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MATHEMATIZATION IN INTRODUCTORY PHYSICS

By

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## ABSTRACT OF THE DISSERTATION

Mathematization in Introductory Physics

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Mathematization is central to STEM disciplines as a cornerstone of the quantitative reasoning that characterizes these fields. Introductory physics is required for most STEM majors in part so that students develop expert-like mathematization. This dissertation describes coordinated research and curriculum development for strengthening mathematization in introductory physics; it blends scholarship in physics and mathematics education in the form of three papers.

The first paper explores mathematization in the context of physics, and makes an original contribution to the measurement of physics students' struggle to mathematize. Instructors naturally assume students have a conceptual mastery of algebra before embarking on a college physics course because these students are enrolled in math courses beyond algebra. This paper provides evidence that refutes the validity of this assumption and categorizes some of the barriers students commonly encounter with quantification and representing ideas symbolically.

The second paper develops a model of instruction that can help students progress from their starting points to their instructor's desired endpoints. Instructors recognize that the introductory physics course introduces new ideas at an astonishing rate. More than most physicists

realize, however, the way that mathematics is used in the course is foreign to a large portion of class. This paper puts forth an instructional model that can move *all* students toward better quantitative and physical reasoning, despite the substantial variability of those students' initial states.

The third paper describes the design and testing of curricular materials that foster mathematical creativity to prepare students to better understand physics reasoning. Few students enter introductory physics with experience generating equations in response to specific challenges involving unfamiliar quantities and units, yet this generative use of mathematics is typical of the thinking involved in doing physics. It contrasts with their more common experience with mathematics as the practice of specified procedures to improve efficiency. This paper describes new curricular materials based on invention instruction provide students with opportunities to generate mathematical relationships in physics, and the paper presents preliminary evidence of the effectiveness of this method with mathematically underprepared engineering students.

## **Dedication**

*This work is dedicated to Betty and Don who never stopped encouraging this dream, and to Djamel, Laila and Yasmine, for inspiring me every day to finish. May your dreams never fade.*

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As with most scientific endeavors, this work is part of the bounty from a rich collaboration. I would like to acknowledge the significant contribution of Andrew Boudreaux, at Western Washington State University, and Stephen Kanim, at New Mexico State University, to the intellectual development of this dissertation work. The vast majority of the writing in this dissertation is my own, but there are occasional paragraphs that were largely written by my collaborators. The subject matter is the product of our group's professional activity. The portions of the dissertation that were written by my collaborators (or mostly by my collaborators) include:

Chapter 2:

Section II.a- Physics habits of mind was written by Stephen Kanim

Chapter 4:

Section I - The description of the squareness invention task and data was written by Stephen Kanim.

Sections II and III.a. were written collaboratively with Andrew Boudreaux.

For the experiments conducted at institutions other than Rutgers, each researcher described the experiment, data collection and analysis for his home institution.

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## **CHAPTER 1: Introduction**

The process of *mathematization* is central to the STEM disciplines as a cornerstone of the quantitative reasoning characteristic of these fields. Mathematizing involves representing ideas symbolically, defining problems quantitatively, producing solutions, and checking for coherence, all in coordinated service of building understanding of how the world works <sup>(1,2)</sup>. To mathematize in physics means to go back and forth between the physical and the symbolic worlds.

Introductory physics is required for most STEM majors in part so that students will develop expert-like mathematization. Yet this goal is not being met for most of the students taking physics. This dissertation describes coordinated research and curriculum development for developing student mathematization in the introductory physics course.

A condition essential for meaningful progress is a set of well-articulated links between scholarship in physics and mathematics education. This dissertation, which consists of three interrelated papers, establishes such links. As the dissertation is a part of an ongoing collaborative project, there are three co-authors on all papers (S. Brahmia, A. Boudreaux, and S. Kanim) with nearly all of the writing being done by me. In the remainder of the introduction, I will be using the pronoun “We” when referring to the joint work and the pronoun “I” when referring to my own conjectures and suggestions.

Success at meeting the broad goal of developing more expert-like mathematization will benefit from:

- (1) Exploring what mathematization means in the context of physics, and developing a more realistic understanding of the preparation of our students to think this way.**

Instructors naturally assume students have a conceptual mastery of arithmetic and algebra before embarking on a university physics course because physics students are typically

enrolled in math courses beyond algebra. The first paper questions the validity of these assumptions and categorizes some of the stumbling blocks students commonly encounter.

**(2) Developing a model of instruction that will take students from their starting points to our desired endpoints.**

Experienced instructors recognize that the introductory physics course introduces new ideas at an astonishing rate. More than most physicists realize, however, the way that mathematics is used in the course is foreign to a large fraction of students. The second paper puts forth an instructional model that can move *all* students toward better quantitative and physical reasoning, despite the substantial variability of those students' initial states.

**(3) Designing and testing curricular materials that foster mathematical creativity to prepare students to better understand physics reasoning.**

Few of the students entering introductory physics have had experience with mathematics as an endeavor in which equations are invented in response to specific challenges. This approach, referred to as the *generative* use of mathematics, contrasts with the much more common student experience with mathematics as the practice of specified procedures to improve efficiency. Providing students with opportunities to generate even the very simplest relationships in physics can start to address deficits in students' mathematical facility. The third paper of this dissertation describes new curricular materials that provide this experience.

As with any interdisciplinary domain, it is important to establish the language of discourse. In order to keep the subject readable to the broad community of instructors who teach math and physics, every attempt has been made to use the simplest terms while describing theoretical ideas related to scholarship on learning. Following the advice of Albert Einstein, the language has been made as simple as possible, but no simpler.

The literature contains clusters of terms that describe the same fundamental ideas. The terms employed here were chosen to be as close as possible to our intended meaning while retaining flexibility, recognizing that there are families of synonyms that could also be used. Clarity through repetitive use of a limited number of terms has been chosen over rendering the prose more interesting through the use of synonyms. In the descriptions of the three papers that follows, I define the terms in the context of their first appearance. Appendix A includes a glossary of terms for the reader.

### Obstacles to Mathematization

The first paper in this dissertation discusses specific obstacles to mathematization in introductory physics. In order to make sense of student reasoning, we make use of established, fine-grained models of analytical, procedural, conceptual, and epistemological reasoning <sup>(3-5)</sup>. We use Hammer's general term *resources* to refer to any of the constructs employed by these models <sup>(4)</sup> and Wittman's notion of a *coordinated set of resources* <sup>(6)</sup> to describe reasoning that involves activation of multiple resources. We address the common assumption that the mathematics preparation of incoming students will allow them to successfully mathematize in physics contexts because they have taken the prerequisite mathematics courses. An examination of the research literature in mathematics education indicates students often use basic arithmetic and algebraic tools in ways quite different than what university and college physics instructors might expect. The focus of this paper is to study this mismatch.

We base our description of expert-like mathematization on studies of successful engineering and physics majors <sup>(7,8)</sup> and use Sherin's terminology to describe target characteristics. Students are *flexible* when they spontaneously activate productive mathematical reasoning independent of context, and *generative* when they are able to create a novel symbolic solution to a problem. We describe common instabilities in the resources and the coordinated sets of resources that are necessary for mathematization in introductory physics.

Through this project, we have developed and tested a suite of questions to assess student facility with proportional reasoning, which appear both in the text of the paper and extracted in Appendix B for easy reference. Among other student populations, responses have been collected from 1,200 freshmen and sophomore engineering majors pre and post instruction at Rutgers University. We found that many of the issues that physics students struggle with fall under three areas of learning previously articulated in mathematics education research:

- *generalized structural reasoning*, the spontaneous use of mathematical structure (e.g. the functional dependence of force on distance for a  $1/r^2$  force) to guide thinking regardless of the context,
- *quantification*, the process of conceptualizing the attribute of object such that it has a unit of measure that has a proportional relationship with its magnitude, and
- *symbolizing*, the back and forth translation between symbols and meaning to facilitate discourse.

We discuss how these areas relate to the key components of mathematization in physics, and the implications our results have for physics instruction.

### Model of Mathematization

In the second paper we question to what extent the instructional models (both explicit and implicit) that underlie the introductory physics course influence students' development of effective mathematization. Two striking findings from the physics education research literature are presented as motivation:

- (1) On attitude diagnostics that include items related to views about the use of mathematics, student responses are less expert-like at the end of an algebra- or calculus-based physics course than they are at the beginning of the course. For example, between one-third and one-half of students typically agree with the

statement “I do not expect physics equations to help my understanding of the ideas; they are just for doing calculations” *after* having completed their physics course.

This result has been documented with the Colorado Learning Attitudes in Science Survey (CLASS) and the Maryland Physics Expectations Survey (MPEX) in a variety of courses at multiple institutions.

- (2) Notable exceptions to this trend occur in physics courses for pre-service elementary teachers using either the *Physics and Everyday Thinking* curriculum or *Physics by Inquiry*. These courses have no mathematics prerequisite. The curricula are comprehensive, student-centered, and lab-based, and are used coherently throughout all parts of the course.

In order to account for these findings, we explore theories from mathematics education and the learning sciences about the development of mathematical knowledge. The learning progression of Gray&Tall <sup>(9)</sup> suggests that *proceptual understanding* of arithmetic and algebra (which integrates **pro**cedural mastery and **concept**ual understanding) is an essential starting point for mathematizing in algebra- and calculus-based physics. We use their term *proceptual divide*, which provides useful way to distinguish “strong” and “weak” students in physics. Some students who have difficulty learning algebra may become entrenched in a procedural approach focused on memorization and execution of algorithms. More successful students develop flexibility by seeing procedures as closely linked to the mathematical objects they produce, and by compressing operations into thinkable mathematical concepts. Students on the procedure-oriented side of the divide use the bulk of their working memory struggling with mathematical procedures, while students who have integrated procedural skill with understanding reduce the load on their working memory and leave some “space” for physics. These divergent ways of approaching mathematics can lead to a physics achievement gap. This paper presents examples of common curricular practices designed to improve the accuracy of students’ procedural problem solving in math and science, and argues that though well intentioned, these practices can actually



widen the proceptual divide by interfering with the development of mathematization in the pre-college classroom.

A framework we refer to as the *quadrant of mathematization* is introduced to account for the difficulty many students have with basic algebraic reasoning in college physics. The framework, which is consistent with the described results from mathematics education research, is also used to explain improvements in mathematization measured by attitudinal diagnostics for a few select courses. We hypothesize that instruction that engages students explicitly and actively in mathematization in physics contexts will help all students meet the instructional goal of increasing sophistication in their mathematizing. The quadrant of mathematization framework is used to motivate an instructional model for developing mathematization.

### Physics Invention Tasks




The third paper presented in this dissertation describes a curricular intervention designed to foster the flexible and generative use of mathematics necessary for meaningful learning in introductory algebra- and calculus-based physics. Many students entering introductory physics have had little experience with mathematics as a *generative* endeavor, that is, an endeavor in which quantities and equations are invented in response to specific challenges. Instead, much of students' mathematical experience has focused on procedural practice. Students are often presented with fully formulated procedures and methods, shown the types of problems these procedures are suited for, and then provided with opportunities to practice in order to develop efficiency. This approach leaves little room for students to develop their own procedures when they encounter a novel problem.

Physics Invention Tasks are designed based on a *preparation for future learning* (PFL) transfer paradigm <sup>(10)</sup>. The tasks prime students to make sense of the mathematical reasoning that they will encounter in subsequent formal physics instruction. The tasks call on existing (and, perhaps, dormant) mathematical resources while also building and strengthening coordinated sets

of resources. We cite Sherin's *symbolic forms* as an example of the kind of coordinated set of resources that are important in introductory physics. We hypothesize that as students progress, these symbolic forms become the resources of further learning<sup>(7,8)</sup>. It is through this mechanism that the invention tasks prepare students for future learning. We describe three sample invention sequences, which appear in their entirety in Appendix C. We present our findings from a study comparing two calculus-based introductory physics courses at Rutgers University; one that used physics invention tasks as part of its curriculum and one that did not. We report on differences in student performance on the Force Concept Inventory (FCI) and CLASS.

Table I summarizes concepts central to the trio of papers contained here. Together, the set of papers presented in this dissertation articulates a problem, presents a theoretical framework for understanding the problem and designing a solution, describes the implementation of a practical intervention, and presents preliminary evidence of the effectiveness of the intervention. The ideas that have been brought together here hail from a variety of areas of scholarship, and represent an interdisciplinary, research-based approach for responding to a fundamental challenge to learning and teaching in introductory physics.

**Table I:** Key mathematics education theory terms applied in the context of a physics example

	 <b>Proceptual development</b>		
Quantification  (including symbolizing)	Procedures and process	Generalized structural reasoning	Expert-like mathematization
<i>Example: steadily slowing down</i>			
A braking force causes an object that is spinning freely to slow down. The quantities of interest are the braking force, in Newtons, the rate of slowing down (a time rate, so should be velocity units divided by a time unit) and the mass in kg; sensible symbols are $F$ , $a$ and $M$ .	The relationship between the quantities is described by $a=F/M$ . We note that there can be more than one force acting, and that more generally $F$ is $F_{\text{net}}$ , or the vector sum of all of the forces acting on the object.	For a given net force , if we vary the mass we see that the acceleration of the object is inversely proportional to the mass of the object. If instead we look at a given fixed mass and vary the force, we see that the acceleration is proportional to the net force.	Since the acceleration is proportional to $1/m$ , doubling the mass results in an acceleration that is half as big because that is how <b>inverse proportional relationships</b> always work. If we double both the mass and the net force, then the acceleration will not vary because the two changes cancel each other out: one doubling is directly proportional and the other is inversely proportional. The net force-to-mass ratio is unchanged.
<b>Preparation for future learning</b>			
<b>Coordinated set of resources</b>			
<i>Smallest</i>			<i>Largest</i>

1. Freudenthal, *Mathematics as an educational task*, (D. Reidel, Dordrecht, 1973)
2. Treffers, A., and Vonk, H., *Three dimensions: A model of goal and theory description in mathematics instruction-the wiskobas project*, (Reidel Dordrecht, 1987)
3. DiSessa, A. A., "Toward an epistemology of physics", *Cognition and Instruction*, 10, pp.105-225, (1993).
4. Hammer, D., "Student resources for learning introductory physics", *American Journal of Physics*, 68, pp.S52-S59, (2000).
5. Hammer, D., Elby, A., Scherr, R. E., & Redish, E. F. Resources, framing, and transfer. In J. Mestre (Ed.), *Transfer of Learning from a Modern Multidisciplinary Perspective*, (Greenwich, CT: Information Age Publishing, 2005), pp. 89-120.
6. Wittmann, M. C., "Understanding coordinated sets of resources: An example from quantum tunneling", *Proceedings of the Fermi School in Physics Education Research*, Varenna, Italy, (2003).
7. Bing, T. J., and Redish, E. F., "Epistemic complexity and the journeyman-expert transition", *Physical Review Special Topics-Physics Education Research*, 8, pp.010105, (2012).
8. Sherin, B. L., "How students understand physics equations", *Cognition and Instruction*, 19, pp.479-541, (2001).
9. Gray, E. M., and Tall, D. O., "Duality, ambiguity, and flexibility: A" proceptual" view of simple arithmetic", *Journal for Research in Mathematics Education*, pp.116-140, (1994).
10. Bransford, J. D., and Schwartz, D. L., "Rethinking transfer: A simple proposal with multiple implications", in A. P. D. Iran Nejad Pearson Ed., *Review of research in education*, (Washington, DC: American Educational Research Association, 1999) pp.61-100.

## **CHAPTER 2: Obstacles to mathematization in introductory physics**

### **Abstract**

Experts in physics and other STEM disciplines develop and communicate ideas through *mathematization*, or the mental habit of going back and forth between the physical world and the symbolic world. Mathematization is an important instructional goal in introductory physics. Introductory physics is one of the few introductory courses broadly required outside of mathematics in which this kind of thinking about the physical world is foundational. This paper asserts that the *flexible* and *generative* use of mathematics associated with mathematization is central to the value of physics instruction for all students, and that there are significant obstacles preventing most students from developing these intellectual characteristics. Instructors commonly assume that the mathematics preparation of incoming students will allow them to successfully mathematize in physics contexts, yet examination of the research literature in mathematics education indicates that most students use basic arithmetic and algebraic tools in ways quite different than what university and college physics instructors might expect. In this study we embark on an exploration of this mismatch. We developed a suite of questions that assess students' proportional reasoning skills and administered tests to over 1,500 freshmen and sophomore engineering majors pre- and post- instruction at Rutgers University in the fall of 2013. We report on our findings in the context of what is known from prior research, and we discuss the implications our results have for physics instruction.

### **I. Introduction**

When prompted on an exam to determine the orbital period of Neptune given the Earth-Sun mean distance, and the Neptune-Sun mean distance, a student dutifully applies Keplers 3<sup>rd</sup> law and comes up with  $10^{15}$  years, then moves on to the next question. When he gets his exam back,

he chides himself for making a dumb mistake with his calculator, having multiplied instead of divided. It is all too common in physics courses that students make algebra mistakes resulting in absurd answers and never bat an eye. While physics majors may make a similar calculation error, they are less likely to ignore it because they expect physics to make sense, and they have the mental resources to check and see if it does.

*Mathematizing* involves representing ideas symbolically, defining problems quantitatively, solving problems, and verifying that it all makes sense in order to understand how the world works <sup>(1,2)</sup>. Simply stated, to mathematize in physics means going back and forth between the physical world and the symbolic world. Mathematization is an important goal in introductory physics because physicists develop and communicate ideas this way. The student described above was not mathematizing, he was simply responding to a prompt to calculate.

Sherin <sup>(3)</sup> describes two essential characteristics of expert-like mathematization in the introductory course: *flexible* when students spontaneously activate productive mathematical reasoning independent of context, and *generative* when they are able to create a novel symbolic solution to a problem. We adopt Sherin's criteria and consider effective mathematization in physics to involve both *flexibility* and *generativity* with mathematics in the context of problem solving.

We agree with other researchers <sup>(4,5)</sup> that mathematizing in physics involves a tight blending of mathematics and physics concepts <sup>(6)</sup>. Take for example the phenomena of thunder and lightning. The situation triggers thinking about time rates, one unimaginably fast (so, assumed instantaneous) and the other plausibly slow enough that we can notice the travel time. Speed of sound implies the division of a distance by a time (and not multiplication, subtraction etc.) in order to answer questions about these coupled phenomena. And the question that comes to mind is *how far away is the lightning?*, so using a rate to answer a question about a related quantity is also important. Mathematization is not just doing division to find a quantity, but includes

reflection on the interrelationship between the situation, the mathematical objects given, the actions, and validation that each makes sense, as well as the conscious judgment that division is appropriate here.

Introductory physics is a required course for most STEM majors in part to help students develop more expert-like mathematization. Instructors model these habits of mind for their students from the first days of the course. From the instructor's perspective, the mathematics consists mainly of straightforward algebraic ideas that students have encountered repeatedly in middle and high school mathematics. However, while most students will have mastered the procedural skills necessary to calculate from a formula, many struggle to assimilate the quantitative reasoning. There is growing evidence from attitudinal diagnostics that most medium- to large-enrollment introductory physics courses are ineffective at advancing students' mathematization in a more expert-like direction, a result that appears to be ubiquitous across a variety of instructional approaches <sup>(7,8)</sup>.

Physics courses have an important role to play in helping students become more productive thinkers. Our success at meeting this broad goal will benefit from exploring how mathematics as used by physicists is different from the ways that mathematics is taught by mathematicians. We begin this paper with a summary of some of these differences that become important in introductory physics courses, which we describe in detail in a related paper <sup>(9)</sup>. We devote the remainder of this manuscript to developing an accurate understanding of the preparation of our students, which depends strongly on their previous instruction. Instructors naturally assume students have a conceptual mastery of arithmetic and algebra before embarking on a physics course because physics students are typically enrolled in math courses beyond algebra. We question the validity of this assumption.

Most students taking introductory physics do not show evidence of a flexible and generative use of mathematics. Bing and Redish <sup>(10)</sup>, and Sherin <sup>(3)</sup>, describe the development of these

characteristics for more advanced students as part of their development of physics expertise. However, since fewer than 3% of the students taking introductory physics end up majoring in physics, it is also useful to focus on the mathematical development of the rest. In Tuminaro's study of algebra-based physics students<sup>(11)</sup>, he points out that if students do not have the expectation that conceptual knowledge of mathematics is connected to what they are doing when solving physics problems, then they are likely to frame their problem-solving activities in terms of plug-and-chug manipulations or of intuitive sense-making that is primarily qualitative. For these students, sense-making is not part of calculating.

Using math generatively requires different skills than using math procedurally. Students must be willing to try things without knowing whether they will work, which requires courage. They must learn how to check to see whether their intuitions match the mathematics that they generate, and how to iterate toward better solutions.

Consider the following scenarios common in the introductory courses that exemplify a lack of flexibility and generativity. We categorize them based on known difficulties from mathematics education research:

- Students struggle to choose (unprompted) basic algebraic tools that they have become proficient with in math classes to solve physics problems. For example, figuring out that a wavespeed is the ratio of the wavelength to the period of a sine wave without using a formula. Correspondingly, students become proficient with solving end-of-chapter problems that require effective manipulation of equations but don't recognize the connections between the physics techniques and the mathematics that they have previously learned (e.g., the kinematics equations and general arithmetic progressions in math.) This is a weakness with *generalizing structural reasoning*.<sup>(12)(3)</sup>
- Many introductory physics students, even those with high SAT math scores, often reason in puzzling ways. Their decision-making is frequently fundamentally flawed



algebraically (e.g., adding unlike quantities, or using the same symbol to represent different physical quantities). In addition, the likelihood for inconsistent or seemingly senseless decisions increases if the numeric quantities involve decimals and fractions instead of whole numbers. This is a weakness with *quantification* <sup>(13, 14)</sup>.

- Students struggle with the various uses of the equals sign, zero, positive and negative sign, constants and variables. They tend to adhere to a single definition of these symbols, even at the college level, and commonly do not recognize the context dependence of their meaning. (e.g. thinking that accelerating in the negative direction means slowing down even if the motion is in the negative direction, assuming zero net force means no force) This is a weakness with *flexible symbolizing* <sup>(15-19)</sup>.

In the next section we expand on the descriptions of these obstacles based on prior research in mathematics and physics education. In Section II we describe the study we have undertaken to better understand some of the obstacles to developing mathematization in the context of physics. In Section III we present the findings of our study, which provide evidence of the pervasiveness of the obstacles listed above. We follow in Section IV with a discussion of the results of our study, where we present our position that these obstacles all stem from the same problem. And finally, we end the paper with a discussion in Section IV of the implications this study has for helping students to better develop the habits of mind common to physicists.

## II. Background

This study has been guided by two questions for research:

1. What specific difficulties with algebraic reasoning do students encounter in introductory physics courses?
2. What factors associated with reasoning about physical quantities impact student ability to reason successfully?

In order to address the first question, we establish target reasoning goals, or physics “habits of mind”, that characterize the way that mathematizing happens in the context of physics. We then explore the three main categories of obstacles along the path of development of mathematization: structural reasoning, quantification and symbolizing.

### II.a. Physics habits of mind

In this section, we summarize our work (Table II) from a related paper <sup>(9)</sup> that provides a preliminary account of the ways that physicists use mathematics that is foreign to many of their students – even those who have been consistently successful in mathematics courses. This list builds on previous discussions of these differences by Redish <sup>(20, 21)</sup> and Tuminaro <sup>(11)</sup>, and is not intended to be exhaustive. We then devote the remainder of this section discussing findings from the literature that inform our effort to better understand the preparation of our students to mathematize in physics.

**Table II:** Physics habits of mind (from Kanim, Brahmia and Boudreaux<sup>(9)</sup>)

<i>Habit</i>	<i>Brief Description</i>
Connecting mathematics to physical phenomena	Physicists use mathematical models to predict and explain the behavior of physical objects and interactions with the help of invariant relationships, and in turn they use what they know about the objects and interactions to generate mathematical equations.
Ability to distinguish variables, constants, and units and distinguish between their various meanings	Physicists are adept at manipulating equations that contain multiple variables and constants of many forms. Their distinction between variable and constant is fluid, and will depend on what they are trying to do.

Use of context to interpret symbols	Physicists use symbols in ways that must be interpreted based on context, and we are often not explicit that the symbols mean different things.
Consistency between functional forms, physical changes and graphical representations	Physicists reason about covariation of variables based on the form of the equation. Associated with this skill is a fluency between reasoning about equations and about their graphical representation.
Use of units and dimensions	Units are central to how physicists think about quantities, and physicists use units to reason about how quantities are related to one another. Inclusion of units serves multiple purposes for physicists as they seek to connect physical situations and processes with their mathematical representations, sometimes reasoning based on units alone.
Use of grouping of symbols and quantities to make meaning	Appropriate grouping of symbols, values, and numbers helps physicists to connect the mathematics they use to the physical situations. They know when to combine two mathematical entities into one and when to keep them separate. (e.g. although “ $ma$ ” has the units of force when grouped, it’s not thought of as a force, but as always as mass $\times$ acceleration)

## **II.b. Generalized structural reasoning**

Generalized structural reasoning involves recognizing a mathematical structure (e.g. a  $1/r^2$  force) and using it to guide thinking, regardless of the context. While common practice for physicists, this kind of generalized reasoning is novel for students even though they may have

solved many inverse square problems using Newton's Universal Law of Gravitation and Coulomb's law. Rebello, Cui, Bennett, Zollman and Ozimek <sup>(22)</sup> describe students' capacity to generalize reasoning and methods that they learned in trigonometry and calculus to physics contexts in end-of-chapter textbook problems as part of their description of horizontal transfer, or the mapping of new information onto an already existing knowledge structure. The horizontal/vertical metaphor was introduced by Treffers and Vonk <sup>(2)</sup> where they describe horizontal mathematizing (leaving the symbolic world) and vertical mathematizing (within the symbolic world); horizontal mathematizing most closely matches the process in physics.

According to Rebello *et al.* horizontal transfer is analogous to the Sherin notion of flexibility. In a study that focused on measuring horizontal transfer between prerequisite math courses and subsequent physics courses, Rebello *et al.* <sup>(22)</sup> interviewed calculus-based physics students at Kansas State University, asking them to solve physics problems that involved simple integration or differentiation similar to ones they had already solved in homework. While the students were able to procedurally do the calculus required in a physics problem when it was prompted (as it typically is in many end-of-chapter problems) they were largely unsuccessful in the interviews at setting up and solving problems that involved mathematical decision-making that involved calculus. The researchers also surveyed algebra-based students before and after instruction using both a trigonometry survey and a set of trigonometry-based physics problems, and found little evidence of horizontal transfer of trigonometry from math to physics for these students. The authors report "...we had assumed that the problems would involve horizontal transfer and therefore be perceived as relatively straightforward by the students. It appears however that this was not the case with most students." Schoenfield <sup>(23)</sup> describes math students' *belief systems* (*i.e.* expectations) as rigid due to their strong belief in the context specificity of problem-solving approaches (*e.g.*, they use deductive argumentation in geometry proof problems but do not use it in any other contexts.) We propose that the lack of horizontal transfer observed by Rebello *et al.*

may be due to rigidity in students' belief systems exemplified in mathematics. It is not part of their beliefs about learning math that they would spontaneously use mathematical reasoning unprompted in math class, and certainly not outside of math class. We do not see evidence of a robust and generalizable mathematical structure for algebra (let alone calculus) for most students in introductory courses, and thereby avoid Rebello's use of the term "transfer." We prefer to think in terms of Hammer's resource framework, and Wittman's coordinated set of resources. Rebello's students are not spontaneously activating calculus or trigonometry resources in the physics context unless they are prompted to do so.

Bassok & Holyoak <sup>(24)</sup> explored generalized reasoning of two structurally analogous topics, the kinematic equations for constant acceleration motion taught in a physics course and arithmetic progressions taught in the high school Algebra II course<sup>1</sup>. In each experiment they taught two groups that were enrolled in either physics or Algebra II– the algebra students learned the math topic first and then solved physics problems, and the physics students learned the physics topic first and then were asked to solve arithmetic progression problems related to topics outside of physics. The authors observed an asymmetry in the ability of students to subsequently solve structurally isomorphic problems in the other context. Specifically, the students who learned about progressions in the physics context of constant acceleration motion were much less effective at spontaneously recognizing the overarching structural reasoning about mathematical progressions in other contexts. Physics instruction prepared the students to solve a class of problems, but it did not prepare them to use their knowledge in a flexible way.

Sherin <sup>(3)</sup> reports also on the opacity of the underlying mathematical structure in the kinematics equations, even for students who are quite strong in physics. All of the students in his study failed to recognize that the arithmetic progression that they very readily invoked to

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<sup>1</sup> In the US this course precedes a precalculus course and involves extensive coverage of logarithms, complex numbers, high order polynomials and matrices in addition to review of topics taught in Algebra 1.

describe the mass of a sandpile as a function of time with sand falling onto it at a steady rate was exactly the same reasoning they used to describe  $v(t)$  in constant acceleration motion. The students were perplexed as to why he would ask them such a simple question about sand, yet could not figure out why the “correction” to  $v_o$  should be  $at$ .

Each of these examples describes a reasoning structure that experts readily activate in a variety of contexts. We think of the reasoning structure as a coordinated set of resources, that calls on a robust set of individual resources to be readily accessible in order to activate. In the case of Sherin’s (very strong) physics students, activation of their coordinated arithmetic progression set was perhaps context dependent, and sand was a context that activated the reasoning. In the absence of activities that challenge the students to broaden their coordinated set to include physics quantities, they did not generalize the reasoning.

### **II.c. Quantification**

Physical quantities are the building blocks of physics, and in our introductory course students encounter  $\sim 10^2$  new quantities. Physical quantities involve both a number and a unit, which are often made up of other units, and many have a direction associated with them. In order to begin to quantify efficiently in physics, students must have mastered numbers of all kinds (including positive and negative numbers), the use of units (regardless how abstract they seem), and have a conceptual understanding of the arithmetic involved in combining quantities. As they become more adept, they must also master vector arithmetic in order to combine quantities.

Mastery of number may seem trivial and well out of the domain of college physics. But *mastering number in the context of physical quantities* is not the same as mastering number in math class, where units are rarely involved. In physics, the numeric value of a physical quantity carries a deep meaning that is strongly associated with its unit, which differs from math and is also very different from the use of numbers in everyday life. Most people care about the actual value of numbers in very few contexts (e.g. money, time, distance); outside of these contexts

numbers are not used specifically for the meaning of their value. They are commonly used to define thresholds (a blood pressure of 120/80 means don't worry) or to rank or compare (my amp goes up to 11), where the meaning of the numbers is irrelevant beyond ordering. For a physicist, a steady speed of 25 mph means that you will travel 25 miles in one hour, while for most students it is just slower than they want to drive.

Researchers in mathematics education have identified *quantification* as a significant challenge to students who are learning to mathematize. Thompson <sup>(13)</sup>, who has researched and written extensively on this topic over the past two decades, defines quantification to be “the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship ... with its unit.” He considers quantification to be “a root of mathematical thinking”, and argues that learners develop their mathematics from reasoning about quantities.

The complexity of the numeric values can produce a cognitive strain on students as they learn new physics. Decimals and fractions are much more common than whole numbers in the physical science context of measurement, while whole numbers are more commonly used when students learn algebra. Similarly, large and small numbers, which are ubiquitous in physics, pose even greater difficulties and are not frequently used in algebra courses.

Quantities in physics include compound quantities that result from multiplying and dividing other quantities (e.g. momentum, kinetic energy.) Procedurally the arithmetic involved in creating new quantities is not a challenge for most students. Yet deciding when and why the arithmetic makes sense poses a very big challenge <sup>(13)</sup>. Some physical quantities are ratio or product quantities, which require the ability to conceptualize multiplication and division in order to understand what they represent. Some are conserved quantities, which require a conceptual understanding of summation. Many are rates of change, which require a conceptual understanding both of differences, and ratios. Some, such as flux or electric field, combine already very poorly understood quantities (electric force, charge, surface area.)

Vergnaud <sup>(25)</sup> makes a strong case that multiplication, division, fraction, ratios, proportions, linear functions, dimensional analysis and vector space are not mathematically independent, and should be included in the domain he named multiplicative structures. Students have difficulties conceptualizing arithmetic and the simplest multiplicative structures in physics <sup>(11)</sup>. The operations of addition, subtraction, multiplication and division have clear rules that apply when dealing with physical quantities, but that don't apply when dealing with the pure numbers of math class. When reasoning about quantities, physicists recognize that addition and subtraction can be carried out only with like quantities expressed in like units, and that multiplying and dividing quantities creates something completely different from either of the multiplied quantities. Students are not taught to reason about multiplicative structures with physical quantities in math class, but are expected to know how to reason this way in physics.

Students can progress through the mathematics curriculum up to precalculus using the (usually effective) concept of multiplication as repeated addition, and they have had little incentive to conceptually understand the product of quantities. Similarly, if students think of division as an operation that creates parts from a whole, then they are ill prepared to conceptualize the ratio quantities that are ubiquitous in the physical sciences (density, velocity, acceleration, heat capacity, etc.). Understanding rates of change, intensive properties of a system and flux all depend on a strong conceptual understanding of division, and all are foundational ideas in introductory physics.

#### **II.d. Symbolizing**

Mathematics education researchers have been grappling with learning theories related to symbolizing and communicating for decades. The context dependence of symbol use in physics is nuanced, and often not part of students' mathematics preparation. In this section we explore the roles some fundamental symbols play in the physicists development of concepts, and provide



evidence for some ways that students struggle. We focus on the symbols that make up the equations used in physics: signed numbers, zero, equals sign, and variables.

Students first encounter signed numbers as a straightforward extension of whole numbers in physical science typically at the middle school level or early high school, and usually within a year of learning about them in math class. Physics instructors generally expect that students who have encountered integers in math class should understand the various meanings signed numbers can have in science. The instructor may not appreciate how much of a struggle it is for students to reason with negative numbers in such varied contexts. For example, “-5” units of charge packs into its meaning “five units of one type of charge that is opposite to the only other type of charge.” The naming is arbitrarily related to the concepts of positive and negative numbers i.e., mathematical objects that quantify opposites. This use is very different from “-5” degrees Celsius, which quantifies relative to an arbitrarily determined zero value. Similarly, new meanings crop up with negative velocity, negative work and so on. Physics contexts are rich with information packed into the symbols of signed numbers and zero. The critical concept of summing up integers to determine a net effect (e.g. net charge, net force) often escapes students who are still developing flexibility with the procedures of adding integers. For students learning in math class that zero represents absence, a zero net force, or zero net charge is commonly misinterpreted as no force or no charge.

A confounding feature of integer use in physics is that the operations of addition and subtraction (represented by the symbols “+” and “-“) can easily be confused with the descriptors, positive and negative, that can characterize the opposite natures of some physical quantities (position, charge, velocity, etc.) For example, a representation of the statement that five units of negative charge are taken away from an electrically neutral object is written as  $0 - (-5 \text{ charge units}) = +5 \text{ charge units}$ . Here zero represents a balance of opposites, -5 represents the kind of charge, the minus sign represents the action of removing, and +5 represents the amount and type of net charge after the event. When we use a minus sign it sometimes means “subtract” or “same

number, opposite sign” but can also mean “in the other direction” or “same as the charge of an electron.” Experts habitually use context clues and keep mental track of negative signs.

Sherin <sup>(3)</sup> refers to the conceptual use of integers in physics as a symbolic form “competing terms cluster”, which includes the notion of zero to represent balance, and positive and negative quantities as competing terms in an expression. This cluster is dependent on a stable coordinate set of resources that includes a conceptual understanding of signed numbers and zero. He observed that flexibility with this symbolic form is a feature of expert problem solving in introductory physics.

Zero carries novel meanings in physics when it represents balance (*e.g.*, net force, net charge), a reference point (*e.g.* freezing temperature of water, potential energy at infinite separation), in addition to its commonly understood meaning of absence (*e.g.* zero absolute pressure.) These other uses in physics require inferences that are often missed by students who are used to math instruction where zero represents the absence of everything, and integers are learned in the context of positions on a number line.

Developing flexibility with negative numbers is a known challenge in math education. Vlassis <sup>(26)</sup> used written diagnostic questions as well as interviews to investigate the understanding of negative numbers by Belgian students taking algebra. She found that in order for students to fully understand the concept of a negative number, they had to develop a flexibility with the various ways in which negative numbers are used in context. The most challenging context is the way we often use them in physics – to quantify opposites.

Even for students in calculus-based physics, negative quantities pose challenges. Bajracharya, Wemyss, and Thompson <sup>(27)</sup> investigated student understanding of integration in the context of  $P - V$  diagrams in introductory physics. Their results suggest an incomplete understanding of the criteria that determine the sign of a definite integral. Students struggle with the concept of a negative area, and with the concept of positive and negative directions of integration.

In addition to signed numbers, students do not necessarily understand the equals sign in the way we expect. High school and college students commonly use the equals sign inappropriately as they solve equations or evaluate expressions in algebra and calculus<sup>(28, 29)</sup>. Students commonly understand the equals sign to be a prompt for calculation (e.g.,  $v = 3 \text{ m/s} \times 5 \text{ s}$ ), which is one possible use: Far fewer understand it to be the relational symbol of mathematical equivalence (e.g.,  $(F_{\text{a on b}} + F_{\text{c on b}})/m_b = a_b$ ). It is likely that students mostly interpret statements with an equals sign as a prompt to calculate.

In a study that looked closely at student use of variables to represent an English statement, Cohen and Kanim<sup>(30)</sup> revisited an older study of the “reversal error” by Clement, *et al.*<sup>(29)</sup>. The original study posed the “students-and-professors” problem to explore students’ difficulties converting a sentence into a mathematical expression. Students were asked to write an equation using  $S$  for the number of students and  $P$  for the number of professors to represent the statement, “There are six times as many students as professors at this university.” Clement *et al.* found that students taking calculus-based introductory physics found this task challenging, and commonly put the 6 on the wrong side of the equation. Cohen and Kanim<sup>(30)</sup> explored this “reversal error” in depth by changing sentence structure and the choice of symbols and found consistently that roughly half of the calculus-based physics students at their institution got this question incorrect.

The equations typically encountered in the algebra questions posed in the research cited here are not nearly as symbol-varied and symbol-rich as the equations we use in physics. In a study at University of Illinois, researchers<sup>(31)</sup> gave students identical problems on their final exams in the introductory physics course for engineers; one set had only numeric values and its partner had only symbolic values. The authors report on a differential in scores of up to 65% on these paired questions. They observed that the ability for students to solve exam questions that relied on accurate manipulation of symbolic representations correlated to course grades, and the strongest correlation was for the bottom quartile of the students. Students who manipulate and reason poorly in the face of symbolic quantities are much more likely to do poorly in an introductory

physics course. They lack the flexibility and generativity necessary to succeed, and likely do not understand what their instructors are communicating.

### **III. Methods and Materials**

#### **III.a. Question development**

We have developed a large number of written questions to gauge student facility with proportional reasoning, as described in detail in Boudreaux, Kanim and Brahmia<sup>(32)</sup>. Some of these questions were drawn directly from mathematics and physics research literature, while others were developed as part of the current investigation. Most of the questions were designed to focus on a single reasoning sub-skill of proportional reasoning. The items assess reasoning rather than computational skill; in all cases neither a calculator nor multistep problem solving are required.

The items included in this study underwent repeated cycles of validation and modification over a 3-year period. They are presented in this paper in their final form. Initially, all items asked students to explain their reasoning and show their work. We collected written responses from over 1000 students enrolled in a variety of courses (general education physics, introductory calculus-based and algebra-based physics, and physics for pre-service elementary teachers) taught at Rutgers University, NMSU (New Mexico State University), and WWU (Western Washington University). We created scoring rubrics based on student responses. Three coders initially scored tests to establish inter-rater reliability before we analyzed the data. We then created multiple-choice versions of the questions, with distractors based on the student difficulties identified during the analysis of written work and interviews. We established question validity through interviews conducted with individual students at WWU. Results from the interviews and from the analysis of written responses were used to guide modifications to improve not only the clarity but also the effectiveness of the distractors in characterizing student reasoning.

The interviews were conducted at WWU with student volunteers from calculus-based introductory physics courses, general education physics courses, and a physics course for pre-service elementary teachers. We conducted over 20 interviews with each interview lasting about one hour. The interviews were videotaped for later transcription and analysis. We used a semi-structured protocol. The interviewer posed specific proportional reasoning questions and asked the interview subject to “think out loud.” The interviewer clarified the question as needed, prompted the subject to explain his or her thinking after periods of sustained silence, and asked the subject to elaborate on statements that were brief or unclear. The interviewer did *not* offer hints or guiding questions. At the close of some of the interviews, the interviewer asked the subject to reflect on how difficult he or she felt answering each item had been, and why.

Throughout the development of assessment items, we have observed that variation in student reasoning cannot be explained solely through the lens of the proportional reasoning sub-skills that we had developed <sup>(32)</sup>. Student performance seems sensitive to the physical context, the level of abstraction of the ratio or product quantity, and the numerical complexity of the quantities involved. This basic finding, consistent with previous studies of student reasoning about ratio <sup>(33-35)</sup> led us to develop multiple versions of many of the assessment questions. These versions differ in only a single surface feature, in an attempt to isolate causes of variation in student reasoning. We report on our findings in the next section of this paper as examples of obstacles to mathematization encountered by students in introductory physics.

### **III.b. Large-scale study**

The results presented in this paper derive from a large-scale study involving the freshman and sophomore non-honors engineering students at a Rutgers University (N = 2,115 pretests and 1,784 post tests administered in all). The students in this study were comparatively well-prepared mathematically; their mean mathematics SAT (2011/2012 test version) score was 680.

We administered a suite of multiple-choice proportional reasoning items as an ungraded quiz under exam conditions during the first week of the introductory, calculus-based physics courses and the general chemistry course in fall 2013. The testing conditions were the same in the physics courses for the post-test, which was administered 10 days before the end of the semester just before Thanksgiving break. There was a sizeable drop in test taking on the post test in Chemistry due to the fact that the test was administered online outside of class. (see Table III).

**Table III:** Assessment administration Fall 2013 Rutgers University (engineering students)

<i>Class</i>	<i>Subject</i>	<i>Items used in this study</i>	<i>PreTest</i>	<i>PostTest</i>	<i>#versions</i>
Freshmen	Mechanics	Rice Squareness Heffalumps and Woozles	Supervised  In class	Supervised  In class	8
Freshman	General Chemistry	Traxolene Olive oil Force Vector	Supervised  In class	Unsupervised  Online	6
Sophomore	Electricity & Magnetism	Mass Density Charge Density	Supervised  In class	Supervised  In class	6

The suite contained 19 items in all distributed on three tests in three different subjects: seven items were administered on the test for mechanics students ( $n_{\text{pre}}=759$  and  $n_{\text{post}}=769$ ), seven on the test for chemistry students ( $n_{\text{pre}}=692$  and  $n_{\text{post}}=494$ ), and five on the test for the E & M students ( $n_{\text{pre}}=664$  and  $n_{\text{post}}=521$ ). 613 students took both the mechanics and the chemistry course simultaneously. The proportional reasoning items in each case were bundled with a standardized concept inventory (the Force Concept Inventory <sup>(36)</sup> in the freshman physics course, the Conceptual Survey of Electricity and Magnetism <sup>(37)</sup> in the sophomore physics course, and the Chemical Concepts Inventory <sup>(38)</sup> in the chemistry course). In a single sitting, students first completed the proportional reasoning suite, followed immediately by the concept inventory. The students were not constrained by time and were awarded credit for participation. The students

took an identical posttest, administered under the same conditions (with the exception of Chemistry, see Table III), at the end of the course.

In order to test the effect of surface features on student reasoning, we administered matched versions of selected proportional reasoning items on different versions the suite (see Table III.) The suite was administered in the recitation section of the course, and within a given recitation the versions were assigned randomly. We administered the identical versions of the suite on the pretest and the posttest to each student in the study.

We report on the percentage of students who selected the correct answer, the standard error of the binomial distribution, and the percentage of students who selected each of the incorrect answers. We use the pooled standard error to compare student performance on two different questions, and in comparing pre to post test scores.

## **IV. Findings**

### **IV.a. Generalized structural reasoning**

To illustrate student difficulty with generalizing structural reasoning from our data, we present results from four paired, structurally isomorphic questions in our study (see Fig. 1 and Fig. 2). Because ratio reasoning is ubiquitous in the physics classroom, one might expect that students would spontaneously create a ratio as a way of thinking productively about comparisons. To test this, we created the pair of structurally isomorphic questions shown in Fig. 1. While both questions require the formation of a ratio representing a unit rate, one unit rate is familiar (a density) and can be obtained using a known equation, while the other is in a familiar context but requires students to generate a unit rate. The two question versions were randomly assigned to students at the end of the first semester of the freshman General Chemistry for Engineers course. In the context for which students could invoke the density equation 65% of 188 students

answered correctly; for the unit rate olive oil question with no ready formula available 56% of 161 students answered correctly (effect size<sup>2</sup> 2.4).

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**Figure 1:** Structurally isomorphic diagnostic questions structural generalizability of the unit rate,  $\sigma_{\text{pooled}}=3.7\%$  (freshmen engineering students, administered in the chemistry course, post test)

a) Traxolene Question: Chemistry Context - well known formula (N=188,  $\sigma=3.5\%$ )

You are part of a team that has invented a new, high-tech material called “traxolene.” One gram of traxolene has a volume of 0.41 cubic centimeters. For a laboratory experiment, you are working with a piece of traxolene that has a volume of 3 cubic centimeters. Which of the following expressions helps figure out the mass of this piece of traxolene (in grams)?

- a.  $\frac{3}{0.41}$       **(65% correct)**    b.  $\frac{0.41}{3}$     c.  $3 \cdot 0.41$     d.  $(3+1) \cdot 0.41$     e. none of these

b) Olive Oil Question: Everyday Context – no formula (N=161,  $\sigma=3.9\%$ )

You go to the farmer’s market to buy olive oil. When you arrive you realize that you have only one dollar in your pocket. The clerk sells you 0.26 pints of olive oil for one dollar. You plan next week to buy 3 pints of olive oil. Which of the following expressions helps figure out how much this will cost (in dollars)?

- a.  $\frac{3}{0.26}$       **(56% correct)**    b.  $\frac{0.26}{3}$     c.  $3 \cdot 0.26$     d.  $(3 + 1) \cdot 0.26$     e. none of these
- 

In the second set of paired questions, students were expected to spontaneously construct a ratio of the sides of a square (see Fig. 2). We present data for a matched set of students who were enrolled both in the chemistry course and in the physics course. One question is asked in the context of the components of a building rooftop, and the paired question is asked in the context of a force vector. When the force vector question was asked in an open response format in prior years some students answered by resorting to inverse trigonometric functions and attempting to grind through the algebra. We did not see analogous responses on the open response version of the building rooftop question. Our results on the paired multiple-choice questions shown in Fig. 2 indicate that students are more likely to respond with this right-triangle thinking if asked in

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<sup>2</sup> effect size = (%correct in question a - %correct in question b)/pooled standard deviation



a physics context. We report on the post-test and note that even after a semester of physics, they are not even very efficient with the brute force trigonometric approach. In the context for which they felt prompted to invoke trigonometry the students scored 22% after one semester of mechanics, and scored 17% in the context for which they have memorized no formula (effect size 2.6).

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**Figure 2:** Structurally isomorphic diagnostic questions testing ratio formation; the correct answer is choice (a) in both cases;  $\sigma_{\text{pooled}}=1.9\%$   $n_{\text{matched}}=408$

a) Square Buildings Question (everyday context) (freshmen engineering students, administered in the mechanics course, post test)

You are riding in an airplane. Below you see three rectangular buildings with the rooftop dimensions:

*Building A: 77 ft by 93 ft*  
*Building B: 51 ft by 64 ft*  
*Building C: 96 ft by 150 ft*

You are interested in how close the shapes of the rooftops of the buildings are to being square. You decide to rank them by “squareness,” from *most* square to *least* square. Which of the following choices is the best ranking?

a. A, B, C (**17% correct**)   b. B, A, C (**72%**)   c. C, A, B (**4%**)   d. C, B, A (**3%**)   e. B, C, A (**4%**)

b) Force Vector Question (physics context)

(freshmen engineering students, administered in the chemistry course, post test)

Each of three different objects (A, B, C) experience two forces, one in the +x direction and one in the +y direction. Rank each object according to how close the direction of the net force is to a 45° angle between the x-direction and the y-dir, from *closest* to 45° to *farthest* from 45°.

Object	Force in +x-direction	Force in +y-direction
A	77 Newtons	93 Newtons
B	51 Newtons	64 Newtons
C	96 Newtons	150 Newtons

a. A, B, C (**22% correct**)   b. B, A, C (**51%**)   c. C, A, B (**20%**)   d. C, B, A (**5%**)   e. B, C, A (**2%**)

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Student performance on these questions suggests that this pair of questions is significantly more challenging than the previous pair due to the absence of a readily accessible algorithm; most students selected the incorrect answer (b), which is found by minimizing the difference of the side lengths rather than forming a ratio.

#### IV.b. Quantification

We administered a set of isomorphically paired questions that probed the effect of number complexity. In Figure 3 we compare two versions of a pre/post test administered to freshmen engineering students in their first semester of study. The questions were identical in every way except for the complexity of the number. Figure 3 shows the question, and the results broken down by version. Although they are engineering students, the students struggle more when reasoning about decimal quantities than they do about whole numbers. Comparing the versions on the pretest, we see a significant difference in scores with an effect size of 6.7. The gap remains even after instruction, with an effect size of 3.7.

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**Figure 3:** Structurally isomorphic diagnostic questions testing effect of numeric complexity, before and after one semester of analytical physics,  $n=268 \pm 5$  for all cases;  $\sigma_{\text{pooled}}=2.7\%$  (freshmen engineering students, administered in mechanics course). The quantities in square brackets represent the quantities used in version 2. (There was no statistical difference between student performance using decimals and a third version using fractions.)

Bartholomew is making rice pudding using his grandmother's recipe. For three servings of pudding the ingredients include 4 [0.75] pints of milk and 2 [0.5] cups of rice. Bartholomew looks in his refrigerator and sees he has one pint of milk. Given that he wants to use all of the milk, which of the following expressions will help Bartholomew figure out how many cups of rice he should use?

- a)  $2/4$  [0.5/0.75]
- b)  $4/2$  [0.75/0.5]
- c)  $4 \times 2$  [0.5 x 0.75]
- d)  $(2 + 1) \times 4$  [(0.5 + 1) x 0.75]
- e) none of these

Version	Pre(%)	Post(%)
Whole numbers	79	77
Decimals	61	67

---

In addition to numerical complexity, the abstractness of the physical quantities can interfere with students' reasoning. The paired questions shown in Fig. 4 were administered to sophomore engineering students; most had already completed the freshman course General Chemistry for Engineers and have been using the density equation since middle school. The mass-density

question can be answered using the above equation, and in prior years when prompted to explain their reasoning many students used it, or evoked factual evidence the “the density is constant because it is an intrinsic quantity.” It was striking how many students used this sentence (or one nearly identical), which figures prominently in most chemistry and physical science textbooks. All of the students had just completed a traditionally taught course in calculus-based introductory electricity and magnetism, in which charge density is a fundamental quantity. In the context for which they could invoke the density equation as memorized they scored 80% after one semester of electromagnetism, and scored 55% in the context for which they have no memorized formula (effect size 9.3). These results are consistent with previous work by Kanim<sup>(39)</sup>.

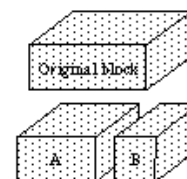
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**Figure 4:** Structurally isomorphic diagnostic paired questions (from Kanim<sup>(39)</sup>) testing generalizability of density,  $\sigma_{\text{pooled}}=2.7\%$ . (sophomore engineering students, post test)

a) Physical Science Context - well known formula (N=262,  $\sigma=2.2\%$ )

A uniform block of cheese is cut into two unequal pieces, labeled A and B. A correct ranking of the mass densities (from largest to smallest) of the original block, the largest piece (A), and the smallest piece (B) is:

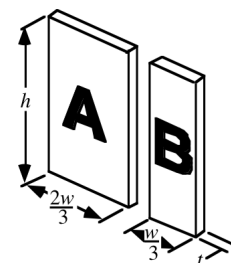
- (a) Original block, largest piece, smallest piece.
- (b) Smallest piece, largest piece, original block.
- (c) Largest piece, smallest piece, original block
- (d) All mass densities are the same. **(80% correct)**
- (e) Not possible to determine without additional information.



b) Physics Context – less commonly known formula (N=258,  $\sigma=3.1\%$ )

A plastic block of width  $w$ , height  $h$ , and thickness  $t$  has a positive charge  $+Q_0$  distributed uniformly throughout its volume. The block is then broken into two pieces, A and B as shown. Which is a correct ranking of the charge densities of the original block ( $\rho_0$ ), piece A ( $\rho_A$ ), and piece B ( $\rho_B$ )?

- (a)  $\rho_0 > \rho_A > \rho_B$
- (b)  $\rho_0 = \rho_A = \rho_B$  **(55% correct)**
- (c)  $\rho_0 < \rho_A < \rho_B$
- (d)  $\rho_A > \rho_B > \rho_0$
- (e) There is not enough information to compare the charge densities.



#### IV.c. Symbolizing

Physics students also struggle with variable as a general quantity and in a functional relationship. As part of our study, a version of the rice question was asked replacing one of the decimal values with a variable. The results are shown in Fig. 5(a). Even after a semester of rigorous problem solving in physics, students do not tend to improve their ability to reason about variables as a general quantity. In fact, they get worse (effect size of the variable version, pre to post=3.4).

**Figure 5:** Questions probing numeric complexity

a) RICE QUESTIONS: Structurally isomorphic diagnostic questions testing effect of numeric complexity, before and after a semester of analytical physics,  $\sigma_{\text{pooled}}=2.9\%$  (freshmen engineering students, administered in mechanics course). The quantities in square brackets represent the quantities used in version 2 and 3, respectively. There was no statistical difference between student performance using decimals, and a fourth version using fractions. The number of students who took each test is shown in parentheses in the table below.

Bartholomew is making rice pudding using his grandmother's recipe. For three servings of pudding the ingredients include 4 [0.75,  $N$ ] pints of milk and 2 [0.5,  $5/8$ ] cups of rice. Bartholomew looks in his refrigerator and sees he has one pint of milk. Given that he wants to use all of the milk, which of the following expressions will help Bartholomew figure out how many cups of rice he should use?

- a)  $2/4$  [0.5/0.75,  $(5/8)/N$ ]  
 b)  $4/2$  [0.75/0.5,  $N/(5/8)$ ]  
 c)  $4 \times 2$  [0.5  $\times$  0.75,  $N \times (5/8)$ ]  
 d)  $(2 + 1) \times 4$  [(0.5 + 1)  $\times$  0.75,  $((5/8)+1) \times N$ ]  
 e) none of these

Version	Pre(%)	Pos(%)
Whole numbers	79 <sup>(273)</sup>	77 <sup>(262)</sup>
Decimals	61 <sup>(264)</sup>	67 <sup>(270)</sup>
One variable	44 <sup>(33)</sup>	34 <sup>(227)</sup>

#### b) HEFFALUMPS AND WOZZLES:

Consider the following statement about Winnie the Pooh's dream: "*There are three times as many heffalumps as woozles.*"

Some students were asked to write an equation to represent this statement, using  $h$  for the number of heffalumps and  $w$  for the number of woozles. Which of the following is correct?

- a.  $3h/w$  (3%) b.  $3h = w$  (36%) c.  $3h + w$  (2%) d.  $h = 3w$  (49% correct) e. a & b (9%)

In our study of calculus-based physics students we asked one of the Cohen and Kanim questions (see Fig. 5(b)). Generating a functional relationship between two variables is clearly a challenge even for the students in the most mathematical courses of introductory physics.

## V. Discussion of the findings

In the context of structural reasoning, there is a difference between the reasoning associated with the density of the Traxolene and the olive oil unit price (Fig. 1). Since there is a fairly large effect size, we conclude that students find it more challenging to use ratio reasoning in a context that is novel, like purchasing olive oil at a market. We also note that neither context was particularly simple (over 1/3 of the students were incorrect); this indicates that spontaneously forming a ratio is a challenge, even for engineering students. The difficulty that the students have with the Square Buildings and Force Vector (Fig. 2) questions demonstrate just how challenging it is for students to spontaneously form a ratio as a tool for comparison. Over half of the students in both contexts selected the answer that corresponds to taking a difference rather than a ratio. While for physicists ratios are usually the ‘go to’ tool for making comparisons, students have not had practice with deciding whether to use ratios or differences. Instead, most of their mathematics instruction has been application of given ratios. We note that these tasks especially, but also the tasks in Fig. 1, require generativity. These results are fairly clean measures of the struggle that students go through to develop the generativity that characterizes expertise.

The quantification results demonstrate to what extent student reasoning is sensitive to surface features of the quantities involved. In the case of the rice question (Fig. 3), we get significantly different results by changing just the complexity of the numeric value of the quantity. Students apparently are very strongly distracted in their reasoning if values are decimal. Similarly, tasks that involve more abstract quantities (like charge density) seem to inhibit the triggering of the coordinated set of resources that enable effective reasoning in less abstract contexts (like mass

density) (Fig 4). We note that students appear to lack the flexibility in their ratio reasoning to be successful in all contexts and with all types of quantity.

In the case of symbolizing we see two results with important consequences (Fig 5.) The first is that the use of even just one letter in the place of numeric values, i.e. a generalized variable, involves even more suppression of reasoning than the decimal value does. Consistent with other published results on our version of the students and professors problem, the Heffulumps and Woozles problem results show less than half of the class getting this question correct (with the most common incorrect answer corresponding to making the reversal error.) Given these results, and the strength of the signal, it is clear that students in introductory physics struggle to understand variables in the varied ways we use them.

## **VI. Implications for instruction**

Many students who might otherwise pursue a path towards expertise in physics get lost because instructors underestimate the mathematical challenges that are presented in the introductory course. We have reported on just a few of them in this report; there are certainly others. Instructors unintentionally create barriers to success for most students in the introductory course because they unwittingly use highly coordinated sets of resources to make mathematical sense of physics without attending to teaching that process explicitly to their students. Tuminaro<sup>(11)</sup> states that “There is an ontological discord between the mathematics taught in introductory, college-level math courses and introductory, college-level physics courses. By an ontological discord, I simply mean that the mathematical objects used in introductory, college-level physics courses are often more complex than the mathematical objects used in introductory, college-level math courses.”

We began Section II of this paper cataloging habits of mind that are foundational for expert physicists. We also categorized the obstacles commonly encountered by introductory students.

Table IV below describes how those relate to one another. Although the habits of mind were generated through observation and reflection, and the students' difficulties are empirical, it is not surprising that there is significant overlap. Articulating and reflecting on these habits and difficulties provides a framework for discussion; it brings us closer to understanding how to teach such that all students learn them as a result of a physics course. While we are still very far from being able to teach these skills to all students who take physics, we believe that the explicit focus we have given them thus far has been very productive. There appears to be an increased interest from the physics education research community in the past few years on issues of students' mathematical reasoning as well as an uptick in communication between physics education researchers and mathematics education researchers. We hope that this increased interest leads to a more explicit understanding of how we use mathematics in physics, of how this use is different from what students have experienced, and of curricula that bridges this gap.

**Table IV:** Expert habits of mind and observed areas of student difficulty

<b>Physics Habit of Mind</b>	<i>Connecting mathematics to physical phenomena</i>	<i>Ability to distinguish variables, constants, and units and distinguish between their various meanings</i>	<i>Use of context to interpret symbols</i>	<i>Consistency between functional forms, physical changes and graphical representations</i>	<i>Use of units and dimension</i>	<i>Use of grouping of symbols and quantities to make meaning</i>
<b>Area of Student Weakness</b>	Structural reasoning, Quantification Symbolizing	Quantification, Symbolizing	Symbolizing	Structural reasoning	Quantific- ation	Symbolizing

## **VII. Conclusion**

We address a problem that is becoming increasingly clear as more researchers turn their focus to mathematical reasoning in physics. We adopt a resources framework and describe the obstacles that students' encounter as they progress through an introductory physics course. We claim that developing flexibility and generativity in physics requires activation of coordinated sets of resources. Many of the resources are in place for many of our students, but the activation and the coordination appears to be fragile and dependent on surface features of the assessments. We have highlighted three broad areas of weakness, and supported our claims based on results from large scale testing of engineering students.

We conjecture that while structural reasoning, quantification and symbolizing are different aspects of mathematization in introductory physics, weakness in any combination of them impedes students' capacity to mathematize effectively. Each is a symptom of a bigger problem; specifically, that students are not taught to mathematize in the context of physics, even though reasoning in physics is based on a conceptual understanding of the mathematics that is used. It is commonly assumed that students who take a course as a prerequisite automatically conceptualize the subject matter of that course. We challenge this assumption and claim that the consequence of not making this assumption is that physics instruction will devise its own ways of fostering mathematization to move in a more expert-like direction as a result of a well-structured course.

In a subsequent paper we develop a theoretical instructional model for developing mathematization in physics instruction, and use this model to explain the areas of weakness reported on in this paper. And in another paper we present an instructional model that we have found useful as we attempt to understand student progress in the development of mathematization skills as a part of their physics learning.



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### **CHAPTER 3: An instructional model for developing mathematization in introductory physics**

#### **Abstract**

Experts in physics develop and communicate ideas through *mathematization*, the process of evaluating novel contexts by generating descriptions based on mathematical principles. Introductory physics is required for most STEM majors, in part so that all students will develop more expert-like mathematization. There is growing evidence from attitudinal diagnostics that most medium- to large-enrollment introductory physics courses independently of their instructional approaches are ineffective at advancing students' mathematization in a more expert-like direction. In addition to the lack of sophistication in mathematizing, college students have difficulties with seemingly simple algebraic reasoning. We wonder to what extent the instructional models (and the associated assumptions) that form the basis of the introductory physics course influence students' development of effective mathematization. To answer this question we first explore leading theories from mathematics education and from the learning sciences that explain student development of mathematical knowledge. We then show how common curricular practices that are designed to improve the accuracy of students' procedural problem solving in math and science might actually interfere with the development of mathematization in the pre-college classroom. We hypothesize that students need explicit, sequenced instruction in mathematization within the physics curriculum to increase their sophistication this area. This paper presents a framework that synthesizes existing research to account for the difficulty many students have with apparently simple algebraic reasoning in physics. The framework further suggests an instructional model that can lead to mathematical flexibility and generativity in physics.

## I. Introduction

Physicists develop and communicate ideas through *mathematization*, or the process of using mathematical ideas to characterize and reason about the physical world. Therefore one of the important instructional goals of required introductory physics courses for most STEM majors is to help students become more expert-like in their mathematization. An example of mathematization can be seen in the analysis of the motion of a marble falling through a jar of honey. We could just divide the displacement of the marble by the time of motion to find the speed but real mathematization is not just doing division to find a rate, but includes reflection on the interrelationship between the situation, the mathematical objects given, and the actions, as well as the conscious judgment that displacement and time are measurable and that the operation of division is appropriate here. Physics courses have an important role to play in helping students to become more productive mathematizers.

Instructors model their own expert mathematization for their students from the first days of the mechanics course, when they introduce the concept of acceleration and throughout the remainder of the year and the many quantities that follow. From the instructor's perspective, the mathematics used in an introductory physics course consists mainly of straightforward algebraic ideas that students should have encountered repeatedly in middle and high school mathematics and thus this mathematics does not deserve additional instructional time; it is the physics that students need to learn. However, while most students have the procedural skills necessary to calculate from a formula, many struggle to become more expert-like in their mathematization. The instructors model much more than just doing calculations, perhaps not recognizing how many mathematical decisions they make automatically. And it is those automatic decisions that most students lack.

Students whose mathematics background has included sense-making and the concepts associated with quantification may be able to bridge this mismatch by emulating the skills modeled by the physicist. With repeated exposure, these students will gradually see value in and

begin to mimic the way their instructors reason with mathematics, and may ultimately embrace the idea that such reasoning is appropriate for generating and making sense of equations that describe real world phenomena.

Research shows that most students have limited flexibility using mathematics in introductory physics <sup>(1,2)</sup>. While the past two decades have seen considerable development in physics instruction to help students better conceptually understand physical quantities and their relation to each other (see Meltzer and Thornton <sup>(3)</sup> for a literature review), students still struggle to connect mathematical thinking with conceptual understanding in physics <sup>(4-6) (7) (8,9)</sup>. Contemporary physics instruction often involves the use of multiple representations to help conceptualize physics <sup>(10)</sup>. Once students are able to connect a physical situation to a representation (force diagram, energy bar charts, verbal description, etc.) instructors often assume that connecting this representation to mathematical equations and subsequent problem solution is relatively straightforward and well within the abilities of their students. However, this connection often involves *generation* of mathematical equations based on the specifics of the representation, and many students' experience of mathematics has included little or no exposure to mathematics as a creative activity involving abstract quantities.

For these students, the expectation that they *create* an equation represents an epistemological barrier – particularly if they need to create a variable at the same time. Furthermore, they have little experience with making sense of equations and are ill equipped to determine whether a generated equation is correct <sup>(5, 11, 12)</sup>.

The mathematization goal is not being met for in the majority of courses that teach the majority of students: algebra-based and calculus-based physics. Important results from physics education research show on attitudinal diagnostics such as MPEX <sup>(9)</sup> and CLASS <sup>(8)</sup>, student attitudes about learning physics often suffer as a result of taking an introductory course in physics. The results are so robust that it doesn't seem to matter whether the course was taught with a very traditional pedagogy or with interactive engagement techniques. Of particular relevance is that

student responses associated with mathematization are almost invariably less expert-like at the end of a physics course than they were at the beginning, with typically between one-third to one-half of the students agreeing with statements like “I do not expect physics equations to help my understanding of the ideas; they are just for doing calculations.” Although many instructors make a special point of demonstrating mathematization as they lecture, unfortunately the mathematics preparation of most students enrolling in introductory physics is limited largely to the rehearsal of algorithmic procedures <sup>(13)</sup>. Even students who have been successful in math – as measured by courses taken and grades received – may have internalized a view of mathematics as a set of algorithms, and may regard equations as recipes for obtaining numerical answers rather than as tools for sense-making.

Recent studies <sup>(14-16)</sup> have highlighted the ways that mathematics as used in physics is different from the ways that mathematics is used by mathematicians, and is different from the ways that it is learned by students in mathematics classes. Most of this research has occurred in the context of advanced introductory or upper-division classes, thus with students who developed more expert-like mathematization in the introductory courses. The situation is quite different in introductory courses. When Torigoe investigated students’ use of numbers and variables in the introductory course he found that when the students have to use variables rather than numbers in multiple choice test questions their performance drops significantly <sup>(4-6, 17)</sup>. Brahmia, Kanim and Boudreax <sup>(1)</sup> administered a suite of proportional reasoning items in the physics and chemistry courses for freshmen and sophomore engineering students and observed that, even after a year of calculus-based physics, students do not have the proportional reasoning skills that instructors assume they have mastered in middle school math. Their reasoning is even less reliable on questions in which one of the numbers is replaced with one variable. As part of his development of a theoretical framework for analyzing students’ mathematical reasoning in physics, Tuminaro <sup>(2)</sup> investigated the kinds of mistakes that students enrolled in an algebra-based course made and

found that they usually were not due to mathematical difficulties, but were instead due to a failure to use their mathematical knowledge appropriately in physics contexts.

Given the growing evidence from attitudinal diagnostics that most medium- to large-enrollment introductory physics courses are ineffective at advancing students' learning attitudes about problem solving in a more expert-like direction, we wonder to what extent do the instructional model and associated assumptions influence meeting this goal in the introductory physics course? What is it about the following exceptions that have resulted in positive gains in the CLASS problem solving categories while other courses that are also highly effective by measures of conceptual learning do not? Otero and Gray <sup>(18)</sup> reported significant gains across multiple university classes using the Physics for Everyday Thinking (PET) or Physical Science for Everyday Thinking (PSET) curricula. In a six-institution study of PbI (Physics by Inquiry), Lindsey, Hsu, Sadaghiana, Taylor, and Cummings <sup>(19)</sup> also reported on significant positive CLASS gains. The PET, PSET, and PbI curricula are all guided-inquiry-based curricula designed for pre-service elementary teachers. If we compare the pre-service teachers to the calculus-based students, and focus on the fraction of students who continue to hold naïve views we see an interesting trend. The pre-service teachers are understandably much less expert-like to begin with; even with these strong reported gains about one-third of the class continues to hold naïve views in the CLASS problem solving categories after instruction. As a comparison, it is fairly typical in the majority of calculus-based courses for approximately one-third of the class to hold naïve views in the CLASS problem-solving categories after instruction, but they had more expert-like views initially and then those views deteriorated as a result of instruction. So most calculus-based physics students after having taken a physics course hold views about problem solving in physics that are as expert-like as the pre-service elementary teachers do that were reported on by Otero and Gray and Lindsey *et al.*

The two calculus-based introductory courses that are the exceptions in the literature are Brewe, Traxler, de la Garza, and Kramer <sup>(20)</sup>, who reported on positive gains on the CLASS in



their two-semester calculus-based introductory physics sequence at Florida International University using Modeling Instruction <sup>(21)</sup> with small enrollment courses (<30 students), and Brahmia, Boudreaux and Kanim <sup>(22)</sup>, who reported on positive gains in a two-semester introductory medium enrollment (<150) calculus-based sequence using Invention Tasks <sup>(22)</sup> and ISLE <sup>(23)</sup>. These courses observe gains in mathematization categories between 5%-15%, with an average ~73% of their students giving expert-like responses on the CLASS problem solving categories after instruction. Redish and Hammer <sup>(24)</sup> also report improvements in their algebra-based course as measured by the MPEX.

There is a disparity between the goals of developing expert-like mathematization and the CLASS results for all but the few courses in introductory physics mentioned here; although the populations are different, all of these courses succeed at advancing students' mathematization in a more expert-like direction. We argue that the assumptions about students conceptualization of mathematics interferes largely with creating an effective design for a course in which a strong majority of students develop expert-like mathematization. In the courses for future elementary teachers, there is no assumption of students' conceptual understanding of mathematics and the highly effective curricula mentioned provide students with opportunities to develop their own mathematical ideas. The calculus-based courses mentioned are examples of two programs that do not assume a conceptual understanding of the way physicists use even basic algebra. Physics courses will benefit from an instructional model that attends to the goals associated with physics content and practices while also planning for instruction that succeeds in meeting the mathematizations goals.

In the next section of this paper we discuss prior research in the learning sciences, in mathematics education and in physics education. We begin by exploring leading theories from mathematics education and from the learning sciences concerning developing pre-college mathematical knowledge. We then describe examples of common pre-college curricular practices that are designed to improve the accuracy of students' procedural problem solving in math and

science, and we argue that these well-intentioned practices actually interfere with the development of mathematization in the pre-college classroom. We end this section situating the importance of physics problem solving in the process of mathematization by discussing the current theoretical models that describe the problem-solving process in physics. In Section III we present our model. We hypothesize that explicit, sequenced instruction in mathematization within the physics curriculum, beyond demonstration, will help students meet the instructional goal of increasing sophistication in their mathematizing. Based on the current theoretical work discussed in section II, we present our *quadrant of mathematization* as a theoretical framework for articulating current deficits, for explaining the CLASS gains in courses that do not make assumptions about students' prior conceptual understanding of algebra and for proposing a better model for instruction. And in section IV we articulate suggested teaching approaches based on our instructional model that could help facilitate the development of a flexible and generative understanding of mathematics in physics for all students. We hope that the crude model we present forms a useful but tentative basis for future research by physics education researchers.

## **II. Mathematization: Background Information**

Theories about how mathematization is developed inform our understanding of mathematization in the context of physics. Sherin <sup>(16)</sup> writes about his observations of high-achieving 3<sup>rd</sup> semester engineering majors at an elite university in his development of symbolic forms. His discussion is an excellent starting point as we work backwards to better understand the mental models used by the broader community of incoming introductory students with diverse preparations. We follow with the work of mathematics education theorists who have written extensively about the development of the fundamental algebraic reasoning that is assumed in introductory physics. We then proceed to the challenges to developing a conceptual understanding of algebra for all students, and finally describe an instructional opportunity that is currently not sufficiently exploited.

## II.a. Models of precollege mathematization

Mathematics education researchers recognize the importance of generating mathematical descriptions as part of learning about the world. Gravemeijer, Cobb, Bowers and Whitenack <sup>(25)</sup> conjecture that “symbolizing is integral to the creation of new mathematical entities and thus to the knowledge construction.” Schoenfeld <sup>(26)</sup> describes a *mathematical point of view* that includes being motivated to mathematize and to develop competence with the goal of understanding underlying structure. Sherin <sup>(16)</sup> describes mathematization in the context of physics problem solving as a *flexible* and *generative* use of mathematics, where flexibility means mathematics used in a variety of ways that may differ from the way one was taught (for example, the use of a negative sign to represent both a type of charge and a vector direction in the same statement), and generativity describes creating mathematical expressions from scratch to describe a physical system.

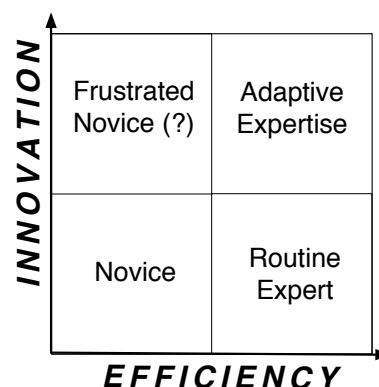
There are several theories of learning that can provide a foundation for the discussion of developing mathematization; we have selected those that most closely inform our experiences. Our model is most strongly influenced by the models developed by Schwartz, Bransford and Sears <sup>(27)</sup>, and Gray and Tall <sup>(13)</sup>.

We first ask, what are the characteristics of expertise in any discipline? Schwartz *et al.* <sup>(27)</sup> isolate and describe separate aspects of learning, *efficiency* and *innovation*, and they argue that both are necessary for the development of adaptive expertise. Efficiency refers to the learner’s ability to rapidly recall and apply her knowledge in the new situation while innovation is her ability to restructure her thinking so that it becomes more tractable than before. Becoming efficient frees the working memory to focus on other tasks. They site as an example of efficiency a doctor who has frequently performed a specific type of surgery; she is able to perform it without making a mistake even under the most stressful conditions. Innovation, by contrast, is the rearrangement of environments and of thinking to handle new types of problems or information;

an innovative field nurse might call on his culinary experience to improvise a sterilizer using a camping stove and a pressure cooker from the mess supplies.

Schwartz *et al.* <sup>(27)</sup> argue that, especially in the United States, the instructional emphasis in mathematics and the assessments that are used are focused largely on developing efficiency alone. A focus on efficiency alone produces “routine experts” <sup>(28, 29)</sup> who are very good at a particular set of problems, but who do not necessarily continue to learn throughout their lifetimes and who are unprepared for unexpected circumstances or for the creation of new knowledge.

Innovation, argue Schwartz *et al.* <sup>(27)</sup>, complements efficiency and can lead to adaptability when the two are blended effectively. They argue that in order to develop adaptive expertise we must prepare learners to engage in both kinds of learning rather than one at the expense of the other (see Fig. 6).

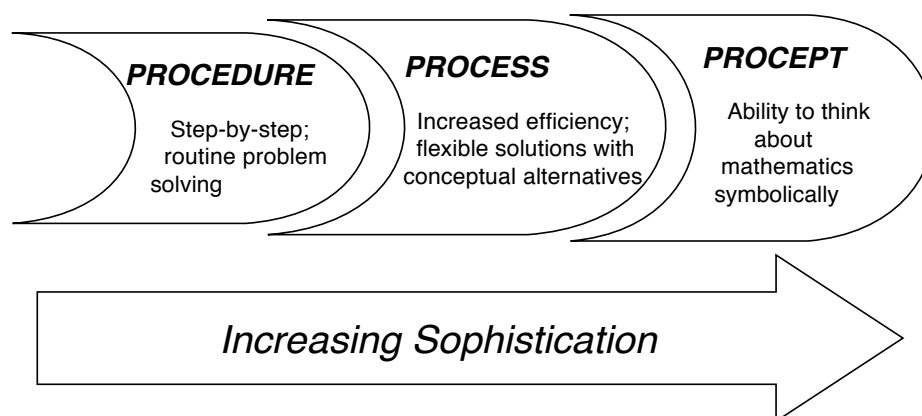


**Figure 6:** Efficiency and Innovation: (From Schwartz, Bransford and Sears <sup>(27)</sup>)

In practice, expert mathematization in physics contains elements of both procedural mastery (efficiency) and thinking in terms of underlying mathematical structures (innovation). Gray and Tall <sup>(13)</sup> (see Fig 7 for an abridged version) describe a model of the development of mathematization in which procedural mastery is an early stage that builds to a proceptual understanding (i.e. **procedural** flexibility combined with **conceptual** understanding and symbolic representation.) Like Schwartz *et al.*, Gray and Tall claim that procedural learning is a necessary but not sufficient part of proceptual understanding. The distinction between these levels can perhaps be best understood with an example: Young students asked to find 47-35 might first use a number line, starting at 47 and following the procedure for subtraction of counting to the left 35 places to land on 12. As they develop flexibility, these students will process the operation and recognize that it is equivalent, and more efficient, to start at 35 and move to the right while counting the places necessary to arrive at 47. They come to think of the concept of subtraction as

thing unto itself and decide that it can have a symbolic representation independent of the numbers involved. Finally a proceptual understanding is realized with a conceptual understanding that  $x = a - b$  can be thought of as  $a = x + b$ , for any  $x$ ,  $a$  and  $b$ . In the context of mathematics learning, innovation is woven into the development of proceptual understanding, and it starts with learning procedures. Efficiency is a hallmark of processing in Gray and Tall's model, and we see parallels between innovation and proceptual understanding.

**Figure 7:** Proceptual development, adapted from Gray and Tall <sup>(13)</sup>



## II.b. Obstacles to developing mathematization

### (i) Instruction practices in mathematics that sabotage

In practice, the development of flexibility in mathematics instruction is sometimes short-circuited because learners are given procedures to memorize rather than opportunities to develop a proceptual understanding of mathematical notions. This type of instruction results in improved efficiency, but does little to encourage innovation. For example, the “butterfly method” ( see Fig. 8) is commonly used in school math instruction <sup>(30)</sup>. This method is intended to help students learn

to become more efficient at mastering a particularly challenging skill – the addition of fractions with unlike denominators.

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**Figure 8:** The butterfly method of fraction addition

- Draw wings as shown
- Draw body as shown
- “x” for tail
- “+” for antenna
- Multiply the wings and add together for the head
- Multiply denominators for the tail

$$\begin{array}{c} 96 \quad 12 \quad 7 \quad 91 \\ \diagdown \quad \diagup \\ 13 \quad + \quad 8 \\ \diagup \quad \diagdown \\ 104 \quad 104 \end{array} = \frac{187}{104}$$

---

While the butterfly method allows students to be successful if they follow the rules, they are completely inflexible in their thinking. The consequence of this focus on efficiency at the expense of innovation can be seen in the NAEP (National Assessment of Educational Progress) results described by Carpenter *et al.*<sup>(31)</sup>. One test item asks students to estimate the answer to  $12/13 + 7/8$ :

- (a) 1                      (b) 2                      (c) 19                      (d) 21                      (e) I don't know.

Of approximately 20,000 12<sup>th</sup> graders taking the NAEP, only 37% answered this question correctly, while 36% gave either choice (c), the sum of the numerators, or (d), the sum of the denominators. In the language of Gray and Tall and Schwartz *et al.* the students never achieve a proceptual understanding nor do they develop adaptive expertise.

Tall *et al.*<sup>(32)</sup> argue that when students who learn mathematics using inflexible procedural approaches such as the butterfly method described above, they actually end up working harder

than those who develop a proceptual understanding of mathematical tools. Schwartz and Moore<sup>(33)</sup> propose that: a) mathematical models are constructed in working memory; b) there are specific constraints that help alleviate working memory demand; and c) mathematical tools can aid in model construction by determining specific relationships and thereby reduce working memory demands.

Mathematics instruction is sequential, and previously learned procedures are often incorporated or assumed within the symbols used in newer, more complicated procedures. According to Tall, inflexible thinkers actually do qualitatively more difficult mathematics, because they are burdened in the long run with high demands on their working memory. While students with a proceptual understanding of the elements of new procedures can treat symbols as mathematical objects, their counterparts with only a procedural understanding must continue to work through the procedures. As a result, the mathematics achievement gap widens as the “strugglers” accumulate experience with relatively more cumbersome and meaningless mathematical procedures, and many eventually give up.

Struggling students are commonly taught mnemonic devices and tricks that bypass the process and proceptual stages of mathematical learning illustrated in Fig. 7. These students are trained to become adept at algorithmic performance, thereby leaving them with an impression of mathematics as an inflexible set of rules. The cognitive load of memorizing procedures leaves little working memory for reflection. When it becomes essential to make decisions in higher-level math courses, these students have little decision-making experience to call on.

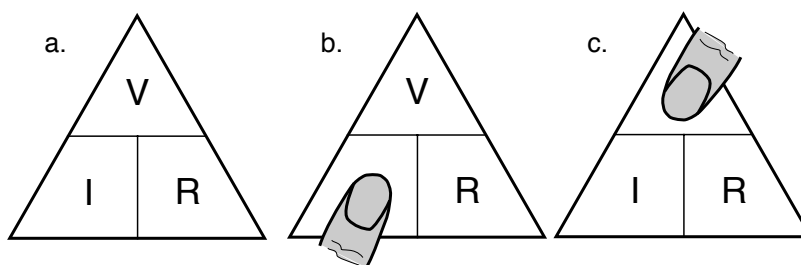
On the other hand the student who senses early on that mathematics is a flexible tool sees subsequent math as an extension of already learned concepts. Because they are not trying to hold algorithms in their working memory the “succeeders” have the mental space available to reflect while they learn. They are able thereby to learn more efficiently. They are ready to use mathematics flexibly and to generate procedures as needed in higher-level math courses.

Schoenfeld <sup>(26)</sup> describes mathematically powerful students as “flexible thinkers with a broad repertoire of techniques and perspectives for dealing with novel problems and situations.”

## (ii) Instruction practices in physical science that sabotage

The introduction of physical quantities and measurement in physical science (which typically occurs in upper elementary and middle school in the US) is a context in which quantitative algebraic and arithmetic reasoning is fundamental. Flexible reasoning about the quantities in physical science requires conceptual understanding of arithmetic <sup>(34)</sup>.

**Figure 9:** The equation triangle. Students are instructed to (a) create a triangle, and then (b) or (c) cover up the quantity they are solving for. Whatever they see gives the quantities and the operation they need to find their answer. In addition to  $V=IR$ , it also works for  $F=ma$ ,  $v=f\lambda$ , etc.



But the systemic drive for efficiency in physical science classes mitigates against forming strong connections between concepts and their associated mathematical expression, and towards algorithmic learning. For example, a commonly used shortcut procedure that facilitates efficiency is the equation triangle (see Fig. 9). Rather than expecting an understanding of the symbols and their relationships in physical science, students are provided with a handy tool that enables answer-making rather than sense-making.

Measurement and manipulation of quantities in physical science classes present opportunities for students to develop a firmer conceptual understanding of multiplication and division <sup>(35)</sup>, something that only the mathematically flexible may have attained in math class. Math contexts usually involve at most one physical quantity (e.g.  $3 \times 4 = 12$ ,  $6 \text{ pies} / 3 = 2 \text{ pies}$ ) so the quotients and



products found there are almost never completely different from the factors and dividends. In addition, in physical science contexts there are many opportunities to reconceptualize multiplicative relationships. We note that the equations as written in Fig. 9 are how they tend to appear in every physical science textbook, yet a simple algebraic rearrangement of the equations provides an opportunity to discuss cause and effect ( $a=F/m$ ,  $I=V/R$  and  $\lambda=v/F$ ). In science classes students could come to understand that by multiplying and dividing physical quantities, they are creating something that is completely different from the factors and dividends. However, the introduction of procedural games that dissociate the mathematics from the reasoning squanders opportunities for students to adapt earlier conceptions of multiplication and division, or to learn more sophisticated quantitative reasoning. As a result, many students simply memorize formulas in order to “learn”, for example memorizing that density equals mass divided by volume without developing a sense of the meaning of the quantity  $6 \text{ kg} / 4 \text{ m}^3$ .

Curriculum developers could design science materials in middle and high school to focus on ensuring that students get practical experience with creating and understanding symbols, and developing a conceptual understanding of arithmetic and basic algebra. As physical science is commonly taught, though, it is assumed either that students already have a conceptual understanding of arithmetic and are ready to reason symbolically (thereby leaving the strugglers behind), or more commonly that students need a shortcut in order to reason algebraically about physical quantities. Introductory college physics courses as typically taught require that students enter the course equipped with a *proceptual* understanding of relevant mathematical procedures. While some students have had the kinds of mathematics and science instruction that allows them to transition smoothly to the mathematical thinking in a physics context that is expected, most students have not <sup>(1, 2, 5, 8)</sup>.

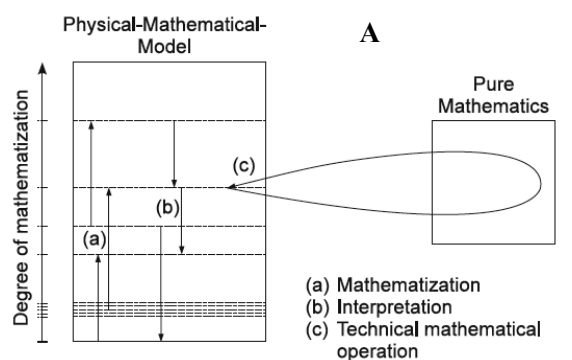
### II.c. Models of problem solving in physics

Problem solving is an important instructional mathematizing activity in a physics course, so we consider here current cognitive models for problem solving in physics to help inform our instructional model of mathematization.

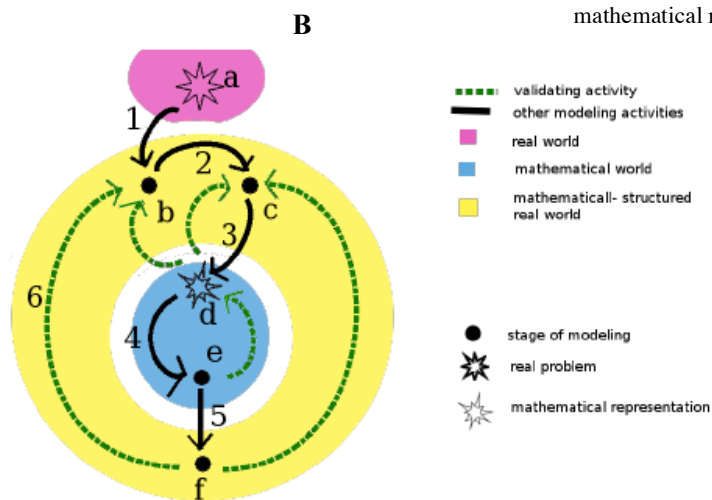
Uhden, Karam, Pietrocola, and Pospiech <sup>(36)</sup> present a problem solving mental model based on earlier models <sup>(37)(38)</sup> that includes both mathematization and physics reasoning. They use historical and philosophical considerations to argue for more homogeneity in the blending of mathematical and physics reasoning than existed in earlier models (see Fig. 10(A)). In their model, there are two components of heterogeneity that stand out: First, they make a clear distinction in mental modes between structural and technical skills (procedure), arguing that structural reasoning is based on mathematical relations (for example, inverse proportions), while procedure is pure math, and occupies a mental space in isolation from the physical reasoning (c in their diagram.) This distinction is in contrast to the notion of adaptive expertise <sup>(27)</sup> and proceptual <sup>(13)</sup> representations of functional mathematical knowledge. In addition, they represent mathematization (a) and physical interpretation (b) as competing processes, indicating that interpretation decreases the level of mathematization.

In contrast to the purely theoretical model of Uhden *et al.*, we describe two empirically developed models; one is from physics education research and the other is from mathematics education research. Both represent even more tightly blended mathematics and physics reasoning, and neither makes the distinction of procedure being "pure math". In the physics study, Hull, Kuo, Gupta and Elby <sup>(39)</sup> present their analysis of interview data. They present two students as contrasting cases of successful problem solvers: One demonstrates expertise through a proceptual understanding while the other is highly efficient and strictly procedural. Both earn high scores on their solutions using current problem-solving rubrics, but it is clear that the rubrics don't, and should, reward proceptual understanding in some way. The researchers argue that the

"opportunistic blending of conceptual and formal mathematical reasoning is a part of problem-solving expertise..." and should thus be rewarded. In the study from mathematics education, Czocher<sup>(40)</sup> conducted interviews throughout the quarter of four sophomore engineering majors enrolled in a differential equations course. The students were observed solving problems in everyday contexts that required generating mathematical descriptions from a variety of branches of mathematics, including differential equations. She describes a much finer-grained blending of mathematical reasoning and physical sense-making than is represented in the Uhden *et al.* model. She proposes a problem-solving model that reflects her observations of this blending (see Fig. 10(B)). The frequent interplay between the mathematics and the sense-making advocated by Czocher and Hull *et al.* is reflected in the validating activities that occur at every stage in her model.



**Figure 10:** Problem solving models  
(A): From Uhden *et al.* (2012)<sup>(36)</sup>  
(B): The Czocher (2013)<sup>(40)</sup> model stages are labeled [a] through [e]: real world situation [a], situation model [b], real model [c], mathematical representation [d], mathematical results [e], and real results [f]. The transitions labeled [1] - [6] represent the mathematical modeling cycle (understanding [1], simplifying/structuring [2], mathematizing [3], working mathematically [4], interpreting [5], and validating [6]) and the unlabeled transitions correspond to the kinds of validating activity observed in her research. The double star at stage [d] is the mathematical representation.



The butterfly technique, the equation triangle, and techniques like them are practical solutions to a real problem: teachers are charged with preparing students to pass high stakes exams that target specific procedural skills. Recent research on problem solving in physics highlights the importance for student success of the *interplay* between mathematical procedural efficiency and physical sense-making. When instructors focus on (easily measured) procedural skills, they may inadvertently short-circuit the intellectual effort necessary to understand what it means to perform these procedures. This may in turn contribute to a sense among students that mathematics is nothing more than a set of procedures to be memorized. These students are then ill equipped to make sense of physics, or to develop, through their study of physics, powerful modes of quantitative reasoning. In the next sections we describe our investigation of the consequences of an excessive focus on efficiency as it manifests itself in algebraic reasoning in the introductory physics course.

### **III. Instructional model of mathematization in physics**

Much of the mathematics that students struggle with in introductory physics is taught in middle school. The focus of instruction in these courses favors efficiency over conceptual understanding, an emphasis that is aligned with the format and content of national and state exams. It is rare for high school mathematics or physics curricula to address the development of flexibility and generativity. Many students are so focused on procedure that they miss the opportunity to conceptually understand what they are doing when they solve math and physics problems. While some learn mathematization, it is rarely taught.

In prior work <sup>(1)</sup> we identified that students' conceptual understanding of mathematics in physics is strongly tied to issues of general structural reasoning, quantification, symbolizing, and of student epistemologies about what it means to do mathematics. In concert with that investigation, we experimented with instructional approaches and curricular modifications

intended to promote innovation and mathematization <sup>(41)</sup>. Our considerations of what has worked, and what hasn't, have led us to model the development of mathematization in physics as requiring a reconsideration of mathematical procepts in physics contexts. This model for instruction views the conceptual understanding of mathematics and physics understanding as inseparable. It reflects the observations of Czoher and Hull *et al.* and targets developing qualities consistent with the descriptions of successful students in Bing & Redish <sup>(14)</sup>, Sherin <sup>(16)</sup>, and Torigoe and Gladding <sup>(5)</sup>. In addition, in contrast to the model of Uhden *et al.* we view mathematization and physical interpretation as complementary, not competing, processes.

Our model is based on the following observations:

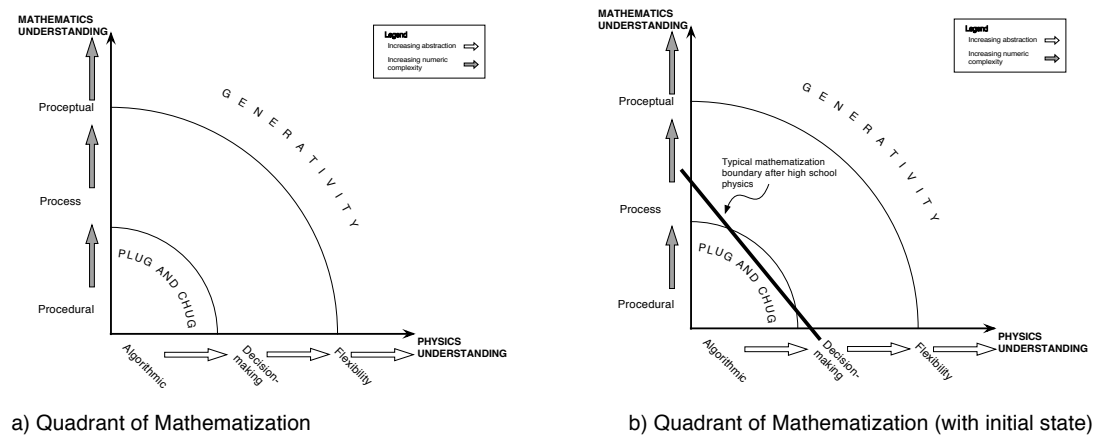
1. Procedural mastery of problem solving in both physics and mathematics precedes the ability to reason about the structure underlying a class of problems.
2. Decision-making requires self-efficacy, or the belief that you are able to succeed if you work hard and long enough at something, which comes from having had prior success.
3. Flexibility combines efficiency with quick and sensible decision-making.
4. Generativity is dependent on developing a proceptual understanding of the mathematics involved in conjunction with developing flexibility in the physics contexts.

### **III.a. Quadrant of mathematization**

Figure 11a displays our *quadrant of mathematization* in physics framework, which is consistent with the quantitative mental model proposed by Schwartz and Moore <sup>(33)</sup>. In this representation, new knowledge in physics exists in the first quadrant of the plane defined by a mathematics understanding axis and a physics understanding axis, reflecting our belief that successful mathematization requires a blend of proceptual mastery of mathematics along with increased flexibility with and conceptual understanding of physics. The effect of mathematical

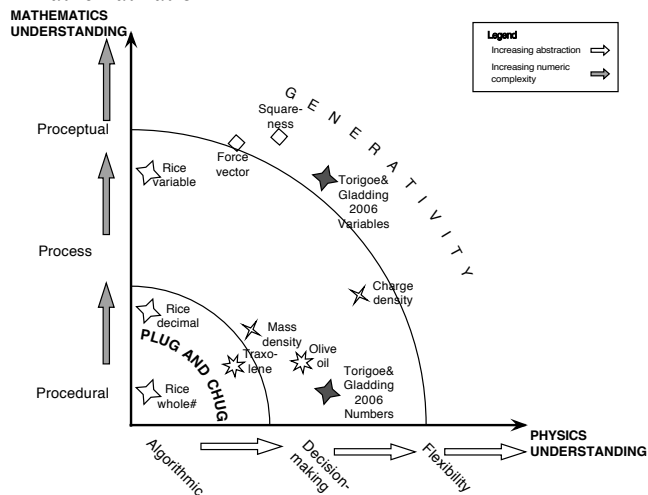
numerical complexity and physics context (reported in Brahmia *et al.* <sup>(1)</sup>) is represented as arrows along the respective axes between the stages.

**Figure 11:** Quadrant of mathematization I: a) Physics mathematization as a function of mathematics understanding; b) functional representation of pre-instruction for engineering students.



At any given time, students' progress can be described in this model by a line that connects a point on the mathematical understanding axis with one on the horizontal physical understanding axis (See Fig. 11b). We emphasize here that the model we are proposing is both crude and preliminary: Further development of this model might include a refinement of the

**Figure 12:** Assessments items mapped onto Quadrant of mathematization



stages represented along both axes.

In Fig. 12 we include approximate locations within the quadrant of mathematization of the paired assessment items described in Brahmia *et al.*<sup>(1)</sup> (see Appendix B). For comparison we also include the paired questions asked by Torigoe and Gladding<sup>(17)</sup> (see Appendix B). Note that the interaction of numerical complexity with reasoning in the cases of the Rice questions (see Appendix B) and the Torigoe and Gladding questions; we argue that students can efficiently move procedurally through the numeric questions but that their efficiency is diminished and they have to attend to more decision making when more complex numbers are involved. Similarly, the charge context of the density paired questions suppresses student reasoning which is readily activated when the question is asked in a less abstract context of mass density. While the Traxolene and Olive Oil questions can be answered using the equivalent-fraction method of solving ratio problems that students master in preparation for the SAT, the Force Vector and Squareness problems require students to generate a ratio on their own and are thus much more challenging.

### **III.b. Effect of instruction: two perspectives**

The assumptions that we, as physics instructors typically make of the initial state of students in introductory physics are that they are “like us” in their mathematical understanding, since they have taken the required math courses. Thus we see our task as helping the students to increase their physics understanding, exploiting this mathematical background. In our instructional model such task corresponds to the transition shown in Fig. 13(a) and 13(b).

However, the results of our pretesting<sup>(1)</sup> show that students do not start with the proceptual understanding that is implicit in traditional instructional approaches. Instead, typical students have some process skills in addition to a solid procedural understanding of the requisite

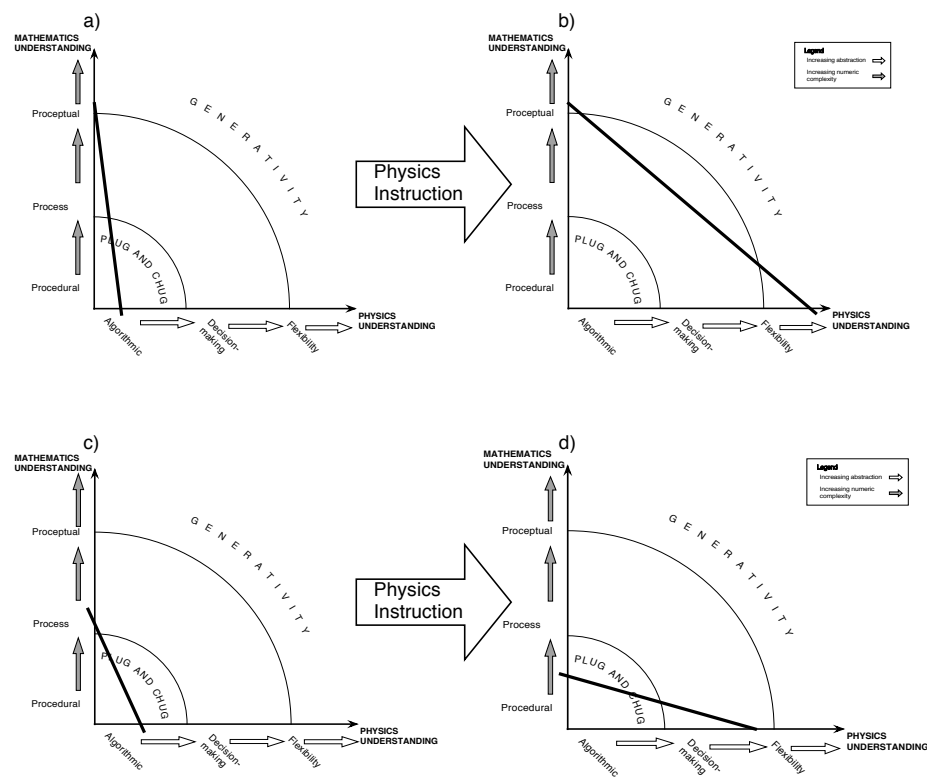
mathematics, but lack the proceptual understanding that is necessary to progress at the pace that is expected.

**Figure 13:** Modeling the effect of instruction

a) before instruction, assuming students have a conceptual understanding of the prerequisite mathematics

b) after instruction, based on typical instructional goals; c) before instruction, loosely based on pretest results

d) after instruction, loosely based on post test results.



Our post-test results on measures of proportional reasoning show very little change after a semester of physics instruction. Moreover, for the majority of the items reported on in Brahmia, Kanim and Boudreaux <sup>(1)</sup>, we see statistically significant deterioration as suggested in Fig. 13(d). It is possible that for students whose mathematical understanding is fragile at the beginning of the



course, the additional cognitive load imposed by using this mathematics in a physics context prompts a retreat to more algorithmic and procedural approaches to the associated mathematics. Consistent with the typical shift away from expert-like mathematization measured by the CLASS in physics we observed that algebraic reasoning of many students deteriorates as they move through traditional physics instruction.

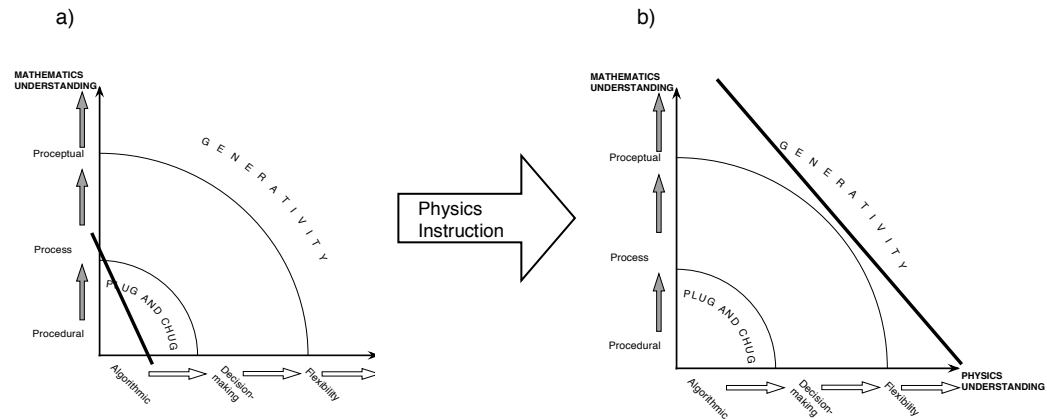
When we assume that students already mathematize in the same way as we do as instructors, and focus on developing the procedural skills necessary to successfully answer traditional end-of-chapter physics problems, we achieve a solution to a real problem: how to get the majority of students, with a disparate set of math skills, to perform sufficiently well on problem-solving tests in introductory physics. However, we substitute the goal of mastering the (easily measured) set of procedural skills for the more difficult to teach and to measure crosscutting quantitative skills that are an important primary purpose for requiring introductory physics. When instruction does not make that assumption, as is the case with the PET, PSET, and PbI curricula and in courses at FIU and Rutgers in calculus-based physics, the results are strikingly different; students become more expert-like in CLASS responses associated with mathematization.

### **III.c. Instructional model for expert-like mathematization**

We believe that meeting an instructional goal of flexible and generative understanding of mathematics in the context of physics requires explicit instruction in the mathematization of physics. Our goal is to have the line in the quadrant of mathematization move diagonally away from the origin as shown below in Fig. 14, increasing students' flexibility with physics concepts while at the same time promoting mathematical understanding from strictly procedural to proceptual in a physics context.

**Figure 14:** Instructional model for developing expert-like mathematization

The diagonal bold line represents students' starting state. a) Initial state of class; b) Intended final state of class. Instruction that meets the goal of developing generativity and flexibility sees that line moving diagonally away from the origin.



#### IV. Implications for Instruction

In what follows, we make suggestions for instructional modification based on what is known from our and others' research for changes that will help instruction realize an increase in expert-like mathematization. We adopt the theoretical framework of PFL (preparation for future learning) <sup>(27,33,42)</sup> that is based on priming students to be both innovative (generative and flexible) and efficient (competent and flexible) in a situation of activating mathematical knowledge in a physics context. PFL in our model occurs regularly throughout the course; it becomes part of the course culture.

Our suggestions overlap with the *necessity principle* coined by Harel <sup>(43,44)</sup> that says, interpreted in the context of physics, that students' mathematization should come from an intellectual necessity and provide resolution to a dilemma. We conjecture that doing math in a

particular context (solving physics problems using algebra, or just solving algebra problems) is not a priming activity that prepares students to think conceptually about the mathematics that they use because there is no real necessity. An essential feature to this priming is to situate students in the role of invoking math tools to make sense instead of just “finding the right equation” and following a procedure. We have found that we have more success when we require students to generate mathematical quantities in an everyday context first, as we will describe in the next section.

We have had promising initial results with curricular modifications that we believe are contributing to students’ becoming more flexible and generative in their uses of mathematics in introductory physics <sup>(22)</sup>. These modifications follow, and their success has generated, the three broad guidelines below. See Brahmia *et al.* <sup>(22)</sup> for details of the curricula we have developed and some results from measures of students’ mathematical fluency, physics content development, and attitudinal changes. The guidelines here are based on work that we consider promising but preliminary; we articulate them in the context of our experience.

### **1. Anticipate students’ areas of weakness and plan accordingly.**

We expected that students would struggle with units, and with the arithmetic involved. As a result, we were particularly careful to explain our reasoning about the mathematics to our students (*e.g.*, why does it make sense that describing fastness would involve a ratio of displacement and a time interval rather than the product of the two?) In turn, we expected them to justify their mathematical reasoning when they did algebra with physical quantities.

Students struggle with both numeric structural difficulty and context abstraction. We offered activities that balanced these two competing domains of difficulty to create a bridge that helped students focus on the instructional goal for the activity. For example, when teaching a conceptually abstract concept like electric field, we started with units that allowed for the use of whole numbers less than 10. Once students developed some mastery of the concept, we

provided experiences for them to reason with increasingly complex numbers and symbols.

We have found that in order of increasing difficulty, the number representations are: whole numbers, decimals and fractions, familiar variables, strange variables (Greek letters, non-mnemonic letters).

## **2. Take advantage of productive representations, and avoid unproductive ones.**

The process of finding consistency between productive representations, which has been shown to be a characteristic of expert problem solving, develops as students make sense of information communicated in variety of representations. We expected that many students entered our courses equipped with “effort-saving” unproductive representations that ultimately work as obstacles to the goal of becoming efficient and flexible. While equation triangles and butterflies are unproductive representations, there are productive mathematical and graphical representations that have been shown to improve student understanding (*e.g.*, free-body diagrams, Bar Charts <sup>(45)</sup>, P-V diagrams, motion diagrams). To break student dependence on unproductive representations and to help them recognize the difference between productive and unproductive representations in physics we followed the lead of curriculum developers <sup>(23, 45)</sup> who build their mathematical developments on the use of effective multiple representations.

## **3. Put students in the role of being generative with mathematics in the context of physics.**

Sherin <sup>(46)</sup> argues that “intuitive knowledge can be refined within the context of problem solving. More generally... that conceptual understanding and problem solving need not be separate, either in instruction or research.” Schwartz and Moore, and Schwartz, Martin and Pfaffman <sup>(33, 42)</sup> observed in middle school students that coupling a conceptual development of mathematics to the context of physics helped the students to better understand both the math and the physics. We designed our instructional modifications based on the expectation that in

order for students to learn to solve problems and make sense of them, they would need to learn what it means to make sense in physics. Philip<sup>(47)</sup> suggests that if students are given the opportunity to reflect on the different symbolic representations in physics, they are more likely to develop a conceptual understanding of variable.

In practice, many effective physics instructors already do this in the advanced courses. They take time to help their students understand when and why we choose to use particular math tools (Taylor expansion, parameterization, etc.) because instructors recognize that these tools will be challenging. But instructors often have not reflected on why they sum energies, or multiply a force and a time interval, and often don't share the when and why of arithmetic and algebra with their introductory students. We have tried to extend this careful description of mathematical tools to the less obviously challenging ones that students have already seen by explicitly describing how they are used differently in physics. Often this requires that we as instructors reflect more carefully on how and why we use mathematics.

Our own use of invention instruction<sup>(22)</sup> based on the instructional model developed in the context of mathematics by Schwartz, Chase, Oppezzo and Chin<sup>(48)</sup>, is yielding very encouraging early results with mathematically underprepared engineering students. We see significantly better FCI gains and CLASS scores than traditional instruction yields with better-prepared peers, especially in the categories most closely aligned with appreciating and understanding the use of equations in physics<sup>(22)</sup>

## V. Conclusion

Students who do not develop a proceptual understanding of the mathematics that they learn in K-12 classes are at a distinct disadvantage in introductory physics courses, as they are rapidly introduced to more sophisticated and sometimes idiosyncratic treatments of these mathematical ideas in a new and challenging context. Sometimes physics instruction compounds

this challenge by unintentionally reinforcing an adherence to perceived shortcuts, like memorizing equations instead of understanding them. The mental burden of accumulated shortcuts leads some physics learners to give up without ever experiencing mathematization while those who have been mathematically prepared with more focus on a conceptual understanding of procedures don't have to work as hard to understand these new uses of familiar procedures. The *Matthew effect* describes accumulated advantage – the rich get richer. In physics, we often ignore the variation in the preparation of our students. Students typically come into our class at the procedural level in algebra, and even in relatively selective student populations there is a rigidity in approach to mathematical procedures that is in sharp contrast to the flexible procedures in physics<sup>(1, 11)</sup>. We believe that, in the absence of explicit focus on the conceptualization of algebra in the context of physics, accumulated abundance in understanding for only the strongest and best-prepared students reigns supreme. Essentially this Matthew effect in physics selects future physicists based not on potential, but on past mathematical preparation.

By making crosscutting procepts explicit and available to all students beginning with their first physics course, we believe that a more diverse group of learners will emerge, and that all students will begin to mathematize in a more expert-like physics way. We have found it useful to model the instructional changes we have initiated in terms of a quadrant of mathematization, which allows us to consider both the physics and mathematics implications of the changes we make, and to keep in mind the goal of increased flexibility and generativity.

Inevitably, developing a depth of proceptual understanding will require sacrificing some breadth of “coverage”. We assert emphatically and unapologetically that this trade-off is worth it. Explicit ongoing instruction in crosscutting procepts is necessary in order for all students to have equal access to the flexible and generative understanding of mathematical reasoning in physics that is necessary for learning in physics.

The development and understanding of crosscutting mathematical concepts in the physical sciences can be compared to the development of expository critical thinking in the social

sciences. Implicit in people's understanding of why one would take a history course is that beyond understanding history, one learns to think critically and effectively about the human condition. Implicit in the reason many STEM majors require physics is the expectation that beyond memorizing conservation laws and the procedures for solving physics problems, a physics course teaches one to think critically and effectively about natural laws. We may in fact consider these two broad crosscutting skills as integral to critical thinking; in a rough sense one skill is language based and the other equation based. Physics, and in particular the introductory physics course, has an important role to play in developing the habits of mind that underlie the broader educational goals of the university.

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## **CHAPTER 4: Developing Mathematization with Physics Invention Tasks**

### **Abstract**

Experts in physics and other STEM disciplines develop and communicate ideas through *mathematization*, or the mental habit of going back and forth between the physical world and the symbolic world. Instructors commonly assume that the mathematics preparation of incoming students will allow them to successfully mathematize in physics contexts, yet examination of the research literature in mathematics education and physics education indicates that most students struggle with nuanced mathematization in physics. Invention instruction has shown considerable promise as a curricular framework for improving students flexibility and generativity with mathematics. In this paper we describe our curricular intervention, *physics invention tasks*, designed to develop mathematical creativity and thereby prepare students to better understand the reasoning in subsequent formal instruction. We describe their implementation at Rutgers University, WWU and NMSU. And finally, we share results from a preliminary study that compares FCI (Force Concept Inventory) gains, result on some MBT (Mechanics Baseline Test questions) and CLASS (Colorado Learning Attitudes about Science Survey) scores for two calculus-based courses at Rutgers, one that used *physics invention tasks* and one that did not.

### **I Introduction**

*“One factor that has remained constant through all the twists and turns of the history of physics is the decisive importance of the mathematical imagination.”*

*- Freeman Dyson*

*“Imagination does not become great until human beings, given the courage and the strength, use it to create.”*

*- Maria Montessori*

As a result of a first exposure to physics, we would like students to learn physics concepts and the connections of these concepts to each other and to their mathematical expression. We would like students to be able to solve physics problems in a manner that, within the constraints of their limited physics knowledge, nonetheless looks like problem solving as practiced by physicists. Finally, we would like them to see physics as we see it – as an exciting endeavor that allows creativity and innovation within the framework imposed by the requirement that the products of this creativity model the physical world.

Physicists clearly view doing physics as a creative endeavor, but do introductory physics students also perceive what they do in a physics course as mathematically creative? A significant and important result from physics education research is that the answer to this question is a resounding “no”, and, even more discouragingly, that the effect of taking a physics course only exacerbates this view: Students see physics as even less creative mathematically at the end of introductory physics than they did when they started the course <sup>(1,2)</sup>. With few exceptions, these findings appear to hold whether the mode of instruction is fairly traditional or whether it has components of active inquiry.

Many students entering introductory physics have had little experience with mathematics as a *generative* endeavor involving abstract quantities, where equations and quantities are invented to solve specific problems. Instead, much of their experience with mathematics has been procedural practice intended to improve efficiency, and usually they are told which procedure to use, and word problems with everyday quantities. As a result, students often have had little experience with considering the procedures they have learned in light of their potential for solving specific problems. The following example highlights students’ struggle to generate a novel and effective

arithmetic quantity. We posed a question on a multiple-choice diagnostic that asked freshmen engineering students at Rutgers University to rank rectangles in order of “squareness” given the rectangles’ dimensions <sup>(3)</sup>. Only 18% of the 763 students tested selected the answer that corresponded to forming a ratio; the vast majority (71%) selected the answer that corresponded to taking a difference of the sides (which is consistent with the much more common comparison operation learned in school mathematics courses.) We’ve asked this question to engineering freshman at WWU (Western Washington University) and at NMSU (New Mexico State University); both institutions yielded similar results.

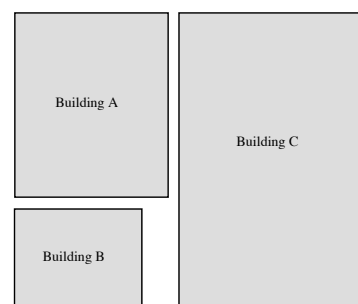
We believe that an introductory physics course is ideally situated to develop mathematical imaginations, and to ask students to reconsider the mathematics they already know in light of its potential to solve problems. Introductory physics is a required course for most STEM majors in part so that students will develop more expert-like *mathematization*. In the context of physics, *mathematization* is the representation of ideas with the help of symbols, finding problems, solving problems, and experimenting with symbolic means, in order to become better acquainted with the physical world <sup>(4,5)</sup>. Simply stated, to mathematize means to go from the physical world to the symbolic world. Physics courses have an important role to play in helping students to become more productive thinkers, and currently only the top performing students benefit from that outcome.

Using math generatively requires different skills than using math procedurally. Students must be willing to try things without knowing whether they will work, which requires a courage that must be nurtured. They must learn how to check to see whether their intuitions match the mathematics that they generate, and how to iterate toward better solutions. We would like to create opportunities for the development of these skills within the constraints of the introductory course.

To better explore students’ creative potential using the “squareness” question, we asked it again but in a different assessment context, this time as an *invention task*. Rather than evaluating

student work in a sequestered problem-solving situation, as was the case on our multiple-choice diagnostic, we instead asked them the following in a small group setting:

You are flying in an airplane, and you see the rooftops of three buildings labeled *A*, *B*, and *C* directly below you as shown. You find yourself wondering which building is the most square.



Invent a *squareness index* that allows you to characterize the squareness of the three buildings. Larger numbers on your index should correspond to squarer buildings. Then *rank* the buildings in order of decreasing squareness.

Students in a second-semester calculus-based physics course ( $N=99$ ) at NMSU worked in groups on this task during the first week of class. About 84% constructed an index using a ratio, and only 7 students constructed a ratio using a difference (seven students gave other, more complicated procedures.) For this same group of students, only 22% gave an answer consistent with a ratio ranking on the pretest described previously. We also tried this invention task in a conceptual physics course at NMSU with less success: of 43 students participating, 7 gave no answer, 10 constructed an index using a difference, and 20 used a ratio (7 gave answers that were not clear). For this group of students, only 5 (about 12%) chose an answer consistent with a ratio on a pretest taken before the squareness exercise, and 28 students (about 65%) gave answers suggesting a difference strategy. We include this example because it reflects our belief that by providing increased opportunities for students to develop their mathematical imagination as part of an introductory physics course, we can promote more effective use of mathematics both in physics and in subsequent coursework.

In this paper we describe our curricular intervention, *physics invention tasks*, designed to develop mathematical creativity and thereby prepare students to better understand the reasoning in subsequent formal instruction. We describe their implementation at Rutgers University, WWU and NMSU. And finally, we share results from a preliminary study that compares FCI (Force Concept Inventory) <sup>(6)</sup> gains, results on some MBT (Mechanics Baseline Test) questions <sup>(7)</sup>, and CLASS (Colorado Learning Attitudes about Science Survey) <sup>(1)</sup> scores for two calculus-based courses at Rutgers, one that used *physics invention tasks* and one that did not.

## II Theoretical Underpinnings

A significant outcome of PER is that instructional strategies have been developed that have been demonstratively effective at strengthening students' conceptual understanding <sup>(8,9)</sup> and at developing scientific ways of knowing <sup>(10,11)</sup>. However, many students who show evidence that they understand an introduced concept often struggle to reason in mathematical ways about that concept <sup>(12)</sup>. There has been little research done in PER into designing and assessing methods specifically focused on developing the building blocks of expert-like mathematization in introductory physics.

We use Sherin's <sup>(13)</sup> characterization of the expert-like mathematization that he observed in high-achieving students to create some teaching goals for the introductory course; he describes students as *flexible* if they spontaneously activate productive mathematical reasoning independent of context, and *generative* if they are able to create a novel symbolic solution to a problem. We consider these to be roughly analogous to the use by Schwartz, Bransford and Sears <sup>(14)</sup> of *efficiency* and *innovation* in the learning sciences. Effective mathematization in physics involves both flexibility and generativity.



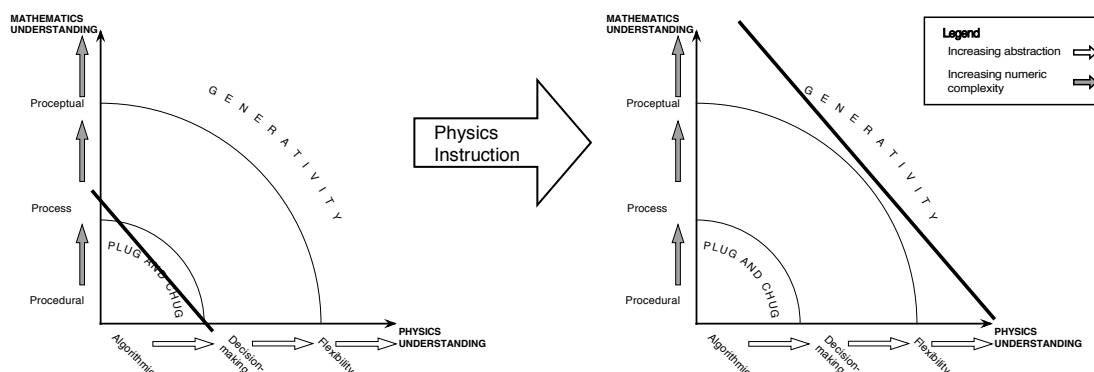
## II.a. Development of expertise

We agree with other researchers <sup>(15, 16)</sup> that mathematizing in physics involves a tight blending of mathematical and physical concepts <sup>(3)</sup>. Take for example a ball rolling across a floor at a steady speed. The situation triggers thinking about rates, which implies the division of a distance by a time (and not multiplication, subtraction, etc.) in order to answer questions about this motion. Mathematization is not just doing division to find a rate, but includes reflection on the interrelationship between the situation, the mathematical objects given, and the actions, as well as the conscious judgment that division is appropriate here.

### (i) Mathematization in physics instruction

In Fig. 15 we describe a *quadrant of mathematization* <sup>(3)</sup> as an instructional design tool in which students develop generativity based on an interdependency of mathematical proceptual development and conceptual flexibility in physics.

**Figure 15:** Quadrant of mathematization II: The diagonal bold line represents students' starting state. a) Initial state of class; b) Intended final state of class. Instruction that meets the goal of developing generativity and flexibility sees that line moving diagonally away from the origin.



Theories about how mathematization is developed inform our understanding of mathematization in the context of physics. Sherin <sup>(13)</sup> writes about his observations of high

achieving 3<sup>rd</sup> semester engineering majors at Northwestern University as they collaboratively solved physics problems. The data set includes 27 hours of video in all. Sherin categorizes characteristics of reasoning exhibited by the student groups when they are successful. He claims students learn to understand physics equations in terms of a limited number of elements that he calls *symbolic forms*. Each symbolic form associates a simple conceptual schema with a pattern of symbols in an equation. For example he describes the “ratio” form ( $x/y$ ) as a “comparison of a quantity in the numerator and denominator. ...In most cases, the quantities have the same units. Utterances involve whether  $x$  or  $y$  is greater and whether the ratio is greater than, equal to, or less than one.” This particular form is clearly one element of reasoning we wish our students were learning, and then learning to activate, in an introductory physics course. Sherin’s collection of symbolic forms is an excellent starting point as we work backwards to better understand the mental models used by the broader community of incoming introductory students with diverse preparations.

## **(ii) Procedural skills and conceptual understanding**

Procedural mastery of mathematics and conceptual understanding are not the same, and much has been written on this in math education <sup>(17, 18)</sup>. For example, from Gray and Tall <sup>(17)</sup> “the symbol  $3/4$  stands for both the process of division and the concept of fraction”. They refer to the target learning goal as a *proceptual* understanding – where *procedural* mastery and *conceptual* understanding coexist. In this example a student with a proceptual understanding would be able to move fluidly between the procedure of dividing 3 by 4, and recognizing what the fraction  $\frac{3}{4}$  represents as a mathematical idea.

Some students who have difficulty may become entrenched in a procedural approach, perhaps even reaching a stage that can lead to procedural efficiency. Other students develop greater flexibility by seeing processes as a whole and compressing operations into thinkable mathematical concepts, thereby reducing the load on their working memory. These divergent ways of approaching mathematics can lead to an achievement gap between those who perform

procedurally and those who develop greater flexibility. In arithmetic, Gray & Tall <sup>(17)</sup> called this the *proceptual divide*. They claim that the methods of symbolism used by students on the two sides of this divide are starkly different; those on the efficient end use symbols effectively to reduce the load on their working memory, while the others get mired down in distracting details.

Those who approach math problems procedurally very likely approach solving physics problems in the same way, and their framing of what they are doing as students becomes an answer-making frame. Their tendency toward answer-making, as opposed to sense-making, is reflected in their CLASS responses. It is unlikely that additional practice with procedure (while perhaps pointing out the overall concept we are aiming at) will finally get them to a proceptual algebraic reasoning.

### **(iii) Quantification and models of physics reasoning**

In order to visualize reasoning it is perhaps helpful to consider a structural model. Several researchers have introduced small-scale models of reasoning which act as building blocks of physics reasoning <sup>(19-22)</sup>. We use Hammer's general term "resources" to refer to any of these constructs <sup>(20, 21)</sup> and Wittman's term of a *coordinated set of resources* <sup>(23)</sup> to describe reasoning that requires activation of multiple resources. We cite Sherin's symbolic forms as an example of a the kind of coordinated set of resources that are a desired outcome of introductory physics, and note that as students progress these symbolic forms become the resources of future learning <sup>(13, 24)</sup>. It is through this mechanism that the invention tasks prepare students for future learning.

The objects of mathematical reasoning in physics are always physical quantities. We introduce  $\sim 10^2$  physical quantities in a typical introductory physics courses. In many cases, the words we use for those quantities (for example, work or force) have different everyday meanings than they do in physics, and students have to reconcile these meanings with the physics definitions. The names of other quantities (moment of inertia, latent heat, electromotive force) are

historical artifacts and present additional barriers to learning. Our hope in the introductory course is that students develop both a conceptual understanding of these physical quantities (and of the relevant connections between them) and a proceptual understanding of the underlying mathematics that will allow them to solve problems.

Because of this focus on quantity, physics has an important role to play in helping students develop a proceptual understanding of the math we use. The Dutch mathematician Freudenthal<sup>(25)</sup> claims that “there is no mathematics without mathematizing.” Researchers in mathematics education have identified *quantification* as a significant challenge to students who are learning to mathematize. Thompson<sup>(26)</sup>, who has researched and written extensively on this topic over the past two decades, defines quantification to be “the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute’s measure entails a proportional relationship (linear, bilinear, or multi-linear) with its unit.” He considers quantification to be “a root of mathematical thinking”, and argues that learners develop their mathematics from reasoning about quantities. Certain modes of structural reasoning are ubiquitous in physics, as articulated in Sherin’s symbolic forms. Ellis<sup>(27)</sup> claims that these modes are more likely to develop when students practice with quantities, rather than the strictly numerical patterns and algorithms common to school mathematics.

Quantification in physics requires quite a bit of mathematical judgment, and frequently some knowledge of history. For example quantifying the motion quality of an object entails recognizing: (1) that both mass and velocity matter, that they matter in the same way (doubling the mass doubles the amount of motion-ness or doubling the velocity doubles the amount of motion-ness) and (2) that the motion has a direction that is determined by the velocity, and so motion-ness must be a vector with units that are the product of the mass units and velocity units. Finally, this quantity gets a name and a symbolic representation; physics has chosen momentum (from Latin, circa 1782), and the symbol  $p$  is used because the word "impetus" formally in place of "momentum" comes from the latin, "petere," to go towards or rush upon. Practicing

quantification in physics could provide students with opportunities to develop mathematical judgment, and better connect with nicknames that allow them to make sense of the physics quantity.

Reasoning mathematically about quantities is no less of a challenge for physics students than it is for math students. Brahmia, Kanim & Boudreaux<sup>(3)</sup> make the case that students may not reason algebraically about quantities the way we expect them to, even after a year of calculus-based introductory physics. They found that traditional instruction can actually shift students to reason less expertly about quantity in some contexts, a finding consistent with CLASS results related to students' perceptions of the role of mathematics in physics. Physics instruction typically presents new quantities for students to master rather than engaging them in the processes of creating them, which we suggest means that students are not spontaneously activating their existing mathematical reasoning resources because they are too busy trying to absorb the pre-packaged sets of resources presented to them.

#### **(iv) Motivating and developing expert-like mathematization**

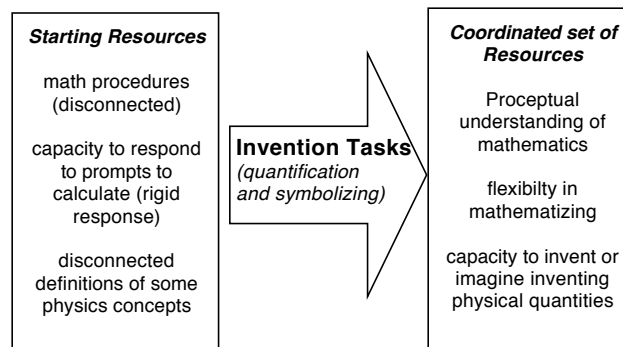
We hypothesize that attending to students' intrinsic motivation to learn and conceptualize quantities in physics can facilitate the development of expert-like mathematization. For those who have succeeded and progressed in physics, motivation likely came out of a desire to understand physics coupled to a belief that understanding would come with effort. These students become more expert-like regardless; they are on the productive side of the proceptual divide. The students on the other side of that divide in the introductory course are not strongly motivated to understand physics<sup>(28)</sup> and many believe that they may not be able to understand even with effort. These students may adopt "survival strategies" that maximize performance on algorithmic tasks without necessarily leading to understanding.

Research-based curricula can provide opportunities for all students to explore, and even develop, some of the foundational relationships between physical quantities, *e.g.*, between force

and acceleration, and can also provide guided practice reasoning about quantities in specific contexts. For example, curricula that

guides a student to recognize that the acceleration of a ball is directed down a ramp as the ball moves up the ramp, turns around, and moves back down has been shown to effectively help students differentiate between velocity and

**Figure 16:** The effect of Invention Tasks



acceleration. What is often lacking even in high quality curricula, however, is activation of students' math resources that prime them to engage mathematically with the process of creating the quantities themselves. And in many curricula the students do not create anything at all, they simply learn what the different quantities are. Invention instruction <sup>(29-31)</sup> attempts to fill this gap by addressing quantification itself as a motivation and preparation for future learning (See Fig. 16) An important quality of the process of invention in learning physics is that students develop physics judgment. When we give a definition and expect students to learn it, students learn what something is, but not what it isn't. This may seem trivial but it makes their knowledge fragile. One of the key features of some effective research-validated initiatives <sup>(10, 32-35)</sup> is that students are often forced to decide what is not relevant. When students struggle together to invent a quantity, they do the same thing. When students learn that a square has sides of equal length, the definition doesn't help them to decide how to judge squareness, as we have seen. The invention format requires that they make these judgments.

## **II.b. Invention Instruction**

The concept of invention instruction is used in a variety of research-based physics curricula and in that context it is described elsewhere <sup>(10, 11, 36, 37)</sup>. We describe here the theoretical

underpinnings of the approach specifically in the context of priming students mathematically as a preparation for future learning <sup>(38)</sup>.

### **(i) Invention tasks explained**

The physics invention tasks presented in this paper are based on an instructional approach developed by Dan Schwartz and colleagues at Stanford University. In their invention work, which focuses as we do on developing mathematization of a situation, students are presented with a data set of some kind, and asked to make a judgment or decision. As an example, Schwartz and Martin describe a task in which students are shown four grids, each of which represents a test of a baseball-pitching machine. The grids have a central “X”, the target, and several marks surrounding the target, representing locations where pitches landed. Students are asked to develop a formula for a numerical index that will help consumers judge the reliability of the machines.

Invention tasks are open-ended, with little or no guidelines provided to students about how to resolve the issue. The goal is not for students to arrive at a canonical solution (e.g., the formula for mean deviation in the example above), but rather, to prepare students to learn efficiently in response to subsequent instruction. Invention instruction is consistent with the *necessity principle* of Harel <sup>(39, 40)</sup>, which asserts that productive mathematization flows from the intellectual necessity of resolving a dilemma. In the pitching machine example, students must mathematize in order to decide which pitching machine is best. The focal data set in a well-designed invention task draws student attention to essential features of the context while also obliging them to make judgments about the relevancy of extra information. Then, through their exertion, students articulate the significance of these features, and transcend familiar ways of responding to the situation. These ingrained approaches may impede a student’s ability to understand a novel solution. (Student prior knowledge plays a strong role in what can be learned and how instruction is interpreted. Schwartz and Bransford <sup>(29)</sup> relate the example of how young children, who “know” that the world

is flat, respond to being told that it is round by conceiving of the world as round like a pancake). Through inventing, students come into contact with the deep structure of the situation at hand. It is this structure that must be grappled with in order to craft a solution that is concise, powerful, and general. The familiarity students gain with this structure prepares them to appreciate and assimilate future direct instruction in a way that would not otherwise be possible.

## **(ii) Contrasting cases and learning**

A key feature of invention tasks is their use of contrasting cases. These are instances at once recognizable as belonging to the same phenomena or context, while at the same time having important distinguishing features. Schwartz and Martin <sup>(30)</sup> note that these contrasting cases “are a powerful way to help people discern differentiating properties.” In the pitching machine example, two of the grids have four recorded locations of pitches landing, while the other two grids have five. Students quickly recognize that any method of generating a reliability index must allow for comparisons between tests that involved different numbers of pitches. Although most groups will not resolve this challenge satisfactorily, the stage is set for the students to appreciate during subsequent instruction how the formula for mean deviation does so.

To evaluate the efficacy of this approach, Schwartz and Martin, Schwartz, Martin and Pfaffman, and Schwartz, Chase, Oppizzo, and Chin <sup>(30, 31, 41)</sup> have conducted a series of teaching experiments involving a variety of topics (e.g., statistics, algebra, physical science). Invention instruction has repeatedly been shown to lead to significantly higher performance in future learning when compared with a variety of other instructional methods. Time on task has been controlled for in these experiments; it seems that the initial time spent on student invention work is generally regained in higher efficiency of students’ future learning.



### **III Invention tasks in physics**

In this section we present detailed descriptions of three exemplary invention sequences developed over the course of the project. The first, which is a kinematics index, is a direct adoption with minor modification of Dan Schwartz's group<sup>(31)</sup> for the popping and speeding up indices, and they appear here with their permission. The other sequences, one related to work and the other to rotational kinematics, were developed in response to observed student learning behaviors, and are representative the generative nature of this project. We selected these sequences because they characterize the trajectory this project has followed. What started out as a project to help students develop creativity with ratio reasoning broadened to include reasoning about quantities formed from all arithmetic operations, and eventually to include invention of equations.

#### **III.a. Examples of physics invention tasks**

We have developed and classroom tested over 50 invention tasks in mechanics, electricity and magnetism, and thermodynamics. A core of about 20 of these tasks, in kinematics, Newtonian dynamics, electrostatics, and calorimetry, have undergone extensive use and revision in multiple courses at Rutgers, Western Washington University, and New Mexico State University. Some tasks are posed in physics contexts, and involve quantities such as displacement, change in velocity, or electric force. In these cases, students typically invent ratio or product quantities that a physics instructor would recognize as, for instance, velocity, acceleration, or electric field. Other invention tasks make use of everyday contexts, for example, popcorn kernels popping to fill a bowl, or teams of workers washing cars. In these cases, students invent ratio or product quantities that are similar in logical structure to quantities from physics, but generally are not part of the standard canon of quantities presented in an introductory physics course. Examples include the number of kernels popped during each second and the number of person-minutes required to wash each car. In many cases, invention tasks are used in sequences to build

from everyday contexts to more physics-oriented contexts. In an example presented below, students start with a task in which they invent popping rate as a way to compare different brands of popcorn, and then immediately work on a task in which they “invent” the speed ratio, to compare different sports cars. Having wrestled with the strengths and weaknesses of a variety of approaches, students are poised to appreciate and understand the target procedure. The intent is to activate students’ everyday sense-making resources, and then to provide opportunities for students to extend those resources to more formal scientific contexts.

In many of the existing reports of curriculum development projects involving invention work, students have been presented with challenges that are quite open-ended. In these situations, student groups generally invent very different approaches and procedures, with the “target” solution (i.e., the solution that a scientist or mathematician would recognize as standard) rarely represented. This general framework is believed to play a large role in the effectiveness of invention work in preparing students for subsequent direct instruction.

The physics invention tasks reported here are somewhat less open-ended. In our experience, within a classroom of 10 student groups there will often be only two or three different solutions, with the standard solution almost always represented. In some cases, all student groups converge on the target procedure. While we recognize the value of open-ended challenges and the diverse approaches they generate, we believe that even relatively closed-form invention tasks contribute strongly to the development of student reasoning. In traditional instruction, students are presented with procedures and tools and expected to use them to solve problems that may have little connection to students’ real world experiences. Students have scant opportunity to engage in or appreciate the creative process underlying the procedures and tools. Deep engagement with the relevant reasoning may be inhibited, leading students to understand physics as a set of algorithms. Even closed-form invention work provides students with some opportunity to respond to challenges that arise authentically in context. Increased motivation can lead to deeper engagement; the subsequent productive sensemaking then contributes to enhanced self-efficacy.

By motivating the need for quantitative analysis, rather than imposing it from an authority source, invention work helps create student identity as participant in the intellectual work and owner of the resulting knowledge.

The physics invention tasks, like the context-rich problems<sup>(32,33)</sup>, put students at the center of the action. Students are faced with the need to make a choice or decision of some kind. They must identify the appropriate criteria from among other potentially irrelevant features of the various cases, as well as a method for evaluating the cases or options along those criteria, is up to the students. Students generally work in collaborative groups of three or four. In smaller courses, we have used portable whiteboards to structure full class discussions in which small groups share their approaches, with a goal of reaching consensus on a single “best” approach. In larger courses, this process can occur in an instructor led discussion. After the initial invention work, students are asked to apply the invented quantity in one or more new situations. This provides opportunities for students to practice proportional reasoning in context. At this point, the course instructor often presents the target procedure; i.e., the procedure or tool that has been accepted by the physics community to make comparisons of the type involved in the invention context. Below, we illustrate this general framework with three specific examples, one involving a sequence of three invention tasks leading to the concept of acceleration, and a second involving three tasks leading to the concept of work, and a third that develops the quantities and kinematics equations associated with rotational motion. All of these invention sequences are shown in full in the appendix. Other invention tasks and sequences can be found at the Physics Invention Task website<sup>(42)</sup>.

#### **(i) Kinematics sequence**

In the kinematics sequence, first invention activity students encounter in most introductory mechanics courses, students generate the concept of time rate of change, and create an index using its ratio representation before receiving formal instruction on velocity. The sequence

consists of three invention tasks: Popcorn Popping Index, Fastness Index, and Speeding Up Index. Students build from inventing a simple ratio in an everyday context to inventing a compound ratio (in which the numerator, change in velocity, is itself a ratio) in one-dimensional kinematics contexts.

The popcorn task, developed originally by Dan Schwarz and colleagues<sup>(31)</sup> at Stanford University, opens by framing *index* as a number that helps people make comparisons. *Miles per gallon*, *batting average*, and *school grades* are provided as examples. Students are told that three different companies make popcorn that pops faster or slower, and are asked to invent a popping index to let consumers know how fast each brand pops. Students are given three rules to constrain their work: 1) a given brand of popcorn pops at the same speed, so each brand gets only a single popping index; 2) the same procedure must be used for each brand to find its index; and 3) a big index value should mean that the popcorn pops faster. Students are then presented with a time sequence of sketches showing each of the three different brands of popcorn filling a bowl.

During small group discussions, students typically notice that Hot Pops has the most kernels in the bowl, but also takes the greatest amount of time. By counting kernels, students often then notice patterns in a given set of sketches, for example that Hot Pops has 10 corns in the first 20 seconds, then 5 corns in the next 10 seconds, and 10 more in the next 20 seconds. Some groups notice that the final sketch appears the same for each brand, with the bowl completely full, and wonder if this means that all brands pop equally fast. In some cases, the group will call over an instructor to ask whether or not they are “supposed to actually count kernels.” In other cases, students resolve this issue by realizing that the time at which each brand finishes popping is unknown. After 15-20 minutes, (more time than most instructors think this will take) most groups have invented the ratio of number of corns popped to elapsed time as a means of comparing the brands.

Students then move on to a task in which they are asked to invent a *fastness* index for cars. Students are shown motion diagrams for each of six different cars that drip oil once per second.

A car company makes cars that all have the same fastness, but students are not told how many companies are represented. Students are asked to decide which cars are from the same company. As in the popcorn task, students initially notice that cars travel different amounts of distance in different amounts of time. Some students are initially confused after observing that the cars have different starting locations. In some cases, groups debate whether oil drops that are more closely spaced or more widely spread indicate a higher speed. It is common for groups to measure a time interval by counting the number of oil drops, rather than the number of gaps between drops. After resolving these or other difficulties through discussion, most groups decide to use the distance between consecutive oil drops as a fastness index to compare the cars.

These two invention tasks are followed by a short set of sense-making and application questions, which provide students practice with a variety of distinct modes of proportional reasoning, such as interpreting ratios in context, constructing ratios from measured values, and applying ratios to make quantitative predictions. (For a detailed discussion of these and other modes, or subskills of proportional reasoning, see Boudreaux, Kanim and Brahmia <sup>(43)</sup>). Depending on the student population and instructional context, the third task in the kinematics sequence may follow *Fastness* directly, or may come after intervening instruction on velocity (or other aspects of kinematics).

The Speeding Up Index invention task uses the same context as the Fastness task: a set of cars that each drips oil once per second. In this case, however, speedometer readings are included for some of the instants corresponding to oil drops. Students are asked to make a speeding up index for each car, with a bigger index meaning that a car speeds up more rapidly. Students must use their index to decide which cars come from the same company. (A company always makes its cars speed up in the same way.)

The difficulty mentioned above, in which student compute a time interval by counting drops instead of gaps, is particularly prevalent here. Many groups do readily extend their work from the Fastness task to this new setting, computing a change in velocity, rather than a displacement, and

combing it with elapsed time to form a ratio corresponding to acceleration. In fact, our experience suggests that some groups initially engage less deeply with the Speeding Up task than they did with Fastness, perhaps recognizing an algorithm along the lines of “to form the index, compute a change and divide by a time interval.” The questions following the invention task, in which students interpret and apply their invented index, can push students back toward sense-making. One question asks students to interpret the index value in everyday language. Some groups engage in considerable discussion to arrive at something along the lines of “the change in speed that occurs in each second.” Additional questions ask students to carefully articulate the difference between *fast* and *speeding up quickly*, and between *slow* and *speeding up slowly*, designed to help students connect their proceptual understanding of the ratios to the conceptual understanding of the physics quantities.

The kinematics invention sequence introduces students gently to the process of being creative with mathematics, and builds up to inventing acceleration, which is well known to be fraught with conceptual difficulties in physics <sup>(44, 45)</sup>. Subsequent formal instruction on kinematics introduces students to velocity and acceleration as formal names for the fastness index and the speeding up index. These new quantities thereby occupy less space in students’ working memory allowing them to focus on the structural reasoning that is an important instructional goal for kinematics.

## **(ii) Work sequence**

The shift from the *Fastness Index* invention task to the *Speeding Up Index* task requires flexibility in applying the concept of rate of change to a new situation. We find that students (as well as physics faculty in professional development workshops) soon begin to reach for ratio as a measure when they are confronted with new challenges, as they should since that reasoning has served them well thus far. When confronted with the scenario of the first product quantity in the

curriculum, the *Car Washing Inefficiency Index* task, many quickly invent the ratio of number of minutes to number of persons on the team. Typical conversations involve the (correct) interpretation of this ratio as “the number of minutes per person.” Groups will often agree that this is a good solution to the problem of judging the relative inefficiency of the different teams. They then find that the index values for different teams at the same site do not match, in violation of the rules that have been externally imposed. This realization generally causes strong cognitive dissonance and a return to sense-making.

To show just how challenging it is to create a product quantity, we include below an excerpt from a dialog between 4 physics faculty members (P1, P2, P3, P4) attending an AAPT workshop during which they collaboratively worked through the *Car Washing Inefficiency Index* task after having completed the Kinematics Sequence described above.

P2: Two different teams from each location-

P3: -Yeah, so that's the left picture and the right picture.

P2: Okay, so there are two teams... and we want an index for each location-

P2: Each location.

P3: So we don't want the index per team-

P2: -Right.

P3:-Necessarily.

P2: Well, they're supposed to be the same.

P1: Are they the same?

P2: It's supposed to be. Teams from the same locations should have the same-

P2: Aaah, okay- okay. That-yeah, alright.

P4: So I feel like time per person.

P3: So you want- so you feel more time-that's less efficient?

P4: Yeah.

P1: And if it takes more people were supposed to (inaudible)?

P2: Oh, wait.

P3: But she's doing time per person. So if it's 10.2 minutes for two people-

P4: Yeah, wait but that-that isn't the same.

P2: It's not?

P3: It's not? So you get 10 minutes for two people versus 6.8 for three?

P4: Right.

P2: So that's 3.4.

P3: 10 for two.

P2: No, 10 point two-

P2: 10 per two is 5.1.

P4: Yeah.

P3: And it's per person?

P2: Yeah, that doesn't-that's not good.

P3: And 6.8 per three. So that's two-

P2: -And, and-

P4: -Oh right because that wouldn't-

P1: -And we want time per people, right?-

P2:-And we want-and we-yeah.

P3: Wait, what did you- I didn't get it. You said oh right.

P1: Time-time times people.

P2: Time times people. Yeah, that's right.

P4: That would make sense, yes.

P1: Right, exactly.

P2: And that should also get us back to this (paper?) values should correspond to less efficient.



P4: Yeah.

P3: Wait, why did you say that makes sense? Because I didn't get the-

P4: The number. So-

P2: -You want man hours.

P1: (inaudible)

P4: Right, you spend 20.4 minutes washing the first car and then-

P3: -So what is the number of minutes per person telling me?-

P4:-you pay for it.

P2: The number of minutes per person isn't telling me anything.

P3: It's a useless number, we can (inaudible) everything?

P4: Yeah, it's not... (group laughs).

P2: Well, it's probably not useless. It's just not getting at what we want here.

Part of competency is the conscious attention paid to searching for useful invariants, and productive intuition about which quantities are likely to be invariant. Experts tried and discarded ratio reasoning here because it was in conflict with what they knew had to be invariant, and the invariant reasoning won. Actual students struggle much longer than these faculty experts did; their appreciation of invariance is not nearly as deeply ingrained.

The resistance that students demonstrate to letting go of reasoning that has been productive in the past is an indication of a lack of *generativity* in the use of mathematics. In a situation requiring invention of a new mathematical approach, student participants invariably return to what is familiar, not recognizing that the ratio is no longer appropriate for making the required comparison. Students who have already demonstrated flexibility at spontaneously using ratio reasoning in a variety of contexts struggle when generativity is needed.

The *Weightlifting Index* in this sequence is much easier for students than the *Car Washing Inefficiency Index* is. Students typically engage in dialog in which they compare ratio reasoning

to product reasoning, and fairly quickly converge on the merits of the product because they recognize that bigger is better for both the mass and for the height.

The *Job Difficulty Index* poses difficulty with numeric complexity. Brahmia, Kanim and Boudreaux <sup>(3)</sup> have shown that student reasoning on algebraic tasks in physics can be sensitive to the complexity of the numbers involved. We include large numbers, and variables which renders this task much more challenging for students than the *Weightlifting Index*. The subsequent follow up questions pertaining to interpreting the value of the product is much more challenging for the students than it is for the ratio quantities. There is in fact no one correct interpretation. There are two convenient ones, namely force necessary to move the object one meter or the distance the object would move if pushed by a force of one Newton, but there are also an infinite number of others. There is no “unit product” analogous to a unit rate. The exercise helps students to appreciate that a product of physical quantities is not sequential addition as it is with pure numbers, but that it is in fact something else.

Subsequent formal instruction in Energy includes work, kinetic energy, spring and gravitational potential energies and an increase in internal energy <sup>(46)</sup>, are all product quantities. Students have a tendency to focus on one of the quantities and ignore others <sup>(47)</sup>. The work invention sequence primes students to focus on more than one of the quantities that make up a product quantity.

### **(iii) Rotational kinematics sequence**

The observed struggles with generativity and with the use of variable in Brahmia, Kanim and Boudreaux <sup>(3)</sup> led us to develop a *spinning disk sequence* that engages students in the process of inventing quantities that describe rotational displacement, rotational velocity and rotational acceleration and then use them to invent the kinematic equations for rotational motion with constant angular acceleration. We were surprised at how much the students were able to generate

on their own, and how creative their symbolizing was (e.g., one group used stars and smileys to represent angular displacement and velocity, another circles and triangles).

This sequence is the first to engage the students in symbolizing in a significant way. The students engage and struggle with notions of covariational representations <sup>(26)</sup>, and distinguishing between parameters and variables in an expression <sup>(48)</sup> that they have generated.

The most challenging part of the sequence is the last question, in which students were asked to compare measurements that were not identical but were in fact in agreement. We note that students continue to struggle with this well-known difficulty <sup>(49-52)</sup>. It is a reminder to us that students resist flexible interpretations of numeric quantities, and that developing flexibility regarding quantification and measurement is a slow process.

### **III.b. Sociocultural considerations**

Invention tasks generate a well-defined set of course norms: struggle is a communal, there are no dumb ideas, creativity is valued, and the answer can't be Googled. Goldberg, Otero and Robinson <sup>(36)</sup> describe these norms as being amongst those that are conducive to effective physics instruction. After the very first task students realize that they are all going to struggle together and that the kinds of things people learned by excelling in traditional (high school) physics classes were not going to be particularly helpful. In fact one student, who happens to be Latina from an urban socioeconomically disadvantaged high school, reported early on,

*“I wish I hadn’t taken AP physics in high school because it ruined my thinking. I want to get better at inventing things in physics.”*

This particular student did eventually become very creative, but it took more than a semester to see evidence of this in her test-taking. The universal novelty of the tasks develops self-efficacy (i.e., the belief in one's own capacity to succeed); the tasks are challenging, inviting and doable. Self-efficacy is an important characteristic of successful learners in physics <sup>(53)</sup>. In addition, the

fact that there are usually a few equally useful indices that are created by student groups in response to any given task, a classroom norm develops that physics learning is fundamentally both a creative and flexible process. Flexibility is a characteristic of expert-like mathematization<sup>(13, 24)</sup>.

We note that social consensus-building norms have been reported on by Ross & Otero<sup>(54)</sup> as characteristic of effective learning environments for physics students in the economically disadvantaged high school they studied. The researchers report that students positively identified with physics instruction “that valued their naïve and developing understandings, and that shifted authority for validating science knowledge from the instructor...to social consensus.” We share results in Section 5 of this paper from our preliminary study that support our claim that physics invention tasks are effective for students for whom inclusion involves valuing naïve views and social consensus.

#### **IV Implementing invention tasks**

We have used invention tasks in a variety of small and medium enrollment (100-200 students) learning environments. They are regularly used at Rutgers in a first year calculus-based course (enrollment ~150) in recitation sections of ~20 students each, and as part of the curricular materials provided to physics teachers through the Graduate School of Education. They are also used regularly as lecture activities at WWU in courses that include calculus-based, algebra-based and during small group meetings (~20 students) in the general education physics courses in addition to episodic use in conjunction with the PET<sup>(36)</sup> curriculum for future teachers, and they are used episodically at NMSU in the lecture of some of the calculus-based physics courses.

We emphasize that invention tasks are an add-before activity designed to complement curricular materials. They take generally 20-30 mins for an entire sequence initially, but students become more efficient with inventing and we usually schedule ~20 minutes for invention as we introduce new physical quantities, usually ~ one sequence every two weeks.

Inevitably, developing a depth of proceptual understanding requires sacrificing some breadth of “coverage”. We assert emphatically and unapologetically that this trade-off is worth it. There is a certain amount of timesaving in the subsequent formal instruction because students have been primed to understand the lecture. As an example we note that the subsequent lecture on rotational kinematics this year was facilitated so much by the new invention task described above that we will need to rearrange the syllabus next year. We gained nearly one full lecture of formal lecture time by not having to psychologically prepare students for the barrage of Greek symbols; they already understood that we had to pick something so why not Greek letters, they already had the experience of using their prior knowledge with strange symbols and were primed to view the kinematics equations as a structural template, not a specific set of formulas to memorize.

## **V     Assessing student learning**

### **V.a. Research Design**

We compare the performance of students enrolled in the two first semester introductory physics (mechanics) courses for engineering students, EAP I (Extended Analytical Physics I <sup>(55,</sup> <sup>56)</sup>) and AP I (Analytical Physics I). Table V highlights the demographic differences between the course populations.

Student placement in EAP I is based on low math placement test scores (concurrent placement in pre-calculus). Approximately 30% of the EAP I students each year are in the Educational Opportunity Fund program, which provides financial and other support services to first-generation, economically disadvantaged students. Many of these students are from under-represented minority groups.

**Table V:** Demographic comparison between EAP I and AP Fall 2013

	<b>EAP I</b> <i>Underprepared</i>	<b>AP I</b> <i>Mainstream</i>
# of students	~140	~700
Mean Mathematics SAT (2013 test)	610	680
% African American or Latino	40%	12%
% female	30%	21%
Median Mean Household Income of sending district <sup>(57)</sup>	\$73,000	\$104,000

EAP I has the additional goal of helping underprepared students catch up to their better prepared peers by the time they finish their first year of study. Each week the students meet twice for recitation and twice for lecture in EAP I, thus spending twice as much time in class as the AP I students do. There is no required lab in the first year for either course. Recitations are run by graduate teaching assistants in both courses, and are limited to an enrollment of 24 students in EAP I, and 30 students in AP I. The total enrollment in EAP I ranges between 120-170 students each year, while the AP I enrollment is 600-800 students. A rough rule of thumb to compare the structures of EAP I to AP I is half as many students in lecture and twice as much contact time.

EAP I was created with the intention of developing mathematical reasoning skills to compensate for the weak mathematical preparation as measured by the Math Placement Test at Rutgers. EAP I is a calculus-based course and the students don't start calculus until their second semester. An important component to developing an inclusive scientific community is the development of scientific abilities <sup>(49)</sup>. The Investigative Science Learning Environment (ISLE) <sup>(10)</sup>curriculum got its Rutgers' start in EAP I in 2001, and it provides a setting for students to use

the processes of science and cognitive strategies that help them construct physics and mathematics ideas. They learn to apply these processes to real-world problems. We have refined and adapted features of ISLE in the context of this course, but in the absence of a lab EAP I is not a full-blown ISLE course. The current curriculum can be thought of as a hybrid of ISLE infused with Invention Tasks.

The API course has fairly traditional lectures, but uses clickers to encourage student engagement. The recitations are a combination of a hands-on minilab <sup>(58)</sup>, a quiz and traditional instructor problem solving at the board. The syllabi of AP I and EAP I include the same chapters from the textbook over the course of the year, and include thermodynamics and waves topics in addition to mechanics.

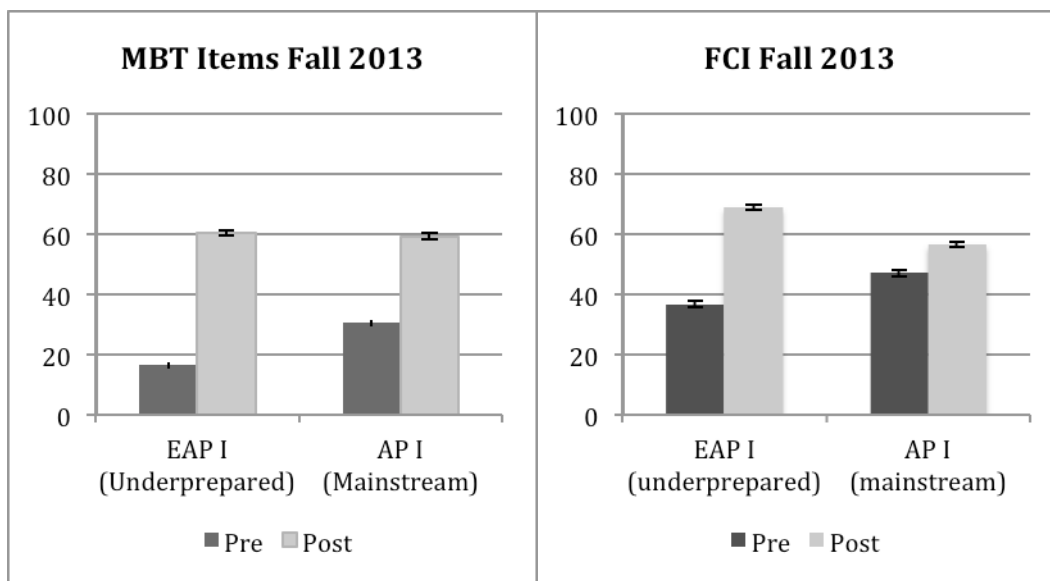
The quantitative results presented in this paper derive from several measures of the students enrolled in EAP I and AP I. The FCI (Force Concept Inventory) <sup>(6)</sup> was bundled with selected problems from the MBT (Mechanics Baseline Test) <sup>(7)</sup> and a suite of multiple-choice proportional reasoning items from PRAS <sup>(3,43)</sup>. The faculty teaching AP I selected 7 problems for the pretest from the MBT that were consistent with their teaching goals and the instructors from both courses agreed to put those questions on the final exam. The pretest was administered as an ungraded quiz under exam conditions at the beginning of the course. In a single sitting, students first completed the proportional reasoning suite, followed immediately by the Force Concept Inventory, and then the selected MBT questions. The students were not constrained by time and were awarded credit for earnest effort. The students took the post test, administered under the same conditions, during the second to last week of the course. The posttest was identical with the exception of one change, the MBT questions were administered as part of their final exam two weeks later.

In addition to the in-class testing, the students took the CLASS physics online as a pretest <sup>(1)</sup> during the first week of class, and as a post test during the last week of class. Completion of the pre and post online survey replaced a quiz grade in both courses. Many students in API either did not participate, or did not agree to have their data be part of this study. We attribute this in part to

survey fatigue. We report on matched pre/post results, and we have eliminated any tests that showed patterns of low effort (long strings of same answer).

### V.b. Findings

In Figure 17a we present the results on the MBT questions (#s). The questions were selected by the instructors of AP I to measure the problem solving capacity of the students, reflecting an important goal for both courses. We present matched pre/post data in both cases. We note that the students who did not take the final at the regularly scheduled time are not included in MBT sample. On the MBT questions the mean and the standard deviation of the mean for EAP I (n=114) is  $16.4\% \pm 1.4\%$  on the pre test and  $60.4\% \pm 1.6\%$  on the post test, and the for AP I (n=624) is  $30.4\% \pm 0.9\%$  in the pre test and  $59.2\% \pm 1.0\%$  on the post test. While there is a measurable difference on the pretest, there is no measurable difference on the post test.



**Figure 17:** MBT and FCI results: a) Mechanics Baseline Test items;  $\sigma_{\text{mean}}$ : EAP I (n=114) 1.4%(pre), 1.6%(post); AP I (n=624) 0.9%(pre), 1.0%(post) b) Force Concept Inventory;  $\sigma_{\text{mean}}$ : EAP I (n=108) 1.4%(pre), 1.5%(post); AP I (n=576) 0.8%(pre), 0.8%(post)



Figure 17b shows the results for the FCI. We present matched pre/post data in both cases. The mean and the standard deviation of the mean for EAP I (n=135) is  $36.6\% \pm 1.4\%$  on the pre test and  $68.9\% \pm 1.5\%$  on the post test, and the for AP I (n=757) is  $47.1\% \pm 0.8\%$  in the pre test and  $56.6\% \pm 0.8\%$  on the post test, which correspond to normalized gains of 0.51 and 0.22, respectively. We compare this gain to a baseline measurement taken in EAP I prior to the introduction of the invention tasks (see Table VI) in order to separate out the effects of invention instruction.

Although there were minor changes over 10 years (different students, different TAs, different textbook), the only major change to the course was the introduction of the physics invention tasks. We emphasize that the invention tasks are an add-before activity and do not represent a curriculum on their own. We believe that the gains we see are a case of the whole being greater than the sum of its parts. We recommend using invention tasks as preparation for learning with research-validated curricula, like ISLE, Modeling, or Tutorials, to name a few. An important result that we see in EAP I is the positive gains made on the CLASS, especially in categories that directly reflect student beliefs about the role of mathematics in physics (see Fig. 18).

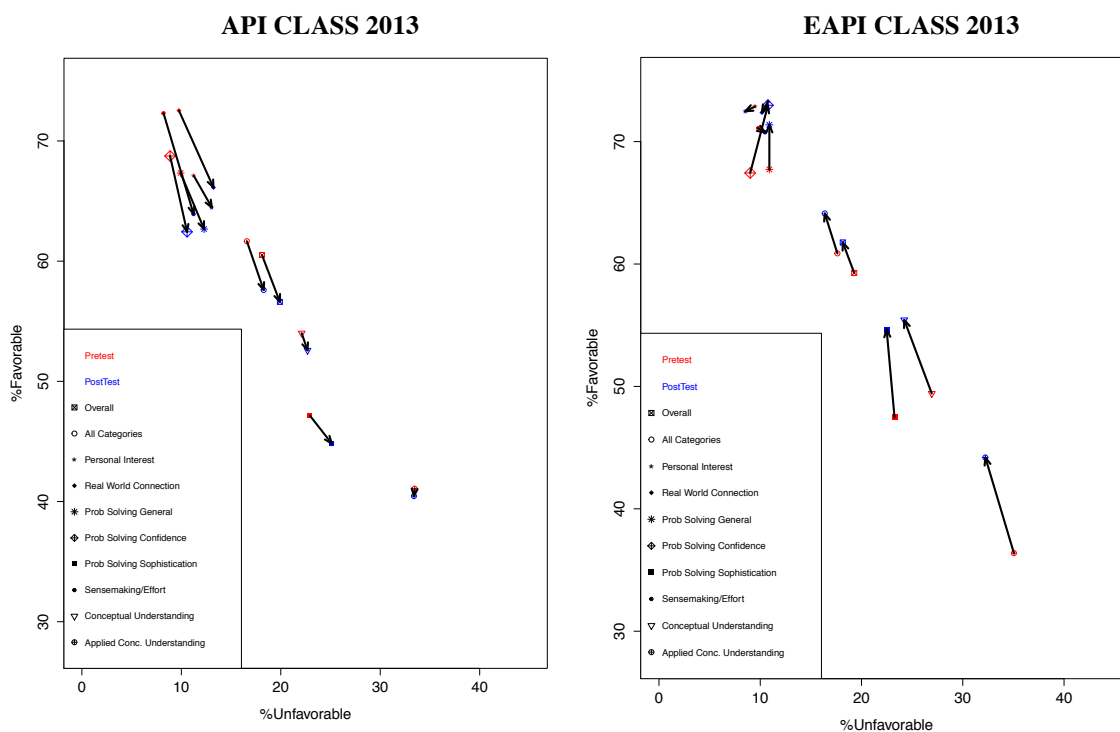
**Table VI:** FCI comparison before and after introduction of invention instruction; the pooled standard deviation of the mean is 1.8%.

Year	Method	Pre	Post	Normalized gain
2003 (n=123)	Modified ISLE	40%	59%	0.32
2013 (n=135)	Modified ISLE + invention instruction	37%	69%	0.51

It is particularly challenging to have anything but negative gains on the CLASS, even in courses using interactive engagement. The gains seen here are comparable to those reported on in highly

successful lower enrollment calculus-based courses<sup>(59)</sup>. Although we do not have EAP I baseline data for comparison on the CLASS, we compare these results to an algebra-based course with a lab and with a comparable enrollment taught at Rutgers University; we note that the CLASS gains for that course are fairly neutral, similar to results reported using other research-validated curricular materials in medium to large-enrollment algebra-based courses. We attribute the positive CLASS gains in EAP I to the combination of a modified ISLE curriculum and the Invention Tasks used as a preparation for learning.

**Figure 18:** CLASS result, by category. The favorable scores are plotted on the vertical axis and the unfavorable on the horizontal axis. Improvement is represented by movement up and to the left.  $n_{API}=270$ ,  $n_{EAPI}=77$



## VI Conclusion

Mathematical imagination, or a flexible and generative use of mathematics, is the context of physics is an instructional goal that resonates with physicists. We do not want to give the impression that it is fully formed in our students and just waiting to be unleashed. Using math

generatively requires different skills than using math procedurally. Students must be willing to try math without knowing whether it will be productive. Developing mathematical imagination not only takes courage, but it also takes effort. The invention tasks are based on instructional design principles that : (1) establish sociocultural norms conducive to building courage, and (2) engage students in such a way that they are motivated to put out the necessary effort.

It is well known in mathematics education research that quantification is challenging for students; we claim that it poses a significant challenge in the introductory physics where students are fire hosed with new and abstract quantities throughout the course. We argue that even curricula shown to be highly effective at meeting many of the instructional goals of introductory physics are not as effective as they could be at activating reasoning resources that students have already developed, and these curricula generally do not result in students becoming more expert-like in their thinking about math in physics. We propose that student algebraic reasoning resources are fairly primitive, and that physics is an optimum discipline for developing them further. We propose that the instructional model of preparation for future learning developed by Bransford and Schwartz can be effective in the context of quantification in the introductory physics course.

We presented our solution to this problem in the form of Invention Tasks in physics, which we argue are effective precisely because they engage students creatively in physics at a very fundamental level. We presented preliminary data, which indicates that students are learning by every measure we used, and they are developing more expert-like attitudes about the use of mathematics in physics. Generating something new requires a courage that we foster with invention tasks. Students develop their capacity to check to see whether their intuitions match the mathematics that they generate, and through trial and error get better at iterating toward better solutions. When instructors see the invention tasks, especially the first in the series, they cannot believe that their students will struggle. We stress unequivocally that they will struggle, and that

their struggle becomes their motivation to understand. A 10-year veteran NJ high school physics teacher who attended a PUM workshop later blogs:

*"I confess I didn't think the popcorn popping index to fastness index progression was going to lead to such vibrant conversations and there were actually a few exclamations of "Oh, I get it!" as we went through the activity. I skipped over those activities with my honors classes, but then went back to them after seeing how well they worked in my College Prep class."*

We have used these materials as part of instruction with over a thousand students, and every time the experience is similar. We have workshopped these materials with hundreds of college faculty and high school teachers, and the participants wrestle and struggle every time to make meaning in this way. This kind of struggle is precisely what is necessary to motivate students to both want to understand, and to believe that they can understand. We assert emphatically that vibrant conversation and collective, engaging struggle is foundational to creating a productive learning environment for all students in physics. We have provided a method that does both, and invite you to explore for yourselves <sup>(42)</sup>.

There is still much work to do in exploring student understanding of quantification in physics, and the role it plays in developing expert-like mathematization. Maybe the documented difficulties students have with acceleration, momentum, etc. are all connected deep down by difficulty with the nature/notion of quantity. How might other aspects of mathematization, such as representations and methods, develop more effectively if students engage in invention activities involving the development of a method (as in the spinning disk Invention Task) or in a new representation? At the outset of this work, we knew it would be exploratory. We hope it will also be generative as other researchers view their work through a lens of mathematization, and all that it entails for our students.

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## **CHAPTER 5: Summary of the thesis**

A desirable outcome of a physics course is to help students build understanding that links concepts to each other and to their mathematical expression so that students are able to approach authentic physics problems in mathematically sophisticated ways. While some students achieve this goal, a large fraction of those who take introductory physics do not.

The papers that make up this dissertation tell the story of a research-based approach to developing expert-like mathematization in the introductory physics course. The work itself is foundational as it builds bridges between physics education and mathematics education scholarship specifically in the following areas: assessing the cognitive resources associated with mathematization, developing a model for instructional design specifically for meeting enhanced mathematization as a teaching goal, and designing curricula for developing mathematization in the introductory physics course.

The story this dissertation tells begins with a study of the students' preparation to mathematize in the context of physics. We develop a suite of questions that assess the coordinated set of resources associated with students' proportional reasoning skills and find that many of the issues that physics students struggle with fall under three main categories: *generalized structural reasoning*, *quantification*, and *symbolizing*. We offer and use this categorization as a guide for instruction and curriculum development.

Next we use our measures of student preparation to estimate a starting "state" of a typical student in a calculus-based physics course and describe a model of instruction that takes students from their starting state to our desired endpoints. We build a model in terms of a quadrant of mathematization, which allows us to consider both the physics and mathematics implications of the changes we make, and to achieve the goal of increased flexibility and generativity.

Finally we design curricular materials that foster mathematical creativity to prepare students for better understanding of physics reasoning. We present physics invention tasks that provide opportunity for students to gain experience with mathematics as a generative endeavor, in which quantities and equations are invented to solve specific problems. We share our promising findings of a preliminary study of the effectiveness of physics invention tasks at improving student reasoning, problem solving and learning attitudes about mathematics in physics.

This work provides a base from which a rich body of research can emerge. There is much work to be done in measuring the initial states of students' mathematization. Our assessment items are the first to probe deeply into this area of learning, and our work is quite preliminary. Our instructional model of mathematization is crude, and will likely see much

iteration in the coming years. Invention tasks are one solution to a problem; there are many more out there waiting to be created. And finally, mathematization doesn't stop in the introductory course. The pathway to expert-like mathematization in the physics courses beyond the introductory level is an active field of physics education research. Our approach to researching mathematization can be extended, perhaps fruitfully, into exploring the further development of mathematization in physics. The potential for invention tasks in these physics courses is promising and is already drawing interest from researchers in this area.

The story told represents the articulation of a problem, the design of a method of its assessment, the modeling of a solution, the design and implementation of a practical intervention and a preliminary measurement of the effectiveness of the intervention. The ideas that have been brought together here hail from a variety of areas of scholarship and represent an interdisciplinary, research-based approach to addressing an important problem in introductory physics. By making crosscutting procepts explicit and available to all students beginning with their first physics course, we hope to produce a more diverse group of learners, and that all students will begin to mathematize in a more expert-like physics way.

### **APPENDIX A: Glossary of Terms**

<b>coordinated set of resources</b>	reasoning that requires activation of multiple resources
<b>flexible</b>	having the capacity to spontaneously activate productive mathematical reasoning independent of context
<b>generalized structural reasoning</b>	recognizing a mathematical structure (e.g. a $1/r^2$ force) and using it to guide thinking, regardless of the context
<b>generative</b>	having the ability to create a novel symbolic solution to a problem
<b>proceptual divide</b>	intellectual gap between students who master procedural mathematics and those who master proceptual mathematics
<b>proceptual understanding</b>	both a procedural mastery and a conceptual understanding
<b>quantification</b>	the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship with its unit
<b>resources</b>	small-scale models of reasoning which act as building blocks of analytical, procedural, conceptual, and epistemological reasoning
<b>symbolic forms</b>	simple conceptual mental schema associated with a pattern of symbols in an equation (example: $\square + \square \dots = \square + \square \dots$ represents "balancing")
<b>symbolizing</b>	back/forth shifting between symbols and meaning to facilitate discourse
<b>CLASS</b>	Colorado Learning Attitudes about Science Survey
<b>FCI</b>	Force Concept Inventory
<b>MPEX</b>	Maryland Physics Expectations survey
<b>PFL</b>	Preparation for Future Learning

## APPENDIX B: Assessment Items

### Four questions involving generalizing structure:

**Traxolene Question and Olive Oil Question:** Structurally isomorphic diagnostic questions structural generalizability of the unit rate,  $\sigma_{\text{pooled}}=3.7\%$  (freshmen engineering students, post test)

a) Chemistry Context - well known formula (N=188,  $\sigma=3.5\%$ )

You are part of a team that has invented a new, high-tech material called “traxolene.” One gram of traxolene has a volume of 0.41 cubic centimeters. For a laboratory experiment, you are working with a piece of traxolene that has a volume of 3 cubic centimeters. Which of the following expressions helps figure out the mass of this piece of traxolene (in grams)?

- a.  $\frac{3}{0.41}$  **(65% correct)**      b.  $\frac{0.41}{3}$       c.  $3 \cdot 0.41$       d.  $(3+1) \cdot 0.41$       e. none of these

b) Everyday Context – no formula (N=161,  $\sigma=3.9\%$ )

You go to the farmer’s market to buy olive oil. When you arrive you realize that you have only one dollar in your pocket. The clerk sells you 0.26 pints of olive oil for one dollar. You plan next week to buy 3 pints of olive oil. Which of the following expressions helps figure out how much this will cost (in dollars)?

- a.  $\frac{3}{0.26}$  **(56% correct)**      b.  $\frac{0.26}{3}$       c.  $3 \cdot 0.26$       d.  $(3 + 1) \cdot 0.26$       e. none of these

**Square Buildings Question and Force Vector Question:** Structurally isomorphic diagnostic questions testing ratio formation, correct answer is choice (a) in both cases;  $\sigma_{\text{pooled}}=1.9\%$   $n_{\text{matched}}=408$

**a) Square Buildings Question: Everyday Context**

(freshmen engineering students, administered in the mechanics course, post test)

You are riding in an airplane. Below you see three rectangular buildings with the rooftop dimensions:

*Building A: 77 ft by 93 ft*  
*Building B: 51 ft by 64 ft*  
*Building C: 96 ft by 150 ft*

You are interested in how close the shapes of the rooftops of the buildings are to being square. You decide to rank them by “squareness,” from *most* square to *least* square. Which of the following choices is the best ranking?

- a. A, B, C (17% correct)   b. B, A, C (72%)   c. C, A, B (4%)   d. C, B, A (3%)   e. B, C, A (4%)

**b) Force Vector Question: Physics Context**

(freshmen engineering students, administered in the chemistry course, post test)

Each of three different objects (A, B, C) experience two forces, one in the +x direction and one in the +y direction.

Rank each object according to how close the direction of the net force is to a 45° angle between the x-direction and the y-direction, from *closest* to 45° to *farthest* from 45°.

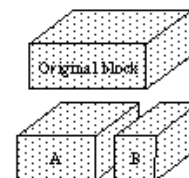
Object	Force in +x-direction	Force in +y-direction
A	77 Newtons	93 Newtons
B	51 Newtons	64 Newtons
C	96 Newtons	150 Newtons

- a. A, B, C (22% correct)   b. B, A, C (51%)   c. C, A, B (20%)   d. C, B, A (5%)   e. B, C, A (2%)

**Questions involving context: Mass Density Question and Charge Density Question:** Structurally isomorphic diagnostic paired questions (from Kanim {Kanim, 1999, #57950}) testing generalizability of density,  $\sigma_{\text{pooled}}=2.7\%$ .(sophomore engineering students)

a) Physical Science Context - well known formula (N=262,  $\sigma=2.2\%$ )

A uniform block of cheese is cut into two unequal pieces, labeled A and B. A correct ranking of the mass densities (from largest to smallest) of the original block, the largest piece (A), and the smallest piece (B) is:

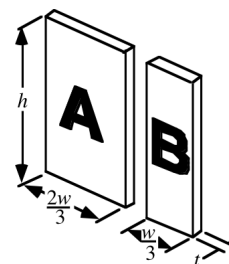


- (a) Original block, largest piece, smallest piece.
- (b) Smallest piece, largest piece, original block.
- (c) Largest piece, smallest piece, original block
- (d) All mass densities are the same. **(80% correct)**
- (e) Not possible to determine without additional information.

b) Physics Context – less commonly known formula (N=258,  $\sigma=3.1\%$ )

A plastic block of width  $w$ , height  $h$ , and thickness  $t$  has a positive charge  $+Q_0$  distributed uniformly throughout its volume. The block is then broken into two pieces, A and B as shown. Which is a correct ranking of the charge densities of the original block ( $\rho_0$ ), piece A ( $\rho_A$ ), and piece B ( $\rho_B$ )?

- (a)  $\rho_0 > \rho_A > \rho_B$
- (b)  $\rho_0 = \rho_A = \rho_B$  **(55% correct)**
- (c)  $\rho_0 < \rho_A < \rho_B$
- (d)  $\rho_A > \rho_B > \rho_0$
- (e) There is not enough information to compare the charge densities.



**Questions involving the use of variables and numeric complexity: Rice Questions and Heffalumps and woozles:**

a) **The Rice Questions:** Structurally isomorphic diagnostic questions testing effect of numeric complexity, before and after a semester of analytical physics,  $\sigma_{\text{pooled}}=2.9\%$ .(freshmen engineering students, administered in mechanics course). The quantities in square brackets represent the quantities used in version 2 and 3, respectively. There was no statistical difference between student performance using decimals, and a fourth version using fractions. The number of students who took each test is shown in curly brackets in the table below.

Bartholomew is making rice pudding using his grandmother's recipe. For three servings of pudding the ingredients include 4 [0.75,  $N$ ] pints of milk and 2 [0.5,  $5/8$ ] cups of rice. Bartholomew looks in his refrigerator and sees he has one pint of milk. Given that he wants to use all of the milk, which of the following expressions will help Bartholomew figure out how many cups of rice he should use?

- a)  $2/4$  [0.5/0.75,  $(5/8)/N$ ]  
 b)  $4/2$  [0.75/0.5,  $N/(5/8)$ ]  
 c)  $4 \times 2$  [0.5  $\times$  0.75,  $N \times (5/8)$ ]  
 d)  $(2 + 1) \times 4$  [(0.5 + 1)  $\times$  0.75,  $((5/8)+1) \times N$ ]  
 e) none of these

Version	Pre(%)	Pos(%)
Whole numbers	79 <sup>(273)</sup>	77 <sup>(262)</sup>
Decimals	61 <sup>(264)</sup>	67 <sup>(270)</sup>
One variable	44 <sup>(33)</sup>	34 <sup>(227)</sup>

**b) Heffalumps and woozles:**

Consider the following statement about Winnie the Pooh's dream: "*There are three times as many heffalumps as woozles.*"

Some students were asked to write an equation to represent this statement, using  $h$  for the number of heffalumps and  $w$  for the number of woozles. Which of the following is correct?

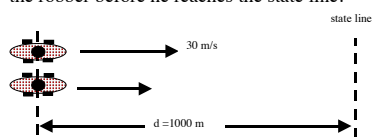
- a.  $3h/w$  (3%) b.  $3h = w$  (36%) c.  $3h + w$  (2%) d.  $h = 3w$  (49% correct) e. a & b (9%)

From: Torigoe, E., and Gladding, G., "Same to us, different to them: Numeric computation versus symbolic representation", AIP Conference Proceedings 883, pp.153, (2007).

The correct answer is bold and the percentage of students who selected each answer is shown in parentheses.

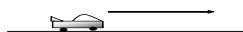
### Final 1

A bank robber is racing towards the state line at a constant speed of 30 m/s and passes a cop who is parked 1000 m from the state line. At the moment the robber passes him, the cop accelerates from rest with a constant acceleration to try to catch the robber before he reaches the state line.



6. What is the minimum acceleration that the cop must have in order to catch the robber before he reaches the state line? (Ignore any reaction time.)

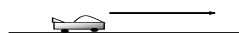
- a.  $0.23 \text{ m/s}^2$  (< 1%)   b.  $0.45 \text{ m/s}^2$  (7%)   c.  $0.90 \text{ m/s}^2$  (6%)  
**d.  $1.80 \text{ m/s}^2$  (85%)**   e.  $3.60 \text{ m/s}^2$  (< 1%)  
 (More than 1 choice: 2%)



7. A car can go from 0 to 60 m/s in 8 seconds. At what distance,  $d$ , from the start (at rest) is the car traveling 30 m/s? (Assume a constant acceleration.)

- a. 30 m (< 1%)   **b. 60 m (94%)**  
 c. 90 m (< 1%) (No corresponding choice)  
 d. 120 m (2%)   e. 240 m (2%)  
 (More than 1 choice: 0%)

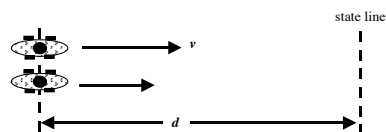
### Final 2



6. A car can go from 0 to a speed  $v_f$  in  $t$  seconds. At what distance,  $d$ , from the start (at rest) is the car traveling  $(v_f/2)$ ? (Assume constant acceleration.)

- a.  $d = v_f t$  (< 1%) (No corresponding choice)  
 b.  $d = \frac{v_f t}{2}$  (25%)   c.  $d = \frac{v_f t}{4}$  (27%)  
**d.  $d = \frac{v_f t}{8}$  (45%)**   e.  $d = \frac{v_f t}{16}$  (< 1%)  
 (More than 1 choice: 3%)

7. A bank robber is racing towards the state line at a constant speed of  $v$  and passes a cop who is parked  $d$  from the state line. At the moment the robber passes him, the cop accelerates from rest with a constant acceleration to try to catch the robber before he reaches the state line.



What is the minimum acceleration that the cop must have in order to catch the robber before he reaches the state line? (ignore any reaction time).

- a.  $a_{\min} = (2v)^2/d$  (3%)   **b.  $a_{\min} = 2v^2/d$  (57%)**  
 c.  $a_{\min} = v^2/d$  (6%)   d.  $a_{\min} = v^2/(2d)$  (29%)  
 e.  $a_{\min} = v^2/(4d)$  (< 1%)  
 (More than 1 choice: 4%)



## APPENDIX C: Invention Tasks

### **Kinematics Sequence: *Popcorn popping, Fastness, and Speeding Up***

An *index* is a number that helps people compare things.

- *Miles per gallon* is an index of how well a car uses gas.
- *Batting average* is an index of how well a baseball player hits.
- *Grades* are an index of how well students perform on a test.

We want you to invent a procedure for computing an index that helps make comparisons.

#### A. Popping Index

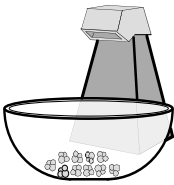
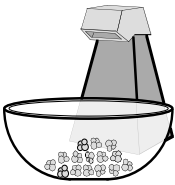
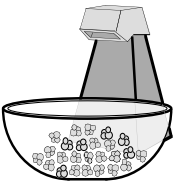
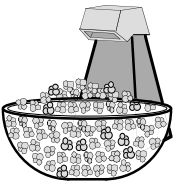
Three companies make popcorn. They use different types of corn so the popping is fast or slow.

Invent a procedure for computing a “popping index” to let consumers know how fast each brand pops.

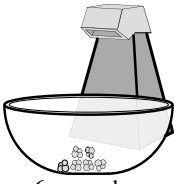
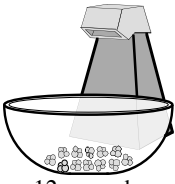
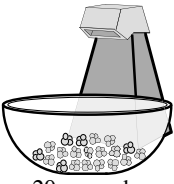
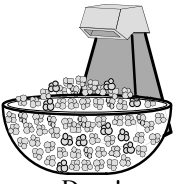
Rules for the Index:

1. The same brand of popcorn pops at the same speed. So a brand of popcorn only gets a single popping index.
2. You have to use the same procedure for each brand to find its index.
3. A big index value should mean that the popcorn pops faster. A small index value should mean that the popcorn pops slower.

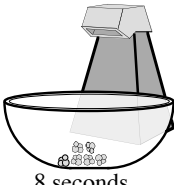
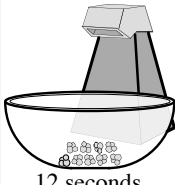
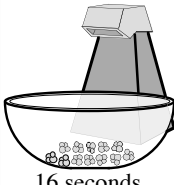
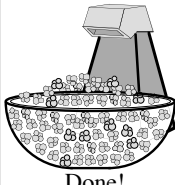
Hot Pops Index: \_\_\_\_\_

 <p>20 seconds</p>	 <p>30 seconds</p>	 <p>50 seconds</p>	 <p>Done!</p>
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Hip Hop Popping Corn Index: \_\_\_\_\_

 <p>6 seconds</p>	 <p>12 seconds</p>	 <p>20 seconds</p>	 <p>Done!</p>
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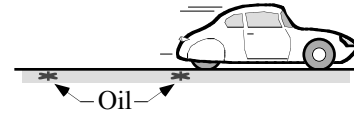
Poppomatic Popcorn Index: \_\_\_\_\_

 <p>8 seconds</p>	 <p>12 seconds</p>	 <p>16 seconds</p>	 <p>Done!</p>
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## B. Fastness Index

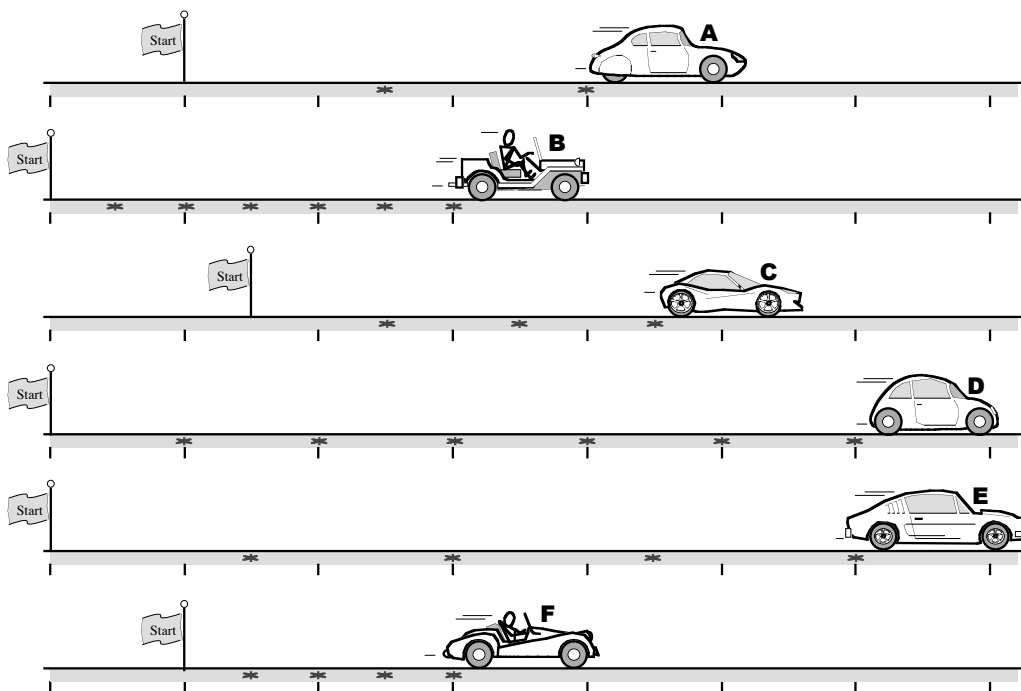
Let's look at another kind of index. Your task this time is to come up with a *fastness index* for cars with dripping oil.

You will see several cars, and will need to come up with one number to stand for each car's "fastness." There is no watch or clock to tell you how long each car has been traveling. However, all the cars drip oil once every second. (They are not very good cars!)



Some relevant information:

- A company makes cars that all have the same fastness.
- We will not tell you how many companies there are.
- You have to decide which cars are from the same company. They may look different! To show cars that are from the same company, draw a line connecting them.



## C. Follow up questions

1. Which popcorn is fastest? Which car is fastest? Explain.
2. For each question below, explain your reasoning:
  - A full bowl of popcorn has 60 popped corns in it. Determine the amount of time for the fastest popcorn to fill a bowl.
  - Write an expression for the time required for the fastest car to travel  $B$  blocks.
3. Another company, “Acme,” has an index of 2.5.
  - a. Let’s say that Acme makes popcorn. Using everyday language, describe the specific information that the number 2.5 tells about this their popcorn.
  - b. Now let’s say that Acme makes cars. Using everyday language, describe the specific information that the number 2.5 tells about their car.
4. Suppose that a friend of yours is having trouble deciding whether to compute the index for Car #1 as  $3/2 = 1.5$ , or  $2/3 = 0.67$ . Describe one way of convincing your friend of the correct answer.

## D. Speeding up index

Now you will invent a *speeding up index* for cars with dripping oil. You will see several cars, and must create a number to stand for each car’s speeding up. (*See the diagram on the following page.*)

As before, each car drips oil once every second. The speedometer reading tells you how fast the car is going when the oil drips.

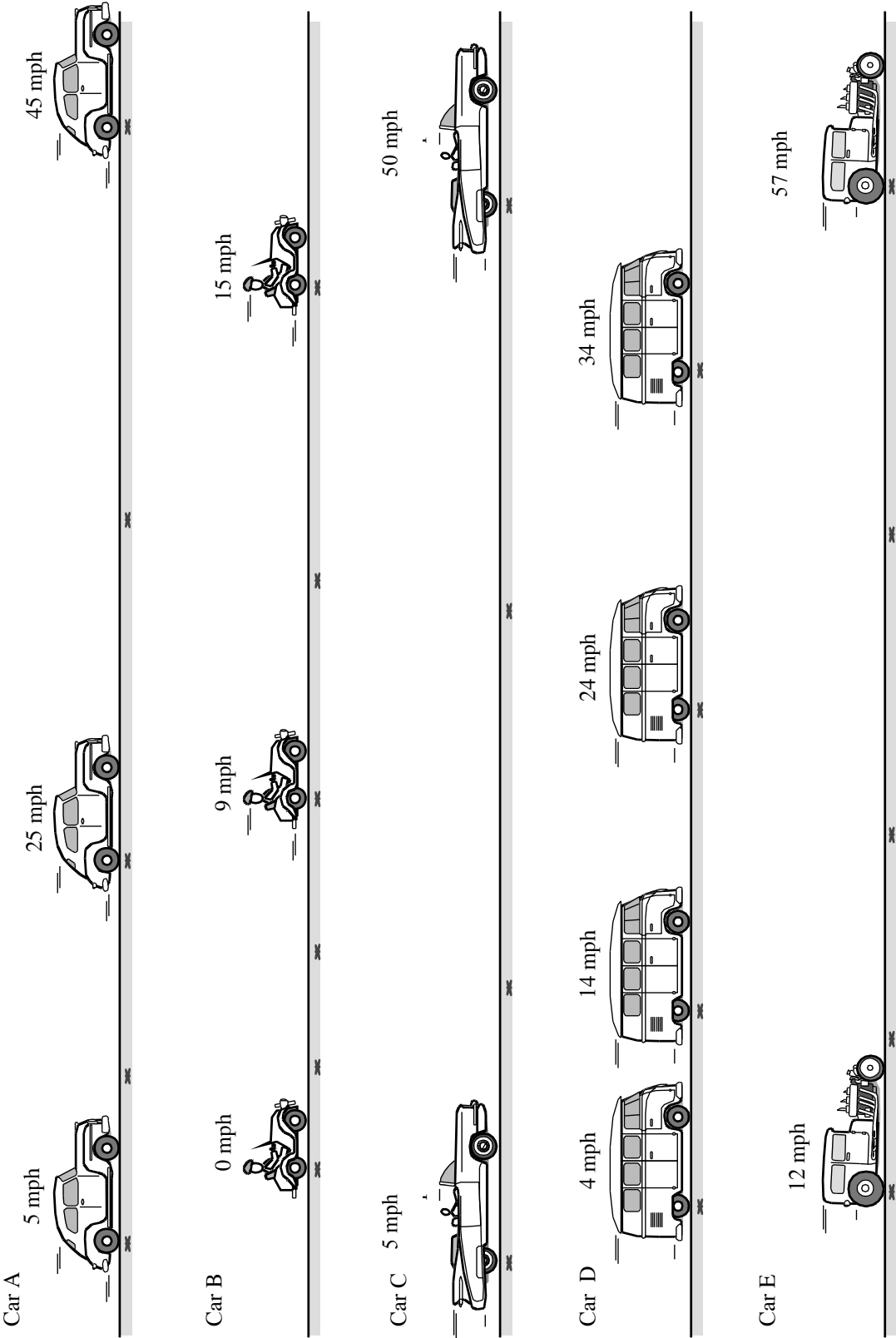
***This task might be a little harder than before!***

Some relevant information:

- A company always makes its cars speed up in the same way.
- We will not tell you how many companies there are.
- You have to decide which cars are from the same company. They may look different!

E. Follow up questions. (Complete these once you have invented a speeding up index for the situations shown in the diagram on the next page.)

1. How many car companies are there? Why were there less than 5 car companies, even though there were 5 different diagrams describing the car companies? (Choose each reason that could apply).
  - *because some cars went as fast as others in each second.*
  - *because some cars sped up as much as others in each second.*
  - *because some cars sped up as much as others during the entire trip shown*
2. Explain, in your own words, the difference between *fast*, *slow*, *speeding up quickly*, and *speeding up slowly*. Can a car be going fast and speed up slowly? Can a car be going slow and speed up quickly? Explain.



## Work Sequence: *Car Washing, Weight Lifting, and Plain Old Hard Work*

### A. Car Washing Inefficiency Index

You're the manager of a chain of four Irvine-area car washes in which teams of employees wash the cars by hand. You want to find out which locations are the most inefficient so that the teams there can be retrained. The teams don't all have the same number of people, however, so how can you determine which location is the most inefficient?

Shown below are times for how long it took to wash a Toyota Camry. You have data for two different teams from each of the four locations. Invent a procedure for computing a *car washing inefficiency index*. Bigger index values should correspond to more inefficient teams. Teams from the same location should have the same index value.



Wash Time: 10.2 minutes



Wash Time: 6.8 minutes

**Shimmy Shine**



Wash Time: 9 minutes



Wash Time: 12 minutes

**Suds R Us**



Wash Time: 11.8 minutes



Wash Time: 5.9 minutes

**Klean Whips**



Wash Time: 8.4 minutes

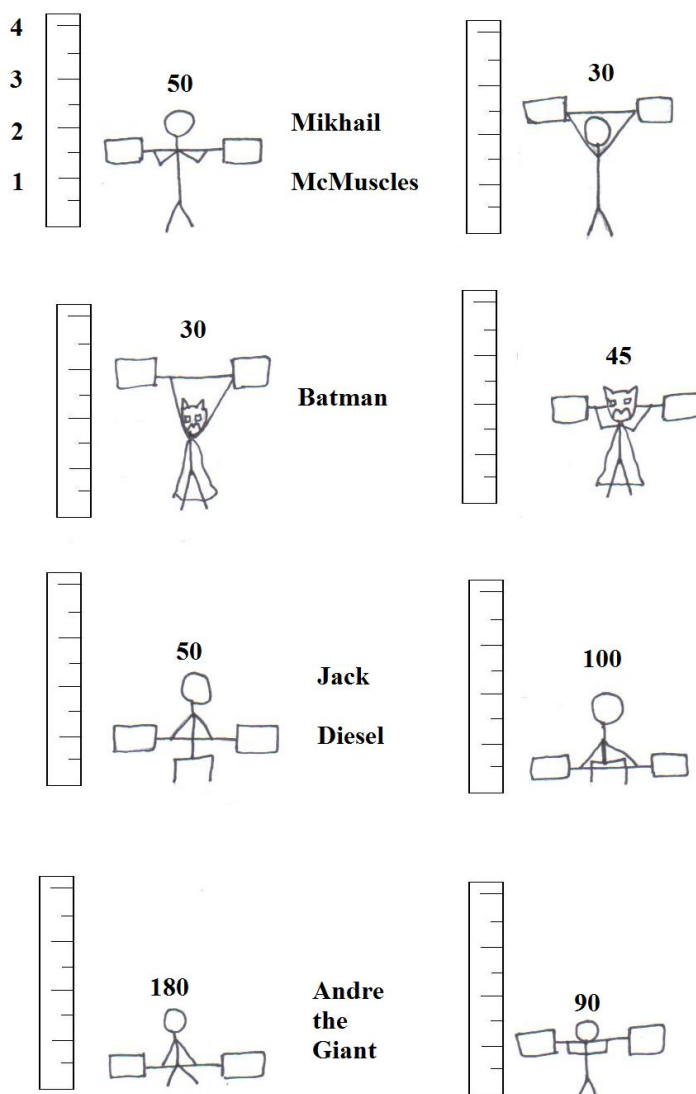


Wash Time: 10.5 minutes

**Sooper Sponges**

## B. Weightlifting Index


Several of your friends have been bragging about how strong they are. To settle the matter, they have decided to hold a weightlifting contest. The problem is that they don't agree on how to score the competition, since they all lift different amounts of weight and they lift the weights to different heights. So, you've been recruited as a judge. Invent a procedure for computing a *weightlifting index* for each competitor. Remember: a bigger index value should correspond to a stronger competitor.





### C. Job Difficulty Index

Some friends are complaining about their summer jobs. Each person thinks he works the hardest out of everyone. All their jobs are different, so it's difficult to tell. You've been asked to decide once and for all whose job is the most difficult. Invent a job difficulty index for the workers, and rank them from hardest to easiest job. Some information has already been provided in the table below.

Name	Job	Force (Monday)	Distance (Monday)	Force (Tuesday)	Distance Tuesday)	Job Difficulty Index
<b>Burley</b>	Lifts ice blocks in a meat packing plant	$1.8 \times 10^3$	70.5	$1.41 \times 10^3$	90	
<b>Lug</b>	Pushes a luggage cart at the airport		360	$6.0 \times 10^3$		$1.44 \times 10^5$
<b>Monty</b>	Rolls old computers to a storage facility	$2.5 \times 10^3$	100		500	
<b>Rollo</b>	Rolls oil drums up a ramp to a truck bed		270		180	$4.455 \times 10^5$
<b>Missing Number</b>		$2.5 \times 10^4$			120	N

D. Follow up questions:

1. Discuss in everyday language what the numeric value of the weightlifting index tells you about each competitor. (Hint: Think about what the number means if the competitor lifts only 1 unit of weight, or, alternatively, or if he lifts only one unit of distance?)
2. Which competitor(s) won the competition?
3. If Batman had attempted to lift 150 weight units how far would he have lifted it?
4. What weight could Mikhail McMuscles lift to 1.25 height units?
5. How do you know which summer job is the most difficult? Explain in everyday language how you decided whose was the most difficult job.
6. The ability to change a system, as discussed in lecture, is the product of the amount of force exerted times the distance the force is exerted. This is called **work** in physics. How are the weightlifting and job difficulty indices similar to work?
7. The weightlifting competitors are thinking of trying out for the cheerleading team. Applicants must be able to lift 30 weight units to a height of 3 height units. Which applicants should go to the tryouts and which should look for other activities?
8. At Burley's meat-packing plant they decided to double the width of the ice blocks, making them twice as heavy. Assuming Burley's job difficulty index doesn't change, what else will change about his job? How will it change?
9. Discuss the meaning of the value of Missing Number's job difficulty index.
10. Discuss in everyday language what the numeric value of *job difficulty index* tells you about the job.

## Spinning Disk Sequence: *rotational quantities, rotational kinematics, applications*

As a hipster, you've landed a dream summer internship working for a small company that manufactures retro record players for vinyls. Your group is an engineering team of interns doing quality control. You've been hired for your physics prowess and are expected to work independently using the knowledge you've gained in your first year courses.

You are put to task to measure the constant rotational speed of the turntables, and to measure the constant rate at which they speed up when they are turned on, and the constant rate at which they slow down when they are turned off.

You need to develop a model for the turntable's motion for each of the three cases:

- Rotating at a steady speed
- Speeding up at a steady rate
- Slowing down at a steady rate

You have a solid grasp of steady speed linear motion, and speeding up and slowing down along a line, but you don't recall having studied this for the case of a spinning object.

1. You know you have to invent new models here so you start with the **physical quantities**.

You figure the best bet is to reason by analogy with linear motion. Work with your teammates to invent physical quantities that will allow you to describe the spinning motion of the record player (the turntables at your tables will help you visualize.) Pick whatever symbols make the most sense to your group. As long as you all agree to use the same symbols, they can be anything you want. You can organize your work using the table provided.

Linear Quantity	Symbolic represent -ation and Units	Verbal Description	Rotational Quantity	Symbolic represent -ation and Units	Verbal Description
<i>Linear Displac- ement</i>	$x-x_0$ <i>measured in meters</i>	<i>Difference between the final position and the initial position</i>			

2. You're on a roll and are pretty confident that if you can reason the same way mathematically about spinning that you have about moving along a line, you'll be able to develop your own set of equations that will allow you describe the rotating motion of the turntable. Using your new quantities, write an equation that describes how they change with time for the turntable when it is:
  - a. rotating at a steady speed
  - b. speeding up at a steady rate
  - c. slowing down at a steady rate
3. In the linear kinematic equations, which are our models that describe linear motions with constant or zero acceleration, we describe how the linear position and linear velocity depend on time. In your analogous rotational models, are the rotational versions of position, velocity and acceleration variables or parameters? What about time, is it a variable or a parameter? Ponder this question for each of the three cases above and be prepared to defend your choices.
4. Now that you have a model, you will want to test it. Describe a testing experiment you can do to try and disprove your models. You happen to have in your workplace a turntable that is used as a standard that has known values for the steady speed at which it rotates, and for the rate at which it speeds up and slows down. You also have a stopwatch.
5. You measure that when spinning at a steady speed, the turntable makes  $1.48 \pm 0.03$  revolutions in  $2.0 \pm .04$  seconds. Does this turntable pass the quality control for 33 rpm, 45 rpm, 78 rpm or none of them? Explain your decision as you would to your boss.