SOME RESULTS IN THE REPRESENTATION THEORY OF STRONGLY GRADED VERTEX ALGEBRAS

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ABSTRACT OF THE DISSERTATION

Some results in the representation theory of strongly graded vertex algebras

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In the first part of this thesis, we study strongly graded vertex algebras and their strongly graded modules, which are conformal vertex algebras and their modules with a second, compatible grading by an abelian group satisfying certain grading restriction conditions. We consider a tensor product of strongly graded vertex algebras and its tensor product strongly graded modules. We prove that a tensor product of strongly graded irreducible modules for a tensor product of strongly graded vertex algebras is irreducible, and that such irreducible modules, up to equivalence, exhaust certain naturally defined strongly graded irreducible modules for a tensor product of strongly graded vertex algebras. We also prove that certain naturally defined strongly graded modules for the tensor product strongly graded vertex algebra are completely reducible if and only if every strongly graded module for each of the tensor product factors is completely reducible. These results generalize the corresponding known results for vertex operator algebras and their modules.

In the second part, we derive certain systems of differential equations for matrix elements of products and iterates of logarithmic intertwining operators among strongly graded generalized modules for a strongly graded conformal vertex algebra under suitable assumptions. Using these systems of differential equations, we verify the convergence and extension property needed in the logarithmic tensor category theory for such strongly graded generalized modules developed by Huang, Lepowsky and Zhang.

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Dedication

I dedicate my dissertation work to my beloved parents who have supported me all the way since the beginning of my study. I will never forget my father, who is no longer of this world since January 16th in 2014. May you find peace and happiness in the paradise!

I also dedicate my thesis to my wife—Yang Wen, who has been a great source of encouragement and support.

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Chapter 1

Introduction

Vertex operator algebras, as defined in [FLM], and more generally, *vertex algebras*, as defined in [B1], form a fundamental class of algebraic structures. The representation theory of vertex (operator) algebras plays deep roles in the construction and study of infinite-dimensional Lie algebra representations, in the development of structure underlying sporadic finite simple groups, in string theory, in the theory of modular functions, and in many other areas.

The difference between the terminology vertex operator algebra and vertex algebra is that a vertex operator algebra amounts to a vertex algebra with a conformal vector such that the eigenspaces of the operator $L(0)$ are all finite dimensional with (integral) eigenvalues that are truncated from below (cf. [LL]). In [HLZ1] and [HLZ2], the authors use a notion of *conformal vertex algebra*, which is a vertex algebra with a conformal vector and with an $L(0)$-eigenspace decomposition, and a notion of *strongly graded conformal vertex algebra*, which is a conformal vertex algebra with a second, compatible grading by an abelian group satisfying certain grading restriction conditions.

In this thesis, we explore two important aspects of the representation theory of strongly graded vertex algebras. In the first part of this work, we consider tensor products of algebras and tensor products of modules for the respective tensor factors. The second part of this work is motivated by tensor product functors for modules for an algebra, and the corresponding tensor categories of modules, a much different, and more sophisticated, aspect of the theory than tensor products of algebras and their corresponding modules. Thus in this thesis, “tensor products” are actually used in two completely different senses.

In a series of papers ([HL1]–[HL4], [H1]), Yi-Zhi Huang and James Lepowsky developed a theory of braided tensor categories, and more precisely, of “vertex tensor categories,” for the module category of what they called a “finitely reductive” vertex operator algebra satisfying certain additional conditions; finitely reductive means that the module category is semisimple and that certain finiteness conditions hold. But it is just as natural and important to develop a theory for non-semisimple module categories in vertex operator algebra theory as it is in the Lie theory. In [HLZ1] and [HLZ2], this tensor product theory is generalized to a larger family of categories of strongly graded modules for a conformal vertex algebra, under suitably relaxed conditions.

We would like to investigate vertex tensor categories in the sense of [HL1], but more generally, in the setting of [HLZ1], associated with the tensor product of strongly graded vertex algebras, and this motivates the problem of determining the irreducible modules for this tensor product algebra. This problem is solved in the first part of this thesis, Chapter 2.

To develop the representation theory of vertex operator algebras that are not finitely reductive, it is necessary to consider certain generalized modules that are not completely reducible and the logarithmic intertwining operators among them. In [HLZ2], the authors prove that under certain conditions, matrix elements of products and iterates of logarithmic intertwining operators among generalized modules for a vertex operator algebra satisfy certain systems of differential equations. Using this result, they verify the convergence and extension property, an important sufficient condition, introduced in its original form in [H1], for constructing vertex tensor categories for generalized modules for a vertex operator algebra. In the second part of this thesis, Chapter 3, we generalize these arguments to logarithmic intertwining operators among generalized modules for a strongly graded vertex algebra.

Now we proceed to give more details.

In the first part of this thesis, Chapter 2, we prove that a tensor product of strongly graded irreducible modules for a tensor product of strongly graded vertex algebras is irreducible, and that conversely, such irreducible modules, up to equivalence, exhaust certain naturally defined strongly graded irreducible modules for a tensor product of strongly graded vertex algebras. (These terms are defined in Chapter 2.) As a consequence, we determine all the strongly graded irreducible modules for the tensor product of the moonshine module vertex operator algebra $V^\natural$ with a vertex algebra associated with a self-dual even lattice, in particular, the two-dimensional Lorentzian lattice.

The moonshine conjecture of Conway and Norton in [CN] included the conjecture that there should exist an infinite-dimensional representation V of the (not yet constructed) Fischer-Griess Monster sporadic finite simple group $\mathbb{M}$ such that the McKay-Thompson series T_g for $g \in \mathbb{M}$ acting on V should have coefficients that are equal to the coefficients of the q-series expansions of certain modular functions. In particular, this conjecture incorporated the McKay-Thompson conjecture, which asserted that there should exist a (suitably nontrivial) $\mathbb{Z}$-graded $\mathbb{M}$-module $V = \coprod_{i \geq -1} V_{-i}$ with graded dimension equal to the elliptic modular function $j(\tau) - 744 = \sum_{i \geq -1} c(i) q^i$, where we write q for $e^{2\pi i \tau}$, τ in the upper half-plane. Such an $\mathbb{M}$-module, the "moonshine module," denoted by $V^\natural$, was constructed in [FLM], and in fact, the construction of [FLM] gave a vertex operator algebra structure on $V^\natural$ equipped with an action of $\mathbb{M}$. In [FLM], the authors also gave an explicit formula for the McKay-Thompson series of any element of the centralizer of an involution of type 2**B** of $\mathbb{M}$; the case of the identity element of $\mathbb{M}$ proved the McKay-Thompson conjecture.

Borcherds then showed in [B2] that the rest of the McKay-Thompson series for the elements of $\mathbb{M}$ acting on $V^\natural$ are the expected modular functions. He obtained recursion formulas for the coefficients of McKay-Thompson series for $V^\natural$ from the Euler-Poincaré identity for certain homology groups associated with a special Lie algebra, the "monster Lie algebra," which he constructed using the tensor product of the moonshine module vertex operator algebra $V^\natural$ and a natural vertex algebra associated with the two-dimensional Lorentzian lattice. The importance of this tensor product vertex algebra motivates the first part of this thesis, Chapter 2.

In a series of papers ([HL1]–[HL4], [H1]), the authors developed a tensor product theory for modules for a vertex operator algebra under suitable conditions. A structure called "vertex tensor category structure," which is much richer than braided tensor category structure, has thereby been established for many important categories of modules for classes of vertex operator algebras (see [HL1]). It is expected that a vertex tensor category together with certain additional structures determines uniquely (up to isomorphism) a vertex operator algebra such that the vertex tensor category constructed from a suitable category of modules for it is equivalent (in the sense of vertex tensor categories) to the original vertex tensor category. In [HLZ1] and [HLZ2], this tensor product theory is generalized to a larger family of categories of "strongly graded modules" for a conformal vertex algebra, under suitably relaxed conditions. We want to investigate the vertex tensor category in the sense of [HL1], but in the setting of [HLZ1], associated with the tensor product of the moonshine module vertex operator algebra $V^\natural$ and the vertex algebra associated with the two-dimensional Lorentzian lattice. The first step in thinking about this is to determine the irreducible modules for this algebra.

For the vertex operator algebra case, it is proved in [FHL] that a tensor product module $W_1 \otimes \cdots \otimes W_p$ for a tensor product vertex algebra $V_1 \otimes \cdots \otimes V_p$ (where W_i is a V_i-module) is irreducible if and only if each W_i is irreducible. The proof uses a version of Schur's Lemma and also the density theorem [J]. It is also proved in [FHL] that these irreducible modules W are (up to equivalence) exactly all the irreducible modules for the tensor product algebra $V_1 \otimes \cdots \otimes V_p$. The proof uses the fact that each homogeneous subspace of W is finite dimensional. In this paper, we generalize the arguments in [FHL] to prove similar, more general results for strongly graded modules for strongly graded conformal vertex algebras.

For the strongly graded conformal vertex algebra case, the homogeneous subspaces of a strongly graded module are no longer finite dimensional. However, by using the fact that each doubly homogeneous subspace (homogeneous with respect to both gradings) of a strongly graded conformal vertex algebra is finite dimensional, we prove a suitable version of Schur's Lemma for strongly graded modules under the assumption that the abelian group that gives the second grading of the strongly graded algebra is countable.

To avoid unwanted flexibility in the second grading such as a shifting of the grading by an element of the abelian group, we suppose that the grading abelian groups A for a strongly graded conformal vertex algebra and $\tilde{A}$ (which includes A as a subgroup) for its strongly graded modules are always determined by a vector space, which we typically call $\mathfrak{h}$, consisting of operators induced by V. We call this kind of strongly graded conformal vertex algebra a "strongly $(\mathfrak{h}, A)$-graded conformal vertex algebra" and its strongly graded modules "strongly $(\mathfrak{h}, \tilde{A})$-graded modules." Important examples of strongly $(\mathfrak{h}, A)$-graded conformal vertex algebras and their strongly $(\mathfrak{h}, \tilde{A})$-graded modules are the vertex algebras associated with nondegenerate even lattices and their modules.

For strongly $(\mathfrak{h}_i, \tilde{A}_i)$-graded modules W_i for strongly $(\mathfrak{h}_i, A_i)$-graded conformal vertex algebras V_i, we construct a tensor product strongly $(\oplus_{i=1}^p \mathfrak{h}_i, \oplus_{i=1}^p \tilde{A}_i)$-graded module $W_1 \otimes \cdots \otimes W_p$ for the tensor product strongly graded conformal vertex algebra $V_1 \otimes \cdots \otimes V_p$. Then we prove that this tensor product module $W_1 \otimes \cdots \otimes W_p$ is irreducible if and only if each W_i is irreducible, under the assumption that each grading abelian group A_i for V_i is a countable group.

To determine all the irreducible strongly graded modules (up to equivalence) for the tensor product strongly graded conformal vertex algebra $V_1 \otimes \cdots \otimes V_p$, the main difficulty is that we need to deal with the second grading by the abelian groups. For the strongly $(\oplus_{i=1}^p \mathfrak{h}_i, \tilde{A})$-graded modules W for the tensor product strongly $(\oplus_{i=1}^p \mathfrak{h}_i, \oplus_{i=1}^p A_i)$-graded vertex algebra $V_1 \otimes \cdots \otimes V_p$, we assume there is a decomposition $\tilde{A} = \tilde{A}_1 \oplus \cdots \oplus \tilde{A}_p$, such that W is an $(\mathfrak{h}_i, \tilde{A}_i)$-graded module (that is, a strongly graded module except for the grading restriction conditions) when viewed as a V_i-module. We call this kind of strongly $(\oplus_{i=1}^{p}\mathfrak{h}_i, \tilde{A})$-graded module a strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded module. In the main theorem, we prove that if such a module is irreducible, then it is a tensor product of strongly graded irreducible modules. Then, as a corollary of the main theorem, we classify the strongly graded modules for the tensor product strongly graded conformal vertex algebra $V^{\natural} \otimes V_L$, where L is an even lattice, and in particular, where L is the (self-dual) two-dimensional Lorentzian lattice.

It is proved in [DMZ] that every module for the tensor product vertex operator algebra $V_1 \otimes \cdots \otimes V_p$ is completely reducible if and only if every module for each vertex operator algebra V_i is completely reducible. We also generalize the argument in [DMZ] to prove a similar result for tensor product strongly $(\mathfrak{h}, A)$-graded conformal vertex algebras.

In the second part of this thesis, Chapter 3, we generalize the arguments in [H3] and [HLZ2] to prove that for a strongly graded conformal vertex algebra V, matrix elements of products and iterates of logarithmic intertwining operators among triples of strongly graded generalized V-modules under suitable assumptions satisfy certain systems of differential equations and that the prescribed singular points are regular. Using these differential equations, we verify the *convergence and extension property* needed in the theory of logarithmic tensor categories for strongly graded generalized V-modules in [HLZ2]. Consequently, under certain assumptions on the strongly graded generalized modules for a strongly graded conformal vertex algebra V, we obtain a natural structure of braided tensor category on the category of strongly graded generalized V-modules using the main result of [HLZ2].

It was proved in [H3] that if every module W for a vertex operator algebra $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ satisfies the C_1-*cofiniteness condition*, that is, dim $W/C_1(W) < \infty$, where $C_1(W)$ is the subspace of W spanned by elements of the form $u_{-1}w$

for $u \in V_+ = \coprod_{n>0} V_{(n)}$ and $w \in W$, then matrix elements of products and iterates of intertwining operators among triples of V-modules satisfy certain systems of differential equations. Moreover, for prescribed singular points, there exist such systems of differential equations such that the prescribed singular points are regular. In Section 11 of [HLZ2] (Part VII), using the same argument as in [H3], certain systems of differential equations were derived for matrix elements of products and iterates of logarithmic intertwining operators among triples of generalized V-modules. In the second part of my thesis, we prove similar, more general results for matrix elements of products and iterates of logarithmic intertwining operators among triples of strongly graded generalized modules for a strongly graded vertex algebra.

In the second part of the thesis, Chapter 3, we generalize the C_1-cofiniteness condition for generalized modules for a vertex operator algebra to a *C_1-cofiniteness condition with respect to $\tilde{A}$* for strongly $\tilde{A}$-graded generalized modules for a strongly graded vertex algebra. That is, every strongly graded generalized $\tilde{A}$-module W for a strongly A-graded vertex algebra V satisfies the condition that for $\beta \in \tilde{A}$, $\dim W^{(\beta)}/(C_1(W))^{(\beta)} < \infty$, where $W^{(\beta)}$ and $(C_1(W))^{(\beta)}$ are the $\tilde{A}$-homogeneous subspace of W and $C_1(W)$ with $\tilde{A}$-grading β, respectively. Furthermore, for a strongly graded vertex subalgebra V_0 of V, the C_1-cofiniteness condition for W as a V_0-module implies the C_1-cofiniteness condition for W as a V-module. In particular, the case that W satisfies the C_1-cofiniteness condition as a module for $V^{(0)}$—the A-homogeneous subspace of V with A-weight 0—is the same as the case that W satisfies the C_1-cofiniteness condition as a vertex operator algebra module.

The key step in deriving systems of differential equations in [H3] is to construct a finitely generated $R = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, (z_1 - z_2)^{-1}]$-module that is a quotient module of the tensor product of R and a quadruple of modules for a vertex operator algebra. However, for a strongly graded conformal vertex algebra, the quotient module constructed in the same way is not finitely generated since there can be infinitely many $\tilde{A}$-homogeneous subspaces in the strongly graded generalized modules. In order to obtain a finitely generated quotient module, we assume that fusion rules for triples of certain $\tilde{A}$-homogeneous subspaces of strongly graded generalized V-modules viewed as $V^{(0)}$-modules are zero for all but finitely many triples of such $\tilde{A}$-homogeneous subspaces.

Under the assumption on the fusion rules for triples of certain $\tilde{A}$-homogeneous subspaces and the C_1-cofiniteness condition with respect to $\tilde{A}$ for the strongly $\tilde{A}$-graded generalized modules, we construct a natural map from a finitely generated R-module to the set of matrix elements of products and iterates of logarithmic intertwining operators among triples of strongly graded generalized V-modules. The images of certain elements under this map provide systems of differential equations for the matrix elements of products and iterates of logarithmic intertwining operators, as a consequence of the $L(-1)$-derivative property for the logarithmic intertwining operators. Moreover, for any prescribed singular point, we derive certain systems of differential equations such that this prescribed singular point is regular. Using these systems of differential equations, we verify the convergence and extension property needed in the construction of associativity isomorphism for the logarithmic tensor category structure developed in [HLZ2]. Consequently, if all the assumptions mentioned above are satisfied, we obtain a braided tensor category structure on the category of strongly graded generalized V-modules.

The material in Chapters 2 and 3 is contained in [Y1] and [Y2], respectively.

It would be valuable to construct interesting examples of strongly graded modules other than modules for vertex algebras associated with non-positive-definite even lattices. Not only would such examples broaden the applicability of the differential equations constructed in the second part of this thesis, but they would also shed further light on Huang-Lepowsky's vertex tensor category theory.

Chapter 2

Tensor products of strongly graded vertex algebras and their strongly graded modules

2.1 Strongly graded vertex algebras and their strongly graded modules

We recall the following four definitions from [HLZ1].

Definition 2.1.1 A *conformal vertex algebra* is a $\mathbb{Z}$-graded vector space

$$V = \coprod_{n\in\mathbb{Z}} V_{(n)} \tag{2.1}$$

(for $v \in V_{(n)}$, we say the *weight* of v is n and we write $\mathrm{wt}\, v = n$) equipped with a linear map $V \otimes V \to V[[x, x^{-1}]]$, or equivalently,

$$\begin{aligned} V &\to (\mathrm{End}\, V)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1} \ \ (\text{where } v_n \in \mathrm{End}\, V), \end{aligned} \tag{2.2}$$

$Y(v, x)$ denoting the *vertex operator associated with* v, and equipped also with two distinguished vectors $\mathbf{1} \in V_{(0)}$ (the *vacuum vector*) and $\omega \in V_{(2)}$ (the *conformal vector*), satisfying the following conditions for $u, v \in V$: the *lower truncation condition*:

$$u_n v = 0 \quad \text{for } n \text{ sufficiently large} \tag{2.3}$$

(or equivalently, $Y(u, x)v \in V((x))$); the *vacuum property*:

$$Y(\mathbf{1}, x) = 1_V; \tag{2.4}$$

the *creation property*:

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x\to 0} Y(v, x)\mathbf{1} = v \tag{2.5}$$

(that is, $Y(v,x)\mathbf{1}$ involves only nonnegative integral powers of x and the constant term is v); the *Jacobi identity* (the main axiom):

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)Y(v,x_2)-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(v,x_2)Y(u,x_1)$$
$$=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(u,x_0)v,x_2) \tag{2.6}$$

(note that when each expression in (2.6) is applied to any element of V, the coefficient of each monomial in the formal variables is a finite sum; on the right-hand side, the notation $Y(\cdot,x_2)$ is understood to be extended in the obvious way to $V[[x_0,x_0^{-1}]]$); the *Virasoro algebra relations*:

$$[L(m),L(n)]=(m-n)L(m+n)+\frac{1}{12}(m^3-m)\delta_{n+m,0}c \tag{2.7}$$

for $m,n\in\mathbb{Z}$, where

$$L(n)=\omega_{n+1}\quad\text{for}\quad n\in\mathbb{Z},\quad\text{i.e.,}\quad Y(\omega,x)=\sum_{n\in\mathbb{Z}}L(n)x^{-n-2}, \tag{2.8}$$

$$c\in\mathbb{C} \tag{2.9}$$

(the *central charge* or *rank* of V);

$$\frac{d}{dx}Y(v,x)=Y(L(-1)v,x) \tag{2.10}$$

(the $L(-1)$-*derivative property*); and

$$L(0)v=nv=(\text{wt }v)v\quad\text{for}\quad n\in\mathbb{Z}\quad\text{and}\quad v\in V_{(n)}. \tag{2.11}$$

This completes the definition of the notion of conformal vertex algebra. We will denote such a conformal vertex algebra by $(V,Y,\mathbf{1},\omega)$.

Definition 2.1.2 Given a conformal vertex algebra $(V,Y,\mathbf{1},\omega)$, a *module* for V is a $\mathbb{C}$-graded vector space

$$W=\coprod_{n\in\mathbb{C}}W_{(n)} \tag{2.12}$$

(graded by *weights*) equipped with a linear map $V \otimes W \to W[[x, x^{-1}]]$, or equivalently,

$$\begin{array}{rcl} V & \to & (\mathrm{End}\, W)[[x, x^{-1}]] \\ v & \mapsto & Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \quad (\text{where } \ v_n \in \mathrm{End}\, W) \end{array} \tag{2.13}$$

(note that the sum is over $\mathbb{Z}$, not $\mathbb{C}$), $Y(v, x)$ denoting the *vertex operator on* W *associated with* v, such that all the defining properties of a conformal vertex algebra that make sense hold. That is, the following conditions are satisfied: the lower truncation condition: for $v \in V$ and $w \in W$,

$$v_n w = 0 \quad \text{for } \ n \ \text{ sufficiently large} \tag{2.14}$$

(or equivalently, $Y(v, x)w \in W((x))$); the vacuum property:

$$Y(\mathbf{1}, x) = 1_W; \tag{2.15}$$

the Jacobi identity for vertex operators on W: for $u, v \in V$,

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y(v, x_2)Y(u, x_1) \\ = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y(Y(u, x_0)v, x_2) \end{aligned} \tag{2.16}$$

(note that on the right-hand side, $Y(u, x_0)$ is the operator on V associated with u); the Virasoro algebra relations on W with scalar c equal to the central charge of V:

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{n+m,0}c \tag{2.17}$$

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \quad \text{for } n \in \mathbb{Z}, \ \text{ i.e., } \ Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}; \tag{2.18}$$

$$\frac{d}{dx}Y(v, x) = Y(L(-1)v, x) \tag{2.19}$$

(the $L(-1)$-derivative property); and

$$(L(0) - n)w = 0 \quad \text{for } \ n \in \mathbb{C} \ \text{ and } \ w \in W_{(n)}, \tag{2.20}$$

where $n = \mathrm{wt}\ w$.

This completes the definition of the notion of module for a conformal vertex algebra.

Definition 2.1.3 Let A be an abelian group. A conformal vertex algebra

$$V = \coprod_{n\in\mathbb{Z}} V_{(n)}$$

is said to be *strongly graded with respect to* A (or *strongly* A*-graded*, or just *strongly graded* if the abelian group A is understood) if it is equipped with a second gradation, by A,

$$V = \coprod_{\alpha\in A} V^{(\alpha)},$$

such that the following conditions are satisfied: the two gradations are compatible, that is,

$$V^{(\alpha)} = \coprod_{n\in\mathbb{Z}} V_{(n)}^{(\alpha)} \ \ (\text{where } V_{(n)}^{(\alpha)} = V_{(n)} \cap V^{(\alpha)}) \ \ \text{for any} \ \ \alpha \in A;$$

for any $\alpha, \beta \in A$ and $n \in \mathbb{Z}$,

$$V_{(n)}^{(\alpha)} = 0 \quad \text{for} \ \ n \ \ \text{sufficiently negative}; \tag{2.21}$$

$$\dim V_{(n)}^{(\alpha)} < \infty; \tag{2.22}$$

$$\mathbf{1} \in V_{(0)}^{(0)}; \tag{2.23}$$

$$\omega \in V_{(2)}^{(0)}; \tag{2.24}$$

$$v_l V^{(\beta)} \subset V^{(\alpha+\beta)} \quad \text{for any} \ \ v \in V^{(\alpha)}, \ l \in \mathbb{Z}. \tag{2.25}$$

This completes the definition of the notion of strongly A-graded conformal vertex algebra.

For modules for a strongly graded algebra we will also have a second grading by an abelian group, and it is natural to allow this group to be larger than the second grading group A for the algebra. (Note that this already occurs for the *first* grading group, which is $\mathbb{Z}$ for algebras and $\mathbb{C}$ for modules.)

Definition 2.1.4 Let A be an abelian group and V a strongly A-graded conformal vertex algebra. Let $\tilde{A}$ be an abelian group containing A as a subgroup. A V-module

$$W = \coprod_{n\in\mathbb{C}} W_{(n)}$$

is said to be *strongly graded with respect to* $\tilde{A}$ (or *strongly* $\tilde{A}$*-graded*, or just *strongly graded* if the abelian group $\tilde{A}$ is understood) if it is equipped with a second gradation, by $\tilde{A}$,

$$W = \coprod_{\beta\in\tilde{A}} W^{(\beta)}, \tag{2.26}$$

such that the following conditions are satisfied: the two gradations are compatible, that is, for any $\beta \in \tilde{A}$,

$$W^{(\beta)} = \coprod_{n\in\mathbb{C}} W^{(\beta)}_{(n)} \quad (\text{where } \ W^{(\beta)}_{(n)} = W_{(n)} \cap W^{(\beta)})$$

for any $\alpha \in A$, $\beta \in \tilde{A}$ and $n \in \mathbb{C}$,

$$W^{(\beta)}_{(n+k)} = 0 \text{ for } \ k \in \mathbb{Z} \ \text{ sufficiently negative;} \tag{2.27}$$

$$\dim W^{(\beta)}_{(n)} < \infty \tag{2.28}$$

$$v_l W^{(\beta)} \subset W^{(\alpha+\beta)} \quad \text{for any } \ v \in V^{(\alpha)},\ l \in \mathbb{Z}. \tag{2.29}$$

This completes the definition of the notion of strongly $\tilde{A}$-graded module for a strongly A-graded conformal vertex algebra.

Remark 2.1.5 It is always possible that there are different gradings on W by $\tilde{A}$, such as by shifting by an element in $\tilde{A}$. However, in this paper, we shall fix one particular $\tilde{A}$-grading on W.

In order to study strongly graded V-modules for tensor product algebras, we shall need the following generalization:

Definition 2.1.6 In the setting of Definition 2.1.4 (the definition of "strongly graded module"), a V-module (not necessarily strongly graded, of course) is *doubly graded with respect to* $\tilde{A}$ if it satisfies all the conditions in Definition 2.1.4 except perhaps for (2.27) and (2.28).

Example 2.1.7 Note that the notion of conformal vertex algebra strongly graded with respect to the trivial group is exactly the notion of vertex operator algebra. Let V be a vertex operator algebra, viewed (equivalently) as a conformal vertex algebra strongly graded with respect to the trivial group. Then the V-modules that are strongly graded with respect to the trivial group (in the sense of Definition 2.1.4) are exactly the ($\mathbb{C}$-graded) modules for V as a vertex operator algebra, with the grading restrictions as follows: For $n \in \mathbb{C}$,

$$W_{(n+k)} = 0 \quad \text{for} \;\; k \in \mathbb{Z} \;\; \text{sufficiently negative} \tag{2.30}$$

and

$$\dim W_{(n)} < \infty. \tag{2.31}$$

Example 2.1.8 An important source of examples of strongly graded conformal vertex algebras and modules comes from the vertex algebras and modules associated with even lattices. We recall the following construction from [FLM]. Let L be an even lattice, i.e., a finite-rank free abelian group equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, not necessarily positive definite, such that $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ for all $\alpha \in L$. Let $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$. Then $\mathfrak{h}$ is a vector space with a nonsingular bilinear form $\langle \cdot, \cdot \rangle$, extended from L. We form a Heisenberg algebra

$$\widehat{\mathfrak{h}}_{\mathbb{Z}} = \coprod_{n \in \mathbb{Z},\; n \neq 0} \mathfrak{h} \otimes t^n \oplus \mathbb{C}c.$$

Let $(\widehat{L}, {}^-)$ be a central extension of L by a finite cyclic group $\langle \kappa \mid \kappa^s = 1 \rangle$. Fix a primitive sth root of unity, say ω, and define the faithful character

$$\chi : \langle \kappa \rangle \to \mathbb{C}^*$$

by the condition

$$\chi(\kappa) = \omega.$$

Denote by $\mathbb{C}_\chi$ the one-dimensional space $\mathbb{C}$ viewed as a $\langle \kappa \rangle$-module on which $\langle \kappa \rangle$ acts according to χ:

$$\kappa \cdot 1 = \omega,$$

and denote by $\mathbb{C}\{L\}$ the induced $\widehat{L}$-module

$$\mathbb{C}\{L\} = \mathrm{Ind}_{\langle\kappa\rangle}^{\widehat{L}} \mathbb{C}_\chi = \mathbb{C}[\widehat{L}] \otimes_{\mathbb{C}[\langle\kappa\rangle]} \mathbb{C}_\chi.$$

Then

$$V_L = S(\widehat{\mathfrak{h}}_{\mathbb{Z}}^-) \otimes \mathbb{C}\{L\}$$

has a natural structure of conformal vertex algebra; see [B1] and Chapter 8of [FLM]. For $\alpha \in L$, choose an $a \in \widehat{L}$ such that $\bar{a} = \alpha$. Define

$$\iota(a) = a \otimes 1 \in \mathbb{C}\{L\}$$

and

$$V_L^{(\alpha)} = \mathrm{span}\ \{h_1(-n_1)\cdots h_k(-n_k) \otimes \iota(a)\},$$

where $h_1, \ldots, h_k \in \mathfrak{h}$, $n_1, \ldots, n_k > 0$, and where $h(n)$ is the operator associated with $h \otimes t^n$ via the $\hat{\mathfrak{h}}_{\mathbb{Z}}$-module structure of V_L. Then V_L is equipped with a natural second grading given by L itself. Also for $n \in \mathbb{Z}$, we have

$$(V_L)_{(n)}^{(\alpha)} = \mathrm{span}\ \{h_1(-n_1)\cdots h_k(-n_k) \otimes \iota(a) |\ \bar{a} = \alpha, \sum_{i=1}^{k} n_i + \frac{1}{2}\langle\alpha, \alpha\rangle = n\},$$

making V_L a strongly L-graded conformal vertex algebra in the sense of Definition 2.1.3. When the form $\langle\cdot,\cdot\rangle$ on L is also positive definite, then V_L is a vertex operator algebra, that is, as in Example 2.1.7, V_L is a strongly A-graded conformal vertex algebra for A the trivial group. In general, a conformal vertex algebra may be strongly graded for several choices of A.

Any sublattice M of the "dual lattice" L° of L containing L gives rise to a strongly M-graded module for the strongly L-graded conformal vertex algebra (see Chapter 8 of [FLM]; cf. [LL]). In fact, any irreducible V_L-module is equivalent to a V_L-module of the form $V_{L+\beta} \subset V_{L^\circ}$ for some $\beta \in L^\circ$ and any V_L-module W is equivalent to a direct sum of irreducible V_L-modules, i.e.,

$$W = \coprod_{\gamma_i \in L^\circ,\ i=1,\ldots,n} V_{\gamma_i + L},$$

where γ_i's are arbitrary elements of L°, and $n \in \mathbb{N}$ (see [D1], [DLM]; cf. [LL]). In general, a module for a strongly graded vertex algebra may be strongly graded for several choices of $\tilde{A}$.

Notation 2.1.9 In the remainder of this section, without further assumption, we will let A be an abelian group and V be a strongly A-graded conformal vertex algebra. Also, we will let $\tilde{A}$ be an abelian group containing A and W be a doubly graded V-module with respect to $\tilde{A}$. When we need W to be strongly graded, we will say it explicitly.

Definition 2.1.10 The subspaces $V_{(n)}^{(\alpha)}$ for $n \in \mathbb{Z}$, $\alpha \in A$ in Definition 2.1.6 are called the *doubly homogeneous subspaces* of V. The elements in $V_{(n)}^{(\alpha)}$ are called *doubly homogeneous* elements. Similar definitions can be used for $W_{(n)}^{(\beta)}$ in the module W.

Notation 2.1.11 Let v be a doubly homogeneous element of V. Let $\mathrm{wt}\ v_n$, $n \in \mathbb{Z}$, refer to the weight of v_n as an operator acting on W, and let A-$\mathrm{wt}\ v_n$ refer to the A-weight of v_n on W.

Lemma 2.1.12 *Let $v \in V_{(n)}^{(\alpha)}$, for $n \in \mathbb{Z}$, $\alpha \in A$. Then for $m \in \mathbb{Z}$,* $\mathrm{wt}\ v_m = n - m - 1$ *and* A-$\mathrm{wt}\ v_m = \alpha$.

Proof. The first equation is standard from the theory of graded conformal vertex algebras and the second follows easily from the definitions. □

Definition 2.1.13 The algebra $A(V;W)$ *associated with* V *and* W is defined to be the algebra of operators on W induced by V, i.e., the algebra generated by the set

$$\{v_n \mid v \in V,\ n \in \mathbb{Z}\}.$$

For a subspace V' of V, we use $A(V';W)$ to denote the subalgebra of $A(V;W)$ generated by the set

$$\{v_n \mid v \in V',\ n \in \mathbb{Z}\}.$$

For a subspace $W^{'}$ of W, we use $A(V;W^{'})$ to denote the subalgebra of $A(V;W)$ preserving $W^{'}$. Similarly for $V^{'}$ and $W^{'}$, we use $A(V^{'};W^{'})$ to denote the subalgebra of $A(V;W)$ generated by the operators on $W^{'}$ induced by $V^{'}$.

Remark 2.1.14 When $W^{'}$ is a submodule of W, there are two possible definitions for $A(V;W^{'})$ in Definition 2.1.13. One is as an algebra associated with V and $W^{'}$, the other is as a subalgebra of $A(V;W)$. But it does not matter because they are both algebras of operators on $W^{'}$ generated by the set

$$\{v_n \mid v \in V,\ n \in \mathbb{Z}\}.$$

Similar comments hold for $V^{'}$ a subalgebra of V.

The following lemma follows easily from Lemma 2.1.12:

Lemma 2.1.15 *The algebra $A(V;W)$ is doubly graded by $\mathbb{Z}$ and A. Moreover for $n \in \mathbb{Z}$,*

$$A(V;W)_{(n)} = \text{span } \{(v_1)_{j_1} \cdots (v_m)_{j_m} \mid \sum_{i=1}^{m} \text{wt } (v_i)_{j_i} = n,$$
$$\text{where } m \in \mathbb{N},\ v_i \in V,\ j_i \in \mathbb{Z},\ \text{for } i = 1, \ldots, m\}$$

and for $\alpha \in A$,

$$A(V;W)^{(\alpha)} = \text{span } \{(v_1)_{j_1} \cdots (v_m)_{j_m} \mid \sum_{i=1}^{m} A\text{-wt } (v_i)_{j_i} = \alpha,$$
$$\text{where } m \in \mathbb{N},\ v_i \in V,\ j_i \in \mathbb{Z},\ \text{for } i = 1, \ldots, m\}.$$

Proposition 2.1.16 *Let W be an irreducible doubly graded V-module with respect to $\tilde{A}$. Then we have the following results:*

(a) Each weight subspace $W_{(h)}$ ($h \in \mathbb{C}$) is irreducible under the algebra $A(V;W_{(h)})$.

(b) Each $\tilde{A}$-homogeneous subspace $W^{(\beta)}$ ($\beta \in \tilde{A}$) is irreducible under the algebra $A(V;W^{(\beta)})$.

(c) Each doubly homogeneous subspace $W_{(h)}^{(\beta)}$ ($h \in \mathbb{C}$, $\beta \in \tilde{A}$) is irreducible under the algebra $A(V;W_{(h)}^{(\beta)})$.

Proof. We only prove statement (a), the proofs of statements (b) and (c) being similar. If $W_{(h)}$ is not irreducible, we can find a nontrivial proper submodule U of $W_{(h)}$ under the algebra $A(V; W_{(h)})$. This submodule cannot generate all W under the action by the algebra $A(V; W)$, since by Lemma 2.1.15,

$$A(V; W)U = \coprod_{n\in\mathbb{Z}} A(V; W)_{(n)}U \subset U \oplus \coprod_{m\in\mathbb{Z}, m\neq h} W_{(m)}.$$

This contradicts the irreducibility of W. □

Remark 2.1.17 A V-module W decomposes into submodules corresponding to the congruence classes of its weights modulo $\mathbb{Z}$: For $\mu \in \mathbb{C}/\mathbb{Z}$, let

$$W_{(\mu)} = \coprod_{\bar{n}=\mu} W_{(n)}, \tag{2.32}$$

where $\bar{n}$ denotes the equivalence class of $n \in \mathbb{C}$ in $\mathbb{C}/\mathbb{Z}$. Then

$$W = \coprod_{\mu\in\mathbb{C}/\mathbb{Z}} W_{(\mu)} \tag{2.33}$$

and each $W_{(\mu)}$ is a V-submodule of W. Thus if a module W is indecomposable (in particular, if it is irreducible), then all complex numbers n for which $W_{(n)} \neq 0$ are congruent modulo $\mathbb{Z}$ to each other.

Definition 2.1.18 Let W_1 and W_2 be doubly graded V-modules with respect to $\tilde{A}$. A *module homomorphism* from W_1 to W_2 is a linear map ψ such that

$$\psi(Y(v, x)w) = Y(v, x)\psi(w) \text{ for } v \in V,\ w \in W_1,$$

and such that ψ preserves the grading by $\tilde{A}$. An *isomorphism* is a bijective homomorphism. An *endomorphism* is a homomorphism from W to itself, we denote the endomorphism ring by $\mathrm{End}_V^{\tilde{A}}(W)$.

Remark 2.1.19 Suppose V, W_1, W_2, ψ are as in Definition 2.1.18. Then ψ is compatible with both gradings:

$$\psi((W_1)^{(\beta)}_{(h)}) \subset (W_2)^{(\beta)}_{(h)},\ h \in \mathbb{C},$$

because ψ commutes with $L(0)$ (see Section 4.5 of [LL]), and because ψ preserves the grading by $\tilde{A}$.

Remark 2.1.20 The endomorphism ring $\mathrm{End}_V^{\tilde{A}}(W)$ is a subring of the commuting ring

$$\begin{aligned} & \mathrm{End}_V(W) \\ \triangleq\ & \{\text{linear maps } \psi : W \to W |\ \psi(Y(v,x)w) = Y(v,x)\psi(w), \text{for } v \in V, w \in W\}. \end{aligned}$$

Proposition 2.1.21 *Suppose W is an irreducible strongly $\tilde{A}$-graded V-module. Then* $\mathrm{End}_V^{\tilde{A}}(W) = \mathbb{C}$.

Proof. For any $\lambda \in \mathbb{C}$, $\psi \in \mathrm{End}_V^{\tilde{A}}(W)$, let W_λ^ψ be the λ-eigenspace of ψ. Then W_λ^ψ is a V-submodule of W. Because W is irreducible, $W_\lambda^\psi = 0$ or W. We still need to show $W_\lambda^\psi \neq 0$, for some $\lambda \in \mathbb{C}$.

Choose $h \in \mathbb{C}$, $\beta \in \tilde{A}$ such that $W_{(h)}^{(\beta)} \neq 0$. Then by Remark 2.1.19, ψ preserves $W_{(h)}^{(\beta)}$. Since dim $W_{(h)}^{(\beta)} < \infty$ and we are working over $\mathbb{C}$, ψ has an eigenvector in $W_{(h)}^{(\beta)}$. Therefore $W_\lambda^\psi \neq 0$ for some $\lambda \in \mathbb{C}$. ∎

Proposition 2.1.22 *Suppose A is a countable abelian group. Then* $\mathrm{End}_V(W) = \mathbb{C}$.

Proof. From Definition 2.1.3, $V_{(n)} = \coprod_{\alpha \in A} V_{(n)}^{(\alpha)}$, where each doubly homogeneous subspace $V_{(n)}^{(\alpha)}$ has finite dimension. Since A is a countable group, there are countably many such doubly homogeneous subspaces $V_{(n)}^{(\alpha)}$, and hence V has countable dimension. Since W is irreducible, from Proposition 4.5.6 of [LL], we know

$$W = \mathrm{span}\{v_n w \mid v \in V, n \in \mathbb{Z}\},$$

for any nonzero element w in W. Since V has countable dimension, so does W. Then the result follows from Dixmier's Lemma, which says that if S is an irreducible set of operators on a vector space W of countable dimension over $\mathbb{C}$, then the commuting ring of S on W consists of the scalars (cf. Lemma 2.2 in [L], and [W], p.11), where we take S to be $A(V;W)$. ∎

2.2 Tensor products of strongly graded vertex algebras and their strongly graded modules

In this section, we are going to introduce the notion of tensor product of finitely many strongly graded conformal vertex algebras and their modules (see [FHL], [LL], cf. [M1]).

Let $A_1, \dots, A_p$ be abelian groups, and let $V_1, \dots, V_p$ be strongly $A_1, \dots, A_p$-graded conformal vertex algebras with conformal vectors $\omega^1, \dots, \omega^p$, respectively.

Let

$$A = A_1 \oplus \cdots \oplus A_p.$$

Then the vector space

$$V = V_1 \otimes \cdots \otimes V_p$$

becomes a strongly A-graded conformal vertex algebra, which we shall call the *tensor product strongly A-graded conformal vertex algebra*, with the following structure:

$$Y(v^{(1)} \otimes \cdots \otimes v^{(p)}, x) = Y(v^{(1)}, x) \otimes \cdots \otimes Y(v^{(p)}, x)$$

for $v^{(i)} \in V_i$ and the vacuum vector is

$$\mathbf{1} = \mathbf{1} \otimes \cdots \otimes \mathbf{1}.$$

(Here we use the notation $\mathbf{1}$ for the vacuum vectors of V and each V_i.) The conformal vector is

$$\omega = \omega^1 \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \cdots + \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \omega^p.$$

Then

$$L(n) = L_1(n) \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes L_p(n)$$

or $n \in \mathbb{Z}$. (Here we use the notation $L_i(n)$ for the operators on V_i associated with ω^i, $i = 1, \ldots, p$.) The A-grading of V is given by

$$V = \coprod_{\alpha \in A} V^{(\alpha)},$$

with

$$V^{(\alpha)} = V_1^{(\alpha_1)} \otimes \cdots \otimes V_p^{(\alpha_p)},$$

where $\alpha_i \in A_i$, $i = 1, \ldots, p$, are such that $\alpha_1 + \cdots + \alpha_p = \alpha$. The $\mathbb{Z}$-grading of V is given by

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)},$$

where

$$V_{(n)} = \coprod_{n_1 + \cdots + n_p = n} (V_1)_{(n_1)} \otimes \cdots \otimes (V_p)_{(n_p)}.$$

(It follows that the $\mathbb{Z}$-grading is given by $L(0)$ defined above.)

Proposition 2.2.1 *The tensor product of finitely many strongly graded conformal vertex algebras is a strongly graded conformal vertex algebra whose central charge is the sum of the central charges of the tensor factors.*

Proof. The grading restrictions (2.21) and (2.22) clearly hold. The Jacobi identity follows from the weak commutativity and weak associativity properties, as in Section 3.4 of [LL]. ∎

Notation 2.2.2 For each $i = 1, \ldots, p$, we identify V_i with the subspace $1 \otimes \cdots \otimes 1 \otimes V_i \otimes 1 \otimes \cdots \otimes 1$ of V. The strongly graded conformal vertex algebra V_i is a vertex subalgebra of V. However, it is not a conformal vertex subalgebra of V because the conformal vector of V and V_i do not match.

Remark 2.2.3 From the definition of tensor product strongly graded conformal vertex algebra, we see that

$$Y((1 \otimes \cdots \otimes 1 \otimes v^{(i)} \otimes 1 \otimes \cdots \otimes 1, x) = 1_{V_1} \otimes \cdots \otimes 1_{V_{i-1}} \otimes Y(v^{(i)}, x) \otimes 1_{V_{i+1}} \otimes \cdots \otimes 1_{V_p},$$

for $v^{(i)} \in V_i$. In particular, we have

$$[Y(V_i, x_1), Y(V_j, x_2)] = 0,$$

for $i, j = 1, \ldots, p$ and $i \neq j$.

Lemma 2.2.4 *For all $n \in \mathbb{Z}$, $(v^{(1)} \otimes \cdots \otimes v^{(p)})_n$ can be expressed as a linear combination, finite on any given vector, of operators of the form $(v^{(1)} \otimes 1 \otimes \cdots \otimes 1)_{i_1} \cdots (1 \otimes \cdots \otimes 1 \otimes v^{(p)})_{i_p}$.*

Proof. We prove the result as in [FHL] by induction. When $p = 2$, taking Res_{x_1} and the constant term in x_0 of the Jacobi identity, we find that

$$\begin{aligned}
Y(v^{(1)} \otimes v^{(2)}, x_2) &= \mathrm{Res}_{x_0} x_0^{-1} Y(Y(v^{(1)} \otimes 1, x_0)(1 \otimes v^{(2)}), x_2) \\
&= \mathrm{Res}_{x_1}(x_1 - x_2)^{-1} Y(v^{(1)} \otimes 1, x_1) Y(1 \otimes v^{(2)}, x_2) \\
&\quad - \mathrm{Res}_{x_1}(-x_2 + x_1)^{-1} Y(1 \otimes v^{(2)}, x_2) Y(v^{(1)} \otimes 1, x_1),
\end{aligned}$$

so that for all $n \in \mathbb{Z}$, $(v^{(1)} \otimes v^{(2)})_n$ can be expressed as a linear combination, finite on any given vector, of operators of the form $(v^{(1)} \otimes 1)_{n_1}(1 \otimes v^{(2)})_{n_2}$.(Note that we don't need operators of the form $(1 \otimes v^{(2)})_{n_2}(v^{(1)} \otimes 1)_{n_1}$ because of the Remark 2.2.3.)

For general p, taking Res_{x_1} and the constant term in x_0 of the Jacobi identity, we have

$$\begin{aligned}
&Y(v^{(1)} \otimes \cdots \otimes v^{(p)}, x_2) \\
&= \mathrm{Res}_{x_0} x_0^{-1} Y(Y(v^{(1)} \otimes \cdots \otimes v^{(p-1)} \otimes 1, x_0)(1 \otimes \cdots \otimes 1 \otimes v^{(p)}), x_2) \\
&= \mathrm{Res}_{x_1}(x_1 - x_2)^{-1} Y(v^{(1)} \otimes \cdots \otimes v^{(p-1)} \otimes 1, x_1) Y(1 \otimes \cdots \otimes 1 \otimes v^{(p)}, x_2) \\
&\quad - \mathrm{Res}_{x_1}(-x_2 + x_1)^{-1} Y(1 \otimes \cdots \otimes 1 \otimes v^{(p)}, x_2) Y(v^{(1)} \otimes \cdots \otimes v^{(p-1)} \otimes 1, x_1).
\end{aligned}$$

It follows that $(v^{(1)} \otimes \cdots \otimes v^{(p)})_n$ is a linear combination of the operators $(v^{(1)} \otimes \cdots \otimes v^{(p-1)} \otimes 1)_{n_1} \cdot (1 \otimes \cdots \otimes 1 \otimes v^{(p)})_{n_2}$. Thus the lemma holds by the induction hypothesis. ∎

Now we define the notion of tensor product module for tensor product strongly $A = A_1 \oplus \cdots \oplus A_p$-graded conformal vertex algebra $V = V_1 \otimes \cdots \otimes V_p$ with the notions above. Let $\tilde{A}_1, \dots, \tilde{A}_p$ be abelian groups containing $A_1, \dots, A_p$ as subgroups, respectively, and let $W_1, \dots, W_p$ be strongly $\tilde{A}_1, \dots, \tilde{A}_p$-graded modules for $V_1, \dots, V_p$, respectively.

Let

$$\tilde{A} = \tilde{A}_1 \oplus \cdots \oplus \tilde{A}_p,$$

Then we can construct the *tensor product strongly $\tilde{A}$-graded module*

$$W = W_1 \otimes \cdots \otimes W_p$$

for the tensor product strongly A-graded algebra V by means of the definition

$$Y(v^{(1)} \otimes \cdots \otimes v^{(p)}, x) = Y(v^{(1)}, x) \otimes \cdots \otimes Y(v^{(p)}, x) \quad \text{for } v^{(i)} \in V_i,\ i = 1, \dots, p,$$

$$L(n) = L_1(n) \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes L_p(n) \quad \text{for } n \in \mathbb{Z}.$$

(Here we use the notation $L_i(n)$ for the operators associated with ω^i on W_i, $i = 1, \dots, p$.) The $\tilde{A}$-grading of W is defined as

$$W = \coprod_{\beta \in \tilde{A}} W^{(\beta)},$$

with

$$W^{(\beta)} = W_1^{(\beta_1)} \otimes \cdots \otimes W_p^{(\beta_p)},$$

where $\beta_i \in \tilde{A}_i$, $i = 1, \dots, p$, are such that $\beta_1 + \cdots + \beta_p = \beta$. The $\mathbb{C}$-grading of W is defined as

$$W = \coprod_{n \in \mathbb{C}} W_{(n)},$$

where

$$W_{(n)} = \sum_{n_1 + \cdots + n_p = n} (W_1)_{(n_1)} \otimes \cdots \otimes (W_p)_{(n_p)}.$$

It follows that the $\mathbb{C}$-grading is given by the operator $L(0)$ on W defined above. It is clear that the algebra V is also a module for itself.

Proposition 2.2.5 *The structure W constructed above is a strongly $\tilde{A}$-graded module for the tensor product strongly A-graded conformal vertex algebra V.*

Assumption 2.2.6 In the remainder of this section, we always assume that A, and that each A_i $(i = 1, \cdots, p)$ is a countable abelian group.

Using Proposition 2.1.22, we now prove:

Theorem 2.2.7 *Let $W = W_1 \otimes \cdots \otimes W_p$ be a strongly $\tilde{A} = \tilde{A}_1 \oplus \cdots \oplus \tilde{A}_p$-graded V-module, with the notations as above. Then W is irreducible if and only if each W_i is irreducible.*

Proof. The "only if" part is trivial. For the "if" part, for simplicity of notation, we take $p = 2$ without losing any essential content. Take a nonzero submodule $W \subset W_1 \otimes W_2$, let $w_1^{(1)}, \ldots, w_n^{(1)} \in W_1$ and $w_1^{(2)}, \ldots, w_n^{(2)} \in W_2$ be linearly independent such that $\Sigma_{j=1}^n a_j (w_j^{(1)} \otimes w_j^{(2)}) \in W$, where each $a_j \neq 0$. Take any $w^{(1)} \in W_1$, $w^{(2)} \in W_2$. By Proposition 2.1.22, the commuting ring consists of the scalars for W_1 and W_2. Thus by the density theorem (see for example Section 5.8 of [J]), there are $b_1 \in A(V_1; W_1 \otimes W_2)$, $b_2 \in A(V_2; W_1 \otimes W_2)$ such that

$$b_1 \cdot w_1^{(1)} = w^{(1)},\ b_1 \cdot w_i^{(1)} = 0,\ \text{for } i = 2, \ldots, n.$$

$$b_2 \cdot w_1^{(2)} = w^{(2)},\ b_2 \cdot w_i^{(2)} = 0,\ \text{for } i = 2, \ldots, n.$$

Then

$$(b_1 b_2) \cdot \Sigma_{j=1}^n\ a_j (w_j^{(1)} \otimes w_j^{(2)}) = a_1 (w^{(1)} \otimes w^{(2)})\ \in W.$$

Hence $w^{(1)} \otimes w^{(2)} \in W$, and so $W = W_1 \otimes W_2$. ∎

2.3 Strongly $(\mathfrak{h}, A)$-graded vertex algebras and their strongly $(\mathfrak{h}, \tilde{A})$-graded modules

For some strongly A-graded vertex algebras V, there is a vector space $\mathfrak{h}$ consisting of mutually commuting operators induced by V such that the A-grading of V is given by $\mathfrak{h}$ in the following way: for $\alpha \in A$, $V^{(\alpha)}$ is the weight space of $\mathfrak{h}$ of weight α. Here is an example:

Example 2.3.1 Consider the strongly L-graded conformal vertex algebra V_L in Example 2.1.8. For $h \in \mathfrak{h}$, there is an operator $h(0)$ on V_L such that

$$h(0) \cdot V_L^{(\alpha)} = \langle h, \alpha \rangle V_L^{(\alpha)}.$$

We identify $\mathfrak{h}$ with the set of operators

$$\{h(0) \;=\; (h(-1) \cdot \mathbf{1})_0 \mid h \in \mathfrak{h}\}$$

(see Chapter 8 of [FLM]). Since the symmetric bilinear form $\langle \cdot, \cdot \rangle$ is nondegenerate, $V_L^{(\alpha)}$ is characterized as the weight space of $\mathfrak{h}$ of weight α.

Consider the tensor algebra $T(V[t, t^{-1}])$ over the vector space $V[t, t^{-1}]$. Then any V-module W, in particular, V itself, can be regarded as a $T(V[t, t^{-1}])$-module uniquely determined by the condition that for $v \in V$, $n \in \mathbb{Z}$, $v \otimes t^n$ acts on W as v_n. In the following definitions, we consider a particular subspace of $T(V[t, t^{-1}])$ acting on V and W.

Definition 2.3.2 A strongly A-graded vertex algebra equipped with a vector subspace

$$\mathfrak{h} \subset T(V[t, t^{-1}])$$

is called *strongly ($\mathfrak{h}$, A)-graded* if there is a nondegenerate pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathfrak{h} \times A &\longrightarrow \mathbb{C} \\ (h, \alpha) &\longmapsto \langle h, \alpha \rangle \end{aligned}$$

linear in the first variable and additive in the second variable, such that $\mathfrak{h}$ acts commutatively on V and

$$V^{(\alpha)} = \{v \in V \mid h \cdot v = \langle h, \alpha \rangle v, \text{ for all } h \in \mathfrak{h}\}.$$

By Definition 2.3.2, the strongly graded conformal vertex algebra V_L in Example 2.3.1 is strongly $(\mathfrak{h}, L)$-graded, where $\mathfrak{h}$ is the set of operators $\{(h(-1)\cdot\mathbf{1})_0 \mid h \in L \otimes_{\mathbb{Z}} \mathbb{C}\}$.

For a strongly $(\mathfrak{h},\ A)$-graded vertex algebra V, a natural module category is the category of strongly $\tilde{A}$-graded V-modules W with an action of $\mathfrak{h}$, such that the $\tilde{A}$-grading on W is given by weight spaces of $\mathfrak{h}$. Here is an example:

Example 2.3.3 As in Example 2.1.8, any sublattice M of L° containing L gives rise to a strongly M-graded V_L-module V_M. Take $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$ and identify $\mathfrak{h}$ as the set of operators $\{(h(-1)\cdot\mathbf{1})_0 \mid h \in \mathfrak{h}\}$ as in Example 2.3.1. Then for $\beta \in M$,

$$V_M^{(\beta)} = \{w \in V_M \mid h \cdot w = \langle h, \beta\rangle w, \text{ for all } h \in \mathfrak{h}\}.$$

so that we have examples of the following:

Definition 2.3.4 A strongly $\tilde{A}$-graded module for a strongly $(\mathfrak{h},\ A)$-graded vertex algebra is said to be *strongly* $(\mathfrak{h},\ \tilde{A})$*-graded* if there is a nondegenerate pairing

$$\begin{aligned}\langle\cdot,\cdot\rangle : \mathfrak{h} \times \tilde{A} &\longrightarrow \mathbb{C}\\ (h, \beta) &\longmapsto \langle h, \beta\rangle\end{aligned}$$

linear in the first variable and additive in the second variable, such that the operators in $\mathfrak{h}$ act commutatively on W and

$$W^{(\beta)} = \{w \in W \mid h \cdot w = \langle h, \beta\rangle w, \text{ for all } h \in \mathfrak{h}\}.$$

Remark 2.3.5 Submodules and quotient modules of strongly $(\mathfrak{h}, \tilde{A})$-graded conformal modules are also strongly $(\mathfrak{h}, \tilde{A})$-graded modules. Irreducible strongly $(\mathfrak{h}, \tilde{A})$-graded modules are strongly $(\mathfrak{h}, \tilde{A})$-graded modules without nontrivial submodules. Strongly $(\mathfrak{h}, \tilde{A})$-graded module homomorphisms are strongly $\tilde{A}$-graded module homomorphisms which commute with the actions of $\mathfrak{h}$.

The following propositions are natural analogues of Proposition 2.2.1 and Proposition 2.2.5.

Proposition 2.3.6 *Let $V_1, \ldots, V_p$ be strongly $(\mathfrak{h}_1, A_1), \ldots, (\mathfrak{h}_p, A_p)$-graded conformal vertex algebras, respectively. Let $A = A_1 \oplus \cdots \oplus A_p$, $\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_p$, and let $\langle \cdot, \cdot \rangle_i$ denote the pairing between $\mathfrak{h}_i$ and A_i, for $i = 1, \ldots, p$. Then the tensor product algebra $V = V_1 \otimes \cdots \otimes V_p$ becomes a strongly $(\mathfrak{h}, A)$-graded conformal vertex algebra, where the nondegenerate pairing is given by:*

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathfrak{h} \times A &\longrightarrow \mathbb{C} \\ (h, \alpha) &\longmapsto \sum_{i=1}^{p} \langle h_i, \alpha_i \rangle_i, \end{aligned}$$

where $h = h_1 + \cdots + h_p$, $\alpha = \alpha_1 + \cdots + \alpha_p$, for $h_i \in \mathfrak{h}_i$, $\alpha_i \in A_i$, $i = 1, \ldots, p$, and

$$V^{(\alpha)} = V_1^{(\alpha_1)} \otimes \cdots \otimes V_p^{(\alpha_p)} = \{v \in V_1 \otimes \cdots \otimes V_p \mid h \cdot v = \langle h, \alpha \rangle v, \text{ for all } h \in \mathfrak{h}\}.$$

Proof. It is easy to see that the pairing defined above is nondegenerate, and $V^{(\alpha)}$ is characterized uniquely as the eigenspace of $\mathfrak{h}$. ∎

Proposition 2.3.7 *Let $W_1, \ldots, W_p$ be strongly $(\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p)$-graded conformal modules for strongly $(\mathfrak{h}_1, A_1), \ldots, (\mathfrak{h}_p, A_p)$-graded conformal vertex algebras $V_1, \ldots, V_p$, respectively. Let $\tilde{A} = \tilde{A}_1 \oplus \cdots \oplus \tilde{A}_p$, $\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_p$, and let $\langle \cdot, \cdot \rangle_i$ denote the pairing between $\mathfrak{h}_i$ and $\tilde{A}_i$, for $i = 1, \ldots, p$. Then the tensor product module $W = W_1 \otimes \cdots \otimes W_p$ becomes a strongly $(\mathfrak{h}, \tilde{A})$-graded module for the strongly graded vertex algebra V, where the nondegenerate pairing is given by:*

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathfrak{h} \times \tilde{A} &\longrightarrow \mathbb{C} \\ (h, \beta) &\longmapsto \sum_{i=1}^{p} \langle h_i, \beta_i \rangle_i, \end{aligned}$$

where $h = h_1 + \cdots + h_p$, $\beta = \beta_1 + \cdots + \beta_p$, for $h_i \in \mathfrak{h}_i$, $\beta_i \in \tilde{A}_i$, $i = 1, \ldots, p$, and

$$W^{(\beta)} = W_1^{(\beta_1)} \otimes \cdots \otimes W_p^{(\beta_p)} = \{w \in W_1 \otimes \cdots \otimes W_p \mid h \cdot w = \langle h, \beta \rangle w, \text{ for all } h \in \mathfrak{h}\}.$$

The following proposition is an analogue and consequence of Theorem 2.2.7.

Theorem 2.3.8 *Let $W = W_1 \otimes \cdots \otimes W_p$ be a strongly $(\mathfrak{h}, \tilde{A})$-graded module constructed in Proposition 2.3.7. Then W is irreducible if and only if each W_i is irreducible.*

2.4 Irreducible modules for tensor product strongly graded algebra

Our goal is to determine all the strongly $(\mathfrak{h}, \tilde{A})$-graded irreducible modules for the tensor product strongly $(\mathfrak{h}, A)$-graded conformal vertex algebra constructed in Proposition 2.3.6. To do this, we need to define a more specific kind of strongly $(\mathfrak{h}, \tilde{A})$-graded modules as follows:

Definition 2.4.1 Let $V_1, \ldots, V_p, V$ be strongly $(\mathfrak{h}_1, A_1), \ldots, (\mathfrak{h}_p, A_p), (\mathfrak{h}, A)$-graded conformal vertex algebras, respectively, as in the setting of Proposition 2.3.6. Let W be a strongly $(\mathfrak{h}, \tilde{A})$-graded V-module, where $\tilde{A}$ is an abelian group containing A as a subgroup, so that in particular, for $\beta \in \tilde{A}$,

$$W^{(\beta)} = \{w \in W \mid h \cdot w = \langle h, \beta \rangle w, \text{ for all } h \in \mathfrak{h}\}.$$

Assume that there exists an abelian subgroup $\tilde{A}_i$ of $\tilde{A}$ containing A_i as a subgroup for each $i = 1, \ldots, p$ such that

$$\tilde{A} = \tilde{A}_1 \oplus \cdots \oplus \tilde{A}_p,$$
$$\langle \mathfrak{h}_i, \tilde{A}_j \rangle = 0 \text{ if } i \neq j$$

and such that W is a doubly graded V_i-module with respect to $\tilde{A}_i$ and the $\tilde{A}_i$-grading is given by $\mathfrak{h}_i$ in the following way: For $\beta_i \in \tilde{A}_i$,

$$W^{(\beta_i)} = \{w \in W \mid h_i \cdot w = \langle h_i, \beta_i \rangle w, \text{ for all } h_i \in \mathfrak{h}_i\}.$$

Then W is called a *strongly* $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$*-graded* V*-module.*

Remark 2.4.2 Submodules and quotient modules of strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded V-modules are also strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded modules. Irreducible strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded modules are strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded modules without nontrivial submodules. Strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded module homomorphisms are strongly $(\mathfrak{h}, \tilde{A})$-graded V-module homomorphisms.

Example 2.4.3 The strongly $(\mathfrak{h}, \tilde{A})$-graded tensor product module $W_1 \otimes \cdots \otimes W_p$ constructed in Proposition 2.3.7 is a strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded $V_1 \otimes \cdots \otimes V_p$-module.

From Example 2.1.8, we can see that any V_L-module is a strongly L°-graded module. Based on this fact, it is easy to show that the following example satisfies the conditions in Definition 2.4.1.

Example 2.4.4 Let $V^\natural$ be the moonshine module constructed in [FLM], which is a strongly $(\langle 0\rangle, \langle 0\rangle)$-graded conformal vertex algebra as in Example 2.1.7; let V_L be the conformal vertex algebra associated with the even 2-dimensional unimodular Lorentzian lattice L, which is a strongly $(\mathfrak{h}, L)$-graded conformal vertex algebra as constructed in Example 2.1.8. Then any strongly $(\mathfrak{h}, L)$-graded module for $V^\natural \otimes V_L$ is strongly $((\langle 0\rangle, \langle 0\rangle), (\mathfrak{h}, L))$-graded (note that L is a self-dual lattice, i.e., $L^\circ = L$).

Notation 2.4.5 For $\beta_1 \in \tilde{A}_1, \ldots, \beta_p \in \tilde{A}_p$, we let $W^{(\beta_1,\ldots,\beta_p)}$ denote the following common weight space of $\mathfrak{h}_1, \ldots, \mathfrak{h}_p$, i.e.,

$$W^{(\beta_1,\ldots,\beta_p)} := \{w \in W \mid h_i \cdot w = \langle h_i, \beta_i\rangle w, \text{ for all } h_i \in \mathfrak{h}_i,\ i = 1, \ldots, p\}.$$

Next we assume W to be a strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded $V_1 \otimes \cdots \otimes V_p$-module, with the notation as in Definition 2.4.1.

Proposition 2.4.6 *Suppose that W is irreducible. Then for $\beta_1 \in \tilde{A}_1, \ldots, \beta_p \in \tilde{A}_p$, $W^{(\beta_1,\ldots,\beta_p)}$ is irreducible under the algebra of operators $A(V_1 \otimes \cdots \otimes V_p; W^{(\beta_1,\ldots,\beta_p)})$.*

Proof. The proof is similar to the proof of Proposition 2.1.16. ∎

Lemma 2.4.7 *For $\beta \in \tilde{A}$, we have*

$$W^{(\beta)} = W^{(\beta_1,\ldots,\beta_p)},$$

where $\beta = \beta_1 + \cdots + \beta_p$.

Proof. This is a consequence of Definition 2.4.1. ∎

Theorem 2.4.8 *Let W be a strongly $((\mathfrak{h}_1, \tilde{A}_1),\ \dots,\ (\mathfrak{h}_p, \tilde{A}_p))$-graded irreducible $V_1 \otimes \cdots \otimes V_p$-module, with the notions as in Definition 2.4.1. Then W is a tensor product of irreducible strongly $(\mathfrak{h}_i, \tilde{A}_i)$-graded V_i-modules, for $i = 1, \dots, p$.*

Proof. For simplicity of notation, we take $p = 2$, as above. Since W is irreducible, by Remark 2.1.17, $W = \coprod_{\bar{n}=\mu} W_{(n)}$ for some $\mu \in \mathbb{C}/\mathbb{Z}$, where $\bar{n}$ denotes the equivalent class of $n \in \mathbb{C}$ in $\mathbb{C}/\mathbb{Z}$. Choose $\beta \in \tilde{A}$ such that $W^{(\beta)} \neq 0$. Then there exists $n_0 \in \mathbb{C}$ such that $W^{(\beta)}_{(n_0)}$ is the lowest weight space of $W^{(\beta)}$. Since $W^{(\beta)}_{(n_0)}$ is finite dimensional and we are working over $\mathbb{C}$, there exists a simultaneous eigenvector $w_0 \in W^{(\beta)}_{(n_0)}$ for the commuting operators $L_i(0)$ and the operators in $\mathfrak{h}_i,\ i = 1, 2$. Denote by $n_1, n_2 \in \mathbb{Z}$ the corresponding eigenvalues for $L_1(0),\ L_2(0)$. Then we have $n_0 = n_1 + n_2$. Denote by $\beta_1 \in \tilde{A}_1,\ \beta_2 \in \tilde{A}_2$ the corresponding weights for $\mathfrak{h}_1,\ \mathfrak{h}_2$. By Lemma 2.4.7, we have $W^{(\beta)} = W^{(\beta_1,\beta_2)}$, and $\beta = \beta_1 + \beta_2$.

Now the $L(-1)$-derivative condition and the $L(0)$-bracket formula imply that

$$[L_1(0), Y(v^{(1)} \otimes 1, x)] = Y(L_1(0)(v^{(1)} \otimes 1), x) + x\frac{d}{dx}Y(v^{(1)} \otimes 1, x)$$

for $v^{(1)} \in V_1$. Thus for doubly homogeneous vector $v^{(1)}$ and $n \in \mathbb{Z}$,

$$\mathrm{wt}_1 (v^{(1)} \otimes 1)_n = \mathrm{wt}_1 (v^{(1)} \otimes 1) - n - 1,$$

where wt_1 refers to $L_1(0)$-eigenvalue on both $V_1 \otimes V_2$ and the space of operators on W. In particular, $(v^{(1)} \otimes 1)_n$ permutes $L_1(0)$-eigenspaces. Moreover, since $(1 \otimes v^{(2)})_n$, for $v^{(2)} \in V_2$, commutes with $L_1(0)$, it preserves $L_1(0)$-eigenspaces. Of course, similar statements hold for $L_2(0)$, $\mathfrak{h}_1(0)$, $\mathfrak{h}_2(0)$.

By Lemma 2.4.6, $W^{(\beta_1,\beta_2)}$ is irreducible under the algebra of the operators $A(V_1 \otimes V_2; W^{(\beta_1,\beta_2)})$. Then $W^{(\beta_1,\beta_2)}$ is generated by w_0 by the irreducibility, and is spanned by elements of the form

$$(v_1^{(1)} \otimes 1)_{m_1} \cdots (v_k^{(1)} \otimes 1)_{m_k} (1 \otimes v_1^{(2)})_{n_1} \cdots (1 \otimes v_l^{(2)})_{n_l} w_0$$

where $v_i^{(1)} \in V_1$ and $v_j^{(2)} \in V_2$, $v_i^{(1)}, v_j^{(2)}$ are doubly homogeneous, and the A-weights of $\sum_{i=1}^m v_i^{(1)}$ and $\sum_{j=1}^n v_j^{(2)}$ are 0.

Hence $W^{(\beta_1,\beta_2)}$ is the direct sum of its simultaneous eigenspaces for $L_i(0)$ and $\mathfrak{h}_i$, for $i = 1, 2$, and the $L_1(0)$, $L_2(0)$-eigenvalues are bounded below by n_1, n_2, respectively. It follows that the lowest weight space $W^{(\beta_1,\beta_2)}_{(n_0)}$ is filled up by the simultaneous eigenspace for the operators $L_i(0)$ with eigenvalues n_i. To be more precise, we use $W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$ to denote the subspace $W^{(\beta_1,\beta_2)}_{(n_0)}$. By a similar argument as in Proposition 2.4.6, $W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$ is irreducible under the algebra of operators $A(V_1 \otimes V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$.

By the density theorem, the algebra $A(V_1 \otimes V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$ fills up End $W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$. Because $A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$ and $A(V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$ are commuting algebras of operators and $A(V_1 \otimes V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$ is generated by $A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$ and $A(V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$, we see that

$$\text{End } W^{(\beta_1,\beta_2)}_{(n_1,n_2)} = A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})A(V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)}).$$

Choose an irreducible $A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$-submodule M_1 of $W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$. Then $A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$ acts faithfully on M_1 since any element of $A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$ annihilating M_1 annihilates

$$\begin{aligned}
A(V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)}) \cdot M_1 &= A(V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)}) \cdot M_1 \\
&= (\text{End } W^{(\beta_1,\beta_2)}_{(n_1,n_2)})M_1 \\
&= W^{(\beta_1,\beta_2)}_{(n_1,n_2)}.
\end{aligned}$$

Thus $A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$ restricts faithfully to End M_1 and hence is isomorphic to a full matrix algebra. Similarly, $A(V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$ is isomorphic to a full matrix algebra. It follows that

$$\text{End } W^{(\beta_1,\beta_2)}_{(n_1,n_2)} = A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)}) \otimes A(V_2; W^{(\beta_1,\beta_2)}_{(n_1,n_2)}).$$

Then $W_{(n_1,n_2)}^{(\beta_1,\beta_2)}$ has the structure

$$W_{(n_1,n_2)}^{(\beta_1,\beta_2)} = M_1 \otimes M_2$$

as an irreducible $A(V_1; W_{(n_1,n_2)}^{(\beta_1,\beta_2)}) \otimes A(V_2; W_{(n_1,n_2)}^{(\beta_1,\beta_2)})$-module. Here, as an irreducible $A(V_i; W_{(n_1,n_2)}^{(\beta_1,\beta_2)})$-submodule of $W_{(n_1,n_2)}^{(\beta_1,\beta_2)}$, M_i has $\tilde{A}_i$-grading β_i induced by $\mathfrak{h}_i$, and has $\mathbb{C}$-grading n_i induced by $L_i(0)$, respectively, for $i = 1, 2$.

Let

$$w^0 = y_1 \otimes y_2$$

(where $y_i \in M_i$, for $i = 1, 2$) be a nonzero decomposable tensor in $W_{(n_1,n_2)}^{(\beta_1,\beta_2)}$. Let W_i be the doubly graded V_i-submodule of W generated by w^0. Then the module W_1 has a strongly $(\mathfrak{h}_1, \tilde{A}_1)$-graded V_1-module structure such that

$$W_1 = \coprod_{n \in \mathbb{C},\ \gamma \in \tilde{A}} (W_1)_{(n)}^{(\gamma)},$$

where

$$\begin{aligned}
(W_1)_{(n)}^{(\gamma)} \quad = \quad & \mathrm{span}\big\{(v_1^{(1)} \otimes \mathbf{1})_{s_1} \cdots (v_p^{(1)} \otimes \mathbf{1})_{s_p} w^0 \ \big| \\
& \mathrm{wt}\ v_1^{(1)} - s_1 - 1 + \cdots + \mathrm{wt}\ v_p^{(1)} - s_p - 1 = n - n_1, \\
& A\text{-wt}\ v_1^{(1)} + \cdots + A\text{-wt}\ v_p^{(1)} = \gamma - \beta_1, \\
& \text{where} \quad v_1^{(1)}, \ldots, v_p^{(1)} \in V_1,\ s_1, \ldots, s_p \in \mathbb{Z}\big\}.
\end{aligned}$$

This module we constructed satisfies the grading restrictions (2.27) and (2.28) in Definition 2.1.4, which follows from the fact that W is a strongly graded $V_1 \otimes V_2$-module and each doubly homogeneous subspace of W_1 lies in the doubly homogeneous subspace of W. Also, $W_1^{(\gamma)}$ is the weight space of $\mathfrak{h}_1$ with weight γ, hence by Definition 2.3.4, W_1 is a strongly $(\mathfrak{h}_1, \tilde{A}_1)$-graded V_1-module.

We claim that W_1 is V_1-irreducible (and similarly for W_2). In fact, consideration of the abelian group grading shows that any nonzero V_1-submodule of W_1 not intersecting $W^{(\beta_1,\beta_2)}$ will give rise to a nonzero $V_1 \otimes V_2$-submodule of W

not intersecting $W^{(\beta_1,\beta_2)}$. Thus any nonzero V_1-submodule of W_1 must intersect $W^{(\beta_1,\beta_2)}$. Then consideration of the weight shows that the (β_1,β_2)-subspace of any nonzero V_1-submodule of W_1 not intersecting $W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$ would give rise to a nonzero $A(V_1\otimes V_2; W^{(\beta_1,\beta_2)})$-submodule of $W^{(\beta_1,\beta_2)}$ not intersecting $W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$. Thus any nonzero V_1-submodule of W_1 must intersect $W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$. But the irreducible $A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})$-module $A(V_1; W^{(\beta_1,\beta_2)}_{(n_1,n_2)})\cdot w^0$ is the full intersection of W_1 and $W^{(\beta_1,\beta_2)}_{(n_1,n_2)}$, so that the V_1-submodule must contain w^0 and hence be all of W_1. This proves the V_1-irreducibility of W_1.

Finally, to show that W is isomorphic to $W_1\otimes W_2$, consider the abstract tensor product $V_1\otimes V_2$-module $W_1\otimes W_2$, where W_i is the strongly $\tilde{A}_i$-graded V_i-module defined above, for $i=1,2$. Define a linear map

$$\varphi : W_1\otimes W_2 \to W$$

$$b_1\cdot w^0\otimes b_2\cdot w^0 \mapsto b_1b_2\cdot w^0,$$

where b_i is any operator induced by V_i. Then φ is well defined and is a $V_1\otimes V_2$-module homomorphism. Since $W_1\otimes W_2$ is irreducible by Theorem 2.2.7, φ is a module isomorphism. ∎

Example 2.4.9 Let V_{L_i} be the conformal vertex algebra associated with an even lattice L_i as in Example 2.1.8, where $i=1,\ldots,p$. Let $V_{L_1}\otimes\cdots\otimes V_{L_p}$ be the tensor product strongly graded vertex algebra of $V_{L_1},\ldots,V_{L_p}$. By the construction of a lattice vertex algebra in Example 2.1.8, we have

$$V_{L_1}\otimes\cdots\otimes V_{L_p} = V_{L_1\oplus\cdots\oplus L_p},$$

and every irreducible $V_{L_1\oplus\cdots\oplus L_p}$-module is equivalent to a module of the form

$$V_{L_1+\gamma_1\oplus\cdots\oplus L_p+\gamma_p} = V_{L_1+\gamma_1}\otimes\cdots\otimes V_{L_p+\gamma_p},$$

for some $\gamma_i\in L_i^\circ$, $i=1,\ldots,p$. This example illustrates Theorem 2.4.8.

Now we can describe our main examples:

Corollary 2.4.10 *The only irreducible strongly* $(\mathfrak{h}, L)$*-graded module of* $V^\natural \otimes V_L$*, where* L *is the unique even 2-dimensional unimodular Lorentzian lattice and* $\mathfrak{h} = \{(h(-1)\cdot 1)_0 \mid h \in L \otimes_{\mathbb{Z}} \mathbb{C}\}$*, up to equivalence, is itself.*

Proof. Let W be an irreducible strongly $(\mathfrak{h}, L)$-graded module of $V^\natural \otimes V_L$. Then by Example 2.4.4, W is a strongly $((\langle 0\rangle, \langle 0\rangle), (\mathfrak{h}, L))$-graded module of $V^\natural \otimes V_L$. By Theorem 2.4.8, it is a tensor product of an irreducible strongly $(\langle 0\rangle, \langle 0\rangle)$-graded $V^\natural$-module with an irreducible strongly $(\mathfrak{h}, L)$-graded V_L-module. By [D2], $V^\natural$ is its only irreducible module, up to equivalence. Also, by [D1] (cf. [LL], Example 2.1.8), V_L is its only irreducible module because L is self-dual. Therefore

$$W = V^\natural \otimes V_L$$

as claimed. ∎

Remark 2.4.11 Thanks to Prof. Haisheng Li, we provide another proof for Corollary 2.4.10. Let W be an irreducible module for $V^\natural \otimes V_L$. Then W can be viewed as a weak $V^\natural$-module. Since every weak module for $V^\natural$ is completely reducible and the only irreducible module for $V^\natural$ is itself up to isomorphism (see [D2] and [DLM]), W is a direct sum of $V^\natural$ up to isomorphism, i.e.,

$$W = \coprod V^\natural.$$

From Remark 4.7.1 in [FHL], Schur's Lemma for irreducible modules holds for irreducible modules of $V^\natural$ over $\mathbb{C}$. Also, since the operators on W induced from $V^\natural$ and V_L commute with each other, $\mathrm{Hom}_{V^\natural}(V^\natural, W)$ can be viewed as a V_L-module. We have the natural $V^\natural \otimes V_L$-module isomorphism

$$\begin{aligned} V^\natural \otimes \mathrm{Hom}_{V^\natural}(V^\natural, W) &= V^\natural \otimes \mathrm{Hom}_{V^\natural}(V^\natural, \coprod V^\natural) \\ &= V^\natural \otimes \coprod \mathrm{Hom}_{V^\natural}(V^\natural, V^\natural) \simeq \coprod V^\natural = W. \end{aligned}$$

(See also Lemma 4.13 and Proposition 4.14 in [Li]). Since W is irreducible, $\mathrm{Hom}_{V^\natural}(V^\natural, W)$ has to be an irreducible V_L-module, which is V_L itself, up to isomorphism. So $W \simeq V^\natural \otimes V_L$ as a $V^\natural \otimes V_L$-module.

Remark 2.4.12 In Corollary 2.4.10, the 2-dimensional self-dual Lorentzian lattice can of course be generalized to any self-dual nondegenerate even lattice.

2.5 Complete reducibility

Definition 2.5.1 Let V be a strongly $(\mathfrak{h}, A)$-graded conformal vertex algebra. Then a strongly $(\mathfrak{h}, \tilde{A})$-graded V-module is called *completely reducible* if it is a direct sum of irreducible strongly $(\mathfrak{h}, \tilde{A})$-graded V-modules.

Notation 2.5.2 In the remainder of this section, we will always let $A = A_1 \oplus \cdots \oplus A_p$, $\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_p$, and $V = V_1 \otimes \cdots \otimes V_p$.

Definition 2.5.3 A strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded module for the tensor product conformal vertex algebra V is called *completely reducible* if it is a direct sum of irreducible strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded V-modules.

Theorem 2.5.4 *Let $V_1, \ldots, V_p$ be strongly $(\mathfrak{h}_1, A_1), \ldots, (\mathfrak{h}_p, A_p)$-graded conformal vertex algebras, respectively, and let V be their tensor product strongly $(\mathfrak{h}, A)$-graded conformal vertex algebra. Then every strongly $((\mathfrak{h}_1, \tilde{A}_1), \ldots, (\mathfrak{h}_p, \tilde{A}_p))$-graded V-module is completely reducible if and only if every strongly $(\mathfrak{h}_i, \tilde{A}_i)$-graded V_i-module is completely reducible.*

Proof. It suffices to prove the result for $n = 2$. Let W be a strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V = V_1 \otimes V_2$-module. Then by Proposition 2.4.7, we can take $w \in W^{(\beta_1, \beta_2)}_{(n_1, n_2)}$, where $\beta_i \in \tilde{A}_i$, $n_i \in \mathbb{C}$, for $i = 1, 2$.

Let M be the strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V_1 \otimes V_2$-submodule of W generated by w, i.e., M is spanned by elements of the form

$$(v_1^{(1)} \otimes \mathbf{1})_{s_1} \cdots (v_p^{(1)} \otimes \mathbf{1})_{s_p} (\mathbf{1} \otimes v_1^{(2)})_{t_1} \cdots (\mathbf{1} \otimes v_q^{(2)})_{t_q} w$$

where $v_1^{(1)}, \ldots, v_p^{(1)}$ are doubly homogeneous elements in V_1 and $v_1^{(2)}, \ldots, v_q^{(2)}$ are doubly homogeneous elements in V_2, respectively, and $s_1, \ldots, s_p, t_1, \ldots, t_q \in \mathbb{Z}$.

Let M_i be the doubly graded V_i-submodule of M generated by w. Then M_i is a strongly $(\mathfrak{h}_i, \tilde{A}_i)$-graded V_i-module, respectively, for $i = 1, 2$, in an obvious way as in the proof of Theorem 2.4.8.

By Proposition 2.3.7 and Example 2.4.3, $M_1 \otimes M_2$ is strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded. Moreover, we have a strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V_1 \otimes V_2$-module epimorphism from $M_1 \otimes M_2$ to M by sending $b_1 w \otimes b_2 w \mapsto b_1 b_2 w$, where b_i is an operator induced by V_i, for $i = 1, 2$. If every strongly $(\mathfrak{h}_i, \tilde{A}_i)$-graded V_i-module is completely reducible, then M_i is a direct sum of irreducible strongly $(\mathfrak{h}_i, \tilde{A}_i)$-graded V_i-modules and therefore $M_1 \otimes M_2$ is a direct sum of irreducible strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V_1 \otimes V_2$-modules (see Theorem 2.3.8). Then as a quotient module of $M_1 \otimes M_2$, M is also a direct sum of irreducible strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V_1 \otimes V_2$-modules, and consequently, W is a direct sum of irreducible strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V_1 \otimes V_2$-modules.

Conversely, assume that every strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V_1 \otimes V_2$-module W is completely reducible. We first observe that $V_1 \otimes V_2$ is strongly $((\mathfrak{h}_1, A_1), (\mathfrak{h}_2, A_2))$-graded, hence a $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V_1 \otimes V_2$-module itself by Proposition 2.3.6 and Example 2.4.3, and hence is a direct sum of irreducible strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded modules. Let W be an irreducible strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V_1 \otimes V_2$-module. Then W is a tensor product of an irreducible strongly $(\mathfrak{h}_1, \tilde{A}_1)$-graded module for V_1 and an irreducible strongly $(\mathfrak{h}_2, \tilde{A}_2)$-graded module for V_2 by Theorem 2.4.8. In particular, V_1 has irreducible strongly $(\mathfrak{h}_1, \tilde{A}_1)$-graded modules and V_2 has irreducible strongly $(\mathfrak{h}_2, \tilde{A}_2)$-graded modules, respectively.

Let W_1 be a strongly $(\mathfrak{h}_1, \tilde{A}_1)$-graded V_1-module and W_2 be an irreducible strongly $(\mathfrak{h}_2, \tilde{A}_2)$-graded V_2-module. Since every strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V_1 \otimes V_2$-module is completely reducible, $W_1 \otimes W_2$ is a direct sum of irreducible strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded modules:

$$W_1 \otimes W_2 = \coprod_i M_i$$

where each M_i is an irreducible strongly $((\mathfrak{h}_1, \tilde{A}_1), (\mathfrak{h}_2, \tilde{A}_2))$-graded $V_1 \otimes V_2$-module. Fix i and let $x_1^{(i)}, \ldots, x_n^{(i)} \in W_1$ and $y_1^{(i)}, \ldots, y_n^{(i)} \in W_2$ be linearly independent doubly homogeneous elements such that $\sum_j c_j x_j^{(i)} \otimes y_j^{(i)} \in M_i$, where $c_j \in \mathbb{C}, c_j \neq 0$. By the density theorem (as in the proof of Theorem 2.2.7), each $x_j^{(i)} \otimes y_j^{(i)} \in M_i$. Let W_{i1} be the doubly graded V_1-submodule of W_1 generated by $x_{j_0}^{(i)}$, for some $j_0 \in \{1, 2, \ldots, n\}$. Then W_{i1} is a strongly $(\mathfrak{h}_1, \tilde{A}_1)$-graded V_1-submodule as in the proof of Theorem 2.4.8. By the irreducibility of M_i, we see that $M_i = W_{i1} \otimes W_2$ and that W_{i1} is an irreducible strongly $(\mathfrak{h}_1, \tilde{A}_1)$-graded V_1-submodule of W_1. Therefore, $W_1 \otimes W_2 = (\coprod_i W_{i1}) \otimes W_2$. By the density theorem, for any nonzero $w_2 \in W_2$, $W_1 \otimes w_2 = (\coprod_i W_{i1}) \otimes w_2$. Hence as a V_1-module, $W_1 \cong (\coprod_i W_{i1})$, and thus W_1 is completely reducible. Similarly for V_2. ∎

Example 2.5.5 Let V_{L_i} be the conformal vertex algebra associated with an even lattice L_i as in Example 2.1.8, where $i = 1, \ldots, p$. Let $V_{L_1} \otimes \cdots \otimes V_{L_p}$ be the tensor product strongly graded vertex algebra of $V_{L_1}, \ldots, V_{L_p}$. By the construction of a lattice vertex algebra as in Example 2.1.8, we have

$$V_{L_1} \otimes \cdots \otimes V_{L_p} = V_{L_1 \oplus \cdots \oplus L_p}.$$

As in Example 2.1.8, every module for $V_{L_1 \oplus \cdots \oplus L_p}$, hence for $V_{L_1} \otimes \cdots \otimes V_{L_p}$, is completely reducible. This example illustrates Theorem 2.5.4.

Corollary 2.5.6 *Every strongly $(\mathfrak{h}, L)$-graded module for the strongly $(\mathfrak{h}, L)$-graded conformal vertex algebra $V^\natural \otimes V_L$, where L is the unique even 2-dimensional unimodular Lorentzian lattice and $\mathfrak{h} = \{(h(-1) \cdot 1)_0 \mid h \in L \otimes_{\mathbb{Z}} \mathbb{C}\}$, is completely reducible.*

Chapter 3

Differential equations and logarithmic intertwining operators for strongly graded vertex algebras

3.1 Strongly graded generalized modules

In this section, we will recall the following definitions from [HLZ1]:

Definition 3.1.1 A generalized module for a conformal vertex algebra is defined in the same way as a module for a conformal vertex algebra except that in the grading (2.12), each space $W_{(n)}$ is replaced by $W_{[n]}$, where $W_{[n]}$ is the generalized $L(0)$-eigenspace corresponding to the generalized eigenvalue $n \in \mathbb{C}$; that is, (2.12) and (2.20) in the definition are replaced by

$$W = \coprod_{n \in \mathbb{C}} W_{[n]}$$

and

$$\text{for } n \in \mathbb{C} \text{ and } w \in W_{[n]},\ (L(0) - n)^k w = 0,\ \text{for } k \in \mathbb{N} \text{ sufficiently large},$$

respectively. For $w \in W_{[n]}$, we still write wt $w = n$ for the generalized weight of w.

Definition 3.1.2 Let A be an abelian group and V a strongly A-graded conformal vertex algebra. Let $\tilde{A}$ be an abelian group containing A as a subgroup. A generalized V-module

$$W = \coprod_{n \in \mathbb{C}} W_{[n]}$$

is said to be *strongly graded with respect to* $\tilde{A}$ (or *strongly* $\tilde{A}$*-graded*, or just *strongly graded*) if the abelian group $\tilde{A}$ is understood) if it is equipped with a second gradation, by $\tilde{A}$,

$$W = \coprod_{\beta \in \tilde{A}} W^{(\beta)},$$

such that the following conditions are satisfied: the two gradations are compatible, that is, for any $\beta \in \tilde{A}$,

$$W^{(\beta)} = \coprod_{n \in \mathbb{C}} W^{(\beta)}_{[n]}, \quad \text{where} \quad W^{(\beta)}_{[n]} = W_{[n]} \cap W^{(\beta)};$$

for any $\alpha \in A$, $\beta \in \tilde{A}$ and $n \in \mathbb{C}$,

$$W^{(\beta)}_{[n+k]} = 0) \quad \text{for} \quad k \in \mathbb{Z} \ \text{ sufficiently negative;} \tag{3.1}$$

$$\dim W^{(\beta)}_{[n]} < \infty;$$

$$v_l W^{(\beta)} \subset W^{(\alpha+\beta)} \quad \text{for any} \quad v \in V^{(\alpha)}, \ l \in \mathbb{Z}.$$

A strongly $\tilde{A}$-graded (generalized) V-module W is said to be *lower bounded* if instead of (3.1), it satisfies the stronger condition that for any $\beta \in \tilde{A}$,

$$W^{(\beta)}_{(n)} = 0 \ \text{ (respectively, } W^{(\beta)}_{[n]} = 0) \ \text{ for } \ n \in \mathbb{C} \ \text{ and } \ \Re(n) \ \text{ sufficiently negative.}$$

In this chapter, we will derive systems of differential equations for matrix elements of products and iterates of logarithmic intertwining operators among generalized modules for a strongly graded vertex algebra. For this purpose, first we need the notion of contragredient module for a strongly graded (generalized) module. With the strong gradedness condition on a (generalized) module, we can indeed define the corresponding notion of contragredient module.

Definition 3.1.3 Let $W = \coprod_{\beta \in \tilde{A}, n \in \mathbb{C}} W^{(\beta)}_{[n]}$ be a strongly $\tilde{A}$-graded generalized module for a strongly A-graded conformal vertex algebra. For each $\beta \in \tilde{A}$ and $n \in \mathbb{C}$, let us identify $(W^{(\beta)}_{[n]})^*$ with the subspace of W^* consisting of the linear function on W vanishing on each $W^{(\gamma)}_{[n]}$ with $\gamma \neq \beta$ or $m \neq n$. We define W' to be the $(\tilde{A} \times \mathbb{C})$-graded vector subspaces of W^* given by

$$W' = \coprod_{\beta \in \tilde{A}, n \in \mathbb{C}} (W')^{(\beta)}_{[n]}, \quad \text{where} \quad (W')^{(\beta)}_{[n]} = (W^{(-\beta)}_{[n]})^*.$$

The *adjoint vertex operators* $Y'(v,z)$ $(v \in V)$ on W' is defined in the same way as vertex operator algebra in section 5.2 in [FHL] (see Section 2 of [HLZ1]). The pair (W', Y') carries a strongly graded module structure as follows:

Proposition 3.1.4 *Let $\tilde{A}$ be an abelian group containing A as a subgroup and V a strongly A-graded conformal vertex algebra. Let (W,Y) be a strongly $\tilde{A}$-graded V-module (respectively, generalized V-module). Then the pair (W',Y') carries a strongly $\tilde{A}$-graded V-module (respectively, generalized V-module) structure. If W is lower bounded, so is W'.*

Definition 3.1.5 The pair (W',Y') is called the *contragredient module* of (W,Y).

3.2 C_1-cofiniteness condition

In this section, we will let V denote a strongly A-graded conformal vertex algebra and let W denote a strongly $\tilde{A}$-graded lower bounded (generalized) V-module, where A, $\tilde{A}$ are abelian groups such that A is an abelian subgroup of $\tilde{A}$.

In the following definition, we generalize the C_1-cofiniteness condition for the (generalized) modules for a vertex operator algebra to a C_1-cofiniteness condition with respect to $\tilde{A}$ for the strongly $\tilde{A}$-graded (generalized) modules for a strongly graded conformal vertex algebra.

Definition 3.2.1 Let $C_1(W)$ be the subspace of W spanned by elements of the form $u_{-1}w$ for

$$u \in V_+ = \coprod_{n>0} V_{(n)}$$

and $w \in W$. The $\tilde{A}$-grading on W induces an $\tilde{A}$-grading on $W/C_1(W)$:

$$W/C_1(W) = \coprod_{\beta \in \tilde{A}} (W/C_1(W))^{(\beta)},$$

where

$$(W/C_1(W))^{(\beta)} = W^{(\beta)}/(C_1(W))^{(\beta)}$$

for $\beta \in \tilde{A}$. If dim $(W/C_1(W))^{(\beta)} < \infty$ for $\beta \in \tilde{A}$, we say that W is C_1-*cofinite with respect to* $\tilde{A}$ or W satisfies the C_1-*cofiniteness condition with respect to* $\tilde{A}$.

Remark 3.2.2 Let V_0 be a conformal vertex subalgebra of V strongly graded with respect to an abelian subgroup A_0 of A. If W is C_1-cofinite with respect to $\tilde{A}$ as a strongly graded (generalized) V_0-module, then W is C_1-cofinite with respect to $\tilde{A}$ as a strongly graded (generalized) V-module.

Example 3.2.3 Let V_L be the conformal vertex algebra associated with a nondegenerate even lattice L and let W be a strongly M-graded (generalized) V_L-module for a sublattice M of L° containing L as in Example 2.1.8. Then W satisfies the C_1-cofiniteness condition with respect to M as a $V_L^{(0)}$-module. Thus W is also C_1-cofinite with respect to M as a strongly graded V_L-module.

3.3 Logarithmic intertwining operators

Logarithmic intertwining operators were introduced and studied in [M2]. We first recall the relevant definitions from [M2] [M3], [HLZ2]; we use the versions in [HLZ2].

Throughout this section, we shall use $x, x_0, x_1, x_2, \ldots$ to denote commuting formal variables and $z, z_0, z_1, z_2, \ldots$ to denote complex variables or complex numbers.

Definition 3.3.1 Let (W_1, Y_1), (W_2, Y_2) and (W_3, Y_3) be generalized modules for a conformal vertex algebra V. A *logarithmic intertwining operator of type* $\binom{W_3}{W_1\, W_2}$ is a linear map

$$\mathcal{Y}(\cdot, x)\cdot : W_1 \otimes W_2 \to W_3[\log x]\{x\}, \tag{3.2}$$

or equivalently,

$$w_{(1)} \otimes w_{(2)} \mapsto \mathcal{Y}(w_{(1)}, x)w_{(2)} = \sum_{n\in\mathbb{C}} \sum_{k\in\mathbb{N}} {w_{(1)}}^{\mathcal{Y}}_{n;\,k} w_{(2)} x^{-n-1} (\log x)^k \in W_3[\log x]\{x\} \tag{3.3}$$

for all $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, such that the following conditions are satisfied: the *lower truncation condition*: for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $n \in \mathbb{C}$,

$$w_{(1)}{}^{\mathcal{Y}}_{n+m;\,k}w_{(2)} = 0 \quad \text{for} \quad m \in \mathbb{N} \ \text{ sufficiently large, independently of } k; \tag{3.4}$$

the *Jacobi identity*:

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)&Y_3(v,x_1)\mathcal{Y}(w_{(1)},x_2)w_{(2)} \\ &-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\mathcal{Y}(w_{(1)},x_2)Y_2(v,x_1)w_{(2)} \\ =\ x_2^{-1}\delta&\left(\frac{x_1-x_0}{x_2}\right)\mathcal{Y}(Y_1(v,x_0)w_{(1)},x_2)w_{(2)} \end{aligned} \tag{3.5}$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ (note that the first term on the left-hand side is meaningful because of (3.4)); the $L(-1)$-*derivative property:* for any $w_{(1)} \in W_1$,

$$\mathcal{Y}(L(-1)w_{(1)},x) = \frac{d}{dx}\mathcal{Y}(w_{(1)},x). \tag{3.6}$$

Definition 3.3.2 In the setting of Definition 3.3.1, suppose in addition that V and W_1, W_2 and W_3 are strongly graded. A logarithmic intertwining operator $\mathcal{Y}$ as in Definition 3.3.1 is a *grading-compatible logarithmic intertwining operator* if for $\beta, \gamma \in \tilde{A}$ and $w_1 \in W_1^{(\beta)}$, $w_2 \in W_2^{(\gamma)}$, $n \in \mathbb{C}$ and $k \in \mathbb{N}$, we have

$$(w_1)_{n;k}w_2 \in W_3^{(\beta+\gamma)}.$$

Definition 3.3.3 In the setting of Definition 3.3.2, the grading-compatible logarithmic intertwining operators of a fixed type $\binom{W_3}{W_1\,W_2}$ form a vector space, which we denote by $\mathcal{V}^{W_3}_{W_1W_2}$. We call the dimension of $\mathcal{V}^{W_3}_{W_1W_2}$ the *fusion rule* for W_1, W_2 and W_3 and denote it by $N^{W_3}_{W_1W_2}$.

Let V be a strongly A-graded vertex algebra and V_0 be a strongly A_0-graded vertex subalgebra of V, where A is an abelian group and A_0 is an abelian subgroup of A. Let $\tilde{A}$ be an abelian group containing A as its subgroup.

We shall use the following two sets in the next section: For $\beta_i \in \tilde{A}$, $i = 1, 2, 3$, set

$$\tilde{I}^{(\beta_1,\beta_2,\beta_3)} = (\beta_1 + A_0) \times (\beta_2 + A_0) \times (\beta_3 + A_0).$$

For any strongly $\tilde{A}$-graded generalized V-modules W_i $(i = 0, 1, \ldots, 4)$ and any logarithmic intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of type $\binom{W_0'}{W_1\, W_4}$ and $\binom{W_4}{W_2\, W_3}$, respectively, set

$$I^{(\beta_1,\beta_2,\beta_3)}_{\mathcal{Y}_1,\mathcal{Y}_2} = \left\{ (\widetilde{\beta_1}, \widetilde{\beta_2}, \widetilde{\beta_3}) \in \tilde{I}^{(\beta_1,\beta_2,\beta_3)} \;\middle|\; \begin{array}{c} \exists \quad w_i \in W_i^{(\widetilde{\beta_i})} \ (i = 1, 2, 3) \ \text{ s.t.} \\ \mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(w_2, x_2)w_3 \neq 0 \end{array} \right\}.$$

For brevity, we will use $I^{(\beta_1,\beta_2,\beta_3)}$ to denote the set $I^{(\beta_1,\beta_2,\beta_3)}_{\mathcal{Y}_1,\mathcal{Y}_2}$ in the rest of this paper.

Lemma 3.3.4 *Suppose that every strongly $\tilde{A}$-graded V-module satisfies C_1-cofiniteness condition with respect to $\tilde{A}$ as a V_0-module and that for any two fixed elements β_1 and β_2 in $\tilde{A}$ and any triple of strongly graded generalized V-modules M_1, M_2 and M_3, the fusion rule*

$$N^{M_3^{(\widetilde{\beta_1}+\widetilde{\beta_2})}}_{M_1^{(\widetilde{\beta_1})}\, M_2^{(\widetilde{\beta_2})}} \neq 0$$

for only finitely many pairs $(\widetilde{\beta_1}, \widetilde{\beta_2}) \in (\beta_1 + A_0) \times (\beta_2 + A_0)$. Then the set $I^{(\beta_1,\beta_2,\beta_3)}$ defined above is a finite set.

Proof. Since for the triple of strongly graded generalized modules (W_1, W_2, W_3), the fusion rules $N^{W_3^{(\widetilde{\beta_1}+\widetilde{\beta_2})}}_{W_1^{(\widetilde{\beta_1})}\, W_2^{(\widetilde{\beta_2})}} \neq 0$ for only finitely many pairs $(\widetilde{\beta_1}, \widetilde{\beta_2}) \in (\beta_1 + A_0) \times (\beta_2 + A_0)$, the logarithmic intertwining operator $\mathcal{Y}_2(w_2, x_2)w_3$, where $w_2 \in W_2^{(\widetilde{\beta_2})}$ and $w_3 \in W_3^{(\widetilde{\beta_3})}$, have to be 0 except for finitely many pairs $(\widetilde{\beta_2}, \widetilde{\beta_3}) \in (\beta_2 + A_0) \times (\beta_3 + A_0)$, and then there are only finitely many triples $(\widetilde{\beta_1}, \widetilde{\beta_2}, \widetilde{\beta_3}) \in \tilde{I}^{(\beta_1,\beta_2,\beta_3)}$ such that the products of logarithmic intertwining operators

$$\mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(w_2, x_2)w_3 \neq 0,$$

where $w_1 \in W_1^{(\widetilde{\beta_1})}$, $w_2 \in W_2^{(\widetilde{\beta_2})}$ and $w_3 \in W_3^{(\widetilde{\beta_3})}$. Thus the set $I^{(\beta_1,\beta_2,\beta_3)}$ is a finite set. □

Remark 3.3.5 In the case that A_0 is a finite subgroup of A, the assumption in Lemma 3.3.4 holds automatically.

Example 3.3.6 Let W be a strongly M-graded (generalized) module for the lattice vertex algebra V_L as in Example 3.2.3. Then W satisfies the assumption in Lemma 3.3.4 because $V_0 = V_L^{(0)}$ and A_0 is the trivial group in this case.

3.4 Differential equations

In this section, we assume that V is a strongly A-graded vertex algebra with a vertex subalgebra V_0 strongly graded with respect to an abelian subgroup A_0 of A, and we assume that every strongly graded $\tilde{A}$-(generalized) V-module is $\mathbb{R}$-graded, lower bounded and satisfies C_1-cofiniteness condition with respect to $\tilde{A}$ as a V_0-module.

Let W_i be strongly $\tilde{A}$-graded generalized V-modules for $i = 0, 1, \ldots, 4$ and let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be logarithmic intertwining operators of type $\binom{W_0'}{W_1\, W_4}$ and $\binom{W_4}{W_2\, W_3}$, respectively. Let $\tilde{I}^{(\beta_1,\beta_2,\beta_3)}$ and $I^{(\beta_1,\beta_2,\beta_3)}$ be the two sets defined in the previous section.

Let $R = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, (z_1 - z_2)^{-1}]$, β_1, β_2 and β_3 be three fixed elements in $\tilde{A}$. Set

$$\widetilde{T}^{(\beta_1,\beta_2,\beta_3)} = \coprod_{(\widetilde{\beta_1},\widetilde{\beta_2},\widetilde{\beta_3}) \in \tilde{I}^{(\beta_1,\beta_2,\beta_3)}} R \otimes W_0^{(\widetilde{\beta_1}+\widetilde{\beta_2}+\widetilde{\beta_3})} \otimes W_1^{(\widetilde{\beta_1})} \otimes W_2^{(\widetilde{\beta_2})} \otimes W_3^{(\widetilde{\beta_3})}$$

and

$$T_{\mathcal{Y}_1,\mathcal{Y}_2}^{(\beta_1,\beta_2,\beta_3)} = \coprod_{(\widetilde{\beta_1},\widetilde{\beta_2},\widetilde{\beta_3}) \in I^{(\beta_1,\beta_2,\beta_3)}} R \otimes W_0^{(\widetilde{\beta_1}+\widetilde{\beta_2}+\widetilde{\beta_3})} \otimes W_1^{(\widetilde{\beta_1})} \otimes W_2^{(\widetilde{\beta_2})} \otimes W_3^{(\widetilde{\beta_3})}.$$

Then $\widetilde{T}^{(\beta_1,\beta_2,\beta_3)}$ and $T_{\mathcal{Y}_1,\mathcal{Y}_2}^{(\beta_1,\beta_2,\beta_3)}$ have natural R-module structures. For convenience, in the rest of this paper, we will use $T^{(\beta_1,\beta_2,\beta_3)}$ to denote $T_{\mathcal{Y}_1,\mathcal{Y}_2}^{(\beta_1,\beta_2,\beta_3)}$.

For simplicity, we shall omit one tensor symbol to write $f(z_1, z_2) \otimes w_0 \otimes w_1 \otimes w_2 \otimes w_3$ as $f(z_1, z_2)w_0 \otimes w_1 \otimes w_2 \otimes w_3$ in $\widetilde{T}^{(\beta_1,\beta_2,\beta_3)}$ and $T^{(\beta_1,\beta_2,\beta_3)}$. For a strongly $\tilde{A}$-graded generalized V-module W, let (W', Y') be the contragredient module of W (recall definition 3.1.5). In particular, for $u \in V$ and $n \in \mathbb{Z}$, we have the operators u_n on W'. Let $u_n^* : W \to W$ be the adjoint of $u_n : W' \to W'$. Note that since wt u_n = wt $u-n-1$, we have wt $u_n^* = -$wt $u+n+1$. Also, A-wt u_n^* $= -(A$-wt $u_n)$.

Let $(\widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\beta}_3) \in \tilde{I}^{(\beta_1,\beta_2,\beta_3)}$ and let $\widetilde{\beta}_0 = \widetilde{\beta}_1 + \widetilde{\beta}_2 + \widetilde{\beta}_3$. For $u \in (V_0)_+$ and $w_i \in W_i^{(\widetilde{\beta}_i)}$ $(i = 0,1,2,3)$, let $J^{(\beta_1,\beta_2,\beta_3)}$ be the submodule of $\widetilde{T}^{(\beta_1,\beta_2,\beta_3)}$ generated by elements of the form

$$\begin{aligned}
&\mathcal{A}(u, w_0, w_1, w_2, w_3)\\
= &\sum_{k\geq 0}\binom{-1}{k}(-z_1)^k u^*_{-1-k}w_0\otimes w_1\otimes w_2\otimes w_3 - w_0\otimes u_{-1}w_1\otimes w_2\otimes w_3\\
&-\sum_{k\geq 0}\binom{-1}{k}(-(z_1-z_2))^{-1-k}w_0\otimes w_1\otimes u_k w_2\otimes w_3\\
&-\sum_{k\geq 0}\binom{-1}{k}(-z_1)^{-1-k}w_0\otimes w_1\otimes w_2\otimes u_k w_3,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{B}(u, w_0, w_1, w_2, w_3)\\
= &\sum_{k\geq 0}\binom{-1}{k}(-z_2)^k u^*_{-1-k}w_0\otimes w_1\otimes w_2\otimes w_3\\
&-\sum_{k\geq 0}\binom{-1}{k}(-(z_1-z_2))^{-1-k}w_0\otimes u_k w_1\otimes w_2\otimes w_3\\
&-w_0\otimes w_1\otimes u_{-1}w_2\otimes w_3 - \sum_{k\geq 0}\binom{-1}{k}(-z_2)^{-1-k}w_0\otimes w_1\otimes w_2\otimes u_k w_3,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{C}(u, w_0, w_1, w_2, w_3)\\
= \;& u^*_{-1}w_0\otimes w_1\otimes w_2\otimes w_3 - \sum_{k\geq 0}\binom{-1}{k}z_1^{-1-k}w_0\otimes u_k w_1\otimes w_2\otimes w_3\\
&-\sum_{k\geq 0}\binom{-1}{k}z_2^{-1-k}w_0\otimes w_1\otimes u_k w_2\otimes w_3 - w_0\otimes w_1\otimes w_2\otimes u_{-1}w_3,
\end{aligned}$$

$$
\begin{aligned}
&\mathcal{D}(u, w_0, w_1, w_2, w_3)\\
= \ & u_{-1}w_0 \otimes w_1 \otimes w_2 \otimes w_3\\
&-\sum_{k\geq 0} \binom{-1}{k} z_1^{k+1} w_0 \otimes e^{z_1^{-1}L(1)}(-z_1^2)^{L(0)} u_k (-z_1^{-2})^{L(0)} e^{-z_1^{-1}L(1)} w_1 \otimes w_2 \otimes w_3\\
&-\sum_{k\geq 0} \binom{-1}{k} z_2^{k+1} w_0 \otimes w_1 \otimes e^{z_2^{-1}L(1)}(-z_2^2)^{L(0)} u_k (-z_2^{-2})^{L(0)} e^{-z_2^{-1}L(1)} w_2 \otimes w_3\\
&-w_0 \otimes w_1 \otimes w_2 \otimes u^*_{-1} w_3.
\end{aligned}
$$

We shall also need a submodule $S^{(\beta_1,\beta_2,\beta_3)}_{\mathcal{Y}_1,\mathcal{Y}_2}$ of $\tilde{T}^{(\beta_1,\beta_2,\beta_3)}$ generated by elements of the form

$$w_0 \otimes w_1 \otimes w_2 \otimes w_3$$

for $w_i \in W_i^{(\widetilde{\beta}_i)} (i = 0, 1, 2, 3)$, $(\widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\beta}_3) \in \tilde{I}^{(\beta_1,\beta_2,\beta_3)} \setminus I^{(\beta_1,\beta_2,\beta_3)}$. For simplicity, we denote $S^{(\beta_1,\beta_2,\beta_3)}_{\mathcal{Y}_1,\mathcal{Y}_2}$ by $S^{(\beta_1,\beta_2,\beta_3)}$.

Lemma 3.4.1 *Let $\beta_i \in \tilde{A}$. Then*

$$\widetilde{T}^{(\beta_1,\beta_2,\beta_3)} = T^{(\beta_1,\beta_2,\beta_3)} \oplus S^{(\beta_1,\beta_2,\beta_3)}.$$

We shall find an R-submodule of $\tilde{T}^{(\beta_1,\beta_2,\beta_3)}$ such that its complement in $T^{(\beta_1,\beta_2,\beta_3)}$ is finitely generated. For this purpose, we use the following R-submodule of $\tilde{T}^{(\beta_1,\beta_2,\beta_3)}$:

$$\tilde{J}^{(\beta_1,\beta_2,\beta_3)} = J^{(\beta_1,\beta_2,\beta_3)} \oplus S^{(\beta_1,\beta_2,\beta_3)}.$$

For $r \in R$, we can define the R-submodules $T^{(\beta_1,\beta_2,\beta_3)}_{(r)}$, $F_r(T^{(\beta_1,\beta_2,\beta_3)})$ and $F_r(\tilde{J}^{(\beta_1,\beta_2,\beta_3)})$ as in [H3]. Note that $F_r(T^{(\beta_1,\beta_2,\beta_3)})$ is a finitely generated R-module since $I^{(\beta_1,\beta_2,\beta_3)}$ is a finite set by Lemma 3.3.4.

Proposition 3.4.2 *Let W_i be strongly $\tilde{A}$-graded generalized V-modules and let $\beta_i \in \tilde{A}$ for $i = 0, 1, 2, 3$. Then there exists $M \in \mathbb{Z}$ such that for any $r \in \mathbb{R}$, $F_r(T^{(\beta_1,\beta_2,\beta_3)}) \subset F_r(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) + F_M(T^{(\beta_1,\beta_2,\beta_3)})$. In particular, $T^{(\beta_1,\beta_2,\beta_3)} \subset \tilde{J}^{(\beta_1,\beta_2,\beta_3)} + F_M(T^{(\beta_1,\beta_2,\beta_3)})$.*

Proof. For $\widetilde{\beta}_i \in \tilde{A}$, let $\widetilde{\beta}_0$ denote $\widetilde{\beta}_1+\widetilde{\beta}_2+\widetilde{\beta}_3$ and let $(C_1(W_i))^{(\widetilde{\beta}_i)}$ be the subspace of W_i spanned by elements of the form $u_{-1}w_i \in W_i^{(\widetilde{\beta}_i)}$, where

$$u \in (V_0)_+ = \coprod_{n>0} (V_0)_{(n)}.$$

Since dim $W_i^{(\widetilde{\beta}_i)}/(C_1(W_i))^{(\widetilde{\beta}_i)} < \infty$ for $i = 0, 1, 2, 3$, there exists $M \in \mathbb{Z}$ such that

$$\begin{aligned} \coprod_{n>M} T_{(n)}^{(\beta_1,\beta_2,\beta_3)} \subset \coprod_{(\widetilde{\beta_1},\widetilde{\beta_2},\widetilde{\beta_3}) \in I^{(\beta_1,\beta_2,\beta_3)}} & R((C_1(W_0))^{(\widetilde{\beta_0})} \otimes W_1^{(\widetilde{\beta_1})} \otimes W_2^{(\widetilde{\beta_2})} \otimes W_3^{(\widetilde{\beta_3})}) \\ + \quad & R(W_0^{(\widetilde{\beta_0})} \otimes (C_1(W_1))^{(\widetilde{\beta_1})} \otimes W_2^{(\widetilde{\beta_2})} \otimes W_3^{(\widetilde{\beta_3})}) \\ + \quad & R(W_0^{(\widetilde{\beta_0})} \otimes W_1^{(\widetilde{\beta_1})} \otimes (C_1(W_2))^{(\widetilde{\beta_2})} \otimes W_3^{(\widetilde{\beta_3})}) \\ + \quad & R(W_0^{(\widetilde{\beta_0})} \otimes W_1^{(\widetilde{\beta_1})} \otimes W_2^{(\widetilde{\beta_2})} \otimes (C_1(W_3))^{(\widetilde{\beta_3})}). \end{aligned} \tag{3.7}$$

We use induction on $r \in \mathbb{R}$. If r is equal to M, $F_M(T^{(\beta_1,\beta_2,\beta_3)}) \subset F_M(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) + F_M(T^{(\beta_1,\beta_2,\beta_3)})$. Now we assume that $F_r(T^{(\beta_1,\beta_2,\beta_3)}) \subset F_r(\tilde{J}^{(\beta_1,\beta_2,\beta_3)})+F_M(T^{(\beta_1,\beta_2,\beta_3)})$ for $r < s$ where $s > M$. We want to show that any homogeneous element of $T_{(s)}^{(\beta_1,\beta_2,\beta_3)}$ can be written as a sum of an element of $F_s(\tilde{J}^{(\beta_1,\beta_2,\beta_3)})$ and an element of $F_M(T^{(\beta_1,\beta_2,\beta_3)})$. Since $s > M$, by (3.7), any element of $T_{(s)}^{(\beta_1,\beta_2,\beta_3)}$ is an element of the right hand side of (3.7). We shall discuss only the case that this element is in $R(W_0^{(\widetilde{\beta_0})} \otimes (C_1(W_1))^{(\widetilde{\beta_1})} \otimes W_2^{(\widetilde{\beta_2})} \otimes W_3^{(\widetilde{\beta_3})})$; the other cases are completely similar.

We need only discuss elements of the form $w_0 \otimes u_{-1}w_1 \otimes w_2 \otimes w_3$, where $w_i \in W_i^{(\widetilde{\beta}_i)}$ for $i = 0, 2, 3$, $u_{-1}w_1 \in (C_1(W_1))^{(\widetilde{\beta_1})}$ and $u \in (V_0)_+$. We see from Lemma 3.4.1 that the elements $u^*_{-1-k}w_0 \otimes w_1 \otimes w_2 \otimes w_3$, $w_0 \otimes w_1 \otimes u_k w_2 \otimes w_3$ and $w_0 \otimes w_1 \otimes w_2 \otimes u_k w_3$ for $k \geq 0$ are either in $S^{(\beta_1,\beta_2,\beta_3)}$ or in $T^{(\beta_1,\beta_2,\beta_3)}$. By assumption, the weight of $w_0 \otimes u_{-1}w_1 \otimes w_2 \otimes w_3$ is s, then the weight of $u^*_{-1-k}w_0 \otimes w_1 \otimes w_2 \otimes w_3$, $w_0 \otimes w_1 \otimes u_k w_2 \otimes w_3$ and $w_0 \otimes w_1 \otimes w_2 \otimes u_k w_3$ for $k \geq 0$, are all less than s. Thus these elements either lie in $F_s(\tilde{J}^{(\beta_1,\beta_2,\beta_3)})$ or in

$F_{s-1}(T^{(\beta_1,\beta_2,\beta_3)})$. Also, since $\mathcal{A}(u, w_0, w_1, w_2, w_3) \in F_s(\tilde{J}^{(\beta_1,\beta_2,\beta_3)})$, we see that

$$\begin{aligned}
&w_0 \otimes u_{-1}w_1 \otimes w_2 \otimes w_3\\
&\quad= \mathcal{A}(u, w_0, w_1, w_2, w_3) + \sum_{k\geq 0} \binom{-1}{k} (-z_1)^k u^*_{-1-k} w_0 \otimes w_1 \otimes w_2 \otimes w_3\\
&\qquad - \sum_{k\geq 0} \binom{-1}{k} (-(z_1 - z_2))^{-1-k} w_0 \otimes w_1 \otimes u_k w_2 \otimes w_3\\
&\qquad - \sum_{k\geq 0} \binom{-1}{k} (-z_1)^{-1-k} w_0 \otimes w_1 \otimes w_2 \otimes u_k w_3
\end{aligned}$$

can be written as a sum of an element of $F_s(\tilde{J}^{(\beta_1,\beta_2,\beta_3)})$ and elements of $F_{s-1}(T^{(\beta_1,\beta_2,\beta_3)})$. Thus by the induction assumption, the element $w_0 \otimes u_{-1}w_1 \otimes w_2 \otimes w_3$ can be written as a sum of an element of $F_s(\tilde{J}^{(\beta_1,\beta_2,\beta_3)})$ and an element of $F_M(T^{(\beta_1,\beta_2,\beta_3)})$.

Now we have

$$\begin{aligned}
T^{(\beta_1,\beta_2,\beta_3)} &= \coprod_{r\in\mathbb{R}} F_r(T^{(\beta_1,\beta_2,\beta_3)})\\
&\subset \coprod_{r\in\mathbb{R}} F_r(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) + F_M(T^{(\beta_1,\beta_2,\beta_3)})\\
&= \tilde{J}^{(\beta_1,\beta_2,\beta_3)} + F_M(T^{(\beta_1,\beta_2,\beta_3)}). \qquad \square
\end{aligned}$$

We immediately obtain the following:

Corollary 3.4.3 *The quotient R-module $T^{(\beta_1,\beta_2,\beta_3)}/(T^{(\beta_1,\beta_2,\beta_3)} \cap \tilde{J}^{(\beta_1,\beta_2,\beta_3)})$ is finitely generated.*

Proof. We have the following R-module isomorphism:

$$T^{(\beta_1,\beta_2,\beta_3)}/(T^{(\beta_1,\beta_2,\beta_3)} \cap \tilde{J}^{(\beta_1,\beta_2,\beta_3)}) \simeq (T^{(\beta_1,\beta_2,\beta_3)} + \tilde{J}^{(\beta_1,\beta_2,\beta_3)})/\tilde{J}^{(\beta_1,\beta_2,\beta_3)}.$$

By the previous Proposition, the R-module $(T^{(\beta_1,\beta_2,\beta_3)} + \tilde{J}^{(\beta_1,\beta_2,\beta_3)})/\tilde{J}^{(\beta_1,\beta_2,\beta_3)}$ is a submodule of

$$\begin{aligned}
&(\tilde{J}^{(\beta_1,\beta_2,\beta_3)} + F_M(T^{(\beta_1,\beta_2,\beta_3)}))/\tilde{J}^{(\beta_1,\beta_2,\beta_3)}\\
&\simeq F_M(T^{(\beta_1,\beta_2,\beta_3)})/(F_M(T^{(\beta_1,\beta_2,\beta_3)}) \cap \tilde{J}^{(\beta_1,\beta_2,\beta_3)}),
\end{aligned}$$

which is finitely generated. $\square$

For an element $\mathcal{W} \in T^{(\beta_1,\beta_2,\beta_3)}$, we shall use $[\mathcal{W}]$ to denote the equivalence class in
$T^{(\beta_1,\beta_2,\beta_3)}/T^{(\beta_1,\beta_2,\beta_3)} \cap \tilde{J}^{(\beta_1,\beta_2,\beta_3)}$ containing $\mathcal{W}$. We also have:

Corollary 3.4.4 *Let W_i be strongly $\tilde{A}$-graded generalized V-modules for $i = 0, 1, 2, 3$. For any $\tilde{A}$-homogeneous elements $w_i \in W_i$ $(i = 0, 1, 2, 3)$, let M_1 and M_2 be the R-submodules of*
$T^{(\beta_1,\beta_2,\beta_3)}/T^{(\beta_1,\beta_2,\beta_3)} \cap \tilde{J}^{(\beta_1,\beta_2,\beta_3)}$ generated by $[w_0 \otimes L(-1)^j w_1 \otimes w_2 \otimes w_3]$, $j \geq 0$, and by
$[w_0 \otimes w_1 \otimes L(-1)^j w_2 \otimes w_3]$, $j \geq 0$, respectively. Then M_1, M_2 are finitely generated. In particular, for any $\tilde{A}$-homogeneous elements $w_i \in W_i$ $(i = 0, 1, 2, 3)$, there exist $a_k(z_1, z_2)$, $b_l(z_1, z_2) \in R$ for $k = 1, \dots, m$ and $l = 1, \dots, n$ such that

$$\begin{aligned}[] [w_0 \otimes L(-1)^m w_1 \otimes w_2 \otimes w_3] + a_1(z_1, z_2)[w_0 \otimes L(-1)^{m-1} w_1 \otimes w_2 \otimes w_3] \\ + \cdots + a_m(z_1, z_2)[w_0 \otimes w_1 \otimes w_2 \otimes w_3] = 0, \end{aligned} \tag{3.8}$$

$$\begin{aligned}[] [w_0 \otimes w_1 \otimes L(-1)^n w_2 \otimes w_3] + b_1(z_1, z_2)[w_0 \otimes w_1 \otimes L(-1)^{n-1} w_2 \otimes w_3] \\ + \cdots + b_n(z_1, z_2)[w_0 \otimes w_1 \otimes w_2 \otimes w_3] = 0. \end{aligned} \tag{3.9}$$

Now we establish the existence of systems of differential equations:

Theorem 3.4.5 *Suppose that every strongly $\tilde{A}$-graded V-module satisfies C_1-cofiniteness condition with respect to $\tilde{A}$ as a V_0-module and suppose that for any two fixed elements β_1 and β_2 in $\tilde{A}$ and any triple of strongly graded generalized V-modules M_1, M_2 and M_3, the fusion rule*

$$N^{M_3^{(\widetilde{\beta_1}+\widetilde{\beta_2})}}_{M_1^{(\widetilde{\beta_1})} M_2^{(\widetilde{\beta_2})}} \neq 0$$

for only finitely many pairs $(\widetilde{\beta_1}, \widetilde{\beta_2}) \in (\beta_1 + A_0) \times (\beta_2 + A_0)$. Let W_i be strongly $\tilde{A}$-graded generalized V-modules for $i = 0, 1, 2, 3, 4$ and let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be logarithmic *intertwining operators of type* $\binom{W_0'}{W_1\,W_4}$, $\binom{W_4}{W_2\,W_3}$. *Then for any* $\tilde{A}$*-homogeneous elements* $w_i \in W_i$ $(i = 0, 1, 2, 3)$*, there exist*

$$a_k(z_1, z_2), b_l(z_1, z_2) \in \mathbb{C}[z_1^{\pm}, z_2^{\pm}, (z_1 - z_2)^{-1}]$$

for $k = 1, \ldots, m$ *and* $l = 1, \ldots, n$ *such that the series*

$$\langle w_0, \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle, \tag{3.10}$$

satisfying the expansions of the system of differential equations

$$\frac{\partial^m \varphi}{\partial z_1^m} + a_1(z_1, z_2)\frac{\partial^{m-1}\varphi}{\partial z_1^{m-1}} + \cdots + a_m(z_1, z_2)\varphi = 0, \tag{3.11}$$

$$\frac{\partial^n \varphi}{\partial z_2^n} + b_1(z_1, z_2)\frac{\partial^{n-1}\varphi}{\partial z_2^{n-1}} + \cdots + b_n(z_1, z_2)\varphi = 0 \tag{3.12}$$

in the region $|z_1| > |z_2| > 0$.

Proof. The proof is similar to the proof of Theorem 1.4 in [H3] except for the difference in the R-module $\tilde{J}^{(\beta_1,\beta_2,\beta_3)}$. We sketch the proof as follows:

Let $\Delta = \mathrm{wt}\ w_0 - \mathrm{wt}\ w_1 - \mathrm{wt}\ w_2 - \mathrm{wt}\ w_3$. For $(\widetilde{\beta_1}, \widetilde{\beta_2}, \widetilde{\beta_3}) \in I^{(\beta_1,\beta_2,\beta_3)}$, let $\widetilde{\beta_0} = \widetilde{\beta_1} + \widetilde{\beta_2} + \widetilde{\beta_3}$. Let $\mathbb{C}(\{x\})$ be the space of all series of the form $\sum_{n\in\mathbb{R}} a_n x^n$ for $n \in \mathbb{R}$ such that $a_n = 0$ when the real part of n is sufficiently negative.

Consider the map

$$\phi_{\mathcal{Y}_1,\mathcal{Y}_2} : T^{(\beta_1,\beta_2,\beta_3)} \longrightarrow z_1^{\Delta}\mathbb{C}(\{z_2/z_1\})[z_1^{\pm 1}, z_2^{\pm 1}]$$

defined by

$$\begin{aligned} &\phi_{\mathcal{Y}_1,\mathcal{Y}_2}(f(z_1, z_2)w_0 \otimes w_1 \otimes w_2 \otimes w_3) \\ &= \iota_{|z_1|>|z_2|>0}(f(z_1, z_2))\langle w_0, \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle, \end{aligned}$$

where

$$\iota_{|z_1|>|z_2|>0} : R \quad\longrightarrow\quad \mathbb{C}[[z_2/z_1]][z_1^{\pm 1}, z_2^{\pm 1}]$$

is the map expanding elements of R as series in the regions $|z_1| > |z_2| > 0$.

Using the Jacobi identity for the logarithmic intertwining operators, we have that elements of $J^{(\beta_1,\beta_2,\beta_3)}$ are in the kernel of $\phi_{\mathcal{Y}_1,\mathcal{Y}_2}$. The elements of $S^{(\beta_1,\beta_2,\beta_3)}$ are in the kernel by the construction of the set $I^{(\beta_1,\beta_2,\beta_3)}$. From Lemma 3.4.1, we have

$$\phi_{\mathcal{Y}_1,\mathcal{Y}_2}(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) = 0.$$

Thus the map $\phi_{\mathcal{Y}_1,\mathcal{Y}_2}$ induces a map

$$\bar{\phi}_{\mathcal{Y}_1,\mathcal{Y}_2} : T^{(\beta_1,\beta_2,\beta_3)}/T^{(\beta_1,\beta_2,\beta_3)} \cap \tilde{J}^{(\beta_1,\beta_2,\beta_3)} \longrightarrow z_1^{\Delta}\mathbb{C}(\{z_2/z_1\})[z_1^{\pm 1}, z_2^{\pm 1}].$$

Applying $\bar{\phi}_{\mathcal{Y}_1,\mathcal{Y}_2}$ to (3.8) and (3.9) and then use the $L(-1)$-derivative property for logarithmic intertwining operators, we see that (3.10) indeed satisfies the expansions of the system of differential equations in the regions $|z_1| > |z_2| > 0$. $\square$

Remark 3.4.6 Note that in the theorems above, $a_k(z_1; z_2)$ for $k = 1, \ldots, m-1$ and $b_l(z_1; z_2)$ for $l = 1, \ldots, l-1$, and consequently the corresponding system, depend on the logarithmic intertwining operators $\mathcal{Y}_1$, $\mathcal{Y}_2$.

The following result can be proved by the same method, so we omit the proof.

Theorem 3.4.7 *Suppose that every strongly $\tilde{A}$-graded V-module satisfies C_1-cofiniteness condition with respect to $\tilde{A}$ as a V_0-module and suppose that for any two fixed elements β_1 and β_2 in $\tilde{A}$ and any triple of strongly graded generalized V-modules M_1, M_2 and M_3, the fusion rules*

$$N^{M_3^{(\widetilde{\beta_1}+\widetilde{\beta_2})}}_{M_1^{(\widetilde{\beta_1})} M_2^{(\widetilde{\beta_2})}} \neq 0$$

for only finitely many pairs $(\widetilde{\beta_1}, \widetilde{\beta_2}) \in (\beta_1 + A_0) \times (\beta_2 + A_0)$. Let W_i be strongly $\tilde{A}$-graded generalized V-modules for $i = 0, \ldots, n+1$. For any generalized V-modules $\widetilde{W_1}, \ldots, \widetilde{W_{n-1}}$, let

$$\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_{n-1}, \mathcal{Y}_n$$

be logarithmic intertwining operators of types

$$\binom{W_0}{W_1\,\widetilde{W_1}}, \binom{\widetilde{W_1}}{W_2\,\widetilde{W_2}}, \dots, \binom{\widetilde{W_{n-2}}}{W_{n-1}\,\widetilde{W_{n-1}}}, \binom{\widetilde{W_{n-1}}}{W_n\,W_{n+1}},$$

respectively. Then for any $\tilde{A}$-homogeneous elements $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1, \dots,$ $w_{(n+1)} \in W_{n+1}$, there exist

$$a_{k,l}(z_1, \dots, z_n) \in \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}, (z_1 - z_2)^{-1}, (z_1 - z_3)^{-1}, \dots, (z_{n-1} - z_n)^{-1}]$$

for $k = 1, \dots, m$ and $l = 1, \dots, n$ such that the series

$$\langle w'_{(0)}, \mathcal{Y}_1(w_{(1)}, z_1) \cdots \mathcal{Y}_n(w_{(n)}, z_n) w_{(n+1)} \rangle$$

satisfies the system of differential equations

$$\frac{\partial^m \varphi}{\partial z_l^m} + \sum_{k=1}^{m} a_{k,l}(z_1, \dots, z_n) \frac{\partial^{m-k} \varphi}{\partial z_l^{m-k}} = 0, \quad l = 1, \dots, n \tag{3.13}$$

in the region $|z_1| > \cdots > |z_n| > 0$.

Remark 3.4.8 Under the same condition as in the Theorem 3.4.5, it follows from the same argument in this section that matrix elements of iterates of logarithmic intertwining operators

$$\langle w'_{(0)}, \mathcal{Y}_1(\mathcal{Y}_2(w_1, z_1 - z_2), z_2) w_2 \rangle \tag{3.14}$$

also satisfy the expansions of the system of differential equations of the form (3.11) and (3.12) in the region $|z_2| > |z_1 - z_2| > 0$.

Example 3.4.9 Let V_L be the conformal vertex algebra associated with a nondegenerate even lattice L. Then any strongly M-graded generalized V_L-module W (in this example, all the generalized modules are modules) satisfies the assumption in Theorem 3.4.5 and the series (3.10), (3.14) satisfies the expansions of the system of differential equations (3.11) and (3.12) in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively.

3.5 The regularity of the singular points

We first recall the definition for *regular singular points* for a system of differential equations given in [K]. For the system of differential equations of form (3.13), *a singular point*

$$z_0 = (z_0^{(1)}, \dots, z_0^{(n)})$$

is an isolated singular point of the coefficient matrix

$$a_{k,l}(z_1, \dots, z_n) \in \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}, (z_1 - z_2)^{-1}, (z_1 - z_3)^{-1}, \dots, (z_{n-1} - z_n)^{-1}]$$

for $k = 1, \dots, m$ and $l = 1, \dots, n$. For $s = (s_1, \dots, s_n) \in \mathbb{Z}_+^n$, set

$$|s| = \sum_{i=0}^{n} s_i$$

and

$$(\log(z - z_0))^s = (\log(z_1 - z_0^{(1)}))^{s_1} \cdots (\log(z_n - z_0^{(n)}))^{s_n}.$$

For $t = (t^{(1)}, \dots, t^{(n)}) \in \mathbb{C}^n$, set

$$(z - z_0)^t = (z_1 - z_0^{(1)})^{t^{(1)}} \cdots (z_n - z_0^{(n)})^{t^{(n)}}.$$

A singular point z_0 for the system of differential equations of form (3.13) is *regular* if every solution in the punctured disc $(D^\times)^n$

$$0 < |z_i - z_0^{(i)}| < a_i$$

with some $a_i \in \mathbb{R}_+$ $(i = 1, \dots, n)$ is of the form

$$\varphi(z) = \sum_{i=1}^{r} \sum_{|m|<M} (z - z_0)^{t_i} (\log(z - z_0))^m f_{t_i,m}(z - z_0)$$

with $M, r \in \mathbb{Z}_+$ and each $f_{t_i,m}(z - z_0)$ holomorphic in $(D^\times)^n$. Theorem B.16 in [K] gives a sufficient condition for a singular point of a system of differential equations to be regular.

As in [H3], for $r \in \mathbb{R}$, we define the R-modules $F_r^{(z_1=z_2)}(R)$, $F_r^{(z_1=z_2)}(T^{(\beta_1,\beta_2,\beta_3)})$ and $F_r^{(z_1=z_2)}(\widetilde{T}^{(\beta_1,\beta_2,\beta_3)})$, which provide filtration associated to the singular point $z_1 = z_2$ on R, R-modules $T^{(\beta_1,\beta_2,\beta_3)}$ and $\widetilde{T}^{(\beta_1,\beta_2,\beta_3)}$, respectively.

For convenience, we shall use $\widetilde{\beta_0}$ to denote $\widetilde{\beta_1}+\widetilde{\beta_2}+\widetilde{\beta_3}$ for $\widetilde{\beta_i}\in\beta_i+A_0$ $(i=1,2,3)$. We shall also consider the ring $\mathbb{C}[z_1^{\pm},z_2^{\pm}]$ and the $\mathbb{C}[z_1^{\pm},z_2^{\pm}]$-module

$$(T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)}=\coprod_{(\widetilde{\beta_1},\widetilde{\beta_2},\widetilde{\beta_3})\in I^{(\beta_1,\beta_2,\beta_3)}}\mathbb{C}[z_1^{\pm},z_2^{\pm}]\otimes W_0^{(\widetilde{\beta_0})}\otimes W_1^{(\widetilde{\beta_1})}\otimes W_2^{(\widetilde{\beta_2})}\otimes W_3^{(\widetilde{\beta_3})}.$$

Let $(T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)}_{(r)}$ be the space of elements of $(T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)}$ of weight r for $r\in\mathbb{R}$. Let $F_r((T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)})=\coprod_{s\leq r}(T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)}_{(s)}$. These subspaces give a filtration of $(T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)}$ in the following sense: $F_r((T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)})\subset F_s((T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)})$ for $r\leq s$ and $(T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)}=\coprod_{r\in\mathbb{R}}F_r((T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)})$.

Let $F_r^{(z_1=z_2)}(\tilde{J}^{(\beta_1,\beta_2,\beta_3)})=F_r^{(z_1=z_2)}(\widetilde{T}^{(\beta_1,\beta_2,\beta_3)})\cap\tilde{J}^{(\beta_1,\beta_2,\beta_3)}$ for $r\in\mathbb{R}$. We have the following lemma:

Lemma 3.5.1 *For any $r\in\mathbb{R}$, $F_r((T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)})\subset F_r^{(z_1=z_2)}(\tilde{J}^{(\beta_1,\beta_2,\beta_3)})+F_M(T^{(\beta_1,\beta_2,\beta_3)})$.*

Proof. The proof is similar to the proof of Proposition 3.4.2 except for some slight differences. We discuss elements of the form $w_0\otimes u_{-1}w_1\otimes w_2\otimes w_3$ with weight s, where $w_i\in W_i^{(\widetilde{\beta_i})}$ for $i=0,1,2,3$ and $u\in(V_0)_+$. By definition of the element $\mathcal{A}(u,w_0,w_1,w_2,w_3)$ in the R-submodule $\tilde{J}^{(\beta_1,\beta_2,\beta_3)}$, we have

$$\begin{aligned}
&w_0\otimes u_{-1}w_1\otimes w_2\otimes w_3\\
&\quad=\sum_{k\geq 0}\binom{-1}{k}(-z_1)^k u^*_{-1-k}w_0\otimes w_1\otimes w_2\otimes w_3-\mathcal{A}(u,w_0,w_1,w_2,w_3)\\
&\qquad-\sum_{k\geq 0}\binom{-1}{k}(-(z_1-z_2))^{-1-k}w_0\otimes w_1\otimes u_kw_2\otimes w_3\\
&\qquad-\sum_{k\geq 0}\binom{-1}{k}(-z_1)^{-1-k}w_0\otimes w_1\otimes w_2\otimes u_kw_3.
\end{aligned}$$

We know from Lemma 3.4.1 that the elements $u^*_{-1-k}w_0\otimes w_1\otimes w_2\otimes w_3$, $w_0\otimes w_1\otimes u_kw_2\otimes w_3$ and $w_0\otimes w_1\otimes w_2\otimes u_kw_3$ for $k\geq 0$ are either in $S^{(\beta_1,\beta_2,\beta_3)}\subset\tilde{J}^{(\beta_1,\beta_2,\beta_3)}$ or in $T^{(\beta_1,\beta_2,\beta_3)}$ with weights less than the weight of $w_0\otimes u_{-1}w_1\otimes w_2\otimes w_3$.

In the first case, since elements of the form $w_0 \otimes w_1 \otimes u_k w_2 \otimes w_3$ are in $F_{s-k-1}^{(z_1=z_2)}(\tilde{J})$, $(-(z_1-z_2))^{-1-k} w_0 \otimes w_1 \otimes u_k w_2 \otimes w_3 \in F_s^{(z_1=z_2)}(\tilde{J})$. Thus in this case, $w_0 \otimes u_{-1} w_1 \otimes w_2 \otimes w_3$ is an element of $F_s^{(z_1=z_2)}(\tilde{J}^{(\beta_1,\beta_2,\beta_3)})$.

In the second case, by induction assumption, $u^*_{-1-k} w_0 \otimes w_1 \otimes w_2 \otimes w_3$, $w_0 \otimes w_1 \otimes w_2 \otimes u_k w_3 \in F_s^{(z_1=z_2)}(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) + F_M(T^{(\beta_1,\beta_2,\beta_3)})$ and $w_0 \otimes w_1 \otimes u_k w_2 \otimes w_3 \in F_{s-k-1}^{(z_1=z_2)}(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) + F_M(T^{(\beta_1,\beta_2,\beta_3)})$. Hence the element $(-(z_1-z_2))^{-1-k} w_0 \otimes w_1 \otimes u_k w_2 \otimes w_3 \in F_s^{(z_1=z_2)}(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) + F_M(T^{(\beta_1,\beta_2,\beta_3)})$. Thus in this case, $w_0 \otimes u_{-1} w_1 \otimes w_2 \otimes w_3$ can be written as a sum of an element of $F_s^{(z_1=z_2)}(\tilde{J}^{(\beta_1,\beta_2,\beta_3)})$ and an element of $F_M(T^{(\beta_1,\beta_2,\beta_3)})$. $\square$

Using Lemma 3.5.1, we get the following refinement of proposition 3.4.2:

Proposition 3.5.2 *For any $r \in \mathbb{R}$,*

$$F_r^{(z_1=z_2)}(T^{(\beta_1,\beta_2,\beta_3)}) \subset F_r^{(z_1=z_2)}(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) + F_M(T^{(\beta_1,\beta_2,\beta_3)}).$$

In particular,

$$F_r^{(z_1=z_2)}(T^{(\beta_1,\beta_2,\beta_3)}) = F_r^{(z_1=z_2)}(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) \cap T^{(\beta_1,\beta_2,\beta_3)} + F_M(T^{(\beta_1,\beta_2,\beta_3)}).$$

Proof. It is a consequence of the decomposition:

$$F_r^{(z_1=z_2)}(T^{(\beta_1,\beta_2,\beta_3)}) = \coprod_{i=0}^{r} (z_1-z_2)^{-i} F_{r-i}((T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)})$$

and Lemma 3.5.1. $\square$

Let $w_i \in W_i^{(\widetilde{\beta}_i)}$ for $i = 0, 1, 2, 3$ and $(\widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\beta}_3) \in I^{(\beta_1,\beta_2,\beta_3)}$. Then by Proposition 3.5.2,

$$w_0 \otimes w_1 \otimes w_2 \otimes w_3 = \mathcal{W}_1 + \mathcal{W}_2$$

where $\mathcal{W}_1 \in F_\sigma^{(z_1=z_2)}(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) \cap T^{(\beta_1,\beta_2,\beta_3)} = F_\sigma^{(z_1=z_2)}(T^{(\beta_1,\beta_2,\beta_3)}) \cap \tilde{J}^{(\beta_1,\beta_2,\beta_3)}$ and $\mathcal{W}_2 \in F_M(T^{(\beta_1,\beta_2,\beta_3)})$. Using the same proof as Lemma 2.2 in [H3], we have the following lemma:

Lemma 3.5.3 *For any $s \in [0,1)$, there exist $S \in \mathbb{R}$ such that $s + S \in \mathbb{Z}_+$ and for any $w_i \in W_i$, $i = 0,1,2,3$, satisfying $\sigma \in s + \mathbb{Z}$, $(z_1 - z_2)^{\sigma+S}\mathcal{W}_2 \in (T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)}$.*

Theorem 3.5.4 *Suppose that every strongly $\tilde{A}$-graded V-module satisfies C_1-cofiniteness condition with respect to $\tilde{A}$ as a V_0-module and suppose that for any two fixed elements β_1 and β_2 in $\tilde{A}$ and any triple of strongly graded generalized V-modules M_1, M_2 and M_3, the fusion rule*

$$N^{M_3^{(\widetilde{\beta_1}+\widetilde{\beta_2})}}_{M_1^{(\widetilde{\beta_1})}M_2^{(\widetilde{\beta_2})}} \neq 0$$

for only finitely many pairs $(\widetilde{\beta_1}, \widetilde{\beta_2}) \in (\beta_1 + A_0) \times (\beta_2 + A_0)$. Let W_i, $w_i \in W_i$ for $i = 0,1,2,3,4$, $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be the same as in Theorem 3.4.5. For any possible singular point of the form $(z_1 = 0, z_2 = 0, z_1 = \infty, z_2 = \infty, z_1 = z_2)$, $z_1^{-1}(z_1 - z_2) = 0$, or $z_2^{-1}(z_1 - z_2) = 0$, there exist

$$a_k(z_1, z_2), b_l(z_1, z_2) \in \mathbb{C}[z_1^{\pm}, z_2^{\pm}, (z_1 - z_2)^{-1}]$$

for $k = 1, \ldots, m$ and $l = 1, \ldots, n$, such that this singular point of the system (3.11) and (3.12) satisfied by (3.10) is regular.

Proof. The proof is the same as the proof of Theorem 2.3 in [H3] except that we use Proposition 3.5.2 and Lemma 3.5.3 here. □

We can prove the following theorem using the same method, so we omit the proof here.

Theorem 3.5.5 *For any set of possible singular points of the system (3.13) in Theorem 3.4.7 of the form $z_i = 0$ or $z_i = \infty$ for some i or $z_i = z_j$ for some $i \neq j$, the $a_{k,l}(z_1, \ldots, z_n)$ in Theorem 3.4.7 can be chosen for $k = 1, \ldots, m$ and $l = 1, \ldots, n$ so that these singular points are regular.*

3.6 Braided tensor category structure

In the logarithmic tensor category theory developed in [HLZ1] and [HLZ2], the convergence and expansion property for the logarithmic intertwining operators are needed in the construction of the associativity isomorphism. In this section, we will recall the definition of convergence and expansion property for products and iterates of logarithmic intertwining operators and then follow [HLZ2] to give sufficient conditions for a category to have these properties.

Throughout this section, we will let $\mathcal{M}_{sg}$ (respectively, $\mathcal{GM}_{sg}$) denote the category of the strongly $\tilde{A}$-graded (respectively, generalized) V-modules. We are going to study the subcategory $\mathcal{C}$ of $\mathcal{M}_{sg}$ (respectively, $\mathcal{GM}_{sg}$) satisfying the following assumptions.

Assumption 3.6.1 We shall assume the following:

- A_0, A and $\tilde{A}$ are abelian groups satisfying $A_0 \leq A \leq \tilde{A}$.

- V is a strongly A-graded conformal vertex algebra with a strongly A_0-graded vertex subalgebra V_0 and V is an object of $\mathcal{C}$ as a V-module.

- All (generalized) V-modules are lower bounded, satisfy the C_1-cofiniteness condition with respect to $\tilde{A}$ as V_0-modules and for any two fixed elements β_1 and β_2 in $\tilde{A}$ and any triple of strongly graded generalized V-modules M_1, M_2 and M_3, the fusion rule
$$N^{M_3^{(\widetilde{\beta_1}+\widetilde{\beta_2})}}_{M_1^{(\widetilde{\beta_1})} M_2^{(\widetilde{\beta_2})}} \neq 0$$
for only finitely many pairs $(\widetilde{\beta_1}, \widetilde{\beta_2}) \in (\beta_1 + A_0) \times (\beta_2 + A_0)$.

- For any object of $\mathcal{C}$, the (generalized) weights are real numbers and in addition there exist $K \in \mathbb{Z}$ such that $(L(0) - L(0)_s)^K = 0$ on the generalized module.

- $\mathcal{C}$ is closed under images, under the contragredient functor, under taking finite direct sums.

Given objects W_1, W_2, W_3, W_4, M_1 and M_2 of the category $\mathcal{C}$, let $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}^1$ and $\mathcal{Y}^2$ be logarithmic intertwining operators of types $\binom{W_4}{W_1\, M_1}$, $\binom{M_1}{W_2\, W_3}$, $\binom{W_4}{M_2\, W_3}$ and $\binom{M_2}{W_1\, W_2}$, respectively. We recall the following definitions and theorems from Section 11 in [HLZ2] (part VII):

Convergence and extension property for products For any $\beta \in \tilde{A}$, there exists an integer N_β depending only on $\mathcal{Y}_1$ and $\mathcal{Y}_2$ and β, and for any doubly homogeneous elements $w_{(1)} \in (W_1)^{(\beta_1)}$ and $w_{(2)} \in (W_2)^{(\beta_2)}$ $(\beta_1, \beta_2 \in \tilde{A})$ and any $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$ such that

$$\beta_1 + \beta_2 = -\beta,$$

there exist $M \in \mathbb{N}$, $r_k, s_k \in \mathbb{R}$, $i_k, j_k \in \mathbb{N}$, $k = 1, \ldots, M$, and analytic functions $f_k(z)$ on $|z| < 1$, $k = 1, \ldots, M$, satisfying

$$\mathrm{wt}\ w_{(1)} + \mathrm{wt}\ w_{(2)} + s_k > N_\beta,\ \ k = 1, \ldots, M,$$

such that

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle_{W_4}\big|_{x_1=z_1,\ x_2=z_2}$$

is absolutely convergent when $|z_1| > |z_2| > 0$ and can be analytically extended to the multivalued analytic function

$$\sum_{k=1}^{M} z_2^{r_k}(z_1 - z_2)^{s_k}(\log z_2)^{i_k}(\log(z_1 - z_2))^{j_k} f_k\Big(\frac{z_1 - z_2}{z_2}\Big)$$

(here $\log(z_1 - z_2)$ and $\log z_2$, and in particular, the powers of the variables, mean the multivalued functions, not the particular branch we have been using) in the region $|z_2| > |z_1 - z_2| > 0$.

Convergence and extension property without logarithms for products When $i_k = j_k = 0$ for $k = 1, \ldots, M$, we call the property above the *convergence and extension property without logarithms for products.*

Convergence and extension property for iterates For any $\beta \in \tilde{A}$, there exists an integer $\tilde{N}_\beta$ depending only on $\mathcal{Y}^1$ and $\mathcal{Y}^2$ and β, and for any doubly homogeneous elements $w_{(1)} \in (W_1)^{(\beta_1)}$ and $w_{(2)} \in (W_2)^{(\beta_2)}$ $(\beta_1, \beta_2 \in \tilde{A})$ and any $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$ such that

$$\beta_1 + \beta_2 = -\beta,$$

there exist $\tilde{M} \in \mathbb{N}$, $\tilde{r_k}, \tilde{s_k} \in \mathbb{R}$, $\tilde{i_k}, \tilde{j_k} \in \mathbb{N}$, $k = 1, \ldots, \tilde{M}$, and analytic functions $\tilde{f_k}(z)$ on $|z| < 1$, $k = 1, \ldots, M$, satisfying

$$\mathrm{wt}\ w_{(1)} + \mathrm{wt}\ w_{(2)} + \tilde{s_k} > \tilde{N}_\beta,\ k = 1, \ldots, \tilde{M},$$

such that

$$\langle w'_{(0)}, \mathcal{Y}_1(\mathcal{Y}_2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle_{W_4}|_{x_0 = z_1 - z_2,\ x_2 = z_2}$$

is absolutely convergent when $|z_2| > |z_1 - z_2| > 0$ and can be analytically extended to the multivalued analytic function

$$\sum_{k=1}^{\tilde{M}} z_1^{\tilde{r_k}} z_2^{\tilde{s_k}} (\log z_1)^{\tilde{i_k}} (\log z_2)^{\tilde{j_k}} \tilde{f_k}(\frac{z_2}{z_1})$$

(here $\log z_1$ and $\log z_2$, and in particular, the powers of the variables, mean the multivalued functions, not the particular branch we have been using) in the region $|z_1| > |z_2| > 0$.

Convergence and extension property without logarithmic for iterates When $i_k = j_k = 0$ for $k = 1, \ldots, M$, we call the property above the *convergence and extension property without logarithms for iterates.*

If the convergence and extension property (with or without logarithms) for products holds for any objects W_1, W_2, W_3, W_4 and M_1 of $\mathcal{C}$ and any logarithmic intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of the types as above, we say that *the convergence and extension property for products holds in $\mathcal{C}$.* We similarly define the meaning of the phrase *the convergence and extension property for iterates holds in $\mathcal{C}$.*

The following theorem generalizes Theorem 11.8 in [HLZ2] to the strongly graded generalized modules for a strongly graded conformal vertex algebra:

Theorem 3.6.2 *Let V be a strongly graded conformal vertex algebra. Then*

1. *The convergence and extension properties for products and iterates hold in $\mathcal{C}$. If $\mathcal{C}$ is in $\mathcal{M}_{sg}$ and if every object of $\mathcal{C}$ is a direct sum of irreducible objects of $\mathcal{C}$ and there are only finitely many irreducible objects of $\mathcal{C}$ (up to equivalence), then the convergence and extension properties without logarithms for products and iterates hold in $\mathcal{C}$.*

2. *For any $n \in \mathbb{Z}_+$, any objects $W_1, \dots, W_{n+1}$ and $\widetilde{W_1}, \dots, \widetilde{W_{n-1}}$ of $\mathcal{C}$, any logarithmic intertwining operators*

$$\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_{n-1}, \mathcal{Y}_n$$

of types

$$\binom{W_0}{W_1\, \widetilde{W_1}}, \binom{\widetilde{W_1}}{W_2\, \widetilde{W_2}}, \dots, \binom{\widetilde{W_{n-2}}}{W_{n-1}\, \widetilde{W_{n-1}}}, \binom{\widetilde{W_{n-1}}}{W_n\, W_{n+1}},$$

respectively, and any $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1, \dots, W_{(n+1)} \in W_{n+1}$, the series

$$\langle w'_{(0)}, \mathcal{Y}_1(w_{(1)}, z_1) \cdots \mathcal{Y}_n(w_{(n)}, z_n) w_{(n+1)} \rangle$$

is absolutely convergent in the region $|z_1| > \cdots > |z_n| > 0$ and its sum can be analytically extended to a multivalued analytic function on the region given by $z_1 \neq 0$, $i = 1, \dots, n$, $z_i \neq z_j$, $i \neq j$, such that for any set of possible singular points with either $z_i = 0, z_i = \infty$ or $z_i = z_j$ for $i \neq j$, this multivalued analytic function can be expanded near the singularity as a series having the same form as the expansion near the singular points of a solution of a system of differential equations with regular singular points.

Proof. The first statement in the first part and the statement in the second part of the theorem follow directly from Theorem 3.4.7 and Theorem 3.5.5 and the theorem of differential equations with regular singular points. The second statement in the first part can be proved using the same method in [H3]. $\square$

In order to construct braided tensor category on the category of strongly graded generalized V-modules, we need the following assumption on $\mathcal{C}$ (see Assumption 10.1, Theorem 11.4 of [HLZ2]).

Assumption 3.6.3 *Suppose the following two conditions are satisfied:*

1. *$\mathcal{C}$ is closed under $P(z)$-tensor products for some $z \in \mathbb{C}^{\times}$.*

2. *Every finite-generated lower bounded doubly graded generalized V-module is an object of $\mathcal{C}$.*

Conjecture 3.6.4 We conjectured that the category of certain strongly graded generalized V-modules satisfying the first condition in Assumption 3.6.3. The case for the vertex operator algebra was proved in [H2].

Under Assumption 3.6.1 and Assumption 3.6.3 on the category $\mathcal{C} \subset \mathcal{GM}_{sg}$, we generalize the main result (Theorem 12.15) of [HLZ2] to the category of strongly graded generalized modules for a strongly graded vertex algebra:

Theorem 3.6.5 *Let V be a strongly graded conformal vertex algebra. Then the category $\mathcal{C}$, equipped with the tensor product bifunctor $\boxtimes$, the unit object V, the braiding isomorphism $\mathcal{R}$, the associativity isomorphism $\mathcal{A}$, and the left and right unit isomorphisms l and r in [HLZ2], is an additive braided tensor category.*

In the case that $\mathcal{C}$ is an abelian category, we have:

Corollary 3.6.6 *If the category $\mathcal{C}$ is an abelian category, then $\mathcal{C}$, equipped with the tensor product bifunctor $\boxtimes$, the unit object V, the braiding isomorphism $\mathcal{R}$, the associativity isomorphism $\mathcal{A}$, and the left and right unit isomorphisms l and r in [HLZ2], is a braided tensor category.*

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