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**DEDUCTIBLE INSURANCE AND THE  
TRANSFER OF RISK**

by

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And approved by

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**ABSTRACT OF THE DISSERTATION**  
**Deductible Insurance and the Transfer of Risk**  
**by CHRISTOPHER GAFFNEY**

**Dissertation Director:**  
**Adi Ben-Israel**

This thesis develops a deepened understanding of insurance and its benefits, focusing on practical aspects of insurance coverage and risk reduction. It is easy to see that the purchase of insurance increases the expected loss suffered by the insured, otherwise the insurer's expected profit would be negative. In view of this, we show that insurance is a variance-reducing mechanism. We first prove that the customer's variance is less than the variance that would be experienced if insurance was not purchased, and further show that the variance of an insured loss  $\mathbf{X}$  is equal to the sum of the covariances of the insured and insurer losses with  $\mathbf{X}$ .

As insurance increases the insured's expected loss and decreases the variance, we develop a mean-variance model of insurance demand, showing how the relationship between the premium and the insured's risk preference defines the demand for insurance. We verify Arrow's classical (utility-based) result that the optimal policy has full coverage above a non-zero deductible and consider the insurer's perspective, showing that the customer can be induced to purchase the insurer's optimal policy.

Next, we consider different forms of coinsurance. We show that the optimal straight coinsurance policy is inferior to the optimal deductible policy, while coinsurance combined with either a stop-loss limit or a deductible is equivalent to a deductible policy (in the optimum). We also show that, in each of the coinsurance cases, the optimal policy involves partial coverage if the premium exceeds

the expected reimbursement.

Finally, we consider the system composed of the insured and insurer, discussing how certain customers may receive discounted premiums that are subsidized by other customers and showing how the customer and the company share in the variance. We then discuss a benefit of insurance; in the case of a single insurer and a single customer, the sum of their individual variances is less than the variance in the uninsured case, and in the case of a single insurer and multiple insured, the variance of this system is smaller than the sum of the individual uninsured variances if the insurer reimbursements are sufficiently uncorrelated.

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# 1 Introduction

The theory of optimal insurance coverage has been studied for over 50 years. While considerable attention has been paid to the expected benefit of insurance in terms of utility theory and its generalizations, little attention has been paid to the practical benefits of insurance in terms of variance reduction. In this thesis we study these benefits and how the major components of deductible insurance (premium, coverage level, deductible) relate to each other and to the reduction of risk.

When an insurance contract is being negotiated, both sides of the agreement (insured and insurer) have their own self-interest to protect. The insured wishes to minimize shocks from large losses while incurring as small of an upfront expense (i.e., premium) as possible. Conversely, the insurer wishes to maximize profit while (perhaps) maintaining a standard of low variance. Central to these conflicting concerns is the deductible  $D$  and the coverage limit  $C$ , which together determine a premium, set by the insurance company, which gives a price for the  $(C, D)$  pairs that may be chosen by the insured.

There are several objectives used in determining the optimal insurance plan. The bulk of the literature considers the problem as one of maximizing the expected utility of final wealth, while some contemporary approaches consider non-expected utility preferences such as stochastic dominance. Under a wide variety of circumstances, partial insurance coverage in the form of a deductible is shown to be the optimal insurance plan. We consider the problem from a Markowitzian mean-variance perspective, and also explore various forms of coinsurance.

Insurance can be thought of as a zero-sum game - the insured and the insurer share in the value of a random loss. It is simple to see that, for the insured, the purchase of insurance increases the insured's expected loss; otherwise the insurer would have no motive to offer insurance. The insured and insurer also share the variance of the random variable, and so we explore the question of whether or not

insurance can be thought of as a zero-sum game in this context. We also show that insurance reduces the variance of the customer and thus, insurance creates a loss-increasing, but risk-reducing system.

## 1.1 Terminology and Assumptions

We begin by describing some commonly used terms in insurance theory. The *insured* or the *customer* is the person or entity purchasing insurance protection from the *insurer* or *insurance company*. The *uninsured case* refers to the situation that the customer would experience if he did not purchase insurance. The *reimbursement* or *indemnity* paid from the insurer to the insured is the compensation paid for a loss. *Full coverage* provides reimbursement for the entire loss of the insured, while *partial coverage* provides reimbursement for only a portion of the insured's loss.

*Deductible insurance* is a form of partial insurance where the *deductible* is the value  $D$  for which any loss greater than  $D$  results in a reimbursement from the insurer. Any loss less than  $D$  is not covered under the insurance plan, and so is the responsibility of the insured. For example, suppose that a loss  $x$  occurs. If  $x \leq D$ , then there is no reimbursement, but if  $x > D$  the insurer provides a reimbursement of  $x - D$ . Insurers typically offer plans with an upper limit  $C$  on *coverage*, so that the maximum reimbursement that will be made is  $C - D$ . We sometimes refer to full coverage above the deductible, meaning that the insured pays  $D$ , and the insurer pays any remaining loss.

*Coinsurance* is another form of partial coverage, where the reimbursement is a fixed percentage of some portion of the loss. When reimbursement occurs as a percentage of the entire loss we refer to the policy as *straight coinsurance*. Coinsurance is often combined with a *stop-loss*, a level of loss above which the insurer provides full reimbursement. Coinsurance can also be combined with deductible insurance to form a policy where the insurer pays a percentage of the

loss above a deductible.

The *actuarial* or *actuarially fair* value of an insurance policy is the expected value of the payment made by the insurer to the insured as a reimbursement for loss. An insurer that sets the premium equal to the actuarial value of the policy would therefore have an expected profit of zero. An insurance premium is said to be *loaded* if it is computed as a fixed percentage of the actuarial value of the policy. Loaded premiums are widely found in the literature, for example, in [2], [14], [41], and [46].

Our model is set in a single period, single risk environment, where the random loss variable  $\mathbf{X}$  is assumed to be nonnegative with no upper bound. To simplify matters, we assume that  $\mathbf{X}$  is modeled by an absolutely continuous probability distribution with density  $f(x)$  and cumulative distribution  $F(x)$ , and that its variance is finite. In many of the calculations we make use of Leibniz's rule for differentiation under the integral sign: If  $F(x) = \int_{a(x)}^{b(x)} f(x, t) dt$ , then

$$\frac{d}{dx} F(x) = f(x, b(x)) b'(x) - f(x, a(x)) a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) dt.$$

Many of the figures show contour plots of various functions. In all of these figures, darker shading represents a higher function value and lighter shading (i.e, white) represents smaller function values.

We do not consider the problem of modeling the distribution of the loss variable. See, for example, [23], for a discussion on insurance event modeling. We also do not consider common difficulties associated with insurance such as moral hazard and adverse selection. See Chapters 6 and 7 in [17] for a discussion of these matters.

## 1.2 Literature Review

The modern literature on optimal insurance coverage began in the 1960's with the work of Arrow [1], Mossin [46], and Smith [59]. The work of these authors, and the bulk of the subsequent literature on optimal insurance, is done from a von Neumann-Morgenstern [63] utility perspective, where the insured seeks to maximize the expected utility of final wealth. Utility functions are usually assumed to display risk aversion in the sense of Arrow [2] and Pratt [50]. Frequent use of this approach persists to this day in the insurance literature.

The initial results focused on determining when full or partial coverage is preferable. As insurance premiums are generally priced above the expected reimbursement provided by the insurance company, partial coverage is generally optimal. A further determination concerning the form of partial coverage (e.g., deductible, coinsurance, or a combination of the two) was then necessary, and deductible insurance has been shown to be the optimal structure in many cases.

In particular, Mossin [46] and Smith [59] independently published similar results concerning the optimal insurance structure. The major result is a cornerstone of modern insurance theory, and is known as Mossin's Theorem:

*If the premium is actuarially fair or discounted, then a risk averse expected utility maximizer will purchase full coverage. If the premium is unfair then partial coverage is optimal.*

Mossin's theorem is generalized in Schlesinger [57] for upper limit policies and in Hong et. al. [29] to account for random initial wealth levels.

Arrow [1], [2], [3] also showed that partial insurance coverage is optimal. In particular, he proved the second cornerstone of modern insurance theory:

*If the premium is a fixed percentage ( $> 100\%$ ) of the expected reimbursement, then the policy that maximizes a risk averse customer's expected utility of final wealth is full coverage above a nonzero deductible*

The intuition behind Arrow's result is as follows: An insurance purchase consti-

tutes a trade-off in expected utility - there is a decrease in expected utility due to the premium and an increase due to the indemnity paid by the insurer to cover losses. Large losses correspond with the lowest utility of final wealth, and due to the concavity of the utility curve, avoiding these losses provides the greatest benefit to utility. Therefore, large loss values should be insured first. Because the premium is unfair there is a point where the added utility from insurance is less than the utility lost from paying the premium, and so full coverage cannot be optimal, i.e., there is a point  $D$  where coverage should cease.

The next major area is optimal deductible levels. Schlesinger [55] provided key results in this area, providing optimality conditions for expected utility maximizers and proving that the optimal deductible level decreases as risk aversion increases.

After these early important works, many papers consider cases where standard assumptions do not hold. For instance, Lian and Schlesinger [39] consider the case of asymmetric information on loss size (in the insurer's favor) under the expected utility framework, while Bond and Crocker [8] consider the case when the insured is privy to extra information. Background risks are considered by Eeckhoudt and Kimball [20], while Meyer and Meyer [45] consider cases where certain types of losses are excluded from reimbursement. Dreze and Schokkaert [21] consider the affects of health insurance with ex post moral hazard on Arrow's theorem.

Arrow's theorem showed that coverage limits below the full value of the loss variable are not optimal, and so these limits have received a relatively small amount of attention in the literature. Doherty et. al. [19] show that deductible policies with a coverage limit are the optimal contracts when losses are not verifiable. Cummins and Mahul [14] also considered an upper limit on coverage, proving the optimality of deductible policies, while Zhou et. al. [64] consider a similar problem, an upper limit on insurer reimbursement. The optimality of coverage limits is shown in cases where third party insurance exists (Huberman et.

al. [30]) and as a response to moral hazard in property insurance (Garratt and Marshall [25]).

In the late 1980s through the 1990s the research program focused on the robustness of the results of Arrow and Mossin under less restrictive non-expected utility preferences such as stochastic dominance (see [37] and [54]). Machina [41] extends Mossin's theorem and the first order optimality conditions of Schlesinger to non-expected utility preferences. Gollier and Schlesinger [26] and [56] prove that Arrow's theorem holds under risk aversion within any decision theoretic framework that satisfies second order stochastic dominance.

The insurance literature is dominated by expected utility theory and generalizations such as stochastic dominance, however, there are a host of other measures that may be appropriate for insurance decisions. Krokmal et. al. [35] summarize risk measurement, emphasizing mean-risk measures (e.g., mean-variance and semivariance) as well as utility theory and stochastic dominance. Steinbach [61] provides a detailed review of mean-variance analysis, discussing among other things, downside risk measures and multiperiod models.

Markowitzian mean-variance models [42] [43] have long been used in financial portfolio modeling, and offer a clear delineation between risk and reward. Levy [37] provides a discussion of stochastic dominance that points out its relation with mean-variance models, while, in a portfolio theory context, Levy and Markowitz [38] and Kroll et. al. [36] compare utility maximizing portfolios with mean-variance efficient portfolios, showing that the utility maximizing, mean-variance efficient, portfolio produces a near-optimal (with respect to expected utility) portfolio.

Within insurance, mean-variance models have been used to investigate the robustness of Mossin's Theorem in Hong et. al. [29], while Doherty [18] considers Arrow's results in a mean-variance context for a portfolio of risky assets, considering the role that uninsurable risks play in optimal insurance protection.



The portfolio context is also examined in [44], and [58], although the insurance plans discussed are limited to coinsurance. Another mean-variance approach is found in [33], however, the discussion focuses on premium pricing of reinsurance contracts.

Our results differ from the literature on mean-variance insurance models in several ways. Because the literature considers mean-variance approaches in a portfolio context, the chief concern lies in the relation between the portfolio risks, while the particulars of the actual policy (i.e., the optimality conditions for coverage and deductible) are not examined. We provide such an analysis, and we also discuss the importance of the insurer's choice of premium loading and the insured's risk tolerance, pointing out how their relation determines whether or not insurance will be demanded.

An alternative to mean-variance or mean-standard deviation approaches is to consider the *downside measures*, where the full range of a random variable  $\mathbf{Z}$  is not taken into consideration, instead only the disagreeable portion of  $\mathbf{Z}$  is considered. Before discussing two such approaches, we first note that, in our framework there is no “upside”, i.e., the random variable  $\mathbf{X}$  considered here refers only to loss values. Nonetheless, downside measures do have an appeal, as it is likely that customers are more concerned with large losses than with small.

The most common downside measure is *Value at Risk (VaR)* (see, for example, [4] or [16]), which is heavily used in finance. A similar, but coherent measure of risk (in the sense of [4]), is conditional value at risk (CVaR) (see [53]). In an insurance setting both of these measures would give a value based on the probability that a loss exceeds a given threshold. In Chi and Tan [12] both VaR and CVaR are used to determine, from the insurer's perspective, optimal reinsurance, while a practical application of CVaR in insurance is given by Liu et. al. [40]. Consiglio et. al. [13] provide a model that optimizes CVaR for insurance policies that offer a guaranteed rate of return. A related measure with applications

to insurance is the optimized or recourse certainty equivalent introduced by Ben-Tal et. al. ([5], [6]).

Ogryczak and Ruszczyński [48] [49] introduce a mean-semideviation model for decisions under uncertainty, and show how such a model can serve as a bridge between mean-risk and stochastic dominance approaches. It was shown that mean-semideviation is consistent with stochastic dominance, i.e, if a pair  $(C, D)$  minimizes the mean-semideviation, then it cannot be inferior, in a stochastic dominance sense, to another policy. Due to the wealth of stochastic dominance results in the insurance literature, the application of this measure to insurance may provide interesting results. For further discussion of deviation measures see Rockafellar et. al. [52].

Several other methodologies have been used in assessing insurance policies. For example, Bernard et. al. [7] consider optimal insurance coverage under rank-dependent expected utility, showing, in contrast to classical results, that there are cases where small losses should be insured. Harel and Harpaz [27] discuss fair actuarial values in the case of insurer's with parameter uncertainty that is updated through Bayesian learning. Sung et. al. [62] apply cumulative prospect theory [34] to explain the prevalence of (relatively) low coverage policies.

Another route taken in determining optimal policies is to consider *Pareto Optimality*. Studies of this type began with the early work of Borch [9] and Arrow [1], and were generalized by Raviv [51]. The main results show that the Pareto optimal policy involves full coverage above a nonzero deductible if the insurer is risk neutral, and coinsurance above a nonzero deductible if the insurer is risk averse. Contemporary work also makes use of this conception, for example, Dana and Scarsini [15] consider the case of background risk, while Jouini et. al. [32] define an *optimal risk sharing* by an allocation of risk that is both Pareto optimal and individually rational (there is no loss of utility by entering into the allocation), and discuss such allocations for various value at risk formulations.

With the passage of the Affordable Care Act in the United States, government intervention in insurance markets has become a heavily debated topic. The RAND corporation recently published a research report [22] investigating this law and its effects on insurance premiums. Due to tax incentives and the presumed inclusion of a higher number of healthy customers, they predict that average insurance premiums will see little change with the implementation of the law. A discussion of government intervention for catastrophe markets can be found in [11], where the avoidance of insurer insolvency is the primary goal.

The portions of this work concerning separable premiums, optimality conditions for expected loss minimization, and the reduction of variance for the insured have been published in [24].

### 1.3 Outline

We begin in Chapter 2 by describing the deductible insurance model and discussing the insurance premium, focusing the discussion on real-world insurance situations. The insured loss, insurer profit, indemnity, and premium are all given as functions of the coverage and deductible. As the model does not include expected or non-expected utility, the final wealth level need not be the objective. Instead, we first focus on minimizing the magnitude of expected loss.

We note that, in most practical situations, the purchase of insurance will result in an increase to expected loss for the insured (relative to the uninsured case). As such, we consider the insured's variance, and show that it is smaller than the variance in the uninsured case. Conversely, the insurer's variance increases by entering into an insurance contract. We also show that the loss variable's variance,  $\text{Var } \mathbf{X}$ , is equal to the sum of the covariance of the insured's loss with  $\mathbf{X}$  and the covariance of the insurer's loss with  $\mathbf{X}$ .

In Chapter 3 we discuss a mean-variance approach to determine the insured's optimal policy. We confirm the classical utility-based results of Arrow and Mossin

for mean-variance minimizers, showing that full coverage above a non-zero deductible is optimal if the premium is priced above the policy's actuarial value, while a 0 deductible and full coverage is optimal for premiums at or below the policy's actuarial value. We also give a condition that must be satisfied in order for the insured's mean-variance to decrease from the uninsured case. This condition provides a specific inequality relating the premium loading and risk tolerance with the coverage and deductible.

We next consider mean-variance preferences for the insurer, and show that the insured's optimal policy differs from the insurer's. We show through an example that there exists a specific set of  $(C, D)$  policies where the mutually optimal  $(C, D)$  pair is the insurer's optimal policy. Finally, we show that our mean-variance results also hold for a mean-standard deviation framework. Chapter 3 concludes with a discussion showing that the mean-variance results will also hold under a mean-standard deviation model.

Chapter 4 discusses coinsurance policies in the mean-variance framework. We begin by considering a straight coinsurance policy, and then extend the model to include an upper stop-loss limit and a deductible. Optimality conditions for these three cases are derived, and we show that partial coverage is optimal when the premium is priced above the policy's actuarial value. If the premium is priced below the policy's actuarial value, then full coverage is optimal. We then show that the optimal coinsurance with stop-loss and the optimal coinsurance with deductible policies are both equivalent to the optimal deductible policy of Chapter 3, and furthermore, we prove that these policies are superior to the optimal straight coinsurance policy.

Chapter 5 discusses the insurance system defined by the insurer and the insured. We first discuss how the insurer might respond to the situation where new, risky, customers are introduced to the system. Such responses can include raising the premiums of those currently insured or adding new, less risky, customers.

We then discuss the variances experienced in the insurance system, specifically showing that, in a single customer, single insurer framework, the sum of the insured variance and the insurer variance is always less than the uninsured variance  $\text{Var } \mathbf{X}$ . This result shows that insurance reduces the risk that is actually experienced in the world, and expresses a social benefit of insurance. We then consider a multiple insured - single insurer situation, and show that the sum of these variances is reduced from the uninsured case if the covariances among the insurer reimbursements are sufficiently low.

## 2 The Insurance Model

We present a deductible insurance model and investigate the connections between coverage, deductible, and premium on the one hand, and expectation and variance on the other. In section 2.1 we describe the insurance model and consider the problem of minimizing the insured's expected loss. We discuss the insurance premium in 2.2, considering the commonly used conception of a loaded premium as well as a more general premium function that we fit to real-world insurance data. Section 2.4 discuss the purchase of insurance under budgetary constraints.

We discuss the effect of insurance on loss variance in 2.5, showing that, relative to the uninsured case, insurance decreases the variance of the insured and increases the insurer variance. Additionally, we show that the variance of the loss variable  $\mathbf{X}$  (the asset being insured) is equal to the sum of the two covariances relating the insured and the insurer to  $\mathbf{X}$ .

### 2.1 The Model

We offer a straightforward model of deductible insurance with a (potentially infinite) coverage limit and a premium depending on the coverage and the deductible. Upon the execution of an insurance contract, a loss-sharing system is created. The insured seeks to control his loss level, while the insurer hopes to generate profit. When a loss occurs, its value is shared between the two entities. Let  $\mathbf{X}$  be the random variable describing the value of the loss that is being insured and  $x$  be a particular realization of  $\mathbf{X}$ . The coverage and deductible are given by  $C$  and  $D$ , respectively, and the insurance premium is a function of  $C$  and  $D$ , denoted  $p(C, D)$ . For a given loss  $x$ , the reimbursement,  $I(x|C, D)$ , paid from the insurer to the insured is given by

$$I(x|C, D) = \begin{cases} 0, & \text{if } 0 \leq x < D; \\ x - D, & \text{if } D \leq x < C; \\ C - D, & \text{if } C \leq x. \end{cases} \quad (2.1)$$

The loss of the insured is

$$L(x|C, D) = p(C, D) + x - I(x|C, D) = p(C, D) + \begin{cases} x, & \text{if } 0 \leq x < D; \\ D, & \text{if } D \leq x < C; \\ x + D - C, & \text{if } C \leq x, \end{cases} \quad (2.2)$$

and the corresponding profit of the insurer is

$$R(x|C, D) = p(C, D) - I(x|C, D) = p(C, D) - \begin{cases} 0, & \text{if } x < D; \\ x - D, & \text{if } D \leq x < C; \\ C - D, & \text{if } C \leq x. \end{cases} \quad (2.3)$$

In particular, we see that, when  $C = D$  (the case of no insurance coverage),  $I(x|C, D) = 0$ . If, additionally,  $p(k, k) = 0$  for all  $k \geq 0$ , then we also have  $L(x|C, D) = x$  and  $R(x|C, D) = 0$ .

The expected loss of the insured is by (2.2),

$$\mathbb{E} L(\mathbf{X}|C, D) = p(C, D) + \mathbb{E} \mathbf{X} + \int_D^C (D - x) f(x) dx + (D - C) \int_C^\infty f(x) dx. \quad (2.4)$$

Similarly, the expected profit of the insurer is,

$$\mathbb{E} R(\mathbf{X}|C, D) = p(C, D) - \int_D^C (x - D) f(x) dx - (C - D) \int_C^\infty f(x) dx. \quad (2.5)$$

Going forward, we assume that  $C$  and  $D$  are given, and do not write them if they are not needed explicitly, thus  $L(\mathbf{X}|C, D)$  is abbreviated  $L(\mathbf{X})$  and  $R(\mathbf{X}|C, D)$  is written  $R(\mathbf{X})$ .

It follows from (2.2) and (2.3) that

$$L(x) = x + R(x), \text{ for all } x \geq 0, \quad (2.6)$$

and consequently, the expected loss of the insured equals the (uninsured) expected loss plus the expected profit of the insurance company,

$$E L(\mathbf{X}) = E \mathbf{X} + E R(\mathbf{X}), \quad (2.7)$$

as can be seen also from (2.4) and (2.5). Equation (2.7) gives the intuitive result that the insured pays (in expectation) both the value of the random loss variable and the insurer's profit.

If the insurance company is profitable, as is generally the case, an insurance policy as above does not reduce the expected cost to the insured. Indeed, under the assumption that  $ER(\mathbf{X}) > 0$ , we obtain  $EL(\mathbf{X}) > E\mathbf{X}$ , i.e., that the purchase of insurance increases the expected loss of the insured.

Figure 2.1 shows the loss of the insured and the insurer as a function of a random loss  $x$ . The insured is responsible for the loss  $x$  as it increases from 0 to  $D$ . The insurer profit in this region is constant, and equal to the premium  $p(C, D)$ . For losses between  $D$  and  $C$ , the insured pays  $D$ , with the remaining  $x - D$  paid by the insurer. The insurer profit decreases in this region, while the insured loss is constant. For losses larger than  $C$ , the insurer pays  $C - D$  and the remaining loss,  $D + x - C$  is paid by the insured.

## 2.2 Premium

The premium  $p(C, D)$  is set by the insurer and chosen by the insured. The insurer typically offers a menu of coverage-deductible pairs that the insured can choose from (see Table 2.1 and Example 2.1 below). At the most general level we



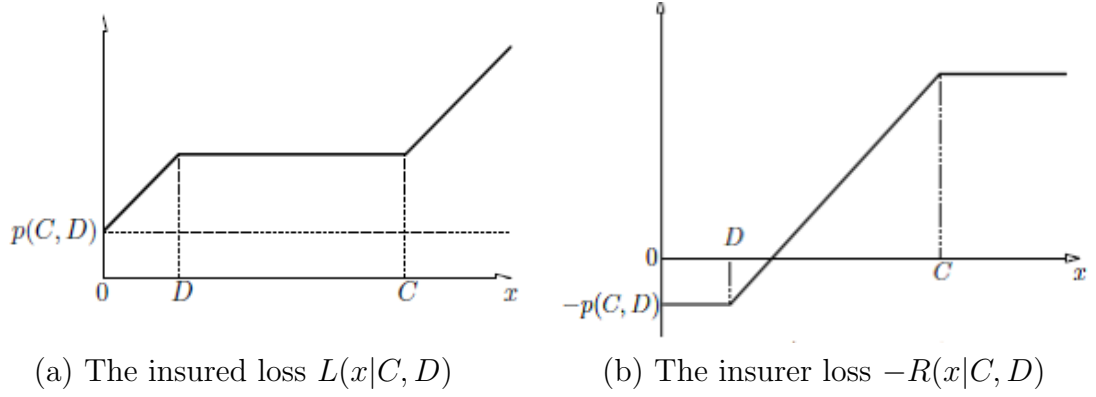


Figure 2.1: Insured and Insurer Loss

assume that  $p(C, D)$  is monotonically increasing in  $C$  and decreasing in  $D$ . Thus, the premium increases as the range of covered losses expands.

For each  $(C, D)$  pair there is a corresponding premium that is *actuarially fair*, where the insurer's expected profit is 0 and the insured's expected loss is equal to the expected value of the loss variable being insured. We expect that most premiums are unfair, so as to ensure that the insured's expected profit is positive.

### 2.2.1 Loaded Premium

One type of premium that fits our general conception is the *loaded premium*, which is given by:

$$p(C, D) = \lambda E I(\mathbf{X}) \text{ for } C > D \geq 0, \quad (2.8)$$

where  $\lambda > 0$  is the *loading factor*. The dependence on  $C$  and  $D$  is a result of the dependence of  $I(\mathbf{X})$  on  $C$  and  $D$ . Due to its intuitive appeal, loaded premiums are the most common form of premiums found in the literature.

### 2.2.2 Separable Premium

Although the loaded premium will be the focus of this work, we will now briefly introduce a second type of premium which allows for greater flexibility in the premium function.

Coverage ( $C$ )	Premium (%)	Deductible ( $D$ )	Premium (%)
\$1 million	134%		
\$250,000	100%	\$250	100%
\$150,000	97%	\$500	87%
\$75,000	95%	\$1,000	80%
\$50,000	92%	\$2,000	77%
\$15,000	87%	\$2,500	74%

Table 2.1: PIP premium as function of coverage and deductible, Source: NJ Auto Insurance Buyer’s Guide, [47]

$$p(C, D) = \begin{cases} 0, & \text{if } C = D; \\ \alpha(D)\beta(C), & \text{if } 0 \leq D < C, \end{cases} \quad (2.9)$$

where  $\alpha(D)$  is a positive, monotonely decreasing function of  $D$ , and  $\beta(C)$  is positive and monotonely increasing in  $C$ . In this way we ensure that higher coverage levels give higher premiums, while premiums decrease as deductibles increase.

The following example from auto–insurance in New Jersey shows the premium as a function of  $C$  and  $D$ . Note that not all values are permissible, in particular there are minimum coverage and deductible.

**Example 2.1. (Auto–insurance).** Standard auto insurance policies in the State of New Jersey contain Personal Injury Protection (PIP), with premiums depending on the deductible  $D$  and coverage  $C$  as shown in Table 2.1. The table lists 6 possible coverages and 5 deductibles; with minimum values of  $C = \$15,000$ , and  $D = \$250$ . The standard premium (100% in Table 2.1 is for  $C = \$250,000$  and  $D = \$250$ , and changes depending on the insurance company. There is also a 20% co–payment for losses between the deductible selected (by the buyer) and \$5,000. See [47] for details.

For example, Jane chose the minimum coverage  $C = \$15,000$  (resulting in a reduction of 13% from the premium for the standard coverage  $C = \$250,000$ ) and a deductible  $D = \$2,500$  (getting a 26% reduction from the standard premium

for  $D = \$250$ ). She pays  $0.87 \cdot 0.74 = 0.64$  times the standard premium.

If Jane has an accident resulting in \$10,000 of medical expenses, she pays the first \$2,500 as deductible, and an additional \$500 copayment (20% of the \$2,500 that is left of the first \$5,000). The insurance pays the remaining \$7,000.

Analyzing the data of Table 2.1, we approximate the premium  $p(C, D)$  by (2.9). The data suggests that the cost of coverage  $\beta(C)$  is affine in  $C$ , say

$$\beta(C) = \beta(0) + m C, \text{ for some } \beta(0) \geq 0, m > 0,$$

and the premium is therefore

$$p(C, D) = \alpha(D) (\beta(0) + m C).$$

Zero coverage is typically not allowed, (for example, in New Jersey the minimum coverage for auto-insurance is \$15,000, see Table 2.1), and therefore the term  $\beta(0)$  is just a device for expressing  $\beta(C)$  in its domain. The data of Table 2.1 is interpolated by (2.9) with

$$\alpha(D) = a D^{-b}, \quad a = 1.94, \quad b = 0.123, \quad (250 \leq D \leq 2500), \quad (2.10a)$$

$$\beta(C) = \beta(0) + m C, \quad \beta(0) = 0.9, \quad m = 4.5 \cdot 10^{-7}, \quad (15,000 \leq C \leq 10^6), \quad (2.10b)$$

up to a multiplicative constant, and plotted in Figure 2.2, where darker color indicates higher premium. The error of the interpolations (2.10a)–(2.10b) at the given points is  $O(10^{-2})$ .

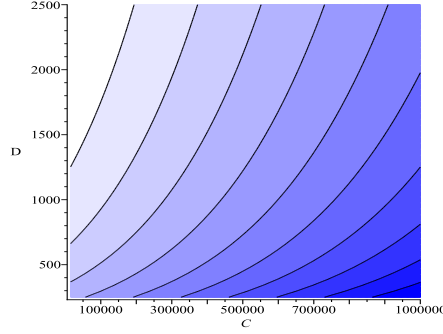


Figure 2.2: Contour plot of the PIP premium in Table 2.1 (Darker shading for higher premiums)

### 2.2.3 Actuarial Fairness

Recall that a  $(C, D)$  policy is called *actuarially fair* if the insurer's expected profit,  $E R(\mathbf{X})$ , is zero. By (2.3), this means that a policy is actuarially fair if

$$E I(\mathbf{X}) = p(C, D).$$

If  $p(C, D) > E I(\mathbf{X})$  we say the premium is *unfair*, while if  $p(C, D) < E I(\mathbf{X})$  we say the premium is *discounted*.

With the standard loaded premium function,  $p(C, D) = \lambda E I(\mathbf{X})$ , and within a single insurer single insured framework, it is simple to see that the insurer can only achieve a positive expectation if  $\lambda > 1$ . We now generalize this result for any premium depending on  $C$  and  $D$ .

**Theorem 2.1.** For any pair  $(C, D)$ , an insurance plan is actuarially fair if the premium  $p(C, D)$  satisfies

$$p(C, D) = \int_D^C (1 - F(x)) dx. \quad (2.11)$$

*Proof.* Using integration by parts we obtain

$$\begin{aligned}
\int_D^C (1 - F(x))dx &= C - D - \int_D^C F(x)dx \\
&= C - D + \int_D^C [xf(x) - xF(x)]_D^C \\
&= \int_D^C xf(x)dx + C(1 - F(C)) - D(1 - F(D)) \\
&= \int_D^C (x - D)f(x)dx + \int_C^\infty (C - D)f(x)dx. \tag{2.12}
\end{aligned}$$

For  $E R(\mathbf{X}) = 0$  we need, by (2.5),

$$p(C, D) = \int_D^C (x - D)f(x)dx + \int_C^\infty (C - D)f(x)dx, \tag{2.13}$$

and the proof follows by noting that the right sides of (2.12) and (2.13) are equal.  $\square$

**Remark 2.1.** Assuming the expected profit to the insurer is nonnegative, it follows from Theorem 2.1 that, for any pair  $(C, D)$ , we must have

$$p(C, D) \geq \int_D^C (1 - F(x)) dx, \tag{2.14}$$

a condition that the premium must satisfy in order to guarantee an expected profit (or break-even) for the insurer. For the case of a loaded premium, equation (2.14) becomes

$$\lambda E I(\mathbf{X}) \geq \int_D^C (1 - F(x)) dx.$$

Using equations (2.1) and (2.12), this is equivalent to

$$\lambda E I(\mathbf{X}) \geq \int_D^C (x - D)f(x)dx + \int_C^\infty (C - D)f(x)dx = E I(\mathbf{X}),$$

which holds if  $\lambda \geq 1$ .

Although we usually expect premiums to be priced above their actuarial value, i.e., we expect unfair premiums, where  $p(C, D) > E I(\mathbf{X})$ , there are cases where one may obtain an actuarially fair or even discounted premium. We discuss this in section 5.1.

## 2.3 The Insurance Budget

The *budget*  $B$  available for buying insurance imposes conditions on the coverage  $C$  and deductible  $D$ . In this section we discuss the conditions for both the generalized premium  $p(C, D) = \alpha(D)\beta(C)$  and the loaded premium  $p(C, D) = \lambda E I(\mathbf{X})$ .

### 2.3.1 Separable Premium

For the case where  $p(C, D) = \alpha(D)\beta(C)$ , we first assume that the entire budget is spent. We then have

$$\alpha(D) \beta(C) = B, \quad (2.15)$$

an equation that can be solved for  $D$  as a function of  $C$ ,

$$D = \alpha^{-1}(B/\beta(C)), \quad (2.16)$$

an increasing function, i.e. buying, with a fixed budget, more coverage makes it necessary to increase the deductible. Because  $D$  is nonnegative, the smallest possible coverage corresponds to  $D = 0$  in (2.15), i.e.,  $C$  must satisfy,

$$C \geq \beta^{-1}(B/\alpha(0)), \quad (2.17)$$

the right-hand side is the smallest possible coverage (corresponding to zero deductible) for the given budget  $B$ .

**Example 2.2. ( $\alpha(\cdot)$  exponential).** In the exponential case  $\alpha(D) = \alpha(0)e^{-\delta D}$ ,

the deductible (2.16) becomes

$$D = -\frac{1}{\delta} \log \left( \frac{B}{\alpha(0) \beta(C)} \right), \quad (2.18)$$

where  $D \geq 0$  if (2.17) holds, with  $D = 0$  if  $B = \alpha(0) \beta(C)$ . Some of the curves (2.18) are shown in Figure 2.3 for  $\beta(C) = C$ ,  $\alpha(0) = 0.6$  and  $\delta = 0.005$ . Higher curves (i.e. greater deductible) correspond to lower budgets  $B$ .

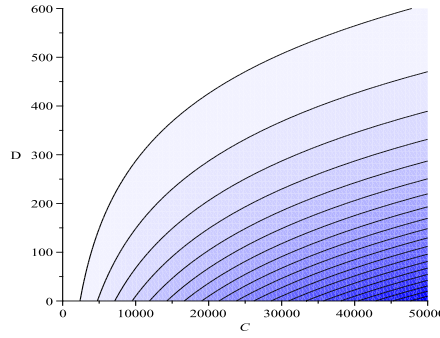


Figure 2.3: Illustration of Example 2.2

### 2.3.2 Loaded Premium

We next consider the case of the loaded premium. The assumption that the entire budget is spent means that

$$E I(\mathbf{X}) = \frac{B}{\lambda}. \quad (2.19)$$

It is difficult to obtain an analytic expression for  $D$  or  $C$  from (2.19), however, we can use the derivatives of  $E I(\mathbf{X})$  to gain insight into (2.19).

$$\begin{aligned} \frac{\partial}{\partial C} E I(\mathbf{X}) &= \int_C^\infty f(x) dx > 0 \\ \frac{\partial}{\partial D} E I(\mathbf{X}) &= - \int_D^\infty f(x) dx < 0 \end{aligned}$$

Therefore, for fixed  $\lambda$ , the budget must increase to accommodate an increase in coverage or a decrease in deductible, while a decreased budget can be accommo-

dated by either a decrease in coverage or an increase in deductible.

**Example 2.3.** Figure 2.4 shows possible  $(C, D)$  pairs for various budget lines for an exponentially distributed loss with mean 100 and a loading factor  $\lambda = 1.25$ . The curves are similar to those in Example 2.2, with low deductibles corresponding to high premiums. We note that the contours flatten out quickly along the horizontal axis, while there is a larger differentiation among different values of  $D$ .

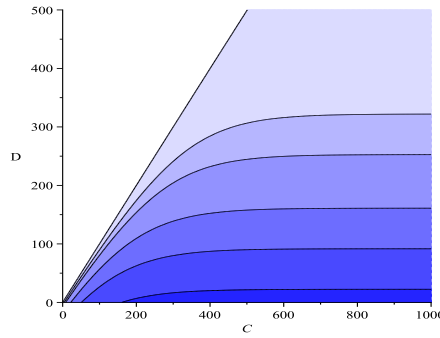


Figure 2.4: Illustration of Example 2.3

## 2.4 First Order Optimality Conditions

We derive optimality conditions for the problem of minimizing the insured's expected loss. In contrast to the expected utility based results of Arrow [2], we show that a policy with a 0 deductible can be optimal. Since  $EL(\mathbf{X})$  is the sum of  $ER(\mathbf{X})$  and a constant  $E\mathbf{X}$ , these conditions also hold for the insurer. We assume throughout that the support of the distribution  $F$  is not contained in  $[D, C]$ , i.e.,

$$\int_D^C f(x) dx < 1. \quad (2.20)$$

Indeed if  $\int_D^C f(x) dx = 1$  then by (2.4) the insured cost  $L(x) = p(C, D) + D$  with certainty. The expected value (2.4) is unchanged if there is a positive probability that no loss occurs (i.e.  $x = 0$ ). Indeed,  $E\mathbf{X}$  requires then a Stieltjes integral, but the contribution of the value  $x = 0$  is zero.



The respective derivatives of  $E L(\mathbf{X})$  give rise to the first order optimality conditions

$$\frac{\partial}{\partial C} E L(\mathbf{X}) = \frac{\partial}{\partial C} p(C, D) - \int_C^\infty f(x) dx = 0 \quad (2.21a)$$

$$\frac{\partial}{\partial D} E L(\mathbf{X}) = \frac{\partial}{\partial D} p(C, D) + \int_D^\infty f(x) dx = 0, \text{ if } D > 0, \quad (2.21b)$$

$$\frac{\partial}{\partial D} E L(\mathbf{X}) = \frac{\partial}{\partial D} p(C, D) + \int_D^\infty f(x) dx \geq 0, \text{ if } D = 0. \quad (2.21c)$$

#### 2.4.1 Insured Optimality Conditions for the Separable Premium

If the premium is given by  $p(C, D) = \alpha(D)\beta(C)$  conditions (2.21a), (2.21b), and (2.21c) become

$$\frac{\partial}{\partial C} E L(\mathbf{X}) = \alpha(D) \beta'(C) - \int_C^\infty f(x) dx = 0 \quad (2.22a)$$

$$\frac{\partial}{\partial D} E L(\mathbf{X}) = \alpha'(D) \beta(C) + \int_D^\infty f(x) dx = 0, \text{ if } D > 0, \quad (2.22b)$$

$$\frac{\partial}{\partial D} E L(\mathbf{X}) = \alpha'(D) \beta(C) + \int_D^\infty f(x) dx \geq 0, \text{ if } D = 0. \quad (2.22c)$$

To provide a more detailed analysis, we further assume that  $\beta(C) = \beta(0) + mC$  as in Example 2.1.

**Remark 2.2.** Writing (2.22a) as

$$\int_C^\infty f(x) dx = m \alpha(D), \quad (2.23)$$

we note that the left side of (2.23) is decreasing in  $C$ , and for fixed  $C$ , the right side is decreasing in  $D$ , by the assumption on  $\alpha(\cdot)$ . It follows that  $C$  and  $D$  move in the same direction, a higher deductible  $D$  corresponds to a higher coverage  $C$ .

If  $D$  is a differentiable function of  $C$  we can use implicit differentiation and

(2.22a) to obtain

$$\frac{dD}{dC} = -\frac{\frac{\partial^2 L(\mathbf{X})}{\partial C^2}}{\frac{\partial^2 L(\mathbf{X})}{\partial D \partial C}} = -\frac{f(C)}{m \alpha'(D)},$$

which is positive by the assumptions on the premium function  $\alpha(\cdot)$ .

**Remark 2.3.** The first-order optimality conditions (2.22a)–(2.22b),

$$\text{Prob}\{\mathbf{X} \geq C\} = \int_C^\infty f(x) dx = m \alpha(D) \quad (2.24a)$$

$$\text{Prob}\{\mathbf{X} \geq D\} = \int_D^\infty f(x) dx = -\alpha'(D) \beta(C) \quad (2.24b)$$

can be solved for  $D$  and  $C$ . For  $D = 0$  the optimality condition (2.22c) gives the inequality

$$\alpha'(0) \beta(C) + 1 \geq 0$$

or by (2.10)

$$C(0) \leq -\frac{1}{m} \left( \frac{1}{\alpha'(0)} + \beta(0) \right), \quad (2.25)$$

an upper bound on the coverage, corresponding to a zero deductible. There may be an external upper bound  $C_{\max}$  on coverage, say

$$C \leq C_{\max}. \quad (2.26)$$

For example, in the auto-insurance data of Example 2.1,  $C_{\max} = \$1$  million.

**Corollary 2.1.** A sufficient condition for a positive deductible is

$$\int_{-\frac{1}{m} \left( \frac{1}{\alpha'(0)} + \beta(0) \right)}^\infty f(x) dx > m \alpha(0). \quad (2.27)$$

*Proof.* It follows from Remark 2.2 that the lowest possibly optimal coverage  $C(0)$  corresponds to  $D = 0$ .  $C(0)$  is determined from (2.24a),

$$\int_{C(0)}^{\infty} f(x) dx = m \alpha(0),$$

and in addition must satisfy (2.25). These two conditions are satisfied only if

$$\int_{-\frac{1}{m} \left( \frac{1}{\alpha'(0)} + \beta(0) \right)}^{\infty} f(x) dx \leq m \alpha(0), \quad (2.28)$$

which is then a necessary condition for  $D = 0$ . The reverse inequality, (2.27), is therefore sufficient for a positive deductible.  $\square$

The next example illustrates, for the commonly occurring Gamma distribution (see [10], [28], [31], and [60]), the calculation of  $C$  and  $D$  satisfying the necessary optimality conditions.

**Example 2.4.** Let the random variable  $\mathbf{X}$  have the **Gamma distribution** with **scale**  $1/\lambda$  and **shape**  $k \geq 1$ ,

$$f(x) = \frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda x}, \quad x \geq 0, \quad (2.29)$$

in particular,

$$\Gamma(k) = (k-1)!,$$

if  $k$  is integer, in which case (2.29) is the **Erlang distribution**. Let  $\alpha(D)$  be given by,

$$\alpha(D) = \alpha(0) e^{-\delta D}, \quad D \geq 0.$$

**Zero deductible.** For  $D = 0$  to be optimal we must have, by (2.28),

$$\int_S^\infty f(x) dx = \frac{\lambda^k}{\Gamma(k)} \int_S^\infty x^{k-1} e^{-\lambda x} dx = e^{-\lambda S} \sum_{i=0}^{k-1} (i!)^{-1} (\lambda S)^i \leq m \alpha(0),$$

where

$$S = -\frac{1}{m} \left( \frac{1}{\alpha'(0)} + \beta(0) \right). \quad (2.30)$$

**Positive deductible.** Equations (2.24a)–(2.24b) become

$$\begin{aligned} e^{-\lambda C} \sum_{i=0}^{k-1} (i!)^{-1} (\lambda C)^i &= m \alpha(0) e^{-\delta D}, \\ e^{-\lambda D} \sum_{i=0}^{k-1} (i!)^{-1} (\lambda D)^i &= \alpha(0) \beta(C) \delta e^{-\delta D} \end{aligned}$$

or,

$$e^{\delta D - \lambda C} = \frac{m \alpha(0)}{\sum_{i=0}^{k-1} (i!)^{-1} (\lambda C)^i}, \quad (2.32a)$$

$$e^{(\delta - \lambda) D} = \frac{\alpha(0) \beta(C) \delta}{\sum_{i=0}^{k-1} (i!)^{-1} (\lambda D)^i}. \quad (2.32b)$$

From (2.32a) we get  $D$  as a function of  $C$ ,

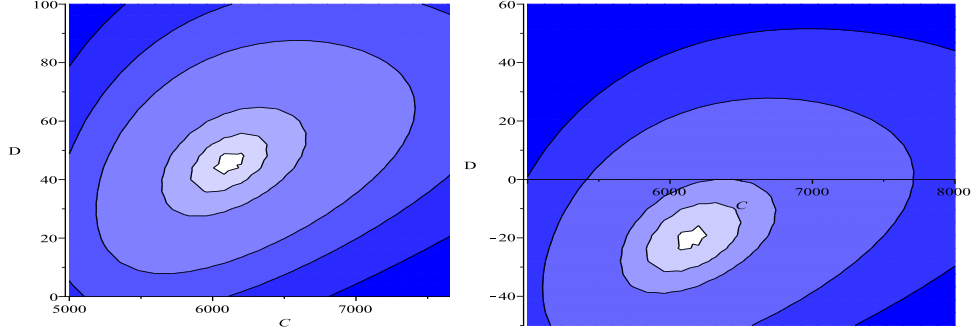
$$D = \frac{1}{\delta} \left( \log \left( \frac{m \alpha(0)}{\sum_{i=0}^{k-1} (i!)^{-1} (\lambda C)^i} \right) + \lambda C \right)$$

which can be substituted in (2.32b) to give an equation for  $C$ .

We show in Figure 2.5 the contour lines of the expected loss (2.4) for the Gamma distribution with  $k = 2$  and  $\lambda = 10^{-3}$ . For the premium (2.10a, 2.10b) we use  $\delta = 10^{-2}$  and  $\alpha(0) = 0.5$ , and  $\beta(0) = 50$ . In Figure 2.5(a) the slope is  $m = 0.25$ , and in Figure 2.5(b),  $m = 0.125$ .

In these figures, lighter shading corresponds with lower costs, and the optimal pair  $(C, D)$  is indicated by the brightest spot. Figure 2.5(b) illustrates that a zero

deductible may be optimal, since the unconstrained minimum gives a negative deductible. The optimal deductible  $D = 0$  lies on the closest contour (where  $D \geq 0$ ) to the theoretical optimum.



(a) Example with optimal deductible: positive (b) zero.

Figure 2.5: Contour lines of (2.4), the expected cost

#### 2.4.2 Insured Optimality Conditions for a Loaded Premium

If the premium is loaded,  $p(C, D) = \lambda E I(\mathbf{X})$ , the optimality conditions become

$$\begin{aligned} \frac{\partial}{\partial C} E L(\mathbf{X}) &= \lambda (E I(\mathbf{X}))' - \int_C^\infty f(x) dx = \lambda \int_C^\infty f(x) dx - \int_C^\infty f(x) dx \\ &= (\lambda - 1) \int_C^\infty f(x) dx = 0, \end{aligned} \quad (2.33a)$$

$$\begin{aligned} \frac{\partial}{\partial D} E L(\mathbf{X}) &= \lambda (E I(\mathbf{X}))' + \int_D^\infty f(x) dx = -\lambda \int_D^\infty f(x) dx + \int_D^\infty f(x) dx \\ &= (1 - \lambda) \int_D^\infty f(x) dx = 0 \text{ if } D > 0, \end{aligned} \quad (2.33b)$$

$$\frac{\partial}{\partial D} E L(\mathbf{X}) = (1 - \lambda) \int_D^\infty f(x) dx \geq 0, \text{ if } D = 0. \quad (2.33c)$$

To solve for the optimal levels of  $C$  and  $D$  we set the left sides of (2.33a) and (2.33b) equal to each other:

$$(\lambda - 1) \int_C^\infty f(x) dx = (1 - \lambda) \int_D^\infty f(x) dx,$$

which has no solution if  $\lambda \neq 1$  and no unique solution if  $\lambda = 1$ . For  $\lambda = 1$  (the ac-

tuarily fair case), the premium is exactly equal to the expected reimbursement, and so every policy is equivalent to the uninsured case.

**Remark 2.4.** If  $\lambda > 1$ , then  $p(C, D) > E I(\mathbf{X})$ , and so the optimal decision is to remain uninsured. On the other hand, if  $\lambda < 1$ , then  $p(C, D) < E I(\mathbf{X})$ . In this case  $\frac{\partial}{\partial C} E L(\mathbf{X}) < 0$  and  $\frac{\partial}{\partial D} E L(\mathbf{X}) > 0$ , therefore the optimal decision is to increase  $C$  and decrease  $D$  as much as possible, i.e., to purchase full coverage.

### 2.4.3 Optimality Conditions for the Insurer

It follows from (2.7) that the insurer's first order optimality conditions are the same as those of the insured, which is to be expected since this is a zero-sum game, and the best outcome for one player is the worst for the other. Indeed, the derivatives of  $E R(\mathbf{X})$  with respect to  $C$  and  $D$  are

$$\frac{\partial}{\partial C} E R(\mathbf{X}) = \frac{\partial}{\partial C} E L(\mathbf{X}) = \frac{\partial}{\partial C} p(C, D) - \int_C^\infty f(x) dx \quad (2.34a)$$

$$\frac{\partial}{\partial D} E R(\mathbf{X}) = \frac{\partial}{\partial D} E L(\mathbf{X}) = \frac{\partial}{\partial D} p(C, D) + \int_D^\infty f(x) dx \quad (2.34b)$$

It follows that the insurer may not offer the optimal policy desired by the insured, who must then settle for a non-optimal plan.

## 2.5 Variance

Because the purchase of insurance results in an increase expected loss to the insured, we consider variance reduction as the primary incentive for insurance. In this section we show that insurance decreases the variance of the insured, while increasing the insurer's variance.

### 2.5.1 Insured Variance

Applying (2.6), the variance for the insured is given by

$$\begin{aligned}\text{Var } L(\mathbf{X}) &= \text{Var } \mathbf{X} + \text{Var } R(\mathbf{X}) + 2 \text{Cov } [R(\mathbf{X}), \mathbf{X}] \\ &= \text{Var } \mathbf{X} + \text{Var } I(\mathbf{X}) - 2 \text{Cov } [I(\mathbf{X}), \mathbf{X}].\end{aligned}\quad (2.35)$$

The second line follows from the first because, by (2.3),

$$\begin{aligned}\text{Var } R(\mathbf{X}) &= \text{E } R(\mathbf{X})^2 - (\text{E } R(\mathbf{X}))^2 \\ &= p(C, D)^2 - 2p(C, D)\text{E } I(\mathbf{X}) + \text{E } I(\mathbf{X})^2 - (p(C, D) - \text{E } I(\mathbf{X}))^2 \\ &= \text{E } I(\mathbf{X})^2 - (\text{E } I(\mathbf{X}))^2 = \text{Var } I(\mathbf{X})\end{aligned}\quad (2.36)$$

and

$$\begin{aligned}\text{Cov } [R(\mathbf{X}), \mathbf{X}] &= \text{E } (R(\mathbf{X}) \cdot \mathbf{X}) - \text{E } R(\mathbf{X}) \cdot \text{E } \mathbf{X} \\ &= \text{E } (p(C, D) \cdot \mathbf{X}) - \text{E } (I(\mathbf{X}) \cdot \mathbf{X}) - \text{E } (p(C, D) \cdot \mathbf{X}) + \text{E } I(\mathbf{X}) \cdot \text{E } \mathbf{X} \\ &= -\text{Cov } [I(\mathbf{X}), \mathbf{X}]\end{aligned}\quad (2.37)$$

**Remark 2.5.** We note that the variance can be calculated without reference to the premium, as  $p(C, D)$  appears in  $\text{E } L(\mathbf{X})$  and  $\text{E } R(\mathbf{X})$ , but not in  $\text{E } I(\mathbf{X})$ . This is expected, as  $p(C, D)$  acts as a constant after  $C$  and  $D$  are chosen, and the addition of a constant to a random variable does not change its variance.

In the uninsured case,  $C = D$ ,  $\text{E } I(\mathbf{X})$  and  $\text{Var } I(\mathbf{X})$  are identically 0. We therefore obtain from (2.35) that

$$\text{Var } L(\mathbf{X}|k, k) = \text{Var } \mathbf{X} \text{ for all } k \geq 0. \quad (2.38)$$

The following theorem relates the variance  $\text{Var } L(\mathbf{X})$  to the coverage  $C$  and

deductible  $D$ .

**Theorem 2.2.** Let  $F, C, D$  satisfy (2.20),  $\int_D^C f(x) dx < 1$ . Then

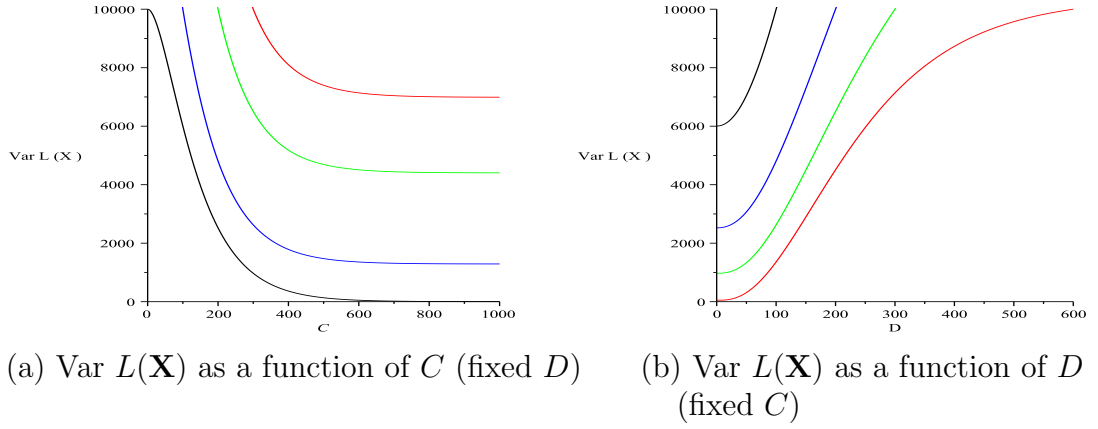
$$\frac{\partial}{\partial C} \text{Var } L(\mathbf{X}|C, D) < 0, \quad (2.39)$$

and

$$\frac{\partial}{\partial D} \text{Var } L(\mathbf{X}|C, D) > 0. \quad (2.40)$$

See Appendix A for the proof.

The variance  $\text{Var } L(\mathbf{X}|C, D)$  is thus a decreasing function of  $C$  (for fixed  $D$ ) and an increasing function of  $D$  (for fixed  $C$ ). We illustrate this in Figure 2.6.



Line Color	(a)	(b)
Black	$D = 0$	$C = 100$
Blue	100	200
Green	200	300
Red	300	600

(c) Key

Figure 2.6:  $\text{Var } L(\mathbf{X})$  as a function of (a)  $C$  and (b)  $D$

The following result establishes that insurance acts as a variance reducing mechanism, i.e., that the purchase of insurance provides a decrease in variance from the uninsured case.



**Corollary 2.2.**

(a) Let  $0 < D < C_1 < C_2$ , and let  $\int_D^{C_2} f(x) dx < 1$ . Then

$$\text{Var } L(\mathbf{X}|C_2, D) < \text{Var } L(\mathbf{X}|C_1, D) < \text{Var } \mathbf{X}. \quad (2.41)$$

(b) Let  $0 < D_1 < D_2 < C$ , and let

$$\int_{D_1}^C f(x) dx < 1.$$

Then

$$\text{Var } L(\mathbf{X}|C, D_1) < \text{Var } L(\mathbf{X}|C, D_2) < \text{Var } \mathbf{X}. \quad (2.42)$$

*Proof.* It follows from (2.38) that for  $C = D$ ,

$$\text{Var } L(\mathbf{X}|D, D) = \text{Var } \mathbf{X}. \quad (2.43)$$

(a) The left inequality in (2.41) follows from (2.39) and the right inequality from (2.43), writing  $\text{Var } \mathbf{X}$  as  $\text{Var } L(\mathbf{X}|D, D)$ . (b) is similarly proved.  $\square$

**Corollary 2.3.** To minimize variance, it is optimal to purchase full coverage,  $D = 0$  and  $C = \infty$ .

*Proof.* From Theorem 2.2, variance is decreasing in  $C$  and increasing in  $D$ . It is therefore optimal to increase  $C$  as much as possible and decrease  $D$  as much as possible.  $\square$

**Example 2.5.** Consider the Gamma distribution (2.29) with  $k = 2$  and  $\lambda = 10^{-3}$ . Figure 2.7 shows contour lines of  $\text{Var } L(\mathbf{X})$  in the  $(C, D)$ -plane. For convenience we represent these contour lines as the curves  $\beta - \text{Var } L(\mathbf{X}) = 0$  for several values of  $\beta$ . The red line is  $D = C$ , corresponding to the uninsured case (where the variance is  $\text{Var } \mathbf{X}$ ). Lower curves correspond to lower values of  $\text{Var } L(\mathbf{X})$ . These contour lines illustrate the flattening out of the variance, and the fact that above

a certain threshold value, increasing the coverage  $C$  has a negligible effect on the variance.

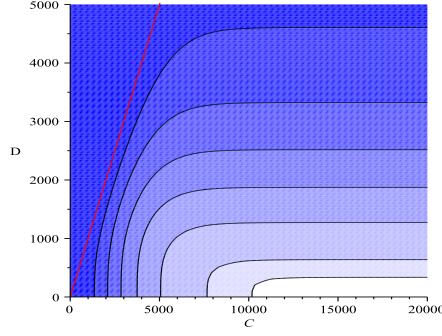


Figure 2.7: Contour lines of  $L(\mathbf{X}|C, D)$ , the insured variance

### 2.5.2 Insurer Variance

We derive similar results for the insurer as were found for the insured. The insurer variance is, by (2.36),

$$\text{Var } R(\mathbf{X}) = \text{Var } I(\mathbf{X}), \quad (2.44)$$

In the uninsured case the insurer's variance is 0. We show that the insurer's variance increases by entering into an insurance contract, but it cannot exceed the variance of the random variable that is being insured.

**Theorem 2.3.** Let  $F, C, D$  satisfy (2.20),  $\int_D^C f(x) dx < 1$ . Then

$$\frac{\partial}{\partial C} \text{Var } R(\mathbf{X}) > 0, \quad (2.45)$$

and

$$\frac{\partial}{\partial D} \text{Var } R(\mathbf{X}) < 0. \quad (2.46)$$

See Appendix B for the proof.

**Corollary 2.4.**

(a) Let  $0 < D < C_1 < C_2$ , and let  $\int_D^{C_2} f(x) dx < 1$ . Then

$$0 < \text{Var } R(\mathbf{X}|C_1, D) < \text{Var } R(\mathbf{X}|C_2, D) < \text{Var } \mathbf{X}. \quad (2.47)$$

(b) Let  $0 < D_1 < D_2 < C$ , and let  $\int_{D_1}^C f(x) dx < 1$ . Then

$$0 < \text{Var } R(\mathbf{X}|C, D_2) < \text{Var } R(\mathbf{X}|C, D_1) < \text{Var } \mathbf{X}. \quad (2.48)$$

*Proof.* It follows from (2.3) that  $R(\mathbf{X}|\infty, 0) - p(\infty, 0) = \mathbf{X}$ , and from Remark (2.5) that  $\text{Var } [R(\mathbf{X}) - p(C, D)] = \text{Var } R(\mathbf{X})$ . Therefore, the middle and right inequalities in (2.47) follow from (2.45). The left inequality is trivial. (b) is similarly proved.  $\square$

The following result establishes the relationship between the loss variable  $\mathbf{X}$ 's variance and the insured and insurer.

**Theorem 2.4.**  $\text{Var } \mathbf{X} = \text{Cov } [L(\mathbf{X}), \mathbf{X}] + \text{Cov } [-R(\mathbf{X}), \mathbf{X}]$

*Proof.*

$$\begin{aligned} \text{Cov } [L(\mathbf{X}), \mathbf{X}] + \text{Cov } [-R(\mathbf{X}), \mathbf{X}] &= \text{E } [L(\mathbf{X}) \cdot \mathbf{X}] - \text{E } L(\mathbf{X})\text{E } \mathbf{X} \\ &\quad + \text{E } [-R(\mathbf{X}) \cdot \mathbf{X}] - \text{E } [-R(\mathbf{X})]\text{E } \mathbf{X} \\ &= \text{E } [\mathbf{X}(L(\mathbf{X}) - R(\mathbf{X}))] - \text{E } \mathbf{X} [\text{E } L(\mathbf{X}) \\ &\quad - \text{E } R(\mathbf{X})] \end{aligned}$$

Substituting  $\mathbf{X} + R(\mathbf{X})$  for  $L(\mathbf{X})$ , we have

$$\text{Cov } [L(\mathbf{X}), \mathbf{X}] - \text{Cov } [R(\mathbf{X}), \mathbf{X}] = \text{E } \mathbf{X}^2 - (\text{E } \mathbf{X})^2 = \text{Var } \mathbf{X}$$

$\square$

**Remark 2.6.** We have established that the insured's variance ranges from 0 (full coverage) to  $\text{Var } \mathbf{X}$  (no coverage) and the insurer's variance increases from 0 (no coverage) to  $\text{Var } \mathbf{X}$  (full coverage). The variances  $\text{Var } L(\mathbf{X})$  and  $\text{Var } R(\mathbf{X})$  are equal if the associated losses covary with  $\mathbf{X}$  equally. To see this, recall that  $L(\mathbf{X}) = \mathbf{X} + R(\mathbf{X})$ . Then  $\text{Var } L(\mathbf{X}) = \text{Var } R(\mathbf{X})$  if

$$\text{Var } [\mathbf{X} + R(\mathbf{X})] = \text{Var } R(\mathbf{X}).$$

For two random variables  $\mathbf{Z}_1$  and  $\mathbf{Z}_2$ ,  $\text{Var } (\mathbf{Z}_1 + \mathbf{Z}_2) = \text{Var } \mathbf{Z}_1 + \text{Var } \mathbf{Z}_2 + 2 \text{Cov } (\mathbf{Z}_1, \mathbf{Z}_2)$ . The insured and insurer variances are then equal if  $\text{Var } \mathbf{X} + \text{Var } R(\mathbf{X}) + 2 \text{Cov } (R(\mathbf{X}), \mathbf{X}) = \text{Var } R(\mathbf{X})$ . Simplifying, we obtain that  $\text{Var } L(\mathbf{X}) = \text{Var } R(\mathbf{X})$  if  $\text{Var } \mathbf{X} + 2 \text{Cov } (R(\mathbf{X}), \mathbf{X}) = 0$ , or equivalently, by (5.3a) below,

$$\text{Cov } [L(\mathbf{X}), \mathbf{X}] = -\text{Cov } (R(\mathbf{X}), \mathbf{X}). \quad (2.49)$$

Applying Theorem 2.4 to equation (2.49), the insured and insurer have equal variances if

$$\text{Cov } [L(\mathbf{X}), \mathbf{X}] = \frac{1}{2} \text{Var } \mathbf{X}.$$

### 3 Mean-Variance Optimization

Because insurance reduces the variance of the costs incurred by the insured, while increasing their expected value, a Markowitzian mean–variance model, [42]–[43] is natural. The insured seeks to compensate for the increased value of their expected loss with the decrease in variance. The mean-variance approach is used here because it provides a simple, understandable, and tractable objective that is grounded in the fact that insurance is a mean-increasing and variance-reducing mechanism. Furthermore, mean-variance optimization provides a specific formula to determine the optimal coverage and deductible, and it clearly shows the interaction between the insurer’s attitude (expressed by the loading parameter  $\lambda$ ) and the insured’s attitude (expressed by the risk parameter  $\delta$  discussed below). Mean-variance also has a fundamental relationship with many other measures (e.g., standard and semi-deviation measures), and serves as a logical starting point for mean-risk approaches in insurance theory.

A major criticism of mean-variance approaches questions the use of variance as a risk measure. Put simply, variance similarly penalizes both gains and losses. This criticism is questionable in our framework, however, since the “gains” are really just the small values of  $L(\mathbf{X})$ , and the “losses” are the larger values of  $L(\mathbf{X})$ . In this case, we have the less egregious situation where large and small losses are being similarly penalized. Nonetheless, future work would do well to consider the various downside risk measures (e.g., semi-deviation and value-at-risk measures) to determine if the results derived below for mean-variance models, and in the literature for utility and non-expected utility models, still hold.

In 3.1 we describe the mean-variance model. Of particular interest here is the parameter  $\delta$ , which is an indication of the insured’s risk tolerance. We show how the demand for insurance is affected by the insured’s choice of  $\delta$  and its relationship to a critical level  $\delta^*$ .

Arrow’s classical result on optimal insurance coverage showed that full cover-

age above a nonzero deductible is the optimal policy for a risk averse insured with expected utility preferences and an actuarially unfair premium. If the premium is actuarially fair or discounted, then full coverage with a zero deductible is optimal. In section 3.2 we verify that these results also hold for a mean-variance minimizer paying a loaded premium. We then show that the insurer's optimal policy does not match the insured's, i.e., we show that there is no pair  $(C, D)$  that minimizes both the insured's and the insurer's mean-variance, however, the insurer may be able to induce the insured to purchase the insurer's optimal policy. Section 3.3 shows that the results of 3.2 are also valid for a mean-standard deviation model.

### 3.1 The Mean-Variance Model and Insurance Demand

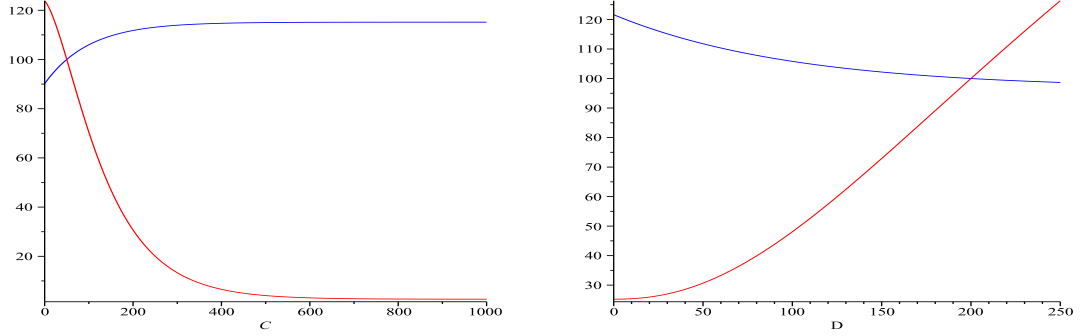
We utilize a mean-variance model where the customer seeks to minimize the sum of the expected insured loss and a multiple of the variance. In Figure 3.1 we graph the insured's mean and variance as a function of (a)  $C$  and (b)  $D$ . In this, and all other figures and examples of this chapter, we use a loaded premium with  $\lambda = 1.25$  and an exponentially distributed loss with mean 100 and variance 10,000. We observe that the range of values taken by the variance is much larger than the range of values taken by the expected insured loss. A mean-variance minimizer is thus likely to accept an increase in expected loss for the potentially large decrease in variance.

#### 3.1.1 General Principles

For a random loss  $\mathbf{Z}$ , the objective of the insured is:

$$\min_{C, D: C > D} [EL(\mathbf{Z}) + \delta \text{Var } L(\mathbf{Z})]. \quad (3.1)$$

The parameter  $\delta$  expresses the tradeoff between the mean (expected loss) and variance, and represents the decision maker's attitude towards risk. As  $\delta$  increases



The insured mean (blue) and variance (red) as a function of (a)  $C$  for  $D = 50$  and (b)  $D$  for  $C = 200$

Figure 3.1: The insured mean and variance as a function of (a)  $C$  and (b)  $D$

the insured places greater weight on the variance term, and so a decision maker with a large  $\delta$  value is likely to demand more insurance coverage than a decision maker with a smaller  $\delta$ .

There is a critical value  $\delta^*$  where a Markowitzian decision maker would be indifferent between buying or not buying insurance.  $\delta^*$  is found by setting the uninsured mean-variance equal to the insured mean-variance:

$$E L(\mathbf{X}) + \delta^* \text{Var } L(\mathbf{X}) = E \mathbf{X} + \delta^* \text{Var } \mathbf{X}.$$

Rearranging this equation we obtain a direct expression for  $\delta^*$ ,

$$\delta^* = \frac{E L(\mathbf{X}) - E \mathbf{X}}{\text{Var } \mathbf{X} - \text{Var } L(\mathbf{X})}. \quad (3.2)$$

The numerator of (3.2) reflects the increase in the insured's expected loss due to the purchase of insurance, while the denominator reflects the decrease in variance due to the purchase of insurance.  $\delta^*$  is thus the cost-benefit ratio of a given  $(C, D)$  policy. As the insured variance and expected loss are dependent on  $C$  and  $D$ , so is  $\delta^*$ , and it may then be written as  $\delta^*(C, D)$ .

Note that  $\delta^*$  is implicitly defined by the insurer. Indeed, the value  $E L(\mathbf{X}) - E \mathbf{X}$ , is a direct result of (the insurer's choice of) the premium, and therefore when

setting the premium, the value  $\delta^*$  is also defined. On the other hand,  $\delta$  is chosen by the insured, and so the relationship between  $\delta$  and  $\delta^*$  is key to determining whether or not insurance will be demanded. It is possible that some decision makers will refuse insurance coverage if this relationship is unfavorable.

For  $\delta > \delta^*$  it is optimal to buy insurance. To see this, rewrite (3.2) as

$$\delta^*(\text{Var } \mathbf{X} - \text{Var } L(\mathbf{X})) = E L(\mathbf{X}) - E \mathbf{X}. \quad (3.3)$$

Since  $\text{Var } \mathbf{X} > \text{Var } L(\mathbf{X})$ , the left side of (3.3) is positive, and we have for  $\delta > \delta^*$  that

$$\delta(\text{Var } \mathbf{X} - \text{Var } L(\mathbf{X})) > \delta^*(\text{Var } \mathbf{X} - \text{Var } L(\mathbf{X})) = E L(\mathbf{X}) - E \mathbf{X},$$

which gives

$$E L(\mathbf{X}) + \delta \text{Var } L(\mathbf{X}) < E \mathbf{X} + \delta \text{Var } \mathbf{X}.$$

Thus, the mean-variance is smaller in the insured case than in the uninsured case. Conversely, for  $\delta < \delta^*$  insurance cannot be justified in the Markowitz model, although in many cases it is required by law.

In general,  $\delta^*$  will change as  $C$  and  $D$  change. Thus, a decision maker (with a constant risk preference  $\delta$ ) may demand insurance for some  $(C, D)$  policies but not for others.

### 3.1.2 Loaded Premium

Suppose the premium is loaded,  $p(C, D) = \lambda E I(\mathbf{X})$ , where  $\lambda > 0$ . Then by (2.4),

$$\begin{aligned} E L(\mathbf{X}) - E \mathbf{X} &= \lambda E I(\mathbf{X}) + \int_D^C (D - x) f(x) dx + (D - C) \int_C^\infty f(x) dx \\ &= \lambda E I(\mathbf{X}) - E I(\mathbf{X}) = (\lambda - 1) E I(\mathbf{X}), \end{aligned}$$



therefore equation (3.2) becomes

$$\delta^* = \frac{(\lambda - 1)E I(\mathbf{X})}{\text{Var } \mathbf{X} - \text{Var } L(\mathbf{X})}.$$

Insurance will be demanded if the  $\delta$  of the potential insured is greater than  $\delta^*$ :

$$\delta \geq \frac{(\lambda - 1)E I(\mathbf{X})}{\text{Var } \mathbf{X} - \text{Var } L(\mathbf{X})} \quad (3.4)$$

In Figure 3.2 we give the contour lines of  $\delta^*$ , showing that it decreases as  $C$  and  $D$  increase. The demand for low deductible/low coverage policies should then be lower than the demand for higher coverage and deductible policies, as the higher coverage/higher deductible policies serve a greater range of  $\delta$ 's.

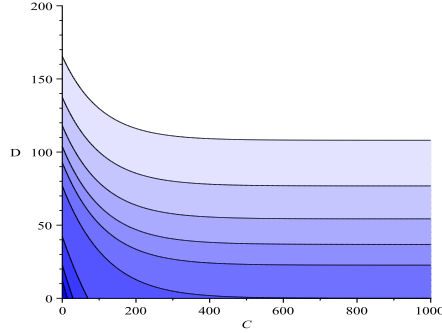


Figure 3.2: Contour lines of  $\delta^*$

We rewrite (3.4) with the fixed values  $\delta$  and  $\lambda$  on one side of the inequality and the variable values  $C$  and  $D$  on the other:

$$\frac{\lambda - 1}{\delta} \leq \frac{\text{Var } \mathbf{X} - \text{Var } L(\mathbf{X})}{E I(\mathbf{X})} \quad (3.5)$$

The choice of whether or not to purchase insurance is determined by (3.5) for all possible policies  $(C, D)$ . Note that the left side of (3.5) is dependent on the attitudes of both the insured ( $\delta$ ) and the insurer ( $\lambda$ ).

**Example 3.1.** Consider an exponentially distributed loss  $\mathbf{X}$  with parameter  $\kappa$ .

Then

$$\text{Var } \mathbf{X} = \kappa^{-2},$$

$$\text{Var } L(\mathbf{X}) = 2\kappa^{-2}(1 - e^{-\kappa D} + e^{-\kappa C}) + 2D\kappa^{-1}(e^{-\kappa C} - e^{-\kappa D}) - \kappa^{-2}(1 - e^{-\kappa D} + e^{-\kappa C})^2,$$

and

$$\text{E } I(\mathbf{X}) = \kappa^{-1}(e^{-\kappa D} - e^{-\kappa C}).$$

Therefore, equation (3.5) becomes

$$\frac{\lambda - 1}{\delta} \leq 2D + \kappa^{-1}(e^{-\kappa D} - e^{-\kappa C})$$

In Tables 3.1 and 3.2 we show the values of  $2D + \kappa^{-1}(e^{-\kappa D} - e^{-\kappa C})$  and  $\frac{\lambda-1}{\delta}$  for  $\kappa = 0.2$  and several values of  $C$ ,  $D$ ,  $\delta$ , and  $\lambda$ . It is apparent that, unless the risk preference  $\delta$  is very small or the premium loading  $\lambda$  is very high, (3.5) is satisfied, and thus insurance will be demanded.

For example, suppose the insurer holds a risk preference  $\delta = 0.001$  and pays a loading factor  $\lambda = 1.25$ . The left side of (3.5) is then equal to 250, and is less than the right side if any of the policies (500, 250), (1000, 250), or (1000, 500) are chosen. Increasing  $\delta$  from 0.001 will allow for a greater variety of possible  $(C, D)$  pairs.

If  $\kappa$  is decreased from 0.2 (thus increasing the expected loss and variance), the values of table 3.1 are increased, thus making (3.5) more easily satisfied. Therefore, in this example, a person's demand for insurance increases with their expected loss.

### 3.2 Optimality Conditions for the Mean-Variance Problem

We now derive the optimality conditions for the mean-variance minimization problem (3.1). We begin by developing general conditions that hold for any

	$D=0$	10	25	50	100	250	500
$C=25$	19.7	30.6	-	-	-	-	-
50	31.6	42.5	61.9	-	-	-	-
100	43.2	54.2	73.6	111.6	-	-	-
250	49.7	60.6	80.0	118.1	206.4	-	-
500	50.0	60.9	80.3	118.4	206.8	500.3	-
1000	50.0	60.9	80.3	118.4	206.8	500.3	1000.0

Table 3.1: Values of  $2D + \kappa^{-1}(e^{-\kappa D} - e^{-\kappa C})$  for assorted values of  $C$  and  $D$

	$\lambda=1$	1.1	1.25	1.5	2
$\delta=0.001$	0	100	250	500	1000
0.01	0	10	25	50	100
0.05	0	2	5	10	20
0.1	0	1	2.5	5	10
0.25	0	0.4	1	2	4
0.5	0	0.2	0.5	1	2
1	0	0.1	0.25	0.5	1
2	0	0.05	0.125	0.25	0.5

Table 3.2: Values of  $\frac{\lambda-1}{\delta}$  for assorted values of  $\lambda$  and  $\delta$

premium, and then derive specific conditions for the case of a loaded premium.

### 3.2.1 General Principles

To determine first-order optimality conditions for the mean-variance minimization problem (3.1) we calculate, with respect to  $C$  and  $D$ , the derivatives of the expected loss and variance. The derivatives of the expected loss are

$$\frac{\partial}{\partial C} \mathbb{E} L(\mathbf{X}) = \frac{\partial}{\partial C} p(C, D) - \int_C^\infty f(x) dx \quad (3.6)$$

$$\frac{\partial}{\partial D} \mathbb{E} L(\mathbf{X}) = \frac{\partial}{\partial D} p(C, D) + \int_D^\infty f(x) dx, \quad (3.7)$$

and the variance derivatives are (see Appendix A for the derivation of these derivatives):

$$\begin{aligned} \frac{\partial}{\partial C} \text{Var } L(\mathbf{X}) &= 2 \int_C^\infty f(x) dx \int_0^D (x - D) f(x) dx \\ &\quad + 2 \int_0^C f(x) dx \int_C^\infty (C - x) f(x) dx \end{aligned} \quad (3.8)$$

$$\begin{aligned} \frac{\partial}{\partial D} \text{Var } L(\mathbf{X}) &= 2 \int_0^D f(x) dx \int_C^\infty (x - C) f(x) dx \\ &\quad + 2 \int_0^D (D - x) f(x) dx \int_D^\infty f(x) dx. \end{aligned} \quad (3.9)$$

The first-order optimality conditions are determined by setting the derivatives  $\frac{\partial}{\partial C}[\text{EL}(\mathbf{X}) + \delta \text{Var } L(\mathbf{X})]$  and  $\frac{\partial}{\partial D}[\text{EL}(\mathbf{X}) + \delta \text{Var } L(\mathbf{X})]$  to zero. Moving the terms  $\frac{\partial}{\partial C} \text{EL}(\mathbf{X})$  and  $\frac{\partial}{\partial D} \text{EL}(\mathbf{X})$  to the opposite sides of the equations, we obtain the optimality conditions

$$\begin{aligned} -\frac{\partial}{\partial C} p(C, D) + \int_C^\infty f(x) dx &= 2\delta \int_C^\infty f(x) dx \int_0^D (x - D) f(x) dx \\ &\quad + 2\delta \int_0^C f(x) dx \int_C^\infty (C - x) f(x) dx \end{aligned} \quad (3.10a)$$

$$\begin{aligned} -\frac{\partial}{\partial D} p(C, D) - \int_D^\infty f(x) dx &= 2\delta \int_0^D f(x) dx \int_C^\infty (x - C) f(x) dx \\ &\quad + 2\delta \int_0^D (D - x) f(x) dx \int_D^\infty f(x) dx \end{aligned} \quad (3.10b)$$

Equations (3.10a) and (3.10b) can be simultaneously solved to determine the optimal coverage and deductible for the mean-variance minimizing insured.

**Proposition 3.1.**

(a) A sufficient condition for infinite coverage is

$$\frac{\partial}{\partial C} p(C, D) \leq \text{Prob}(\mathbf{X} > C) \text{ for all } C. \quad (3.11)$$

(b) A sufficient condition for a zero deductible is

$$-\frac{\partial}{\partial D}p(C, D) \leq \text{Prob}(\mathbf{X} > D) \text{ for all } D. \quad (3.12)$$

*Proof.* (a) Infinite coverage is optimal if  $\frac{\partial}{\partial C}(\text{E } L(\mathbf{X}) + \delta \text{Var } L(\mathbf{X})) = \frac{\partial}{\partial C}\text{E } L(\mathbf{X}) + \delta \frac{\partial}{\partial C}\text{Var } L(\mathbf{X}) \leq 0$  for all  $C$ . By Theorem 2.2 we have that  $\delta \frac{\partial}{\partial C}\text{Var } L(\mathbf{X}) < 0$ . Using equation (3.6),  $\frac{\partial}{\partial C}\text{E } L(\mathbf{X}) \leq 0$  if  $\frac{\partial}{\partial C}p(C, D) \leq \int_C^\infty f(x) dx$ , which is equivalent to (3.11).

(b) Similarly, a 0-deductible is optimal if  $\frac{\partial}{\partial D}\text{E } L(\mathbf{X}) + \delta \frac{\partial}{\partial D}\text{Var } L(\mathbf{X}) \geq 0$  for all  $D \geq 0$ . Applying Theorem 2.2 and equation (3.7),  $\frac{\partial}{\partial D}\text{E } L(\mathbf{X}) \geq 0$  if  $-\frac{\partial}{\partial D}p(C, D) \leq \int_D^\infty f(x) dx$ , which is equivalent to (3.12).  $\square$

### 3.2.2 Loaded Premium

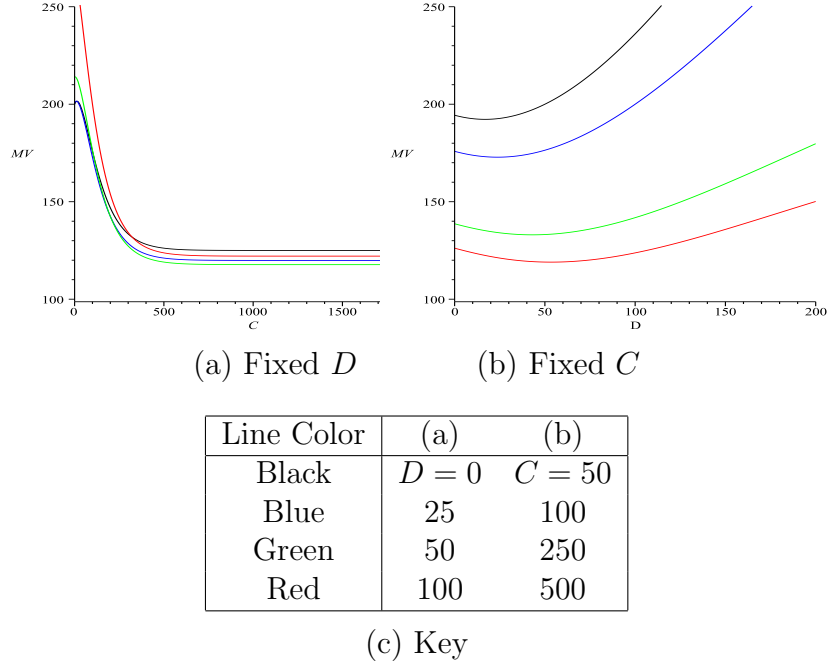
Consider the loaded premium  $p(C, D) = \lambda \text{E } I(\mathbf{X})$ . The mean loss in this case is

$$\text{E } L(\mathbf{X}) = (\lambda - 1)\text{E } I(\mathbf{X}) + \text{E } \mathbf{X}$$

and the variance is given by (2.35).

In Figure 3.3 we plot the mean-variance as a function of (a)  $C$  for several values of  $D$ , and (b)  $D$  for several values of  $C$ . The figure uses the risk preference  $\delta = 0.01$ . In (a) it is apparent that increasing  $C$  beyond a certain threshold does very little in decreasing the mean-variance. Figure (b) shows the mean-variance as convex in  $D$  with a nonzero minimum.

The mean-variance derivatives for a loaded premium policy are, by (2.33a),

Figure 3.3: Plot of  $MV(\mathbf{X}) = \mathbb{E} L(\mathbf{X}) + \delta \text{Var} L(\mathbf{X})$ 

(2.33b), (3.8), and (3.9),

$$\begin{aligned}
\frac{\partial}{\partial C} [\mathbb{E} L(\mathbf{X}) + \delta \text{Var} L(\mathbf{X})] &= (\lambda - 1) \int_C^\infty f(x) dx \\
&+ 2\delta \int_C^\infty f(x) dx \int_0^D (x - D) f(x) dx \\
&+ 2\delta \int_0^C f(x) dx \int_C^\infty (C - x) f(x) dx
\end{aligned} \tag{3.13a}$$

$$\begin{aligned}
\frac{\partial}{\partial D} [\mathbb{E} L(\mathbf{X}) + \delta \text{Var} L(\mathbf{X})] &= (1 - \lambda) \int_D^\infty f(x) dx \\
&+ 2\delta \int_0^D f(x) dx \int_C^\infty (x - C) f(x) dx \\
&+ 2\delta \int_0^D (D - x) f(x) dx \int_D^\infty f(x) dx,
\end{aligned} \tag{3.13b}$$

and the optimality conditions (3.10a) and (3.10b) become

$$\begin{aligned}
& (1 - \lambda) \int_C^\infty f(x) dx \\
&= 2\delta \left( \int_C^\infty f(x) dx \int_0^D (x - D) f(x) dx + \int_0^C f(x) dx \int_C^\infty (C - x) f(x) dx \right)
\end{aligned} \tag{3.14a}$$

$$\begin{aligned}
& (\lambda - 1) \int_D^\infty f(x) dx \\
&= 2\delta \left( \int_0^D f(x) dx \int_C^\infty (x - C) f(x) dx + \int_0^D (D - x) f(x) dx \int_D^\infty f(x) dx \right)
\end{aligned} \tag{3.14b}$$

We now show that the classical insurance results, full coverage for fair or discounted premiums and full coverage above a nonzero deductible for unfair premiums, are valid in the mean-variance framework.

**Theorem 3.1.** If  $\lambda \leq 1$ , i.e., if the insurance premium is fair or discounted, then the optimal insurance policy has a zero deductible and full (infinite) coverage.

*Proof.* Suppose  $\lambda \leq 1$ . Then

$$\begin{aligned}
\frac{\partial}{\partial C} p(C, D) &= \lambda \int_C^\infty f(x) dx \leq \int_C^\infty f(x) dx = \text{Prob}\{\mathbf{X} > C\} \\
-\frac{\partial}{\partial D} p(C, D) &= \lambda \int_D^\infty f(x) dx \leq \int_D^\infty f(x) dx = \text{Prob}\{\mathbf{X} > D\}.
\end{aligned}$$

The sufficient conditions of Proposition 3.1 are therefore satisfied and full coverage ( $D = 0$  and  $C = \infty$ ) is optimal.  $\square$

**Theorem 3.2.** Suppose that  $\delta > 0$  and  $\lambda > 1$  satisfy (3.4) (i.e., that insurance will be demanded). Then the optimal policy for the mean-variance minimizer is  $C = \infty$  and  $D$  satisfying equation (3.17).

*Proof.* If  $D$  is infinite we are in the uninsured case, therefore, by the assumption that (3.4) is satisfied, we know that  $D$  is finite. We further assume, for the time-being, that  $C$  is finite. Then, dividing the optimality conditions (3.14a) and (3.14b) by  $-2\delta \int_C^\infty f(x) dx$  and  $2\delta \int_D^\infty f(x) dx$ , respectively, we obtain the equivalent conditions

$$\frac{\lambda - 1}{2\delta} = \frac{F(C)}{1 - F(C)} \int_C^\infty (x - C) f(x) dx + \int_0^D (D - x) f(x) dx \quad (3.15a)$$

$$\frac{\lambda - 1}{2\delta} = \frac{F(D)}{1 - F(D)} \int_C^\infty (x - C) f(x) dx + \int_0^D (D - x) f(x) dx \quad (3.15b)$$

Combining (3.15a) and (3.15b) we obtain the condition

$$\frac{F(C)}{1 - F(C)} = \frac{F(D)}{1 - F(D)} \quad (3.16)$$

Since  $C \geq D$ , it follows that  $F(C) \geq F(D)$  and  $1 - F(C) \leq 1 - F(D)$ . Therefore the numerator of the left side of (3.16) is greater than or equal to the numerator of the right side, and the denominator of the left side is less than or equal to the denominator of the right side, so  $\frac{F(C)}{1 - F(C)} \geq \frac{F(D)}{1 - F(D)}$ . Hence, the only solutions to equations (3.15a) and (3.15b) are the points  $C = D$ , i.e., the uninsured case.

We next consider the cases where  $C$  or  $D$  take on endpoint values.

- i) If either  $C = 0$  or  $D = \infty$ , then we are in the uninsured case, which is non-optimal by the assumption that (3.4) is satisfied.
- ii) If  $C = \infty$  then we optimize with respect to  $D$ , i.e., we let  $C = \infty$  in (3.15b). Applying L'Hospital's rule, we obtain the condition for  $D$

$$\frac{\lambda - 1}{2\delta} = \int_0^D (D - x) f(x) dx \quad (3.17)$$

- iii) If  $D = 0$  then we optimize with respect to  $C$ , i.e., we choose  $C$  so that (3.15a) is satisfied with  $D = 0$ . However, if (3.15a) is satisfied, then, because



$\frac{F(C)}{1-F(C)} > \frac{F(D)}{1-F(D)}$  for  $C > D$ , we have that the relationship in (3.15b) is “ $>$ ”, which means that  $\frac{\partial}{\partial D}[E L(\mathbf{X}) + \delta \text{Var } L(\mathbf{X})] < 0$  for all  $D$ . Hence, the point  $D = 0$  is not optimal.

Assuming that (3.4) is satisfied, the optimal policy is therefore given by case ii),  $C = \infty$  and  $D$  satisfying (3.17).  $\square$

**Corollary 3.1.** Let the optimal deductible be given by  $D^*$ . Then for any  $\lambda \geq 0$ ,  $D^*$  can be found by solving

$$\text{Max} \left( \frac{\lambda - 1}{2\delta}, 0 \right) = \int_0^D (D - x) f(x) dx. \quad (3.18)$$

If  $\lambda > 1$  we also have that  $D^* > 0$ .

*Proof.* If  $\lambda \leq 1$ , then  $\text{Max} \left( \frac{\lambda - 1}{2\delta}, 0 \right) = 0$ , and the only solution to  $0 = \int_0^D (D - x) f(x) dx$  is  $D = 0$ , which by Theorem 3.1 is optimal for  $\lambda \leq 1$ . If  $\lambda > 1$ , then  $\text{Max} \left( \frac{\lambda - 1}{2\delta}, 0 \right) = \frac{\lambda - 1}{2\delta}$ , and so (3.18) becomes (3.17) from Theorem 3.2. Furthermore, if  $D = 0$ , then the right side of (3.18) is equal to 0, but since  $\lambda > 1$  and  $\delta > 0$ , the left side is positive, and so we cannot have  $D = 0$ . Therefore, we must have  $D^* > 0$ .  $\square$

Assuming that  $\delta$  and  $\lambda$  are such that there exists points  $(C, D)$  that satisfy (3.5), the combination of Theorem 3.2 and Corollary 3.1 replicates the classical insurance result of Arrow and Mossin that, in the case of an unfair premium, partial coverage (non-zero deductible with infinite coverage) is optimal.

**Example 3.2.** Figure 3.4 illustrates Theorem 3.2 and Corollary 3.1 for the exponential example, using a risk preference  $\delta = 0.01$ .

From Corollary 3.1, the optimal value  $D$  is the  $D$  that satisfies

$$\frac{\lambda - 1}{2\delta} = \int_0^D (D - x) f(x) dx.$$

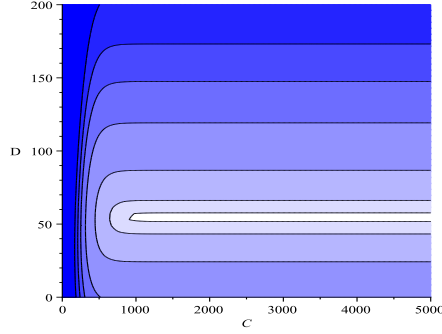


Figure 3.4: Contour lines of  $E L(\mathbf{X}) + \delta \text{Var } L(\mathbf{X})$

Letting  $\lambda = 1.25$  and  $\delta = 0.01$  in the above equation, we obtain the (positive) solution  $D = 54.54$ . The mean-variance of an infinite coverage, 54.54-deductible policy,  $E L(\mathbf{X}|\infty, 54.54) + \delta \text{Var } L(\mathbf{X}|\infty, 54.54)$ , is 117.67.

The mean-variance in the uninsured case,  $E L(\mathbf{X}|C = D) + \delta \text{Var } L(\mathbf{X}|C = D)$ , is 200, and thus the purchase of insurance is optimal in this example.

### 3.2.3 Insurer Mean-Variance

We now consider the insurer's perspective. We assume for the insurer a mean-variance minimization problem

$$\min_{(C,D):C>D} MV_R(\mathbf{X}) = \min_{(C,D):C>D} [E [-R(\mathbf{X})] + \delta_R \text{Var } R(\mathbf{X})],$$

where  $\delta_R$  is the parameter describing the insurer's risk tolerance. Recall that  $R(\mathbf{X})$  is the insurer's profit, so  $-R(\mathbf{X})$  represents the insurer loss and is given by  $-R(\mathbf{X}) = -p(C, D) + I(\mathbf{X})$ . We also have that  $\text{Var } R(\mathbf{X}) = \text{Var } I(\mathbf{X})$ , and we assume a loaded premium with  $\lambda > 1$ . The insurer mean-variance problem can then be written as:

$$\min_{(C,D):C>D} MV_R(\mathbf{X}) = \min_{(C,D):C>D} [(1 - \lambda)E I(\mathbf{X}) + \delta_R \text{Var } I(\mathbf{X})] \quad (3.19)$$

If  $\delta_R = 0$  we say that the insurer is *Risk Neutral*, and if  $\delta_R > 0$  the insurer is

*Risk Averse.* We first establish that policies exist that reduce the insurer's mean-variance. In the uninsured case, the insurer's mean and variance are both 0, and so the insurer has incentive to offer insurance if there exists a  $(C, D)$  for which  $MV_R(\mathbf{X}|C, D) < 0$ . In the risk neutral case ( $\delta_R = 0$ ) the insurer's mean-variance is

$$MV_R(\mathbf{X}) = (1 - \lambda)E I(\mathbf{X}).$$

Since  $\lambda > 1$ , it follows that the mean-variance is always less than 0, and the insurer's mean-variance is therefore decreased for all policies. To find the optimal policy we consider the insurer loss derivatives, which are given by

$$\frac{\partial}{\partial C}[-R(\mathbf{X})] = (1 - \lambda) \int_C^\infty f(x) dx,$$

and

$$\frac{\partial}{\partial D}[-R(\mathbf{X})] = (\lambda - 1) \int_D^\infty f(x) dx.$$

For  $\lambda > 1$ , the insurer's mean loss is thus decreasing in  $C$  and increasing in  $D$ . Therefore, the optimal policy from the insurer's perspective is full coverage, whereas the insured will seek a non-zero deductible.

Next consider the risk averse ( $\delta_R > 0$ ) case.  $MV_R(\mathbf{X}) < 0$  if

$$\frac{\lambda - 1}{\delta_R} > \frac{\text{Var } I(\mathbf{X})}{E I(\mathbf{X})}, \quad (3.20)$$

i.e., the insurer's mean-variance will decrease for any  $(C, D)$  pair for which (3.20) holds. To determine the insurer's optimal policy we consider the mean-variance derivatives. The insurer loss derivatives were computed above, and the variance derivatives are given by (B5) and (B6) in Appendix B:

$$\frac{\partial}{\partial C} \text{Var } I(\mathbf{X}) = 2 \left( C \int_0^C f(x) dx - D \int_0^D f(x) dx - \int_D^C x f(x) dx \right) \int_C^\infty f(x) dx$$

$$\frac{\partial}{\partial D} \text{Var } I(\mathbf{X}) = 2 \left( D \int_D^\infty f(x) dx - \int_D^C x f(x) dx - C \int_C^\infty f(x) dx \right) \int_0^D f(x) dx$$

The mean-variance derivatives are then

$$\begin{aligned} \frac{\partial}{\partial C} MV_R(\mathbf{X}) &= (1 - \lambda) \int_C^\infty f(x) dx \\ &\quad + 2\delta_R \left( C \int_0^C f(x) dx - D \int_0^D f(x) dx - \int_D^C x f(x) dx \right) \int_C^\infty f(x) dx \end{aligned} \quad (3.21a)$$

$$\begin{aligned} \frac{\partial}{\partial D} MV_R(\mathbf{X}) &= (\lambda - 1) \int_D^\infty f(x) dx \\ &\quad + 2\delta_R \left( D \int_D^\infty f(x) dx - \int_D^C x f(x) dx - C \int_C^\infty f(x) dx \right) \int_0^D f(x) dx \end{aligned} \quad (3.21b)$$

Setting (3.21a) and (3.21b) equal to 0, we obtain the first order optimality conditions

$$\begin{aligned} \frac{\lambda - 1}{2\delta_R} &= C \int_0^C f(x) dx - D \int_0^D f(x) dx - \int_D^C x f(x) dx \\ &= (C - D) \int_0^D f(x) dx + \int_D^C (C - x) f(x) dx \end{aligned} \quad (3.22a)$$

$$\begin{aligned} \frac{\lambda - 1}{2\delta_R} &= \frac{\int_0^D f(x) dx}{\int_D^\infty f(x) dx} \left( C \int_C^\infty f(x) dx + \int_D^C x f(x) dx - D \int_D^\infty f(x) dx \right) \\ &= \frac{\int_0^D f(x) dx}{\int_D^\infty f(x) dx} \left( (C - D) \int_C^\infty f(x) dx + \int_D^C (x - D) f(x) dx \right) \end{aligned} \quad (3.22b)$$

We again assume that  $C$  and  $D$  are finite (the case of infinite  $C$  is addressed below, infinite  $D$  corresponds with the uninsured case). Combining the right sides of (3.22a) and (3.22b) we obtain the optimality condition

$$\begin{aligned} \int_D^\infty f(x) dx &\left( (C - D) \int_0^D f(x) dx + \int_D^C (C - x) f(x) dx \right) \\ &= \int_0^D f(x) dx \left( (C - D) \int_C^\infty f(x) dx + \int_D^C (x - D) f(x) dx \right). \end{aligned}$$

Simplifying, we have the equivalent condition

$$C \int_D^C f(x) dx = \int_D^C x f(x) dx,$$

which has no solution unless  $C = D$ . Therefore, there is no finite  $C$ ,  $C > D \geq 0$ , where both mean-variance derivatives are equal to 0. Assuming that there exists a  $(C, D)$  for which  $MV_R(\mathbf{X}) < 0$ , the optimal policy for the insurer must then have either  $C = \infty$  and  $\frac{\partial}{\partial D} MV_R(\mathbf{X}) = 0$ , or  $D = 0$  and  $\frac{\partial}{\partial C} MV_R(\mathbf{X}) = 0$ .

Our focus here is not on determining the particulars of the insurer's optimal policy; instead we wish to analyze the behavior of the insurer's mean-variance in the context of the insured's decisions. We note that  $\frac{\partial}{\partial C} MV_R(\mathbf{X}) > 0$  if and only if

$$\frac{\lambda - 1}{2\delta_R} < C \int_0^C f(x) dx - D \int_0^D f(x) dx - \int_D^C x f(x) dx, \quad (3.23)$$

and observe that, for  $C > D$ ,

$$\begin{aligned} C \int_0^C f(x) dx - D \int_0^D f(x) dx - \int_D^C x f(x) dx \\ > C \int_0^C f(x) dx - C \int_0^D f(x) dx - C \int_D^C f(x) dx = 0. \end{aligned}$$

The right side of (3.23) is therefore always positive, but since

$$\lim_{D \rightarrow C^-} \left( C \int_0^C f(x) dx - D \int_0^D f(x) dx - \int_D^C x f(x) dx \right) = 0,$$

it can be made arbitrarily close to 0.

Suppose  $D$  is fixed. Then, because  $\frac{\lambda-1}{2\delta_R} > 0$ , there is always a  $\bar{C}$  for which (3.23) does not hold for any  $C < \bar{C}$  and for which (3.23) does hold for all  $C > \bar{C}$ . Therefore, the mean-variance derivative with respect to  $C$  has both negative and

positive values:

$$\frac{\partial}{\partial C} MV_R(\mathbf{X}) < 0 \text{ for all } C < \bar{C} \quad (3.24a)$$

$$\frac{\partial}{\partial C} MV_R(\mathbf{X}) > 0 \text{ for all } C > \bar{C} \quad (3.24b)$$

In particular, let  $D$  be given by (3.18), i.e. let  $D$  be optimal for the insured. Then we can find a finite  $\bar{C}$ , for which (3.24b) holds. Raising the coverage above  $\bar{C}$  is unfavorable for the insurer, but by Theorem 3.2, the insured prefers  $C \rightarrow \infty$ . The optimal policies of the insured and the risk averse insurer are therefore different.

We have shown that for both risk neutral and risk averse insurers, there is an inherent disparity between the preferred policies of the insured and the insurer if they each hold mean-variance preferences. There is, however, a balance of power between the insured and the insurer.

In the insurer's favor is i) it can choose the policies, i.e., the  $(C, D)$  pairs, that it offers and ii) individuals are often forced by law to obtain insurance (e.g., health and automobile insurance in the United States), thus allowing insurer's to only offer sub-optimal (from the insured's perspective) policies.

On the other hand, the insured is free to choose from a number of different insurers, thus creating competition among insurance companies, and perhaps forcing particular insurers to offer sub-optimal (from their perspective) policies.

The case i) deserves special attention. Although there is no "natural" point  $(C, D)$  where the insured and insurer's mean-variances are both minimal, the fact that the insurer chooses the policies to be offered opens the possibility of a mutually optimal solution in this insurer defined restricted  $(C, D)$  space.

The following example considers the set of policies offered by the insurer, showing how the insurer may induce the insured to choose the policy that minimizes the insurer's mean-variance.

**Example 3.3.** In Example 3.2 we considered an exponential loss with a mean of 100, and showed that the insured's optimal policy is infinite coverage and a deductible of 54.54. As discussed above, the insurer's optimal solution has either infinite coverage or a zero deductible. Assuming that the insurer's risk preference is  $\delta_R = 0.0005$ , we investigate the two possibilities.

Suppose the coverage is infinite, then letting  $C = \infty$  in (3.21b) we obtain

$$\frac{\partial}{\partial D} MV_R(\mathbf{X})|_{C=\infty} = (\lambda - 1) \int_D^\infty f(x) dx + 2\delta_R \int_D^\infty (D - x) f(x) dx \int_0^D f(x) dx.$$

Setting this equation to 0 we find that there is no solution with  $D \geq 0$ , and so the insurer's optimal policy must have a deductible  $D = 0$ . To find the optimal coverage  $C$  we let  $D = 0$  in (3.21a):

$$\frac{\partial}{\partial C} MV_R(\mathbf{X})|_{D=0} = (1 - \lambda) \int_C^\infty f(x) dx + 2\delta_R \int_0^C (C - x) f(x) dx \int_C^\infty f(x) dx,$$

Setting this equation to 0, we obtain the optimal coverage of  $C = 346.89$ .

To attain it's optimal policy, the insurer must carefully choose the policies that it offers the insured. Let  $\Psi$  be the set of policies that the insured finds inferior to the insurer's optimal policy:

$$\begin{aligned} \Psi &= \{(C, D) : MV(\mathbf{X}|C, D) > MV(\mathbf{X}|346.89, 0)\} \\ &= \{(C, D) : MV(\mathbf{X}|C, D) > 130.35\} \end{aligned}$$

The set of policies that the insurer should offer then takes the form  $\{(346.89, 0)\} \cup \Phi$ , where  $\Phi \subseteq \Psi$ . In Figure 3.5 we show the contour line of  $MV(\mathbf{X}) = 130.35$ . The colored area represents  $\Psi$ , the points where  $MV(\mathbf{X}) > 130.35$ , and the white part is the set of policies that the insured would prefer to the optimal insurer policy.

The insurer may, for example, offer only two policies:  $(346.89, 0)$  and  $(2000, 200)$ .

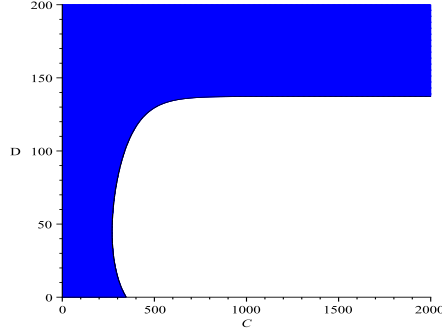


Figure 3.5: Contour line of  $MV(\mathbf{X})$  for the insurer's optimal policy

If the insurer correctly estimated the insured's risk preference  $\delta$ , then the insured will prefer the  $(346.89, 0)$  policy to the  $(2000, 200)$  policy, and the insurer will attain its optimal policy. We note that the point  $(346.89, 0)$  satisfies (3.5), i.e., the insured's mean-variance at this point is smaller than his mean-variance in the uninsured case.

**Remark 3.1.** In example 3.3 we considered the case where the insurer has mean-variance preferences, however, the lesson remains the same no matter what type of preferences the insurer holds - if the insurer can properly estimate the insured's risk preference, it can determine a set of policies to offer that will result in the insured selecting the insurer's optimal policy.

### 3.3 A Mean-Standard Deviation Formulation

In this section we consider an alternative mean-risk formulation, using the standard deviation as a risk measure instead of the variance. We assume throughout this section a loaded premium,  $p(C, D) = \lambda E I(\mathbf{X})$ , for  $\lambda > 0$ .

The standard deviation,  $\sigma(\mathbf{X}) = \sqrt{\text{Var } \mathbf{X}}$ , has as its main advantage that it is in the same unit scale as the mean (as opposed to the variance, where the units are squared). The value of the expected loss is not affected by the risk measure,



and so it is the same as above. The standard deviation is

$$\sigma[L(\mathbf{X})] = [\text{Var } L(\mathbf{X})]^{1/2} \quad (3.25)$$

with derivatives

$$\begin{aligned} \frac{\partial}{\partial C} \sigma[L(\mathbf{X})] &= \frac{1}{2} [\text{Var } L(\mathbf{X})]^{-1/2} \frac{\partial}{\partial C} \text{Var } L(\mathbf{X}) \\ &= \frac{1}{2} \frac{\frac{\partial}{\partial C} \text{Var } L(\mathbf{X})}{\sigma[L(\mathbf{X})]} \end{aligned} \quad (3.26)$$

$$\frac{\partial}{\partial D} \sigma[L(\mathbf{X})] = \frac{1}{2} \frac{\frac{\partial}{\partial D} \text{Var } L(\mathbf{X})}{\sigma[L(\mathbf{X})]} \quad (3.27)$$

Since  $\sigma[L(\mathbf{X})] \geq 0$ , the standard-deviation derivatives have the same signs as the variance derivatives. Therefore, Proposition 3.1 and Theorem 3.1 remain valid, and so for fair or discounted premiums, full coverage is optimal.

The mean-standard deviation derivatives are,

$$\begin{aligned} \frac{\partial}{\partial C} [E L(\mathbf{X}) + \delta \sigma[L(\mathbf{X})]] &= (\lambda - 1) \int_C^\infty f(x) dx \\ &\quad + \delta \frac{\int_C^\infty f(x) dx \int_0^D (x - D) f(x) dx + \int_0^C f(x) dx \int_C^\infty (C - x) f(x) dx}{\sigma[L(\mathbf{X})]} \\ \frac{\partial}{\partial D} [E L(\mathbf{X}) + \delta \sigma[L(\mathbf{X})]] &= (1 - \lambda) \int_D^\infty f(x) dx \\ &\quad + \delta \frac{\int_0^D f(x) dx \int_C^\infty (x - C) f(x) dx + \int_0^D (D - x) f(x) dx \int_D^\infty f(x) dx}{\sigma[L(\mathbf{X})]}, \end{aligned}$$

and the first-order optimality conditions can be written as

$$\begin{aligned} \frac{\lambda - 1}{\delta} \sigma[L(\mathbf{X})] &= \frac{F(C)}{1 - F(C)} \int_C^\infty (x - C) f(x) dx + \int_0^D (D - x) f(x) dx \\ \frac{\lambda - 1}{\delta} \sigma[L(\mathbf{X})] &= \frac{F(D)}{1 - F(D)} \int_C^\infty (x - C) f(x) dx + \int_0^D (D - x) f(x) dx. \end{aligned}$$

The right sides of these equations are equal to the right sides of (3.15a) and (3.15b), and the left sides differ by a positive multiple. Similar results as those

of Theorem 3.2 and Corollary 3.1 are then valid for the mean-standard deviation case, with the optimal policy having infinite coverage and  $D$  satisfying  $\frac{\lambda-1}{\delta}\sigma[L(\mathbf{X})] = \int_0^D (D-x) f(x) dx$ , i.e., we have that, for an unfair premium, the optimal policy has full coverage above a nonzero deductible.

## 4 Coinsurance

In this chapter we discuss coinsurance policies in a mean-variance framework for the insured. In section 4.1 we consider straight coinsurance and derive the mean-variance minimizing policy. Section 4.2 introduces a stop-loss limit to the coinsurance model, and proves that the optimal coinsurance with stop-loss policy is equivalent to the optimal deductible policy. We consider a coinsurance policy with a deductible in 4.3, and show that it too is equivalent to the optimal deductible policy. Additionally, we show that the optimal straight coinsurance policy is inferior to the optimal deductible policy (and by extension, the optimal coinsurance with stop-loss or deductible policies). We assume throughout a loaded premium,  $p(C, D) = \lambda E I(\mathbf{X}|\cdot)$  and a mean-variance expression  $MV(\mathbf{X}|\cdot) = E L(\mathbf{X}|\cdot) + \delta \text{Var } L(\mathbf{X}|\cdot)$ , for some  $\delta > 0$ .

### 4.1 Straight Coinsurance

We define *Straight Coinsurance* as a policy where the insurer pays a fixed percentage of the insured loss, and the insured pays the remaining portion of the loss. Let the coinsurance parameter be given by  $\alpha$ , where  $0 \leq \alpha \leq 1$ . Then for any loss random loss  $x$ , the insured pays  $\alpha x$  and the insurer pays  $(1 - \alpha)x$ . The case where  $\alpha = 1$  corresponds with being uninsured, while if  $\alpha = 0$ , the insured is fully covered for any loss. The expected insurer reimbursement is then

$$E I(\mathbf{X}|\alpha) = (1 - \alpha) \int_0^\infty x f(x) dx = (1 - \alpha) E \mathbf{X}$$

with derivative

$$\frac{d}{d\alpha} E I(\mathbf{X}|\alpha) = -E \mathbf{X}.$$

The expected insured loss is

$$\begin{aligned} E L(\mathbf{X}|\alpha) &= \lambda E I(\mathbf{X}|\alpha) + \alpha \int_0^\infty x f(x) dx \\ &= \lambda(1 - \alpha)E \mathbf{X} + \alpha E \mathbf{X} \\ &= (\lambda + (1 - \lambda)\alpha)E \mathbf{X}, \end{aligned}$$

with derivative

$$\frac{d}{d\alpha} E L(\mathbf{X}|\alpha) = (1 - \lambda)E \mathbf{X}.$$

The insured variance is

$$\text{Var } L(\mathbf{X}|\alpha) = \alpha^2 \text{Var } \mathbf{X}$$

with derivative

$$\frac{d}{d\alpha} \text{Var } L(\mathbf{X}|\alpha) = 2\alpha \text{Var } \mathbf{X}.$$

The derivative of the mean-variance  $MV(\alpha) = E L(\mathbf{X}|\alpha) + \delta \text{Var } L(\mathbf{X}|\alpha)$  is then

$$\frac{d}{d\alpha} MV(\mathbf{X}|\alpha) = (1 - \lambda)E \mathbf{X} + 2\alpha\delta \text{Var } \mathbf{X},$$

and so  $\frac{d}{d\alpha} MV(\mathbf{X}|\alpha) = 0$  if

$$\alpha = \frac{(\lambda - 1)E \mathbf{X}}{2\delta \text{Var } \mathbf{X}}. \quad (4.1)$$

If  $\lambda > 1$ , (4.1) will give a positive value. To verify that the minimal value of  $MV(\mathbf{X}|\alpha)$  occurs here, we compute its value at this critical point and compare to its value at the endpoint  $\alpha = 0$ .

$$\begin{aligned} MV(\mathbf{X}|0) &= \lambda E \mathbf{X}, \\ MV\left(\mathbf{X} \middle| \frac{(\lambda - 1)E \mathbf{X}}{2\delta \text{Var } \mathbf{X}}\right) &= \lambda E \mathbf{X} - \frac{1}{4} \frac{(1 - \lambda)^2 (E \mathbf{X})^2}{\delta \text{Var } \mathbf{X}}. \end{aligned}$$

The mean-variance minimizer thus prefers  $\alpha$  given by (4.1) to  $\alpha = 0$ . If  $\alpha = 1$  we are in the uninsured case, and so  $MV(\mathbf{X}|1) = E \mathbf{X} + \delta \text{Var } \mathbf{X}$ . A

straight coinsurance policy will then provide a reduction in mean-variance from the uninsured case if  $\lambda E \mathbf{X} - \frac{1}{4} \frac{(1-\lambda)^2 (E \mathbf{X})^2}{\delta \text{Var} \mathbf{X}} < E \mathbf{X} + \delta \text{Var} \mathbf{X}$ , or

$$(\lambda - 1)E \mathbf{X} < \delta \text{Var} \mathbf{X} + \frac{1}{4} \frac{(1 - \lambda)^2 (E \mathbf{X})^2}{\delta \text{Var} \mathbf{X}}.$$

For the case of a fair or discounted premium,  $\lambda \leq 1$ , note that  $\frac{d}{d\alpha} MV(\alpha)|_{\alpha=0} = (1 - \lambda)E \mathbf{X} \geq 0$ . Therefore, the insured never wants to increase  $\alpha$ , and so we must have that full coverage ( $\alpha = 0$ ) is optimal.

**Example 4.1.** Figure 4.1 plots the mean-variance for  $0 \leq \alpha \leq 1$  for an exponentially distributed loss with mean 100, loading  $\lambda = 1.25$ , and risk preference  $\delta = 0.01$ . Inserting the parameters  $\lambda$  and  $\delta$  into equation (4.1) we obtain  $\alpha = 0.125$

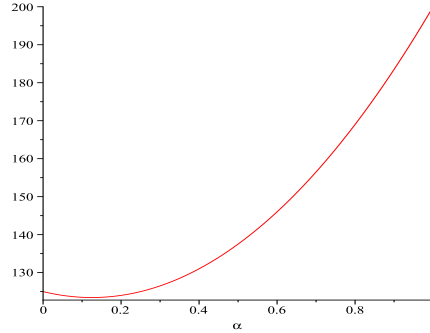


Figure 4.1: Plot of  $E L(\mathbf{X}|\alpha) + \delta \text{Var} L(\mathbf{X}|\alpha)$

as the optimal point.  $MV(0.125) = (1.25 + (1 - 1.25) \cdot 0.125) \cdot 100 + 0.01 \cdot 0.125^2 \cdot 10,000 = 123.44$ .

Note that the parameters in this example are the same as those in example 3.2, where the minimum mean-variance value was 117.67. In this case then, the optimal deductible policy provides a smaller mean-variance than the optimal straight coinsurance policy.

## 4.2 Coinsurance with a Stop-Loss Limit

We now add a stop-loss limit  $S$  to the coinsurance model. Losses  $x$  between 0 and  $S$  are partially paid by the insurer at the level  $(1 - \alpha)x$ , with the insured responsible for the remaining  $\alpha x$ . The insurer pays the entirety of any losses over the stop-loss limit  $S$  so that the insured is only responsible for  $\alpha S$ . The premium is dependent on  $\alpha$  and  $S$ , and is given by  $p(\alpha, S)$ . In Figure 4.2 we provide an illustration of the insured loss as a function of the total loss  $x$  (with  $\alpha = 0.25$  and  $S = 100$ ). Notice that the slope of the loss is 0.25 for  $x < 100$  and 0 otherwise.

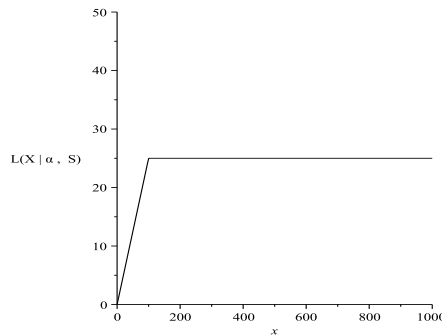


Figure 4.2: Plot of the insured loss,  $L(\mathbf{X}|\alpha, S)$ , for  $\alpha = 0.25$  and  $S = 100$ .

**Straight Stop-Loss** Before we fully describe the coinsurance with stop-loss model, it will be helpful to consider a straight stop-loss model, where losses below  $S$  are paid entirely by the insured and losses above  $S$  are fully paid by the insurer. This is the same as having the coinsurance parameter  $\alpha = 1$  in the coinsurance with stop-loss model. The expected reimbursement is

$$\begin{aligned} E I(\mathbf{X}|S) &= \int_S^\infty (x - S) f(x) dx, \\ &= E \mathbf{X} - \int_0^S x f(x) dx - S \int_S^\infty f(x) dx \end{aligned}$$

and the expected insured loss is

$$\begin{aligned}
E L(\mathbf{X}|S) &= \lambda E I(\mathbf{X}|S) + \int_0^S x f(x) dx + S \int_S^\infty f(x) dx \\
&= \lambda E \mathbf{X} - \lambda \left( \int_0^S x f(x) dx + S \int_S^\infty f(x) dx \right) + \int_0^S x f(x) dx \quad (4.2) \\
&\quad + S \int_S^\infty f(x) dx \\
&= \lambda E \mathbf{X} + (1 - \lambda) \left( \int_0^S x f(x) dx + S \int_S^\infty f(x) dx \right).
\end{aligned}$$

Note that, if  $\lambda > 1$ , then  $E L(\mathbf{X}|S) > E \mathbf{X}$ . Indeed,  $E L(\mathbf{X}|S) - E \mathbf{X} = (\lambda - 1)E \mathbf{X} + (1 - \lambda) \left( \int_0^S x f(x) dx + S \int_S^\infty f(x) dx \right) = (\lambda - 1) \int_S^\infty (x - S) f(x) dx > 0$  if  $\lambda > 1$ .

The insured variance for a straight stop-loss policy is

$$\begin{aligned}
\text{Var } L(\mathbf{X}|S) &= \int_0^S x^2 f(x) dx + S^2 \int_S^\infty f(x) dx \\
&\quad - \left( \int_0^S x f(x) dx + S \int_S^\infty f(x) dx \right)^2. \quad (4.3)
\end{aligned}$$

We observe that for  $S = 0$ , we have  $\text{Var } L(\mathbf{X}|0) = 0$ . Also, note that this model is equivalent to a deductible policy with  $D = S$  and infinite coverage.

**Coinsurance with Stop-Loss** Returning to the coinsurance with stop-loss model, the reimbursement  $I(\alpha, S)$  paid by the insurer to the insured is given by

$$I(x|\alpha, S) = \begin{cases} (1 - \alpha)x, & \text{if } 0 \leq x < S; \\ (1 - \alpha)S + x - S, & \text{if } S \leq x \end{cases}$$

and the insured loss is

$$L(x|\alpha, S) = p(\alpha, S) + \begin{cases} \alpha x, & \text{if } 0 \leq x < S; \\ \alpha S, & \text{if } S \leq x. \end{cases}$$

**Remark 4.1.** In the coinsurance with stop-loss model, the cases  $S = 0$  and

$\alpha = 0$  are both equivalent to having full insurance coverage for any loss. The case  $S = \infty$  is equivalent to straight coinsurance, while the case  $\alpha = 1$  is equivalent to the straight stop-loss model. If we have both  $S = \infty$  and  $\alpha = 1$  we are in the uninsured case.

The expected insured loss is

$$\begin{aligned} EL(\mathbf{X}|\alpha, S) &= p(\alpha, S) + \alpha \int_0^S x f(x) dx + \alpha S \int_S^\infty f(x) dx \\ &= \lambda \left( (1 - \alpha) \int_0^S x f(x) dx + \int_S^\infty (x - \alpha S) f(x) dx \right) \\ &\quad + \alpha \left( \int_0^S x f(x) dx + S \int_S^\infty f(x) dx \right) \end{aligned}$$

and the insured variance is

$$\begin{aligned} \text{Var } L(\mathbf{X}|\alpha, S) &= \alpha^2 \int_0^S x^2 f(x) dx + \alpha^2 S^2 \int_S^\infty f(x) dx \\ &\quad - \left( \alpha \int_0^S x f(x) dx + \alpha S \int_S^\infty f(x) dx \right)^2 \\ &= \alpha^2 \text{Var } L(\mathbf{X}|S) \end{aligned}$$

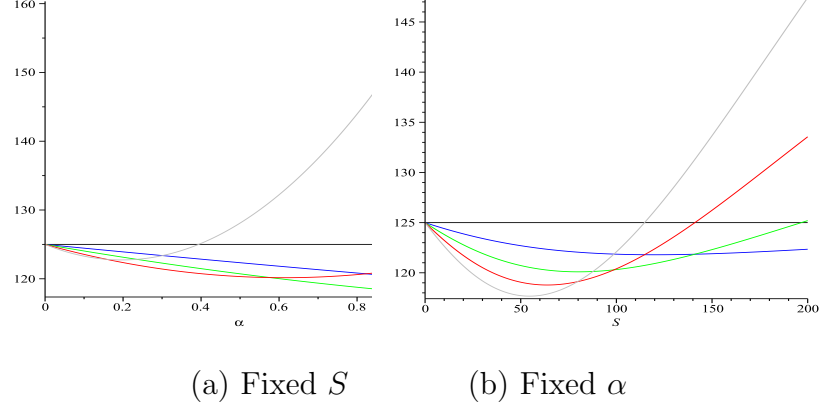
The mean-variance is given by  $MV(\alpha, S) = E L(\mathbf{X}|\alpha, S) + \delta \text{Var } L(\mathbf{X}|\alpha, S)$ , and insurance will be demanded if it provides an improvement to the uninsured case, i.e., if

$$MV(\mathbf{X}|\alpha, S) \leq E \mathbf{X} + \delta \text{Var } \mathbf{X} = MV(\mathbf{X}|1, \infty). \quad (4.4)$$

Figure 4.3 plots the mean-variance as (a) a function of  $\alpha$  for fixed  $S$  and (b) a function of  $S$  for fixed  $\alpha$ . We see in both figures that the curves for different fixed values of  $\alpha$  and  $S$  often cross, and notice that as both  $S$  and  $\alpha$  increase (the gray line in each figure), the mean-variance experiences heavy growth. When  $\alpha$  is maximal ( $\alpha = 1$ ) and  $S \rightarrow \infty$ , we approach the uninsured case, and thus it appears that insurance is always optimal in this example. The lowest mean-



variance occurs in both figures at the point  $(1, 50)$ .



Line Color	(a)	(b)
Black	$S = 0$	$\alpha = 0$
Blue	25	0.25
Green	50	0.50
Red	100	0.75
Gray	250	1

(c) Key

Figure 4.3: Plot of  $MV(\alpha, S) = E L(\mathbf{X}|\alpha, S) + \delta \text{Var } L(\mathbf{X}|\alpha, S)$

To assist in determining the minimal policy, we first calculate the derivatives of  $E L(\mathbf{X}|\alpha, S)$ ,  $\text{Var } L(\mathbf{X}|\alpha, S)$ , and  $MV(\mathbf{X}|\alpha, S)$  with respect to  $\alpha$  and  $S$ :

$$\frac{\partial}{\partial \alpha} E L(\mathbf{X}|\alpha, S) = (1 - \lambda) \left( \int_0^S x f(x) dx + S \int_S^\infty f(x) dx \right)$$

$$\frac{\partial}{\partial \alpha} \text{Var } L(\mathbf{X}|\alpha, S) = 2\alpha \text{Var } L(\mathbf{X}|S)$$

The mean-variance derivative with respect to  $\alpha$  is then

$$\frac{\partial}{\partial \alpha} MV(\mathbf{X}|\alpha, S) = (1 - \lambda) \left( \int_0^S x f(x) dx + S \int_S^\infty f(x) dx \right) + 2\delta \alpha \text{Var } L(\mathbf{X}|S) \quad (4.5)$$

The critical value of  $\alpha$  is given by  $\frac{\partial}{\partial \alpha} MV(\mathbf{X}|\alpha, S) = 0$ , with solution

$$\alpha = \frac{(\lambda - 1) \left( \int_0^S x f(x) dx + S \int_S^\infty f(x) dx \right)}{2\delta \text{Var } L(\mathbf{X}|S)} \text{ for } S > 0. \quad (4.6)$$

If  $S = 0$ , then  $\frac{\partial}{\partial \alpha} MV(\mathbf{X}|\alpha, S) = 0$  for any  $\alpha$ .

We also have

$$\frac{\partial}{\partial S} E L(\mathbf{X}|\alpha, S) = \alpha(1 - \lambda) \int_S^\infty f(x) dx$$

and

$$\begin{aligned} \frac{\partial}{\partial S} \text{Var } L(\mathbf{X}|\alpha, S) &= 2\alpha^2 S \int_S^\infty f(x) dx \\ &\quad - 2\alpha^2 \left( \int_0^S x f(x) dx + S \int_S^\infty f(x) dx \right) \int_S^\infty f(x) dx \\ &= 2\alpha^2 \int_S^\infty f(x) dx \int_0^S (S - x) f(x) dx. \end{aligned}$$

Therefore,

$$\frac{\partial}{\partial S} MV(\mathbf{X}|\alpha, S) = \alpha \int_S^\infty f(x) dx \left( 1 - \lambda + 2\alpha\delta \int_0^S (S - x) f(x) dx \right)$$

and  $\frac{\partial}{\partial S} MV(\mathbf{X}|\alpha, S) = 0$  if either

$$\frac{\lambda - 1}{2\delta} = \alpha \int_0^S (S - x) f(x) dx. \quad (4.7)$$

or

$$\alpha = 0. \quad (4.8)$$

The objective of the insured is to minimize the mean-variance:  $MV(\mathbf{X}|\alpha, S) = E L(\mathbf{X}|\alpha, S) + \delta \text{Var } L(\mathbf{X}|\alpha, S)$ , where  $0 \leq \alpha \leq 1$ . To solve this optimization problem we form the Karush-Kuhn-Tucker (KKT) conditions. The only constraint,

$\alpha \leq 1$ , is linear, and so the constraint qualification is met. Let  $\mu$  be the multiplier associated with this constraint, the KKT conditions are then:

$$\begin{aligned} (1) \quad & \frac{\partial}{\partial \alpha} MV(\mathbf{X}|\alpha, S) + \mu = 0 \\ (2) \quad & \frac{\partial}{\partial S} MV(\mathbf{X}|\alpha, S) = 0 \\ (3) \quad & \mu(\alpha - 1) = 0 \\ (4) \quad & \alpha \leq 1 \\ (5) \quad & \alpha, \mu, S \geq 0 \end{aligned}$$

To solve the KKT conditions we consider the complementary slackness condition (3),  $\mu(\alpha - 1) = 0$ , and the two associated possibilities; 1)  $\mu \neq 0$  and 2)  $\mu = 0$ .

**1)  $\mu \neq 0$  :**

If  $\mu \neq 0$ , then  $\alpha = 1$ . The KKT conditions become

$$\begin{aligned} (1') \quad & \frac{\partial}{\partial \alpha} MV(\mathbf{X}|\alpha, S)|_{\alpha=1} + \mu = 0 \\ (2') \quad & \frac{\partial}{\partial S} MV(\mathbf{X}|\alpha, S)|_{\alpha=1} = 0 \\ (3') \quad & \mu, S \geq 0, \end{aligned}$$

We first consider condition (1'). Letting  $\alpha = 1$  in (4.5) we have

$$\frac{\partial}{\partial \alpha} MV(\mathbf{X}|\alpha, S)|_{\alpha=1} = (1 - \lambda) \left( \int_0^S x f(x) dx + S \int_S^\infty f(x) dx \right) + 2\delta \text{Var } L(\mathbf{X}|S),$$

which is less than 0 if

$$\frac{\lambda - 1}{2\delta} > \frac{\text{Var } L(\mathbf{X}|S)}{\int_0^S x f(x) dx + S \int_S^\infty f(x) dx} \text{ and } S > 0, \quad (4.9)$$

and equal to 0 if (4.9) holds with equality or if  $S = 0$ .

If (4.9) does not hold, then from (1') we must have  $\mu \leq 0$ . In this case the system (1') – (3') cannot be solved, as either the nonnegativity constraint (3') is violated or the assumption of this case,  $\mu \neq 0$ , is violated. We now show that if (2') holds then (4.9) must also hold.

**Lemma 4.1.** If  $\frac{\partial}{\partial S}MV(\mathbf{X}|\alpha, S)|_{\alpha=1} = 0$  then  $\frac{\partial}{\partial \alpha}MV(\mathbf{X}|\alpha, S)|_{\alpha=1} < 0$

*Proof.* Suppose that  $\frac{\partial}{\partial S}MV(\alpha, S)|_{\alpha=1} = 0$ . By (4.7), we have the following equation that can be solved for  $S$ .

$$\frac{\lambda - 1}{2\delta} = \int_0^S (S - x) f(x) dx \quad (4.10)$$

Combining (4.9) and (4.10) and writing the variance term as in (4.3), we have that  $\frac{\partial}{\partial \alpha}MV(\mathbf{X}|\alpha, S)|_{\alpha=1} < 0$  if

$$\begin{aligned} & \int_0^S (S - x) f(x) dx \left( \int_0^S x f(x) dx + S \int_S^\infty f(x) dx \right) \\ & \geq \int_0^S x^2 f(x) dx + S^2 \int_S^\infty f(x) dx - \left( \int_0^S x f(x) dx + S \int_S^\infty f(x) dx \right)^2 \end{aligned}$$

Expanding each side of the inequality and canceling like terms, we obtain

$$S \int_0^S f(x) dx \int_0^S x f(x) dx \geq \int_0^S x^2 f(x) dx - S \int_0^S x f(x) dx \int_S^\infty f(x) dx \quad (4.11)$$

Inequality (4.11) always holds since

$$\begin{aligned} S \int_0^S f(x) dx \int_0^S x f(x) dx &= S \int_0^S x f(x) dx \left( 1 - \int_S^\infty f(x) dx \right) \\ &> \int_0^S x^2 f(x) dx - S \int_0^S x f(x) dx \int_S^\infty f(x) dx. \end{aligned}$$

and so we have  $\frac{\partial}{\partial \alpha}MV(\mathbf{X}|\alpha, S)|_{\alpha=1} \leq 0$ . □

Lemma 4.1 ensures that,  $\mu \geq 0$  when  $\alpha = 1$ , and so the case  $\mu \neq 0$ ,  $\alpha = 1$ , and

$S$  given by (4.10) is feasible. The right side of (4.10) is equal to 0 when  $S = 0$ , and increases continuously with  $S$ . Since the left side is constant, there will be a unique nonnegative solution to (4.10) if  $\lambda > 1$ .

The mean-variance for this case is given by

$$\begin{aligned} MV(\mathbf{X}|1, S) &= E L(\mathbf{X}|S) + \delta \text{Var } L(\mathbf{X}|S) \\ &= \lambda E \mathbf{X} + (1 - \lambda) \left( \int_0^S x f(x) dx + S \int_S^\infty f(x) dx \right) + \delta \text{Var } L(\mathbf{X}|S) \end{aligned} \quad (4.12)$$

**2)  $\mu = 0$  :**

We now consider the second possibility for the complementary slackness condition,  $\mu = 0$ . The KKT conditions become

$$\begin{aligned} (1'') \quad & \frac{\partial}{\partial \alpha} MV(\alpha, S) = 0 \\ (2'') \quad & \frac{\partial}{\partial S} MV(\alpha, S) = 0 \\ (3'') \quad & \alpha \leq 1 \\ (4'') \quad & \alpha, S \geq 0, \end{aligned}$$

and so (4.6) and (4.7) must both be satisfied. If  $S = 0$  (the case of full coverage), then (4.6) is satisfied, and if, in addition,  $\alpha = 0$ , then (4.7) is satisfied. Conditions (1'')-(4'') are thus satisfied, and the full coverage case is feasible.

For the case  $S > 0$  we use  $\alpha$  from (4.6) in (4.7) to obtain

$$\text{Var } L(\mathbf{X}|S) = \int_0^S (S - x) f(x) dx \left( \int_0^S x f(x) dx + S \int_S^\infty f(x) dx \right), \quad (4.13)$$

which has no solution with  $S > 0$ . To see this, note that

$$\begin{aligned} \text{Var } L(\mathbf{X}|S) &= \int_0^S x^2 f(x) dx + S^2 \int_S^\infty f(x) dx \\ &\quad - \left( \int_0^S x f(x) dx + S \int_S^\infty f(x) dx \right)^2. \end{aligned}$$

Adding the second term of this equation to both sides of (4.13) we have

$$\begin{aligned} &\int_0^S x^2 f(x) dx + S^2 \int_S^\infty f(x) dx \\ &= \int_0^S x f(x) dx \left( \int_0^S (S-x) f(x) dx + \int_0^S x f(x) dx + S \int_S^\infty f(x) dx \right) \\ &\quad + S \int_S^\infty f(x) dx \left( \int_0^S (S-x) f(x) dx + \int_0^S x f(x) dx + S \int_S^\infty f(x) dx \right) \\ &= S \left( \int_0^S x f(x) dx + S \int_S^\infty f(x) dx \right). \end{aligned}$$

The solution to (4.13) is therefore given by

$$\int_0^S x^2 f(x) dx = S \int_0^S x f(x) dx,$$

where the only solution is  $S = 0$ . The case  $S > 0$  is therefore infeasible.

The optimal coinsurance with stop loss policy in this case is therefore full coverage. The mean-variance is

$$MV(\mathbf{X}|\alpha, 0) = \lambda E \mathbf{X} \tag{4.14}$$

We therefore have two potential solutions to the mean-variance problem;  $(1, S)$ , where  $S$  is given by (4.7), and  $(\alpha, 0)$ , where the value of  $\alpha$  is irrelevant since all losses are fully covered in this case. The following theorem describes the circumstances under which one or the other is optimal.

**Theorem 4.1.** Suppose that there is an  $\alpha$  and  $S$  such that (4.4) holds, i.e., that

there is a coinsurance with stop-loss policy for which the mean-variance is reduced from the uninsured case. Then

a) If the premium is fair or discounted,  $\lambda \leq 1$ , then the optimal coinsurance with stop-loss policy is full coverage,  $S = 0$ .

b) If the premium is unfair,  $\lambda > 1$ , then the optimal policy is  $(\alpha, S) = (1, S)$ , where  $S$  is the solution to (4.10):  $\frac{\lambda-1}{2\delta} = \int_0^S (S-x) f(x) dx$

*Proof.* In the discussion of the KKT conditions we showed that the two possibly optimal policies are  $(1, S)$  and  $(\alpha, 0)$ . By (4.12) and (4.14), we have that  $MV(\mathbf{X}|1, S) > MV(\mathbf{X}|\alpha, 0)$  if

$$(1 - \lambda) \left( \int_0^S x f(x) dx + S \int_S^\infty f(x) dx \right) + \delta \text{Var } L(\mathbf{X}|S) > 0, \quad (4.15)$$

where  $S$  is given by (4.10). Additionally, we note that the solution to (4.10) has  $S > 0$  and that, since (4.10) holds, Lemma 4.1 applies and therefore (4.9) is satisfied.

a) If  $\lambda \leq 1$ , then inequality (4.15) is immediately satisfied, and we obtain an optimal policy of  $(\alpha, 0)$ .

b) Suppose that  $\lambda > 1$ . Since  $S > 0$  we can rewrite (4.15) as

$$\frac{\lambda - 1}{\delta} < \frac{\text{Var } L(\mathbf{X}|S)}{\int_0^S x f(x) dx + S \int_S^\infty f(x) dx} \quad (4.16)$$

From (4.10) we have  $\frac{\lambda-1}{2\delta} = \int_0^S (S-x) f(x) dx$ , and so (4.16) becomes

$$2 \int_0^S (S-x) f(x) dx \leq \frac{\text{Var } L(X|S)}{\int_0^S x f(x) dx + S \int_S^\infty f(x) dx}.$$

From (4.9) and (4.10) we also have

$$\int_0^S (S-x) f(x) dx \geq \frac{\text{Var } L(X|S)}{\int_0^S x f(x) dx + S \int_S^\infty f(x) dx},$$

and so (4.16) is not satisfied, which implies that  $MV(\mathbf{X}|1, S) < MV(\mathbf{X}|\alpha, 0)$ .  $\square$

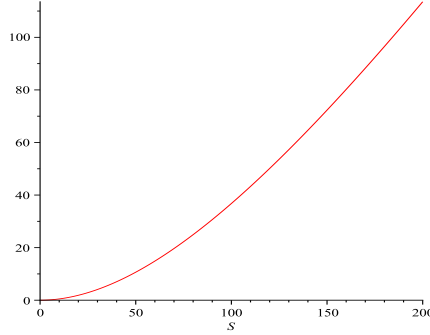


Figure 4.4: Plot of (4.10)

We have again shown that partial coverage is optimal for unfair premiums and full coverage is optimal for fair or discounted premiums. Indeed, if  $\lambda > 1$ , the optimal coverage for a coinsurance with stop-loss policy has the coinsurance parameter  $\alpha = 1$ , i.e., fractional coverage below a particular level is never optimal. This policy,  $(1, S)$ , is equivalent to a deductible policy with infinite coverage, as the insured pays all losses up to a certain level ( $S$  here,  $D$  in a deductible policy) and nothing for losses above that level. Figure 4.4 is an illustration of equation (4.10). The horizontal axis gives the optimal values of  $S$  corresponding to  $\frac{\lambda-1}{2\delta}$  (the vertical axis).

For the case  $\lambda \leq 1$ , we again have that full coverage,  $S = 0$ , is optimal.

**Example 4.2.** In Figure 4.5 we show the mean-variance contour lines for coinsurance policies with stop loss, where the same parameters as Example 4.1 are used - exponentially distributed loss with mean 100,  $\delta = 0.01$ , and  $\lambda = 1.25$ . The optimal point has  $\alpha = 1$  and  $S = 54.54$ . Note that this is the same solution as given in Example 3.2, where the problem was to minimize the mean-variance of a deductible policy.



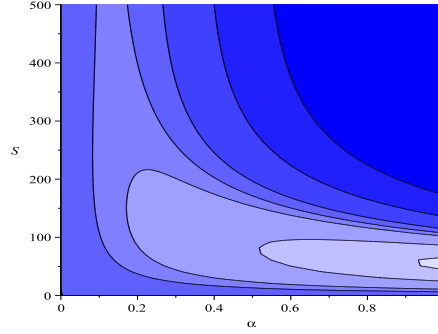


Figure 4.5: Plot of  $E L(\mathbf{X}|\alpha, S) + \delta \text{Var } L(\mathbf{X}|\alpha, S)$

### 4.3 Coinsurance with a Deductible

In the previous section we showed that fractional coverage ( $0 < \alpha < 1$ ) is never optimal when it is applied to losses that are lower than a given level. We now consider the opposite problem, where fractional coverage is allowed above a given level. To this end, we develop a mean-variance model for policies that include both coinsurance and deductibles.

Let the deductible be given by  $D \geq 0$ , and the parameter  $\alpha$  as above. The insured pays all losses up to  $D$ , while losses above  $D$  are partially paid by the insurer at the rate  $(1 - \alpha)$ . The insurer reimbursement is therefore

$$I(x|\alpha, D) = \begin{cases} 0, & \text{if } 0 \leq x \leq D; \\ (1 - \alpha)(x - D), & \text{if } D < x \end{cases}$$

and the insured loss is

$$L(x|\alpha, D) = p(\alpha, D) + \begin{cases} x, & \text{if } 0 \leq x \leq D; \\ D + \alpha(x - D), & \text{if } D < x. \end{cases}$$

In Figure 4.6 we give an example of the insured loss as a function of the total loss  $x$ . The insured loss is piecewise linear in  $x$ , with the slope reduced at the point  $x = D$ .

The premium is  $p(\alpha, D) = \lambda E I(\mathbf{X}) = \lambda(1 - \alpha) \int_D^\infty (x - D) f(x) dx$ , and the

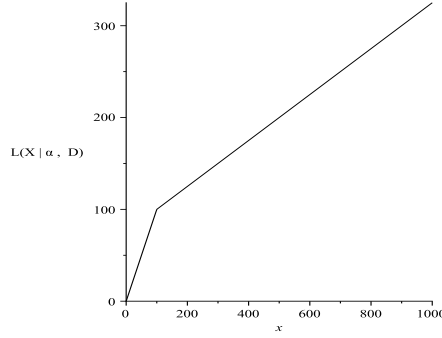


Figure 4.6: Plot of the insured loss,  $L(\mathbf{X}|\alpha, D)$ , for  $\alpha = 0.25$  and  $D = 100$ .

expected insured loss and insured variance are:

$$\begin{aligned}
 E L(\mathbf{X}|\alpha, D) &= \lambda(1 - \alpha) \int_D^\infty (x - D) f(x) dx + \int_0^D x f(x) dx \\
 &\quad + D \int_D^\infty f(x) dx + \alpha \int_D^\infty (x - D) f(x) dx \\
 &= \int_0^D x f(x) dx + D \int_D^\infty f(x) dx \\
 &\quad + (\lambda(1 - \alpha) + \alpha) \int_D^\infty (x - D) f(x) dx \\
 \text{Var } L(\mathbf{X}|\alpha, D) &= \int_0^D x^2 f(x) dx + \int_D^\infty (D + \alpha(x - D))^2 f(x) dx \\
 &\quad - \left( \int_0^D x f(x) dx + D \int_D^\infty f(x) dx + \alpha \int_D^\infty (x - D) f(x) dx \right)^2
 \end{aligned}$$

The mean-variance is given by  $MV(\mathbf{X}|\alpha, D) = E L(\mathbf{X}|\alpha, D) + \delta \text{Var } L(\mathbf{X}|\alpha, D)$ . Before we determine the optimal  $(\alpha, D)$  pair, we first consider the behavior of the mean-variance as a function of a single variable. Figure 4.7 plots the mean-variance as (a) a function of  $\alpha$  for fixed values of  $D$  and (b) a function of  $D$  for fixed values of  $\alpha$ . In this figure we use  $\delta = 0.01$ ,  $\lambda = 1.25$ , and  $f(x)$  an exponential distribution with parameter 0.01.

It is apparent from (a) that, as  $\alpha$  approaches 1, the value of  $D$  becomes inconsequential. This is consistent with expectations, as the case  $\alpha = 1$  corresponds with the uninsured case. We also observe that  $MV(\mathbf{X}|\alpha, D)$  generally appears to increase with  $\alpha$ , although for small values of  $D$  the mean-variance is decreasing

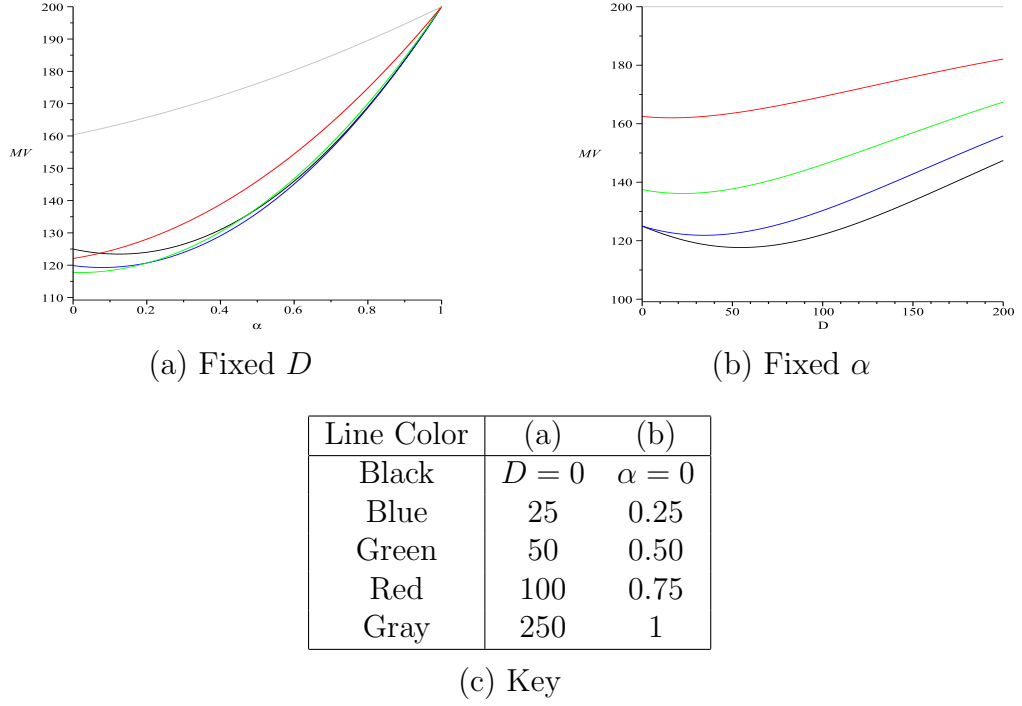


Figure 4.7: Plot of  $E L(\mathbf{X}|\alpha, D) + \delta \text{Var } L(\mathbf{X}|\alpha, D)$

in  $\alpha$  when  $\alpha$  is small. The minimal value of  $MV(\alpha, D)$  occurs when  $D = 50$  and  $\alpha = 0$ .

Figure 4.7(b) shows a greater separation in mean-variance values. It is consistent that raising  $D$  above a particular level will always raise the mean-variance, and we also note that raising the deductible above 0 provides a mean-variance reduction.

To determine the mean-variance minimizing  $(\alpha, D)$  pair, we first calculate the mean-variance derivatives. The derivatives with respect to  $\alpha$  are

$$\frac{\partial}{\partial \alpha} E L(\mathbf{X}|\alpha, D) = (1 - \lambda) \int_D^\infty (x - D) f(x) dx$$

and

$$\begin{aligned} \frac{\partial}{\partial \alpha} \text{Var } L(\mathbf{X}|\alpha, D) &= 2\alpha \int_D^\infty (x - D)^2 f(x) dx + 2D \int_D^\infty (x - D) f(x) dx \\ &\quad - 2v(\alpha, D) \int_D^\infty (x - D) f(x) dx, \end{aligned}$$

where

$$v(\alpha, D) = \int_0^D x f(x) dx + D \int_D^\infty f(x) dx + \alpha \int_D^\infty (x - D) f(x) dx. \quad (4.17)$$

The mean-variance derivative with respect to  $\alpha$ ,  $\frac{\partial}{\partial \alpha} MV(\mathbf{X}|\alpha, D)$   
 $= \frac{\partial}{\partial \alpha} E L(\mathbf{X}|\alpha, D) + \delta \frac{\partial}{\partial \alpha} \text{Var } L(\mathbf{X}|\alpha, D)$ , is then

$$\begin{aligned} \frac{\partial}{\partial \alpha} MV(\alpha, D) &= (1 - \lambda) \int_D^\infty (x - D) f(x) dx + 2\delta \alpha \int_D^\infty (x - D)^2 f(x) dx \\ &\quad + 2\delta(D - v(\alpha, D)) \int_D^\infty (x - D) f(x) dx \end{aligned} \quad (4.18)$$

The derivatives with respect to  $D$  are

$$\frac{\partial}{\partial D} E L(\mathbf{X}|\alpha, D) = (1 - \lambda)(1 - \alpha) \int_D^\infty f(x) dx$$

and

$$\begin{aligned} \frac{\partial}{\partial D} \text{Var } L(\mathbf{X}|\alpha, D) &= 2D(\alpha - 1)^2 \int_D^\infty f(x) dx + 2\alpha(1 - \alpha) \int_D^\infty x f(x) dx \\ &\quad - 2(1 - \alpha)v(\alpha, D) \int_D^\infty f(x) dx. \end{aligned}$$

The mean-variance derivative with respect to  $D$  is then

$$\begin{aligned} \frac{\partial}{\partial D} MV(\mathbf{X}|\alpha, D) &= (1 - \lambda)(1 - \alpha) \int_D^\infty f(x) dx + 2\delta \alpha(1 - \alpha) \int_D^\infty x f(x) dx \\ &\quad + 2\delta(D(\alpha - 1)^2 - (1 - \alpha)v(\alpha, D)) \int_D^\infty f(x) dx \end{aligned} \quad (4.19)$$

with  $v(\alpha, D)$  as in (4.17).

To minimize the mean-variance, we again form the KKT conditions. The only constraints in the problem are  $\alpha \leq 1$  and nonnegativity, so if we let  $\mu$  be the constraint multiplier, the KKT conditions are:

$$\begin{aligned} (1) \quad & \frac{\partial}{\partial \alpha} MV(\mathbf{X}|\alpha, D) + \mu = 0 \\ (2) \quad & \frac{\partial}{\partial D} MV(\mathbf{X}|\alpha, D) = 0 \\ (3) \quad & \mu(\alpha - 1) = 0 \\ (4) \quad & \alpha \leq 1 \\ (5) \quad & \alpha, \mu, D \geq 0 \end{aligned}$$

We again begin by considering the two possibilities for the complementary slackness condition (3),  $1)\mu \neq 0$  and  $2)\mu = 0$ .

**1)  $\mu \neq 0$  :**

First, suppose that  $\mu \neq 0$ . Then  $\alpha = 1$ , and the situation is equivalent to the uninsured case, as the insured pays everything both below and above the deductible.

The mean-variance in this case is

$$MV(\mathbf{X}|1, D) = E \mathbf{X} + \delta \text{Var } \mathbf{X} \quad (4.20)$$

**2)  $\mu = 0$  :**

Next, consider the case  $\mu = 0$ . Furthermore, assume that  $D$  is finite (if it is not

we are in the uninsured case). The KKT conditions become

$$(1') \quad \frac{\partial}{\partial \alpha} MV(\alpha, D) = 0$$

$$(2') \quad \frac{\partial}{\partial D} MV(\alpha, D) = 0$$

$$(3') \quad \alpha \leq 1$$

$$(4') \quad \alpha, S \geq 0$$

The first condition is satisfied when

$$\frac{\lambda - 1}{2\delta} = \frac{\alpha \int_D^\infty (x - D)^2 f(x) dx}{\int_D^\infty (x - D) f(x) dx} + D - v(\alpha, D), \quad (4.21)$$

and the second condition is satisfied when  $\alpha = 1$  or

$$\frac{\lambda - 1}{2\delta} = \frac{\alpha \int_D^\infty x f(x) dx}{\int_D^\infty f(x) dx} + D - v(\alpha, D) - \alpha D, \quad \alpha < 1. \quad (4.22)$$

If  $\alpha = 1$ , we are in the uninsured case as discussed in 1) above. We therefore consider  $0 \leq \alpha < 1$ , and the two subcases  $\lambda \leq 1$  and  $\lambda > 1$ .

**2.1)  $\mu = 0$  and  $\lambda \leq 1$  :**

Suppose that  $\lambda \leq 1$ . Since  $D - v(\alpha, D) = \int_0^D (D - x) f(x) dx - \alpha \int_D^\infty (x - D) f(x) dx$ , (4.21) is equivalent to

$$\frac{\lambda - 1}{2\delta} = \frac{\alpha \left( \int_D^\infty (x - D)^2 f(x) dx - \left( \int_D^\infty (x - D) f(x) dx \right)^2 \right)}{\int_D^\infty (x - D) f(x) dx} + \int_0^D (D - x) f(x) dx \quad (4.23)$$

and (4.22) can be written as

$$\begin{aligned}
\frac{\lambda - 1}{2\delta} &= \frac{\alpha \int_D^\infty (x - D) f(x) dx}{\int_D^\infty f(x) dx} + \frac{\alpha D \int_D^\infty f(x) dx}{\int_D^\infty f(x) dx} + \int_0^D (D - x) f(x) dx \\
&\quad - \alpha \int_D^\infty (x - D) f(x) dx - \alpha D \\
&= \frac{\alpha \int_D^\infty (x - D) f(x) dx \int_0^D f(x) dx}{\int_D^\infty f(x) dx} + \int_0^D (D - x) f(x) dx. \quad (4.24)
\end{aligned}$$

By inspection, we see that the right sides of (4.23) and (4.24) are strictly positive. Therefore, neither (4.21) nor (4.22) have a solution if  $\lambda \leq 1$ , since the left sides of (4.21) and (4.22) are both negative in this case. In other words, we have  $\frac{\partial}{\partial \alpha} MV(\mathbf{X}|\alpha, D) > 0$  and  $\frac{\partial}{\partial D} MV(\mathbf{X}|\alpha, D) > 0$ , and it is therefore optimal to decrease  $\alpha$  and  $D$  as much as possible – to the point  $(0, 0)$ .

## 2.2) $\mu = 0$ and $\lambda > 1$ :

We now consider the case  $\lambda > 1$ . Combining (4.21) and (4.22), we obtain the necessary condition

$$\frac{\alpha \int_D^\infty (x - D)^2 f(x) dx}{\int_D^\infty (x - D) f(x) dx} = \frac{\alpha \int_D^\infty x f(x) dx}{\int_D^\infty f(x) dx} - \alpha D \quad (4.25)$$

We consider the two cases  $\alpha = 0$  and  $0 < \alpha < 1$ .

### 2.2.1) $\mu = 0$ , $\lambda > 1$ , and $\alpha = 0$ :

If  $\alpha = 0$ , then equation (4.25) reduces to  $0 = 0$ . To find  $D$  we let  $\alpha = 0$  in (4.22) to obtain the equation

$$\frac{\lambda - 1}{2\delta} = D - v(0, D) = \int_0^D (D - x) f(x) dx. \quad (4.26)$$

This equation is equivalent to (4.10) from the previous section. This is expected because  $\alpha = 0$  is the same as a stop-loss coinsurance policy that has  $\alpha = 1$ , which was the case used to derive (4.10).

### 2.2.2) $\mu = 0$ , $\lambda > 1$ , and $0 < \alpha < 1$ :

If  $0 < \alpha < 1$ , we divide each side of (4.25) by  $\alpha$  and form a common denominator on the right side to obtain the condition

$$\frac{\int_D^\infty (x - D)^2 f(x) dx}{\int_D^\infty (x - D) f(x) dx} = \frac{\int_D^\infty (x - D) f(x) dx}{\int_D^\infty f(x) dx},$$

which can be simplified to

$$\int_D^\infty f(x) dx \int_D^\infty x^2 f(x) dx - \left( \int_D^\infty x f(x) dx \right)^2 = 0. \quad (4.27)$$

Recall the Cauchy Schwarz inequality, which can be written in the form

$$\left( \int_a^b u(x)v(x) dx \right)^2 \leq \int_a^b u^2(x) dx \int_a^b v^2(x) dx.$$

Suppose the limits of integration are  $b = \infty$  and  $a = D$ , and let  $u(x) = x\sqrt{f(x)}$  and  $v(x) = \sqrt{f(x)}$ . Then we obtain

$$\left( \int_D^\infty x f(x) dx \right)^2 \leq \int_D^\infty x^2 f(x) dx \int_D^\infty f(x) dx,$$

with equality only holding in the limiting case  $D \rightarrow \infty$ . Thus, there is no finite solution to (4.27), and so this case is infeasible.

To summarize, the KKT conditions have four terminal cases that are dependent on the values of  $\lambda$ ,  $\mu$ , and  $\alpha$ . In Table 4.1 we provide a summary of these cases.

Case	$\mu$	$\lambda$	$\alpha$	$D$	Description
1	$\neq 0$	Any	1	Undefined	The uninsured case
2.1	0	$\leq 1$	0	0	Full coverage
2.2.1	0	$> 1$	0	Solve (4.26)	Equivalent to optimal stop-loss coinsurance policy
2.2.2	0	$> 1$	$> 0$	Finite	Infeasible

Table 4.1: Terminal cases for the KKT conditions



Assuming that  $\lambda$  and  $\delta$  are such that the insured case is preferable to the uninsured case, the optimal policy for  $\lambda \leq 1$  is full coverage,  $\alpha = D = 0$ . If  $\lambda > 1$  the optimal policy has  $\alpha = 0$  and  $D$  satisfying  $\frac{\lambda-1}{2\delta} = \int_0^D (D-x) f(x) dx$ . The optimal mean-variance in this case is

$$\begin{aligned} MV(\mathbf{X}|0, D) = & \int_0^D x f(x) dx + D \int_D^\infty f(x) dx + \lambda \int_D^\infty (x-D) f(x) dx \\ & + \delta \left( \int_0^D x^2 f(x) dx + D^2 \int_D^\infty f(x) dx - v(0, D)^2 \right). \end{aligned}$$

**Corollary 4.1.** Let  $\lambda > 1$ . Then the optimal straight coinsurance policy has a greater mean-variance than the optimal deductible policy. Moreover, the mean-variance in the optimal deductible policy is equal to the mean-variance of both the optimal coinsurance with stop-loss and the optimal coinsurance with deductible policies.

*Proof.* Let  $MV^*(C, D)$ ,  $MV^*(\alpha)$ ,  $MV^*(\alpha, S)$ , and  $MV^*(\alpha, D)$  denote the minimal mean-variance values for a deductible, straight coinsurance, coinsurance with stop-loss, and coinsurance with deductible policy, respectively. By Theorem 4.1 we have that,  $MV^*(C, D) = MV^*(\alpha, S)$ . It also must be the case that  $MV^*(\alpha, S) < MV^*(\alpha)$ , since a straight coinsurance policy is equivalent to an  $(\alpha, \infty)$  policy, but we have  $S$  finite in  $MV^*(\alpha, S)$ . Thus,  $MV^*(C, D) < MV^*(\alpha)$ . To complete the proof, note that the KKT discussion above showed that the optimal coinsurance with deductible policy has  $\alpha = 0$ , so there is no coinsurance, and  $D$  found by equation (4.26), which is precisely the equation used to find  $D$  in the deductible case. Thus  $MV^*(\alpha, D) = MV(C, D)$ .  $\square$

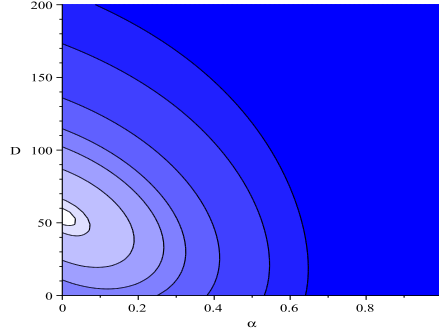


Figure 4.8: Plot of  $E L(\mathbf{X}|\alpha, D) + \delta \text{Var } L(\mathbf{X}|\alpha, D)$

**Example 4.3.** We again consider an exponential distribution with mean 100, a loading factor  $\lambda = 1.25$ , and a risk preference  $\delta = 0.01$ . Figure 4.8 shows the contour lines of the mean-variance, with the optimal point  $(0, 54.53765)$ , the same as in the prior examples for deductible policies and coinsurance with stop-loss policies. This was expected, as the equation (4.26) that is solved to find  $D$  is the same as the equations to find  $D$  and  $S$  in the deductible and coinsurance with stop-loss policies, respectively.

## 5 The Insurance System

When considering an insurance purchaser's optimal coverage it is natural to consider the one-to-one relationship between the insured and insurer. However, from the insurer's perspective, the profit and variance from a single customer is only a piece of a larger group of customers. In this way, we can consider an insurance system consisting of the single insurer and a group of insured customers.

In the first section we introduce the insurance system and describe the insurance company's options in setting individual premiums in a group setting. Section 5.2 discusses some properties of the covariances between  $\mathbf{X}$ ,  $L(\mathbf{X})$ , and  $R(\mathbf{X})$ , and the following section considers the sum of variances within the system, i.e, the sum of the variances experienced by the insurance customers and the insurer. We show that this sum of variances is smaller than the uninsured sum of variances - a possible social benefit of insurance. Section 5.4 extends this concept to the case of multiple customers and a single insured, showing that the sum of variances is less than the sum of uninsured variances if the insurer reimbursements have sufficiently low covariances. We conclude by discussing the interpretation of the sum of variances measure.

### 5.1 Premium

Under a system framework, the insurer no longer needs to price each individual loan above its actuarial value. Instead, the insurer can place customers in groups and set premiums so that the total premium for the group exceeds the expected indemnity that it will be paid. This allows the insurer flexibility to price policies to meet governmental obligations or other enterprise considerations.

In particular, suppose that there are  $k$  insured, purchasing insurance to cover random variables  $\mathbf{X}_i$ ,  $i = 1 \dots k$ , with a corresponding premium  $p_i(C, D)$ . In order to guarantee an expected profit, the insurance company must set the premiums

so that the total of all premiums is greater than the total of all expected payouts:

$$\sum_{i=1}^k p_i(C, D) > \sum_{i=1}^k E I(\mathbf{X}_i) \quad (5.1)$$

Suppose that the premium is loaded,  $p(C, D) = \lambda E I(\mathbf{X})$ . Then a sufficient condition for the satisfaction of (5.1) is that  $\lambda_i > 1$  for each customer  $i$ . The system viewpoint allows for this condition to be weakened. Some loading factors may be discounted ( $\lambda_i < 1$ ) as long as the loading factors of the other customers is large enough to compensate for the expected loss induced by the discounted customers. In particular, the expected profit from the customers paying a loading  $\lambda > 1$  must outweigh the expected loss from the customers paying a loading  $\lambda < 1$ :

$$\sum_{i: \lambda_i > 1} \lambda_i E I(\mathbf{X}_i) > \sum_{j: \lambda_j < 1} \lambda_j E I(\mathbf{X}_j)$$

Consider an insurer whose current customer base exclusively pays premiums with  $\lambda > 1$ . If the insurer has reason to introduce new customers to the system who pay discounted premiums, there are options for the insurer to ensure continued profitability. For example, the premium loadings of existing customers can be raised, or new customers with loadings greater than one can be introduced.

**Example 5.1.** Suppose that an insurer has 10 customers in a group, each with a loading factor  $\lambda = 1.20$  and an expected payout \$400. The insurer's expected profit is therefore  $(\lambda - 1)\$4,000 = \$800$ . An additional 10 customers are to be added to the group with loadings  $\lambda = 0.95$  and expected payouts of \$2,000. The insurer then faces an expected loss of  $(1 - 0.95) \cdot \$20,000 + (1 - 1.20) \cdot \$4,000 = \$200$ .

If the insurer chooses to equally increase the loading of each of the existing customers, it must do so by at least  $z$ , where  $z$  satisfies  $(1.20 - 1 + z) \cdot \$4000 - (1 - 0.95) \cdot \$20,000 \geq 0$ , i.e., the new loading must increase by  $z = 0.05$ , from 1.20 to 1.25.

Alternatively, suppose the insurer instead wishes to add a group of  $t$  customers, and that there are three such groups,  $t_1$ ,  $t_2$ , and  $t_3$  to choose from. The total expected payouts for these groups are \$2,000, \$4,000, and \$8,000, respectively. The minimum required average loadings are:

$$\begin{aligned}\lambda_{t_1} &= 1 + \frac{1}{\$2,000} \cdot \$200 = 1.10 \\ \lambda_{t_2} &= 1 + \frac{1}{\$4,000} \cdot \$200 = 1.05 \\ \lambda_{t_3} &= 1 + \frac{1}{\$8,000} \cdot \$200 = 1.025\end{aligned}$$

As a final possibility, suppose that the insurer will add a group of customers with a mean loss of \$250 and one of three loadings:  $\lambda_1 = 1.10$ ,  $\lambda_2 = 1.20$ , or  $\lambda_3 = 1.30$ . The number  $k$  of customers needed for each loading must satisfy

$$\begin{aligned}\lambda_1 = 1.10: k &> \frac{1}{\$250(1.10 - 1)} \cdot \$200 = 8 \\ \lambda_1 = 1.20: k &> \frac{1}{\$250(1.20 - 1)} \cdot \$200 = 4 \\ \lambda_1 = 1.30: k &> \frac{1}{\$250(1.30 - 1)} \cdot \$200 = 2\frac{2}{3}\end{aligned}$$

These cases are analogous to the current situation in the United States with the Affordable Care Act. Indeed, if insurer's are required to offer coverage to all people, then the insurance system will be a mix of those with higher expected loss levels (e.g., those with pre-existing conditions) and those with lower expected losses (e.g., healthy individuals).

## 5.2 Covariance

In this section we provide some useful properties of the covariances among  $L(\mathbf{X})$ ,  $R(\mathbf{X})$ , and  $\mathbf{X}$ . It is intuitively clear that  $L(\mathbf{X})$  and  $\mathbf{X}$  increase and decrease together, while  $R(\mathbf{X})$  moves in the opposite direction. The following results for-

malize these intuitions. First, we establish a general property of random variables.

**Lemma 5.1.** Let  $\mathbf{X}$ ,  $\mathbf{Y}$ , and  $\mathbf{Z}$  be random variables. Then

$$\text{a) } \text{Cov}(\mathbf{X} + \mathbf{Y}, \mathbf{Z}) = \text{Cov}(\mathbf{X}, \mathbf{Z}) + \text{Cov}(\mathbf{Y}, \mathbf{Z})$$

$$\text{b) } \text{Cov}(\mathbf{X} - \mathbf{Y}, \mathbf{Z}) = \text{Cov}(\mathbf{X}, \mathbf{Z}) - \text{Cov}(\mathbf{Y}, \mathbf{Z})$$

*Proof.*

$$\begin{aligned} \text{a) } \text{Cov}(\mathbf{X} + \mathbf{Y}, \mathbf{Z}) &= \mathbb{E}[(\mathbf{X} + \mathbf{Y})\mathbf{Z}] - \mathbb{E}(\mathbf{X} + \mathbf{Y})\mathbb{E}\mathbf{Z} \\ &= \mathbb{E}\mathbf{X}\mathbf{Z} + \mathbb{E}\mathbf{Y}\mathbf{Z} - \mathbb{E}\mathbf{X}\mathbb{E}\mathbf{Z} - \mathbb{E}\mathbf{Y}\mathbb{E}\mathbf{Z} \\ &= \text{Cov}(\mathbf{X}, \mathbf{Z}) + \text{Cov}(\mathbf{Y}, \mathbf{Z}) \end{aligned}$$

$$\begin{aligned} \text{b) } \text{Cov}(\mathbf{X} - \mathbf{Y}, \mathbf{Z}) &= \mathbb{E}[(\mathbf{X} - \mathbf{Y})\mathbf{Z}] - \mathbb{E}(\mathbf{X} - \mathbf{Y})\mathbb{E}\mathbf{Z} \\ &= \mathbb{E}\mathbf{X}\mathbf{Z} - \mathbb{E}\mathbf{Y}\mathbf{Z} - \mathbb{E}\mathbf{X}\mathbb{E}\mathbf{Z} + \mathbb{E}\mathbf{Y}\mathbb{E}\mathbf{Z} \\ &= \text{Cov}(\mathbf{X}, \mathbf{Z}) - \text{Cov}(\mathbf{Y}, \mathbf{Z}) \end{aligned}$$

□

An immediate consequence of Lemma 5.1 is that

$$\text{Cov}(\mathbf{X} + \mathbf{Y}, \mathbf{Y}) = \text{Cov}(\mathbf{X}, \mathbf{Y}) + \text{Cov}(\mathbf{Y}, \mathbf{Y}) = \text{Cov}(\mathbf{X}, \mathbf{Y}) + \text{Var } \mathbf{Y} \quad (5.2a)$$

$$\text{Cov}(\mathbf{X} - \mathbf{Y}, \mathbf{Y}) = \text{Cov}(\mathbf{X}, \mathbf{Y}) - \text{Cov}(\mathbf{Y}, \mathbf{Y}) = \text{Cov}(\mathbf{X}, \mathbf{Y}) - \text{Var } \mathbf{Y}, \quad (5.2b)$$

Equations (2.6), (5.2a), and (5.2b) therefore give:

$$\text{Cov}[L(\mathbf{X}), \mathbf{X}] = \text{Cov}[R(\mathbf{X}), \mathbf{X}] + \text{Var } \mathbf{X} \quad (5.3a)$$

$$\text{Cov}[R(\mathbf{X}), \mathbf{X}] = \text{Cov}[L(\mathbf{X}), \mathbf{X}] - \text{Var } \mathbf{X} \quad (5.3b)$$

$$\text{Cov}[L(\mathbf{X}), R(\mathbf{X})] = \text{Cov}[R(\mathbf{X}), \mathbf{X}] + \text{Var } R(\mathbf{X}), \quad (5.3c)$$

which can be used to show the following covariance relations in our model:

**Corollary 5.1.**

$$\text{Cov} [R(\mathbf{X}), \mathbf{X}] \leq \text{Cov} [L(\mathbf{X}), R(\mathbf{X})] \leq \text{Cov} [L(\mathbf{X}), \mathbf{X}] \quad (5.4)$$

*Proof.* Since  $0 \leq \text{Var } R(\mathbf{X})$ , the left inequality of (5.4) is immediate from (5.3c).

Substituting (5.3b) into (5.3c) we obtain

$$\text{Cov}[L(\mathbf{X}), R(\mathbf{X})] = \text{Cov} [L(\mathbf{X}), \mathbf{X}] + \text{Var } R(\mathbf{X}) - \text{Var } \mathbf{X}$$

The right inequality of (5.4) follows since  $\text{Var } R(\mathbf{X}) \leq \text{Var } \mathbf{X}$  (Corollary 2.4).  $\square$

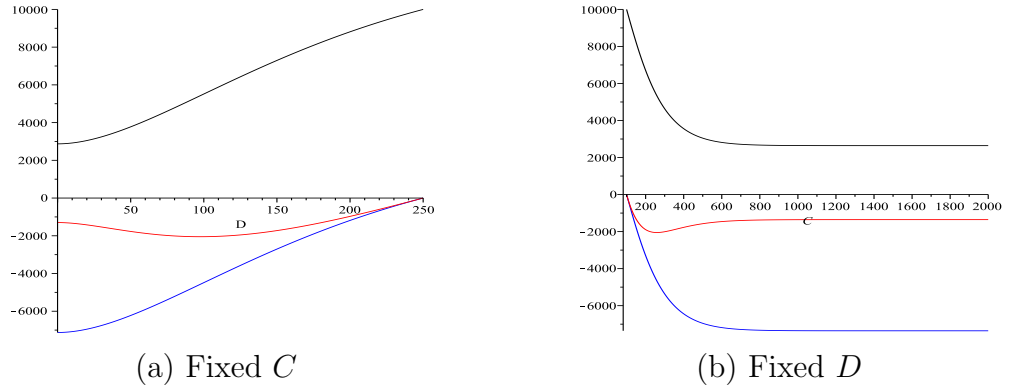


Figure 5.1: Plot of  $\text{Cov}(L(\mathbf{X}), \mathbf{X})$  [Black],  $\text{Cov}(R(\mathbf{X}), \mathbf{X})$  [Red],  $\text{Cov}(L(\mathbf{X}), R(\mathbf{X}))$  [Blue] for (a)  $C = 250$  and (b)  $D = 100$ .

Figure 5.1 illustrates Corollary 5.1 for an exponential distribution with mean 100. In (a)  $C$  is set to 250 and  $D$  varies, and so the case  $D = 250$  corresponds with the uninsured case. At this point  $\text{Cov}[L(\mathbf{X}), \mathbf{X}] = \text{Var } \mathbf{X} = 10,000$ , while  $\text{Cov}[R(\mathbf{X}), \mathbf{X}]$  and  $\text{Cov}[L(\mathbf{X}), R(\mathbf{X})]$  are both equal to zero. We note that  $\text{Cov}[L(\mathbf{X}), \mathbf{X}]$  and  $\text{Cov}[R(\mathbf{X}), \mathbf{X}]$  are parallel, whereas  $\text{Cov}[L(\mathbf{X}), R(\mathbf{X})]$  has a slightly parabolic shape. We observe similar phenomena in figure (b).

**Corollary 5.2.**  $\text{Cov}[R(\mathbf{X}), \mathbf{X}] < 0$  if and only if  $\text{Cov}[L(\mathbf{X}), \mathbf{X}] < \text{Var } \mathbf{X}$

*Proof.* Follows directly from 5.3a □

**Theorem 5.1.**  $\text{Cov} [L(\mathbf{X}), R(\mathbf{X})] < 0$

The proof is given in Appendix D.

Figure 5.2 gives the contour plot for  $\text{Cov} [L(\mathbf{X}), R(\mathbf{X})]$  for an exponential distribution with mean 100. The plot identifies a minimum value of the covariance at the approximate point  $(C, D) = (251, 102)$ . We note that the contours flatten as  $C$  increases, while changes to  $D$  result in changing covariance values.

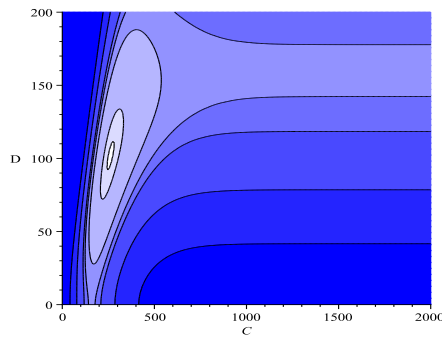


Figure 5.2: Contour plot of  $\text{Cov} [L(\mathbf{X}), R(\mathbf{X})]$ .

**Corollary 5.3.**  $\text{Cov} [R(\mathbf{X}), \mathbf{X}] < 0$

*Proof.* Follows from Corollary 5.1 and Theorem 5.1. □

**Corollary 5.4.**  $\text{Cov} [L(\mathbf{X}), \mathbf{X}] > 0$

*Proof.* By the Cauchy-Schwarz inequality we have

$$|\text{Cov} [R(\mathbf{X}), \mathbf{X}]| < \sqrt{\text{Var } R(\mathbf{X}) \cdot \text{Var } \mathbf{X}}$$

From Corollary 2.4 we have that  $\text{Var } R(\mathbf{X}) < \text{Var } \mathbf{X}$ , and so we can write

$$|\text{Cov} [R(\mathbf{X}), \mathbf{X}]| < \sqrt{\text{Var } \mathbf{X} \cdot \text{Var } \mathbf{X}} = \text{Var } \mathbf{X}.$$



Therefore,  $-\text{Cov} [R(\mathbf{X}), \mathbf{X}] < \text{Var } \mathbf{X}$ , or equivalently

$$\text{Cov} [-R(\mathbf{X}), \mathbf{X}] < \text{Var } \mathbf{X} \quad (5.5)$$

From Theorem 2.4 we have that  $\text{Var } \mathbf{X} = \text{Cov} [L(\mathbf{X}), \mathbf{X}] + \text{Cov} [-R(\mathbf{X}), \mathbf{X}]$ , and so

$$\text{Cov} [L(\mathbf{X}), \mathbf{X}] = \text{Var } \mathbf{X} - \text{Cov} [-R(\mathbf{X}), \mathbf{X}] > 0,$$

where the inequality follows from (5.5).  $\square$

Figure 5.3 gives the contour plots of  $\text{Cov} [L(\mathbf{X}), \mathbf{X}]$  and  $\text{Cov} [R(\mathbf{X}), \mathbf{X}]$  for an exponentially distributed loss. Both covariances are minimized when  $D = 0$  and  $C \rightarrow \infty$ , however, all values in (a) are nonnegative, with a minimal value of 0, while in (b) all values are non-positive, with a minimal value of  $-10,000$ .

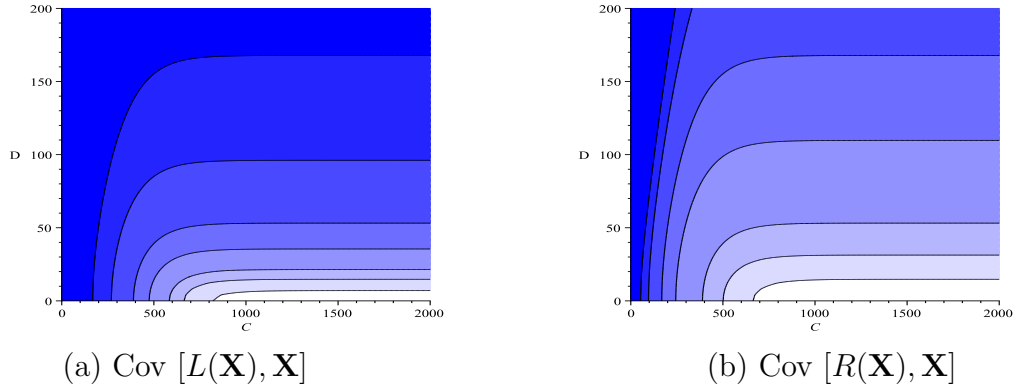


Figure 5.3: Contour plots of  $\text{Cov} [L(\mathbf{X}), \mathbf{X}]$  and  $\text{Cov} [R(\mathbf{X}), \mathbf{X}]$ .

### 5.3 System Sum of Variances

In this section we provide evidence that insurance may provide a social benefit by reducing the overall level of risk experienced by the insured and insurer. We have established that the purchase of insurance creates a risk-sharing system, where the insurer takes on a portion of a random loss variable's variance in exchange for

a premium. Equation (2.7) established the relationship between expected insured loss, expected insurer profit, and the expectation of the random loss variable:  $EL(\mathbf{X}) = ER(\mathbf{X}) + E\mathbf{X}$ . Using this equation we can equivalently express the insurance relationship as

$$EL(\mathbf{X}) + E[-R(\mathbf{X})] = E\mathbf{X} \quad (5.6)$$

Note that  $-R(\mathbf{X})$  is the loss of the insurer. The insurance system is thus the individual losses experienced by the insured and the insurer.

We now consider the variances in the insurance system and compare to the variance of the random loss variable, looking for a counterpart to equation (5.6). We will first consider the case of a single insured and single insurer, and then extend to the case of multiple insured in section 5.4.

The system sum of variances (SSV) is defined as the sum of the insured and insurer variances,  $\text{Var } L(\mathbf{X}) + \text{Var } [-R(\mathbf{X})]$ . Because  $L(\mathbf{X})$  and  $R(\mathbf{X})$  are dependent random variables, the SSV is not a variance. Nonetheless, we believe that it is a meaningful measure in an economic or social sense (see Section 5.5 for a discussion of this) We now show that the SSV is less than the uninsured variance:

**Theorem 5.2.** The system sum of variances is less than the uninsured variance:

$$\text{Var } L(\mathbf{X}) + \text{Var } [-R(\mathbf{X})] < \text{Var } \mathbf{X} \quad (5.7)$$

*Proof.* First, recall that the variance of the sum of two random variables  $\mathbf{X}$  and  $\mathbf{Y}$  is given by  $\text{Var } (\mathbf{X} + \mathbf{Y}) = \text{Var } \mathbf{X} + \text{Var } \mathbf{Y} + 2 \text{Cov } (\mathbf{X}, \mathbf{Y})$ . Since  $L(\mathbf{X}) = \mathbf{X} + R(\mathbf{X})$ , we have

$$\text{Var } L(\mathbf{X}) = \text{Var } \mathbf{X} + \text{Var } R(\mathbf{X}) + 2 \text{Cov } [R(\mathbf{X}), \mathbf{X}]. \quad (5.8)$$

Using (5.8) and the fact that  $\text{Var} [-R(\mathbf{X})] = \text{Var} R(\mathbf{X})$ , we obtain

$$\begin{aligned}\text{Var} L(\mathbf{X}) + \text{Var} [-R(\mathbf{X})] &= \text{Var} L(\mathbf{X}) + \text{Var} R(\mathbf{X}) \\ &= \text{Var} \mathbf{X} + 2 (\text{Var} R(\mathbf{X}) + \text{Cov} [R(\mathbf{X}), \mathbf{X}])\end{aligned}$$

From (5.3c), we have that  $\text{Cov} [R(\mathbf{X}), \mathbf{X}] = \text{Cov} [L(\mathbf{X}), R(\mathbf{X})] - \text{Var} R(\mathbf{X})$ , and so

$$\text{Var} L(\mathbf{X}) + \text{Var} [-R(\mathbf{X})] = \text{Var} \mathbf{X} + 2 \text{Cov} [L(\mathbf{X}), R(\mathbf{X})] \quad (5.9)$$

Since  $\text{Var} \mathbf{X} > 0$ , it follows that the SSV is less than the uninsured variance if  $\text{Cov} [L(\mathbf{X}), R(\mathbf{X})] < 0$ , which is guaranteed by Theorem 5.1.  $\square$

This theorem shows that the variance experienced by the insured and the insurer is less than the variance that would be experienced in the uninsured case. As can be seen from equation (5.9), the remainder of the uninsured variance,  $\text{Var} \mathbf{X} - \text{Var} L(\mathbf{X}) - \text{Var} [-R(\mathbf{X})]$ , is made up of the covariance between the insured and the insurer.

We next derive the first order optimality conditions for system variance minimization. In equation (5.9) it was shown that the system variance is comprised of the variance of  $\mathbf{X}$  and the covariance of  $L(\mathbf{X})$  and  $R(\mathbf{X})$ . The system variance is therefore minimized when  $\text{Cov} [L(\mathbf{X}), R(\mathbf{X})]$  is minimized. Figure 5.4 shows the System Variance contour lines. We note that this figure is identical to Figure 5.2, with the values of the contour lines shifted.

The derivatives of  $\text{Cov} [L(\mathbf{X}), R(\mathbf{X})]$  are given by (see Appendix D for the

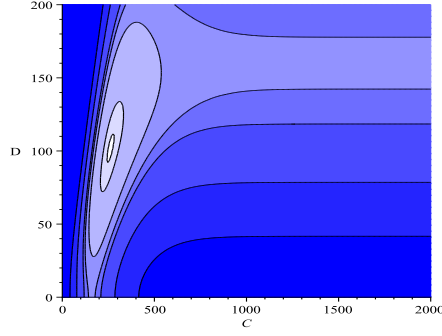


Figure 5.4: Contour plot of the SSV,  $\text{Var } L(\mathbf{X}) + \text{Var } [-R(\mathbf{X})]$ .

calculation):

$$\begin{aligned}
 \frac{\partial}{\partial C} \text{Cov } [L(\mathbf{X}), R(\mathbf{X})] &= - \int_0^D f(x) dx \int_C^\infty (x + D - C) f(x) dx \\
 &\quad + (C - D) \int_0^D f(x) dx \int_C^\infty f(x) dx \\
 &\quad + \int_0^D x f(x) dx \int_C^\infty f(x) dx \\
 &\quad + \int_D^C f(x) dx \int_C^\infty (C - x) f(x) dx \\
 &\quad + \int_D^C (C - x) f(x) dx \int_C^\infty f(x) dx
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{\partial}{\partial D} \text{Cov } [L(\mathbf{X}), R(\mathbf{X})] &= \int_0^D f(x) dx \int_C^\infty (x + D - C) f(x) dx \\
 &\quad + \int_C^\infty f(x) dx \int_0^D (D - C - x) f(x) dx \\
 &\quad + \int_D^C f(x) dx \int_0^D (D - x) f(x) dx \\
 &\quad + \int_D^C (D - x) f(x) dx \int_0^D f(x) dx.
 \end{aligned}$$

Setting these derivatives equal to 0 we obtain the first-order optimality conditions:

$$\begin{aligned}
& \int_C^\infty f(x) dx \left( 2(C - D) \int_0^D f(x) dx + \int_0^D x f(x) dx \right) \\
& - \int_0^D f(x) dx \int_C^\infty x f(x) dx \\
& = \int_D^C f(x) dx \int_C^\infty (x - C) f(x) dx - \int_D^C (C - x) f(x) dx \int_C^\infty f(x) dx
\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}
& \int_C^\infty f(x) dx \left( 2(C - D) \int_0^D f(x) dx + \int_0^D x f(x) dx \right) \\
& - \int_0^D f(x) dx \int_C^\infty x f(x) dx \\
& = \int_D^C f(x) dx \int_0^D (D - x) f(x) dx - \int_D^C (x - D) f(x) dx \int_0^D f(x) dx
\end{aligned} \tag{5.11}$$

Notice that the left sides of (5.10) and (5.11) are equal. We then have the necessary condition

$$\begin{aligned}
& \int_D^C f(x) dx \int_C^\infty (x - C) f(x) dx - \int_D^C (C - x) f(x) dx \int_C^\infty f(x) dx \\
& = \int_D^C f(x) dx \int_0^D (D - x) f(x) dx - \int_D^C (x - D) f(x) dx \int_0^D f(x) dx,
\end{aligned}$$

which can be solved for  $C$  and  $D$  to find potentially optimal solutions.

## 5.4 SSV with Multiple Customers

We now consider the case of an insurance system with a single insurer and multiple insured. The SSV in this case is the sum of the individual variances of each customer and the insurer variance. If the total number of customers is  $n$ , then we can write the insurer variance as  $\text{Var} \left( \sum_{i=1}^n [-R(\mathbf{X}_i)] \right)$ . The multi-customer

SSV is then given by

$$\sum_{i=1}^n \text{Var } L(\mathbf{X}_i) + \text{Var} \left( \sum_{i=1}^n [-R(\mathbf{X}_i)] \right) \quad (5.12)$$

**Proposition 5.1.** Suppose there are  $n$  customers with insured losses  $L(\mathbf{X}_i)$ ,  $i = 1 \dots n$ , and that the insurer loss is  $\sum_{i=1}^n [-R(\mathbf{X}_i)]$ . Then the multiple customer SSV (5.12) is less than the sum of uninsured variances,  $\sum_{i=1}^n \text{Var } \mathbf{X}_i$  if and only if

$$\sum_{1 \leq i < j \leq n} \text{Cov } [R(\mathbf{X}_i), R(\mathbf{X}_j)] < \sum_{i=1}^n \text{Cov } [L(\mathbf{X}_i), -R(\mathbf{X}_i)] \quad (5.13)$$

*Proof.* The multiple customer system variance is less than the total uninsured variance if

$$\sum_{i=1}^n \text{Var } L(\mathbf{X}_i) + \text{Var} \left( \sum_{i=1}^n [-R(\mathbf{X}_i)] \right) < \sum_{i=1}^n \text{Var } \mathbf{X}_i \quad (5.14)$$

The insurer variance can be written as

$$\begin{aligned} \text{Var} \sum_{i=1}^n [-R(\mathbf{X}_i)] &= \sum_{i=1}^n \text{Var} [-R(\mathbf{X}_i)] + 2 \sum_{1 \leq i < j \leq n} \text{Cov} [-R(\mathbf{X}_i), -R(\mathbf{X}_j)], \\ &= \sum_{i=1}^n \text{Var} [-R(\mathbf{X}_i)] + 2 \sum_{1 \leq i < j \leq n} \text{Cov} [R(\mathbf{X}_i), R(\mathbf{X}_j)] \end{aligned}$$

and so (5.14) becomes

$$2 \sum_{1 \leq i < j \leq n} \text{Cov} [R(\mathbf{X}_i), R(\mathbf{X}_j)] < \sum_{i=1}^n (\text{Var } \mathbf{X}_i - \text{Var } L(\mathbf{X}_i) - \text{Var} [-R(\mathbf{X}_i)]) \quad (5.15)$$

From (5.9), we have that, for any customer  $j$ ,

$$\text{Var } \mathbf{X}_j - \text{Var } L(\mathbf{X}_j) - \text{Var} [-R(\mathbf{X}_j)] = -2 \text{Cov} [L(\mathbf{X}_j), R(\mathbf{X}_j)]. \quad (5.16)$$

The result (5.13) follows by combining (5.15) with (5.16).

□

Note that, by Theorem 5.1, the right side of (5.13) is positive. Therefore, if the customer set is chosen so that  $\sum_{1 \leq i < j \leq n} \text{Cov} [R(\mathbf{X}_i), R(\mathbf{X}_j)] \leq 0$ , the SSV is guaranteed to be less than the sum of the uninsured variances. The requirement that  $\text{Cov} [R(\mathbf{X}_i), R(\mathbf{X}_j)]$  is small (relative to  $\text{Cov} [L(\mathbf{X}_i), -R(\mathbf{X}_i)]$ ) fits with a likely goal of the insurer; keeping the correlations between the customer reimbursements small (or negative) is consistent with the insurer avoiding large shocks and maintaining solvency.

## 5.5 Interpretation

Our conception of the system sum of variances should not be taken as a variance in the statistical sense. Indeed, when computing the variance of a collection of dependent random variables one considers the sum of these variables. For example, let  $\mathbf{X}$  and  $\mathbf{Y}$  be dependent random variables. The variance of their sum is given by

$$\text{Var} (\mathbf{X} + \mathbf{Y}) = \text{Var} \mathbf{X} + \text{Var} \mathbf{Y} + 2 \text{Cov} (\mathbf{X}, \mathbf{Y}). \quad (5.17)$$

The relationship between  $\mathbf{X}$  and  $\mathbf{Y}$  is thus taken into consideration; if they tend to move in the same direction then  $\text{Var} (\mathbf{X} + \mathbf{Y}) > \text{Var} \mathbf{X} + \text{Var} \mathbf{Y}$ , while if they tend to move in different directions  $\text{Var} (\mathbf{X} + \mathbf{Y}) < \text{Var} \mathbf{X} + \text{Var} \mathbf{Y}$ . In other words,  $\text{Var} \mathbf{X} + \text{Var} \mathbf{Y}$  is a sort of baseline for  $\text{Var} (\mathbf{X} + \mathbf{Y})$ , adjusted by the covariance of the random variables.

In the case of the random variables  $L(\mathbf{X}|C, D)$  and  $-R(\mathbf{X}|C, D)$ , equation (5.17) becomes

$$\text{Var} [L(\mathbf{X}) - R(\mathbf{X})] = \text{Var} [\mathbf{X} + R(\mathbf{X}) - R(\mathbf{X})] = \text{Var} \mathbf{X}.$$

The variance of  $L(\mathbf{X}|C, D)$  plus  $R(\mathbf{X}|C, D)$  is thus equal to the variance of the

random variable being insured. This is expected since the insured and insurer losses make up the entire random variable loss. However, the insured and insurer variances do not make up the entire random variable variance - this is shown by Theorems 2.4 and 5.2.

The SSV,  $\text{Var } L(\mathbf{X}) + \text{Var } [-R(\mathbf{X})]$ , considers the addition of the variances of dependent random variables. We argue that this measure is meaningful in an *economic sense*. When dealing with uncertainty one should examine if it makes sense to consider covariance as a relevant component of variability. The covariance is certainly relevant in cases where two dependent random variables  $\mathbf{X}$  and  $\mathbf{Y}$  are experienced together or by the same agent. In such cases the agent is really producing a new random variable,  $\mathbf{Z}$ , that is made up of  $\mathbf{X}$  and  $\mathbf{Y}$ . The relationship between  $\mathbf{X}$  and  $\mathbf{Y}$  is clearly vital to the variation involved in  $\mathbf{Z}$ . On the other hand, if  $\mathbf{X}$  and  $\mathbf{Y}$  are not experienced together as a single random variable, i.e., if they are experienced separately by two agents  $A_X$  and  $A_Y$ , then  $\text{Cov } (\mathbf{X}, \mathbf{Y})$  is relevant to neither  $A_X$  nor  $A_Y$ , as they only experience the individual variances of  $\mathbf{X}$  and  $\mathbf{Y}$ , respectively.

The intuition behind the sum of the variances being less than the uninsured variance is that the support of the loss distribution shrinks. Prior to the purchase of insurance the insured's liability spanned the entire loss distribution, while upon the purchase of insurance the liability is limited to the tails of the distribution with the insurer assuming liability for mid-distribution losses. Although the entire loss distribution is still covered, by splitting it among two agents the potential losses are contracted, and the overall variability is reduced.

The key idea is that the conception of SSV considers the variance experienced by all participants in the loss (the insured and the insurer). The fact that the SSV is less than the uninsured variance means that less variance is actually experienced in the world, and as such, points toward a socially beneficial conception of insurance.



## 6 Conclusion

### 6.1 Summary

We introduced a model of deductible insurance with a coverage limit, establishing the intuitive idea that insurance provides a benefit to the insured by reducing his variance at a cost of an increased expected loss. This leads naturally to a mean-variance model, which we considered in detail, showing that full coverage is optimal if the premium is fair or discounted, while in the case of unfair premiums, full coverage above a non-zero deductible is optimal. We also showed that the insurer can offer to the insured a set of  $(C, D)$  policies that will result in the insured choosing the insurer's optimal policy.

The next set of results concerned mean-variance optimization of coinsurance policies. Optimality conditions for straight coinsurance, coinsurance with a stop-loss limit, and coinsurance with a deductible were derived, and it was shown that the optimal coinsurance with stop-loss policy and the optimal coinsurance with deductible policies are equivalent to the optimal deductible policy. The optimal straight coinsurance policy is inferior to all of these policies. We also showed that in all of these coinsurance cases, the optimal policy has full coverage if premiums are actuarially fair or discounted, and partial coverage in the case of unfair premiums.

Finally, we discussed the insurance system. We demonstrated how one subset of insurance customers can subsidize the premiums of another subset, and provided several results relating the insurer, the insured, and the random loss variable  $\mathbf{X}$ . We then discussed the system sum of variances, showing that, in a single insurer single insured framework, the total of the insured and insurer variances is less than the variance that would be experienced in the uninsured case. This relation also holds in the case of multiple customers if the covariances relating the customer reimbursements are sufficiently low.

## 6.2 Recommendations for Future Research

Within the deductible model that we presented there are some further avenues that can be considered. For one, we can consider the *quantity* of insurance,  $C - D$ , and the differences to mean, variance, and mean-variance that result from changes to the quantity of insurance or from changes in the location of insurance (i.e., the difference between a (100,0) policy and a (200,100) policy. We also would like to investigate the possibility of insurer-insured equilibria, perhaps following the optimal risk sharing conception from [15].

In our model we considered the risk of a single random variable to the insured. A complete mean-variance treatment of insurance should consider the entire set of risks that one faces, as correlations between risks can affect insurance choices. As such, an analysis of background risks building on [18] is recommended.

We also plan to elaborate on the comparative results relating the optimal deductible, coinsurance, coinsurance with stop-loss, and coinsurance with deductible policies. In particular, we will investigate the robustness of the idea that straight coinsurance is inferior to deductible policies and coinsurance with stop-loss/deductible policies, considering non-optimal points.

Further consideration of optimal insurance under different mean-risk measures is also recommended. Prior to doing so, it would be appropriate to consider exactly what is desired from insurance. In the mean-variance case modeled here we considered the entire range of potential losses, while the various shortfall risk measures (e.g., CVaR and mean-semideviation) would be useful in describing insurance coverage when we are mostly concerned with *large* losses. Given its relation to stochastic dominance, an adaption of the mean-semideviation approach as given in [48] and [49] may provide insightful results.

## 7 Appendix

### A Proof of Theorem 2.2

We prove (2.39)

$$\frac{\partial}{\partial C} \text{Var } L(\mathbf{X}) < 0$$

and (2.40),

$$\frac{\partial}{\partial D} \text{Var } L(\mathbf{X}) > 0$$

given that (2.20) holds,

$$\int_D^C f(x) dx < 1. \quad (\text{A1})$$

From (2.1)

$$\mathbb{E} I(\mathbf{X}) = \int_D^C (x - D) f(x) dx + (C - D) \int_C^\infty f(x) dx. \quad (\text{A2})$$

By (2.35), the insured variance is

$$\begin{aligned} \text{Var } L(\mathbf{X}) &= \text{Var } \mathbf{X} + \text{Var } I(\mathbf{X}) - 2 \text{Cov } [I(\mathbf{X}), \mathbf{X}] \\ &= \mathbb{E} \mathbf{X}^2 - (\mathbb{E} \mathbf{X})^2 + \mathbb{E} I(\mathbf{X})^2 - (\mathbb{E} I(\mathbf{X}))^2 - 2 \mathbb{E} I(\mathbf{X}) \cdot \mathbf{X} + 2 \mathbb{E} I(\mathbf{X}) \mathbb{E} \mathbf{X} \\ &= \mathbb{E} (\mathbf{X} - I(\mathbf{X}))^2 - (\mathbb{E} \mathbf{X} - \mathbb{E} I(\mathbf{X}))^2 \\ &= \int_0^D x^2 f(x) dx + D^2 \int_D^C f(x) dx + \int_C^\infty (x + D - C)^2 f(x) dx \\ &\quad - \left( \mathbb{E} \mathbf{X} - \int_D^C (x - D) f(x) dx - (C - D) \int_C^\infty f(x) dx \right)^2 \end{aligned} \quad (\text{A3})$$

Differentiating with respect to  $C$  we have

$$\begin{aligned} \frac{\partial}{\partial C} \text{Var } L(\mathbf{X}) &= 2 \int_C^\infty (C - x - D) f(x) dx \\ &\quad + 2 \int_C^\infty f(x) dx \left( \mathbb{E} \mathbf{X} - \int_D^C (x - D) f(x) dx - (C - D) \int_C^\infty f(x) dx \right), \end{aligned}$$

which can be rearranged to obtain

$$\begin{aligned} \frac{\partial}{\partial C} \text{Var } L(\mathbf{X}) &= 2 \int_C^\infty f(x) dx \left( (C - D) \int_0^D f(x) dx + \int_D^C (C - x) f(x) dx \right) \\ &\quad + 2 \left( \mathbf{E} \mathbf{X} \int_C^\infty f(x) dx - \int_C^\infty x f(x) dx \right). \end{aligned} \quad (\text{A4})$$

Since

$$\begin{aligned} \mathbf{E} \mathbf{X} \int_C^\infty f(x) dx - \int_C^\infty x f(x) dx &= \int_C^\infty f(x) dx \left( \mathbf{E} \mathbf{X} - \int_C^\infty x f(x) dx \right) \\ &\quad - \int_0^C f(x) dx \int_C^\infty x f(x) dx \\ &= \int_C^\infty f(x) dx \int_0^C x f(x) dx \\ &\quad - \int_0^C f(x) dx \int_C^\infty x f(x) dx, \end{aligned}$$

$\frac{\partial}{\partial C} \text{Var } L(\mathbf{X})$  can be rewritten as

$$\begin{aligned} \frac{\partial}{\partial C} \text{Var } L(\mathbf{X}) &= 2 \int_C^\infty f(x) dx \left( \int_0^D (x + C - D) f(x) dx + C \int_D^C f(x) dx \right) \\ &\quad - 2 \int_0^C f(x) dx \int_C^\infty x f(x) dx \\ &= 2 \left( \int_C^\infty f(x) dx \int_0^D (x - D) f(x) dx + C \int_0^C f(x) dx \int_C^\infty f(x) dx \right) \\ &\quad - 2 \int_0^C f(x) dx \int_C^\infty x f(x) dx \\ &= 2 \int_C^\infty f(x) dx \int_0^D (x - D) f(x) dx \\ &\quad + 2 \int_0^C f(x) dx \int_C^\infty (C - x) f(x) dx \end{aligned} \quad (\text{A5})$$

If (A1) holds, then we have  $\int_0^D (x - D) f(x) dx \leq 0$  and  $\int_C^\infty (C - x) f(x) dx \leq 0$  with at least one of the inequalities strict. The result  $\frac{\partial}{\partial C} \text{Var } L(\mathbf{X}) < 0$  then follows.

**Remark A.1.** If (A1) does not hold, then  $\int_0^D (x - D) f(x) dx = 0$  and  $\int_C^\infty (C - x) f(x) dx = 0$ , and so  $\frac{\partial}{\partial C} \text{Var } L(\mathbf{X}) = 0$ .

We next show that  $\frac{\partial}{\partial D} \text{Var } L(\mathbf{X}) > 0$ . Using (A3) we calculate the derivative with respect to  $D$ :

$$\begin{aligned}
\frac{\partial}{\partial D} \text{Var } L(\mathbf{X}) &= 2D \int_D^\infty f(x) dx + 2 \int_C^\infty (x - C) f(x) dx \\
&\quad - 2 \int_D^\infty f(x) dx \left( \mathbb{E} \mathbf{X} - \int_D^C (x - D) f(x) dx - (C - D) \int_C^\infty f(x) dx \right) \\
&= 2 \int_D^\infty f(x) dx \left( D - D \int_D^\infty f(x) dx \right) + 2 \int_C^\infty x f(x) dx \\
&\quad + 2C \int_C^\infty f(x) dx \left( -1 + \int_D^\infty f(x) dx \right) \\
&\quad - 2 \int_D^\infty f(x) dx \left( \int_0^D x f(x) dx + \int_C^\infty x f(x) dx \right) \\
&= 2 \int_0^D f(x) dx \left( D \int_D^\infty f(x) dx - C \int_C^\infty f(x) dx \right) \\
&\quad + 2 \int_C^\infty x f(x) dx \left( 1 - \int_D^\infty f(x) dx \right) - 2 \int_0^D x f(x) dx \int_D^\infty f(x) dx \\
&= 2 \int_0^D f(x) dx \int_C^\infty (x - C) f(x) dx \\
&\quad + 2 \int_0^D (D - x) f(x) dx \int_D^\infty f(x) dx
\end{aligned} \tag{A6}$$

If (A1) holds, then we have  $\int_0^D (D - x) f(x) dx \geq 0$  and  $\int_C^\infty (x - C) f(x) dx \leq 0$  with at least one of the inequalities strict. The result  $\frac{\partial}{\partial D} \text{Var } L(\mathbf{X}) > 0$  then follows.

**Remark A.2.** If (A1) does not hold, then  $\int_0^D (D - x) f(x) dx = 0$  and  $\int_C^\infty (x - C) f(x) dx = 0$ , and so  $\frac{\partial}{\partial D} \text{Var } L(\mathbf{X}) = 0$ .

□

## B Proof of Theorem 2.3

We prove (2.45)

$$\frac{\partial}{\partial C} \text{Var } R(\mathbf{X}) > 0, \tag{B1}$$

and (2.46)

$$\frac{\partial}{\partial D} \text{Var } R(\mathbf{X}) < 0. \tag{B2}$$

given that (2.20) holds,

$$\int_D^C f(x) dx < 1. \quad (\text{B3})$$

From (2.36), we have that  $\text{Var } R(\mathbf{X}) = \text{Var } I(\mathbf{X})$ , and

$$\begin{aligned} \text{Var } I(\mathbf{X}) &= \text{E } [I(\mathbf{X})]^2 - (\text{E } [I(\mathbf{X})])^2 \\ &= \int_D^C (x - D)^2 f(x) dx + (C - D)^2 \int_C^\infty f(x) dx \\ &\quad - \left( \int_D^C (x - D) f(x) dx + (C - D) \int_C^\infty f(x) dx \right)^2. \end{aligned} \quad (\text{B4})$$

The derivative of  $\text{Var } I(\mathbf{X})$  with respect to  $C$  is

$$\begin{aligned} \frac{\partial}{\partial C} \text{Var } I(\mathbf{X}) &= 2(C - D) \int_C^\infty f(x) dx - 2 \int_D^C (x - D) f(x) dx \int_C^\infty f(x) dx \\ &\quad - 2(C - D) \left( \int_C^\infty f(x) dx \right)^2 \\ &= 2 \int_C^\infty f(x) dx \left( C - D - \int_D^C (x - D) - (C - D) \int_C^\infty f(x) dx \right) \\ &= 2 \int_C^\infty f(x) dx \left( C \int_0^C f(x) dx - D \int_0^D f(x) dx - \int_D^C x f(x) dx \right) \end{aligned} \quad (\text{B5})$$

Since

$$\begin{aligned} C \int_0^C f(x) dx - D \int_0^D f(x) dx - \int_D^C x f(x) dx \\ > C \int_0^C f(x) dx - C \int_0^D f(x) dx - \int_D^C C f(x) dx = 0, \end{aligned}$$

we obtain  $\frac{\partial}{\partial C} \text{Var } I(\mathbf{X}) > 0$ .

We next differentiate with respect to  $D$ :

$$\begin{aligned}
\frac{\partial}{\partial D} \text{Var } I(\mathbf{X}) &= 2 \int_D^C (D-x) f(x) dx - 2(C-D) \int_C^\infty f(x) dx \\
&\quad + 2 \int_D^\infty f(x) dx \left( \int_D^C (x-D) f(x) dx + (C-D) \int_C^\infty f(x) dx \right) \\
&= 2 \int_D^C (x-D) f(x) dx \left( -1 + \int_D^\infty f(x) dx \right) \\
&\quad + 2(C-D) \int_C^\infty f(x) dx \left( -1 + \int_D^\infty f(x) dx \right) \\
&= -2 \int_0^D f(x) dx \left( \int_D^C (x-D) f(x) dx + (C-D) \int_C^\infty f(x) dx \right) \quad (\text{B6})
\end{aligned}$$

Therefore  $\frac{\partial}{\partial D} \text{Var } I(\mathbf{X}) < 0$ .

□

## C Proof of Theorem 5.1

We prove Theorem 5.1,  $\text{Cov}(L(\mathbf{X}), R(\mathbf{X})) < 0$ . We have that

$$\begin{aligned}
\text{Cov}(L(\mathbf{X}), R(\mathbf{X})) &= \text{Var } R(\mathbf{X}) + \text{Cov}(R(\mathbf{X}), \mathbf{X}) \\
&= \text{E } R(\mathbf{X})^2 - (\text{E } R(\mathbf{X}))^2 + \text{E } (\mathbf{X} \cdot R(\mathbf{X})) - \text{E } \mathbf{X} \text{E } R(\mathbf{X}). \quad (\text{C1})
\end{aligned}$$

To derive an expression for  $\text{Cov}(L(\mathbf{X}), R(\mathbf{X}))$ , we consider the term pairs  $\text{E } R(\mathbf{X})^2 + \text{E } (\mathbf{X} \cdot R(\mathbf{X}))$  and  $-(\text{E } R(\mathbf{X}))^2 - \text{E } \mathbf{X} \text{E } R(\mathbf{X})$ .

$$\begin{aligned}
\text{E } R(\mathbf{X})^2 + \text{E } \mathbf{X} \cdot R(\mathbf{X}) &= \text{E } (R(\mathbf{X}))((R(\mathbf{X}) + \mathbf{X})) \\
&= D \int_D^C (D-x) f(x) dx + (D-C) \int_C^\infty (x+D-C) f(x) dx \\
&\quad - (\text{E } R(\mathbf{X}))^2 - \text{E } \mathbf{X} \text{E } R(\mathbf{X}) = -\text{E } R(\mathbf{X})(\text{E } R(\mathbf{X}) + \text{E } \mathbf{X}) \\
&= -\text{E } R(\mathbf{X}) \left( \int_0^D x f(x) dx + D \int_D^C f(x) dx + \int_C^\infty (x+D-C) f(x) dx \right)
\end{aligned}$$

Therefore,

$$\begin{aligned} \text{Cov } (L(\mathbf{X}), R(\mathbf{X})) = & D \int_D^C (D-x) f(x) dx - \text{ER} \left( \int_0^D x f(x) dx + D \int_D^C f(x) dx \right) \\ & + (D-C - \text{E } R(\mathbf{X})) \int_C^\infty (x+D-C) f(x) dx. \end{aligned} \quad (\text{C2})$$

Because

$$\begin{aligned} D-C - \text{ER} &= \int_0^D (D-C) f(x) + \int_D^C (D-C - (D-x)) f(x) dx \\ &+ \int_C^\infty (D-C - (D-C)) f(x) dx \\ &= (D-C) \int_0^D f(x) dx + \int_D^C (x-C) f(x) dx, \end{aligned}$$

the second term of (C2) becomes

$$(D-C) \int_0^D f(x) dx \int_C^\infty (x+D-C) f(x) dx + \int_D^C (x-C) f(x) dx \int_C^\infty (x+D-C) f(x) dx. \quad (\text{C3})$$

Writing  $\text{E } R(\mathbf{X})$  as  $\int_D^C (D-x) f(x) dx + (D-C) \int_C^\infty f(x) dx$  in the third term of (C2) and multiplying through we obtain

$$\begin{aligned} & \int_D^C (x-D) f(x) dx \int_0^D x f(x) dx + D \int_D^C (x-D) f(x) dx \int_D^C f(x) dx \\ & + (C-D) \int_C^\infty f(x) dx \int_0^D x f(x) dx + D(C-D) \int_C^\infty f(x) dx \int_D^C f(x) dx. \end{aligned} \quad (\text{C4})$$

The covariance can then be written as

$$\text{Cov } (L(\mathbf{X}), R(\mathbf{X})) = P_1 + P_2 + P_3 + P_4, \quad (\text{C5})$$



where

$$\begin{aligned}
P_1 &= (D - C) \int_0^D f(x) dx \int_C^\infty (x + D - C) f(x) dx \\
&\quad + (C - D) \int_C^\infty f(x) dx \int_0^D x f(x) dx, \\
P_2 &= D \int_D^C (D - x) f(x) dx + D \int_D^C (x - D) f(x) dx \int_D^C f(x) dx \\
P_3 &= D(C - D) \int_C^\infty f(x) dx \int_D^C f(x) dx \\
P_4 &= \int_D^C (x - D) f(x) dx \int_0^D x f(x) dx + \int_D^C (x - C) f(x) dx \int_C^\infty (x + D - C) f(x) dx
\end{aligned}$$

To carry out the proof we show that  $P_1 < 0$  and  $P_2 + P_3 + P_4 < 0$ .

Factoring out  $(D - C)$  in  $P_1$  we obtain

$$\begin{aligned}
P_1 &= (D - C) \left( \int_0^D f(x) dx \int_C^\infty (x + D - C) f(x) dx - \int_C^\infty f(x) dx \int_0^D x f(x) dx \right) \\
&= (D - C) \int_C^\infty (D - C) f(x) dx \int_0^D f(x) dx \tag{C6a} \\
&\quad + (D - C) \left( \int_C^\infty x f(x) dx \int_0^D f(x) dx - \int_C^\infty f(x) dx \int_0^D x f(x) dx \right). \tag{C6b}
\end{aligned}$$

Since

$$(D - C) \left( \int_C^\infty x f(x) dx \int_0^D f(x) dx \right) < (D - C) \left( C \int_C^\infty f(x) dx \int_0^D f(x) dx \right)$$

and

$$(D - C) \left( - \int_C^\infty f(x) dx \int_0^D x f(x) dx \right) < (D - C) \left( -D \int_C^\infty f(x) dx \int_0^D f(x) dx \right),$$

expression (C6b) is less than

$$-(D - C)^2 \int_C^\infty f(x) dx \int_0^D f(x) dx.$$

Therefore

$$P_1 < (D-C) \int_C^\infty (D-C) f(x) dx \int_0^D f(x) dx - (D-C)^2 \int_C^\infty f(x) dx \int_0^D f(x) dx = 0$$

We now show that  $P_2 + P_3 + P_4 < 0$ . We have

$$\begin{aligned} P_2 &= D \left( \int_D^C (D-x)f(x) dx + \int_D^C (x-D)f(x) dx \int_D^C f(x) dx \right) \\ &= D \left( D \int_D^C f(x) dx \left( 1 - \int_D^C f(x) dx \right) + \int_D^C x f(x) dx \left( \int_D^C f(x) dx - 1 \right) \right) \end{aligned} \quad (C7)$$

Adding  $P_3$  to (C7) we obtain

$$P_3 + P_6 + P_7 = D \left( D \int_D^C f(x) dx \int_0^D f(x) dx - \int_D^C x f(x) dx \int_0^D f(x) dx \right) \quad (C8a)$$

$$\begin{aligned} &+ D \left( D \int_D^C f(x) dx \int_C^\infty f(x) dx - \int_D^C x f(x) dx \int_C^\infty f(x) dx \right) \\ &+ D(C-D) \int_D^C f(x) dx \int_C^\infty f(x) dx \end{aligned} \quad (C8b)$$

Line (C8a) can be rewritten as

$$D \int_0^D f(x) dx \int_D^C (D-x)f(x) dx \quad (C9)$$

and line (C8b) is equal to

$$D \int_C^\infty f(x) dx \int_D^C (D-x+C-D)f(x) dx = D \int_C^\infty f(x) dx \int_D^C (C-x)f(x) dx \quad (C10)$$

Adding lines (C9) and (C10) to  $P_4$  we obtain:

$$P_2 + P_3 + P_4 \tag{C11}$$

$$\begin{aligned} &= \int_D^C (x - C) f(x) dx \int_C^\infty (x + D - C - D) f(x) \\ &+ \int_D^C (x - D) f(x) dx \int_0^D (x - D) f(x) dx \\ &= \int_D^C (x - C) f(x) dx \int_C^\infty (x - C) f(x) + \int_D^C (x - D) f(x) dx \int_0^D (x - D) f(x) dx < 0, \end{aligned} \tag{C12}$$

since the first and fourth integrals of line (C12) are negative and the second and third integrals are positive. Therefore, we have  $\text{Cov} (L(\mathbf{X}), R(\mathbf{X})) = \sum_{i=1}^4 p_i < 0$ , and the proof is complete.

## D Calculation of the Derivatives of $\text{Cov} [L(\mathbf{X}), R(\mathbf{X})]$

Writing  $\text{Cov} [L(\mathbf{X}), R(\mathbf{X})] = P_1 + P_2 + P_3 + P_4$  as in (C5), we calculate the derivatives as follows:

$$\begin{aligned} \frac{\partial}{\partial D} P_1 &= \int_0^D f(x) dx \int_C^\infty (x + D - C) f(x) dx + (C - D) f(D) \int_C^\infty (C - x) f(x) dx \\ &+ \int_C^\infty f(x) dx \int_0^D (D - C - x) f(x) dx \\ \frac{\partial}{\partial D} (P_2 + P_3 + P_4) &= \int_D^C f(x) dx \int_0^D (D - x) f(x) dx + \int_D^C (D - x) f(x) dx \int_0^D f(x) dx \\ &- (C - D) f(D) \int_C^\infty (C - x) f(x) dx \end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial}{\partial D} \text{Cov} [L(\mathbf{X}), R(\mathbf{X})] &= \int_0^D f(x) dx \int_C^\infty (x + D - C) f(x) dx \\
&+ \int_C^\infty f(x) dx \int_0^D (D - C - x) f(x) dx \\
&+ \int_D^C f(x) dx \int_0^D (D - x) f(x) dx \\
&+ \int_D^C (D - x) f(x) dx \int_0^D f(x) dx
\end{aligned} \tag{D1}$$

Similarly,

$$\begin{aligned}
\frac{\partial}{\partial C} P_1 &= - \int_0^D f(x) dx \int_C^\infty (x + D - C) f(x) dx + (C - D) \int_0^D f(x) dx \int_C^\infty f(x) dx \\
&+ \int_0^D x f(x) dx \int_C^\infty f(x) dx + (C - D) f(C) \int_0^D (D - x) f(x) dx \\
\frac{\partial}{\partial C} (P_2 + P_3 + P_4) &= (C - D) f(C) \int_0^D (x - D) f(x) dx + \int_D^C f(x) dx \int_C^\infty (C - x) f(x) dx \\
&+ \int_D^C (C - x) f(x) dx \int_C^\infty f(x) dx
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial}{\partial C} \text{Cov} [L(\mathbf{X}), R(\mathbf{X})] &= - \int_0^D f(x) dx \int_C^\infty (x + D - C) f(x) dx \\
&+ (C - D) \int_0^D f(x) dx \int_C^\infty f(x) dx \\
&+ \int_0^D x f(x) dx \int_C^\infty f(x) dx + \int_D^C f(x) dx \int_C^\infty (C - x) f(x) dx \\
&+ \int_D^C (C - x) f(x) dx \int_C^\infty f(x) dx
\end{aligned}$$

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