THE LEIBNIZ FORMULA FOR DIVIDED DIFFERENCE OPERATORS ASSOCIATED TO KAC-MOODY ROOT SYSTEMS

BY

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ABSTRACT OF THE DISSERTATION

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In this dissertation we present a new Leibniz formula (i.e. generalized product rule) for the type of divided difference operators first introduced by Bernšteĭn, Gel′fand, and Gel′fand. The formula applies for divided difference operators associated to the geometric representation of the Coxeter system of any Kac-Moody group, be it finite-dimensional or infinite-dimensional. Our formula shows that in order to study the structure of the equivariant cohomology ring there is no need to actually construct it at all because the structure constants are encoded in our Leibniz formula for divided difference operators. The formula may be used to compute the structure constants and prove general results about them. In the future our results may be useful in finding Littlewood-Richardson rules in equivariant cohomology and may make the study of certain problems in Schubert calculus more accessible to researchers who are not necessarily well-versed in algebro-geometric or topological methods.

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I am indebted to my wife, without whom life would be dreary, and to my mother who has been nothing but kind and supportive. Though he is no longer with us, it can't hurt to also thank my father for his support and guidance over the years. I thank my siblings and the rest of my family as well for caring that I exist, which is more important than it may sound.

Dedication

To my father, Richard Irving Samuel. He always said that nobody does anything that is hard for them. Anybody who has tried to get a PhD in mathematics would disagree. There is, however, some truth to it: a problem is hard until you solve it. Once you solve it, it may still be hard, but it becomes infinitely easier.

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Chapter 1

Introduction

Let us begin our discussion with an enumerative geometry problem in three-dimensional projective space. Suppose we are given two lines L_1 and L_2 as well as a plane R_3 and a point $P_3 \in R_3$. How many ways are there to choose a point P, a line L, and a plane R such that $P \in L \subset R$, $P \in L_1$, $L_2 \subset R$, $P_3 \in L$, and $L \subset R_3$? Under generic conditions, the answer is that there is exactly one way. In this case we may reason directly. Note that R must be the unique plane passing through P_3 and L_2, note that L must be equal to the intersection of R and R_3 and P must be the only point in the intersection of L and L_1. This shows uniqueness. One can imagine that in a more general problem the direct approach could get complicated. As in most enumerative problems it would be preferable to have a general algorithm for counting the configurations without having to actually find them all.

In 1879 in [22] Schubert presented a new approach to enumerative geometry that is now known as Schubert calculus. In what is now classical Schubert calculus one works in a Grassmannian manifold, which as a set is just the collection of all planes (vector subspaces) of a fixed dimension in a finite-dimensional vector space. If V is a vector space of dimension n, then a (complete) *flag* is an ascending chain $V_1 \subset V_2 \subset \cdots \subset V_n = V$ of subspaces such that $\dim(V_i) = i$. Schubert defined what are now known as *Schubert cells* as sets of planes that intersect the subspaces in a fixed flag in subspaces of specified dimensions. A *Schubert variety* is the closure (in the topological sense) of a Schubert cell in the manifold. The solutions to many enumerative geometry problems involving incidence of planes can be realized as the points contained in the intersection of multiple Schubert varieties in Grassmannians corresponding to different flags. The problem in the above paragraph is of a more general sort in which we instead consider the manifold consisting of all flags in the vector space, which also has Schubert varieties that may be defined by incidence conditions. More generally, we may choose an integer k with $2 \leq k \leq n$ and fixed integers $1 \leq i_1 < i_2 < \cdots < i_k = n$ and consider the partial flag variety consisting of all chains of subspaces $V_{i_1} \subset V_{i_2} \subset \cdots \subset V_{i_{k-1}} \subset V_{i_k} = V$ such that $\dim(V_{i_j}) = i_j$ for all $1 \leq j \leq k$; Grassmannians are the case where $k = 2$, while the complete flag variety is the case where $k = n$.

Being able to express the solution of a problem as the intersection of Schubert varieties is only useful if it is easier to compute the number of points in the intersection of Schubert varieties. Luckily, if we are working in complex vector spaces then cohomology provides a way. Each Schubert variety has an associated class in the cohomology ring of its flag variety, and if the intersection of some number of Schubert varieties in general position is a finite set of points then the number of points will be the coefficient of the class of a point in the product of the corresponding Schubert classes. Indeed, these intersection counts form the structure constants for the cohomology ring.

To be more precise, we note that the Schubert varieties in a complete flag variety are indexed by elements of the Weyl group W. Let the Schubert variety corresponding to $u \in W$ be denoted by X_u. The corresponding Schubert class will be denoted by $[X_u]$. W has a complementation operation (namely left multiplication by the longest element) that we denote by $u \mapsto u^{\perp}$. Then in the cohomology ring of the complete flag variety we have the identity

$$[X_u] \cdot [X_v] = \sum_{w \in W} c^w_{u,v} [X_w]$$

where $c^w_{u,v}$ counts the number of points in the intersections of translations of the Schubert varieties X_u, X_v, and $X_{w^{\perp}}$ into general position.

One of the first general computational results for the Grassmannian was Pieri's formula. Pieri's formula allows us to easily express the product of an arbitrary Schubert class with certain special Schubert classes. These special Schubert classes generate the cohomology ring; hence the problem is reduced to solving systems of linear equations with integer coefficients. A similar formula for the complete flag variety is given by the Chevalley-Monk formula [19], and another formula, which is a bona fide generalization of Pieri's rule, is given by Sottile in [23].

In [17], Littlewood and Richardson provided a more appetizing way of computing the coefficients than solving systems of equations with their famed *Littlewood-Richardson rule*, which allows one to compute the coefficients in the case of the Grassmannian. The Littlewood-Richardson rule expresses the coefficient directly as the cardinality of a set of computable objects that are known as *Young tableaux.* In recent times the term "Littlewood-Richardson rule" has come to describe any formula expressing some number in a ring related to Schubert calculus (of which singular cohomology of flag varieties is only one example) as the cardinality of a set of computable ("combinatorial") objects, or at least as a sum of positive numbers. The Grassmannian is a flag variety with only one step, so we may ask if such a formula exists for general flag varieties. This problem has proved to be surprisingly difficult. In [20], Purbhoo and Sottile provided a formula for multiplying certain elements called *Grassmannian elements.* A rule for equivariant cohomology (to be described in a moment) has been proved recently for two-step flag varieties by Buch [7].

Schubert calculus has grown considerably in 135 years. There are more rings to consider than just the cohomology rings of partial flag varieties. Let G be a complex semisimple linear algebraic group. A subgroup $B \subset G$ is called a *Borel subgroup* if B is a maximal Zariski closed, connected, and solvable subgroup of G. The set G/B of left cosets of B in G is called a *generalized flag variety*; we obtain the classical complete flag variety when we take $G = SL(n, \mathbb{C})$. Even more generally, we may let G be a Kac-Moody group, defined by Tits in [24]. Since Kac-Moody groups are not in general algebraic varieties, Borel subgroups must be defined differently; the definition of a Borel subgroup in the general, possibly infinite-dimensional, Kac-Moody case may also be found in [24].

After their introduction, general methods were sought for studying the cohomology rings of these generalized flag varieties. One such general method was presented by Borel in [6]: Borel proved that the cohomology ring of a generalized partial flag variety is isomorphic to the quotient of a polynomial ring by the ideal generated by non-constant homogeneous polynomials that are invariant under a certain action of the

Weyl group. Bernšteĭn, Gel′fand, and Gel′fand [1] identified specific polynomial representatives of the Schubert classes. In doing this they introduced certain operators known as *divided difference operators*, which we will be denoting by ∂_v for $v \in W$. Lascoux and Schützenberger went further in [16] and defined specific polynomials in type A, which are known as *Schubert polynomials*, that have nonnegative integer coefficients and represent the Schubert classes when the quotient is taken. Macdonald in [18] studied the relationship between Schubert polynomials and divided difference operators in detail, and Billey and Haiman in [2] generalized the concept of Schubert polynomials to the other infinite families of linear algebraic groups (types B, C, and D).

A *torus* in the context of this dissertation is a group that is isomorphic to the direct product $(\mathbb{C}^*)^m$ of m copies of the multiplicative group of nonzero complex numbers for some integer $m \geq 1$. If G is a connected complex linear algebraic group, then a maximal abelian Lie subgroup is called a *maximal torus*, and is indeed a torus as the name indicates. In the case of a Kac-Moody group, we define a maximal torus to be the intersection $B \cap N$ of the elements of a (B, N) pair for G (defined in [24]). Being a subgroup of G, a maximal torus acts on a complete flag variety G/B.

Any topological space X that has a torus action can be assigned a *torus-equivariant cohomology ring*, which, intuitively speaking, takes into account the torus action. For example, the maximal Grassmannians in types B and D are isomorphic as varieties and hence have the same cohomology rings; however, the torus actions on these two varieties are different, and the equivariant cohomology ring shows this, distinguishing the two varieties. Rather than integers, the coefficients in the equivariant cohomology ring are polynomials. Goresky-Kottwitz-MacPherson theory [10] provides a way to realize the equivariant cohomology ring as a subring of a direct product of polynomial rings where the components of the elements in the subring satisfy certain divisibility conditions relative to each other. The specific polynomial rings are the equivariant cohomology rings of fixed points of the torus action, and one can identify the Schubert classes by computing the image of the Schubert class in these polynomial rings, which is known as the *restriction* to the fixed point. In [3] Billey provided a formula (indeed, a positive formula) for the restriction of a Schubert class to a fixed point. Kostant and Kumar in [15] showed that the equivariant cohomology ring is, in a certain sense, dual to the ring of divided difference operators (more precisely, the nil-Hecke algebra) and gave an explicit formula for the coefficients in the cohomology ring.

We denote the Schubert classes in $H_T^*(G/B)$ by $[X_v]_T$ for $v \in W$. Since the Schubert classes form a basis for $H_T^*(G/B)$ as a module over a polynomial ring, we may write

$$[X_u]_T \cdot [X_v]_T = \sum_{w \in W} c_{u,v}^w [X_w]_T,$$

where in general the structure constant $c_{u,v}^w$ is a polynomial. The structure constants $c_{u,v}^w$ in equivariant cohomology generalize the coefficients in ordinary cohomology in that $c_{u,v}^w$ is the same as in ordinary cohomology whenever the coefficient in ordinary cohomology is not zero. We therefore use the same notation for the structure constants in equivariant cohomology as for the structure constants in ordinary cohomology.

In this dissertation we connect divided difference operators to equivariant cohomology in a previously unknown way via a formula that we call "the Leibniz rule." The original Leibniz rule is a generalized product rule for derivatives. In [18] Macdonald provided a product rule for divided difference operators that he named after the Leibniz rule due to its similarity, and we are following suit.

Our formula is as follows.

Theorem 1 (The Leibniz rule)**.** *Suppose (W, S) is a Kac-Moody Coxeter system and $\{\partial_w | w \in W\}$ are the divided difference operators associated to a root system for (W, S). Then for all rational functions p we have the formula*

$$\partial_w p = \sum_{u,v \in W} c_{u,v}^w \partial_u(p) \partial_v$$

in the nil-Hecke ring, where the $c_{u,v}^w$ are the structure constants in equivariant cohomology. In particular, if q is another rational function then

$$\partial_w(pq) = \sum_{u,v \in W} c_{u,v}^w \partial_u(p) \partial_v(q).$$

Macdonald's Leibniz rule, which applies in type A, involves a generalization of divided difference operators called *skew divided difference operators* that depend on two Weyl group elements. One may obtain our formula from Macdonald's by expressing the skew divided difference operators in terms of ordinary divided difference operators with polynomial coefficients.

Our formula, especially in the latter form, shows that equivariant cohomology has been present in the theory of Schubert calculus for longer than has been known. Even if one wishes only to study ordinary cohomology with ordinary Schubert polynomials, the divided difference operators used to define Schubert polynomials are intrinsically linked to equivariant cohomology in an elementary way.

Chapter 2

Coxeter groups and flag varieties

2.1 General definitions

2.1.1 Flag varieties and Schubert varieties

The most general definition of a finite- or infinite-dimensional *flag variety* over the complex numbers is that it is the quotient of a Kac-Moody group over the complex numbers by a Borel subgroup, as defined in [24]. Fix such a group G and a Borel subgroup $B \subset G$, so that the flag variety is G/B. An element of G/B is therefore a coset xB for some $x \in G$.

Let $N \subset G$ be a subgroup such that (B, N) is a Tits system, and let (B^-, N) be the opposite Tits system. Define $T = B \cap N$; we refer to T as the *maximal torus.* T is normal in N. N/T, is called the *Weyl group* and is denoted by W. For each $w \in W$ the double coset $B^- wB$ is well-defined (i.e. the orbit of the left action by B^- depends only on the elements of N it contains, and only on the images of those elements in N/T) and is called the *Schubert cell* corresponding to w, which will be denoted by X_w^0. The closure of X_w^0 will be denoted by X_w and is called the *Schubert variety* corresponding to w. These definitions may be found in [4] in slightly less generality.

2.1.2 The Weyl group and the root system

Our main object of study will be the Weyl group W together with its action on the dual of the Lie algebra of the torus. W is a *Coxeter group*, which means there is a generating set $S \subset W$ such that W has a presentation involving only relations of the form

$$(ss')^k = 1$$

for pairs $s, s' \in S$ such that $k = 1$ if $s = s'$ and $k \geq 2$ otherwise. This means in particular that $s^2 = 1$ for all $s \in S$. The choice of S is not usually unique, but it will generally be fixed. Hence we will refer to the pair (W, S) as a *Coxeter system* as is usually done in the literature. In the case where W is the Weyl group of a Kac-Moody group, S is determined by a choice of Borel subgroup. The elements of S will be called *simple reflections*. Elements of W that are conjugates of elements of S are called *reflections*. Proofs of the results we state here on Coxeter groups can be found, for example, in [5].

Every element $w \in W$ has an associated nonnegative integer called its *length*, which will be denoted by $\ell(w)$ and is equal to the minimal length of a sequence $(s_1, s_2, \ldots, s_n)$ of elements of S such that $s_1 s_2 \cdots s_n = w$. For each $s \in S$ and $w \in W$ either $\ell(ws) = \ell(w) - 1$ or $\ell(ws) = \ell(w) + 1$. The set of elements $s \in S$ such that $\ell(ws) = \ell(w) - 1$, which are known as *right descents* of w, will be denoted by $D_R(w)$. Similarly, for each $s \in S$ either $\ell(sw) = \ell(w) - 1$ or $\ell(sw) = \ell(w) + 1$, and we denote the set of left descents of w by $D_L(w)$.

One may find the following definition of a root system in [4]. The dual $\mathrm{Lie}(T)^*$ of the Lie algebra of T has certain elements called *roots*. To define roots we identify $\mathrm{Lie}(T)$ with a Cartan subalgebra of $\mathrm{Lie}(G)$. A nonzero element $x \in \mathrm{Lie}(T)^*$ is called a root if the subspace of $\mathrm{Lie}(G)$ consisting of all elements $a \in \mathrm{Lie}(G)$ such that for all $h \in \mathrm{Lie}(T)$ we have the equation

$$[h, a] = x(h)a$$

is not zero. Roots thus are essentially generalized eigenvalues. The set of all roots is called the *root system* of G and will be denoted by R. The action of W on $\mathrm{Lie}(T)^*$ permutes the root system. A choice of Borel subgroup, in addition to determining S, also imposes a division of R into two disjoint subsets R^+ and R^-. R^+ is called the set of *positive roots*, while R^- is the set of *negative roots*. The terms “positive” and “negative” reflect the fact that if y is a root, then $y \in R^+$ if and only if $-y \in R^-$.

Each element of S is associated to a unique positive root x_s defined to be the element of R^+ such that $s \cdot x_s \notin R^+$. Indeed, $s \cdot x_s = -x_s$. The roots x_s for $s \in S$ are called *simple roots*. Each reflection t is associated to a unique positive root as well, namely the root $x_t \in R^+$ such that $t \cdot x_t = -x_t$. If $t = wsw^{-1}$ with $\ell(w)$ minimal, then $x_t = w(x_s)$.

For each $w \in W$ and each reflection t, either $\ell(tw) > \ell(w)$ or $\ell(tw) < \ell(w)$. In the latter case we call x_t a (left) *inversion* of w, and we denote the set of all left inversions of w by $L(w)$. $L(w)$ always has $\ell(w)$ elements. We may find those elements as follows.

A sequence $e = (e_1, e_2, \ldots, e_n)$ such that $e_i \in S$ for all i will be called a *word* in S. If $e_1 e_2 \cdots e_n \in W$ has length n, then e will be called a *reduced word*. The inversions of $w = e_1 e_2 \cdots e_n$ can be obtained from a reduced word. For each i define

$$r_i(e) = e_1 e_2 \cdots e_{i-1} \cdot x_{e_i}.$$

Then

$$L(w) = \{r_i(e) | 1 \leq i \leq \ell(w)\}.$$

There is a partial ordering on W known as *Bruhat order* that we will denote by $\leq$. One way to define Bruhat order is to first define a relation $\rightarrow$ by declaring that $u \rightarrow tu$ if t is a reflection such that $\ell(tu) > \ell(u)$, then define $\leq$ to be the reflexive, transitive closure of this relation. An equivalent and often much easier to use definition is to say that $u \leq v$ if and only if for some (equivalently, any) reduced word for v there exists a subsequence that is a reduced word for u.

Coxeter systems have a property known as the *exchange property*, which may be stated as follows: if $(s_1, s_2, \ldots, s_n)$ is a word for an element $w \in W$ and $s \in S$ is such that $\ell(ws) < \ell(w)$, then there exists an index i such that $(s_1, s_2, \ldots, s_{i-1}, \widehat{s_i}, s_{i+1}, \ldots, s_n)$ is a word for the element ws, where the hat over s_i indicates omission. This property in fact characterizes Coxeter systems.

2.2 Cohomology and equivariant cohomology

Let X be a flag variety. In the case where X is finite-dimensional, we may use Borel-Moore homology to associate to each Schubert variety X_w for $w \in W$ a class $[X_w]$ in the cohomology ring $H^*(X)$ (see, for instance, [9]). If X is infinite-dimensional, a similar method of associating a Schubert variety to a class in $H^*(X)$ applies when we approximate X with finite-dimensional spaces (see, for instance, [11]). The classes $[X_w]$

for $w \in W$ form a basis for $H^*(X)$. We may therefore write

$$[X_u] \cdot [X_v] = \sum_w c_{u,v}^w [X_w]$$

for some integers $c_{u,v}^w$.

To define torus-equivariant cohomology, we use a contractible space ET on which the torus T acts freely (e.g. $(\mathbb{C}^\infty - \{0\})^n$ with the componentwise action) and declare that the ring $H_T^*(X)$ is the ordinary cohomology ring of the space $ET \times_T X$ defined as the quotient of $ET \times X$ by the equivalence relation $(e, x) \sim (et, t^{-1}x)$. This can be defined for any space with a torus action, including a point. If pt is a point, then $H_T^*(pt)$ is isomorphic to a polynomial ring in indeterminates $t_1, t_2, \ldots, t_n$. The indeterminates t_i may be regarded as algebraic generators of the character ring of the torus as they are the Chern classes of equivariant line bundles on $(\mathbb{P}^\infty)^n$. The roots may be expressed in terms of the t_i when we consider the t_i as characters.

As a module over $H_T^*(pt)$, $H_T^*(X)$ has Schubert classes $[X_w]_T$ as a basis. We therefore may write

$$[X_u]_T \cdot [X_v]_T = \sum_w c_{u,v}^w [X_w]_T$$

where each $c_{u,v}^w$ is an element of $H_T^*(pt)$, i.e. a polynomial. We do not notationally distinguish the coefficients $c_{u,v}^w$ in equivariant cohomology from those in ordinary cohomology because if $\ell(u) + \ell(v) = \ell(w)$, then the coefficients are the same as ordinary cohomology, and we will be using only the coefficients in equivariant cohomology from now on.

The action of T on X has one fixed point x_v in each Schubert cell X_v^0. The inclusion $x_v \to X$ induces a restriction homomorphism $H_T^*(X) \to H_T^*(x_v)$. We may combine these restriction homomorphisms into a ring homomorphism into the direct product

$$H_T^*(X) \to \prod_{v \in W} H_T^*(x_v).$$

This homomorphism is injective (see [8]).

Chapter 3

Derivation of the Leibniz rule

We parallel the construction of Kostant and Kumar in [15] by defining a ring associated to a linear representation of a completely general group, then use the construction to derive our Lebniz formula. The proofs of the results about the nil-Hecke ring that we present are simpler than the original proofs. For the results that are not original, we will indicate where they first arose.

3.1 The skew group ring

Let G be a group and let V be a finite-dimensional vector space over a field F with a linear action of G on the left (i.e. a representation of G). Let $\mathrm{Sym}(V)$ be the symmetric algebra of V; then if $\{x_1, x_2, \ldots, x_n\}$ is any basis of V, $\mathrm{Sym}(V)$ is isomorphic to the polynomial ring $F[x_1, x_2, \ldots, x_n]$. There is a left action of G on $\mathrm{Sym}(V)$ that arises from the action on V given by $g \cdot f(x_1, x_2, \ldots, x_n) = f(g \cdot x_1, g \cdot x_2, \ldots, g \cdot x_n)$ for $g \in G$ and $f \in \mathrm{Sym}(V)$. G acts on $\mathrm{Sym}(V)$ by ring automorphisms (indeed, F-algebra automorphisms), which may be deduced from the universal property of polynomial rings.

The subset H of $\mathrm{Sym}(V)$ consisting of all nonzero homogeneous polynomials is multiplicatively closed. Let $J(V) = H^{-1}\mathrm{Sym}(V)$ be the localization of $\mathrm{Sym}(V)$ in H; then $J(V)$ is a graded ring where the homogeneous elements of degree m can be written as quotients $\frac{p}{q}$ such that $p, q \in H$ and $\deg(p) - \deg(q) = m$. G acts on the left on $J(V)$ via F-algebra automorphisms by declaring that for each $g \in G$ we have

$$g \cdot \frac{p}{q} = \frac{g \cdot p}{g \cdot q}$$

where $p \in \mathrm{Sym}(V)$ and $q \in H$. By definition of the action of G, $g \cdot H = H$ for all

$g \in G$. The action on $J(V)$ is well defined since if $b \in H$ and $a \in \mathrm{Sym}(V)$ are two other elements such that $aq = bp$, then $(g \cdot a)(g \cdot q) = (g \cdot b)(g \cdot p)$, so by definition of the localization we have $\frac{g \cdot p}{g \cdot q} = \frac{g \cdot a}{g \cdot b}$.

We define an F-algebra $G(V)$, which is *the skew group ring*, to be the free left $J(V)$-module with basis G, where the product is given by $(pg) \cdot (qh) = p(g \cdot q)gh$ for $p, q \in J(V)$ and $g, h \in G$ and is extended distributively. This is a special case of the skew group rings originally defined in [12]. Let $p, q, r \in J(V)$ and $f, g, h \in G$. The computations $(pf) \cdot ((qg) \cdot (rh)) = (pf) \cdot (q(g \cdot r)gh) = p(f \cdot q)(fg \cdot r)fgh$ and $((pf) \cdot (qg)) \cdot (rh) = (p(f \cdot q)fg) \cdot (rh) = p(f \cdot q)(fg \cdot r)fgh$ show that this product turns $G(V)$ into an associative F-algebra, though it is not a $J(V)$-algebra. We turn $G(V)$ into a right $J(V)$-module (hence a $J(V)$-bimodule) by declaring that $(pg)q = p(g \cdot q)g$. $G(V)$ is also free as a right $J(V)$-module with basis G since if $p_g \in J(V)$ for $g \in G$ are such that

$$\sum_{g \in G} g p_g = 0,$$

then

$$\sum_{g \in G} (g \cdot p_g) g = 0,$$

hence $g \cdot p_g = 0$ for all $g \in G$, implying that $p_g = 0$ for all $g \in G$. $G(V)$ is sometimes referred to as the "smash product" of the group ring and $J(V)$.

We note that $G(V)$ is a graded ring where the elements of G are homogeneous of degree 0 and the elements of $J(V)$ have the same degrees in $G(V)$ as in $J(V)$.

3.2 The coproduct and the dual

We wish to turn $G(V)$ into a $J(V)$-bialgebra. Since $G(V)$ is not even a $J(V)$-algebra, this is not technically possible. However, we may define a coproduct in a certain sense by dropping the condition that the coefficient ring be central. We denote by

$$G(V) \ {}_{J(V)}{\otimes}\ G(V),$$

as is usually done, the tensor product of $G(V)$ with itself in the category of left $J(V)$-modules, which means that if $p \in J(V)$ and $g, h \in G$, then

$$pg \otimes h = g \otimes ph.$$

We also define

$$G(V)\ {}_{J(V)}\!\otimes_{J(V)} G(V)$$

to be the tensor product in the category of $J(V)$-bimodules, where

$$gp \otimes h = g \otimes ph.$$

The formal definition of $G(V)\ {}_{J(V)}\!\otimes_{J(V)} G(V)$ is via a universal property. If M is a right $J(V)$-module and N is a left $J(V)$ module, a function $F : M \times N \to A$ from $M \times N$ into an abelian group A is called a $J(V)$-*middle-linear map* if for all $m, m' \in M$, $n, n' \in N$, and $c \in J(V)$ we have the three equations

$$F(m + m', n) = F(m, n) + F(m', n)$$

$$F(m, n + n') = F(m, n) + F(m, n')$$

$$F(mc, n) = F(m, cn)$$

The tensor product $M\ {}_{J(V)}\!\otimes_{J(V)} N$ is an abelian group such that there is a universal $J(V)$-middle linear map $T : M \times N \to M\ {}_{J(V)}\!\otimes_{J(V)} N$ such that given any middle linear map $F : M \times N \to A$ there is a unique homomorphism $F' : M\ {}_{J(V)}\!\otimes_{J(V)} N \to A$ such that $F = F' \circ T$.

The *product* map is the $J(V)$-bimodule homomorphism

$$\nabla : G(V)\ {}_{J(V)}\!\otimes_{J(V)} G(V) \to G(V)$$

given for elements $g, h \in G$ by

$$\nabla(g \otimes h) = gh.$$

This is the associative, distributive product in $G(V)$ that we have already defined.

The *coproduct* map is the left $J(V)$-module homomorphism

$$\Delta : G(V) \to G(V)\ {}_{J(V)}\!\otimes G(V)$$

given for elements $g \in G$ by

$$\Delta(g) = g \otimes g$$

Due to noncommutativity with the module coefficients, componentwise multiplication does not quite make sense in general in $G(V)\ {}_{J(V)}\!\otimes G(V)$. However, we will shortly see that Δ is multiplicative in a certain sense.

We may use the coproduct to define the *dual* as follows. Let

$$G^*(V) = \mathrm{Hom}_{J(V)-\mathrm{Mod}}(G(V), J(V))$$

be the left dual of $G(V)$, which consists of all linear functions $f : G(V) \to J(V)$. Note that we do not put any sort of restrictions on the functions; this is the bare dual of a free module of possibly infinite rank. $G^*(V)$ is a left $J(V)$-module, where for $a \in J(V)$, $f \in G^*(V)$, and $g \in G(V)$ we have

$$(af)(g) = f(ag).$$

The coproduct Δ induces a left $J(V)$-module homomorphism

$$\Delta^* : \mathrm{Hom}_{J(V)-\mathrm{Mod}}(G(V)\ {}_{J(V)}\!\otimes G(V), J(V)) \to G^*(V)$$

defined for $f'' \in \mathrm{Hom}_{J(V)-\mathrm{Mod}}(G(V)\ {}_{J(V)}\!\otimes G(V), J(V))$ and $w \in G(V)$ by

$$(\Delta^*(f''))(w) = f''(\Delta(w)).$$

If G is infinite, then $\mathrm{Hom}_{J(V)-\mathrm{Mod}}(G(V)\,{}_{J(V)}\!\otimes G(V), J(V))$ is not isomorphic to $G^*(V)\,{}_{J(V)}\!\otimes G^*(V)$. However, we may imbed $G^*(V)\ {}_{J(V)}\!\otimes G^*(V)$ into $\mathrm{Hom}_{J(V)-\mathrm{Mod}}(G(V)\ {}_{J(V)}\!\otimes G(V), J(V))$ quite naturally as follows. A pair of elements $f, f' \in G^*(V)$ determines a left $J(V)$-bilinear function $D_{f,f'} : G(V) \times G(V) \to J(V)$ by the rule

$$D_{f,f'}(w, w') = f(w)f'(w')$$

for $w, w' \in G(V)$. This induces a unique homomorphism $E_{f,f'} : G(V)\ {}_{J(V)}\!\otimes G(V) \to J(V)$ such that

$$E_{f,f'}(w \otimes w') = f(w)f'(w').$$

The assignment $(f, f') \mapsto E_{f,f'}$ is itself a left $J(V)$-bilinear function $G^*(V) \times G^*(V) \to \mathrm{Hom}_{J(V)-\mathrm{Mod}}(G(V)\,{}_{J(V)}{\otimes} G(V), J(V))$, so there is an induced homomorphism $G^*(V)\,{}_{J(V)}{\otimes}\, G^*(V) \to \mathrm{Hom}_{J(V)-\mathrm{Mod}}(G(V)\,{}_{J(V)}{\otimes}\, G(V), J(V))$, and this is the imbedding we use. Despite the complicated construction, the resulting evaluation map behaves in the same way as for free modules of finite rank, namely

$$(f \otimes f')(w \otimes w') = f(w)f'(w').$$

We therefore use this homomorphism in order to treat $G^*(V)\,{}_{J(V)}{\otimes}\, G^*(V)$ as a subset of $\mathrm{Hom}_{J(V)-\mathrm{Mod}}(G(V)\,{}_{J(V)}{\otimes}\, G(V), J(V))$ and do not introduce any special notation for this construction. Having established this, we may deduce that Δ^* restricts to a product

$$\nabla' : G^*(V)\,{}_{J(V)}{\otimes}\, G^*(V) \to G^*(V)$$

which is the left $J(V)$-module homomorphism such that

$$(\nabla'(f \otimes f'))(g) = (f \otimes f')(\Delta(g)) = (f \otimes f')(g \otimes g) = f(g)f'(g)$$

for all $g \in G$. The dual is therefore the associative, commutative $J(V)$-algebra of all functions $f : G \to J(V)$ with pointwise multiplication as the product. This is also the dual of the group algebra $J(V)[G]$.

$G^*(V)$ may be considered as a right $J(V)$-module, hence a $J(V)$-bimodule, by declaring that for $p \in J(V)$, $f \in G^*(V)$, and $u \in G$ we have

$$(f \cdot p)(u) = f(up) = (u \cdot p)f(u).$$

∇ has a dual

$$\nabla^* : G^*(V) \to \mathrm{Hom}_{J(V)-\mathrm{Mod}}(G(V)\,{}_{J(V)}{\otimes}_{J(V)}\, G(V), J(V))$$

such that for any element $f \in G^*(V)$ and each $w'' \in G(V)\,{}_{J(V)}{\otimes}_{J(V)}\, G(V)$ we have

$$(\nabla^*(f))(w'') = f(\nabla(w'')).$$

If G is infinite, ∇^* would not normally be called a coproduct since the range of ∇^* is not a tensor square of $G^*(V)$. We will see, however, that in the case of a certain subring we are concerned with when G is a Coxeter group it induces a coproduct via the following construction.

We construct a $J(V)$-bimodule homomorphism

$$G^*(V) \ {}_{J(V)}\!\otimes_{J(V)} G^*(V) \to \mathrm{Hom}_{J(V)-\mathrm{Mod}}(G(V) \ {}_{J(V)}\!\otimes_{J(V)} G(V), J(V)),$$

which we denote by $a \mapsto \overline{a}$ for $a \in G^*(V) \ {}_{J(V)}\!\otimes_{J(V)} G^*(V)$, as follows. The construction is similar to the imbedding of $G^*(V) \ {}_{J(V)}\!\otimes G^*(V)$ into $\mathrm{Hom}_{J(V)-\mathrm{Mod}}(G(V) \ {}_{J(V)}\!\otimes G(V), J(V))$. Indeed, it is so similar that the overline notation is necessary to distinguish the two constructions. For any pair of elements $f, f' \in G^*(V)$ the function $F_{f,f'} : G(V) \times G(V) \to J(V)$ defined by

$$F_{f,f'}(w, w') = f(wf'(w'))$$

is a $J(V)$-middle linear map. By the universal property of $G(V) \ {}_{J(V)}\!\otimes_{J(V)} G(V)$, $F_{f,f'}$ induces a group homomorphism $P_{f,f'} : G(V) \ {}_{J(V)}\!\otimes_{J(V)} G(V) \to J(V)$ such that

$$P_{f,f'}(w \otimes w') = f(wf'(w'))$$

for all $w, w' \in G(V)$. This homomorphism of abelian groups is also a left $J(V)$-module homomorphism. The assignment $(f, f') \mapsto P_{f,f'}$ is a function $G^*(V) \times G^*(V) \to \mathrm{Hom}_{J(V)-\mathrm{Mod}}(G(V) \ {}_{J(V)}\!\otimes_{J(V)} G(V), J(V))$. Since $P_{f \cdot p,f'}(w \otimes w') = f(wpf'(w')) = P_{f,pf'}(w \otimes w')$, the assignment is a $J(V)$-middle linear map, hence there is an induced homomorphism $G^*(V) \ {}_{J(V)}\!\otimes_{J(V)} G^*(V) \to \mathrm{Hom}_{J(V)-\mathrm{Mod}}(G(V) \ {}_{J(V)}\!\otimes_{J(V)} G(V), J(V))$, and this is the map $a \mapsto \overline{a}$ that we set out to define. By definition we can see that evaluation using this construction has the property that $\overline{(f \otimes f')}(w \otimes w') = f(wf'(w'))$ for all $f, f' \in G^*(V)$ and all $w, w' \in G(V)$. The construction will be used in Section 3.4.

Note that there is an obvious action of $G(V)$ on $J(V)$ given by

$$\left(\sum_{g \in G} p_g g \right) \cdot q = \sum_{g \in G} p_g (g \cdot q).$$

Suppose M and N are both left $G(V)$-modules. Then it is clear that M and N are also left $J(V)$-modules. We define an action of $G(V)$ on $M \ {}_{J(V)}\!\otimes N$, called the *semi-diagonal* *action*, by declaring that for $p \in J(V)$, $g \in G$, $m \in M$, and $n \in N$ we have

$$pg \cdot (m \otimes n) = p(g \cdot m) \otimes (g \cdot n)$$

and extending linearly. Then we have the following result.

Proposition 3.2.1. *Suppose M and N are left $G(V)$-modules and $w \in G(V)$ is an element. Suppose we have an expansion of $\Delta(w)$ of the form*

$$\Delta(w) = \sum_{i=1}^{k} u_i \otimes v_i$$

for some $u_i, v_i \in G(V)$, $1 \leq i \leq k$. Then for all $m \in M$ and $n \in N$ we have the formula

$$w \cdot (m \otimes n) = \sum_{i=1}^{k} (u_i \cdot m) \otimes (v_i \cdot n)$$

where $M \ {}_{J(V)}\!\otimes N$ is given the semi-diagonal action of $G(V)$.

Proof. Let $a_{i,g}, b_{i,h} \in J(V)$ for $1 \leq i \leq k$ and $g, h \in G$ be such that

$$u_i = \sum_{g \in G} a_{i,g} g$$

and

$$v_i = \sum_{h \in G} b_{i,h} h.$$

Let $c_{g'} \in J(V)$ for $g' \in G$ be defined by

$$w = \sum_{g' \in G} c_{g'} g'.$$

Then

$$\Delta(w) = \sum_{i,g,h} a_{i,g} b_{i,h} g \otimes h = \sum_{g' \in G} c_{g'} g' \otimes g'.$$

Thus for $g, h \in G$ we have

$$\sum_{i=1}^{k} a_{i,g} b_{i,h} = \delta_{g,h} c_g.$$

We thus have the formula

$$\sum_{i=1}^{k} (u_i \cdot m) \otimes (v_i \cdot n) = \sum_{i,g,h} a_{i,g} b_{i,h} (g \cdot m) \otimes (h \cdot n) = \sum_{g' \in G} c_{g'} (g' \cdot m) \otimes (g' \cdot n) = w \cdot (m \otimes n)$$

as desired. $\square$

Corollary 3.2.2. *If $w \in G(V)$ and $p \in J(V)$ then for every expansion of $\Delta(w)$ of the form*

$$\Delta(w) = \sum_{i=1}^{k} u_i \otimes v_i$$

we have the formula

$$wp = \sum_{i=1}^{k} (u_i \cdot p) v_i.$$

Proof. Note that $G(V)$ is isomorphic to $J(V) \ {}_{J(V)}\!\otimes G(V)$ as a left $J(V)$-module via the isomorphism determined by the rule $p \otimes qg \mapsto pqg$ for $p, q \in J(V)$ and $g \in G$. With this isomorphism, the semi-diagonal action of $G(V)$ on $J(V) \ {}_{J(V)}\!\otimes G(V)$ corresponds to left multiplication in $G(V)$. Apply Proposition 3.2.1 with $M = J(V)$, $N = G(V)$, $m = p$, and $n = 1$. □

Corollary 3.2.3. *Let $w, w' \in G(V)$. Suppose*

$$\Delta(w) = \sum_{i=1}^{k} u_i \otimes v_i$$

for some $u_i, v_i \in G(V)$ and

$$\Delta(w') = \sum_{j=1}^{\ell} u'_j \otimes v'_j$$

for some $u'_j, v'_j \in G(V)$. Then

$$\Delta(ww') = \sum_{i=1}^{k} \sum_{j=1}^{\ell} u_i u'_j \otimes v_i v'_j.$$

Proof. Note that

$$\Delta(ww') = w \cdot \Delta(w')$$

where $G(V)$ acts on $G(V) \ {}_{J(V)}\!\otimes G(V)$ via the semi-diagonal action; this can be seen by expanding w and w' in terms of elements G. Apply Proposition 3.2.1 with $M = G(V)$, $N = G(V)$, $m = u'_j$, and $n = v'_j$ as j ranges from 1 to ℓ. □

We specialize now to the case of Kac-Moody Coxeter systems and their crystallographic geometric representations, where there is a lot to say.

3.3 Divided difference operators

Let G be a Kac-Moody group, let $B \subset G$ be a Borel subgroup, let $N \subset G$ be a subgroup such that (B, N) is a Tits system for G, let $T = B \cap N$ be the corresponding maximal torus, and let $V = \mathrm{Lie}(T)^*$. Let $W = N/T$ be the Weyl group of G with generating set $S \subset W$ determined by B such that (W, S) is a Coxeter system. Let $R \subset \mathrm{Lie}(T)^*$ be the root system of G, and let $\{x_s | s \in S\} \subset R$ be the set of simple roots determined by B. $W(V)$ and its coproduct structure are defined and used in [15].

We define elements $\partial_s \in W(V)$ for $s \in S$ by the equation

$$\partial_s = \frac{1}{x_s}(1 - s).$$

The following is proved in [15, Proposition 4.2].

Proposition 3.3.1. *Let $e = (e_1, e_2, \ldots, e_n)$ be a word in S. If e is not reduced, then*

$$\partial_{e_1}\partial_{e_2}\cdots\partial_{e_n} = 0.$$

If e is reduced, then $\partial_{e_1}\partial_{e_2}\cdots\partial_{e_n}$ depends only on the product $e_1e_2\cdots e_n$.

Proof. It will suffice to show the following:

(1) For each $s \in S$ we have

$$\partial_s^2 = 0.$$

(2) If $s, s' \in S$ satisfy $(ss')^2 = 1$, then

$$\partial_s\partial_{s'} = \partial_{s'}\partial_s.$$

(3) If $s, s' \in S$ satisfy $(ss')^3 = 1$, then

$$\partial_s\partial_{s'}\partial_s = \partial_{s'}\partial_s\partial_{s'}.$$

(4) If $s, s' \in S$ satisfy $(ss')^4 = 1$, then

$$\partial_s\partial_{s'}\partial_s\partial_{s'} = \partial_{s'}\partial_s\partial_{s'}\partial_s.$$

(5) If $s, s' \in S$ satisfy $(ss')^6 = 1$, then

$$\partial_s \partial_{s'} \partial_s \partial_{s'} \partial_s \partial_{s'} = \partial_{s'} \partial_s \partial_{s'} \partial_s \partial_{s'} \partial_s.$$

(1) is a trivial calculation. The other identities need only be checked in the root systems of rank 2 generated by the corresponding simple roots x_s and $x_{s'}$. The root system of a Kac-Moody group is *crystallographic*, meaning every root is an integral linear combination of simple roots. By the classification of finite crystallographic root systems, an account of which may be found in [13], the following are the only cases that need to be checked: (2) needs to be checked in type $A_1 \times A_1$, (3) needs to be checked in type A_2, (4) needs to be checked in type $B_2 = C_2$, and (5) needs to be checked in type G_2. These calculations are necessary but unenlightening. What we find is that if $R^+_{s,s'}$ is the set of positive roots of the root system spanned by x_s and $x_{s'}$ and $W_{s,s'}$ is the dihedral group generated by s and s', then the products in question are both equal to

$$\prod_{y \in R^+_{s,s'}} y^{-1} \sum_{w \in W_{s,s'}} (-1)^{\ell(w)} w$$

and hence equal to each other. □

We define $\mathrm{Diff}(V)$ to be the left $\mathrm{Sym}(V)$-module generated by $\{\partial_w | w \in W\}$ where $\partial_w = \partial_{e_1} \partial_{e_2} \cdots \partial_{e_n}$ for any reduced word e for w. The elements ∂_w are known as *divided difference operators.* It is not immediately clear that $\mathrm{Diff}(V)$ is closed under right multiplication by elements of $\mathrm{Sym}(V)$.

Lemma 3.3.2. *If $s \in S$ and $p \in \mathrm{Sym}(V)$, then $\partial_s \cdot p \in \mathrm{Sym}(V)$.*

Proof. Since the simple reflections act as reflections in Euclidean space, if p is of degree 1 then $(1 - s) \cdot p$ is divisible by x_s, so $\partial_s \cdot p \in \mathrm{Sym}(V)$. If $p = p_1 p_2 \cdots p_k$ is a product of linear factors, then the formula

$$p_1 p_2 \cdots p_k - (s \cdot p_1)(s \cdot p_2) \cdots (s \cdot p_k) = \sum_{i=1}^{k} (s \cdot p_1)(s \cdot p_2) \cdots (s \cdot p_{i-1})(p_i - (s \cdot p_i)) p_{i+1} p_{i+2} \cdots p_k,$$

a proof of which may be found in [21], shows that $(1 - s) \cdot p$ is divisible by x_s. Thus for all $p \in \mathrm{Sym}(V)$ we have that $(1 - s) \cdot p$ is divisible by x_s, which implies that $\partial_s \cdot p \in \mathrm{Sym}(V)$. □

This allows us to prove the following.

Lemma 3.3.3. Diff(V) *is a subring of* $W(V)$*. Furthermore, if* $d \in \text{Diff}(V)$ *then there exist an integer* k *and elements* $u_i, v_i \in \text{Diff}(V)$ *for* $1 \leq i \leq k$ *such that*

$$\Delta(d) = \sum_{i=1}^{k} u_i \otimes v_i.$$

Proof. To prove that Diff(V) is a subring of $W(V)$, it will suffice to show that for all $w \in W$ and $p \in \text{Sym}(V)$ we have the containment

$$\partial_w p \in \text{Diff}(V).$$

We show this by induction on $\ell(w)$. We have the formula

$$\Delta(\partial_s) = \partial_s \otimes 1 + s \otimes \partial_s$$

Hence

$$\partial_s p = (\partial_s \cdot p) + (s \cdot p)\partial_s \in \text{Diff}(V)$$

by Corollary 3.2.2 and Lemma 3.3.2. Hence the result holds for $w \in S$. Let n be an integer and assume the inductive hypothesis that for all sequences of elements $s_1, s_2, \ldots, s_{n-1} \in S$ and all $q \in J(V)$ we have

$$\partial_{s_1}\partial_{s_2}\cdots\partial_{s_{n-1}} q \in \text{Diff}(V).$$

Suppose then that $(s_1, s_2, \ldots, s_n)$ is a reduced word for an element $w \in W$. Then

$$\partial_w p = \partial_{s_1}\partial_{s_2}\cdots\partial_{s_{n-1}}\partial_{s_n} p = \partial_{s_1}\partial_{s_2}\cdots\partial_{s_{n-1}}((\partial_{s_n} \cdot p) + (s_n \cdot p)\partial_{s_n}).$$

Since right multiplication by ∂_{s_n} preserves Diff(V), it follows by associativity of multiplication in Diff(V) and the inductive hypothesis that $\partial_w p \in \text{Diff}(V)$. Therefore, it follows by induction that Diff(V) is a subring of $W(V)$.

Now we show by induction on $\ell(w)$ that for all $w \in W$ there exist a k and elements $u_i, v_i \in \text{Diff}(V)$ for $1 \leq i \leq k$ such that

$$\Delta(\partial_w) = \sum_{i=1}^{k} u_i \otimes v_i.$$

The result is true for $w \in S$. Assume by induction that the result holds for elements u with $\ell(u) < \ell(w)$. Suppose $s \in D_L(w)$. Then by the inductive hypothesis there exist an ℓ and elements $u_i', v_i' \in \mathrm{Diff}(V)$ for $1 \leq i \leq \ell$ such that

$$\Delta(\partial_{sw}) = \sum_{i=1}^{\ell} u_i' \otimes v_i'.$$

By Corollary 3.2.3, we have

$$\Delta(\partial_w) = \Delta(\partial_s \partial_{sw}) = \sum_{i=1}^{\ell} \partial_s u_i' \otimes v_i' + \sum_{i=1}^{\ell} s u_i' \otimes \partial_s v_i'$$

so the result holds by induction. □

We note that $\mathrm{Diff}(V)$ has the property that if $p \in \mathrm{Sym}(V)$ and $v \in \mathrm{Diff}(V)$, then $v \cdot p \in \mathrm{Sym}(V)$.

The following lemma will be necessary. It is essentially [15, Theorem 4.6] combined with Proposition 4.3 in the same paper.

Lemma 3.3.4. $\mathrm{Diff}(V)$ *is a free left* $\mathrm{Sym}(V)$*-module with basis* $\{\partial_w\}_{w \in W}$*. For any element* $v \in W$ *we have the formula*

$$v = \sum_{v' \leq v} C_{v'}(v) \partial_{v'}$$

for some elements $C_{v'}(v) \in \mathrm{Sym}(V)$*.*

Proof. We first consider the expansion of ∂_w for $w \in W$ in terms of elements of W. Write ∂_w as

$$\partial_w = \partial_{s_1} \partial_{s_2} \cdots \partial_{s_n}.$$

where $s_i \in S$ for all i. Then

$$\partial_w = \frac{1}{x_{s_1}}(1 - s_1) \frac{1}{x_{s_2}}(1 - s_2) \cdots \frac{1}{x_{s_n}}(1 - s_n).$$

By the exchange property of Coxeter systems, the only elements of W that occur when we expand this product have reduced words that are subwords of the reduced word $(s_1, s_2, \ldots, s_n)$. Hence

$$\partial_w = \sum_{w' \leq w} d_{w'}^{w} w'$$

for some coefficients $d^w_{w'} \in J(V)$, where the coefficient $d^w_w \neq 0$. This shows that the set of divided difference operators indexed by W is lower triangular in the basis W of $W(V)$ and hence freely generates a left $J(V)$-submodule. It follows that $\mathrm{Diff}(V)$ is freely generated by the divided difference operators as a left module over $\mathrm{Sym}(V)$ since any nontrivial relation over $\mathrm{Sym}(V)$ would also hold over $J(V)$. We know that the transition matrix is invertible and the inverse matrix has all of its entries in $\mathrm{Sym}(V)$ because each generator $s \in S$ may be written as $1 - x_s\partial_s$, which implies that every element of W is contained in $\mathrm{Diff}(V)$. The inverse matrix is also lower triangular, so the elements $C_{v'}(v)$ must exist. □

Lemma 3.3.5. $\mathrm{Diff}(V)$ *is a graded left* $\mathrm{Sym}(V)$*-module such that* ∂_w *is homogeneous of degree* $-\ell(w)$ *for all* $w \in W$ *and is a graded subring of* $W(V)$*. The grading is respected by* Δ*, where* $\mathrm{Diff}(V)\ {}_{\mathrm{Sym}(V)}\!\otimes \mathrm{Diff}(V)$ *is given the induced grading, in the sense that if* $a \in \mathrm{Diff}(V)$ *is homogeneous of degree* m*, then* $\Delta(a)$ *is homogeneous of degree* m*.*

Proof. Since $\deg(\partial_s) = -1$ for $s \in S$, it follows that $\deg(\partial_w) = -\ell(w)$ for all $w \in W$. The result about Δ follows from the fact that $\Delta : W(V) \to W(V)\ {}_{J(V)}\!\otimes W(V)$ respects the grading. □

3.4 Formulas in $H^*_T(X)$

We now wish to consider the graded left $\mathrm{Sym}(V)$-dual of $\mathrm{Diff}(V)$ (where each divided difference operator ∂_w is considered to have degree $-\ell(w)$, as in the previous section), which we denote by $\mathrm{Diff}^*(V) = \mathrm{Hom}_{\mathrm{Sym}(V)-\mathrm{Mod}}(\mathrm{Diff}(V), \mathrm{Sym}(V))$. We let $\{C_u | u \in W\}$ be the $\mathrm{Sym}(V)$-basis of $\mathrm{Diff}^*(V)$ satisfying

$$C_u(\partial_v) = \delta_{u,v}.$$

for all $u, v \in W$. The evaluations $C_{v'}(v)$ for $v, v' \in W$ are the elements of $\mathrm{Sym}(V)$ asserted to exist in Lemma 3.3.4, hence $C_u(v) = 0$ unless $u \leq v$.

There is a natural $\mathrm{Sym}(V)$-module homomorphism $\alpha : \mathrm{Diff}^*(V) \to W^*(V)$ obtained by declaring that if $f \in \mathrm{Diff}^*(V)$ and $w \in W$ then $(\alpha(f))(w) = f(w)$. Equivalently, for all $w \in W$ we declare that $(\alpha(f))(\partial_w) = f(\partial_w)$. We claim that α is an isomorphism onto the submodule $D \subset W^*(V)$ consisting of all functions f' such that for all $u \in W$ we have that $f'(\partial_u) \in \mathrm{Sym}(V)$ and $f'(\partial_u) = 0$ for all but finitely many such u. The fact that α has its image in D follows from the definition of $\mathrm{Diff}^*(V)$ as the graded left $\mathrm{Sym}(V)$-dual of $\mathrm{Diff}(V)$. α is injective because if $f, f' \in \mathrm{Diff}^*(V)$ satisfy $f(\partial_u) \neq f'(\partial_u)$ for some $u \in W$, then $(\alpha(f))(\partial_u) \neq (\alpha(f'))(\partial_u)$. Finally, surjectivity follows from the fact that if $f'' \in D$, we may define $f \in \mathrm{Diff}^*(V)$ by the rule $f(\partial_u) = f''(\partial_u)$ for all $u \in W$, and $\alpha(f) = f''$.

We show now that the product $\nabla' : W^*(V)\ {}_{J(V)}\!\otimes W^*(V) \to W^*(V)$, which we recall is given by pointwise multiplication of functions, restricts to a product on $\mathrm{Diff}^*(V)$. If $u, w, w' \in W$, then

$$(\nabla'(C_w \otimes C_{w'}))(\partial_u) = (C_w \otimes C_{w'})(\Delta(\partial_u))$$

It is clear that $(C_w \otimes C_{w'})(\Delta(\partial_u))$ is always an element of $\mathrm{Sym}(V)$. Write

$$\Delta(\partial_u) = \sum_{v,v' \in W} a_{v,v'} \partial_v \otimes \partial_{v'}$$

where $a_{v,v'} \in \mathrm{Sym}(V)$ for all $v, v' \in W$. Application of $C_w \otimes C_{w'}$ to a term of the form $p\partial_v \otimes \partial_{v'}$ for $p \in \mathrm{Sym}(V)$ yields 0 unless $v = w$ and $v' = w'$. If $a_{w,w'} \neq 0$, then $\ell(u) \leq \ell(w) + \ell(w')$. To see this, note that $\Delta(\partial_u)$ is homogeneous of degree $-\ell(u)$ by Lemma 3.3.5, so any term $a_{w,w'}\partial_w \otimes \partial_{w'}$ with $a_{w,w'} \neq 0$ must be homogeneous of degree $-\ell(u)$, hence $-\ell(u) = \deg(a_{w,w'}\partial_w \otimes \partial_{w'}) = \deg(a_{w,w'}) - \ell(w) - \ell(w') \geq -\ell(w) - \ell(w')$. Since there are only finitely many elements of W of a given length, there are only finitely many u such that the result is nonzero, hence $\mathrm{Diff}^*(V)$ is closed under ∇'.

Now that we have a basis and a product operation, we may define structure constants $c_{u,v}^w$ by

$$C_u C_v = \sum_{w \in W} c_{u,v}^w C_w.$$

We will see that it is not an accident that we use the same notation for these structure constants as for the ones in equivariant cohomology of flag varieties. In the remainder of this section we find a formula for $C_u(v)$ for $u, v \in W$ and obtain an isomorphism of $\mathrm{Diff}^*(V)$ with $H_T^*(X)$.

We will need to apply the overline homomorphism

$$W^*(V)\ {}_{J(V)}\otimes_{J(V)}\ W^*(V) \to \mathrm{Hom}_{J(V)-\mathrm{Mod}}(W(V)\ {}_{J(V)}\otimes_{J(V)}\ W(V), J(V))$$

given by

$$a \mapsto \overline{a}$$

and defined in Section 3.2 to elements of $\mathrm{Diff}^*(V)\ {}_{\mathrm{Sym}(V)}\otimes_{\mathrm{Sym}(V)}\ \mathrm{Diff}^*(V)$. To do this we imbed $\mathrm{Diff}^*(V)\ {}_{\mathrm{Sym}(V)}\otimes_{\mathrm{Sym}(V)}\ \mathrm{Diff}^*(V)$ in $W^*(V)\ {}_{J(V)}\otimes_{J(V)}\ W^*(V)$. Recall that $H \subset \mathrm{Sym}(V)$ consists of all nonzero homogeneous polynomials and $\alpha : \mathrm{Diff}^*(V) \to W^*(V)$ is the inclusion. The imbedding we desire is a composition

$$\begin{array}{c}
\mathrm{Diff}^*(V)\ {}_{\mathrm{Sym}(V)}\otimes_{\mathrm{Sym}(V)}\ \mathrm{Diff}^*(V) \\
\Big\downarrow {\scriptstyle f\mapsto 1f} \\
H^{-1}(\mathrm{Diff}^*(V)\ {}_{\mathrm{Sym}(V)}\otimes_{\mathrm{Sym}(V)}\ \mathrm{Diff}^*(V)) \\
\Big\downarrow {\scriptstyle \frac{1}{q}(C_u\otimes C_v)\mapsto(\frac{1}{q}C_u)\otimes C_v} \\
H^{-1}\mathrm{Diff}^*(V)\ {}_{J(V)}\otimes_{J(V)}\ H^{-1}\mathrm{Diff}^*(V) \\
\Big\downarrow {\scriptstyle H^{-1}\alpha\otimes H^{-1}\alpha} \\
W^*(V)\ {}_{J(V)}\otimes_{J(V)}\ W^*(V)
\end{array}$$

The first map is injective because $\mathrm{Diff}^*(V)\ {}_{\mathrm{Sym}(V)}\otimes_{\mathrm{Sym}(V)}\ \mathrm{Diff}^*(V)$ is a torsion-free left $\mathrm{Sym}(V)$-module. The second map is readily seen to be an isomorphism by noting that both modules are isomorphic to the free left $J(V)$-module with a basis consisting of the elements $C_u \otimes C_v$ for $u, v \in W$. The third map is injective because all modules involved are torsion-free and α is injective.

Proposition 3.4.1. *The coproduct*

$$\Delta' : \mathrm{Diff}^*(V) \to \mathrm{Diff}^*(V)\ {}_{\mathrm{Sym}(V)}\otimes_{\mathrm{Sym}(V)}\ \mathrm{Diff}^*(V)$$

defined by

$$\Delta'(C_w) = \sum_{w_{(1)},w_{(2)}} C_{w_{(1)}} \otimes C_{w_{(2)}}$$

for all $w \in W$, where we sum the right-hand side over all pairs $w_{(1)}, w_{(2)} \in W$ such that $\ell(w_{(1)}) + \ell(w_{(2)}) = \ell(w)$ and $w_{(1)}w_{(2)} = w$, satisfies the equation

$$\overline{\Delta'} = \nabla^*$$

on $\mathrm{Diff}^*(V)$*, where* $\nabla^* : W^*(V) \to \mathrm{Hom}_{J(V)-\mathrm{Mod}}(W(V)\ {}_{J(V)}\otimes_{J(V)} W(V), J(V))$ *is the dual of* ∇*.*

Proof. We have the equation

$$(\nabla^*(C_w))(\partial_u \otimes \partial_v) = C_w(\partial_u \partial_v).$$

$\nabla^*(C_w)$ is thus uniquely determined by the fact that $(\nabla^*(C_w))(\partial_u \otimes \partial_v) = 0$ unless both of the conditions $\ell(u) + \ell(v) = \ell(w)$ and $uv = w$ are satisfied, in which case $(\nabla^*(C_w))(\partial_u \otimes \partial_v) = 1$. Note that for $u, v, u', v' \in W$ we have the equation

$$\overline{(C_u \otimes C_{u'})}(\partial_v \otimes \partial_{v'}) = C_u(\partial_v C_{u'}(\partial_{v'})) = \delta_{u,v}\delta_{u',v'}.$$

Thus we can see by examining the definition of $\Delta'(C_w)$ that $\overline{(\Delta'(C_w))}(\partial_u \otimes \partial_v) = 0$ unless both of the conditions $\ell(u) + \ell(v) = \ell(w)$ and $uv = w$ are satisfied, in which case $\overline{(\Delta'(C_w))}(\partial_u \otimes \partial_v) = 1$. It follows that

$$\overline{\Delta'} = \nabla^*$$

on $\mathrm{Diff}^*(V)$. □

Corollary 3.4.2. *We have the formula*

$$C_w(uv) = \sum_{w_{(1)}, w_{(2)}} C_{w_{(1)}}(u)(u \cdot C_{w_{(2)}}(v))$$

for all $u, v, w \in W$*, where the notation is as in Proposition 3.4.1.*

Proof. We know that

$$C_w(uv) = \overline{(\Delta'(C_w))}(u \otimes v).$$

Proposition 3.4.1 allows us to write

$$C_w(uv) = \sum_{w_{(1)}, w_{(2)}} C_{w_{(1)}}(uC_{w_{(2)}}(v)) = \sum_{w_{(1)}, w_{(2)}} C_{w_{(1)}}(u)(u \cdot C_{w_{(2)}}(v))$$

as desired. □

The following formula is proved by Billey in [3].

Theorem 2 (Billey's restriction formula)**.** *Suppose $u \in W$ satisfies $\ell(u) = k$ and e is a reduced word for an element $w \in W$. Then*

$$C_u(w) = \sum_{\substack{e_{i_1} e_{i_2} \cdots e_{i_k} = u \\ i_1 < i_2 < \cdots < i_k}} (-r_{i_1}(e))(-r_{i_2}(e)) \cdots (-r_{i_k}(e)).$$

Proof. Suppose $\ell(w) = n$. We obtain the result by iterating Corollary 3.4.2 $n-1$ times, corresponding to the decomposition of w as the product $e_1 e_2 \cdots e_n$. This gives us the equation

$$C_u(e_1 e_2 \cdots e_n) = \sum C_{u_{(1)}}(e_1)(e_1 \cdot C_{u_{(2)}}(e_2))(e_1 e_2 \cdot C_{u_{(3)}}(e_3)) \ldots (e_1 e_2 \cdots e_{n-1} \cdot C_{u_{(n)}}(e_n)),$$

where the right-hand side is summed over all decompositions $u_{(1)} u_{(2)} u_{(3)} \cdots u_{(n)} = u$ such that $\ell(u_{(1)}) + \ell(u_{(2)}) + \cdots + \ell(u_{(n)}) = k$.

If a particular term in the sum is nonzero then each $u_{(i)}$ satisfies either $u_{(i)} = 1$ or $u_{(i)} = e_i$ by Lemma 3.3.4. We can see directly from the expansion

$$e_i = 1 - x_{e_i} \partial_{e_i}$$

that $C_1(e_i) = 1$ and $C_{e_i}(e_i) = -x_{e_i}$, so $e_1 e_2 \cdots e_{i-1} \cdot C_{e_i}(e_i) = -r_i(e)$. The result follows. ☐

This allows us to obtain an isomorphism with the equivariant cohomology ring. The following is proved in [15, Theorem 5.12].

Corollary 3.4.3. *The left* $\mathrm{Sym}(V)$*-linear map determined by $C_u \mapsto [X_u]$ is a ring isomorphism between* $\mathrm{Diff}^*(V)$ *and the torus-equivariant cohomology ring of the flag variety with root system R.*

Proof. $C_u(v)$ is the same as the restriction of $[X_u]$ to the fixed point x_v. For all $v \in W$ define a ring homomorphism $\iota_v : \mathrm{Diff}^*(V) \to H_T^*(x_v)$ by linearly extending the rule $C_u \mapsto C_u(v)$ for all $u \in W$. The ring homomorphism

$$\prod_{v \in W} \iota_v : \mathrm{Diff}^*(V) \to \prod_{v \in W} H_T^*(x_v)$$

can be composed with the inverse of the fixed-point restriction homomorphism to obtain the desired isomorphism. ☐

3.5 The Leibniz rule and skew divided difference operators

The following proposition follows from [15, Proposition 4.15] combined with Proposition 4.24 in the same paper.

Proposition 3.5.1. *The coproduct in* Diff(V) *is given by*

$$\Delta(\partial_w) = \sum_{u,v \in W} c^w_{u,v} \partial_u \otimes \partial_v$$

Proof. This is easily seen by the equation

$$(C_u \otimes C_v)(\Delta(\partial_w)) = (C_u C_v)(\partial_w) = c^w_{u,v}.$$

□

We may now prove our Leibniz rule.

Proof of Theorem 1. The result is a direct consequence of Proposition 3.5.1 and Corollary 3.2.2. □

It is natural now to define elements ∂^w_u by

$$\partial^w_u = \sum_{v \in W} c^w_{u,v} \partial_v.$$

The coproduct can thus be expressed more compactly as

$$\Delta(\partial_w) = \sum_{u \in W} \partial_u \otimes \partial^w_u = \sum_{u \in W} \partial^w_u \otimes \partial_u.$$

and the Leibniz formula is given by

$$\partial_w p = \sum_{u \in W} (\partial_u \cdot p) \partial^w_u = \sum_{u \in W} (\partial^w_u \cdot p) \partial_u$$

The elements ∂^w_u will be called *skew divided difference operators.* The elements $\partial_{w/u}$ defined by

$$\partial_{w/u} = u^{-1} \partial^w_u$$

were studied by Macdonald in [18] and used to prove the following Leibniz rule in type A:

$$\partial_w(pq) = \sum_{u \leq w} u \partial_{w/u}(p) \partial_u(q).$$

Macdonald defines $\partial_{w/u}$ by an explicit formula involving divided difference operators ∂_s and simple reflections s for $s \in S$.

The context of Macdonald's work is the study of *Schubert polynomials*, which are introduced in [16] and may be defined as follows. In this situation $W = S_n$ is the symmetric group, and we are working in a polynomial ring $\mathbb{Z}[t_1, t_2, \ldots, t_n]$ on which a permutation $w \in S_n$, considered as a bijection from $\{1, 2, \ldots, n\} = [n] \to [n]$, acts by the rule

$$w(t_i) = t_{w(i)}.$$

This action is the same as when we consider the t_i as generating the character ring of the torus. The Coxeter generators of S_n are the simple transpositions $s_1, s_2, \ldots, s_{n-1}$ defined by $s_i(i) = i+1$, $s_i(i+1) = i$, and $s_i(j) = j$ if $j \neq i, i+1$. The simple root x_{s_i} corresponding to s_i is then given by

$$x_{s_i} = t_i - t_{i+1}.$$

The divided difference operator ∂_{s_i} preserves $\mathbb{Z}[t_1, t_2, \ldots, t_n]$ and acts on $f \in \mathbb{Z}[t_1, t_2, \ldots, t_n]$ by the rule

$$(\partial_{s_i} \cdot f)(t_1, t_2, \ldots, t_n) = \frac{f(t_1, t_2, \ldots, t_n) - f(t_1, t_2, \ldots, t_{i-1}, t_{i+1}, t_i, t_{i+2}, \ldots, t_n)}{t_i - t_{i+1}}.$$

Schubert polynomials are indexed by elements of W and will be denoted by S_v for $v \in W$. Schubert polynomials are defined by first defining S_{w_0}, where $w_0 \in S_n$ is the reversal permutation, meaning $w_0(i) = n + 1 - i$ for all i. For that definition we have

$$S_{w_0} = t_1^{n-1} t_2^{n-2} \cdots t_{n-2}^2 t_{n-1}.$$

Then we define the Schubert polynomial S_v for $v \in S_n$ by the rule

$$S_v = \partial_{v^{-1}w_0}(S_{w_0}).$$

In [14] Kirillov shows in this scenario that if $\ell(u) + \ell(v) = \ell(w)$, then

$$\partial_{w/u} \cdot S_v = c_{u,v}^w.$$

Note that since in this case $c_{u,v}^w$ is constant, it is also true that $\partial_u^w \cdot S_v = c_{u,v}^w$. This implies that the coefficient of ∂_v in ∂_u^w is equal to $c_{u,v}^w$, which is obvious from the way we define ∂_u^w. The new result in our case is identification of the rest of the coefficients of the divided difference operators in the expansion of ∂_u^w as the structure constants in equivariant cohomology. Our formula also applies for completely general Kac-Moody groups rather than just in type A.

Obviously, if one wishes to obtain a Leibniz formula for a product of more than two rational functions one may iteratively apply Theorem 1. However, we may obtain a more elegant formula as follows. For elements $u_1, u_2, \ldots, u_k \in W$ and $w \in W$ define $c^w_{u_1,u_2,\ldots,u_k}$ by the following formula:

$$C_{u_1}C_{u_2}\cdots C_{u_k} = \sum_{w\in W} c^w_{u_1,u_2,\ldots,u_k} C_w.$$

This definition is valid since the product in $\mathrm{Diff}^*(V)$ is associative.

Lemma 3.5.2. *For all $u_1, u_2, \ldots, u_k \in W$ and $w \in W$ we have the formula*

$$c^w_{u_1,u_2,\ldots,u_k} = \sum_{w'\in W} c^w_{u_1,u_2,\ldots,u_{k-2},w'} c^{w'}_{u_{k-1},u_k}.$$

Proof. By definition we have the formula

$$C_{u_{k-1}}C_{u_k} = \sum_{w'\in W} c^{w'}_{u_{k-1},u_k} C_{w'}.$$

We also have the formula

$$C_{u_1}C_{u_2}\cdots C_{u_{k-2}}C_{u_{k-1}}C_{u_k} = \sum_{w'\in W} C_{u_1}C_{u_2}\cdots C_{u_{k-2}} c^{w'}_{u_{k-1},u_k} C_{w'} = \sum_{w,w'\in W} c^w_{u_1,u_2,\ldots,u_{k-2},w'} c^{w'}_{u_{k-1},u_k} C_w$$

which implies the result. □

We define generalized skew divided difference operators as follows:

$$\partial^w_{u_1,u_2,\ldots,u_k} = \sum_{v\in W} c^w_{u_1,u_2,\ldots,u_k,v}\partial_v.$$

It is not immediately clear that this sum is finite. However, the following proposition combined with an easy induction and the fact that the coproduct is well-defined implies that it is. This is also implied by the fact that $c^w_{u_1,u_2,\ldots,u_k,v} = 0$ unless $u_i \le w$ for all i and $v \le w$.

Proposition 3.5.3. *For any elements* $u_1, \ldots, u_{k-1} \in W$ *and* $w \in W$ *we have the formula*

$$\Delta(\partial^w_{u_1,\ldots,u_{k-1}}) = \sum_{u_k \in W} \partial_{u_k} \otimes \partial^w_{u_1,u_2,u_3,\ldots,u_k}$$

Proof. We have

$$\Delta(\partial^w_{u_1,\ldots,u_{k-1}}) = \sum_{v \in W} c^w_{u_1,\ldots,u_{k-1},v} \Delta(\partial_v) = \sum_{u',u'',v \in W} c^w_{u_1,\ldots,u_{k-1},v} c^v_{u',u''} \partial_{u'} \otimes \partial_{u''}.$$

By Lemma 3.5.2 this is equal to

$$\sum_{u',u'' \in W} c^w_{u_1,\ldots,u_{k-1},u',u''} \partial_{u'} \otimes \partial_{u''} = \sum_{u',u'' \in W} \partial_{u'} \otimes c^w_{u_1,\ldots,u_{k-1},u',u''} \partial_{u''} = \sum_{u' \in W} \partial_{u'} \otimes \partial^w_{u_1,\ldots,u_{k-1},u'}$$

and setting $u_k = u'$ gives us the result. □

Corollary 3.5.4 (The generalized Leibniz rule)**.** *For any integer* $k > 0$*, any element* $w \in W$*, and any rational functions* $p_1, p_2, \ldots, p_k$ *we have the formula*

$$\partial_w p_1 p_2 \cdots p_k = \sum_{u_1,u_2,\ldots,u_k \in W} \partial_{u_1}(p_1) \partial_{u_2}(p_2) \cdots \partial_{u_k}(p_k) \partial^w_{u_1,u_2,\ldots,u_k}.$$

Proof. The base case of $k = 1$ has already been proved. Assuming by induction that

$$\partial_w p_1 p_2 \cdots p_{k-1} p_k = \sum_{u_1,u_2,\ldots,u_{k-1} \in W} \partial_{u_1}(p_1) \partial_{u_2}(p_2) \cdots \partial_{u_{k-1}}(p_{k-1}) \partial^w_{u_1,u_2,\ldots,u_{k-1}} p_k$$

Proposition 3.5.3 implies via Corollary 3.2.2 that

$$\partial_w p_1 p_2 \cdots p_{k-1} p_k = \sum_{u_1,u_2,\ldots,u_{k-1},u_k \in W} \partial_{u_1}(p_1) \partial_{u_2}(p_2) \cdots \partial_{u_{k-1}}(p_{k-1}) \partial_{u_k}(p_k) \partial^w_{u_1,u_2,\ldots,u_{k-1},u_k}$$

which is the desired result. □

Vita

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