

**A PRODUCT FORMULA FOR CERTAIN
LITTLEWOOD-RICHARDSON COEFFICIENTS FOR
JACK AND MACDONALD POLYNOMIALS**

by

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ABSTRACT OF THE DISSERTATION

A product formula for certain Littlewood-Richardson coefficients for Jack and Macdonald polynomials

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Jack polynomials generalize several classical families of symmetric polynomials, including Schur polynomials, and are further generalized by Macdonald polynomials. In 1989, Richard Stanley conjectured that if the Littlewood-Richardson coefficient for a triple of Schur polynomials is 1, then the corresponding coefficient for Jack polynomials can be expressed as a product of weighted hooks of the Young diagrams associated to the partitions indexing the coefficient. We prove a special case of this conjecture in which the partitions indexing the Littlewood-Richardson coefficient have at most 3 parts. We also show that this result extends to Macdonald polynomials.

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Introduction

Jack polynomials $J_\lambda(\alpha; x)$ are a one parameter family of symmetric functions indexed by an integer partition λ . They were first introduced by Henry Jack [6] in 1969 as generalizations of spherical functions over $\mathrm{GL}(n, \mathbb{F})/\mathrm{U}(n, \mathbb{F})$, where $\alpha = 1/2, 1, 2$ correspond to the cases of $\mathbb{F} = \mathbb{H}, \mathbb{C}, \mathbb{R}$. Jack polynomials can be characterized in several ways. They appear as simultaneous eigenfunctions of certain Laplace-Beltrami type differential operators [14]. In addition, they form an orthogonal basis for the ring of symmetric functions over the field of rational functions in α . Jack polynomials were further generalized in 1988 by Macdonald polynomials $J_\lambda(q, t; x)$ [13], which are a two parameter family of polynomials that reduce to Jack polynomials under a special limit.

The $\alpha = 1$ specialization gives us scalar multiples of the well-known Schur polynomials [7, 18], which play a central role in the representation theory of S_n as well as $\mathrm{GL}(n, \mathbb{C})$. These polynomials are also indexed by partitions, and can be described combinatorially in terms of Young tableaux. Moreover, the coefficients that arise when a product of two Schur functions is decomposed into a sum of Schur functions have a combinatorial description known as the Littlewood-Richardson Rule (see [14, 5]), given by counting the number of skew tableaux of a certain type. These Littlewood-Richardson coefficients also appear in various other fields outside of representation theory, such as in the study of Grassmanians and sums of Hermitian matrices (see [5, 2]).

It is a continuing area of interest to find appropriate generalizations of these results for Schur polynomials in the context of Jack and Macdonald polynomials. Various works [19, 14, 10, 3, 15] establish several combinatorial properties of these polynomials and conjecture others. It is also possible to compute the Littlewood-Richardson coefficients for such polynomials (see [16, 17]), but currently there are no combinatorial formulas for these coefficients in the style of the Littlewood-Richardson Rule. In this work,

we prove a special case of one of Richard Stanley's conjectures [19, Conj. 8.5] which proposes a combinatorial description for certain Littlewood-Richardson coefficients for Jack polynomials. In particular, this conjecture generalizes the Littlewood-Richardson Rule for triples of partitions (λ, μ, ν) such that the corresponding coefficient for Schur polynomials indexed by this triple is 1. We prove that this conjecture is true when the triple (λ, μ, ν) is restricted to having at most 3 parts and extend this result to coefficients of Macdonald polynomials as well.

In Chapter 1, we provide some background about the combinatorics of partitions and symmetric functions. In Chapter 2, we give a precise statement of Stanley's conjecture for Littlewood-Richardson coefficients of Jack polynomials and state our main theorem regarding the special case of this conjecture in which the coefficients are indexed by partitions with at most 3 parts. In order to prove this theorem, we first classify all the partitions that satisfy the hypothesis of Stanley's conjecture in Chapter 3. Then, in Chapter 4, we use this classification to experimentally obtain formulas for the coefficients in each case of our classification, and then develop some general algebraic results which we use to verify each of the experimental formulas. In Chapter 5, we extend our result from coefficients for Jack polynomials to coefficients for Macdonald polynomials. Finally, we describe some ongoing work and further directions relating to our results in Chapter 6.

Chapter 1

Preliminaries

In this section, we present some basic definitions and background information pertaining to the theory of partitions and symmetric functions. We refer the reader to [14] for a more detailed treatment of this material.

1.1 Partitions

Definition 1.1.1. A *partition* λ is a sequence $(\lambda_1, \lambda_2, \dots, \lambda_n)$ of non-negative integers listed in weakly decreasing order:

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0.$$

Each nonzero λ_i is called a *part* of λ . We will sometimes write a partition λ in the form $(i_1^{m_1}, i_2^{m_2}, \dots, i_k^{m_k})$, where $i_j^{m_j}$ denotes m_j parts equal to i_j . We call m_j the *multiplicity* of i_j in λ .

The *length* $\ell(\lambda)$ of a partition λ is the number of parts of λ . Let \mathcal{P}_n denote the set of partitions of length at most n . We think of $\lambda \in \mathcal{P}_n$ as an n -tuple, with $\lambda_i = 0$ for $i > \ell(\lambda)$.

The *weight* $|\lambda|$ of λ is the sum of its parts:

$$|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_n.$$

If $|\lambda| = n$, then we say λ is a *partition of n* .

Given any two partitions λ and μ , we can define $\lambda + \mu$ as the partition obtained by taking the sum of λ and μ as sequences:

$$(\lambda + \mu)_i = \lambda_i + \mu_i.$$

Given two partitions λ, μ of n , we say $\mu \leq \lambda$ if for all $i \in \{1, \dots, n\}$,

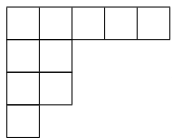
$$\mu_1 + \dots + \mu_i \leq \lambda_1 + \dots + \lambda_i.$$

The relation \leq defines a partial order, known as the *dominance order*, on the set of all partitions of n .

Partitions are commonly represented diagrammatically.

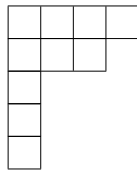
Definition 1.1.2. The *Young diagram* of a partition λ is a left justified array of boxes such that there are λ_i boxes in row i . (We will use the same symbol λ to denote both the partition and its Young diagram.)

Example 1.1.3. Let $\lambda = (5, 2, 2, 1)$. Then the corresponding Young diagram is:



The *conjugate* λ' of a partition λ is the partition whose diagram is the transpose of the diagram of λ , where the transpose is obtained by reflecting across the main diagonal and thus interchanging rows and columns.

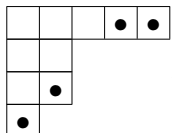
Example 1.1.4. If $\lambda = (5, 2, 2, 1)$ (as in Example 1.1.3), then the transpose of its Young diagram is:



and so $\lambda' = (4, 3, 1, 1, 1)$.

We say $\lambda \supset \mu$ if the diagram of λ contains the diagram of μ . Let $\lambda - \mu$ be the set theoretic difference between the two diagrams, which we call a *skew diagram*.

Example 1.1.5. If $\lambda = (5, 2, 2, 1)$ and $\mu = (3, 2, 1)$, then the skew diagram λ/μ is denoted by the marked boxes in the diagram below:



If the skew diagram consists of $r = |\lambda| - |\mu|$ boxes and has at most one box in each column (respectively, row), we refer to it as a *horizontal r -strip* (respectively *vertical r -strip*). In Example 1.1.5, λ/μ is a horizontal 4-strip. However, it is not a vertical strip since the first row of the skew diagram contains two boxes.

A *skew tableau* T is obtained by filling each box of a skew diagram λ/μ with a positive number, where $\lambda - \mu$ is called the *shape* of T . If m_i denotes the number of times i appears in the skew tableau, we say (m_1, \dots, m_r) is the *weight* of the T , and the *word* $w(T)$ of T is the sequence obtained by reading the entries of T from right to left in each row.

Example 1.1.6. Let T be the skew tableau given by

			1	2
	1	1	3	3
1	2			
3				

Then:

- the shape of T is $(5, 5, 2, 1) - (3, 1)$.
- the weight of T is $(4, 2, 3)$.
- the word of T is $w(T) = (2, 1, 3, 3, 1, 1, 2, 1, 3)$.

A skew tableau T is said to be *semistandard* if the entries of T weakly increase across rows (from left to right) and strongly increase down columns. We say that T satisfies the Yamanouchi word condition if the number of occurrences of an integer i never exceeds the number of occurrences of $i - 1$ for any initial segment of $w(T)$.

Definition 1.1.7. A *Littlewood-Richardson* tableau is a semistandard skew tableau T that satisfies the Yamanouchi word condition.

Example 1.1.8. The skew tableau

			1	1
	1	1	2	
2	3			

is a Littlewood-Richardson tableau.

We will call any filling of a skew diagram that gives a Littlewood-Richardson tableau an *LR filling*.

1.2 Symmetric Functions

Let $\mathbb{Z}[x_1, \dots, x_n]$ denote the ring of polynomials in n independent variables x_1, \dots, x_n with integer coefficients. Let S_n be the symmetric group on n letters. Then S_n acts on $\mathbb{Z}[x_1, \dots, x_n]$ by permuting the variables, and a polynomial is called *symmetric* if it is unchanged under this action. The symmetric polynomials form a subring:

$$\Lambda_n = \mathbb{Z}[x_1, \dots, x_n]^{S_n}.$$

For each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we can define the monomial

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Then we can define the *monomial symmetric function* m_λ , where λ is a partition of length at most n , by

$$m_\lambda(x_1, \dots, x_n) = \sum_{\alpha \in S_n \cdot \lambda} x^\alpha,$$

where $S_n \cdot \lambda$ is the orbit of λ under the action of S_n . The monomial symmetric functions form a \mathbb{Z} -basis for Λ_n .

For a partition λ , we can also define the skew-symmetric polynomial a_λ by

$$a_\lambda(x_1, \dots, x_n) = \sum_{w \in S_n} \epsilon(w) x^{w(\lambda)},$$

where $\epsilon(w)$ is the sign of the permutation $w \in S_n$. Let δ be the partition $(n-1, n-2, \dots, 1, 0)$. Then $a_{\lambda+\delta}$ is divisible by a_δ , and the quotient

$$s_\lambda(x_1, \dots, x_n) = \frac{a_{\lambda+\delta}}{a_\delta},$$

called the *Schur polynomial*, is a symmetric function. The s_λ , $\ell(\lambda) \leq n$ also form a basis for Λ_n .

Schur polynomials appear as spherical functions over $\mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n, \mathbb{C})$. Spherical functions over $\mathrm{GL}(n, \mathbb{F})/\mathrm{U}(n, \mathbb{F})$ are further generalized by Jack polynomials $J_\lambda(\alpha; x_1, \dots, x_n)$, where $\alpha = 1/2, 1, 2$ correspond to the case of $\mathbb{F} = \mathbb{H}, \mathbb{C}, \mathbb{R}$, respectively.

To define Jack polynomials, we must first define the operator $D(\alpha)$ on $\Lambda \otimes \mathbb{Q}(\alpha)$ by

$$D(\alpha) = \frac{\alpha}{2} \sum_i x_i^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i \neq j} \frac{x_i^2}{x_i - x_j} \frac{\partial}{\partial x_i}.$$

Then $D(\alpha)$ is upper triangular on the basis of monomial symmetric functions m_λ , ie

$$D(\alpha)m_\lambda = \sum_{\mu \leq \lambda} b_{\lambda,\mu} m_\mu.$$

Definition 1.2.1. The *monic* Jack polynomials

$$P_\lambda = P_\lambda(\alpha; x_1, \dots, x_n) = \sum_{\mu \leq \lambda} v_{\lambda,\mu} m_\mu$$

are the eigenfunctions of $D(\alpha)$ such that $v_{\lambda,\lambda} = 1$.

Note that $P_\lambda(1) = s_\lambda$. We will also find it convenient to consider the following scalar multiples of J_λ :

Definition 1.2.2. The *integral* Jack polynomials

$$J_\lambda = J_\lambda(\alpha; x_1, \dots, x_n) = \sum_{\mu \leq \lambda} v_{\lambda,\mu} m_\mu$$

are the eigenfunctions of $D(\alpha)$ such that if $|\lambda| = m$, then $v_{\lambda,(1^m)} = m!$.

Jack polynomials are further generalized by Macdonald polynomials which are eigenfunctions of the operator $D(q, t)$ on $\Lambda \otimes \mathbb{Q}(q, t)$ defined by:

$$D(q, t) = \sum_i \left(\prod_{i \neq j} \frac{tx_i - x_j}{x_i - x_j} T_{q,i} \right),$$

where

$$T_{q,i} f(x_1, \dots, x_n) = f(x_1, \dots, qx_i, \dots, x_n).$$

Then, once again,

$$D(q, t)m_\lambda = \sum_{\mu \leq \lambda} b_{\lambda,\mu} m_\mu.$$

Definition 1.2.3. The Macdonald polynomials

$$P_\lambda = P_\lambda(q, t; x_1, \dots, x_n) = \sum_{\mu \leq \lambda} v_{\lambda,\mu} m_\mu$$

are the eigenfunctions of $D(q, t)$ such that $v_{\lambda,\lambda} = 1$.

We can recover the Jack polynomials from the Macdonald polynomials by taking the limit as q, t go to 1, where the parameter α signifies the direction along which this limit is taken. Thus,

$$\lim_{t \rightarrow 1} P_\lambda(t^\alpha, t) = P_\lambda(\alpha).$$

1.3 The Littlewood-Richardson Rule

Schur functions can be interpreted combinatorially, by the following theorem.

Theorem 1.3.1.

$$s_\lambda = \sum_T x^{\theta(T)},$$

where T is a tableau of shape λ , and $\theta(T)$ is the weight of T .

Example 1.3.2. $s_{(2,1)} \in \Lambda_3$:

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array}$$

$$s_{(2,1)} = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

This also leads to a way of combinatorially interpreting the coefficients that appear when a product of Schur polynomials is expanded as a sum of Schur polynomials. This was developed using two major results. We start with a theorem that tells us how to expand such a product when one of the polynomials in the product is indexed by a partition of length 1.

Theorem 1.3.3 (Pieri Rule).

$$s_\mu s_{(r)} = \sum_\lambda s_\lambda,$$

where λ/μ is a horizontal r -strip.

Example 1.3.4. $\mu = (3, 1), r = 2$

$$\begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & & \\ \hline & & & \\ \hline 1 & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline & & \\ \hline & 1 & 1 \\ \hline & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline & & \\ \hline & 1 & \\ \hline & & \\ \hline 1 & & \\ \hline \end{array}$$

$$s_{(3,1)} s_{(2)} = s_{(5,1)} + s_{(4,2)} + s_{(4,1,1)} + s_{(3,3)} + s_{(3,2,1)}$$

Note that we can also consider the transpose of each of the indexing partitions, to get a way of multiplying two Schur polynomials when one of them is indexed by a partition consisting of a single column.

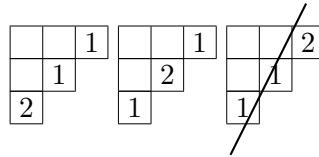
Finally, we can extend this result to products of two Schur polynomials indexed by general partitions. This is done using the Littlewood-Richardson rule.

Theorem 1.3.5 (Littlewood-Richardson Rule).

$$s_\mu s_\nu = \sum_\lambda c_{\mu,\nu}^\lambda s_\lambda,$$

where $c_{\mu,\nu}^\lambda$ is the number of Littlewood-Richardson tableaux T of shape λ/μ and weight ν .

Example 1.3.6. $\mu = (2, 1), \nu = (2, 1), \lambda = (3, 2, 1)$



$$c_{(2,1),(2,1)}^{(3,2,1)} = 2$$

Chapter 2

Stanley's Conjecture

2.1 Statement of Conjecture

We wish to generalize the Littlewood-Richardson rule to obtain a description of the coefficients that appear when a product of Jack or Macdonald polynomials is expanded as a sum of the respective polynomials. While it is possible to compute these coefficients recursively, there is currently no combinatorial result that clearly reduces to the Littlewood-Richardson rule as we take the appropriate limit of the Jack or Macdonald polynomials to recover the corresponding Schur polynomials. However, in [19], Stanley made some observations and conjectures that give us some steps towards this goal. While Stanley discusses only the case of Jack polynomials in his paper, all results can be generalized to Macdonald polynomials as well.

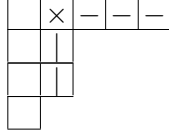
In order to state Stanley's Conjecture [19, Conj. 8.5], we must first define the hook length for a box in a Young diagram and some of its analogues. The *hook-length* $h_\lambda(b)$ of a box b in the partition λ is obtained by counting all the boxes to the right of b (called the *arm*, denoted $a_\lambda(b)$) and all the boxes below b (called the *leg*, denoted $\ell(b)$) along with b itself.

$$a_\lambda(i, j) = \lambda_i - j$$

$$\ell_\lambda(i, j) = \lambda'_j - i$$

$$h_\lambda(i, j) = a_\lambda(i, j) + \ell_\lambda(i, j) + 1$$

Example 2.1.1. $\lambda = (5, 2, 2, 1), b = (1, 2)$



$$h_\lambda(b) = 3 + 2 + 1 = 6$$

We can define 2 α -generalizations of $h_\lambda(b)$:

- *upper hook-length*: $h_\lambda^*(b) = \alpha(a(b) + 1) + \ell(b)$
- *lower hook-length*: $h_*^\lambda(b) = \alpha(a(b)) + \ell(b) + 1$

In effect, the upper hook treats the corner box as part of the arm, whereas the lower treats it as part of the leg.

We define the following products of hook lengths:

$$H_*^\lambda = \prod_{b \in \lambda} h_*^\lambda(b)$$

$$H_\lambda^* = \prod_{b \in \lambda} h_\lambda^*(b)$$

$$j_\lambda = H_*^\lambda \cdot H_\lambda^*$$

Then we can relate the integral and the monic Jack polynomials as follows:

$$J_\lambda(\alpha) = H_*^\lambda P_\lambda(\alpha).$$

We can also define the dual $J_\lambda^*(\alpha)$ of $J_\lambda(\alpha)$ under the canonical inner product by:

$$J_\lambda^*(\alpha) = j_\lambda^{-1} J_\lambda(\alpha).$$

Finally, we consider the following expansions:

$$J_\mu J_\nu = \sum_\lambda g_{\mu\nu}^\lambda(\alpha) J_\lambda^*,$$

$$P_\mu P_\nu = \sum_\lambda c_{\mu\nu}^\lambda(\alpha) P_\lambda.$$

Then

$$g_{\mu\nu}^\lambda(\alpha) = H_\lambda^* H_*^\mu H_*^\nu c_{\mu,\nu}^\lambda(\alpha).$$

We are now ready to state Stanley's conjecture.

Conjecture 2.1.2 (Stanley, 1989). *Given partitions λ, μ, ν such that $c_{\mu, \nu}^{\lambda}(1) = 1$, then for all α ,*

$$g_{\mu, \nu}^{\lambda}(\alpha) = \left(\prod_{b \in \lambda} \tilde{h}_{\lambda}(b) \right) \left(\prod_{b \in \mu} \tilde{h}_{\mu}(b) \right) \left(\prod_{b \in \nu} \tilde{h}_{\nu}(b) \right), \quad (2.1.1)$$

where $\tilde{h}_{\xi}(b)$ is either $h_{\xi}^*(b)$ or $h_{\xi}^{\#}(b)$. Moreover, we can choose these hooks such that there is an equal number of upper and lower hooks.

Unfortunately, while this conjecture states that such a choice is always possible, there is no canonical way to make such a choice, and no conjecture for an assignment that might work in general. In fact, as Stanley himself notes in [19], there is often more than one assignment of upper and lower hooks that would satisfy this conjecture.

Example 2.1.3. $\lambda = (2, 2, 2, 1, 1), \mu = (2, 1, 1), \nu = (2, 1, 1)$

l	l	u	?	u	?
l	l	l		l	
u	?	?		?	
l				?	
?					

Of the 6 boxes marked “?”, 5 must be taken to be upper hooks and 1 to be a lower hook, so there are 6 possible ways to obtain the correct coefficient.

Since we can get $c_{\mu, \nu}^{\lambda}$ by dividing $g_{\mu, \nu}^{\lambda}$ by all the upper hooks in λ and all the lower hooks in μ and ν , we will call such hooks *standard hooks* and boxes assigned to have standard hooks in Equation 2.1.1 to be *standard boxes*. On the other hand, we will call lower hooks in λ and upper hooks in μ and ν *flipped hooks* and boxes with such an assignment in Equation 2.1.1 *flipped boxes*. If $g_{\mu, \nu}^{\lambda}(\alpha)$ is given by a product of only standard hooks, then $c_{\mu, \nu}^{\lambda}(\alpha) = 1$ for all α . In general, $c_{\mu, \nu}^{\lambda}(\alpha)$ can be regarded as a product over flipped boxes of the ratio of the flipped hook to the standard hook. When $\alpha = 1$, the upper and lower hooks have the same value, and so any such product reduces to 1, in agreement with the hypothesis $c_{\mu, \nu}^{\lambda}(1) = 1$.

We will call any triple (λ, μ, ν) of partitions that satisfy the hypothesis $c_{\mu, \nu}^{\lambda}(1) = 1$ an *extremal triple*. Such triples correspond to the case of a unique Littlewood-Richardson tableau of shape $\lambda - \mu$ with weight ν , but it remains difficult to generate all such triples in general. Extremal triples lie on the boundary of Horn cones, which are given by the

eigenvalues of Hermitian matrices A, B, C such that $A + B + C = 0$. (However, note that not all boundary triples are extremal.) Extremal triples also play a prominent role in Fulton's conjecture, which states that an extremal triple remains extremal under a scaling of all three partitions by the same factor. (A proof of Fulton's conjecture is given by Knutson, Tao and Woodward in [12].)

2.2 Main Theorem

In this work, we prove the following special case of Stanley's conjecture.

Theorem 2.2.1. *Stanley's conjecture is true for $\lambda, \mu, \nu \in \mathcal{P}_3$.*

We will show this by first classifying all extremal triples of partitions in \mathcal{P}_3 , which we do in Chapter 3. We thus divide the problem into several cases and develop an experimental formula in the form of Equation 2.1.1 for $c_{\mu, \nu}^\lambda$ in each case. A complete list of these is given in Section 4.3. In Section 4.6, we verify that our experimental formulas indeed give the correct coefficient, thus completing the proof of Theorem 2.2.1. In Chapter 5, we extend this theorem to get Theorem 5.0.9, which shows that the coefficient for the corresponding Macdonald polynomials can also be obtained for extremal triples of partitions in \mathcal{P}_3 using the same system of upper and lower hook assignments using a suitable generalization of hook-lengths.

2.3 The Pieri Rule for Jack Polynomials

By Theorem 1.3.3, we see that if ν consists of a single row (or column), λ, μ, ν must be an extremal triple. In fact, we have an analogue of this theorem that gives a proof of Stanley's conjecture when ν falls into this special case.

Theorem 2.3.1 (Pieri Rule for columns [9, Thm 6.3]). *If λ/μ is a vertical r -strip and $\nu = (1^r)$, then*

$$c_{\mu, \nu}^\lambda(\alpha) = \prod_{s \in X(\lambda/\mu)} \frac{h_*^\lambda(s) h_\mu^*(s)}{h_\lambda^*(s) h_*^\mu(s)},$$

where $X(\lambda/\mu)$ denotes all the boxes $(i, j) \in \mu$ such that $\mu_i = \lambda_i$ and $\mu'_j < \lambda'_j$.

Example 2.3.2. $\lambda = (4, 2, 2), \mu = (3, 2, 1), \nu = (1, 1)$

$$\begin{array}{|c|c|c|c|} \hline u & u & u & u \\ \hline u & l & & \\ \hline u & u & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline l & l & l \\ \hline l & u & \\ \hline l & & \\ \hline \end{array} \quad \begin{array}{|c|} \hline l \\ \hline l \\ \hline \end{array}$$

$$c_{\mu,\nu}^{\lambda} = \frac{2\alpha}{1+\alpha}$$

$$g_{\mu,\nu}^{\lambda} = 32\alpha^5(3+2\alpha)(1+2\alpha)^2(2+\alpha)^2(2+3\alpha)$$

We define

$$b_{\lambda}(\alpha) = \frac{H_{*}^{\lambda}(\alpha)}{H_{\lambda}^{*}(\alpha)}.$$

Thus, we can think of $b_{\lambda}(\alpha)$ as an operator that switches upper and lower hooks. This gives us the following equation:

$$c_{\mu',\nu'}^{\lambda'}\left(\frac{1}{\alpha}\right) = \frac{c_{\mu,\nu}^{\lambda}(\alpha)b_{\mu}(\alpha)b_{\nu}(\alpha)}{b_{\lambda}(\alpha)}. \quad (2.3.1)$$

Therefore, if λ, μ, ν is an extremal triple and we transpose all 3 partitions, the resulting Littlewood-Richardson coefficient corresponds to swapping all the upper and lower hooks. This allows us to use the Pieri rule for columns as a rule for rows as well.

Example 2.3.3. We consider the triple obtained by transposing the partitions in Example 2.3.2:

$\lambda = (3, 3, 1, 1), \mu = (3, 2, 1), \nu = (2)$

$$\begin{array}{|c|c|c|} \hline l & l & l \\ \hline l & u & l \\ \hline l & & \\ \hline l & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline u & u & u \\ \hline u & l & \\ \hline u & & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline u & u \\ \hline \end{array}$$

$$c_{\mu,\nu}^{\lambda} = \frac{16\alpha^2(1+2\alpha)}{3(1+\alpha)^4}$$

$$g_{\mu,\nu}^{\lambda} = 32\alpha^5(2+3\alpha)(1+2\alpha)^2(2+\alpha)^2(3+2\alpha)$$

Chapter 3

Classification

We present a classification of all extremal triples (λ, μ, ν) consisting of partitions in \mathcal{P}_3 . In particular, we show that such triples correspond to each face of co-dimension one of the $n = 3$ Horn cone (see [12]). It turns out that this correspondence is no longer true if we allow partitions of greater length, in which case extremal triples form a proper subset of the triples that lie on boundary faces of the associated Horn cone.

3.1 Horn's Inequalities

Horn cones were defined by [4] to answer the following problem: given two $n \times n$ Hermitian matrices A and B with eigenvalues $\mu = (\mu_1, \dots, \mu_n)$ and $\nu = (\nu_1, \dots, \nu_n)$ (arranged in weakly decreasing order), we wish to determine the possible eigenvalues $\lambda = (\lambda_1, \dots, \lambda_n)$ of the sum $C = A + B$. Horn conjectured a list of inequalities involving λ, μ, ν that, together with the condition $|\lambda| = |\mu| + |\nu|$, determine all possible combinations. These inequalities were verified by the works of Klyachko [8] and of Knutson and Tao [11], which also show that the Littlewood-Richardson coefficient $c_{\mu, \nu}^{\lambda}$ is nonzero if and only if (λ, μ, ν) lie in the Horn cone \mathcal{H}_n . Later, Knutson, Tao and Woodward [12] determined the minimal necessary list of such inequalities that determines this cone.

Using this list of inequalities for \mathcal{H}_3 , we have that the Littlewood-Richardson coefficient $c_{\mu, \nu}^{\lambda}$ is nonzero if the partitions $\lambda, \mu, \nu \in \mathcal{P}_3$ are such that $|\lambda| = |\mu| + |\nu|$, and they satisfy all of the inequalities in Table 3.1.1 below.

It is known that extremal triples (λ, μ, ν) all lie on a union of some faces of the Horn cone (see [1, 12]). We will refer to a face of codimension one as a *facet*. Since each facet is obtained by changing one of the defining inequalities to an equality, for \mathcal{H}_3 , we will refer to each facet by the same number as the corresponding inequality as above.

Table 3.1.1: Defining Inequalities for \mathcal{H}_3

1. $\mu_3 \leq \mu_2$	7. $\lambda_1 \leq \mu_1 + \nu_1$	13. $\lambda_3 \geq \mu_3 + \nu_3$
2. $\mu_2 \leq \mu_1$	8. $\lambda_2 \leq \mu_1 + \nu_2$	14. $\lambda_2 \geq \mu_3 + \nu_2$
3. $\nu_3 \leq \nu_2$	9. $\lambda_2 \leq \mu_2 + \nu_1$	15. $\lambda_2 \geq \mu_2 + \nu_3$
4. $\nu_2 \leq \nu_1$	10. $\lambda_3 \leq \mu_1 + \nu_3$	16. $\lambda_1 \geq \mu_3 + \nu_1$
5. $\lambda_3 \leq \lambda_2$	11. $\lambda_3 \leq \mu_2 + \nu_2$	17. $\lambda_1 \geq \mu_2 + \nu_2$
6. $\lambda_2 \leq \lambda_1$	12. $\lambda_3 \leq \mu_3 + \nu_1$	18. $\lambda_1 \geq \mu_1 + \nu_3$

In general, not every facet of \mathcal{H}_n contains extremal triples. However, this does hold for \mathcal{H}_3 , and so one can check triples (λ, μ, ν) on the interior of each face, and determine that every single facet does indeed give an extremal triple. In the next section, we present a direct combinatorial proof of this fact.

3.2 Littlewood-Richardson Tableaux

We will show that each facet of \mathcal{H}_3 contains extremal triples by classifying the possible Littlewood-Richardson tableaux of shape λ/μ of weight ν in the case that $\lambda, \mu, \nu \in \mathcal{P}_3$. The cases presented in this proof were also used to determine the experimentally obtained formulas for $c_{\mu, \nu}^\lambda(\alpha)$ presented in Section 4.3.

Theorem 3.2.1. *For partitions $\lambda, \mu, \nu \in \mathcal{P}_3$, we have $c_{\mu, \nu}^\lambda(1) = 1$ if and only if λ, μ, ν lie on a facet of the Horn cone \mathcal{H}_3 .*

Proof. A skew diagram of shape λ/μ consists of at most three rows. Therefore, if ν has length 3, then any LR filling of λ/μ of weight ν must consist of at least ν_3 occurrences of i in row i . We therefore only need to consider the remaining boxes, and we can thus assume, without loss of generality, that ν has length at most 2. By symmetry, we can also assume the same for μ .

Now let T be a Littlewood-Richardson tableau of shape λ/μ with weight ν . Then

$w(T)$ must be a sequence of 1's and 2's of the form $(1^{a_1}, 2^{b_2}, 1^{b_1}, 2^{c_2}, 1^{c_1})$, where i^m denotes m consecutive occurrences of i . In order to satisfy the Yamanouchi word condition, we must require that $a_1 \geq b_2$ and $a_1 + b_1 \geq b_2 + c_2$. For instance, if T_1 is the following diagram:

$$\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & 1 & 1 & 1 \\ \hline & & & 1 & 1 & 2 & 2 & & \\ \hline 1 & 2 & 2 & 2 & & & & & \\ \hline \end{array},$$

then $w(T_1) = (1^3, 2^2, 1^2, 2^3, 1^1)$. Note, however, that in this case, a filling of this skew diagram of weight $(6, 5)$ is not unique. We must therefore determine which restrictions on the set $(a_1, b_2, b_1, c_2, c_1)$ of multiplicities in $w(T)$ lead to an extremal triple (λ, μ, ν) .

First, suppose every column in λ/μ consists of a single box, so that λ/μ is a horizontal $|\nu|$ -strip:

$$\begin{array}{|c|c|c|c|c|} \hline & & & & 1 \\ \hline & & 1 & 2 & \\ \hline 1 & 2 & & & \\ \hline \end{array}$$

In order to have a unique LR filling, either $b_2 = 0$ (type B) or $c_1 = 0$ (type C). To see this, consider the case of an LR filling in which both b_2 and c_1 are nonzero, as in the diagram above. Then the last 1 in the third row can be swapped with the first 2 in the second row to get another LR filling,

$$\begin{array}{|c|c|c|c|c|} \hline & & & & 1 \\ \hline & & 1 & 1 & \\ \hline 2 & 2 & & & \\ \hline \end{array}$$

so $c_{\mu, \nu}^\lambda(1)$ must be greater than 1 in this case. However, as this second diagram illustrates, requiring that the filling be of type B or C is not sufficient to give an extremal triple, even though it is a necessary condition. Specifically, in the absence of any additional restrictions, it may be possible to swap a 2 in the third row with a 1 in the second row.

Therefore, for each type, B or C, we require one of the following restrictions:

- I. $c_2 = 0$
- II. $b_1 = 0$
- III. $a_1 = b_2$
- IV. $a_1 + b_1 = b_2 + c_2$.

Conditions I and II remove one of the quantities that would have been involved in such a swap to get a new LR filling with the same weight. Conditions III and IV imply that any such swap would violate the Yamanouchi word condition, since the swap would have the effect of increasing b_2 while leaving a_1, b_1 and c_2 unchanged.

Finally, we consider the case in which λ/μ is no longer necessarily a horizontal strip. Then every column in the skew diagram could have up to two boxes, and whenever it does contain two boxes, the filling must be a 1 in the upper box and a 2 in the lower box. We could have an overlap between the first and second rows (denoted by type o_1) or an overlap between the second and third rows (denoted type o_2). In the case that o_i does not occur, we denote the number of columns in the gap between the rows of the skew diagram by g_i . Thus, we have 32 cases in all (type B or C, type I-IV, type o_1 or g_1 , and type o_2 or g_2).

We will use o_i and g_i not only as a label for each type, but also a count (analogous to a_1, b_i, c_i) of the number of overlapping columns in the skew diagram, or the number of columns in the gap between rows of the skew diagram. Therefore, in general, the parts of λ, μ, ν are given by:

$$\nu_1 = a_1 + b_1 + c_1 + o_1 + o_2 + \nu_3$$

$$\nu_2 = b_2 + c_2 + o_1 + o_2 + \nu_3$$

$$\mu_1 = b_1 + b_2 + g_1 + o_2 + c_1 + c_2 + g_2 + \mu_3$$

$$\mu_2 = c_1 + c_2 + g_2 + \mu_3$$

$$\lambda_1 = b_1 + b_2 + g_1 + o_2 + c_1 + c_2 + g_2 + a_1 + o_1 + \mu_3 + \nu_3$$

$$\lambda_2 = b_1 + b_2 + o_2 + c_1 + c_2 + g_2 + o_1 + \mu_3 + \nu_3$$

$$\lambda_3 = o_2 + c_1 + c_2 + \mu_3 + \nu_3$$

Therefore, each of the 32 cases corresponds to a restriction on the partitions λ, μ, ν . For instance, B.I. g_1g_2 means that $b_2 = c_2 = o_1 = o_2 = 0$, and therefore $\nu_2 = \nu_3$. Similarly, B.II. g_1o_2 means that $b_2 = c_2 = o_1 = g_2 = 0$, and so we get that $\mu_2 + \nu_2 = \lambda_3$. We give a complete list of restrictions in Table 3.2.1 below, where each number refers to the facet of \mathcal{H}_3 determined by the correspondingly numbered Horn inequality above.

Table 3.2.1: Extremal triples of partitions in \mathcal{P}_3

Type	g_1g_2	g_1o_2	o_1g_2	o_1o_2
B.I	(3)	(11)	(8)	(16)
B.II	(15)	(5)	(2)	(10)
B.III	(18)	(18)	(6)	(6)
B.IV	(12)	(12)	(17)	(17)
C.I	(13)	(1)	(13)	(1)
C.II	(7)	(14)	(7)	(14)
C.III	(9)	(9)	(9)	(9)
C.IV	(4)	(4)	(4)	(4)

We thus verify that each facet of \mathcal{H}_3 appears in this table, and therefore each of them must contain only extremal triples.

□

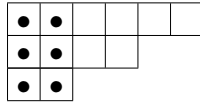
Chapter 4

Proof of Main Theorem

4.1 Division Numbers

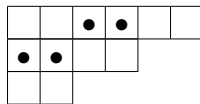
Every partition λ can be divided into rectangular *blocks* consisting of all columns of the same height. We will use ω_i^λ to denote the block $((\lambda_i - \lambda_{i+1})^i)$. Then if $\ell(\lambda) = n$, we can decompose λ as the sum $\omega_1^\lambda + \omega_2^\lambda + \cdots + \omega_n^\lambda$ of all its blocks.

Example 4.1.1. Let $\lambda = (6, 4, 2)$. The block ω_3^λ is highlighted in the Young diagram below.



We refer to each part of a block ω_i^λ as a *strip*. Thus, each strip consists of a row within a block.

Example 4.1.2. Let $\lambda = (6, 4, 2)$. The strips $(\omega_3^\lambda)_2$ and $(\omega_2^\lambda)_1$ are highlighted in the Young diagram below.



It turns out that for an extremal triple (λ, μ, ν) of partitions in \mathcal{P}_3 , it is possible to obtain $c_{\mu, \nu}^\lambda$ by an assignment of upper and lower hooks in the corresponding diagrams such that within each strip, all the upper hooks that occur appear to the left of all the lower hooks that occur. (Note that a strip may contain only upper hooks or only lower hooks.) We can thus encode the coefficient $c_{\mu, \nu}^\lambda$ by a system of *division numbers*, which are numbers for each strip in λ, μ, ν indicating the transition point between upper and lower hooks. By convention, we use the division numbers to count the flipped hooks in each strip, ie the lower hooks in each strip of λ and the upper hooks in each strip of μ and ν .

For each partition, we write the division numbers in a matrix style array, arranged in the same order (left to right, top to bottom) as the strip to which they correspond. Note that the division number symbols differ from matrices in that they contain no entries below the off-diagonal. Moreover, as we prove in Lemma 4.4.2 below, all hooks in the blocks ω_3^μ and ω_3^ν can be taken to be lower hooks, corresponding to division numbers of 0 for all the strips in those blocks. Therefore, we will write the division numbers for λ within a 3×3 array and those for μ and ν within a 2×2 array.

Example 4.1.3. $\lambda = (8, 7, 4), \mu = (6, 3), \nu = (5, 5)$

$$\begin{array}{|c|c|c|c|c|c|c|c|} \hline u & l & l & l & u & l & l & u \\ \hline u & u & l & l & u & l & l & \\ \hline u & l & l & l & & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|c|} \hline u & u & l & u & u & u \\ \hline u & u & l & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|} \hline u & l & l & l & l \\ \hline u & u & u & u & l \\ \hline \end{array}$$

$c_{\mu,\nu}^\lambda$ is encoded by the division numbers given as follows.

$$\lambda : \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & \\ 3 & & \end{bmatrix} \quad \mu : \begin{bmatrix} 2 & 3 \\ 2 & \end{bmatrix} \quad \nu : \begin{bmatrix} 1 & 0 \\ 4 & \end{bmatrix}.$$

4.2 Algebraic Structures

To compute $c_{\mu,\nu}^\lambda$ from the division numbers, we require the following notation.

Let β be a multiset. We regard β as the set of vanishing points (counted with multiplicity) of a polynomial. Therefore, let $\phi(x; \beta)$ be the smallest degree polynomial in x such that $\phi(b; \beta) = 0$ for all nonzero $b \in \beta$ and $\phi(0; \beta) = 1$. In particular, we have:

$$\phi(x; \beta) := \prod_{b \in \beta, b \neq 0} \left(\frac{b-x}{b} \right).$$

Such polynomials give us a natural way to write the coefficients $c_{\mu,\nu}^\lambda(\alpha)$ for extremal triples. Given an upper hook h_λ^* , we can write the ratio of the corresponding lower hook to the upper hook as

$$\frac{h_\lambda^* - (\alpha - 1)}{h_\lambda^*}.$$

Also, given a lower hook h_*^μ , we can write the ratio of the corresponding upper hook to the lower hook as

$$\frac{-h_*^\mu - (\alpha - 1)}{-h_*^\mu}.$$

Thus, each $c_{\mu,\nu}^\lambda(\alpha)$ can be written as $\phi(\alpha - 1; \mathcal{F}(\lambda, \mu, \nu))$, where $\mathcal{F}(\lambda, \mu, \nu)$ is the set of standard hooks of flipped boxes in λ and negatives of standard hooks of flipped boxes in μ and ν .

We will also find it convenient to write our hooks in terms of $r = 1/\alpha$. In this case, we regard our hook-lengths as

$$\begin{aligned} h_\lambda^*(b) &= a(b) + 1 + \ell(b)r \\ h_*^\lambda(b) &= a(b) + (\ell(b) + 1)r \end{aligned}$$

and we have that

$$\phi(\alpha - 1; m\alpha + n) = \frac{(m-1)\alpha + n + 1}{m\alpha + n} = \frac{(m-1) + (n+1)r}{m + nr} = \phi(1 - r; m + nr).$$

Successive flipped r -hooks within a single strip differ by 1, and so we require an effective way to describe such products. To do this, we will first define the following notation:

$$\langle x; a \rangle_j = \phi(x; \{a, \dots, a + j - 1\}).$$

When x is fixed and clear from context, we will suppress it and simply write $\langle a \rangle_j$.

We will make use of two main identities involving such terms.

For the first identity, observe that if $j = j_1 + j_2$, then

$$\langle a \rangle_{j_1} \langle a + j_1 \rangle_{j_2} = \langle a \rangle_j = \langle a \rangle_{j_2} \langle a + j_2 \rangle_{j_1}$$

and so

$$\frac{\langle a \rangle_{j_1}}{\langle a + j_2 \rangle_{j_1}} = \frac{\langle a \rangle_{j_2}}{\langle a + j_1 \rangle_{j_2}}. \quad (4.2.1)$$

For the second identity, note that if $a + b = x$ then

$$\left(\frac{a-x}{a}\right) \left(\frac{b-x}{b}\right) = \left(\frac{a-x}{a}\right) \left(\frac{-a}{x-a}\right) = 1$$

and more generally that

$$\langle a \rangle_j \langle b - j + 1 \rangle_j = 1, \quad (4.2.2)$$

where the i^{th} term in the first product cancels with the $(j-i+1)^{\text{th}}$ term in the second product, since $a + (i-1) + (b-j+1) + (j-i) = a + b = x$. Note that this is equivalent to saying that $\langle a \rangle_j \langle b \rangle_j = 1$ whenever $a + b = x - j + 1$.

Now note that such terms can be used to describe the product of flipped hooks within a single strip. We will use the notation:

$$[b; n] = \langle 1 - r; b + 1 \rangle_n = \phi(1 - r; \{b + 1, b + 2, \dots, b + n\}).$$

Let h_λ^{ij} denote $h_\lambda^*(i, 1) - h_\lambda^*(j, 1)$. Then given partitions λ, μ, ν and a set of division numbers \mathbf{n} for each strip in these partitions, we define $\mathbf{d}_{\mu, \nu}^\lambda(\mathbf{n})$ to be the product:

$$\mathbf{d}_{\mu, \nu}^\lambda(\mathbf{n}) = \prod_{i \leq j} [h_\lambda^{ij}; n_{ij}^\lambda] \cdot [-h_\mu^{ij}; n_{ij}^\mu] \cdot [-h_\nu^{ij}; n_{ij}^\nu],$$

where n_{ij}^λ is the division number corresponding to the i^{th} strip in ω_j^λ , and n_{ij}^ξ is the division number corresponding to the i^{th} strip in ω_{j-1}^ξ for $\xi \in \{\mu, \nu\}$. We will refer to the starting point $b = \pm h_\xi^{ij}$ in each term of the form $[b; n]$ as the *anchor* for the corresponding strip.

Using equation 4.2.1, we can determine how changes to the anchors or division numbers affect $\mathbf{d}_{\mu, \nu}^\lambda(\mathbf{n})$. In particular, we have that

$$\frac{[h_\xi^{ij}; n_{ij}^\xi - t]}{[h_\xi^{ij}; n_{ij}^\xi]} = \frac{1}{[h_\xi^{ij} + n_{ij}^\xi - t; t]}, \quad (4.2.3)$$

$$\frac{[h_\xi^{ij} + t; n_{ij}^\xi]}{[h_\xi^{ij}; n_{ij}^\xi]} = \frac{[h_\xi^{ij} + n_{ij}^\xi; t]}{[h_\xi^{ij}; t]}. \quad (4.2.4)$$

4.3 Division Numbers for Extremal Triples in \mathcal{P}_3

Let \mathfrak{d}_{ijk} encode the quantity $|\lambda_i + \mu_j - \nu_k|$, and let ξ_{ij} denote $\xi_i - \xi_j$ for any partition ξ . Let \mathfrak{p} be the positive part of $\lambda_3 - \mu_2 - \nu_3$, so that $\mathfrak{p} = o_2 = \max(\lambda_3 - \mu_2 - \nu_3, 0)$. Finally, let $x^\pm = x \pm \mathfrak{p}$, where x is either some \mathfrak{d}_{ijk} or some ξ_{ij} .

We present a complete list of division number formulas below for extremal triples in \mathcal{P}_3 . These formulas are grouped according to facets of the Horn cone defined by the inequalities in Table 3.1.1. For each case, we present the division numbers for λ, μ, ν , and list the proposition in which this formula is verified. These propositions all appear in Section 4.6.

Table 4.3.1: Division numbers for extremal triples in \mathcal{P}_3

	Case:	λ :	μ :	ν :	Prop.
1.	$\mu_3 = \mu_2$	$\begin{bmatrix} \mathfrak{d}_{333} & \mathfrak{d}_{222} & 0 \\ \mathfrak{d}_{333} & \mathfrak{d}_{222} \\ \mathfrak{d}_{333} \end{bmatrix}$	$\begin{bmatrix} 0 & \mathfrak{d}_{111} \\ 0 \end{bmatrix}$	$\begin{bmatrix} \mathfrak{d}_{333} & \mathfrak{d}_{222} \\ \mathfrak{d}_{333} \end{bmatrix}$	4.6.1
2.	$\mu_2 = \mu_1$	$\begin{bmatrix} \mathfrak{d}_{111} & \mathfrak{d}_{213} & 0 \\ \mathfrak{d}_{222} & \mathfrak{d}_{213} \\ \mathfrak{d}_{111} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ \mathfrak{d}_{333} \end{bmatrix}$	$\begin{bmatrix} \mathfrak{d}_{213} & \mathfrak{d}_{111} \\ \mathfrak{d}_{213} \end{bmatrix}$	4.6.2
3.	$\nu_3 = \nu_2$	$\begin{bmatrix} \mathfrak{d}_{333} & \mathfrak{d}_{222} & 0 \\ \mathfrak{d}_{333} & \mathfrak{d}_{222} \\ \mathfrak{d}_{333} \end{bmatrix}$	$\begin{bmatrix} \mathfrak{d}_{333} & \mathfrak{d}_{222} \\ \mathfrak{d}_{333} \end{bmatrix}$	$\begin{bmatrix} 0 & \mathfrak{d}_{111} \\ 0 \end{bmatrix}$	4.6.1
4.	$\nu_2 = \nu_1$	$\begin{bmatrix} \mathfrak{d}_{111} & \mathfrak{d}_{231} & 0 \\ \mathfrak{d}_{222} & \mathfrak{d}_{231} \\ \mathfrak{d}_{111} \end{bmatrix}$	$\begin{bmatrix} \mathfrak{d}_{231} & \mathfrak{d}_{111} \\ \mathfrak{d}_{231} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ \mathfrak{d}_{333} \end{bmatrix}$	4.6.2
5.	$\lambda_3 = \lambda_2$	$\begin{bmatrix} \mathfrak{d}_{111} & 0 & 0 \\ \mathfrak{d}_{333} & 0 \\ \mathfrak{d}_{223} \end{bmatrix}$	$\begin{bmatrix} \mathfrak{d}_{232} & \mathfrak{d}_{223} \\ \mathfrak{d}_{222} \end{bmatrix}$	$\begin{bmatrix} \mathfrak{d}_{223} & \mathfrak{d}_{232} \\ \mathfrak{d}_{223} \end{bmatrix}$	4.6.6
6.	$\lambda_2 = \lambda_1$	$\begin{bmatrix} \mathfrak{d}_{333} & \mathfrak{d}_{322}^- & 0 \\ \mathfrak{p} & \mathfrak{d}_{231} \\ \mathfrak{d}_{221} \end{bmatrix}$	$\begin{bmatrix} \mathfrak{d}_{212}^- & \mu_{12} \\ \mathfrak{d}_{221} \end{bmatrix}$	$\begin{bmatrix} \mathfrak{d}_{231}^+ & \mathfrak{d}_{221} \\ \mathfrak{d}_{231} \end{bmatrix}$	4.6.7
7.	$\lambda_1 = \mu_1 + \nu_1$	$\begin{bmatrix} 0 & 0 & 0 \\ \mathfrak{d}_{333} & 0 \\ \mathfrak{d}_{333} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ \mathfrak{d}_{333} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ \mathfrak{d}_{333} \end{bmatrix}$	4.6.4
8.	$\lambda_2 = \mu_1 + \nu_2$	$\begin{bmatrix} \mathfrak{d}_{333} & 0 & 0 \\ 0 & 0 \\ \mathfrak{d}_{333} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ \mathfrak{d}_{333} \end{bmatrix}$	$\begin{bmatrix} \mathfrak{d}_{333} & \mathfrak{d}_{111} \\ 0 \end{bmatrix}$	4.6.4
9.	$\lambda_2 = \mu_2 + \nu_1$	$\begin{bmatrix} \mathfrak{d}_{333} & 0 & 0 \\ 0 & 0 \\ \mathfrak{d}_{333} \end{bmatrix}$	$\begin{bmatrix} \mathfrak{d}_{333} & \mathfrak{d}_{111} \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ \mathfrak{d}_{333} \end{bmatrix}$	4.6.4

10.	$\lambda_3 = \mu_1 + \nu_3$	$\begin{bmatrix} \mathfrak{d}_{111}^+ & \mathfrak{d}_{223} & 0 \\ \mathfrak{d}_{221}^+ & \mathfrak{d}_{223} & \\ \mathfrak{p} & & \end{bmatrix}$	$\begin{bmatrix} \mathfrak{p} & 0 \\ \mathfrak{d}_{223}^+ & \end{bmatrix}$	$\begin{bmatrix} \mathfrak{d}_{111} & \mathfrak{d}_{223}^+ \\ \mathfrak{d}_{221} & \end{bmatrix}$	4.6.4
11.	$\lambda_3 = \mu_2 + \nu_2$	$\begin{bmatrix} \mathfrak{d}_{333} & \mathfrak{d}_{222} & 0 \\ \mathfrak{d}_{333} & \mathfrak{d}_{222} & \\ 0 & & \end{bmatrix}$	$\begin{bmatrix} \mathfrak{d}_{332} & \mathfrak{d}_{223} \\ 0 & \end{bmatrix}$	$\begin{bmatrix} \mathfrak{d}_{323} & \mathfrak{d}_{232} \\ 0 & \end{bmatrix}$	4.6.3
12.	$\lambda_3 = \mu_3 + \nu_1$	$\begin{bmatrix} \mathfrak{d}_{111}^+ & \mathfrak{d}_{223} & 0 \\ \mathfrak{d}_{221}^+ & \mathfrak{d}_{223} & \\ \mathfrak{p} & & \end{bmatrix}$	$\begin{bmatrix} \mathfrak{d}_{111} & \mathfrak{d}_{223}^+ \\ \mathfrak{d}_{221} & \end{bmatrix}$	$\begin{bmatrix} \mathfrak{p} & 0 \\ \mathfrak{d}_{223}^+ & \end{bmatrix}$	4.6.4
13.	$\lambda_3 = \mu_3 + \nu_3$	$\begin{bmatrix} 0 & \mathfrak{d}_{111} & 0 \\ 0 & \mathfrak{d}_{111} & \\ 0 & & \end{bmatrix}$	$\begin{bmatrix} 0 & \mathfrak{d}_{111} \\ 0 & \end{bmatrix}$	$\begin{bmatrix} 0 & \mathfrak{d}_{111} \\ 0 & \end{bmatrix}$	4.6.5
14.	$\lambda_2 = \mu_3 + \nu_2$	$\begin{bmatrix} \mathfrak{d}_{323} & 0 & 0 \\ \mathfrak{d}_{333} & 0 & \\ \mathfrak{d}_{323} & & \end{bmatrix}$	$\begin{bmatrix} \mathfrak{d}_{323} & 0 \\ \mathfrak{d}_{333} & \end{bmatrix}$	$\begin{bmatrix} 0 & \mathfrak{d}_{111} \\ 0 & \end{bmatrix}$	4.6.5
15.	$\lambda_2 = \mu_2 + \nu_3$	$\begin{bmatrix} \mathfrak{d}_{332} & 0 & 0 \\ \mathfrak{d}_{333} & 0 & \\ \mathfrak{d}_{332} & & \end{bmatrix}$	$\begin{bmatrix} 0 & \mathfrak{d}_{111} \\ 0 & \end{bmatrix}$	$\begin{bmatrix} \mathfrak{d}_{332} & 0 \\ \mathfrak{d}_{333} & \end{bmatrix}$	4.6.5
16.	$\lambda_1 = \mu_3 + \nu_1$	$\begin{bmatrix} \mathfrak{d}_{333} & \mathfrak{d}_{232} & 0 \\ \mathfrak{d}_{323} & \mathfrak{d}_{222} & \\ \mathfrak{d}_{323} & & \end{bmatrix}$	$\begin{bmatrix} 0 & \mu_{12} \\ 0 & \end{bmatrix}$	$\begin{bmatrix} \mathfrak{d}_{333} & \mathfrak{d}_{232} \\ \mathfrak{d}_{323} & \end{bmatrix}$	4.6.5
17.	$\lambda_1 = \mu_2 + \nu_2$	$\begin{bmatrix} \mathfrak{d}_{111} & \mathfrak{d}_{223}^- & 0 \\ \mathfrak{d}_{222}^+ & \mathfrak{d}_{113} & \\ \mathfrak{d}_{121}^+ & & \end{bmatrix}$	$\begin{bmatrix} \mathfrak{d}_{112}^- & 0 \\ \mathfrak{d}_{221} & \end{bmatrix}$	$\begin{bmatrix} \mathfrak{d}_{213}^+ & \mathfrak{d}_{121} \\ \lambda_{23}^+ & \end{bmatrix}$	4.6.8
18.	$\lambda_1 = \mu_1 + \nu_3$	$\begin{bmatrix} \mathfrak{d}_{333} & \mathfrak{d}_{223} & 0 \\ \mathfrak{d}_{332} & \mathfrak{d}_{222} & \\ \mathfrak{d}_{332} & & \end{bmatrix}$	$\begin{bmatrix} \mathfrak{d}_{333} & \mathfrak{d}_{223} \\ \mathfrak{d}_{332} & \end{bmatrix}$	$\begin{bmatrix} 0 & \nu_{12} \\ 0 & \end{bmatrix}$	4.6.5

Note that all division numbers that appear in Table 4.3.1 are positive and do not exceed the size of the strip in which they appear.

4.4 Extremal Paths

In order to verify the proposed formulas for the coefficient $c_{\mu,\nu}^\lambda$, we will typically use induction on $|\lambda| - |\mu|$. In order to do this, we decompose ν into two pieces ν' and ν'' , and compute the coefficients obtained when we expand the product $P_\mu P_{\nu'} P_{\nu''}$ as a sum. Using associativity, we can expand this product in 2 different ways.

Lemma 4.4.1. *For fixed $\lambda, \mu, \zeta, \epsilon$,*

$$\sum_{\kappa \subset \lambda} c_{\mu,\zeta}^\kappa \cdot c_{\kappa,\epsilon}^\lambda = \sum_{\eta \subset \lambda} c_{\zeta,\epsilon}^\eta \cdot c_{\mu,\eta}^\lambda.$$

Proof. We use the associativity of product $P_\mu P_\zeta P_\epsilon$ to expand the coefficient of P_λ in this product as a sum in 2 different ways:

$$\begin{aligned} (P_\mu P_\zeta) P_\epsilon &= \left(\sum_{\kappa} c_{\mu,\zeta}^\kappa P_\kappa \right) P_\epsilon \\ &= \sum_{\xi} \sum_{\kappa} c_{\mu,\zeta}^\kappa \cdot c_{\kappa,\epsilon}^\xi P_\xi \\ P_\mu (P_\zeta P_\epsilon) &= P_\mu \left(\sum_{\eta} c_{\zeta,\epsilon}^\eta P_\eta \right) \\ &= \sum_{\xi} \sum_{\eta} c_{\zeta,\epsilon}^\eta \cdot c_{\mu,\eta}^\xi P_\xi. \end{aligned}$$

Picking out the coefficient of P_λ in this expression tells us:

$$\sum_{\kappa \subset \lambda} c_{\mu,\zeta}^\kappa \cdot c_{\kappa,\epsilon}^\lambda = \sum_{\eta \subset \lambda} c_{\zeta,\epsilon}^\eta \cdot c_{\mu,\eta}^\lambda.$$

□

It turns out that for extremal triples (λ, μ, ν) of partitions in \mathcal{P}_3 , we can always decompose ν (or, equivalently, μ) into subpartitions ν' and ν'' such that all the coefficients that appear in the expression

$$\sum_{\kappa \subset \lambda} c_{\mu,\nu'}^\kappa \cdot c_{\kappa,\nu''}^\lambda = \sum_{\eta \subset \lambda} c_{\mu,\eta}^\lambda \cdot c_{\nu',\nu''}^\eta$$

are indexed by extremal triples. We can solve this equation for $c_{\mu,\nu}^\lambda$, and we call the resulting expression an *extremal path*.

In particular, we can pick ν'' to consist of a single row or column. In this case, coefficients involving ν'' can be obtained using the Pieri rule. Since ν' is strictly smaller than ν , we can apply our inductive hypothesis or results of a previous case to compute $c_{\mu,\nu}^\kappa$ for $\kappa \subset \lambda$. On the other hand, since $|\eta| = |\nu|$, we use the following lemma to simplify coefficients of the form $c_{\mu,\eta}^\lambda$ for $\eta \subset \lambda, \eta \neq \nu$.

Lemma 4.4.2. $c_{\mu,\nu}^\lambda = c_{\mu-\omega_3^\mu, \nu-\omega_3^\nu}^{\lambda-\omega_3^\mu-\omega_3^\nu}$.

Proof. We first use Lemma 4.4.1 with $\epsilon = (1^3)$ and $\zeta = \nu - \epsilon$, to get $c_{\mu,\nu}^\lambda \cdot c_{\zeta,\epsilon}^\nu = c_{\mu,\zeta}^{\lambda-\epsilon} \cdot c_{\lambda-\epsilon,\epsilon}^\lambda$. By the Pieri rule, $c_{\zeta,\epsilon}^\nu = c_{\lambda-\epsilon,\epsilon}^\lambda = 1$, and so we get $c_{\mu,\nu}^\lambda = c_{\mu,\zeta}^{\lambda-\epsilon}$. We can then iterate this to get $c_{\mu,\nu}^\lambda = c_{\mu,\nu-\omega_3^\nu}^{\lambda-\omega_3^\nu}$. Finally, we use the symmetry between μ and ν to obtain our identity. □

Since, in general, $\eta_3 > \nu_3$ and thus $|\omega_3^\eta| > |\omega_3^\nu|$, this lemma allows us to reduce $c_{\mu,\eta}^\lambda$ such that it can also be computed by our inductive hypothesis or results of a previous case.

4.5 Main Lemmas

We will reduce our extremal path expressions to one of the following 2 identities, depending on whether ν'' is taken to be a row or a column in our decomposition of ν .

We will use the notation $[n]$ to denote $\{1, \dots, n\}$.

Lemma 4.5.1. *Let n be fixed, and let $I = [n]$. Given sets $\sigma = \{\sigma_i\}, \tau = \{\tau_i\}$ indexed by $i \in I$, we can define*

$$\begin{aligned} \beta_j(\sigma, \tau) &= \{\sigma_i - \sigma_j\} \cup \{\tau_i + \sigma_j\}, \\ \phi_j(x; \sigma, \tau) &= \phi(x; \beta_j(\sigma, \tau)), \\ \Phi(x; \sigma, \tau) &= \sum_{j \in I} \phi_j(x; \sigma, \tau) \end{aligned}$$

Then for all x, σ, τ ,

$$\Phi(x; \sigma, \tau) \equiv \Phi(x; \tau, \sigma).$$

Proof. We use induction on n . If $n = 1$, then

$$\beta_1(\sigma, \tau) = \beta_1(\tau, \sigma) = \{0, \tau_1 + \sigma_1\},$$

and so

$$\Phi(x; \sigma, \tau) = \Phi(x; \tau, \sigma) = \frac{\tau_1 + \sigma_1 - x}{\tau_1 + \sigma_1}.$$

For greater n , we note that each $\phi_j(x; \sigma, \tau)$ and $\phi_j(x; \tau, \sigma)$ is a polynomial of degree $2n - 1$ in x . We will show that the expression $\Phi(x; \sigma, \tau) - \Phi(x; \tau, \sigma)$ vanishes at all points of the form $x_{kl} = (\sigma_k + \tau_l)$, $k, l \in I$, and therefore must be identically 0. If we fix some k and l in I , we see that

$$\phi_k(x_{kl}; \sigma, \tau) = \phi_l(x_{kl}; \tau, \sigma) = 0.$$

If $j \neq k$, then

$$\phi_j(x_{kl}; \sigma, \tau) = \prod_{i \neq j} \frac{\sigma_i - \sigma_j - \sigma_k - \tau_l}{\sigma_i - \sigma_j} \prod_i \frac{\tau_i + \sigma_j - \sigma_k - \tau_l}{\tau_i + \sigma_j}.$$

We factor out the $i = k$ term from the first product and the $i = l$ term from the second product to get

$$\phi_j(x_{kl}; \sigma, \tau) = \frac{(-\sigma_j - \tau_l)(\sigma_j - \sigma_k)}{(\sigma_k - \sigma_j)(\tau_l + \sigma_j)} \left(\prod_{i \neq j, k} \frac{\sigma_i - \sigma_j - \sigma_k - \tau_l}{\sigma_i - \sigma_j} \prod_{i \neq l} \frac{\tau_i + \sigma_j - \sigma_k - \tau_l}{\tau_i + \sigma_j} \right).$$

Since

$$\frac{(-\sigma_j - \tau_l)(\sigma_j - \sigma_k)}{(\sigma_k - \sigma_j)(\tau_l + \sigma_j)} = 1,$$

we get that

$$\begin{aligned} \phi_j(x_{kl}; \sigma, \tau) &= \prod_{i \neq l} \frac{\tau_i + \sigma_j + \sigma_k - \tau_l}{\tau_i + \sigma_j} \prod_{i \neq j, k} \frac{\sigma_i - \sigma_j - \sigma_k - \tau_l}{\sigma_i - \sigma_j} \\ &= \phi_j(x_{kl}; \sigma_{i \neq k}, \tau_{i \neq l}). \end{aligned}$$

By a similar calculation, we have

$$\phi_j(x_{kl}; \tau, \sigma) = \phi_j(x_{kl}; \tau_{i \neq l}, \sigma_{i \neq k}).$$

Therefore

$$\Phi(x_{kl}; \sigma, \tau) - \Phi(x_{kl}; \tau, \sigma) = \Phi(x_{kl}; \sigma_{i \neq k}, \tau_{i \neq l}) - \Phi(x_{kl}; \tau_{i \neq l}, \sigma_{i \neq k}),$$

which is identically 0, by the inductive hypothesis. □

Lemma 4.5.2. Fix n and let $\sigma = (\sigma_1, \sigma_2), \tau = (\tau_1, \tau_2)$. Let σ_i^k denote $\sigma_i + k$. Let

$$\begin{aligned}\beta_t^n(j; \sigma, \tau) &= \bigcup_{\substack{i \in [2] \\ k \in [n-t]}} \{-k\} \cup \left\{ \sigma_i^{k-1} - \sigma_j^t : i \neq j \right\} \cup \left\{ \tau_i^{k-1} + \sigma_j^t \right\}, \\ \phi_t^n(x; \sigma, \tau) &= \phi(x; \beta_t^n(1; \sigma, \tau)) \cdot \phi(x; \beta_{n-t}^n(2; \sigma, \tau)), \\ \Phi_n(x; \sigma, \tau) &= \sum_{t=0}^n \phi_t^n(x; \sigma, \tau)\end{aligned}$$

Then for all x, σ, τ ,

$$\Phi_n(x; \sigma, \tau) \equiv \Phi_n(x; \tau, \sigma).$$

Proof. We prove this identity by induction on n . When $n = 1$, the result follows from Lemma 4.5.1.

For general n , we note that each $\phi_t^n(x; \sigma, \tau)$ and $\phi_t^n(x; \tau, \sigma)$ is a polynomial of degree $4n$ in x , so we must show that $\Phi_n(x; \sigma, \tau) - \Phi_n(x; \tau, \sigma)$ vanishes at $4n + 1$ points. Note that all terms vanish at $x = 1$ since 1 is contained in at least one of the sets $[n - t]$ or $[t]$. We will show that $\Phi_n(x; \sigma, \tau) - \Phi_n(x; \tau, \sigma)$ also vanishes at the $4n$ points given by $x = (\sigma_l + \tau_m + k - 1)$, $l, m \in [2], k \in [n]$. Since a transposition of σ_1 and σ_2 takes $\phi_t^n(x; \sigma, \tau)$ to $\phi_{n-t}^n(x; \sigma, \tau)$ and keeps $\phi_t^n(x; \tau, \sigma)$ fixed, we can assume without loss of generality that $l = m = 1$, and so we let $x_k = (\sigma_1 + \tau_1 + k - 1)$.

We claim that

$$\phi_t^n(x; \sigma, \tau) = \begin{cases} 0 & \text{if } t < k \\ \phi_{t-k}^{n-k}(x_k; \sigma + k\mathbf{e}_1, \tau + k\mathbf{e}_1) \cdot c_k^n & \text{if } t \geq k \end{cases} \quad (4.5.1)$$

where $\mathbf{e}_1 = (1, 0)$ and c_k^n is a term that does not depend on t , and is symmetric in σ and τ .

We note that if $t < k$, then for $j = k - t$,

$$\tau_1^{j-1} + \sigma_1^t - x_k = t + j - k = 0.$$

Since $k \in [n]$, j must be in $[n - t]$, and so this implies that for $t < k$

$$\phi_t^n(x; \sigma, \tau) = 0.$$

Now assume that $t \geq k$. By definition, we have that

$$\phi_t^n(x; \sigma, \tau) = \phi(x; \beta_t^n(1; \sigma, \tau)) \cdot \phi(x; \beta_{n-t}^n(2; \sigma, \tau)),$$

and so we work with each of these two factors separately.

Let $\langle a \rangle_j = \langle x_k; a \rangle_j$. We have that

$$\begin{aligned} \phi(x_k; \beta_t^n(1; \sigma, \tau)) &= \langle -n + t \rangle_{n-t} \langle \sigma_2 - \sigma_1 - t \rangle_{n-t} \\ &\quad \cdot \langle \tau_2 + \sigma_1 + t \rangle_{n-t} \langle \tau_1 + \sigma_1 + t \rangle_{n-t}, \\ \phi(x_k; \beta_{t-k}^{n-k}(1; \sigma + k\mathbf{e}_1, \tau + k\mathbf{e}_1)) &= \langle -n + t \rangle_{n-t} \langle \sigma_2 - \sigma_1 - t \rangle_{n-t} \\ &\quad \cdot \langle \tau_2 + \sigma_1 + t \rangle_{n-t} \langle \tau_1 + \sigma_1 + k + t \rangle_{n-t}. \end{aligned}$$

Note that the first three factors on the right hand side are the same in both lines.

Therefore, if we divide the first expression by the second, we can simplify the ratio using 4.2.1 to obtain:

$$\begin{aligned} \frac{\phi(x_k; \beta_t^n(1; \sigma, \tau))}{\phi(x_k; \beta_{t-k}^{n-k}(1; \sigma + k\mathbf{e}_1, \tau + k\mathbf{e}_1))} &= \frac{\langle \tau_1 + \sigma_1 + t \rangle_{n-t}}{\langle \tau_1 + \sigma_1 + k + t \rangle_{n-t}} \\ &= \frac{\langle \tau_1 + \sigma_1 + t \rangle_k}{\langle \tau_1 + \sigma_1 + n \rangle_k}. \end{aligned} \quad (4.5.2)$$

On the other hand, we have that:

$$\begin{aligned} \phi(x_k; \beta_t^n(2; \sigma, \tau)) &= \langle -t \rangle_t \cdot \langle \sigma_1 - \sigma_2 - n + t \rangle_t \\ &\quad \cdot \langle \tau_1 + \sigma_2 + n - t \rangle_t \cdot \langle \tau_2 + \sigma_2 + n - t \rangle_t, \\ \phi(x_k; \beta_{t-k}^{n-k}(2; \sigma + k\mathbf{e}_1, \tau + k\mathbf{e}_1)) &= \langle -t + k \rangle_{t-k} \cdot \langle \sigma_1 - \sigma_2 + k - n + t \rangle_{t-k} \\ &\quad \cdot \langle \tau_1 + \sigma_2 + k + n - t \rangle_{t-k} \cdot \langle \tau_2 + \sigma_2 + n - t \rangle_{t-k}. \end{aligned}$$

Therefore if we divide the first expression by the second, and once again use 4.2.1 to simplify the ratio, we get

$$\begin{aligned} \frac{\phi(x_k; \beta_t^n(2; \sigma, \tau))}{\phi(x_k; \beta_{t-k}^{n-k}(2; \sigma + k\mathbf{e}_1, \tau + k\mathbf{e}_1))} &= \langle -t \rangle_k \langle \sigma_1 - \sigma_2 - n + t \rangle_k \langle \tau_1 + \sigma_2 + n - t \rangle_k \langle \tau_2 + \sigma_2 + n - k \rangle_k. \end{aligned} \quad (4.5.3)$$

By 4.2.2, we can rewrite the first term on the right hand side of this expression as

$$\langle -t \rangle_k = \frac{1}{\langle \tau_1 + \sigma_1 + t \rangle_k}$$

since $\tau_1 + \sigma_1 + t - t = x_k + k - 1$. We can also simplify the the middle two terms using the same identity. Since

$$(\sigma_1 - \sigma_2 - n + t) + (\tau_1 + \sigma_2 + n - t) = x_k - k + 1,$$

by 4.2.2, we get that

$$\langle \sigma_1 - \sigma_2 - n + t \rangle_k \langle \tau_1 + \sigma_2 + n - t \rangle_k = 1.$$

Therefore, we can reduce 4.5.3 to

$$\frac{\phi(x_k; \beta_t^n(2; \sigma, \tau))}{\phi(x_k; \beta_{t-k}^{n-k}(2; \sigma + k\mathbf{e}_1, \tau + k\mathbf{e}_1))} = \frac{\langle \tau_2 + \sigma_2 + n - k \rangle_k}{\langle \tau_1 + \sigma_1 + t \rangle_k}. \quad (4.5.4)$$

Finally, we multiply the expressions in 4.5.2 and 4.5.4 to get that

$$\begin{aligned} \frac{\phi_t^n(x_k; \sigma, \tau)}{\phi_{t-k}^{n-k}(x_k; \sigma + k\mathbf{e}_1, \tau + k\mathbf{e}_1)} &= \frac{\langle \tau_1 + \sigma_1 + t \rangle_k}{\langle \tau_1 + \sigma_1 + n \rangle_k} \cdot \frac{\langle \tau_2 + \sigma_2 + n - k \rangle_k}{\langle \tau_1 + \sigma_1 + t \rangle_k} \\ &= \frac{\langle \tau_2 + \sigma_2 + n - k \rangle_k}{\langle \tau_1 + \sigma_1 + n \rangle_k}, \end{aligned}$$

which completes our proof of equation 4.5.1, with

$$c_k = \frac{\langle \tau_2 + \sigma_2 + n - k \rangle_k}{\langle \tau_1 + \sigma_1 + n \rangle_k}.$$

It is easy to see that this c_k does not depend on t and is symmetric in σ and τ .

Since each ϕ_t^n contains this factor of c_k whenever $t \geq k$, it follows that

$$\Phi_n(x_k; \sigma, \tau) = \Phi_{n-k}(x_k; \sigma + k\mathbf{e}_1, \tau + k\mathbf{e}_1)c_k.$$

Similarly, by transposing σ and τ , we get

$$\Phi_n(x_k; \tau, \sigma) = \Phi_{n-k}(x_k; \tau + k\mathbf{e}_1, \sigma + k\mathbf{e}_1)c_k.$$

Thus, by the inductive hypothesis:

$$\Phi_n(x_k; \sigma, \tau) - \Phi_n(x_k; \tau, \sigma) = 0.$$

□

4.6 Verification of Division Number Formulas

As in Section 4.3, we will use the following notation:

$$\begin{aligned} \mathfrak{d}_{ijk} &= |\lambda_i + \mu_j - \nu_k|, \\ \xi_{ij} &= \xi_i - \xi_j, \\ \mathfrak{p} &= \max(\lambda_3 - \mu_2 - \nu_3, 0), \\ x^\pm &= x \pm \mathfrak{p}. \end{aligned}$$

Given a set of division numbers \mathbf{n} , recall that

$$\mathbf{d}_{\mu,\nu}^\lambda(\mathbf{n}) = \prod_{i \leq j} [h_\lambda^{ij}; n_{ij}^\lambda] \cdot [-h_\mu^{ij}; n_{ij}^\mu] \cdot [-h_\nu^{ij}; n_{ij}^\nu], \quad (4.6.1)$$

where n_{ij}^λ is the division number corresponding to the i^{th} strip in ω_j^λ , and n_{ij}^ξ is the division number corresponding to the i^{th} strip in ω_{j-1}^ξ for $\xi \in \{\mu, \nu\}$.

For each extremal triple (λ, μ, ν) we have a set of division numbers $\mathbf{n}(\lambda, \mu, \nu)$, as listed in Table 4.3.1. Let

$$d_{\mu,\nu}^\lambda = \mathbf{d}_{\mu,\nu}^\lambda(\mathbf{n}(\lambda, \mu, \nu)).$$

We verify that for each case in Table 4.3.1, $c_{\mu,\nu}^\lambda = d_{\mu,\nu}^\lambda$.

In order to do this, for each fixed triple (λ, μ, ν) , we first decompose ν into two subpartitions ζ and ϵ , such that ϵ consists of a single row or column. By Lemma 4.4.1, we then get an extremal path of the form:

$$\sum_i c_{\mu,\zeta}^{\kappa(i)} \cdot c_{\kappa(i),\epsilon}^\lambda = \sum_i c_{\zeta,\epsilon}^{\eta(i)} \cdot c_{\mu,\eta(i)}^\lambda, \quad (4.6.2)$$

where for some i , $\eta(i) = \nu$. For such a path, we can use either a previously established result or induction to get that each $c_{\xi_2,\xi_3}^{\xi_1}$ equals $d_{\xi_2,\xi_3}^{\xi_1}$ for all the triples $(\xi_1, \xi_2, \xi_3) \neq (\lambda, \mu, \nu)$. Therefore, to show that $c_{\mu,\nu}^\lambda = d_{\mu,\nu}^\lambda$, it suffices to instead verify the analogous identity for $d_{\mu,\nu}^\lambda$:

$$\sum_i d_{\mu,\zeta}^{\kappa(i)} \cdot d_{\kappa(i),\epsilon}^\lambda = \sum_i d_{\zeta,\epsilon}^{\eta(i)} \cdot d_{\mu,\eta(i)}^\lambda. \quad (4.6.3)$$

When the sums on either side of equation 4.6.3 consist of more than one term, we can prove this identity by writing each $d_{\mu,\zeta}^{\kappa(i)} \cdot d_{\kappa(i),\epsilon}^\lambda$ and $d_{\zeta,\epsilon}^{\eta(i)} \cdot d_{\mu,\eta(i)}^\lambda$ as the product

of $d_{\mu,\nu}^\lambda \cdot d_{\zeta,\epsilon}^\nu$ with additional terms produced by changes to the anchors and division numbers, as given by equation 4.2.1. We will then show that the additional factors produced satisfy equation 4.6.3 by showing that they fall into the form of Lemma 4.5.1 (if ϵ is a single column) or 4.5.2 (if ϵ is a single row).

In the proofs below, we will also use the fact that since $c_{\mu,\nu}^\lambda = c_{\nu,\mu}^\lambda$, any division number formula that we prove for a coefficient based on a condition involving μ and ν can be subsequently be used for the condition obtained by interchanging μ and ν , as long as one also interchanges the role of μ and ν in the formula.

Proposition 4.6.1. *If $\mu_3 = \mu_2$, then $c_{\mu,\nu}^\lambda$ is given by*

$$\lambda : \begin{bmatrix} \mathfrak{d}_{333} & \mathfrak{d}_{222} & 0 \\ \mathfrak{d}_{333} & \mathfrak{d}_{222} & \\ \mathfrak{d}_{333} & & \end{bmatrix} \quad \mu : \begin{bmatrix} 0 & \mathfrak{d}_{111} \\ 0 & \end{bmatrix} \quad \nu : \begin{bmatrix} \mathfrak{d}_{333} & \mathfrak{d}_{222} \\ \mathfrak{d}_{333} & \end{bmatrix}$$

Proof. By Lemma 4.4.2, $c_{\mu,\nu}^\lambda = c_{\mu-\omega_3^\mu,\nu}^{\lambda-\omega_3^\mu}$. Since, $\mu - \omega_3^\mu$ consists of a single part, we can use the Pieri rule to compute $c_{\mu-\omega_3^\mu,\nu}^{\lambda-\omega_3^\mu}$. The Pieri rule for rows can be obtained from Theorem 2.3.1 and equation 2.3.1. In particular, we get that $c_{\mu-\omega_3^\mu,\nu}^{\lambda-\omega_3^\mu}$ is obtained by treating all the hooks in λ, μ, ν as flipped hooks, except those corresponding to boxes $(i, j) \in \nu, \lambda$ such that $\nu'_i = \lambda'_i$ and $\nu_j < \lambda_j$. Finally, we note that the \mathfrak{d}_{131} flipped hooks in ω_1^λ can be exchanged with the last \mathfrak{d}_{131} flipped hooks in ω_1^μ , leaving no flipped hooks in ω_1^λ and only $\mu_{13} - \mathfrak{d}_{131} = \mathfrak{d}_{111}$ flipped hooks in ω_1^μ . \square

Proposition 4.6.2. *If $\mu_2 = \mu_1$, then $c_{\mu,\nu}^\lambda$ is given by:*

$$\lambda : \begin{bmatrix} \mathfrak{d}_{111} & \mathfrak{d}_{213} & 0 \\ \mathfrak{d}_{222} & \mathfrak{d}_{213} & \\ \mathfrak{d}_{111} & & \end{bmatrix} \quad \mu : \begin{bmatrix} 0 & 0 \\ \mathfrak{d}_{333} & \end{bmatrix} \quad \nu : \begin{bmatrix} \mathfrak{d}_{213} & \mathfrak{d}_{111} \\ \mathfrak{d}_{213} & \end{bmatrix}$$

Proof. Let $d_{\mu,\nu}^\lambda$ by the hypothesized formula. We use induction on $(\nu_2 - \nu_3)$ to show that $c_{\mu,\nu}^\lambda = d_{\mu,\nu}^\lambda$. If $\nu_2 = \nu_3$, we can determine $c_{\mu,\nu}^\lambda$ using Prop. 4.6.1 with the roles of μ and ν reversed in the following way. First, note that by Horn inequality (8),

$\lambda_2 \leq \mu_1 + \nu_2 = \mu_2 + \nu_2$, and by inequality (15), $\lambda_2 \geq \mu_2 + \nu_3 = \mu_2 + \nu_2$, so $\lambda_2 = \mu_2 + \nu_2$. Therefore, $\mathfrak{d}_{222} = 0$, and so $\mathfrak{d}_{213} = \mathfrak{d}_{222} = 0$. Since $|\lambda| = |\mu| + |\nu|$, it also follows that $\mathfrak{d}_{111} = \mathfrak{d}_{333}$. Finally, note that since $\lambda_2 - \mu_3 - \nu_3 = \mu_1 - \mu_3$ in this case, we can exchange the lower hooks in the second strip of ω_3^λ with the upper hooks in the first strip of ω_2^μ , so that Prop. 4.6.1 gives us the following division numbers:

$$\lambda : \begin{bmatrix} \mathfrak{d}_{111} & 0 & 0 \\ 0 & 0 & \\ \mathfrak{d}_{111} & & \end{bmatrix} \quad \mu : \begin{bmatrix} 0 & 0 \\ \mathfrak{d}_{333} & \end{bmatrix} \quad \nu : \begin{bmatrix} 0 & \mathfrak{d}_{111} \\ 0 & \end{bmatrix}$$

We can verify that this is the same as $d_{\mu,\nu}^\lambda$ in this case.

If $\nu_2 > \nu_3$ then we can decompose ν into $\epsilon = (1, 1)$ and $\zeta = \nu - \epsilon$. Then by Lemma 4.4.1, we have

$$\sum_{1 \leq i \leq 3} c_{\mu,\eta(i)}^\lambda c_{\zeta,\epsilon}^{\eta(i)} = \sum_{1 \leq i \leq 3} c_{\mu,\zeta}^{\kappa(i)} c_{\kappa(i),\epsilon}^\lambda, \quad (4.6.4)$$

where $\eta(i) = \zeta + (1, 1, 1) - \mathbf{e}_i$ and $\kappa(i) = \lambda - (1, 1, 1) + \mathbf{e}_i$, where \mathbf{e}_i is a triple consisting of a 1 in the i^{th} position and 0's elsewhere. Note that

$$\eta(3) = \nu.$$

We will show that our hypothesized coefficients satisfy equation 4.6.4. We can determine $d_{\kappa(i),\epsilon}^\lambda$ and $d_{\zeta,\epsilon}^{\eta(i)}$ by the Pieri rule, since ϵ consists of a single column. In particular, we have that

$$\begin{aligned} d_{\kappa(i),\epsilon}^\lambda &= \phi \left(1 - r; \left\{ h_\lambda^{ij} + 1, -h_{\kappa(i)}^{ij} + 1 : j > i \right\} \right), \\ d_{\zeta,\epsilon}^{\eta(i)} &= \phi \left(1 - r; \left\{ h_{\eta(i)}^{ij} + 1, -h_\zeta^{ij} + 1 : j > i \right\} \right). \end{aligned}$$

Note that by this definition, $d_{\zeta,\epsilon}^\nu = d_{\zeta,\epsilon}^{\eta(3)} = 1$.

We can also write out $d_{\mu,\zeta}^{\kappa(i)}$ and $d_{\mu,\eta(i)}^\lambda$, by comparing them to $d_{\mu,\nu}^\lambda$, since $\kappa(i)$ is obtained by modifying the parts of λ and $\eta(i)$ and ζ are obtained by modifying the parts of ν . Therefore, $d_{\mu,\zeta}^{\kappa(i)}$ and $d_{\mu,\eta(i)}^\lambda$ can be determined by examining how these changes to λ and ν change the anchors and division numbers for each strip.

In the symbols below, the entries denote how the corresponding anchor for each strip must be changed for that coefficient compared to the anchor of $d_{\mu,\nu}^\lambda$, and a * indicates a change of -1 to the corresponding division number.

$$\begin{aligned}
\kappa(1) : \begin{pmatrix} +1^* & +1^* & 0 \\ 0 & 0^* & \\ 0^* & & \end{pmatrix} & \quad \mu : \begin{pmatrix} 0 & 0 \\ 0^* & \end{pmatrix} & \quad \zeta : \begin{pmatrix} +1^* & 0^* \\ +1^* & \end{pmatrix}, \\
\kappa(2) : \begin{pmatrix} 0 & -1 & 0 \\ +1^* & 0 & \\ 0 & & \end{pmatrix} & \quad \mu : \begin{pmatrix} 0 & 0 \\ 0^* & \end{pmatrix} & \quad \zeta : \begin{pmatrix} +1 & 0 \\ +1 & \end{pmatrix}, \\
\kappa(3) : \begin{pmatrix} -1 & 0^* & 0 \\ -1 & 0^* & \\ 0 & & \end{pmatrix} & \quad \mu : \begin{pmatrix} 0 & 0 \\ 0 & \end{pmatrix} & \quad \zeta : \begin{pmatrix} +1^* & 0 \\ +1^* & \end{pmatrix}, \\
\lambda : \begin{pmatrix} 0^* & 0^* & 0 \\ 0 & 0^* & \\ 0^* & & \end{pmatrix} & \quad \mu : \begin{pmatrix} 0 & 0 \\ 0^* & \end{pmatrix} & \quad \eta(1) : \begin{pmatrix} +2^* & +1^* \\ +1^* & \end{pmatrix}, \\
\lambda : \begin{pmatrix} 0 & 0^* & 0 \\ 0^* & 0^* & \\ 0 & & \end{pmatrix} & \quad \mu : \begin{pmatrix} 0 & 0 \\ 0^* & \end{pmatrix} & \quad \eta(2) : \begin{pmatrix} +1^* & -1 \\ +2^* & \end{pmatrix}.
\end{aligned}$$

This allows us to determine each summand $d_{\mu,\zeta}^{\kappa(i)} d_{\kappa(i),\epsilon}^\lambda$ and $d_{\mu,\eta(i)}^\lambda d_{\zeta,\epsilon}^{\eta(i)}$ of equation 4.6.4 compared to $d_{\mu,\nu}^\lambda$, since we can use equations 4.2.3 and 4.2.4 to write terms of the form $[b+r; n+s]$ as a product of $[b; n]$ and some additional factors.

In particular, we factor out $d_{\mu,\nu}^\lambda$ from each of these terms. In addition, we factor out terms that appear in a majority of the six summands. Note that these terms come from the blocks ω_2^ξ for each partition ξ . Thus, we factor out $\frac{1}{\mathcal{X}}$ from each expression, where

$$\mathcal{X} = \phi(1-r; \lambda_{12} + r + \mathfrak{d}_{213}, \mathfrak{d}_{213}, -\mu_{23} - r + \mathfrak{d}_{333}, 1 - \nu_{13} - 2r, 1 - \nu_{23} - r).$$

Using the notation of Lemma 4.5.1, we can rewrite \mathcal{X} as $\phi_3(1-r; \sigma, \tau)$, where

$$\begin{aligned}\tau_1 &= \lambda_{12} + \mathfrak{d}_{213} + r = \lambda_1 - \mu_1 - \nu_3 + r \\ \tau_2 &= \mathfrak{d}_{213} = \lambda_2 - \mu_1 - \nu_3 \\ \tau_3 &= \mathfrak{d}_{333} - \mu_{23} - r = \lambda_3 - \mu_1 - \nu_3 - r \\ \sigma_1 &= 1 - \nu_{13} - 2r \\ \sigma_2 &= 1 - \nu_{23} - r \\ \sigma_3 &= 0.\end{aligned}$$

This allows us to write each term $d_{\mu, \zeta}^{\kappa(i)} d_{\kappa(i), \epsilon}^\lambda$ and $d_{\mu, \eta(i)}^\lambda d_{\zeta, \epsilon}^{\eta(i)}$ as a product of the form

$$\frac{d_{\mu, \nu}^\lambda}{\mathcal{X}} \phi(1-r; \{a_1, \dots, a_n\}).$$

In the table below, we present the elements of the set \mathcal{A} corresponding to each term. We will use $\kappa(i)$ to indicate terms corresponding to $d_{\mu, \zeta}^{\kappa(i)} d_{\kappa(i), \epsilon}^\lambda$ and $\eta(i)$ to indicate terms corresponding $d_{\mu, \eta(i)}^\lambda d_{\zeta, \epsilon}^{\eta(i)}$.

	a_1	a_2	a_3	a_4	a_5
$\kappa(1)$	$\lambda_{12} + \mathfrak{d}_{213} + r$	$1 - \mathfrak{d}_{111} - r$	$-\lambda_{13} - 2r$	$-\lambda_{12} - r$	$1 + \nu_{12} - \mathfrak{d}_{111}$
$\kappa(2)$	\mathfrak{d}_{213}	$-\lambda_{23} - r$	$\lambda_{12} + r$	$\mathfrak{d}_{213} + 1 - \nu_{13} - 2r$	$1 - \nu_{23} - r$
$\kappa(3)$	$\lambda_{13} + 2r$	$\lambda_{23} + r$	$\mathfrak{d}_{333} - \mu_{23} - r$	$1 - \lambda_{13} - \mathfrak{d}_{111} - 3r$	$1 - \lambda_{23} - \mathfrak{d}_{222} - 2r$
$\eta(1)$	$1 - \lambda_{13} - \mathfrak{d}_{111} - 3r$	$1 - \mathfrak{d}_{111} - r$	$\nu_{13} + 2r - 1$	$\nu_{12} + r$	$\mathfrak{d}_{213} + 1 - \nu_{13} - 2r$
$\eta(2)$	$1 - \lambda_{23} - \mathfrak{d}_{222} - 2r$	$1 - \mathfrak{d}_{111} + \nu_{12}$	$\nu_{23} + r - 1$	$-\nu_{12} - r$	$\mathfrak{d}_{213} + 1 - \nu_{23} - r$

Using the σ_i and τ_i defined above, and once again using the notation of Lemma 4.5.1, one can check that

$$\begin{aligned}d_{\mu, \zeta}^{\kappa(i)} d_{\kappa(i), \epsilon}^\lambda &= \frac{d_{\mu, \nu}^\lambda}{\mathcal{X}} \phi_i(1-r; \tau, \sigma), \\ d_{\mu, \eta(i)}^\lambda d_{\zeta, \epsilon}^{\eta(i)} &= \frac{d_{\mu, \nu}^\lambda}{\mathcal{X}} \phi_i(1-r; \sigma, \tau).\end{aligned}$$

Therefore, we have that

$$\begin{aligned}\sum_{i=1}^3 d_{\mu, \zeta}^{\kappa(i)} c_{\kappa(i), \epsilon}^\lambda &= \frac{d_{\mu, \nu}^\lambda}{\mathcal{X}} \Phi(1-r; \tau, \sigma), \\ \sum_{i=1}^3 d_{\mu, \eta(i)}^\lambda c_{\zeta, \epsilon}^{\eta(i)} &= \frac{d_{\mu, \nu}^\lambda}{\mathcal{X}} \Phi(1-r; \sigma, \tau).\end{aligned}$$

and so by Lemma 4.5.1, we have that both sums are equal, showing that $d_{\mu, \nu}^\lambda$ satisfies equation 4.6.4.

By the inductive hypothesis, each $c_{\mu,\zeta}^{\kappa(i)} = d_{\mu,\zeta}^{\kappa(i)}$, since μ is unchanged and $\zeta_2 - \zeta_3 = \nu_2 - \nu_3 - 1$. Similarly, $c_{\mu,\eta(i)}^\lambda = d_{\mu,\eta(i)}^\lambda$ for $i = 1, 2$, since $\eta(1)_2 - \eta(1)_3 = \nu_2 - \nu_3 - 1$ and $\eta(2)_2 - \eta(2)_3 = \nu_2 - \nu_3 - 2$. Therefore, we get that the remaining term $c_{\mu,\eta(i)}^\lambda = c_{\mu,\nu}^\lambda$ must equal $d_{\mu,\nu}^\lambda$.

□

These last two propositions are instrumental in proving all of the remaining cases, since they allow us to form extremal paths by decomposing ν into two pieces ζ and ϵ such that ϵ consists of a single part, and ζ is such that either $\zeta_2 = \zeta_3$ or $\zeta_1 = \zeta_2$. Then we can determine coefficients involving ϵ using the Pieri rule for rows, and coefficients involving ζ using either Proposition 4.6.1 or 4.6.2.

In particular, Prop 4.6.1 allows us to form an extremal path by decomposing ν into $\epsilon = (\nu_2 - \nu_3)$ and $\zeta = (\nu_1, \nu_3, \nu_3)$.

$$\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline & & & & & & & \\ \hline & & \bullet & \bullet & & & & \\ \hline & & & & & & & \\ \hline & & & & & & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline & & & & & & & \\ \hline & & & & & & & \\ \hline & & & & & & & \\ \hline & & & & & & & \\ \hline & & & & & & & \\ \hline \end{array} + \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \end{array}$$

Prop 4.6.2 allows us to form an extremal path by decomposing ν into $\epsilon = \omega_1^\nu$ and $\zeta = \nu - \epsilon = \omega_3^\nu + \omega_2^\nu$.

$$\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & \bullet & \bullet & \bullet \\ \hline & & & & & & & & \\ \hline & & & & & & & & \\ \hline & & & & & & & & \\ \hline & & & & & & & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|} \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \end{array}$$

Proposition 4.6.3. *If $\lambda_3 = \mu_2 + \nu_2$, then $c_{\mu,\nu}^\lambda$ is given by the following division numbers:*

$$\lambda : \begin{bmatrix} \mathfrak{d}_{333} & \mathfrak{d}_{222} & 0 \\ \mathfrak{d}_{333} & \mathfrak{d}_{222} & \\ 0 & & \end{bmatrix} \quad \mu : \begin{bmatrix} \mathfrak{d}_{332} & \mathfrak{d}_{223} \\ 0 & \end{bmatrix} \quad \nu : \begin{bmatrix} \mathfrak{d}_{323} & \mathfrak{d}_{232} \\ 0 & \end{bmatrix}.$$

Proof. We will first use Lemma 4.4.1 to show that

$$c_{\mu,\nu}^\lambda c_{\zeta,\epsilon}^\nu = c_{\mu,\zeta}^\kappa c_{\kappa,\epsilon}^\lambda,$$

where $\epsilon = (\nu_2 - \nu_3)$, $\zeta = (\nu_1, \nu_3, \nu_3)$ and $\kappa = \lambda - (0, 0, \nu_2 - \nu_3)$. For any η such that $c_{\epsilon,\zeta}^\eta \neq 0$, we have that $\eta_2 \leq \nu_2$, with equality holding only if $\eta = \nu$. By Horn inequality

(11), we know that if $\eta_2 + \mu_2 < \nu_2 + \mu_2 = \lambda_3$, then $c_{\mu,\eta}^\lambda = 0$. On the other hand, if $\kappa_3 > \mu_2 + \nu_3$ then since $\zeta_2 = \zeta_3 = \nu_3$, we have that $\kappa_3 > \mu_2 + \zeta_2$, which would imply that $c_{\mu,\zeta}^\kappa = 0$.

Using Prop 4.6.1 and the fact that $\kappa = (\lambda_1, \lambda_2, \lambda_3 - \nu_{23})$ and $\zeta = (\nu_1, \nu_3, \nu_3)$, we have that $c_{\mu,\zeta}^\kappa$ is given by the division numbers:

$$\kappa : \begin{bmatrix} \mathfrak{d}_{333} - \nu_{23} & \mathfrak{d}_{222} + \nu_{23} & 0 \\ \mathfrak{d}_{333} - \nu_{23} & \mathfrak{d}_{222} + \nu_{23} & \\ \mathfrak{d}_{333} - \nu_{23} & & \end{bmatrix} \quad \mu : \begin{bmatrix} \mathfrak{d}_{333} - \nu_{23} & \mathfrak{d}_{222} + \nu_{23} \\ \mathfrak{d}_{333} - \nu_{23} & \end{bmatrix} \quad \zeta : \begin{bmatrix} 0 & \mathfrak{d}_{111} \\ 0 & \end{bmatrix}$$

where \mathfrak{d}_{ijk} still refers to $|\lambda_i - \mu_j - \nu_k|$. Therefore,

$$\begin{aligned} c_{\mu,\zeta}^\kappa &= [\lambda_{13} + \nu_{23} + 2r; \mathfrak{d}_{333} - \nu_{23}][\lambda_{23} + \nu_{23} + r; \mathfrak{d}_{333} - \nu_{23}][0; \mathfrak{d}_{333} - \nu_{23}] \\ &\quad \cdot [\lambda_{12} + r; \mathfrak{d}_{222} - \nu_{23}][0; \mathfrak{d}_{222} - \nu_{23}][-\mu_{13} - 2r; \mathfrak{d}_{333} - \nu_{23}] \\ &\quad \cdot [-\mu_{23} - r; \mathfrak{d}_{333} - \nu_{23}][-\mu_{12} - r; \mathfrak{d}_{222} - \nu_{23}][-\nu_{13} - r; \mathfrak{d}_{111}]. \end{aligned}$$

Note that since $\lambda_3 = \mu_2 + \nu_2$ in this case, we also have that $\mathfrak{d}_{333} - \nu_{23} = \mu_{23}$. This implies that

$$[-\mu_{23} - r; \mathfrak{d}_{333} - \nu_{23}][0; \mathfrak{d}_{333} - \nu_{23}] = 1,$$

and so these terms can be removed from the product above.

Next, we compute $c_{\kappa,\epsilon}^\lambda$ and $c_{\zeta,\epsilon}^\nu$ by the Pieri rule to get that

$$\begin{aligned} c_{\kappa,\epsilon}^\lambda &= [\lambda_{13} + 2r; \nu_{23}][\lambda_{23} + r; \nu_{23}][-\lambda_{13} + \nu_{23} + r; \nu_{23}][-\lambda_{23} + \nu_{23}; \nu_{23}], \\ c_{\zeta,\epsilon}^\nu &= [\nu_{12} + r; \nu_{23}][-\nu_{13} - r; \nu_{23}]. \end{aligned}$$

Therefore, we get that

$$\begin{aligned} c_{\mu,\zeta}^\kappa c_{\kappa,\epsilon}^\lambda &= [\lambda_{13} + \nu_{23} + 2r; \mathfrak{d}_{333} - \nu_{23}][\lambda_{23} + \nu_{23} + r; \mathfrak{d}_{333} - \nu_{23}][\lambda_{12} + r; \mathfrak{d}_{222} - \nu_{23}] \\ &\quad \cdot [0; \mathfrak{d}_{222} - \nu_{23}][-\mu_{13} - 2r; \mathfrak{d}_{333} - \nu_{23}][-\mu_{12} - r; \mathfrak{d}_{222} - \nu_{23}] \\ &\quad \cdot [-\nu_{13} - r; \mathfrak{d}_{111}][\lambda_{13} + 2r; \nu_{23}][\lambda_{23} + r; \nu_{23}] \\ &\quad \cdot [-\lambda_{13} + \nu_{23} + r; \nu_{23}][-\lambda_{23} + \nu_{23}; \nu_{23}] \\ &= [\lambda_{13} + 2r; \mathfrak{d}_{333}][\lambda_{23} + r; \mathfrak{d}_{333}][\lambda_{12} + r; \mathfrak{d}_{222}][0; \mathfrak{d}_{222}][-\mu_{13} - 2r; \mathfrak{d}_{332}] \\ &\quad \cdot [-\mu_{12} - r; \mathfrak{d}_{223}][-\nu_{13} - r; \mathfrak{d}_{111}], \end{aligned}$$

and so dividing by $c_{\zeta, \epsilon}^{\nu}$ gives us that

$$c_{\mu, \nu}^{\lambda} = [\lambda_{13} + 2r; \mathfrak{d}_{333}][\lambda_{23} + r; \mathfrak{d}_{333}][\lambda_{12} + r; \mathfrak{d}_{222}][0; \mathfrak{d}_{222}][-\mu_{13} - 2r; \mathfrak{d}_{332}] \\ \cdot [-\mu_{12} - r; \mathfrak{d}_{223}][-\nu_{13} - r; \mathfrak{d}_{232}][-\nu_{12}; \nu_{23}].$$

This expression corresponds to the desired division numbers. \square

Proposition 4.6.4. *If $\lambda_i = \mu_i + \nu_1$, then $c_{\mu, \nu}^{\lambda} = c_{\mu, \zeta}^{\kappa} c_{\kappa, \epsilon}^{\lambda}$, where $\epsilon = \omega_1^{\nu}$, $\zeta = \nu - \epsilon$ and $\kappa = \lambda - |\epsilon| \mathbf{e}_i$.*

Proof. We apply Lemma 4.4.1. For any η such that $c_{\epsilon, \zeta}^{\eta} \neq 0$, we have that $\eta_1 \leq \nu_1$, with equality holding only if $\eta = \nu$. By Horn inequalities (7),(9) and (12), we know that if $\eta_1 + \mu_i < \nu_1 + \mu_i = \lambda_i$, then $c_{\mu, \eta}^{\lambda} = 0$. On the other hand, if $\kappa_i > \lambda_i - |\epsilon|$, then $\zeta_1 + \mu_i = \nu_1 - |\epsilon| + \mu_i = \lambda_i - |\epsilon| < \kappa_i$, which would imply that $c_{\mu, \zeta}^{\kappa} = 0$. \square

Proposition 4.6.5. *If $\lambda_i = \mu_i + \nu_3$, then $c_{\mu, \nu}^{\lambda} = c_{\mu, \zeta}^{\kappa} c_{\kappa, \epsilon}^{\lambda}$, where $\epsilon = \omega_1^{\nu}$, $\zeta = \nu - \epsilon$ and $\kappa = \mu + (\nu_3^3) + (\nu_2^3) - \nu_2 \mathbf{e}_i$.*

Proof. We apply Lemma 4.4.1. For any η such that $c_{\epsilon, \zeta}^{\eta} \neq 0$, we have that $\eta_3 \geq \nu_3$, with equality holding only if $\eta = \nu$. By Horn inequalities (13), (15) and (18), we know that if $\eta_3 + \mu_i > \nu_3 + \mu_i = \lambda_i$, then $c_{\mu, \eta}^{\lambda} = 0$. On the other hand, if $\kappa_i < \mu_i + \nu_3$, then $\kappa_i < \mu_i + \zeta_3$, which would imply that $c_{\mu, \zeta}^{\kappa} = 0$. \square

Note that Propositions 4.6.4 and 4.6.5 completely determine the coefficients in those cases, since $c_{\kappa, \epsilon}^{\lambda}$ can be determined by Proposition 4.6.1 and $c_{\mu, \zeta}^{\kappa}$ can be determined by Proposition 4.6.2. The details are similar to the proof of Proposition 4.6.3.

The remaining cases all make use of Lemma 4.5.2.

Proposition 4.6.6. *If $\lambda_3 = \lambda_2$, then $c_{\mu, \nu}^{\lambda}$ is given by*

$$\lambda : \begin{bmatrix} \mathfrak{d}_{111} & 0 & 0 \\ \mathfrak{d}_{333} & 0 & \\ \mathfrak{d}_{223} & & \end{bmatrix} \quad \mu : \begin{bmatrix} \mathfrak{d}_{232} & \mathfrak{d}_{223} \\ \mathfrak{d}_{222} & \end{bmatrix} \quad \nu : \begin{bmatrix} \mathfrak{d}_{223} & \mathfrak{d}_{232} \\ \mathfrak{d}_{223} & \end{bmatrix}$$

Proof. Let $d_{\mu,\nu}^\lambda$ be the expression given by the proposition. We use induction on $k = \mathfrak{d}_{222} = \nu_2 + \mu_2 - \lambda_2$ to show that $c_{\mu,\nu}^\lambda = d_{\mu,\nu}^\lambda$. If $k = 0$, then $\nu_2 + \mu_2 = \lambda_2 = \lambda_3$, and so $c_{\mu,\nu}^\lambda$ can be obtained from Prop. 4.6.3 as follows. We observe that the fact that $\lambda_2 = \lambda_3$ implies that $\mathfrak{d}_{2jk} = \mathfrak{d}_{3jk}$ for all j, k , and the fact that $\mathfrak{d}_{222} = 0$ implies that $\mathfrak{d}_{111} = \mathfrak{d}_{333}$ and $\mathfrak{d}_{223} = \nu_{23}$. Thus, Prop 4.6.3 gives us

$$\lambda : \begin{bmatrix} \mathfrak{d}_{111} & 0 & 0 \\ \mathfrak{d}_{333} & 0 & \\ 0 & & \end{bmatrix} \quad \mu : \begin{bmatrix} \mathfrak{d}_{232} & \mathfrak{d}_{223} \\ 0 & \end{bmatrix} \quad \nu : \begin{bmatrix} \mathfrak{d}_{223} & \mathfrak{d}_{232} \\ 0 & \end{bmatrix}.$$

We can verify that this is the same as the desired formula, by noting that since $\mathfrak{d}_{223} = \nu_{23}$, we can swap all the lower hooks in the third strip of ω_3^λ with all the upper hooks in the second strip of ω_2^ν .

For $k > 0$, let

$$\epsilon = (\nu_2 - \nu_3),$$

$$\zeta = (\nu_1, \nu_3, \nu_3),$$

and

$$\eta(t) = \nu - (-t, t),$$

$$\kappa(t) = (\mu_1 + k - t, \lambda_2 - \nu_3, \mu_3 + t).$$

Note that $\eta(0) = \nu$.

By Lemma 4.4.1, we have that

$$\sum_{t=0}^k c_{\mu,\epsilon}^{\kappa(t)} \cdot c_{\kappa(t),\zeta}^\lambda = \sum_{t=0}^k c_{\zeta,\epsilon}^{\eta(t)} \cdot c_{\mu,\eta(t)}^\lambda. \quad (4.6.5)$$

We show that our hypothesized coefficients satisfy equation 4.6.5.

We can determine $d_{\mu,\epsilon}^{\kappa(t)}$ and $d_{\zeta,\epsilon}^{\eta(t)}$ using the Pieri rule, to get

$$\begin{aligned} d_{\mu,\epsilon}^{\kappa(t)} &= [\mu_{12} + k - 2t + 2r; t][\mathfrak{d}_{333} - t + r; t][0; t][\mathfrak{d}_{213} + k - t + r; \nu_{23} - k][0; \nu_{23} - k] \\ &\quad \cdot [-\mu_{13} - 2r; t][-\mu_{23} - r; t][-\mu_{12} - r; \nu_{23} - k][-\nu_{23} + r; \nu_{23} - k + t], \end{aligned}$$

$$d_{\zeta,\epsilon}^{\eta(t)} = [\nu_{12} + 2t + r; \nu_{23} - t][0; \nu_{23} - t][-\nu_{13} - r; \nu_{23} - t][-\nu_{23} - r; \nu_{23} - t].$$

We can obtain $d_{\kappa(t),\zeta}^\lambda$ using Proposition 4.6.1, which gives us

$$d_{\kappa(t),\zeta}^\lambda = [\lambda_{13} + 2r; \mathfrak{d}_{333} - t][r; \mathfrak{d}_{333} - t][-\mu_{13} - k + 2t - 2r; \mathfrak{d}_{333} - t][-\nu_{13} - r; \mathfrak{d}_{333} - t].$$

Finally, we can determine each $d_{\mu,\eta(t)}^\lambda$ by modifying the $d_{\mu,\nu}^\lambda$ to get that:

$$\begin{aligned} d_{\mu,\eta(t)}^\lambda &= [\lambda_{13} + 2r; \mathfrak{d}_{111} + t][r; \mathfrak{d}_{333}][0; \mathfrak{d}_{223}][-\mu_{13} - 2r; \mathfrak{d}_{232} + t][-\mu_{23} - r; \mathfrak{d}_{222} - t] \\ &\quad \cdot [-\mu_{12} - r; \mathfrak{d}_{223}][-\nu_{13} - t - 2r; \mathfrak{d}_{223}][-\nu_{23} + t - r; \mathfrak{d}_{223}] \\ &\quad \cdot [-\nu_{12} - 2t - r; \mathfrak{d}_{232} + t]. \end{aligned}$$

Therefore, using equations 4.2.3 and 4.2.4, we get that:

$$\begin{aligned} \frac{d_{\mu,\eta(t)}^\lambda d_{\zeta,\epsilon}^{\eta(t)}}{d_{\mu,\nu}^\lambda d_{\zeta,\epsilon}^\nu} &= \frac{[\mathfrak{d}_{311} + 2r; t][-\mathfrak{d}_{212} - 2r; t][\mathfrak{d}_{221} + r; t][-\nu_{12} - 2t - r; t][-k - r; t]}{[-\mu_{23} - r + k - t; t][-\mathfrak{d}_{331} - t - r; t][\nu_{12} + r; t][\nu_{12} + t + r; t][-t - r; t]}, \\ \frac{d_{\mu,\epsilon}^{\kappa(t)} d_{\kappa(t),\zeta}^\lambda}{d_{\mu,\nu}^\lambda d_{\zeta,\epsilon}^\nu} &= \frac{[-\mu_{13} - 2r; t][\mu_{12} + k - 2t + 2r; t][-\mu_{23} - r; t][-k - r; t]}{[\mathfrak{d}_{212} + r - t; t][-\mu_{13} - k + t - 2r; t][-\mathfrak{d}_{331} - t - r; t][-t - r; t]} \\ &\quad \cdot \frac{[\mathfrak{d}_{311} + 2r; k - t][\mu_{23} - k; k]}{[-\mu_{13} - 2r; k - t][\nu_{12} + r - k; k]}. \end{aligned}$$

Thus, using the notation of Lemma 4.5.2, we have that:

$$\begin{aligned} d_{\mu,\eta(t)}^\lambda \cdot d_{\zeta,\epsilon}^{\eta(t)} &= d_{\mu,\nu}^\lambda \cdot d_{\zeta,\epsilon}^\nu \cdot \frac{\phi_t^k(x; \sigma, \tau)}{\phi_0^k(x; \sigma, \tau)}, \\ d_{\mu,\epsilon}^{\kappa(t)} \cdot d_{\kappa(t),\zeta}^\lambda &= d_{\mu,\nu}^\lambda \cdot d_{\zeta,\epsilon}^\nu \cdot \frac{\phi_t^k(x; \tau, \sigma)}{\phi_0^k(x; \sigma, \tau)}, \end{aligned}$$

where $\sigma = (0, \mathfrak{d}_{321} + r)$ and $\tau = (-\mathfrak{d}_{331} - r, -\mathfrak{d}_{311} - 3r)$.

This implies that

$$\sum_{t=0}^k d_{\zeta,\epsilon}^{\eta(t)} \cdot d_{\mu,\eta(t)}^\lambda = \frac{d_{\mu,\nu}^\lambda \cdot d_{\zeta,\epsilon}^\nu}{\phi_0^k(x; \sigma, \tau)} \cdot \Phi_k(x; \sigma, \tau), \quad (4.6.6)$$

$$\sum_{t=0}^k d_{\mu,\epsilon}^{\kappa(t)} \cdot d_{\kappa(t),\zeta}^\lambda = \frac{d_{\mu,\nu}^\lambda \cdot d_{\zeta,\epsilon}^\nu}{\phi_0^k(x; \sigma, \tau)} \cdot \Phi_k(x; \tau, \sigma). \quad (4.6.7)$$

Therefore, by Lemma 4.5.2, we have that the hypothesized coefficients satisfy equation 4.6.5.

Finally, we note that $c_{\mu,\eta(t)}^\lambda = d_{\mu,\eta(t)}^\lambda$ for $t > 0$ using the inductive hypothesis, since $\eta_2 < \nu_2$. Therefore, we also have that $c_{\mu,\nu}^\lambda = d_{\mu,\nu}^\lambda$.

□

Proposition 4.6.7. *If $\lambda_2 = \lambda_1$, then $c_{\mu,\nu}^\lambda$ is given by*

$$\lambda : \begin{bmatrix} \mathfrak{d}_{333} & \mathfrak{d}_{322}^- & 0 \\ \mathfrak{p} & \mathfrak{d}_{231} & \\ \mathfrak{d}_{221} & & \end{bmatrix} \quad \mu : \begin{bmatrix} \mathfrak{d}_{212}^- & \mu_{12} \\ \mathfrak{d}_{221} & \end{bmatrix} \quad \nu : \begin{bmatrix} \mathfrak{d}_{231}^+ & \mathfrak{d}_{221} \\ \mathfrak{d}_{231} & \end{bmatrix}$$

Proof. Let $d_{\mu,\nu}^\lambda$ be the expression given by the proposition. We use induction on $n = \mu_1 + \nu_2 - \lambda_1 - s$ to show that $c_{\mu,\nu}^\lambda = d_{\mu,\nu}^\lambda$. If $n = 0$, then $\mu_1 + \nu_2 = \lambda_1 - \mathfrak{p} = \lambda_2$. If $\mathfrak{p} = 0$, this case reduces to that of Prop 4.6.4 (with the role of μ and ν switched). If $\mathfrak{p} = \lambda_3 - \mu_2 - \nu_3$, then $p = \lambda_1 - \nu_1 - \mu_3$, and so the condition $n = 0$ implies that $\nu_1 + \mu_3 = \lambda_1$, which reduces this case to that of Prop 4.6.5 (once again with the role of μ and ν switched).

For $n > 0$, let

$$\epsilon = (\nu_2 - \nu_3),$$

$$\zeta = (\nu_1, \nu_3, \nu_3),$$

and

$$\eta(t) = \nu - (-t, t),$$

$$\kappa(t) = \lambda - (0, n - t, t).$$

Note that $\eta(0) = \nu$.

By Lemma 4.4.1, we have that

$$\sum_{t=0}^n c_{\mu,\epsilon}^{\kappa(t)} \cdot c_{\kappa(t),\zeta}^\lambda = \sum_{t=0}^n c_{\zeta,\epsilon}^{\eta(t)} \cdot c_{\mu,\eta(t)}^\lambda.$$

We show that our hypothesized coefficients satisfy this equation.

We can determine $d_{\mu,\epsilon}^{\kappa(t)}$ and $d_{\zeta,\epsilon}^{\eta(t)}$ using the Pieri rule, $d_{\kappa(t),\zeta}^\lambda$ using Proposition 4.6.1, and $d_{\mu,\eta(t)}^\lambda$ by the proposed formula for $d_{\mu,\nu}^\lambda$. Using analogous calculations to those in the proof of Proposition 4.6.6 and using the notation of Lemma 4.5.2, one can check that:

$$\begin{aligned} d_{\mu,\eta(t)}^\lambda \cdot d_{\zeta,\epsilon}^{\eta(t)} &= d_{\mu,\nu}^\lambda \cdot d_{\zeta,\epsilon}^\nu \cdot \frac{\phi_t^n(x; \sigma, \tau)}{\phi_0^n(x; \sigma, \tau)}, \\ d_{\mu,\epsilon}^{\kappa(t)} \cdot d_{\kappa(t),\zeta}^\lambda &= d_{\mu,\nu}^\lambda \cdot d_{\zeta,\epsilon}^\nu \cdot \frac{\phi_t^n(x; \tau, \sigma)}{\phi_0^n(x; \sigma, \tau)}, \end{aligned}$$

where $\sigma = (2\mathbf{p} + \mathfrak{d}_{323} + r, \mu_{13} - n + 2r)$ and $\tau = (-\mathbf{p} - \mu_{23} - r, 0)$.

The result follows from Lemma 4.5.2. \square

Proposition 4.6.8. *If $\lambda_1 = \mu_2 + \nu_2$, then $c_{\mu,\nu}^\lambda$ is given by*

$$\lambda : \begin{bmatrix} \mathfrak{d}_{111} & \mathfrak{d}_{223}^- & 0 \\ \mathfrak{d}_{222}^+ & \mathfrak{d}_{113} & \\ \mathfrak{d}_{121}^+ & & \end{bmatrix} \quad \mu : \begin{bmatrix} \mathfrak{d}_{112}^- & 0 \\ \mathfrak{d}_{221} & \end{bmatrix} \quad \nu : \begin{bmatrix} \mathfrak{d}_{213}^+ & \mathfrak{d}_{121} \\ \lambda_{23}^+ & \end{bmatrix}$$

Proof. Let $d_{\mu,\nu}^\lambda$ be the expression given by the proposition. We use induction on $n = \mu_1 + \nu_2 - \lambda_1 - \mathbf{p}$ to show that $c_{\mu,\nu}^\lambda = d_{\mu,\nu}^\lambda$. Suppose $n = 0$. Then either $\mu_1 = \mu_2$ (if $\mathbf{p} = 0$), so that this case reduces to that of Prop. 4.6.2, or $\lambda_2 = \mu_3 + \nu_1$ (if $\mathbf{p} = \lambda_3 - \mu_2 - \nu_3$), which implies that $\lambda_3 = \mu_1 + \nu_3$, and so this case reduces to that of Prop 4.6.4 with the roles of μ and ν switched.

For $n > 0$, let

$$\begin{aligned} \epsilon &= \omega_1', \\ \zeta &= \omega_3' + \omega_2', \end{aligned}$$

and

$$\begin{aligned} \eta(t) &= \nu - (-t, t), \\ \kappa(t) &= \lambda - (0, n - t, n), \end{aligned}$$

Note that $\eta(0) = \nu$.

By Lemma 4.4.1, we have that

$$\sum_{t=0}^n c_{\mu,\zeta}^{\kappa(t)} \cdot c_{\kappa(t),\epsilon}^\lambda = \sum_{t=0}^n c_{\zeta,\epsilon}^{\eta(t)} \cdot c_{\mu,\eta(t)}^\lambda.$$

We show that our hypothesized coefficients satisfy this equation.

We can determine $d_{\kappa(t),\epsilon}^\lambda$ and $d_{\zeta,\epsilon}^{\eta(t)}$ using the Pieri rule, $d_{\mu,\zeta}^{\kappa(t)}$ using Proposition 4.6.2, and $d_{\mu,\eta(t)}^\lambda$ for $t > 0$ using the inductive hypothesis. Note that by the Pieri rule, $d_{\zeta,\epsilon}^{\nu} = d_{\zeta,\epsilon}^{\eta(0)} = 1$.

Thus, once again by an argument similar to that in the proof of Proposition 4.6.6, we have that:

$$d_{\mu, \eta(t)}^\lambda \cdot d_{\zeta, \epsilon}^{\eta(t)} = d_{\mu, \nu}^\lambda \cdot \frac{\phi_t^n(x; \sigma, \tau)}{\phi_0^n(x; \sigma, \tau)} \text{ if } t > 0,$$

$$d_{\mu, \zeta}^{\kappa(t)} \cdot d_{\kappa(t), \epsilon}^\lambda = d_{\mu, \nu}^\lambda \cdot \frac{\phi_t^n(x; \tau, \sigma)}{\phi_0^n(x; \sigma, \tau)},$$

where $\sigma = (0, n - \mu_{13} - 2r)$ and $\tau = (\mathfrak{d}_{333} - \mathfrak{p} + r, \mathfrak{d}_{233} + 2r)$.

The result follows from Lemma 4.5.2. □

This completes our proof of Theorem 2.2.1.

Chapter 5

Macdonald Polynomials

Jack polynomials $P_\lambda(\alpha; x)$ are generalized by Macdonald polynomials $P_\lambda(q, t; x)$, where

$$\lim_{t \rightarrow 1} P_\lambda(t^\alpha, t; x) = P_\lambda(\alpha; x).$$

Stanley's conjecture can be extended to Macdonald polynomials in a very straightforward way. Just as we had defined α -generalizations of hook-length earlier, we can also define two (q, t) -generalizations:

- *upper hook-length*: $h_\lambda^*(b) = 1 - q^{a(b)+1}t^{\ell(b)}$
- *lower hook-length*: $h_\lambda^*(b) = 1 - q^{a(b)}t^{\ell(b)+1}$

In fact, using these hook lengths, we can apply the Pieri rule (as stated in Theorem 2.3.1) to Macdonald polynomials as well (see [14, IV.6.24]). We can also get the appropriate analogue of Equation 2.3.1 by defining

$$b_\lambda(q, t) = \frac{H_\lambda^*(q, t)}{H_\lambda^*(q, t)},$$

which gives us

$$c_{\mu', \nu'}^{\lambda'}(t, q) = \frac{c_{\mu, \nu}^\lambda(q, t)b_\mu(q, t)b_\nu(q, t)}{b_\lambda(q, t)}.$$

Finally, we use the definitions and theorem above to get the following extension of Theorem 2.2.1.

Theorem 5.0.9. *For an extremal triple (λ, μ, ν) of partitions in \mathcal{P}_3 , $c_{\mu, \nu}^\lambda(q, t)$ can be expressed by the same assignment of upper and lower hooks as $c_{\mu, \nu}^\lambda(\alpha)$, using the corresponding (q, t) -hooks instead of α -hooks.*

Proof. We can still use the same classification as before, as well as the subsequent division number notation to give an assignment of upper and lower hooks. We can also once again express the ratio of a flipped to a standard (q, t) -hook in terms of ϕ , in the following way. Given a (q, t) -hook $h_\lambda(b)$ (which could be either upper or lower), we define

$$\hat{h}_\lambda(b) = \frac{h_\lambda(b)}{q^{a(b)}t^{\ell(b)}}.$$

Then

$$\begin{aligned} \phi\left(t - q; \left\{ \hat{h}_\lambda^*(b) \right\}\right) &= \frac{\hat{h}_\lambda^*(b) - (t - q)}{\hat{h}_\lambda^*(b)} \\ &= \frac{h_\lambda^*(b) - (t - q)(q^{a(b)}t^{\ell(b)})}{h_\lambda^*(b)} \\ &= \frac{h_\lambda^\lambda(b)}{h_\lambda^*(b)}, \end{aligned}$$

and

$$\begin{aligned} \phi\left(t - q; \left\{ -\hat{h}_*^\lambda(b) \right\}\right) &= \frac{-\hat{h}_*^\lambda(b) - (t - q)}{-\hat{h}_*^\lambda(b)} \\ &= \frac{h_*^\lambda(b) - (q - t)(q^{a(b)}t^{\ell(b)})}{h_*^\lambda(b)} \\ &= \frac{h_\lambda^*(b)}{h_\lambda^\lambda(b)}. \end{aligned}$$

We can thus express our coefficients in terms of the same ϕ functions, but this time using the modified hook $\hat{h}(q, t)$ instead of the corresponding hook $h(\alpha)$, and using $x = t - q$ instead of $x = \alpha - 1$. We can also prove that these expressions give the correct coefficient using the same lemmas as before with the same modification to x and the hooks in σ and τ . \square

To see that $\lim_{t \rightarrow 1} c_{\mu, \nu}^\lambda(t^\alpha, t) = c_{\mu, \nu}^\lambda(\alpha)$, we note that

$$\lim_{t \rightarrow 1} \frac{1 - t^{\alpha(a(b))}t^{\ell(b)+1}}{1 - t^{\alpha(a(b)+1)}t^{\ell(b)}} = \frac{\alpha(a(b)) + \ell(b) + 1}{\alpha(a(b) + 1) + \ell(b)},$$

as desired.

Chapter 6

Further Directions

6.1 Extremal triples in \mathcal{P}_n with $n > 3$

The algebraic identities we obtain can be used in higher dimensions, but they relate to a system of extremal paths. As n gets larger, the classification problem becomes much more complicated, both for unique LR fillings and for faces of Horn cones that correspond to extremal triples. Given the role played by the codimension one faces of Horn cones when $n = 3$, one might wonder if Stanley's conjecture can be extended from extremal triples to all boundary triples on this cone. However, this is not true, as demonstrated by the example below.

Example 6.1.1. Let $\lambda = (5, 3, 2, 1)$, $\mu = (3, 2, 1)$, $\nu = (2, 2, 1)$. This lies on the codimension one face given by $\lambda_1 = \mu_1 + \nu_1$. However, in this case $c_{\mu,\nu}^\lambda(1) = 2$ and

$$g_{\mu,\nu}^\lambda(\alpha) = 48\alpha^6(1 + 3\alpha)(3 + 5\alpha)(3 + \alpha)(1 + 2\alpha)^2(3 + 2\alpha)(2 + \alpha)^2(2\alpha^2 + 11\alpha + 2).$$

We also note that when $n = 3$, our extremal paths typically involved decomposing μ or ν into rectangular blocks, since all triples involving a rectangle are extremal in this case. However, when $n > 3$, it is possible to have a non-extremal triple even if ν is a rectangular partition.

Example 6.1.2. Let $\lambda = (4, 3, 2, 1)$, $\mu = (3, 2, 1)$, $\nu = (2, 2)$. Then $c_{\mu,\nu}^\lambda(1) = 2$.

We therefore need a more efficient technique for determining extremal triples and for finding ways to expand them as extremal paths.

6.2 Non-extremal triples in \mathcal{P}_3

In general, if $c_{\mu,\nu}^\lambda(1) = k > 1$, we cannot write it as the sum of k different hook assignments for λ, μ, ν , each multiplied by a power of α . Stanley and Hanlon demonstrated this with the following example.

Example 6.2.1. $\lambda = (4, 2, 1), \mu = (3, 1), \nu = (2, 1)$

$$g_{\mu,\nu}^\lambda(\alpha) = 8\alpha^5(9 + 97\alpha + 294\alpha^2 + 321\alpha^3 + 131\alpha^4 + 12\alpha^5)$$

One can verify that any two expressions $f_1(\alpha)$ and $f_2(\alpha)$ given by hook assignments must share a common linear factor not equal to α or a common integer factor not equal to 8. However, the above expression for $g_{\mu,\nu}^\lambda$ has no rational zeros besides 0, and no integer factors besides 8.

However, if we can expand the coefficients for non-extremal triples as an extremal path, we might obtain a way to write coefficient as the sum of k positive terms, each of which factors into linear factors in α , given by k terms of the form found in Lemmas 4.5.1 or 4.5.2. In particular, we could write:

$$c_{\mu,\nu}^\lambda \cdot c_{\nu',\nu''}^\nu + \sum_{t=k+1}^n c_{\mu,\eta(t)}^\lambda \cdot c_{\nu',\nu''}^{\eta(t)} = \sum_{t=1}^n c_{\mu,\nu'}^{\kappa(t)} \cdot c_{\kappa(t),\nu''}^\lambda,$$

and then write $c_{\mu,\nu}^\lambda$ in this form as long as the remaining coefficients are sufficient to find appropriate choices for σ and τ . This would give a combinatorial description of such coefficients, and also show that they are positive expressions in α , as predicted by another conjecture of Stanley [19, Conj. 8.3].

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