ON THE STRUCTURE OF PRINCIPAL SUBSPACES OF STANDARD MODULES FOR AFFINE LIE ALGEBRAS OF TYPE A

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ABSTRACT OF THE DISSERTATION

On the structure of principal subspaces of standard modules for affine Lie algebras of type A

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Using the theory of vertex operator algebras and intertwining operators, we obtain presentations for the principal subspaces of all the standard $\widehat{\mathfrak{sl}(3)}$-modules. Certain of these presentations had been conjectured and used in work of Calinescu to construct exact sequences leading to the graded dimensions of certain principal subspaces. We prove the conjecture in its full generality for all standard $\widehat{\mathfrak{sl}(3)}$-modules. We then provide a conjecture for the case of $\widehat{\mathfrak{sl}(n)}$, $n \geq 4$. In addition, we construct completions of certain universal enveloping algebras and provide a natural setting for families of defining relations for the principal subspaces of standard modules for untwisted affine Lie algebras. We also use the theory of vertex operator algebras and intertwining operators, along with conjecturally assumed presentations for certain principal subspaces, to construct exact sequences among principal subspaces of certain standard $\widehat{\mathfrak{sl}(n)}$-modules, $n \geq 3$. As a consequence, we obtain the multigraded dimensions of the principal subspaces $W(k_1\Lambda_1 + k_2\Lambda_2)$ and $W(k_{n-2}\Lambda_{n-2} + k_{n-1}\Lambda_{n-1})$. This generalizes earlier work by Calinescu on principal subspaces of standard $\widehat{\mathfrak{sl}(3)}$-modules, where similar assumptions were made.

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Dedication

To my parents.

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Chapter 1

Introduction

There is a long-standing connection between the theories of vertex operators and vertex operator algebras ([B], [FLM], [LL], etc.) and affine Lie algebras (cf. [K]) on the one hand, and Rogers-Ramanujan-type combinatorial identities (cf. [A]) on the other hand ([LM], [LW1]–[LW4], [LP1]–[LP2], and many other references). In this introduction, we briefly sketch the results found in this thesis about several closely related problems. The first of these problems concerns finding presentations for principal subspaces of standard modules. In particular, we give new results concerning presentations of the principal subspaces of the standard $\widehat{\mathfrak{sl}(3)}$-modules. This set of results is the subject of Chapter 4 and can also be found in [S1]. The second of these problems concerns providing a natural setting for families of operators from which such presentations arise. Using a completion of certain universal enveloping algebras, we provide such a setting, and reformulate all known and conjectured presentations in this context. This is carried out in Chapter 5 and the Appendix. The third and final of these problems concerns finding families of exact sequences and q-difference equations. In Chapter 6, we derive such exact sequences, and obtain the multigraded dimensions of certain principal subspaces, which are related to the sum sides of Rogers-Ramanujan-type combinatorial identities. The results concerning the latter two problems can also be found in [S2]. We begin this introduction by first sketching a brief history of some of the connections between affine Lie algebras, vertex operator algebras, and Rogers-Ramanujan-type combinatorial identities.

Many difference-two type partition conditions have been interpreted and obtained by the study of certain natural substructures of standard (i.e., integrable highest weight) modules for affine Lie algebras. In particular, in [FS1]–[FS2], Feigin and Stoyanovsky, motivated by the earlier work by Lepowsky and Primc [LP2], introduced the notion of "principal subspace" of a standard module for an affine Lie algebra, and in the case of $A_1^{(1)} (= \widehat{\mathfrak{sl}(2)})$ and $A_2^{(1)} (= \widehat{\mathfrak{sl}(3)})$ obtained, under certain assumptions (presentations for these principal subspaces in terms of generators and relations) the multigraded dimensions ("characters") of the principal subspaces of the "vacuum" standard modules. Interestingly enough, these multigraded dimensions were related to the Rogers-Ramanujan partition identities, and more generally, the Gordon-Andrews identities, but in a different setting than the original vertex-algebraic interpretation of these identities in [LW2]–[LW4]. A more general case was considered by Georgiev in [G], where combinatorial bases were constructed for the principal subspaces associated to certain standard $A_n^{(1)}$-modules. Using these bases, Georgiev obtained the multigraded dimensions of these principal subspaces. More recently, combinatorial bases have been constructed for principal subspaces in more general lattice cases ([P], [MiP]), for the principal subspaces of the vacuum standard modules for the affine Lie algebras $B_2^{(1)}$ [Bu], for principal subspaces in the quantum $\widehat{\mathfrak{sl}(n+1)}$-case [Ko], and for certain natural substructures of principal subspaces ([Pr], [J1]–[J3], [T1]–[T4], [Ba], [JPr]).

In [CLM1]–[CLM2], the authors addressed the problem of vertex-algebraically interpreting the classical Rogers-Ramanujan recursion and, more generally, the Rogers-Selberg recursions (cf. [A]) by using intertwining operators among modules for vertex operator algebras to construct exact sequences leading to these recursions. In particular, the solutions of these recursions gave the graded dimensions of the principal subspaces of the standard $A_1^{(1)}$-modules. In [CLM1]–[CLM2] (as in [FS1]–[FS2]), the authors assumed certain presentations for the principal subspaces of the standard $\widehat{\mathfrak{sl}(2)}$-modules (presentations that can be derived from [LP2]; the nontrivial part is the completeness of the relations). In [CalLM1]–[CalLM2], the authors gave an a priori proof, again using intertwining operators, of the completeness of the presentations assumed in [FS1]-[FS2] and [CLM1]–[CLM2]. These results were extended to the level 1 standard $\widehat{\mathfrak{sl}(n+1)}$-modules by Calinescu in [C4], and later to the level 1 standard modules for the untwisted affine Lie algebras of type ADE in [CalLM3]. The desired presentations were proved, and exact sequences were obtained leading to recursions and the graded dimensions of the principal subspaces of the level 1 standard modules. In [CalLM4], the authors have initiated the study of principal subspaces for standard modules for twisted affine Lie algebras, extending the past work of [CLM1]–[CLM2], [CalLM1]–[CalLM3] to the case of the level 1 standard module for the twisted affine Lie algebra $A_2^{(2)}$.

In the work [C3], Calinescu considered the principal subspaces of certain higher level standard $\widehat{\mathfrak{sl}(3)}$-modules. In this work, she conjecturally assumed presentations for certain principal subspaces, and using the theory of vertex operator algebras and intertwining operators, she constructed exact sequences among these principal subspaces. Using these exact sequences, along with the multigraded dimensions in [G], Calinescu was able to find the multigraded dimensions of principal subspaces which had not previously been studied.

A different variant of principal subspace was considered in [AKS] and [FFJMM]. In [AKS], the authors cite well-known presentations for standard modules, and use these to provide (without proof) a set of defining relations for each principal subspace. In [FFJMM], in which the authors consider $A_2^{(1)}$, they do indeed prove that certain relations form a set of defining relations for their variant of principal subspace. In the case of the vacuum modules, the principal subspaces in [FFJMM] are essentially identical to the principal subspaces considered in the present work, and their defining relations indeed agree with those in Chapter 4 of the present work. For the non-vacuum modules, the principal subspaces considered in the present work can be viewed as proper substructures of those considered in [FFJMM], and correspondingly, the defining relations we obtain are different. Our method for proving the completeness of our defining relations is completely different from the method in [FFJMM].

We now give a brief overview of the structure of this thesis. In Chapter 2, we recall certain vertex-algebraic constructions of standard $\widehat{\mathfrak{sl}(n+1)}$-modules and of intertwining operators among these modules. In Chapter 3, we recall the notion of principal subspace of a standard module and prove certain useful properties of principal subspaces. We now very briefly recall some of these notions. Given a complex semisimple Lie algebra $\mathfrak{g}$, a fixed Cartan subalgebra $\mathfrak{h}$, a fixed set of positive roots Δ_+, and a root vector x_α for each $\alpha \in \Delta_+$, consider the subalgebra $\mathfrak{n} = \coprod_{\alpha\in\Delta_+} \mathbb{C}x_\alpha \subset \mathfrak{g}$ spanned by the positive root vectors. The affinization $\bar{\mathfrak{n}} = \mathfrak{n} \otimes \mathbb{C}[t, t^{-1}]$ of $\mathfrak{n}$ is a subalgebra of the affine Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$. Let $L(\Lambda)$ be the standard module of $\widehat{\mathfrak{g}}$ with highest weight Λ and level k, a positive integer, and let $v_\Lambda \in L(\Lambda)$ be a highest weight vector. The principal subspace of $L(\Lambda)$ is defined by

$$W(\Lambda) = U(\bar{\mathfrak{n}}) \cdot v_\Lambda, \tag{1.1}$$

where $U(\cdot)$ is the universal enveloping algebra. We also have natural surjective maps

$$f_\Lambda : U(\bar{\mathfrak{n}}) \to W(\Lambda). \tag{1.2}$$

By presentation of $W(\Lambda)$, we mean a complete description of $\ker f_\Lambda$ in terms of its generators.

Chapter 4 in this thesis is another step forward in the spirit of [CalLM1]-[CalLM3]. We exploit intertwining operators among vertex operator algebra modules to solve the problem of giving an a priori proof of presentations for the principal subspaces of all the standard modules for $A_2^{(1)} (= \widehat{\mathfrak{sl}(3)})$, including those assumed conjecturally and used in [C3]. The methods used in the proof of these presentations are similar to those in [CalLM1]–[CalLM3], in that certain minimal counterexamples are postulated and shown not to exist. However, in the general case, we needed to introduce certain new ideas to prove our presentations. We then proceed to formulate the presentations for principal subspaces of all the standard modules for $A_n^{(1)}$ as a conjecture. In particular, we take $\mathfrak{g} = \mathfrak{sl}(3)$. We precisely determine $\mathrm{Ker} f_\Lambda$ in terms of certain natural left ideals of $U(\bar{\mathfrak{n}})$. Specifically, in terms of the fundamental weights of $A_2^{(1)}$, which we label Λ_0, Λ_1, and Λ_2, we may express Λ as

$$\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2,$$

for some nonnegative integers k_0, k_1, and k_2. We define an ideal $I_{k\Lambda_0}$ in terms of left ideals generated by the coefficients of certain vertex operators associated with singular vectors in a natural way. This left ideal is then used to define a larger left ideal

$$I_\Lambda = I_{k\Lambda_0} + U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^{k_0+k_2+1} + U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k_0+k_1+1} + U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-1)^{k_0+1},$$

where we use $x(n)$ to denote the action of $x \otimes t^n \in \widehat{\mathfrak{g}}$ for $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$. We then proceed to show that

$$\mathrm{Ker} f_\Lambda = I_\Lambda.$$

The proof of this result is similar in structure to the proof of the presentations in [CalLM2]. Considering all dominant integral weights together, we choose minimal counterexamples (certain elements in $\mathrm{Ker} f_\Lambda \setminus I_\Lambda$) and show that a contradiction is reached for each Λ. Certain maps used in [CalLM3] are also generalized and used in the proof, but these ideas do not extend to the most general case. We develop a method for reaching the desired contradictions for each Λ which "rebuilds" the minimal counterexample to show that it is in fact an element of I_Λ. This "rebuilding" technique can also be used to show all of the presentations proved in the works [CalLM1]-[CalLM3] in the type A case with suitable modifications (see remarks at the end of Section 4).

In [C3], certain of these presentations were conjectured and used to construct exact sequences among principal subspaces. Using these exact sequences, Calinescu obtained the previously unknown graded dimensions for principal subspaces whose highest weights are of the form $k_1\Lambda_1 + k_2\Lambda_2$, where k_1, k_2 are positive integers. The problem of constructing exact sequences for more general highest weights is still unsolved.

Chapter 5 of this thesis, along with the Appendix, focuses on providing a more natural setting for the annihilating ideals which give presentations of the principal subspaces of the standard modules. In [CLM1]–[CLM2] and [CalLM1]–[CalLM3], the annihilator of the highest weight vector of each principal subspace is written in terms of certain elements of $U(\widehat{\mathfrak{g}})$ which, when viewed as operators, annihilate the highest weight vector. An important set of these operators arises from certain null vector identities given by powers of vertex operators and are written as infinite formal sums of elements of $U(\widehat{\mathfrak{g}})$ also viewed as operators. The ideals which annihilate the highest weight vectors can be expressed using operators defined by certain truncations of these formal sums, in order to view these operators as elements of $U(\widehat{\mathfrak{g}})$. We provide the details of the construction of a completion of the universal enveloping algebra $U(\bar{\mathfrak{n}})$ to give more natural presentations (without such truncations) for the defining annihilating ideals of principal subspaces. This completion was discussed in [C1]–[C2] and [CalLM3], but the details of this construction were not supplied. We prove various properties of this completion and the defining ideals for principal subspaces, including their more natural definition inside this completion. These completions may be generalized to the twisted setting used in [CalLM4] (as in [LW3], where similar completions were originally constructed in a general twisted or untwisted setting).

Our main result in Chapter 6 is a natural generalization of [C3] to the case of $\widehat{\mathfrak{sl}(n+1)}$, $n \geq 2$. Although our methods recover the same information as in [CLM1]–[CLM2] when $n = 1$, we take $n \geq 2$ for notational convenience. In the case where $n = 2$, we recover the results in [C3] with a slight variant of the methods. As in [C3], we conjecturally assume presentations for certain principal subspaces, and use these to provide exact sequences among principal subspaces of certain standard $\widehat{\mathfrak{sl}(n+1)}$-modules. Using these exact sequences, along with the multigraded dimensions found in [G], we give previously unknown multigraded dimensions of principal subspaces. To state the main result of this chapter, we let $\Lambda_0, \ldots, \Lambda_n$ denote the fundamental weights of $\widehat{\mathfrak{sl}(n+1)}$. The dominant integral weights Λ of $\widehat{\mathfrak{sl}(n+1)}$ are $k_0\Lambda_0 + \cdots + k_n\Lambda_n$ for $k_0, \ldots, k_n \in \mathbb{N}$, and we use $L(\Lambda)$ to denote the standard module with highest weight Λ, $W(\Lambda)$ to denote its principal subspace, and $\chi'_{W(\Lambda)}(x_1, \ldots, x_n, q)$ to denote its multigraded dimension. Our result states:

Theorem 1.0.1 *Let $k \geq 1$. For any i with $1 \leq i \leq n-1$ and $k_i, k_{i+1} \in \mathbb{N}$ such that $k_i + k_{i+1} = k$, the sequences*

$$\begin{aligned} &W(k_i\Lambda_i + k_{i+1}\Lambda_{i+1}) \xrightarrow{\phi_i} \\ &\quad W(k_i\Lambda_0 + k_{i+1}\Lambda_i) \xrightarrow{1^{\otimes k_i - 1} \otimes \mathcal{Y}_c(e^{\lambda_i}, x) \otimes 1^{\otimes k_{i+1}}} \\ &\qquad W((k_i - 1)\Lambda_0 + (k_{i+1}+1)\Lambda_i) \longrightarrow 0 \end{aligned} \tag{1.3}$$

when $k_i \geq 1$, and

$$\begin{aligned} &W(k_i\Lambda_i + k_{i+1}\Lambda_{i+1}) \xrightarrow{\psi_i} \\ &\quad W(k_{i+1}\Lambda_0 + k_i\Lambda_{i+1}) \xrightarrow{1^{\otimes k_{i+1} - 1} \otimes \mathcal{Y}_c(e^{\lambda_{i+1}}, x) \otimes 1^{\otimes k_i}} \\ &\qquad W((k_{i+1} - 1)\Lambda_0 + (k_i+1)\Lambda_{i+1}) \longrightarrow 0 \end{aligned} \tag{1.4}$$

when $k_{i+1} \geq 1$, are exact.

The maps ϕ_i, ψ_i, and $\mathcal{Y}_c(e^{\lambda_i}, x)$ are maps naturally arising from the lattice construction of the level 1 standard modules and intertwining operators among these modules. As a consequence of this theorem, we obtain results about multigraded dimensions when the first map ϕ_i or ψ_i is injective, and we have the following theorem and its corollary:

Theorem 1.0.2 *Let $k \geq 1$. Let $k_1, k_2, k_{n-1}, k_n \in \mathbb{N}$ with $k_1 \geq 1$ and $k_n \geq 1$, such that $k_1 + k_2 = k$ and $k_{n-1} + k_n = k$. Then*

$$\begin{aligned} &\chi'_{W(k_1\Lambda_1+k_2\Lambda_2)}(x_1, \ldots, x_n, q) = \\ &= \quad x_1^{-k_1}\chi'_{W((k_1-1)\Lambda_0+(k_2+1)\Lambda_1)}(x_1q^{-1}, x_2q, x_3 \ldots, x_n, q) \\ &\qquad -x_1^{-k_1}\chi'_{W(k_1\Lambda_0+k_2\Lambda_1)}(x_1q^{-1}, x_2q, x_3, \ldots, x_n, q) \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} &\chi'_{W(k_{n-1}\Lambda_{n-1}+k_n\Lambda_n)}(x_1, \ldots, x_n, q) = \\ &= \quad x_n^{-k_n}\chi'_{W((k_n-1)\Lambda_0+(k_{n-1}+1)\Lambda_n)}(x_1, \ldots, x_{n-1}q, x_nq^{-1}, q) \\ &\qquad -x_n^{-k_n}\chi'_{W(k_n\Lambda_0+k_{n-1}\Lambda_n)}(x_1, \ldots, x_{n-1}q, x_nq^{-1}, q). \end{aligned} \tag{1.6}$$

Theorem 1.0.2 immediately gives us:

Corollary 1.0.3 *In the setting of Theorem 1.0.2, we have that*

$$\chi'_{W(k_1\Lambda_1+k_2\Lambda_2)}(x_1, \ldots, x_n, q) =$$

$$= \sum \left(\frac{q^{{r_1^{(1)}}^2+\ldots+{r_1^{(k)}}^2+\sum_{t=k_1+1}^{k} r_1^{(t)}+\sum_{t=1}^{k} r_2^{(t)}-r_1^{(t)}}(1-q^{r_1^{(k_1)}})}{(q)_{r_1^{(1)}-r_1^{(2)}}\cdots(q)_{r_1^{(k-1)}-r_1^{(k)}}(q)_{r_1^{(k)}}} \right)\times$$

$$\times\left(\frac{q^{{r_2^{(1)}}^2+\ldots+{r_2^{(k)}}^2-r_2^{(1)}r_1^{(1)}-\ldots-r_2^{(k)}r_1^{(k)}}}{(q)_{r_2^{(1)}-r_2^{(2)}}\cdots(q)_{r_2^{(k-1)}-r_2^{(k)}}(q)_{r_2^{(k)}}} \right)\times$$

$$\times\cdots\times\left(\frac{q^{{r_n^{(1)}}^2+\ldots+{r_n^{(k)}}^2-r_n^{(1)}r_{n-1}^{(1)}-\ldots-r_n^{(k)}r_{n-1}^{(k)}}}{(q)_{r_n^{(1)}-r_n^{(2)}}\cdots(q)_{r_n^{(k-1)}-r_n^{(k)}}(q)_{r_n^{(k)}}} \right) x_1^{-k_1+\sum_{i=1}^{k} r_1^{(i)}}\cdots x_n^{\sum_{i=1}^{n} r_n^{(i)}}$$

and

$$\chi'_{W(k_{n-1}\Lambda_{n-1}+k_n\Lambda_n)}(x_1, \ldots, x_n, q) =$$

$$= \sum \left(\frac{q^{{r_1^{(1)}}^2+\ldots+{r_1^{(k)}}^2}}{(q)_{r_1^{(1)}-r_1^{(2)}}\cdots(q)_{r_1^{(k-1)}-r_1^{(k)}}(q)_{r_1^{(k)}}} \right) \left(\frac{q^{{r_2^{(1)}}^2+\ldots+{r_2^{(k)}}^2-r_2^{(1)}r_1^{(1)}-\ldots-r_2^{(k)}r_1^{(k)}}}{(q)_{r_2^{(1)}-r_2^{(2)}}\cdots(q)_{r_2^{(k-1)}-r_2^{(k)}}(q)_{r_2^{(k)}}} \right) \times$$

$$\times\cdots\times \left(\frac{q^{{r_n^{(1)}}^2+\ldots+{r_n^{(k)}}^2-r_n^{(1)}r_{n-1}^{(1)}-\ldots-r_n^{(k)}r_{n-1}^{(k)}+\sum_{t=k_n+1}^{k} r_n^{(t)}}}{(q)_{r_n^{(1)}-r_n^{(2)}}\cdots(q)_{r_n^{(k-1)}-r_n^{(k)}}(q)_{r_n^{(k)}}} \right) \times$$

$$\times q^{\sum_{t=1}^{k} r_{n-1}^{(t)}-r_n^{(t)}}(1-q^{r_n^{(k_n)}})x_1^{\sum_{i=1}^{k} r_1^{(i)}}\cdots x_n^{-k_n+\sum_{i=1}^{n} r_n^{(i)}}$$

where the sums are taken over decreasing sequences $r_j^{(1)} \geq r_j^{(2)} \geq \cdots \geq r_j^{(k)} \geq 0$ *for each* $j = 1, \ldots, n$.

The expressions in Corollary 1.0.3 can also be written as follows: As in [G], for $s = 1, \ldots, k-1$ and $i = 1, \ldots, n$, set $p_i^{(s)} = r_i^{(s)} - r_i^{(s+1)}$, and set $p_i^{(k)} = r_i^{(k)}$. Also, let $(A_{lm})_{l,m=1}^{n}$ be the Cartan matrix of $\mathfrak{sl}(n+1)$ and $B^{st} := \min\{s,t\}$, $1 \leq s,t \leq k$. Then,

$$\chi'_{W(k_1\Lambda_1+k_2\Lambda_2)}(x_1,\ldots,x_n,q) =$$

$$\sum_{\substack{p_1^{(1)},\ldots,p_1^{(k)}\geq 0\\ \vdots \\ p_n^{(1)},\ldots,p_n^{(k)}\geq 0}} \frac{q^{\frac{1}{2}\sum_{l,m=1,\ldots,n}^{s,t=1,\ldots,l} A_{lm}B^{st}p_l^{(s)}p_m^{(t)}}}{\prod_{i=1}^{n}\prod_{s=1}^{k}(q)_{p_i{}^{(s)}}} q^{\widetilde{p_1}} q^{\sum_{t=1}^{k} p_2^{(t)}+\cdots+p_2^{(k)}-p_1^{(t)}-\cdots-p_1^{(k)}} \times$$

$$\times(1-q^{p_1^{(k_1)}+\cdots+p_1^{(k)}})x_1^{-k_1}\prod_{i=1}^{n} x_i^{\sum_{s=1}^{k} sp_i^{(s)}}$$

where $\widetilde{p_1} = p_1^{(k_1+1)} + 2p_1^{(k_1+2)} + \cdots + k_2 p_1^{(k)}$ and

$$\chi'_{W(k_{n-1}\Lambda_{n-1}+k_n\Lambda_n)}(x_1,\ldots,x_n,q) =$$

$$\sum_{\substack{p_1^{(1)},\ldots,p_1^{(k)}\geq 0\\ \vdots \\ p_n^{(1)},\ldots,p_n^{(k)}\geq 0}} \frac{q^{\frac{1}{2}\sum_{l,m=1,\ldots,n}^{s,t=1,\ldots,l} A_{lm}B^{st}p_l^{(s)}p_m^{(t)}}}{\prod_{i=1}^{n}\prod_{s=1}^{k}(q)_{p_i{}^{(s)}}} q^{\widetilde{p_n}} q^{\sum_{t=1}^{k} p_{n-1}^{(t)}+\cdots+p_{n-1}^{(k)}-p_n^{(t)}-\cdots-p_n^{(k)}} \times$$

$$\times(1-q^{p_n^{(k_n)}+\cdots+p_n^{(k)}})x_n^{-k_n}\prod_{i=1}^{n} x_i^{\sum_{s=1}^{k} sp_i^{(s)}}$$

where $\widetilde{p_n} = p_n^{(k_n+1)} + 2p_n^{(k_n+2)} + \cdots + k_{n-1} p_1^{(k)}$.

Similar multigraded dimensions for different variants of principal subspaces have been studied in [AKS] and [FFJMM]. Modularity properties of certain multigraded dimensions, in the context of principal subspaces of standard modules, have been studied in [St], [WZ], and more recently in [BCFK].

Chapter 2

Preliminaries

2.1 Constructions in the $\widehat{\mathfrak{sl}(n+1)}$ case

We begin by recalling certain vertex-algebraic constructions for the untwisted affine Lie algebra $\widehat{\mathfrak{sl}(n+1)}$, n a positive integer. We shall be working in the setting of [FLM] and [LL].

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(n+1)$. Also fix a set of roots Δ, a set of simple roots $\{\alpha_1, \ldots, \alpha_n\}$, and a set of positive roots Δ_+. Let $\langle \cdot, \cdot \rangle$ denote the Killing form, rescaled so that $\langle \alpha, \alpha \rangle = 2$ for each $\alpha \in \Delta$. Using this form, we identify $\mathfrak{h}$ with $\mathfrak{h}^*$. Let $\lambda_1, \ldots, \lambda_n \in \mathfrak{h} \simeq \mathfrak{h}^*$ denote the fundamental weights of $\mathfrak{sl}(n+1)$. Recall that $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$ for each $i, j = 1, \ldots, n$. Denote by $Q = \sum_{i=1}^n \mathbb{Z}\alpha_i$ and $P = \sum_{i=1}^n \mathbb{Z}\lambda_i$ the root lattice and weight lattice of $\mathfrak{sl}(n+1)$, respectively.

For each root $\alpha \in \Delta$, we have a root vector $x_\alpha \in \mathfrak{sl}(n+1)$ (recall that $[h, x_\alpha] = \langle \alpha, h \rangle x_\alpha$ for each $h \in \mathfrak{h}$). We define

$$\mathfrak{n} = \sum_{\alpha \in \Delta_+} \mathbb{C} x_\alpha,$$

a nilpotent subalgebra of $\mathfrak{sl}(n+1)$.

We have the corresponding untwisted affine Lie algebra given by

$$\widehat{\mathfrak{sl}(n+1)} = \mathfrak{sl}(n+1) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

where c is a non-zero central element and

$$[x \otimes t^m, y \otimes t^p] = [x, y] \otimes t^{m+p} + m\langle x, y \rangle \delta_{m+p,0} c$$

for any $x, y \in \mathfrak{sl}(n+1)$ and $m, p \in \mathbb{Z}$. If we adjoin the degree operator d, where

$$[d, x \otimes t^m] = m x \otimes t^m$$

$$[d, c] = 0,$$

we obtain the affine Kac-Moody Lie algebra $\widetilde{\mathfrak{sl}(n+1)} = \widehat{\mathfrak{sl}(n+1)} \oplus \mathbb{C}d$ (cf. [K]). We define two important subalgebras of $\widehat{\mathfrak{sl}(n+1)}$:

$$\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$

and the Heisenberg subalgebra

$$\widehat{\mathfrak{h}}_{\mathbb{Z}} = \coprod_{m \in \mathbb{Z} \setminus \{0\}} \mathfrak{h} \otimes t^m \oplus \mathbb{C}c$$

(in the notation of [FLM], [LL]). We extend our form $\langle \cdot, \cdot \rangle$ to $\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ by defining

$$\begin{aligned} \langle c, c \rangle &= 0 \\ \langle d, d \rangle &= 0 \\ \langle c, d \rangle &= 1. \end{aligned}$$

Using this form, we may identify $\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ with $(\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d)^*$. The simple roots of $\widehat{\mathfrak{sl}(n+1)}$ are α_0, $\alpha_1, \ldots, \alpha_n$ and the fundamental weights of $\widehat{\mathfrak{sl}(n+1)}$ are $\Lambda_0, \Lambda_1, \ldots, \Lambda_n$, given by

$$\alpha_0 = c - (\alpha_1 + \alpha_2 + \cdots + \alpha_n)$$

and

$$\Lambda_0 = d, \ \Lambda_i = \Lambda_0 + \lambda_i$$

for each $i = 1, \ldots, n$.

An $\widehat{\mathfrak{sl}(n+1)}$-module V is said to have level $k \in \mathbb{C}$ if the central element c acts as multiplication by k (i.e. $c \cdot v = kv$ for all $v \in V$). Any standard (i.e. irreducible integrable highest weight) module $L(\Lambda)$ with $\Lambda \in (\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d)^*$ has nonnegative integral level, given by $\langle \Lambda, c \rangle$ (cf. [K]). Let $L(\Lambda_0)$, $L(\Lambda_1), \ldots, L(\Lambda_n)$ denote the standard $\widehat{\mathfrak{sl}(n+1)}$-modules of level 1 with v_{Λ_0}, $v_{\Lambda_1}, \ldots, v_{\Lambda_n}$ as highest weight vectors, respectively.

Continuing to work in the setting of [FLM] and [LL], we now recall the lattice vertex operator construction of the level 1 standard modules for $\widehat{\mathfrak{sl}(n+1)}$. We use $U(\cdot)$

to denote the universal enveloping algebra. The induced module

$$M(1) = U(\widehat{\mathfrak{h}}) \otimes_{U(\mathfrak{h}\otimes\mathbb{C}[t]\oplus\mathbb{C}c)} \mathbb{C}$$

has a natural $\widehat{\mathfrak{h}}$-module structure, where $\mathfrak{h} \otimes \mathbb{C}[t]$ acts trivially and c acts as identity on the one-dimensional module $\mathbb{C}$. Let $s = 2(n+1)^2$. We fix a primitive s^{th} root of unity ν_s, and a central extension $\widehat{P}$ of the weight lattice P by the finite cyclic group $\langle\kappa\rangle = \langle\kappa \mid \kappa^s = 1\rangle$ of order s,

$$1 \to \langle\kappa\rangle \to \widehat{P} \xrightarrow{-} P \to 1$$

with associated commutator map $c_0 : P \times P \longrightarrow \mathbb{Z}/s\mathbb{Z}$, defined by $aba^{-1}b^{-1} = \kappa^{c_0(\bar{a},\bar{b})}$ for $a, b \in \widehat{P}$. Let $c : P \times P \longrightarrow \mathbb{C}^\times$ denote the alternating $\mathbb{Z}$-bilinear map defined by $c(\lambda,\mu) = \nu_s^{c_0(\lambda,\mu)}$ for $\lambda, \mu \in P$. We require that

$$c(\alpha,\beta) = (-1)^{\langle\alpha,\beta\rangle} \quad \text{for} \quad \alpha,\beta \in Q.$$

Such a central extension $\widehat{P}$ of P does indeed exist (see Remark 6.4.12 in [LL]).

We define the faithful character $\chi : \langle\kappa\rangle \longrightarrow \mathbb{C}^\times$ by $\chi(\kappa) = \nu_s$. Let $\mathbb{C}_\chi$ be the one dimensional $\langle\kappa\rangle$-module, where the action of κ is given by $\kappa \cdot 1 = \nu_s$, and form the induced $\widehat{P}$-module

$$\mathbb{C}\{P\} = \mathbb{C}[\widehat{P}] \otimes_{\mathbb{C}[\langle\kappa\rangle]} \mathbb{C}_\chi.$$

For any subset $E \subset P$, we define $\widehat{E} = \{a \in P | \bar{a} \in E\}$, and we form $\mathbb{C}\{E\}$ in the obvious way. Then, the space

$$V_Q = M(1) \otimes \mathbb{C}\{Q\}$$

carries a natural vertex operator algebra structure, with 1 as vacuum vector, and the space

$$V_P = M(1) \otimes \mathbb{C}\{P\}$$

is naturally a V_Q-module. We now recall some important details of this construction (cf. [LL]).

Choose a section

$$\begin{aligned} e : P &\longrightarrow \widehat{P} \\ \alpha &\mapsto e_\alpha, \end{aligned} \tag{2.1}$$

(i.e. a map which satisfies $^{-} \circ e = 1$) such that $e_0 = 1$. Let $\epsilon_0 : P \times P \longrightarrow \mathbb{Z}/s\mathbb{Z}$ the corresponding 2-cocycle, defined by the condition $e_\alpha e_\beta = \kappa^{\epsilon_0(\alpha,\beta)} e_{\alpha+\beta}$ for $\alpha, \beta \in P$ and define the map $\epsilon : P \times P \longrightarrow \mathbb{C}^\times$ by $\epsilon(\alpha,\beta) = \nu_s^{\epsilon_0(\alpha,\beta)}$ For any $\alpha, \beta \in P$ we have

$$\epsilon(\alpha,\beta)/\epsilon(\beta,\alpha) = c(\alpha,\beta) \tag{2.2}$$

and

$$\epsilon(\alpha,0) = \epsilon(0,\alpha) = 1. \tag{2.3}$$

We use this choice of section (2.1) identify $\mathbb{C}\{P\}$ and the group algebra $\mathbb{C}[P]$. In particular, we have a vector space isomorphism given by

$$\begin{aligned} \mathbb{C}[P] &\longrightarrow \mathbb{C}\{P\} \\ e^\alpha &\mapsto \iota(e_\alpha) \end{aligned} \tag{2.4}$$

for $\alpha \in P$, where, for $a \in \widehat{P}$, we set $\iota(a) = a \otimes 1 \in \mathbb{C}\{P\}$. By restriction, we also have the identification $\mathbb{C}[Q] \simeq \mathbb{C}\{Q\}$. There is a natural action $\widehat{P}$ on $\mathbb{C}[P]$ given by

$$\begin{aligned} e_\alpha \cdot e^\beta &= \epsilon(\alpha,\beta) e^{\alpha+\beta}, \\ \kappa \cdot e^\beta &= \nu_s e^\beta \end{aligned}$$

for $\alpha, \beta \in P$. As operators on $\mathbb{C}[P] \simeq \mathbb{C}\{P\}$ we have

$$e_\alpha e_\beta = \epsilon(\alpha,\beta) e_{\alpha+\beta}. \tag{2.5}$$

We make the identifications

$$V_P = M(1) \otimes \mathbb{C}[P],$$

$$V_Q = M(1) \otimes \mathbb{C}[Q]$$

and we set

$$V_Q e^{\lambda_i} = M(1) \otimes \mathbb{C}[Q] e^{\lambda_i}, \quad i = 1, \ldots, n$$

Given a Lie algebra element $a \otimes t^m \in \widehat{\mathfrak{sl}(n+1)}$, where $a \in \mathfrak{sl}(n+1), m \in \mathbb{Z}$, we will denote its action on an $\widehat{\mathfrak{sl}(n+1)}$-module using the notation $a(m)$. In particular, for $h \in \mathfrak{h}$ and $m \in \mathbb{Z}$, we have the operators $h(m)$ on V_P:

$$h(0)(v \otimes \iota(e_\alpha)) = \langle h, \alpha \rangle (v \otimes \iota(e_\alpha))$$

$$h(m)(v \otimes \iota(e_\alpha)) = (h(m)v \otimes \iota(e_\alpha)).$$

For a formal variable x and $\lambda \in P$, we define the operator x^λ by

$$x^\lambda(v \otimes \iota(e_\mu)) = x^{\langle \lambda, \mu \rangle}(v \otimes \iota(e_\mu))$$

for $v \in M(1)$ and $\mu \in P$. For each $\lambda \in P$, we define the vertex operators

$$Y(\iota(e_\lambda), x) = E^-(-\lambda, x)E^+(-\lambda, x)e_\lambda x^\lambda, \tag{2.6}$$

where

$$E^\pm(-\lambda, x) = \exp\left(\sum_{\pm n > 0} \frac{-\lambda(n)}{n} x^{-n} \right) \in (\mathrm{End}\ V_P)[[x, x^{-1}]]$$

Using the identification (2.4) we write $Y(e^\lambda, x)$ for $Y(\iota(e_\lambda), x)$. In particular, for any root $\alpha \in \Delta$ we have the operators $x_\alpha(m)$ defined by

$$Y(e^\alpha, x) = \sum_{m \in \mathbb{Z}} x_\alpha(m) x^{-m-1}. \tag{2.7}$$

It is easy to see that

$$x^\lambda e_\mu = x^{\langle \lambda, \mu \rangle} e_\mu x^\lambda \tag{2.8}$$

and

$$\lambda(m) e_\mu = e_\mu \lambda(m) \tag{2.9}$$

for all $\lambda, \mu \in P$ and $m \in \mathbb{Z}$. Using (2.2), (2.5) and (2.6)-(2.9) we obtain, for $\alpha \in \Delta$, $\mu \in P$,

$$x_\alpha(m) e_\mu = c(\alpha, \mu) e_\mu x_\alpha(m + \langle \alpha, \mu \rangle). \tag{2.10}$$

Along with the action of $\widehat{\mathfrak{h}}$, the operators $x_\alpha(m)$, $m \in \mathbb{Z}$, give V_P a $\widehat{\mathfrak{sl}(n+1)}$-module structure. In particular, we have that

$$V_P = V_Q \oplus V_Q e^{\lambda_1} \oplus \cdots \oplus V_Q e^{\lambda_n}$$

and that V_Q, $V_Qe^{\lambda_1}, \ldots, V_Qe^{\lambda_n}$ are the level 1 basic representations of $\widehat{\mathfrak{sl}(n+1)}$ with highest weights Λ_0, $\Lambda_1, \ldots, \Lambda_n$ and highest weight vectors $v_{\Lambda_0} = 1 \otimes 1$, $v_{\Lambda_1} = 1 \otimes e^{\lambda_1}, \ldots, v_{\Lambda_n} = 1 \otimes e^{\lambda_n}$, respectively. We make the identifications

$$L(\Lambda_0) = V_Q$$

$$L(\Lambda_i) = V_Qe^{\lambda_i}$$

for each $i = 1, \ldots, n$. Moreover, taking

$$\omega = \frac{1}{2}\sum_{i=1}^{n} u^{(i)}(-1)^2 v_{\Lambda_0}$$

to be the standard conformal vector, where $\{u^{(1)}, \ldots, u^{(n)}\}$ is an orthonormal basis of $\mathfrak{h}$, the operators $L(m)$ defined by

$$Y(\omega, x) = \sum_{m\in\mathbb{Z}} L(m)x^{-m-2} \tag{2.11}$$

provide a representation of the Virasoro algebra of central charge n. The vertex operators (2.6) and (2.22) give $L(\Lambda_0)$ the structure of a vertex operator algebra whose irreducible modules are precisely $L(\Lambda_0)$,$L(\Lambda_1), \ldots, L(\Lambda_n)$. We shall write

$$v_{\Lambda_0} = \mathbf{1}, \quad v_{\Lambda_1} = e^{\lambda_1}, \ldots, v_{\Lambda_n} = e^{\lambda_n}. \tag{2.12}$$

As in [G], [CLM1]–[CLM2], [C3]–[C4], and [CalLM1]–[CalLM3], we need certain intertwining operators among standard modules. We recall some facts from [FHL] and [DL] about intertwining operators and, in particular, the intertwining operators between $L(\Lambda_0)$, $L(\Lambda_1), \ldots, L(\Lambda_n)$.

Given modules W_1, W_2 and W_3 for the vertex operator algebra V, an intertwining operator of type

$$\left(\begin{array}{c} W_3 \\ W_1 \quad W_2 \end{array}\right)$$

is a linear map

$$\begin{aligned} \mathcal{Y}(\cdot\,, x) : W_1 &\longrightarrow \mathrm{Hom}(W_2, W_3)\{x\} \\ w &\mapsto \mathcal{Y}(w, x) = \sum_{n\in\mathbb{Q}} w_n x^{-n-1} \end{aligned}$$

such that all the axioms of vertex operator algebra which make sense hold (see [FHL]). The main axiom is the Jacobi identity:

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) & Y(u,x_1)\mathcal{Y}(w_{(1)},x_2)w_{(2)} \\ & -x_0^{-1}\delta\left(\frac{x_2-x_1}{x_0}\right)\mathcal{Y}(w_{(1)},x_2)Y(u,x_1)w_{(2)} \\ = x_2^{-1}\delta & \left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}(Y(u,x_0)w_{(1)},x_2)w_{(2)} \end{aligned}$$

for $u \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$.

Define the operators $e^{i\pi\lambda}$ and $c(\cdot\,,\lambda)$ on V_P by:

$$e^{i\pi\lambda}(v \otimes e^{\beta}) = e^{i\pi\langle\lambda,\beta\rangle} v \otimes e^{\beta},$$

$$c(\cdot\,,\lambda)(v \otimes e^{\beta}) = c(\beta,\lambda) v \otimes e^{\beta},$$

for $v \in M(1)$ and $\beta, \lambda \in P$. We have that

$$\begin{aligned} \mathcal{Y}(\cdot,x) : L(\Lambda_r) & \longrightarrow \mathrm{Hom}(L(\Lambda_s),L(\Lambda_p))\{x\} \\ w & \mapsto \mathcal{Y}(w,x) = Y(w,x)e^{i\pi\lambda_r}c(\cdot,\lambda_r) \end{aligned} \tag{2.13}$$

defines an intertwining operator of type

$$\left(\begin{array}{c} L(\Lambda_p) \\ L(\Lambda_r) \quad L(\Lambda_s) \end{array}\right) \tag{2.14}$$

if and only if $p \equiv r+s \bmod (n+1)$ (cf. [DL]).

If we take $u = e^{\alpha}$ and $w_1 = e^{\lambda_r}$ (for $r = 1, \ldots, n$) in the Jacobi identity (2.13) and apply Res_{x_0} (the formal residue operator, giving us the coefficient of x_0^{-1}), we have

$$[Y(e^{\alpha},x_1),\mathcal{Y}(e^{\lambda_r},x_2)] = 0, \tag{2.15}$$

whenever $\alpha \in \Delta_+$, which means that each coefficient of the series $\mathcal{Y}(e^{\lambda_r},x)$ commutes with the action of $x_\alpha(m)$ for positive roots α.

Given such an intertwining operator, we define a map

$$\mathcal{Y}_c(e^{\lambda_r},x) : L(\Lambda_s) \longrightarrow L(\Lambda_p)$$

by

$$\mathcal{Y}_c(e^{\lambda_r}, x) = \mathrm{Res}_x x^{-1-\langle \lambda_r, \lambda_s \rangle} \mathcal{Y}(e^{\lambda_r}, x)$$

and by (2.15) we have

$$[Y(e^{\alpha}, x_1), \mathcal{Y}_c(e^{\lambda_r}, x_2)] = 0, \tag{2.16}$$

which implies

$$[x_{\alpha}(m), \mathcal{Y}_c(e^{\lambda_r}, x_2)] = 0 \tag{2.17}$$

for each $m \in \mathbb{Z}$.

Consider the space

$$V_P^{\otimes k} = \underbrace{V_P \otimes \cdots \otimes V_P}_{k \text{ times}}. \tag{2.18}$$

We extend the operators e_{λ}, $\lambda \in P$, to operators on $V_p^{\otimes k}$, k a positive integer, by defining:

$$e_{\lambda}^{\otimes k} = e_{\lambda} \otimes \cdots \otimes e_{\lambda} : V_P^{\otimes k} \to V_P^{\otimes k}.$$

For any standard $\widehat{\mathfrak{sl}(n+1)}$-module $L(\Lambda)$ of positive integral level k, its highest weight Λ is of the form

$$\Lambda = k_0\Lambda_0 + \cdots + k_n\Lambda_n$$

for some nonnegative integers $k_0, \ldots, k_n$ satisfying $k_0 + \cdots + k_n = k$. Any standard $\widehat{\mathfrak{sl}(n+1)}$-module $L(\Lambda)$ of positive integral level k, may be realized as an $\widehat{\mathfrak{sl}(n+1)}$-submodule of $V_P^{\otimes k}$. Indeed, let

$$v_{i_1,\ldots,i_k} = v_{\Lambda_{i_1}} \otimes \cdots \otimes v_{\Lambda_{i_k}} \in V_P^{\otimes k}, \tag{2.19}$$

where exactly k_i indices are equal to i for each $i = 0, \ldots, n$. Then, we have that $v_{i_1,\ldots,i_k}$ is a highest weight vector for $\widehat{\mathfrak{sl}(n+1)}$, and

$$L(\Lambda) \simeq U(\widehat{\mathfrak{sl}(n+1)}) \cdot v_{i_1,\ldots,i_k} \subset V_P^{\otimes k} \tag{2.20}$$

(cf. [K]). Here, the action of $\widehat{\mathfrak{sl}(n+1)}$ on $V_P^{\otimes k}$ is given by the usual diagonal action of a Lie algebra on a tensor product of modules:

$$a \cdot v = \Delta(a)v = (a \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes a)v \tag{2.21}$$

for $a \in \widehat{\mathfrak{sl}(n+1)}$, $v \in V_P^{\otimes k}$ and is extended to $U(\widehat{\mathfrak{sl}(n+1)})$ in the usual way. We also have a natural vertex operator algebra structure on $L(k\Lambda_0)$ and $L(k\Lambda_0)$-module structure on $L(\Lambda)$ given by:

Theorem 2.1.1 *([FZ], [DL], [Li1]; cf. [LL]) The standard module $L(k\Lambda_0)$ has a natural vertex operator algebra structure. The level k standard $\widehat{\mathfrak{sl}(n+1)}$-modules provide a complete list of irreducible $L(k\Lambda_0)$-modules.*

Let ω denote the Virasoro vector in $L(k\Lambda_0)$. We have a natural representation of the Virasoro algebra on each $L(\Lambda)$ given by

$$Y_{L(\Lambda)}(\omega, x) = \sum_{m\in\mathbb{Z}} L(m)x^{-m-2} \tag{2.22}$$

The operators $L(0)$ defined in (2.22) provide each $L(\Lambda)$ of level k with a grading, which we refer to as the *weight* grading:

$$L(\Lambda) = \coprod_{s\in\mathbb{Z}} L(\Lambda)_{(s+h_\Lambda)} \tag{2.23}$$

where $h_\Lambda \in \mathbb{Q}$ and depends on Λ. In particular, we have the grading

$$L(k\Lambda_0) = \coprod_{s\in\mathbb{Z}} L(\Lambda)_{(s)}. \tag{2.24}$$

We denote the weight of an element $a \cdot v_\Lambda \in W(\Lambda)$ by $\mathrm{wt}(a \cdot v_\Lambda)$. We will also write

$$\mathrm{wt}(x_\alpha(m)) = -m,$$

where we view $x_\alpha(m)$ both as an operator and as an element of $U(\bar{\mathfrak{n}})$.

We also have n distinct *charge* gradings on each $L(\Lambda)$ of level k, given by the eigenvalues of the operators $\lambda_i(0)$ for $i = 1, \ldots, n$:

$$L(\Lambda) = \coprod_{r_i\in\mathbb{Z}} L(\Lambda)_{[r_i+\langle\lambda_i,\Lambda\rangle]}. \tag{2.25}$$

We call these the λ_i-*charge* gradings. An element of $L(\Lambda)$ with λ_i-charges n_i for $i = 1, \ldots, n$ has *total charge* $\sum_{i=1}^n n_i$. The gradings (2.23) and (2.25) are compatible, and we have that

$$L(\Lambda) = \coprod_{r_1,\ldots,r_n,s\in\mathbb{Z}} L(\Lambda)_{r_1+\langle\lambda_1,\Lambda\rangle,\ldots,r_n+\langle\lambda_n\Lambda\rangle;s+h_\Lambda}. \tag{2.26}$$

2.2 The case of $\mathfrak{g} = \mathfrak{sl}(3)$

In chapter 4 we work in the case where $n = 2$, and we recall some important details. The finite-dimensional simple Lie algebra $\mathfrak{sl}(3)$ has a standard basis

$$\{h_{\alpha_1},\ h_{\alpha_2},\ x_{\pm\alpha_1},\ x_{\pm\alpha_2},\ x_{\pm(\alpha_1+\alpha_2)}\};$$

we do not need to normalize the root vectors. We fix the Cartan subalgebra

$$\mathfrak{h} = \mathbb{C}h_{\alpha_1} \oplus \mathbb{C}h_{\alpha_2}$$

of $\mathfrak{sl}(3)$. Under our identification of $\mathfrak{h}$ with $\mathfrak{h}^*$, we have

$$\alpha_1 = h_{\alpha_1} \text{ and } \alpha_2 = h_{\alpha_2}.$$

We also have the fundamental weights $\lambda_1, \lambda_2 \in \mathfrak{h}^*$ of $\mathfrak{sl}(3)$, given by the condition $\langle \lambda_i, \alpha_j \rangle = \delta_{i,j}$ for $i, j = 1, 2$. In particular, we have

$$\lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2 \text{ and } \lambda_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$$

and

$$\alpha_1 = 2\lambda_1 - \lambda_2 \text{ and } \alpha_2 = -\lambda_1 + 2\lambda_2.$$

The level 1 standard modules of $\widehat{\mathfrak{sl}(3)}$ are $L(\Lambda_0)$, $L(\Lambda_1)$, and $L(\Lambda_2)$. Given the intertwining operators (2.13), we have that

$$\mathcal{Y}_c(e^{\lambda_i}, x)v_{\Lambda_0} = r_1 v_{\Lambda_i} \tag{2.27}$$

$$\mathcal{Y}_c(e^{\lambda_i}, x)v_{\Lambda_i} = r_2 x_{\alpha_i}(-1) \cdot v_{\Lambda_j} = r_2' e_{\lambda_i} \cdot v_{\Lambda_i} \tag{2.28}$$

$$\mathcal{Y}_c(e^{\lambda_i}, x)v_{\Lambda_j} = r_3 x_{\alpha_1+\alpha_2}(-1) \cdot v_{\Lambda_0} = r_3' e_{\lambda_i} \cdot v_{\Lambda_j} \tag{2.29}$$

for $i, j = 1, 2$, $i \neq j$ and some constants $r_1, r_2, r_3, r_2', r_3' \in \mathbb{C}^\times$.

For any level k standard $\widehat{\mathfrak{sl}(3)}$-module $L(\Lambda)$, its highest weight Λ is of the form

$$\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$$

for some nonnegative integers k_0, k_1, k_2 satisfying $k_0 + k_1 + k_2 = k$. We now give a realization of these modules. Consider the space

$$V_P^{\otimes k} = \underbrace{V_P \otimes \cdots \otimes V_P}_{k \text{ times}}, \tag{2.30}$$

and let

$$v_{i_1,\dots,i_k} = v_{\Lambda_{i_1}} \otimes \cdots \otimes v_{\Lambda_{i_k}} \in V_P^{\otimes k}, \tag{2.31}$$

where exactly k_0 indices are equal to 0, k_1 indices are equal to 1 and k_2 indices are equal to 2. Then, we have that $v_{i_1,\dots,i_k}$ is a highest weight vector for $\widehat{\mathfrak{sl}(3)}$, and

$$L(\Lambda) \simeq U(\widehat{\mathfrak{sl}(3)}) \cdot v_{i_1,\dots,i_k} \subset V_P^{\otimes k} \tag{2.32}$$

(cf. [K]).

The operators $L(0)$ defined in (2.22) provide each $L(\Lambda)$ of level k with a grading, which we refer to as the *weight* grading:

$$L(\Lambda) = \coprod_{s\in\mathbb{Z}} L(\Lambda)_{(s+h_\Lambda)} \tag{2.33}$$

where

$$h_\Lambda = \frac{\langle \Lambda, \Lambda + \alpha_1 + \alpha_2 \rangle}{2(k+3)}.$$

In particular, we have the grading

$$L(k\Lambda_0) = \coprod_{s\in\mathbb{Z}} L(\Lambda)_{(s)}. \tag{2.34}$$

We denote the weight of an element $a \cdot v_\Lambda \in W(\Lambda)$ by $\mathrm{wt}(a \cdot v_\Lambda)$. We will also write

$$\mathrm{wt}(x_\alpha(m)) = -m,$$

where we view $x_\alpha(m)$ both as an operator and as an element of $U(\bar{\mathfrak{n}})$.

We also have two different *charge* gradings on each $L(\Lambda)$ of level k, given by the eigenvalues of the operators $\lambda_1(0)$ and $\lambda_2(0)$:

$$L(\Lambda) = \coprod_{r_i\in\mathbb{Z}} L(\Lambda)_{[r_i+\langle\lambda_i,\Lambda\rangle]} \tag{2.35}$$

for each $i = 1, 2$. We call these the λ_1-*charge* and λ_2-*charge* gradings, respectively. An element of $L(\Lambda)$ with λ_1-charge n_1 and λ_2-charge n_2 is said to have *total charge* n_1+n_2. The gradings (2.23) and (2.25) are compatible, and we have that

$$L(\Lambda) = \coprod_{r_1,r_2,s\in\mathbb{Z}} L(\Lambda)_{r_1+\langle\lambda_1,\Lambda\rangle, r_2+\langle\lambda_2\Lambda\rangle; s+h_\Lambda}. \tag{2.36}$$

Chapter 3

Principal subspaces of standard modules

3.1 General definitions

We are now ready to define our main object of study. Consider the $\widehat{\mathfrak{sl}(n+1)}$-subalgebra

$$\bar{\mathfrak{n}} = \mathfrak{n} \otimes \mathbb{C}[t, t^{-1}]. \tag{3.1}$$

The Lie algebra $\bar{\mathfrak{n}}$ has the following important subalgebras:

$$\bar{\mathfrak{n}}_- = \mathfrak{n} \otimes t^{-1}\mathbb{C}[t^{-1}]$$

and

$$\bar{\mathfrak{n}}_+ = \mathfrak{n} \otimes \mathbb{C}[t]$$

Let $U(\bar{\mathfrak{n}})$ be the universal enveloping algebra of $\bar{\mathfrak{n}}$. We recall that $U(\bar{\mathfrak{n}})$ has the decomposition:

$$U(\bar{\mathfrak{n}}) = U(\bar{\mathfrak{n}}_-) \oplus U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+. \tag{3.2}$$

Given a $\widehat{\mathfrak{sl}(n+1)}$-module $L(\Lambda)$ of positive integral level k with highest weight vector v_Λ, the *principal subspace* of $L(\Lambda)$ is defined by:

$$W(\Lambda) = U(\bar{\mathfrak{n}}) \cdot v_\Lambda.$$

$W(\Lambda)$ inherits the grading (2.36), and we have that

$$W(\Lambda) = \coprod_{r_1,\ldots,r_n,s\in\mathbb{Z}} W(\Lambda)_{r_1+\langle\lambda_1,\Lambda\rangle,\ldots,r_n+\langle\lambda_n\Lambda\rangle;s+h_\Lambda} \tag{3.3}$$

For convenience, we will use the notation

$$W(\Lambda)'_{r_1,\ldots,r_n;s} = W(\Lambda)_{r_1+\langle\lambda_1,\Lambda\rangle,\ldots,r_n+\langle\lambda_n\Lambda\rangle;s+h_\Lambda}$$

As in [CLM1]-[CLM2], [CalLM3], [C1]-[C2], define the *multigraded dimension* of $W(\Lambda)$ by:

$$\chi_{W(\Lambda)}(x_1,\ldots,x_n,q) = tr_{W(\Lambda)} x_1^{\lambda_1}\cdots x_n^{\lambda_n} q^{L(0)}$$

and its modification

$$\chi'_{W(\Lambda)}(x_1,\ldots,x_n,q) = x^{-\langle\lambda_1,\Lambda\rangle}\ldots x^{-\langle\lambda_n,\Lambda\rangle} q^{-h_\Lambda}\chi_{W(\Lambda)}(x_1,\ldots,x_n,q) \in \mathbb{C}[[x_1,\ldots x_n,q]]$$

In particular, we have that

$$\chi'_{W(\Lambda)}(x_1,\ldots,x_n,q) = \sum_{r_1,\ldots,r_n,s\in\mathbb{N}} \dim(W(\Lambda)'_{r_1,\ldots,r_n;s}) x^{r_1}\cdots x^{r_n} q^s.$$

For each such Λ, we have a surjective map

$$\begin{aligned} F_\Lambda : U(\widehat{\mathfrak{g}}) &\longrightarrow L(\Lambda) \\ a &\mapsto a\cdot v_\Lambda \end{aligned} \tag{3.4}$$

and its surjective restriction f_Λ:

$$\begin{aligned} f_\Lambda : U(\bar{\mathfrak{n}}) &\longrightarrow W(\Lambda) \\ a &\mapsto a\cdot v_\Lambda. \end{aligned} \tag{3.5}$$

A precise description of the kernels $\mathrm{Ker} f_\Lambda$ for every each $\Lambda = \sum_{i=0}^n k_i\Lambda_i$ gives a presentation of the principal subspaces $W(\Lambda)$ for $\widehat{\mathfrak{sl}(n+1)}$, as we will now discuss.

For each $\lambda \in P$ and character $\nu : Q \longrightarrow \mathbb{C}^\times$, we define a map $\tau_{\lambda,\nu}$ on $\bar{\mathfrak{n}}$ by

$$\tau_{\lambda,\nu}(x_\alpha(m)) = \nu(\alpha)x_\alpha(m - \langle\lambda,\alpha\rangle)$$

for $\alpha \in \Delta_+$ and $m \in \mathbb{Z}$. It is easy to see that $\tau_{\lambda,\nu}$ is an automorphism of $\bar{\mathfrak{n}}$. In the special case when ν is trivial (i.e., $\nu = 1$), we set

$$\tau_\lambda = \tau_{\lambda,1}.$$

The map $\tau_{\lambda,\nu}$ extends canonically to an automorphism of $U(\bar{\mathfrak{n}})$, also denoted by $\tau_{\lambda,\nu}$, given by

$$\tau_{\lambda,\nu}(x_{\beta_1}(m_1)\cdots x_{\beta_r}(m_r)) = \nu(\beta_1+\cdots+\beta_r)x_{\beta_1}(m_1-\langle\lambda,\beta_1\rangle)\cdots x_{\beta_r}(m_r-\langle\lambda,\beta_r\rangle) \tag{3.6}$$

for $\beta_1, \ldots, \beta_r \in \Delta_+$ and $m_1, \ldots, m_r \in \mathbb{Z}$. Notice that if $\lambda = \lambda_i$ for $i = 1, \ldots, n$, we have that

$$\mathrm{wt}(\tau_{-\lambda}(a)) \leq \mathrm{wt}(a)$$

for each $a \in U(\bar{\mathfrak{n}})$. We will use this fact frequently without mention.

Define the formal sums

$$R_t^i = \sum_{m_1+\cdots+m_n=-t} x_{\alpha_i}(m_1)x_{\alpha_i}(m_2)\cdots x_{\alpha_i}(m_{k+1}) \tag{3.7}$$

and their truncations

$$R_{M,t}^i = \sum_{\substack{m_1+\cdots+m_{k+1}=-t,\\ m_1,\ldots,m_{k+1}\leq M}} x_{\alpha_i}(m_1)\cdots x_{\alpha_i}(m_{k+1}) \tag{3.8}$$

for $t \in \mathbb{Z}$, $M \in \mathbb{Z}$ and $i = 1, \ldots, n$. Note that each $R_{M,t}^i \in U(\bar{\mathfrak{n}})$ and the infinite sum $R_t^i \notin U(\bar{\mathfrak{n}})$, but R_t^i is still well-defined as an operator on $W(\Lambda)$, since, when acting on any element of $W(\Lambda)$, only finitely many of its terms are nonzero. Let J be the left ideal of $U(\bar{\mathfrak{n}})$ generated by the elements $R_{-1,t}^i$ for $t \geq k+1$ and $i = 1, 2$:

$$J = \sum_{i=1}^{n} \sum_{t\geq k+1} U(\bar{\mathfrak{n}})R_{-1,t}^i. \tag{3.9}$$

Define a left ideal of $U(\bar{\mathfrak{n}})$ by:

$$I_{k\Lambda_0} = J + U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+$$

and for each $\Lambda = \sum_{i=0}^{n} k_i\Lambda_i$, define

$$I_\Lambda = I_{k\Lambda_0} + \sum_{\alpha\in\Delta_+} U(\bar{\mathfrak{n}})x_\alpha(-1)^{k+1-\langle\alpha,\Lambda\rangle}.$$

Conjecture 3.1.1 *For each $\Lambda = k_0\Lambda_0 + \cdots + k_n\Lambda_n$ with $k_0, \ldots, k_n, k \in \mathbb{N}$, $k \geq 1$, and $k_0 + \cdots + k_n = k$, we have that*

$$\mathrm{Ker} f_\Lambda = I_\Lambda$$

In particular,

$$\mathrm{Ker} f_{k_0\Lambda_0+k_i\Lambda_i} = I_{k\Lambda_0} + U(\bar{\mathfrak{n}})x_{\alpha_i}(-1)^{k_0+1} \tag{3.10}$$

In the case that $\mathfrak{g}$ is of type ADE and $k = 1$ or $\mathfrak{g} = \mathfrak{sl}(2)$ or $\mathfrak{g} = \mathfrak{sl}(3)$ and $k \geq 1$, this conjecture has been proved. The presentations (3.10) are suggested by the bases found in [G], but an a priori proof is lacking. This proof will be the focus of future work.

3.2 Details for the $\widehat{\mathfrak{sl}(3)}$ case

We define certain operators that will be needed for the proof of Conjecture 3.1.1 when $n = 2$. These operators have natural generalization for $n \geq 2$ and generalize the $\tau_{\lambda,\nu}$ maps above. Define the injective maps

$$\begin{aligned} \tau_{\lambda_1,\nu}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2} : U(\bar{\mathfrak{n}}) &\longrightarrow U(\bar{\mathfrak{n}}) \\ a &\mapsto \tau_{\lambda_1,\nu}(a) x_{\alpha_1}(-1)^{k_1} x_{\alpha_1+\alpha_2}(-1)^{k_2}. \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \tau_{\lambda_2,\nu}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2} : U(\bar{\mathfrak{n}}) &\longrightarrow U(\bar{\mathfrak{n}}) \\ a &\mapsto \tau_{\lambda_2,\nu}(a) x_{\alpha_2}(-1)^{k_2} x_{\alpha_1+\alpha_2}(-1)^{k_1}. \end{aligned} \tag{3.12}$$

Let $\omega_i = \alpha_i - \lambda_i \in P$ for $i = 1, 2$. Generalizing the idea of [CalLM3], we define, for each character $\nu : Q \to \mathbb{C}^\times$, injective linear maps

$$\begin{aligned} \sigma_{\omega_1,\nu}^{k_1\Lambda_1+k_2\Lambda_2} : U(\bar{\mathfrak{n}}) &\longrightarrow U(\bar{\mathfrak{n}}) \\ a &\mapsto \tau_{\omega_1,\nu}(a) x_{\alpha_1}(-1)^{k_1}. \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \sigma_{\omega_2,\nu}^{k_1\Lambda_1+k_2\Lambda_2} : U(\bar{\mathfrak{n}}) &\longrightarrow U(\bar{\mathfrak{n}}) \\ a &\mapsto \tau_{\omega_2,\nu}(a) x_{\alpha_2}(-1)^{k_2}. \end{aligned} \tag{3.14}$$

The following facts about $U(\bar{\mathfrak{n}})$ will be useful:

Lemma 3.2.1 *Given $r, k \in \mathbb{N}$ and root vectors $x_\alpha, x_\beta \in \mathfrak{sl}(3)$ with $\alpha, \beta, \alpha + \beta \in \Delta_+$*

and $[x_\alpha, x_\beta] = C_{\alpha,\beta} x_{\alpha+\beta}$ *for some constant* $C_{\alpha,\beta} \in \mathbb{C}^\times$, *we have*

$$\begin{aligned} & x_\beta(m_1)\dots x_\beta(m_r)x_\alpha(-1)^k \\ = & \sum_{p=0}^{k} x_\alpha(-1)^{k-p} \sum_{\substack{j_1,\cdots,j_p=1 \\ j_1<\cdots<j_p}}^{r} C_{j_1,\dots,j_p} x_\beta(m_1)\cdots \\ & \cdots x_{\alpha+\beta}(m_{j_1}-1)\dots x_{\alpha+\beta}(m_{j_p}-1)\cdots x_\beta(m_r) \end{aligned} \tag{3.15}$$

for some constants $C_{j_1,\dots j_p} \in \mathbb{C}$. *The constants* $C_{j_1,\dots j_p}$ *are understood to be* 0 *when* $p > r$.

Proof: We induct on $k \in \mathbb{N}$. For $k = 1$ we have:

$$x_\beta(m_1)\cdots x_\beta(m_r)x_\alpha(-1)$$

$$= x_\alpha(-1)x_\beta(m_1)\cdots x_\beta(m_r) \tag{3.16}$$

$$+\sum_{j=1}^{r} C_{\beta,\alpha} x_\beta(m_1)\cdots x_{\alpha+\beta}(m_j-1)\cdots x_\beta(m_r) \tag{3.17}$$

and so our claim is true for $k = 1$. Assume that our claim is true for some $k \geq 1$. Then we have:

$$
\begin{aligned}
&x_\beta(m_1)\cdots x_\beta(m_r)x_\alpha(-1)^{k+1}\\
&= \sum_{p=0}^{k} x_\alpha(-1)^{k-p} \sum_{\substack{j_1,\dots,j_p=1\\ j_1<\cdots<j_p}}^{r} \Big(C_{j_1,\dots,j_p} x_\beta(m_1)\cdots x_{\alpha+\beta}(m_{j_1}-1)\cdots\\
&\qquad\qquad \cdots x_{\alpha+\beta}(m_{j_p}-1)\cdots x_\beta(m_r)x_\alpha(-1)\Big)\\
&= \sum_{p=0}^{k} x_\alpha(-1)^{k-p+1} \sum_{\substack{j_1,\dots,j_p=1\\ j_1<\cdots<j_p}}^{r} \Big(C_{j_1,\dots,j_p} x_\beta(m_1)\cdots\\
&\qquad\qquad \dots x_{\alpha+\beta}(m_{j_1}-1)\cdots x_{\alpha+\beta}(m_{j_p}-1)\cdots x_\beta(m_r)\Big)\\
&+ \sum_{p=0}^{k} x_\alpha(-1)^{k-p} \sum_{\substack{s\neq j_q, s=1\\ q=1,\dots,p}}^{r} \sum_{\substack{j_1,\dots,j_p=1\\ j_1<\cdots<j_p}}^{r} \Big(C_{j_1,\dots,j_p} C_{\beta,\alpha} x_\beta(m_1)\cdots x_{\alpha+\beta}(m_{j_1}-1)\dots\\
&\qquad\qquad \cdots x_{\alpha+\beta}(m_s-1)\cdots\\
&\qquad\qquad \dots x_{\alpha+\beta}(m_{j_p}-1)\cdots x_\beta(m_r)\Big)\\
&= \sum_{p=0}^{k+1} x_\alpha(-1)^{k+1-p} \sum_{\substack{j_1,\dots,j_p=1\\ j_1<\cdots<j_p}}^{r} C'_{j_1,\dots,j_p} x_\beta(m_1)\cdots x_{\alpha+\beta}(m_{j_1}-1)\cdots\\
&\qquad\qquad \cdots x_{\alpha+\beta}(m_{j_p}-1)\cdots x_\beta(m_r)
\end{aligned}
$$

for some constants $C'_{j_1,\dots,j_p}\in\mathbb{C}$, concluding our proof.

Corollary 3.2.2 *For $0\le m\le k$ and simple roots $\alpha_i,\alpha_j\in\Delta_+$ such that $\alpha_i+\alpha_j\in\Delta_+$, we have*

$$
\begin{aligned}
&R^i_{-1,t}x_{\alpha_j}(-1)^m\\
&= x_{\alpha_j}(-1)^m R^i_{-1,t} + r_1 x_{\alpha_j}(-1)^{m-1}[R^i_{-1,t+1}, x_{\alpha_j}(0)] + \dots\\
&\quad + r_m[\dots[R^i_{-1,t+m}, x_{\alpha_j}(0)],\dots,x_{\alpha_j}(0)] + b x_{\alpha_i+\alpha_j}(-1) + c
\end{aligned}
$$

for some $r_1\dots r_m\in\mathbb{C}, b\in U(\bar{\mathfrak{n}})$, and $c\in U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+$. In particular, we have that

$$R^i_{-1,t}x_{\alpha_j}(-1)^m \in I_{k\Lambda_0} + U(\bar{\mathfrak{n}})x_{\alpha_i+\alpha_j}(-1).$$

Moreover, if $a\in I_{k\Lambda_0}$ then

$$a x_{\alpha_j}(-1)^m \in I_{k\Lambda_0} + U(\bar{\mathfrak{n}})x_{\alpha_i+\alpha_j}(-1).$$

The next two lemmas show that the maps $\tau_{\lambda_i,\nu}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$ and $\sigma_{\omega_i,\nu}^{k_1\Lambda_1+k_2\Lambda_2}$, $i=1,2$, allow us to move between the left ideals we have defined.

Lemma 3.2.3 *For every character ν, we have that*

$$\tau_{\lambda_1,\nu}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(I_{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}) \subset I_{k_2\Lambda_0+k_0\Lambda_1+k_1\Lambda_2}$$

and

$$\tau_{\lambda_2,\nu}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(I_{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}) \subset I_{k_1\Lambda_0+k_2\Lambda_1+k_0\Lambda_2}.$$

Proof: We prove the claim for $\tau_{\lambda_1,\nu}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$. The claim for $\tau_{\lambda_2,\nu}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$ follows similarly. Since $I_{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$ is a homogeneous ideal, it suffices to prove our claim for $\nu=1$.

We have that

$$\begin{aligned}
&\tau_{\lambda_1}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(R^1_{-1,t}) \\
&= \tau_{\lambda_1}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}\left(\sum_{m_1+\ldots+m_{k+1}=-t,\ m_i\leq -1} x_{\alpha_1}(m_1)\cdots x_{\alpha_1}(m_{k+1})\right) \\
&= \sum_{m_1+\cdots+m_{k+1}=-t,\ m_i\leq -1} \tau_{\lambda_1}\Big(x_{\alpha_1}(m_1)\cdots x_{\alpha_1}(m_{k+1})\Big)x_{\alpha_1}(-1)^{k_1}x_{\alpha_1+\alpha_2}(-1)^{k_2} \\
&= \sum_{m_1+\cdots+m_{k+1}=-t,\ m_i\leq -1} x_{\alpha_1}(m_1-1)\cdots x_{\alpha_1}(m_{k+1}-1)x_{\alpha_1}(-1)^{k_1}x_{\alpha_1+\alpha_2}(-1)^{k_2} \\
&= x_{\alpha_1}(-1)^{k_1}x_{\alpha_1+\alpha_2}(-1)^{k_2}R^1_{-1,t+(k+1)} + a x_{\alpha_1}(-1)^{k_1+1}x_{\alpha_1+\alpha_2}(-1)^{k_2} \\
&= x_{\alpha_1}(-1)^{k_1}x_{\alpha_1+\alpha_2}(-1)^{k_2}R^1_{-1,t+(k+1)} + b[x_{\alpha_2}(0),\ldots[x_{\alpha_2}(0),x_{\alpha_1}(-1)^{k_1+k_2+1}]\ldots]
\end{aligned}$$

for some $a,b\in U(\bar{\mathfrak{n}})$. Clearly

$$\begin{aligned}
&x_{\alpha_1}(-1)^{k_1}x_{\alpha_1+\alpha_2}(-1)^{k_2}(-1)R^1_{-1,t+(k+1)} \\
&\quad + s[x_{\alpha_2}(0),\ldots[x_{\alpha_2}(0),x_{\alpha_1}(-1)^{k_1+k_2+1}]\ldots] \in I_{k_2\Lambda_0+k_0\Lambda_1+k_1\Lambda_2}
\end{aligned}$$

and so

$$\tau_{\lambda_1}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(R^1_{-1,t}) \in I_{k_2\Lambda_0+k_0\Lambda_1+k_1\Lambda_2}.$$

We also have

$$\begin{aligned}
&\tau_{\lambda_1}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(R^2_{-1,t}) \\
&= \tau_{\lambda_1}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}\left(\sum_{m_1+\ldots+m_{k+1}=-t,\ m_i\leq -1} x_{\alpha_2}(m_1)\ldots x_{\alpha_2}(m_{k+1})\right) \\
&= \sum_{m_1+\cdots+m_{k+1}=-t,\ m_i\leq -1} \tau_{\lambda_1}\left(x_{\alpha_2}(m_1)\cdots x_{\alpha_2}(m_{k+1})\right)x_{\alpha_1}(-1)^{k_1}x_{\alpha_1+\alpha_2}(-1)^{k_2} \\
&= \sum_{m_1+\cdots+m_{k+1}=-t,\ m_i\leq -1} x_{\alpha_2}(m_1)\cdots x_{\alpha_2}(m_{k+1})x_{\alpha_1}(-1)^{k_1}x_{\alpha_1+\alpha_2}(-1)^{k_2} \\
&= R^2_{-1,t}x_{\alpha_1}(-1)^{k_1}x_{\alpha_1+\alpha_2}(-1)^{k_2} \\
&= a + bx_{\alpha_1+\alpha_2}(-1)^{k_2+1}
\end{aligned}$$

for some $a \in I_{k\Lambda_0}$ and $b \in U(\bar{\mathfrak{n}})$, with the last equality following from Corollary 3.2.2. So we have that

$$\tau_{\lambda_1,\nu}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(R^2_{-1,t}) \in I_{k_2\Lambda_0+k_0\Lambda_1+k_1\Lambda_2}.$$

Since J is the left ideal of $U(\bar{\mathfrak{n}})$ generated by $R^1_{-1,t}$ and $R^2_{-1,t}$, we have that

$$\tau_{\lambda_1}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(J) \subset I_{k_2\Lambda_0+k_0\Lambda_1+k_1\Lambda_2}.$$

We now show that $\tau_{\lambda_1}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+) \subset I_{k_2\Lambda_0+k_0\Lambda_1+k_1\Lambda_2}$. If $m \in \mathbb{N}$, we have have that

$$\begin{aligned}
\tau_{\lambda_1}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(x_{\alpha_1}(m)) &= x_{\alpha_1}(m-1)x_{\alpha_1}(-1)^{k_1}x_{\alpha_1+\alpha_2}(-1)^{k_2} \\
&= x_{\alpha_1}(-1)^{k_1}x_{\alpha_1+\alpha_2}(-1)^{k_2}x_{\alpha_1}(m-1) \in U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+
\end{aligned}$$

if $m > 0$ and

$$\begin{aligned}
\tau_{\lambda_1}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(x_{\alpha_1}(m)) &= x_{\alpha_1}(-1)^{k_1+1}x_{\alpha_1+\alpha_2}(-1)^{k_2} \\
&= r[x_{\alpha_2}(0),\ldots,[x_{\alpha_2}(0),x_{\alpha_1}(-1)^{k_1+k_2+1}]\ldots] \\
&\in I_{k_2\Lambda_0+k_0\Lambda_1+k_1\Lambda_2}
\end{aligned}$$

for some $r \in \mathbb{C}$ if $m = 0$. We also have

$$\begin{aligned}
\tau_{\lambda_1}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(x_{\alpha_2}(m)) &= x_{\alpha_2}(m)x_{\alpha_1}(-1)^{k_1}x_{\alpha_1+\alpha_2}(-1)^{k_2} \\
&= x_{\alpha_1}(-1)^{k_1}x_{\alpha_1+\alpha_2}(-1)^{k_2}x_{\alpha_2}(m) \\
&\quad +rx_{\alpha_1}(-1)^{k_1-1}x_{\alpha_1+\alpha_2}(-1)^{k_2}x_{\alpha_1+\alpha_2}(m-1)
\end{aligned}$$

for some $r \in \mathbb{C}$ and so

$$\tau_{\lambda_1}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(x_{\alpha_2}(m)) \in I_{k_2\Lambda_0+k_0\Lambda_1+k_1\Lambda_2}.$$

Finally, we have that, for $m \geq 0$,

$$\begin{aligned}
\tau_{\lambda_1}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(x_{\alpha_1+\alpha_2}(m)) &= x_{\alpha_1+\alpha_2}(m-1)x_{\alpha_1}(-1)^{k_1}x_{\alpha_1+\alpha_2}(-1)^{k_2} \\
&= x_{\alpha_1}(-1)^{k_1}x_{\alpha_1+\alpha_2}(-1)^{k_2}x_{\alpha_1+\alpha_2}(m-1) \\
&\in I_{k_2\Lambda_0+k_0\Lambda_1+k_1\Lambda_2}.
\end{aligned}$$

Since $U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+$ is a left ideal of $U(\bar{\mathfrak{n}})$, we have that

$$\tau_{\lambda_1,\nu}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+) \subset I_{k_2\Lambda_0+k_0\Lambda_1+k_1\Lambda_2}$$

and so we have

$$\tau_{\lambda_1,\nu}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(I_{k\Lambda_0}) \subset I_{k_2\Lambda_0+k_0\Lambda_1+k_1\Lambda_2}.$$

We now check the remaining terms. We have

$$\begin{aligned}
&\tau_{\lambda_1}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(x_{\alpha_1}(-1)^{k_0+k_2+1}) \\
&= x_{\alpha_1}(-2)^{k_0+k_2+1}x_{\alpha_1}(-1)^{k_1}x_{\alpha_1+\alpha_2}(-1)^{k_2} \\
&= cx_{\alpha_1+\alpha_2}(-1)^{k_2}R^1_{-1,2(k_0+k_2+1)+k_1} + a_1x_{\alpha_1}(-1)^{k_1+1}x_{\alpha_1+\alpha_2}(-1)^{k_2} \\
&= cx_{\alpha_1+\alpha_2}(-1)^{k_2}R^1_{-1,2(k_0+k_2+1)+k_1} + a_2[x_{\alpha_2}(0),\dots[x_{\alpha_2}(0),x_{\alpha_1}(-1)^{k_1+k_2+1}] \\
&\in I_{k_2\Lambda_0+k_0\Lambda_1+k_1\Lambda_2}
\end{aligned}$$

for some $c, a_1, a_2 \in U(\bar{\mathfrak{n}})$. So, since $U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^{k_0+k_2+1}$ is the left ideal of $U(\bar{\mathfrak{n}})$ generated by $x_{\alpha_1}(-1)^{k_0+k_2+1}$, we have that

$$\tau_{\lambda_1,\nu}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(U(\bar{\mathfrak{n}})x_{\alpha}(-1)^{k_0+k_2+1}) \subset I_{k_2\Lambda_0+k_0\Lambda_1+k_1\Lambda_2}.$$

By Lemma 3.2.1 we have

$$
\begin{aligned}
&\tau_{\lambda_1}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(x_{\alpha_2}(-1)^{k_0+k_1+1}) \\
&= x_{\alpha_2}(-1)^{k_0+k_1+1}x_{\alpha_1}(-1)^{k_1}x_{\alpha_1+\alpha_2}(-1)^{k_2} \\
&= x_{\alpha_1}(-1)^{k_1}x_{\alpha_2}(-1)^{k_0+k_1+1}x_{\alpha_1+\alpha_2}(-1)^{k_2} \\
&\quad +r_1x_{\alpha_1}(-1)^{k_1-1}x_{\alpha_1+\alpha_2}(-2)x_{\alpha_2}(-1)^{k_0+k_1}x_{\alpha_1+\alpha_2}(-1)^{k_2} \\
&\quad +\cdots+r_{k_1}x_{\alpha_1+\alpha_2}(-2)^{k_1}x_{\alpha_2}(-1)^{k_0+1}x_{\alpha_1+\alpha_2}(-1)^{k_2} \\
&= r_0'x_{\alpha_1}(-1)^{k_1}[x_{\alpha_1}(0),\dots[x_{\alpha_1}(0),R^2_{-1,k+1}]\dots] \\
&\quad +r_1'x_{\alpha_1}(-1)^{k_1-1}[x_{\alpha_1}(0),\dots[x_{\alpha_1}(0),R^2_{-1,k+2}]\dots] \\
&\quad +\dots r_{k_1}'[x_{\alpha_1}(0),\dots[x_{\alpha_1}(0),R_{-1,2k_1+k_0+k_2+1}]\dots]+ax_{\alpha_1+\alpha_2}(-1)^{k_2+1} \\
&\in I_{k_2\Lambda_0+k_0\Lambda_1k_1\Lambda_2}
\end{aligned}
$$

for some $a\in U(\bar{\mathfrak{n}})$ and $r_0',r_1,r_1',\dots,r_{k_1},r_{k_1}'\in\mathbb{C}$. So, since $U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k_0+k_1+1}$ is the left ideal of $U(\bar{\mathfrak{n}})$ generated by $x_{\alpha_2}(-1)^{k_0+k_1+1}$, we have that

$$\tau_{\lambda_1}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k_0+k_1+1})\subset I_{k_2\Lambda_0+k_0\Lambda_1+k_1\Lambda_2}.$$

Finally, we have that

$$
\begin{aligned}
&\tau_{\lambda_1}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(x_{\alpha_1+\alpha_2}(-1)^{k_0+1}) \\
&= x_{\alpha_1+\alpha_2}(-2)^{k_0+1}x_{\alpha_1}(-1)^{k_1}x_{\alpha_1+\alpha_2}(-1)^{k_2} \\
&= r[x_{\alpha_2}(0),\dots[x_{\alpha_2}(0),R^1_{-1,2k_0+2+k_1+k_2}],\dots]+ax_{\alpha_1+\alpha_2}(-1)^{k_2+1}
\end{aligned}
$$

for some $a\in U(\bar{\mathfrak{n}})$ and some constant $r\in\mathbb{C}$. So, since $U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-1)^{k_0+1}$ is the left ideal of $U(\bar{\mathfrak{n}})$ generated by $x_{\alpha_1+\alpha_2}(-1)^{k_0+1}$, we have that

$$\tau_{\lambda_1}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}(U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-1)^{k_0 1})\subset I_{k_2\Lambda_0+k_0\Lambda_1+k_1\Lambda_2}.$$

This concludes our proof.

Lemma 3.2.4 *For every character ν, we have that*

$$\sigma_{\omega_1,\nu}^{k_1\Lambda_1+k_2\Lambda_2}(I_{k_1\Lambda_1+k_2\Lambda_2})\subset I_{k_1\Lambda_0+k_2\Lambda_1}$$

and

$$\sigma_{\omega_2,\nu}^{k_1\Lambda_1+k_2\Lambda_2}(I_{k_1\Lambda_1+k_2\Lambda_2})\subset I_{k_2\Lambda_0+k_1\Lambda_2}.$$

Proof: We prove the claim for $\sigma_{\omega_1,\nu}^{k_1\Lambda_1+k_2\Lambda_2}$. The claim for $\sigma_{\omega_2,\nu}^{k_1\Lambda_1+k_2\Lambda_2}$ follows similarly. Since $I_{k_1\Lambda_1+k_2\Lambda_2}$ is a homogeneous ideal, it suffices to prove our claim for $\nu = 1$. We have that

$$\begin{aligned}
&\sigma_{\omega_1}^{k_1\Lambda_1+k_2\Lambda_2}(R^1_{-1,t}) \\
&= \sigma_{\omega_1}^{k_1\Lambda_1+k_2\Lambda_2}\Bigg(\sum_{m_1+\cdots+m_{k+1}=-t,\ m_i\leq -1} x_{\alpha_1}(m_1)\cdots x_{\alpha_1}(m_{k+1})\Bigg) \\
&= \sum_{m_1+\cdots+m_{k+1}=-t,\ m_i\leq -1} \sigma_{\omega_1}\Big(x_{\alpha_1}(m_1)\cdots x_{\alpha_1}(m_{k+1})\Big)x_{\alpha_1}(-1)^{k_1} \\
&= \sum_{m_1+\cdots+m_{k+1}=-t, m_i\leq -1} x_{\alpha_1}(m_1-1)\cdots x_{\alpha_1}(m_{k+1}-1)x_{\alpha_1}(-1)^{k_1} \\
&= R^1_{-1,t+(k+1)} + a x_{\alpha_1}(-1)^{k_1+1}
\end{aligned}$$

for some $a \in U(\bar{\mathfrak{n}})$ and so $\sigma_{\omega_1}^{k_1\Lambda_1+k_2\Lambda_2}(R^1_{-1,t}) \in I_{k_1\Lambda_0+k_2\Lambda_1}$. We also have, by Lemma 3.2.1, that

$$\begin{aligned}
&\sigma_{\omega_1}^{k_1\Lambda_1+k_2\Lambda_2}(R^2_{-1,t}) \\
&= \sigma_{\omega_1}^{k_1\Lambda_1+k_2\Lambda_2}\Bigg(\sum_{m_1+\cdots+m_{k+1}=-t,\ m_i\leq -1} x_{\alpha_2}(m_1)\cdots x_{\alpha_2}(m_{k+1})\Bigg) \\
&= \sum_{m_1+\cdots+m_{k+1}=-t,\ m_i\leq -1} \sigma_{\omega_1}\Big(x_{\alpha_2}(m_1)\cdots x_{\alpha_2}(m_{k+1})\Big)x_{\alpha_1}(-1)^{k_1} \\
&= \sum_{m_1+\cdots+m_{k+1}=-t,\ m_i\leq -1} x_{\alpha_2}(m_1+1)\cdots x_{\alpha_2}(m_{k+1}+1)x_{\alpha_1}(-1)^{k_1} \\
&= \sum_{m_1+\cdots+m_{k+1}=-t,\ m_i\leq -1}\ \sum_{p=0}^{k_1} x_{\alpha_1}(-1)^{k_1-p} \sum_{\substack{j_1,\ldots,j_p=1 \\ j_1<\cdots<j_p}}^{k+1} \Big(C_{j_1,\ldots,j_p} x_{\alpha_2}(m_1+1)\cdots \\
&\cdots \qquad x_{\alpha_1+\alpha_2}(m_{j_1})\cdots x_{\alpha_1+\alpha_2}(m_{j_p})\cdots x_{\alpha_2}(m_{k+1}+1)\Big) \\
&= \sum_{p=0}^{k_1} x_{\alpha_1}(-1)^{k_1-p}[\ldots[R^2_{-1,t-(k+1-p)}, x_{\alpha_1}(0)],\ldots, x_{\alpha_1}(0)] + b x_{\alpha_2}(0)
\end{aligned}$$

for some $b \in U(\bar{\mathfrak{n}})$ and constants $C_{j_1,\ldots,j_p} \in \mathbb{C}$. Since J is the left ideal of $U(\bar{\mathfrak{n}})$ generated by $R^1_{-1,t}$ and $R^2_{-1,t}$, we have that

$$\sigma_{\omega_1,\nu}^{k_1\Lambda_1+k_2\Lambda_2}(J) \subset I_{k_1\Lambda_0+k_2\Lambda_1}.$$

We now show that $\sigma_{\omega_1}^{k_1\Lambda_1+k_2\Lambda_2}(U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+) \subset I_{k_1\Lambda_0+k_2\Lambda_1}$. We have

$$\sigma_{\omega_1}^{k_1\Lambda_1+k_2\Lambda_2}(x_{\alpha_1}(m)) = x_{\alpha_1}(m-1)x_{\alpha_1}(-1)^{k_1} \in U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+ + U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^{k_1+1}$$

$$\begin{aligned}
\sigma_{\omega_1}^{k_1\Lambda_1+k_2\Lambda_2}(x_{\alpha_2}(m)) &= x_{\alpha_2}(m+1)x_{\alpha_1}(-1)^{k_1} \\
&= cx_{\alpha_1}(-1)^{k_1-1}x_{\alpha_1+\alpha_2}(m)+x_{\alpha_1}(-1)^{k_1}x_{\alpha_2}(m+1) \\
&\in U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{\omega_1}^{k_1\Lambda_1+k_2\Lambda_2}(x_{\alpha_1+\alpha_2}(m)) &= x_{\alpha_1+\alpha_2}(m)x_{\alpha_1}(-1)^{k_1} \\
&= x_{\alpha_1}(-1)^{k_1}x_{\alpha_1+\alpha_2}(m) \\
&\in U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+
\end{aligned}$$

for $m \geq 0$. Since $U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+$ is the left ideal of $U(\bar{\mathfrak{n}})$ generated by $\bar{\mathfrak{n}}_+$, we have that

$$\sigma_{\omega_1}^{k_1\Lambda_1+k_2\Lambda_2}(U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+) \subset I_{k_1\Lambda_0+k_2\Lambda_1}$$

and so we have

$$\sigma_{\omega_1,\nu}^{k_1\Lambda_1+k_2\Lambda_2}(I_{k\Lambda_0}) \subset I_{k_1\Lambda_0+k_2\Lambda_1}.$$

We now check the remaining terms. We have

$$\begin{aligned}
\sigma_{\omega_1}^{k_1\Lambda_1+k_2\Lambda_2}(x_{\alpha_1}(-1)^{k_2+1}) &= x_{\alpha_1}(-2)^{k_2+1}x_{\alpha_1}(-1)^{k_1} \\
&= rR^1_{-1,2k_2+2+k_1}+ax_{\alpha_1}(-1)^{k_1+1}
\end{aligned}$$

for some $a \in U(\bar{\mathfrak{n}})$ and $r \in \mathbb{C}$, and so

$$\sigma_{\omega_1}^{k_1\Lambda_1+k_2\Lambda_2}(x_{\alpha_1}(-1)^{k_2+1}) \in I_{k_1\Lambda_0+k_2\Lambda_1}.$$

We also have, by Lemma 3.2.1,

$$\begin{aligned}
\sigma_{\omega_1}^{k_1\Lambda_1+k_2\Lambda_2}(x_{\alpha_2}(-1)^{k_1+1}) &= x_{\alpha_2}(0)^{k_1+1}x_{\alpha_1}(-1)^{k_1} \\
&= x_{\alpha_1}(-1)^{k_1}x_{\alpha_2}(0)^{k_1+1} \\
&\quad +r_1x_{\alpha_1}(-1)^{k_1-1}x_{\alpha_1+\alpha_2}(-1)x_{\alpha_2}(0)^{k_1} \\
&\quad +\cdots+r_{k_1}x_{\alpha_1+\alpha_2}(-1)^{k_1}x_{\alpha_2}(0)
\end{aligned}$$

for some constants $r_1,\ldots,r_{k_1} \in \mathbb{C}$, and so

$$\sigma_{\omega_1}^{k_1\Lambda_1+k_2\Lambda_2}(x_{\alpha_2}(-1)^{k_1+1}) \in U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+ \subset I_{k_1\Lambda_0+k_2\Lambda_1}.$$

Finally,

$$\begin{aligned}\sigma_{\omega_1}^{k_1\Lambda_1+k_2\Lambda_2}(x_{\alpha_1+\alpha_2}(-1)) &= x_{\alpha_1+\alpha_2}(-1)x_{\alpha_1}(-1)^{k_1} \\ &= r[x_{\alpha_2}(0), x_{\alpha_1}(-1)^{k_1+1}]\end{aligned}$$

for some constant $r \in \mathbb{C}$. So we have that

$$\sigma_{\omega_1}^{k_1\Lambda_1+k_2\Lambda_2}(x_{\alpha_1+\alpha_2}(-1)^{k_1+1}) \in U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^{k_1+1} \subset I_{k_1\Lambda_0+k_2\Lambda_1}.$$

This concludes our proof.

Remark 3.2.5 Lemmas 3.2.3 and 3.2.4 here directly generalize Lemma 3.1 and Lemma 3.2 in [CalLM3], respectively. Lemma 3.2.4 in this paper does not have an analogue for $I_{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$, and will be the main reason our proof of the presentations needs ideas other than those found in [CalLM1]-[CalLM3].

Remark 3.2.6 Note that $\tau_{\lambda_i,\nu}^{k\Lambda_0} = \tau_{\lambda_i,\nu}$, so that, as in [CalLM1]-[CalLM3], we have

$$\tau_{\lambda_i,\nu}(I_{k\Lambda_0}) \subset I_{k\Lambda_i}.$$

For any $\lambda \in P$ we have the linear isomorphism

$$e_\lambda : V_P \longrightarrow V_P.$$

In particular, for $i, j = 1, 2$ with $i + j = 3$ we have

$$e_{\lambda_i} \cdot v_{\lambda_0} = v_{\lambda_i}$$

$$e_{\lambda_i} \cdot v_{\lambda_i} = \epsilon(\lambda_i, \lambda_i)x_{\alpha_i}(-1) \cdot v_{\lambda_j}$$

$$e_{\lambda_i} \cdot v_{\lambda_j} = \epsilon(\lambda_i, \lambda_j)x_{\alpha_1+\alpha_2}(-1) \cdot v_{\lambda_0}$$

Since

$$e_{\lambda_i}x_\alpha(m) = c(\alpha, -\lambda_i)x_\alpha(m - \langle\alpha, \lambda_i\rangle)e_{\lambda_i} \quad \text{for } \alpha \in \Delta_+ \text{ and } m \in \mathbb{Z}$$

we have that

$$e_{\lambda_i}(a \cdot v_{\lambda_0}) = \tau_{\lambda_i, c_{-\lambda_i}}(a) \cdot v_{\lambda_i}, \quad a \in U(\bar{\mathfrak{n}}). \tag{3.18}$$

For any $\lambda \in P$, we define linear isomorphisms on $V_P^{\otimes k}$ by

$$e_\lambda^{\otimes k} = \underbrace{e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_1}}_{k \text{ times}} : V_P^{\otimes k} \longrightarrow V_P^{\otimes k}.$$

In particular, we have

$$\begin{aligned} & e_{\lambda_1}^{\otimes k}(\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_0 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}}) \\ = \ & \epsilon(\lambda_1, \lambda_1)^{k_1} \epsilon(\lambda_1, \lambda_2)^{k_2} \frac{1}{k_1!} \frac{1}{k_2!} x_{\alpha_1}(-1)^{k_1} x_{\alpha_1+\alpha_2}(-1)^{k_2} \\ & \cdot(\underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_0 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_2 \text{ times}}). \end{aligned}$$

and

$$\begin{aligned} & e_{\lambda_2}^{\otimes k}(\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_0 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}}) \\ = \ & \epsilon(\lambda_2, \lambda_1)^{k_1} \epsilon(\lambda_2, \lambda_2)^{k_2} \frac{1}{k_1!} \frac{1}{k_2!} x_{\alpha_2}(-1)^{k_2} x_{\alpha_1+\alpha_2}(-1)^{k_1} \\ & \cdot(\underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_0 \text{ times}} \otimes \underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_2 \text{ times}}). \end{aligned}$$

This, along with the fact that

$$e_{\lambda_i}^{\otimes k} x_\alpha(m) = c(\alpha, -\lambda_i) x_\alpha(m - \langle \alpha, \lambda_i \rangle) e_{\lambda_i}^{\otimes k} \quad \text{for } \ \alpha \in \Delta_+ \ \ , i = 1, 2, \text{and } \ m \in \mathbb{Z}$$

gives us

$$\begin{aligned} & e_{\lambda_1}^{\otimes k}(a \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_0 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}})) \\ = \ & \epsilon(\lambda_1, \lambda_1)^{k_1} \epsilon(\lambda_1, \lambda_2)^{k_2} \frac{1}{k_1!} \frac{1}{k_2!} \tau_{\lambda_1, c_{-\lambda_1}}^{k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2}(a) \\ & \cdot(\underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_0 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_2 \text{ times}}). \end{aligned}$$

and

$$
\begin{aligned}
e_{\lambda_2}^{\otimes k}(a \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_0 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}})) \\
= \quad \epsilon(\lambda_1, \lambda_2)^{k_1} \epsilon(\lambda_2, \lambda_2)^{k_2} \frac{1}{k_1!} \frac{1}{k_2!} \tau_{\lambda_2, c_{-\lambda_2}}^{k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2}(a) \\
\cdot (\underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_0 \text{ times}} \otimes \underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_2 \text{ times}}).
\end{aligned}
$$

In particular, the above gives us:

Theorem 3.2.7 *For any $k_0, k_1, k_2 \in \mathbb{N}$ such that $k = k_0 + k_1 + k_2$, we have injective maps*

$$e_{\lambda_1}^{\otimes k} : W(k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2) \longrightarrow W(k_2\Lambda_0 + k_0\Lambda_1 + k_2\Lambda_2)$$

and

$$e_{\lambda_2}^{\otimes k} : W(k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2) \longrightarrow W(k_1\Lambda_0 + k_2\Lambda_1 + k_0\Lambda_2).$$

Remark 3.2.8 Notice that the maps $e_{\lambda_i}^{\otimes k}$, $i = 1, 2$, cyclically permute the weights.

Remark 3.2.9 The maps $e_{\lambda_1}^{\otimes k}$ and $e_{\lambda_2}^{\otimes k}$ were the motivation for the definitions of $\tau_{\lambda_1,\nu}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$ and $\tau_{\lambda_2,\nu}^{k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$, respectively.

Recall that $\omega_i = \alpha_i - \lambda_i$ for $i = 1, 2$. For $i, j = 1, 2$ with $i + j = 3$ we have maps

$$e_{\omega_i} \cdot v_{\lambda_i} = \epsilon(\omega_i, \lambda_i) x_{\alpha_i}(-1) \cdot v_{\lambda_0}$$

$$e_{\omega_i} \cdot v_{\lambda_j} = \epsilon(\omega_i, \lambda_j) v_{\lambda_i}$$

As operators, we have that

$$e_{\omega_i} x_\alpha(m) = c(\alpha, -\omega_i) x_\alpha(m - \langle \alpha, \omega_i \rangle) e_{\omega_i} \quad \text{for } \ \alpha \in \Delta_+ \ \text{ and } \ m \in \mathbb{Z}$$

For any such $\omega_i \in P$, we define linear isomorphisms on $V_P^{\otimes k}$ by

$$e_{\omega_i}^{\otimes k} = \underbrace{e_{\omega_i} \otimes \cdots \otimes e_{\omega_i}}_{k \text{ times}} : V_P^{\otimes k} \longrightarrow V_P^{\otimes k}.$$

In particular, we have

$$e_{\omega_1}^{\otimes k}(\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_1\ \text{times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{k_2\ \text{times}})$$
$$= \epsilon(\omega_1,\lambda_1)^{k_1}\epsilon(\omega_1,\lambda_2)^{k_2}\frac{1}{k_1!}x_{\alpha_1}(-1)^{k_1}\cdot(\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_1\ \text{times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_2\ \text{times}}).$$

and

$$e_{\omega_2}^{\otimes k}(\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_1\ \text{times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{k_2\ \text{times}})$$
$$= \epsilon(\omega_2,\lambda_1)^{k_1}\epsilon(\omega_2,\lambda_2)^{k_2}\frac{1}{k_2!}x_{\alpha_2}(-1)^{k_2}\cdot(\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{k_1\ \text{times}}\otimes\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_2\ \text{times}}).$$

This, along with the fact that

$$e_{\omega_i}^{\otimes k}x_\alpha(m)=c(\alpha,-\omega_i)x_\alpha(m-\langle\alpha,\omega_i\rangle)e_{\omega_i}^{\otimes k}\quad\text{for}\ \ \alpha\in\Delta_+\ \ ,i=1,2,\text{and}\ \ m\in\mathbb{Z}$$

give us

$$e_{\omega_1}^{\otimes k}(a\cdot(\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_1\ \text{times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{k_2\ \text{times}}))$$
$$= \epsilon(\omega_1,\lambda_1)^{k_1}\epsilon(\omega_1,\lambda_2)^{k_2}\frac{1}{k_1!}\sigma_{\omega_1,c_{-\omega_1}}^{k_1\Lambda_1+k_2\Lambda_2}(a)\cdot(\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_1\ \text{times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_2\ \text{times}}).$$

and

$$e_{\omega_2}^{\otimes k}(a\cdot(\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_1\ \text{times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{k_2\ \text{times}}))$$
$$= \epsilon(\omega_2,\lambda_1)^{k_1}\epsilon(\omega_2,\lambda_2)^{k_2}\frac{1}{k_2!}\sigma_{\omega_2,c_{-\omega_2}}^{k_1\Lambda_1+k_2\Lambda_2}(a)\cdot(\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{k_1\ \text{times}}\otimes\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_2\ \text{times}}).$$

In particular, the above gives us:

Theorem 3.2.10 *For any $k_1,k_2\in\mathbb{N}$ such that $k=k_1+k_2$, we have injective maps*

$$e_{\omega_1}^{\otimes k}:W(k_1\Lambda_1+k_2\Lambda_2)\longrightarrow W(k_1\Lambda_0+k_2\Lambda_1)$$

and

$$e_{\omega_2}^{\otimes k} : W(k_1\Lambda_1 + k_2\Lambda_2) \longrightarrow W(k_1\Lambda_1 + k_2\Lambda_0).$$

Remark 3.2.11 The maps $e_{\omega_1}^{\otimes k}$ and $e_{\omega_2}^{\otimes k}$ were the motivation for the definitions of $\sigma_{\omega_1,\nu}^{k_1\Lambda_1+k_2\Lambda_2}$ and $\sigma_{\omega_2,\nu}^{k_1\Lambda_1+k_2\Lambda_2}$, respectively.

Chapter 4

Presentations of the principal subspaces of the standard $\widehat{\mathfrak{sl}(3)}$-modules

4.1 A proof of the presentations

We are now ready to prove Conjecture 3.1.1 in the case where $n = 2$. Recall that for each $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$, we defined

$$I_\Lambda = I_{k\Lambda_0} + \sum_{\alpha\in\Delta_+} U(\bar{\mathfrak{n}})x_\alpha(-1)^{k+1-\langle\alpha,\Lambda\rangle}.$$

Theorem 4.1.1 *Let $k \in \mathbb{N}$. For every $k_0, k_1, k_2 \in \mathbb{N}$ such that $k_0 + k_1 + k_2 = k$ and weight $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$, we have that*

$$\mathrm{Ker} f_\Lambda = I_\Lambda.$$

Proof: The fact that $I_\Lambda \subset \mathrm{Ker} f_\Lambda$ is clear. Indeed, the $(k+1)$-power of each vertex operator $Y(e^{\alpha_j}, x)$, $j = 1, \dots, n$, is equal to 0 on $V_P^{\otimes k}$, and so we have $Y(e^{\alpha_j}, x)^{k+1} = 0$ on each $W(\Lambda)$ of level k. In particular, we have

$$Y(e^{\alpha_j}, x)^{k+1} = \sum_{t\in\mathbb{Z}} R_t^j x^{t-(k+1)}$$

which implies

$$\mathrm{Res}_x x^{-t+k} Y(e^{\alpha_j}, x)^{k+1} \cdot v_\Lambda = R_{-1,t}^j \cdot v_\Lambda = 0,$$

and so we have that $J \subset \mathrm{Ker} f_\Lambda$. The fact that $U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+ \subset \mathrm{Ker} f_\Lambda$ is clear. Finally, the fact that

$$x_{\alpha_1}(-1)^{k_0+k_2+1} \cdot v_\Lambda = 0$$

$$x_{\alpha_2}(-1)^{k_0+k_1+1} \cdot v_\Lambda = 0$$

$$x_{\alpha_1+\alpha_2}(-1)^{k_0+1} \cdot v_\Lambda = 0$$

follow from the fact that, in the level 1 case, we have

$$x_{\alpha_1}(-1) \cdot v_{\Lambda_1} = 0$$

$$x_{\alpha_2}(-1) \cdot v_{\Lambda_2} = 0$$

and

$$Y(e^{\alpha}, x)^2 = 0$$

on V_P for each $\alpha \in \Delta_+$. Thus, we have that $I_\Lambda \subset \mathrm{Ker} f_\Lambda$.

It remains to show that $\mathrm{Ker} f_\Lambda \subset I_\Lambda$. We proceed by contradiction. Consider the set of elements

$$\{a \in U(\bar{\mathfrak{n}}) | a \in \mathrm{Ker} f_\Lambda \text{ and } a \notin I_\Lambda \text{ for some } \Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2\}. \tag{4.1}$$

We may and do assume that homogeneous elements of (4.1) have positive weight (otherwise, if a is such an element with non-positive weight, we have $a \in U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}+ \subset I_{k\Lambda_0}$, by our decomposition (3.2)). Among all elements in (4.1), we look at those of lowest total charge. Among all elements of lowest total charge in (4.1), we choose a nonzero element of lowest weight. We call this element a.

First, we show that $\Lambda \neq k\Lambda_1$ (that is, we show that $a \notin (\mathrm{Ker} f_{k\Lambda_1}) \setminus I_{k\Lambda_1}$). Indeed, suppose that $\Lambda = k\Lambda_1$. Then

$$a \cdot (v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}) = 0$$

We have two cases to consider: when the λ_1-charge of a is nonzero, and when the λ_1-charge of a is zero. If the λ_1-charge of a is nonzero, then we have:

$$a \cdot (v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}) = e_{\lambda_1}^{\otimes k}(\tau_{\lambda_1, c_{-\lambda_1}}^{-1}(a) \cdot (v_{\lambda_0} \otimes \cdots \otimes v_{\lambda_0})) = 0$$

and so, by the injectivity of $e_{\lambda_1}^{\otimes k}$, we have

$$\tau_{\lambda_1, c_{-\lambda_1}}^{-1}(a) \cdot (v_{\lambda_0} \otimes \cdots \otimes v_{\lambda_0}) = 0.$$

Now, we have $\mathrm{wt}(\tau_{\lambda_1, c_{-\lambda_1}}^{-1}(a)) < \mathrm{wt}(a)$, and so by assumption on a, we have that

$$\tau_{\lambda_1, c_{-\lambda_1}}^{-1}(a) \in I_{k\Lambda_0}.$$

But then, by Lemma 3.2.3, we have that $\tau_{\lambda_1,c_{-\lambda_1}}(\tau^{-1}_{\lambda_1,c_{-\lambda_1}}(a)) = a \in I_{k\Lambda_1}$, a contradiction. So a cannot have positive λ_1-charge. Suppose that a has λ_1-charge equal to 0, and so the λ_2-charge of a is positive. In this case, we have that $\tau^{-1}_{\lambda_1,c_{-\lambda_1}}(a)$ is a nonzero constant multiple of a. As before, we have

$$a \cdot (v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}) = e^{\otimes k}_{\lambda_1}(\tau^{-1}_{\lambda_1,c_{-\lambda_1}}(a) \cdot (v_{\lambda_0} \otimes \cdots \otimes v_{\lambda_0})) = 0$$

which implies

$$e^{\otimes k}_{\lambda_1}(a \cdot (v_{\lambda_0} \otimes \cdots \otimes v_{\lambda_0})) = 0.$$

By the injectivity of $e^{\otimes k}_{\lambda_1}$, we have that

$$a \cdot (v_{\lambda_0} \otimes \cdots \otimes v_{\lambda_0}) = 0.$$

Applying the map $\mathcal{Y}_c(e^{\lambda_2}, x)^{\otimes k}$, we have that

$$a \cdot (v_{\lambda_2} \otimes \cdots \otimes v_{\lambda_2}) = 0.$$

This gives that

$$a \cdot (v_{\lambda_2} \otimes \cdots \otimes v_{\lambda_2}) = e^{\otimes k}_{\lambda_2}(\tau^{-1}_{\lambda_2,c_{-\lambda_2}}(a) \cdot (v_{\lambda_0} \otimes \cdots \otimes v_{\lambda_0})) = 0.$$

By injectivity of $e^{\otimes k}_{\lambda_2}$, we have that

$$\tau^{-1}_{\lambda_2,c_{-\lambda_2}}(a) \cdot (v_{\lambda_0} \otimes \cdots \otimes v_{\lambda_0}) = 0$$

Since $\mathrm{wt}(\tau^{-1}_{\lambda_2,c_{-\lambda_2}}(a)) < \mathrm{wt}(a)$, we have that $\tau^{-1}_{\lambda_2,c_{-\lambda_2}}(a) \in I_{k\Lambda_0}$, and so

$$\tau_{\lambda_2,c_{-\lambda_2}}(\tau^{-1}_{\lambda_2,c_{-\lambda_2}}(a)) = a \in I_{k\Lambda_2} = I_{k\Lambda_0} + U(\bar{\mathfrak{n}})x_{\alpha_2}(-1).$$

So we may write

$$a = b_1 + c_1 x_{\alpha_2}(-1)$$

for some $b_1 \in I_{k\Lambda_0}$ and $c_1 \in U(\bar{\mathfrak{n}})$. Since $a \notin I_{k\Lambda_1}$, we have that $c_1 \neq 0$ (otherwise, $a = b_1 \in I_{k\Lambda_0} \subset I_{k\Lambda_1}$, which is a contradiction). So we have

$$c_1 x_{\alpha_2}(-1) \cdot (v_{\lambda_0} \otimes \cdots \otimes v_{\lambda_0}) = (a - b_1) \cdot (v_{\lambda_0} \otimes \cdots \otimes v_{\lambda_0}) = 0$$

Applying $\mathcal{Y}_c(e_1^{\lambda}, x) \otimes \mathcal{Y}_c(e^{\lambda_2}, x)^{\otimes k-1}$, we have

$$c_1 x_{\alpha_2}(-1) \cdot (v_{\lambda_1} \otimes v_{\lambda_2} \otimes \cdots \otimes v_{\lambda_2}) = 0$$

which implies

$$c_1 \cdot (e_{\lambda_2} v_{\lambda_2} \otimes v_{\lambda_2} \otimes \cdots \otimes v_{\lambda_2}) = e_{\lambda_2}^{\otimes k}(\tau_{\lambda_2, c_{-\lambda_2}}^{-1}(c_1) \cdot (v_{\lambda_2} \otimes v_{\lambda_0} \otimes \cdots \otimes v_{\lambda_0})) = 0.$$

By the injectivity of $e_{\lambda_2}^{\otimes k}$, we have that

$$\tau_{\lambda_2, c_{-\lambda_2}}^{-1}(c_1) \cdot (v_{\lambda_2} \otimes v_{\lambda_0} \otimes \cdots \otimes v_{\lambda_0}) = 0.$$

Since the total charge of $\tau_{\lambda_2, c_{-\lambda_2}}^{-1}(c_1)$ is less than the total charge of a, we have that

$$\tau_{\lambda_2, c_{-\lambda_2}}^{-1}(c_1) \in I_{(k-1)\Lambda_0 + \Lambda_2}.$$

Applying $\tau_{\lambda_2, c_{-\lambda_2}}^{(k-1)\Lambda_0 + \Lambda_2}$, we have

$$\begin{aligned} \tau_{\lambda_2, c_{-\lambda_2}}^{(k-1)\Lambda_0 + \Lambda_2}(\tau_{\lambda_2, c_{-\lambda_2}}^{-1}(c_1)) &= c_1 x_{\alpha_2}(-1) \\ &\in I_{\Lambda_1 + (k-1)\Lambda_2} \end{aligned}$$

by Lemma 3.2.3. So we have that

$$c_1 x_{\alpha_2}(-1) \in I_{k\Lambda_0} + U(\bar{\mathfrak{n}}) x_{\alpha_2}(-1)^2 + U(\bar{\mathfrak{n}}) x_{\alpha_1}(-1)^k + U(\bar{\mathfrak{n}}) x_{\alpha_1+\alpha_2}(-1).$$

Since the λ_1-charge of a is 0, we may write

$$c_1 x_{\alpha_2}(-1) = b' + c' x_{\alpha_2}(-1)^2$$

for some $b' \in I_{k\Lambda_0}$ and $c' \in U(\bar{\mathfrak{n}})$. Thus, we have that

$$a = b + b' + c' x_{\alpha_2}(-1)^2 = b_2 + c_2 x_{\alpha_2}(-1)^2,$$

where we set $b_2 = b + b' \in I_{k\Lambda_0}$ and $c_2 = c'$. Continuing in this way, and applying the operator $\mathcal{Y}_c(e^{\lambda_1}, x)^{\otimes j} \otimes \mathcal{Y}_c(e^{\lambda_2}, x)^{\otimes (k-j)}$ at each step, we eventually obtain that

$$a = b_k + c_k x_{\alpha_2}(-1)^k$$

for some $b_k \in I_{k\Lambda_0}$ and $c_k \in U(\bar{\mathfrak{n}})$. We have that

$$c_k x_{\alpha_2}(-1)^k \cdot (v_{\lambda_0} \otimes \cdots \otimes v_{\lambda_0}) = (a - b) \cdot (v_{\lambda_0} \otimes \cdots \otimes v_{\lambda_0}) = 0,$$

which implies

$$e_{\omega_2}^{\otimes k} e_{\lambda_2}^{\otimes k}(\tau_{\alpha_2,c_{-\alpha_2}}^{-1}(c_k)\cdot(v_{\lambda_0}\otimes\cdots\otimes v_{\lambda_0}))$$
$$= \epsilon(\omega_2,\lambda_2)^k c_k x_{\alpha_2}(-1)^k\cdot(v_{\lambda_0}\otimes\cdots\otimes v_{\lambda_0}) = 0.$$

By injectivity of both $e_{\lambda_2}^{\otimes k}$ and $e_{\omega_2}^{\otimes k}$, we have that $\tau_{\alpha_2,c_{-\alpha_2}}^{-1}(c_k)(v_{\lambda_0}\otimes\cdots\otimes v_{\lambda_0}) = 0$. Since $\tau_{\alpha_2,c_{-\alpha_2}}^{-1}(c_k)$ is of lower total charge than a, we have that $\tau_{\alpha_2,c_{-\alpha_2}}^{-1}(c_k) \in I_{k\Lambda_0}$. Thus, by Lemmas 3.2.3 and 3.2.4, we have that

$$(\sigma_{\omega_2,c_{-\omega_2}}\circ\tau_{\lambda_2,c_{-\lambda_2}})(\tau_{\alpha_2,c_{-\alpha_2}}^{-1}(c_k)) = c_k x_{\alpha_2}(-1)^k \in I_{k\Lambda_0}.$$

Hence

$$a = b_k + c_k x_{\alpha_2}(-1)^k \in I_{k\Lambda_0} \subset I_{k\Lambda_1},$$

a contradiction. So we have that $\Lambda \neq k\Lambda_1$. Similarly, we have that $\Lambda \neq k\Lambda_2$.

We now show that $\Lambda \neq k_0\Lambda_0 + k_1\Lambda_1$ for some $k_0, k_1 \in \mathbb{N}$ with $k_0 + k_1 = k$. We've already shown this for $k_0 = 0$, so we proceed by induction on k_0 as k_0 ranges from 0 to k. Suppose we've shown that $\Lambda \neq k_0\Lambda_0 + k_1\Lambda_1$ for some $k_0, k_1 \in \mathbb{N}$ with $k_0 + k_1 = k$ and $k_1 > 1$, and suppose $\Lambda = (k_0+1)\Lambda_0 + (k_1-1)\Lambda_1$. In this case, we have that

$$a\cdot(\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_0+1\ \text{times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_1-1\ \text{times}}) = 0.$$

Applying the operator $1^{\otimes k_0}\otimes\mathcal{Y}_c(e^{\lambda_1},x)\otimes 1^{\otimes k_1-1}$, we have that

$$a\cdot(\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_0\ \text{times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_1\ \text{times}}) = 0.$$

By our inductive hypothesis, we have that

$$a \in I_{k_0\Lambda_0+k_1\Lambda_1} = I_{k\Lambda_0} + U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^{k_0+1},$$

so we may write

$$a = b + cx_{\alpha_1}(-1)^{k_0+1}$$

for some $b \in I_{k\Lambda_0}$ and $c \in U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^{k_0+1}$. Here, $c \neq 0$ (otherwise, $a = b \in I_{k\Lambda_0}$, a contradiction). So we have that

$$\begin{aligned} & c x_{\alpha_1}(-1)^{k_0+1} \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_0+1 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1-1 \text{ times}}) \\ = \quad & (a-b) \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_0+1 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1-1 \text{ times}}) \\ = \quad & 0 \end{aligned}$$

and this, using the diagonal action (2.21) of $x_{\alpha_1}(-1)^{k_0+1}$, implies that

$$c \cdot (\underbrace{x_{\alpha_1}(-1)v_{\Lambda_0} \otimes \cdots \otimes x_{\alpha_1}(-1)v_{\Lambda_0}}_{k_0+1 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1-1 \text{ times}}) = 0.$$

We may rewrite this as

$$\begin{aligned} & c \cdot (\underbrace{x_{\alpha_1}(-1)v_{\Lambda_0} \otimes \cdots \otimes x_{\alpha_1}(-1)v_{\Lambda_0}}_{k_0+1 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1-1 \text{ times}}) \\ = \quad & \epsilon(\omega_1, \lambda_1)^{-(k_0+1)} \epsilon(\omega_1, \lambda_2)^{-(k_1-1)} e_{\omega_1}^{\otimes k} \Bigg(\tau_{\omega_1, c_{-\omega_1}}^{-1}(c) \cdot \\ & \qquad \cdot (\underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_0+1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_1-1 \text{ times}}) \Bigg) \\ = \quad & 0 \end{aligned}$$

and so, by the injectivity of $e_{\omega_1}^{\otimes k}$, we have that

$$\tau_{\omega_1, c_{-\omega_1}}^{-1}(c) \cdot (\underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_0+1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_1-1 \text{ times}}) = 0.$$

Now, since the total charge of $\tau_{\omega_1, c_{-\omega_1}}^{-1}(c)$ is less than the total charge of a, we have that $\tau_{\omega_1, c_{-\omega_1}}^{-1}(c) \in I_{(k_0+1)\Lambda_1 + (k_1-1)\Lambda_2}$. Using Lemma 3.2.4, we obtain

$$\sigma_{\omega_1, c_{-\omega_1}}^{(k_0+1)\Lambda_1 + (k_1-1)\Lambda_2}(\tau_{\omega_1, c_{-\omega_1}}^{-1}(c)) = c x_{\alpha_1}(-1)^{k_0+1} \in I_{(k_0+1)\Lambda_0 + (k_1-1)\Lambda_1}$$

and so

$$a = b + c x_{\alpha_1}(-1)^{k_0+1} \in I_{(k_0+1)\Lambda_0 + (k_1-1)\Lambda_1},$$

a contradiction. Hence, $\Lambda \neq (k_0+1)\Lambda_0 + (k_1-1)\Lambda_1$, completing our induction.

This gives us that $\Lambda \neq k_0\Lambda_0 + k_1\Lambda_1$ for any choice of $k_0, k_1 \in \mathbb{N}$ with $k_0 + k_1 = k$. A similar argument shows that $\Lambda \neq k_0\Lambda_0 + k_2\Lambda_2$ for any choice of $k_0, k_2 \in \mathbb{N}$ with $k_0 + k_2 = k$. We now proceed to show that $\Lambda \neq k_1\Lambda_1 + k_2\Lambda_2$ for any choice of $k_1, k_2 \in \mathbb{N}$ with $k_1 + k_2 = k$.

Suppose, for contradiction, that $\Lambda = k_1\Lambda_1 + k_2\Lambda_2$ for some $k_1, k_2 \in \mathbb{N}$ with $k_1 + k_2 = k$. We show by induction that, given $1 \leq j \leq k_2$, a can be written in the form

$$a = b + cx_{\alpha_1}(-1)^j$$

for some $b \in I_{k_1\Lambda_1 + k_2\Lambda_2}$ and $c \in U(\bar{\mathfrak{n}})$. First, we prove the claim for $j = 1$. We have that

$$a \cdot (\underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}}) = 0.$$

Applying the operator $1^{\otimes k_1} \otimes \mathcal{Y}_c(e^{\lambda_1}, x)^{\otimes k_2}$, we have that

$$a \cdot (\underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{e_{\lambda_1}v_{\Lambda_2} \otimes \cdots \otimes e_{\lambda_1}v_{\Lambda_2}}_{k_2 \text{ times}}) = 0$$

so that

$$\begin{aligned} & a \cdot (\underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{e_{\lambda_1}v_{\Lambda_2} \otimes \cdots \otimes e_{\lambda_1}v_{\Lambda_2}}_{k_2 \text{ times}}) \\ = \; & e_{\lambda_1}^{\otimes k}(\tau_{\lambda_1, c_{-\lambda_1}}^{-1}(a) \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}})) \\ = \; & 0. \end{aligned}$$

Since $e_{\lambda_1}^{\otimes k}$ is injective, we have that

$$\tau_{\lambda_1, c_{-\lambda_1}}^{-1}(a) \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}}) = 0.$$

Now, $\tau_{\lambda_1, c_{-\lambda_1}}^{-1}(a)$ has the same total charge as a, and satisfies $\mathrm{wt}(\tau_{\lambda_1, c_{-\lambda_1}}^{-1}(a)) \leq \mathrm{wt}(a)$. Since we've shown that $\Lambda = k_1\Lambda_0 + k_2\Lambda_2$ does not give the element smallest weight among those of smallest total charge in (4.1), we have that

$$\tau_{\lambda_1, c_{-\lambda_1}}^{-1}(a) \in I_{k_1\Lambda_0 + k_2\Lambda_2} = I_{k\Lambda_0} + U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k_1+1}.$$

Applying $\tau_{\lambda_1,c_{-\lambda_1}}$ and using Lemma 3.2.3, we see that

$$a \in I_{k\Lambda_1} + U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k_1+1} = I_{k\Lambda_0} + U(\bar{\mathfrak{n}})x_{\alpha_1}(-1) + U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k_1+1}.$$

So we may write

$$a = b + cx_{\alpha_1}(-1)$$

for some $b \in I_{k\Lambda_0} + U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k_1+1} \subset I_{k_1\Lambda_1+k_2\Lambda_2}$ and $c \in U(\bar{\mathfrak{n}})$ and our claim holds for $j = 1$. Now, suppose for induction that we may write

$$a = b + cx_{\alpha_1}(-1)^j$$

for some $b \in I_{k_1\Lambda_1+k_2\Lambda_2}, c \in U(\bar{\mathfrak{n}}), 1 \leq j \leq k_2 - 1$. We have

$$\begin{aligned} & cx_{\alpha_1}(-1)^j \cdot (\underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}}) \\ = \quad & (a-b) \cdot (\underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}}) \\ = \quad & 0. \end{aligned}$$

Applying the operator $1^{\otimes k_1} \otimes \mathcal{Y}_c(e^{\lambda_1}, x)^{k_2-j} \otimes 1^{\otimes j}$, we have

$$cx_{\alpha_1}(-1)^j \cdot (\underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{e_{\lambda_1}v_{\Lambda_2} \otimes \cdots \otimes e_{\lambda_1}v_{\Lambda_2}}_{k_2-j \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{j \text{ times}}) = 0$$

which, using the diagonal action (2.21), implies

$$c \cdot (\underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{e_{\lambda_1}v_{\Lambda_2} \otimes \cdots \otimes e_{\lambda_1}v_{\Lambda_2}}_{k_2-j \text{ times}} \otimes \underbrace{e_{\lambda_1}v_{\Lambda_1} \otimes \cdots \otimes e_{\lambda_1}v_{\Lambda_1}}_{j \text{ times}}) = 0$$

so that

$$\begin{aligned} & c \cdot (\underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{e_{\lambda_1}v_{\Lambda_2} \otimes \cdots \otimes e_{\lambda_1}v_{\Lambda_2}}_{k_2-j \text{ times}} \otimes \underbrace{e_{\lambda_1}v_{\Lambda_1} \otimes \cdots \otimes e_{\lambda_1}v_{\Lambda_1}}_{j \text{ times}}) \\ = \quad & e_{\lambda_1}^{\otimes k}(\tau_{\lambda_1,c_{-\lambda_1}}^{-1}(c) \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2-j \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{j \text{ times}})) \\ = \quad & 0. \end{aligned}$$

Since $e_{\lambda_1}^{\otimes k}$ is injective, we have

$$\tau_{\lambda_1,c_{-\lambda_1}}^{-1}(c) \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2-j \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{j \text{ times}}) = 0.$$

Now, since the total charge of $\tau^{-1}_{\lambda_1, c_{-\lambda_1}}(c)$ is less than the total charge of a, we have that

$$\tau^{-1}_{\lambda_1, c_{-\lambda_1}}(c) \in I_{k_1\Lambda_0 + j\Lambda_1 + (k_2 - j)\Lambda_2}.$$

Recall

$$\begin{aligned} I_{k_1\Lambda_0 + j\Lambda_1 + (k_2-j)\Lambda_2} &= I_{k\Lambda_0} + U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^{k_1+k_2-j+1} + U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k_1+j+1} \\ &\quad + U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-1)^{k_1+1} \end{aligned}$$

so that, applying $\tau_{\lambda_1, c_{-\lambda_1}}$, Lemma 3.2.3 gives us

$$c \in I_{k\Lambda_1} + U(\bar{\mathfrak{n}})x_{\alpha_1}(-2)^{k_1+k_2-j+1} + U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k_1+j+1} + U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-2)^{k_1+1}.$$

So we may write

$$c = c_1 + c_2 x_{\alpha_1}(-1) + c_3 x_{\alpha_1}(-2)^{k_1+k_2-j+1} + c_4 x_{\alpha_2}(-1)^{k_1+j+1} + c_5 x_{\alpha_1+\alpha_2}(-2)^{k_1+1}$$

so that

$$\begin{aligned} c x_{\alpha_1}(-1)^j &= c_1 x_{\alpha_1}(-1)^j + c_2 x_{\alpha_1}(-1)^{j+1} + c_3 x_{\alpha_1}(-2)^{k_1+k_2-j+1} x_{\alpha_1}(-1)^j \\ &\quad + c_4 x_{\alpha_2}(-1)^{k_1+j+1} x_{\alpha_1}(-1)^j + c_5 x_{\alpha_1+\alpha_2}(-2)^{k_1+1} x_{\alpha_1}(-1)^j \end{aligned}$$

for some $c_1 \in I_{k\Lambda_0}$ and $c_2, c_3, c_4, c_5 \in U(\bar{\mathfrak{n}})$. We now analyze each of these terms. By Corollary 3.2.2, we have that $c_1 x_{\alpha_1}(-1)^j$ is of the form $c_1' + c_1'' x_{\alpha_1+\alpha_2}(-1)$ for some $c_1' \in I_{k\Lambda_0}, c_1'' \in U(\bar{\mathfrak{n}})$, so that in particular $c_1 x_{\alpha_1}(-1)^j \in I_{k_1\Lambda_1 + k_2\Lambda_2}$. Clearly $c_2 x_{\alpha_1}(-1)^{j+1}$ is of the desired form. For $c_3 x_{\alpha_1}(-2)^{k_1+k_2-j+1} x_{\alpha_1}(-1)^j$, we have that

$$c_3 x_{\alpha_1}(-2)^{k_1+k_2-j+1} x_{\alpha_1}(-1)^j = c_3' R^1_{-1, 2(k_1+k_2-j+1)+j} + c_3'' x_{\alpha_1}(-1)^{j+1}$$

for some $c_3', c_3'' \in U(\bar{\mathfrak{n}})$ which means that $c_3 x_{\alpha_1}(-2)^{k_1+k_2-j+1} x_{\alpha_1}(-1)^j$ is of the desired form. For $c_4 x_{\alpha_2}(-1)^{k_1+j+1}$, we use Lemma 3.2.1 to obtain

$$\begin{aligned} & c_4 x_{\alpha_2}(-1)^{k_1+j+1} x_{\alpha_1}(-1)^j \\ &= c_4 (x_{\alpha_1}(-1)^j x_{\alpha_2}(-1)^{k_1+j+1} + n_1 x_{\alpha_1}(-1)^{j-1} x_{\alpha_1+\alpha_2}(-2) x_{\alpha_2}(-1)^{k_1+j} \\ &\quad + \cdots + n_j x_{\alpha_1+\alpha_2}(-1)^j x_{\alpha_2}(-1)^{k_1+1}) \end{aligned}$$

for some constants $n_1, \dots, n_j \in \mathbb{C}$, which is clearly an element of $I_{k_1\Lambda_1+k_2\Lambda_2}$. Finally, for $c_5 x_{\alpha_1+\alpha_2}(-2)^{k_1+1} x_{\alpha_1}(-1)^j$, again using Lemma 3.2.1, we have that

$$\begin{aligned}
&c_5 x_{\alpha_1+\alpha_2}(-2)^{k_1+1} x_{\alpha_1}(-1)^j \\
= \;& c_5(x_{\alpha_1}(-1)^{k_1+1+j} x_{\alpha_2}(-1)^{k_1+1} - x_{\alpha_2}(-1)^{k_1+1} x_{\alpha_1}(-1)^{k_1+1+j} \\
& + n_1 x_{\alpha_2}(-1)^{k_1} x_{\alpha_1+\alpha_2}(-2) x_{\alpha_1}(-1)^{k_1+j} + \dots \\
& + n_{k_1} x_{\alpha_2}(-1) x_{\alpha_1+\alpha_2}(-2)^{k_1} x_{\alpha_1}(-1)^{j+1})
\end{aligned}$$

for some constants $n_1, \dots, n_{k_1} \in \mathbb{C}$, which is clearly of the desired form. This completes our induction. In particular, we have that a has the form $b + c x_{\alpha_1}(-1)^{k_2}$ for some $b \in I_{k_1\Lambda_1+k_2\Lambda_2}$ and $c \in U(\bar{\mathfrak{n}})$. This gives us

$$\begin{aligned}
& c x_{\alpha_1}(-1)^{k_2} \cdot (\underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}}) \\
= \;& (a-b) \cdot (\underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}}) \\
= \;& 0.
\end{aligned}$$

which, using the diagonal action (2.21), implies that

$$\begin{aligned}
& c \cdot (\underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{e_{\lambda_1} v_{\Lambda_1} \otimes \cdots \otimes e_{\lambda_1} v_{\Lambda_1}}_{k_2 \text{ times}}) \\
= \;& e_{\lambda_1}^{\otimes k} (\tau_{\lambda_1, c_{-\lambda_1}}^{-1}(c) \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_2 \text{ times}})) \\
= \;& 0.
\end{aligned}$$

By the injectivity of $e_{\lambda_1}^{\otimes k}$, we have that

$$\tau_{\lambda_1, c_{-\lambda_1}}^{-1}(c) \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_2 \text{ times}}) = 0$$

Now, since $\tau_{\lambda_1, c_{-\lambda_1}}^{-1}(c)$ has lower total charge than a, we have that

$$\tau_{\lambda_1, c_{-\lambda_1}}^{-1}(c) \in I_{k_1\Lambda_0+k_2\Lambda_1} = I_{k\Lambda_0} + U(\bar{\mathfrak{n}}) x_{\alpha_1}(-1)^{k_1+1}.$$

So, by Lemma 3.2.3, we have that

$$c \in I_{k\Lambda_1} + U(\bar{\mathfrak{n}}) x_{\alpha_1}(-2)^{k_1+1}.$$

We may thus write

$$c = c_1 + c_2 x_{\alpha_1}(-1) + c_3 x_{\alpha_1}(-2)^{k_1+1}$$

for some $c_1 \in I_{k\Lambda_0}$ and $c_2, c_3 \in U(\bar{\mathfrak{n}})$ and so

$$cx_{\alpha_1}(-1)^{k_2} = c_1 x_{\alpha_1}(-1)^{k_2} + c_2 x_{\alpha_1}(-1)^{k_2+1} + c_3 x_{\alpha_1}(-2)^{k_1+1} x_{\alpha_1}(-1)^{k_2}.$$

By Lemma 3.2.2, we have that

$$c_1 x_{\alpha_1}(-1)^{k_2} \in I_{k\Lambda_0} + U(\bar{\mathfrak{n}}) x_{\alpha_1+\alpha_2}(-1) \subset I_{k_1\Lambda_1+k_2\Lambda_2}.$$

Clearly, $c_2 x_{\alpha_1}(-1)^{k_2+1} \in I_{k_1\Lambda_1+k_2\Lambda_2}$. Finally,

$$c_3 x_{\alpha_1}(-2)^{k_1+1} x_{\alpha_1}(-1)^{k_2} = c_3' R^1_{-1,2(k_1+1)+k_2} + c_3'' x_{\alpha_1}(-1)^{k_2+1} \in I_{k_1\Lambda_1+k_2\Lambda_2}$$

for some $c_3', c_3'' \in U(\bar{\mathfrak{n}})$. So we have that

$$a = b + cx_{\alpha_1}(-1)^{k_2} \in I_{k_1\Lambda_1+k_2\Lambda_2},$$

which is a contradiction. Hence, we have that $\Lambda \neq k_1\Lambda_1 + k_2\Lambda_2$ for some $k_1, k_2 \in \mathbb{N}$ such that $k_1 + k_2 = k$.

We now proceed to show, via induction, that $\Lambda \neq k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$ for all $k_0, k_1, k_2 \in \mathbb{N}$ such that $k_0 + k_1 + k_2 = k$. We've already shown this in the case when $k_0 = 0$. Suppose now for induction that we've shown $\Lambda \neq k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$ for some $0 \leq k_0 < k$ and for all $k_1, k_2 \in \mathbb{N}$ such that $k_0 + k_1 + k_2 = k$. Consider the weight $\Lambda = (k_0+1)\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$ such that $k_0 + k_1 + k_2 + 1 = k$. In this case, we have that

$$a \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_0+1 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}}) = 0$$

We now claim that, for any $m, n \in \mathbb{N}$ satisfying $1 \leq m \leq k_0 + 1$ and $0 \leq n \leq k_0 + k_2 + 2 - m$, there exist $b \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$ and $c, d \in U(\bar{\mathfrak{n}})$ such that

$$a = b + cx_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^m + dx_{\alpha_1+\alpha_2}(-1)^{m+1}.$$

To show this, we use a nested induction. First, we show the claim for $m = 1$ and $n = 0$, and then proceed to show it is true for $m = 1$ and all $0 \leq n \leq k_0 + k_2 + 1$.

Applying the operators $1^{\otimes k_0} \otimes \mathcal{Y}_c(e^{\lambda_1}, x) \otimes 1^{\otimes(k_1+k_2)}$, we obtain

$$a \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_0 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1+1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}}) = 0. \tag{4.2}$$

By our inductive hypothesis, we have that

$$\begin{aligned} a \in I_{k_0\Lambda_0+(k_1+1)\Lambda_1+k_2\Lambda_2} &= I_{k\Lambda_0} + U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^{k_0+k_2+1} + U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k_0+k_1+2} \\ &\quad + U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-1)^{k_0+1}. \end{aligned}$$

So we may write

$$a = a_1 + a_2 x_{\alpha_1}(-1)^{k_0+k_2+1} + a_3 x_{\alpha_2}(-1)^{k_0+k_1+2} + a_4 x_{\alpha_1+\alpha_2}(-1)^{k_0+1}$$

for some $a_1 \in I_{k\Lambda_0}$ and $a_2, a_3, a_4 \in U(\bar{\mathfrak{n}})$. Clearly,

$$a_1, a_3 x_{\alpha_2}(-1)^{k_0+k_1+2} \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}.$$

We thus have that

$$\begin{aligned} &(a_2 x_{\alpha_1}(-1)^{k_0+k_2+1} + a_4 x_{\alpha_1+\alpha_2}(-1)^{k_0+1}) \\ &\qquad \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_0+1 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}}) \\ &= (a - a_1 - a_3 x_{\alpha_2}(-1)^{k_0+k_1+2}) \\ &\qquad \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_0+1 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}}) \\ &= 0. \end{aligned}$$

Applying the operator $1^{\otimes k_0} \otimes \mathcal{Y}_c(e^{\lambda_2}, x) \otimes 1^{\otimes k_1+k_2}$, we obtain

$$\begin{aligned} &(a_2 x_{\alpha_1}(-1)^{k_0+k_2+1} + a_4 x_{\alpha_1+\alpha_2}(-1)^{k_0+1}) \\ &\quad \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_0 \text{ times}} \otimes v_{\Lambda_2} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}}) \\ &= a_2 x_{\alpha_1}(-1)^{k_0+k_2+1} \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_0 \text{ times}} \otimes v_{\Lambda_2} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}}) \\ &= 0, \end{aligned}$$

since $a_4 x_{\alpha_1+\alpha_2}(-1)^{k_0+1} \in I_{k_0\Lambda_0+k_1\Lambda_1+(k_2+1)\Lambda_2}$. In particular, we may write

$$a_2 x_{\alpha_1}(-1)^{k_0+k_2+1} \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_0 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2+1 \text{ times}}) = 0.$$

Applying the operator $\mathcal{Y}_c(e^{\lambda_2}, x)^{\otimes k_0} \otimes 1^{\otimes k_1+k_2+1}$, we have that

$$a_2 x_{\alpha_1}(-1)^{k_0+k_2+1} \cdot (\underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_0 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2+1 \text{ times}}) = 0$$

which, using the diagonal action (2.21), implies

$$\begin{aligned} a_2 \cdot (&\underbrace{x_{\alpha_1}(-1)v_{\Lambda_2} \otimes \cdots \otimes x_{\alpha_1}(-1)v_{\Lambda_2}}_{k_0 \text{ times}} \otimes \\ &\otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{x_{\alpha_1}(-1)v_{\Lambda_2} \otimes \cdots \otimes x_{\alpha_1}(-1)v_{\Lambda_2}}_{k_2+1 \text{ times}}) = 0. \end{aligned}$$

From this, we have that

$$a_2 \cdot (\underbrace{e_{\lambda_1}v_{\Lambda_1} \otimes \cdots \otimes e_{\lambda_1}v_{\Lambda_1}}_{k_0 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{e_{\lambda_1}v_{\Lambda_1} \otimes \cdots \otimes e_{\lambda_1}v_{\Lambda_1}}_{k_2+1 \text{ times}}) = 0$$

which implies that

$$e_{\lambda_1}^{\otimes k}(\tau_{\lambda_1,c_{-\lambda_1}}^{-1}(a_2) \cdot (\underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_0 \text{ times}} \otimes \underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_2+1 \text{ times}})) = 0.$$

Now, by injectivity of $e_{\lambda_1}^{\otimes k}$, we have that

$$\tau_{\lambda_1,c_{-\lambda_1}}^{-1}(a_2) \cdot (\underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_0 \text{ times}} \otimes \underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_2+1 \text{ times}}) = 0.$$

Since the total charge of $\tau_{\lambda_1,c_{-\lambda_1}}^{-1}(a_2)$ is less than the total charge of a, we have that

$$\tau_{\lambda_1,c_{-\lambda_1}}^{-1}(a_2) \in I_{k_1\Lambda_0+(k_0+k_2+1)\Lambda_1} = I_{k\Lambda_0} + U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^{k_1+1}.$$

So, in particular, applying $\tau_{\lambda_1,c_{-\lambda_1}}$, we obtain

$$a_2 \in I_{k\Lambda_1} + U(\bar{\mathfrak{n}})x_{\alpha_1}(-2)^{k_1+1}.$$

So we may write $a_2 = a_{2,1} + a_{2,2}x_{\alpha_1}(-1) + a_{2,3}x_{\alpha_1}(-2)^{k_1+1}$ for some $a_{2,1} \in I_{k\Lambda_0}$ and $a_{2,2}, a_{2,3} \in U(\bar{\mathfrak{n}})$. So we have that

$$\begin{aligned} &a_2 x_{\alpha_1}(-1)^{k_0+k_2+1} \\ &\quad = \quad a_{2,1}x_{\alpha_1}(-1)^{k_0+k_2+1} + a_{2,2}x_{\alpha_1}(-1)^{k_0+k_2+2} + a_{2,3}x_{\alpha_1}(-2)^{k_1+1}x_{\alpha_1}(-1)^{k_0+k_2+1}. \end{aligned}$$

By Lemma 3.2.2, we have that

$$a_{2,1}x_{\alpha_1}(-1)^{k_0+k_2+1} \in I_{k\Lambda_0} + U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-1).$$

Clearly,

$$a_{2,2}x_{\alpha_1}(-1)^{k_0+k_2+2} \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}.$$

Finally,

$$\begin{aligned} a_{2,3}x_{\alpha_1}(-2)^{k_1+1}x_{\alpha_1}(-1)^{k_0+k_2+1} &= rR_{-1,2(k_1+1)+k_0+k_2+1} + a'_{2,3}x_{\alpha_1}(-1)^{k_0+k_2+1} \\ &\in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2} \end{aligned}$$

for some constant $r \in \mathbb{C}$. Thus, we have that

$$a_2x_{\alpha_1}(-1)^{k_0+k_2+1} \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2} + U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-1).$$

Clearly

$$a_1, a_3x_{\alpha_1}(-1)^{k_0+k_2+1}, a_4x_{\alpha_1+\alpha_2}(-1)^{k_0+1} \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2} + U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-1)$$

and so we may write

$$a = b + cx_{\alpha_1+\alpha_2}(-1)$$

for some $b \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$ and $c \in U(\bar{\mathfrak{n}})$, and so our claim holds for $m = 1, n = 0$ (notice here the d term is 0).

Now, we assume our claim holds for $m = 1$ and for some $n \in \mathbb{N}$ satisfying $0 \le n \le k_0 + k_2$. We show that it holds for $n + 1$. Suppose

$$a = b + cx_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1) + dx_{\alpha_1+\alpha_2}(-1)^2$$

for some $b \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$ and $c, d \in U(\bar{\mathfrak{n}})$. We must now consider the cases when $0 \le n < k_2$ and $k_2 \le n \le k_0 + k_2$ separately. First assume $0 \le n < k_2$. In this case, we have

$$a \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_0+1 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}}) = 0$$

which implies

$$\begin{aligned}
&(cx_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1) + dx_{\alpha_1+\alpha_2}(-1)^2)\\
&\qquad\qquad\cdot(\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_0+1 \text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_1 \text{ times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{k_2 \text{ times}})\\
= &\ (a-b)\cdot(\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_0+1 \text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_1 \text{ times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{k_2 \text{ times}})\\
= &\ 0.
\end{aligned}$$

Applying the operator $1\otimes\mathcal{Y}_c(e^{\lambda_1},x)^{\otimes k_0}\otimes 1^{\otimes k_1}\otimes\mathcal{Y}_c(e^{\lambda_1},x)^{\otimes k_2-n}\otimes 1^{\otimes n}$, we obtain

$$\begin{aligned}
&(cx_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1) + dx_{\alpha_1+\alpha_2}(-1)^2)\\
&\quad\cdot(v_{\Lambda_0}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_0+k_1 \text{ times}}\otimes\underbrace{e_{\lambda_1}v_{\Lambda_2}\otimes\cdots\otimes e_{\lambda_1}v_{\Lambda_2}}_{k_2-n \text{ times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{n \text{ times}})\\
= &\ cx_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)\\
&\quad\cdot(v_{\Lambda_0}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_0+k_1 \text{ times}}\otimes\underbrace{e_{\lambda_1}v_{\Lambda_2}\otimes\cdots\otimes e_{\lambda_1}v_{\Lambda_2}}_{k_2-n \text{ times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{n \text{ times}})\\
= &\ 0.
\end{aligned}$$

This implies that

$$c\cdot(e_{\lambda_1}v_{\Lambda_2}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_0+k_1 \text{ times}}\otimes\underbrace{e_{\lambda_1}v_{\Lambda_2}\otimes\cdots\otimes e_{\lambda_1}v_{\Lambda_2}}_{k_2-n \text{ times}}\otimes\underbrace{e_{\lambda_1}v_{\Lambda_1}\otimes\cdots\otimes e_{\lambda_1}v_{\Lambda_1}}_{n \text{ times}})=0$$

which gives us

$$e_{\lambda_1}^{\otimes k}(\tau^{-1}_{\lambda_1,c_{-\lambda_1}}(c)\cdot(v_{\Lambda_2}\otimes\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_0+k_1 \text{ times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{k_2-n \text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{n \text{ times}}))=0.$$

By the injectivity of $e_{\lambda_1}^{\otimes k}$, we have that

$$\tau^{-1}_{\lambda_1,c_{-\lambda_1}}(c)\cdot(v_{\Lambda_2}\otimes\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_0+k_1 \text{ times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{k_2-n \text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{n \text{ times}})=0.$$

Since $\tau^{-1}_{\lambda_1,c_{-\lambda_1}}(c)$ is of lower total charge than a, we have that

$$\begin{aligned}
\tau^{-1}_{\lambda_1,c_{-\lambda_1}}(c) &\in I_{(k_0+k_1)\Lambda_0+n\Lambda_1+(k_2-n+1)\Lambda_2}\\
&= I_{k\Lambda_0}+U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^{k_0+k_1+k_2-n+2}+U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k_0+k_1+n+1}\\
&\quad +U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-1)^{k_0+k_1+1}.
\end{aligned}$$

Applying the map $\tau_{\lambda_1, c_{-\lambda_1}}$, Lemma 3.2.3 gives us

$$\begin{aligned} c &\in I_{k\Lambda_1} + U(\bar{\mathfrak{n}})x_{\alpha_1}(-2)^{k_0+k_1+k_2-n+2} + U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k_0+k_1+n+1} \\ &\quad + U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-2)^{k_0+k_1+1}. \end{aligned}$$

So we may write

$$\begin{aligned} c = c_1 + c_2 x_{\alpha_1}(-1) + c_3 x_{\alpha_1}(-2)^{k_0+k_1+k_2-n+2} \\ + c_4 x_{\alpha_2}(-1)^{k_0+k_1+n+1} + c_5 x_{\alpha_1+\alpha_2}(-2)^{k_0+k_1+1} \end{aligned}$$

for some $c_1 \in I_{k\Lambda_0}$ and $c_2, c_3, c_4, c_5 \in U(\bar{\mathfrak{n}})$. This gives us

$$\begin{aligned} & c x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1) \\ = \ & c_1 x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1) + c_2 x_{\alpha_1}(-1)^{n+1} x_{\alpha_1+\alpha_2}(-1) \\ & + c_3 x_{\alpha_1}(-2)^{k_0+k_1+k_2-n+2} x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1) \\ & + c_4 x_{\alpha_2}(-1)^{k_0+k_1+n+1} x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1) \\ & + c_5 x_{\alpha_1+\alpha_2}(-2)^{k_0+k_1+1} x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1). \end{aligned}$$

By Lemma 3.2.2, we may write

$$c_1 x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1) = c_1' + c_1'' x_{\alpha_1+\alpha_2}(-1)^2$$

for some $c_1' \in I_{k\Lambda_0}$ and $c_1'' \in U(\bar{\mathfrak{n}})$. Clearly $c_2 x_{\alpha_1}(-1)^{n+1} x_{\alpha_1+\alpha_2}(-1)$ is already of the desired form. We may write

$$\begin{aligned} & c_3 x_{\alpha_1}(-2)^{k_0+k_1+k_2-n+2} x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1) \\ = \ & c_3' R^1_{-1,2(k_0+k_1+k_2-n+2)+n} + c_3'' x_{\alpha_1}(-1)^{n+1} x_{\alpha_1}(-1)^{n+1} x_{\alpha_1+\alpha_2}(-1) \end{aligned}$$

for some $c_3', c_3'' \in U(\bar{\mathfrak{n}})$, which gives us an element of the desired form. Using Lemma 3.2.1, we may write

$$\begin{aligned}
&c_4 x_{\alpha_2}(-1)^{k_0+k_1+n+1} x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1) \\
&= c_4 (x_{\alpha_1}(-1)^n x_{\alpha_2}(-1)^{k_0+k_1+n+1} x_{\alpha_1+\alpha_2}(-1) \\
&\quad + m_1 x_{\alpha_1}(-1)^{n-1} x_{\alpha_1+\alpha_2}(-2) x_{\alpha_2}(-1)^{k_0+k_1+n} x_{\alpha_1+\alpha_2}(-1) \\
&\quad + \cdots + m_n x_{\alpha_1+\alpha_2}(-2)^n x_{\alpha_2}(-1)^{k_0+k_1+1} x_{\alpha_1+\alpha_2}(-1)) \\
&= c_4 (m_1' x_{\alpha_1}(-1)^n [x_{\alpha_1}(0), \ldots [x_{\alpha_1}(0), x_{\alpha_2}(-1)^{k_0+k_1+j+2}] \ldots] \\
&\quad + \cdots + m_n' x_{\alpha_1+\alpha_2}(-2)^n [x_{\alpha_1}(0), x_{\alpha_2}(-1)^{k_0+k_1+2}] \\
&\in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}
\end{aligned}$$

for some constants $m_1, \ldots m_n, m_1' \ldots m_n' \in \mathbb{C}$,which is an element of the desired form. Finally, we have that

$$\begin{aligned}
&c_5 x_{\alpha_1+\alpha_2}(-2)^{k_0+k_1+1} x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1) \\
&= c_5' [x_{\alpha_1}(0), \ldots [x_{\alpha_1}(0), x_{\alpha_2}(-1)^{k_0+k_1+2+n}] \ldots] \\
&\in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}
\end{aligned}$$

for some $c_5' \in U(\bar{\mathfrak{n}})$, which gives us an element of the desired form. So we have that

$$\begin{aligned}
c x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1) \in\ & I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2} + U(\bar{\mathfrak{n}}) x_{\alpha_1}(-1)^{n+1} x_{\alpha_1+\alpha_2}(-1) \\
& + U(\bar{\mathfrak{n}}) x_{\alpha_1+\alpha_2}(-1)^2.
\end{aligned}$$

So we may conclude that

$$a \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2} + U(\bar{\mathfrak{n}}) x_{\alpha_1}(-1)^{n+1} x_{\alpha_1+\alpha_2}(-1) + U(\bar{\mathfrak{n}}) x_{\alpha_1+\alpha_2}(-1)^2$$

proving our claim for $n+1$ when $0 \le n < k_2$.

We now need to consider the case that $k_2 \le n \le k_0 + k_2$. In this case, we again have

$$a \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_0+1 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}}) = 0$$

which implies

$$
\begin{aligned}
&(cx_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1) + dx_{\alpha_1+\alpha_2}(-1)^2)\\
&\qquad \cdot(\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_0+1 \text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_1 \text{ times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{k_2 \text{ times}})\\
= \;& (a-b)\cdot(\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_0+1 \text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_1 \text{ times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{k_2 \text{ times}}))\\
= \;& 0.
\end{aligned}
$$

Applying the operator $1\otimes\mathcal{Y}_c(e^{\lambda_1},x)^{\otimes k_0+k_2-n}\otimes\mathcal{Y}_c(e^{\lambda_2},x)^{\otimes n-k_2}\otimes 1^{\otimes k_1+k_2}$, we obtain

$$
\begin{aligned}
&(cx_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1) + dx_{\alpha_1+\alpha_2}(-1)^2)\\
&\quad \cdot(v_{\Lambda_0}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_0+k_2-n \text{ times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{n-k_2 \text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_1 \text{ times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{k_2 \text{ times}})\\
= \;& cx_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)\\
&\quad \cdot(v_{\Lambda_0}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_0+k_2-n \text{ times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{n-k_2 \text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_1 \text{ times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{k_2 \text{ times}})\\
= \;& 0.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&c\cdot(e_{\lambda_1}v_{\Lambda_2}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_0+k_2-n \text{ times}}\otimes\underbrace{e_{\lambda_1}v_{\Lambda_1}\otimes\cdots\otimes e_{\lambda_1}v_{\Lambda_1}}_{n-k_2 \text{ times}}\otimes\\
&\qquad\qquad \otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_1 \text{ times}}\otimes\underbrace{e_{\lambda_1}v_{\Lambda_1}\otimes\cdots\otimes e_{\lambda_1}v_{\Lambda_1}}_{k_2 \text{ times}}) = 0
\end{aligned}
$$

which gives

$$
\begin{aligned}
&e_{\lambda_1}^{\otimes k}(\tau^{-1}_{\lambda_1,c_{-\lambda_1}}(c)\cdot(v_{\Lambda_2}\otimes\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_0+k_2-n \text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{n-k_2 \text{ times}}\otimes\\
&\qquad\qquad \otimes\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_1 \text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_2 \text{ times}})) = 0.
\end{aligned}
$$

Since $e_{\lambda_1}^{\otimes k}$ is injective, we have that

$$
\begin{aligned}
&\tau^{-1}_{\lambda_1,c_{-\lambda_1}}(c)\cdot(v_{\Lambda_2}\otimes\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_0+k_2-n \text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{n-k_2 \text{ times}}\otimes\\
&\qquad\qquad \otimes\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_1 \text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_2 \text{ times}}) = 0.
\end{aligned}
$$

Now, since the total charge of $\tau^{-1}_{\lambda_1,c_{-\lambda_1}}(c)$ is less than the total charge of a, we have that

$$\begin{aligned}\tau^{-1}_{\lambda_1,c_{-\lambda_1}}(c) &\in I_{(k_0+k_1+k_2-n)\Lambda_0+n\Lambda_1+\Lambda_2}\\ &= I_{k\Lambda_0}+U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^{k_0+k_1+k_2-n+2}+U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k}\\ &\quad +U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-1)^{k_0+k_1+k_2-n+1}.\end{aligned}$$

Applying $\tau_{\lambda_1,c_{-\lambda_1}}$, Lemma 3.2.3 gives us

$$\begin{aligned}c\in I_{k\Lambda_1}&+U(\bar{\mathfrak{n}})x_{\alpha_1}(-2)^{k_0+k_1+k_2-n+2}\\ &+U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k}+U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-2)^{k_0+k_1+k_2-n+1}.\end{aligned}$$

So we may write

$$\begin{aligned}c = c_1&+c_2x_{\alpha_1}(-1)+c_3x_{\alpha_1}(-2)^{k_0+k_1+k_2-n+2}\\ &+c_4x_{\alpha_2}(-1)^{k}+c_5x_{\alpha_1+\alpha_2}(-2)^{k_0+k_1+k_2-n+1}\end{aligned}$$

for some $c_1\in I_{k\Lambda_0}$ and $c_2,c_3,c_4,c_5\in U(\bar{\mathfrak{n}})$, so that

$$\begin{aligned}&cx_{\alpha_1}(-1)^{n}x_{\alpha_1+\alpha_2}(-1)\\ &= c_1x_{\alpha_1}(-1)^{n}x_{\alpha_1+\alpha_2}(-1)+c_2x_{\alpha_1}(-1)^{n+1}x_{\alpha_1+\alpha_2}(-1)\\ &\quad +c_3x_{\alpha_1}(-2)^{k_0+k_1+k_2-n+2}x_{\alpha_1}(-1)^{n}x_{\alpha_1+\alpha_2}(-1)\\ &\quad +c_4x_{\alpha_2}(-1)^{k}x_{\alpha_1}(-1)^{n}x_{\alpha_1+\alpha_2}(-1)\\ &\quad +c_5x_{\alpha_1+\alpha_2}(-2)^{k_0+k_1+k_2-n+1}x_{\alpha_1}(-1)^{n}x_{\alpha_1+\alpha_2}(-1).\end{aligned}$$

As before, we have that

$$\begin{aligned}&c_1x_{\alpha_1}(-1)^{n}x_{\alpha_1+\alpha_2}(-1),\, c_2x_{\alpha_1}(-1)^{n+1}x_{\alpha_1+\alpha_2}(-1)\\ &\in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}+U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^{n+1}x_{\alpha_1+\alpha_2}(-1)+U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-1)^{2}\end{aligned}$$

and so have the desired form. We also have that

$$\begin{aligned}&c_3x_{\alpha_1}(-2)^{k_0+k_1+k_2-n+2}x_{\alpha_1}(-1)^{n}x_{\alpha_1+\alpha_2}(-1)\\ &= c_3'R^{1}_{-1,2(k_0+k_1+k_2-n+2)+n}+c_3''x_{\alpha_1}(-1)^{n+1}x_{\alpha_1+\alpha_2}(-1)\end{aligned}$$

for some $c_3', c_3'' \in U(\bar{\mathfrak{n}})$, which is of the desired form. Using Lemma 3.2.1, we have

$$\begin{aligned}
&c_4 x_{\alpha_2}(-1)^k x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1) \\
&\quad = \quad c_4(x_{\alpha_1}(-1)^n x_{\alpha_2}(-1)^k x_{\alpha_1+\alpha_2}(-1) \\
&\qquad\qquad + m_1 x_{\alpha_1}(-1)^{n-1} x_{\alpha_1+\alpha_2}(-2) x_{\alpha_2}(-1)^{k-1} x_{\alpha_1+\alpha_2}(-1) \\
&\qquad\qquad + \cdots + m_n x_{\alpha_1+\alpha_2}(-2)^n x_{\alpha_2}(-1)^{k-n} x_{\alpha_1+\alpha_2}(-1)) \\
&\quad = \quad c_4(m_0' x_{\alpha_1}(-1)^n [x_{\alpha_1}(0), R^2_{-1,k+1}] + m_1' x_{\alpha_1}(-1)^{n-1}[x_{\alpha_1}(0), [x_{\alpha_1}(0), R^2_{-1,k+2}]] \\
&\qquad\qquad + \cdots + m_n' [x_{\alpha_1}(0), \ldots [x_{\alpha_1}(0), R^2_{-1,k+n+1}] \ldots]) + c_4' x_{\alpha_1+\alpha_2}(-1)^2
\end{aligned}$$

for some $m_1, \ldots m_n, m_0', \ldots m_n' \in \mathbb{C}$ and $c_4' \in U(\bar{\mathfrak{n}})$, which is of the desired form. Finally,

$$\begin{aligned}
&c_5 x_{\alpha_1+\alpha_2}(-2)^{k_0+k_1+k_2-n+1} x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1) \\
&\quad = \quad c_5[x_{\alpha_2}(0), \ldots [x_{\alpha_2}(0), R^1_{-1,2(k-n)+n+1}] \ldots] + c_5' x_{\alpha_1}(-1)^{n+1} x_{\alpha_1+\alpha_2}(-1)
\end{aligned}$$

for some $c_5' \in U(\bar{\mathfrak{n}})$. So we have that

$$\begin{aligned}
c x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1) \quad \in \quad & I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2} + U(\bar{\mathfrak{n}}) x_{\alpha_1+\alpha_2}(-1)^{n+1} x_{\alpha_1+\alpha_2}(-1) \\
& + U(\bar{\mathfrak{n}}) x_{\alpha_1+\alpha_2}(-1)^2,
\end{aligned}$$

which gives us that

$$a \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2} + U(\bar{\mathfrak{n}}) x_{\alpha_1+\alpha_2}(-1)^{n+1} x_{\alpha_1+\alpha_2}(-1) + U(\bar{\mathfrak{n}}) x_{\alpha_1+\alpha_2}(-1)^2,$$

completing our induction on n. We've thus shown that

$$a = b + c x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^m + d x_{\alpha_1+\alpha_2}(-1)^{m+1}$$

for some $b \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$ and $c, d \in U(\bar{\mathfrak{n}})$ holds when $m = 1$ and $0 \le n \le k_0 + k_2 + 2 - m$.

Now, we induct on m. Assume that we've shown, for some $m \in \mathbb{N}$ satisfying $1 \le m \le k_0$ and all $0 \le n \le k_0 + k_2 + 2 - m$ that there exist $b \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$ and $c, d \in U(\bar{\mathfrak{n}})$ such that

$$a = b + c x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^m + d x_{\alpha_1+\alpha_2}(-1)^{m+1}. \tag{4.3}$$

We show that this holds for $m+1$ as well. We have by (4.3) that

$$a = b + cx_{\alpha_1}(-1)^{k_0+k_2+2-m}x_{\alpha_1+\alpha_2}(-1)^m + dx_{\alpha_1+\alpha_2}(-1)^{m+1}$$

for some $b \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$ and $c, d \in U(\bar{\mathfrak{n}})$. Here, we have that

$$\begin{aligned} cx_{\alpha_1}(-1)^{k_0+k_2+2-m}x_{\alpha_1+\alpha_2}(-1)^m &= c'[x_{\alpha_2}(0), \ldots [x_{\alpha_2}(0), x_{\alpha_1}(-1)^{k_0+k_2+1}] \ldots] \\ &\in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}, \end{aligned}$$

for some $c' \in U(\bar{\mathfrak{n}})$, so $a \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2} + U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-1)^{m+1}$. In this case, we have that

$$a = b' + c'x_{\alpha_1}(-1)^0 x_{\alpha_1+\alpha_2}(-1)^{m+1} + d'x_{\alpha_1+\alpha_2}(-1)^{m+2}$$

for some $b' \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$ and $c', d' \in U(\bar{\mathfrak{n}})$ (here, $d = 0$). So our claim holds for $m+1$ and $n = 0$. Now, assume that we have shown

$$a = b + cx_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1} + dx_{\alpha_1+\alpha_2}(-1)^{m+2}$$

for some $b \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$, $c, d \in U(\bar{\mathfrak{n}})$ and n satisfying $0 \leq n \leq k_0 + k_2 - m$. As with $m = 1$, we need to consider the cases $0 \leq n < k_2$ and $k_2 \leq n \leq k_0 + k_2 - m$ separately. First, suppose $0 \leq n < k_2$. Here, we have that

$$\begin{aligned} &(cx_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1} + dx_{\alpha_1+\alpha_2}(-1)^{m+2}) \\ &\qquad \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_0+1 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}}) \\ &= (a-b) \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_0+1 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}}) \\ &= 0. \end{aligned}$$

Applying the operator $1^{\otimes m+1} \otimes \mathcal{Y}_c(e^{\lambda_1}, x)^{\otimes k_0-m} \otimes 1^{\otimes k_1} \otimes \mathcal{Y}_c(e^{\lambda_1}, x)^{\otimes k_2-n} \otimes 1^{\otimes n}$, we have

$$\begin{aligned} cx_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1} \cdot (&\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{m+1 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_0+k_1-m \text{ times}} \otimes \\ &\underbrace{e_{\lambda_1}v_{\Lambda_2} \otimes \cdots \otimes e_{\lambda_1}v_{\Lambda_2}}_{k_2-n \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{n \text{ times}}) = 0, \end{aligned}$$

which implies that

$$c\cdot(\underbrace{e_{\lambda_1}v_{\Lambda_2}\otimes\cdots\otimes e_{\lambda_1}v_{\Lambda_2}}_{m+1\text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_0+k_1-m\text{ times}}\otimes$$
$$\otimes\underbrace{e_{\lambda_1}v_{\Lambda_2}\otimes\cdots\otimes e_{\lambda_1}v_{\Lambda_2}}_{k_2-n\text{ times}}\otimes\underbrace{e_{\lambda_1}v_{\Lambda_1}\otimes\cdots\otimes e_{\lambda_1}v_{\Lambda_1}}_{n\text{ times}})=0$$

and so we have

$$e_{\lambda_1}^{\otimes k}(\tau_{\lambda_1,c_{-\lambda_1}}^{-1}(c)\cdot(\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{m+1\text{ times}}\otimes\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_0+k_1-m\text{ times}}\otimes$$
$$\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{k_2-n\text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{n\text{ times}}))=0.$$

Since $e_{\lambda_1}^{\otimes k}$ is injective, we have that

$$\tau_{\lambda_1,c_{-\lambda_1}}^{-1}(c)\cdot(\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{m+1\text{ times}}\otimes\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_0+k_1-m\text{ times}}\otimes$$
$$\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{k_2-n\text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{n\text{ times}})=0.$$

Now, since $\tau_{\lambda_1,c_{-\lambda_1}}^{-1}(c)$ has lower total charge than a, we have that

$$\begin{aligned}\tau_{\lambda_1,c_{-\lambda_1}}^{-1}(c) &\in I_{(k_0+k_1-m)\Lambda_0+n\Lambda_1+(k_2-n+m+1)\Lambda_2}\\ &= I_{k\Lambda_0}+U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^{k-n+1}+U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k_0+k_1-m+n+1}\\ &\quad +U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-1)^{k_0+k_1-m+1}.\end{aligned}$$

Applying the map $\tau_{\lambda_1,c_{-\lambda_1}}$, by Lemma 3.2.3 we have that

$$\begin{aligned}&c\in I_{k\Lambda_1}+U(\bar{\mathfrak{n}})x_{\alpha_1}(-2)^{k-n+1}\\ &\quad +U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k_0+k_1-m+n+1}+U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-2)^{k_0+k_1-m+1}.\end{aligned}$$

So we may write

$$\begin{aligned}&c=c_1+c_2x_{\alpha_1}(-1)+c_3x_{\alpha_1}(-2)^{k-n+1}\\ &\quad +c_4x_{\alpha_2}(-1)^{k_0+k_1-m+n+1}+c_5x_{\alpha_1+\alpha_2}(-2)^{k_0+k_1-m+1}\end{aligned}$$

for some $c_1 \in I_{k\Lambda_0}$ and $c_2, c_3, c_4, c_5 \in U(\bar{\mathfrak{n}})$, and we have

$$cx_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1} = c_1 x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1}$$
$$+c_2 x_{\alpha_1}(-1)^{n+1} x_{\alpha_1+\alpha_2}(-1)^{m+1} + c_3 x_{\alpha_1}(-2)^{k-n+1} x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1}$$
$$+c_4 x_{\alpha_2}(-1)^{k_0+k_1-m+n+1} x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1}$$
$$+c_5 x_{\alpha_1+\alpha_2}(-2)^{k_0+k_1-m+1} x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1}.$$

By Lemma 3.2.2, we have that

$$c_1 x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1} \in I_{k\Lambda_0} + U(\bar{\mathfrak{n}}) x_{\alpha_1+\alpha_2}(-1)^{m+2}.$$

The summand $c_2 x_{\alpha_1}(-1)^{n+1} x_{\alpha_1+\alpha_2}(-1)^{m+1}$ is clearly of the desired form. We have that

$$\begin{aligned}
& c_3 x_{\alpha_1}(-2)^{k-n+1} x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1} \\
= \quad & c_3' x_{\alpha_1+\alpha_2}(-1)^{m+1} R^1_{-1,2(k-n+1)+n} + c_3'' x_{\alpha_1}(-1)^{n+1} x_{\alpha_1+\alpha_2}(-1)^{m+1}
\end{aligned}$$

for some $c_3', c_3'' \in U(\bar{\mathfrak{n}})$, which is clearly of the desired form. For the next summand, by Lemma 3.2.1 we have that

$$\begin{aligned}
& c_4 x_{\alpha_2}(-1)^{k_0+k_1-m+n+1} x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1} \\
= \quad & c_4(x_{\alpha_1}(-1)^n x_{\alpha_2}(-1)^{k_0+k_1-m+n+1} x_{\alpha_1+\alpha_2}(-1)^{m+1} \\
& +m_1 x_{\alpha_1}(-1)^{n-1} x_{\alpha_1+\alpha_2}(-2) x_{\alpha_2}(-1)^{k_0+k_1-m+n} x_{\alpha_1+\alpha_2}(-1)^{m+1} \\
& +\cdots + m_n x_{\alpha_1+\alpha_2}(-2)^n x_{\alpha_2}(-1)^{k_0+k_1-m+1} x_{\alpha_1+\alpha_2}(-1)^{m+1}) \\
= \quad & c_4(m_0' x_{\alpha_1}(-1)^n [x_{\alpha_1}(0), \dots [x_{\alpha_1}(0), x_{\alpha_2}(-1)^{k_0+k_1+2+n}] \dots] \\
& +m_1' x_{\alpha_1}(-1)^{n-1} x_{\alpha_1+\alpha_2}(-2) [x_{\alpha_1}(0), \dots [x_{\alpha_1}(0), x_{\alpha_2}(-1)^{k_0+k_1+n+1}] \dots] \\
& +\cdots + m_n' x_{\alpha_1+\alpha_2}(-1)^n [x_{\alpha_1}, \dots [x_{\alpha_1}(0), x_{\alpha_2}(-1)^{k_0+k_1+2}] \dots])
\end{aligned}$$

for some $m_1, \dots m_n, m_0', \dots m_n' \in \mathbb{C}$, which is an element of $I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$. Finally, applying Lemma 3.2.1, we have that

$$\begin{aligned}
& c_5 x_{\alpha_1+\alpha_2}(-2)^{k_0+k_1-m+1} x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1} \\
= \quad & c_5(m_0[x_{\alpha_1}(-1)^{k_0+k_1-m+1+n}, x_{\alpha_2}(-1)^{k_0+k_1-m+1}] \\
& +m_1 x_{\alpha_2}(-1) x_{\alpha_1+\alpha_2}(-2)^{k_0+k_1-m} x_{\alpha_1}(-1)^{n+1} \\
& +\cdots + m_{k_0+k_1-m} x_{\alpha_2}(-1)^{k_0+k_1-m} x_{\alpha_1+\alpha_2}(-2) x_{\alpha_1}(-1)^{k_0+k_1-m+n}) x_{\alpha_1+\alpha_2}(-1)
\end{aligned}$$

for some $m_0, \dots, m_{k_0+k_1-m} \in \mathbb{C}$. Notice, all the terms in the right hand side are elements of

$$U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^{n+1}x_{\alpha_1+\alpha_2}(-1)^{m+1}$$

except for the term

$$m_0c_5x_{\alpha_1}(-1)^{k_0+k_1-m+1+n}x_{\alpha_2}(-1)^{k_0+k_1-m+1}x_{\alpha_1+\alpha_2}(-1)^{m+1}.$$

This term, however, may be written as

$$\begin{aligned} & m_0c_5x_{\alpha_1}(-1)^{k_0+k_1-m+1+n}x_{\alpha_2}(-1)^{k_0+k_1-m+1}x_{\alpha_1+\alpha_2}(-1)^{m+1} \\ = \quad & c_5'x_{\alpha_1}(-1)^{k_0+k_1-m+1+n}[x_{\alpha_1}(0), \dots [x_{\alpha_1}(0), x_{\alpha_2}(-1)^{k_0+k_1+2}] \dots] \\ \in \quad & I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2} \end{aligned}$$

for some $c_5' \in U(\bar{\mathfrak{n}})$ Hence, we have that

$$\begin{aligned} cx_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1} \quad \in \quad & I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2} + U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^{n+1}x_{\alpha_1+\alpha_2}(-1)^{m+1} \\ & + U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-1)^{m+2} \end{aligned}$$

and is of the desired form. From this, we conclude that we may write

$$a = b' + c'x_{\alpha_1}(-1)^{n+1}x_{\alpha_1+\alpha_2}(-1)^{m+1} + d'x_{\alpha_1+\alpha_2}(-1)^{m+2}$$

for some $b' \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$ and $c', d' \in U(\bar{\mathfrak{n}})$, completing our induction for $0 \leq n < k_2$.

We now assume that $k_2 \leq n \leq k_0 + k_2 - m$. As before, we have that

$$a = b + cx_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1} + dx_{\alpha_1+\alpha_2}(-1)^{m+2}$$

for some $b \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$ and $c, d \in U(\bar{\mathfrak{n}})$. So, as before, we have that

$$\begin{aligned} & (cx_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1} + dx_{\alpha_1+\alpha_2}(-1)^{m+2}) \\ & \qquad \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_0+1 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}}) \\ = \quad & (a-b) \cdot (\underbrace{v_{\Lambda_0} \otimes \cdots \otimes v_{\Lambda_0}}_{k_0+1 \text{ times}} \otimes \underbrace{v_{\Lambda_1} \otimes \cdots \otimes v_{\Lambda_1}}_{k_1 \text{ times}} \otimes \underbrace{v_{\Lambda_2} \otimes \cdots \otimes v_{\Lambda_2}}_{k_2 \text{ times}}) \\ = \quad & 0. \end{aligned}$$

Applying the operator $1^{\otimes(m+1)}\otimes\mathcal{Y}_c(e^{\lambda_1},x)^{\otimes(k_0+k_2-m-n)}\otimes\mathcal{Y}_c(e^{\lambda_2},x)^{\otimes(n-k_2)}\otimes 1^{\otimes(k_1+k_2)}$, we have that

$$\begin{aligned}
&(cx_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1}+dx_{\alpha_1+\alpha_2}(-1)^{m+2})\\
&\qquad\cdot(\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{m+1\text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_0+k_2-m-n\text{ times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{n-k_2\text{ times}}\otimes\\
&\qquad\qquad\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_1\text{ times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{k_2\text{ times}})\\
=\;&(cx_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1})\\
&\qquad\cdot(\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{m+1\text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_0+k_2-m-n\text{ times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{n-k_2\text{ times}}\otimes\\
&\qquad\qquad\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_1\text{ times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{k_2\text{ times}})\\
=\;&0.
\end{aligned}$$

This implies that

$$\begin{aligned}
&cx_{\alpha_1}(-1)^n\cdot(\underbrace{e_{\lambda_1}v_{\Lambda_2}\otimes\cdots\otimes e_{\lambda_1}v_{\Lambda_2}}_{m+1\text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_0+k_2-m-n\text{ times}}\otimes\\
&\qquad\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{n-k_2\text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_1\text{ times}}\otimes\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{k_2\text{ times}})=0
\end{aligned}$$

which implies

$$\begin{aligned}
&c\cdot(\underbrace{e_{\lambda_1}v_{\Lambda_2}\otimes\cdots\otimes e_{\lambda_1}v_{\Lambda_2}}_{m+1\text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_0+k_2-m-n\text{ times}}\otimes\\
&\qquad\otimes\underbrace{e_{\lambda_1}v_{\Lambda_1}\otimes\cdots\otimes e_{\lambda_1}v_{\Lambda_1}}_{n-k_2\text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_1\text{ times}}\otimes\underbrace{e_{\lambda_1}v_{\Lambda_1}\otimes\cdots\otimes e_{\lambda_1}v_{\Lambda_1}}_{k_2\text{ times}})=0.
\end{aligned}$$

From this, we have that

$$\begin{aligned}
&e_{\lambda_1}^{\otimes k}(\tau^{-1}_{\lambda_1,c_{-\lambda_1}}(c)\cdot(\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{m+1\text{ times}}\otimes\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_0+k_2-m-n\text{ times}}\otimes\\
&\qquad\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{n-k_2\text{ times}}\otimes\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_1\text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_2\text{ times}}))=0.
\end{aligned}$$

Since $e_{\lambda_1}^{\otimes k}$ is injective, we have that

$$\tau_{\lambda_1,c_{-\lambda_1}}^{-1}(c)\cdot(\underbrace{v_{\Lambda_2}\otimes\cdots\otimes v_{\Lambda_2}}_{m+1\text{ times}}\otimes\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_0+k_2-m-n\text{ times}}\otimes$$
$$\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{n-k_2\text{ times}}\otimes\underbrace{v_{\Lambda_0}\otimes\cdots\otimes v_{\Lambda_0}}_{k_1\text{ times}}\otimes\underbrace{v_{\Lambda_1}\otimes\cdots\otimes v_{\Lambda_1}}_{k_2\text{ times}})=0.$$

Since the total charge of $\tau_{\lambda_1,c_{-\lambda_1}}^{-1}(c)$ is less than the total charge of a, we have that

$$\begin{aligned}\tau_{\lambda_1,c_{-\lambda_1}}^{-1}(c) &\in I_{(k_0+k_1+k_2-m-n)\Lambda_0+n\Lambda_1+(m+1)\Lambda_2}\\ &= I_{k\Lambda_0}+U(\bar{\mathfrak{n}})x_{\alpha_1}(-1)^{k_0+k_1+k_2-n+2}+U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k_0+k_1+k_2-m+1}\\ &\quad +U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-1)^{k_0+k_1+k_2-m-n+1}.\end{aligned}$$

Applying $\tau_{\lambda_1,c_{-\lambda_1}}$ to both sides, Lemma 3.2.3 gives us

$$\begin{aligned}c &\in I_{k\Lambda_1}+U(\bar{\mathfrak{n}})x_{\alpha_1}(-2)^{k_0+k_1+k_2-n+2}+U(\bar{\mathfrak{n}})x_{\alpha_2}(-1)^{k_0+k_1+k_2-m+1}\\ &\quad +U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-2)^{k_0+k_1+k_2-m-n+1}.\end{aligned}$$

So we may write

$$\begin{aligned}c &= c_1+c_2x_{\alpha_1}(-1)+c_3x_{\alpha_1}(-2)^{k_0+k_1+k_2-n+2}\\ &\quad +c_4x_{\alpha_2}(-1)^{k_0+k_1+k_2-m+1}\\ &\quad +c_5x_{\alpha_1+\alpha_2}(-2)^{k_0+k_1+k_2-m-n+1}\end{aligned}$$

for some $c_1\in I_{k\Lambda_0}$ and $c_2,c_3,c_4,c_5\in U(\bar{\mathfrak{n}})$, which gives

$$\begin{aligned}&cx_{\alpha_1}(-1)^nx_{\alpha_1+\alpha_2}(-1)^{m+1}\\ &= c_1x_{\alpha_1}(-1)^nx_{\alpha_1+\alpha_2}(-1)^{m+1}\\ &\quad +c_2x_{\alpha_1}(-1)^{n+1}x_{\alpha_1+\alpha_2}(-1)^{m+1}\\ &\quad +c_3x_{\alpha_1}(-2)^{k_0+k_1+k_2-n+2}x_{\alpha_1}(-1)^nx_{\alpha_1+\alpha_2}(-1)^{m+1}\\ &\quad +c_4x_{\alpha_2}(-1)^{k_0+k_1+k_2-m+1}x_{\alpha_1}(-1)^nx_{\alpha_1+\alpha_2}(-1)^{m+1}\\ &\quad +c_5x_{\alpha_1+\alpha_2}(-2)^{k_0+k_1+k_2-m-n+1}x_{\alpha_1}(-1)^nx_{\alpha_1+\alpha_2}(-1)^{m+1}.\end{aligned}$$

As before, we analyze each summand and show that it is of the desired form. By Corollary 3.2.2, we have that

$$c_1x_{\alpha_1}(-1)^nx_{\alpha_1+\alpha_2}(-1)^{m+1}\in I_{k\Lambda_0}+U(\bar{\mathfrak{n}})x_{\alpha_1+\alpha_2}(-1)^{m+2}.$$

The summand $c_2 x_{\alpha_1}(-1)^{n+1} x_{\alpha_1+\alpha_2}(-1)^{m+1}$ is clearly of the desired form. For

$$c_3 x_{\alpha_1}(-2)^{k_0+k_1+k_2-n+2} x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1}$$

we have that

$$\begin{aligned} & c_3 x_{\alpha_1}(-2)^{k_0+k_1+k_2-n+2} x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1} \\ = \quad & c_3' R^1_{-1,2(k_0+k_1+k_2+2-n)+n} + c_3'' x_{\alpha_1}(-1)^{n+1} x_{\alpha_1+\alpha_2}(-1)^{m+1} \end{aligned}$$

for some $c_3', c_3'' \in U(\bar{\mathfrak{n}})$, which is of the desired form. We also have, by Lemma 3.2.1, that

$$\begin{aligned} & c_4 x_{\alpha_2}(-1)^{k_0+k_1+k_2-m+1} x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1} \\ = \quad & c_4 (x_{\alpha_1}(-1)^n x_{\alpha_2}(-1)^{k_0+k_1+k_2-m+1} x_{\alpha_1+\alpha_2}(-1)^{m+1} \\ & + m_1 x_{\alpha_1}(-1)^{n-1} x_{\alpha_1+\alpha_2}(-2) x_{\alpha_2}(-1)^{k_0+k_1+k_2-m} x_{\alpha_1+\alpha_2}(-1)^{m+1} \\ & + \cdots + m_n x_{\alpha_1+\alpha_2}(-2)^n x_{\alpha_2}(-1)^{k_0+k_1+k_2-m-n+1} x_{\alpha_1+\alpha_2}(-1)^{m+1}) \\ = \quad & c_4 (m_0' x_{\alpha_1}(-1)^n [x_{\alpha_1}(0), \ldots [x_{\alpha_1}(0), R^2_{-1,k+1}] \ldots] \\ & + m_1' x_{\alpha_1}(-1)^{n-1} [x_{\alpha_1}(0), \ldots [x_{\alpha_1}(0), R^2_{-1,k+2}] \ldots] \\ & + \cdots + m_n' [x_{\alpha_1}(0), \ldots [x_{\alpha_1}(0), R^2_{-1,k+n+1}] \ldots]) + c_4' x_{\alpha_1+\alpha_2}(-1)^{m+2} \end{aligned}$$

for some constants $m_1, \ldots m_n, m_0', \ldots m_n' \in \mathbb{C}$ and $c_4' \in U(\bar{\mathfrak{n}})$, which is of the desired form. Finally, we have that

$$\begin{aligned} & c_5 x_{\alpha_1+\alpha_2}(-2)^{k_0+k_1+k_2-m-n+1} x_{\alpha_1}(-1)^n x_{\alpha_1+\alpha_2}(-1)^{m+1} \\ = \quad & c_5' [x_{\alpha_2}(0), \ldots [x_{\alpha_2}(0), R^1_{2(k_0+k_1+k_2-m-n+1)+n+m+1}] \ldots] \\ & + c_5'' x_{\alpha_1}(-1)^{n+1} x_{\alpha_1+\alpha_2}(-1)^{m+1} \end{aligned}$$

for some $c_5', c_5'' \in U(\bar{\mathfrak{n}})$, which is of the desired form. Hence, we may write

$$a = b' + c' x_{\alpha_1}(-1)^{n+1} x_{\alpha_1+\alpha_2}(-1)^{m+1} + d' x_{\alpha_1+\alpha_2}(-1)^{m+2}$$

for some $b' \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$ and $c', d' \in U(\bar{\mathfrak{n}})$, completing our induction.

So, in particular, we may find $b \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$ and $c, d \in U(\bar{\mathfrak{n}})$ such that

$$a = b + c x_{\alpha_1}(-1)^{k_2+1} x_{\alpha_1+\alpha_2}(-1)^{k_0+1} + d x_{\alpha_1+\alpha_2}(-1)^{k_0+2}.$$

Clearly we have that $b, dx_{\alpha_1+\alpha_2}(-1)^{k_0+2} \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$. We also have that

$$\begin{aligned} & cx_{\alpha_1}(-1)^{k_2+1}x_{\alpha_1+\alpha_2}(-1)^{k_0+1} \\ = \; & c'[x_{\alpha_2}(0), \ldots [x_{\alpha_2}(0), x_{\alpha_1}(-1)^{k_0+k_2+2}] \ldots] \end{aligned}$$

which is an element of $I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$. So we have that $a \in I_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$, a contradiction, and so $\Lambda \neq (k_0+1)\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$, completing our proof.

Remark 4.1.2 The proof of the main theorem is similar in structure to the proof found in [CalLM2], in that we show that our "minimal counterexample" element a cannot be in $(\mathrm{Ker} f_\Lambda) \setminus I_\Lambda$ for each Λ by eliminating each Λ in a certain order. In [CalLM2], it was shown that $\Lambda \neq k\Lambda_1$, $\Lambda \neq \Lambda_0 + (k-1)\Lambda_1$, $\Lambda \neq 2\Lambda_0 + (k-2)\Lambda_1, \ldots, \Lambda \neq k\Lambda_0$, in that order. This choice was incredibly important in the proof. In the proof above, the order in which the possible Λ are eliminated is equally important, but more choices have to be made. We show that $\Lambda \neq k\Lambda_i$, $\Lambda \neq \Lambda_0 + (k-1)\Lambda_i$, $\Lambda \neq 2\Lambda_0 + (k-2)\Lambda_i, \ldots, \Lambda \neq k\Lambda_0$ for $i = 1, 2$. Then, we proceed to show that $\Lambda \neq k_1\Lambda_1 + k_2\Lambda_2$ for each $k_1, k_2 \in \mathbb{N}$ satisfying $k_1 + k_2 = k$, and this can only be shown once all $\Lambda = k_0\Lambda_0 + k_i\Lambda_i$ with $k_0, k_i \in \mathbb{N}$ satisfying $k_0 + k_i = k$, $i = 1, 2$ have been ruled out. Once we have $\Lambda \neq k_1\Lambda_1 + k_2\Lambda_2$, we then proceed to show that $\Lambda \neq k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$ for all remaining choices of $k_0, k_1, k_2 \in \mathbb{N}$ with $k_0 + k_1 + k_2 = k$ by taking $k_0 = 1, k_0 = 2, \ldots$ until all remaining choices of Λ have been eliminated.

Remark 4.1.3 The part of the proof which considers weights of the form $k\Lambda_i$ and $k_0\Lambda_0 + k_i\Lambda_i$ for $i = 0, 1, 2$ uses generalizations of ideas of [CalLM1]-[CalLM3]. These ideas no longer work in the general case, so a new method was developed to handle the remaining cases. This method "rebuilds" the element the "minimal counterexample" element a in order to reach certain desired contradictions. This new method works equally well for weights of the form $k\Lambda_i$ and $k_0\Lambda_0 + k_i\Lambda_i$.

Remark 4.1.4 The method developed in the above proof, which "rebuilds" our "minimal counterexample" element a to show that it is actually in the ideal I_Λ can be used to show all the presentations considered in [CalLM1]–[CalLM3] in the type A case. In this sense, it is a unifying method, and should be able to be generalized to prove presentations for the principal subspaces of all the standard $\widehat{\mathfrak{sl}(n+1)}$-modules, but at this stage it is not completely clear how this can be done in the case where $n > 2$ and $k > 1$.

Remark 4.1.5 Unlike in [CalLM2] and [C3], we use only intertwining operators for level 1 standard modules to prove certain inclusions of kernels inside other kernels. For example, when we we wanted to show that

$$\mathrm{Ker} f_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2} \subset \mathrm{Ker} f_{k_0\Lambda_0+(k_1+1)\Lambda_1+k_2\Lambda_2}$$

in (4.2), we simply applied the operator $1^{\otimes k_0} \otimes \mathcal{Y}_c(e^{\lambda_1}, x) \otimes 1^{\otimes k_1+k_2}$ to

$$a \cdot v_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2} = 0$$

to obtain

$$a \cdot v_{k_0\Lambda_0+(k_1+1)\Lambda_1+k_2\Lambda_2} = 0,$$

so that if $a \in \mathrm{Ker} f_{(k_0+1)\Lambda_0+k_1\Lambda_1+k_2\Lambda_2}$ then $a \in \mathrm{Ker} f_{k_0\Lambda_0+(k_1+1)\Lambda_1+k_2\Lambda_2}$. Such methods work equally well in the cases considered in [CalLM2] and [C3] in showing similar inclusions. Using these types of maps gives an alternate technique for proving such inclusions which does not require use of intertwining operators for higher level standard modules as in [CalLM2] and [C3].

Chapter 5

Presentations of principal subspaces of standard modules and a completion of $U(\bar{\mathfrak{n}})$

5.1 A reformulation of the presentation problem

In this section we reformulate Conjecture 3.1.1, along with all known presentations of principal subspaces, in terms of a natural completion of $U(\bar{\mathfrak{n}})$, which we denote by $\widetilde{U(\bar{\mathfrak{n}})}$. A version of this completion was constructed in [LW3], and we recall this construction, suitably adapted to our present setting, in the appendix. In this section only, for $\alpha \in \Delta$ and $n \in \mathbb{Z}$, will use the notation $x_\alpha(n)$ for completion elements $X_\alpha(n)$ from the appendix, and no confusion should arise.

We may define a natural "lifting" of the maps f_Λ:

$$\begin{aligned} \widetilde{f_\Lambda} : \widetilde{U(\bar{\mathfrak{n}})} &\longrightarrow W(\Lambda) \\ a &\mapsto a \cdot v_\Lambda. \end{aligned} \tag{5.1}$$

Indeed, given $a \in \widetilde{U(\bar{\mathfrak{n}})}$, we may uniquely express a as $a = b + c$ for some $b \in U(\bar{\mathfrak{n}}_-)$ and $c \in \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}$ (by (7.7)), and define $a \cdot v_\Lambda = b \cdot v_\Lambda$. That is, we let c act as 0.

We now reformulate Conjecture 3.1.1 in terms of finding $\mathrm{Ker}\widetilde{f_\Lambda}$. Recall the formal sums

$$R^i_t = \sum_{m_1+\cdots+m_{k+1}=-t} x_{\alpha_i}(m_1)\cdots x_{\alpha_i}(m_{k+1}),$$

which are well defined as operators on each $W(\Lambda)$. It is important to note that each R^i_t is not an element of $\widetilde{U(\bar{\mathfrak{n}})}$, so we seek natural representatives for R^i_t in $\widetilde{U(\bar{\mathfrak{n}})}$, in the sense that, when viewed as operators on $W(\Lambda)$, these representatives are equal to R^i_t.

Let $\mathcal{A}$ denote the set of finite sequences of integers. Given a sequence of integers

$A = (m_1, \dots, m_{k+1}) \in \mathcal{A}$, define a function

$$\begin{array}{rcl} \# : \mathbb{Z} \times A & \longrightarrow & \mathbb{N} \\ (n, A) & \mapsto & \text{number of occurrences of } \; n \; \text{ in } A. \end{array} \tag{5.2}$$

For any sequence in $(m_1, ..., m_{k+1}) \in \mathcal{A}$, define

$$A_{m_1,\dots,m_{k+1}} = \{\#(n, A) | n \in \mathbb{Z}\} \setminus \{0\} = \{n_1, \dots, n_j\}$$

where $n_1, \dots, n_j$ are positive integers and $n_1 + \cdots + n_j = k + 1$. Define integers

$$c_{m_1,\dots,m_{k+1}} = \binom{k+1}{n_1, ..., n_j} = \frac{(k+1)!}{(n_1)! \dots (n_j)!}.$$

We define

$$\mathcal{R}^i_t = R^i_{-1,t} + \sum_{\substack{m_1 \leq \cdots \leq m_{k+1}, \\ m_1 + \cdots + m_{k+1} = -t, \\ m_{k+1} \geq 0}} c_{m_1,\dots,m_{k+1}} x_{\alpha_i}(m_1) \cdots x_{\alpha_i}(m_{k+1}), \tag{5.3}$$

which is clearly in $\widetilde{U(\bar{\mathfrak{n}})}$. We may also write, for each $\mathcal{R}^i_t$,

$$\mathcal{R}^i_t = R^i_{M,t} + \sum_{\substack{m_1 \leq \cdots \leq m_{k+1}, \\ m_1 + \cdots + m_{k+1} = -t, \\ m_{k+1} \geq M+1}} c_{m_1,\dots,m_{k+1}} x_{\alpha_i}(m_1) \cdots x_{\alpha_i}(m_{k+1}), \tag{5.4}$$

and, as elements of $\widetilde{U(\bar{\mathfrak{n}})}$, (5.3) and (5.4) are equal.

Remark 5.1.1 As mentioned above, the formal sums

$$R^i_t = \sum_{m_1 + \cdots + m_{k+1} = -t} x_{\alpha_i}(m_1) \cdots x_{\alpha_i}(m_{k+1}).$$

are not elements of $\widetilde{U(\bar{\mathfrak{n}})}$. Informally, R^i_t is in a sense a "limit" of (5.4), i.e.

$$R^i_t = \lim_{M \to \infty} \Bigg(R^i_{M,t} + \sum_{\substack{m_1 \leq \cdots \leq m_{k+1}, \\ m_1 + \cdots + m_{k+1} = -t, \\ m_{k+1} \geq M+1}} c_{m_1,\dots,m_{k+1}} x_{\alpha_i}(m_1) \cdots x_{\alpha_i}(m_{k+1}) \Bigg),$$

where infinitely many relations in $\widetilde{I}$ need to be applied to obtain R_t^i from $\mathcal{R}_t^i$. However, as operators on $W(\Lambda)$, $\mathcal{R}_t^i$ and R_t^i are equal.

Lemma 5.1.2 *Let $\alpha \in \Delta_+$ and $m \in \mathbb{N}$. Then, for any $i = 1, \ldots, n$ and $t \in \mathbb{Z}$ we have that*

$$\mathcal{R}_t^i x_\alpha(-m) = x_\alpha(-m)\mathcal{R}_t^i + x_\alpha(0)R_{t+m}^i + c$$

for some $c \in \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}$. In particular,

$$\mathcal{R}_t^i x_\alpha(-m) \in I_{k\Lambda_0} + \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}.$$

Proof: First, suppose that $\alpha + \alpha_i \in \Delta_+$. We may write

$$\mathcal{R}_t^i = R_{m,t}^i + \sum_{\substack{m_1 \leq \cdots \leq m_{k+1}, \\ m_1 + \cdots + m_{k+1} = -t, \\ m_{k+1} \geq m+1}} c_{m_1,\ldots,m_{k+1}} x_{\alpha_i}(m_1) \cdots x_{\alpha_i}(m_{k+1}).$$

By definition of $\widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}$,

$$\sum_{\substack{m_1 \leq \cdots \leq m_{k+1}, \\ m_1 + \cdots + m_{k+1} = -t, \\ m_{k+1} \geq m+1}} c_{m_1,\ldots,m_{k+1}} x_{\alpha_i}(m_1) \cdots x_{\alpha_i}(m_{k+1}) x_\alpha(-m) \in \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}.$$

For $R^i_{m,t}x_\alpha(-m)$, we may write

$$\begin{aligned}
&R^i_{m,t}x_\alpha(-m)\\
&= \sum_{\substack{m_1+\cdots+m_{k+1}=-t,\\ m_1,\ldots,m_{k+1}\le m}} x_{\alpha_i}(m_1)\cdots x_{\alpha_i}(m_{k+1})x_\alpha(-m)\\
&= \sum_{j=1}^{k+1}\sum_{\substack{m_1+\cdots+m_{k+1}=-t,\\ m_1,\ldots,m_{k+1}\le m}} C_{\alpha_i,\alpha}x_{\alpha_i}(m_1)\cdots x_{\alpha_i+\alpha}(m_j-m)\cdots x_{\alpha_i}(m_{k+1})\\
&\quad +x_\alpha(-m)R^i_{m,t}\\
&= \sum_{j=1}^{k+1}\sum_{\substack{m_1+\cdots+m_{k+1}=-t-m,\\ m_1,\ldots,m_{k+1}\le m}} C_{\alpha_i,\alpha}x_{\alpha_i}(m_1)\cdots x_{\alpha_i+\alpha}(m_j)\cdots x_{\alpha_i}(m_{k+1})\\
&\quad +b+x_\alpha(-m)R^i_{m,t}
\end{aligned}$$

for some $b\in U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+$. We have that

$$\begin{aligned}
&\sum_{j=1}^{k+1}\sum_{\substack{m_1+\cdots+m_{k+1}=-t-m,\\ m_1,\ldots,m_{k+1}\le m}} C_{\alpha_i,\alpha}x_{\alpha_i}(m_1)\cdots\\
&\qquad\qquad \cdots x_{\alpha_i+\alpha}(m_j)\cdots x_{\alpha_i}(m_{k+1})+b+x_\alpha(-m)R^i_{m,t}\\
&= [x_\alpha(0),R^i_{m+t}]+b+x_\alpha(-m)R^i_{m,t},
\end{aligned}$$

establishing our claim when $\alpha+\alpha_i\in\Delta_+$. If $\alpha+\alpha_i\notin\Delta_+$ the claim is clear since

$$\mathcal{R}^i_t x_\alpha(-m)=x_\alpha(-m)\mathcal{R}^i_t\in I_{k\Lambda_0}+\widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+},$$

concluding our proof.

Using an almost identical argument, we have that

Corollary 5.1.3 *If $a\in U(\bar{\mathfrak{n}}_-)$ and $t\in\mathbb{Z}$, we have that*

$$\mathcal{R}^i_t a\in I_{k\Lambda_0}+\widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}.$$

Proof: It suffices to show that the claim holds for monomials

$$x_{\beta_1}(-m_1)\ldots x_{\beta_j}(-m_j) \in U(\bar{\mathfrak{n}}).$$

This follows immediately using the same argument as above, and writing

$$\mathcal{R}_t^i = R^i_{m_1+\cdots+m_j,t} + \sum_{\substack{m_1 \leq \cdots \leq m_{k+1}, \\ m_1+\cdots+m_{k+1}=-t, \\ m_{k+1} \geq m_1+\cdots+m_j+1}} c_{m_1,\ldots,m_{k+1}} x_{\alpha_i}(m_1)\cdots x_{\alpha_i}(m_{k+1}).$$

As in [C1]-[C2] and [CalLM3], let $\mathcal{J}$ be the two sided ideal of $\widetilde{U(\bar{\mathfrak{n}})}$ generated by the $\mathcal{R}_t^i$, $i = 1, \ldots, n$ and $t \geq k+1$. As in [CalLM3], we have the following theorem:

Theorem 5.1.4 *We may describe $I_{k\Lambda_0}$ by:*

$$I_{k\Lambda_0} \equiv \mathcal{J} \quad \text{modulo} \quad \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}. \tag{5.5}$$

and moreover, for I_Λ, we have:

$$I_\Lambda \equiv \mathcal{J} + \sum_{\alpha\in\Delta_+} U(\bar{\mathfrak{n}}) x_\alpha(-1)^{k+1-\langle\alpha,\Lambda\rangle} \quad \text{modulo} \quad \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}. \tag{5.6}$$

Proof: We first show that

$$I_{k\Lambda_0} \subset \mathcal{J} \text{ modulo } \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}.$$

Indeed, any element $a \in I_{k\Lambda_0}$ may be written as

$$a = \sum_{i=1}^n a_i R^i_{-1,t} + b$$

for some $a_i \in U(\bar{\mathfrak{n}})$ and $b \in U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+$. It suffices to show that each $a_i R^i_{-1,t} \in \mathcal{J} + \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}$. Indeed, we may write $R^i_{-1,t} = \mathcal{R}^i_t + c$ for some $c \in \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}$, and we clearly have that $a_i R^i_{-1,t} = a_i R^i_t + a_i c \in \mathcal{J} + \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}$.

It remains to show that

$$\mathcal{J} \subset I_{k\Lambda_0} \text{ modulo } \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}.$$

It suffices to prove

$$a\mathcal{R}^i_t b \in I_{k\Lambda_0} + \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}$$

for all $a, b \in \widetilde{U(\bar{\mathfrak{n}})}$. By (7.7), we may write $b = b_1 + b_2$ for some $b_1 \in U(\bar{\mathfrak{n}}_-)$ and $b_2 \in \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}$. Clearly $a\mathcal{R}_t^i b_2 \in \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}$, and so

$$a\mathcal{R}_t^i b \equiv a\mathcal{R}_t^i b_1 \text{ modulo } \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}.$$

By Corollary 5.1.3, we have that

$$\mathcal{R}_t^i b_1 \in I_{k\Lambda_0} + \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+},$$

so it suffices to show that

$$aR_{-1,t}^i \in I_{k\Lambda_0} + \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}.$$

Using the notation from the appendix, we have that $a = [\mu]$ for some $\mu \in F(\Delta_+)$, and we may write

$$\mu = \sum_{c \in \mathrm{Supp}(\mu)} \mu(c)X(c) = \sum_{c \in \mathrm{Supp}_t(\mu)} \mu(c)X(c) + \sum_{c \in \mathrm{Supp}(\mu) \setminus \mathrm{Supp}_t(\mu)} \mu(c)X(c).$$

The sum $\sum_{c \in \mathrm{Supp}_t} \mu(c)X(c)$ is finite, so we have that

$$\sum_{c \in \mathrm{Supp}_t(\mu)} [\mu(c)X(c)]R_{-1,t}^i \in I_{k\Lambda_0},$$

and by definition of $\widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}$ we have that

$$\sum_{c \in \mathrm{Supp}(\mu) \setminus \mathrm{Supp}_t(\mu)} [\mu(c)X(c)]R_{-1,t}^i \in \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+},$$

establishing

$$I_{k\Lambda_0} \equiv \mathcal{J} \text{ modulo } \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}.$$

The fact that

$$I_\Lambda \equiv \mathcal{J} + \sum_{\alpha \in \Delta_+} U(\bar{\mathfrak{n}})x_\alpha(-1)^{k+1-\langle \alpha, \Lambda \rangle} \quad \text{modulo} \quad \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}$$

follows immediately, establishing our theorem.

As a consequence of Theorem 5.1.4, along with the results of [CalLM1] - [CalLM3] and Chapter 4, we have that:

Theorem 5.1.5 *In the case where $\mathfrak{g} = \mathfrak{sl}(n+1)$ with:*

- $n = 1$ *and* $\Lambda = k_0\Lambda_0 + k_1\Lambda_1$ *with* $k_0 + k_1 = k \geq 1$
- $n = 2$ *and* $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$ *with* $k_0 + k_1 + k_2 = k \geq 1$
- $n \geq 3$ *and* $\Lambda = \Lambda_i$ *with* $i = 0, \ldots, n$

or $\mathfrak{g}$ *is of type* D *or* E *with* $k = 1$ *we have that*

$$Ker f_\Lambda \equiv \widetilde{I_\Lambda} \quad \text{modulo} \quad \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}.$$

We reformulate Conjecture 3.1.1 as follows:

Conjecture 5.1.6 *Suppose* $\mathfrak{g} = \mathfrak{sl}(n+1)$, $k_0, \ldots, k_n, k \in \mathbb{N}$ *with* $k \geq 1$ *and* $k_0 + \cdots + k_n = k$. *For each* $\Lambda = k_0\Lambda_0 + \cdots + k_n\Lambda_n$, *we have that*

$$Ker f_\Lambda \equiv \widetilde{I_\Lambda} \quad \text{modulo} \quad \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+}$$

or that

Conjecture 5.1.7 *In the context of Conjecture 5.1.6, for each* $\Lambda = k_0\Lambda_0 + \cdots + k_n\Lambda_n$, *we have that*

$$Ker \widetilde{f_\Lambda} = \widetilde{I_\Lambda}.$$

Chapter 6

Exact sequences and multigraded dimensions

6.1 Exact sequences

In this section, we construct exact sequences among the principal subspaces of certain standard modules, and use these to find multigraded dimensions.

Given $\lambda \in P$ and character $\nu : Q \longrightarrow \mathbb{C}^*$, we define a map $\tau_{\lambda,\nu}$ on $\bar{\mathfrak{n}}$ by

$$\tau_{\lambda,\nu}(x_\alpha(m)) = \nu(\alpha)x_\alpha(m - \langle\lambda, \alpha\rangle)$$

for $\alpha \in \Delta_+$ and $m \in \mathbb{Z}$. It is easy to see that $\tau_{\lambda,\nu}$ is an automorphism of $\bar{\mathfrak{n}}$. The map $\tau_{\lambda,\nu}$ extends canonically to an automorphism of $U(\bar{\mathfrak{n}})$, also denoted by $\tau_{\lambda,\nu}$, given by

$$\tau_{\lambda,\nu}(x_{\beta_1}(m_1)\cdots x_{\beta_r}(m_r)) = \nu(\beta_1+\cdots+\beta_r)x_{\beta_1}(m_1-\langle\lambda, \beta_1\rangle)\cdots x_{\beta_r}(m_r-\langle\lambda, \beta_r\rangle) \quad (6.1)$$

for $\beta_1, \ldots, \beta_r \in \Delta_+$ and $m_1, \ldots, m_r \in \mathbb{Z}$. In particular, we have that

$$e_\lambda^{\otimes k}(a \cdot v_\Lambda) = \tau_{\lambda,c_{-\lambda}}(a) \cdot e_\lambda^{\otimes k} v_\Lambda \quad (6.2)$$

where $\lambda \in P$, Λ is a dominant integral weight of $\widehat{\mathfrak{sl}(n+1)}$, and $c_{-\lambda}(\alpha) = c(-\lambda, \alpha)$ for all $\alpha \in \Delta_+$.

For each $j = 1, \ldots, n$, set $\omega_j = \alpha_j - \lambda_j$. For each $1 \le i \le n-1$ and $k_i, k_{i+1} \in \mathbb{N}$ with $k_i + k_{i+1} = k \ge 1$, define maps

$$\phi_i = e_{\omega_i}^{\otimes k} \circ (1^{\otimes k_i} \otimes \mathcal{Y}_c(e^{\lambda_{i-1}}, x)^{\otimes k_{i+1}})$$

$$\psi_i = e_{\omega_{i+1}}^{\otimes k} \circ (1^{\otimes k_i} \otimes \mathcal{Y}_c(e^{\lambda_{i+2}}, x)^{\otimes k_{i+1}})$$

In the case that $i = 1$, we take $\phi_1 = e_{\omega_1}^{\otimes k}$ and in the case that $i = n-1$ we take $\psi_{n-1} = e_{\omega_n}^{\otimes k}$.

Theorem 6.1.1 *For every $k_i, k_{i+1} \in \mathbb{N}$ with $k_i + k_{i+1} = k$ and $k \geq 1$, we have*

$$\phi_i : W(k_i\Lambda_i + k_{i+1}\Lambda_{i+1}) \to W(k_i\Lambda_0 + k_{i+1}\Lambda_i) \tag{6.3}$$

and

$$\psi_i : W(k_i\Lambda_i + k_{i+1}\Lambda_{i+1}) \to W(k_{i+1}\Lambda_0 + k_i\Lambda_{i+1}) \tag{6.4}$$

Moreover, for $r_1, \ldots, r_n, s \in \mathbb{Z}$,

$$\phi_i : W(k_i\Lambda_i + k_{i+1}\Lambda_{i+1})'_{r_1,\ldots,r_n;s} \tag{6.5}$$

$$\to W(k_i\Lambda_0 + k_{i+1}\Lambda_i)'_{r_1,\ldots,r_i+k_i,\ldots,r_n;s-r_{i-1}+r_i-r_{i+1}+k_i} \tag{6.6}$$

and

$$\psi_i : W(k_i\Lambda_i + k_{i+1}\Lambda_{i+1})'_{r_1,\ldots,r_n;s} \tag{6.7}$$

$$\to W(k_{i+1}\Lambda_0 + k_i\Lambda_{i+1})'_{r_1,\ldots,r_{i+1}+k_{i+1},\ldots,r_n;s-r_i+r_{i+1}-r_{i+2}+k_i}, \tag{6.8}$$

where we take $r_0 = r_{n+1} = 0$.

Proof: We prove only (6.3) since (6.4) follows analogously. Let $a \cdot v_{k_i\Lambda_i+k_{i+1}\Lambda_{i+1}} \in W(k_i\Lambda_i + k_{i+1}\Lambda_{i+1})$ for some $a \in U(\bar{\mathfrak{n}})$. We have that

$$\begin{aligned}
&\phi_i(a \cdot v_{k_i\Lambda_i+k_{i+1}\Lambda_{i+1}}) \\
&= \phi_i(a \cdot (\underbrace{e^{\lambda_i} \otimes \cdots \otimes e^{\lambda_i}}_{k_i-times} \otimes \underbrace{e^{\lambda_{i+1}} \otimes \cdots \otimes e^{\lambda_{i+1}}}_{k_{i+1}-times})) \\
&= \left(e_{\omega_i}^{\otimes k} \circ (1^{\otimes k_i} \otimes \mathcal{Y}_c(e^{\lambda_{i-1}}, x)^{\otimes k_{i+1}})\right)(a \cdot (\underbrace{e^{\lambda_i} \otimes \cdots \otimes e^{\lambda_i}}_{k_i-times} \otimes \underbrace{e^{\lambda_{i+1}} \otimes \cdots \otimes e^{\lambda_{i+1}}}_{k_{i+1}-times})) \\
&= e_{\omega_i}^{\otimes k}(a \cdot (\underbrace{e^{\lambda_i} \otimes \cdots \otimes e^{\lambda_i}}_{k_i-times} \otimes \underbrace{e_{\lambda_{i-1}}e^{\lambda_{i+1}} \otimes \cdots \otimes e_{\lambda_{i-1}}e^{\lambda_{i+1}}}_{k_{i+1}-times})) \\
&= c_1\tau_{\omega_i,c_{-\omega_i}}(a) \cdot (\underbrace{e^{\alpha_i} \otimes \cdots \otimes e^{\alpha_i}}_{k_i-times} \otimes \underbrace{e^{\lambda_i} \otimes \cdots \otimes e^{\lambda_i}}_{k_{i+1}-times}) \\
&= c_2\tau_{\omega_i,c_{-\omega_i}}(a)x_{\alpha_i}(-1)^{k_i} \cdot (\underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{k_i-times} \otimes \underbrace{e^{\lambda_i} \otimes \cdots \otimes e^{\lambda_i}}_{k_{i+1}-times}) \\
&\in W(k_i\Lambda_0 + k_{i+1}\Lambda_i)
\end{aligned}$$

for some constants $c_1, c_2 \in \mathbb{C}$. The fourth equality follows from the fact that $\lambda_{i-1} + \lambda_{i+1} + \omega_i = \lambda_i$. This concludes our proof.

Using the presentations (3.10), we construct exact sequences which give the multi-graded dimensions of certain principal subspaces (compare to [C1]).

Theorem 6.1.2 *Let $k \geq 1$. For any i with $1 \leq i \leq n-1$ and $k_i, k_{i+1} \in \mathbb{N}$ such that $k_i + k_{i+1} = k$, the sequences:*

$$\begin{aligned} &W(k_i\Lambda_i + k_{i+1}\Lambda_{i+1}) \xrightarrow{\phi_i} \\ &\quad W(k_i\Lambda_0 + k_{i+1}\Lambda_i) \xrightarrow{1^{\otimes k_i-1} \otimes \mathcal{Y}_c(e^{\lambda_i},x) \otimes 1^{\otimes k_{i+1}}} \\ &\qquad W((k_i-1)\Lambda_0 + (k_{i+1}+1)\Lambda_i) \longrightarrow 0, \end{aligned} \tag{6.9}$$

when $k_i \geq 1$, and

$$\begin{aligned} &W(k_i\Lambda_i + k_{i+1}\Lambda_{i+1}) \xrightarrow{\psi_i} \\ &\quad W(k_{i+1}\Lambda_0 + k_i\Lambda_{i+1}) \xrightarrow{1^{\otimes k_{i+1}-1} \otimes \mathcal{Y}_c(e^{\lambda_{i+1}},x) \otimes 1^{\otimes k_i}} \\ &\qquad W((k_{i+1}-1)\Lambda_0 + (k_i+1)\Lambda_{i+1}) \longrightarrow 0, \end{aligned} \tag{6.10}$$

when $k_{i+1} \geq 1$, are exact.

Proof: We prove that (6.9) is exact. The exactness of (6.10) can be proved analogously.

We first show that $\mathrm{Im}\phi_i \subset \mathrm{Ker}(1^{\otimes k_i-1} \otimes \mathcal{Y}_c(e^{\lambda_i},x) \otimes 1^{\otimes k_{i+1}})$. Suppose that $w \in \mathrm{Im}\phi_i$. We have that

$$(1^{\otimes k_i-1} \otimes \mathcal{Y}_c(e^{\lambda_i},x) \otimes 1^{\otimes k_{i+1}})(w) = vx_{\alpha_i}(-1)^{k_i} \cdot v_{k_i\Lambda_0+k_{i+1}\Lambda_i} = 0$$

for some $v \in U(\bar{\mathfrak{n}})$, and so $w \in \mathrm{Ker}(1^{\otimes k_i-1} \otimes \mathcal{Y}_c(e^{\lambda_i},x) \otimes 1^{\otimes k_{i+1}})$. Hence $\mathrm{Im}\phi_i \subset \mathrm{Ker}(1^{\otimes k_i-1} \otimes \mathcal{Y}_c(e^{\lambda_i},x) \otimes 1^{\otimes k_{i+1}})$.

We now show that $\mathrm{Ker}(1^{\otimes k_i-1} \otimes \mathcal{Y}_c(e^{\lambda_i},x) \otimes 1^{\otimes k_{i+1}}) \subset \mathrm{Im}\phi_i$ by characterizing the elements of each set. If $w \in \mathrm{Ker}(1^{\otimes k_i-1} \otimes \mathcal{Y}_c(e^{\lambda_i},x) \otimes 1^{\otimes k_{i+1}})$, we may write $w = f_{k_i\Lambda_0+k_{i+1}\Lambda_i}(u)$ for some $u \in U(\bar{\mathfrak{n}})$. We have that

$$(1^{\otimes k_i-1} \otimes \mathcal{Y}_c(e^{\lambda_i},x) \otimes 1^{\otimes k_{i+1}})(f_{k_i\Lambda_0+k_{i+1}\Lambda_{k_i}}(u)) = 0 \text{ iff } f_{(k_i-1)\Lambda_0+(k_{i+1}+1)\Lambda_i}(u) = 0$$

and by (3.10) we have

$$f_{(k_i-1)\Lambda_0+(k_{i+1}+1)\Lambda_i}(u) = 0 \text{ iff } u \in I_{(k_i-1)\Lambda_0+(k_{i+1}+1)\Lambda_i}$$

so that

$$w = f_{k_i\Lambda_0+k_{i+1}\Lambda_i}(u) \in \mathrm{Ker}(1^{\otimes k_i-1} \otimes \mathcal{Y}_c(e^{\lambda_i}, x) \otimes 1^{\otimes k_{i+1}}) \text{ iff } u \in I_{(k_i-1)\Lambda_0+(k_{i+1}+1)\Lambda_i}.$$

On the other hand, if $w \in \mathrm{Im}\phi_i$, we may write

$$w = vx_{\alpha_i}(-1)^{k_i} \cdot v_{k_i\Lambda_0+k_{i+1}\Lambda_i} = f_{k_i\Lambda_0+k_{i+1}\Lambda_i}(vx_{\alpha_i}(-1)^{k_i})$$

for some $v \in U(\bar{\mathfrak{n}})$. We may also write

$$w = f_{k_i\Lambda_0+k_{i+1}\Lambda_i}(u)$$

for some $u \in U(\bar{\mathfrak{n}})$. Putting these together, we have that

$$(u - vx_{\alpha_i}(-1)^{k_i}) \cdot v_{k_i\Lambda_0+k_{i+1}\Lambda_i} = 0$$

which implies

$$u - vx_{\alpha_i}(-1)^{k_i} \in I_{k_i\Lambda_0+k_{i+1}\Lambda_i}.$$

We therefore have that

$$w = f_{k_i\Lambda_0+k_{i+1}\Lambda_i}(u) \in \mathrm{Im}\phi_i \text{ iff } u \in I_{k_i\Lambda_0+k_{i+1}\Lambda_i} + U(\bar{\mathfrak{n}})x_{\alpha_i}(-1)^{k_i}$$

Noticing that

$$I_{(k_i-1)\Lambda_0+(k_{i+1}+1)\Lambda_i} = I_{k\Lambda_0} + U(\bar{\mathfrak{n}})x_{\alpha_i}(-1)^{k_i} \subset I_{k_i\Lambda_0+k_{i+1}\Lambda_i} + U(\bar{\mathfrak{n}})x_{\alpha_i}(-1)^{k_i},$$

we have that

$$\begin{aligned} w = f_{k_i\Lambda_0+k_{i+1}\Lambda_i}(u) &\in \mathrm{Ker}(1^{\otimes k_i-1} \otimes \mathcal{Y}_c(e^{\lambda_i}, x) \otimes 1^{\otimes k_{i+1}}) \\ &\Leftrightarrow \quad u \in I_{(k_i-1)\Lambda_0+(k_{i+1}+1)\Lambda_i} \\ &\Rightarrow \quad u \in I_{k_i\Lambda_0+k_{i+1}\Lambda_i} + U(\bar{\mathfrak{n}})x_{\alpha_i}(-1)^{k_i} \\ &\Leftrightarrow \quad w = f_{k_i\Lambda_0+k_{i+1}\Lambda_i}(u) \in \mathrm{Im}\phi_i, \end{aligned}$$

completing our proof.

Remark 6.1.3 Notice that, in general, the first map in each exact sequence is not injective like it is in [CLM1]–[CLM2],[C1]–[C2], and [CalLM1]–[CalLM3]. In fact, there are only a few cases where injectivity holds, given in the corollaries below.

Corollary 6.1.4 *In the setting of Theorem 6.1.2, following sequences are exact:*

$$
\begin{aligned}
&0 \longrightarrow W(k_1\Lambda_1 + k_2\Lambda_2) \stackrel{e_{\omega_1}^{\otimes k}}{\longrightarrow} \\
&\qquad W(k_1\Lambda_0 + k_2\Lambda_1) \stackrel{1^{\otimes k_i - 1}\otimes \mathcal{Y}_c(e^{\lambda_i},x)\otimes 1^{\otimes k_{i+1}}}{\longrightarrow} \\
&\qquad\qquad W((k_1-1)\Lambda_0 + (k_2+1)\Lambda_1) \longrightarrow 0
\end{aligned}
\tag{6.11}
$$

and

$$
\begin{aligned}
&0 \longrightarrow W(k_{n-1}\Lambda_{n-1} + k_n\Lambda_n) \stackrel{e_{\omega_n}^{\otimes k}}{\longrightarrow} \\
&\qquad W(k_n\Lambda_0 + k_{n-1}\Lambda_n) \stackrel{1^{\otimes k_i - 1}\otimes \mathcal{Y}_c(e^{\lambda_i},x)\otimes 1^{\otimes k_{i+1}}}{\longrightarrow} \\
&\qquad\qquad W((k_n-1)\Lambda_0 + (k_{n-1}+1)\Lambda_n) \longrightarrow 0
\end{aligned}
\tag{6.12}
$$

Remark 6.1.5 It is important to note that (6.9), (6.10), (6.11), and (6.12) are fundamentally different from the exact sequences used in [C1] and [CalLM2]. In [C1] and [CalLM2], exact sequences are constructed using intertwining operators among level k standard modules. The sequences (6.9), (6.10), (6.11), and (6.12) only require intertwining operators among level 1 standard modules and recover the same information about multigraded dimensions, as we will see below.

Corollary 6.1.6 *For each $i = 1, \ldots, n$ the following sequences are exact:*

$$
0 \longrightarrow W(k\Lambda_i) \stackrel{e_{\omega_i}^{\otimes k}}{\longrightarrow} W(k\Lambda_0) \stackrel{1^{\otimes k-1}\otimes \mathcal{Y}_c(e^{\lambda_i},x)}{\longrightarrow} W((k-1)\Lambda_0 + \Lambda_i) \longrightarrow 0
$$

Remark 6.1.7 These exact sequences in Corollary 6.1.6 are the level k analogues of the exact sequences found in [CalLM3].

6.2 Multigraded dimensions

We now use the exact sequences (6.11) and (6.12) to obtain the multigraded dimensions $\chi'_{W(k_1\Lambda_1+k_2\Lambda_2)}(x_1, \ldots, x_n, q)$ and $\chi'_{W(k_{n-1}\Lambda_{n-1}+k_n\Lambda_n)}(x_1, \ldots, x_n, q)$.

Theorem 6.2.1 *Let $k \geq 1$. Let $k_1, k_2, k_{n-1}, k_n \in \mathbb{N}$ with $k_1 \geq 1$ and $k_n \geq 1$, such that*

$k_1+k_2=k$ and $k_{n-1}+k_n=k$. Then

$$\begin{aligned}
&\chi'_{W(k_1\Lambda_1+k_2\Lambda_2)}(x_1,\dots,x_n,q)= \\
&= \; x_1^{-k_1}\chi'_{W((k_1-1)\Lambda_0+(k_2+1)\Lambda_1)}(x_1q^{-1},x_2q,x_3\dots,x_n,q) \\
&\quad -x_1^{-k_1}\chi'_{W(k_1\Lambda_0+k_2\Lambda_1)}(x_1q^{-1},x_2q,x_3,\dots,x_n,q)
\end{aligned} \tag{6.13}$$

and

$$\begin{aligned}
&\chi'_{W(k_{n-1}\Lambda_{n-1}+k_n\Lambda_n)}(x_1,\dots,x_n,q)= \\
&= \; x_n^{-k_n}\chi'_{W((k_n-1)\Lambda_0+(k_{n-1}+1)\Lambda_n)}(x_1,\dots,x_{n-1}q,x_nq^{-1},q) \\
&\quad -x_n^{-k_n}\chi'_{W(k_n\Lambda_0+k_{n-1}\Lambda_n)}(x_1,\dots,x_{n-1}q,x_nq^{-1},q).
\end{aligned} \tag{6.14}$$

Proof: It is easy to see that the maps used in (6.11) and (6.12) have the property that:

$$W(k_1\Lambda_0+k_2\Lambda_1)'_{r_1,\dots,r_n,s} \overset{1^{\otimes k_1-1}\otimes\mathcal{Y}_c(e^{\lambda_1},x)\otimes 1^{\otimes k_2}}{\longrightarrow} W((k_1-1)\Lambda_0+(k_2+1)\Lambda_1)'_{r_1,\dots,r_n,s}$$

and

$$W(k_n\Lambda_0+k_{n-1}\Lambda_n)'_{r_1,\dots,r_n,s} \overset{1^{\otimes k_n-1}\otimes\mathcal{Y}_c(e^{\lambda_n},x)\otimes 1^{\otimes k_{n-1}}}{\longrightarrow} W((k_n-1)\Lambda_0+(k_{n-1}+1)\Lambda_n)'_{r_1,\dots,r_n,s}.$$

Combining this fact with the exactness of (6.11) and (6.12), along with (6.5) and (6.7) give

$$\begin{aligned}
&\chi'_{W(k_1\Lambda_0+k_2\Lambda_1)}(x_1,\dots,x_n,q)= \\
&\quad x_1^{k_1}q^{k_1}\chi'_{W(k_1\Lambda_1+k_2\Lambda_2)}(x_1q,x_2q^{-1},x_3\dots,x_n,q) \\
&\quad +\chi'_{W((k_1-1)\Lambda_0+(k_2+1)\Lambda_1)}(x_1,\dots,x_n,q)
\end{aligned} \tag{6.15}$$

and

$$\begin{aligned}
&\chi'_{W(k_n\Lambda_0+k_{n-1}\Lambda_n)}(x_1,\dots,x_n,q)= \\
&\quad x_n^{k_n}q^{k_n}\chi'_{W(k_{n-1}\Lambda_{n-1}+k_n\Lambda_n)}(x_1,\dots,x_{n-1}q^{-1},x_nq,q) \\
&\quad +\chi'_{W((k_n-1)\Lambda_0+(k_{n-1}+1)\Lambda_n)}(x_1,\dots,x_n,q).
\end{aligned} \tag{6.16}$$

which may be rewritten as

$$\begin{aligned}
&\chi'_{W(k_1\Lambda_1+k_2\Lambda_2)}(x_1q,x_2q^{-1},x_3\dots,x_n,q)= \\
&= \; x_1^{-k_1}q^{-k_1}\chi'_{W(k_1\Lambda_0+k_2\Lambda_1)}(x_1,\dots,x_n,q) \\
&\quad -x_1^{-k_1}q^{-k_1}\chi'_{W((k_1-1)\Lambda_0+(k_2+1)\Lambda_1)}(x_1,\dots,x_n,q)
\end{aligned} \tag{6.17}$$

and

$$\begin{aligned}\chi'_{W(k_{n-1}\Lambda_{n-1}+k_n\Lambda_n)}(x_1,\ldots,x_{n-1}q^{-1},x_nq,q) =& \\ = \quad x_n^{-k_n}q^{-k_n}\chi'_{W(k_n\Lambda_0+k_{n-1}\Lambda_n)}(x_1,\ldots,x_n,q)& \\ -x_n^{-k_n}q^{-k_n}\chi'_{W((k_n-1)\Lambda_0+(k_{n-1}+1)\Lambda_n)}(x_1,\ldots,x_n,q).&\end{aligned}\tag{6.18}$$

Making the substitutions

$$x_1 \mapsto x_1q^{-1},\ x_2 \mapsto x_2q$$

in (6.17) and

$$x_n \mapsto x_nq^{-1},\ x_{n-1} \mapsto x_{n-1}q$$

in (6.18) immediately proves our theorem.

We now use Theorem 6.2.1 to write down explicit expressions for $\chi'_{W(k_1\Lambda_1+k_2\Lambda_2)}(x_1,\ldots,x_n,q)$ and $\chi'_{W(k_{n-1}\Lambda_{n-1}+k_n\Lambda_n)}(x_1,\ldots,x_n,q)$. In [G], Georgiev obtained:

$$\begin{aligned}&\chi'_{W(k_0\Lambda_0+k_j\Lambda_j)}(x_1,\ldots,x_n,q) = \\ &= \sum \left(\frac{q^{{r_1^{(1)}}^2+\ldots+{r_1^{(k)}}^2+\sum_{t=1}^k r_1^{(t)}\delta_{1,j_t}}}{(q)_{r_1^{(1)}-r_1^{(2)}}\cdots(q)_{r_1^{(k-1)}-r_1^{(k)}}(q)_{r_1^{(k)}}}\right)\times \\ &\times\left(\frac{q^{{r_2^{(1)}}^2+\ldots+{r_2^{(k)}}^2-r_2^{(1)}r_1^{(1)}-\ldots-r_2^{(k)}r_1^{(k)}+\sum_{t=1}^k r_2^{(t)}\delta_{2,j_t}}}{(q)_{r_2^{(1)}-r_2^{(2)}}\cdots(q)_{r_2^{(k-1)}-r_2^{(k)}}(q)_{r_2^{(k)}}}\right)\times \\ &\times\cdots\times\left(\frac{q^{{r_n^{(1)}}^2+\ldots+{r_n^{(k)}}^2-r_n^{(1)}r_{n-1}^{(1)}-\ldots-r_n^{(k)}r_{n-1}^{(k)}+\sum_{t=1}^k r_n^{(t)}\delta_{n,j_t}}}{(q)_{r_n^{(1)}-r_n^{(2)}}\cdots(q)_{r_n^{(k-1)}-r_n^{(k)}}(q)_{r_n^{(k)}}}\right)x_1^{\sum_{i=1}^k r_1^{(i)}}\cdots x_n^{\sum_{i=1}^n r_n^{(i)}}\end{aligned}$$

where the sums are taken over decreasing sequences $r_j^{(1)} \geq r_j^{(2)} \geq \cdots \geq r_j^{(k)} \geq 0$ for each $j = 1,\ldots,n$ and $j_t = 0$ for $0 \leq t \leq k_0$ and $j_t = j$ for $k_0 < t \leq k$, $j = 1,\ldots,n$, where $(q)_r = \prod_{i=1}^r(1-q^i)$ and $(q)_0 = 1$. In particular, we have that

$$\begin{aligned}&\chi'_{W(k_1\Lambda_0+k_2\Lambda_1)}(x_1,\ldots,x_n,q) = \\ &= \sum_{\substack{r_1^{(1)}\geq\ldots\geq r_1^{(k)}\geq 0\\ \vdots \\ r_n^{(1)}\geq\ldots\geq r_n^{(k)}\geq 0}} \left(\frac{q^{{r_1^{(1)}}^2+\ldots+{r_1^{(k)}}^2+\sum_{t=k_1+1}^k r_1^{(t)}}}{(q)_{r_1^{(1)}-r_1^{(2)}}\cdots(q)_{r_1^{(k-1)}-r_1^{(k)}}(q)_{r_1^{(k)}}}\right)\times\end{aligned}$$

$$\times\left(\frac{q^{{r_2^{(1)}}^2+\ldots+{r_2^{(k)}}^2-r_2^{(1)}r_1^{(1)}-\ldots-r_2^{(k)}r_1^{(k)}}}{(q)_{r_2^{(1)}-r_2^{(2)}}\cdots(q)_{r_2^{(k-1)}-r_2^{(k)}}(q)_{r_2^{(k)}}}\right)\times$$

$$\times\cdots\times\left(\frac{q^{{r_n^{(1)}}^2+\ldots+{r_n^{(k)}}^2-r_n^{(1)}r_{n-1}^{(1)}-\ldots-r_n^{(k)}r_{n-1}^{(k)}}}{(q)_{r_n^{(1)}-r_n^{(2)}}\cdots(q)_{r_n^{(k-1)}-r_n^{(k)}}(q)_{r_n^{(k)}}}\right)x_1^{\sum_{i=1}^k r_1^{(i)}}\cdots x_n^{\sum_{i=1}^n r_n^{(i)}}$$

and

$$\chi'_{W(k_n\Lambda_0+k_{n-1}\Lambda_n)}(x_1,\ldots,x_n,q)=$$

$$=\sum_{\substack{r_1^{(1)}\geq\ldots\geq r_1^{(k)}\geq 0\\ \vdots\\ r_n^{(1)}\geq\ldots\geq r_n^{(k)}\geq 0}}\left(\frac{q^{{r_1^{(1)}}^2+\ldots+{r_1^{(k)}}^2}}{(q)_{r_1^{(1)}-r_1^{(2)}}\cdots(q)_{r_1^{(k-1)}-r_1^{(k)}}(q)_{r_1^{(k)}}}\right)\times$$

$$\times\left(\frac{q^{{r_2^{(1)}}^2+\ldots+{r_2^{(k)}}^2-r_2^{(1)}r_1^{(1)}-\ldots-r_2^{(k)}r_1^{(k)}+}}{(q)_{r_2^{(1)}-r_2^{(2)}}\cdots(q)_{r_2^{(k-1)}-r_2^{(k)}}(q)_{r_2^{(k)}}}\right)\times$$

$$\times\cdots\times\left(\frac{q^{{r_n^{(1)}}^2+\ldots+{r_n^{(k)}}^2-r_n^{(1)}r_{n-1}^{(1)}-\ldots-r_n^{(k)}r_{n-1}^{(k)}+\sum_{t=k_n+1}^k r_n^{(t)}}}{(q)_{r_n^{(1)}-r_n^{(2)}}\cdots(q)_{r_n^{(k-1)}-r_n^{(k)}}(q)_{r_n^{(k)}}}\right)x_1^{\sum_{i=1}^k r_1^{(i)}}\cdots x_n^{\sum_{i=1}^n r_n^{(i)}}.$$

Applying these two expressions to Theorem 6.2.1 immediately gives:

Corollary 6.2.2 *In the setting of Theorem 6.2.1, we have*

$$\chi'_{W(k_1\Lambda_1+k_2\Lambda_2)}(x_1,\ldots,x_n,q)=$$

$$=\sum\left(\frac{q^{{r_1^{(1)}}^2+\ldots+{r_1^{(k)}}^2+\sum_{t=k_1+1}^k r_1^{(t)}+\sum_{t=1}^k r_2^{(t)}-r_1^{(t)}}(1-q^{r_1^{(k_1)}})}{(q)_{r_1^{(1)}-r_1^{(2)}}\cdots(q)_{r_1^{(k-1)}-r_1^{(k)}}(q)_{r_1^{(k)}}}\right)\times$$

$$\times\left(\frac{q^{{r_2^{(1)}}^2+\ldots+{r_2^{(k)}}^2-r_2^{(1)}r_1^{(1)}-\ldots-r_2^{(k)}r_1^{(k)}}}{(q)_{r_2^{(1)}-r_2^{(2)}}\cdots(q)_{r_2^{(k-1)}-r_2^{(k)}}(q)_{r_2^{(k)}}}\right)\times$$

$$\times\cdots\times\left(\frac{q^{{r_n^{(1)}}^2+\ldots+{r_n^{(k)}}^2-r_n^{(1)}r_{n-1}^{(1)}-\ldots-r_n^{(k)}r_{n-1}^{(k)}}}{(q)_{r_n^{(1)}-r_n^{(2)}}\cdots(q)_{r_n^{(k-1)}-r_n^{(k)}}(q)_{r_n^{(k)}}}\right)x_1^{-k_1+\sum_{i=1}^k r_1^{(i)}}\cdots x_n^{\sum_{i=1}^n r_n^{(i)}}$$

and

$$\chi'_{W(k_{n-1}\Lambda_{n-1}+k_n\Lambda_n)}(x_1,\ldots,x_n,q)=$$

$$=\sum\left(\frac{q^{{r_1^{(1)}}^2+\ldots+{r_1^{(k)}}^2}}{(q)_{r_1^{(1)}-r_1^{(2)}}\cdots(q)_{r_1^{(k-1)}-r_1^{(k)}}(q)_{r_1^{(k)}}}\right)\left(\frac{q^{{r_2^{(1)}}^2+\ldots+{r_2^{(k)}}^2-r_2^{(1)}r_1^{(1)}-\ldots-r_2^{(k)}r_1^{(k)}+}}{(q)_{r_2^{(1)}-r_2^{(2)}}\cdots(q)_{r_2^{(k-1)}-r_2^{(k)}}(q)_{r_2^{(k)}}}\right)\times$$

$$\times\cdots\times\left(\frac{q^{{r_n^{(1)}}^2+\ldots+{r_n^{(k)}}^2-r_n^{(1)}r_{n-1}^{(1)}-\ldots-r_n^{(k)}r_{n-1}^{(k)}+\sum_{t=k_n+1}^k r_n^{(t)}}}{(q)_{r_n^{(1)}-r_n^{(2)}}\cdots(q)_{r_n^{(k-1)}-r_n^{(k)}}(q)_{r_n^{(k)}}}\right)\times$$

$$\times q^{\sum_{t=1}^{k} r_{n-1}^{(t)} - r_n^{(t)}} (1 - q^{r_n^{(k_n)}}) x_1^{\sum_{i=1}^{k} r_1^{(i)}} \cdots x_n^{-k_n + \sum_{i=1}^{n} r_n^{(i)}}$$

where the sums are taken over decreasing sequences $r_j^{(1)} \geq r_j^{(2)} \geq \cdots \geq r_j^{(k)} \geq 0$ *for each* $j = 1, \ldots, n$.

Remark 6.2.3 Corollary 6.2.2 above is $\widehat{\mathfrak{sl}(n+1)}$-analogue of Corollary 4.1 in [C1]. The multigraded dimension for $\chi'_{W(k_1\Lambda_1+k_2\Lambda_2)}$ in [C1] can be recovered from the expression above for $\chi'_{W(k_1\Lambda_1+k_2\Lambda_2)}$ by taking $n = 2$.

Remark 6.2.4 Throughout this work we have been assuming that $n \geq 2$ for notational convenience. In the case that $n = 1$ (that is, when $\mathfrak{g} = \mathfrak{sl}(2)$), the above results recover the recursions and multigraded dimensions found in [CLM1]-[CLM2].

Remark 6.2.5 The expressions in Corollary 6.2.2 can also be written as follows: As in [G], for $s = 1, \ldots, k-1$ and $i = 1, \ldots, n$, set $p_i^{(s)} = r_i^{(s)} - r_i^{(s+1)}$, and set $p_i^{(k)} = r_i^{(k)}$. Also, let $(A_{lm})_{l,m=1}^{n}$ be the Cartan matrix of $\mathfrak{sl}(n+1)$ and $B^{st} := \min\{s, t\}$, $1 \leq s, t \leq k$. Then,

$$\chi'_{W(k_1\Lambda_1+k_2\Lambda_2)}(x_1, \ldots, x_n, q) =$$

$$\sum_{\substack{p_1^{(1)},\ldots,p_1^{(k)} \geq 0 \\ \vdots \\ p_n^{(1)},\ldots,p_n^{(k)} \geq 0}} \frac{q^{\frac{1}{2}\sum_{l,m=1,\ldots,n}^{s,t=1,\ldots,l} A_{lm} B^{st} p_l^{(s)} p_m^{(t)}}}{\prod_{i=1}^{n} \prod_{s=1}^{k} (q)_{p_i{}^{(s)}}} q^{\widetilde{p_1}} q^{\sum_{t=1}^{k} p_2^{(t)} + \cdots + p_2^{(k)} - p_1^{(t)} - \cdots - p_1^{(k)}} \times$$

$$\times (1 - q^{p_1^{(k_1)} + \cdots + p_1^{(k)}}) x_1^{-k_1} \prod_{i=1}^{n} x_i^{\sum_{s=1}^{k} s p_i^{(s)}}$$

where $\widetilde{p_1} = p_1^{(k_1+1)} + 2p_1^{(k_1+2)} + \cdots + k_2 p_1^{(k)}$ and

$$\chi'_{W(k_{n-1}\Lambda_{n-1}+k_n\Lambda_n)}(x_1, \ldots, x_n, q) =$$

$$\sum_{\substack{p_1^{(1)},\ldots,p_1^{(k)} \geq 0 \\ \vdots \\ p_n^{(1)},\ldots,p_n^{(k)} \geq 0}} \frac{q^{\frac{1}{2}\sum_{l,m=1,\ldots,n}^{s,t=1,\ldots,l} A_{lm} B^{st} p_l^{(s)} p_m^{(t)}}}{\prod_{i=1}^{n} \prod_{s=1}^{k} (q)_{p_i{}^{(s)}}} q^{\widetilde{p_n}} q^{\sum_{t=1}^{k} p_{n-1}^{(t)} + \cdots + p_{n-1}^{(k)} - p_n^{(t)} - \cdots - p_n^{(k)}} \times$$

$$\times(1-q^{p_n^{(k_n)}+\cdots+p_n^{(k)}})x_n^{-k_n}\prod_{i=1}^{n}x_i^{\sum_{s=1}^{k}sp_i^{(s)}}$$

where $\widetilde{p_n}=p_n^{(k_n+1)}+2p_n^{(k_n+2)}+\cdots+k_{n-1}p_1^{(k)}$.

Chapter 7

Appendix

7.1 A completion of the universal enveloping algebra of certain nilpotent Lie algebras

Above, we needed the construction of a completion of a certain universal enveloping algebra. In this section, we work in a natural generality and recall a construction in [LW3] and use it to construct a completion of the universal enveloping algebra of certain subalgebras of affine Lie algebras associated to a finite dimensional semisimple Lie algebras.

Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, a set of roots Δ, a set of simple roots $\Pi = \{\alpha_1, ..., \alpha_n\}$, a set of positive roots Δ_+, and a symmetric invariant nondegenerate bilinear form $\langle \cdot, \cdot \rangle$, normalized so that $\langle \alpha, \alpha \rangle = 2$ for long roots $\alpha \in \Delta$. For each $\alpha \in \Delta_+$, let $x_\alpha \in \mathfrak{g}$ be a root vector associated to the root α. We have that

$$[x_\alpha, x_\beta] = C_{\alpha,\beta} x_{\alpha+\beta} \tag{7.1}$$

for some constants $C_{\alpha,\beta} \in \mathbb{C}$. Let $S \subset \Delta_+$ be a nonempty set of positive roots such that if $\alpha, \beta \in S$ and $\alpha + \beta \in \Delta_+$, then $\alpha + \beta \in S$. Define the nilpotent subalgebra $\mathfrak{n}_S \subset \mathfrak{g}$ by

$$\mathfrak{n}_S = \sum_{\alpha \in S} \mathbb{C} x_\alpha.$$

In the case that $S = \Delta_+$, we write $\mathfrak{n}_S = \mathfrak{n}$.

We have the corresponding untwisted affine Lie algebra given by

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

where c is a non-zero central element and

$$[x \otimes t^m, y \otimes t^p] = [x, y] \otimes t^{m+p} + m\langle x, y\rangle \delta_{m+p,0} c$$

for any $x, y \in \mathfrak{g}$ and $m, p \in \mathbb{Z}$ and

$$\bar{\mathfrak{n}}_S = \mathfrak{n}_S \otimes \mathbb{C}[t, t^{-1}],$$

a Lie subalgebra of $\widehat{\mathfrak{g}}$. The Lie algebra $\bar{\mathfrak{n}}_S$ has the following important subalgebras:

$$\bar{\mathfrak{n}}_{S-} = \mathfrak{n}_S \otimes t^{-1}\mathbb{C}[t^{-1}]$$

and

$$\bar{\mathfrak{n}}_{S+} = \mathfrak{n}_S \otimes \mathbb{C}[t].$$

Let $U(\bar{\mathfrak{n}}_S)$ be the universal enveloping algbera of $\bar{\mathfrak{n}}_S$. Using the Poincare-Birkhoff-Witt theorem, it is easy to see that $U(\bar{\mathfrak{n}}_S)$ has the decomposition

$$U(\bar{\mathfrak{n}}_S) = U(\bar{\mathfrak{n}}_{S-}) \oplus U(\bar{\mathfrak{n}}_S)\bar{\mathfrak{n}}_{S+}. \tag{7.2}$$

Let $M(S)$ denote the free monoid on $\mathbb{Z} \times S$. We may write

$$M(S) = \cup_{n \geq 0} M(S)_n$$

where

$$M(S)_n = \mathbb{Z}^n \times S^n$$

and composition of elements $\circ$ is given by juxtaposition:

$$\begin{aligned}(n_1, \ldots, n_k; \gamma_1, \ldots \gamma_k) &\circ (m_1, \ldots, m_l; \beta_1, \ldots, \beta_l) \\ &= (n_1, \ldots, n_k, m_1, \ldots m_l; \gamma_1, \ldots \gamma_k, \beta_1, \ldots \beta_l)\end{aligned}$$

where

$$(n_1, \ldots, n_k; \gamma_1, \ldots \gamma_k) \in M(S)_k,$$

$$(m_1, \ldots, m_l; \beta_1, \ldots, \beta_l) \in M(S)_l,$$

and

$$(n_1, \ldots, n_k, m_1, \ldots m_l; \gamma_1, \ldots \gamma_k, \beta_1, \ldots \beta_l) \in M(S)_{k+l}.$$

As in [LW3], define, for $n \geq 0$, a map

$$\begin{aligned} \tau : \mathbb{Z}^n &\longrightarrow \mathbb{Z}^n \\ (i_1, \ldots, i_n) &\mapsto (i_1 + \cdots + i_n, i_2 + \cdots + i_n, \ldots, i_n). \end{aligned} \tag{7.3}$$

For any $b = (n_1, \ldots, n_k; \beta_1, \ldots, \beta_k) \in M(S)_k$ and $i \in \mathbb{Z}$, we write

$$b \leq i \text{ if } \tau(n_1, \ldots, n_k) \leq (i, \ldots, i).$$

In other words, we have

$$n_1 + \cdots + n_k \leq i,$$

$$n_2 + \cdots + n_k \leq i,$$

$$\vdots$$

$$n_k \leq i.$$

The set $\mathrm{Map}(M(S), \mathbb{C})$ of all functions

$$f : M(S) \longrightarrow \mathbb{C}$$

has the structure of of an algebra given by taking the identity element to be the function which is 1 on $M(S)_0$ and 0 elsewhere, and by setting

$$(r\mu)(a) = r(\mu(a)),$$

$$(\mu_1 + \mu_2)(a) = \mu_1(a) + \mu_2(a),$$

and

$$(\mu_1\mu_2)(a) = \sum_{a=b\circ c} \mu_1(b)\mu_2(c)$$

for $r \in \mathbb{C}$, $\mu, \mu_1, \mu_2 \in \mathrm{Map}(M(S), \mathbb{C})$, and $a \in M(S)$. As in [LW3], for each $\mu \in \mathrm{Map}(M(S), \mathbb{C})$ and $i \in \mathbb{Z}$, we define sets

$$\mathrm{Supp}(\mu) = \{a \in M(S) | \mu(a) \neq 0\}$$

and

$$\mathrm{Supp}_i(\mu) = \{a \in M(\Delta_+) | a \leq i\} \cap \mathrm{Supp}(\mu).$$

Note that if $i \leq j$ then

$$\mathrm{Supp}_i(\mu) \subset \mathrm{Supp}_j(\mu)$$

and that

$$\mathrm{Supp}(\mu) = \cup_{i\in\mathbb{Z}}\mathrm{Supp}_i(\mu).$$

Define $F(S) \subset \mathrm{Map}(M(S), \mathbb{C})$ by

$$F(S) := \{\mu : M(S) \longrightarrow \mathbb{C} \mid \mathrm{Supp}_i(\mu) \text{ is finite for all } i \in \mathbb{Z}\}$$

and $F_0(S) \subset F(S)$ by

$$F_0(S) := \{\mu \in F(S) \mid \mathrm{Supp}(\mu) \text{ is finite}\}$$

We have that

Proposition 7.1.1 *([LW3]) $F(S)$ is a subalgebra of Map$(M(S), \mathbb{C})$, and $F_0(S) \subset F(S)$ is a subalgebra of $F(S)$. Moreover, $F_0(S)$ is the free algebra on $\mathbb{Z} \times S$.*

For each $a \in M(S)$, define maps $X(a) \in F_0(S)$ by

$$X(a)(b) = \delta_{a,b}.$$

In particular, for $(n; \beta) \in M(S)_1$, write

$$X_\beta(n) = X((n; \beta))$$

and extend this so that for any $a = (n_1, \ldots, n_k; \beta_1, \ldots, \beta_k) \in M(S)$

$$X(a) = X_{\beta_1}(n_1) \ldots X_{\beta_k}(n_k).$$

For any $\mu \in \mathrm{Map}(M(S), \mathbb{C})$, we may write

$$\mu = \sum_{a\in\mathrm{Supp}(\mu)} \mu(a)X(a).$$

Consider the ideal I_S of $F_0(S)$ generated by

$$[X_\alpha(n), X_\beta(m)] - C_{\alpha,\beta}X_{\alpha+\beta}(m + n)$$

for $\alpha, \beta \in S$ and $m, n \in \mathbb{Z}$, where $C_{\alpha,\beta}$ are the structure constants (7.1). We have the following proposition:

Proposition 7.1.2 $U(\bar{\mathfrak{n}}_S) \simeq F_0(S)/I_S$

Proof: Let $T(\bar{\mathfrak{n}}_S)$ denote the tensor algebra on $\bar{\mathfrak{n}}_S$. Let ϕ be the bijection

$$\begin{aligned} \phi : \mathbb{Z} \times S \quad &\longrightarrow \quad \bar{\mathfrak{n}}_S \\ (n, \beta) \quad &\mapsto \quad x_\beta(n). \end{aligned} \tag{7.4}$$

Since $F_0(S)$ is the free algebra on $\mathbb{Z} \times \Delta_+$ and $T(\bar{\mathfrak{n}}_S)$ is the free algebra on $\bar{\mathfrak{n}}_S$, we extend ϕ to a map of free algebras

$$\begin{aligned} \phi : F_0(S) \quad &\longrightarrow \quad T(\bar{\mathfrak{n}}_S) \\ X_{\beta_1}(n_1) \dots X_{\beta_k}(n_k) \quad &\mapsto \quad x_{\beta_1}(n_1) \dots x_{\beta_k}(n_k), \end{aligned} \tag{7.5}$$

extended linearly to all of $F_0(S)$. The fact that ϕ is an algebra isomorphism is clear. The proposition follows immediately.

We now impose similar natural relations on $F(S)$. Consider the ideal $\widetilde{I_S}$ of $F(S)$ generated by

$$[X_\alpha(n), X_\beta(m)] - C_{\alpha,\beta} X_{\alpha+\beta}(m+n)$$

for $\alpha, \beta \in S$ and $m, n \in \mathbb{Z}$, where $C_{\alpha,\beta}$ are the structure constants (7.1).

Definition 7.1.3 *Define the* completion *of* $U(\bar{\mathfrak{n}}_S)$ *by:*

$$\widetilde{U(\bar{\mathfrak{n}}_S)} := F(S)/\widetilde{I_S}. \tag{7.6}$$

Denote by $[\mu]$ *the coset of* $\mu \in F(S)$ *in* $\widetilde{U(\bar{\mathfrak{n}}_S)}$.

We now introduce some important substructures of $\widetilde{U(\bar{\mathfrak{n}}_S)}$ and prove some useful facts about these substructures. Let

$$M(S)_- = \{(m_1, \dots, m_k; \beta_1, \dots, \beta_k) \in M(S) | k \in \mathbb{N}, m_i \leq -1 \text{ for each } i = 1, \dots, k\}.$$

Define

$$\widetilde{U(\bar{\mathfrak{n}}_{S-})} = \{a \in \widetilde{U(\bar{\mathfrak{n}}_S)} \mid a = [\mu] \text{ for some } \mu \in F(S) \text{ with } \mathrm{Supp}(\mu) \subset M(S)_-\}.$$

Lemma 7.1.4 *We have that*

$$\widetilde{U(\bar{\mathfrak{n}}_{S-})} \simeq U(\bar{\mathfrak{n}}_{S-}).$$

Proof: Suppose $[\mu] \in \widetilde{U(\bar{\mathfrak{n}}_{S-})}$ for some μ with $\mathrm{Supp}(\mu) \subset M(S)_-$. We may write

$$\mu = \sum_{a \in \mathrm{Supp}(\mu)} \mu(a) X(a)$$

and so

$$[\mu] = \sum_{a \in \mathrm{Supp}(\mu)} [\mu(a) X(a)].$$

By definition, $\mathrm{Supp}_{-1}(\mu)$ is finite, so that there are finitely many

$$a = (m_1, \ldots, m_n; \beta_1, \ldots, \beta_n) \in \mathrm{Supp}(\mu), \;\; k \in \mathbb{N}$$

such that

$$m_1 + \cdots + m_k \leq -1$$

$$m_2 + \cdots + m_k \leq -1$$

$$\vdots$$

$$m_k \leq -1.$$

Since each such $m_i \leq -1$, $i = 1, \ldots k$, have have that $\mathrm{Supp}_n(\mu) = \mathrm{Supp}_{-1}(\mu)$ for all $n \geq 0$. In particular, we have that

$$\mathrm{Supp}(\mu) = \cup_{n \in \mathbb{Z}} \mathrm{Supp}_n(\mu) = \mathrm{Supp}_{-1}(\mu)$$

and so $\mathrm{Supp}(\mu)$ is finite and $\mu \in F_0(S)$. By the proof of Proposition 7.1.2, we have that

$$\widetilde{U(\bar{\mathfrak{n}}_{S-})} \simeq U(\bar{\mathfrak{n}}_{S-}),$$

concluding our proof.

Let

$$M(S)_+ = \{(m_1, \ldots, m_k; \beta_1, \ldots, \beta_k) \in M(S) \mid k \in \mathbb{N} \text{ and } \exists i \leq k \text{ with } m_i + \cdots + m_k \geq 0\}$$

We define

$$\widetilde{U(\bar{\mathfrak{n}}_S)\bar{\mathfrak{n}}_{S+}} = \{a \in \widetilde{U(\bar{\mathfrak{n}}_S)} \mid a = [\mu] \text{ for some } \mu \in F(S) \text{ with } \mathrm{Supp}(\mu) \subset M(S)_+\}.$$

Remark 7.1.5 The space $\widetilde{U(\bar{\mathfrak{n}}_S)\bar{\mathfrak{n}}_{S+}}$ is the collection of all elements of $\widetilde{U(\bar{\mathfrak{n}}_S)}$ which have at least one representation as "infinite sums" of elements of $U(\bar{\mathfrak{n}}_S)\bar{\mathfrak{n}}_{S+}$. Indeed, any element $X(a) \in U(\bar{\mathfrak{n}}_S)$ with $a \in M(S)_+$ can be written as

$$X(a) = X(b)X(c),$$

where

$$b = (m_1, \ldots, m_{i-1}; \beta_1, \ldots, \beta_{i-1}),$$

$$c = (m_i, \ldots, m_k; \beta_1, \ldots, \beta_k),$$

and $m_i + \ldots m_k \geq 0$. By (7.2), $X(c) \in U(\bar{\mathfrak{n}}_S)\mathfrak{n}_{S+}$ and $X(b) \in U(\bar{\mathfrak{n}}_S)$, and so $X(a) \in U(\bar{\mathfrak{n}}_S)\mathfrak{n}_{S+}$.

Proposition 7.1.6 *$\widetilde{U(\bar{\mathfrak{n}}_S)}$ has the decomposition*

$$\widetilde{U(\bar{\mathfrak{n}}_S)} = U(\bar{\mathfrak{n}}_{S-}) \oplus \widetilde{U(\bar{\mathfrak{n}}_S)\bar{\mathfrak{n}}_{S+}} \tag{7.7}$$

Proof: Given any $u \in U(\bar{\mathfrak{n}}_S)$, using (7.2) we may write

$$u = u_1 + u_2$$

where $u_1 \in U(\bar{\mathfrak{n}}_{S-})$ and $u_2 \in U(\bar{\mathfrak{n}}_S)\bar{\mathfrak{n}}_{S+}$. Suppose $[\mu] \in \widetilde{U(\bar{\mathfrak{n}}_S)}$ for some $\mu \in F(S)$. Writing

$$\mu = \sum_{a \in \operatorname{Supp}(\mu)} \mu(a)X(a),$$

we have

$$[\mu] = \sum_{a \in \operatorname{Supp}(\mu)} [\mu(a)X(a)]$$

and each $[\mu(a)X(a)] \in U(\bar{\mathfrak{n}}_S)$. Since $\mu \in F(S)$, there are only finitely many $a \in \operatorname{Supp}(\mu)$ such that $a \in \operatorname{Supp}_{-1}(\mu)$, so that, ranging over all $k \in \mathbb{Z}$, there are only finitely many $a = (m_1, \ldots, m_k; \beta_1, \ldots, \beta_k)$ with

$$m_1 + \cdots + m_k \leq -1$$

$$m_2 + \cdots + m_k \leq -1$$

$$\vdots$$

$$m_k \leq -1.$$

For these finitely many $a \in \mathrm{Supp}_{-1}(\mu)$, we write

$$[\mu(a)X(a)] = [\mu_{1,a}] + [\mu_{2,a}]$$

for some $[\mu_{1,a}] \in U(\bar{\mathfrak{n}}_{S-})$ and $[\mu_{2,a}] \in U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_{S+}$. By definition of $U(\bar{\mathfrak{n}}_{S-})$, we have that

$$\sum_{a \in \mathrm{Supp}_{-1}(\mu)} [\mu_{1,a}] \in U(\bar{\mathfrak{n}}_{S-})$$

since the sum is finite, and

$$\sum_{a \in \mathrm{Supp}_{-1}(\mu)} [\mu_{2,a}] + \sum_{a \in \mathrm{Supp}(\mu) \setminus \mathrm{Supp}_{-1}(\mu)} [\mu(a)X(a)] \in \widetilde{U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_{S+}},$$

since

$$\mathrm{Supp}(\mu) \setminus \mathrm{Supp}_{-1}(\mu) \subset M(S)_+.$$

This shows $[\mu] \in U(\bar{\mathfrak{n}}_{S-}) + \widetilde{U(\bar{\mathfrak{n}}_S)\bar{\mathfrak{n}}_{S+}}$. The fact that $U(\bar{\mathfrak{n}}_{S-}) \cap \widetilde{U(\bar{\mathfrak{n}}_S)\bar{\mathfrak{n}}_{S+}} = 0$ follows from the fact that $U(\bar{\mathfrak{n}}_{S-}) \cap U(\bar{\mathfrak{n}}_S)\bar{\mathfrak{n}}_{S+} = 0$, proving our proposition.

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