SECONDARY PRE-SERVICE TEACHERS' RECOGNITION OF STUDENTS' MATHEMATICAL REASONING

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ABSTRACT OF DISSERATION

Maher and colleagues have developed a professional development model to help teachers learn to attend to student reasoning (Maher, Landis, & Palius, 2010). This model was built on the idea that teachers first should improve their own reasoning skills in order to be better prepared to attend to student reasoning. In-service interventions using this model have consisted of teachers solving a variety of mathematical problems that previously were given to students to solve, teacher analysis of written student problem solutions, teacher analysis of student solutions from video, and an analysis of student solutions by the teacher after a planned classroom implementation of the problem-solving activity.

A variation on the above-mentioned intervention model has been described for use with for secondary pre-service mathematics teachers (Palius, M. F. & Maher, C. A., 2011). An interesting vantage point of pre-service teachers as a study population is that they have not yet been influenced by a school's agenda or faculty room discourse. Thus, this study was conducted in a mathematics education course at a large public university, in which the subjects were the pre-service secondary mathematics teachers enrolled for the academic semester. The intervention itself consisted of five 80-minute sessions, which was less than one fifth of the total class periods in the course.

The purpose of this research has been to (1) determine if there was a change in pre-service teachers' recognition of student arguments after the intervention, (2) determine if there was a change in pre-service teachers' beliefs after the intervention and (3) determine if there were any connections between pre-service teachers' solutions to problem-solving tasks and their recognition of student reasoning. A careful analysis was conducted on the pre- and post-assessment data and on the pre-service teachers' written work to determine if the pre-service teachers showed any change in their ability to better analyze students' reasoning, as well as if any change occurred in their beliefs.

It was found that even a short intervention could influence the pre-service teachers' ability to recognize student reasoning in both written work assessment and video assessments. Furthermore, the short intervention also resulted in data that indicates change in the pre-services teachers' beliefs about student mathematics learning, mathematics, and mathematics teaching.

Findings from this study indicate that an intervention that involves problem-solving, video analysis of students and analysis of student work can help improve pre-service teachers' ability to better attend to student reasoning. This study also indicates that beliefs can change even over a short intervention. Further studies may evaluate the influence of different lengths of time for this type of intervention, as well as examine whether a replication of this study in other secondary pre-service class settings generates the same findings.

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CHAPTER 1: INTRODUCTION

1.1 Background

Currently, Maher and colleagues have developed a professional development model to help teachers learn to attend to student reasoning (Maher, Landis, & Palius, 2010). This model was built on the idea that teachers first should improve their own reasoning skills in order to be better prepared to attend to student reasoning. Current in-service interventions using this model contain four main components: teachers doing math, teachers studying video-taped episodes that illustrate students' reasoning when engaged in open-ended investigations, teachers implementing similar tasks in their classrooms, and teachers analyzing their students' work. The videos used in the interventions came from longitudinal and cross-sectional research[1] and have been collected over the past two decades by researchers at The Robert B. Davis Institute for Learning at Rutgers University.

1.2 The Study

My participation in this larger project focuses on one component of this work, namely secondary pre-service mathematics teachers enrolled in a mathematics education course at a large public university. Data collected by Maher and colleagues from a three-week intervention include participants' study of videos of children's mathematical reasoning. One way this intervention is unique is that the intervention takes place with secondary pre-service teachers and has a relatively short time period in which it was conducted. The class met twice a week for 80-minute periods for a total of 5 class sessions in which the prospective teachers worked on the mathematical tasks. There was also an online Sakai site where students watched videos, responded to questions regarding the videos, and completed some problem solving on their own.

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[1] Longitudinal study in Kenilworth schools (both grants) with cross-sectional studies in Colts Neck and New Brunswick (first grant only) is MDR-9053597 and REC-9814846. Informal Math Learning research is REC-0309062.

Another reason this intervention is unique is that the population of this intervention contained secondary pre-service teachers. The population of the secondary pre-service teachers gives an interesting vantage point to look at the current reforms in education as they have not yet been influenced by a school's agenda or faculty room discourse. An important goal of teacher education is to provide pre-service teachers experiences in which they can question their own beliefs about teaching and learning.

Therefore, this research will analyze whether a short intervention can still influence the views of the pre-service teachers in two ways. First, the study examines if pre-service teachers can learn to recognize more forms of student reasoning after participating in the intervention activities which include discussing problems solving tasks with peers and watching videos of children solving problems. Second, the study examines if the intervention had any influence on their beliefs about students' mathematical learning, mathematics, and teaching mathematics.

1.3 Research Questions

In light of these areas of examination, I am investigating three questions regarding this intervention:

1. What evidence, if any, indicates there was a change in pre-service teachers' recognition of student reasoning after the intervention?
2. What evidence, if any, indicates there was a change in pre-service teachers' beliefs after the intervention?
3. What connections, if any, are there between the pre-service teachers' solutions to problem-solving tasks and their attention to student reasoning?

CHAPTER 2: REVIEW OF LITERATURE

2.1 Introduction to Noticing

2.1.1 What is Noticing?

The NCTM's Principles and Standards for School Mathematics (2000) has been calling for a standardization in educational practices in mathematics. NCTM does not provide a "right" formula for teaching. Instead, NCTM states that "[e]ffective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well" (p. 16). Thus, the student should play a large role in both the teacher's planning and decisions within the classroom. The National Academy of Education (2005) indicates that discovering what students know and how they reason about a subject can be a powerful tool in targeting instruction. In order to teach students, "we need to know what our students are thinking, how they produce the chain of little marks we see on their papers, and what they can do (or want to do) with the material we present to them" (Noddings, 1990, p. 15). Thus, the teacher needs to know students' current mathematical understandings in order to design instructional activities that allow students to challenge their existing knowledge. Therefore, teachers should be able to recognize student reasoning and, through this reasoning, assess what understandings the students may or may not have.

Further, the teacher should take on a role as a facilitator of mathematical discussions. In order to do this, the teacher needs to listen to the students' reasoning and help guide students on a path that can lead to appropriate conceptualizations. Yackel and Cobb (1996) recognize that

> teachers [can] capitalize on the learning opportunities that arise…as they begin to listen to their students' explanations. The increasingly sophisticated way [teachers] select tasks and respond to children's solutions, shows their own developing understanding of the students' mathematical activity and conceptual development (np, 1996).

Thus, teachers can better modify and enhance lessons if they can better understand what their students understand.

2.1.2 Noticing and the Classroom

Current research on teachers' ability to attend to student reasoning (also known as noticing) focuses on what teachers are attending to in their observation of students. Maher, Landis, and Palius (2010) contend that there are pre-requisites for teacher's knowledge in order for them to attend to student reasoning. These prerequisites include "a deep knowledge of the underlying mathematics that is taught, how students learn the mathematics, and how classroom environments can be designed to motivate and support children's learning" (p. 3). Furthermore, Van Es and Sherin (2002) identify three main features of noticing: identifying what is important in a teaching situation, making connections between classroom interactions and the broader concepts they represent, and teachers using their repertoire of knowledge to reason about a given situation. Jacobs, Lamb, and Phillip (2010) comment on the complex nature of professional noticing by highlighting that

> attending to children's strategies requires not only the ability to focus on important features in a complex environment but also knowledge about what is mathematically significant and skill in finding those mathematically significant indicators in children's messy, and often incomplete, strategy explanation (p. 194)

Furthermore, they add that there is a complexity of professional noticing about student reasoning in the classroom to which teachers must attend, interpret, and respond almost simultaneously. Before teachers can perform such a complicated task, they first must learn how to notice children's mathematical behavior and ways of reasoning and perhaps this might occur first outside their own classrooms.

2.2 Research on Noticing and Professional Development

Educational reforms are now beginning to focus on how pre-service and in-service teachers learn to recognize and build on student reasoning. A part of teaching professional development on noticing involves immersing teachers in environments that allow teachers to analyze students reasoning such as observations in actual classrooms, email exchanges, studying copies of student work, or the viewing videos of students (Berlinger et al, 1998; Kazemi and Frank, 2004; Maher et al, 2010; Sherin &

Van Es, 2005; Star & Strickland, 2007; Tunc-Pekkan, 2008/2009). Four ideas have been considered in recent research studies that focus on the impact of teacher reasoning on student reasoning: (1) the role of problem solving; (2) the role of attending to student reasoning, (3) the role that watching videos of children solving mathematical problems, and (4) the role that beliefs can play in helping teachers notice children's mathematical reasoning.

2.2.1 Role of Problem Solving

According to the NCTM (2000), problem solving is an integral part of all mathematics learning. Thus, several interventions have immersed pre-service and in-service teachers in the learning process by having participants work on the same problems and discuss their solutions (Bell et al, 2010; Chamberlin, 2004; Jacob et al, 2010; Tunc-Pekkam et al, 2008/2009; Maher et al, 2010). One goal of these cooperative problem-solving activities would be to prepare pre-service and in-service teachers "to 'understand' other people's solutions and to rely more on their own mathematical abilities, as well as the importance of building new mathematical knowledge through their own efforts" (Tunc-Pekkan, 2008/2009, p. 6). Since problem-solving activities occur in groups, it is hoped that teachers not only further their own mathematical understanding of the problem, but also learn to listen to the ideas being expressed by individuals in the group. In order for teachers to be able to recognize their students' reasoning, teachers must learn how to observe and listen to their students' ideas.

In their work with twenty-seven pre-service teachers, Tunc-Pekkan and D'Ambrosio (2008/2009) wanted to see the effect in-class problem solving had on pre-service teacher's abilities to respond to sixth-grade students working on similar problems via an email exchange. The findings indicate that only four of the 27 pre-service teachers were able to effectively respond to the child's reasoning. The remaining pre-service teachers either told the sixth grade students procedures on how to solve the problem using the pre-service teacher's method or just did not respond. This study shows the process of problem solving can provide some assistance in learning to notice student reasoning but is not effective by itself.

2.2.2 Role of Listening

In fact, the study by Tunc-Pekkan and D'Ambrosio also brings in the role of communication in being able to attend to student reasoning. In particular, teachers must learn how to attend to student reasoning by both listening to what the student is saying as well as being able to interpret student work. Maher and Davis (1990) describe an episode where a teacher failed to understand her students' representation of the problem (which was correct) and actually had the students solve the problem the teacher's way (which in this case was incorrect). Maher and Davis emphasize that

> paying attention to mathematical thinking of students engaged in active mathematical constructions, and trying to make sense of what students are doing and why they are doing it is prerequisite…into gaining insight into the nature of the development of children's representations" (p. 89).

Furthermore, by listening, teachers can challenge and extend students' thinking on a problem as well as use the feedback to modify and create lessons. In order to do this, pre-service and in-service teachers must develop what Brent Davis calls hermeneutic listening (Davis, 1997). In hermeneutic listening, there is no prescribed trajectory on how to solve the problem. Instead, hermeneutic listening makes learning a social process in which the teacher's role is one "of participating, of interpreting, of transforming, of interrogating – in short, of listening" instead of just a formative evaluation or information seeking process (p. 371). In Davis' case study, Davis follows a teacher named Wendy. In her classroom, there is an absence of a structured format and prescribed set of learning outcomes. She does not try to converge all learners to a particular understanding at the end of the class session. Instead, Wendy participates in the exploration of mathematics with the class. The learning occurs within the community with all students participating and asking questions that lead the discussions.

Furthermore, since humans are social beings, teachers must also try to understand what students are thinking as students try to represent and discuss their ideas. Ball (1991) compares teacher communication with students to cross-cultural communication. Since students express their ideas in their own language, teachers must learn to mediate through students' words, pictures, analogies, and speech. In a study by Francisco and Maher (2011), teachers had opportunities to watch students solve problems in an afterschool program. While watching the students solve several mathematical tasks, teachers began to turn their attention to the different types of reasoning the students used while solving the problem tasks. In fact, the teachers were impressed by the language the students developed to describe and represent their ideas. This study is just one example of how teachers can begin to learn how to attend to student reasoning through observation and discussion.

2.2.3 Role of Videos

For pre-service and in-service teachers to learn how to notice students' mathematical behavior and ways of reasoning, researchers have been making use of tools such as videos and students' collected work to analyze student reasoning (Berlinger et al, 1998; Maher et al, 2010; Sherin & Van Es, 2005; Star & Strickland, 2007). Videos provide teachers with a new window by which they can observe, re-observe, and study classroom interactions.

Berlinger and colleagues (1998) suggested that experienced teachers are better able to evaluate videos of classrooms, focusing more on the content and student reasoning as opposed to novice and beginning teachers who tend to focus on pedagogy and the teacher moves. However, Sherin and Van Es (2005) found that pre-service and in-service teachers who watch and reflect on episodes of their own classroom teaching have improved their ability to notice events of students reasoning. In a yearlong study with in-service teachers, teachers worked in what Sherin coined as "video clubs" where teachers discuss and watch episodes of their own classrooms. From these video club meetings, there was a change in focus of the teachers from pedagogical issues "towards attending to student thinking" (p. 482). Meanwhile, their study with pre-service teachers using a Video Analysis Support Tool (VAST) found that VAST accelerated pre-service teachers' ability to notice classroom events that pertained to teacher discourse and student mathematical ideas. Furthermore, in both studies pre- and in-service teachers began interpreting the video they were watching, opposed to evaluating them. In other words, the teachers did not jump to judge the events they were watching; instead, they were better able to discuss influential factors that impacted teacher decisions and student understanding.

Furthermore, Star and Strickland's research has shown that pre-service education courses can help guide pre-service teachers on noticing specific features of the classroom (2007). In a methods course with prospective teachers that used videos, Star and Strickland found that by the end of the course the pre-service teachers were better able to notice classroom events in four categories: classroom environment, tasks, mathematical content, and communication. Like Sherin and Van Es's research, Star and Strickland suggest that noticing can be taught through video with support. In particular, Star and Strickland emphasize two influential support systems that help these pre-service teachers notice. First, there was a pre-assessment which hinted to the prospective teachers what to look at when they viewed the video a second time. Second, the prospective teachers were provided with an observation framework with which to guide their viewing. The implications of this study suggest that repeated viewings of videos enable teachers to deeply analyze specific features of the classroom and that experience with the videos, along with some observation framework, can improve the teacher's ability to notice specific features of the classroom as well as student reasoning.

2.2.4 Role of Beliefs

A teacher's beliefs towards mathematics can strongly influence his or her ability to notice student reasoning. In a study by Kazemi and Frank (2004) teachers met in monthly workgroup meetings throughout the year to discuss student work on specific problems given by the teachers in their classrooms. During the workgroup meetings led by facilitator, the teachers were asked to discuss student work and observations teachers made about student problem solving during their classroom time. While initial discussions consisted of focusing on correct and incorrect answers, later discussions led to a focus on student reasoning. Kazemi and Frank note that "the shift in noting children's sophisticated reasoning is an important marker of teacher learning because the comments teachers offered in the discussions showed they were impressed with their own students' thinking" (p. 219). This then led the teachers to discuss the student work with students instead of looking only at the student work submissions alone. Therefore, teachers can learn more about what their students understand by looking at the reasoning of the students and not just the correctness of an answer.

Furthermore, a teacher's beliefs about student ability can also have an impact on how they teach. For example, Battey and Franke (2008) studied two different teachers, Mrs. Brown and Mr. Gray, in professional development workshops and within their classrooms. Mrs. Brown expressed the belief that, in teaching mathematics, a teacher should focus on one specific set of procedures to arrive at a "correct" solution. Meanwhile, Mr. Gray focused on a "correct" solution and opened the classroom to a discussion of the different methods to reach the "correct" solution. In both cases, these teachers were focused on a "correct" answer rather than on the student reasoning that produced the correct answer. In another study, Jaberg and colleagues (2002) followed a teacher in a unit of teaching fractions to sixth-grade students. Data on this teacher's beliefs were collected via observations, interviews with the teacher, and journal entries submitted by the teacher to the researchers. Unlike the teachers in the study by Battey and Frank, the teacher in this study focused on the development of student reasoning. The teacher opened the classroom up to discussion and allowed class discussions to explore new ideas. Jaberg and colleagues report that through their observations of this teacher, the teacher took on the role of a questioner and the students took on the role of problem solvers. Thus, because of this teacher's focus on reasoning, her students learned both mathematical content and reasoning.

2.3 Research on the Beliefs of Pre-Service Teachers

Case studies have documented how teachers change and regress in their views of teaching and tend to hold both constructivist and traditional views on education simultaneously (Jaberg, Lubkinski, and Yazujian, 2002; Clarke, 1997). For example, Sandra Frid (2000) looked at pre-service teacher's beliefs to see if the constructivist theory she was providing in class transferred to her students' internal beliefs and classroom practices and completed two studies. In the first study, Frid studied a class of pre-service teachers who were in the first year, second semester of a three-year program towards a Bachelor of Teaching degree. Frid's goal in the first year class was to empower her students (the pre-service teachers) by giving them opportunities to experience how mathematics could be taught in meaningful ways and explore teaching and learning ideas through the use of manipulatives and discussions of theory. A questionnaire and interview were given to pre-service teachers both at the beginning of the semester and the end of the semester in order to compare and contrast pre-service teachers' beliefs before and after the course. At the end of the semester, Frid found that the pre-service teachers were beginning to develop some constructivist views of about mathematics teaching and learning while still holding on to their traditional views.

In the second study, Frid was curious as to whether these newly developed constructivist views were reflected in teaching practice and studied a practicum class composed of student teachers who were in their third year of the program. Frid and colleagues did informal observations of their classrooms and used practicum time for reflection on student teachers' lessons. Frid found that the student teachers operated primarily within a direct instruction approach to learning. She also noted that student teachers who tried to provide student-centered activities in their classrooms mistook hands-on activities and group work as definitions of student-centered classrooms. In fact, after discussing with the student teacher why these activities were planned, the activities tended to be more teacher-centered. In the discussions, there was little mention on the role the activity could play on the child's mathematical learning. Therefore, Frid's studies indicate these pre-service teachers still held their initial (traditional) assumptions about teaching and learning and that their beliefs remained durable over time. While the pre-service teachers are indicating that they hold some constructivist ideas in their belief system, they are still constructing these ideas and weaving them into their original beliefs on teaching and learning.

In a different study involving pre-service teachers, Van Es and Conroy (2009) case studied four pre-service teachers from a California University. Data were collected via journal entries, lesson plan submissions, and videotaped classroom time. In particular, the researches claim that these four pre-service teachers appeared to have "unclear and vague notions of what it means to engage students in mathematical discourse that promotes reasoning" (p. 99). While the teachers had written in their lesson plans that they wanted to engage the students in the lesson to help promote mathematical understanding, the questions they asked the students were either yes/no questions, questions involving a numerical response, or questions that asked to students to state the next step in the problem solving process. While this study only showcases four pre-service teachers, it does show similar findings to Frid's study in that these four pre-service teachers are still trying to take the ideas they learned about teaching and learn how they can become weaved into their own teaching practices.

The research presented in this paper seeks to make a contribution to the current research on pre-service teachers: what they notice in students' reasoning and their beliefs.

This research also seeks to contribute to the current research by providing an example of how a short intervention could still influence what pre-service teachers notice in students' reasoning and challenge their beliefs.

CHAPTER 3: METHODOLOGY

3.1 Setting

The study took place at large public university during an academic semester in a mathematics education course. This class is a requirement for pre-service secondary mathematics teachers in the Mathematics Education program at the university. The subjects of this study are the pre-service secondary mathematics teachers enrolled in this class. Eligibility for this class is a declared major in mathematics and a B+ average for admission into the 5-year Ed.M. mathematics education program. The pre-service teachers of this study underwent a two-and-a-half-week intervention starting the fourth week of the semester. During this time, the class met twice a week; each session was 80 minutes in length.

Prior to the start of the intervention, the pre-service teachers were asked to complete a pre-assessment prior to the start of the intervention. (The pre-assessment will be described in detail in Section 3.3.)

Class meetings were interactive, involving both group discussions and whole class discussions. Groups worked together to solve and discuss the solutions to the Towers Problem, Ankur's Challenge, and the Pizza with Halves Problem (see Appendix A). The intervention included the following activities summarized in Table 3.1.

Table 3. 1 - Timeline of Intervention and Activities

Session	Class Meeting
1	1. Pre-Service teachers worked on the Pizza with Halves Problem in groups.
2	1. Groups presented their solutions to the Pizza with Halves Problems to the whole class. 2. Pre-Service teachers worked on Towers 5-Tall Problem and Ankur's Challenge Problems in groups. 3. Homework assigned was to watch Brandon interview and PUP Math videos.

3	1. Groups presented their solutions to the Towers 5-Tall Problem to the whole class. 2. Groups continued their work on Ankur's challenge.
4	1. Groups presented their solutions to Ankur's challenge problem to the whole class. 2. Groups worked on the Taxicab Problem. 3. Homework assigned to watch the 11^{th} Grade Pizza Videos and Romina's Proof.
5	1. Groups presented their solutions to the Taxicab Problem to the whole class. 2. Class discussion on similarities and differences between all the problems discussed during the intervention. 3. Homework assigned to complete the post-assessment.

After completion of the intervention, the pre-service teachers were asked to complete a post-assessment that was identical to the pre-assessment that was given before the intervention.

3.2 Data Collection

The classroom sessions were planned and facilitated first, by Dr. Carolyn Maher (Sessions 1-3) and Dr. Alice Alston (Sessions 4-5). Data were collected by: (1) online submissions for the pre-assessment, post-assessment, and homework via an interface known as Sakai and (2) collection of in-class work (e.g., overhead transparencies used by pre-service teachers in their solution presentations).

It should be noted here that while the pre-service teachers were encouraged to complete all pre- and post-assessments, there were a few cases where the pre-service teacher did not complete one of them due to reasons outside of the intervention's control. Thus, the data analyzed for each individual assessment from this intervention only contains an analysis of those pre-service teachers who completed both the pre- and post-assessments.

3.3 Data Sources

The data for the study come from the analysis of pre-service teachers' answers to the pre- and post-assessments. Their homework and overhead transparencies were also analyzed for those individual pre-service teachers who completed all pre- and post- assessments.

The pre- and post- assessment data from the present study come from three primary sources. The first is a Student-Work Assessment (Appendix B) in which the pre-service teachers evaluated the solutions of middle-school aged children's written work on both the Towers Problem and the Pizza Problem. The pre-service teachers were asked to indicate, using a 5- point Likert scale, whether the children's solutions to these tasks were Not at all Convincing, Not Very Convincing, Undecided, Somewhat Convincing, or Convincing. In addition, the pre-service teachers were asked to provide reasons as to why they considered the solutions of the children convincing or not by providing an explanation.

The second source of assessment data were collected from a Video Assessment (Appendix C) in which the pre-service teachers watched the Gang of Four Video[2]. The video showcases four 4th grade students in a group discussing their solutions for the 3-tall Tower Problem when selecting from two colors of unifix cubes. (A Transcript of the Gang of Four Video can be found in Appendix D.) During this video, the children in the video used many different problem-solving strategies to solve and explain how they were sure they found all possible towers of a particular height. In total there were 6 different problem-solving strategies used by the students in the video: solving by cases, controlling for the variable, using figures, using patterns, using a doubling strategy, and doing numerical operations (to be defined in the next section). After viewing the video, the pre-service teachers were asked to identify and

[2] Smithsonian Astrophysical Observatory. (2000). PUP Math: Gang of four [video]. Retrieved from http://hdl.rutgers.edu/1782.1/rucore00000001201.Video.000064470

evaluate the strategies that the students used in the video and also state which strategies they may have found convincing or not convincing.

Third, assessment data were collected on the pre-service's beliefs. The Beliefs Assessment (Appendix E) consisted of 34 statements about mathematics teaching and learning in which the pre-service teachers rated each statement as Strongly Disagree, Disagree, Undecided, Agree, or Strongly Agree.

There were two secondary sources of data. One came from the transparencies the pre-service teachers created during in class problem-solving activities. The transparencies contained group solutions to Towers Problem, Ankur's Challenge, and the Pizza with Halves Problem. The other secondary source of data came from the pre-service teachers' homework assignments in which they were asked to discuss and comment on the problem-solving strategies they saw in the videos they watched (online or in class) as well as the group presentations of problem solutions that occurred during class presentations.

3.4 Method of Analysis

3.4.1 Student Work Assessment

The Student Work Assessment (Appendix B) comprised of 16 problems, in which the pre-service teacher had an opportunity to analyze middle school-aged children's work on Tower and Pizza Problems. The Student Work Assessment provided two forms of data: (1) a rating of the children's work and justification as Not at all Convincing, Not Very Convincing, Undecided, Somewhat Convincing, or Convincing and (2) justification as to why the pre-service teacher gave the student a specific rating.

3.4.1.1 Coding of Justification

During the coding process, responses to the pre-assessment and post-assessment data were blinded and mixed. In order to blind the responses of each pre-service teacher, each pre-service teacher that served as a subject in the study was given a subject number between 01 and 25. In order to be able to identify the difference between a pre-assessment and a post-assessment, pre-assessments had a prefix of 1 before the subject number and post-assessments had a prefix of 2 before the subject number. This allowed the data sets to be paired after the coding process. After general review of the responses, 11 response themes were identified to help aid the interpretation of the data. Each problem-solving strategy and its specific definition are given in Table 3.2. Data were then analyzed to determine the frequency in which the pre-service teachers identified the various problem-solving strategies used by students. To aid in the analysis, a check off rubric (Appendix G) was made to identify if the pre-service teacher had found at least one incidence of any of the 11 problem-solving strategies through the assessment.

Table 3. 2 – Problem Solving Strategies with Definitions

Problem Solving Strategy	**Definition**
1. Cases	Identification of the use of grouping by cases methods to solve problems.
2. Controlling for the Variable	Identification that the student made some variables constant.
3. Figures	Identification of the use of drawings, other than tree diagrams, to assist in problem solving.
4. Tree Diagram	Identification of the use of a branching tree to help solve the problem.
5. Patterns	Identification of the use of a type of non-numerical patterns (such as opposites or staircase patterns).
6. Exhaustion	Identification of listing of all possible answers to the problem.
7. Guess and Check	Identification of random guessing and continuous checking of results.
8. Doubling Strategy	Identification of the use of doubling the solution to case (n-1) to determine the solution to case n.
9. Simpler Problem	Identification of the use a problem that is identical to the current problem except in numerical size of the sample space.
10. Similar Problem	Identification of the use of a problem isomorphic to current problem.
11. Numerical Operations	Identification of use of basic mathematical operations and formulas to arrive at a solution.

3.4.1.2 Analysis between Pre- and Post- Assessment Data

In analyzing the data, the researcher noted the number of distinct problem-solving strategies the pre-service teacher identified in the pre-assessment and compared this information with the number of distinct problem-solving strategies the pre-service teacher identified in the post-assessment results. In particular, the researcher was interested in finding out if the pre-service teacher was more cognizant of additional student problem-solving strategies after the intervention. Instances were flagged where the pre-service teacher identified one or more problem-solving strategies that were not identified during the pre-assessment.

3.4.2 Video Assessment

The Video Assessment (Appendix C) required the pre-service teachers to watch a video of ten-year old children discussing their solutions to the 3-Tall Towers Problem. The transcript of the video (Appendix D) was available to the pre-service teachers, who were asked to write a description of the forms of reasoning and problem-solving strategies they observed in the video and to state if any the observed forms of reasoning or problem-solving strategies were convincing or not convincing.

3.4.2.1 Coding of Justification

A scoring rubric was developed from earlier studies to analyze the data found from the video assessment. During the scoring process, responses to the pre-assessment and post-assessment data were blinded and mixed. Reliability of the scoring was verified by having at least two researchers grade the same rubrics and was computed to be 89.9%. The rubric, found in Appendix H, contained a check off list with the first 8 questions focusing on the identification of specific forms of reasoning and problem-solving strategies. The last four questions focused on whether a particular form of reasoning or problem-solving strategy was convincing. Based on the first 8 questions, 6 themes were identified to help aid the interpretation of the data. Data were analyzed to determine if the pre-service teachers identified at least one instance of any of the 6 problem-solving strategies. The strategies identified were given the same definitions as in the Student Work Assessment, reproduced in Table 3.3, targeting only the problem-solving strategies that were identified in the Video Assessment.

Table 3. 3 – Problem Solving Strategies with Definitions

Problem Solving Strategy	**Definition**
1. Cases	Identification of the use of grouping by cases methods to solve problems.
2. Controlling for the Variable	Identification that the student made some variables constant.
3. Figures	Identification of the use of drawings, other than tree diagrams, to assist in problem solving.
4. Patterns	Identification of the use of a type of non-numerical patterns (such as opposites or staircase patterns).
5. Doubling Strategy	Identification of the use of doubling the solution to case (n-1) to determine the solution to case n.
6. Numerical Operations	Identification of use of basic mathematical operations and formulas to arrive at a solution.

3.4.2.2 Analysis between Pre- and Post- Assessment Data

It was of interest to know if, after the intervention, a pre-service teacher recognized any problem-solving strategies that were not identified in the pre-assessment. Thus, it was noted which problem-solving strategies were identified in the pre-assessment and these were compared with those identified in the post-assessment. In particular, the researcher flagged instances where the pre-service teacher identified additional problem-solving strategies in the post-assessment that were not identified on the pre-assessment.

3.4.3 Beliefs Assessment

The Beliefs Assessment (found in Appendix E) comprised of 34 statements on beliefs about student learning and teaching (Maher, C. A., Landis, J. H., & Palius, M. F. , 2010). The pre-service teacher was asked to score each belief statement on the following 5-point Likert scale: Strongly Disagree, Disagree, Undecided, Agree, or Strongly Agree. The pre-service teachers were to check only one response.

3.4.3.1 Coding of Beliefs Statements

In analyzing the beliefs data, the researcher tagged each statement in the beliefs assessment as one of four categories: beliefs about student mathematics learning, beliefs about mathematics, beliefs about teaching mathematics, and general beliefs. General beliefs were not coded in the later analysis as there were no opportunities during the intervention to address those beliefs.

3.4.3.2 Analysis between Pre- and Post- Beliefs Assessment Data

In the analysis of belief statement responses, each pre-service teacher Beliefs Questionnaire response was given a binary code of "Yes" or "No." Questions in which a response was consistent with beliefs held by NCTM (2000) reform efforts were coded as "Yes." Meanwhile, questions in which a response was undecided or inconsistent with beliefs held by current reform efforts were coded as "No." In particular, the researcher was interested in finding out in which beliefs statements the pre-service teacher's beliefs changed from the pre-assessment to the post-assessment and which beliefs stayed consistent from the pre-assessment to the post-assessment. A McNemar test was preformed to test for statistical significance of the pre- and post- assessment transitions on each belief statement.

CHAPTER 4: STUDENT WORK ASSESSMENT RESULTS

In the Student Work Assessment, the pre-service teachers were asked to read through student solutions of Towers Problems, Ankur's Challenge, and the Pizza Problem. The assessment was used to determine which types of heuristic arguments the pre-service teachers were able to identify in each of student's explanation on the solution to the problem. Polya defines a heuristic argument as "reasoning not regarded as final and strict but as provisional and plausible only, whose purpose is to discover the solution of the present problem" (p. 113).

As a pre-assessment each pre-service study participant was asked to: (1) read each individual solution and rate the solution as: Not at all Convincing, Not Very Convincing, Undecided, Somewhat Convincing, or Convincing and (2) provide a brief explanation as to the rationale for their specific rating. This chapter examines the pre-service teacher's rationale discussion and identifies the student's heuristic argument types that were mentioned by the pre-service teacher in their justification statement. The heuristic argument types that the pre-service teachers used to describe student solutions to these three problems are: Cases, Controlling for the Variable, Figures, Tree Diagrams, Patterns, Exhaustive, Guess and Check, Doubling Strategy, Simpler Problem, Similar Problem, and Numerical Operations. It should be noted that some student arguments use multiple heuristic arguments.

Following a research intervention course of study, for the post-assessment, each pre-service teacher was asked to repeat the pre-assessment assignment. Thus, for clarification it should be pointed out that description of a transition on each heuristic argument type and the associated transition diagram does not distinguish between: (1) a pre-service teacher who mentions the heuristic argument type on the pre-assessment and fails to mention this heuristic argument type at all on the post-assessment and (2) a pre-service teacher who mentions the heuristic argument type on the pre-assessment and mentions it again on the post-assessment, but with the same or lower level of convincingness. In the transition diagrams both case (1) and case (2) are counted as a non-transition.

4.1 Results by Heuristic Argument Type

4.1.1 Cases

A cases heuristic argument is defined as a heuristic argument that involves solving a problem by partitioning the objects in the problem into subcategories. In particular, a pre-service teacher was scored as noticing a cases heuristic argument if the pre-service teacher stated the child "grouped or organized" the towers/pizzas by specifically defined categories such as the number of blue cubes in a tower or number of toppings on a half of a pizza. Examples of statements coded as a cases heuristic argument include:

- "The student used one red in the tower, then two reds in the tower, then three and finally 4."
- "Samantha solved this problem by cases. She first started with one topping then two, and so on."
- "Considered the amount of topping which then they can focus on how many different solutions for a certain number of toppings."
- "He groups all the towers together based on the colors on the bottom on the tower."

On the pre-assessment, 7 out of 16 pre-service teachers identified a student using cases to solve at least one of the problems and 9 out of the 16 did not. Of the 9 who did not identify a cases argument on the pre-assessment 6 of them did so on the post-assessment. Thus, the frequency of a transition from lack of identification of a student cases heuristic argument to identification of student use of a cases heuristic argument is 6/9 or 66.67%. Figure 4.1 illustrates the frequency growth of a cases heuristic argument from the pre-assessment to the post-assessment.

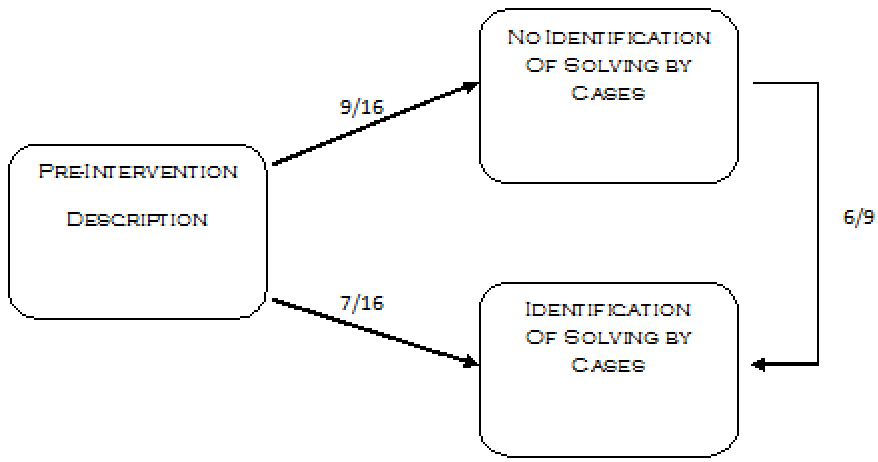

Figure 4. 1 – Transition Diagram on Cases Heuristic Argument

4.1.2 Controlling for the Variable

A controlling for the variable heuristic argument was defined as a heuristic argument in which specific variables are controlled in the solution set and solutions under these controlled situations are found and later generalized. In particular, a pre-service teacher was scored as describing a controlling for the variable heuristic argument when the pre-service teacher was able to specifically describe the process the child used to create particular groups of towers/pizzas such as describing a staircase pattern or if the pre-service teacher was able to indicate which variables were being held constant to create specific groups of towers such as indicating how the towers were created by changing the position of specific block within a particular tower. Examples of statements coded as a control for variable heuristic argument include:

- "Alan starts with a blue base and constructs all the possible towers."
- "This method is convincing because there is a pattern and holding variables constant."
- "The student knew that if a tower only had 4 blocks and needed 3 different colors then one color needs to have two blocks in each tower. She found the maximum number of solutions for a tower with two blocks of one color then multiplied it by 3 describing her solution thoroughly."
- "The last group was found by controlling for the variable. They put two of the same color on top and moved it down one at a time until they found all possibilities. "

On the pre-assessment, 6 out of 16 pre-service teachers identified the student controlling for a variable to solve at least one of the problems and 10 out of 16 did not. Of the 10 pre-service teachers who did not refer to a controlling for variable student argument on the pre-assessment, 4 of the 10 did so on the post-assessment. Thus, the frequency of a transition from a lack of identification of a student controlling for a variable to identification of student controlling for a variable is 4/10 or 40%. Figure 4.2 illustrates the frequency growth of a controlling for variables heuristic argument from the pre-assessment to the post-assessment.

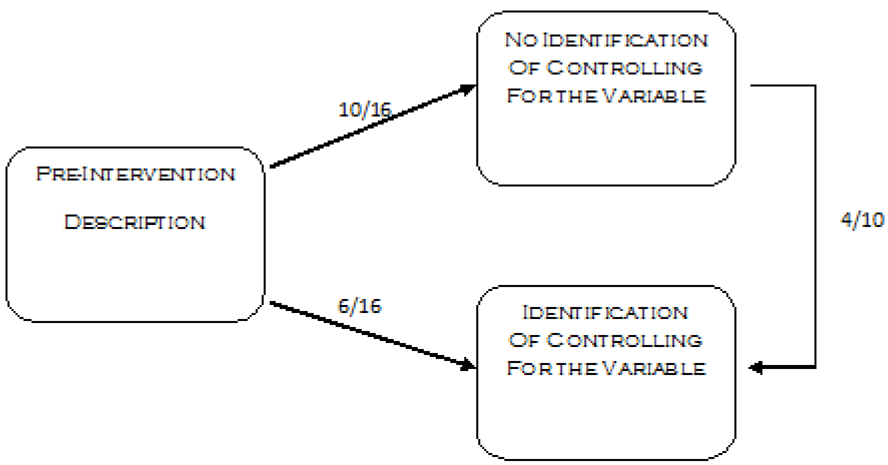

Figure 4. 2 – Transition Diagram on Controlling for the Variable Heuristic Argument

4.1.3 Figures

A heuristic argument that involves drawing figures is a heuristic argument that uses visual images other than a tree diagram to express the solution to a problem. In particular, a pre-service teacher was scored as noticing a heuristic argument using figures if the pre-service teacher made a comment regarding that the child using diagrams, charts, or pictures other than a tree diagram. Examples of statements coded as a heuristic argument that used figures include:

- "The student gives a thorough explanation through the use of diagrams on how to solve the problem."

- "Jamie did find all the possible combinations and you can see it in the drawing."
- "Great full color visual representation of all the different combinations. "

On the pre-assessment, 14 out of 16 pre-service teachers identified the student using figures (other than a tree diagram) to solve at least one of the problems and 2 of the 16 pre-service teachers did not. Of the 2 who did not identify a student as using a figure on the pre-assessment, 1 of the 2 pre-service teachers did so on the post-assessment. Thus, the frequency of a transition from a lack of identification of use of figures to identification of the use of figures is ½ or 50%. Figure 4.3 illustrates the frequency growth of identification of a figure heuristic argument from the pre-assessment to the post-assessment.

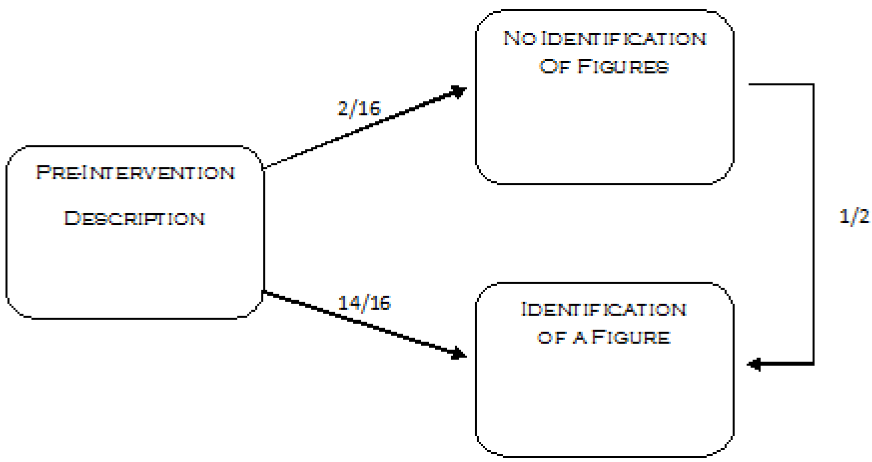

Figure 4. 3 - Transition Diagram on Figures Heuristic Argument

4.1.4 Tree Diagram

A heuristic argument that utilizes a tree diagram is defined as a graphic organizer resembling the branches of a tree that is used to list all possibilities of a solution in a systematic way. In particular, a pre-service teacher was scored as noticing a tree heuristic argument if the teacher referred to the use of a tree diagram or was able to describe the organization of the tree diagram itself. Examples of statements coded as a tree diagram heuristic argument include:

- "The tree diagram shows all the different combinations and ensures that none of them are duplicates."
- "The tree helps to visualize it all."
- "Possibilities are all branched in this different diagram."

- "They found all possible towers by starting with one color and branching off to see what possible combinations they can make."

On the pre-assessment, 7 out of 16 pre-service teachers identified the student using a tree diagram to solve at least one of the problems and 9 pre-service teachers did not. Of the 9 pre-service teachers who did not refer to a tree diagram solution on the pre-assessment 5 did so on the post-assessment. Thus, the frequency of a transition from lack of identification of a student tree diagram heuristic argument on the pre-assessment to identification of student use of a tree diagram heuristic argument on the post-assessment is 5/9 or is 55.56%. Figure 4.4 illustrates the frequency growth of identification a tree diagram heuristic argument from the pre-assessment to the post-assessment results.

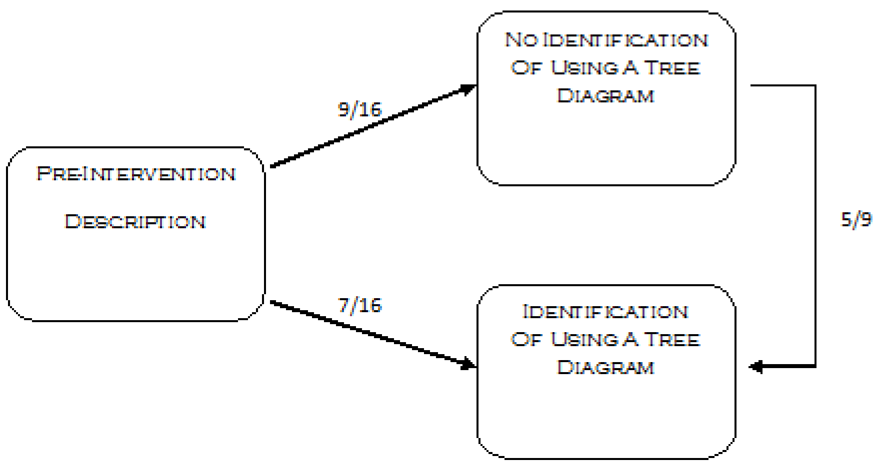

Figure 4. 4 – Transition Diagram on Tree Diagram Heuristic Argument

4.1.5 Patterns

A heuristic argument that uses patterns is identified as a heuristic argument that uses geometric or numeric patterns to solve the problem. In particular, a pre-service teacher is scored as noticing a heuristic argument that uses patterns if the pre-service teacher stated that the students used a "pattern" without specific descriptions of the pattern being used or described the patterns used by the students such as opposites. Examples of statements coded as a heuristic argument using patterns include:

- "It follows a pattern that can be replicated."
- "Her grouping by opposites includes both inverses and reflections."
- "Cathy combined drawing patterns with an explanation to support her heuristic argument. The teacher explained what kind of pattern the student used and backed it up with extra information. The patterns are logical."

On the pre-assessment, 15 out of 16 pre-service teachers identified the student as using a pattern to solve at least one of the problems and 1 out of 16 did not. The one pre-service teacher who did not identify the use a pattern solution by a student on the pre-assessment did so on the post-assessment. Thus, the frequency of a transition from lack of identification of a student using patterns to identification of a student using patterns is 1/1 or 100%. Figure 4.4 illustrates the frequency growth of identification of a pattern heuristic argument from the pre-assessment to the post-assessment.

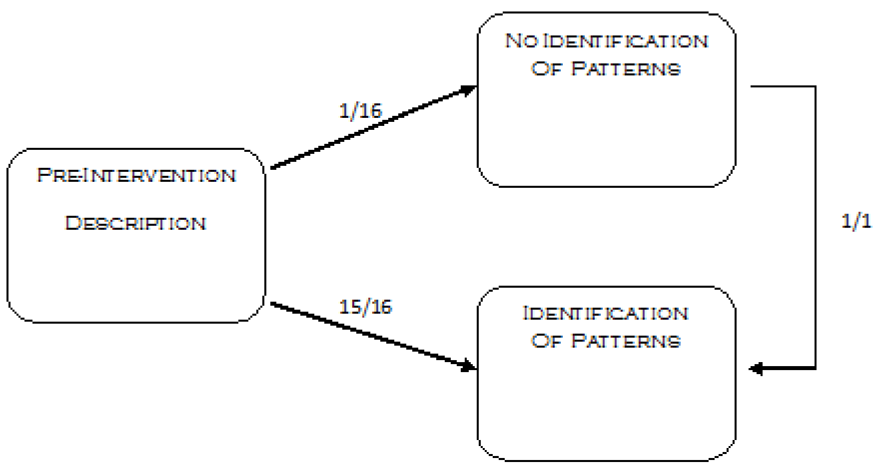

Figure 4. 5 - Transition Diagram on Patterns Heuristic Argument

4.1.6 Exhaustive

An exhaustive heuristic argument is defined as a heuristic argument in which the problem is solved by listing all solutions without any use of patterns, cases, opposites, or tree diagrams. In particular, a pre-service teacher was scored as noticing an exhaustive heuristic argument if they described the student as having "shown all" or "listed all" possible towers or pizzas but not indicate a haphazard way of creating the list such as a guess and check type method. Examples of statements coded as an exhaustive heuristic argument include:

- "The student organized them in a way that accounted for all towers and made it clear that all combinations were exhausted."
- "He listed all possible combinations."
- "Tony used a proof by exhaustion."

On the pre-assessment, 6 out of 16 pre-service teachers identified the student using an exhaustive list to solve at least one of the problems and 10 did not. Of the 10 pre-service teachers who did not refer to an exhaustive student argument on the pre-assessment, 3 of the 10 did so on the post-assessment. Thus, the frequency of a transition from lack of identification of a student using an exhaustive list on the pre-assessment to identification of student using an exhaustive list on the post-assessment is 3/10 or 30%. Figure 4.6 illustrates the frequency growth of identification of an exhaustive heuristic argument from the pre-assessment to the post-assessment.

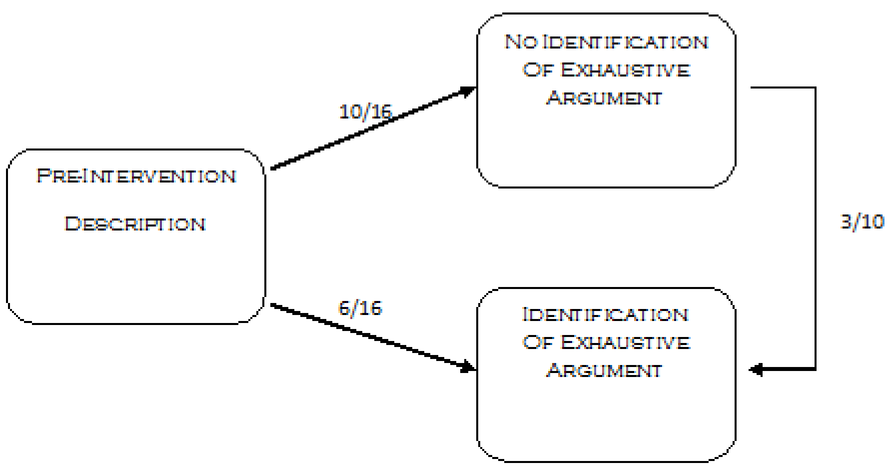

Figure 4. 6 – Transition Diagram on Exhaustive Heuristic Argument

4.1.7 Guess and Check

An argument that uses a guess and check strategy is defined as a heuristic argument that is characterized by repeated, varied attempts which are continued until success, or until the student stops trying. It is an unsystematic method that does not employ insight, theory or an organized methodology.

In particular, a pre-service teacher was scored as noticing a guess and check argument if the pre-service teacher indicated that the student was making or guessing random tower arrangements and then possibly checking for duplicates. Examples of statements coded as a guess and check argument include:

- "I don't see anything beyond trial and error in the ordering, but as the others she did final and list all the towers."
- "It is convincing because the student did a guess and check approach."
- "They just guessed to find all possibilities. They even had to erase because they had ones that didn't work or were duplicates."

On the pre-assessment, 13 out of 16 pre-service teachers identified the student using guess and check to solve at least one of the problems and 3 did not. Of the 3 pre-service teachers who did not mention a guess and check heuristic student argument on the pre-assessment one of these pre-service teachers did so on the post-assessment. Thus, the frequency of a transition from lack of identification of using guess and check on the pre-assessment to identification of a student using guess and check on the post-assessment is 1/3 or 33.33%. Figure 4.7 illustrates the frequency growth of identification of a guess and check heuristic argument from the pre-assessment to the post-assessment.

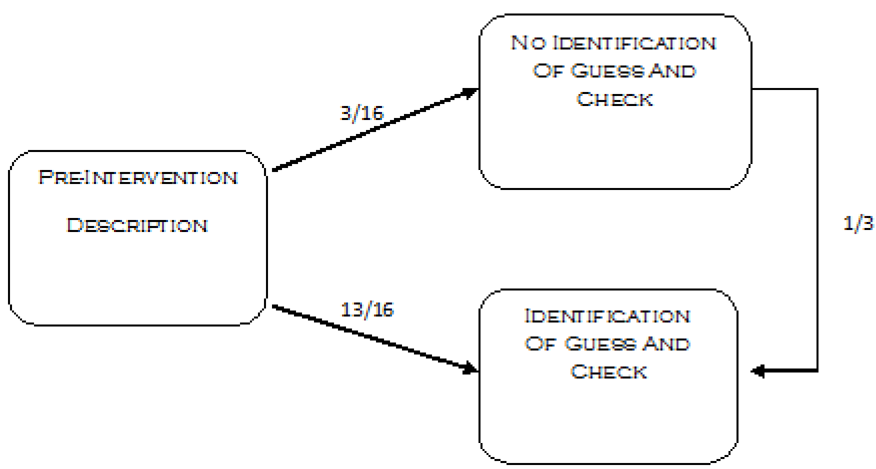

Figure 4. 7 – Transition Diagram on Guess and Check Heuristic Argument

4.1.8 Doubling Strategy

An argument that uses a doubling strategy is defined as a heuristic argument in which the problem with sample size **n** was solved by doubling the solution of the problem based upon a sample size of **n-1**. In particular, a pre-service teacher is scored as noticing the use of a doubling strategy if the teacher specifically indicated the use of a doubling heuristic argument or described a portion of an inductive process such as building towers 2-tall by adding one of the two colors to each member of a 1-tall tower solution. Examples of statements coded as a doubling strategy heuristic argument include:

- "They first started with one cube and added on all possible additions and continued on the process."
- "Brian seems to be half way to making an inductive argument. He states clearly the possibilities for a 1 block tower, and then builds on those 2 cases."

On the pre-assessment, 1 out of 16 pre-service teachers identified the student using a doubling heuristic argument to solve at least one of the problems and 15 pre-service teachers did not. Of the 15 pre-service teachers who did not mention a doubling heuristic argument on the pre-assessment 2 of the 15 did so on the post-assessment. Thus, the frequency of a transition from lack of identification of a student using a doubling heuristic argument on the pre-assessment to identification of student using a doubling heuristic argument on the post-assessment is 2/15 or 13.33%. Figure 4.8 illustrates the frequency growth of identification of a doubling heuristic argument from the pre-assessment to the post-assessment.

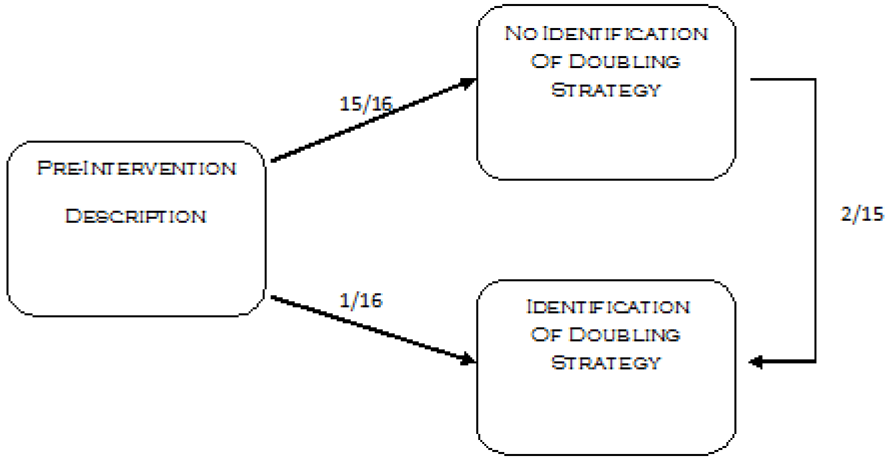

Figure 4. 8 – Transition Diagram on Doubling Heuristic Argument

4.1.9 Interference from a Simpler Problem

A simpler problem heuristic argument is defined as a heuristic argument in which the problem is solved by changing the size of the sample space in order to find similarities between different sample spaces and their respective solutions. In particular, a pre-service teacher is scored as noticing a heuristic argument that used a simpler problem if the pre-service teacher indicated the student solved the problem by using the results from a previously solved problem with an almost identical task. Examples of statements coded as a simpler problem heuristic argument include:

- "It seems to make sense because they are following what happened for the last odd number they found with the three high towers."
- "In this at least he gives an example of his logic. However, he is over-generalizing the pattern he saw in the n=3 case."
- "Danny suggested that it was either 25 or 24 based on previous information about 3 cubes high. Also, it is fairly reasonable to say that if for 3 cubes the answer was (3*3)-1, then for 5 cubes it might also be (5*5)-1. "

On the pre-assessment, 8 out of 16 pre-service teachers identified the student using a simpler problem to solve at least one of the problems. Figure 4.9 shows that of the 8 pre-service teachers who did not mention a simpler problem heuristic on the pre-assessment none of them did so on the post-assessment.

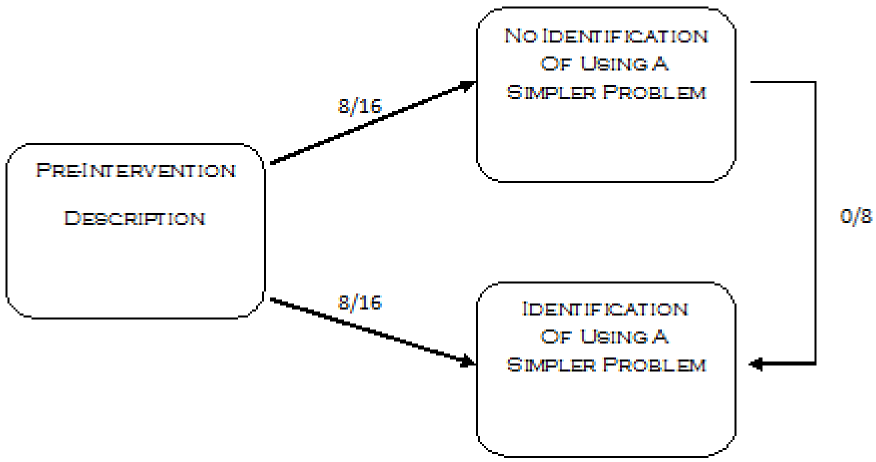

Figure 4. 9 – Transition Diagram on Use of Simpler Heuristic Argument

4.1.10 Inference from a Similar Problem

A similar problem heuristic argument is defined as a heuristic argument in which the problem was solved by comparing the problem solution to another problem that the student believed was isomorphic in nature. In particular, a pre-service teacher is scored as noticing a heuristic argument that used inference from a similar problem if they noted students used the solution from a towers problem, for example, to help with finding the solution to a pizza with halves problem. Examples of statements coded as a similar problem heuristic argument include:

- "The student uses a truth table to determine how many choices of pizzas there are and relates it back to the tower problem."
- "The student related the problem to the blocks problem, which makes sense since they both relate to the same concepts."

On the pre-assessment, 6 out of 16 pre-service teachers identified the student using the results of a similar problem to solve at least one of the problems and 10 of the 16 did not. Of the 10 pre-service teachers who did not mention a simpler problem heuristic on the pre-assessment 2 of them did so on the post-assessment. Thus, the frequency of a transition from lack of identification of a student using a similar problem on the pre-assessment to identification of student using a similar problem on the post-assessment is 2/10 or 20%. Figure 4.10 illustrates the frequency growth of identification of a simpler problem heuristic argument from the pre-assessment to the post-assessment.

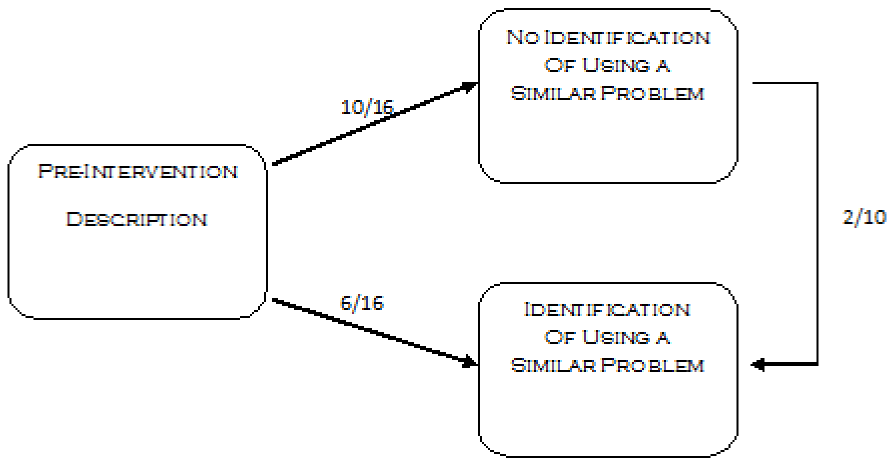

Figure 4. 10 – Transition Diagram on Similar Heuristic Argument

4.1.11 Numerical Operations

A numerical operations heuristic argument is defined as a heuristic argument in which the problem is solved by using addition, subtraction, multiplication, or division operations. In particular, a pre-service teacher is scored as noticing a numerical heuristic argument if the teacher mentioned a particular mathematical operation or numerical pattern that the student used to solve the problem. Examples of statements coded as a numerical heuristic argument include:

- "He says he multiplied 4*4."
- "He also told us why he multiplied 5x5 and subtracted one."
- "The student seemed to have an idea on how to use multiplication to solve the problem."
- "The student squared 4 to find the answer."

On the pre-assessment, 13 out of 16 pre-service teachers identified the student using numerical operations to solve at least one of the problems and 3 of the 16 did not. Of the 3 pre-service teachers who did not mention a numerical operation heuristic argument of the pre-assessment, one did so on the post-assessment. Thus, the frequency of a transition from lack of identification of using numerical operations on the pre-assessment to identification of student using numerical operations on the post-assessment is 1/3 or 33.33%. Figure 4.11 illustrates the frequency growth of identification of a numerical operation heuristic argument from the pre-assessment to the post-assessment.

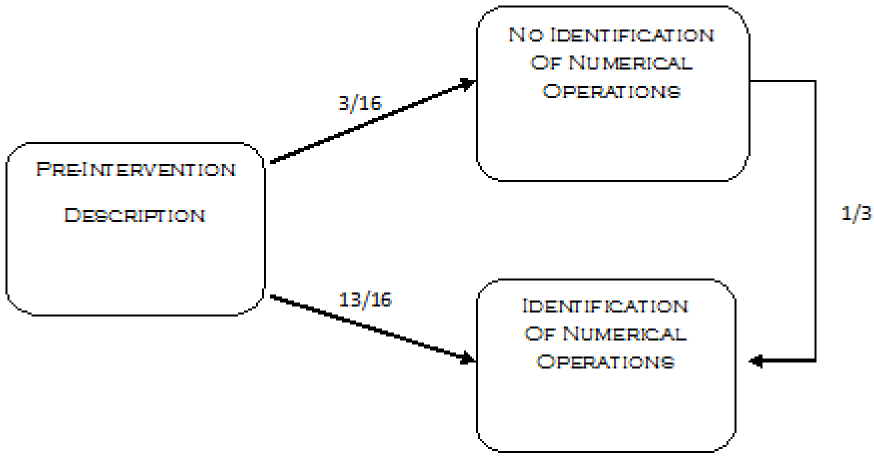

Figure 4. 11 – Transition Diagram on Numerical Operations Heuristic Argument

4.1.12 Summary of the Heuristic Arguments

A summary of the frequency of the 11 problem-solving strategies mentioned by the pre-service teachers in the description of student problem solutions is contained in Table 4.1. The cases heuristic argument exhibited the greatest post-assessment change for 6/16 of the pre-service teachers, followed by the tree diagram heuristic argument for 5/16 of the pre-service teachers, the controlling for variables heuristic argument for 4 of the 16 teachers, the exhaustive heuristic argument for 3 of the 16 teachers and the doubling and similar problem heuristic argument types each exhibiting change for 2 of the 16 teachers. Post-assessment change for the five other heuristic argument types was noted for 1 or fewer of the 16 pre-service teachers in the study.

Table 4. 1 – Student Work Assessment: Heuristic Argument Types Mentioned by Pre-Service Teachers

Heuristic Argument Type	Number Teachers on Pre-Assessment	Number Teachers on Post-Assessment	Number Teachers Exhibiting Growth	Growth Rate (%)
Cases	7	13	6	66.7
Control for Variables	6	10	4	40.0
Figures	14	15	1	50.0
Tree Diagrams	7	12	5	55.6
Patterns	15	16	1	100.0
Exhaustive	6	9	3	30.0
Guess and Check	13	14	1	33.3
Doubling	1	3	2	13.3
Simpler Problem	8	8	0	0
Similar Problem	6	8	2	20.0
Numerical Operation	13	14	1	33.3
Overall Mean	8.73	11.09	2.36	32.45

4.2 Results of Post-Assessment Transition on Multiple Heuristic arguments

Of the 16 pre-service teachers who participated in the pre- and post- student work assessment, 13 out of 16 pre-service teachers showed post-assessment growth for at least one of the 11 heuristic argument types. Figure 4.12 shows the total number of distinct heuristics argument types for the pre-service teachers on the pre-assessment and post-assessments. For the pre-assessment, the mean number of heuristic argument types is 6, the median is 6.5, the standard deviation is 2.28, and a 95% mean confidence interval for number of argument types is 4.78 to 7.22. On the post-assessment, the mean number of heuristic argument types is 7.56, the median is 8, the standard deviation is 1.90, and the 95% mean confidence interval for the number of argument types is 6.55 to 8.57.

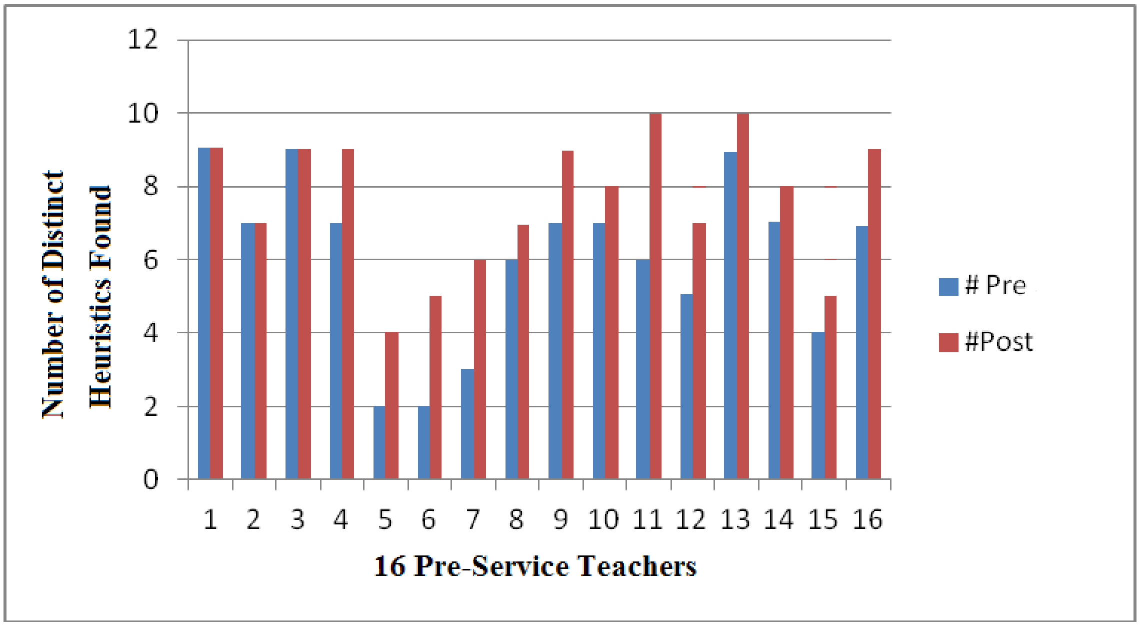

Figure 4. 12 – Individual Pre- and Post- Assessment Results

To determine if the growth in the number of heuristic argument types is statistically significant, a paired student-t-test was performed on the post-assessment heuristic count versus the pre-assessment heuristic count. The null hypothesis is: the mean transition growth is zero versus the alternative hypothesis: the mean transition growth is greater than zero. In examining all 16 pre-service teacher participants in the study, the difference between the post-assessment count and pre-assessment count is 7.56 – 6.0 = 1.56 which is statistically significant based up the Student t statistic of $t(15) = 5.42$ and $p < 0.0001$. Thus, the data provide evidence of a statistically significant transition growth with a mean growth estimate of 1.56 heuristic argument types and a 95% mean growth confidence interval of 0.95 to 2.18 heuristic argument types.

CHAPTER 5: VIDEO ASSESSMENT RESULTS

In the Video Pre- and Post- Assessment, the pre-service teachers were asked to watch a group of young children discussing their solutions to the Towers Problems. The assessment was used to determine which types of heuristic arguments the pre-service teachers were able to identify in each of student's explanation on the solution to the problem. After watching the video, the pre-service teachers were asked to write a short summary of the heuristic arguments they noticed and which they found convincing. The pre-service teachers were also given a transcript of the video. Since a "convincing" heuristic argument could mean different things to different people, this chapter focuses only on an analysis of types of heuristic arguments used by the pre-service teacher to describe the student solution on the video. For clarification it should also be pointed out that description of transition on each heuristic argument type and the associated transition diagram does not distinguish between: (1) a pre-service teacher who mentions the heuristic argument type on the pre-assessment and fails to mention this heuristic argument type at all on the post-assessment and (2) a pre-service teacher who mentions the heuristic argument type on the pre-assessment and mentions it again on the post-assessment, but with the same or lower level of convincingness. In the transition diagrams both case (1) and case (2) described above are counted as a non-transition.

5.1 Results by Heuristic Argument Type

5.1.1 Cases

A cases heuristic argument is defined as a heuristic argument that involves solving a problem by dividing the objects in the problem into subcategories that are non-overlapping and exhaustive. In particular, a pre-service teacher was scored as noticing a case heuristic argument if the pre-service teacher wrote a response that identified how the child was able to group the towers with such groups as only blue cubes, one red and two blue cubes, with two red and one blue cube, two blue cubes are stuck apart, two blue cubes separated by one red cube, and so on. On the pre-assessment, 9 out of 14 pre-service teachers mentioned a cases heuristic argument and 5 did not. Of the 5 pre-service teachers who did not mention a cases argument on the pre-assessment none of them did so on the post-assessment. Thus, the frequency of a transition from lack of identification of use of a cases argument on the pre-assessment to identification of a cases argument on the post-assessment is 0/5 or 0.0%. Figure 5.1 illustrates the frequency growth of identification of a cases heuristic argument from the pre-assessment to the post-assessment.

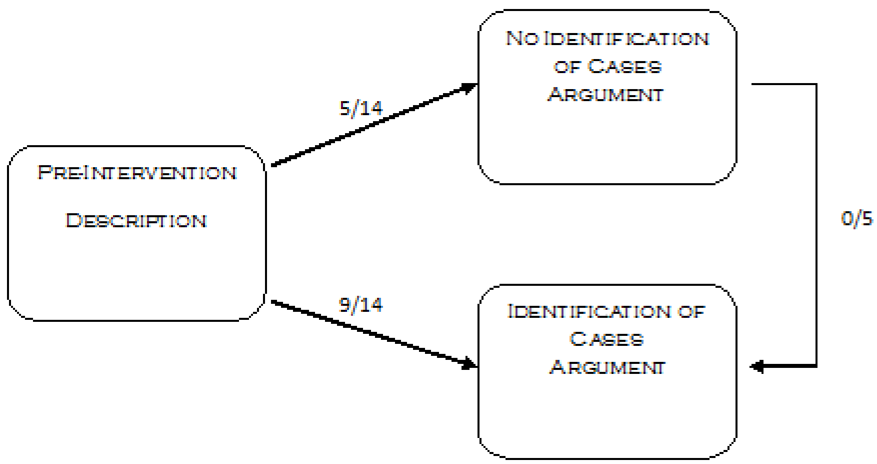

Figure 5. 1 – Transition Diagram on Cases Heuristic Argument

5.1.2 Controlling for the Variable

A controlling for the variable heuristic argument was defined as a heuristic argument in which a specific variable is fixed, that is, controlled in the solution set and solutions under these controlled situations are found and later generalized. In particular, a pre-service teacher was scored as noticing a heuristic argument in which the student controlled for the variable if they identified towers or groups of towers, by the logical placement of cubes of one color or the other such as a staircase pattern of a single cube of one color (or more than one cube of the same color) being moved in consecutively lower or higher in position within the tower. On the pre-assessment, 3 out of 14 pre-service teachers mentioned a controlling for the variable heuristic argument and 11 did not. Of the 11 pre-service teachers who did not mention a controlling for variables argument on the pre-assessment 2 did so on the post-assessment. Thus, the frequency of a transition from lack of identification of a controlling for variable argument on the pre-assessment to identification of a controlling for variable argument on the post-assessment is 2/11 or 18.18%. Figure 5.2 illustrates the frequency growth of identification of a controlling for variable heuristic argument from the pre-assessment to the post-assessment.

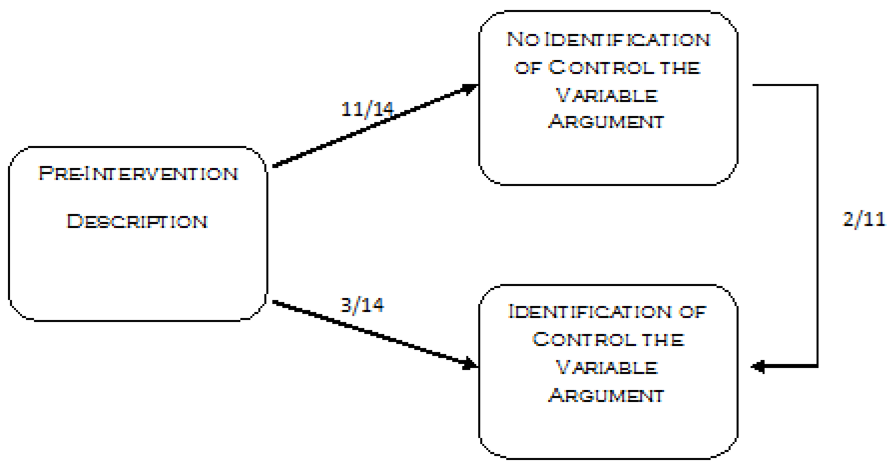

Figure 5. 2 – Transition Diagram on Controlling for the Variable Heuristic Argument

5.1.3 Figures

A heuristic argument that involves figures is a heuristic argument that uses visual images to express the solution to a problem. In particular, pre-service teachers were scored as noticing a heuristic argument that uses figures if they provided a description of towers the children in the video made and described how the children built towers or if the pre-service teachers indicated the use of diagrams used by students. On the pre-assessment, 10 out of 14 pre-service teachers mentioned a figures heuristic argument and 4 did not. Of the 4 pre-service teachers who did not mention a figures argument on the pre-assessment 3 did so on the post-assessment. Thus, the frequency of a transition from lack of identification of a figures argument on the pre-assessment to identification of a figures argument on the post-assessment is 3/4 or 75.0%. Figure 5.3 illustrates the frequency growth of identification of a figures heuristic argument from the pre-assessment to the post-assessment.

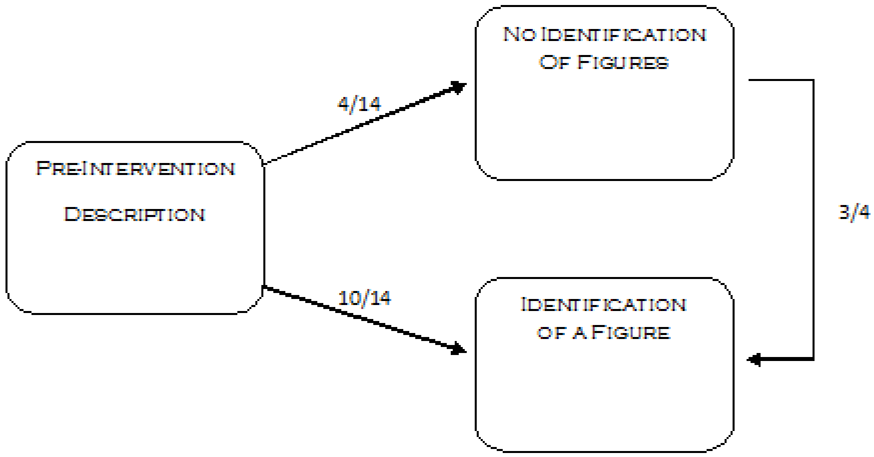

Figure 5. 3 - Transition Diagram on Figures Heuristic Argument

5.1.4 Patterns

A heuristic argument that uses patterns was identified as an exhaustive heuristic argument that used a geometric organization of the problem trying to be solved. In particular, a pre-service teacher was scored as noticing a heuristic argument that uses patterns if the pre-service teacher stated that the students used a "pattern" without specific descriptions of the pattern being used or described the patterns used by the students such as opposites. On the pre-assessment, 9 out of 14 pre-service teachers noticed a patterns heuristic argument and 5 did not. Of the 5 pre-service teachers who did not mention a patterns argument on the pre-assessment 2 did so on the post-assessment. Thus, the frequency of a transition from lack of identification of a patterns argument on the pre-assessment to identification of a patterns argument on the post-assessment is 2/5 or 40.0%. Figure 5.4 illustrates the frequency growth of identification of a patterns heuristic argument from the pre-assessment to the post-assessment.

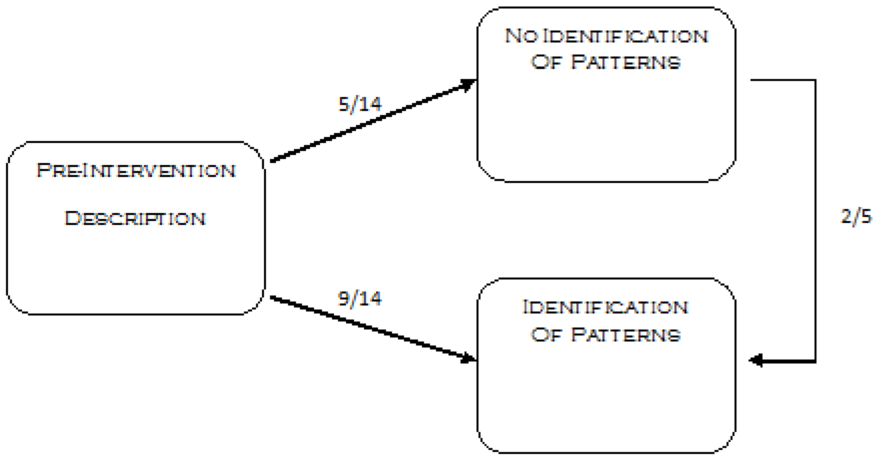

Figure 5. 4 - Transition Diagram on Patterns Heuristic Argument

5.1.5 Doubling Strategy

An argument that uses a doubling strategy was defined as a heuristic argument in which the problem with sample size **n** was solved by doubling the outcome of the problem with sample size **n-1**. In particular, a pre-service teacher was scored as noticing a doubling heuristic argument if the pre-service teacher made a comment regarding that the child was able to create a tower of a specific height by adding of one of two colors to the top of the tower of previous height . For example, the pre-service teacher could have noted that the child added a red and blue to all towers of 2-tall to create unique towers of 3-tall. On the pre-assessment, 7 out of 14 pre-service teachers noticed a doubling heuristic argument and 7 did not. Of the 7 pre-service teachers who did not mention a doubling strategy argument on the pre-assessment 1 did so on the post-assessment. Thus, the frequency of a transition from lack of identification of a doubling strategy argument on the pre-assessment to identification of a doubling strategy argument on the post-assessment is 1/7 or 14.28%. Figure 5.5 illustrates the frequency growth of identification of a doubling strategy heuristic argument from the pre-assessment to the post-assessment.

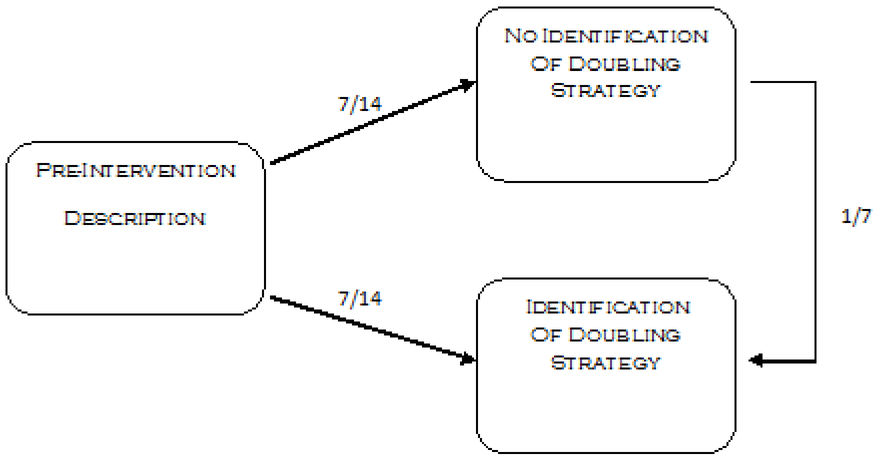

Figure 5. 5 – Transition Diagram on Doubling Heuristic Arguments

5.1.6 Numerical Operations

A numerical operations heuristic argument was defined as a heuristic argument in which the problem was solved by using an arithmetic operation, e.g., addition, subtraction, multiplication, or division operations. In particular, a pre-service teacher was scored as noticing a numerical operations heuristic argument if they provided a description that mentioned particular patterns involving numbers including additive properties (2,4,6,..), doubling or multiplication properties (2,4,8,..), or base squared properties (1,4, 9,…). On the pre-assessment, 7 out of 14 pre-service teachers noticed a numerical operations heuristic argument and 7 did not. Of the 7 pre-service teachers who did not mention a numerical operations argument on the pre-assessment 1 did so on the post-assessment. Thus, the frequency of a transition from lack of identification of a numerical operations argument on the pre-assessment to identification of a numerical operations argument on the post-assessment is 1/7 or 14.28%. Figure 5.6 illustrates the frequency growth of identification of a numerical operations heuristic argument from the pre-assessment to the post-assessment.

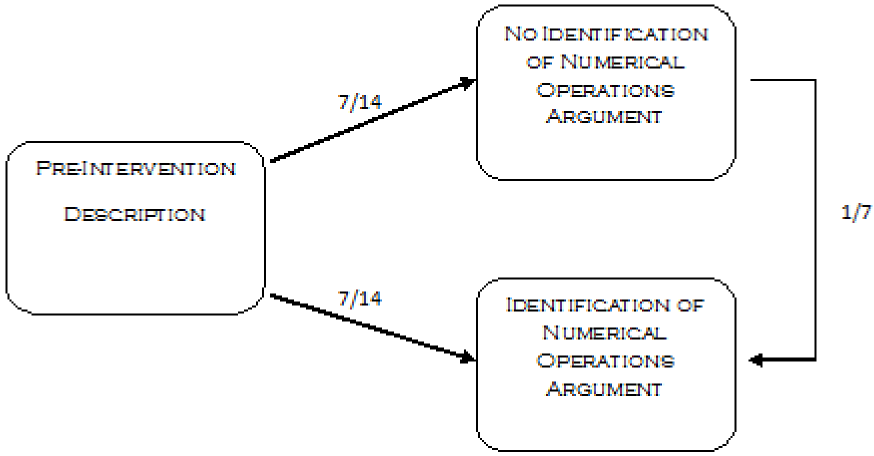

Figure 5. 6 – Transition Diagram on Numerical Operations Heuristic Argument

5.1.7 Summary of the Heuristic Arguments

A summary of all 6 heuristic argument types in this assessment is provided in Table 5.1 below. Heuristic arguments that used figures exhibited the greatest growth for 3/14 of the pre-service teachers. This is followed by heuristic argument of controlling for the variable and patterns with 2/14 of the pre-service teachers exhibiting growth. Then, 1/14 of the pre-service teachers showed growth with heuristic arguments that used doubling strategies and numerical operations heuristic arguments. Finally, there was no growth at all for the heuristic argument of cases.

Table 5. 1 – Use of Video for Student Problem Solving Assessment:
Heuristic Argument Types Mentioned by Pre-Service Teachers

Heuristic Argument Type	Number Teachers on Pre-Assessment	Number Teachers on Post-Assessment	Number Exhibiting Growth	Growth Rate (%)
Cases	9	9	0	0.0
Control for Variable	3	5	2	18.2
Figures	10	13	3	75.0
Patterns	9	11	2	40.0
Doubling Strategy	7	8	1	14.3
Numerical Operation	7	8	1	14.3
Overall Mean	7.5	9.0	1.5	23.1

5.2 Results of Post-Assessment Transition on Multiple Heuristic Arguments

Of the 14 pre-service teachers who participated in the pre- and post- video assessment, 43% of the teachers showed a post-assessment transition. Figure 5.7 shows the number of distinct heuristic arguments the pre-service teachers mentioned on the pre-assessment and post-assessments. For the pre-assessment, the mean number of distinct heuristic arguments mentioned is 3.43, the median is 4, the standard deviation is 1.74, and the 95% mean confidence interval of (2.42, 4.43). On the post-assessment, the mean number of distinct heuristic arguments mentioned is 4.29, the median is 4, the standard deviation is 1.54, and the 95% mean confidence interval of (3.40, 5.18).

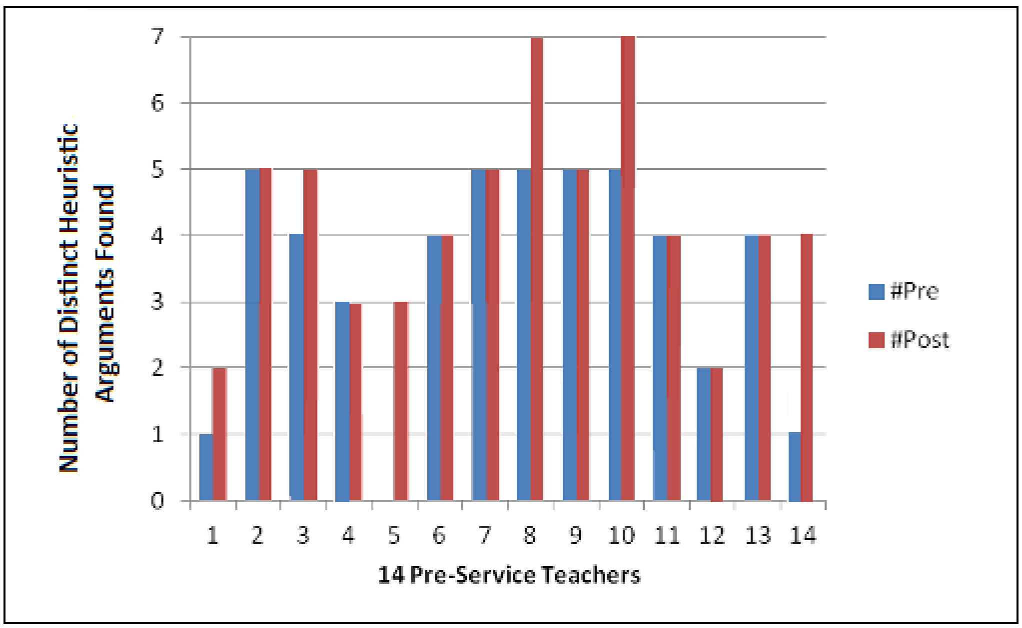

Figure 5. 7 – Individual Pre- and Post- Assessment Results[3]

To determine if the growth of the group itself is statistically significant, a paired student t-test was performed on the post-assessment heuristic argument count minus the pre-assessment heuristic argument count under the null hypothesis: The mean transition growth is zero versus the alternative hypothesis: The mean transition growth is greater than zero. In examining all 14 pre-service teacher

[3] The pre-service teacher with ID of 5 had a 0 pre-assessment score.

participants in the study, the difference between the post-assessment count and pre-assessment count is 4.286 – 3.429 = 0.857 which is statistically significant based up the Student t statistic of $t(13) = 2.747$ and $p < 0.01$. Thus, the data provide evidence of a statistically significant transition growth with a mean growth estimate of 0.86 argument types and a 95% mean growth confidence interval of 0.18 to 1.53 argument types.

CHAPTER 6: BELIEFS ASSESSMENT RESULTS

Sixteen pre- and post- Beliefs Assessment Questionnaire responses were determined as appropriate to evaluate the changes in beliefs of 17 secondary pre-service teachers. The Beliefs Questionnaire items encompassed three areas: (1) beliefs about students' mathematics learning (2) beliefs about what it means to do mathematics and (3) beliefs about effective mathematics teaching. (See Appendix E for the Original Beliefs Assessment and Appendix F for the 16 subset of beliefs).

In the analysis of the Beliefs Questionnaire response data, questionnaire responses that were consistent with beliefs held by current reform efforts were coded as "Yes" and responses that were inconsistent with beliefs held by current reform efforts were coded as "No." Furthermore, each question on the belief survey was grouped in a category based on the types of beliefs it questioned: Students Mathematical Learning, Mathematics, or Teaching Mathematics. To analyze each beliefs questionnaire item for evidence of pre-to-post-assessment statistically significant growth, a McNemar Test for the significance of change was performed.

6.1 Beliefs about Students' Mathematics Learning Results

Pre-service teachers' beliefs about student mathematics learning were assessed on the basis of the pre-service teachers' responses to the six items on the Beliefs Questionnaire listed in Table 6.1. In Q1, Q7, Q15, and Q18, responses of Agree and Strongly Agree were coded as "Yes" since those beliefs are consistent current reform standards. Meanwhile, in Q11 and Q29, a code of "Yes" was given to responses of Disagree and Strongly Disagree.

Table 6. 1 – Beliefs on Student Math Learning Taken From the Beliefs Questionnaire

Q1.	Learners generally understand more mathematics than their teachers or parents expect.
Q7.	All students are capable of working on complex math tasks.
*Q11.	Young children must master math facts before starting to solve problems.
Q15.	Learners generally have more flexible solution strategies than their teachers or parents expect.
Q18.	Learners can solve problems in novel ways before being taught to solve such problems.
*Q29.	Only the most talented students can learn math with understanding.

- Responses to items Q11 and Q29 were analyzed with Strongly Disagree and Disagree coded as consistent with current reform standards. In contrast, responses of Strongly Agree and Agree to the other listed items were coded as consistent with current reform standards.

To determine the rate of transition to beliefs consistent with current reform standards, Table 6.2 summarizes the fraction of the pre-service teachers whose beliefs changed from the pre-assessment to post-assessment for each belief statement and the fraction of the pre-service teachers whose beliefs remained the same from the pre-assessment to the post-assessment for each belief statement.

For Q1, 1 out of 3 pre-service teachers who held this belief consistent with standards on the pre-assessment also held this belief consistent with standards on the post- assessments whereas 10 out of 14 pre-service teachers with beliefs inconsistent with standards on the pre-assessment transitioned to beliefs consistent with standards on the post-assessment.

For Q7, 7 out of 9 pre-service teachers who held this belief consistent with standards on the pre-assessment also held this belief consistent with standards on the post- assessments whereas 6 out of 8 pre-service teachers with beliefs inconsistent with standards on the pre-assessment transitioned to beliefs consistent with standards on the post-assessment.

For Q11, 1 out of 4 pre-service teachers who held this belief consistent with standards on the pre-assessment also held this belief consistent with standards on the post-assessment whereas 5 out of 13 pre-service teachers with beliefs inconsistent with standards on the pre-assessment transitioned to beliefs consistent with standards on the post-assessment.

For Q15, 4 out of 7 pre-service teachers who held this belief consistent with standards on the pre- assessment also held this belief consistent with standards on the post-assessment whereas 8 out of 10 pre-service teachers with beliefs inconsistent with standards on the pre-assessment transitioned to beliefs consistent with standards on the post-assessment.

For Q18, 9 out of 10 pre-service teachers who held beliefs consistent with standards on both the pre-assessment also held this belief consistent with standards on the post-assessment whereas 5 out of

7 pre-service teachers with beliefs inconsistent with standards on the pre-assessment transitioned to beliefs consistent with standards on the post-assessment.

For Q29, all 16 pre-service teachers who held beliefs consistent with standards on the pre-assessment also held this belief consistent with standards on the post-assessment.

On average, 79.2% of the pre-service teachers who had beliefs consistent with standards on the pre-assessment continued to have beliefs consistent to the standards on the post-assessment. In contrast, 64.2% of the pre-service teachers who had beliefs inconsistent with standards on the pre-assessment transitioned to beliefs consistent to the standards on the post-assessment.

Table 6. 2 – Student Math Learning: Rate of Transition to Beliefs Consistent with Standards

Beliefs Consistent With Reform Standards?		Fraction of Participants Who Transition to Beliefs Consistent With Standards						Weighted Average (%)
Pre	Post	Q1	Q7	*Q11	Q15	Q18	*Q29	
Yes	Yes	1/3	7/9	1/4	4/7	9/10	16/16	38/48 = 79.2%
	No	2/3	2/9	3/4	3/7	1/10	0/16	10/48 = 20.8%%
No	Yes	10/14	6/8	5/13	8/10	5/7	0/1	34/53 = 64.2%
	No	4/14	2/8	8/13	2/10	2/7	1/1	18/47 = 35.8%

* Results based upon reverse coding of belief responses to this questionnaire item

To determine if the beliefs questionnaire responses listed in Table 6.2 provide evidence of a statistically significant change in beliefs, a McNemar Test for significance of changes was performed. The Null Hypothesis: There was no post-assessment transition growth in the teacher's beliefs concerning student math learning versus the Alternative Hypothesis: There was statistically significant post-assessment transition growth in the teacher's beliefs concerning student math learning. Table 6.3 summarizes the number of total pre-service teachers with a belief transition for each belief statement. The third column in Table 6.3 indicates the number of pre-service teachers with a belief transition to current reform standards. The last column in Table 6.3 indicates the statistical significance p-score for the McNemar test for each Belief Questionnaire item in Table 6.1. In examining the responses to Q1 in Table 6.3 we note that of the teachers who made a pre-to-post beliefs transition, 10 out of 12 or 83.3% made a transition to a belief consistent with standards. Based upon the McNemar Test this result provides evidence of statistically significant growth at the 0.019 significance level in the teacher's belief that learners generally understand more mathematics than their teachers or parents expect. In contrast, in examining responses to Belief Questionnaire items Q7, Q11, Q15, Q18, and Q19 we note that of the teachers who made a pre-to-post transition, on the average, 72.7% made a transition to a belief consistent with current standards. However, due to the limited sample size the McNemar Test does not provide statistically significant evidence of transition growth for teacher beliefs Q7, Q11, Q15, Q18, and Q19.

Table 6. 3 – McNemar Test of Change for Teacher's Beliefs on Student Math Learning

Beliefs Statement	McNemar Test For Change		
	Pre-Service Teachers with Beliefs Transition	Pre-Service Teachers with Transition to Current Belief Standards	p-Score
Q1	12	10	0.019
Q7	8	6	0.145
Q11	8	5	0.363
Q15	11	8	0.113
Q18	6	5	0.109
Q29	0	0	N.S.

6.2 Beliefs about Mathematics Results

Pre-service teachers' beliefs about mathematics were assessed on the basis of pre-service teachers' responses to five related items on the Beliefs Questionnaire, shown in Table 6.4. In Q8 and Q19 responses of Agree and Strongly Agree were coded as "Yes" since those beliefs are consistent current reform standards. In contrast, for Q5, Q17, and Q20, a code of "Yes" was given to responses of Disagree and Strongly Disagree.

Table 6. 4 – Beliefs on Mathematics Taken From Beliefs Questionnaire

*Q5.	Math is primarily about learning procedures.
Q8.	Math is primarily about identifying patterns.
*Q17.	Manipulatives cannot be used to justify a solution to a problem.
Q19.	Understanding math concepts is more powerful than memorizing procedures.
*Q20.	Diagrams are not to be accepted as justifications for procedures.

- Responses to items Q5, Q17, and Q20 were analyzed with Strongly Disagree and Disagree coded as consistent with current reform standards. In contrast, responses of Strongly Agree and Agree to the other listed items were coded as consistent with current reform standards.

To determine the rate of transition to beliefs consistent with current reform standards, Table 6.5 summarizes the fraction of the pre-service teachers whose beliefs changed from the pre-assessment to post-assessment for each belief statement and the fraction of the pre-service teachers whose beliefs remained the same from the pre-assessment to the post-assessment for each belief statement.

For Q5, 10 out of 11 pre-service teachers who held this belief consistent with standards on the pre- assessment also held this belief consistent with standards on the post- assessments whereas 5 out of 6 pre-service teachers with beliefs inconsistent with standards on the pre-assessment transitioned to beliefs consistent with standards on the post-assessment.

For Q8, 5 out of 8 pre-service teachers who held this belief consistent with standards on the pre-assessment also held this belief consistent with standards on the post- assessments whereas 3 out of 9 pre-service teachers with beliefs inconsistent with standards on the pre-assessment transitioned to beliefs consistent with standards on the post-assessment.

For Q17, 4 out of 6 pre-service teachers who held this belief consistent with standards on the pre- assessment also held this belief consistent with standards on the post- assessments whereas 6 out of 11 pre-service teachers with beliefs inconsistent with standards on the pre-assessment transitioned to beliefs consistent with standards on the post-assessment.

For Q19, 14 out of 14 pre-service teachers who held this belief consistent with standards on the pre- assessment also held this belief consistent with standards on the post- assessments whereas 3 out of 3 pre-service teachers with beliefs inconsistent with standards on the pre-assessment transitioned to beliefs consistent with standards on the post-assessment.

For Q20, 11 out of 12 pre-service teachers who held this belief consistent with standards on the pre- assessment also held this belief consistent with standards on the post- assessments whereas 1 out of 5 pre-service teachers with beliefs inconsistent with standards on the pre-assessment transitioned to beliefs consistent with standards on the post-assessment.

On average, 86.3% of the pre-service teachers who had beliefs consistent with standards on the pre-assessment continued to have beliefs consistent to the standards on the post-assessment. In contrast, 55.9% of the pre-service teachers who had beliefs inconsistent with standards on the pre-assessment transitioned to beliefs consistent to the standards on the post-assessment.

Table 6. 5 – Mathematics: Rate of Transition to Beliefs Consistent with Standards

Beliefs Consistent With Reform Standards?		Fraction of Participants That Transition to Beliefs Consistent With Standards					Weighted Average (%)
Pre	Post	*Q5	Q8	*Q17	Q19	*Q20	
Yes	Yes	10/11	5/8	4/6	14/14	11/12	44/51 = 86.3%
	No	1/11	3/8	2/6	0/14	1/12	7/51 = 13.7%
No	Yes	5/6	3/9	6/11	3/3	1/5	19/34 = 55.9%
	No	1/6	6/9	5/11	0/3	4/5	15/34 = 44.1%

* Results based upon reverse coding of belief responses to this item

To determine if the beliefs questionnaire responses listed in Table 6.5 provide evidence of a statistically significant change in beliefs, a McNemar Test for significance of changes was performed. The Null Hypothesis: There was no post-assessment transition growth in the teacher's beliefs concerning mathematics versus the Alternative Hypothesis: There was statistically significant post-assessment transition growth in the teacher's beliefs concerning mathematics. Table 6.6 summarizes the number of total pre-service teachers with a belief transition for each belief statement. The third column in Table

6.6 indicates the number of pre-service teachers with a belief transition to current reform standards. The last column in Table 6.6 indicates the statistical significance p-score for the McNemar test for each Belief Questionnaire item in Table 6.4.

In examining responses to Belief Questionnaire items Q5, Q8, Q17, Q19, and Q20 listed in Table 6.5 we note that of the teachers who made a pre-to-post transition, on the average, 72.0% made a transition to a belief consistent with current standards. However, due to the limited sample size the McNemar Test does not provide statistically significant evidence of transition growth for teacher beliefs Q5, Q8, Q17, Q19, and Q20.

Table 6. 6 – McNemar Test of Change for Teacher's Beliefs on Mathematics

Beliefs Statement	Mc Nemar Test For Change		
	Pre-Service Teachers with Beliefs Transition	Pre-Service Teachers with Transition to Current Belief Standards	p-Score
Q5	6	5	0.109
Q8	6	3	0.6556
Q17	8	6	0.145
Q19	3	3	0.125
Q20	2	1	0.50

6.3 Beliefs about Teaching Mathematics Results

Pre-service teachers' beliefs about teaching were assessed on the basis of pre-service teachers' responses to five related items on the Beliefs Questionnaire, shown in Table 6.7. In Q9, Q12, Q21, and Q33, responses of Agree and Strongly Agree were coded as "Yes" since those beliefs are consistent current reform standards. Meanwhile, in Q6, a code of "Yes" was given to responses of Disagree and Strongly Disagree.

Table 6. 7 – Beliefs on Teaching Mathematics Taken From Beliefs Questionnaire

*Q6.	Students will get confused if you show them more than one way to solve a problem.
Q9.	If students learn math concepts before they learn procedures, they are more likely to understand the concepts.
Q12.	Teachers should show students multiple ways of solving a problem.
Q21.	If students learn math concepts before they learn procedures, they are more likely to understand the procedures.
Q33.	Teachers should intervene as little as possible when students are working on open-ended mathematical problems.

- Responses to item Q6 were analyzed with Strongly Disagree and Disagree coded as consistent with current reform standards. In contrast, responses of Strongly Agree and Agree to the other listed items were coded as consistent with current reform standards.

To determine the rate of transition to beliefs consistent with current reform standards, Table 6.8 summarizes the fraction of the pre-service teachers whose beliefs changed from the pre-assessment to post-assessment for each belief statement and the fraction of the pre-service teachers whose beliefs remained the same from the pre-assessment to the post-assessment for each belief statement.

For Q6, 8 out of 10 pre-service teachers who held this belief consistent with standards on the pre- assessment also held this belief consistent with standards on the post- assessments whereas 3 out of 7 pre-service teachers with beliefs inconsistent with standards on the pre-assessment transitioned to beliefs consistent with standards on the post-assessment.

For Q9, 11 out of 11 pre-service teachers who held this belief consistent with standards on the pre- assessment also held this belief consistent with standards on the post- assessments whereas 3 out of 6 pre-service teachers with beliefs inconsistent with standards on the pre-assessment transitioned to beliefs consistent with standards on the post-assessment.

For Q12, 15 out of 16 pre-service teachers who held this belief consistent with standards on the pre- assessment also held this belief consistent with standards on the post- assessments whereas 1 out of 1 pre-service teachers with beliefs inconsistent with standards on the pre-assessment transitioned to beliefs consistent with standards on the post-assessment.

For Q21, 11 out of 14 pre-service teachers who held this belief consistent with standards on the pre- assessment also held this belief consistent with standards on the post- assessments whereas 3 out of 3 pre-service teachers with beliefs inconsistent with standards on the pre-assessment transitioned to beliefs consistent with standards on the post-assessment.

For Q33, 3 out of 6 pre-service teachers who held this belief consistent with standards on the pre- assessment also held this belief consistent with standards on the post- assessments whereas 5 out of 11 pre-service teachers with beliefs inconsistent with standards on the pre-assessment transitioned to beliefs consistent with standards on the post-assessment.

On average, 84.2% of the pre-service teachers who had beliefs consistent with standards on the pre-assessment continued to have beliefs consistent to the standards on the post-assessment. In contrast, 53.6% of the pre-service teachers who had beliefs inconsistent with standards on the pre-assessment transitioned to beliefs consistent to the standards on the post-assessment.

Table 6. 8 – Teaching Mathematics: Rate of Transition to Beliefs Consistent with Standards

Beliefs Consistent With Reform Standards?		Fraction of Participants That Transition to Beliefs Consistent With Standards					Weighted Average (%)
Pre	Post	*Q6	Q9	Q12	Q21	Q33	
Yes	Yes	8/10	11/11	15/16	11/14	3/6	48/57 = 84.2%
	No	2/10	0/11	1/16	3/14	3/6	9/57 =15.8%
No	Yes	3/7	3/6	1/1	3/3	5/11	15/28 = 53.6%
	No	4/7	3/6	0/1	0/3	6/11	13/28 = 46.4%

* Results based upon reverse coding of belief responses to this item

To determine if the beliefs questionnaire responses listed in Table 6.8 provide evidence of a statistically significant change in beliefs, a McNemar Test for significance of changes was performed. The Null Hypothesis: There was no post-assessment transition growth in the teacher's beliefs concerning teaching mathematics versus the Alternative Hypothesis: There was statistically significant post-assessment transition growth in the teacher's beliefs concerning teaching mathematics. Table 6.9 summarizes the number of total pre-service teachers with a belief transition for each belief statement. The third column in Table 6.9 indicates the number of pre-service teachers with a belief transition to current reform standards. The last column in Table 6.9 indicates the statistical significance p-score for the McNemar test for each Belief Questionnaire item in Table 6.7.

In examining responses to Belief Questionnaire items Q6, Q9, Q12, Q21, and Q33 listed in Table 6.8 we note that of the teachers who made a pre-to-post transition, on the average, 62.5% made a transition to a belief consistent with current standards. However, due to the limited sample size the McNemar Test does not provide statistically significant evidence of transition growth for teacher beliefs Q6, Q9, Q12, Q21, and Q33.

Table 6. 9 – McNemar Test of Change for Teacher's Beliefs on Teaching Mathematics

Beliefs Statement	Mc Nemar Test For Change		
	Pre-Service Teachers with Beliefs Transition	Pre-Service Teachers with Transition to Current Belief Standards	p-Score
Q6	5	3	0.50
Q9	3	3	0.125
Q12	2	1	0.50
Q21	6	3	0.656
Q33	8	5	0.363

CHAPTER 7: CASE STUDIES RESULTS

For a subset of pre-service teachers in the study, five participated in all pre- and post-assessments. These five study participants have been chosen for a case study analysis. This case study analysis attempts to look at the overall picture of the intervention from the pre-assessment to post-assessment as well as look at the work the pre-service teachers did during the intervention as well. The work that was analyzed includes pre- and post- student work assessments, pre- and post- video assessments, pre- and post- beliefs assessments, in class prepared transparencies of solutions to problems solved in class, and typed up homework assignments in which the students commented on topics discussed in class and seen in videos they watched at home.

7.1 Pre-Service Teacher Subject 03

7.1.1 Subject 03's Participation in the Intervention

Based on the pre-assessment results, pre-service teacher Subject 03 was able to recognize 6 out of 11 heuristic arguments used by the students. He identified convincing heuristic arguments used by students containing figures, patterns, exhaustive lists, and tree diagrams. He also recognized heuristic arguments using numerical operations and simpler problems even though he did not find these particular heuristic arguments convincing. Most of the explanations he provided for why he thought a heuristic argument was convincing were based on whether the mathematics used was appropriate while solving the problem.

During the intervention, Subject 03 had opportunities to solve problems and watch videos of children solving the same problems he worked on in class. The first problem Subject 03 worked on was the Pizza with Halves Problem. To solve this problem, Subject 03 and his group used heuristic arguments that employed controlling for the variable and exhaustive lists. From the transparency his group made, shown in Figure 7.1A, it appears that Subject 03's group first created an exhaustive list of all possible toppings for a half of pizza. His use of exhaustive lists at the beginning of the problem is consistent with his pre-assessment discussion indicating exhaustive lists can be convincing heuristic arguments. Utilizing this exhaustive list Subject 03's group described how one can pick any topping from the list to be on one half of pizza and pair it with all remaining possible toppings on the second half of the pizza. Thus, we note that the group used the controlling for variable heuristic by asserting that for each type of topping on one half of the pizza you could have any choice of a topping from the exhaustive list on the other half of the pizza.

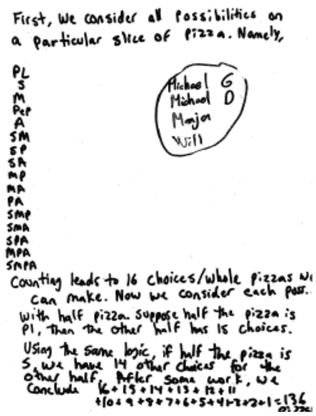

Figure 7.1A – Subject 03's Group Transparency on the Pizza with Halves Problem

Next, Subject 03 began work on the Towers 5-Tall Problem. In solving the 5-Tall Problem, shown in Figure 7.1B, Subject 03 used a tree diagram heuristic argument to find the number of towers selected from two colors that are 5-tall. Again, his use of a tree diagram is consistent with his pre-assessment narrative indicating that exhaustive lists can be a convincing heuristic argument.

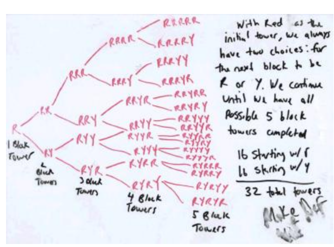

Figure 7.1B – Subject 03's Group Transparency on the Pizza with Halves Problem

Following these in-class activities, Subject 03 had several opportunities to learn about how others solved these problems including watching the presentations by his classmates as well as watching videos of young children and 11th grade students solving these same problems. For homework, Subject 03 was asked to comment on the forms of reasoning he saw either in class or in the videos. Subject 03's focus was mainly on the children in the video. Subject 03 first responded to the Pizza with Halves Problem. He noted that he was:

> "particularly interested to see the students attempt to list all the possibilities, but get held up when attempting to decide the best way to organize the information. It was apparent that they wanted to use more mathematical techniques (combinations/ permutations) but did not fully understand how to implement such an approach…I also think the younger children I watched in the other videos were so willing to list all the combinations because they were unaware of an alternative method."

Subject 03 appears to have focused on the mathematics that he felt the students did not know of any alternative problem solving method. While he also started the problem with an exhaustive list, he commented that his group was able to solve the problem more quickly than the students in the video because they had a better understanding of combinations. He also commented that once the students realized the Pizza with Halves Problem was similar to the Towers problem, the students in the video appeared to better understand the problem.

Subject 03 also commented on a video of Romina solving Ankur's Challenge. In his write-up he noticed her use of controlling for the variable since the first color (top of the tower) could be one of three colors. While he found her heuristic argument convincing, he stated his heuristic argument was more convincing because he used a visual which he described:

> "I started with one color cube and then found all possible combinations which are 12 towers. I then multiplied by three because I could have started with any of the 3 colors…[M]y solution explicitly shows you all possibilities. However, I do think it would be useful to use my solution to support Romina's strategy."

Essentially, Subject 03 wrote out the entire list of all possible combinations a person could have with a particular color on top in accordance with the Ankur's Challenge's conditions. This is different from Romina's more symbolic approach to the problem as she did not create an overall list. Thus, Subject 03 appeared to weigh visuals and lists highly as he claimed that this would help Romina's Proof become more convincing.

Subject 03's final homework assignment asked him to comment on the heuristic arguments he found most convincing from the intervention. Subject 03 stated:

"I think any of the argument[s] that provided a list and then some sort of generalization to be the most effective proof. This is because you could actually see all possibilities."

Thus, after the intervention ended, Subject 03's focus on the post-assessment changed from just focusing on the mathematics to also focusing on visuals and explanations. Subject 03's post-assessment focus changed relative to his pre-assessment even though he did not identify any new heuristic arguments on the post-assessment.

7.1.2 Comparing Subject 03's Problem Solving Strategies to What He Notices

During the intervention, Subject 03 mainly used the strategies of controlling for the variable, exhaustive lists, and tree diagrams during the intervention's problem-solving tasks. For the most part, he did notice student problem-solving strategies that were similar to his own problem-solving strategies. However, he found problem-solving strategies that were different from his own strategies such as using a simpler problem and patterns. Thus, Subject 03 was able to look beyond his own problem-solving strategies to find other strategies used by students. Since Subject 03 found problem-solving strategies that used lists and visuals as more effective, he often found these types of strategies more convincing than the other strategies employed by the students.

7.2 Pre-Service Teacher Subject 07

7.2.1 Subject 07's Participation in the Intervention

Based on the pre-assessment results, Subject 07 was able to recognize 3 out of 11 mathematical heuristic arguments used by the students. Prior to the intervention, Subject 07 was able to find convincing heuristic arguments used by students that utilized patterns and numerical operations. He was also able to recognize heuristic arguments that used figures even though he did not find these particular heuristic arguments convincing. Most of the explanations he provided for why he thought a heuristic argument was convincing were based on the quality of the explanation.

During the intervention, Subject 07 had opportunities to solve problems and watch videos of children solving the same problems he worked on in class. The first problem Subject 07 worked on was the Pizza with Halves Problem, focusing on the problem where there are two topping choices since his group struggled initially with the 4-toppings problem. To solve this problem, Subject 07 and his group used a combination of strategies: a controlling for variable strategy to establish two cases (similar or non-similar toppings on each half) and then an exhaustive lists strategy to count the number of possible pizzas for each of these two cases. From the transparency his group made, shown in Figure 7.2A, Subject 07's group showed the two cases: "whole pies" and "half pies." Under the "whole pie" case, there is an exhaustive list of all possible toppings on a whole pie. In other words, they listed all pies in which each half of the pie had the same topping. Then, under the "half pies" case, the group created another exhaustive list for all possible pies where the halves contained different toppings.

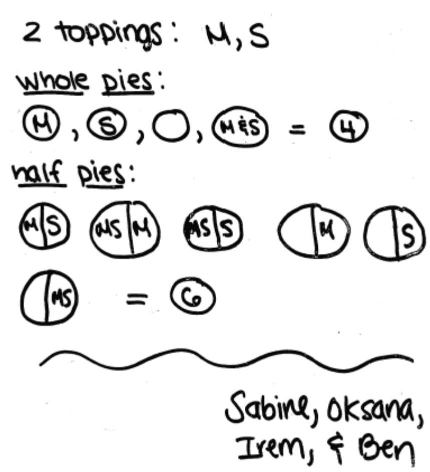

Figure 7.2A – Subject 07's Group Transparency on the Pizza with Halves Problem

Next, Subject 07 began working on the Towers 5-Tall Problem. In solving the 5-Tall Problem, shown in Figure 7.2B, Subject 07 appears to have used a doubling strategy, building up the towers from one tall to five tall by adding a cube (1 of each color) to the top of each of the previous towers of cubes.

At the end of the problem, his group used some mathematical notation to generalize their results. They noticed that the number of possible towers is 2^n where n is the height of the tower. They verified this conclusion by showing that the formula works for cases n=1 to n=5.

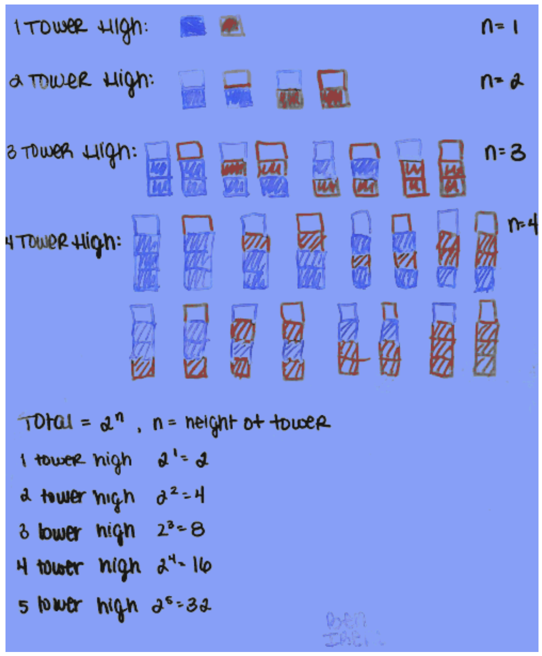

Figure 7.2B – Subject 07's Group Transparency on the Towers 5-Tall Problem

The idea of building towers up from previous created towers also continues as he works on Ankur's problem. His group's solution is shown in Figure 7.2C. However, the doubling approach only got his group halfway through the problem. They begin by listing out all possible cases of towers three tall with three colors. Then, they used a form of solving-by-cases strategy and worked with each 3-tall tower as its own group. Within each group, his group first built towers by placing a new color cube on top, essentially adding in the one duplicate color. However, this method missed cases where the duplicate color is not always the top color. Therefore, it appears Subject 07's group used a form of staircase pattern to list out all the remaining cases where the 4th cube could have been placed while adhering to the color scheme order/stack he described as a particular case.

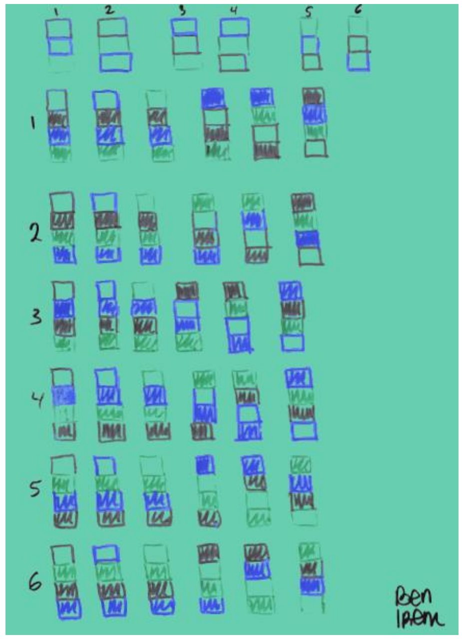

Figure 7.2C – Subject 07's Group Transparency on Ankur's Challenge

During these in class activities, Subject 07 also had several opportunities to watch how others solved these problems by watching the presentations by his classmates and watching videos of young children and 11th grade students solving these same problems. For homework, Subject 07 was asked to comment on the forms of reasoning he saw either in class or in the videos. Subject 07's focus was mainly on the children in the video. Subject 07 first responded to the Pizza with Halves Problem. He noted that the 11th graders "started with fewer options and built their way up to the problem stated." Subject 07 noted that the students were solving the problem in a way similar to his group as they were building up the cases in a doubling type approach, starting with a small sample space of less pizza toppings and building the number of topics one by one. He also noted that some students tried to use tree diagrams but claimed it was inaccurate due to treating plain pizza as a case. To him, the "greatest similarity was that both groups [his and the video groups] sought to find a pattern as to expand the solution into a general form." Even though he commented on the Pizza with Halves Problem, his 5-Tall transparency appears more consistent with what he wrote in his comments. In his 5-tall transparency, his group used a doubling approach and found a general formula for n-tall.

Meanwhile, in response to the Romina's Proof video of Ankur's challenge, Subject 07 considered Romina's solution a case based solution. He noted that she considered all the ways a tower can have two reds, multipled by 2 because there were two remaining colors to place on and the order of these two cubes mattered. Finally, he noted that Romina multipled by 3 because there were two analogous cases with two green and two blue cubes. Subject 07 indicated this was a convincing solution and noted that there is no need to draw a figure because listing the cases is proof enough. Finally, Subject 07 acknowledged that Romina's solution was different from his as they both were created using different case situations. Instead of focusing on the repeated color, he started with 6 towers 3 high using all colors and built an additional block onto each of those towers.

Subject 07's final homework assignment asked him to comment on the heuristic arguments from the intervention that he found most convincing and least convincing. Subject 07 states:

> "The two types of solutions that I had found most convincing were the ones that built of[f] simpler problem[s] and those that used systematic recursion. I do not like arguments that simply tried to list out all possibilities. The reason I do not like any of those solutions is that there was too much of a chance for missing one or even (listing) the same solution twice. I would be appeased with the other solutions if they were a bit more systematic."

Thus, after the intervention ended, Subject 07 was able to notice and be convinced by more student heuristic arguments from the Student Work and Video Post-assessments. Subject 07 was able to recognize 7 out of 11 mathematical heuristic arguments used by the students. On the post-assessment Subject 07 was able to notice and become convinced of many more heuristic argument types than he did on the pre-assessment including: cases, doubling strategy, tree diagrams, and guess and check. In addition to the 6 heuristic arguments he was able to recognize other students using, he employed two additional heuristic arguments involving patterns and exhaustive lists in solving some of the in-class problems.

7.2.2 Comparing Subject 07's Problem Solving Strategies to What He Notices

As strategies on the intervention problem solving tasks, Ben mainly used the strategies of solving by cases, exhaustive lists, doubling strategy, and patterns. For the most part, he noticed student problem-solving strategies that were similar to his own problem-solving strategies. However, he found other problem-solving strategies that were different from his own strategies such as using a simpler problem. Thus, Subject 03 was able to look beyond his own problem-solving strategies to find other strategies used by students.

7.3 Pre-Service Teacher Subject 12

7.3.1 Subject 12's Participation in Intervention

Based on the pre-assessment results, Subject 12 was able to recognize 3 out of 11 heuristic arguments used by the students. Prior to the intervention, Subject 12 was able to find convincing heuristic arguments used by students containing use of patterns and figures. She was also able to recognize arguments that used guess and check strategies even though she did not find these particular arguments convincing. Most of the explanations she provided for why she thought a heuristic argument was convincing were based on the quality of the explanation being used.

During the intervention, Subject 12 had opportunities to solve problems and watch videos of children solving the same problems she worked on in class. The first problem Subject 12 worked on was with Subject 07 on the Pizza with Halves Problem, focusing on the problem where there are two topping choices as this group was struggling with the 4-toppings initially. To solve this problem, Subject 07, Subject 12, and their group used both a cases and exhaustive lists heuristic argument. This transparency is reproduced below in Figure 7.3A. It should be noted that this is the only problem in which Subject 12 worked with Subject 07.

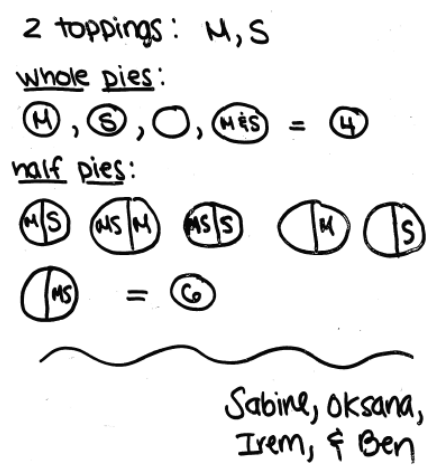

Figure 7.3A – Subject 12's Group Transparency on the Pizza with Halves Problem

Next, Subject 12 began work on the Towers 5-Tall Problem. In solving the 5-Tall Problem, shown in Figure 7.3B, Subject 12 appeared to use controlling for variable and tree diagram heuristic strategies. She used a tree diagram to show all possible cases when the blue cube is on the bottom. Thus, they used similar type reasoning to argue how many towers are possible when green cube is on the bottom. Then, the girls added the 16 towers with blue on the bottom with the 16 towers with green on the bottom.

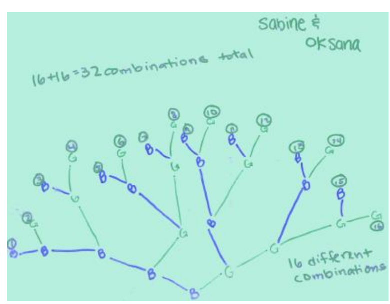

Figure 7.3B – Subject 12's Group Transparency on the Tower 5-Tall Problem

Subject 12 continued to use control for variable and tree diagram heuristic arguments to solve Ankur's Challenge as shown in Figure 7.3C. Like in the Towers 5-Tall problem, the ladies decided on a color for use on the bottom of the tower (controlling for the variable). From there, the ladies actually used a doubling type of approach to build a tree diagram. The ladies first created a 1-tall tower with one blue block. They then created a 2-tall tower by adding a block on top of the blue block. They did this again for the case of 3-tall by building on the 2-tall towers. Finally, they build the 4-tall towers consistent with the conditions set forth by the Ankur's Challenge. It appears they then re-organized this information using a tree diagram. Since they knew a similar approach could be used for situations where the red cube is on the bottom and green cube is on the bottom, they multiplied their solution by 3 to account for all cases.

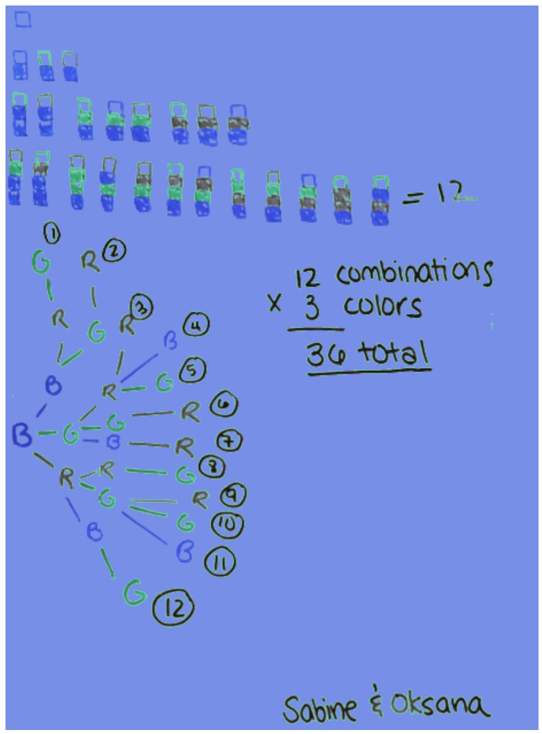

Figure 7.3C – Subject 12's Group Transparency on Ankur's Challenge

Following these in-class activities and during the process of working on Ankur's Challenge, Subject 12 had several opportunities to watch how others solved these same problems by watching the presentations by her classmates, watching videos of young children solving these problems, and watching videos of 11th grades students solving these problems. For homework, Subject 12 was asked to comment on the forms of reasoning she saw either in class or in the videos. Subject 12 first responded to the Pizza with Halves Problem. She stated that the 11th graders' use of "a tree diagram stood out to me because that is how me and my partner started to solve the problem." She also identified that the students in the video used methods that involved the knowledge of combinatorics. Being unable to remember the process, she noted that the students also searched for patterns and tried to connect the towers problem to the pizza problem.

Meanwhile, in response to the Romina's Proof video of Ankur's challenge, Subject 12 described that Romina:

> "found all of the possible double combinations (12) with one color and multiplied it by 3…Me and my partner did it differently by building on from a base but the process Romina used is just as good!"

It appears that Subject 12 understood that there is some control of the variable in both Romina's and her own methods as they both multiplied by 3 to represent the base of the tower could be one of three colors. Subject 12 also recognized that Romina made a list of all possible situations where one can place the two cubes of the same color. She found this equally convincing to her doubling type approach to the problem.

Subject 12's final homework assignment asked her to comment on the heuristic arguments she found most convincing and least convincing from the intervention. Overall, Subject 12's focus on a heuristic argument being convincing has to do with the visuals and the explanation. She noted that she finds Romina's proof as convincing as her own because Romina drew the towers and "was able to explain it" whereas she found that the students who used combinatorical methods:

> "a bit unclear to me maybe because I didn't understand it from the start or maybe the children didn't have a clear understanding of it and couldn't explain it well. Maybe a clearer picture would make more sense to explain where each number came from."

Thus, on the post-assessment, Subject 12 was able to notice and be convinced by more student heuristic arguments from the Student Work and Video than she did on the pre-assessment. In the post-assessments, Subject 12 was able to recognize 6 out of 11 mathematical heuristic arguments used by the students. Besides the heuristic arguments she mentioned on the pre-assessment, Subject 12 was also able to notice and become convinced of many more heuristic argument types on the post-assessment including: cases, tree diagrams, and numerical operations. In addition to the 6 heuristic arguments she was able to recognize other students using, she also used herself one additional heuristic argument of induction in solving some of the in-class problems.

7.3.2 Comparing Subject 12's Problem Solving Strategies to What He Notices

In solving the intervention's problem-solving tasks, Subject 12 mainly used the strategies of controlling for the variable, cases, exhaustive lists, doubling strategy, and tree diagrams. For the most part, she did notice student problem-solving strategies that were similar to her own problem-solving strategies. However, she found problem-solving strategies that were different from her own strategies such as using figures and guess and check by the post-assessment. Thus, Subject 12 was able to look beyond her own problem-solving strategies to find other strategies used by students. However, there were also several strategies Subject 12 used herself but did not identify noticing in the pre- and post-assessments. They were exhaustive lists and doubling strategy.

7.4 Pre-Service Teacher Subject 14

7.4.1 Subject 14's Participation in the Intervention

Based on the pre-assessment results, Subject 14 was able to recognize 8 out of 11 mathematical heuristic arguments used by the students. In the pre-assessment, Subject 14 mentioned several convincing heuristic arguments types used by students: cases, use of figures, use of patterns, use of a similar problem, use of doubling strategy, and controlling for the variable. He was also able to recognize heuristic arguments of guess and check and numerical operations even though he did not find these particular heuristic arguments convincing. Most of the explanations he provided for why he thought a heuristic argument was convincing were based on the mathematics used.

During the intervention, Subject 14 had opportunities to solve problems and watch videos of children solving the same problems he worked on in class. The first problem Subject 14 worked on was the Pizza with Halves Problem. He did not create a transparency of his work for this question as he came late on the day of this in-class activity but did state his solution methods within the homework:

> "In class my group personally went with the approach of focusing on one topping and finding out how many different combinations we could out of the one and multiply it by how many topping there were [four] total minus the repeated ones."

From this description, it appears that Subject 14's group used a cases approach to the problem. Within each case, they created an exhaustive list of all possible pizzas. At the end they multiplied by 4 to account for each of the four toppings as well as did necessary subtraction since their lists had double counted some pizzas.

In solving the 5-Tall Problem, shown in Figure 7.4A, Subject 14 and his partner appear to have used a type of cases approach by grouping towers by the number of blues found within the tower. He then listed all ways that the blue could appear. For example, in his Groups of Two, Subject 14 showed that the blues could be stuck together, 1 block away from each other, or two blocks away from each other. He only listed one example of each of these subcategories and did not list out all the subcategories of each group. However, he did state how many towers are possible to create within each group. Before adding up the total towers, he also accounted for the case of "zero" blues, which he drew as an all red tower.

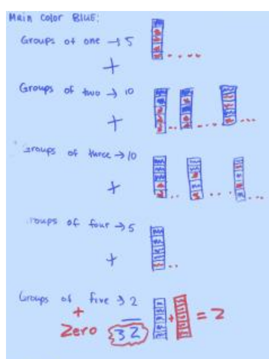

Figure 7.4A – Subject 14's Group Transparency on the Towers 5-Tall Problem

The idea of using cases also appeared in his solution to Ankur's Challenge, shown in Figure 7.4B. Subject 14's group first listed out all possible towers that start with a blue block, then all towers that start with a red block, and finally all towers that start with a yellow block. Under each category, his group created an exhaustive list of all possible towers that could be created under the conditions set forth by Ankur's Challenge. Subject 14's group also provided another heuristic argument for the solution using a tree diagram in which they only choose one color for the top. After creating all the possible branches for the tree, they went through each branch to see which branches applied to the conditions set forth by Ankur's Challenge and crossed out the ones that did not apply to the problem. After this elimination process, they counted that there are 12 possible towers that start with red. Finally, they multiplied by 3 as they realized that the tower could have started with red, yellow, or blue. Hence, they appeared to have used a control for variable strategy by choosing a single color to start with for the tree.

Figure 7.4B – Subject 14's Group Transparency on Ankur's Challenge

Following these in-class activities and during the process of working on Ankur's Challenge, Subject 14 had several opportunities to watch how others solved these problems by watching the presentations by his classmates, watching videos of young children, and watching videos of 11th grades students solving these same problems. For homework, Subject 14 was asked to comment on the forms of reasoning he saw either in class or in the videos. Subject 14 first responded to the Pizza with Halves Problem. After watching the video, Subject 14 noted that the students in the video used combinatorial methods involving factorials and Pascal's Triangle. In terms of the class presentations, Subject 14 noted that many of his classmates used algorithmic methods and "smart ways for organizing the groups."

Meanwhile, in response to the Romina's Proof video of Ankur's challenge, Subject 14 saw his solution as similar yet less direct compared to Romina's solution. He noted that she "used the idea of finding a single combination of each set of blocks and then multiplied it by three because of the different combinations."

Thus, on the post-assessment Subject 14 noticed and was convinced by more student heuristic arguments from the Student Work and Video than he was on the pre-assessment. Subject 14 recognized 10 out of 11 mathematical heuristic arguments used by the students. On the post-assessment Subject 14 used two argument types in his own problem solutions that he did not mention in describing the student solutions: exhaustive lists and tree diagrams.

7.4.2 Comparing Subject 14's Problem Solving Strategies to What He Notices

On the intervention's problem-solving tasks, Subject 14 mainly used the strategies of controlling for the variable, cases, exhaustive lists, numerical operations, and tree diagrams. For the most part, he did notice student problem-solving strategies that were similar to his own problem-solving strategies. However, he found more problem-solving strategies that were different from his own strategies such as using figures and patterns by the post-assessment. Thus, Subject 14 was able look beyond his own problem-solving strategies to find other strategies used by students. However, the one strategy Subject 14 used himself but did not identify noticing in the pre- and post- assessments of other student work was exhaustive lists and tree diagrams.

7.5 Pre-Service Teacher Subject 15

7.5.1 Subject 15's Participation in the Intervention

Based on the pre-assessment results, Subject 15 was able to recognize 9 out of 11 mathematical heuristic arguments used by the students. Prior to the intervention, Subject 15 was able to identify convincing heuristic arguments used by students that include cases, patterns, figures, patterns, exhaustive lists, numerical operations, simpler problems, doubling strategy, and controlling for the variable. He was also able to recognize arguments that used guess and check even though he did not find these particular arguments as convincing. Most of the explanations he provided for why he thought a heuristic argument was convincing were based on the explanation and the mathematics used.

During the intervention, Subject 15 had opportunities to solve problems and watch videos of children solving the same problems he worked on in class. The first problem Subject 15 worked on was the Pizza with Halves Problem. To solve this problem, Subject 15 used a cases approach by grouping the pizza based on the number of toppings on just the first half of the pizza pie. From the transparency he made, shown in Figure 7.5A, Subject 15 started creating exhaustive lists for the case of 1 topping on the first half. He also broke down this list into a set of sub-categories based on the number of toppings on the second half of the pizza. However, by the second half, Subject 15 just stated how many pizzas there were within each subgroup instead of listing them all. At the end of the problem, Subject 15 accounted for the doubles he created when generating his lists. Thus, he subtracted out the pizzas that were counted twice.

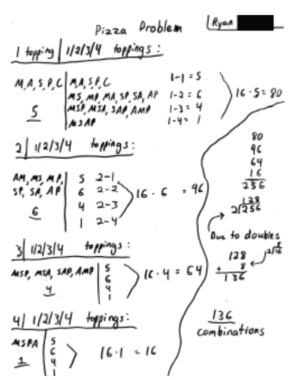

Figure 7.5A – Subject 15's Group Transparency on the Pizza with Halves Problem

Next, Subject 15 worked on the Towers 5-Tall Problem. In solving the 5-Tall Problem, shown in Figure 7.5B, Subject 15 appeared to use a type of doubling approach, building up the towers from one tall to five tall by adding a cube to the bottom of each of the previous tall towers of cubes. He showed a diagram for the cases of 1-tall, 2-tall, and 3-tall. However, while doing this doubling approach, Subject 15's group also appeared to generalize a formula to figure out the towers which are 4-tall and 5-tall instead of drawing them out.

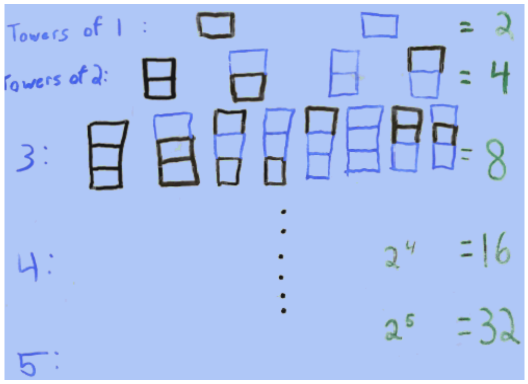

Figure 7.5B – Subject 15's Group Transparency on the Towers 5-Tall Problem

Meanwhile, Subject 15 and his partner used a different approach to solve Ankur's Challenge whose solution is shown in Figure 7.5C. This time, Subject 15 and his group started off with a tree diagram. However, they also used their knowledge from the last problem on 5-tall with 2 colors to come up with a formula for 4-tall selecting from 3 colors. Instead of drawing out the tree completely, Subject 15's group started using the general formula they found in previous problems to count the total towers 4-tall. Since this list could include towers that contain only one or two colors, Subject 15's group subtracted out these cases from their total. Again, they used combinatorial generalized notation to count how many towers fit in categories of one or two color towers which are 4-tall.

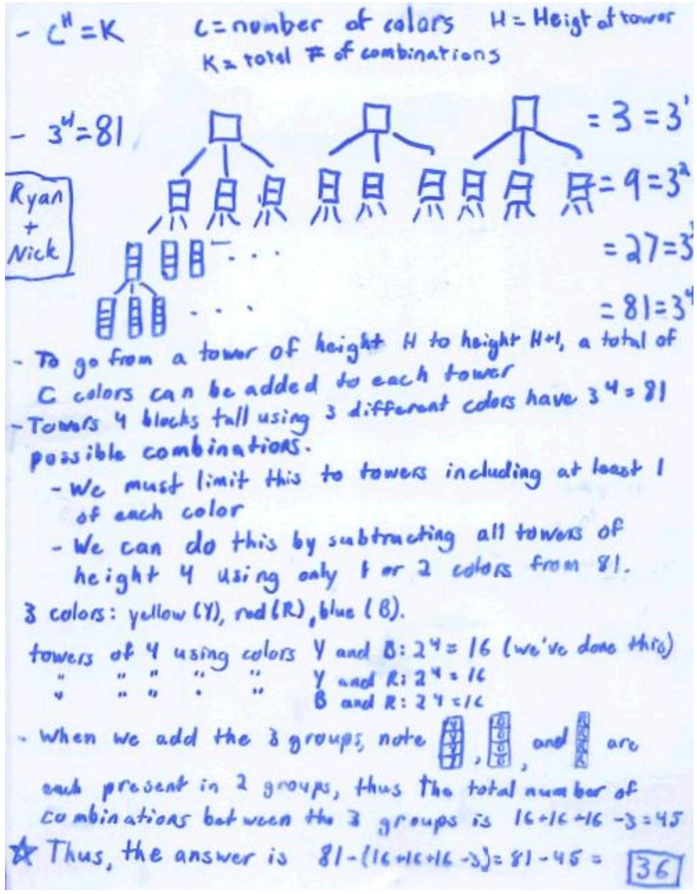

Figure 7.5C – Subject 15's Group Transparency on Ankur's Challenge

Following these in-class activities and during the process of working on Ankur's Challenge, Subject 15 had several opportunities to watch how others solved these problems by watching the presentations by his classmates, watching videos of young children solving these problems, and watching 11th grade students solving these same problems. For homework, Subject 15 was asked to comment on the forms of reasoning he saw either in class or in the videos. Subject 15 first responded to the Pizza with Halves Problem. After watching the videos Subject 15 was able to fully describe the reasoning the students in the video used:

> "They initially became frustrated trying to find an efficient way to solve the problem and suggested just listing the possibilities, comparing their answers, and trying to find an identifiable pattern after knowing the answer. After breaking the possibilities into ordered categories, they see that the combinations relate to Pascal's triangle."

Subject 15 also commented on the methods his classmates used to solve the problem. He noted that his classmates also used similar groupings for the pizzas as the students did initially in the video. He also commented on his classmates' methods to solving the problem.

> "While some students began a guess and check method….[t]he technique of holding one half of a pizza constant while varying the second half was very clever and algebraic. It proved to be very effective. I was impressed by the student in our class who solved the problem this way, creating a summation formula."

A final connection he found between his class discussions and the videos was that both groups were able to draw parallels between the towers and pizza problems.

Meanwhile, in response to the Romina's Proof video of Ankur's challenge, Subject 15 considered Romina's solution a case-based solution. Subject 15 provided a detailed description of what Romina did to solve the problem. Subject 15 wrote:

> "Romina organized the towers as though there were only 2 different kinds of blocks: 1's and not 1's. From this, she found a total of 6 blocks. Every non-1 block was an OX block.

He also noted that Romina controlled where the double color blocks were placed (the 1- blocks) and then was able to figure out all possible towers after the 1-blocks were placed. He wrote how Romina found all possible towers for each controlled situation:

> "As there were two OX blocks per tower and only one X and one O could be present in the tower, the only available choices were to let the first non-1 block be an X and the second non-1 block be an O, and vice versa. As this is two choices per block, each of the 6 combinations can be expressed two ways, thus 6x2=12."

From here, he noted that the 1-blocks could have any of the three colors available. Subject 15 believed Romina's method was very convincing. He also compared Romina's method to those of his classmates. He identified two methods his classmates used: working backwards from 81 possible towers 4-tall and using a pattern to move the 2 blocks of the same color to different locations within a tower. He stated that Romina's strategy was a more efficient method to solve the problem.

Subject 15's final homework assignment asked him to comment on the heuristic arguments he found most convincing and least convincing from the intervention. Overall, there were two types of heuristic arguments that were most convincing to Subject 15: Romina's Proof and doubling heuristic arguments used in the 5-Tall Tower Problem. His reason for finding these convincing was that he felt that the heuristic arguments were easy to visualize. Meanwhile, he stated that heuristic arguments using exhaustive lists and guess and check were not convincing because of their lack of organization. He was not convinced that these types of heuristic arguments prove that all possible towers/pizzas were found. In order to make these unconvincing heuristic arguments more convincing, Subject 15 suggests that the heuristic arguments should include either tree diagrams or organizational grouping.

By the post-assessment Subject 15 recognized 10/11 mathematical heuristic arguments used by the students. He noticed and was convinced by more student heuristic arguments from the Student Work and Video than he was on the pre-assessment including reasoning from a similar problem (such as the isomorphism that can be made between the Towers and Pizza with Halves Problems). In his own problem solving solution Subject 15 used the tree diagram heuristic strategy that he did not describe students using.

7.5.2 Comparing Subject 15's Problem-Solving Strategies to What He Notices

In solutions to the intervention's problem-solving tasks Subject 15 mainly used the strategies of cases, exhaustive lists, numerical operations, doubling strategy, similar problems, and tree diagrams. For the most part, he noticed student problem-solving strategies that were similar to his own problem-solving strategies. However, he found problem-solving strategies that were different from his own strategies such as using patterns and simpler problems by the post-assessment. In addition, Subject 15 used the tree diagram strategy that he did not describe students using. Thus, Subject 15 was able look beyond his own problem-solving strategies to find other strategies used by students.

7.6 Case Studies Summary

Table 7.1 below shows the types of heuristic arguments the pre-service teachers noticed throughout their participation in this research study. There are three possibilities regarding teachers noticing heuristic argument types used by students and teachers using argument heuristic strategies in their own problem solutions: (1) the teacher notices and uses, (2) teacher notices but does not use, and (3) the teacher does not notice use by the student but uses the strategy in his (her) problem solution.

The X1 entry in Table 7.1 means the teacher mentioned that the heuristic was used in a student solution and the teacher used this strategy in his (her) own problem solutions, X2 means the teacher noticed the strategy in a student solution but did not use it, and X3 means the teacher did not notice the heuristic as used in a student solution but the teacher did use the heuristic in his (her) own problem solutions.

The last column of Table 7.1 is the mode of the 5 teacher cell classifications for each of the listed heuristic argument types. You will note that two argument types have two modes. From Table 7.1 we note that 7 of the 11 or 63.6% of the argument types have mode X1, the teacher observed the strategy as used by the students and used the strategy in their own problem solutions; 5 of the 11 or 45.5% have a mode of X2, the teacher observed the strategy in the solution of other students but did not use the strategy in their own problem solutions; and 1 of the 11 or 9.1% has a mode of X3, the teacher used the problem solving strategy but did not report this strategy as used in the student solutions. In summary, we find the general trend is that the teacher problem solution strategies parallel those of the students.

Table 7. 1 – Heuristic Arguments Noticed by Study Participants*

	Subject 03	Subject 07	Subject 12	Subject 14	Subject 15	Modal Response
Cases		X1	X1	X1	X1	X1
Controlling for Variable	X1	X3	X1	X1	X1	X1
Figures	X2	X2	X2	X2	X1	X2
Tree Diagrams	X1	X2	X1	X3	X3	X1 & X3
Patterns	X2	X1	X2	X2	X2	X2
Exhaustive	X1	X1	X3	X3	X1	X1
Guess and Check	X2	X2	X2	X2	X2	X2
Doubling Strategy		X1	X3	X2	X1	X1
Simpler Problem	X2	X2			X2	X2
Similar Problem	X1		X2	X2	X1	X1 & X2
Numerical Operations	X2	X1	X1	X1	X1	X1

* X1 in a cell means the teacher mentioned the heuristic was used in a student solution and used the strategy in his (her) problem solutions, X2 means the teacher noticed the strategy but did not use it, and X3 means the teacher did not notice the heuristic as used in a student solution but the teacher did use the heuristic in his (her) own solution.

CHAPTER 8: FINDINGS/CONCLUSIONS

8.1 Introduction

This study took place at a large public university in a class composed of pre-service mathematics teachers in during an academic term. It took place in a course which is a required course for college math majors in their junior year who intended to become teachers. The intervention that took place over a 2.5 week period in which each week contained two 80-minute class sessions. Class sessions included problem-solving activities, watching videos of children solving problems similar to the ones the teachers were solving in class, and classroom discussions. Furthermore, class sessions were supplemented with homework assignments such as watching videos and solving problems. It should be noted that the intervention itself was not performed by the instructor of the course but by outside researchers. The purpose of the intervention was to see if such a short intervention would influence the pre-service teachers to notice children heuristic arguments as well as influence the beliefs these pre-service teachers had prior to the intervention about students, mathematics, and teaching.

8.2 Findings

8.2.1 Pre-Service Teachers' Noticing on Student Work Assessment

Sixteen pre-service teachers were given the Student Work Assessment before and after the intervention. The assessment posed problems and student written solutions to those problems. The pre-service teachers were asked to state how convinced they were of a particular student's solution and to provide explanations of why they were or were not convinced of the solution. In these explanations, the pre-service teachers indicated the types of heuristic arguments they identified in the student solutions to problems.

There were a total of 11 main heuristic arguments that were analyzed to determine if the pre-service teachers noticed them in the student work assessment: cases, controlling for the variable, figures, tree diagrams, patterns, exhaustive, guess and check, doubling strategy, simpler problems, similar problems, and numerical operations heuristic arguments. The data were analyzed to determine the number of heuristic arguments the pre-service teacher found on the pre-assessment and the post-assessment. On the pre-assessment, there was an average of 5.69 heuristic arguments found and on the post-assessment there was an average of 7.38 heuristic arguments found. The post-assessment increase of 1.69 heuristic arguments higher than the pre-assessment was determined by the Student paired t-test to be statistically significance at the 0.0001 level of significance.

8.2.2 Pre-Service Teachers' Noticing on Video Assessment

Fourteen pre-service teachers participated in the pre- and post- Video Assessment. The assessment required the pre-service teachers to watch students discuss their solutions for the 3-Tall Towers Problem. The pre-service teachers also had access to the transcript of the video when they commented on the student heuristic arguments that they identified from the video clip.

There were a total of 6 main heuristic argument types that were analyzed to determine whether they were noticed by the pre-service teachers from the video assessment: cases, controlling for the variable, figures, patterns, doubling strategy, and numerical operations heuristic argument. An analysis was done to determine the types and frequency of heuristic arguments the pre-service teachers found on the pre- and post-assessments. On the pre-assessment, there was an average of 3.43 heuristic arguments found and on the post-assessment there was an average of 4.29 heuristic arguments found. The post-assessment increase of 0.86 heuristic arguments higher than the pre-assessment was determined by the Student paired t-test to be statistically significance at the 0.01 level of significance.

8.2.3 Types of Heuristic Arguments Noticed Across Assessments

For the student work assessment, the most noticed heuristic arguments on the pre-assessment were figures and patterns whereas numerical operations was the second most noticed heuristic argument type, as shown in Table 8.1 below. The same was true for the post-assessment. The least noticed heuristic argument on the pre-assessment was the doubling strategy and the same was true on the post-assessment. However, the use of a cases heuristic argument exhibited the most growth on this assessment with 6 more pre-service teachers noticing this type of heuristic argument after the intervention. Meanwhile, tree diagrams exhibited the second most growth after the intervention. In summary, 2/3 or 66.7% of the teachers who failed to refer to a cases argument on the pre-assessment did so on the post-assessment and 55.6% of the teachers who failed to refer to a tree diagram argument on the pre-assessment did so on the post-assessment.

Table 8. 1 – Student Work Assessment Noticed Heuristic Arguments and Growth

Heuristic Argument Type	**Percent Included on Pre-Assessment**	**Percent Included on Post-Assessment**	**Percent Exhibiting Growth**
Cases	43.75%	81.25%	37.5%
Control for Variable	37.5%	62.5%	25%
Figures	87.5%	93.75%	6.25%
Tree Diagrams	43.75%	75%	31.25%
Patterns	93.75%	100%	6.25%
Exhaustive	37.5%	56.25%	37.5%
Guess and Check	81.25%	87.5%	6.25%
Doubling Strategy	6.25%	18.75%	12.5%
Simpler Problem	50%	50%	0%
Similar Problem	37.5%	50%	12.5%
Numerical Operation	81.25%	87.5%	6.25%

For the video assessment, there were only 6 heuristic argument types found in the assessment. The most noticed heuristic argument on the pre-assessment was the use of figures whereas patterns and cases were the second most noticed heuristic argument types as shown in Table 8.2 below. Meanwhile, on the post-assessment figures was the most noticed heuristic argument type with patterns being the second most noticed heuristic argument type. However, the use of figures exhibited the most growth on this assessment with 3 more people noticing these types of heuristic argument after the intervention. Meanwhile, controlling for the variable and patterns showed the second most growth after the intervention. In summary, 75.0% of the of the teachers who failed to refer to a figures argument on the pre-assessment did so on the post-assessment, 40.0% of the teachers who failed to refer to a patterns argument on the pre-assessment did so on the post-assessment, and 18.2% of the teachers who failed to refer to controlling for variables argument on the pre-assessment did so on the post-assessment.

Table 8. 2 – Video Assessment Noticed Heuristic Arguments and Growth

Heuristic Argument Type	**Percent Included on Pre-Assessment**	**Percent Included on Post-Assessment**	**Percent Exhibiting Growth**
Cases	64.29%	64.29%	0%
Control for the Variables	21.43%	35.71%	14.28%
Figures	71.43%	92.86%	21.43%
Patterns	64.29%	78.57%	14.28%
Doubling Strategy	50%	57.14%	7.14%
Numerical Operations	50%	57.14%	7.14%

The combination of the student work assessment and video assessment overall post-assessment results can be found in Table 8.3. The first two columns state the number of pre-service teachers who noticed a particular heuristic argument at least once on each assessment. The last column indicates the mean percentage of the time a particular heuristic argument was noticed across the two assessments.

Overall, the most noticed heuristic argument by the end of the intervention was using figures with 93% of the pre-service teachers noticing. The second most noticed heuristic argument was patterns whereas tree diagrams was the third most noticed. Meanwhile, the least noticed heuristic argument was the doubling strategy with 37% of the pre-service teachers noticing.

Table 8. 3 – Overall Noticed Heuristic Arguments and Growth

Heuristic argument Type	**Noticed on Student Work Post-Assessment**	**Noticed on Video Post-Assessment**
Cases	13/16	9/14
Control for the Variables	10/16	5/14
Figures	15/16	13/14

Tree Diagrams	12/16	NA
Patterns	16/16	11/14
Exhaustive	9/16	NA
Guess and Check	14/16	NA
Doubling Strategy	3/16	8/14
Simpler	8/16	NA
Similar	8/16	NA
Numerical operations	14/16	8/14

8.2.4 Pre-Service Teachers' Beliefs

Seventeen pre-service teachers participated in the pre- and post- Beliefs Assessment. The assessment gave the pre-service teachers the chance to read a series of beliefs statements and respond to each statement as: Strongly Agree, Agree, Undecided, Disagree, and Strongly Disagree. The 16 beliefs questions that were examined in this study were categorized into 3 categories: Beliefs about Student Mathematics Learning, Beliefs about Mathematics, and Beliefs about Teaching Mathematics.

In examination of the all the beliefs statements, the McNemar test indicates that the only belief statement that showed a statistically significant belief transition to current reform standards was Q1: "Learners generally understand more mathematics than their teachers or parents expect" with a p-score of 0.019. The remaining 15 beliefs statements show consistency in transitions to current reform standards as shown in Table 8.4. Due to the small sample size the overall transition rates reported in Table 8.4 cannot be declared statistically significant.

Table 8. 4 – Overall Percent Transition to Beliefs Consistent with Reform Standards

Belief Category	**Student Math Learning**	**Mathematics**	**Teaching Mathematics**
Percent Transition to Beliefs Consistent with Reform Standards	55.9%	53.6%	64.2%

8.2.5 Relationship Between Teacher and Student Problem Solving Strategies

Based upon the analysis of Chapter 7 we find that at least 3 of the 5 teachers utilized the following 4 problem solving strategies that they reported students using: cases, controlling for variables, exhaustive list, and numerical operations. At least 3 of the 5 teachers described the following 4 problem solving strategies that students used, but the teacher did not use these strategies in their own problem solving solutions: figures, patterns, guess and check, and simpler problem. Four of the 5 teachers did use one or more problem-solving strategies that they did not report students using. These strategies include controlling for variables (one teacher), doubling strategy (one teacher), tree diagram (2 teachers), and exhaustive list (2 teachers). While 4 of 5 teachers utilized one or more strategies that they did not describe students using, on the average the teachers noted or utilized 90.9% of the strategies that were utilized by the students.

8.3 Significance and Limitations of the Study

This study was a short intervention and is limited to one small class. The findings from this study verify that a pre-service teacher development intervention that involves: (1) teachers solving a variety of rich problems similar to ones that are given to students, (2) teachers analysis of written student problem solutions, (3) teacher analysis of student solutions from video, and (4) analysis of student solutions by the teacher following a planned classroom problem solving activity can help improve pre-service teachers' ability to better attend to student reasoning. Further, the conclusions here imply that growth is possible in a short intervention but does not imply that this growth will occur in other studies. Growth is seen to occur in both the pre-service teachers' attention to student reasoning and the pre-service teachers' beliefs.

The study findings suggest the merit and direction of further studies. Studies similar to this study might want to examine the time factor and assess the effects of the intervention over different lengths of time. It may be important to study whether longer lengths of the intervention will lead to more growth. Another factor that future studies might examine is doing the same interventions among different classes of varying math background and ages, and among varying teacher preparation institutions to ascertain if the results of this study are replicable across different classrooms and different levels of pre-service teachers. Finally, it may be of interest to increase the number and/or variety of problems explored by the teachers to ascertain if the types and variety of problems influence the amount of growth.

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APPENDIX A: STATEMENT OF PROBLEMS SOLVED DURING IN CLASS SESSIONS

Pizza with Halves Problem

A local pizza shop has asked us to help them keep track of certain pizza sales. Their standard "plain" pizza contains cheese. On this cheese pizza, one or two toppings can be added to either half of the plain pie or the whole pie. How many possible choices for pizza do customers have if they can select from two different toppings (sausage and pepperoni) that could be placed on either the whole cheese pizza or half a cheese pizza? List all the possible different selections. Find a way to convince each other that you have accounted for all possibilities.

Towers 5-Tall Problem

Your group has two colors of Unifix Cubes. Work together and make as many different towers five cubes tall as is possible when selecting from two colors. See if you and your partner can plan a good way to find all the towers five cubes tall.

Ankur's Challenge

Find all possible towers that are 4 cubes tall, selecting from cubes available in *three* different colors, so that the resulting towers contain at least one of each color. Convince us that you have found them all.

APPENDIX B: STUDENT WORK ASSESSMENT

EVALUTING STUDENTS' WORK IN RESPONSE TO PROBLEM-SOLVING TASKS

Instructions:

Students in various grade levels have engaged in certain problem-solving tasks, and then have recorded their solutions as written work. You are being asked to evaluate each student's response in a two-step process. First you will select a numerical value that corresponds to how convincing you find each response. Then, in short-answer format, you will give support for the numerical choice you make. The numerical scale is as follows:

-2 = Not at all convincing
-1 = Not very convincing
0 = Undecided
1 = Somewhat convincing
2 = Convincing

The examples of student work are organized according to the problem-solving task they worked on, and you will be presented with the statement of the task that the students themselves were given. Following the statement of task are images of student work on that task.

For each example of student work:

(1) check a box to select a numerical value for how convincing you find it, and
(2) give support for your selection by writing in the space below the check boxes

This assessment is being administered as pretest and posttest to get baseline and end-of-course measures from everyone taking this course. Those measurements serve both as indicators of what you learn in the course and how effective the course activities are in helping students attain desired learning objectives. Your thoughtful responses are much appreciated.

Go to the next page to begin this assessment.

ITEMS 1 – 8: STUDENTS' WORK ON THE TWO-COLOR TOWERS PROBLEM

Statement of task for the two-color towers problem:

You have two different colors of unifix cubes available to build towers. Your task is to make as many different looking towers as possible, each exactly four cubes high. Find a way to convince yourself and others that you have found all possible towers four cubes high, and that you have no duplicates.

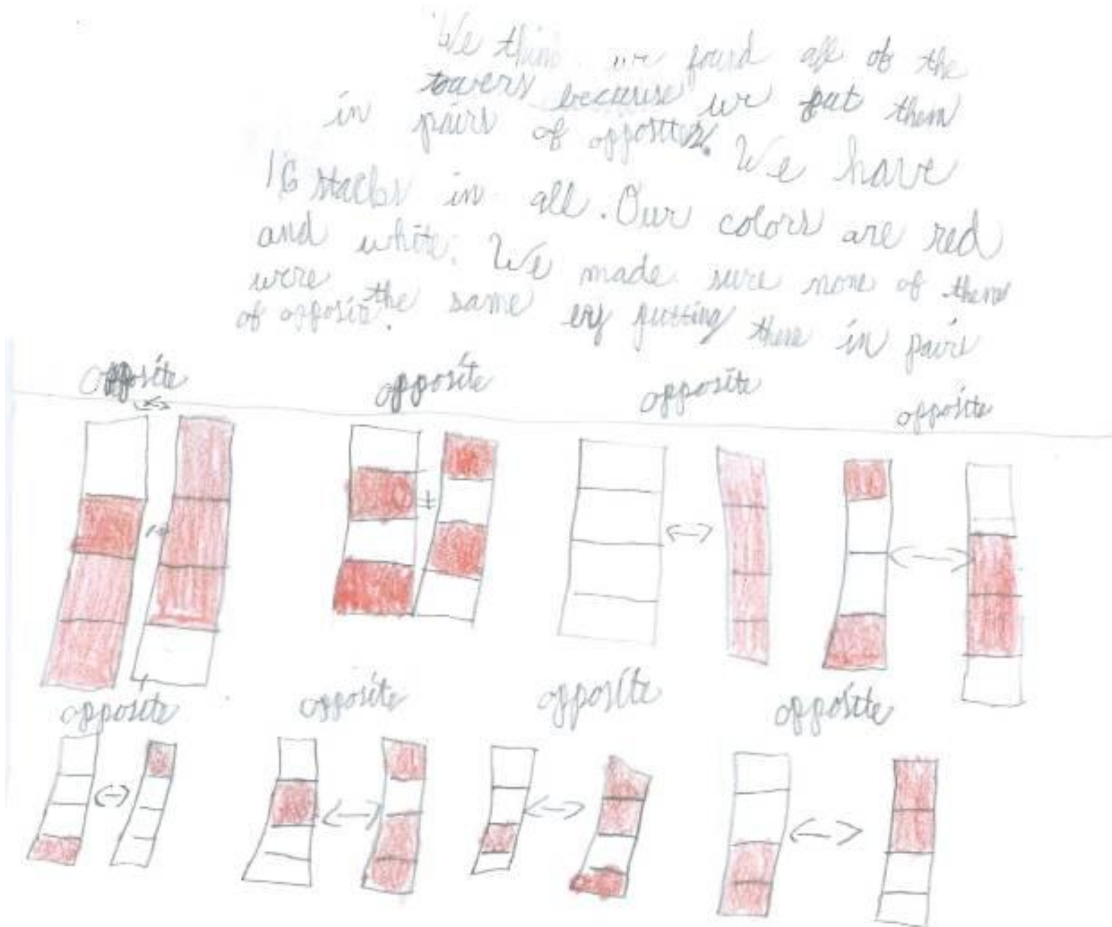
(1) Student work by Tony, a 3rd grader, in response to the two-color towers problem

☐ -2 ☐ -1 ☐ 0 ☐ 1 ☐ 2

Choose a numerical value on scale of -2 (not at all convincing) to 2 (convincing). Give support for your choice in the space below.

(2)

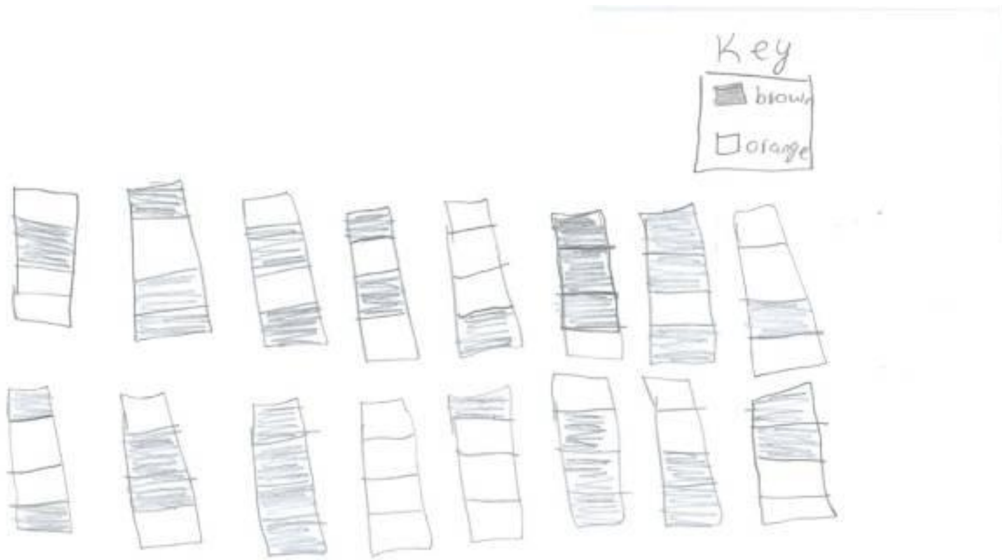
Student work by Kelly, a 3rd grader, in response to the two-color towers problem



☐ -2 ☐ -1 ☐ 0 ☐ 1 ☐ 2

Choose a numerical value on scale of -2 (not at all convincing) to 2 (convincing). Give support for your choice in the space below.

(3) Student work by Jaime, a 7th grader, in response to the two-color towers problem

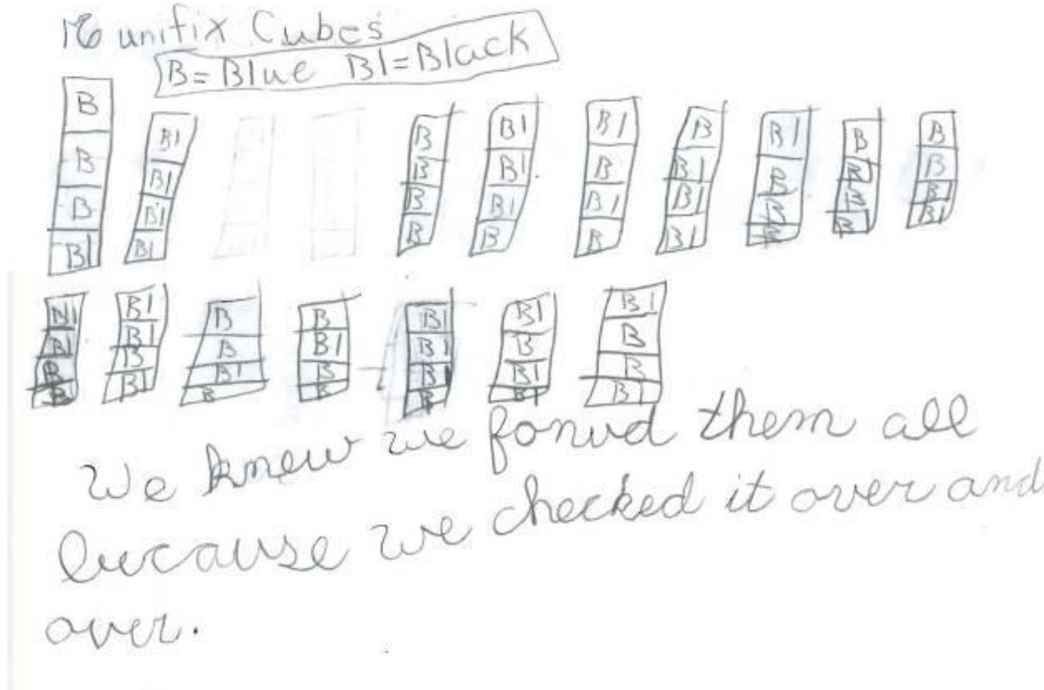


We think that 16 is all the possibilities for the blocks because we had 2 colors, and 2x2 = 4 and 4x4 = 16, and 4 is the number of blocks we needed to have in each tower.

☐ -2 ☐ -1 ☐ 0 ☐ 1 ☐ 2

Choose a numerical value on scale of -2 (not at all convincing) to 2 (convincing). Give support for your choice in the space below.

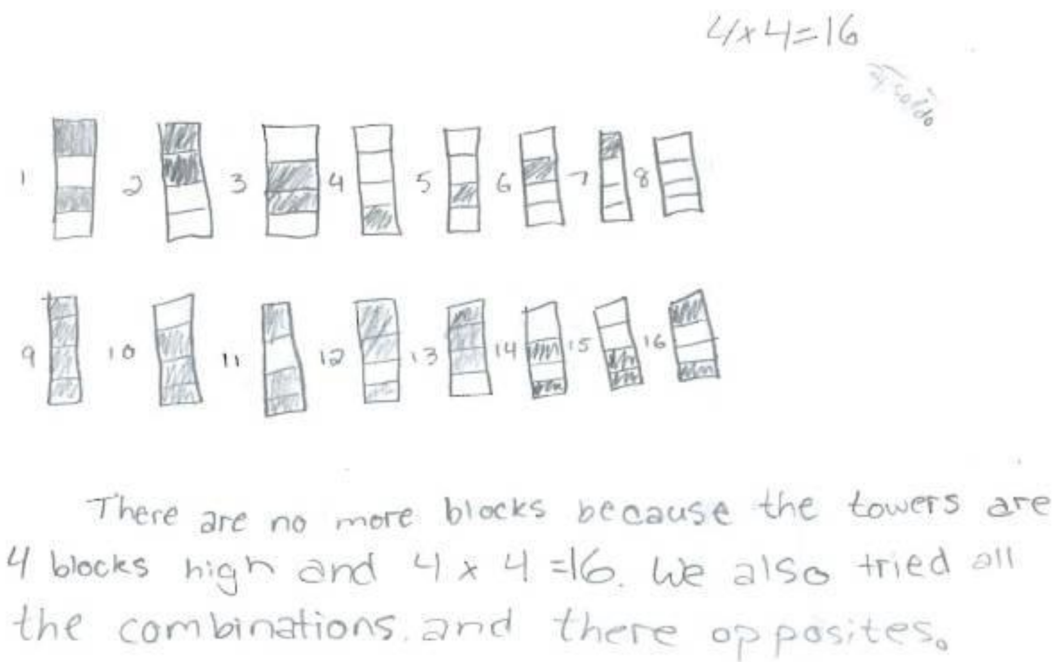
(4) Student work by Laura, a 4th grader, in response to the two-color towers problem



☐ -2 ☐ -1 ☐ 0 ☐ 1 ☐ 2

Choose a numerical value on scale of -2 (not at all convincing) to 2 (convincing). Give support for your choice in the space.

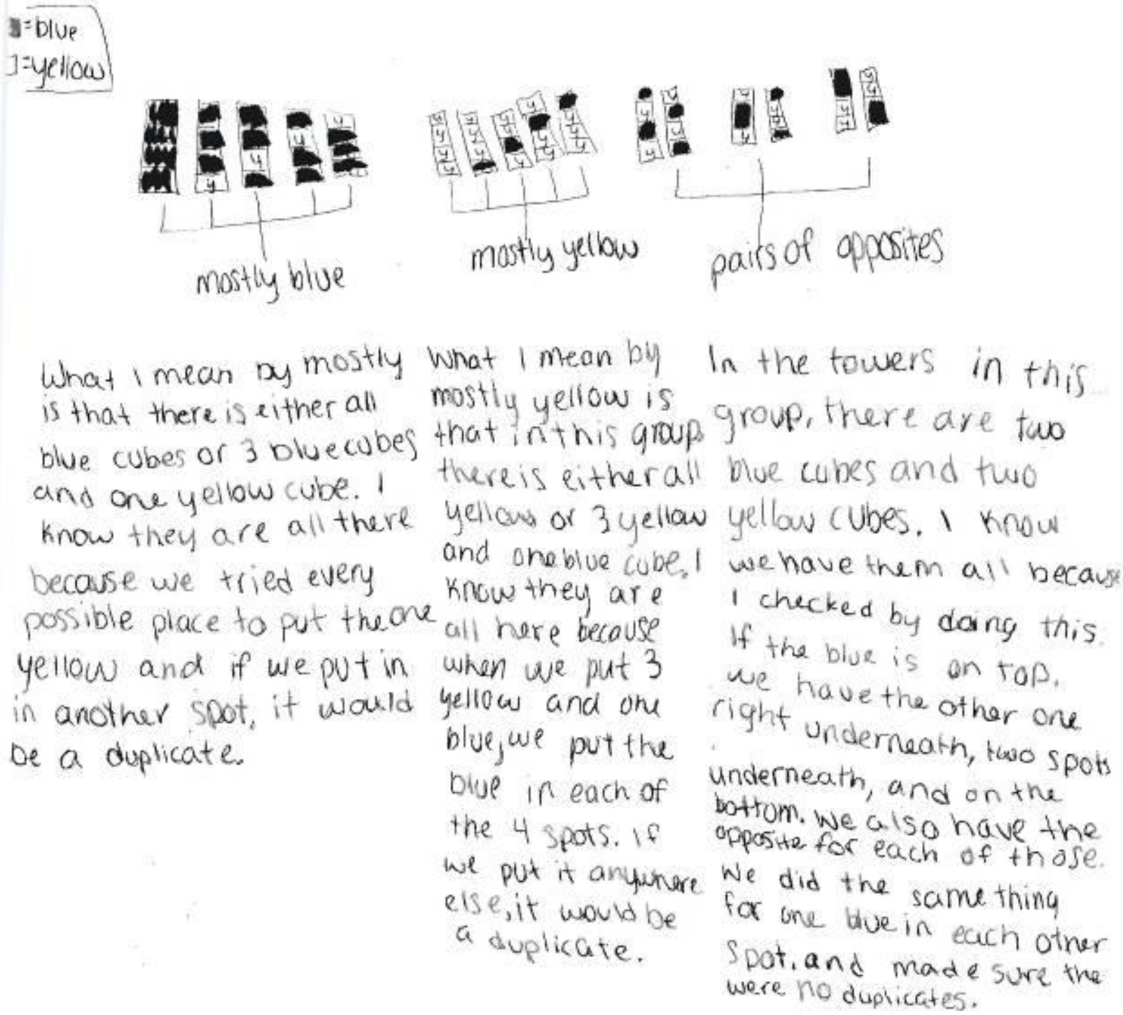
(5) Student work by Danny, a 7th grader, in response to the two-color towers problem



☐ -2 ☐ -1 ☐ 0 ☐ 1 ☐ 2

Choose a numerical value on scale of -2 (not at all convincing) to 2 (convincing). Give support for your choice in the space.

(6) Student work by Cathy, a 6th grader, on the two-color towers problem



What I mean by mostly is that there is either all blue cubes or 3 blue cubes and one yellow cube. I know they are all there because we tried every possible place to put the one yellow and if we put in in another spot, it would be a duplicate.

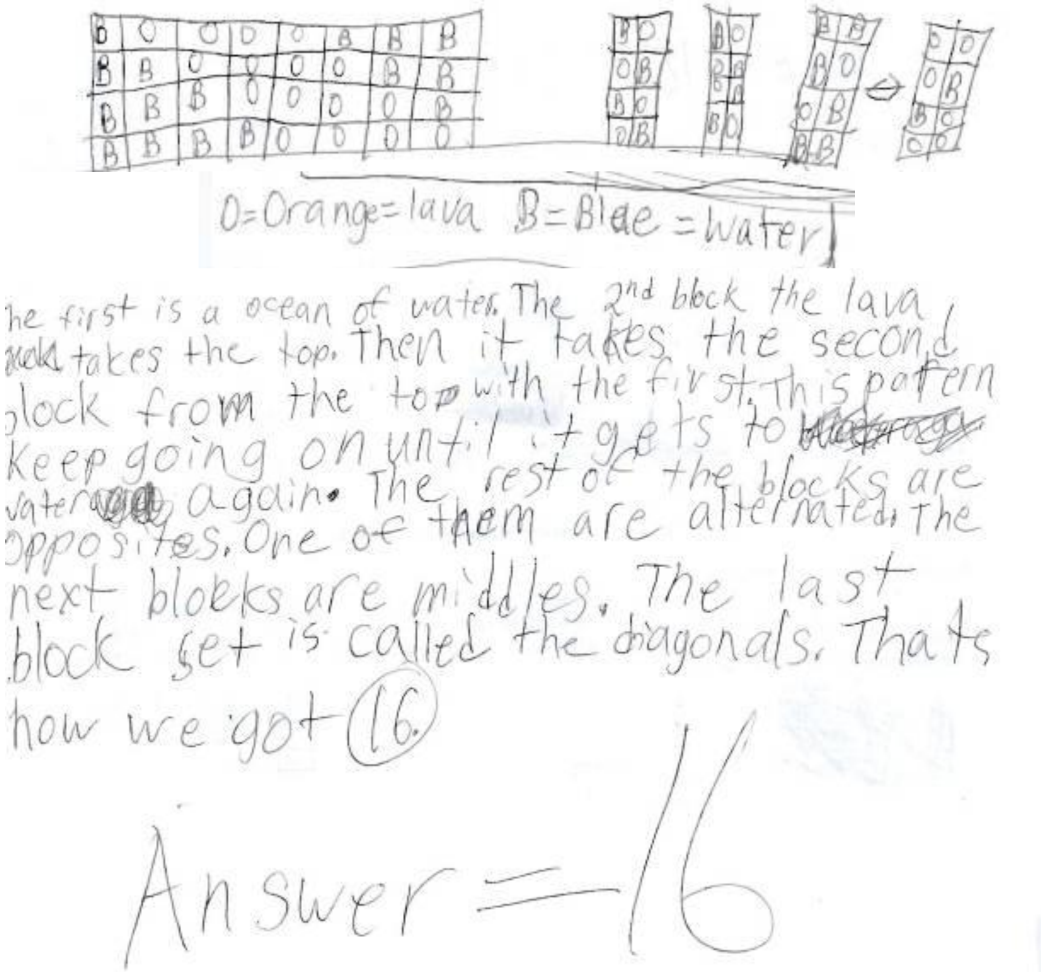
What I mean by mostly yellow is that in this group there is either all yellows or 3 yellow and one blue cube. I know they are all here because when we put 3 yellow and one blue, we put the blue in each of the 4 spots. If we put it anywhere else, it would be a duplicate.

In the towers in this group, there are two blue cubes and two yellow cubes. I know we have them all because I checked by doing this. If the blue is on top, we have the other one right underneath, two spots underneath, and on the bottom. We also have the opposite for each of those. We did the same thing for one blue in each other spot, and made sure the were no duplicates.

☐ -2 ☐ -1 ☐ 0 ☐ 1 ☐ 2

Choose a numerical value on scale of -2 (not at all convincing) to 2 (convincing). Give support for your choice in the space below.

(7) Student work by Prashant, a 4th grader, on the two-color towers problem



O=Orange=lava B=Blue=water

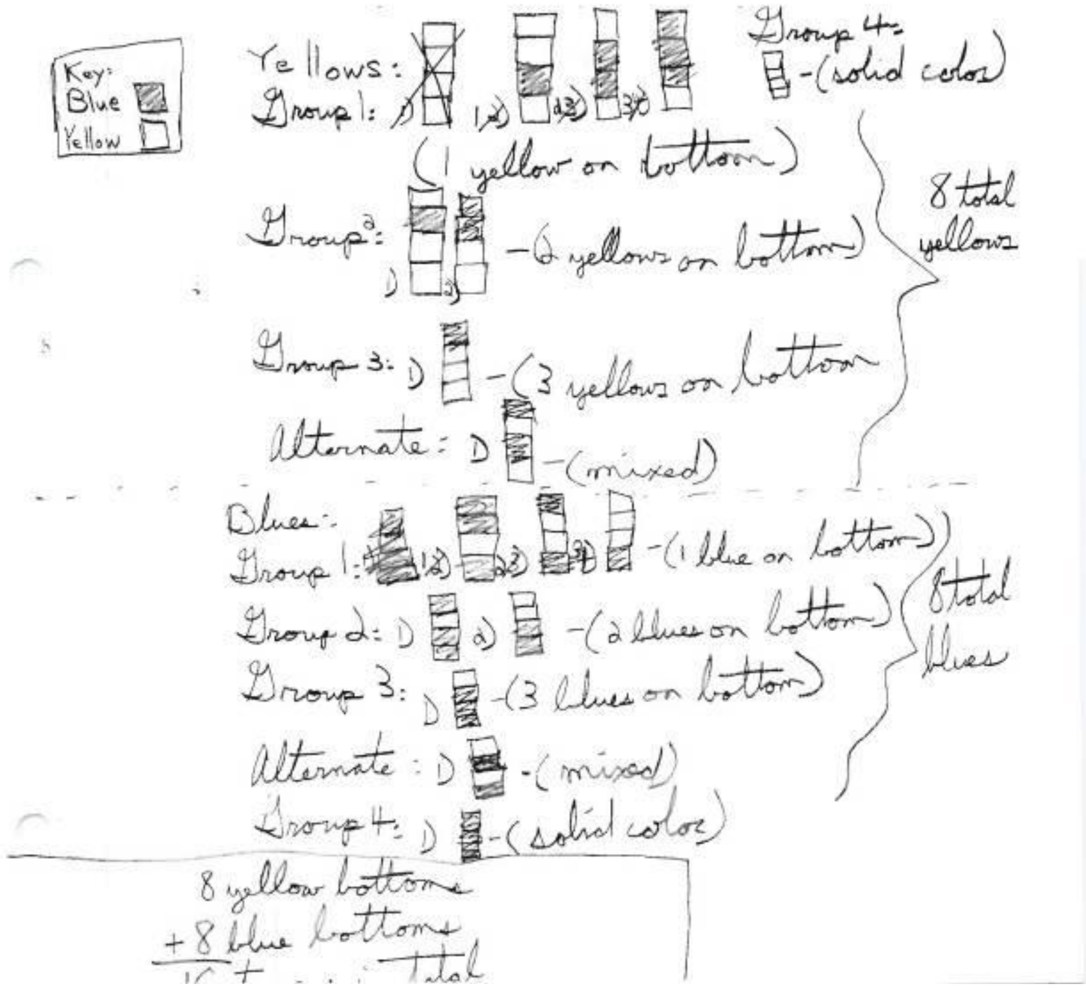
he first is a ocean of water. The 2nd block the lava ~~...~~ takes the top. Then it takes the second block from the top with the first. This patern keep going on until it gets to ~~...~~ water ~~...~~ again. The rest of the blocks are opposites. One of them are alternated. The next blokks are middles. The last block set is called the diagonals. Thats how we got 16.

Answer = 16

☐ -2 ☐ -1 ☐ 0 ☐ 1 ☐ 2

Choose a numerical value on scale of -2 (not at all convincing) to 2 (convincing). Give support for your choice in the space below.

(8) Student work by Brian, a 5th grader, on the two-color towers problem
(Note: his work starts below and continues on the next page)



(Note: cut off portion reads "16 towers total")

Explaintion: There are 16 tower combinations in total. We know there are no duplicates towers, because we organized it so that they were catagorized in groups ~~of~~ based on how many of one color were on the bottom. or other combinations
~~T~~ The only possible way there could be another combination is if we could work 5 unifix cubes instead of 4. Since we can only work with 4 cubes, there can only be a certain number of one color on the bottom.

☐ -2 ☐ -1 ☐ 0 ☐ 1 ☐ 2

Choose a numerical value on scale of -2 (not at all convincing) to 2 (convincing). Give support for your choice in the space below.

ITEMS 9 – 10: STUDENTS' WORK ON THE TWO-COLOR TOWERS EXTENSION PROBLEM

Statement of task for the two-color towers extension problem:

Make a prediction about a solution for finding all possible towers three cubes high when selecting from two colors of unifix cubes (without building them). For instance, do you think there will be more, fewer, or the same number of possible towers as what you found for towers that are four cubes high? Make a prediction about a solution for finding all possible towers five cubes high (without building them). Again, do you think there will be more, fewer, or the same number of possible towers as what you found for towers that are four cubes high? If time permits, you may build the towers to check the accuracy of your predictions.

(9) Student work by Ricky, a 4th grader, on the two-color towers extension problem

We predict that for 5 story towers, there would be 24 combination. We know this because 5x5=25 and 25-1=24 and that is how we think you do it for odd numbers.

Guess - 24 ways

☐ -2 ☐ -1 ☐ 0 ☐ 1 ☐ 2

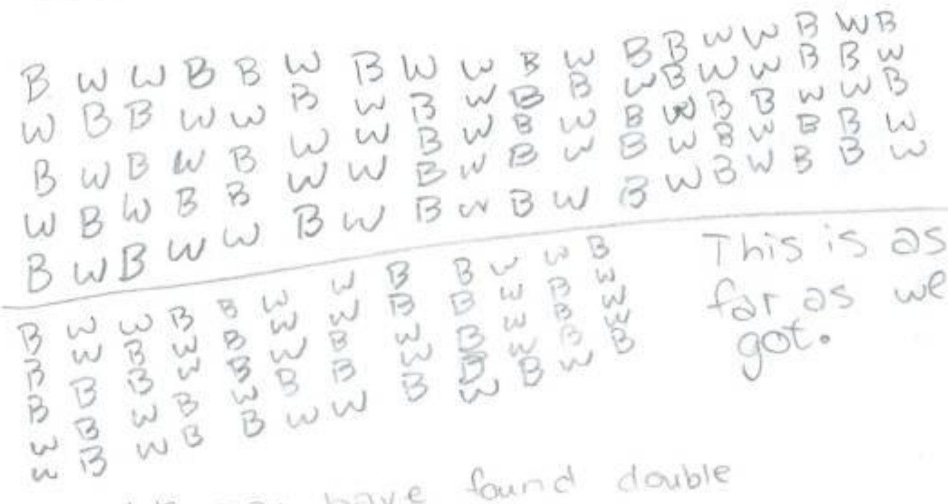
Choose a numerical value on scale of -2 (not at all convincing) to 2 (convincing). Give support for your choice in the space below.

(10) Student work by Danny, a 7th grader, on the two-color towers extension problem

We think there will be 9 towers when they are only 3 high because 3x3=9.

We think we got all of them because we came up with one and found its opposite.

24 or 25 because 5x5=25 and last time it was one less then its square. It might be 24 because when we did 3 blocks then it was 1 less than if you multiplied 3x3=9. but 3 is odd so since 5 is odd I thought it would be 24!



This is as far as we got.

We may have found double

☐ -2 ☐ -1 ☐ 0 ☐ 1 ☐ 2

Choose a numerical value on scale of -2 (not at all convincing) to 2 (convincing). Give support for your choice in the space below.

ITEMS 11 – 12: STUDENTS' WORK ON THE THREE-COLOR TOWERS EXTENSION PROBLEM

Statement of task for the three-color towers extension problem:
Find all possible towers four cubes tall, selecting from cubes available in three different colors so that the resulting towers have at least one of each color. Show your solution and provide a convincing argument that you have found them all.

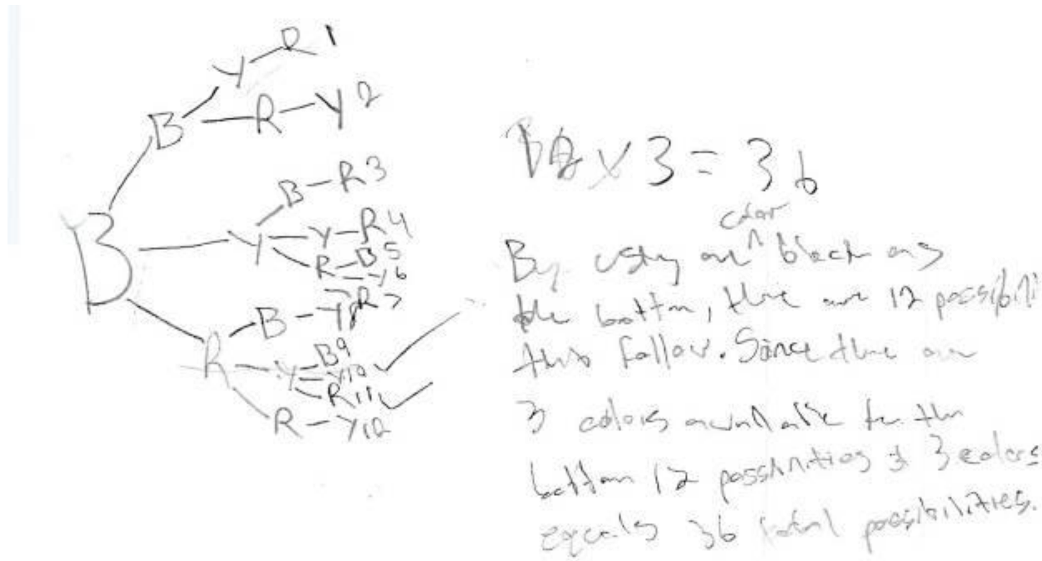
(11) Student work by Cecilia, a 6th grader, on the three-color towers extension problem

We started by finding the towers with double yellows as the 1st and 2nd cubes from the bottom. Then 2nd and 3rd, then 3rd and 4th. Next, we found The towers with the yellow as the 1st + 4th, 1st + 3rd, and 2nd and 3rd. After we found that (12 towers alltogether), we multiplied that by the three colors to get 36 total towers.

☐ -2 ☐ -1 ☐ 0 ☐ 1 ☐ 2

Choose a numerical value on scale of -2 (not at all convincing) to 2 (convincing). Give support for your choice in the space below.

(12) Student work by Alan, a 6th grader, on the three-color towers extension problem



☐ -2 ☐ -1 ☐ 0 ☐ 1 ☐ 2

Choose a numerical value on scale of -2 (not at all convincing) to 2 (convincing). Give support for your choice in the space below.

ITEMS 13 – 14: STUDENTS' WORK ON THE PIZZA PROBLEM

Statement of task for the pizza problem:

Capri Pizza has asked you to help design a form to keep track of certain pizza choices. They offer a standard "plain" pizza with cheese and tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms and pepperoni. How many choices does a customer have? List all possible choices. Find a way to convince each other that you have accounted for all possibilities.

(13) Student work by Subject 15, a 6th grader, on the pizza problem
(Note: his work starts below and continues on the next page)

	Peppers	sausage	mushrooms	pepperoni
√1.	1	1	1	1
√2.	1	0	1	1
√3.	1	0	0	1
√4.	1	0	0	0
√5.	0	1	1	1
√6.	0	1	1	0
√7.	0	1	0	0
√8.	0	0	1	1
9.	0	0	1	0
√10.	0	0	0	0
√11.	1	0	1	0
√12	0	1	0	1
√13.	1	1	0	1
√14.	0	0	0	1
√15.	1	1	0	0
√16.	1	1	1	0

To get out answer it has many steps. First we found all the combinations of 4 which is 1. then we found all the combinations of 1 which is 4

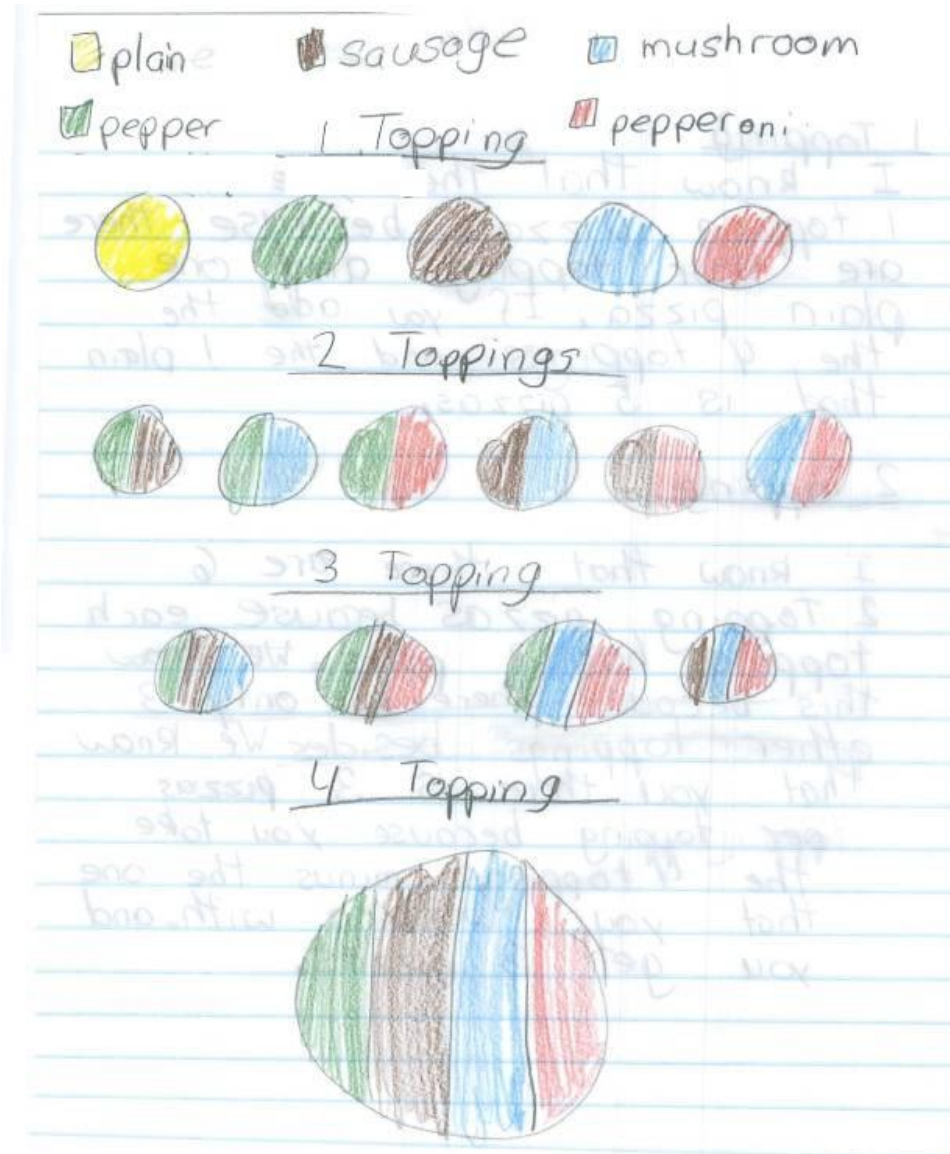
We have noticed that each topping is in 8 combos. Eight is half of 16 which is the total number of combos. Half of eight is 4 which is the amount of toppings. There are 16 combinations in all. There are 4 combos of 1 and 1 combo of 4. There are 6 combos of 2 and 4 combos of 3. There are eight different combinations and eight symmetrical combinations. The amount of combos for 3 is the same as the combos for 1 because for each combination of 1 there is a combination of 3.

This reminds me of the building block problem. It reminds me because we used the same method to find the answer.

☐ -2 ☐ -1 ☐ 0 ☐ 1 ☐ 2

Choose a numerical value on scale of -2 (not at all convincing) to 2 (convincing). Give support for your choice in the space below.

(14) Student work by Samantha, a 6th grader, on the pizza problem
(Note: her work starts below and continues on the next two pages)



1 Topping
I know that there are 5 1 topping pizza's because there are four toppings and one plain pizza. If you add the the 4 toppings and the 1 plain that is 5 pizzas.

2 Topping

I know that there are 6 2 Topping pizzas because each topping has 3 pairs. ~~We know this because there is only 3 other toppings besides~~ We know that you there is 3 pizzas per topping because you take the 4 toppings minus the one that your working with and you get 3 pizzas.

3 topping
I know that there are 4 3 topping pizzas because there is more toppings. We know that there are 3 pizza per topping because you take 4 toppings minus the one that your working with and you get 3 pizzas per topping

4 topping
All the toppings on 1 pizza

☐ -2 ☐ -1 ☐ 0 ☐ 1 ☐ 2

Choose a numerical value on scale of -2 (not at all convincing) to 2 (convincing). Give support for your choice in the space below.

APPENDIX C: VIDEO ASESSMENT PROMPT

Instructions:

This episode is an assessment interview with four 4th grade students, Milin, Michelle, Jeff and Stephanie, for building all possible different towers of a particular height when selecting from two colors of unifix cubes. The children, working in pairs, had built towers four and five cubes tall during class sessions. Each of the children was subsequently interviewed individually and asked to describe how he or she had approached the tasks and to justify any solutions that had been constructed. In this group interview, the students are sharing their ideas about the towers problems, explaining and justifying their solutions to each other. While they consider towers of various heights during the session, they specifically reason about towers that are three cubes tall. Although unifix cubes were available, the children chose not to use them during the interview. The segment begins with short clips from the 4th grade classroom session to provide a background context of the students' building and organizing their towers with unifix cubes.

After viewing the video of the children explaining and justifying their approaches to the problems, please describe as completely as you can:

(1) each example of reasoning that a child puts forth;
(2) whether or not the reasoning forms a valid argument;
(3) whether or not the argument is convincing; and
(4) why or why not you are convinced. Give evidence from the interview to support any claims that you make. You may refer to the attached transcript as needed.

Each response will be evaluated according to the following criteria:

- Recognition of children's heuristic arguments
- Your assessment of the validity or not of children's reasoning
- Evidence to support your claims
- Whether the warrants you give are partial or complete

APPENDIX D: TRANSCRIPT OF THE GANG OF FOUR VIDEO

This episode is an assessment interview with four 4th grade students, Milin, Michelle, Jeff and Stephanie, for building all possible different towers of a particular height when selecting from two colors of unifix cubes. The children, working in pairs, had built towers four and five cubes tall during class sessions. Each of the children was subsequently interviewed individually and asked to describe how he or she had approached the tasks and to justify any solutions that had been constructed. In this group interview, the students are sharing their ideas about the towers problems, explaining and justifying their solutions to each other. While they consider towers of various heights during the session, they specifically reason about towers that are three cubes tall. Although unifix cubes were available, the children chose not to use them during the interview.

The segment begins with short clips from the 4th grade classroom session to provide the background context of the students' building and organizing their towers with unifix cubes.

Transcript

CM:	……You know the towers problems?
All:	Yeah.
CM:	The last one we did in class - Remember what that was about?
Jeff:	Robin Hood? That was the last one we did –
M, M, S:	Towers of 5!
CM:	You remember what you did with those Towers of 5?
All:	Um-hm.
CM:	Um-hm. Tell me about it. What was the problem?
Jeff:	How many –
Michelle:	You had to figure out how many – how many different towers you could make for five blocks up.
CM:	Any five blocks?
All:	No. Two colors.
CM:	Two colors. OK. And did you figure that out?
All:	Yeah.
CM:	And what is it? Do you remember?
All:	32!
CM:	You're sure of it?
All:	Yeah!
CM:	How can you be so sure?
Milin:	We checked!
CM:	How can you be so sure?
Jeff:	Remember when we did all the charts - the thingies – the
Milin:	And then remember –
Jeff:	All the different patterns. Remember, I convinced you up in the -
CM:	Yeah – in the room. OK. But I remember saying to you, Jeff, and I remember saying to you, Michelle – and to you, Stephanie - and Stephanie did try to work on towers of six and I asked all of you if you-
Milin:	So did I.

CM: You did, too? If you were building towers of six, how many would there be?
Jeff: I don't know
Michelle: I did some but I didn't-
CM: But do you know how many?
Stephanie: Yeah.
Milin: Probably 64.
CM: Why do you think 64?
Milin: Well, because there was a pattern.
CM: What's that?
Milin: You just times them by two
CM: Times what by two?
Milin: The towers by two, because one is two, and then we figured out two is two, and then, I mean four, and then -
Jeff: You are not making much sense!
Michelle: See, if you had only one block up and two colors, then you would have two towers, and we figured out that the other day that you keep on doing…
Jeff: Everything is opposites!
Michelle: …like two times two would be four and then…
CM: So four would be for what?
Stephanie: All you have to do-
Michelle: …four for, there would be four towers for two high.
CM: Okay.
Jeff: They are all opposites though.
CM: Okay well, let me hear what Michelle is saying.
Michelle: And then for this three high, you would have eight towers and four high, you would have twelve towers and then you keep on doing it like that.
CM: Do you agree with that?
Jeff: I don't know what you are talking about.
Stephanie: Well. What it is - is-
Michelle: Well - five high would be twenty-five and then -
CM: Okay, lets get a piece of paper and write down what you are saying and see if you all agree. I think Jeff hasn't been with us for a while and he doesn't know what we are talking about. But let's take one at a time. Let's just agree as we are moving along.
Michelle: If you had one high see there is red and blue then you would have two and then if you had -
CM: Okay, write that down. Two. Did you agree with that?
Jeff: Yeah.
CM: Do you know what she is talking about?
Jeff: There is one red and one blue so there is only one way to do it so it's two.
CM: One way you can do it and so it's two.
Jeff: Yeah, you see if you have to make towers of one and there is only two colors
Milin: He keeps on doing that.
CM: All right, let's go on.
Michelle: If you had two towers that would be four, because you have-
Jeff: Yeah I agree with that. Okay.
Michelle: See you would just times it. See two times two
CM: Okay just hold on okay write the four down. Look I don't …Can you explain to me why from two you would get to four? Milin, tell me why.

Milin:	For each one of them you could add one - no two more on because there is a black, I mean a blue, and a red -
Jeff:	What she is doing…
CM:	Let her finish. Okay.
Milin:	See. For that you just put one more for red you put a black on top and a red on top - I mean blue on top instead of black and on blue you put a blue on top and a red on top. You keep on doing that.
CM:	Do you understand what he is talking about?
Stephanie:	Uh- huh!
CM:	You all understand what he is talking about?
Jeff:	Yeah.
CM:	All right. So - so we agree four. What happens if you're building towers three high? What did you say it would be?
All:	It would be eight.
CM:	Write eight down. Can you give me an argument; you don't have to do it. Why we jumped from four to eight?
Michelle:	There's-
CM:	Shhh. That's what Jeff wants to know
Michelle:	There's - there's-
CM:	Go slow. It's Jeff you are convincing not me.
Michelle:	There is two blue. There is two here.
Jeff:	I know that.
Michelle:	And then we went to four so it would have to be times. Two times two equals four and four times two would equal eight.
CM:	That doesn't help Jeff understand. He just knows that we are multiplying two times two
Milin:	I know! I know!
Stephanie:	All right.
Jeff:	If this…
CM:	Okay. One at a time.
Jeff:	If this was like a pattern it would go two - four - six in between the eight.
CM:	Yeah, that's what he is saying.
Milin:	No! No!
Stephanie:	But that's not the pattern we are working on.
CM:	Go ahead Stephanie.
Stephanie:	The pattern that we saw was this. For one block at a time we found two.
Jeff:	We already got two and four
Milin:	Two, four, six-
Stephanie:	I know – two, four and then eight – Right? Two, four and then eight.
CM:	Why eight? That's what Jeffery asked about.
Milin:	I know.
CM:	Go ahead. Let Milin persuade Jeff.
Milin:	If you do that you just have to add for each one of those you have to add
CM:	Each one of what? These four?
Milin:	Yeah. You have to add one more color for each one
CM:	Which way are you adding it? Where are you putting that one more color, Milin?
Milin:	No. Two more colors for each one. See-
CM:	So this one with red on the bottom and blue on the top.
Milin:	You could put another blue or another red.

CM: You agree with that? You can put a blue or red on top and that-

Milin: Yep!

Milin: And that will be two and then on this you could put another red or blue on top that will be four.

Jeff: That is the same right there.

CM: No, this is blue red

Jeff: No. Here look. It's blue oh okay, okay.

Milin: See. Now you see.

CM: Could you find what Milin is saying and now here you could put-

Milin: A red or a blue and same thing here

CM: Do you understand that?

Jeff: Yeah.

CM: So do you see how you get eight?

END OF FIRST SEGMENT / BEGIN NEXT SEGMENT

Stephanie: Yeah, but that's what he is like, that's what is different from mine I just like took the things and went- I just took one and went –

Milin: And kept on-

Stephanie: Here is one red/red/red, blue/blue/blue and then I go like red/blue/blue, blue/red/blue-

CM: So, what I am hearing you say is that you're just…

Milin: Guessing!

CM: …you believe there is eight. But you say guessing. Now, why does that sound like guessing?

Milin: Because what if you could make more?

Stephanie: Okay, this is the three high. Right? And you're convinced you can make eight?

APPENDIX E: ORIGINAL BELIEFS ASSESSMENT

1. Learners generally understand more mathematics than their teachers or parents expect.

1	2	3	4	5
Strongly Agree				Strongly Disagree

2. Teachers should make sure that students know the correct procedure for solving a problem.

1	2	3	4	5
Strongly Agree				Strongly Disagree

3. Calculators can help students learn math facts.

1	2	3	4	5
Strongly Agree				Strongly Disagree

4. It's helpful to encourage student-to-student talking during math activities.

1	2	3	4	5
Strongly Agree				Strongly Disagree

5. Math is primarily about learning the procedures.

1	2	3	4	5
Strongly Agree				Strongly Disagree

6. Students will get confused if you show them more than one way to solve a problem.

1	2	3	4	5
Strongly Agree				Strongly Disagree

7. All students are capable of working on complex math tasks.

1	2	3	4	5
Strongly Agree				Strongly Disagree

8. Math is primarily about identifying patterns.

1	2	3	4	5
Strongly Agree				Strongly Disagree

9. If students learn math concepts before they learn the procedures, they are more likely to understand the concepts.

1	2	3	4	5
Strongly Agree				Strongly Disagree

10. Manipulatives should only be used with students who don't learn from the textbook.

1	2	3	4	5
Strongly Agree				Strongly Disagree

11. Young children must master math facts before starting to solve problems.

1	2	3	4	5
Strongly Agree				Strongly Disagree

12. Teachers should show students multiple ways of solving a problem.

1	2	3	4	5
Strongly Agree				Strongly Disagree

13. Only really smart students are capable of working on complex math tasks.

1	2	3	4	5
Strongly Agree				Strongly Disagree

14. Calculators should be introduced only after students learn math facts.

1	2	3	4	5
Strongly Agree				Strongly Disagree

15. Learners generally have more flexible solution strategies than their teachers or parents expect.

1	2	3	4	5
Strongly Agree				Strongly Disagree

16. Math is primarily about communication.

1	2	3	4	5
Strongly Agree				Strongly Disagree

17. Manipulatives cannot be used to justify a solution to a problem.

1	2	3	4	5
Strongly Agree				Strongly Disagree

18. Learners can solve problems in novel ways before being taught to solve such problems.

1	2	3	4	5
Strongly Agree				Strongly Disagree

19. Understanding math concepts is more powerful than memorizing procedures.

1	2	3	4	5
Strongly Agree				Strongly Disagree

20. Diagrams are not to be accepted as justifications for procedures.

1	2	3	4	5
Strongly Agree				Strongly Disagree

21. If students learn math concepts before procedures, they are more likely to understand the procedures when they learn them.

1	2	3	4	5
Strongly Agree				Strongly Disagree

22. Students are able to tell when their teacher does not like mathematics.

1	2	3	4	5
Strongly Agree				Strongly Disagree

23. Collaborative learning is effective only for those students who actually talk during group work.

1	2	3	4	5
Strongly Agree				Strongly Disagree

24. Students should be corrected by the teacher if their answers are incorrect.

1	2	3	4	5
Strongly Agree				Strongly Disagree

25. Mixed ability groups are effective organizations for stronger students to help slower learners.

1	2	3	4	5
Strongly Agree				Strongly Disagree

26. Collaborative groups work best if students are grouped according to like abilities.

1	2	3	4	5
Strongly Agree				Strongly Disagree

27. Conflicts in learning arise if teachers facilitate multiple solutions.

1	2	3	4	5
Strongly Agree				Strongly Disagree

28. Learning a step-by-step approach is helpful for slow learners.

1	2	3	4	5
Strongly Agree				Strongly Disagree

29. Only the most talented students can learn math with understanding.

1	2	3	4	5
Strongly Agree				Strongly Disagree

30. The idea that students are responsible for their own learning does not work in practice.

1	2	3	4	5
Strongly Agree				Strongly Disagree

31. Teachers need to adjust math instruction to accommodate a range of student abilities.

1	2	3	4	5
Strongly Agree				Strongly Disagree

32. Teacher questioning of students' solutions tends to undermine students' confidence.

1	2	3	4	5
Strongly Agree				Strongly Disagree

33. Teachers should intervene as little as possible when students are working on open-ended mathematics problems.

1	2	3	4	5
Strongly Agree				Strongly Disagree

34. Students should not be penalized for making a computational error when they use the correct procedures for solving a problem.

1	2	3	4	5
Strongly Agree				Strongly Disagree

APPENDIX F: SUBSET OF ORIGINAL BELIEFS ASSESSMENT ANALYZED

Q1. Learners generally understand more mathematics than their teachers or parents expect.

1	2	3	4	5
Strongly Agree				Strongly Disagree

Q5. Math is primarily about learning the procedures.

1	2	3	4	5
Strongly Agree				Strongly Disagree

Q6. Students will get confused if you show them more than one way to solve a problem.

1	2	3	4	5
Strongly Agree				Strongly Disagree

Q7. All students are capable of working on complex math tasks.

1	2	3	4	5
Strongly Agree				Strongly Disagree

Q8. Math is primarily about identifying patterns.

1	2	3	4	5
Strongly Agree				Strongly Disagree

Q9. If students learn math concepts before they learn the procedures, they are more likely to understand the concepts.

1	2	3	4	5
Strongly Agree				Strongly Disagree

Q11. Young children must master math facts before starting to solve problems.

1	2	3	4	5
Strongly Agree				Strongly Disagree

Q12. Teachers should show students multiple ways of solving a problem.

1	2	3	4	5
Strongly Agree				Strongly Disagree

Q15. Learners generally have more flexible solution strategies than their teachers or parents expect.

1	2	3	4	5
Strongly Agree				Strongly Disagree

Q17. Manipulatives cannot be used to justify a solution to a problem.

1	2	3	4	5
Strongly Agree				Strongly Disagree

Q18. Learners can solve problems in novel ways before being taught to solve such problems.

1	2	3	4	5
Strongly Agree				Strongly Disagree

Q19. Understanding math concepts is more powerful than memorizing procedures.

1	2	3	4	5
Strongly Agree				Strongly Disagree

Q20. Diagrams are not to be accepted as justifications for procedures.

1	2	3	4	5
Strongly Agree				Strongly Disagree

Q21. If students learn math concepts before procedures, they are more likely to understand the procedures when they learn them.

1	2	3	4	5
Strongly Agree				Strongly Disagree

Q29. Only the most talented students can learn math with understanding.

1	2	3	4	5
Strongly Agree				Strongly Disagree

Q33. Teachers should intervene as little as possible when students are working on open-ended mathematics problems.

1	2	3	4	5
Strongly Agree				Strongly Disagree

APPENDIX G: STUDENT WORK ASSESMENT SCORING RUBRIC

Subject # Verified by: Date:

	Cases	Control Variable	Figures	Tree Diagram	Pattern	Exhaustive	Guess Check	Doubling	Simpler	Similar	Numerical	None
Q2	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
Q4	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
Q6	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
Q8	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
Q10	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
Q12	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
Q14	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
Q16	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
Q18	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
Q20	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
Q22	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
Q24	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
Q26	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
Q28	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐

APPENDIX H: VIDEO ASSESSMENT SCORING RUBRIC

Instructions for completing the on-line Rubric for scoring a participant response to the Gang of Four assessment video:

Enter the ID number where indicated on the video assessment form. As you scroll through the rubric, mark the appropriate box to indicate the presence or absence of each item in the rubric relative to the participant's description of the children's activity in the video.

A. **The first section of the rubric deals with items referring to the mathematical ideas in the task that may be identified by participants and includes the following four categories: Problem Tasks, Representations, Mathematical Reasoning and Heuristic arguments.**

B. **The second section of the rubric deals with whether or not the participant considered the student(s)' reasoning and heuristic arguments to be convincing. As each response is scored, a list gets generated of those items identified by the scorer in the first section as reasoning that was noted by the participant. For each of these items, the scorer notes whether the participant indicated that he/she found the reasoning to be convincing or not convincing.**

   Note that a participant's remarks about a child's reasoning, argument or behavior should only be scored as "convincing" or "not convincing" if the participant specifically indicates that to be the case somewhere in the response.

Scoring Holistically

Study participants watch a video clip from the "Gang of Four" interview with researcher and four 4th graders: Milin, Michelle, Jeff and Stephanie. In an open-ended format, participants respond to a prompt that asks them to describe as completely as they can: (1) each example of reasoning that a child puts forth; (2) whether or not the reasoning forms a valid argument; (3) whether or not the argument is convincing; and (4) why or why not you are convinced. They are asked to give evidence from the interview to support any claims that they make; and they are provided with copy of transcript for the video clip.

Always begin scoring of an assessment by reading the participant's response in its entirety to get a sense of its scope. Then review it again more carefully to look for written evidence that support scoring of particular rubric items. Because the response format is totally open ended, the participant has freedom to express response in any desired organization. The entire response must be considered, since a participant may respond to one part of the assessment instructions in detail and not repeat this detail in response to the other parts. Indication of convincingness may occur in any portion of the participant's response to the assessment.

Note that the scoring focus is on mathematical reasoning, and of less importance is the language used to express that reasoning. A sophisticated response may name argument type and discuss it only in general form. Other responses may use very informal language. What someone says in his or her response matters more than how it is expressed. Examples can be helpful. Thus, we will use a Wiki to post illustrative (but not exhaustive) examples from participant response data that was scored in previous efforts.

Scorer's Guide to use in responding to the rubric concerning Problem Tasks, Mathematical Representations, Reasoning and Heuristic arguments (Questions 1 through 8):

The following is a list of Problem Tasks, Representations, Mathematical Reasoning and Heuristic arguments referred to by the children during the video. Items have been identified by the research team from studying the transcript as well as the video.

1. **Problem Tasks identified**
    a. **Towers of height 3-cubes with two colors**
    b. **Towers of height 2-cubes with two colors**
    c. **Towers of height 4-cubes with two colors**
    d. **Towers of height 5-cubes with two colors**
    e. **Towers of height 10-cubes with two colors**
    f. **Towers of any height (height "n") with two colors**

    Examples:

    a. Towers of height 1-cube with two colors
    - "how many patterns they could make from towers of 1 block, 2 blocks, 3 blocks, etc.
    - "Since there can only be 2 towers for tower of 1"

    b. Towers of height 2-cubes with two colors
    - "how many patterns they could make from towers of 1 block, 2 blocks, 3 blocks, etc."
    - "…and for towers of 2, they saw that…"

    c. Towers of height 3-cubes with two colors
    - "how many patterns they could make from towers of 1 block, 2 blocks, 3 blocks, etc."
    - "So for 3 high, build towers of all red…"

    d. Towers of height 4-cubes with two colors
    - "…worked together to figure out how many different four and five block combinations a person can make using two different colored unifix cubes.
    - "It also led Michelle to the incorrect conclusion that there are 12 towers with a height of four blocks."

    e. Towers of height 5-cubes with two colors
    - "…worked together to figure out how many different four and five block combinations a person can make using two different colored unifix cubes."
    - "Jeff was able to use the pattern 'times 2' to justify the towers of 5 question…"

    f. Towers of height 10-cubes with two colors
    - "led Stephanie to finally say a ten block tower had 1,024…"
    - "…at the end, Stephanie 'figured it out…' towers of 10 = 1,024."

    g. Towers of any height (height "n") with two colors
    - "They realized that for every one tower of blocks of n height…"
    - "In his own words, he explained why the pattern requires you to multiply 2 as n increases by 1."

2. **Representations constructed or referenced**
    a. **Descriptions (verbal) of towers and how they are built**

- "First she starts with a solid red tower, 2 red blocks. Then does the towers that have one blue so blue/red/red, red/blue/red, red/red/blue
- "…example of reasoning was making opposites. …the student used all of one color (blue) and then all of the other color (red). Next they went onto the top color different than the rest (red, blue, blue) and the opposite of that (blue, red, red)"

b. **Diagrams or charts of towers**
- "She also used a diagram to support her answer."
- "Once they start drawing the patterns down it was easier to see what they were saying."

c. **Numbers or letters used as symbols to represent cubes or towers**
- "Stephanie lists arrangements like r/r/r r/b/b etc."

3. **Numerical Reasoning Patterns identified**

Patterns mentioned by a participant may include only parts of the patterns listed below, but the scorer may be able to infer which pattern is being mentioned.

a. **Additive (2, 4, 6, 8 …..)**

b. **Doubling or "times two" (2, 4, 8, 16 …)**
- "…you just multiply by two."
- "They continue to link the numbers of new towers to two times the previous number."

c. **Base squared (1, 4, 9, 16, 25 ….)**

d. **Alternative or combined (2, 4, 8, 12 …) EXAMPLE (MFP): WHEN PARTICIPANT MENTIONS AT LEAST 8 AND 12, THEN CHECK 3d.**
- "It also leads Michelle to the incorrect conclusion that there are 12 towers with a height of four blocks."
- "*Michelle*: 'for this three high you would have eight towers and four high, you would have twelve towers and then you keep doing it like that…'."

4. **Spatial Reasoning Patterns identified**

a. **The term "pattern" referring to arrangement of colored cubes within a tower.**
- "Showed pattern."
- "She followed a pattern while constructing her diagram: no blues (3 red), one blue…"

b. **Opposites (two towers with corresponding positions having alternate colors)**
- "Jeff says that 'everything is opposites'. He is using the pattern of switching colors."
- "Jeff: 'They're all opposites'."

c. **Identifying towers, or groups of towers, by the "pattern" of how colored cubes are placed (e.g., a "staircase" pattern of a single cube of one color in consecutively lower – or higher – positions in each tower; or towers with patterns of alternating colored cubes; or more than one cube of one color together in consecutively lower or higher positions).**

- "She then puts 1 blue cube at the top of the 2 red blocks and moves the blue's position 'down the stairs.' Next, Step shows all possibilities with 2 blues…"

- "They also argued it would be easier to (when drawing or building these towers) go by the order of how many blocks of each color they are using."

5. Other Reasoning Features noted

a. Direct Answers (unexplained answers for number of towers for certain heights)
NOTE (MFP): 5a is only for those direct answers that do not connect to patterns (3 and 4) or other reasoning (6, 7, and 8).

- "Stephanie 'figured it out…' towers of 10 = 1,024."

b. Guessing

- Milin says this is guessing."

c. Randomly building towers and checking for duplicates

- "Finally a student randomly picked combinations until they thought they had exhausted all of them."

6. Inductive Argument (note that a participant may refer to it as recursive or including recursion). This argument may be expressed with reference to towers of a specific height, as in features (a) and (b) below. It also may be expressed in general form, as in features (c) and (d) below.

a. When building towers selecting from two colors, there are exactly two unique towers of height one. With a single position in the tower, the one cube can be (say) either red or blue.

- "Since there can be only 2 towers for tower of 1…"
- "…because you know the most basic number of towers for one height which is two…"

b. For the two unique towers, one cube in height, cubes of one of the colors can be placed on top of each tower producing two unique towers, 2 cubes high. Cubes of the other color can be placed on top of a duplicate pair of towers one cube high producing two more unique towers. The resulting four towers, 2 cubes high, will contain no duplicates since the two unique pairs differ from each other in the top cube.

- "…and for towers of 2, they saw that they can add either a blue or a red to each of the two towers. So blue + red and blue + blue possible for the first one and red + blue and red + red possible for the second one, so they got 4 towers of 2."

c. Towers of any height, "n", when selecting from two colors, can be generated similarly by taking all the towers with height, "n-1" – that are known to be unique with no duplicates because they were generated recursively from towers one cube high. Cubes of one of the colors (say, red) can be placed on top of each of the towers, producing unique towers "n" cubes high. Cubes of the other color (say, blue) can be placed on top of a duplicate set of towers, "n-1" cubes high, producing a second set of unique towers, "n" cubes high.

- "Milin's argument of adding 'one more color for each one.' He then clarifies it is actually two colors, blue or red which gives two towers per each tower of the previous height."

d. The resulting total set of towers, "n" cubes high, will contain no duplicates since the two generated sets (each of which contained no duplicates) differ from each other in the top cube. This resulting set of towers, "n" cubes high, will always include two times the number of towers as the "n-1" high set.

- "…you could add 1 of each color to the one end for a total of two more combinations. Which would double your answer. So each time you add one more block to the tower, your total number of different combinations doubles."

7. **Stephanie's cases argument for towers 3 cubes high selecting from two colors (blue and red) results in a set of 8 unique towers. A complete argument includes the following cases with the justification for each case. Note that written responses by study participants may well be fragmentary and use much less precise language than the following. Also note that an argument can only be considering a cases argument, rather than the use of a pattern, if the participant clearly defines one or more of the cases; in other words, what it is a case of:**
    a. **All blue cubes or no red cubes resulting in only one tower.**
    Justification – Any other 3 cube high tower that is all blue would be a duplicate of this one.
        - "Stephanie's use of patterns 'one blue, two blues' continuing to three blues."
        - Justification "..starting with all one color…"
    b. **One blue cube and two red cubes resulting in three unique (different) towers.**
    Justification - No more towers can be created with one blue cube and two red cubes because there are only three positions in the tower for the blue cube to occupy. Another position – allowing another tower – would result in a tower 4 cubes high.
        - "Then does the towers that have one blue so blue/red/red, red/blue/red, red/red/blue."
        - Justification "…and then putting in one of the other color in as many different places as possible."
    c. **Two blue cubes stuck together and one red cube resulting in two unique towers.**
    Justification – No more towers can be created of two blue cubes "stuck together" and one red cube in the third position because the two together must be in positions one and two or two and three of the three possible positions in the tower.
        - "She says that she is doing it with the blues stuck together."
        - Justification "Once that option was exhausted, the students went on to two of the other color stuck together as in many different places as possible."
    d. **No blue cubes or all red cubes results in one tower.**
    Justification – Any other 3 cube high tower that is all red would be a duplicate of this one and there can be no more single color towers because there are only two colors.
        - "First she starts with a solid red tower, 3 red blocks."
        Justification "Once this was exhausted the student went to three of the other color."
    e. **Two blue "stuck apart" or separated by one red cube results in one tower.**
    Justification - No more towers can be created by two blue cubes "stuck apart" or separated by the red cube, because, with only three positions, position 2 is the only one that can be considered "in-between" the other two.
        - " …until she got caught up in the issue of whether the 2-blue blocks were stuck together or apart."
        Justification "Finally, the student split up the two of the other color to make her final variation."

8. **An alternate cases argument for towers 3 cubes high selecting from two colors (blue and red) proposed by several of the children. Several of the cases overlap completely with the ones**

articulated by Stephanie and those should be scored in Item 7. <u>Only the portion of cases argument that is different from Stephanie's is to be scored in Item 8.</u>

f. **One red cube and two blue cubes resulting in three unique (different) towers. Justification - No more towers can be created with one red cube and two blue cubes because there are only three positions in the tower for the red cube to occupy. Another position – allowing another tower – would result in a tower 4 cubes high.**
**** Participants may describe argument 8f. as better (preferred, more elegant, etc.) than the way Stephanie organized her cases, which bifurcated 8f. into 7c. and 7e.***
  - "So for 3 high, build towers of all red, one red…"
    Justification "There's red… blue/red/red and you can't make any more in this, so you go on to the next one…"

Scorer's Guide to use in responding to the rubric concerning whether or not the participant considered the student(s)' mathematical reasoning and heuristic arguments to be convincing or not convincing (Questions 9 through 12):

9. **For Question 9, the online rubric is programmed to generate a list including each of the items that the scorer marked positively for Questions 2 through 8. For each of these items, the scorer is to note whether the participant indicated that this particular mathematical reasoning and/or argument by one or more of the children was convincing. The absence of a positive (convincing) response for any item does not necessarily mean that the participant considered this particular item to be NOT convincing.**

10. **The scorer will only mark Question 10 positively if the participant indicates that the children's mathematical reasoning was convincing but gives no specific details about which item of reasoning or piece of an argument was convincing.**

11. **For Question 11, the scorer will consider an identical list of the items marked as present in the participant's description in Questions 2 through 8. However, this time the scorer will only mark an item as present if the participant specifically indicates that it was NOT convincing.**
12. **The scorer will only mark Question 12 as present if the participant indicates that the children's mathematical reasoning was NOT convincing but gives no specific details about which item of reasoning or piece of an argument was convincing.**