{\mathfrak{g}}$ where $\mathfrak{g}$ is a finite-dimensional simple Lie algebra. Suppose $\mathfrak{g}$ has Cartan subalgebra $\mathfrak{h}$ and root system Δ. As in Section 3.1, we take a nondegenerate invariant bilinear form on $\mathfrak{g}$ which is normalized so that

$$\langle \alpha, \alpha \rangle = 2$$

for long roots α. We recall the Cartan integers: for any $\alpha, \beta \in \Delta$,

$$A_{\alpha,\beta} = \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}.$$

For every $\alpha \in \Delta$, we have a unique vector $t_\alpha \in \mathfrak{h}$ which satisfies

$$\langle t_\alpha, h \rangle = \alpha(h)$$

for any $h \in \mathfrak{h}$. We will need the coroots h_α of $\mathfrak{g}$, which are defined by

$$h_\alpha = \frac{2t_\alpha}{\langle \alpha, \alpha \rangle};$$

note that by definition, $\alpha(h_\beta) = A_{\alpha,\beta}$ for any $\alpha, \beta \in \Delta$.

We now recall some properties of a Chevalley basis for $\mathfrak{g}$ (see for example [Hu] Chapter 7 or [FLM2] Chapter 6 for more details). A Chevalley basis contains a root vector x_α for each $\alpha \in \Delta$ and the coroots $h_i = h_{\alpha_i}$, where the α_i form a set of simple roots. Let $\mathfrak{g}_\mathbb{Z}$ denote the $\mathbb{Z}$-span of a Chevalley basis and let $\mathfrak{h}_\mathbb{Z} = \mathfrak{g}_\mathbb{Z} \cap \mathfrak{h}$. Then in fact $\mathfrak{h}_\mathbb{Z}$ is the $\mathbb{Z}$-span of all the coroots of $\mathfrak{g}$. The brackets between elements of $\mathfrak{g}_\mathbb{Z}$ satisfy

$$[h_\alpha, h_\beta] = 0$$

for any $\alpha, \beta \in \Delta$,

$$[h_\alpha, x_\beta] = \beta(h_\alpha)x_\beta = A_{\beta,\alpha}x_\beta$$

for any $\alpha, \beta \in \Delta$, and

$$[x_\alpha, x_\beta] = \begin{cases} \pm x_{\alpha+\beta} & \text{if} \quad \alpha + \beta \in \Delta \\ \pm h_\alpha & \text{if} \quad \alpha + \beta = 0 \\ 0 & \text{if} \quad \alpha + \beta \notin \Delta \cup \{0\} \end{cases}$$

for any $\alpha, \beta \in \Delta$. In particular, $\mathfrak{g}_\mathbb{Z}$ is closed under the bracket. To show that $\mathfrak{g}_\mathbb{Z}$ is an integral form of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$, we also need that $\langle \cdot, \cdot \rangle$ is $\mathbb{Z}$-valued on $\mathfrak{g}_\mathbb{Z}$:

Proposition 3.3.1. *If $\mathfrak{g}_\mathbb{Z}$ is the $\mathbb{Z}$-span of a Chevalley basis for a finite-dimensional simple Lie algebra $\mathfrak{g}$ and if $\langle \cdot, \cdot \rangle$ is scaled so that long roots have square length 2, then $\langle \cdot, \cdot \rangle$ is $\mathbb{Z}$-valued on $\mathfrak{g}_\mathbb{Z}$.*

Proof. We just need to check that $\langle x, y \rangle \in \mathbb{Z}$ if $x, y \in \mathfrak{h}_\mathbb{Z}$ or if $x, y \in \mathfrak{g}_\mathbb{Z}$ are in root spaces corresponding to opposite roots. First, for any $\alpha, \beta \in \Delta$,

$$\langle h_\alpha, h_\beta \rangle = \frac{2\langle t_\alpha, h_\beta \rangle}{\langle \alpha, \alpha \rangle} = \frac{2}{\langle \alpha, \alpha \rangle} A_{\alpha,\beta} \in \mathbb{Z}$$

since $\frac{2}{\langle\alpha,\alpha\rangle} = 1, 2, \text{ or } 3$ according as α is a long root, a short root in types C, D, F, or a short root in type G. Now, for any $\alpha \in \Delta$,

$$\begin{aligned}\langle x_\alpha, x_{-\alpha}\rangle &= \frac{1}{2}\langle [h_\alpha, x_\alpha], x_{-\alpha}\rangle = \frac{1}{2}\langle h_\alpha, [x_\alpha, x_{-\alpha}]\rangle = \pm\frac{1}{2}\langle h_\alpha, h_{-\alpha}\rangle \\ &= \pm\frac{1}{\langle \alpha, \alpha\rangle} A_{\alpha,-\alpha} = \mp 1, \mp 2, \text{ or } \mp 3\end{aligned}$$

according as α is a long root, a short root in types C, D, F, or a short root in type G. This shows $\langle\cdot,\cdot\rangle$ is integral on all of $\mathfrak{g}_{\mathbb{Z}}$. $\square$

Since the $\mathbb{Z}$-span $\mathfrak{g}_{\mathbb{Z}}$ of a Chevalley basis for $\mathfrak{g}$, Proposition 3.2.4 and Corollary 3.2.5 imply that if $\ell \in \mathbb{Z}$, $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ and $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ have integral forms generated by the vectors $a(-1)\mathbf{1}$ for $a \in \mathfrak{g}_{\mathbb{Z}}$. However, when $\mathfrak{g}$ is a finite-dimensional simple Lie algebra, there is another way to obtain integral structure in modules for $\widehat{\mathfrak{g}}$ ([G]; see also [Mit], [P]). The subring $U_{\mathbb{Z}}(\widehat{\mathfrak{g}}) \subseteq U(\widehat{\mathfrak{g}})$ generated by the elements $(x_\alpha \otimes t^n)^k/k!$ for $k \geq 0$, with $n \in \mathbb{Z}$ and x_α a root vector in a Chevalley basis of $\mathfrak{g}$, is an integral form of $U(\widehat{\mathfrak{g}})$ which satisfies (3.5) and (3.6). We describe a basis of $U_{\mathbb{Z}}(\widehat{\mathfrak{g}})$ from [Mit]. Consider the Chevalley basis $\{x_\alpha, h_i\}$ of $\mathfrak{g}$, and suppose θ is the highest root of $\mathfrak{g}$. Then $\widehat{\mathfrak{g}}$ has an integral basis consisting of the vectors

$$x_\alpha \otimes t^n, \;\; h_i \otimes t^n, \;\; -h_\theta \otimes t^0 + \mathbf{k} \tag{3.9}$$

where $n \in \mathbb{Z}$. Given any order of this basis, a basis for $U_{\mathbb{Z}}(\widehat{\mathfrak{g}})$ consists of ordered products of elements of the following forms:

$$\frac{(x_\alpha \otimes t^n)^m}{m!}, \;\; \binom{h_i \otimes t^0 + m - 1}{m}, \;\; \binom{-h_\theta \otimes t^0 + \mathbf{k} + m - 1}{m} \tag{3.10}$$

where $n \in \mathbb{Z}$ and $m \geq 0$, as well as coefficients of powers of x in series of the form

$$\exp\left(\sum_{j\geq 1}(h_i \otimes t^{nj})\frac{x^j}{j}\right) \tag{3.11}$$

for $n \in \mathbb{Z} \setminus \{0\}$.

If U is a finite-dimensional irreducible $\mathfrak{g}$-module with highest weight vector v_0, then $U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_0) \cdot v_0$ is an integral form of U as long as $\mathbf{k}$ acts as an integer ℓ (see [Hu]

Chapter 7). Here $U_{\mathbb{Z}}(\widehat{\mathfrak{g}}_0) = U_{\mathbb{Z}}(\widehat{\mathfrak{g}}) \cap U(\widehat{\mathfrak{g}}_0)$, which has as a basis ordered monomials in elements of the form (3.10) where $n = 0$. Consequently any finite-dimensional $\mathfrak{g}$-module has a $U(\widehat{\mathfrak{g}}_0)$-invariant integral form since finite-dimensional $\mathfrak{g}$-modules are completely reducible. Thus we can apply Theorem 3.2.3 and conclude that for any finite-dimensional $\mathfrak{g}$-module U, the $\widehat{\mathfrak{g}}$-modules $V_{\widehat{\mathfrak{g}}}(\ell, U)$ and $L_{\widehat{\mathfrak{g}}}(\ell, U)$ when $\ell \in \mathbb{Z}$ have $U_{\mathbb{Z}}(\widehat{\mathfrak{g}})$-invariant integral forms which are compatible with the conformal weight gradings.

Proposition 3.3.2. *The integral forms $V_{\widehat{\mathfrak{g}}}(\ell, U)_{\mathbb{Z}}$ and $L_{\widehat{\mathfrak{g}}}(\ell, U)_{\mathbb{Z}}$ generated from an integral form $U_{\mathbb{Z}}$ of U by the action of $U_{\mathbb{Z}}(\widehat{\mathfrak{g}})$ are also generated from $U_{\mathbb{Z}}$ by the action of the operators $x_\alpha(-n)^m/m!$ for $\alpha \in \Delta$ and $n, m > 0$ and the coefficients of powers of x in series of the form*

$$\exp\left(\sum_{j\geq 1} h_i(-nj)\frac{x^j}{j}\right)$$

for $n > 0$.

Proof. This follows immediately from (3.10), (3.11), and an appropriate choice of order on the basis (3.9). □

We can now show that the integral forms obtained using this $U_{\mathbb{Z}}(\widehat{\mathfrak{g}})$ for a finite-dimensional simple Lie algebra $\mathfrak{g}$ have vertex algebraic integral structure:

Theorem 3.3.3. *Suppose $\ell \in \mathbb{Z}$; the integral form $V_{\widehat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$ is the vertex subring of $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ generated by the vectors $\frac{x_\alpha(-1)^k}{k!}\mathbf{1}$ where $k \geq 0$ and x_α is the root vector corresponding to the root α in the chosen Chevalley basis of $\mathfrak{g}$. Moreover, if U is a finite-dimensional $\mathfrak{g}$-module with integral form $U_{\mathbb{Z}}$ and W is $V_{\widehat{\mathfrak{g}}}(\ell, U)$ or $L_{\widehat{\mathfrak{g}}}(\ell, U)$, then $W_{\mathbb{Z}}$ is the $V_{\widehat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$-module generated by $U_{\mathbb{Z}}$.*

Proof. Since $U_{\mathbb{Z}}(\widehat{\mathfrak{g}})$ is generated as a ring by the divided powers $(x_\alpha \otimes t^n)^k/k!$ where α is a root of $\mathfrak{g}$ and $k \geq 0$, we can express $W_{\mathbb{Z}} = U_{\mathbb{Z}}(\widehat{\mathfrak{g}}) \cdot U_{\mathbb{Z}}$ as the $\mathbb{Z}$-span of products of the form

$$\frac{x_{\alpha_1}(m_1)^{k_1}}{k_1!} \cdots \frac{x_{\alpha_n}(m_n)^{k_n}}{k_n!} \cdot u, \tag{3.12}$$

where the α_i are roots of $\mathfrak{g}$, $m_i \in \mathbb{Z}$, $k_i \geq 0$, and $u \in U_{\mathbb{Z}}$ (where U could be $\mathbb{C}1_\ell$, in which case we can take $u = \mathbf{1}$). By Proposition 2.2.4, we need to show that the $\mathbb{Z}$-span of such products equals the $\mathbb{Z}$-span of coefficients of products of the form

$$Y\left(\frac{x_{\alpha_1}(-1)^{k_1}}{k_1!}, x_1\right) \cdots Y\left(\frac{x_{\alpha_n}(-1)^{k_n}}{k_n!}, x_n\right) u. \tag{3.13}$$

First we will analyze the vertex operator associated to a generator $x_\alpha(-1)^k/k!$, where α is any root of $\mathfrak{g}$. First, observe that for any $m, n \in \mathbb{Z}$,

$$[x_\alpha(m), x_\alpha(n)] = [x_\alpha, x_\alpha](m+n) + \langle x_\alpha, x_\alpha \rangle m \delta_{m+n,0} = 0.$$

By (3.3), this means that

$$[Y(x_\alpha(-1)\mathbf{1}, x_1), Y(x_\alpha(-1)\mathbf{1}, x_2)] = 0.$$

Thus for any $k \geq 0$, the product $Y(x_\alpha(-1)\mathbf{1}, x)^k$ is well-defined and equals the normal-ordered product ${}^\circ_\circ Y(x_\alpha(-1)\mathbf{1}, x)^k {}^\circ_\circ$ (see [LL] (3.8.4)). Then we can apply Proposition 3.10.2 in [LL] to conclude that

$$\begin{aligned} Y\left(\frac{x_\alpha(-1)^k}{k!}\mathbf{1}, x\right) &= \frac{Y(x_\alpha(-1)\mathbf{1}, x)^k}{k!} \\ &= \frac{1}{k!}\left(\sum_{n_1 \in \mathbb{Z}} x_\alpha(n_1) x^{-n_1-1}\right) \cdots \left(\sum_{n_k \in \mathbb{Z}} x_\alpha(n_k) x^{-n_k-1}\right) \\ &= \frac{1}{k!} \sum_{l \in \mathbb{Z}} \left(\sum_{n_1+\cdots+n_k=l} x_\alpha(n_1)\cdots x_\alpha(n_k)\right) x^{-l-k}. \end{aligned} \tag{3.14}$$

Consider the coefficient of x^{-l-k} in (3.14) for any $l \in \mathbb{Z}$, that is,

$$\frac{1}{k!} \sum_{n_1+\ldots+n_k=l} x_\alpha(n_1)\cdots x_\alpha(n_k). \tag{3.15}$$

Since the $x_\alpha(n)$ commute with each other, for any $\sigma \in S_n$, $x_\alpha(n_1)\cdots x_\alpha(n_k) = x_\alpha(n_{\sigma(1)})\cdots x_\alpha(n_{\sigma(k)})$. Thus we can collect some of the terms in (3.15).

Take any "partition" of l into exactly k parts, where parts are allowed to be negative or zero, as well as positive. Suppose the distinct parts are $n_1, \ldots, n_m$ where n_j occurs i_j times, that is, $n_j \in \mathbb{Z}$, $n_1 i_1 + \ldots + n_m i_m = l$, and $i_1 + \ldots + i_m = k$. The terms in the sum in (3.15) corresponding to this partition are $x_\alpha(n_1)^{i_1}\cdots x_\alpha(n_m)^{i_m}$ and all permutations. The number of distinct permutations is

$$\binom{k}{i_1}\binom{k-i_1}{i_2}\cdots\binom{k-i_1-\ldots-i_{m-1}}{i_m}$$

(note that $\binom{k-i_1-\ldots-i_{m-1}}{i_m}=\binom{i_m}{i_m}=1$). This equals

$$\frac{k!}{(k-i_1)!\,i_1!}\frac{(k-i_1)!}{(k-i_1-i_2)!\,i_2!}\cdots 1=\frac{k!}{i_1!\,i_2!\cdots i_m!}.$$

Hence the sum of all terms in (3.15) corresponding to this partition is

$$\frac{1}{k!}\frac{k!}{i_1!\,i_2!\cdots i_m!}x_\alpha(n_1)^{i_1}\cdots x_\alpha(n_m)^{i_m}=\frac{x_\alpha(n_1)^{i_1}}{i_1!}\cdots\frac{x_\alpha(n_m)^{i_m}}{i_m!}.$$

Thus the coefficient of x^{-l-k} in $Y(x_\alpha(-1)\mathbf{1},x)^k/k!$ is

$$\sum_{\substack{\text{partitions of } l\\ \text{with } k \text{ parts}}}\frac{x_\alpha(n_1)^{i_1}}{i_1!}\cdots\frac{x_\alpha(n_m)^{i_m}}{i_m!}. \tag{3.16}$$

Considering the case $U=\mathbb{C}1_\ell$, it is clear from (3.12), (3.13) and (3.16) that the vertex subring generated by the $\frac{x_\alpha(-1)^k}{k!}\mathbf{1}$ is contained in $V_{\widehat{\mathfrak{g}}}(\ell,0)_\mathbb{Z}$. On the other hand, we can use induction on k to show that for any $m\in\mathbb{Z}$ and α a root of $\mathfrak{g}$, $x_\alpha(m)^k/k!$ preserves the vertex subring generated by the $\frac{x_\alpha(-1)^k}{k!}\mathbf{1}$. This is trivially true for $k=0$. If $k>0$, take $l=mk$ in (3.16) to conclude that

$$\frac{x_\alpha(m)^k}{k!}=\left(\frac{x_\alpha(-1)^k}{k!}\mathbf{1}\right)_{mk+k-1}-\sum_{\text{partitions}\neq(m,\ldots,m)}\frac{x_\alpha(n_1)^{i_1}}{i_1!}\cdots\frac{x_\alpha(n_m)^{i_m}}{i_m!}. \tag{3.17}$$

Since each $i_j<k$ on the right side, by induction every term on the right side preserves the vertex subring, so $x_\alpha(m)^k/k!$ does as well. Since $\mathbf{1}$ is in any vertex subring, (3.12) with $u=\mathbf{1}$ now implies that $V_{\widehat{\mathfrak{g}}}(\ell,0)_\mathbb{Z}$ is contained in the vertex subring generated by the $\frac{x_\alpha(-1)^k}{k!}\mathbf{1}$. Thus $V_{\widehat{\mathfrak{g}}}(\ell,0)_\mathbb{Z}$ equals the vertex subring generated by the $\frac{x_\alpha(-1)^k}{k!}$.

For general finite-dimensional U, the same argument using (3.16) and (3.17) shows that the $\mathbb{Z}$-span of products of the form (3.12) equals the $\mathbb{Z}$-span of coefficients of products of the form (3.13). This completes the proof of the theorem. $\square$

Corollary 3.3.4. *The integral form $L_{\widehat{\mathfrak{g}}}(\ell,0)_{\mathbb{Z}}$ of $L_{\widehat{\mathfrak{g}}}(\ell,0)$ is the integral form of $L_{\widehat{\mathfrak{g}}}(\ell,0)$ as a vertex algebra generated by the vectors $\frac{x_\alpha(-1)^k}{k!}\mathbf{1}$ where α is a root and $k \geq 0$.*

Proof. We know from the proof of Theorem 3.3.3 that $L_{\widehat{\mathfrak{g}}}(\ell,0)_{\mathbb{Z}}$ is spanned by coefficients of products of the form in (3.13) with $u = \mathbf{1}$, but by Proposition 2.2.4, this is precisely the vertex subring of $L_{\widehat{\mathfrak{g}}}(\ell,0)$ generated by the $\frac{x_\alpha(-1)^k}{k!}\mathbf{1}$. Note that although the Y in (3.13) is the vertex operator for $V_{\widehat{\mathfrak{g}}}(\ell,0)$ acting on $L_{\widehat{\mathfrak{g}}}(\ell,0)$, the vertex operator for $L_{\widehat{\mathfrak{g}}}(\ell,0)$ acting on itself is defined the same way. □

Corollary 3.3.5. *If $\mathfrak{g}$ is of type A, D, or E and ℓ is a non-negative integer, the integral form $L_{\widehat{\mathfrak{g}}}(\ell,0)_{\mathbb{Z}}$ of $L_{\widehat{\mathfrak{g}}}(\ell,0)$ is generated by the vectors $\frac{x_\alpha(-1)^k}{k!}\mathbf{1}$ where α is a root of $\mathfrak{g}$ and $0 \leq k \leq \ell$.*

Proof. This follows from the well-known fact that for any long root α, $x_\alpha(-1)^{\ell+1}\cdot v_0 = 0$, where v_0 is a highest weight vector of a standard irreducible level ℓ $\widehat{\mathfrak{g}}$-module (see for example Proposition 6.6.4 in [LL]). □

Remark 3.3.6. The integral form $L_{\widehat{\mathfrak{g}}}(\ell,0)_{\mathbb{Z}}$ in Corollary 3.3.5 is generally larger than the one constructed using the integral form of $U(\widehat{\mathfrak{g}})$ obtained in Proposition 3.2.2, although they coincide when $\mathfrak{g}$ is a finite-dimensional simple Lie algebra of type A, D, or E and $\ell = 1$.

3.4 Further properties and results

In this section, we assume that $\mathfrak{g}$ is a finite-dimensional simple complex Lie algebra with nondegenerate invariant form $\langle\cdot,\cdot\rangle$ normalized so that long roots have square length 2. Recall from Proposition 3.3.1 that $\langle\cdot,\cdot\rangle$ is integral on the $\mathbb{Z}$-span $\mathfrak{g}_{\mathbb{Z}}$ of a Chevalley basis $\{x_\alpha, h_i\}$ for $\mathfrak{g}$. We shall first apply the results from Chapter 2 on the conformal vector and contragredient modules to the vertex (operator) algebras $V_{\widehat{\mathfrak{g}}}(\ell,0)$ and $L_{\widehat{\mathfrak{g}}}(\ell,0)$ and their modules, where $\ell \in \mathbb{Z}$. Then we shall classify certain integral intertwining operators among $L_{\widehat{\mathfrak{g}}}(\ell, 0)$-modules where ℓ is a non-negative integer. Throughout, we shall use the integral forms introduced in Theorem 3.3.3 and Corollary 3.3.4.

Proposition 3.4.1. *Suppose $V_{\mathbb{Z}}$ is the integral form $V_{\widehat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$ or $L_{\widehat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$ with $\ell \in \mathbb{Z}$. Then ω is in the $\mathbb{Q}$-span $V_{\mathbb{Q}}$ of $V_{\mathbb{Z}}$ and $V_{\mathbb{Z}}$ is generated by lowest weight vectors for the Virasoro algebra, so $V_{\mathbb{Z}}$ may be extended to a larger integral form containing $k\omega$ for any $k \in \mathbb{Z}$ such that $k^2 c \in 2\mathbb{Z}$.*

Proof. If V is the vertex operator algebra $V_{\widehat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$ or $L_{\widehat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$ and $\{u_i\}$ is a Chevalley basis for $\mathfrak{g}_{\mathbb{Z}}$ with dual basis $\{u_i'\}$ with respect to the form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, then

$$\omega = \frac{1}{2(\ell + h)} \sum_{i=1}^{\dim \mathfrak{g}} u_i(-1)u_i'(-1)\mathbf{1}$$

where h is the dual Coxeter number of $\mathfrak{g}$. Since $\ell, h \in \mathbb{Z}$ and $\langle \cdot, \cdot \rangle$ is integral on $\mathfrak{g}_{\mathbb{Z}}$ so that each u_i' is in the $\mathbb{Q}$-span of $\mathfrak{g}_{\mathbb{Z}}$, we have $\omega \in V_{\mathbb{Q}}$. Moreover, $V_{\mathbb{Z}}$ is generated by the vectors $\frac{x_\alpha(-1)^k}{k!}\mathbf{1}$ for α a root, x_α the corresponding root vector in the Chevalley basis of $\mathfrak{g}$, and $k \geq 0$. The commutation relations

$$[L(m), x_\alpha(-1)] = x_\alpha(m-1)$$

for any $m \in \mathbb{Z}$ (see for example [LL] Section 6.2), and the fact that $x_\alpha(m-1)$ commutes with $x_\alpha(-1)$, imply that for $m > 0$ and $k \geq 0$,

$$L(m)\frac{x_\alpha(-1)^k}{k!}\mathbf{1} = \frac{x_\alpha(-1)^{k-1}}{(k-1)!}x_\alpha(m-1)\mathbf{1} = 0.$$

Since also $\frac{x_\alpha(-1)^k}{k!}\mathbf{1}$ is homogeneous of conformal weight k, this means $V_{\mathbb{Z}}$ is generated by lowest weight vectors for the Virasoro algebra. The last assertion now follows directly from Theorem 2.3.2. $\square$

Since $V_{\widehat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$ and $L_{\widehat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$ are generated by lowest weight vectors for the Virasoro algebra, Propositions 2.5.1 and 2.5.2 immediately imply:

Proposition 3.4.2. *The contragredient of a $V_{\widehat{\mathfrak{g}}}(\ell,0)_{\mathbb{Z}}$-module or an $L_{\widehat{\mathfrak{g}}}(\ell,0)_{\mathbb{Z}}$-module is a $V_{\widehat{\mathfrak{g}}}(\ell,0)_{\mathbb{Z}}$-module or an $L_{\widehat{\mathfrak{g}}}(\ell,0)_{\mathbb{Z}}$-module.*

Note that the contragredient of $L_{\widehat{\mathfrak{g}}}(\ell,L_\lambda)_{\mathbb{Z}}$, where λ is a dominant integral weight of $\mathfrak{g}$, is an integral form of $L_{\widehat{\mathfrak{g}}}(\ell,L_{\lambda^*})$ which is not necessarily equal to $L_{\widehat{\mathfrak{g}}}(\ell,L_{\lambda^*})_{\mathbb{Z}}$. In particular, $L_{\widehat{\mathfrak{g}}}(\ell,L_{\lambda^*})'_{\mathbb{Z}}$ is an integral form of $L_{\widehat{\mathfrak{g}}}(\ell,L_\lambda)$ which is not necessarily equal to $L_{\widehat{\mathfrak{g}}}(\ell,L_\lambda)_{\mathbb{Z}}$. Note that the lowest weight space of $L_{\widehat{\mathfrak{g}}}(\ell,L_{\lambda^*})'_{\mathbb{Z}}$ is $(L_{\lambda^*})'_{\mathbb{Z}}$, the $\mathbb{Z}$-dual of $(L_{\lambda^*})_{\mathbb{Z}}$.

Now suppose ℓ is a non-negative integer and consider the simple vertex operator algebra $L_{\widehat{\mathfrak{g}}}(\ell,0)$. We consider intertwining operators among three $L_{\widehat{\mathfrak{g}}}(\ell,0)$-modules $L_{\widehat{\mathfrak{g}}}(\ell,L_{\lambda_i})$ for $i=1,2,3$, where $\langle\lambda_i,\theta\rangle\le\ell$, θ the highest root of $\mathfrak{g}$. In the statement of the following theorem, we use $U_{\mathbb{Z}}(\mathfrak{g})$ to denote $U_{\mathbb{Z}}(\widehat{\mathfrak{g}})\cap U(\mathfrak{g})$, which by (3.10) has as a basis ordered monomials in elements of the forms

$$\frac{x_\alpha^m}{m!},\ \binom{h_i+m-1}{m}$$

where $m\ge 0$ and we identify $x_\alpha = x_\alpha\otimes t^0$ and $h_i = h_i\otimes t^0$.

Theorem 3.4.3. *The lattice of intertwining operators within $V^{L_{\widehat{\mathfrak{g}}}(\ell,L_{\lambda_3})}_{L_{\widehat{\mathfrak{g}}}(\ell,L_{\lambda_1})\,L_{\widehat{\mathfrak{g}}}(\ell,L_{\lambda_2})}$ which are integral with respect to $L_{\widehat{\mathfrak{g}}}(\ell,L_{\lambda_1})_{\mathbb{Z}}$, $L_{\widehat{\mathfrak{g}}}(\ell,L_{\lambda_2})_{\mathbb{Z}}$, and $L_{\widehat{\mathfrak{g}}}(\ell,L_{\lambda_3^*})'_{\mathbb{Z}}$ is isomorphic to*

$$\mathrm{Hom}_{U_{\mathbb{Z}}(\mathfrak{g})}(((L_{\lambda_1})_{\mathbb{Z}}\otimes_{\mathbb{Z}}(L_{\lambda_2})_{\mathbb{Z}})/W_{\mathbb{Z}},(L_{\lambda_3^*})'_{\mathbb{Z}})$$

where $W_{\mathbb{Z}}$ is the $U_{\mathbb{Z}}(\mathfrak{g})$-submodule of $(L_{\lambda_1})_{\mathbb{Z}}\otimes_{\mathbb{Z}}(L_{\lambda_2})_{\mathbb{Z}}$ generated by vectors of the form

$$v_{\lambda_1}\otimes\frac{x_\theta^{\ell-\langle\lambda_1,\theta\rangle+1}}{(\ell-\langle\lambda_1,\theta\rangle+1)!}\cdot w.$$

Here v_{λ_1} is a highest weight vector generating $(L_{\lambda_1})_{\mathbb{Z}}$ as a $U_{\mathbb{Z}}(\mathfrak{g})$-module, and $w\in(L_{\lambda_2})_{\mathbb{Z}}$.

Remark 3.4.4. We use vectors of the form $v_{\lambda_1}\otimes\frac{x_\theta^{\ell-\langle\lambda_1,\theta\rangle+1}}{(\ell-\langle\lambda_1,\theta\rangle+1)!}\cdot w$ to generate $W_{\mathbb{Z}}$, rather than vectors of the form $v_{\lambda_1}\otimes x_\theta^{\ell-\langle\lambda_1,\theta\rangle+1}\cdot w$, to avoid unnecessary torsion in the quotient $((L_{\lambda_1})_{\mathbb{Z}}\otimes_{\mathbb{Z}}(L_{\lambda_2})_{\mathbb{Z}})/W_{\mathbb{Z}}$.

Remark 3.4.5. Note that this theorem determines which intertwining operators in $V_{L_{\widehat{\mathfrak{g}}}(\ell,L_{\lambda_1})\,L_{\widehat{\mathfrak{g}}}(\ell,L_{\lambda_2})}^{L_{\widehat{\mathfrak{g}}}(\ell,L_{\lambda_3})}$ satisfy

$$\langle w'_{(3)}, \mathcal{Y}(w_{(1)},x)w_{(2)}\rangle \in \mathbb{Z}\{x\}$$

for any $w_{(1)} \in L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})_{\mathbb{Z}}$, $w_{(2)} \in L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})_{\mathbb{Z}}$, and $w'_{(3)} \in L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3^*})_{\mathbb{Z}}$, rather than which intertwining operators satisfy

$$\mathcal{Y} : L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})_{\mathbb{Z}} \otimes L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})_{\mathbb{Z}} \to L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3})_{\mathbb{Z}}\{x\}.$$

It is not clear that there will generally be any non-zero intertwining operators which satisfy the latter condition.

Proof. First suppose an intertwining operator $\mathcal{Y}$ is integral with respect to $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})_{\mathbb{Z}}$, $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})_{\mathbb{Z}}$, and $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3^*})'_{\mathbb{Z}}$. Then the map

$$\pi(\mathcal{Y}) : w_{(1)} \otimes w_{(2)} \mapsto w_{(1)}(0)w_{(2)}$$

given in (3.4) sends $(L_{\lambda_1})_{\mathbb{Z}} \otimes_{\mathbb{Z}} (L_{\lambda_2})_{\mathbb{Z}}$ into $(L_{\lambda_3^*})'_{\mathbb{Z}}$ and equals zero on $W_{\mathbb{Z}} \subseteq W = \langle v_{\lambda_1} \otimes x_\theta^{\ell-\langle\lambda_1,\theta\rangle+1} \cdot w\rangle$.

Conversely, suppose

$$f : (L_{\lambda_1})_{\mathbb{Z}} \otimes_{\mathbb{Z}} (L_{\lambda_2})_{\mathbb{Z}} \to (L_{\lambda_3^*})'_{\mathbb{Z}}$$

is a $U_{\mathbb{Z}}(\mathfrak{g})$-homomorphism which equals zero on $W_{\mathbb{Z}}$. Then f extends to a $U(\mathfrak{g})$-homomorphism from $L_{\lambda_1} \otimes L_{\lambda_2}$ to $L'_{\lambda_3^*} \cong L_{\lambda_3}$ which equals zero on W. Thus since π given in (3.4) is an isomorphism, f induces an intertwining operator

$$\mathcal{Y}_f : L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1}) \otimes L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2}) \to L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3})\{x\}$$

which has the property that

$$\langle w'_{(3)}, \mathcal{Y}(w_{(1)},x)w_{(2)}\rangle \in \mathbb{Z}\{x\} \tag{3.18}$$

for $w_{(1)} \in (L_{\lambda_1})_{\mathbb{Z}}$, $w_{(2)} \in (L_{\lambda_2})_{\mathbb{Z}}$, and $w'_{(3)} \in (L_{\lambda_3^*})_{\mathbb{Z}}$. Since $(L_{\lambda_1})_{\mathbb{Z}}$ and $(L_{\lambda_2})_{\mathbb{Z}}$ generate $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_1})_{\mathbb{Z}}$ and $L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_2})$ as $L_{\widehat{\mathfrak{g}}}(\ell, 0)_{\mathbb{Z}}$-modules, respectively, by Theorem

2.4.9 it is enough to show that (3.18) holds for $w_{(1)} \in (L_{\lambda_1})_{\mathbb{Z}}$, $w_{(2)} \in (L_{\lambda_2})_{\mathbb{Z}}$, and $w'_{(3)} \in L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3^*})_{\mathbb{Z}}$. Since (3.18) holds when $w'_{(3)} \in (L_{\lambda_3^*})_{\mathbb{Z}}$, by Proposition 3.3.2 it is enough to show that if it holds for some particular $w'_{(3)} \in L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3^*})_{\mathbb{Z}}$, then it also holds for the coefficients of powers of y in

$$\exp\left(x_\alpha(-n)y\right) \cdot w'_{(3)}$$

where α is a root and $n > 0$, and for coefficients of powers of y in

$$\exp\left(\sum_{j\geq 1} h_i(-nj)\frac{y^j}{j}\right) \cdot w'_{(3)}$$

where $n > 0$.

We will need to use the following commutator formula which holds for any $a \in \mathfrak{g}$ and $m \in \mathbb{Z}$, which follows from the Jacobi identity (2.17) by setting $v = a(-1)\mathbf{1}$ and taking the coefficient of $x_0^{-1}x_1^{-m-1}$: for $w_{(1)} \in L_{\lambda_1}$,

$$[a(m), \mathcal{Y}(w_{(1)}, x)] = x^m \mathcal{Y}(a(0)w_{(1)}, x).$$

Then if also $w_{(2)} \in L_{\lambda_2}$ and $m > 0$,

$$a(m)\mathcal{Y}(w_{(1)}, x)w_{(2)} = x^m \mathcal{Y}(a(0)w_{(1)}, x)w_{(2)}. \tag{3.19}$$

We will also use the fact that for any $L_{\widehat{\mathfrak{g}}}(\ell, 0)$-module W, $w \in W$, $w' \in W'$, $a \in \mathfrak{g}$, and $n \in \mathbb{Z}$,

$$\langle a(n)w', w\rangle = \langle w' - a(-n)w\rangle.$$

This follows from (2.7) and (3.3).

Now suppose (3.18) holds for $w'_{(3)} \in L_{\widehat{\mathfrak{g}}}(\ell, L_{\lambda_3^*})_{\mathbb{Z}}$ and $n > 0$; then by (3.19),

$$\begin{aligned}\langle \exp(x_\alpha(-n)y) \cdot w'_{(3)}, \mathcal{Y}(w_{(1)}, x)w_{(2)} &= \langle w'_{(3)}, \exp(-x_\alpha(n)y)\mathcal{Y}(w_{(1)}, x)w_{(2)}\rangle \\ &= \langle w'_{(3)}, \mathcal{Y}(\exp(-x_\alpha(0)x^n y) \cdot w_{(1)}, x)w_{(2)} \in \mathbb{Z}\{x, y\}\end{aligned}$$

because for any $m \geq 0$, $\frac{x_\alpha(0)^m}{m!} \cdot w_{(1)} \in (L_{\lambda_1})_{\mathbb{Z}}$. Additionally, for each $n > 0$,

$$\left\langle \exp\left(\sum_{j\geq 1} h_i(-nj)\frac{y^j}{j}\right) \cdot w'_{(3)}, \mathcal{Y}(w_{(1)}, x)w_{(2)} \right\rangle =$$

$$\left\langle w'_{(3)}, \exp\left(\sum_{j\geq 1} -h_i(nj)\frac{y^j}{j}\right) \mathcal{Y}(w_{(1)}, x)w_{(2)} \right\rangle =$$

$$\left\langle w'_{(3)}, \mathcal{Y}(\exp\left(-h_i(0)\sum_{j\geq 1}(-1)^j\frac{(-x^n y)^j}{j}\right) \cdot w_{(1)}, x)w_{(2)} \right\rangle =$$

$$\langle w'_{(3)}, \mathcal{Y}(\exp(h_i(0)\log(1 - x^n y)) \cdot w_{(1)}, x)w_{(2)}\rangle =$$

$$\langle w'_{(3)}, \mathcal{Y}((1 - x^n y)^{h_i(0)} \cdot w_{(1)}, x)w_{(2)}\rangle \in \mathbb{Z}\{x, y\}$$

because

$$(1 - x^n y)^{h_i(0)} = \sum_{k\geq 0}(-1)^k \binom{h_i(0)}{k}(x^n y)^k.$$

Note that $\binom{h_i(0)}{k} \cdot w_{(1)} \in (L_{\lambda_1})_{\mathbb{Z}}$ because λ_1 is a dominant integral weight of $\mathfrak{g}$, and thus $h_i(0)$ acts on basis elements of $(L_{\lambda_1})_{\mathbb{Z}}$ as integers. This completes the proof. $\square$

Chapter 4

Integral forms in vertex (operator) algebras and modules based on even lattices

In this chapter we study integral forms in conformal vertex algebras and modules constructed from even lattices. Integral forms in vertex algebras based on even lattices were first studied in [B], but here we provide a new proof of the construction based on generators, and we prove the existence of integral forms in modules. We also show when the standard integral form of a lattice conformal vertex algebra contains the conformal vector ω, and we show the existence of integral intertwining operators among modules for a lattice vertex algebra.

4.1 Conformal vertex algebras based on even lattices

We recall the construction of conformal vertex algebras from even lattices. Suppose L is a nondegenerate even lattice with symmetric bilinear form $\langle\cdot,\cdot\rangle$. Consider also the space $\mathfrak{h} = \mathbb{C}\otimes_{\mathbb{Z}} L$, an abelian Lie algebra with a (trivially) invariant form. Thus we can form the Heisenberg vertex operator algebra $V_{\widehat{\mathfrak{h}}}(1,0)$, which is linearly isomorphic to $S(\widehat{\mathfrak{h}}_-)$, the symmetric algebra on $\widehat{\mathfrak{h}}_-$. We can also construct the twisted group algebra $\mathbb{C}\{L\}$ as follows: given a positive integer s, take a central extension of L by the cyclic group of s elements:

$$1 \to \langle \kappa \mid \kappa^s = 1\rangle \to \widehat{L} \xrightarrow{-} L \to 1. \tag{4.1}$$

If ω_s is a primitive sth root of unity, let $\mathbb{C}_{\omega_s}$ denote the one-dimensional $\langle\kappa\rangle$-module on which κ acts as ω_s. Then the twisted group algebra is the induced $\widehat{L}$-module

$$\mathbb{C}\{L\} = \mathbb{C}[\widehat{L}] \otimes_{\mathbb{C}[\langle\kappa\rangle]} \mathbb{C}_{\omega_s}.$$

It is linearly isomorphic to the group algebra $\mathbb{C}[L]$. We need to choose the integer s and the central extension so that the commutator map c_0, defined by the condition $ab = ba\kappa^{c_0(\bar{a},\bar{b})}$ for $a, b \in \widehat{L}$, satisfies the condition

$$\omega_s^{c_0(\alpha,\beta)} = (-1)^{\langle \alpha,\beta \rangle}$$

for $\alpha, \beta \in L$. For instance, since L is even, we can take $s = 2$ and $c_0(\alpha, \beta) = \langle \alpha, \beta \rangle$ (mod 2). (The isomorphism class of the resulting vertex algebra does not depend on the choices of s and the central extension; see [LL], Proposition 6.5.5.) The $\widehat{L}$-module $\mathbb{C}\{L\}$ is also a module for $\mathfrak{h} = \mathfrak{h} \otimes t^0$:

$$h(0)(a \otimes 1) = \langle h, \bar{a} \rangle (a \otimes 1)$$

for $h \in \mathfrak{h}$ and $a \in \widehat{L}$. Also, for $\alpha \in L$ and x a formal variable, define a map x^α on $\mathbb{C}\{L\}$ by

$$x^\alpha (b \otimes 1) = (b \otimes 1) x^{\langle \alpha, \bar{b} \rangle}$$

for $b \in \widehat{L}$.

We can now extend the vertex operator algebra structure on $S(\widehat{\mathfrak{h}}_-)$ to the larger space

$$V_L = S(\widehat{\mathfrak{h}}_-) \otimes \mathbb{C}\{L\}.$$

For $a \in \widehat{L}$, use $\iota(a)$ to denote the element $1 \otimes (a \otimes 1) \in V_L$. As a vertex algebra, V_L is generated by the $\iota(a)$, which have vertex operators

$$Y(\iota(a), x) = E^-(-\bar{a}, x) E^+(-\bar{a}, x) a x^{\bar{a}} \tag{4.2}$$

where

$$E^{\pm}(\alpha, x) = \exp\left(\sum_{n \in \pm\mathbb{Z}_+} \frac{\alpha(n)}{n} x^{-n} \right)$$

for any $\alpha \in L$, and a denotes the action of $a \in \widehat{L}$ on $\mathbb{C}\{L\}$. The conformal element of V_L is the same as the conformal element in the Heisenberg algebra:

$$\omega = \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{h}} \alpha^{(i)}(-1) \beta^{(i)}(-1) \mathbf{1},$$

where $\{\alpha^{(i)}\}$ is a basis for $\mathfrak{h}$ and $\{\beta^{(i)}\}$ is the corresponding dual basis with respect to $\langle\cdot,\cdot\rangle$. If L is positive definite, V_L is a vertex operator algebra in the sense that the finiteness restrictions on the weight spaces hold. If L is not positive definite, V_L is still a strongly L-graded conformal vertex algebra, where for $\alpha \in L$,

$$V^\alpha = S(\widehat{\mathfrak{h}}_-) \otimes \iota(a),$$

with $a \in \widehat{L}$ such that $\bar{a} = \alpha$.

To obtain modules for V_L, consider L°, the dual lattice of L, and construct the space

$$V_{L^\circ} = S(\widehat{\mathfrak{h}}_-) \otimes \mathbb{C}\{L^\circ\}$$

in the same way we constructed V_L. In particular, we need a central extension of L° by $\langle \kappa \mid \kappa^s = 1\rangle$, s an even integer, with commutator map c_0, having the property that

$$\omega_s^{c_0(\alpha,\beta)} = (-1)^{\langle\alpha,\beta\rangle} \tag{4.3}$$

for $\alpha, \beta \in L$, where ω_s is the sth root of unity used to construct $\mathbb{C}\{L^\circ\}$. Such a central extension always exists (see Remark 6.4.12 in [LL]).

Then V_{L° is a V_L-module with the same action of $\iota(a)$ as in (4.2). For any $S \subseteq L^\circ$, denote by $\mathbb{C}\{S\}$ the subspace of $\mathbb{C}\{L^\circ\}$ spanned by elements of the form $a \otimes 1$ where $\bar{a} \in S$. Then the spaces

$$V_{\beta+L} = S(\widehat{\mathfrak{h}}_-) \otimes \mathbb{C}\{\beta + L\},$$

where β runs over coset representatives of L°/L, exhaust the irreducible V_L-modules up to equivalence.

The vertex algebra V_L (repectively, the module V_{L°) is spanned by vectors of the form

$$\alpha_1(-n_1)\cdots\alpha_k(-n_k)\iota(b)$$

where $\alpha_i \in L$, $n_i \in \mathbb{Z}_+$, and $\bar{b} \in L$ (respectively, $\bar{b} \in L^\circ$). Such a vector has conformal weight

$$n_1 + \ldots + n_k + \frac{\langle \bar{b}, \bar{b}\rangle}{2} \in \mathbb{Q}.$$

The modules V_{L° and $V_{\beta+L}$ where $\beta \in L^\circ$ are modules for V_L as strongly L-graded conformal vertex algebra since they are graded by L°:

$$V_{L^\circ} = \coprod_{\gamma \in L^\circ} V^\gamma$$

where $V^\gamma = S(\widehat{\mathfrak{h}}_-) \otimes \iota(c)$, $c \in L^\circ$ such that $\bar{c} = \gamma$, and $V_{\beta+L}$ is also graded.

We now recall the intertwining operators among V_L-modules from [DL]. For any $\beta \in L^\circ$, we have an intertwining operator

$$\mathcal{Y}_\beta : V_{\beta+L} \otimes V_{L^\circ} \to V_{L^\circ}\{x\},$$

where for any $c \in \widehat{L^\circ}$ such that $\bar{c} \in \beta + L$,

$$\mathcal{Y}_\beta(\iota(c), x) = E^-(-\bar{c}, x)E^+(-\bar{c}, x)cx^{\bar{c}}e^{\pi i\beta}c(\cdot, \beta). \tag{4.4}$$

Here, the operator $e^{\pi i \beta}$ on V_{L° is given by

$$e^{\pi i\beta} \cdot u \otimes \iota(c) = e^{\pi i \langle \beta, \bar{c}\rangle} u \otimes \iota(c)$$

for $u \in S(\widehat{\mathfrak{h}}_-)$, $c \in \widehat{L^\circ}$, and the operator $c(\cdot, \beta)$ is defined by

$$c(\cdot, \beta) \cdot u \otimes \iota(c) = \omega_s^{c_0(\bar{c}, \beta)} u \otimes \iota(c).$$

Then for any $\gamma \in L^\circ$, $\mathcal{Y}_\beta|_{V_{\beta+L}\otimes V_{\gamma+L}}$ is an intertwining operator of type $\binom{V_{\beta+\gamma+L}}{V_{\beta+L}\, V_{\gamma+L}}$. From [DL] we have:

Proposition 4.1.1. *([DL]) For any $\alpha, \beta, \gamma \in L^\circ$,*

$$V_{\beta\,\gamma}^{\alpha} = V_{V_{\beta+L} V_{\gamma+L}}^{V_{\alpha+L}} = \begin{cases} \mathbb{C}\mathcal{Y}_\beta|_{V_{\beta+L}\otimes V_{\gamma+L}} & \text{if} \quad \alpha = \beta+\gamma \\ 0 & \text{if} \quad \alpha \neq \beta + \gamma \end{cases}.$$

The vertex operator algebra V_L has a unique up to scale nondegenerate invariant bilinear form $(\cdot, \cdot)$, since it is simple and $(V_L)_{(0)}/L(1)(V_L)_{(1)}$ is one-dimensional ([Li1]). We normalize this form by setting $(\mathbf{1}, \mathbf{1}) = 1$, and then for any $u, v \in V_L$,

$$(u, v) = \mathrm{Res}_x\, x^{-1}(\mathbf{1}, Y(e^{xL(1)}(-x^{-2})^{L(0)}u, x^{-1})v).$$

In particular, for $a, b \in \widehat{L}$,

$$(\iota(a), \iota(b)) = \mathrm{Res}_x\, x^{-1}(\mathbf{1}, (-x^{-2})^{\langle\bar{a},\bar{a}\rangle/2} E^-(-\bar{a}, x^{-1}) x^{\langle\bar{a},\bar{b}\rangle}\iota(ab))$$
$$= (-1)^{\langle\bar{a},\bar{a}\rangle/2}\delta_{\bar{a}+\bar{b},0}(\mathbf{1}, \iota(ab)),$$

where if $\bar{a}+\bar{b} = 0$, $\iota(ab) = \iota(\kappa^n) = \omega_s^n\mathbf{1}$ for some $n \in \mathbb{Z}$, so that $(\mathbf{1}, \iota(ab)) = \omega_s^n$. Note that since for any $\beta \in L^\circ$, $\mathcal{Y}_\beta|_{V_{\beta+L}\otimes V_{-\beta+L}}$ is a non-zero intertwining operator of type $\binom{V_L}{V_{\beta+L}\, V_{-\beta+L}}$, Proposition 2.5.5 implies that $V'_{\beta+L} \cong V_{-\beta+L}$.

4.2 Construction of integral forms

In order to obtain integral structure in a lattice conformal vertex algebra V_L and its modules, we first need to show that we can choose the central extension so that we have integral structure in the twisted group algebra $\mathbb{C}\{L\}$. Given a central extension $\widehat{L^\circ}$ of L°, we can choose a section $L^\circ \to \widehat{L^\circ}$, denoted $\beta \mapsto e_\beta$ for $\beta \in L^\circ$. We define a 2-cocycle $\varepsilon_0 : L^\circ \times L^\circ \to \mathbb{Z}/s\mathbb{Z}$ by

$$e_\alpha e_\beta = e_{\alpha+\beta}\kappa^{\varepsilon_0(\alpha,\beta)}.$$

Conversely, given a 2-cocycle (such as a bilinear map) $\varepsilon_0 : L^\circ \times L^\circ \to \mathbb{Z}/s\mathbb{Z}$ we can define a central extension of L° by $\langle\kappa \,|\, \kappa^s = 1\rangle$ (see Proposition 5.1.2 in [FLM2]). Given an ε_0, define $\varepsilon(\alpha, \beta) = \omega_s^{\varepsilon_0(\alpha,\beta)}$. The following lemma is an adjustment of Remark 6.4.12 in [LL] (see also Remark 12.17 in [DL]):

Lemma 4.2.1. *There exists a central extension $\widehat{L^\circ}$ satisfying (4.3) and a section $L^\circ \to \widehat{L^\circ}$ such that $\varepsilon(\alpha, \beta) = \pm 1$ for any $\alpha, \beta \in L$.*

Proof. It is possible to choose a base $\{\alpha_1, \ldots, \alpha_l\}$ of L° so that $\{n_1\alpha_1, \ldots, n_l\alpha_l\}$, where the n_i are positive integers, forms a base for L. Since $\langle\cdot,\cdot\rangle$ is $\mathbb{Q}$-valued on L°, there is a positive even integer s such that $\frac{s}{2}\langle\alpha, \beta\rangle \in \mathbb{Z}$ for any $\alpha, \beta \in L^\circ$. Then we have the bilinear 2-cocycle $\varepsilon_0 : L^\circ \times L^\circ \to \mathbb{Z}/s\mathbb{Z}$ determined by its values on the base:

$$\varepsilon_0(\alpha_i, \alpha_j) = \begin{cases} \frac{s}{2}\langle\alpha_i, \alpha_j\rangle + s\mathbb{Z} & \text{if} \quad i < j \\ 0 & \text{if} \quad i \geq j \end{cases}.$$

Using bilinearity, we have for $i < j$

$$\varepsilon_0(n_i\alpha_i, n_j\alpha_j) = n_in_j\varepsilon(\alpha_i, \alpha_j) = n_in_j\frac{s}{2}\langle\alpha_i, \alpha_j\rangle + s\mathbb{Z} = \frac{s}{2}\langle n_i\alpha_i, n_j\alpha_j\rangle + s\mathbb{Z}.$$

Since $\langle\cdot,\cdot\rangle$ is $\mathbb{Z}$-valued and bilinear on L, it follows that $\varepsilon(\alpha, \beta) = \pm 1$ for $\alpha, \beta \in L$.

The 2-cocycle ε_0 corresponds to a section of a central extension $\widehat{L^\circ}$ of L°; we need to show that the corresponding commutator map satisfies (4.3). The commutator map c_0 is given by

$$c_0(\alpha, \beta) = \varepsilon_0(\alpha, \beta) - \varepsilon_0(\beta, \alpha) \tag{4.5}$$

for $\alpha, \beta \in L^\circ$ since

$$e_\alpha e_\beta = e_{\alpha+\beta}\kappa^{\varepsilon_0(\alpha,\beta)} = e_\beta e_\alpha \kappa^{-\varepsilon_0(\beta,\alpha)}\kappa^{\varepsilon_0(\alpha,\beta)}.$$

Thus the commutator map is an alternating $\mathbb{Z}$-bilinear form $L^\circ \times L^\circ \to \mathbb{Z}/s\mathbb{Z}$. Since

$$c_0(n_i\alpha_i, n_j\alpha_j) = \frac{s}{2}\langle n_i\alpha_i, n_j\alpha_j\rangle + s\mathbb{Z}$$

for $i < j$ and since

$$(\alpha, \beta) \mapsto \frac{s}{2}\langle\alpha, \beta\rangle + s\mathbb{Z}$$

is also an alternating bilinear form on L (since L is even), it follows that

$$c_0(\alpha, \beta) = \frac{s}{2}\langle\alpha, \beta\rangle + s\mathbb{Z}$$

for all $\alpha, \beta \in L$. Thus

$$\omega_s^{c_0(\alpha,\beta)} = \omega_s^{s\langle\alpha,\beta\rangle/2} = (-1)^{\langle\alpha,\beta\rangle}$$

for $\alpha, \beta \in L$, as required. $\square$

Throughout the rest of this chapter we will use the central extension and section of Lemma 4.2.1. Moreover, we will assume that $e_0 = 1$, so that $\iota(e_0) = \mathbf{1}$. Thus V_L is generated as vertex algebra by the $\iota(e_\alpha)$ for $\alpha \in L$. This motivates the following definition:

Definition 4.2.2. Set $V_{L,\mathbb{Z}}$ to be the vertex subring of V_L generated by the $\iota(e_\alpha)$ for $\alpha \in L$. Moreover, for any $\beta \in L^\circ$, set $V_{\beta+L,\mathbb{Z}}$ equal to the $V_{L,\mathbb{Z}}$ submodule of $V_{\beta+L}$ generated by $\iota(e_\beta)$. In particular, $V_{L,\mathbb{Z}}$ is the $V_{L,\mathbb{Z}}$ submodule of V_L generated by $\mathbf{1}$.

The algebra part of the following theorem is originally due to Borcherds [B] and has been proved in [P] and [DG]; here we extend the theorem to modules and use a new method of proof:

Theorem 4.2.3. *The vertex subring $V_{L,\mathbb{Z}}$ is an integral form of V_L, and for any $\beta \in L^\circ$, $V_{\beta+L,\mathbb{Z}}$ is an integral form of $V_{\beta+L}$.*

Proof. Since $V_{L,\mathbb{Z}}$ is closed under vertex algebra products and contains $\mathbf{1}$ by definition, we just need to show that for any $\beta \in L^\circ$, $V_{\beta+L,\mathbb{Z}}$ is an integral form of the vector space $V_{\beta+L}$ and that $V_{\beta+L,\mathbb{Z}}$ is compatible with the $L^\circ \times \mathbb{Q}$-gradation of $V_{\beta+L}$, as in (2.12). By Proposition 2.2.4, $V_{\beta+L,\mathbb{Z}}$ is the $\mathbb{Z}$-span of coefficients of products of the form

$$Y(\iota(e_{\alpha_1}), x_1) \dots Y(\iota(e_{\alpha_k}), x_k)\iota(e_\beta) \tag{4.6}$$

where $\alpha_i \in L$. Since $\iota(e_\beta)$ and $\iota(e_\alpha)$ for $\alpha \in L$ are homogeneous in the $L^\circ \times \mathbb{Q}$-gradation of V_{L°, it follows from (2.6) and (2.8) that coefficients of products as in (4.6) are doubly homogeneous. Hence $V_{\beta+L,\mathbb{Z}}$ is compatible with the $L^\circ \times \mathbb{Q}$-gradation:

$$V_{\beta+L,\mathbb{Z}} = \coprod_{\gamma\in\beta+L,\, n\in\mathbb{Q}} V^{\gamma}_{(n)} \cap V_{\beta+L,\mathbb{Z}}. \tag{4.7}$$

To show that $V_{\beta+L,\mathbb{Z}}$ is an integral form of the vector space $V_{\beta+L}$, it is enough to show that for any $\gamma \in \beta + L$, $V^{\gamma}_{(n)} \cap V_{\beta+L,\mathbb{Z}}$ is a lattice in $V^{\gamma}_{(n)}$ whose rank is the dimension of $V^{\gamma}_{(n)}$. Since the vectors $\iota(e_\alpha)$ for $\alpha \in L$ generate V_L as a vertex algebra and since $\iota(e_\beta)$ generates $V_{\beta+L}$ as a V_L-module, $V_{\beta+L,\mathbb{Z}}$ spans $V_{\beta+L}$ over $\mathbb{C}$, and thus by (4.7), $V^{\gamma}_{(n)} \cap V_{\beta+L,\mathbb{Z}}$ spans $V^{\gamma}_{(n)}$ over $\mathbb{C}$ for any $\gamma \in \beta + L$. Thus if $V^{\gamma}_{(n)} \cap V_{\beta+L,\mathbb{Z}}$ is a lattice, its rank is at least the dimension of $V^{\gamma}_{(n)}$.

On the other hand, since $\varepsilon(\alpha, \beta) = \pm 1 \in \mathbb{Q}$ for any $\alpha, \beta \in L$, V_L and $V_{\beta+L}$ have $\mathbb{Q}$-forms, namely, the $\mathbb{Q}$-subalgebra $V_{L,\mathbb{Q}} \subseteq V_L$ generated by the $\iota(e_\alpha)$ for $\alpha \in L$, and the $V_{L,\mathbb{Q}}$-submodule $V_{\beta+L,\mathbb{Q}}$ generated by $\iota(e_\beta)$, respectively. They are spanned over $\mathbb{Q}$ by the vectors

$$\alpha_1(-n_1)\cdots\alpha_k(-n_k)\iota(e_\alpha e_\beta)$$

where $\alpha, \alpha_i \in L$, $n_i \in \mathbb{Z}_+$, and $\beta = 0$ in the algebra case. Now, any set of vectors in $V_{(n)}^\gamma \cap V_{\beta+L,\mathbb{Z}}$ which is linearly independent over $\mathbb{Z}$ is linearly independent over $\mathbb{Q}$ since a dependence relation over $\mathbb{Q}$ reduces to a dependence relation over $\mathbb{Z}$ by clearing denominators. Thus, since $V_{L,\mathbb{Z}} \subseteq V_{L,\mathbb{Q}}$, any set of vectors in $V_{(n)}^\gamma \cap V_{\beta+L,\mathbb{Z}}$ which is linearly independent over $\mathbb{Z}$ is linearly independent over $\mathbb{C}$. This means that if $V_{(n)}^\gamma \cap V_{\beta+L,\mathbb{Z}}$ is a lattice, its rank is no more than the dimension of $V_{(n)}^\gamma$.

Thus we are reduced to showing that for any $\gamma \in \beta + L$, $V_{(n)}^\gamma \cap V_{\beta+L,\mathbb{Z}}$ is a lattice in $V_{(n)}^\gamma$, that is, it is spanned over $\mathbb{Z}$ by a finite set. To show this, we use formula (8.4.22) in [FLM2] to obtain

$$Y(\iota(e_{\alpha_1}), x_1)\cdots Y(\iota(e_{\alpha_k}), x_k)\iota(e_\beta) = {}^\circ_\circ Y(\iota(e_{\alpha_1}), x_1)\cdots Y(\iota(e_{\alpha_k}), x_k){}^\circ_\circ \iota(e_\beta)\cdot \prod_{1\le i<j\le k}(x_i - x_j)^{\langle\alpha_i,\alpha_j\rangle}.$$

where $\alpha_i \in L$. (See also formula (8.6.6) in [FLM2]; we use the normal ordering notation of [FLM2] here.) Since L is even and in particular integral, the binomial product expansions on the right side involve only integer coefficients. Hence coefficients of the non-normal ordered product are integral combinations of coefficients of the corresponding normal ordered product, and vice versa since the coefficients of

$$\prod_{1\le i<j\le k}(x_i - x_j)^{-\langle\alpha_i,\alpha_j\rangle}$$

are also integers. This means that $V_{\beta+L,\mathbb{Z}}$ is the $\mathbb{Z}$-span of coefficients in products of the form

$${}^\circ_\circ Y(\iota(e_{\alpha_1}), x_1)\cdots Y(\iota(e_{\alpha_k}), x_k){}^\circ_\circ \iota(e_\beta) \tag{4.8}$$

where $\alpha_i \in L$. By the definition of normal ordering, (4.8) equals

$$x_1^{\langle\alpha_1,\beta\rangle}\cdots x_k^{\langle\alpha_k,\beta\rangle}E^-(-\alpha_1, x_1)\cdots E^-(-\alpha_k, x_k)\iota(e_{\alpha_1}\cdots e_{\alpha_k}e_\beta).$$

Thus, because $\varepsilon(\alpha,\beta)=\pm 1$ for any $\alpha,\beta\in L$, $V_{\beta+L,\mathbb{Z}}$ is the $\mathbb{Z}$-span of coefficients in products of the form

$$E^-(-\alpha_1,x_1)\cdots E^-(-\alpha_k,x_k)\iota(e_\alpha e_\beta) \tag{4.9}$$

where $\alpha,\alpha_i\in L$. In fact, if $\left\{\alpha^{(1)},\dots,\alpha^{(l)}\right\}$ is a base for L (or any finite spanning set), we may take the α_i in (4.9) to come from $\left\{\pm\alpha^{(1)},\dots,\pm\alpha^{(l)}\right\}$, since if $\alpha=\sum_{i=1}^{l} n_i\alpha^{(i)}\in L$, then by properties of exponentials,

$$E^-(-\alpha,x)=\prod_{i=1}^{l} E^-(-\alpha^{(i)},x)^{n_i},$$

where if n_i is negative, $E^-(-\alpha^{(i)},x)^{n_i}=E^-(\alpha^{(i)},x)^{-n_i}$.

Recall that for $\alpha\in L$,

$$E^-(-\alpha,x)=\exp\left(\sum_{n>0}\frac{\alpha(-n)}{n}x^n\right),$$

so the coefficient of x^m in this operator increases weight by m. Hence the coefficient of any monomial of total degree m in (4.9) is in $V^{\beta+\alpha}_{(m+\langle\beta+\alpha,\beta+\alpha\rangle/2)}$. The coefficients of such monomials, with α fixed, for which $m+\langle\beta+\alpha,\beta+\alpha\rangle/2=n$ span $V_{\beta+L,\mathbb{Z}}\cap V^{\beta+\alpha}_{(n)}$. Since $\left\{\pm\alpha^{(1)},\dots,\pm\alpha^{(l)}\right\}$ is a finite set, there are only a finite number of ways of obtaining coefficients of products of the form (4.9) that lie in $V^{\beta+\alpha}_{(n)}$. This shows that $V_{\mathbb{Z}}\cap V^{\beta+\alpha}_{(n)}$ is spanned over $\mathbb{Z}$ by a finite set of vectors for any $n\in\mathbb{Q},\alpha\in L$, completing the proof. $\square$

Remark 4.2.4. We see from the proof of Theorem 4.2.3 that the integral form $V_{\beta+L,\mathbb{Z}}$ depends on the choice of coset representative β. If $\beta'=\beta+\alpha\in\beta+L$ for some $\alpha\in L$, then

$$\iota(e_{\beta'})=\iota(e_{\alpha+\beta})=\varepsilon(\alpha,\beta)^{-1}\iota(e_\alpha e_\beta).$$

From (4.9), we then see that $V_{\beta'+L,\mathbb{Z}}=\varepsilon(\alpha,\beta)^{-1}V_{\beta+L,\mathbb{Z}}$.

Remark 4.2.5. Borcherds' definition of $V_{L,\mathbb{Z}}$ in [B] does not use the vertex algebra structure of V_L; note that V_L is also an associative algebra with product determined by

$$(\alpha_1(-m_1)\cdots\alpha_j(-m_j)\iota(a))(\beta_1(-n_1)\cdots\beta_k(-n_k)\iota(b)) =$$
$$\alpha_1(-m_1)\cdots\alpha_j(-m_j)\beta_1(-n_1)\cdots\beta_k(-n_k)\iota(ab),$$

where $\alpha_i, \beta_i, \bar{a}, \bar{b} \in L$, $m_i, n_i > 0$. Thus V_L is generated as an associative algebra by the elements $\iota(a)$ for $\bar{a} \in L$ and $\alpha(-n)\mathbf{1}$ where $\alpha \in L$ and $n > 0$. There is a derivation D of this associative algebra structure defined on generators by $D\iota(a) = \bar{a}(-1)\iota(a)$ and $D\alpha(-n)\mathbf{1} = n\alpha(-n-1)\mathbf{1}$. This is precisely the action of $L(-1)$ on these elements, and in fact $D = L(-1)$. In [B], $V_{L,\mathbb{Z}}$ is defined to be the smallest associative subring of V_L containing each $\iota(e_\alpha)$ and invariant under $D^i/i!$ for $i \geq 0$. It is claimed that $V_{L,\mathbb{Z}}$ is then generated as an associative ring by the $\iota(e_\alpha)$ and the coefficients of $E^-(-\alpha, x)\mathbf{1}$, that is, $V_{L,\mathbb{Z}}$ is the $\mathbb{Z}$-span of coefficients of products of the form (4.9) (where β is now 0). From the proof of Theorem 4.2.3, we know that such vectors span $V_{L,\mathbb{Z}}$ as it is defined here. Borcherds' claim has been proven in [P], but we simplify the proof here:

Proposition 4.2.6. *The definition of $V_{L,\mathbb{Z}}$ in [B] agrees with Definition 4.2.2.*

Proof. Let $V^*_{L,\mathbb{Z}}$ denote the structure defined in [B], and $V_{L,\mathbb{Z}}$ the structure of Definition 4.2.2. First, we show $V_{L,\mathbb{Z}} \subseteq V^*_{L,\mathbb{Z}}$. Since $V_{L,\mathbb{Z}}$ is the $\mathbb{Z}$-span of coefficients of products of the form (4.9) and $V^*_{L,\mathbb{Z}}$ is an associative subring, it is enough to show that each $\iota(e_\alpha)$ for $\alpha \in L$ and the coefficients of each $E^-(-\alpha, x)\mathbf{1}$ for $\alpha \in L$ are in $V^*_{L,\mathbb{Z}}$. Now, each $\iota(e_\alpha) \in V^*_{L,\mathbb{Z}}$ by definition; also $V^*_{L,\mathbb{Z}}$ is closed under $L(-1)^i/i!$ for each $i \geq 0$. Thus for any $\alpha \in L$, $V^*_{L,\mathbb{Z}}$ must contain the coefficients of

$$e^{L(-1)x}\iota(e_\alpha) = Y(\iota(e_\alpha), x)\mathbf{1} = E^-(-\alpha, x)\iota(e_\alpha).$$

Recall that in any conformal vertex algebra, $e^{L(-1)x}v = Y(v, x)\mathbf{1}$ for any v (formulas (3.1.29) and (3.1.67) in [LL]). Since $V^*_{L,\mathbb{Z}}$ is an associative subring, it contains the coefficients of

$$(E^-(-\alpha, x)\iota(e_\alpha))(\iota(e_{-\alpha})) = \pm E^-(-\alpha, x)\mathbf{1},$$

since $\varepsilon(\alpha, -\alpha) = \pm 1$.

On the other hand, $V_{L,\mathbb{Z}}$ is an associative subring of V_L (the associative product of any two coefficients of products of the form (4.9) with $\beta = 0$ is again such a coefficient). Also, $V_{L,\mathbb{Z}}$ is preserved by each $L(-1)^i/i!$ since it is closed under vertex operators and $e^{L(-1)x}v = Y(v,x)\mathbf{1}$ for $v \in V_{L,\mathbb{Z}}$. Thus $V_{L,\mathbb{Z}}^* \subseteq V_{L,\mathbb{Z}}$, and $V_{L,\mathbb{Z}} = V_{L,\mathbb{Z}}^*$. $\square$

Remark 4.2.7. If L is the root lattice of a finite-dimensional simple Lie algebra $\mathfrak{g}$ of type A, D, or E, the lattice vertex operator algebra V_L is isomorphic to the level 1 affine Lie algebra vertex operator algebra $L_{\widehat{\mathfrak{g}}}(1,0)$. The isomorphism is determined by

$$\iota(e_\alpha) \mapsto \pm x_\alpha(-1)\mathbf{1},$$

for α a root of $\mathfrak{g}$ and x_α the corresponding root vector in a Chevalley basis for $\mathfrak{g}$. (For the proof of this result see [FLM2] and [DL].) From the definitions, it is clear that the integral forms $V_{L,\mathbb{Z}}$ and $L_{\widehat{\mathfrak{g}}}(1,0)_{\mathbb{Z}}$ correspond under this isomorphism.

4.3 Further properties and results

In this section we exhibit bases for the integral forms of modules for a lattice vertex algebra. We also apply the results on the conformal vector and contragredient modules from Chapter 2 to lattice vertex algebras, and we study integral intertwining operators among modules for lattice vertex algebras. We continue to assume L is an even nondegenerate lattice with dual lattice L°.

The following result on a $\mathbb{Z}$-base for $V_{\beta+L,\mathbb{Z}}$, $\beta \in L^\circ$, has been proved for the algebra case $\beta = 0$ in [DG], but since we will need it later, we include the proof for completeness; assume now that $\left\{\alpha^{(1)}, \ldots \alpha^{(l)}\right\}$ is a base for L:

Proposition 4.3.1. *The distinct coefficients of monomials in products as in (4.9) form a basis for $V_{\beta+L,\mathbb{Z}}$, where the α_i come from $\left\{\alpha^{(1)}, \ldots, \alpha^{(l)}\right\}$ and α is any element of L.*

Proof. The proof of Theorem 4.2.3 shows that the coefficients of monomials in (4.9) span $V_{\beta+L,\mathbb{Z}}$ when the α_i come from $\left\{\pm\alpha^{(1)}, \ldots, \pm\alpha^{(l)}\right\}$. But recall that $E^-(\alpha^{(i)}, x) = E^-(-\alpha^{(i)}, x)^{-1}$. If we expand

$$E^-(-\alpha^{(i)}, x) = 1 + \sum_{j\geq 1} y_{ij} x^j,$$

where y_{ij} is a polynomial in the $\alpha^{(i)}(-k)$, then

$$E^-(\alpha^{(i)}, x) = \frac{1}{1 + \sum_{j\geq 1} y_{ij} x^j} = \sum_{n\geq 0} \left(-\sum_{j\geq 1} y_{ij} x^j\right)^n.$$

Thus the coefficients of $E^-(\alpha^{(i)}, x)$ are polynomials in the coefficients of $E^-(-\alpha^{(i)}, x)$. This shows that the coefficients of monomials in (4.9) span $V_{\beta+L,\mathbb{Z}}$ when the $\alpha_i \in \left\{\alpha^{(1)}, \ldots, \alpha^{(l)}\right\}$.

We also need to show that the indicated coefficients are linearly independent (over $\mathbb{Z}$). In fact, they are linearly independent over $\mathbb{C}$, and to show this, it is sufficient to show that the polynomials y_{ij} are algebraically independent in $S(\widehat{\mathfrak{h}}_-)$. Since

$$E^-(-\alpha^{(i)}, x) = \exp\left(\sum_{n<0} \frac{-\alpha^{(i)}(n)}{n} x^{-n}\right) = \exp\left(\sum_{n>0} \frac{\alpha^{(i)}(-n)}{n} x^n\right),$$

$y_{ij} = \alpha^{(i)}(-j)/j + F_{ij}$, where F_{ij} is a polynomial in the $\alpha^{(i)}(-k)$ with degree greater than 1 and with $k < j$.

Now suppose there is a relation

$$F = \sum c_{i_1\ldots i_k; j_1\ldots j_k} y_{i_1 j_1} \cdots y_{i_k j_k} = 0,$$

where all coefficients $c_{i_1\ldots i_k; j_1\ldots j_k}$ are non-zero. If k_{min} is the smallest degree of any monomial in the y_{ij} in F, then the monomial in F in the $\alpha^{(i)}(-j)$ that has minimal degree is

$$\sum_{k=k_{min}} \frac{c_{i_1\ldots i_k; j_1\ldots j_k}}{j_1 \cdots j_k} \alpha^{(i_1)}(-j_1) \cdots \alpha^{(i_k)}(-j_k).$$

Since the $\alpha^{(i)}(-j)$ are algebraically independent, this sum must equal 0; but then each $c_{i_1\ldots i_k; j_1\ldots j_k} = 0$ for k minimal as well. This contradiction shows that no nontrivial relation $F(\{y_{ij}\}) = 0$ exists, so the y_{ij} are algebraically independent. $\square$

Remark 4.3.2. We can express this basis for $V_{\beta+L,\mathbb{Z}}$ as the elements

$$y_{i_1j_1}\cdots y_{i_kj_k}\iota(e_\alpha e_\beta)$$

where $k\geq 0$, $1\leq i_1\leq\ldots\leq i_k\leq l$, $j_m\leq j_{m+1}$ if $i_m=i_{m+1}$, and $\alpha\in L$.

We now prove the following result on when ω can be added to the integral form $V_{L,\mathbb{Z}}$, which was first observed without proof in [B]:

Proposition 4.3.3. *The integral form $V_{L,\mathbb{Z}}$ of V_L can be extended to an integral form containing ω if the rank of L is even and containing 2ω if the rank of L is odd.*

Proof. If $\{\alpha^{(1)},\ldots,\alpha^{(l)}\}$ is a base for L with dual base $\{\beta^{(1)},\ldots,\beta^{(l)}\}$ for L°, then

$$\omega=\frac{1}{2}\sum_{i=1}^{l}\alpha^{(i)}(-1)\beta^{(i)}(-1)\mathbf{1}.$$

Since $\langle\cdot,\cdot\rangle$ is integral on L, $\beta^{(i)}\in\mathbb{Q}\otimes_{\mathbb{Z}}L$ for any $1\leq i\leq l$, so that ω is in the $\mathbb{Q}$-span of $V_{L,\mathbb{Z}}$. Moreover, $V_{\mathbb{Z}}$ is generated by the lowest weight vectors for the Virasoro algebra $\iota(e_\alpha)$ for $\alpha\in L$. Then by Theorem 2.3.2, $V_{L,\mathbb{Z}}$ can be extended to an integral form containing $k\omega$ for any $k\in\mathbb{Z}$ such that $k^2c\in 2\mathbb{Z}$. Here c is the central charge of V_L, which equals the rank of L. Thus we can take $k=1$ if the rank of L is even and $k=2$ if the rank of L is odd. $\square$

We now determine when $V_{L,\mathbb{Z}}$ already contains ω; the "if" part of the following proposition appeared in [BR1]:

Proposition 4.3.4. *If L is an even lattice, the integral form $V_{L,\mathbb{Z}}$ of V_L contains ω if and only if L is self-dual.*

Proof. Suppose $\left\{\alpha^{(1)},\ldots,\alpha^{(l)}\right\}$ is a base for L. We know from Proposition 4.3.1 that an integral basis for $V_{L,\mathbb{Z}}\cap V^0_{(2)}$ consists of distinct coefficients of monomials of degree 2 in products of the form

$$E^-(-\alpha^{(i)},x_1)E^-(-\alpha^{(j)},x_2)\mathbf{1} \tag{4.10}$$

where $1 \leq i, j \leq l$. Since

$$E^-(-\alpha, x) = \exp\left(\sum_{n>0} \frac{\alpha(-n)}{n} x^n\right)$$
$$= 1 + \alpha(-1)x + \left(\frac{\alpha(-2) + \alpha(-1)^2}{2}\right) x^2 + \ldots, \tag{4.11}$$

the distinct coefficients of monomials of degree 2 in (4.10) are

$$\alpha^{(i)}(-1)\alpha^{(j)}(-1)\mathbf{1}, \quad \frac{\alpha^{(i)}(-2) + \alpha^{(i)}(-1)^2}{2}\mathbf{1} \tag{4.12}$$

where $1 \leq i \leq j \leq l$. We can take these quadratic polynomials as a base for $V_{L,\mathbb{Z}} \cap V^0_{(2)}$.

Now suppose $\left\{\beta^{(1)}, \ldots, \beta^{(l)}\right\}$ is a basis of $\mathfrak{h}$ dual to $\left\{\alpha^{(1)}, \ldots, \alpha^{(l)}\right\}$, and write $\beta^{(i)} = \sum_{j=1}^{l} c_{ji}\alpha^{(j)}$ where $c_{ji} \in \mathbb{C}$. Then

$$\omega = \frac{1}{2}\sum_{i=1}^{l} \alpha^{(i)}(-1)\beta^{(i)}(-1)\mathbf{1} = \frac{1}{2}\sum_{i,j=1}^{l} c_{ji}\alpha^{(i)}(-1)\alpha^{(j)}(-1)\mathbf{1}$$
$$= \sum_{i=1}^{l} \frac{c_{ii}}{2}\alpha^{(i)}(-1)^2\mathbf{1} + \sum_{i<j} \frac{c_{ij} + c_{ji}}{2}\alpha^{(i)}(-1)\alpha^{(j)}(-1)\mathbf{1}. \tag{4.13}$$

In view of the base (4.12) for $V_{L,\mathbb{Z}} \cap V^0_{(2)}$, we see that $\omega \in V_{L,\mathbb{Z}}$ if and only if $c_{ii}, c_{ij} + c_{ji} \in 2\mathbb{Z}$ for all i and $j \neq i$.

Since $\left\{\beta^{(1)}, \ldots, \beta^{(l)}\right\}$ is a dual basis,

$$\left\langle \beta^{(i)}, \beta^{(j)} \right\rangle = \left\langle \beta^{(i)}, \sum_{k=1}^{l} c_{kj}\alpha^{(k)} \right\rangle = c_{ij}.$$

Since $\langle \cdot, \cdot \rangle$ is symmetric, we have $c_{ij} = c_{ji}$. Consequently, $\omega \in V_{\mathbb{Z}}$ if and only if $c_{ii} \in 2\mathbb{Z}$ for all i and $c_{ij} \in \mathbb{Z}$ for $i \neq j$. If L is self-dual, each $\beta^{(i)} \in L$, so each $c_{ij} \in \mathbb{Z}$; also, since L is even, $c_{ii} = \left\langle \beta^{(i)}, \beta^{(i)} \right\rangle \in 2\mathbb{Z}$ for each i. Conversely, if each $c_{ij} \in \mathbb{Z}$, then each $\beta^{(i)} \in L$, so L is self-dual. Thus we see that $\omega \in V_{L,\mathbb{Z}}$ if and only if L is self-dual. $\square$

Example 4.3.5. If L is the root lattice of E_8 or the Leech lattice, then $\omega \in V_{L,\mathbb{Z}}$.

We now apply the results on contragredient modules from Section 2.4 to lattice vertex operator algebras and their modules. Since $V_{L,\mathbb{Z}}$ is generated by the vectors

$\iota(e_\alpha)$ which are lowest weight vectors for the Virasoro algebra, Propositions 2.5.1 and 2.5.2 immediately give us

Proposition 4.3.6. *The contragredient of a $V_{L,\mathbb{Z}}$-module is a $V_{L,\mathbb{Z}}$-module.*

Recall that the contragredient of $V_{\beta+L}$ is $V_{-\beta+L}$. We shall now use an invariant bilinear pairing as in (2.27) to identify $V'_{-\beta+L,\mathbb{Z}}$ as a sublattice of $V_{\beta+L,\mathbb{Z}}$. First we calculate (2.27) with $\mathcal{Y} = \mathcal{Y}_\beta$ and with the form $(\cdot,\cdot)_{V_L}$ on V_L normalized so that $(\mathbf{1},\mathbf{1})_{V_L} = 1$. Thus for $u \in V_{\beta+L}$ and $v \in V_{-\beta+L}$, we have

$$(u,v) = \mathrm{Res}_x(\mathbf{1}, (\mathcal{Y}_\beta)^o_0(u, e^{\pi i}x)e^{xL(1)}v)_{V_L}.$$

In particular, (4.4) implies that for $\gamma \in \beta + L$, $\gamma' \in -\beta + L$,

$$\begin{aligned}
&(\iota(e_\gamma), \iota(e_{\gamma'})) = \\
&\mathrm{Res}_x\, x^{-1}(\mathbf{1}, e^{-\pi i\langle\gamma,\gamma\rangle/2}x^{-\langle\gamma,\gamma\rangle}E^-(-\gamma,-x^{-1})e_\gamma(e^{\pi i}x)^{-\langle\gamma,\gamma'\rangle}e^{\pi i\langle\beta,\gamma'\rangle}c(\gamma',\beta)\iota(e_{\gamma'}))_{V_L} = \\
&\mathrm{Res}_x\, x^{-1}(\mathbf{1}, e^{\pi i(\langle\beta-\gamma,\gamma'\rangle-\langle\gamma,\gamma\rangle/2)}x^{-\langle\gamma,\gamma+\gamma'\rangle}E^-(-\gamma,-x^{-1})c(\gamma',\beta)\varepsilon(\gamma,\gamma')\iota(e_{\gamma+\gamma'}))_{V_L} = \\
&e^{\pi i\langle\gamma-2\beta,\gamma\rangle/2}c(\gamma,\beta)^{-1}\varepsilon(\gamma,\gamma)^{-1}\delta_{\gamma+\gamma',0}. \qquad (4.14)
\end{aligned}$$

Moreover, from (4.14), we see that for any $\alpha \in L$,

$$\begin{aligned}
(\iota(e_\alpha e_\beta), \iota(e_{-\alpha}e_{-\beta})) &= \varepsilon(\alpha,\beta)\varepsilon(-\alpha,-\beta)(\iota(e_{\alpha+\beta}), \iota(e_{-\alpha-\beta})) \\
&= \varepsilon(\alpha,\beta)^2 e^{\pi i\langle\alpha-\beta,\alpha+\beta\rangle/2}c(\alpha+\beta,\beta)^{-1}\varepsilon(\alpha+\beta,\alpha+\beta)^{-1} \\
&= e^{\pi i\langle\alpha,\alpha\rangle/2}\varepsilon(\alpha,\alpha)^{-1}e^{-\pi i\langle\beta,\beta\rangle/2}c(\beta,\beta)^{-1}\varepsilon(\beta,\beta)^{-1},
\end{aligned}$$

since $c(\alpha,\beta) = \varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha)^{-1}$ from (4.5).

If we now renormalize the invariant pairing $(\cdot,\cdot)$ between $V_{\beta+L}$ and $V_{-\beta+L}$ by setting

$$(\cdot,\cdot)_{\mathrm{new}} = e^{\pi i\langle\beta,\beta\rangle/2}c(\beta,\beta)\varepsilon(\beta,\beta)(\cdot,\cdot)_{\mathrm{old}},$$

we see that now

$$(\iota(e_\alpha e_\beta), \iota(e_{\alpha'}e_{-\beta})) = (-1)^{\langle\alpha,\alpha\rangle/2}\varepsilon(\alpha,\alpha)^{-1}\delta_{\alpha+\alpha',0} \in \mathbb{Z}$$

for any $\alpha, \alpha' \in L$. We can use this new invariant bilinear pairing between $V_{\beta+L}$ and $V_{-\beta+L}$, which is the form in (2.27) with

$$\mathcal{Y} = e^{\pi i \langle \beta,\beta\rangle/2} c(\beta,\beta)\varepsilon(\beta,\beta)\mathcal{Y}_\beta|_{V_{\beta+L}\otimes V_{-\beta+L}},$$

to identify $V'_{-\beta+L,\mathbb{Z}}$ as a sublattice of $V_{\beta+L,\mathbb{Z}}$, for any $\beta \in L^\circ$. The proof of Proposition 3.6 of [DG] works in the module generality to give

Proposition 4.3.7. *For any $\beta \in L^\circ$, the integral form $V'_{-\beta+L,\mathbb{Z}}$ is the integral form of $V_{\beta+L}$ integrally spanned by coefficients of products of the form*

$$E^-(-\beta_1, x_1)\cdots E^-(-\beta_k, x_k)\iota(e_\alpha e_\beta)$$

where $\beta_i \in L^\circ$ and $\alpha \in L$.

Note that since $L \subseteq L^\circ$, this proposition shows that $V'_{-\beta+L,\mathbb{Z}}$ is generally larger than $V_{\beta+L,\mathbb{Z}}$. The case $\beta = 0$ gives:

Corollary 4.3.8. *The integral form $V'_{L,\mathbb{Z}}$ is the integral span of coefficients of products of the form*

$$E^-(-\beta_1, x_1)\cdots E^-(-\beta_k, x_k)\iota(e_\alpha)$$

where $\beta_i \in L^\circ$ and $\alpha \in L$.

Remark 4.3.9. Note that the precise identity of $V'_{-\beta+L,\mathbb{Z}}$ depends on the choice of normalization of the invariant bilinear pairing between $V_{\beta+L}$ and $V_{-\beta+L}$. We shall take $V'_{-\beta+L,\mathbb{Z}}$ as described in Proposition 4.3.7 as the official $V'_{-\beta+L,\mathbb{Z}}$, since we have $V_{\beta+L,\mathbb{Z}} \subseteq V'_{-\beta+L,\mathbb{Z}}$ this way.

We conclude this chapter by demonstrating some integral intertwining operators among V_L-modules:

Theorem 4.3.10. *For any $\beta, \gamma \in L^\circ$, there is a rank one lattice of intertwining operators within $\mathcal{V}_{\beta\,\gamma}^{\beta+\gamma}$ integral with respect to $V_{\beta+L,\mathbb{Z}}$, $V_{\gamma+L,\mathbb{Z}}$, and $V'_{-\beta-\gamma+L,\mathbb{Z}}$. Moreover, this lattice is spanned by*

$$\mathcal{Y}_{\beta,\gamma,\mathbb{Z}} = e^{-\pi i\langle \beta,\gamma\rangle}\varepsilon(\gamma,\beta)^{-1}\mathcal{Y}_\beta|_{V_{\beta+L}\otimes V_{\gamma+L}}.$$

Proof. From the definition (4.4) of $\mathcal{Y}_\beta$, we have

$$\begin{aligned}
\mathcal{Y}_\beta(\iota(e_\beta),x)\iota(e_\gamma) &= E^-(-\beta,x)E^+(-\beta,x)x^{\langle\beta,\gamma\rangle}e^{\pi i\langle\beta,\gamma\rangle}c(\gamma,\beta)\iota(e_\beta e_\gamma)\\
&= x^{\langle\beta,\gamma\rangle}E^-(-\beta,x)e^{\pi i\langle\beta,\gamma\rangle}\varepsilon(\gamma,\beta)\varepsilon(\beta,\gamma)^{-1}\varepsilon(\beta,\gamma)\iota(e_{\beta+\gamma})\\
&= x^{\langle\beta,\gamma\rangle}E^-(-\beta,x)e^{\pi i\langle\beta,\gamma\rangle}\varepsilon(\gamma,\beta)\iota(e_{\beta+\gamma}).
\end{aligned}$$

Then

$$\mathcal{Y}_{\beta,\gamma,\mathbb{Z}}(\iota(e_\beta),x)\iota(e_\gamma) = x^{\langle\beta,\gamma\rangle}E^-(-\beta,x)\iota(e_{\beta+\gamma}) \in V'_{-\beta-\gamma+L,\mathbb{Z}}\{x\} \tag{4.15}$$

by Proposition 4.3.7.

Since $\iota(e_\beta)$ and $\iota(e_\gamma)$ generate $V_{\beta+L,\mathbb{Z}}$ and $V_{\gamma+L,\mathbb{Z}}$, respectively, as $V_{L,\mathbb{Z}}$-modules, Theorem 2.4.9 implies that $\mathcal{Y}_{\beta,\gamma,\mathbb{Z}}$ is integral with respect to $V_{\beta+L,\mathbb{Z}}$, $V_{\gamma+L,\mathbb{Z}}$, and $V'_{-\beta-\gamma+L,\mathbb{Z}}$. Moreover, we see from Proposition 4.3.7 and (4.15) that for $c \in \mathbb{C}$,

$$c\mathcal{Y}_{\beta,\gamma,\mathbb{Z}}(\iota(e_\beta),x)\iota(e_\gamma) \in V'_{-\beta-\gamma+L,\mathbb{Z}}\{x\}$$

if and only if $c \in \mathbb{Z}$. Thus $\mathcal{Y}_{\beta,\gamma,\mathbb{Z}}$ spans the lattice of intertwining operators in $V_{\beta\,\gamma}^{\beta+\gamma}$ which are integral with respect to $V_{\beta+L,\mathbb{Z}}$, $V_{\gamma+L,\mathbb{Z}}$, and $V'_{-\beta-\gamma+L,\mathbb{Z}}$. □

Remark 4.3.11. Note that (4.15) shows that for $\beta,\gamma \in L^\circ$, $\mathcal{Y}_{\beta,\gamma,\mathbb{Z}}(\iota(e_\beta),x)\iota(e_\gamma) \notin V_{\beta+\gamma+L,\mathbb{Z}}\{x\}$ since in general $\beta \notin L$. It is not clear that any non-zero integer multiple of $\mathcal{Y}_{\beta,\gamma,\mathbb{Z}}$ is in general integral with respect to $V_{\beta+L,\mathbb{Z}}$, $V_{\gamma+L,\mathbb{Z}}$, and $V_{\beta+\gamma+L,\mathbb{Z}}$.

Chapter 5

Integral forms in tensor powers of the Virasoro vertex operator algebra $L(\frac{1}{2},0)$ and their modules

In this chapter we study integral forms in tensor powers of the Virasoro vertex operator algebra $L(\frac{1}{2},0)$. These vertex operator algebras are significant because, for instance, the vertex operator algebra V_{E_8} based on the E_8 root lattice contains a copy of $L(\frac{1}{2},0)^{\otimes 16}$ and the moonshine module $V^\natural$ contains a copy of $L(\frac{1}{2},0)^{\otimes 48}$ ([DMZ]). Thus studying integral forms in tensor powers of $L(\frac{1}{2},0)$ and their modules may allow the construction of interesting integral forms in V_{E_8} and $V^\natural$.

5.1 Tensor products of vertex operator algebras and modules

We first recall from [FHL] the definition of the tensor product of vertex operator algebras $(U, Y_U, \mathbf{1}_U, \omega_U)$ and $(V, Y_V, \mathbf{1}_V, \omega_V)$. The tensor product vertex operator algebra is the vector space $U \otimes V$ with vertex operator given by

$$Y_{U\otimes V}(u_{(1)} \otimes v_{(1)}, x)(u_{(2)} \otimes v_{(2)}) = Y_U(u_{(1)}, x)u_{(2)} \otimes Y_V(v_{(1)}, x)v_{(2)}, \tag{5.1}$$

vacuum given by

$$\mathbf{1}_{U\otimes V} = \mathbf{1}_U \otimes \mathbf{1}_V$$

and conformal vector given by

$$\omega_{U\otimes V} = \omega_U \otimes \mathbf{1}_V + \mathbf{1}_U \otimes \omega_V.$$

The central charge of $U \otimes V$ is the sum of the central charges of U and V. If W_U is a U-module and W_V is a V-module, then $W_U \otimes W_V$ is a $U \otimes V$-module with vertex operator analogous to (5.1):

$$Y_{W_U\otimes W_V}(u\otimes v,x)(w_U\otimes w_V)=Y_{W_U}(u,x)w_U\otimes Y_{W_V}(v,x)w_V,$$

We have the following result on generating sets for tensor product vertex operator algebras and modules:

Proposition 5.1.1. *If U is generated by S and V is generated by by T, then $U\otimes V$ is generated by*

$$\{s\otimes\mathbf{1},\,\mathbf{1}\otimes t\,|\,s\in S,\,t\in T\}.$$

Moreover, if W_U is a U-module generated by Q and W_V is a V-module generated by R, then $W_U\otimes W_V$ is generated as a $U\otimes V$-module by

$$\{q\otimes r\,|\,q\in Q,\,r\in R\}.$$

Proof. Since U is generated by S and V is generated by T, the subalgebra of $U\otimes V$ generated by the elements $s\otimes\mathbf{1}$ and $\mathbf{1}\otimes t$ contains $U\otimes\mathbf{1}$ and $\mathbf{1}\otimes V$. Then this subalgebra must contain all of $U\otimes V$ because for any $u\in U$, $v\in V$, it contains

$$\mathrm{Res}_x\,x^{-1}Y(u\otimes\mathbf{1},x)(\mathbf{1}\otimes v)=\mathrm{Res}_x\,x^{-1}Y(u,x)\mathbf{1}\otimes v=u\otimes v.$$

Moreover, since Q generates W_U as a U-module, by applying vertex operators of the form $Y(u\otimes\mathbf{1},x)$ for $u\in U$ to vectors of the form $q\otimes r$, we see that the $U\otimes V$-submodule of $W_U\otimes W_V$ generated by the vectors $q\otimes r$ contains $W_U\otimes r$ for any $r\in R$. Then by applying vertex operators of the form $Y(\mathbf{1}\otimes v,x)$, we see that the submodule generated by the vectors $q\otimes r$ equals $W_U\otimes W_V$. □

It is easy to see that if U and V are two vertex operator algebras with integral forms $U_{\mathbb{Z}}$ and $V_{\mathbb{Z}}$, respectively, then $U_{\mathbb{Z}}\otimes_{\mathbb{Z}}V_{\mathbb{Z}}$ is an integral form of $U\otimes V$. Moreover, if W_U is a U-module and W_V is a V-module with integral forms $(W_U)_{\mathbb{Z}}$ and $(W_V)_{\mathbb{Z}}$, respectively, then $(W_U)_{\mathbb{Z}}\otimes_{\mathbb{Z}}(W_V)_{\mathbb{Z}}$ is a $U_{\mathbb{Z}}\otimes_{\mathbb{Z}}V_{\mathbb{Z}}$-module. It is also clear that Proposition 5.1.1 applies to vertex rings and modules.

Remark 5.1.2. All the definitions and results in this section have obvious generalizations to tensor products of more than two algebras or modules.

5.2 Vertex operator algebras based on the Virasoro algebra

Now we recall the construction of vertex operator algebras based on the Virasoro algebra (see for instance [FZ] or [LL] Section 6.1 for more details). Recall the Virasoro Lie algebra

$$\mathcal{L} = \coprod_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}\mathbf{c}$$

with $\mathbf{c}$ central and all other commutation relations given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\mathbf{c} \tag{5.2}$$

for any $m, n \in \mathbb{Z}$. The Virasoro algebra has the decomposition into subalgebras

$$\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_0 \oplus \mathcal{L}_-,$$

where

$$\mathcal{L}_\pm = \coprod_{n\in\mp\mathbb{Z}_+} \mathbb{C}L_n$$

and

$$\mathcal{L}_0 = \mathbb{C}L_0 \oplus \mathbb{C}\mathbf{c}.$$

We also define the subalgebra

$$\mathcal{L}_{\leq 1} = \mathcal{L}_- \oplus \mathcal{L}_0 \oplus \mathbb{C}L_{-1}.$$

For any $\mathcal{L}$-module V, we use $L(n)$ to denote the action of L_n on V.

Now for any complex number ℓ, we have the one-dimensional $\mathcal{L}_{\leq 1}$-module $\mathbb{C}_\ell$ on which $\mathcal{L}_-$, L_0, and L_{-1} act trivially and on which $\mathbf{c}$ acts as the scalar ℓ. Then we form the induced module

$$V(\ell, 0) = U(\mathcal{L}) \otimes_{U(\mathcal{L}_{\leq 1})} \mathbb{C}_\ell,$$

which is a vertex operator algebra with vacuum $\mathbf{1} = 1 \otimes 1$ and generated by its conformal vector $\omega = L(-2)\mathbf{1}$ with vertex operator

$$Y(L(-2)\mathbf{1}, x) = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}.$$

To construct irreducible $V(\ell,0)$-modules, we take a complex number h and consider the one-dimensional $\mathcal{L}_- \oplus \mathcal{L}_0$-module $\mathbb{C}_{\ell,h}$ on which $\mathcal{L}_-$ acts trivially, $\mathbf{c}$ acts as the scalar ℓ and L_0 acts as the scalar h. Then we form the Verma module

$$M(\ell,h) = U(\mathcal{L}) \otimes_{U(\mathcal{L}_- \oplus \mathcal{L}_0)} \mathbb{C}_{\ell,h},$$

which is a $V(\ell,0)$-module. For any $h \in \mathbb{C}$, $M(\ell,h)$ has a unique irreducible quotient $L(\ell,h)$, and these modules $L(\ell,h)$ exhaust the irreducible $V(\ell,0)$-modules up to equivalence.

Note that $V(\ell,0)$ itself is a quotient of $M(\ell,0)$ by the submodule generated by $L(-1)\mathbf{1}$. It is often the case that $V(\ell,0)$ is irreducible as a module for itself and is thus equal to $L(\ell,0)$. In this case, the irreducible $L(\ell,0)$-modules consist of all $L(\ell,h)$. From [W], $V(\ell,0)$ is reducible if and only if

$$\ell = c_{p,q} = 1 - \frac{6(p-q)^2}{pq}$$

where p and q are relatively prime integers greater than 1. In this case, the irreducible $L(c_{p,q},0)$-modules are the modules $L(c_{p,q},h_{m,n})$ where

$$h_{m,n} = \frac{(np-mq)^2 - (p-q)^2}{4pq}$$

for $0 < m < p$ and $0 < n < q$.

Taking $p = 3$ and $q = 4$, we obtain $c_{3,4} = \frac{1}{2}$, and $L(\frac{1}{2},0)$ has the three irreducible modules $L(\frac{1}{2},0)$, $L(\frac{1}{2},\frac{1}{2})$, and $L(\frac{1}{2},\frac{1}{16})$. Moreover, for any positive integer n, the vertex operator algebra $L(\frac{1}{2},0)^{\otimes n}$ is simple, and all its irreducible modules are obtained as tensor products of irreducible modules for $L(\frac{1}{2},0)$ ([FHL]).

We conclude this section by showing the existence of $\mathbb{Q}$-forms inside the irreducible $\mathcal{L}$-modules $L(\ell,h)$ for $\ell, h \in \mathbb{Q}$, using the same kind of argument as in the proof of Theorem 3.2.3:

Proposition 5.2.1. *Suppose $\ell, h \in \mathbb{Q}$ and v_h spans the lowest conformal weight space of $L(\ell,h)$. Then $L(\ell,h)$ has a $\mathbb{Q}$-form $L(\ell,h)_{\mathbb{Q}}$ which is the $\mathbb{Q}$-span of vectors of the* *form*

$$L(-n_1)\cdots L(-n_k)v_h \tag{5.3}$$

where $n_i > 0$.

Proof. Let $U_{\mathbb{Q}}(\mathcal{L})$ denote the $\mathbb{Q}$-subalgebra of $U(\mathcal{L})$ spanned by monomials in the basis elements L_n for $n \in \mathbb{Z}$ and $\mathbf{c}$. Similarly, let $U_{\mathbb{Q}}(\mathcal{L}_\pm)$ denote the $\mathbb{Q}$-subalgebras spanned by monomials in the basis elements $L_{\mp n}$ for $n > 0$, and let $U_{\mathbb{Q}}(\mathcal{L}_0)$ denote the $\mathbb{Q}$-subalgebra spanned by monomials in L_0 and $\mathbf{c}$. Since the structure constants for the basis $\{L_n\}_{n\in\mathbb{Z}} \cup \{\mathbf{c}\}$ given by (5.2) are rational, choosing an appropriate order on the basis and the Poincaré-Birkhoff-Witt Theorem show that $U_{\mathbb{Q}}(\mathcal{L})$ is a $\mathbb{Q}$-form of the associative algebra $U(\mathcal{L})$ and

$$U_{\mathbb{Q}}(\mathcal{L}) = U_{\mathbb{Q}}(\mathcal{L}_+)U_{\mathbb{Q}}(\mathcal{L}_0)U_{\mathbb{Q}}(\mathcal{L}_-).$$

Now we define the $U_{\mathbb{Q}}(\mathcal{L})$-module

$$L(\ell, h)_{\mathbb{Q}} = U_{\mathbb{Q}}(\mathcal{L}) \cdot v_h = U_{\mathbb{Q}}(\mathcal{L}_+) \cdot v_h$$

where the second equality follows because $L(n)v_h = 0$ for $n > 0$, $L(0)v_h = hv_h \in \mathbb{Q}v_h$, and $\mathbf{c} \cdot v_h = \ell v_h \in \mathbb{Q}v_h$. Note that $L(\ell, h)_{\mathbb{Q}}$ is thus given by the $\mathbb{Q}$-span of vectors of the form (5.3), which means that the $\mathbb{C}$-span of $L(\ell, h)_{\mathbb{Q}}$ is all of $L(\ell, h)$. Note also that the intersection of $L(\ell, h)_{\mathbb{Q}}$ with the lowest conformal weight space $\mathbb{C}v_h$ of $L(\ell, h)$ is $\mathbb{Q}v_h$. For any $w \in L(\ell, h)$, we use w_h to denote the component of w in the lowest conformal weight space of $L(\ell, h)$.

Since $L(\ell, h)_{\mathbb{Q}}$ is the $\mathbb{Q}$-span of a spanning set for $L(\ell, h)$ over $\mathbb{C}$, to show that $L(\ell, h)_{\mathbb{Q}}$ is a $\mathbb{Q}$-form of $L(\ell, h)$, we just need to show that any set of vectors in $L(\ell, h)_{\mathbb{Q}}$ that is linearly independent over $\mathbb{Q}$ is also linearly independent over $\mathbb{C}$. To prove this, suppose to the contrary that $\{w_i\}_{i=1}^k \subseteq L(\ell, h)_{\mathbb{Q}}$ is a set of (non-zero) vectors of minimal cardinality which are linearly independent over $\mathbb{Q}$ but linearly dependent over $\mathbb{C}$ (note that $k \geq 2$). This means there is some dependence relation

$$\sum_{i=1}^{k} c_i w_i = 0$$

where $c_i \in \mathbb{C}^\times$.

Now, there is some $y \in U_{\mathbb{Q}}(\mathcal{L}_-)$ such that $(y \cdot w_1)_h \neq 0$, because otherwise w_1 generates a proper $\mathcal{L}$-submodule of $L(\ell, h)$, contradicting the irreducibility of $L(\ell, h)$. Note that for $1 \leq i \leq k$, $(y \cdot w_i)_h = q_i v_h$ where $q_i \in \mathbb{Q}$ and $q_1 \neq 0$, since y preserves $L(\ell, h)_{\mathbb{Q}}$ and $L(\ell, h)_{\mathbb{Q}} \cap \mathbb{C}v_h = \mathbb{Q}v_h$. Thus

$$0 = \left(y \cdot \sum_{i=1}^{k} c_i w_i\right)_h = \sum_{i=1}^{k} c_i (y \cdot w_i)_h = \left(\sum_{i=1}^{k} c_i q_i\right) v_h,$$

or $\sum_{i=1}^{k} c_i q_i = 0$. Then

$$0 = q_1 \sum_{i=1}^{k} c_i w_i - \left(\sum_{i=1}^{k} c_i q_i\right) w_1 = \sum_{i=1}^{k} (c_i q_1 w_i - c_i q_i w_1) = \sum_{i=2}^{k} c_i (q_1 w_i - q_i w_1).$$

Since $q_1 \neq 0$ and the vectors $\{w_i\}_{i=1}^{k}$ are linearly independent over $\mathbb{Q}$, the vectors $\{q_1 w_i - q_i w_1\}_{i=2}^{k}$ are also linearly indepedent over $\mathbb{Q}$. But since they are also dependent over $\mathbb{C}$, this contradicts the minimality of $\{w_i\}_{i=1}^{k}$. $\square$

5.3 Integral forms in $L(\frac{1}{2}, 0)^{\otimes n}$

In this section we construct integral forms in tensor powers of the Virasoro vertex operator algebra $L(\frac{1}{2}, 0)$ using generators. We start with the following basic result:

Proposition 5.3.1. *The vertex operator algebra $L(\frac{1}{2}, 0)$ has an integral form generated by 2ω.*

Proof. Since $\omega = L(-2)\mathbf{1}$ is contained in the $\mathbb{Q}$-form $L(\frac{1}{2}, 0)_{\mathbb{Q}}$ given by Proposition 5.2.1 and ω generates $L(\frac{1}{2}, 0)$ as a vertex operator algebra, Proposition 2.3.5 immediately implies that $L(\frac{1}{2}, 0)$ has an integral form generated by 2ω. $\square$

Now for any positive integer n, we consider $L(\frac{1}{2}, 0)^{\otimes n}$; for any i such that $1 \leq i \leq n$, we set

$$\omega^{(i)} = \mathbf{1}^{\otimes(i-1)} \otimes \omega \otimes \mathbf{1}^{\otimes(n-i)}.$$

Then as an immediate consequence of Propositions 5.1.1 and 5.3.1, we have

Corollary 5.3.2. *For any integer $n \geq 1$, $L(\frac{1}{2},0)^{\otimes n}$ has an integral form generated by the vectors $2\omega^{(i)}$ for $1 \leq i \leq n$.*

Although the preceding corollary gives integral forms in $L(\frac{1}{2},0)^{\otimes n}$ for any n, these forms do not contain ω. We would like to construct integral forms in $L(\frac{1}{2},0)^{\otimes n}$ which do contain ω; from Proposition 2.3.1, this will not be possible unless $n \in 4\mathbb{Z}$, since the central charge of $L(\frac{1}{2},0)^{\otimes n}$ is $\frac{n}{2}$. To facilitate the construction of integral forms containing ω in $L(\frac{1}{2},0)^{\otimes n}$, we recall the definition of a binary linear code (see for example [MS], but we use the notation of [FLM2] Chapter 10).

Let Ω be an n-element set. Then the power set $P(\Omega)$ is a vector space over $\mathbb{F}_2$, the field of two elements, where addition is given by symmetric difference: for $S, T \subseteq \Omega$,

$$S + T = (S \setminus (S \cap T)) \cup (T \setminus (S \cap T)).$$

A *binary linear code* $\mathcal{C}$ is an $\mathbb{F}_2$-subspace of $P(\Omega)$. One example of a binary linear code is the subspace

$$\mathcal{E}(\Omega) = \{S \subseteq \Omega \,|\, |S| \in 2\mathbb{Z}\}$$

of subsets of Ω of even cardinality.

The $\mathbb{F}_2$-vector space $P(\Omega)$ has a nondegenerate $\mathbb{F}_2$-valued bilinear form given by

$$(S,T) \mapsto |S \cap T| + 2\mathbb{Z}$$

for $S, T \subseteq \Omega$. Given a code $\mathcal{C} \subseteq P(\Omega)$, the *dual code* $\mathcal{C}^\perp$ is given by

$$\mathcal{C}^\perp = \{S \subseteq \Omega \,|\, |S \cap T| \in 2\mathbb{Z} \text{ for any } T \in \mathcal{C}\}.$$

A code $\mathcal{C}$ is self-dual if $\mathcal{C} = \mathcal{C}^\perp$. We say that a code $\mathcal{C}$ is a Type II binary linear code if $|\Omega| \in 4\mathbb{Z}$, $\Omega \in \mathcal{C}$, and $|T| \in 4\mathbb{Z}$ for any $T \in \mathcal{C}$. Note that if $\Omega \in \mathcal{C}$, then $\mathcal{C}$ is closed under complements.

We now take $n \geq 1$ and consider $L(\frac{1}{2},0)^{\otimes n}$, and we identify $\Omega = \{1, \ldots, n\}$. Consider a code $\mathcal{C} \subseteq P(\Omega)$ and for any $T \in \mathcal{C}$, we define $\omega_T \in L(\frac{1}{2},0)^{\otimes n}$ by

$$\omega_T = \sum_{i \notin T} \omega^{(i)} - \sum_{i \in T} \omega^{(i)}.$$

The main result of this section is:

Theorem 5.3.3. *Suppose $n \in 4\mathbb{Z}$, $\Omega = \{1, \ldots, n\}$, and $\mathcal{C} \subseteq \mathcal{E}(\Omega)$ is a binary linear code satisfying $\Omega \in \mathcal{C}$ and for any distinct $i, j \in \Omega$ there is some $T_{ij} \in \mathcal{C}$ such that $i \in T_{ij}$ and $j \notin T_{ij}$. Then the vertex subring $L(\frac{1}{2}, 0)^{\otimes n}_{\mathcal{C}}$ generated by the vectors ω_T for $T \in \mathcal{C}$ is an integral form of $L(\frac{1}{2}, 0)^{\otimes n}$ containing ω.*

Proof. It is clear that $\omega \in L(\frac{1}{2}, 0)^{\otimes n}_{\mathcal{C}}$ since $\omega = \omega_{\emptyset}$. Since the vectors $\omega^{(i)}$ are homogeneous of conformal weight 2, so are the vectors ω_T for $T \in \mathcal{C}$, and so by (2.6), $L(\frac{1}{2}, 0)^{\otimes n}_{\mathcal{C}}$ is compatible with the conformal weight gradation of $L(\frac{1}{2}, 0)^{\otimes n}$ in the sense of (2.10). We will show that the $\mathbb{C}$-span of $L(\frac{1}{2}, 0)^{\otimes n}_{\mathcal{C}}$ equals $L(\frac{1}{2}, 0)^{\otimes n}$ by showing that $L(\frac{1}{2}, 0)^{\otimes n}_{\mathcal{C}}$ contains multiples of the generators $\omega^{(i)}$ of $L(\frac{1}{2}, 0)^{\otimes n}$. This will show that the intersection of $L(\frac{1}{2}, 0)^{\otimes n}_{\mathcal{C}}$ with each weight space $L(\frac{1}{2}, 0)^{\otimes n}_{(m)}$ for $m \in \mathbb{Z}$ spans $L(\frac{1}{2}, 0)^{\otimes n}_{(m)}$. Then it will suffice to show that for any $m \in \mathbb{Z}$, $L(\frac{1}{2}, 0)^{\otimes n}_{\mathcal{C}} \cap L(\frac{1}{2}, 0)^{\otimes n}_{(m)}$ is a finitely generated abelian group, that is, a lattice. This is because $L(\frac{1}{2}, 0)^{\otimes n}_{\mathcal{C}} \subseteq L(\frac{1}{2}, 0)^{\otimes n}_{\mathbb{Q}}$, where $L(\frac{1}{2}, 0)_{\mathbb{Q}}$ is the $\mathbb{Q}$-form of Proposition 5.2.1; as in the proofs of Theorems 2.3.2 and 4.2.3, if we can show that $L(\frac{1}{2}, 0)^{\otimes n}_{\mathcal{C}} \cap L(\frac{1}{2}, 0)^{\otimes n}_{(m)}$ is a lattice in $L(\frac{1}{2}, 0)^{\otimes n}_{(m)}$, its rank must equal the dimension of $L(\frac{1}{2}, 0)^{\otimes n}_{(m)}$.

To show that $L(\frac{1}{2}, 0)^{\otimes n}_{\mathcal{C}}$ contains multiples of $\omega^{(i)}$ for $1 \leq i \leq n$, we claim that for any i,

$$\frac{|\mathcal{C}|}{2}\omega^{(i)} = \sum_{T \in \mathcal{C},\, i \notin T} \omega_T \in L(\frac{1}{2}, 0)^{\otimes n}_{\mathcal{C}}.$$

To prove this claim, consider any $j \neq i$, and let $\mathcal{C}_{ij}$ denote the one-dimensional code spanned by $\{i, j\}$. The dual code $\mathcal{C}_{ij}^{\perp}$ has dimension $n-1$ because it contains 2^{n-2} sets containing both i and j and 2^{n-2} sets containing neither i nor j. Since $T_{ij} \in \mathcal{C}$ is not in $\mathcal{C}_{ij}^{\perp}$, $\mathcal{C} + \mathcal{C}_{ij}^{\perp} = P(\Omega)$, so that

$$\dim \mathcal{C} \cap \mathcal{C}_{ij}^{\perp} = \dim \mathcal{C} + \dim \mathcal{C}_{ij}^{\perp} - \dim(\mathcal{C} + \mathcal{C}_{ij}^{\perp}) = \dim \mathcal{C} - 1.$$

This means that half the sets in $\mathcal{C}$ contain neither or both of i and j and half contain exactly one of i and j. Since $\Omega \in \mathcal{C}$, $\mathcal{C}$ is closed under complements. This means that of the sets in $\mathcal{C}$ which do not contain i, half contain j and half do not. By the definition of the vectors ω_T for $T \in \mathcal{C}$, this proves the claim.

Now to finish the proof of the theorem, we just need to show that for any $m \in \mathbb{Z}$, $L(\frac{1}{2},0)^{\otimes n}_{\mathcal{C}} \cap L(\frac{1}{2},0)^{\otimes n}_{(m)}$ is the $\mathbb{Z}$-span of finitely many vectors. For i such that $1 \le i \le n$, we write

$$Y(\omega^{(i)},x) = \sum_{n\in\mathbb{Z}} L^{(i)}(n)x^{-n-2},$$

so that for any i and $n \in \mathbb{Z}$,

$$L^{(i)}(n) = 1^{\otimes(i-1)} \otimes L(n) \otimes 1^{\otimes(n-i)};$$

note that for $m,n \in \mathbb{Z}$, $[L^{(i)}(m), L^{(j)}(n)] = 0$ when $i \neq j$. Also for $T \in \mathcal{C}$, we write

$$Y(\omega_T,x) = \sum_{n\in\mathbb{Z}} L_T(n)x^{-n-2}.$$

By Proposition 2.2.4, $L(\frac{1}{2},0)^{\otimes n}_{\mathcal{C}}$ is the $\mathbb{Z}$-span of vectors of the form

$$L_{T_1}(m_1)\cdots L_{T_k}(m_k)\mathbf{1} \tag{5.4}$$

for $T_i \in \mathcal{C}$ and $m_i \in \mathbb{Z}$.

We now compute the following commutators for $S,T \in \mathcal{C}$ and $m,n \in \mathbb{Z}$:

$$\begin{aligned}
[L_S(m), L_T(n)] &= \left[\sum_{i\notin S} L^{(i)}(m) - \sum_{i\in S} L^{(i)}(m), \sum_{j\notin T} L^{(j)}(n) - \sum_{j\in T} L^{(j)}(n)\right] \\
&= \sum_{i\notin S+T} [L^{(i)}(m), L^{(i)}(n)] - \sum_{i\in S+T} [L^{(i)}(m), L^{(i)}(n)] \\
&=(m-n)\left(\sum_{i\notin S+T} L^{(i)}(m+n) - \sum_{i\in S+T} L^{(i)}(m+n)\right) \\
&\quad + \frac{((n-|S+T|)-|S+T|)(m^3-m)}{24}\delta_{m+n,0} \\
&=(m-n)L_{S+T}(m+n) + \frac{n-2|S+T|}{4}\binom{m+1}{3}\delta_{m+n,0}.
\end{aligned} \tag{5.5}$$

Since $n \in 4\mathbb{Z}$ and $\mathcal{C} \subseteq \mathcal{E}(\Omega)$, and since $L_T(n)\mathbf{1} = 0$ for $n \ge -1$, we can use these commutator relations to straighten products of the form (5.4), so that $L(\frac{1}{2},0)^{\otimes n}_{\mathcal{C}}$ is the $\mathbb{Z}$-span of vectors of the form

$$L_{T_1}(-n_1)\cdots L_{T_k}(-n_k)\mathbf{1}$$

where $T_i \in \mathcal{C}$ and $n_1 \geq \cdots \geq n_k > 1$. Since ω_T has conformal weight 2, $L_T(-n)$ raises conformal weight by n. Since also $\mathcal{C}$ is finite, this shows that for any $m \in \mathbb{Z}$ only finitely many vectors are required to span $L(\frac{1}{2},0)_{\mathcal{C}}^{\otimes n} \cap L(\frac{1}{2},0)_{(m)}^{\otimes n}$. This completes the proof. $\square$

Remark 5.3.4. Note that the generating set $\{\omega_T \,|\, T \in \mathcal{C}\}$ for $L(\frac{1}{2},0)_{\mathcal{C}}^{\otimes n}$ is redundant since if T^c is the complement of $T \in \mathcal{C}$, then $\omega_{T^c} = -\omega_T$.

We record two special cases of Theorem 5.3.3 in the following corollary:

Corollary 5.3.5. *If $n \in 4\mathbb{Z}$ and $\mathcal{C} \subseteq P(\Omega)$ equals $\mathcal{E}(\Omega)$ or is a Type II self-dual code, then $L(\frac{1}{2},0)_{\mathcal{C}}^{\otimes n}$ is an integral form of $L(\frac{1}{2},0)^{\otimes n}$.*

Proof. Clearly $\Omega \in \mathcal{E}(\Omega)$ and Ω is in every Type II code by definition. Thus to apply Theorem 5.3.3 we just need to show that for any distinct $i, j \in \Omega$, there is some $T_{ij} \in \mathcal{C}$ such that $i \in T_{ij}$ and $j \notin T_{ij}$. If $\mathcal{C} = \mathcal{E}(\Omega)$, then we can take $T_{ij} = \{i, k\}$ where k is distinct from i and j (such a k exists because $n \in 4\mathbb{Z}$).

If $\mathcal{C}$ is a Type II self-dual code, suppose for some i and j the desired T_{ij} does not exist. Then since $\Omega \in \mathcal{C}$, $\mathcal{C}$ is closed under complements and every set in $\mathcal{C}$ contains both or neither of i and j. This means $\{i, j\} \in \mathcal{C}^{\perp} = \mathcal{C}$, which is a contradiction since $|T| \in 4\mathbb{Z}$ if T is a set in a Type II code. Hence the desired T_{ij} exists. $\square$

Example 5.3.6. We recall the smallest non-trivial Type II self-dual binary linear code, the Hamming code $\mathcal{H}$ on a set Ω of 8 elements (see [MS] Chapter 1 or [FLM2] Chapter 10). The Hamming code can be described in several ways, but if we identify $\Omega = \{1, 2, \ldots, 8\}$, we can realize $\mathcal{H}$ explicitly as the sets

$$\emptyset,\ \{1,2,3,4\},\ \{1,2,5,6\},\ \{1,2,7,8\},$$
$$\{1,3,5,7\},\ \{2,4,5,7\},\ \{2,3,6,7\},\ \{2,3,5,8\}$$

and their complements.

5.4 Integral forms in modules for $L(\frac{1}{2},0)^{\otimes n}$

We start this section by showing the existence of integral forms in certain modules for $L(\frac{1}{2},0)^{\otimes n}$ for $n \in 4\mathbb{Z}$. We use an integral form $L(\frac{1}{2},0)^{\otimes n}_{\mathcal{C}}$ where $\mathcal{C}$ is a code satisfying the conditions of Theorem 5.3.3. Recall that the irreducible $L(\frac{1}{2},0)$-modules are given by

$$W_H = L(\frac{1}{2},h_1) \otimes \cdots \otimes L(\frac{1}{2},h_n)$$

where $H = (h_1,\ldots,h_n) \in \{0,\frac{1}{2},\frac{1}{16}\}^n$.

Proposition 5.4.1. *Suppose $n \in 4\mathbb{Z}$ and $\mathcal{C} \subseteq \mathcal{E}(\Omega)$ is a code satisfying the conditions of Theorem 5.3.3. Moreover, suppose $H = (h_1,\ldots,h_n) \in \{0,\frac{1}{2}\}^n$ is such that $h_i = \frac{1}{2}$ for an even number of i. If $v_H = v_{h_1} \otimes \cdots \otimes v_{h_n}$ where v_{h_i} spans the lowest conformal weight space of $L(\frac{1}{2},h_i)$, then the $L(\frac{1}{2},0)^{\otimes n}_{\mathcal{C}}$-module $W_{H,\mathcal{C}}$ generated by v_H is an integral form of W_H.*

Proof. By Proposition 5.1.1, W_H is generated as an $L(\frac{1}{2},0)^{\otimes n}$-module by v_H, so the $L(\frac{1}{2},0)^{\otimes n}_{\mathcal{C}}$-module generated by v_H spans W_H over $\mathbb{C}$. Also, $W_{H,\mathcal{C}}$ is compatible with the conformal weight grading of W_H because v_H is homogeneous of weight $h_1+\ldots+h_n$ and the generators ω_T for $T \in \mathcal{C}$ of $L(\frac{1}{2},0)^{\otimes n}_{\mathcal{C}}$ are homogeneous of weight 2. Thus, as in the proof of Theorem 5.3.3, we just need to show that for any conformal weight $n \in \mathbb{C}$, $W_{H,\mathcal{C}} \cap (W_H)_{(n)}$ is spanned over $\mathbb{Z}$ by finitely many vectors.

By Proposition 2.2.4 and the commutation relations (5.5), $W_{H,\mathcal{C}}$ is the $\mathbb{Z}$-span of products of the form

$$L_{T_1}(-n_1)\cdots L_{T_k}(-n_k)v_H \tag{5.6}$$

for $T_i \in \mathcal{C}$ and $n_1 \geq \ldots \geq n_k \geq 0$, as in the proof of Theorem 5.3.3. Now suppose that $S \subseteq \Omega$ is the set such that $h_i = \frac{1}{2}$ for $i \in S$, so that by assumption $|S| \in 2\mathbb{Z}$. Then for $T \in \mathcal{C}$,

$$L_T(0)v_H = \left(\frac{|S|-|S\cap T|}{2} - \frac{|S\cap T|}{2}\right)v_H = \left(\frac{|S|}{2} - |S\cap T|\right)v_H \in \mathbb{Z}v_H.$$

Thus $W_{H,\mathcal{C}}$ is in fact the $\mathbb{Z}$-span of products of the form (5.6) where now $n_1 \geq \ldots n_k > 0$, and we conclude that $W_{H,\mathcal{C}} \cap (W_H)_{(n)}$ is spanned over $\mathbb{Z}$ by finitely many vectors for any $n \in \mathbb{C}$, just as in the proof of Theorem 5.3.3. $\square$

We see that if $H = (h_1, \ldots, h_n) \in \{0, \frac{1}{2}, \frac{1}{16}\}^n$ is such that $h_i = \frac{1}{2}$ for an odd number of i or if $h_i = \frac{1}{16}$ for any i, the $L(\frac{1}{2}, 0)_{\mathcal{C}}^{\otimes n}$-module generated by v_H will not usually be an integral form of W_H. This is because we will generally have $L_T(0)v_H \notin \mathbb{Z}v_h$ for $T \in \mathcal{C}$, even if $\mathcal{C}$ satisfies the conditions of Theorem 5.3.3. For the remainder of this section, we will concentrate on the case $n = 16$, since the lattice vertex operator algebra V_{E_8} based on the E_8 root lattice contains a subalgebra isomorphic to $L(\frac{1}{2}, 0)^{\otimes 16}$ ([DMZ]); thus our work here suggests an alternate approach to obtaining integral structure in V_{E_8}, different from the approach in Chapter 4. Since the moonshine module vertex operator algebra $V^\natural$ similarly contains a subalgebra isomorphic to $L(\frac{1}{2}, 0)^{\otimes 48}$ ([DMZ]), the same ideas may lead to the construction of explicit integral structure in $V^\natural$.

The decomposition of V_{E_8} into submodules for $L(\frac{1}{2}, 0)^{\otimes 16}$ from [DMZ] is as follows:

$$V_{E_8} = \coprod_{\substack{H=(h_1,\ldots,h_{16})\in\{0,\frac{1}{2}\}^{16} \\ h_1+\ldots+h_{16}\in\mathbb{Z}}} W_H \oplus 2^7 W_{(\frac{1}{16},\cdots,\frac{1}{16})}.$$

Remark 5.4.2. Actually, according to [DMZ], the multiplicity of $W_{(\frac{1}{16},\cdots,\frac{1}{16})}$ in V_{E_8} is 2^8. However, this seems to be an error; there are $\binom{16}{2} = 120$ submodules W_H where $H = (h_1, \ldots, h_{16}) \in \{0, \frac{1}{2}\}^{16}$ is such that $h_1 + \ldots + h_{16} = 1$, and then $\binom{16}{2} + 2^7 = 248$ gives the correct dimension of $(V_{E_8})_{(1)}$.

Since we have already obtained integral structure in the modules W_H for which each $h_i \in \{0, \frac{1}{2}\}$ and $h_1 + \ldots + h_{16} \in \mathbb{Z}$, we now find a code $\mathcal{C}_{16}$ on a 16-element set such that $W_{(\frac{1}{16},\cdots,\frac{1}{16})}$ has an integral form generated as an $L(\frac{1}{2}, 0)_{\mathcal{C}_{16}}^{\otimes 16}$-module by $v_{(\frac{1}{16},\cdots,\frac{1}{16})}$.

To construct $\mathcal{C}_{16}$, we consider an 8-element set $\Omega = \{1, 2, \ldots, 8\}$ and an 8-element set $\Omega' = \{1', 2', \ldots, 8'\}$. For any $T \subseteq \Omega$, we use T' to denote the corresponding set in Ω'. Suppose $\mathcal{H}$ is a Hamming code in $P(\Omega)$. Then we define $\mathcal{C}_{16}$ to be the code in $P(\Omega \cup \Omega')$ generated as an $\mathbb{F}_2$-vector space by the sets $T \cup T'$ for $T \in \mathcal{H}$ and Ω'.

Thus $\mathcal{C}_{16}$ is the 5-dimensional code consisting of the 32 sets $T \cup T'$ and $T \cup (T')^c$ for $T \in \mathcal{H}$.

Proposition 5.4.3. *The binary linear code $\mathcal{C}_{16}$ satisfies the conditions of Theorem 5.3.3, and for any $T \in \mathcal{C}_{16}$, $|T| \in 8\mathbb{Z}$.*

Proof. We have $\Omega \cup \Omega' \in \mathcal{C}_{16}$ since $\Omega \in \mathcal{H}$. We need to show that for any distinct $i, j \in \Omega \cup \Omega'$, there is some $T_{ij} \in \mathcal{C}_{16}$ such that $i \in T_{ij}$ but $j \notin T_{ij}$. Now for any $i \in \Omega$ and $j \neq i, i'$, there is some $T \in \mathcal{H}$ such that $i \in T$ but $j \notin T$ (if $j \in \Omega$) or $j \notin T'$ (if $j \in \Omega'$), using Corollary 5.3.5 since $\mathcal{H}$ is a Type II self-dual code. In this case we can take $T_{ij} = T \cup T'$. If $j = i'$, then $i \in T$ for some $T \in \mathcal{H}$, and we can take $T_{ij} = T \cup (T')^c$. The proof if $i \in \Omega'$ is the same.

To prove the second assertion of the proposition, we note that for any $T \in \mathcal{H}$, $|T| \in 4\mathbb{Z}$. If $|T| = 0$, then $|T \cup T'| = 0$ and $|T \cup (T')^c| = 8$. If $|T| = 4$, then both $|T \cup T'| = 8$ and $|T \cup (T')^c| = 8$. If $|T| = 8$, then $|T \cup T'| = 16$ and $|T \cup (T')^c| = 8$. □

Proposition 5.4.4. *Suppose $n = 16$ and $H = (\frac{1}{16}, \cdots, \frac{1}{16})$. Then the $L(\frac{1}{2}, 0)^{\otimes 16}_{\mathcal{C}_{16}}$-submodule $W_{H,\mathcal{C}_{16}}$ of W_H generated by v_H is an integral form of W_H.*

Proof. The proof is the same as the proof of Proposition 5.4.1, except that for $T \in \mathcal{C}_{16}$,

$$L_T(0)v_H = \left(\frac{16 - |T|}{16} - \frac{|T|}{16}\right) v_H = \frac{16 - 2|T|}{16} v_H \in \mathbb{Z}v_H$$

because $|T| \in 8\mathbb{Z}$ by Proposition 5.4.3. □

5.5 Contragredients, integral intertwining operators, and future directions

In this section, for $n \in 4\mathbb{Z}$ and $\mathcal{C} \subseteq \mathcal{E}(\Omega)$ a code on an n-element set Ω satisfying the conditions of Theorem 5.3.3, we consider the contragredients of and intertwining operators among $L(\frac{1}{2}, 0)^{\otimes n}_{\mathcal{C}}$-modules. We also indicate a possible program for obtaining new integral forms in vertex operator algebras containing a subalgebra isomorphic to $L(\frac{1}{2},0)^{\otimes n}$, in particular V_{E_8} and the moonshine module $V^\natural$.

We first observe that since $L(\frac{1}{2},0)_{(0)} = \mathbb{C}\mathbf{1}$, and $L(\frac{1}{2},0)_{(1)} = 0$, [Li1] implies that $L(\frac{1}{2},0)$ has a one-dimensional space of invariant bilinear forms $(\cdot,\cdot)$; since in addition $L(\frac{1}{2},0)$ is a simple vertex operator algebra, any non-zero such form is nondegenerate. In particular, $L(\frac{1}{2},0)$ is self-contragredient. In fact, any irreducible $L(\frac{1}{2},0)$-module is self-contragredient, since the contragredient of an irreducible module is irreducible and the contragredient of a module has the same conformal weights as the module. Note that the conformal weights of $L(\frac{1}{2},h)$ are in $h+\mathbb{N}$ for $h = 0, \frac{1}{2}, \frac{1}{16}$.

For $h = 0, \frac{1}{2}, \frac{1}{16}$, we let $(\cdot,\cdot)_h$ denote the invariant bilinear form on $L(\frac{1}{2},h)$ such that $(v_h, v_h) = 1$, where v_h is a lowest conformal weight vector generating $L(\frac{1}{2},h)$ and $v_0 = \mathbf{1}$. We note that for any $n \geq 1$, irreducible modules for $L(\frac{1}{2},0)^{\otimes n}$ are self-contragredient; for any $H = (h_1, \ldots h_n) \in \{0, \frac{1}{2}, \frac{1}{16}\}^n$, the module $W_H = L(\frac{1}{2},h_1) \otimes \cdots \otimes L(\frac{1}{2},h_n)$ has a nondegenerate invariant bilinear form $(\cdot,\cdot)_H$ determined by

$$(v_1 \otimes \cdots \otimes v_n, w_1 \otimes \cdots \otimes w_n)_H = \prod_{i=1}^{n} (v_i, w_i)_{h_i},$$

where $v_i, w_i \in L(\frac{1}{2},h_i)$ for $i = 1, \ldots, n$.

Now take $n \in 4\mathbb{Z}$, $\Omega = \{1, \ldots, n\}$ and suppose $\mathcal{C} \subseteq \mathcal{E}(\Omega)$ is a code satisfying the conditions of Theorem 5.3.3. Since $L(1)\omega_T = 0$ for any $T \in \mathcal{C}$, Propositions 2.5.1 and 2.5.2 immediately imply:

Proposition 5.5.1. *Contragredients of $L(\frac{1}{2},0)_{\mathcal{C}}^{\otimes n}$-modules are $L(\frac{1}{2},0)_{\mathcal{C}}^{\otimes n}$-modules.*

Moreover, if W_H is an irreducible $L(\frac{1}{2},0)^{\otimes n}$-module such that the $L(\frac{1}{2},0)_{\mathcal{C}}^{\otimes n}$-module $W_{H,\mathcal{C}}$ generated by a lowest conformal weight vector v_H is an integral form of W_H, then we can use the form $(\cdot,\cdot)_H$ to identify $W'_{H,\mathcal{C}}$ with another, generally different, integral form of W_H.

We now consider intertwining operators among $L(\frac{1}{2},0)^{\otimes n}$-modules and state the main result of this section:

Theorem 5.5.2. *Suppose $n \in 4\mathbb{Z}$, $\Omega = \{1, \ldots, n\}$, and $\mathcal{C} \subseteq \mathcal{E}(\Omega)$ is a code satisfying the conditions of Theorem 5.3.3. In addition, suppose $W_{H^{(i)}}$ for $i = 1, 2, 3$ are irreducible $L(\frac{1}{2}, 0)^{\otimes n}$-modules such that the $L(\frac{1}{2}, 0)^{\otimes n}_{\mathcal{C}}$-modules $W_{H^{(i)},\mathcal{C}}$ generated by a lowest conformal weight vector $v_{H^{(i)}}$ are integral forms of $W_{H^{(i)}}$. Then an intertwining operator $\mathcal{Y} \in V^{W_{H^{(3)}}}_{W_{H^{(1)}} W_{H^{(2)}}}$ is integral with respect to $W_{H^{(1)},\mathcal{C}}$, $W_{H^{(2)},\mathcal{C}}$, and $W'_{H^{(3)},\mathcal{C}}$ if and only if the coefficient of $\mathcal{Y}(v_{H^{(1)}}, x)v_{H^{(2)}}$ in the lowest weight space of $W_{H^{(3)}}$ is in $\mathbb{Z}v_{H^{(3)}}$, where we identify $W_{H^{(3)}}$ with $W'_{H^{(3)}}$ using $(\cdot, \cdot)_{H^{(3)}}$.*

We first prove the following general lemma, whose proof uses the cross-brackets of [FLM2] Section 8.9:

Lemma 5.5.3. *Suppose $n \in \mathbb{Z}$, $\Omega = \{1, \ldots, n\}$ and $\mathcal{C} \subseteq P(\Omega)$ is a code. If $\mathcal{Y}$ is an intertwining operator among $L(\frac{1}{2}, 0)^{\otimes n}$-modules $W_{H^{(i)}}$ for $i = 1, 2, 3$, then*

$$[L_T(m), \mathcal{Y}(v_{H^{(1)}}, x)] = x^m[L_T(0), \mathcal{Y}(v_{H^{(1)}}, x)] + mx^m\mathcal{Y}(L_T(0)v_{H^{(1)}}, x) \tag{5.7}$$

for $T \in \mathcal{C}$ and $m \geq 0$.

Proof. In the Jacobi identity (2.17) for intertwining operators, we take $v = \omega_T$ for $T \in \mathcal{C}$ and $w_{(1)} = v_{H^{(1)}}$ and consider the coefficient of x_0^{-2}:

$$\begin{aligned}(x_1 - x_2)[Y(\omega_T, x_1), \mathcal{Y}(v_{H^{(1)}}, x_2)] &= \mathrm{Res}_{x_0} x_0 x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\mathcal{Y}(Y(\omega_T, x_0)v_{H^{(1)}}, x_2)\\ &= \sum_{i\geq 0} \frac{(-1)^i}{i!}\left(\frac{\partial}{\partial x_1}\right)^i\left(x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\right)\mathcal{Y}(L_T(i)v_{H^{(1)}}, x_2)\\ &= x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\mathcal{Y}(L_T(0)v_{H^{(1)}}, x_2),\end{aligned}$$

since $L_T(i)v_{H^{(1)}} = 0$ for $i > 0$. Now taking the coefficient of x_1^{-m-2} and replacing x_2 with x, we obtain

$$[L_T(m + 1), \mathcal{Y}(v_{H^{(1)}}, x)] - x[L_T(m), \mathcal{Y}(v_{H^{(1)}}, x)] = x^{m+1}\mathcal{Y}(L_T(0)v_{H^{(1)}}, x). \tag{5.8}$$

We now prove (5.7) by induction on m. It is certainly true for $m = 0$, and assuming it is true for $m \geq 0$, we see from (5.8) that

$$\begin{aligned}
&[L_T(m+1), \mathcal{Y}(v_{H^{(1)}}, x)] = x[L_T(m), \mathcal{Y}(v_{H^{(1)}}, x)] + x^{m+1}\mathcal{Y}(L_T(0)v_{H^{(1)}}, x) \\
&\quad = x(x^m[L_T(0), \mathcal{Y}(v_{H^{(1)}}, x)] + mx^m\mathcal{Y}(L_T(0)v_{H^{(1)}}, x)) + x^{m+1}\mathcal{Y}(L_T(0)v_{H^{(1)}}, x) \\
&\quad = x^{m+1}[L_T(0), \mathcal{Y}(v_{H^{(1)}}, x)] + (m+1)x^{m+1}\mathcal{Y}(L_T(0)v_{H^{(1)}}, x),
\end{aligned}$$

proving the lemma. □

We now proceed with the proof of Theorem 5.5.2:

Proof. Since we identify $W_{H^{(3)}}$ with $W'_{H^{(3)}}$ using $(\cdot, \cdot)_{H^{(3)}}$ and since $(v_{H^{(3)}}, v_{H^{(3)}})_{H^{(3)}} = 1$, it is clear that if $\mathcal{Y}$ is integral with respect to $W_{H^{(1)},\mathcal{C}}$, $W_{H^{(2)},\mathcal{C}}$, and $W'_{H^{(3)},\mathcal{C}}$, then the coefficient of $\mathcal{Y}(v_{H^{(1)}}, x)v_{H^{(2)}}$ in the lowest weight space of $W_{H^{(3)}}$ must be in $\mathbb{Z}v_{H^{(3)}}$.

Conversely, suppose that the coefficient of $\mathcal{Y}(v_{H^{(1)}}, x)v_{H^{(2)}}$ in the lowest weight space of $W_{H^{(3)}}$ is in $\mathbb{Z}v_{H^{(3)}}$, that is,

$$(\mathcal{Y}(v_{H^{(1)}}, x)v_{H^{(2)}}, v_{H^{(3)}})_{H^{(3)}} \in \mathbb{Z}\{x\}. \tag{5.9}$$

By Theorem 2.4.9, it is enough to prove that

$$(\mathcal{Y}(v_{H^{(1)}}, x)v_{H^{(2)}}, w)_{H^{(3)}} \in \mathbb{Z}\{x\} \tag{5.10}$$

for any $w \in W_{H^{(3)},\mathcal{C}}$. To prove this, we use induction on the conformal weight of w (which is contained in $\sum_{i=1}^n h_i^{(3)} + \mathbb{N}$), the base case given by (5.9). Since we see from the proofs of Propositions 5.4.1 and 5.4.4 that $W_{H^{(3)}}$ is the $\mathbb{Z}$-span of vectors of the form

$$L_{T_1}(-m_1) \cdots L_{T_k}(-m_k)v_{H^{(3)}}$$

where $m_i > 0$, it is enough to show that if (5.10) holds for $w \in W_{H^{(3)},\mathcal{C}}$ of weight less than some fixed N, then it also holds for $L_T(-m)w$ for any $T \in \mathcal{C}$ and $m > 0$.

To prove this, we first observe that since

$$Y^o(\omega_T, x) = Y(e^{xL(1)}(-x^{-2})^{L(0)}\omega_T, x^{-1}) = x^{-4}Y(\omega_T, x^{-1}),$$

the operator $L_T(m)$ is adjoint to $L_T(-m)$ with respect to $(\cdot,\cdot)_{H^{(3)}}$ for any $T \in \mathcal{C}$. Then we use Lemma 5.5.3 and the fact that $L_T(m)v_{H^{(2)}} = 0$ for $m > 0$ to obtain

$$\begin{aligned}
(\mathcal{Y}(v_{H^{(1)}},x)v_{H^{(2)}}, L_T(-m)w)_{H^{(3)}} &= (L_T(m)\mathcal{Y}(v_{H^{(1)}},x)v_{H^{(2)}}, w)_{H^{(3)}} \\
&= ([L_T(m), \mathcal{Y}(v_{H^{(1)}},x)]v_{H^{(2)}}, w)_{H^{(3)}} \\
&= (x^m[L_T(0), \mathcal{Y}(v_{H^{(1)}},x)]v_{H^{(2)}} + mx^m\mathcal{Y}(L_T(0)v_{H^{(1)}},x)v_{H^{(2)}}, w)_{H^{(3)}} \\
&= x^m(\mathcal{Y}(v_{H^{(1)}},x)v_{H^{(2)}}, L_T(0)w)_{H^{(3)}} - x^m(\mathcal{Y}(v_{H^{(1)}},x)L_T(0)v_{H^{(2)}}, w)_{H^{(3)}} \\
&\quad + mx^m\mathcal{Y}(L_T(0)v_{H^{(1)}},x)v_{H^{(2)}}, w)_{H^{(3)}} \in \mathbb{Z}\{x\}
\end{aligned}$$

because $L_T(0)v_{H^{(i)}} \in \mathbb{Z}v_{H^{(i)}}$ for $i = 1,2$ and $L_T(0)w \in W_{H^{(3)},\mathcal{C}}$ has the same weight as w. This proves the theorem. □

We conclude this chapter by outlining a strategy for obtaining interesting integral forms for a vertex operator algebra V which contains a vertex operator subalgebra isomorphic to $L(\frac{1}{2},0)^{\otimes n}$ for $n \in 4\mathbb{Z}$. From [W] and [DMZ], the $L(\frac{1}{2},0)^{\otimes n}$-module V is completely reducible, and has a finite decomposition $V = \coprod W_H$ for some $H = (h_1,\ldots,h_n) \in \{0,\frac{1}{2},\frac{1}{16}\}^n$. If $\Omega = \{1,\ldots n\}$ and $\mathcal{C} \subseteq \mathcal{E}(\Omega)$ is a code satisfying the conditions of Theorem 5.3.3, and if for any W_H appearing in the decomposition of V, the $L(\frac{1}{2},0)^{\otimes n}_{\mathcal{C}}$-submodule $W_{H,\mathcal{C}}$ of W_H generated by a lowest conformal weight vector v_H is an integral form of W_H, then $V_{\mathcal{C}} = \coprod W_{H,\mathcal{C}}$ is an integral form of V as a vector space.

We observe that if $W_{H^{(i)}}$ for $i = 1,2,3$ are three submodules appearing in the decomposition of V, then the vertex operator Y on V restricted to $W_{H^{(1)}} \otimes W_{H^{(2)}}$ and projected onto $W_{H^{(3)}}$ is an intertwining operator of type $\binom{W_{H^{(3)}}}{W_{H^{(1)}}\,W_{H^{(2)}}}$. To show that $V_{\mathcal{C}}$ is an integral form of V as a vertex operator algebra, we would first need to show that we can choose the lowest weight vectors v_H generating $W_{H,\mathcal{C}}$ so that $v_{(0,\ldots,0)} = \mathbf{1}$ and so that for any triple of submodules $W_{H^{(i)}}$ for $i = 1,2,3$ in the decomposition of V, the coefficient of the lowest power of x in the projection of $Y(v_{H^{(1)}},x)v_{H^{(2)}}$ to

$W_{H^{(3)}}$ is in $\mathbb{Z}v_{H^{(3)}}$. If this can be done, then Theorem 5.5.2 shows that

$$Y : V_{\mathcal{C}} \otimes V_{\mathcal{C}} \to V'_{\mathcal{C}}[[x, x^{-1}]],$$

where we use the forms $(\cdot,\cdot)_H$ to identify V with its contragredient as an $L(\frac{1}{2},0)^{\otimes n}$-module (not as a V-module).

However, it is not generally the case that $V_{\mathcal{C}} = V'_{\mathcal{C}}$, so it would also be necessary to show that for any triple of submodules $W_{H^{(i)}}$ for $i = 1, 2, 3$ in the decomposition of V, Y maps $W_{H^{(1)}} \otimes W_{H^{(2)}}$ into $W_{H^{(3)}}$. It is not clear under what circumstances this is the case. It is true for $W_{H^{(1)}} = L(\frac{1}{2},0)^{\otimes n}$ and $W_{H^{(2)}} = W_{H^{(3)}}$. We conclude this dissertation with an example that illustrates the issues involved:

Example 5.5.4. Let Q be the $A_1 \times A_1$ root lattice: $Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ where $\langle \alpha_i, \alpha_j \rangle = 2\delta_{ij}$. Consider the lattice vertex operator algebra V_Q; since $\langle \alpha, \beta \rangle \in 2\mathbb{Z}$ for any $\alpha, \beta \in Q$, we can take the central extension (4.1) of Q to be trivial. For any $\alpha \in Q$, we define

$$\iota(e_\alpha)^{\pm} = \iota(e_\alpha) \pm \iota(e_{-\alpha}).$$

The proof of Theorem 6.3 in [DMZ] shows that V_Q contains a vertex operator subalgebra isomorphic to $L(\frac{1}{2},0)^{\otimes 4}$, where

$$\begin{aligned}
\omega^{(1)} &= \frac{1}{16}(\alpha_1 + \alpha_2)(-1)^2\mathbf{1} + \frac{1}{4}\iota(e_{\alpha_1+\alpha_2})^+,\\
\omega^{(2)} &= \frac{1}{16}(\alpha_1 + \alpha_2)(-1)^2\mathbf{1} - \frac{1}{4}\iota(e_{\alpha_1+\alpha_2})^+,\\
\omega^{(3)} &= \frac{1}{16}(\alpha_1 - \alpha_2)(-1)^2\mathbf{1} + \frac{1}{4}\iota(e_{\alpha_1-\alpha_2})^+,\\
\omega^{(4)} &= \frac{1}{16}(\alpha_1 - \alpha_2)(-1)^2\mathbf{1} - \frac{1}{4}\iota(e_{\alpha_1-\alpha_2})^+.
\end{aligned}$$

It is not hard to check that the decomposition of V_Q into $L(\frac{1}{2},0)^{\otimes 4}$-modules is given by

$$V_Q = \coprod_{\substack{H=(h_1,h_2,h_3,h_4)\in\{0,\frac{1}{2}\}^4\\ h_1+h_2+h_3+h_4\in\mathbb{Z}}} W_H,$$

where the lowest conformal weight vectors generating the submodules in the decomposition can be taken to be

$$\mathbf{1},\ \frac{\iota(e_{\alpha_1})^+ \pm \iota(e_{\alpha_2})^+}{2},\ \frac{\iota(e_{\alpha_1})^- \pm \iota(e_{\alpha_2})^-}{2},\ \frac{(\alpha_1 \pm \alpha_2)(-1)}{2}\mathbf{1},\ \frac{\alpha_1(-1)^2 - \alpha_2(-1)^2}{4}\mathbf{1}.$$

Setting $\Omega = \{1,2,3,4\}$ and $\mathcal{C} = \mathcal{E}(\Omega)$, we take these vectors as the vectors v_H generating $V_{Q,\mathcal{E}(\Omega)}$ as an $L(\frac{1}{2},0)^{\otimes 4}_{\mathcal{E}(\Omega)}$-module. If we try to show that $V_{Q,\mathcal{E}(\Omega)}$ is an integral form containing ω of V_Q as a vertex operator algebra, it is straightforward to calculate the vertex operators $Y(v_{H^{(1)}},x)v_{H^{(2)}}$ to find their lowest coefficients.

However, it is not generally the case here that $W_{H,\mathcal{E}(\Omega)} = W'_{H,\mathcal{E}(\Omega)}$. For example, consider $H = (\frac{1}{2},\frac{1}{2},0,0)$. Then the weight 2 subspace of W_H is the $\mathbb{Z}$-span of the vectors $L_T(-1)v_H$ for $T \in \mathcal{E}(\Omega)$, which is the integral span of the vectors $(L^{(1)}(-1) \pm L^{(2)}(-1))v_H$. But

$$\begin{aligned}
&((L^{(1)}(-1) \pm L^{(2)}(-1))v_H, (L^{(1)}(-1) \pm L^{(2)}(-1))v_H)_H = \\
&(v_H, (L^{(1)}(1) \pm L^{(2)}(1))(L^{(1)}(-1) \pm L^{(2)}(-1))v_H)_H = \\
&(v_H, 2(L^{(1)}(0) + L^{(2)}(0))v_H)_H = 2.
\end{aligned}$$

Similarly,

$$((L^{(1)}(-1) \pm L^{(2)}(-1))v_H, (L^{(1)}(-1) \mp L^{(2)}(-1))v_H)_H = 0,$$

so that the weight 2 subspace of $W'_{H,\mathcal{E}(\Omega)}$ is the integral span of the two vectors $\frac{1}{2}(L^{(1)}(-1) \pm L^{(2)}(-1))v_H$, showing that $(W'_{H,\mathcal{E}(\Omega)})_{(2)} \supsetneq (W_{H,\mathcal{E}(\Omega)})_{(2)}$.

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