# DESIGN OF OBSERVERS FOR SYSTEMS WITH SLOW AND FAST MODES

by

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**A thesis submitted to the**

**Graduate School-New Brunswick**

**Rutgers, The State University of New Jersey**

**In partial fulfillment of the requirements**

**For the degree of**

**Master of Science**

**Graduate Program in Electrical and Computer Engineering**

**Written under the direction of**

**Professor Zoran Gajić**

**And approved by**

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**New Brunswick, New Jersey**

**May, 2014**

## ABSTRACT OF THE THESIS

## Design of Observers for systems with Slow and Fast Modes

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This thesis considers the design of observers for systems with slow and fast modes. The existing design method were able to achieve that goal with $O(\epsilon)$ accuracy. Where $\epsilon$ is a small positive parameter that indicates seperation of system state space variables into slow and fast. We design independent slow and fast reduced-order observer with exact accuracy, and place their closed-loop eigenvalues exactly at the desired location. Furthermore, we apply the two stage method to design the full-order observer. The two stage method results in full-state feedback control for each subsystems which makes the subsystems asymptotic stable. Lastly, the crucial theme of the online savings is the fact that the design allows complete time-scale seperation for both the observer and controller through the complete and exact decomposition into slow and fast time scales. The above method reduces both off-line and on-line computations. In this thesis, we demonstrate the effectiveness of the two stage methods through theoretical and simulation results.

# Acknowledgements

Firstly, I express my deepest and sincerest gratitude to my supervisor, Prof. Zoran Gajić, who has supported me and gave a inspiration thoughout my thesis with his guidance and patience. His constant help gives me better direction whenever I encounter impediment throughout researching this subject.

I wish express thanks to Prof. Quingze. Zou for his valuable teaching last semester. He gave me an important guidance to remind me ideas of controls and the fundamental concepts inspire me to come up with some of new concepts in this thesis.

I would like to thank to my committee members, Prof. Quingze. Zou and Prof. Zingang. Yi, Prof. Zoran Gajić,

I also would like to thank to my senior, Park.Gunyoung, for his help to get some backgrounds for my research.

My deepest gratitude goes to my family for their constant love and support. My wife and son as well as my family in South Korea, have supported me for their love. This thesis is completed with their help and support.

Lastly, I offer my regards and blessings to all of those who supported me in any respect during the completion of the project.

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# Chapter 1

# Introduction

## 1.1 Singularly Perturbed Systems

Large time scale linear systems are encountered frequently in engineering problems. The crucial theme is how to reduce a large time scale system into a reduced form to enhance analysis, design and simulation. Consider a linear time invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1.1}$$

where $x(t) \in R^n$, $u(t) \in R^r$ and $y(t) \in R^p$ are state, control input, and output variables respectively. As proposed by (Anderson, 1982), the above system (1.1) will be classified as a two-time-scale system if the eigenvalues of $A$ matrix, denoted as $\lambda(A)$ can be seperated into two disjoint sets

$$|q_i| \ll |r_i| \; for\ all\ q_i\ in\ Q,\ and\ r_i\ in\ R. \tag{1.2}$$

The small parameter $\epsilon$ is the eigenvalue ratio

$$\epsilon = \frac{\max_i |q_i|}{\min_j |r_i|} \tag{1.3}$$

which presents the system's time scale seperation, and it defines (1.1) as a singularly perturbation system. Now one can regard (1.1) as two coupled subsystems represented by

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \\ \dot{x}_2(t) &= \tilde{A}_{21}x_1(t) + \tilde{A}_{22}x_2(t) + \tilde{B}_2u(t) \\ y(t) &= C_1x_1(t) + C_2x_2(t) \end{aligned} \tag{1.4}$$

where the general system matrices are $A = \begin{bmatrix} A_{11} & A_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_1 \\ \tilde{B}_2 \end{bmatrix}$. Note that $\tilde{A}_{21}$ , $\tilde{A}_{22}$ and $\tilde{B}_2$ have large elements. It can be shown that variations of $x_2(t)$ are fast, which is a characteristic of singularly perturbed systems. Later, we will introduce the Chang transformation to decouple the slow and fast subsystems. Using the spectral norm, $\epsilon$ can be represented by (Anderson, 1982)

$$\frac{\|A_s\|}{\|A_f\|} \leq \epsilon \tag{1.5}$$

where $A_s$ and $A_f$ are matrices obtained after the application of the Chang transformation to the system matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}$. This fact will be discussed in later. Multiplying by $\epsilon$ the second equation in (1.4), the singularly perturbed system is given by

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \\ \epsilon\dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) \\ y(t) &= C_1x_1(t) + C_2x_2(t) \end{aligned} \tag{1.6}$$

where $A_{21} = \epsilon\tilde{A}_{21}$ , $A_{22} = \epsilon\tilde{A}_{22}$ and $B_2 = \epsilon\tilde{B}_2$.

### 1.1.1 Slow and Fast Seperation of Singularly Perturbed Systems

A singularly perturbed system is in the explicit state variable form in which the derivatives of some of the sates are multiplied by a small positive scalar $\epsilon$. Consider the singular perturbed system (1.6), where $\epsilon$ is a small singular perturbation parameter, and $x_1(t) \in \Re^{n_1}$ and $x_2(t) \in \Re^{n_2}$ are state vectors. The system (1.6) may be approximately decomposed into a reduced system showing $n_1$ slow modes and a fast subsystem showing $n_2$ fast modes, (O'Reilly, 1980). The reduced subsystem, in other word, slow subsystem is obtained by setting $\epsilon = 0$ in (1.6), that is

$$\dot{x}_s(t) = A_{11}x_s(t) + A_{12}x_{2s}(t) + B_1u_s(t) \tag{1.7}$$

$$0 = A_{21}x_s(t) + A_{22}x_{2s}(t) + B_2u_s(t) \tag{1.8}$$

$$y_s(t) = C_1x_s(t) + C_2x_{2s}(t) \tag{1.9}$$

$x_{2s}(t)$ is the slow part for $x_2(t)$. Since $x_2(t) = x_{2s}(t) + x_{2f}(t)$, the reduced states $x_{2s}(t)$ can be obtained when $\epsilon$ is set zero. The following assumption should be satisfied for obtaining the slow subsystem.

**Assumption 1.1.1.** $A_{22}^{-1}$ *exists,*

Under Assumption 1.1.1, $x_{2s}(t)$ can be obtained from (1.8) as

$$x_{2s}(t) = -A_{22}^{-1}A_{21}x_s(t) - A_{22}^{-1}B_2u_s(t) \tag{1.10}$$

Substitution of (1.10) into (1.7) results in

$$\begin{aligned} \dot{x}_s(t) &= A_0x_s(t) + B_0u_s(t) \\ y_s(t) &= C_0x_s(t) + D_0u_s(t) \end{aligned} \tag{1.11}$$

with

$$\begin{aligned} &A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21},\ B_0 = B_1 - A_{12}A_{22}^{-1}B_2 \\ &C_0 = C_1 - C_2A_{22}^{-1}A_{21},\ D_0 = -C_2A_{22}^{-1}B_2 \end{aligned} \tag{1.12}$$

The fast subsystem (Kokotovic *et al.*, 1999) is defined by

$$\begin{aligned} \dot{x}_f(\tau) &= A_{22}x_f(\tau) + B_2u_f(\tau) \\ y_f(\tau) &= C_2x_f(\tau) \end{aligned} \tag{1.13}$$

where

$$\tau = \frac{(t - t_0)}{\epsilon} \tag{1.14}$$

In this section, we present how the original singularly perturbed system (1.6) can be decomposed exactly into two sub-systems corresponding to slow and fast variables.

## 1.2 Chang Transformation : Block Triangular Form

The purpose of the Chang transformation is to decouple the fast subsystem from the slow subsystem. The open loop system is considered regarding the Chang transformation in this section. The design of the Chang transformation is achieved in two steps. Firstly, the upper triangular form is achieved and secondly, the block diagonal from is obtained to completely decouple the fast and slow sub-systems.

Consider a general continuous-time linear system which is given by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.15}$$

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, and $A$, $B$ are constant matrices of appropriate dimensions. There exists a similarity transformation defined by

$$\bar{x}(t) = T_1^{-1}x(t) \tag{1.16}$$

where

$$T_1 = \begin{bmatrix} I_n & 0 \\ L & I_m \end{bmatrix} \tag{1.17}$$

which transforms (1.15) into

$$\dot{x}(t) = T_1 T_1^{-1} A T_1 T_1^{-1} x(t) + Bu(t) \tag{1.18}$$

Multiplying from the left by $T_1^{-1}$ both side of (1.18), (1.18) becomes

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t) \tag{1.19}$$

where

$$\bar{A} = T_1^{-1} A T_1, \;\; \bar{B} = T_1^{-1} B \tag{1.20}$$

The above method is a state transformation of a general linear system. Now we apply this method to the singularly perturbed system. Consider the singularly perturbed system.

$$\dot{x} = \tilde{A}x(t) + \tilde{B}u(t), x(t_0) = x_0 \tag{1.21}$$

with

$$\tilde{A} = \begin{bmatrix} A_{11} & A_{12} \\ \frac{1}{\epsilon}A_{21} & \frac{1}{\epsilon}A_{22} \end{bmatrix}, \tilde{B} = \begin{bmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{bmatrix}, x(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{1.22}$$

Using the state tranformation (1.16), the above singularly perturbed system (1.21) has $\bar{A} = T_1 \tilde{A} T_1^{-1}$ represented by

$$\bar{A} = \begin{bmatrix} I_n & 0 \\ L & I_m \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ \frac{1}{\epsilon}A_{21} & \frac{1}{\epsilon}A_{22} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -L & I_m \end{bmatrix} = \begin{bmatrix} A_{11} - A_{12}L & A_{12} \\ f(L) & LA_{12} + \frac{1}{\epsilon}A_{22} \end{bmatrix} \tag{1.23}$$

with

$$f(L) = L(A_{11} - A_{12}L) + \frac{1}{\epsilon}(A_{21} - A_{22}L) \tag{1.24}$$

After the first step of the Chang tranformation, the block-triangular form (1.23) can be obtained if $f(L) = 0$. Note that $A_{21}$ and $A_{22}$ are divided by $\epsilon$ when applying the first step of the Chang tranformation. Furthermore, eigenvalues are preserved under Step 1 of the Chang tranformation. The following $L$ equation is obtained by multiplying (1.24) by $\epsilon$

$$0 = \epsilon L(A_{11} - A_{12}L) + (A_{21} - A_{22}L) \tag{1.25}$$

The newly obtained system will be block-triangular provided the $n_2 \times n_1$ matrix $L$ satisfies (1.25).

### 1.2.1 Recursive Algorithm for Solving L-Equation

We use the fixed-point iteration to find the solution of (1.25), (Gajić and Shen, 1989) Setting $\epsilon = 0$, (1.25) becomes

$$A_{21} - A_{22}L^o = 0 \tag{1.26}$$

The unique solution of (1.25) is

$$L^o = A_{22}^{-1}A_{21} \tag{1.27}$$

The sought value of matrix $L$ is approximated by

$$L = L^0 + O(\epsilon) \tag{1.28}$$

The recursive algorithm proposed by (Gajić and Shen, 1989) is

$$L^{(i+1)} = A_{22}^{-1}(A_{21} + \epsilon L^{(i)}A_{11} - \epsilon L^{(i)}A_{12}L^{(i)}) \tag{1.29}$$

It was shown in Gajic(1986) that algorithm (1.29) converges with the rate of convergence of $O(\epsilon)$, that is, after $i$-iterations, we have

$$\|L^{(i)} - L^0\| = 0|\epsilon^i| \tag{1.30}$$

## 1.3 Chang Transformation : Block Diagonal Form

One additional trasnformation matrix $T_2$ leads to complete seperation of slow and fast states of the singularly perturbed system (1.21). Consider general linear system (1.19). From the upper triangular form, the state $\bar{x}(t)$ changes into the state $\hat{x}(t)$.

$$\hat{x}(t) = T_2^{-1}\bar{x}(t) \tag{1.31}$$

with

$$T_2^{-1} = \begin{bmatrix} I_n & -\epsilon H \\ 0 & I_m \end{bmatrix} \tag{1.32}$$

where $\bar{x}(t)$ are the states with the upper block-triangular form of $\bar{A}$ and $\hat{x}(t)$ are the states corresponding to the block-diagonal form $\hat{A}$. The above state transformation changes (1.19) into

$$\dot{\bar{x}}(t) = T_2 T_2^{-1}\bar{A}T_2 T_2^{-1}\bar{x}(t) + \bar{B}u(t) \tag{1.33}$$

The new coordinates of the state $\hat{x}(t)$ can be obtained by multiplying (1.33) by $T_2$ from the left side.

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \tag{1.34}$$

with

$$\hat{A} = T_2^{-1}\bar{A}T_2, \hat{B} = T_2^{-1}\bar{B} \tag{1.35}$$

In the case of the singularly perturbed system, $\hat{x}(t)$ contains fast and slow states given by

$$\hat{x}(t) = \begin{bmatrix} x_s(t) \\ x_f(t) \end{bmatrix} \tag{1.36}$$

Consider the singularly perturbed system (1.21). After the first step of the Chang transformation, (1.21) becomes

$$\dot{x} = \bar{A}\bar{x}(t) + \bar{B}u(t), \bar{x}(t_0) = \bar{x}_0 \tag{1.37}$$

with

$$\bar{A} = \begin{bmatrix} A_{11} - A_{12}L & A_{12} \\ 0 & LA_{12} + \frac{1}{\epsilon}A_{22} \end{bmatrix} \tag{1.38}$$

$$\bar{B} = T_1^{-1}\tilde{B}$$

In the same way, the block triangular form, obtained using (1.31), produces $\hat{A} = T_2^{-1}\bar{A}T_2$, as

$$\begin{aligned}\hat{A} &= \begin{bmatrix} I_n & -\epsilon H \\ 0 & I_m \end{bmatrix} \begin{bmatrix} A_{11} - A_{12}L & A_{12} \\ 0 & \frac{1}{\epsilon}A_{22} + LA_{12} \end{bmatrix} \begin{bmatrix} I_n & \epsilon H \\ 0 & I_m \end{bmatrix} \\ &= \begin{bmatrix} A_{11} - A_{12}L & \epsilon(A_{11} - A_{12}L)H + A_{12} - \epsilon H(A_{22}/\epsilon + LA_{12}) \\ 0 & \frac{1}{\epsilon}A_{22} + LA_{12} \end{bmatrix}\end{aligned} \tag{1.39}$$

To make (1.39) block-diagonal, the element $\epsilon(A_{11} - A_{12}L)H + A_{12} - \epsilon H(A_{22}/\epsilon + LA_{12})$ should be zero. The $H$ equation is given by

$$0 = \epsilon(A_{11} - A_{12}L)H + A_{12} - H(A_{22} + \epsilon LA_{12}) \tag{1.40}$$

Equation (1.40) is the Sylvester algebraic equation. Its unique solution always exists since matrices $A_{22} + \epsilon LA_{12}$ and $\epsilon(A_{11} - A_{12}L)$ have no eigenvalues in common (Chen, 1999)

### 1.3.1 Recursive Algorithm for Solving the H-Equation

The iterative method is utilized to obtain the solution for $H$. Set $\epsilon = 0$ in (1.40), to obtain $H^o$ as

$$H^o = A_{12}A_{22}^{-1} \tag{1.41}$$

The solution value $H$ has an approximation $H^o$ and the error of $O(\epsilon)$, that is

$$H = H^0 + O(\epsilon) \tag{1.42}$$

In the same process, applying the fixed-point recursive algorithm for $L$, we have

$$\epsilon(A_{12} - A_{22}L)H^{(i)} - H^{(i+1)}A_{22} - \epsilon H^{(i)}LA_{12} + A_{12} = 0 \tag{1.43}$$

Equation (1.43) can be rearranged for $H^{(i+1)}$ as

$$H^{(i+1)} = \epsilon(A_{12} - A_{22}L)H^{(i)}A_{22}^{-1} - \epsilon H^{(i)}LA_{12}A_{22}^{-1} + A_{12}A_{22}^{-1} \tag{1.44}$$

It is no difficult to show that the convergence rate of (1.44) is $O(\epsilon)$.

## 1.4 Chang Transformation : Composite Method

This section presents a combination of two previous sections and fully defined the Chang transformation. The composite state transformation matrices $T_c$ and $T_c^{-1}$ are represented by

$$T_c = T_1 T_2 = \begin{bmatrix} I_n & 0 \\ -L & I_m \end{bmatrix} \begin{bmatrix} I_n & \epsilon H \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} I_n & \epsilon H \\ -L & -L\epsilon H + I_m \end{bmatrix} \tag{1.45}$$

and

$$T_c^{-1} = T_2^{-1} T_1^{-1} = \begin{bmatrix} I_n & -\epsilon H \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ L & I_m \end{bmatrix} = \begin{bmatrix} I_n - \epsilon HL & -\epsilon H \\ L & I_m \end{bmatrix} \tag{1.46}$$

The matrix $T_c$ is said to be the Chang transformation. Consider a general continuous-time linear system (1.15) where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, and $A, B$ are constant matrices of appropriate dimensions. There exists a state transformation defined by

$$\bar{x}(t) = T_c^{-1} x(t) \tag{1.47}$$

where $T_c^{-1}$ is given by (1.46) which transforms a general continuous-time linear system (1.15) into

$$\dot{x}(t) = T_c T_c^{-1} A T_c T_c^{-1} x(t) + Bu(t) \tag{1.48}$$

By multiplying $T_c^{-1}$ on both side of (1.48), (1.48) becomes

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t) \tag{1.49}$$

with

$$\bar{x}(t) = T_c^{-1} x(t), \bar{A} = T_c^{-1} A T_c, \bar{B} = T_c^{-1} B \tag{1.50}$$

The fast and slow states can be decomposed using

$$\begin{bmatrix} x_s(t) \\ x_f(t) \end{bmatrix} = \begin{bmatrix} I_n - \epsilon HL & -\epsilon H \\ L & I_m \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{1.51}$$

Where $x_s(t)$ and $x_f(t)$ are states in the slow and fast coordinates and $x_1(t)$ and $x_2(t)$ are states in the original coordinates. The original state can be recostructed by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} I_n & \epsilon H \\ -L & -L\epsilon H + I_m \end{bmatrix} \begin{bmatrix} x_s(t) \\ x_f(t) \end{bmatrix} \tag{1.52}$$

Note that the complete slow and fast decomposition holds when we only consider unforced response. If we consider the input vector, we need an alternative way to exactly decompose the slow and fast states. This fact will be considered later.

## 1.5 Introduction to Observers and Observer Based Controllers

Sometimes all state space variables are not available for measurements, or it is not practical to measure all of them, or it is too expensive to measure all state space variables. In order to be able to apply the state feedback control to a system, all of its state space variables must be available at all times. Thus, we face the problem of estimating system state space variables.

### 1.5.1 Full-Order Observer Design

Consider a linear time invariant system given as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu_c(t),\ x_{t_0} = x_0 = \ unknown \\ y(t) &= Cx(t) \end{aligned} \tag{1.53}$$

where $x(t) \in \Re^n, u(t) \in \Re^r, y(t) \in \Re^p$ with constant matrices $A, B, C$ having appropriate dimensions. We may construct a full-order observer having the same matrices $A, B, C$ such that

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu_c(t),\ \hat{x}_{t_0} = \hat{x}_0 \\ \hat{y}(t) &= C\hat{x}(t) \end{aligned} \tag{1.54}$$

Then we compare the output $y(t)$ of the system (1.53) and the output $\hat{y}(t)$ of the full-order observer (1.54). These two outputs will be different since in the first case the system initial condition is unknown, and in the second case it has been chosen arbitrarily.

The difference between these two outputs will generate an error signal

$$y(t) - \hat{y}(t) = Cx(t) - C\hat{x}(t) = Ce(t) \tag{1.55}$$

which can be used as the feedback signal to the full-order observer such that the estimation error $e(t)$ is reduced to zero. Considering the feedback signal (1.55), the observer structure is given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu_c(t) + K(y(t) - \hat{y}(t)) \tag{1.56}$$

Note that the observer has the same structure as the system plus the driving feedback term that contain information about the observation error. The observer is implemented on line as a dynamic system driven by the same input as the original system and the measurements coming from the original systems, that is

$$\dot{\hat{x}}(t) = (A - KC)\hat{x}(t) + Bu_c(t) + Ky(t) \tag{1.57}$$

with

$$y(t) = Cx(t),\ \ u_c(t) = F\hat{x}(t) \tag{1.58}$$

This can be realized by proposing the system-observer structure as given in Figure. 1.1.

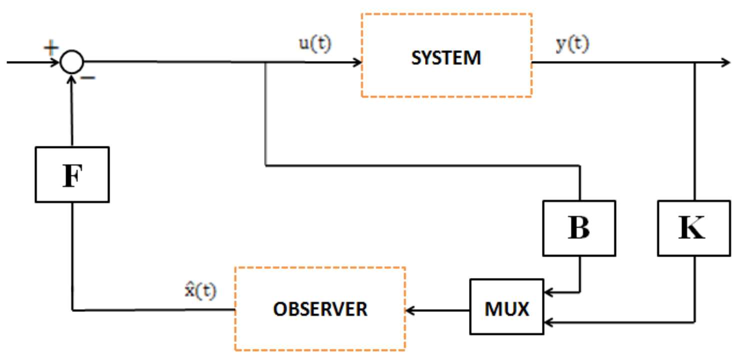


Figure 1.1: Full-order observer-based controller

It is easy to derive an expression for dynamics of the observation error as

$$\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = (A - KC)e(t) \tag{1.59}$$

If the observer gain $K$ is chosen such that the matrix $A - KC$ is asymptotically stable, then the error $e(t)$ can be reduced to zero at steady state. At this point, we need the following assumption.

**Assumption 1.5.1.** *The pair* $(A, C)$ *is observable*

In practice, the observer eigenvalues should be chosen to be about $5-6$ times faster than the system eigenvalues so that the minimal real part of observer eigenvalues to be $5-6$ times bigger than the maximal real part of system eigenvalues, that is

$$|\Re(\lambda_{min})|_{observer} > (5 \; or \; 6) \times |\Re(\lambda_{max})|_{system} \tag{1.60}$$

### 1.5.2 Seperation Principle

This section presents the fact that the observer-based controller preserves the closed-loop system eigenvalues. The system under state feedback control, that is $u(t) = -Fx(t)$ has the closed-loop form as

$$\dot{x}(t) = (A - BF)x(t) \tag{1.61}$$

so that the eigenvalues of the matrix $A - BF$ are the closed-loop system eigenvalues under state feedback. In the case of the observer-based controller, as given in Figure 1.1, the control input signal applied to the observer-based controller is given as

$$u_c(t) = -F\hat{x}(t) = -Fx(t) + Fe(t) \tag{1.62}$$

Substituting equation (1.62) in the full-order observer (1.57) and the system (1.61), we obtain the following augumented closed-loop matrix form

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & -BF \\ KC & A - KC - BF \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \tag{1.63}$$

At this point, we introduce the state transformation matrix given by

$$\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} = T_{aug} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \tag{1.64}$$

Since matrix $T_{aug}$ is nonsingular, we can apply the similarity transformation to the closed-loop matrix form (1.63), which leads to

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A - BF & BF \\ 0 & A - KC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{1.65}$$

It is well known that the similarity transformation preserves the same eigenvalues as in the original system. The state matrix of the system (1.65) is upper block triangular and its eigenvalues are equal to the eigenvalues $\lambda(A - BF) \cup \lambda(A - KC)$, which indicates that the independent placement of observer and controller eigenvalues is possible.

## 1.6 An Observer for Singularly Perturbed Systems

The singularly perturbed system (1.6) may be rewritten as

$$\begin{aligned} \dot{x}(t) &= \tilde{A}x(t) + \tilde{B}u(t), \;\; x(t_0) = x_0 \\ y(t) &= Cx(t) \end{aligned} \tag{1.66}$$

with

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\epsilon} & \frac{A_{22}}{\epsilon} \end{bmatrix}, \;\; \tilde{B} = \begin{bmatrix} B_1 \\ \frac{B_2}{\epsilon} \end{bmatrix} \\ x(t) &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{1.67}$$

The corresponding full-order observer for the singularly perturbed system (1.66) is given as

$$\begin{aligned} \dot{\hat{x}}(t) &= (\tilde{A} - KC)\hat{x}(t) + \tilde{B}u(t) + Ky(t) \\ \hat{y}(t) &= C\hat{x}(t) \end{aligned} \tag{1.68}$$

where $\hat{x}(t)$ is an estimate of the state $x(t)$ in (1.66) and the state error is defined as

$$e(t) = \hat{x}(t) - x(t) \tag{1.69}$$

The role of the observer (1.68) reconstruct the state $x(t)$ of (1.66) in a uniformly asymptotic manner in the sense that

$$\lim_{t \to \infty} e(t) = 0 \tag{1.70}$$

The observability Assumption 1.5.1 is needed for (1.70) to hold

### 1.6.1 An Observer for the Slow Subsystem

An observer for the reduced system (1.11) is given as (O'Reilly, 1979a)

$$\dot{\hat{x}}_s(t) = (A_0 - K_0C_0)\hat{x}_s(t) + K_0y_s(t) + B_0u_s(t) \tag{1.71}$$

where the state reconstruction error is given as

$$e_s(t) = \hat{x}_s(t) - x_s(t) \tag{1.72}$$

The slow error dynamics can be represented by

$$\dot{e}_s(t) = (A_0 - K_0 C_0)e_s(t), e_s(t_0) = e_1(t_0) \tag{1.73}$$

The observer (1.71) will uniformly aymptotically reconstruct the state $x_s(t)$, that is

$$\lim_{t \to \infty} e_s(t) = 0 \tag{1.74}$$

If the following assumption holds

**Assumption 1.6.1.** *The pair* $(A_0, C_0)$ *is observable.*

### 1.6.2 An Observer for the Fast Subsystem

A full-order observer for the fast system (1.13) is given as (O'Reilly, 1979a)

$$\dot{\hat{x}}_f(\tau) = (\frac{1}{\epsilon}A_{22} - \frac{1}{\epsilon}K_2C_2)\hat{x}_f(\tau) + \frac{1}{\epsilon}K_2y_f(\tau) + \frac{1}{\epsilon}B_2u_f(\tau) \tag{1.75}$$

where the state reconstruction error is defined as

$$e_f(\tau) = \hat{x}_f(\tau) - x_f(\tau) \tag{1.76}$$

Similarly, the fast error dynamics can be represented by

$$\dot{e}_f(\tau) = (\frac{1}{\epsilon}A_{22} - \frac{1}{\epsilon}K_2C_2)e_f(\tau),\ e_f(t_0) = \hat{x}_f(0) - x_f(0) \tag{1.77}$$

The observer (1.75) will uniformly aymptotically reconstruct the state $x_f(\tau)$ if

$$\lim_{\tau \to \infty} e_f(\tau) = 0 \tag{1.78}$$

The fast subsystem observability assumption is needed for (1.78) to hold

**Assumption 1.6.2.** *The pair* $(A_{22}, C_2)$ *is observable.*

### 1.6.3 State Reconstruction for the Composite System

This section presents a composite observer design based on the two slow and fast observers (1.71) and (1.75) (O'Reilly, 1979a).

**Lemma 1.6.1.** *If the observer* (1.68) *is coupled to the system* (1.66) *with*

$$K = \begin{bmatrix} K_1 \\ \frac{1}{\epsilon} K_2 \end{bmatrix} \tag{1.79}$$

*where*

$$K_1 = \frac{1}{\epsilon^2} A_{12} A_{22}^{-1} K_2 + K_0 [I - \frac{1}{\epsilon^2} C_2 A_{22}^{-1} K_2] \tag{1.80}$$

*and if* $A_0 + K_0 C_0$ *and* $A_{22} + K_2 C_2$ *are uniformly asymptotically stable, then the eigenvalues related to the error dynamics in the original coordinates satisfy*

$$\begin{aligned} \lambda_i &= \lambda_i (A_0 + K_0 C_0) + O(\epsilon), \;\; i = 1, ..., n_1 \\ \lambda_j &= \lambda_j (\frac{1}{\epsilon} A_{22} + \frac{1}{\epsilon} K_2 C_2) + \frac{O(\epsilon)}{\epsilon}, \;\; i = n_1 + j, \;\; j = 1, ..., n_2 \end{aligned} \tag{1.81}$$

## 1.7 Observer-based Controllers for Singularly Perturbed Systems

A dynamical feedback controllers for the singularly perturbed system (1.66) is given by

$$u_c(t) = F \hat{x}(t) \tag{1.82}$$

where $\hat{x}(t)$ is an estimate of the state $x(t)$ and is generated by the full-order observer (1.68). The overall closed-loop system for the original system (1.66) is given by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} \tilde{A} + \tilde{B}F & -\tilde{B}F \\ 0 & \tilde{A} - KC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{1.83}$$

It is required that the controller (1.83) be uniformly asymptotically stable in the sense that

$$\lim_{t \to \infty} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = 0 \tag{1.84}$$

Obviously, this may be achieved if and only if (1.66) is stabilizable by feedback (1.82) and the observer reconstruction error system, that is

$$\dot{e}(t) = (\tilde{A} - KC) e(t), \;\; e(t_0) = \hat{x}(t_0) - x(t_0) \tag{1.85}$$

The observability Assumption 1.5.1 and the following Assumption is needed for (1.84) to hold

**Assumption 1.7.1.** *The pair* $(\tilde{A}, B)$ *is controllable.*

### 1.7.1 A Controller for the Slow Subsystem

An observer-based controller for the slow subsystem (1.11) is given by (O'Reilly, 1980)

$$u_s(t) = F_0 \hat{x}_s(t) \tag{1.86}$$

where $\hat{x}_s(t)$ is an estimate of the original state $x_s(t)$ generated by the slow reduced-order observer (1.71). The slow state reconstruction error $e_s(t)$ defined by (1.72) satisfies (1.73). Hence, the closed-loop slow subsystem is given by

$$\begin{bmatrix} \dot{x}_s(t) \\ \dot{e}_s(t) \end{bmatrix} = \begin{bmatrix} A_0 + B_0 F_0 & -B_0 F_0 \\ 0 & A_0 - K_0 C_0 \end{bmatrix} \begin{bmatrix} x_s(t) \\ e_s(t) \end{bmatrix} \tag{1.87}$$

The slow subsystem (1.11) is uniformly completely stabilizable by the controller (1.86) and the slow reduced-order observer (1.71) if

$$\lim_{t \to \infty} \begin{bmatrix} x_s(t) \\ e_s(t) \end{bmatrix} = 0 \tag{1.88}$$

The observability Assumption 1.6.1 and the following Assumption is needed for (1.88) to hold

**Assumption 1.7.2.** *The pair* $(A_0, B_0)$ *is controllable.*

### 1.7.2 A Controller for the Fast Subsystem

An observer-based controller for the fast subsystem (1.13) is given in the fast time-scale $\tau$ by

$$u_f(\tau) = F_2 \hat{x}_f(\tau) \tag{1.89}$$

where $\hat{x}_f(\tau)$ is an estimate of the original state $x_f(\tau)$ generated by the fast reduced-order observer (1.75). The slow state reconstruction error $e_f(\tau)$ defined by (1.76) satisfies (1.77). Hence, the closed-loop fast subsystem is given by

$$\begin{bmatrix} \dot{x}_f(\tau) \\ \dot{e}_f(\tau) \end{bmatrix} = \begin{bmatrix} A_{22} + B_2 F_2 & -B_2 F_2 \\ 0 & A_{22} - K_2 C_2 \end{bmatrix} \begin{bmatrix} x_f(\tau) \\ e_f(\tau) \end{bmatrix} \tag{1.90}$$

Furthermore, the fast subsystem (1.13) is uniformly completely stabilizable by the controller (1.89) and the fast reduced-order observer (1.75) if

$$\lim_{\tau \to \infty} \begin{bmatrix} x_f(\tau) \\ e_f(\tau) \end{bmatrix} = 0 \tag{1.91}$$

The observability Assumption 1.6.2 and the following Assumption is needed for (1.91) to hold

**Assumption 1.7.3.** *The pair* $(A_{22}, B_2)$ *is controllable.*

### 1.7.3 A Composite Observer-based Controller

At this point, we need to introduce the observer driven controller proposed by (O'Reilly, 1980).

**Lemma 1.7.1.** *If the observer and controller are coupled to the system* (1.66) *with*

$$F = \begin{bmatrix} F_1 & F_2 \end{bmatrix} \tag{1.92}$$

$$F_1 = [I + K_2 A_{22}^{-1} B_2] F_0 + F_2 A_{22}^{-1} A_{21} \tag{1.93}$$

$$K = \begin{bmatrix} K_1 \\ \frac{1}{\epsilon} K_2 \end{bmatrix} \tag{1.94}$$

$$K_1 = \frac{1}{\epsilon^2} A_{12} A_{22}^{-1} K_2 + K_0 [I - \frac{1}{\epsilon^2} C_2 A_{22}^{-1} K_2] \tag{1.95}$$

*and if the slow subsystem and fast subsystem are each uniformly stabilizable by two observers* (1.71) *and* (1.75) *and controllers* (1.86) *and* (1.89)*, then there exists a positive* $\epsilon^*$ *sufficiently small such that the original system* (1.66) *is uniformly completely stabilizable for any* $\epsilon \in (0, \epsilon^*]$.

This lemma indicates that the state and error dynamic can be reconstructed within $O(\epsilon)$ approximation. There are several papers for observers and observer driven controllers for singularly perturbed systems and all of them did design with $O(\epsilon)$ accuracy (*O′Reilly*, 1980), (*O′Reilly*, 1979*a*),(*O′Reilly*, 1979*b*), (*Gardner JrandCruz Jr*, 1980), and (*Porter*, 1974).

## 1.8 Controller Eigenvalues Assignment

The goal is to find a similarity transformation $P$ and the feedback gain $F$ such that

$$P^{-1}(A - BF)P = \Lambda_{desired} \tag{1.96}$$

with

$$\lambda(A - BF) = \lambda(\Lambda_{desired}) = \lambda_{desired} \tag{1.97}$$

If $(A, B)$ is controllable, $\lambda(A-BF)$ can be arbitrarily located (Chen, 1999). Premultiply (1.96) by $P$ produces

$$AP - BFP = P\Lambda_{desired} \tag{1.98}$$

Rearranging (1.98), it becomes

$$AP - P\Lambda_{desired} = B\bar{F} \tag{1.99}$$

with

$$\bar{F} = FP \tag{1.100}$$

The following lemma was presented in (Chen, 1999, pp. 240).

**Lemma 1.8.1.** *If $A$ and $\Lambda_{desired}$ have no eigenvalues in common, then the unique solution $P$ of $AP - P\Lambda_{desired} = B\bar{F}$ exists if and only if $(A, B)$ is controllable and $(\Lambda_{desired}, \bar{F})$ is observable.*

### 1.8.1 Design Procedure

The procedure for computing the feedback gain through the Lyapunov method is presented in (Chen, 1999) Assume a controllable pair $(A, B)$, where $A$ is $\Re^{n\times n}$ and $B$ is $\Re^{n\times m}$. Find a $\Re^{m\times n}$ real matrix $F$ such that $(A - BF)$ has a set of desired eigenvalues that contains no eigenvalues of $A$.

*Step* 1. Select an $\Re^{n\times n}$ matrix $\Lambda_{desired}$ that has the desired set of eigenvalues. The form of $\Lambda_{desired}$ can be chosen arbitrarily, often it is a diagonal matrix

*Step* 2. Select an arbitrary $\Re^{m\times n}$ vector $\bar{F}$ such that $(\Lambda_{desired}, \bar{F})$ is observable

*Step* 3. Solve the Sylvester(Lyapunov) equation $AP - P\Lambda_{desired} = B\bar{F}$ for the unique

$P$.

*Step* 4. Compute the feedback gain $F = \bar{F}P^{-1}$ if the matrix $P$ is invertible. If $P$ is not invertible, go back to Step 2 and choose another $\bar{F}$.

## 1.9 Observer Eigenvalues Assignment

The corresponding Lyapunov method for obtaining the observer gain is to find the observer gain in the original coordinates. To find the observer gain, we need to transpose matrix $(A - KC)$. Consider the similarity transformation

$$P^{-1}(A^T - C^T K^T)P = \Lambda_{desired}^{obs} \tag{1.101}$$

where

$$\lambda(A^T - C^T K^T) = \lambda(\Lambda_{desired}^{obs}) = \lambda_{desired} \tag{1.102}$$

If $(A, C)$ is observable, $\lambda(A - KC)$ can be arbitrarily located according to (Chen, 1999). It is well known that the closed-loop eigenvalues of the observer should be located $5-6$ times farther to the left from the closed-loop system eigenvalues. Multiplying both side of (1.101) by $P$, (1.101) becomes the following Lyapunov equation

$$A^T P - P\Lambda_{desired}^{obs} = C^T \bar{K}^T \tag{1.103}$$

with

$$\bar{K}^T = K^T P \tag{1.104}$$

### 1.9.1 Design Procedure

For this section we introduce the procedure to compute the observer gain through the Lyapunov method. The following design procedure is presented in (Chen, 1999).Consider the observable pair $(A, C)$, where $A$ is $\Re^{n \times n}$ and $C$ is $\Re^{p \times n}$. Find a $\Re^{n \times p}$ real $K$ such that $(A - KC)$ has any set of desired eigenvalues that contains no eigenvalues of $A$.

*Step* 1. Select an arbitrary matrix $\Lambda_{desired}^{obs}$ that has no common eigenvalues with those of $A$.

*Step* 2. Select an arbitrary $\Re^{p\times n}$ vector $\bar{K}^T$ such that $(\Lambda_{desired}^{obs}, \bar{K}^T)$ is observable.

*Step* 3. Solve for the unique $P$ from the Sylvester equation $A^T P - P\Lambda_{desired}^{obs} = C^T \bar{K}^T$.

*Step* 4. Obtain the transposed observer gain from $K^T = \bar{K}^T P^{-1}$. If $P$ is not invertible, go back to Step 2 and choose another $\bar{K}^T$.

*Step* 5. Obtain the observer gain from $K = (K^T)^T$.

## 1.10 Conclusion

In this thesis, we will show that it is possible to design reduced-order slow and fast controllers exactly and independently in slow and fast time scales.

# Chapter 2

# Two-Time Scale Design for Singularly Perturbed Systems

## 2.1 Introduction

In this section, we will consider how one can place eigenvalues at the desired locations for each subsystems. Since a singular perturbed system has two sets of seperated eigenvalues, some of which are large, it might not be possible to place some of eigenvalues at the desired location in the case of the inaccessible states.

## 2.2 State Feedback Control via the Chang Transformation

This section presents state feedback control via the Chang transformation and shows how to obtain the feedback gain. Firstly, we show that the Chang transformation can't seperate slow and fast states when we consider the input signals.

Consider the singular perturbed system

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t), x_1(t_0) = x_{10} \\ \epsilon\dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t), x_2(t_0) = x_{20} \end{aligned} \tag{2.1}$$

We define matrices $A$ and $B$ as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{2.2}$$

Dividing by $\epsilon$ ,the following matrix form is obtained

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\epsilon} & \frac{A_{22}}{\epsilon} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ \frac{B_2}{\epsilon} \end{bmatrix} u_c(t) \tag{2.3}$$

Similarly, we define matrices $A_{new}$ and $B_{new}$ as

$$A_{new} = \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\epsilon} & \frac{A_{22}}{\epsilon} \end{bmatrix}, B_{new} = \begin{bmatrix} B_1 \\ \frac{B_2}{\epsilon} \end{bmatrix} \tag{2.4}$$

The above general singularly perturbed system (2.3) can be transformed into the following form via the Chang transformation.

$$\begin{bmatrix} \dot{x}_s(t) \\ \dot{x}_f(t) \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & \frac{A_f}{\epsilon} \end{bmatrix} \begin{bmatrix} x_s(t) \\ x_f(t) \end{bmatrix} + \begin{bmatrix} B_s \\ \frac{B_f}{\epsilon} \end{bmatrix} u_c(t) \tag{2.5}$$

System (2.5) can be divided into two subsystems represented by

$$\begin{aligned} \dot{x}_s(t) &= A_s x_s(t) + B_s u_c(t) \\ \dot{x}_f(t) &= \frac{A_f}{\epsilon} x_f(t) + \frac{B_f}{\epsilon} u_c(t) \end{aligned} \tag{2.6}$$

It looks like that the fast states are seperated from the slow states. However, they are coupled through the control signal as

$$u_c(t) = -F_s x_s(t) - F_f x_f(t) \tag{2.7}$$

substitute (2.7) into (2.6)

$$\begin{aligned} \dot{x}_s(t) &= (A_s - B_s F_s) x_s(t) + B_s F_f x_f(t) \\ \dot{x}_f(t) &= \frac{1}{\epsilon}(A_f - B_f F_f) x_f(t) + \frac{1}{\epsilon} B_f F_s x_s(t) \end{aligned} \tag{2.8}$$

From above (2.8), the pure seperation of fast and slow states are impossible, since the first part of equation in (2.8) contains fast states and the second part of equation in (2.8) contains slow states.

Therefore, the different method to seperate fast and slow states are proposed (Kokotovic *et al.*, 1999). Consider the composite state feedback form of a singularly perturbed system,

$$\dot{x}(t) = (A_{new} + B_{new} F) x(t), \quad x(t_0) = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} \tag{2.9}$$

with

$$\begin{aligned} A_{new} &= \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\epsilon} & \frac{A_{22}}{\epsilon} \end{bmatrix}, B_{new} = \begin{bmatrix} B_1(t) \\ \frac{B_2}{\epsilon}(t) \end{bmatrix}, \\ F &= \begin{bmatrix} F_1 & F_2 \end{bmatrix}, x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned} \tag{2.10}$$

The first step of the Chang transformation produces

$$\begin{bmatrix} \dot{x}_s(t) \\ \dot{x}_f(t) \end{bmatrix} = \begin{bmatrix} A_{11f} - A_{12f} L & A_{12f} \\ 0 & L A_{12f} + \frac{1}{\epsilon} A_{22f} \end{bmatrix} \begin{bmatrix} x_s(t) \\ x_f(t) \end{bmatrix} \tag{2.11}$$

with

$$A_{11f} = A_{11} + B_1F_1, A_{12f} = A_{12} + B_1F_2, A_{21f} = A_{21} + B_2F_1, A_{22f} = A_{22} + B_2F_2 \quad (2.12)$$

and

$$\epsilon LA_{11f} - \epsilon LA_{12f}L + A_{21f} - \epsilon A_{22f}L = 0 \quad (2.13)$$

Note that feedback matrices from (2.12) are present in equation (2.13). Applying the second step of the Chang transformation we obtain

$$\begin{bmatrix} \dot{x}_s(t) \\ \dot{x}_f(t) \end{bmatrix} = \begin{bmatrix} (A_{11f}) - (A_{12f})L & 0 \\ 0 & L(A_{12f}) + \frac{A_{22f}}{\epsilon} \end{bmatrix} \begin{bmatrix} x_s(t) \\ x_f(t) \end{bmatrix} \quad (2.14)$$

with

$$(A_{11f} - LA_{12f})\epsilon H + H(\epsilon LA_{12f} - A_{22f}) + A_{12f} = 0 \quad (2.15)$$

Where $A_{11f}, A_{12f}$ and $A_{22f}$ are feedback matrices. If $L$ equation (2.13) and $H$ equation (2.15) are satisfied, it is possible to decouple the slow and fast subsystems represented by equation (2.14)

### 2.2.1 Slow and Fast Time Scales : Uncorrected Method

The seperation of slow and fast sub-systems is discussed in the previous section. In this section, the method to obtain feedback gains for slow and fast subsystems will be considered. The composite input $u_c(t)$ is given by

$$u_c(t) = u_s(t) + u_f(t) = F_0x_s(t) + F_2x_f(t) \quad (2.16)$$

However, a realizable composite control requires that the system states $x_s(t)$ and $x_f(t)$ be expressed in terms of the actual system states $x_1(t)$ and $x_2(t)$ is proposed by Kokotovic, (Kokotovic *et al.*, 1999)

$$\begin{aligned} u_c(t) &= F_0x_1(t) + F_2[x_2(t) + A_{22}^{-1}(A_{21}x_s(t) + B_2F_0x_1(t))] \\ &= F_1x_1(t) + F_2x_2(t) \end{aligned} \quad (2.17)$$

with

$$F_1 = (I_r + F_2A_{22}^{-1}B_2)F_0 + F_2A_{22}^{-1}A_{21} \quad (2.18)$$

In the above design procedure the gain matrices $F_0$ and $F_2$ are seperately designed in the slow and fast time scales. The following theorem is presented by Kokotovic (Kokotovic *et al.*, 1999)

**Lemma 2.2.1.** *Let $F_2$ be designed such that $Re\lambda(A_{22} + B_2F_2) < 0$, then $\exists$ an $\epsilon > 0$ such that the composite control*

$$u_c(t) = F_0x_s(t) + F_2[x_2(t) + A_{22}^{-1}(A_{21}x_s(t) + B_2F_0x_s(t))] \tag{2.19}$$

*applied to the system* (2.3) *, produces the closed-loop system, ( starting from any bounded initial condition $x_{10}$ and $x_{20}$), with the following property*

$$x_1(t) = x_s(t) + O(\epsilon) \tag{2.20}$$

$$x_2(t) = -A_{22}^{-1}(A_{21} + B_2F_0)x_s(t) + x_f(t) + O(\epsilon) \tag{2.21}$$

$$u_c(t) = u_s(t) + u_f(t) + O(\epsilon) \tag{2.22}$$

*for all finite time $t \geq t_0$ and all $\epsilon \in (0, \epsilon^*]$. If in addition $F_0$ is designed such that $Re\lambda(A_0 + B_0F_0) < 0$, there exists an $\epsilon^* > 0$ such that the resulting closed-loop system is asymptotically stable and* (2.20) *-* (2.22) *hold for all $\epsilon \in (0, \epsilon^*$ and $t \in [t_0, \infty)$.*

*Proof.* (Kokotovic *et al.*, 1999) Consider a seperation of fast and slow states through the Chang transformation applied to equation (2.14), which produces

$$\begin{bmatrix} \dot{x}_s \\ \epsilon\dot{x}_f \end{bmatrix} = \begin{bmatrix} (A_{11f}) - (A_{12f})L & 0 \\ 0 & A_{22f} + \epsilon L(A_{12f}) \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} \tag{2.23}$$

with the initial conditions

$$\begin{aligned} x_s^0 &= x_1^0 - \epsilon H x_2^0 \\ x_f^0 &= x_2^0 + Lx_1^0 \end{aligned} \tag{2.24}$$

Where $A_{11f}, A_{12f}, A_{21f}$ and $A_{22f}$ are feedback matrices of (2.12). Instead of using $F_1$ and $F_2$ as design parameters, we use $F_s$ and $F_2$ as design parameters given by

$$F_1 = F_s + F_2L \tag{2.25}$$

Substituting equations (2.12) and (2.25) into $L$ equation (2.13) results in the following equation

$$L = A_{22}^{-1}(A_{21} + B_2F_s) + \epsilon A_{22}^{-1}L[(A_{11} + B_1F_s) - A_{12}L] \tag{2.26}$$

and substituting (2.26) into (2.23), using (2.25) and (2.12), leads to the closed-loop system

$$\begin{bmatrix} \dot{x}_s \\ \epsilon \dot{x}_f \end{bmatrix} = \begin{bmatrix} A_s + B_s F_s & 0 \\ 0 & A_f + B_f F_f \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} \tag{2.27}$$

with

$$\begin{aligned} A_s &= A_0 - \epsilon A_{12} A_{22}^{-1} L (A_{11} - A_{12} L) \\ B_s &= B_0 - \epsilon A_{12} A_{22}^{-1} L B_1 \\ A_f &= A_{22} + \epsilon A_{12}, B_f = B_2 + \epsilon L B_1 \end{aligned} \tag{2.28}$$

Since matrices $A_0, B_0, A_{22}$ and $B_2$ are approximation of matrices $A_s, B_s, A_f$ and $B_f$, we can say that the eigenvalues of matrices $\lambda(A_s + B_s F_s)$ and $\lambda(A_f + B_f F_f)$ are approximated by eigenvalues of $\lambda(A_0 + B_0 F_0)$ and $\lambda(A_{22} + B_2 F_2)$. □

**Eigenvalue Assignment in Slow and Fast Time Scales : Uncorrected Method**

Through the application of the Chang transformation, an $n$-dimensional eigenvalue placement can be reduced to seperate eigenvalue placement problems of dimension $n_1$ and $n_2$. The following theorem is proposed by Anderson (Anderson, 1982)

**Lemma 2.2.2.** *If a $(A, B)$ is controllable, there exists at least one real $m \times n$ dimensional feedback matrix $F$ such that the closed-loop eigenvalues given by $\lambda(A - BF)$ can be placed arbitrary by designer's decision. If the original system is controllable, it can be shown by linear algebra that the slow and fast subsystems in equation* (2.27) *are also controllable.*

The above lemma is considered in the original coordinates. However, one can ask a question about whether a designer can place slow and fast eigenvalues seperately. The singularly perturbed system has different sets of eigenvalues $Q$ and $R$. It can be shown that it is possible to relocate the $n_1$ slow open-loop eigenvalues $Q$ to $n_1$ new eigenvalue location $Q'$. The following theorem is presented by kokotovic (Kokotovic *et al.*, 1999)

**Lemma 2.2.3.** *If $A_{22}^{-1}$ exits and if the slow subsystem pair $(A_0, B_0)$ and the fast system pair $(A_{22}, B_2)$ are both controllable, and feedback gains $F_0, F_2$ are designed to assign distinct eigenvalues $\lambda_i, i = 1, ..., n_1$ and $\lambda_j, j = 1, ..., n_2$, to the matrices $A_0 + B_0 F_0$ and*

*$A_{22} + B_2F_2$ respectively, then $\exists$ an $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*]$ the application of the composite feedback control (uncorrected method) $u_c(t)$ given by*

$$u_c(t) = [(I_r + F_2A_{22}^{-1}B_2)F_0 + F_2A_{22}^{-1}A_{21}]x_1(t) + F_2x_2(t) \tag{2.29}$$

*results in a closed-loop system containing $n_1$ slow eigenvalues $(\lambda_1, \lambda_2, ...., \lambda_{n_1)}$ and $n_2$ fast eigenvalues $(\lambda_{n_1+1}, \lambda_{n_1+2}, ...., \lambda_{n_1+n_2)}$ , which are approximated by*

$$\begin{aligned} \lambda_i &= \lambda_i(A_0 + B_0F_0) + O(\epsilon), i = 1, ..., n_1 \\ \lambda_j &= \frac{[\lambda_j(A_{22} + B_2F_2) + O(\epsilon)]}{\epsilon}, i = n_1 + j, j = 1, ..., n_2 \end{aligned} \tag{2.30}$$

### 2.2.2 Slow and Fast Time Scales : Corrected Method

This method is different from the aforementioned uncorrected method since the state trajectories of the singularly perturbed system (2.1) are approximated by $O(\epsilon^2)$. The following Lemma is proposed by Kokotovic (Kokotovic *et al.*, 1999, pp. 99-101).

**Lemma 2.2.4.** *Using $A_s, B_s$ in (2.28), the corrected slow model is*

$$\begin{aligned} \dot{x}_{sc}(t) &= A_{0c}x_{sc}(t) + B_{0c}u_{sc}(t) \\ A_{0c} &= A_0 - \epsilon A_{12}A_{22}^{-1}L_0(A_{11} - A_{12}L_0) \\ B_{0c} &= B_0 - \epsilon A_{12}A_{22}^{-1}L_0B_1 \end{aligned} \tag{2.31}$$

*with the initial condition*

$$x_{sc}(t_0) = x_1^0 - \epsilon H_0(x_2^0 + L_0x_1^0) \tag{2.32}$$

*Similarly, the corrected fast model in the $\tau = \frac{t-t_0}{\epsilon}$ scale is given by*

$$\begin{aligned} \dot{x}_{fc}(\tau) &= A_{22c}x_{fc}(t) + B_{2c}u_{fc}(t) \\ A_{22c} &= A_{22} + \epsilon L_0A_{12} \\ B_{2c} &= B_2 + \epsilon L_0B_1 \end{aligned} \tag{2.33}$$

*with the initial condition*

$$x_{2fc}(t_0) = x_2^0 + L_1x_1^0 \tag{2.34}$$

*If the composite control*

$$u_c(t) = F_1x_1(t) + F_2x_2(t) \tag{2.35}$$

*with*

$$F_1 = F_{0c} + F_2 L_1 \tag{2.36}$$

*is applied to the system* (2.3) *, and if* $A_{22} + B_2 F_2$ *is a Hurwitz matrix, then there exists an* $\epsilon$ *such that the state and control of the closed-loop system, starting from any bounded initial conditions* $x_1^0$ *and* $x_2^0$*, are approximated based by*

$$x_1(t) = x_{sc}(t) + \epsilon H_0 x_{fc}(\tau) + O(\epsilon^2) \tag{2.37}$$

$$x_2(t) = -L_1 x_{sc}(t) + (I_m - \epsilon L_0 H_0) x_{fc}(\tau) + O(\epsilon^2) \tag{2.38}$$

$$u_c(t) = u_{sc}(t) + u_{fc}(\tau) + O(\epsilon^2) \tag{2.39}$$

*with*

$$\begin{aligned} L_0 &= A_{22}^{-1}(A_{21} + B_2 F_0), \\ L_1 &= A_{22}^{-1}(A_{21} + B_2 F_{0c}) + \epsilon A_{22}^{-1} L_0 (A_0 + B_0 F_0) \\ H_0 &= (A_{12} + B_1 F_2)(A_{22} + B_2 F_2)^{-1} \end{aligned} \tag{2.40}$$

*If* $A_0 + B_0 F_0$ *is also Hurwitz then there exists an* $\epsilon$ *such that the closed-loop system is asymptotically stable.*

The above Lemma shows that state trajectories is approximated by $O(\epsilon^2)$. The aforementioned uncorrected method has states trajectories (2.20) ,(2.21) and (2.22) with $O(\epsilon)$ which indicates corrected method is a better method to approximate states and input.

### Eigenvalues Assignment in Two-time Scale via the Corrected Method

Now that we consider the eigenvalue assignment via the corrected method, firstly, we present the following lemma (Kokotovic *et al.*, 1999)

**Lemma 2.2.5.** *If* $A_{22}^{-1}$ *exists and if the slow subsystem pair* $(A_0, B_0)$ *and the fast subsystem pair* $(A_{22}, B_2)$ *are each controllable then there exists an* $\epsilon^* > 0$ *and gain matrices* $F_{0c}$ *and* $F_2$ *which arbitrarily assign eigenvalues* $\lambda_i, i = 1, ..., n_1$ *and* $\lambda_j, j = 1, ..., n_2$ *to the closed-loop matrices* $A_{0c} + B_{0c} F_{0c}$ *and* $A_{22c} + B_{2c} F_2$*, respectively, such that for all* $\epsilon \in (0, \epsilon^*]$ *the application of the composite feedback control*

$$u_c(t) = [F_{0c} + F_2 L_1] x_1(t) + F_2 x_2(t) \tag{2.41}$$

*with*

$$L_1 = A_{22}^{-1}(A_{21} + B_2F_{0c}) + \epsilon A_{22}^{-1}(A_{21} + B_2F_0)(A_0 + B_0F_0) \tag{2.42}$$

*to the singularly perturbed system* (2.3) *results in the closed-loop system containing* $n_1$ *slow eigenvalues* $(\lambda_1^c, \lambda_2^c, ...., \lambda_{n_1^c})$ *and* $n_2$ *fast eigenvalues* $(\lambda_{n_1+1}^c, \lambda_{n_1+2}^c, ...., \lambda_{n_1+n_2)}^c$ *that are approximated by*

$$\begin{aligned} \lambda_i^c &= \lambda_i(A_{0c} + B_{0c}F_{0c}) + O(\epsilon^2), i = 1, ..., n_1 \\ \lambda_j^c &= \frac{[\lambda_j(A_{22c} + B_{2c}F_{2c}) + O(\epsilon^2)]}{\epsilon}, i = n_1 + j, j = 1, ..., n_2 \end{aligned} \tag{2.43}$$

The fast eigenvalues can be approximated by $O(\epsilon)$, since $O(\epsilon^2)$ approximation of the fast eigenvalues are devided by $\epsilon$.

**Numerical Example : Uncorrected Method**

This section illustrates the uncorrected method, with the accuracies $O(\epsilon)$ and corrected method $O(\epsilon^2)$ specifically. The matrices of the singularly perturbed system are given by

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\epsilon} & \frac{A_{22}}{\epsilon} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ \frac{B_2}{\epsilon} \end{bmatrix}, C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \tag{2.44}$$

For the singularly perturbed system (Kokotovic *et al.*, 1999, pp. 124-125), the matrices $A$,$B$ and $C$ are given by

$$A = \begin{bmatrix} 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.345 & 0 \\ 0 & -\frac{1}{\epsilon}0.524 & -\frac{1}{\epsilon}0.465 & \frac{1}{\epsilon}0.262 \\ 0 & 0 & 0 & -\frac{1}{\epsilon} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\epsilon} \end{bmatrix} \tag{2.45}$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The partitions matrices of $A$ are represented by

$$\begin{aligned} A_{11} &= \begin{bmatrix} 0 & 0.4 \\ 0 & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} 0 & 0 \\ 0.345 & 0 \end{bmatrix}, \\ A_{21} &= \begin{bmatrix} 0 & -0.524 \\ 0 & 0 \end{bmatrix}, A_{22} = \begin{bmatrix} -0.465 & 0.262 \\ 0 & -1 \end{bmatrix} \end{aligned} \tag{2.46}$$

The open-loop system has four eigenvalues at $(0, -0.4282, -4.2218, -10)$. $A_0$ and $B_0$ are defined in (1.12) and given by

$$A_0 = \begin{bmatrix} 0 & 0.4 \\ 0 & -0.3888 \end{bmatrix}$$
$$B_0 = \begin{bmatrix} 0 \\ 0.1944 \end{bmatrix} \tag{2.47}$$

Hence, the uncorrected slow gain $F_0$ is given by

$$F_0 = \begin{bmatrix} -12.8571 & -5.2741 \end{bmatrix} \tag{2.48}$$

The choice of $F_0$ places the slow eigenvalues of $A_0 + B_0 F_0$ at $-0.707 \pm j0.707$. However, the eigenvalues of the actual full system under this feedback control are approximated by $O(\frac{1}{10})$ in the case of $\epsilon = 0.1$. We also place the eigenvalues of the fast modes at $(-7, -8)$. The fast gain matrix $F_2$ is given by

$$F_2 = \begin{bmatrix} -0.3005 & -0.0350 \end{bmatrix} \tag{2.49}$$

The ill-conditioning of computation can be eliminated by this two-time scale method. Once we find the slow gain and the fast gain in the new coordinates, then the composite gain matrix $F$ in the original coordinates is given by

$$F = \begin{bmatrix} F_1 & F_2 \end{bmatrix} \tag{2.50}$$

with

$$F_1 = \begin{bmatrix} -12.8571 & -5.2741 \end{bmatrix}, F_2 = \begin{bmatrix} -0.3005 & -0.0350 \end{bmatrix} \tag{2.51}$$

The poles of $(A + BF)$ are given by

$$\lambda(A + BF) = \begin{bmatrix} -0.6412 + 0.8832i \\ -0.6412 - 0.8832i \\ -4.0273 \\ -9.6904 \end{bmatrix} \tag{2.52}$$

of which the slow eigenvalues are observed to be within $O(\epsilon)$ of the prescribed eigenvalues $(-0.707 \pm j0.707)$ and the fast eigenvalues are also within $O(1)$ of the prescribed eigenvalues $(-7, -8)$.

**Numerical Example : Corrected Method**

Now that we consider the corrected method from the uncorrected method. As previously mentioned, this method assigns eigenvalues with $O(\epsilon^2)$ approximation. Hence, the corrected method improves the uncorrected method regarding the eigenvalues assignment. From (2.31), $A_{0c}$ and $B_{0c}$ are given by

$$A_{0c} = \begin{bmatrix} 0 & 0.4 \\ -1.0162 & -0.6488 \end{bmatrix}$$
$$B_{0c} = \begin{bmatrix} 0 & 0.1944 \end{bmatrix} \tag{2.53}$$

From (2.33), $A_{22c}$ and $B_{2c}$ are given by

$$A_{22c} = \begin{bmatrix} -0.3236 & 0.2620 \\ 0.1820 & -1.0000 \end{bmatrix}$$
$$B_{2c} = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{2.54}$$

The slow gain $F_{0c}$ and the fast gain $F_{2c}$ are given by

$$F_{0c} = \begin{bmatrix} -7.6293 & -3.9367 \end{bmatrix} \tag{2.55}$$

and

$$F_{2c} = \begin{bmatrix} -0.8664 & -0.1764 \end{bmatrix} \tag{2.56}$$

Once we find the slow gain and the fast gain in the slow and fast coordinates through the corrected method, then the gain matrices (2.35)-(2.36) in the original coordinates are given by

$$F = \begin{bmatrix} F_{1c} & F_{2c} \end{bmatrix} \tag{2.57}$$

with

$$F_{1c} = \begin{bmatrix} -15.4838 & -8.2228 \end{bmatrix}, F_{2c} = \begin{bmatrix} -0.8664 & -0.1764 \end{bmatrix} \tag{2.58}$$

The eigenvalues of $(A + BF)$ are

$$\lambda(A + BF) = \begin{bmatrix} -0.7100 + 0.6922i \\ -0.7100 - 0.6922i \\ -7.4970 + 0.8539i \\ -7.4970 - 0.8539i \end{bmatrix} \tag{2.59}$$

The slow eigenvalues are observed to be within $O(\epsilon^2)$ of the prescribed eigenvalues $(-0.707 \pm j0.707)$ and the fast eigenvalues are within $O(\epsilon)$ of the prescribed fast eigenvalues $(-7, -8)$.

## 2.3 Two-Stage Design

The above uncorrected method and the corrected method shows $O(\epsilon)$ and $O(1)$ approximation for the fast eigenvalues, and $O(\epsilon^2)$ and $O(\epsilon)$ approximation for the slow eigenvalues. This section provides the exact assignment of both the slow and fast eigenvalues via state feedback. If the original system is given by

$$\dot{x}(t) = Ax(t) + Bu_c(t) \tag{2.60}$$

with

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\epsilon} & \frac{A_{22}}{\epsilon} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ \frac{B_2}{\epsilon} \end{bmatrix}, \\ x(t) &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{2.61}$$

Then (2.60) becomes the block-diagonal form using the Chang transformation (1.45), that is

$$\begin{bmatrix} x_s \\ x_f \end{bmatrix} = T_c^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.62}$$

Applying the Chang transformation (2.62) to the original singularly perturbed system (2.60), we have

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u_c(t) \tag{2.63}$$

with

$$\begin{aligned} \tilde{A} &= T_c^{-1} A T_c = \begin{bmatrix} A_s & 0 \\ 0 & \frac{A_f}{\epsilon} \end{bmatrix}, \tilde{B} = T_c^{-1} B = \begin{bmatrix} B_s \\ \frac{B_f}{\epsilon} \end{bmatrix} \\ \tilde{x}(t) &= \begin{bmatrix} x_s \\ x_f \end{bmatrix} \end{aligned} \tag{2.64}$$

System (2.63) can be expressed as

$$\begin{aligned} \dot{x}_s &= A_s x_s + B_s u(t) \\ \dot{x}_f &= \frac{A_f}{\epsilon} x_f + \frac{B_f}{\epsilon} u(t) \end{aligned} \tag{2.65}$$

It looks like that the fast states are seperated from the slow state by only considering matrix $\tilde{A}$ in (2.63). Consider the composite input is given by

$$u_c(t) = -F_s x_s(t) - F_f x_f(t) \tag{2.66}$$

Substituting (2.66) to (2.65), so that (2.65) becomes

$$\begin{aligned} \dot{x}_s(t) &= (A_s - B_s F_s) x_s(t) + B_s F_f x_f(t) \\ \dot{x}_f(t) &= \frac{1}{\epsilon}(A_f - B_f F_f) x_f(t) + \frac{1}{\epsilon} B_f F_s x_s(t) \end{aligned} \tag{2.67}$$

It follows from (2.67) that the seperation of fast and slow states is not obtained, since the slow and fast states are mixed again. Hence, the different method is needed to seperate exactly the fast and slow states via feedback control. Suppose, we desire to relocate only $n_1$ slow open-loop eigenvalues to $n_1$ new closed eigenvalues locations. Firstly, we only consider the driving term for the slow states represented by

$$u_c(t) = v(t) - F_s x_s(t) \tag{2.68}$$

Substituting the driving terms from slow states (2.68) in (2.65), the following equations are obtained

$$\begin{aligned} \dot{x}_s(t) &= (A_s - B_s F_s) x_s(t) \\ \dot{x}_f(t) &= \frac{A_f}{\epsilon} x_f(t) + \frac{B_f}{\epsilon} F_s x_s(t) \end{aligned} \tag{2.69}$$

This shows that the slow subsystem has no terms from the fast states. However, the fast subsystem has the terms with the slow state denoted by $\frac{B_f}{\epsilon} F_s x_s$ . Therefore, the second transformation $T_2$ needed to remove the remaining slow terms in $\dot{x}_f(t)$ is presented in the following lemma.

**Lemma 2.3.1.** *(Anderson, 1982)*

$$x_{fnew}(t) = P x_s(t) + x_f(t) \tag{2.70}$$

*and*

$$P(A_s - B_s F_s) - \frac{B_f}{\epsilon} F_s - \frac{A_f}{\epsilon} P = 0 \tag{2.71}$$

*If the above algebraic Lyapunov equation* (2.71) *is satisfied, it is possible to completely decouple the slow and fast states.*

*Proof.* From the second transformation (2.70), we take the derivative of equation (2.70) represented by

$$\begin{aligned}\dot{x}_{fnew}(t) &= P\dot{x}_s(t) + \dot{x}_f(t) \\ &= P(A_s - B_sF_s)x_s(t) + \frac{A_f}{\epsilon}(x_{fnew}(t) - Px_s(t)) - \frac{B_f}{\epsilon}F_sx_s(t) \\ &= [P(A_s - B_sF_s) - \frac{B_f}{\epsilon}F_s - \frac{A_f}{\epsilon}P]x_s(t) + \frac{A_f}{\epsilon}x_{fnew}(t)\end{aligned} \tag{2.72}$$

Therefore, if the Lyapunov equation (2.71) is satisfied, (2.72) becomes

$$\dot{x}_{fnew}(t) = \frac{A_f}{\epsilon}x_{fnew}(t) \tag{2.73}$$

□

This result is quite amazing, since we realize that there is a method to assign the slow eigenvalues without affecting the fast eigenvalues. To summarize the proposed exact relocation of eigenvalues, the state transformation is represented by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = T_c \begin{bmatrix} x_s \\ x_f \end{bmatrix} = T_cT_2 \begin{bmatrix} x_s \\ x_{fnew} \end{bmatrix} = T_{tot} \begin{bmatrix} x_s \\ x_{fnew} \end{bmatrix} \tag{2.74}$$

with

$$\begin{aligned} T_2 &= \begin{bmatrix} I_{n1} & 0 \\ -P & I_{n2} \end{bmatrix} \\ T_c &= \begin{bmatrix} I_{n1} & \epsilon H \\ -L & I_{n2} - \epsilon LH \end{bmatrix} \end{aligned} \tag{2.75}$$

This method is derived so as to obtain the gain $F$ in the original coordinate from the two stage transformed coordinate.

### 2.3.1 Two-Stage Design

This section summurizes previous method to assign exact eigenvalues assignment by proposing the following design algorithm.

*Stage* 1.

*Step* 1.

Apply the Chang transformation to the singularly perturbed system. Then, apply slow states feedback control denoted by $u(t) = v(t) - F_s x_s(t)$. Substituting $u(t) = v(t) - F_s x_s(t)$ into (2.63) produces

$$\begin{bmatrix} \dot{x}_s(t) \\ \dot{x}_f(t) \end{bmatrix} = \begin{bmatrix} A_s - B_s F_s & 0 \\ \frac{B_f}{\epsilon} F_s & \frac{A_f}{\epsilon} \end{bmatrix} \begin{bmatrix} x_s(t) \\ x_f(t) \end{bmatrix} + \begin{bmatrix} B_s \\ \frac{B_f}{\epsilon} \end{bmatrix} v(t) \tag{2.76}$$

From the above equation, we can place slow eigenvalues in the desired location represented by

$$\lambda(A_s - B_s F_s) = \lambda_s^{desired} \tag{2.77}$$

At this point, we need the following assumption.

**Assumption 2.3.1.** *The pair $(A_s, B_s)$ is controllable.*

Under this assumption, according to (Chen, 1999), the eigenvalues of $A_s - B_s F_s$ can be arbitrarily located into the desired locations.

*Step* 2. Introduce another transformation $T_2$

$$\begin{bmatrix} x_s(t) \\ x_{fnew}(t) \end{bmatrix} = T_2^{-1} \begin{bmatrix} x_s(t) \\ x_f(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} x_s(t) \\ x_f(t) \end{bmatrix} \tag{2.78}$$

Apply the second transformation to (2.76) to obtain a block diagonal form as

$$\begin{bmatrix} \dot{x}_s(t) \\ \dot{x}_{fnew}(t) \end{bmatrix} = \begin{bmatrix} A_s - B_s F_s & 0 \\ 0 & \frac{A_f}{\epsilon} \end{bmatrix} \begin{bmatrix} x_s(t) \\ x_{fnew}(t) \end{bmatrix} + \begin{bmatrix} B_s \\ PB_s + \frac{B_f}{\epsilon} \end{bmatrix} v(t) \tag{2.79}$$

Which requires that the algebraic Lyapunov equation be satisfied

$$P(A_s - B_s F_s) - \frac{B_f}{\epsilon} F_s - \frac{A_f}{\epsilon} P = 0 \tag{2.80}$$

*Stage*2.

Firstly, we take $v(t) = F_{f2} x_{fnew}(t)$, then obtain the gain $F_f$ using the eigenvalues placement such that

$$\lambda\left(\frac{A_f}{\epsilon} - (PB_s + \frac{B_f}{\epsilon})F_{f2}\right) = \lambda_f^{desired} \tag{2.81}$$

Similarly, we need the following assumption.

**Assumption 2.3.2.** *The pair* $(A_f, -\epsilon PB_s - B_f)$ *is controllable.*

Taking $v(t) = F_{f2}x_{fnew}(t)$, the subsystem (2.79) can be represented as

$$\dot{\bar{x}}(t) = \bar{A}_{fcl}\bar{x}(t) \tag{2.82}$$

with

$$\begin{aligned} \bar{A}_{fcl} &= \begin{bmatrix} A_s - B_sF_s & B_sF_f \\ 0 & \frac{A_f}{\epsilon} - (PB_s + \frac{B_f}{\epsilon})F_{f2} \end{bmatrix} \\ \bar{x}(t) &= \begin{bmatrix} x_s \\ x_{fnew} \end{bmatrix} \end{aligned} \tag{2.83}$$

From the block triangular form of matrix $\bar{A}_{fcl}$ in (2.82), the independent slow and fast eigenvalues assignment can be represented by

$$\lambda(\bar{A}_{fcl}) = \lambda(A_s - B_sF_s)\bigcup\lambda[A_f - (PB_s + \frac{B_f}{\epsilon})F_{f2}] \tag{2.84}$$

This two stage method shows the independent slow and fast eigenvalues assignment.

### 2.3.2 Exact Eigenvalues Assignment

After obtaining gain $F_s$ and $F_{f2}$ via the two stage design, there is a way to go back to the original coordinates without changing eigenvalues location. We define $u_s(t) = F_sx_s(t)$ and $v(t) = F_fx_{fnew}(t)$. Therefore the input in (2.68) can be represented by

$$u_c(t) = v(t) + F_sx_s(t) = F_fx_{fnew}(t) + F_sx_s(t) \tag{2.85}$$

Formula (2.85) can be expressed in the following matrix form.

$$\begin{aligned} u_c(t) &= \begin{bmatrix} F_s & F_{f2} \end{bmatrix} \begin{bmatrix} x_s(t) \\ x_{fnew}(t) \end{bmatrix} = \begin{bmatrix} F_s & F_{f2} \end{bmatrix} T_2^{-1} \begin{bmatrix} x_s(t) \\ x_f(t) \end{bmatrix} \\ &= \begin{bmatrix} F_s & F_{f2} \end{bmatrix} \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} T_c \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &= \begin{bmatrix} F_s + F_{f2}P & F_{f2} \end{bmatrix} T_c \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned} \tag{2.86}$$

where $T_c$ is the Chang transformation (1.45) and $T_2$ is the second transformation (2.70) . Hence the gains $F_1$ and $F_2$ in the original coordinates are given by

$$
\begin{aligned}
u(t) &= F \begin{bmatrix} x_1(t) \\ x_2 \end{bmatrix} = \begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\
&= \begin{bmatrix} F_s + F_{f2}P & F_{f2} \end{bmatrix} T_c \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
\end{aligned}
\tag{2.87}
$$

and

$$
\begin{aligned}
F_1 &= F_s + F_{f2}P \\
F_2 &= F_{f2}
\end{aligned}
\tag{2.88}
$$

Q.E.D.

The formula (2.88) is the final step to obtain the gain $F$ in the original coordinates after we assign the eigenvalues for each subsystems via the two stage method.

### 2.3.3 Numerical Example : Two Stage Design

The singularly perturbed system is given by (2.45). For the design algorithm Step 1 in Stage 1, we apply the Chang transformation to (2.45). The matrices $A_s, B_s$ and $A_f, B_f$ are given by

$$
\begin{aligned}
A_s &= \begin{bmatrix} 0 & 0.4000 \\ 0 & -0.4282 \end{bmatrix}, B_s = \begin{bmatrix} -0.0325 \\ 0.2489 \end{bmatrix} \\
\frac{A_f}{\epsilon} &= \begin{bmatrix} -4.2218 & 2.6200 \\ 0 & -10.0000 \end{bmatrix}, \frac{B_f}{\epsilon} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}
\end{aligned}
\tag{2.89}
$$

Considering slow state feedback, the matrix form (2.76) can be obtained so that we can assign the slow eigenvalues at the desired location $\lambda[A_s + B_s G_s] = (-0.707 \pm 0.707 \times j)$. The slow gain $F_s$ is given by

$$
F_s = \begin{bmatrix} -11.6731 & -5.4857 \end{bmatrix} \tag{2.90}
$$

For Step 2 in Stage 1, after taking the second transformation (2.78), we solve the algebraic Lyapunov equation (2.80) and matrix $P$ is given by

$$
P = \begin{bmatrix} 9.7886 & 3.9027 \\ 12.8635 & 5.7780 \end{bmatrix} \tag{2.91}
$$

For Stage 2 of the design algorithm, we can assign the fast eigenvalues using $\lambda[A_f + (PB_s + B_f)F_{f2}] = (-7, -8)$ and the fast two-stage transformed gain $F_{f2}$ is given by

$$F_{f2} = \begin{bmatrix} -0.3215 & -0.0516 \end{bmatrix} \tag{2.92}$$

The final step is to compute the gain $F$ in the original coordinate using (2.87) and (2.88)

$$F = \begin{bmatrix} -15.4838 & -8.0664 & -0.8282 & -0.17640 \end{bmatrix} \tag{2.93}$$

The eigenvalues in the original coordinate $\lambda(A + BF)$ are given by

$$\lambda(A + BF) = \begin{bmatrix} -0.7070 + 0.7070i \\ -0.7070 - 0.7070i \\ -7.0000 \\ -8.0000 \end{bmatrix} \tag{2.94}$$

which exactly represent the eigenvalues we intend to place for each subsystem.

# Chapter 3

# Two-Stage Observer Design for Singularly Perturbed Systems

## 3.1 Introduction

The objective of this chapter is to present the design of a full-order observer in slow and fast time scales. This chapter is organized as follows. Section 3.2 briefly reviews the decomposition of a singularly perturbed system, for which a full-order observer will be designed is Section 3.3 using the two-stage design. In Section 3.4, conditions are formulated under which a composite observer reconstructs the state of the original singularly perturbed system. The composite observer is reconstructed from the two independent sub-systems in slow and fast time scales. Numerical example with the exact observer eigenvalue assignment is presented in Sections 3.5 and 3.6. This chapter concludes with a discussion of the duality between the observer and the controller designs for singularly perturbed systems, Section 3.7.

## 3.2 Mode Seperation of Singularly Perturbed Linear Systems

Consider singularly perturbed linear time-invariant system with some inaccessible states represented by

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \\ \epsilon\dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) \\ y(t) &= C_1x_1(t) + C_2x_2(t) \end{aligned} \tag{3.1}$$

where $\epsilon > 0$ is small positive singular perturbation parameter, and $x_1(t) \in \Re^{n_1}, x_2(t) \in \Re^{n_2}$ are state vectors. $u(t) \in \Re^p$ is $p-$ dimensional control vector, and $y(t) \in \Re^q$ is the system output vector. Equation (3.1) may be approximately decomposed into a reduced order system with $n_1$ slow modes and a reduced-order fast subsystem with $n_2$ fast modes. (Kokotovic *et al.*, 1999). By neglecting the fast modes done by setting $\epsilon = 0$ in (3.1), we have

$$\dot{x}_s(t) = A_{11}x_s(t) + A_{12}x_{2s}(t) + B_1u_s(t) \tag{3.2}$$

$$0 = A_{21}x_s(t) + A_{22}x_{2s}(t) + B_2u_s(t) \tag{3.3}$$

$$y_s(t) = C_1x_s(t) + C_2x_{2s}(t) \tag{3.4}$$

Where $x_{2s}(t)$ is the slow state part of $x_2(t)$. Since $x_2(t) = x_{2s}(t) + x_{2f}(t)$ , the reduced states $x_{2s}(t)$ in $x_2(t)$ can be obtained from (3.3). The following assumption has to be satisfied for obtaining the slow subsystem.

Under Assumption 1.1.1, equation (3.3) can be solved for $x_{2s}(t)$, and substituting (3.2) and (3.4), produces the reduced-order slow sub-system

$$\begin{aligned} &\dot{x}_s(t) = A_0x_s(t) + B_0u_s(t) \ , x_s(t_0) = x_{10} \\ &y_s(t) = C_0x_s(t) \end{aligned} \tag{3.5}$$

with

$$\begin{aligned} &A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}, \\ &B_0 = B_1 - A_{12}A_{22}^{-1}B_2 \\ &C_0 = C_1 - C_2A_{22}^{-1}A_{21} \end{aligned} \tag{3.6}$$

Furthermore, the fast subsystem, defined by

$$\begin{aligned} &\dot{x}_f(\tau) = A_{22}x_f(\tau) + B_2u_f(\tau) \ , x_f(t_0) = x_2(t_0) - x_{2s}(t_0) \\ &y_f(\tau) = C_2x_f(\tau) \end{aligned} \tag{3.7}$$

is represented by the fast time scale $\tau$ where

$$\tau = \frac{t - t_0}{\epsilon}, \epsilon = 0 \ at \ t = t_0 \tag{3.8}$$

## 3.3 Two-Stage Design of the Full-Order Observer

The singularly perturbed system (3.1) can be rewritten as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\epsilon} & \frac{A_{22}}{\epsilon} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ \frac{B_2}{\epsilon} \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{3.9}$$

A full-order observer the system (3.9) is given by (Luenberger 1964)

$$\begin{bmatrix} \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\epsilon} & \frac{A_{22}}{\epsilon} \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ \frac{B_2}{\epsilon} \end{bmatrix} u_c(t) + \begin{bmatrix} K_1 \\ \frac{K_2}{\epsilon} \end{bmatrix} (y(t) - \hat{y}(t)) \tag{3.10}$$

with

$$u_c(t) = \begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix}$$
$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} \tag{3.11}$$

When applying the two-stage method to a full-order observer, it will be needed to transpose the full-order observer (3.10), that is

$$\begin{bmatrix} \dot{\hat{x}}_1^T(t) \\ \dot{\hat{x}}_2^T(t) \end{bmatrix} = \begin{bmatrix} A_{11}^T & \frac{A_{21}^T}{\epsilon} \\ A_{12}^T & \frac{A_{22}^T}{\epsilon} \end{bmatrix} \begin{bmatrix} \hat{x}_1^T(t) \\ \hat{x}_2^T(t) \end{bmatrix}$$
$$+ \begin{bmatrix} F_1^T \\ F_2^T \end{bmatrix} \begin{bmatrix} B_1^T & \frac{B_2^T}{\epsilon} \end{bmatrix} \begin{bmatrix} \hat{x}_1^T(t) \\ \hat{x}_2^T(t) \end{bmatrix} + \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix} K^T (x^T(t) - \hat{x}^T(t)) \tag{3.12}$$

with

$$K^T (x^T(t) - \hat{x}^T(t)) = \begin{bmatrix} K_1^T & \frac{K_2^T}{\epsilon} \end{bmatrix} ( \begin{bmatrix} x_1^T(t) \\ x_2^T(t) \end{bmatrix} - \begin{bmatrix} \hat{x}_1^T(t) \\ \hat{x}_2^T(t) \end{bmatrix} ) \tag{3.13}$$

where states $\hat{x}_1^T(t)$ and $\hat{x}_2^T(t)$ are the states in the transposed form of the full-order observer defined in (3.12). The state transformation of Chang is defined by

$$\begin{bmatrix} \hat{x}_1^T(t) \\ \hat{x}_2^T(t) \end{bmatrix} = \begin{bmatrix} I_n & \epsilon H \\ -L & -L\epsilon H + I_m \end{bmatrix} \begin{bmatrix} \hat{x}_s^T(t) \\ \hat{x}_f^T(t) \end{bmatrix} = (T_c^{obs})^{-1} \begin{bmatrix} \hat{x}_s^T(t) \\ \hat{x}_f^T(t) \end{bmatrix} \tag{3.14}$$

where $L$ and $H$ matrices satisfy.

$$\epsilon L(A_{11}^T - A_{21}^T L) + (A_{12}^T - A_{22}^T L) = 0 \tag{3.15}$$

$$\epsilon(A_{11}^T - A_{21}^T L)H + A_{21}^T - H(A_{22}^T + \epsilon L A_{21}^T) = 0 \tag{3.16}$$

When we apply the Chang Transformation (3.14) to the transposed system of the full-order observer (3.12), we obtain

$$\begin{bmatrix} \dot{\hat{x}}_s^T(t) \\ \dot{\hat{x}}_f^T(t) \end{bmatrix} = \begin{bmatrix} A_s^T & 0 \\ 0 & \frac{1}{\epsilon}A_f^T \end{bmatrix} \begin{bmatrix} \hat{x}_s(t) \\ \hat{x}_f(t) \end{bmatrix} + \begin{bmatrix} F_s^T \\ F_f^T \end{bmatrix} \begin{bmatrix} B_s^T & \frac{1}{\epsilon}B_f^T \end{bmatrix} \begin{bmatrix} \hat{x}_s^T(t) \\ \hat{x}_f^T(t) \end{bmatrix} + \begin{bmatrix} C_s^T \\ C_f^T \end{bmatrix} K^T(x^T - \hat{x}^T) \tag{3.17}$$

with

$$\begin{aligned} \begin{bmatrix} A_s^T & 0 \\ 0 & \frac{1}{\epsilon}A_f^T \end{bmatrix} &= T_c^{obs-1} \begin{bmatrix} A_{11}^T & \frac{1}{\epsilon}A_{21}^T \\ A_{12}^T & \frac{1}{\epsilon}A_{22}^T \end{bmatrix} T_c^{obs} \\ \begin{bmatrix} F_s^T \\ F_f^T \end{bmatrix} &= (T_c^{obs})^{-1} \begin{bmatrix} F_1^T \\ F_2^T \end{bmatrix} \\ \begin{bmatrix} C_s^T \\ C_f^T \end{bmatrix} &= (T_c^{obs})^{-1} \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix} \end{aligned} \tag{3.18}$$

The gain term $K^T$ can be regarded as state feedback of the transposed system. Using a memoryless slow state feedback of the form $K^T x^T(t) = v(t) - K_s^T x_s^T(t)$ and $K^T \hat{x}^T = \hat{v}(t) - K_s^T \hat{x}_s^T(t)$ in (3.17), we obtain

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}_s^T(t) \\ \dot{\hat{x}}_f^T(t) \end{bmatrix} &= \begin{bmatrix} A_s^T + C_s^T K_s^T & 0 \\ C_f^T K_s^T & \frac{1}{\epsilon}A_f^T \end{bmatrix} \begin{bmatrix} \hat{x}_s^T(t) \\ \hat{x}_f^T(t) \end{bmatrix} \\ &+ \begin{bmatrix} F_s^T \\ F_f^T \end{bmatrix} \begin{bmatrix} B_s^T & \frac{1}{\epsilon}B_f^T \end{bmatrix} \begin{bmatrix} \hat{x}_s^T(t) \\ \hat{x}_f^T(t) \end{bmatrix} + \begin{bmatrix} C_s^T \\ C_f^T \end{bmatrix} ((v(t) - K_s^T x_s^T(t)) - \hat{v}(t)) \end{aligned} \tag{3.19}$$

At this point, it is possible to locate slow eigenvalues at the desired location given as

$$\lambda(A_s + K_s C_s) = \lambda_s^{desired} \tag{3.20}$$

Now we introduce the second transformation $T_2$ represented by

$$\begin{bmatrix} \hat{x}_s^T(t) \\ \hat{x}_{fnew}^T(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ P_o & I \end{bmatrix} \begin{bmatrix} \hat{x}_s^T(t) \\ \hat{x}_f^T(t) \end{bmatrix} \tag{3.21}$$

where $P$ satisfies the algebraic Lyapunov equation.

$$P_o(A_s^T + C_s^T K_s^T) + C_f^T K_s^T - (\frac{A_f^T}{\epsilon})P_o = 0 \tag{3.22}$$

By applying the second transformation (3.21) to (3.19), we obtain

$$\begin{bmatrix} \dot{\hat{x}}_s^T(t) \\ \dot{\hat{x}}_{fnew}^T(t) \end{bmatrix} = \begin{bmatrix} A_s^T + C_s^T K_s^T & 0 \\ 0 & \frac{1}{\epsilon}A_f^T \end{bmatrix} \begin{bmatrix} \hat{x}_s^T(t) \\ \hat{x}_{fnew}^T(t) \end{bmatrix}$$
$$+ \begin{bmatrix} I & 0 \\ P_o & I \end{bmatrix} \begin{bmatrix} F_s^T \\ F_f^T \end{bmatrix} \begin{bmatrix} B_s^T & \frac{1}{\epsilon}B_f^T \end{bmatrix} \begin{bmatrix} \hat{x}_s^T(t) \\ \hat{x}_{fnew}^T(t) \end{bmatrix} + \begin{bmatrix} I & 0 \\ P_o & I \end{bmatrix} \begin{bmatrix} C_s^T \\ C_f^T \end{bmatrix} ((v(t) - K_s^T x_s^T(t)) - \hat{v}(t)) \tag{3.23}$$

Now that we apply $\hat{v}(t) = -\frac{1}{\epsilon}K_{f2}^T \hat{x}_{fnew}^T(t)$ and $v(t) = -\frac{1}{\epsilon}K_{f2}^T x_{fnew}^T(t)$ to the system (3.23) , to obtain

$$\begin{bmatrix} \dot{\hat{x}}_s^T(t) \\ \dot{\hat{x}}_{fnew}^T(t) \end{bmatrix} = \begin{bmatrix} A_s^T + C_s^T K_s^T & C_s^T \frac{K_{f2}^T}{\epsilon} \\ 0 & (\frac{1}{\epsilon}A_f^T) + (P_o C_s^T + C_f^T)\frac{K_{f2}^T}{\epsilon} \end{bmatrix} \begin{bmatrix} \hat{x}_s^T(t) \\ \hat{x}_{fnew}^T(t) \end{bmatrix}$$
$$+ \begin{bmatrix} F_s^T \\ P_o F_s^T + F_f^T \end{bmatrix} \begin{bmatrix} B_s^T & \frac{1}{\epsilon}B_f^T \end{bmatrix} \begin{bmatrix} \hat{x}_s^T(t) \\ \hat{x}_{fnew}^T(t) \end{bmatrix} + \begin{bmatrix} C_s^T \\ P_o C_s^T + C_f^T \end{bmatrix} (-\frac{1}{\epsilon}K_{f2}^T x_{fnew}^T(t) - K_s^T x_s^T(t)) \tag{3.24}$$

The error dynamic is determined by the following equation (3.24)

$$\begin{bmatrix} \dot{e}_s(t) \\ \dot{e}_{fnew}(t) \end{bmatrix} = A_e \begin{bmatrix} e_s(t) \\ e_{fnew}(t) \end{bmatrix} \tag{3.25}$$

with

$$A_e = \begin{bmatrix} A_s^T + C_s^T K_s^T & C_s^T \frac{K_{f2}^T}{\epsilon} \\ 0 & (\frac{1}{\epsilon}A_f^T) + (P_o C_s^T + C_f^T)\frac{K_{f2}^T}{\epsilon} \end{bmatrix} \tag{3.26}$$

where

$$\begin{aligned} e_s(t) &= e^{(A_s^T + C_s^T K_s^T)} e_s(t_0) \\ e_{fnew}(t) &= e^{((\frac{1}{\epsilon}A_f^T) + (P_o C_s^T + C_f^T)\frac{K_{f2}^T}{\epsilon})} e_{fnew}(t_0) \end{aligned} \tag{3.27}$$

At this point, it is possible to locate fast eigenvalues at the desired location represented by

$$\lambda(\frac{1}{\epsilon}A_f + \frac{K_{f2}}{\epsilon}(P_o C_s^T + C_f^T)^T) = \lambda_f^{desired} \tag{3.28}$$

Note that the matrix $A_e$ indicates the seperation principle $\lambda(A_s^T + C_s^T K_s^T) \bigcup \lambda(\frac{1}{\epsilon}A_f^T + (P_o C_s^T + C_f^T)\frac{K_{f2}^T}{\epsilon})$ which follows from the block triangular form of (3.26) . By transposing (3.24), the observer system can be represented by

$$\begin{bmatrix} \dot{\hat{x}}_s(t) \\ \dot{\hat{x}}_{fnew}(t) \end{bmatrix} = \begin{bmatrix} A_s + K_s C_s & 0 \\ K_{f2} C_s & (\frac{1}{\epsilon}A_f) + \frac{K_{f2}}{\epsilon}(P_o C_s^T + C_f^T)^T \end{bmatrix} \begin{bmatrix} \hat{x}_s(t) \\ \hat{x}_{fnew}(t) \end{bmatrix}$$
$$+ \begin{bmatrix} B_s \\ \frac{1}{\epsilon}B_f \end{bmatrix} \begin{bmatrix} F_s & P_o F_s + F_f \end{bmatrix} \begin{bmatrix} \hat{x}_s(t) \\ \hat{x}_{fnew}(t) \end{bmatrix} + \begin{bmatrix} -K_s \\ -\frac{K_{f2}}{\epsilon} \end{bmatrix} \begin{bmatrix} C_s & (P_o C_s^T + C_f^T)^T \end{bmatrix} \begin{bmatrix} x_s(t) \\ x_{fnew}(t) \end{bmatrix} \tag{3.29}$$

It is required that the two-stage transformed observer (3.29) reconstruct the state $\begin{bmatrix} x_s(t) \\ x_{fnew}(t) \end{bmatrix}$ of the two-stage transformed original system in a uniformly asymptotic manner in the sense that

$$\lim_{t\to\infty} \begin{bmatrix} e_s(t) \\ e_{fnew}(t) \end{bmatrix} = 0 \tag{3.30}$$

The observability assumption is needed for (3.30) to hold

**Assumption 3.3.1.** *The pair* $(A, C)$ *is observable*

Prior to designing a composite observer which will exactly describes the asymptotic state reconstruction of the observer (3.10) with respect to original system (3.9), it is necessary to consider slow and fast observers of (3.29).

### 3.3.1 An Observer for the Slow Subsystem

An observer for the slow subsystem is represented (Luenberger 1964) by

$$\dot{\hat{x}}_s(t) = (A_s + K_s C_s)\hat{x}_s + B_s u(t) - K_s y(t) \tag{3.31}$$

Where the state reconstruction error is represented by

$$e_s(t) = \hat{x}_s(t) - x_s(t) \tag{3.32}$$

From (3.25), the slow error dynamics can be represented by

$$\dot{e}_s(t) = (A_s - K_s C_s)e_s(t),\ e_s(t_0) = \hat{x}_s(t_0) - x_s(t_0) \tag{3.33}$$

By definition, the observer (3.31) will uniformly asymptotically reconstruct the state $x_s(t)$ if

$$\lim_{t\to\infty} e_s(t) = 0 \tag{3.34}$$

The observability assumption is needed for (3.34) to hold

**Assumption 3.3.2.** *The pair $(A_s, C_s)$ is observable*

### 3.3.2 An Observer for the Fast Subsystem

Similarly, a full-order observer for the fast subsystem is represented by

$$\dot{\hat{x}}_{fnew}(\tau) = \frac{1}{\epsilon}A_f + \frac{1}{\epsilon}K_{f2}(P_oC_s^T + C_f^T)^T)\hat{x}_{fnew}(\tau) + \frac{1}{\epsilon}B_f u(\tau) - \frac{1}{\epsilon}K_{f2}y(\tau) \tag{3.35}$$

where the state reconstruction error is defined by

$$e_{fnew}(\tau) = \hat{x}_{fnew}(\tau) - x_{fnew}(\tau) \tag{3.36}$$

From (3.25)

$$\dot{e}_{fnew}(\tau) = \frac{1}{\epsilon}A_f + \frac{1}{\epsilon}K_{f2}(P_oC_s^T + C_f^T)^T)e_{fnew}(\tau) \tag{3.37}$$

The observer (3.35) will reconstruct the state $x_f(\tau)$ in an asymptotic manner if

$$\lim_{\tau\to\infty} e_{fnew}(\tau) = 0 \tag{3.38}$$

The observability assumption is needed for (3.38) to hold

**Assumption 3.3.3.** *The pair $(A_f, C_f + C_sP_o)$ is observable.*

## 3.4 State Reconstruction for the Composite System

In this section we will obtain a composite observer based on the two time-scales observers (3.31) and (3.35) respectively. It is desired that this composite observer exactly describes the original states. Sufficient conditions for an exact reconstruction are given in the following theorem.

**Lemma 3.4.1.** *If the observers* (3.29) *are applied to systems* (3.9) *with*

$$K = (\begin{bmatrix} K_s^T + \frac{1}{\epsilon}K_{f2}^T P_o & \frac{1}{\epsilon}K_{f2}^T \end{bmatrix} T_c^{-1})^T \tag{3.39}$$

*where $T_c$ is the Chang transformation, $P$ is the solution of the algebraic Lyapunov equation* (3.22)*, and if $A_s+K_sC_s$ and $\frac{1}{\epsilon}A_f+\frac{1}{\epsilon}K_{f2}(PC_s^T+C_f^T)^T$ are uniformly asymptotically stable, the eigenvalues related to the error dynamics in the original coordinates satisfy*

$$\begin{aligned}\lambda_i &= \lambda_i(A_s+K_sC_s), i=1,...,n_1\\ \lambda_j &= \frac{[\lambda_j(A_f+K_{f2}(\epsilon P_oC_s^T+C_f^T)^T)]}{\epsilon}, i=n_1+j, j=1,...,n_2\end{aligned} \tag{3.40}$$

*Proof.* Upon appropriate partitioning of the error system in the original coordinate represented by

$$\begin{bmatrix}\dot{e}_1(t)\\ \epsilon\dot{e}_2(t)\end{bmatrix}=\begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ \tilde{A}_{21} & \tilde{A}_{22}\end{bmatrix}\begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix} \tag{3.41}$$

with

$$K=\begin{bmatrix}K_1\\ \frac{K_2}{\epsilon}\end{bmatrix} \tag{3.42}$$

and

$$\begin{aligned}\tilde{A}_{11} &= A_{11}-K_1C_1, \tilde{A}_{12}=A_{12}-K_1C_2\\ \tilde{A}_{21} &= \frac{1}{\epsilon}(A_{21}-K_2C_1), \tilde{A}_{22}=\frac{1}{\epsilon}(A_{22}-K_2C_2)\end{aligned} \tag{3.43}$$

then a seperation of (3.41) into slow and fast mode is facilitated by the following total transformation composed of the Chang transformation $T_c$ and the second transformation $T_2$

$$\begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix}=\begin{bmatrix}I_{n_1} & \epsilon H\\ -L & -\epsilon LH+I_{n_2}\end{bmatrix}\begin{bmatrix}I_{n_1} & 0\\ -P_o & I_{n_2}\end{bmatrix}\begin{bmatrix}e_s(t)\\ e_{fnew}(t)\end{bmatrix} \tag{3.44}$$

where the matrices $L$ , $H$ are solution of the following algebraic equation

$$0=\epsilon L(\tilde{A}_{11}-\tilde{A}_{12}L)+(\tilde{A}_{21}-\tilde{A}_{22}L) \tag{3.45}$$

$$0=\epsilon(\tilde{A}_{11}-\tilde{A}_{12}L)H+\tilde{A}_{12}-H(\tilde{A}_{22}+\epsilon L\tilde{A}_{12}) \tag{3.46}$$

and matrix $P$ satisfies

$$P_o(A_s^T+C_s^TK_s^T)+C_f^TK_s^T-(\frac{A_f^T}{\epsilon})P_o=0 \tag{3.47}$$

Since eigenvalues of the block traiangular matrix of (3.25) are composed of slow and fast eigenvalues, it is possible to place eigenvalues of the slow and fast states independently.

We previously set $K^T\hat{x}^T(t) = v(t) - K_s^T\hat{x}_s^T(t) = -K_s^T\hat{x}_s^T(t) - \frac{1}{\epsilon}K_{f2}^T\hat{x}_{fnew}^T(t)$ in (3.19) and (3.24) , which implies

$$\begin{aligned} K^T\hat{x}^T &= \begin{bmatrix} K_s^T & \frac{1}{\epsilon}K_{f2}^T \end{bmatrix} \begin{bmatrix} \hat{x}_s^T \\ \hat{x}_{fnew}^T \end{bmatrix} \\ &= \begin{bmatrix} K_s^T & \frac{1}{\epsilon}K_{f2}^T \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 \\ P_o & I_{n_2} \end{bmatrix} \begin{bmatrix} \hat{x}_s^T \\ \hat{x}_f^T \end{bmatrix} \\ &= \begin{bmatrix} K_s^T + \frac{1}{\epsilon}K_{f2}^T P_o & \frac{1}{\epsilon}K_{f2}^T \end{bmatrix} T_c^{-1} \begin{bmatrix} \hat{x}_1^T \\ \hat{x}_2^T \end{bmatrix} \end{aligned} \tag{3.48}$$

Hence $\begin{bmatrix} K_s^T + \frac{1}{\epsilon}K_{f2}^T P & \frac{1}{\epsilon}K_{f2}^T \end{bmatrix} T_c^{-1}$ represents transpose of observer gain matrix $K$ in original coordinates. From this fact, observer gain matrix $K$ can be represented by (3.39) □

## 3.5 Design Algorithm for an Observer's Gain

Given that the linear system (3.9) is observable, the following two-time scale design algo-rithm can be applied for the design of a full-order observer of singularly perturbed system.

*Step* 1. Transpose the full-order observer from (3.10) into (3.12)

*Step* 2. Apply the Chang transformation (3.14) to transform (3.12) into (3.17).

*Step* 3. Obtain the submatrices $A_s^T, \frac{A_f^T}{\epsilon}, C_s^T$ and $C_f^T$.

*Step* 4. Obtain slow observer gain $K_s^T$ using eigenvalue placement of matrix $\lambda(A_s^T + C_s^T K_s^T)$.

*Step* 5. Solve Lyapunov equation (3.22) using $A_s^T, \frac{A_f^T}{\epsilon}, C_s^T, C_f^T$ and $K_s^T$.

*Step* 6. Place fast observer eigenvalues in the desired location using the eigenvalue placement for matrix $\lambda(\frac{A_f^T}{\epsilon} + (P_o C_s^T + C_f^T)\frac{K_{f2}^T}{\epsilon})$.

*Step* 7. Go back to the original coordinates using (3.39).

## 3.6 A Numerical Example : $\epsilon = 0.1$

Consider a $4^{th}-$ order system with the system matrices $A, B$ and $C$ are as given below (Kokotovic *et al.*, 1999, pp. 124-125)

$$A = \begin{bmatrix} 0 & 0.4000 & 0 & 0 \\ 0 & 0 & 0.3450 & 0 \\ 0 & -5.2400 & -4.6500 & 2.6200 \\ 0 & 0 & 0 & -10.0000 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The observability matrix has full column rank and therefore the pair $(A, C)$ is observable.

Firstly, we transpose original system matrices $A, B$ and $C$ and obtain $A_T = A^T, B_T = B^T$ and $C_T = C^T$. According to Step 2 and Step 3 of the Chang transformation for the transposed system, the following sub-matrices are obtained

$$A_s^T = \begin{bmatrix} 0 & 0 \\ 0.4406 & -0.4282 \end{bmatrix}, \frac{A_f^T}{\epsilon} = \begin{bmatrix} -4.2218 & 0 \\ 2.7372 & -10.0000 \end{bmatrix}, C_s^T = \begin{bmatrix} 1 & 0 \\ -0.0107 & -1.3813 \end{bmatrix},$$

$$C_f^T = \begin{bmatrix} 0.0077 & 1 \\ 0.0030 & 0 \end{bmatrix}$$

Following Step 4, we place the slow eigenvalues at $(-0.707 \pm 0.707 \times j)$ via the slow feedback gain matrix

$$K_s^T = \begin{bmatrix} -2.8280 & -2.8280 \\ -1.7065 & 1.7593 \end{bmatrix}$$

For Step 5 of the algorithm, we solve the Lyapunov equation and obtain matrix

$$P_o = \begin{bmatrix} 0.7366 & 0.24810 \\ 0.2116 & 0.1793 \end{bmatrix}$$

In Step 6 of the algorithm, we place fast observer's eigenvalues at the desired location. Our $\frac{1}{\epsilon} K_{f2}^T$ of $\lambda(\frac{A_f^T}{\epsilon} + (PC_s^T + C_f^T)\frac{K_{f2}^T}{\epsilon})$ is given by

$$\frac{1}{\epsilon} K_{f2}^T = \begin{bmatrix} -23.7655 & -44.6933 \\ -9.3580 & 50.4362 \end{bmatrix}$$

Step 7. Using (3.39) , matrices $K$ is given by

$$K = \begin{bmatrix} 29.9320 & -2.0620 \\ 15.6874 & -9.0759 \\ 0.6434 & 21.0740 \\ 44.6933 & -50.4362 \end{bmatrix}$$

and check $\lambda(A - KC)$ in the original coordinate given by

$$\lambda(A - KC) = \begin{bmatrix} -2.8280 + 2.8280i \\ -2.8280 - 2.8280i \\ -32.0000 \\ -28.0000 \end{bmatrix}$$

which is the same as we placed the slow and fast eigenvalues in the two time scales.

## 3.7 Observer Controller Duality

In this section, we consider the duality between the full-order observer design of this paper and the stabilizing feedback controller design. Traditionally, the stabilization problem is one of finding the feedback gain $F$ for given $A$ and $B$ such that the closed system

$$\dot{x}(t) = (A - BF)x(t) \tag{3.49}$$

is uniformly asymptotically (exponentially) stable. It is closely related to the state recontruction problem of finding the observer gain $K$ for the given matrices $A$ and $C$ such that the error dynamics

$$\dot{e}(t) = (A - KC)e(t) \tag{3.50}$$

is uniformly asymptotically (exponentially) stable. One problem we have already seen in the two stage design of the full-order observer is that stability of (3.50) is identical to that of the dual system

$$\dot{e}(t) = (A^T - C^T K^T)e(t) \tag{3.51}$$

Examination of the results presented confirms that this observation extends to the full-order observer design and stabilizing feedback controllers for singularly perturbed systems.

The feedback controlled system is defined by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{3.52}$$

with

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\epsilon} & \frac{A_{22}}{\epsilon} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ \frac{B_2}{\epsilon} \end{bmatrix}$$
$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \tag{3.53}$$

The reduced subsystem is

$$\dot{x}_s(t) = A_s x_s(t) + B_s F_s x_s(t)$$
$$y_s(t) = C_s x_s(t) \tag{3.54}$$

The fast subsystem is

$$\dot{x}_{fnew}(t) = \frac{A_f}{\epsilon} x_{fnew}(t) + (P_f B_s + \frac{B_f}{\epsilon}) F_{f2} x_{fnew}(t)$$
$$y_{fnew}(t) = C_{fnew} x_{fnew}(t) \tag{3.55}$$

The controllability pair is

$$(A_s, B_s) \text{ and } (\frac{A_f}{\epsilon}, (P_f B_s + \frac{B_f}{\epsilon})) \tag{3.56}$$

The composite controller gain is

$$F = [F_s + F_{f2} P_f, F_{f2}] T_c \tag{3.57}$$

The algebraic Lyapunov equation is given by

$$P_f(A_s + B_s G_s) + \frac{B_f}{\epsilon} F_s - \frac{A_f}{\epsilon} P_f = 0 \tag{3.58}$$

The dual full-order observer system is

$$\dot{x}^T(t) = A^T x^T(t) + C^T u(t)$$
$$y^T(t) = B^T x^T(t) \tag{3.59}$$

with

$$A^T = \begin{bmatrix} A_{11}^T & (\frac{A_{21}}{\epsilon})^T \\ A_{12}^T & (\frac{A_{22}}{\epsilon})^T \end{bmatrix}, B^T = \begin{bmatrix} B_1^T & (\frac{B_2}{\epsilon})^T \end{bmatrix}$$
$$C^T = \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix} \tag{3.60}$$

The reduced slow subsystem is

$$\begin{aligned} \dot{x}_s^T(t) &= A_s^T x_s^T(t) + C_s^T K_{sT} x_s^T(t) \\ y_s(t) &= B_s^T x_s^T(t) \end{aligned} \tag{3.61}$$

The fast subsystem is

$$\begin{aligned} \dot{x}_{fnew}^T(t) &= \frac{A_f^T}{\epsilon} x_{fnew}^T(t) + (P_o C_s^T + C_f^T)\frac{1}{\epsilon} K_{f2}^T x_{fnew}^T(t) \\ y_f(t) &= B_{fnew} x_{fnew}^T(t) \end{aligned} \tag{3.62}$$

The observability pair is

$$(A_s, C_s) \text{ and } (\frac{A_f}{\epsilon}, (P_o C_s^T + C_f^T)^T) \tag{3.63}$$

The composite observer gain is

$$K = (\begin{bmatrix} K_s^T + \frac{1}{\epsilon} K_{f2}^T P_o & \frac{1}{\epsilon} K_{f2}^T \end{bmatrix} T_c^{-1})^T \tag{3.64}$$

The algebraic Lyapunov equation for $P_o$ is

$$P_o(A_s^T + C_s^T K_s^T) + C_f^T K_s^T - \frac{A_f^T}{\epsilon} P_o = 0 \tag{3.65}$$

## 3.8 Conclusion

In this chapter, the numerically ill-conditioned eigenvalue placement technique of singularly perturbed systems is exactly solved in terms of two slow and fast subsystem (3.31) and (3.35) via slow and fast time scale decomposition. Furthermore, due to the split into two subsystems, eigenvalue placement techniques are applied for each subsystems. We have formulated the observer design problem for linear singularly perturbed systems as one of designing exactly seperate observers for the slow and fast subsystem models. A main result is in Lemma 3.4.1, which presents sufficient conditions under which the state reconstruction of the original singularly perturbed systems is exactly achieved through the composite observer design.

# Chapter 4

# Slow and Fast Observer-based Controller for Singularly Perturbed System

## 4.1 Introduction

In the previous chapter, we design the observer gain $K$ in the original coordinates via the slow and fast time scale decomposition. In the last part of Chapter 2, we have also designed the controller gain $F$ via the two stage design. These two methods provides us with the exact eigenvalues assignment for both the system and the observer.In this chapter, we put them together and consider the observer-based controller for singularly perturbed linear system.

## 4.2 Slow and Fast Observer-Based Controller Derivation

Firstly, we seek a composite observer-based controller through the two stage design for the reduced and fast subsystems (1.11) and (1.13). It is expected that this composite controller stabilize the singularly perturbed system (1.6) with inaccessible state. Sufficient conditions for this stabilization are given in the following lemma.

**Lemma 4.2.1.** *If the reduced system* (1.11) *and the fast system* (1.13) *are each uniformly completely stabilizable by the slow and fast observers* (3.31) *and* (3.35)*, and the controllers* (2.86) *through the two stage, then there exists a* $\epsilon^*$ *sufficiently samll such that the linear singularly perturbed system is uniformly completely stabilizable for any* $\epsilon \in (0, \epsilon^*]$*. Furthermore, such a stabilizing observer's gain and controller's gain are* *given by*

$$F = \begin{bmatrix} F_1 & F_2 \end{bmatrix} = \begin{bmatrix} F_s + F_{f2}P_f & F_{f2} \end{bmatrix} T_c^{con}$$
$$K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = (\begin{bmatrix} K_s^T + \frac{1}{\epsilon}K_{f2}^T P_o & \frac{1}{\epsilon}K_{f2}^T \end{bmatrix} (T_c^{obs})^{-1})^T \tag{4.1}$$

*where the first part of $T_c^{con}$ is the Chang-transformation defined for the controller and the second part of $T_c^{obs}$ is the Chang-Transformation defined for the observer.*

The above lemma shows that we can design a composite observer based controller through the two stage for observer and controller. The observer is driven by the system measurements and control inputs, that is

$$\begin{bmatrix} \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} - K_1C_1 & A_{12} - K_1C_2 \\ \frac{1}{\epsilon}A_{21} - \frac{1}{\epsilon}K_2C_1 & \frac{1}{\epsilon}A_{22} - \frac{1}{\epsilon}K_2C_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{bmatrix} u_c(t) + \begin{bmatrix} K_1 \\ \frac{1}{\epsilon}K_2 \end{bmatrix} y(t) \tag{4.2}$$

It is known from Chapter 1 that there exists a nonsingular transformation $T_{c2}$ defined by (1.45) such that (4.2) is decoupled into pure-slow and pure-fast local observers driven by

$$\begin{bmatrix} \dot{\hat{x}}_s(t) \\ \dot{\hat{x}}_f(t) \end{bmatrix} = \begin{bmatrix} A_{socl} & 0 \\ 0 & \frac{1}{\epsilon}A_{focl} \end{bmatrix} \begin{bmatrix} \hat{x}_s(t) \\ \hat{x}_f(t) \end{bmatrix} + \begin{bmatrix} B_s \\ \frac{1}{\epsilon}B_f \end{bmatrix} u_c(t) + \begin{bmatrix} K_s \\ \frac{1}{\epsilon}K_f \end{bmatrix} y(t) \tag{4.3}$$

with

$$\begin{bmatrix} A_{socl} & 0 \\ 0 & \frac{1}{\epsilon}A_{focl} \end{bmatrix} = T_{c2}^{-1} \begin{bmatrix} A_{11} - K_1C_1 & A_{12} - K_1C_2 \\ \frac{1}{\epsilon}A_{21} - \frac{1}{\epsilon}K_2C_1 & \frac{1}{\epsilon}A_{22} - \frac{1}{\epsilon}K_2C_2 \end{bmatrix} T_{c2}$$
$$\begin{bmatrix} B_s \\ \frac{1}{\epsilon}B_f \end{bmatrix} = T_{c2}^{-1} \begin{bmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{bmatrix} \tag{4.4}$$
$$\begin{bmatrix} K_s \\ \frac{1}{\epsilon}K_f \end{bmatrix} = T_{c2}^{-1} \begin{bmatrix} K_1 \\ \frac{1}{\epsilon}K_2 \end{bmatrix}$$

Thus, these two observers (4.3) can be implemented independently in the slow and fast time scales

$$\dot{\hat{x}}_s(t) = A_{socl}\hat{x}_s(t) + B_s u_c(t) + K_s y(t)$$
$$\dot{\hat{x}}_f(t) = \frac{1}{\epsilon}A_{focl}\hat{x}_f(t) + \frac{1}{\epsilon}B_f u_c(t) + \frac{1}{\epsilon}K_f y(t) \tag{4.5}$$

The control input in the $\hat{x}_s - \hat{x}_f$ coordinates is given by

$$u_c(t) = F \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} = \begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} = \begin{bmatrix} F_s & F_f \end{bmatrix} \begin{bmatrix} \hat{x}_s(t) \\ \hat{x}_f(t) \end{bmatrix} \tag{4.6}$$

where $F_s$ and $F_f$ are obtained from

$$\begin{bmatrix} F_s & F_f \end{bmatrix} = FT_{c2} \tag{4.7}$$

In summary, the procedure to obtain the solution for the slow and fast observer-based controller problem is given by the following design algorithm.

### 4.2.1 Design Algorithm

*Step* 1. Solve (2.81), (2.80), and (2.81) to get $P_f, F_s$ and $F_{f2}$.

*Step* 2. Compute a composite controller gain $F$ in terms of $P_f, F_s$ and $F_{f2}$ using (4.1).

*Step* 3. Solve (3.20), (3.22), and (3.28) to get $P_o, K_s$ and $K_{f2}$.

*Step* 4. Compute a composite observer gain $K$ in terms of $P_o, K_s$ and $K_{f2}$ using (4.1).

*Step* 5. Find the closed-loop state matrices, input matrices, observer and controller gains $A_{socl}, A_{focl}, B_s, B_f$, $K_s, K_f$ and $F_s, F_f$ from (4.4) and (4.7).

*Step* 6. Find the pure-slow and pure-fast observers in the $\hat{x}_s - \hat{x}_f$ coordinates given as (4.5).

The importance of the proposed design procedure is of the fact that it allows the complete and exact decomposition of the feedback control and observer problems into slow and fast time scale subproblems.

### 4.2.2 Numerical Example

Consider a $4^{th}-$ order system with the system matrices $A, B$ and $C$ are in section 1.9. (Kokotovic *et al.*, 1999) The controllability matrix has full row rank and therefore the pair $(A, B)$ is controllable. Furthermore, the observability matrix has full column rank and therefore the pair $(A, C)$ is observable.

The results obtained by using MATLAB are given below. For assigning slow and fast eigenvalues, we locate slow eigenvalues at $\lambda_{cs}^{desired} = (-2, -3)$ and fast eigenvalues at

$\lambda_{cf}^{desired} = (-7, -8)$ for the controller, and slow eigenvalues at $\lambda_{os}^{desired} = 4 \times (-2, -3)$ and fast eigenvalues at $\lambda_{os}^{desired} = 4 \times (-7, -8)$ for the observer.

Following design procedure of Section 4.2.1, the completely decoupled observer in the $x_s - x_f$ coordinates, driven by the system measurements and control inputs, are

$$\dot{\hat{x}}_s(t) = \begin{bmatrix} -52.1 & 0.7 \\ -2679.2 & 32.1 \end{bmatrix} \hat{x}_s(t) + \begin{bmatrix} -0.0611 \\ -2.6711 \end{bmatrix} u_c(t) + \begin{bmatrix} -1.0341 & 0.1828 \\ -69.1001 & 11.4078 \end{bmatrix} y(t)$$

$$\dot{\hat{x}}_f(t) = \begin{bmatrix} -50.0000 & 2.6200 \\ -151.1450 & -10.0000 \end{bmatrix} \hat{x}_f(t) + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u_c(t) + \begin{bmatrix} 1974 & 45 \\ 15349 & 148 \end{bmatrix} y(t)$$

### 4.2.3 Simulation Results

Figures 4.2, 4.3, 4.4 and 4.5 present the results when we simulate the traditional slow and fast observer-based controller as presented in Figure 4.1. The results presented in Figures 4.2, 4.3, 4.4 and 4.5 shows that the states and error convergence are compared with the ones through the composite observer-based controller 1.1 . Figures 4.7 and 4.8 present the results when we simulate the on-line saving using the block diagram in Figure 4.6, which indicate composite parallelism of controller and observer designs in slow and fast time scales.

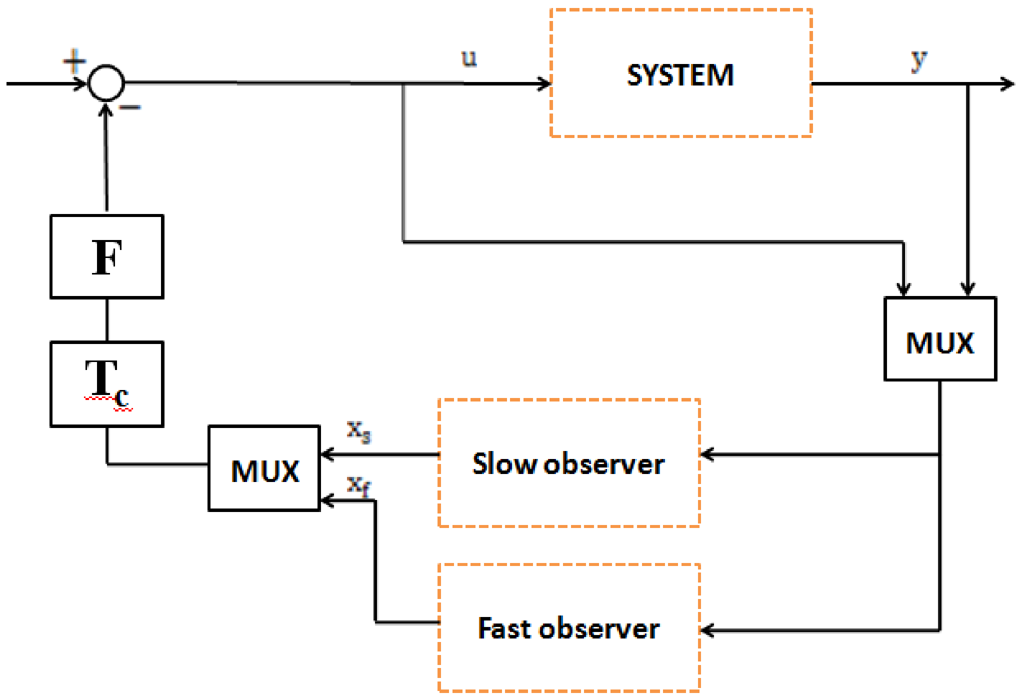


Figure 4.1: Traditional slow and fast observer-based controller

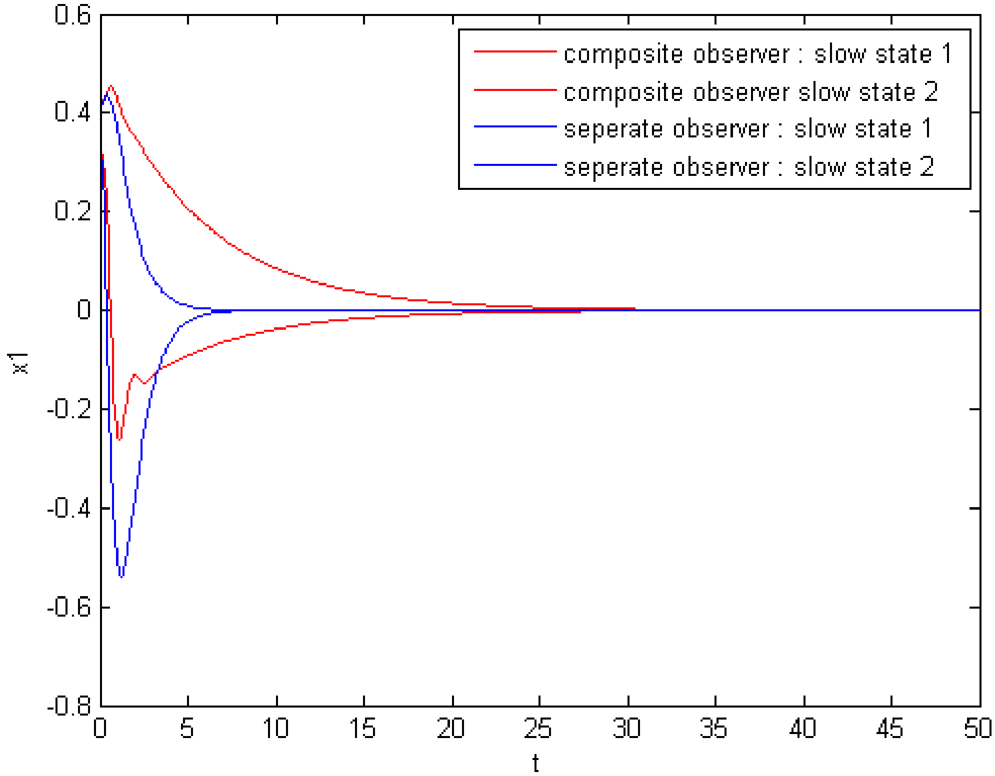


Figure 4.2: Comparison of the slow states $x_1(t) \in \Re^2$

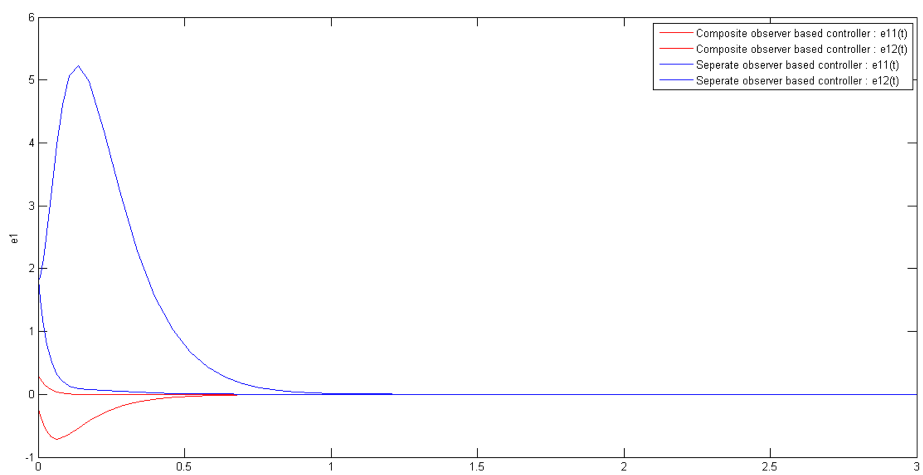


Figure 4.3: Comparison of the slow errors $e_1(t) \in \Re^2$

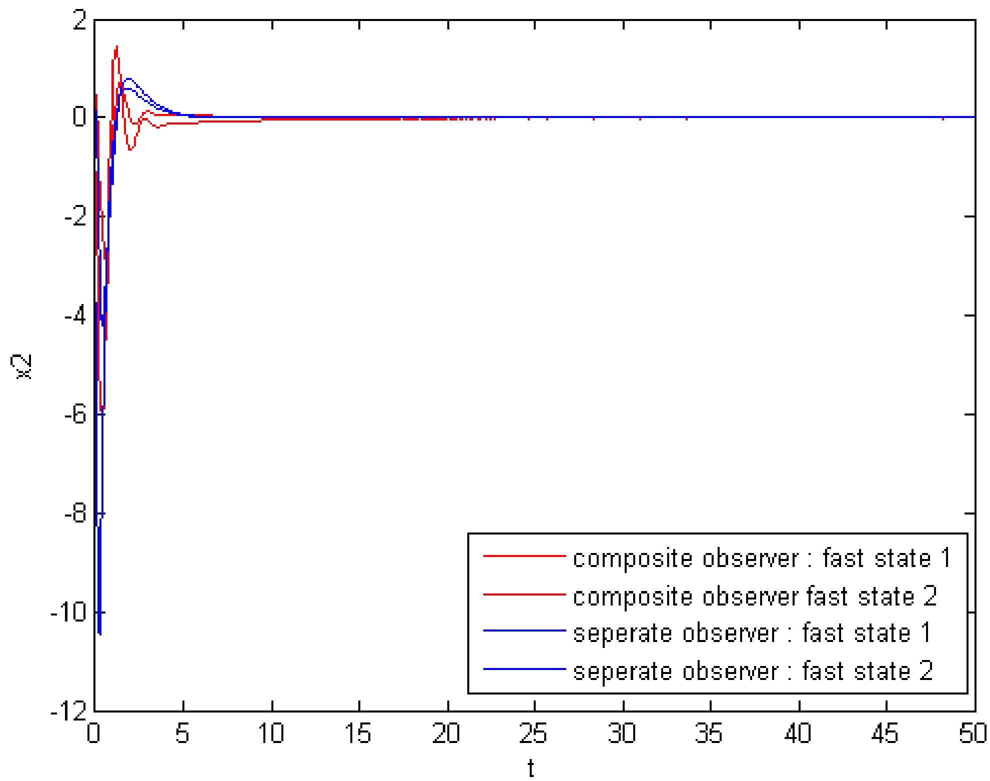


Figure 4.4: Comparison for the fast states $x_2(t) \in \Re^2$

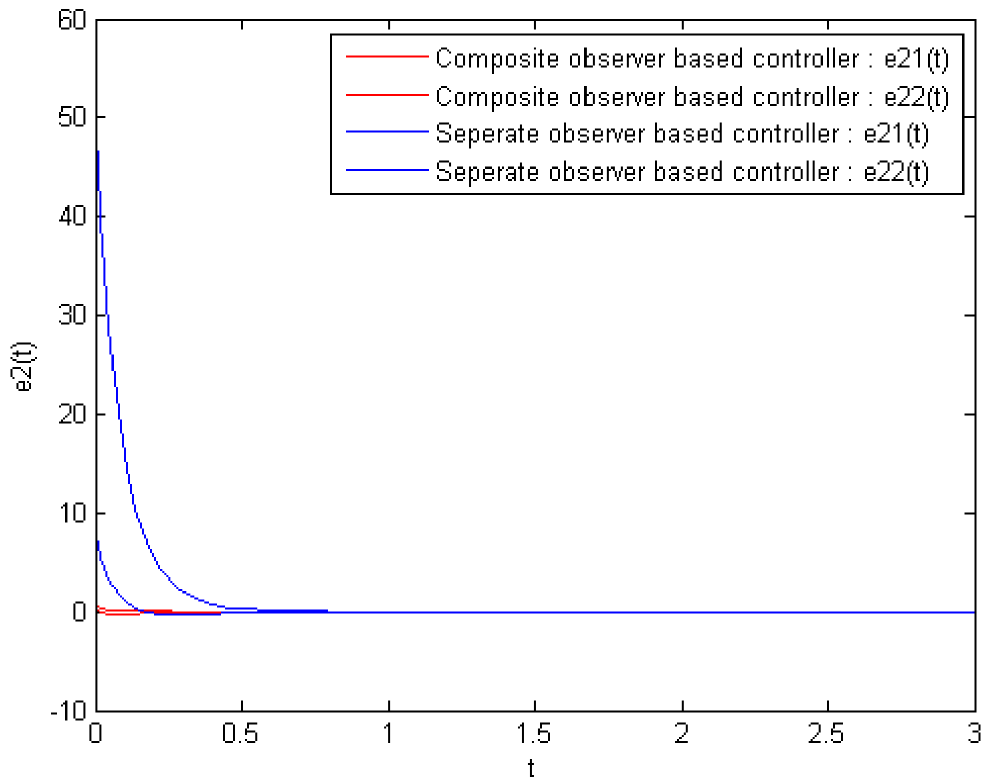


Figure 4.5: Comparison for fast error $e_2(t) \in \Re^2$

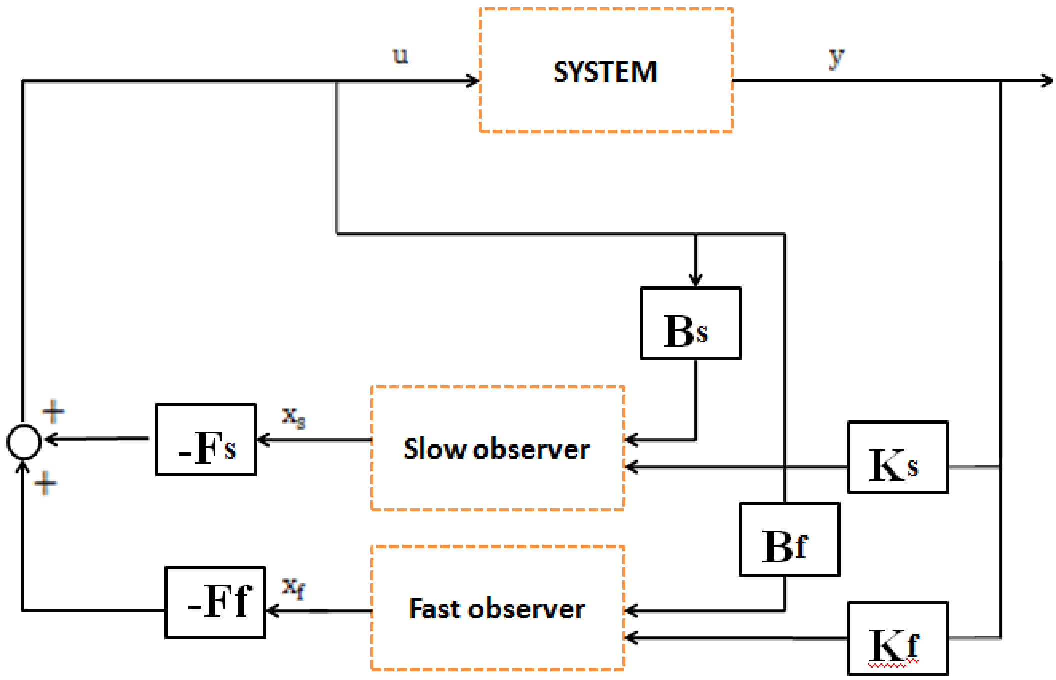


Figure 4.6: Complete parallelism and exact decomposition of the observer-based controller for singularly perturbed linear systems

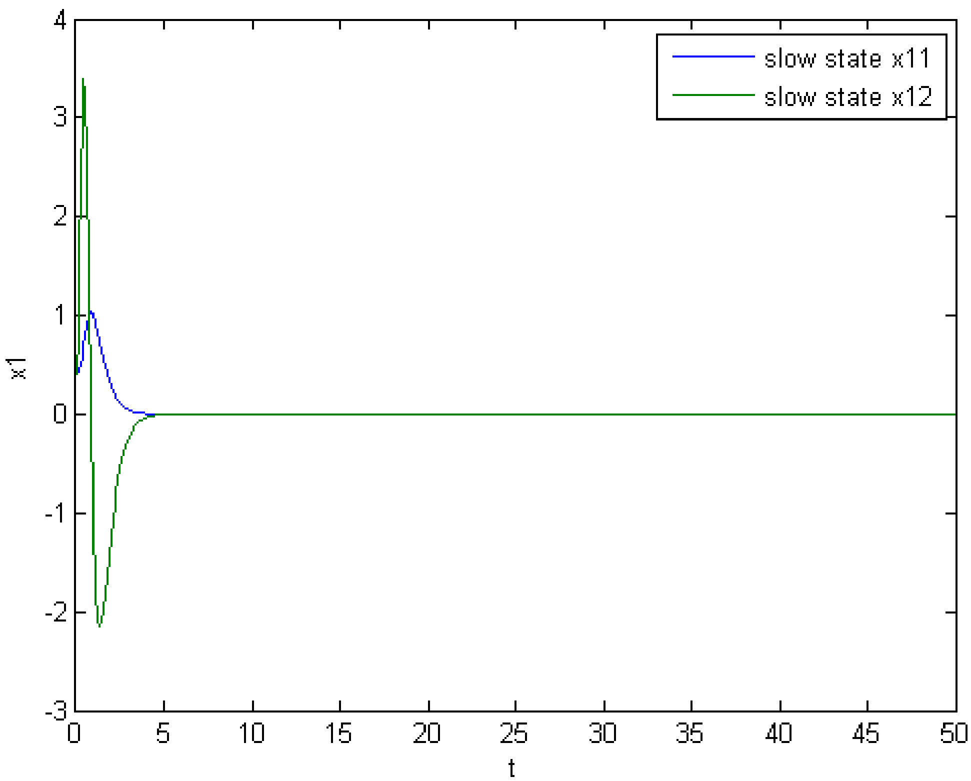


Figure 4.7: Convergence of the slow states $x_1(t) \in \Re^2$

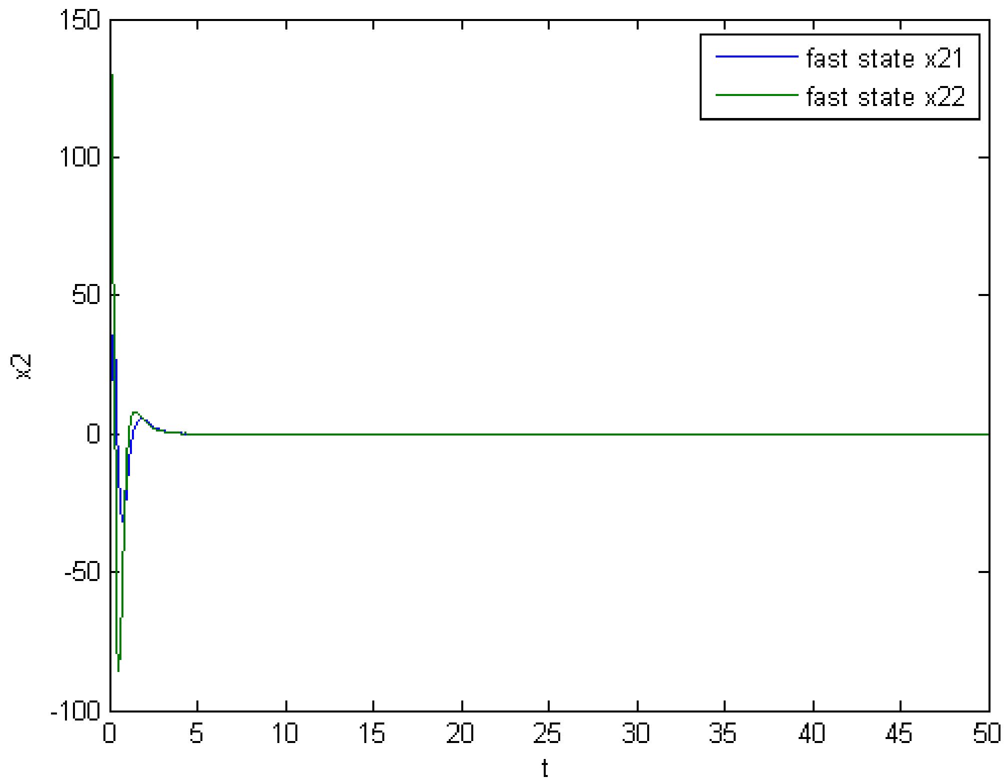


Figure 4.8: Convergence of the fast states $x_2(t) \in \Re^2$

## 4.3 Conclusion

The importance of the proposed method is in the fact that it allows time-scale parallelism of the observer and control tasks through the complete and exact decomposition of the control and observer problems into slow and fast time scales, which reduces both off-line and on-line required computations.

# Chapter 5

# Conclusions and Future Work

## 5.1 Conclusions

We have designed slow and fast observer-based controllers by locating eigenvalues in a composite manner for slow and fast subproblems. The numerically ill-conditioning problem is solved using the two stage method for singularly perturbed linear systems, we can apply the feedback control to each sub-systems. We have demonstrated that the full-order singularly perturbed system can be successfully controlled via the eigenvalue placement technique with the state feedback controllers designed on the subsystem levels. The two stage method is successfully implemented for the full-order observer.

## 5.2 Future Work

It is expected that the two stage method might be applied to a reduced-order observer. Furthermore, we can extend the results of this thesis to the systems composed of $N$ subsystems using the decoupling transformation for $N$ subsystems.

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