INTERACTION AND EXTERNAL FIELD QUANTUM QUENCHES IN THE LIEB-LINIGER AND GAUDIN-YANG MODEL

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ABSTRACT OF THE THESIS

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A review of the solution of the Lieb-Liniger is given. Using the wave function, the dynamics after a quench with a time dependent interaction strength is studied. Directly calculating the overlaps of wave functions, an interaction strength linear in time is examined. Furthermore utilizing those overlaps and the so called Yudson representation a time periodic interaction strength is studied. Moreover the dynamics of the Lieb-Liniger model with an external homogenous field is analyzed. After giving a review of the solution of the Gaudin-Yang model, an outlook on how the wave function for the Gaudin-Yang model in an external homogenous field could be obtained is given.

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Chapter 1

Introduction

In the last years experimental progress in trapping ultracold atoms and realizing various models of one dimensional Bose or Fermi gases in those experiments has led to a renewal of interest in studying those systems. As most physical processes we observe are dynamical, there is especially an interest in nonequilibrium behaviour, for example quantum quenches, which will be defined later.

In those experiments, a quasi one dimensional system can be achieved by applying strong confinement in two transverse directions, therefore only allowing movement in the one residual dimension. This can be realized first and foremost in quantum systems of ultracold atoms, e.g. by means of optical lattices or on atom chips (cf. [1, 2]).

The experimental results obtained by those experiments, thermodynamical as well as nonequilibrium dynamical (cf. [1]), can be very well described by the Lieb-Liniger ([3]) and for higher spin with the Gaudin-Yang model ([4, 5, 6]) and the exact results obtained for those Bethe Ansatz integrable models. Those experiments and theoretical solutions therefore provide a better understanding of the quantum statistical and dynamical effects in quantum many-body physics (cf.[1]).

In the following the Lieb-Liniger and Gaudin-Yang model will be introduced and solved by means of the Bethe-Ansatz, a particular form of a wave function in one dimension introduced by Bethe in 1931 (cf. [7]). Following the review of the Lieb-Liniger model, quantum dynamics after quenches in the interaction strength will be examined utilizing the full overlap of the wave function for different interaction strength and also a different representation for the time evolution of the wave function of an integrable model, the so called Yudson-representation (cf. [8]). Furthermore the wave function and quench dynamics of the Lieb-Liniger model in an external homogenous will be studied. After that the solution of the Gaudin-Yang model will be presented and an outlook on how to solve for the wave function of the Gaudin-Yang model in a constant force potential will be given.

Chapter 2

Lieb-Liniger Model

2.1 Solution of the Lieb-Liniger Model

2.1.1 Preliminaries

The Lieb-Liniger model is a one-dimensional model of a boson gas with point-like interaction of the particles. The model was solved by Lieb and Liniger in 1963 [3].

The model is characterised (cf. [9]) by the Hamiltonian

$$H = \int_{\mathbb{R}} dx \ \left(\partial_x b^\dagger(x,t)\partial_x b(x,t) + c\ b^\dagger(x,t)b^\dagger(x,t)b(x,t)b(x,t)\right) \tag{2.1}$$

with interaction strength $c > 0$ and the quantum fields $b(x,t)$ obeying canonical quantum commutation relations

$$\left[b(x,t), b^\dagger(y,t)\right] = \delta(x-y) \tag{2.2}$$

$$[b(x,t), b(y,t)] = 0 \tag{2.3}$$

$$\left[b^\dagger(x,t), b^\dagger(y,t)\right] = 0 \tag{2.4}$$

As in the following the operators will be at specific times, the time argument will be omitted. Furthermore define the Fock-vacuum $|0\rangle$ and adjoint $\langle 0|$ by

$$b(x)\,|0\rangle = 0 \qquad \langle 0|\,b^\dagger(x) = 0 \qquad \langle 0 \mid 0\rangle = 1 \tag{2.5}$$

2.1.2 Wave function

To look for the eigenfunctions of (2.1) it is convenient to go to the first quantized version of the Hamiltonian at a specific number of particles N with the Ansatz (cf. [9])

$$|\psi\left(k_1,\dots,k_N\right)\rangle = \frac{1}{\sqrt{N!}}\int_{\mathbb{R}^N} d^N x\ \psi_N\left(k_1\dots,k_N|x_1,\dots x_N\right) b^\dagger(x_1)\dots b^\dagger(x_N)\,|0\rangle \tag{2.6}$$

with $\psi\left(k_1\dots,k_N|x_1,\dots x_N\right)$ being symmetric in $x_1,\dots,x_N$. The quantum numbers $k_1,\dots k_N$ are assumed distinct.

Therefore by acting with (2.1) on (2.6) the first quantized version H_N of (2.1) can be found:

$$\begin{aligned}
H\,|\psi\left(k_1,\dots,k_N\right)\rangle &= \frac{1}{\sqrt{N!}}\int_{\mathbb{R}} dy \int_{\mathbb{R}^N} d^N x \left[-\partial_y^2 b^\dagger(y)\sum_i \delta\left(y-x_i\right)\prod_{j\neq i} b^\dagger(x_j)\psi_N(\{k\}|\{x\})\right.\\
&\left.+\frac{c}{\sqrt{N!}}\int_{\mathbb{R}} dy\int_{\mathbb{R}^N} d^N x\ \psi_N(\{k\}|\{x\}) b^\dagger(y)b^\dagger(y)\sum_{i\neq j}\delta\left(y-x_i\right)\delta\left(y-x_j\right)\prod_{k\neq i,j} b^\dagger(x_k)\right]|0\rangle\\
&= \frac{1}{\sqrt{N!}}\int_{\mathbb{R}^N} d^N x\left[-\sum_i\frac{\partial^2}{\partial x_i^2}+c\sum_{i\neq j}\delta(x_i-x_j)\right]\psi_N(\{k\}|\{x\})\prod_i b^\dagger(x_i)\,|0\rangle
\end{aligned}$$

$$\Rightarrow \qquad H_N = -\sum_i\frac{\partial^2}{\partial x_i^2}+2c\sum_{i<j}\delta(x_i-x_j) \tag{2.7}$$

The identities

$$\left[b(y),\prod_{i=1}^N b^\dagger(x_i)\right] = \sum_{i=1}^N\delta\left(y-x_i\right)\prod_{i\neq j} b^\dagger(x_j) \tag{2.8}$$

$$\left[b(y)b(y),\prod_{i=1}^N b^\dagger(x_i)\right]|0\rangle = \sum_{i\neq j}\delta\left(y-x_i\right)\delta\left(y-x_j\right)\prod_{k\neq i,j} b^\dagger(x_k)\,|0\rangle \tag{2.9}$$

have been used. As the wave function $\psi_N(k_1,\dots,k_N|x_1\dots x_N)$ is symmetric in $x_1,\dots,x_N$ it is sufficient to describe the problem in the domain D: $x_1<\dots<x_N$. In this domain the wave function is an eigenfunction of the Hamiltonian

$$H_N^0 = -\sum_i\frac{\partial^2}{\partial x_i^2} \tag{2.10}$$

with eigenvalue E_N. It furthermore has to obey certain continuity conditions, which can be obtained by integrating the Schrödinger equation with the N particle Hamiltonian (2.7) over the relative coordinates in an infinitesimal region:

$$\int_{-\varepsilon}^{+\varepsilon} d(x_{j+1}-x_j)H_N\psi_N = \int_{-\varepsilon}^{+\varepsilon} d(x_{j+1}-x_j)E_N\psi_N \tag{2.11}$$

with the definitions:

$$x = x_{j+1} - x_j \qquad\qquad X = \frac{x_{j+1} + x_j}{2} \tag{2.12}$$

and therefore

$$\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial x_{j+1}^2} = \frac{1}{2}\frac{\partial^2}{\partial X^2} + 2\frac{\partial^2}{\partial x^2} \tag{2.13}$$

the integrals become:

$$\begin{aligned} & -2\frac{\partial}{\partial x}\psi_N\left(x, X, x_1 \ldots x_N\right)\Big|_{-\varepsilon}^{\varepsilon} + 2c\psi_N\left(\{k\}|0, X, x_1 \ldots x_N\right) = 0 \\ \Rightarrow \qquad & \left[\left(\frac{\partial}{\partial x_{j+1}} - \frac{\partial}{\partial x_j}\right) - c\right]_{x_{j+1}=x_j+\varepsilon} \psi_N(k_1, \ldots, k_N|x_1 \ldots x_N) = 0 \end{aligned} \tag{2.14}$$

where the symmetry of the wave function has been used. This condition (2.14) has to be true for all $j \in \{1, \ldots, N-1\}$.

Equations (2.10) and (2.14) are equivalent to (2.1). The solution of these two equations in the domain D can be obtained as follows (cf. [9]). Define the so called Gaudin-operator

$$O_c = \prod_{i>j}\left[\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) + c\right] \tag{2.15}$$

and the totally antisymmetric wave function of free fermions:

$$\psi_f = \det\left(e^{ik_i x_j}\right) \tag{2.16}$$

The eigenfunction to (2.10) satisfying (2.14) is then given by

$$\psi(\{k\}|\{x\}) = \mathcal{N} O_c \psi_f \tag{2.17}$$

where $\mathcal{N}$ is the normalization of the wave function. This is true as O_c in (2.15) commutes with the Hamiltonian H_N^0 in the domain D, therefore (2.17) is eigenfunction of H_N^0. Furthermore (2.17) satisfies (2.14):

$$
\begin{aligned}
\left[\left(\frac{\partial}{\partial x_{j+1}}-\frac{\partial}{\partial x_j}\right)-c\right]_{x_{j+1}=x_j+\varepsilon} O_c\psi_f &= \left[\left(\frac{\partial}{\partial x_{j+1}}-\frac{\partial}{\partial x_j}\right)-c\right]_{x_{j+1}=x_j+\varepsilon}\left[\left(\frac{\partial}{\partial x_{j+1}}-\frac{\partial}{\partial x_j}\right)+c\right] \\
&\quad\times \prod_{\substack{i>k \\ \neg(i=j\wedge k=j+1)}} \left[\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)+c\right]\psi_f \\
&= \left[\left(\frac{\partial}{\partial x_{j+1}}-\frac{\partial}{\partial x_j}\right)^2-c^2\right] \prod_{\substack{i>k \\ \neg(i=j\wedge k=j+1)}} \left[\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)+c\right]\psi_f \Bigg|_{x_{j+1}=x_j+\varepsilon} \\
&= 0
\end{aligned}
\tag{2.18}
$$

The last line follows from the fact that ψ_f and the rest of O_c is totally antisymmetric in x_{j+1} and x_j and the first part in the second line totally symmetric, so at the point $x_{j+1} = x_j$ the whole expression is zero.

Therefore the wave function has been obtained. Applying the operator O_c to ψ_f the wave function in the domain D can be written as:

$$\psi(\{k\}|\{x\}) = \mathcal{N}' \sum_P A(P) e^{i(Pk,x)} \tag{2.19}$$

where P is an element of the symmetric group of order N, $(Pk, x) = \sum_n k_{Pn} x_n$ and

$$A(P) = \prod_{i<j}\left(1 + \frac{ic}{k_{Pi} - k_{Pj}}\right) \tag{2.20}$$

The wave function in the whole domain $\mathbb{R}^N$ can be obtained by symmetrizing (2.19) or equivalently changing the operator O_c to (cf. [10])

$$O_c = \prod_{i>j}\left[\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right) + c\cdot \operatorname{sgn}(x_i - x_j)\right] \tag{2.21}$$

where the sign function compensates for a sign change of the partial derivatives after a permutation of the $\{x_i\}$. The wave function then can be written as

$$\psi(\{k\}|\{x\}) = \mathcal{N}' \sum_P {}_sA(P) e^{i(Pk,x)} \tag{2.22}$$

where

$${}_sA(P) = \prod_{i<j}\left(1 + \frac{ic\cdot \operatorname{sgn}(x_j - x_i)}{k_{Pi} - k_{Pj}}\right) \tag{2.23}$$

The wave function furthermore has to be properly normalized. It can be shown (cf. [10]) that after correct normalization the wave function can be written as

$$\psi(\{k\}|\{x\}) = \frac{1}{N!\sqrt{(2\pi)^N G(k)}} \sum_P {}_sA(P) e^{i(Pk,x)} \tag{2.24}$$

with

$$G(k) = \prod_{i<j} \left(1 + \frac{c^2}{(k_i - k_j)^2}\right) \tag{2.25}$$

and furthermore the following identities hold (cf. [10]):

$$\langle \psi(\{k\}) \mid \psi(\{k'\}) \rangle = \frac{1}{N!} \sum_R \prod_j \delta\left(k_j - k'_{Rj}\right) \tag{2.26}$$

$$\int_{\mathbb{R}^N} d^N k \; \psi(\{k\}|\{x\}) \psi^\star(\{k\}|\{x'\}) = \prod_i \delta(x_i - y_i) \quad \text{for} \quad x, y \in D \tag{2.27}$$

2.1.3 Quench dynamics and Yudson-Representation

Motivated by advances in the field of ultracold atomic or molecular gases (cf. [11]), an important aspect of nonequilibrium dynamics are so-called quantum quenches, where one is interested in the dynamics of a initially stationary state $|\psi_0\rangle$ in the presence of a new Hamiltonian H, the so called quenched Hamiltonian, which is different from the initial Hamiltonian H_0 in for example the interaction constant. This quench can be instantly, i.e. much shorter than any time scale in the problem, or time dependent. The time evoultion is described by

$$|\psi(t)\rangle = e^{-iHt} |\psi_0\rangle \tag{2.28}$$

If the complete set of eigenstates $\{|\lambda\rangle = |\lambda_1, \ldots, \lambda_N\rangle\}$ of the Hamiltonian H is known by means of the Bethe Ansatz this can always be written as:

$$|\psi(t)\rangle = \sum_{\{\lambda\}} e^{-iE(\lambda)t} |\lambda\rangle \langle \lambda \mid \psi_0 \rangle \tag{2.29}$$

(cf. [8]) where $E(\lambda)$ are the corresponding eigenvalues to H and the sum goes over all possible configurations of the $\{\lambda\}$. In general, especially in the case of bound states where one would have to sum over string configurations, this can be very complicated. A different approach, which is much simpler mainly in the case of bound states, is by the use of the Yudson representation ([8]):

$$|\psi(t)\rangle = \int_\Gamma \mathrm{d}\lambda \; e^{-iE(\lambda)t} \, |\lambda\rangle \, (\lambda \mid \psi_0\rangle \tag{2.30}$$

or

$$\mathbb{1} = \int_\Gamma \mathrm{d}\lambda \; |\lambda\rangle \, (\lambda| \tag{2.31}$$

where Γ is a specific path in the complex plane and $(\lambda|$ the Yudson state. Note that there is no sum over the bound states anymore but it is solely an integral representation. For the Lieb-Liniger model the Yudson state can be shown to be (cf. [8],[12]):

$$|\lambda_1, \ldots, \lambda_N) = \frac{(N!)^{1/2}}{(2\pi)^{N/2}} \int \mathrm{d}^N x \; \Theta\left(x_1 < x_2 < \cdots < x_N\right) \prod_{j=1}^{N} e^{i\lambda_j x_j} b^\dagger(x_j) \, |0\rangle \tag{2.32}$$

It is important to state here however, that as the integral is in the complex plane now, the rapidities in the Bethe states $|\lambda_1, \ldots, \lambda_N\rangle$ do not satisfy the boundedness condition for the wave function anymore, that is matrix elements can be divergent. The states are however meaningful when used in a matrix element $\langle F| \, O \, |\lambda\rangle$ with $|F\rangle$ being a state whose configuration space wave function vanishes or oscillates outside a certain domain sufficiently fast. This is for example true for every physical initial state. O can be any local operator (cf. [8]). Therefore when writing $|\lambda\rangle$ in e.g. (2.30) or (2.31) it is implied that those states are subsequently used in a matrix element of the form $\langle F| \, O \, |\lambda\rangle$. Alternatively it is also possible, depending on the integral in question, to choose a path in the complex plane where the integral does not diverge (e.g. in the Lieb-Liniger model cf. [12]).

This can be used in the Lieb-Liniger model. With $|\mathbf{x}\rangle = \frac{1}{\sqrt{N!}} \prod_j b^\dagger(x_j) \, |0\rangle$ as initial state for (2.30) one gets up to normalisation (cf. [12]):

$$|x, t\rangle = \Theta(x_1 > \ldots x_N) \int_\Gamma \prod_j \frac{\mathrm{d}\lambda_j}{2\pi} e^{-iE(\lambda)t} \prod_j e^{-i\lambda_j x_j} \, |\lambda\rangle \tag{2.33}$$

The integration path Γ can chosen to be on the real line for $c > 0$ and parallel to the real line with $\Im(\lambda_j - \lambda_{j-1})$ for $c < 0$ where in the latter case the limited support of physical states is important so that the matrix elements are meaningful.

For two particles the integral can be done analytically for $c \in \mathbb{R}$ and one gets for both cases (cf.

[12]):

$$|x,t\rangle_2 = \int_y \frac{e^{i\frac{(y_1-x_1)^2}{4t}+i\frac{(y_2-x_2)^2}{4t}}}{4\pi it}\left[1-c\sqrt{i\pi t}\Theta(y_1-y_2)e^{\frac{i}{8t}\alpha^2}\text{erfc}\left(\frac{i-1}{4}\frac{i\alpha}{\sqrt{t}}\right)\right]b^\dagger(y_1)b^\dagger(y_2)\left|0\right\rangle \quad (2.34)$$

with $\alpha = 2ct + i(y_1 - x_1) - i(y_2 - x_2)$. For more than two particles (2.33) can be evaluated for large times with a saddle point approximation (cf. [12]).

2.2 Lieb-Liniger Model with time-dependent interaction

2.2.1 Linear time evolution of interaction strength

One example of a quench which is non-instant is changing the interaction strength parameter of the Lieb-Liniger model c constant with time over a finite amount of time.

In recent ultracold atom experiments of particles in atomic traps it was possible to engineer bosons with an effective delta function interaction (cf. [2, 13]). The effective interaction strength is hereby determined by the geometry and frequency of the optical trap or external applied fields (cf. [2, 14]). Therefore quenches with generic time dependent interaction strength are potentially accesible by experiment. This can be of great interest for example in the case where the effective interaction changes sign and exhibits a transition from the repulsive to attractive interaction with bound states in the latter. An easy model for that would be a simply linear time evolution of the interaction strength in the Lieb-Liniger model. The following discussion will be however for a start be focused on the easiest case of two particles and $c > 0$.

Overlap of Bethe states with different c

To calculate the overalp of two Bethe eigenstates $|k_1, k_2\rangle_1$ and $|k_1', k_2'\rangle_2$ to two different Hamiltonian H_1 and H_2 differing in the interaction strength c_1 and c_2 one identity will be particularly useful:

$$2\pi i\ \delta(q_1+q_2)\frac{q_1+q_2+2i\varepsilon}{(q_1+i\varepsilon)(q_2+i\varepsilon)} = (2\pi)^2\prod_{k=1}^{2}\delta(q_k) \quad (2.35)$$

where $\varepsilon \to 0$ is implied. To see this consider for the domain $D:\ x_1 < x_2$ the integral:

$$\sum_P\int_D d^2x\ e^{-i(Pq,x)} = 2\pi i\ \delta(q_1+q_2)\frac{q_1+q_2+2i\varepsilon}{(q_1+i\varepsilon)(q_2+i\varepsilon)} \quad (2.36)$$

where $(q,x) = \sum_{k=1}^{2} q_k x_k$. As the integrand is symmetric one also gets:

$$\sum_P \int_D d^2x \; e^{-i(Pq,x)} = \frac{1}{2} \sum_P \int_{\mathbb{R}^2} d^2x \; e^{-i(Pq,x)} \tag{2.37}$$

$$= (2\pi)^2 \prod_{k=1}^{2} \delta(q_k) \tag{2.38}$$

Therefore the identity is proven. This can be easily extended to many particles.

Now with (2.24) and (2.6) the overlap can be written as:

$$_2\langle k_1' k_2' \mid k_1 k_2 \rangle_1 = \frac{1}{4} \frac{1}{(2\pi)^2} \frac{1}{\sqrt{G_{11}(k) G_{22}(k')}} \int_{\mathbb{R}^2} \mathrm{d}^2x \; \sum_{P,Q} {}_sA_1^*(P,k)\, {}_sA_2(P,k') e^{-i(x,Pk-Qk')} \tag{2.39}$$

$$= \frac{1}{2} \frac{1}{(2\pi)^2} \frac{1}{\sqrt{G_{11}(k) G_{22}(k')}} \underbrace{\int_D \mathrm{d}^2x \; \sum_{P,Q} A_1^*(P,k) A_2(P,k') e^{-i(x,Pk-Qk')}}_{\equiv I} \tag{2.40}$$

with

$$G_{nm}(k) == 1 + \frac{c_n c_m}{(k_1-k_2)^2} = 1 - \frac{\gamma_n \gamma_m}{(k_1-k_2)^2} \tag{2.41}$$

$$_sA_n(P,k) = 1 + \frac{\gamma_n \; \mathrm{sgn}(x_2 - x_1)}{k_{P1} - k_{P2}} \tag{2.42}$$

$$A_n(P,k) = 1 + \frac{\gamma_n}{k_{P1} - k_{P2}} \tag{2.43}$$

Therefore one gets:

$$\begin{aligned} I =& \int_D \mathrm{d}^2x \; \sum_{P,Q} A_1^*(P,k) A_2(P,k') e^{-i(x,Pk-Qk')} \\ =& 2\pi i \; \delta(k_1 + k_2 - k_1' - k_2') \sum_{P,Q} A_1^*(P,k) A_2(P,k') \frac{1}{k_{P1} - k_{Q1}' + i\varepsilon} \\ =& 2\pi i \; \delta(k_1 + k_2 - k_1' - k_2') \left[\frac{k_1 + k_2 - k_1' - k_2' + 2i\varepsilon}{(k_1 - k_1' + i\varepsilon)(k_2 - k_2' + i\varepsilon)} \left(1 - \frac{\gamma_1 \gamma_2}{(k_1-k_2)(k_1'-k_2')} \right) \right. \\ &+ \frac{[\gamma_1(k_1'-k_2') - \gamma_2(k_1-k_2)]\,(k_1 - k_2 - k_1' + k_2')}{(k_1 - k_1' + i\varepsilon)(k_2 - k_2' + i\varepsilon)(k_1'-k_2')(k_1-k_2)} + \frac{k_1 + k_2 - k_1' - k_2' + 2i\varepsilon}{(k_1 - k_2' + i\varepsilon)(k_2 - k_1' + i\varepsilon)} \\ &\times \left. \left(1 + \frac{\gamma_1\gamma_2}{(k_1-k_2)(k_1'-k_2')} \right) + \frac{[\gamma_1(k_1'-k_2') + \gamma_2(k_1-k_2)]\,(k_1 - k_2 + k_1' - k_2')}{(k_1 - k_1' + i\varepsilon)(k_2 - k_2' + i\varepsilon)(k_1'-k_2')(k_1-k_2)} \right] \end{aligned} \tag{2.44}$$

Using the identity this becomes:

$$
\begin{aligned}
I =&(2\pi)^2 G_{12}(k) \sum_P \delta\left(k_1 - k'_{P1}\right) \delta\left(k_2 - k'_{P2}\right) + \\
&2\pi i \ \delta(k_1 + k_2 - k'_1 - k'_2) \left[\frac{\left[\gamma_1(k'_1 - k'_2) - \gamma_2(k_1 - k_2)\right](k_1 - k_2 - k'_1 + k'_2)}{(k_1 - k'_1 + i\varepsilon)(k_2 - k'_2 + i\varepsilon)(k'_1 - k'_2)(k_1 - k_2)} + \right. \\
&\left. \frac{\left[\gamma_1(k'_1 - k'_2) + \gamma_2(k_1 - k_2)\right](k_1 - k_2 + k'_1 - k'_2)}{(k_1 - k'_1 + i\varepsilon)(k_2 - k'_2 + i\varepsilon)(k'_1 - k'_2)(k_1 - k_2)} \right] \\
=&(2\pi)^2 G_{12}(k) \sum_P \delta\left(k_1 - k'_{P1}\right) \delta\left(k_2 - k'_{P2}\right) + I_1
\end{aligned}
\tag{2.45}
$$

where the second part of the sum is defined to be I_1. This can be further simplified:

$$
\begin{aligned}
I_1 =& 2\pi i \ \delta(k_1 + k_2 - k'_1 - k'_2) \left[\frac{\gamma_1 - \gamma_2}{k_1 - k_2} \left(\frac{k_1 + k_2 - k'_1 - k'_2 + 2i\varepsilon}{(k_1 - k'_1 + i\varepsilon)(k_2 - k'_2 + i\varepsilon)} + \right. \right. \\
&\left. \frac{k_1 + k_2 - k'_1 - k'_2 + 2i\varepsilon}{(k_1 - k'_2 + i\varepsilon)(k_2 - k'_1 + i\varepsilon)} \right) + \frac{2(\gamma_1 - \gamma_2)}{k_1 - k_2} \left(\frac{k'_2 - k_2 - i\varepsilon}{(k_1 - k'_1 + i\varepsilon)(k_2 - k'_2 + i\varepsilon)} + \right. \\
&\left. \left. \frac{k'_1 - k_2 - i\varepsilon}{(k_1 - k'_2 + i\varepsilon)(k_2 - k'_1 + i\varepsilon)} \right) \right] \\
=& (2\pi)^2 \frac{\gamma_1 - \gamma_2}{k_1 - k_2} \sum_P \delta\left(k_1 - k'_{P1}\right) \delta\left(k_2 - k'_{P2}\right) - \\
& 2\pi i \frac{2(\gamma_1 - \gamma_2)}{k_1 - k_2} \left(\frac{1}{k_1 - k'_1 + i\varepsilon} + \frac{1}{k_1 - k'_2 + i\varepsilon} \right) \delta(k_1 + k_2 - k'_1 - k'_2)
\end{aligned}
\tag{2.46}
$$

Therefore one gets for the overlap of two Bethe functions to different interaction strengths:

$$
\begin{aligned}
{}_2\langle k'_1 k'_2 \mid k_1 k_2 \rangle_1 =& \frac{\left(1 + \frac{\gamma_1}{k_1 - k_2}\right)\left(1 - \frac{\gamma_2}{k_1 - k_2}\right)}{\sqrt{G_{11}(k) G_{22}(k)}} \frac{1}{2} \sum_P \delta(k_1 - k'_{P1}) \delta(k_2 - k'_{P2}) \\
& - \frac{i}{2\pi} \delta(k_1 + k_2 - k'_1 - k'_2) \frac{\gamma_1 - \gamma_2}{\sqrt{G_{11}(k) G_{22}(k')}} \frac{1}{(k_1 - k'_1 + i\varepsilon)(k_1 - k'_2 + i\varepsilon)}
\end{aligned}
\tag{2.47}
$$

As both Bethe eigenstates to the two different Hamiltonians are a basis to the same Hilbert space and form a complete set, the completeness relation

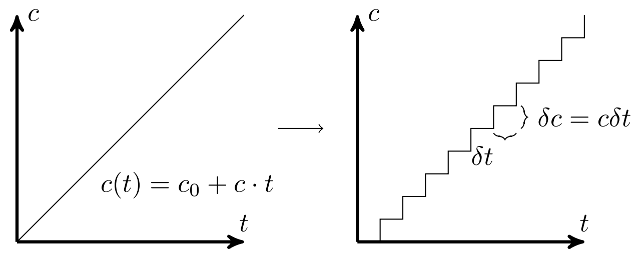
$$
\mathbb{1} = \int \mathrm{d}^2 k \ \ |k_1 k_2\rangle_c \ {}_c\langle k_1 k_2| \tag{2.48}
$$

can be used. This can also be shown explicitly

$$
\begin{aligned}
&\int \mathrm{d}^2k \;\; {}_2\langle k_1'k_2' \mid k_1k_2\rangle_1 \;\; {}_1\langle k_1k_2 \mid k_1''k_2''\rangle_2 \\
=&\int \mathrm{d}^2k \;\; \left[\frac{\left(1+\frac{\gamma_1}{k_1-k_2}\right)\left(1-\frac{\gamma_2}{k_1-k_2}\right)}{\sqrt{G_{11}(k)G_{22}(k)}}\frac{1}{2}\sum_P \delta(k_1-k_{P1}')\delta(k_2-k_{P2}')\right. \\
&\left.-\frac{i}{2\pi}\delta(k_1+k_2-k_1'-k_2')\frac{\gamma_1-\gamma_2}{\sqrt{G_{11}(k)G_{22}(k')}}\frac{1}{(k_1-k_1'+i\varepsilon)(k_1-k_2'+i\varepsilon)}\right] \\
&\left[\frac{\left(1+\frac{\gamma_1}{k_1''-k_2''}\right)\left(1-\frac{\gamma_2}{k_1''-k_2''}\right)}{\sqrt{G_{11}(k'')G_{22}(k'')}}\frac{1}{2}\sum_P \delta(k_1-k_{P1}'')\delta(k_2-k_{P2}'')\right. \\
&\left.-\frac{i}{2\pi}\delta(k_1+k_2-k_1''-k_2'')\frac{\gamma_1-\gamma_2}{\sqrt{G_{11}(k)G_{22}(k'')}}\frac{1}{(k_1''-k_1+i\varepsilon)(k_1''-k_2+i\varepsilon)}\right] \\
=&\frac{1}{4}\left[1+\frac{\left(1+\frac{\gamma_2}{k_1''-k_2''}\right)^2\left(1-\frac{\gamma_1}{k_1''-k_2''}\right)^2}{G_{11}(k'')G_{22}(k'')}\right]\sum_P \delta(k_1'-k_{P1}'')\delta(k_2'-k_{P2}'') \\
&-\frac{i}{2\pi}\frac{\gamma_1-\gamma_2}{\sqrt{G_{22}(k'')G_{22}(k')}}\delta(k_1'+k_2'-k_1''-k_2'')\frac{\left(1+\frac{\gamma_2}{k_1''-k_2''}\right)\left(1-\frac{\gamma_1}{k_1''-k_2''}\right)}{G_{11}(k'')(k_1''-k_1'+i\varepsilon)(k_1''-k_2'+i\varepsilon)} \\
&+\frac{\gamma_1-\gamma_2}{G_{22}(k'')G_{22}(k')}\frac{\left(1+\frac{\gamma_2}{k_1''-k_2''}\right)\left(1-\frac{\gamma_1}{k_1''-k_2''}\right)}{(k_1''-k_2'')}\frac{1}{2}\sum_P \delta(k_1'-k_{P1}'')\delta(k_2'-k_{P2}'') \\
&+\frac{i}{2\pi}\frac{\gamma_1-\gamma_2}{G_{22}(k'')G_{22}(k')}\delta(k_1'+k_2'-k_1''-k_2'')\frac{1-\frac{\gamma_1\gamma_2}{(k_1'-k_2')^2)}}{G_{11}(k'')}\frac{1}{(k_1''-k_1'+i\varepsilon)(k_1''-k_2'+i\varepsilon)} \\
&-\frac{i}{2\pi}\frac{(\gamma_1-\gamma_2)^2}{G_{22}(k'')G_{22}(k')}\delta(k_1'+k_2'-k_1''-k_2'')\left[\frac{4\gamma_1}{G_{11}(k')G_{11}(k'')(k_1'-k_2')^2(k_1''-k_2'')^2}+\right. \\
&\left.\frac{1}{G_{11}(k'')}\frac{1}{(k_1''-k_1'+i\varepsilon)(k_1''-k_2'+i\varepsilon)}\frac{1}{k_1''-k_2''}\right] \\
=&\frac{1}{2}\sum_P \delta(k_1'-k_{P1}'')\delta(k_2'-k_{P2}'') = {}_2\langle k_1'k_2' \mid k_1''k_2''\rangle_2
\end{aligned}
\tag{2.49}
$$

Time evolution operator

To compute the time evolution of the time dependent Hamilton operator $H(t)$ with time dependent interaction strength $c(t) = c_0 + c \cdot t$ one can divide the time into infinitessimal time slices, during which the interaction strength c_n, where the subscript n means the interaction strength in the nth time slice, is assumed constant.



Therefore the time evolution operator $\hat{U}(t)$: $|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle$ can be written as:

$$\hat{U}(t) = e^{-iH_N\delta t} \cdots e^{-iH_1\delta t} \tag{2.50}$$

where the time was divided into N time slices with $N \to \infty$, $\delta t \to 0$ and $N\delta t \to t$. H_n is the Hamilton operator with interaction strength c_n. For the linear time evolution the interaction in the nth time slice c_n has to be $c_n = c_0 + n\, c\, \delta t$.

With the eigenfunctions for the H_n being the Bethe wave functions to interaction strength c_n one can insert the completeness relation (2.48) with appropriate interaction strenth. Therefore one gets:

$$e^{-iH_1t_1} \cdots e^{-iH_Nt_N} = \int_{k^{(1)}} \int_{k^{(N)}} U(k^{(N)}, k^{(1)}) \left| k_1^{(N)} k_2^{(N)} \right\rangle_{N\;1}\!\left\langle k_1^{(1)} k_2^{(1)} \right| \tag{2.51}$$

with

$$\begin{aligned} U(k^{(N)}, k^{(1)}) = \int_{k^{(2)}} \cdots \int_{k^{(N-1)}} \prod_{n=1}^{N-1} &\left[\frac{\left(1 + \frac{\gamma_n}{k_1^{(n)} - k_2^{(n)}}\right)\left(1 - \frac{\gamma_{n+1}}{k_1^{(n)} - k_2^{(n)}}\right)}{\sqrt{G_{n,n}(k) G_{n+1,n+1}(k)}} \right. \\ &\times \frac{1}{2} \sum_P \delta(k_1^{(n)} - k_{P1}^{(n+1)}) \delta(k_2^{(n)} - k_{P2}^{(n+1)}) - \frac{i}{2\pi} \delta(k_1^{(n)} + k_2^{(n)} - k_1^{(n+1)} - k_2^{(n+1)}) \\ &\times \left. \frac{\gamma_n - \gamma_{n+1}}{\sqrt{G_{n,n}(k^{(n)}) G_{n+1,n+1}(k^{(n+1)})}} \frac{1}{(k_1^{(n)} - k_1^{(n+1)} + i\varepsilon)(k_1^{(n)} - k_2^{(n+1)} + i\varepsilon)} \right] \\ &\times e^{-i(k_1^{(1)\,2} + k_2^{(1)\,2})\delta t} \cdots e^{-i(k_1^{(N)\,2} + k_2^{(N)\,2})\delta t} \end{aligned} \tag{2.52}$$

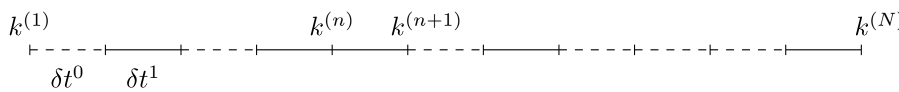
and with $\left| k_1^{(N)} k_2^{(N)} \right\rangle_N$ being the eigenstate to $H(t)$ at time t with $c(t) = c_0 + ct$ and ${}_1\!\left\langle k_1^{(1)} k_2^{(1)} \right|$ the adjoint eigenstate to $H(t)$ at time $t = 0$ with $c(t = 0) = c_0$.

Plugging in the c_n and expanding every factor of the product to order δt one gets:

$$U(k^{(1)}, k^{(N)}) = \int_{k^{(2)}} \cdots \int_{k^{(N-1)}} \prod_{n=1}^{N-1} \Bigg\{ \frac{1}{2} \sum_P \delta(k_1^{(n)} - k_{P1}^{(n+1)}) \delta(k_2^{(n+1)} - k_{P2}^{(n+1)})$$
$$- \delta t \left[i \left(k_1^{(n)\,2} + k_2^{(n)\,2} + \frac{c}{k_1^{(n)} - k_2^{(n)}} \right) \frac{1}{2} \sum_P \delta(k_1^{(n)} - k_{P1}^{(n+1)}) \delta(k_2^{(n)} - k_{P2}^{(n+1)}) \right.$$
$$\left. + \frac{c}{2\pi} \frac{\delta\left(k_1^{(n)} + k_2^{(n)} - k_1^{(n+1)} - k_2^{(n+1)}\right)}{(k_1^{(n)} - k_1^{(n+1)} + i\varepsilon)(k_1^{(n)} - k_2^{(n+1)} + i\varepsilon)} \right] \Bigg\} \tag{2.53}$$

Expanding every factor to first order in δt is sufficient as in a product of N factors $(a + b\delta t + c\delta t^2)^N$ only a and b are important when taking $N \to \infty$ and $\delta t \to 0$. However this expansion is only sensible if the integrals still converge. As will be seen later the expansion of the exponential in (2.52) can only be done up to a certain order in t, otherwise one has to be more careful with the expansion.

The product in (2.53) can be seen as there being $N-1$ places, where each place can either be a δt^0 or a δt^1 insertion connecting $k^{(n)}$ and $k^{(n+1)}$



where represents the summand not proportional to δt in the product of (2.53) and the summand proportional to δt. There is furthermore an integral over every node between two lines.

It is now easy to show that

and

as:

$$\int_{k^{(n)}} \frac{1}{2}\sum_P \delta(k_1^{(n-1)} - k_{P1}^{(n)})\delta(k_2^{(n-1)} - k_{P2}^{(n)}) \left[i\left(k_1^{(n)\,2} + k_2^{(n)\,2} + \frac{c}{k_1^{(n)} - k_2^{(n)}}\right)\right.$$
$$\left. \frac{1}{2}\sum_P \delta(k_1^{(n)} - k_{P1}^{(n+1)})\delta(k_2^{(n)} - k_{P2}^{(n+1)}) + \frac{c}{2\pi}\frac{\delta\left(k_1^{(n)} + k_2^{(n)} - k_1^{(n+1)} - k_2^{(n+1)}\right)}{(k_1^{(n)} - k_1^{(n+1)} + i\varepsilon)(k_1^{(n)} - k_2^{(n+1)} + i\varepsilon)}\right]$$
$$= \left[i\left(k_1^{(n-1)\,2} + k_2^{(n-1)\,2} + \frac{c}{k_1^{(n-1)} - k_2^{(n-1)}}\right)\right.$$
$$\left. \frac{1}{2}\sum_P \delta(k_1^{(n-1)} - k_{P1}^{(n+1)})\delta(k_2^{(n-1)} - k_{P2}^{(n+1)}) + \frac{c}{2\pi}\frac{\delta\left(k_1^{(n-1)} + k_2^{(n-1)} - k_1^{(n+1)} - k_2^{(n+1)}\right)}{(k_1^{(n-1)} - k_1^{(n+1)} + i\varepsilon)(k_1^{(n-1)} - k_2^{(n+1)} + i\varepsilon)}\right] \tag{2.54}$$

and the other way round.

Therefore U can be written as:

$$U(k^{(N)}, k^{(1)}) = \frac{1}{2}\sum_P \delta(k_1^{(1)} - k_{P1}^{(N)})\delta(k_2^{(1)} - k_{P2}^{(N)}) +$$
$$\sum_{m=1}^{N-1} (-\delta t)^m \binom{N-1}{m} \prod_{n=1}^{m} \int_{k^{(n)}} \left[i\left(k_1^{(n)\,2} + k_2^{(n)\,2} + \frac{c}{k_1^{(n)} - k_2^{(n)}}\right)\right.$$
$$\left. \frac{1}{2}\sum_P \delta(k_1^{(n)} - k_{P1}^{(n+1)})\delta(k_2^{(n)} - k_{P2}^{(n+1)}) + \frac{c}{2\pi}\frac{\delta\left(k_1^{(n)} + k_2^{(n)} - k_1^{(n+1)} - k_2^{(n+1)}\right)}{(k_1^{(n)} - k_1^{(n+1)} + i\varepsilon)(k_1^{(n)} - k_2^{(n+1)} + i\varepsilon)}\right] \tag{2.55}$$

with $k^{(m+1)} = k^{(N)}$. For $m = 1$ to $m = 3$ this can be integrated and one gets:

$$U(k^{(N)}, k^{(1)}) = \sum_{m=0}^{3} (-\delta t)^m \binom{N-1}{m} \left\{ \left[i\left(k_1^{(1)\,2} + k_2^{(1)\,2} - \frac{c}{k_1^{(1)} - k_2^{(1)}}\right)\right]^m \right.$$
$$\times \frac{1}{2}\sum_P \delta(k_1^{(1)} - k_{P1}^{(N)})\delta(k_2^{(1)} - k_{P2}^{(N)})$$
$$\left. + \frac{c}{2\pi}\frac{A_m(k^{(1)}, k^{(N)})\delta\left(k_1^{(1)} + k_2^{(1)} - k_1^{(N)} - k_2^{(N)}\right)}{(k_1^{(1)} - k_1^{(N)} + i\varepsilon)(k_1^{(1)} - k_2^{(N)} + i\varepsilon)} \right\} \tag{2.56}$$

with the $A_m(k^{(1)}, k^{(N)})$ satisfying the recursive relation:

$$A_{m+1}(k^{(1)}, k^{(m+1)}) = iA_m(k^{(1)}, k^{(m+1)})\left(k_1^{(m+1)\;2} + k_2^{(m+1)\;2)}\right)$$
$$+ \left[i\left(k_1^{(1)\,2} + k_2^{(1)\,2} - \frac{c}{k_1^{(1)} - k_2^{(1)}}\right)\right]^{m-1} + A_m(k^{(1)}, k^{(1)})\frac{ic}{k_1^{(1)} - k_2^{(1)}} \tag{2.57}$$

For the first three $A_m(k^{(1)}, k^{(m+1)})$ one gets:

$$A_0(k^{(1)}, k) = 0 \tag{2.58}$$

$$A_1(k^{(1)}, k) = 1 \tag{2.59}$$

$$A_2(k^{(1)}, k) = i\left(k_1^{(1)\,2} + k_2^{(1)\,2} + k_1^2 + k_2^2\right) \tag{2.60}$$

$$\begin{aligned} A_3(k^{(1)}, k) =& -\left(k_1^{(1)\,2} + k_2^{(1)\,2} + k_1^2 + k_2^2\right)\left(k_1^2 + k_2^2\right) \\ & - \left(k_1^{(1)\,2} + k_2^{(1)\,2}\right)^2 - \frac{c^2}{\left(k_1^{(1)} - k_2^{(1)}\right)^2} \end{aligned} \tag{2.61}$$

Taking the limit $N \to \infty$ and $\delta t \to 0$ one gets with

$$\binom{N-1}{m} \xrightarrow[N\to\infty]{} N^m \left(\frac{1}{m!} + \frac{m^2 - 3m}{m! N} + \mathcal{O}\left(\frac{1}{N^2}\right)\right) \tag{2.62}$$

the time evolution operator to order t^3:

$$\begin{aligned} U(k^{(N)}, k^{(1)}) =& \sum_{m=0}^{3} \frac{1}{m!}\left[-i\left(k_1^{(1)\,2} + k_2^{(1)\,2} - \frac{c}{k_1^{(1)} - k_2^{(1)}}\right)t\right]^m \\ & \times \frac{1}{2}\sum_P \delta(k_1^{(1)} - k_{P1}^{(N)})\delta(k_2^{(1)} - k_{P2}^{(N)}) \\ & + \frac{c}{2\pi} \frac{\delta\left(k_1^{(1)} + k_2^{(1)} - k_1^{(N)} - k_2^{(N)}\right)}{(k_1^{(1)} - k_1^{(N)} + i\varepsilon)(k_1^{(1)} - k_2^{(N)} + i\varepsilon)} \sum_{m=0}^{3} \frac{(-t)^m}{m!} A_m(k^{(1)}, k^{(N)}) + \mathcal{O}\left(t^4\right) \end{aligned} \tag{2.63}$$

There is a problem however for 4 or more insertions of δt, as in the formula for U for e.g. $m = 4$ there is a divergent integral. To see this one can take U for $m = 3$ and add another insertion of δt. One part of the integral reads:

$$\begin{aligned} \frac{c^2}{(2\pi)^2} \int d^2k^{(3)} & \frac{A_3(k^{(1)}, k^{(3)})\delta\left(k_1^{(1)} + k_2^{(1)} - k_1^{(3)} - k_2^{(3)}\right)}{(k_1^{(1)} - k_1^{(3)} + i\varepsilon)(k_1^{(1)} - k_2^{(3)} + i\varepsilon)} \\ & \times \frac{\delta\left(k_1^{(3)} + k_2^{(3)} - k_1^{(4)} - k_2^{(4)}\right)}{(k_1^{(3)} - k_1^{(4)} + i\varepsilon)(k_1^{(3)} - k_2^{(4)} + i\varepsilon)} \end{aligned} \tag{2.64}$$

which is divergent (compare A_3 in the above table). Therefore one has to be more careful with the expansion of the exponential in (2.52). This can also be seen as the energy of the continuous Lieb-Liniger gas is not bounded and therefore an expansion of $\exp\left(-iE\delta t\right)$ is not valid.

Up to order t^3 however it can be shown with the time-dependent Schrödinger equation that the time evolution operator is correct. As an example the term of order t^0 with $c_0 = 0$ will be shown to satisfy the time dependent SE:

$$\frac{\partial}{\partial t}\int_{k^{(1)}}\int_{k^{(N)}} U(k^{(N)},k^{(1)})\left|k_1^{(N)}k_2^{(N)}\right\rangle_{N\,1}\left\langle k_1^{(1)}k_2^{(1)}\right| = \\ -iH(t)\int_{k^{(1)}}\int_{k^{(N)}} U(k^{(N)},k^{(1)})\left|k_1^{(N)}k_2^{(N)}\right\rangle_{N\,1}\left\langle k_1^{(1)}k_2^{(1)}\right| \quad (2.65)$$

where here $\left|k_1^{(N)}k_2^{(N)}\right\rangle_N$ is the eigenstate to $H(t)$ at time t and ${}_1\left\langle k_1^{(1)}k_2^{(1)}\right|$ the adjoint eigenstate to $H(t)$ at time $t=0$. With

$$\begin{aligned} \left|k_1k_2\right\rangle_c &= \int_{\mathbb{R}^2}\mathrm{d}^2x\ \left|x_1x_2\right\rangle\left\langle x_1x_2\mid k_1k_2\right\rangle_c \\ &= \int_D \mathrm{d}^2x\ \sqrt{2}\left\langle x_1x_2\mid k_1k_2\right\rangle_c b^\dagger(x_1)b^\dagger(x_2)\left|0\right\rangle \\ &= \int_D \mathrm{d}^2x\ \frac{1}{\sqrt{2}}\frac{1}{2\pi}\frac{1}{\sqrt{G_c(k)}}\sum_P A_c(P)e^{-i(Pk,x)}b^\dagger(x_1)b^\dagger(x_2)\left|0\right\rangle \end{aligned} \quad (2.66)$$

and

$$H(t)\left|k_1^{(N)}k_2^{(N)}\right\rangle_N = k_1^{(N)\,2}+k_2^{(N)\,2} \quad (2.67)$$

To order t^0 one gets for the derivative on the time dependent parts of the evolution operator:

$$\begin{aligned} \frac{\partial}{\partial t}U(k^{(1)},k^{(N)}) =& -i\left(k_1^{(1)\,2}+k_2^{(1)\,2}+\frac{c}{k_1^{(1)}-k_2^{(1)}}\right)\frac{1}{2}\sum_P\delta(k_1^{(1)}-k_{P1}^{(N)})\delta(k_2^{(1)}-k_{P2}^{(N)}) \\ &-\frac{c}{2\pi}\frac{\delta(k_1^{(1)}+k_2^{(1)}-k_1^{(N)}-k_2^{(N)})}{(k_1^{(1)}+k_1^{(N)}+i\varepsilon)(k_1^{(1)}+k_2^{(N)}+i\varepsilon)} \end{aligned} \quad (2.68)$$

$$\frac{\partial}{\partial t}\left|k_1^{(N)}k_2^{(N)}\right\rangle_N = -\int_D \mathrm{d}^2x\ \frac{1}{\sqrt{2}}\frac{1}{2\pi}\sum_P\frac{ic}{k_{P1}^{(N)}-k_{P2}^{(N)}}e^{-i(Pk^{(N)},x)}b^\dagger(x_1)b^\dagger(x_2)\left|0\right\rangle \quad (2.69)$$

With that the left side of (2.65) becomes

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_{k^{(1)}}\int_{k^{(N)}} U(k^{(N)},k^{(1)})\,\left|k_1^{(N)}k_2^{(N)}\right\rangle_{N\,1}\left\langle k_1^{(1)}k_2^{(1)}\right| = \\
&\quad\int_{k^{(1)}}\int_{k^{(N)}}\left\{\frac{1}{\sqrt{2}}\frac{1}{2\pi}\int_D \mathrm{d}^2x\;\sum_P e^{-i(Pk^{N},x)}b^\dagger(x_1)b^\dagger(x_2)\left|0\right\rangle\;{}_1\!\left\langle k_1^{(1)}k_1^{(1)}\right|\left[-i\left(k_1^{(1)\,2}+k_2^{(1)\,2}+\right.\right.\right.\\
&\quad\left.\left.\frac{c}{k_1^{(1)}-k_2^{(1)}}\right)\frac{1}{2}\sum_P\delta(k_1^{(1)}-k_{P1}^{(N)})\delta(k_2^{(1)}-k_{P2}^{(N)})-\frac{c}{2\pi}\frac{\delta(k_1^{(1)}+k_2^{(1)}-k_1^{(N)}-k_2^{(N)})}{(k_1^{(1)}+k_1^{(N)}+i\varepsilon)(k_1^{(1)}+k_2^{(N)}+i\varepsilon)}\right]\\
&\quad-\int_D\mathrm{d}^2x\;\frac{1}{\sqrt{2}}\frac{1}{2\pi}\sum_P\frac{ic}{k_{P1}^{(N)}-k_{P2}^{(N)}}e^{-i(Pk^{(N)},x)}b^\dagger(x_1)b^\dagger(x_2)\left|0\right\rangle\;{}_1\!\left\langle k_1^{(1)}k_1^{(1)}\right|\\
&\qquad\qquad\frac{1}{2}\sum_P\delta(k_1^{(1)}-k_{P1}^{(N)})\delta(k_2^{(1)}-k_{P2}^{(N)})\\
&=-\,iH(t)\int_{k^{(1)}}\int_{k^{(N)}} U(k^{(N)},k^{(1)})\,\left|k_1^{(N)}k_2^{(N)}\right\rangle_{N\,1}\left\langle k_1^{(1)}k_2^{(1)}\right|\\
&\quad+\int_{k^{(1)}}\int_D\mathrm{d}^2x\;\frac{1}{\sqrt{2}}\frac{1}{2\pi}\left[-\frac{ic}{k_1^{(1)}-k_2^{(1)}}\left(e^{-i\left(k_1^{(1)}x_1+k_2^{(1)}x_2\right)}+e^{-i\left(k_1^{(1)}x_2+k_2^{(1)}x_1\right)}\right)-\right.\\
&\quad\left.\int_{k^{(N)}}\frac{ic}{k_1^{(N)}-k_2^{(N)}}\left(e^{-i\left(k_1^{(N)}x_1+k_2^{(N)}x_2\right)}+e^{-i\left(k_1^{(N)}x_2+k_2^{(N)}x_1\right)}\right)\frac{1}{2}\sum_P\delta(k_1^{(1)}-k_{P1}^{(N)})\delta(k_2^{(1)}-k_{P2}^{(N)})\right]\\
&\quad b^\dagger(x_1)b^\dagger(x_2)\left|0\right\rangle\;{}_1\!\left\langle k_1^{(1)}k_1^{(1)}\right|-\int_{k^{(1)}}\int_D\mathrm{d}^2x\;\int_{k^{(N)}}\frac{1}{\sqrt{2}}\frac{1}{2\pi}\left(e^{-i\left(k_1^{(N)}x_1+k_2^{(N)}x_2\right)}+\right.\\
&\quad\left.e^{-i\left(k_1^{(N)}x_2+k_2^{(N)}x_1\right)}\right)b^\dagger(x_1)b^\dagger(x_2)\left|0\right\rangle\;{}_1\!\left\langle k_1^{(1)}k_1^{(1)}\right|\frac{c}{2\pi}\frac{\delta(k_1^{(1)}+k_2^{(1)}-k_1^{(N)}-k_2^{(N)})}{(k_1^{(1)}+k_1^{(N)}+i\varepsilon)(k_1^{(1)}+k_2^{(N)}+i\varepsilon)}\\
&=-\,iH(t)\int_{k^{(1)}}\int_{k^{(N)}} U(k^{(N)},k^{(1)})\,\left|k_1^{(N)}k_2^{(N)}\right\rangle_{N\,1}\left\langle k_1^{(1)}k_2^{(1)}\right|
\end{aligned}
\tag{2.70}
$$

Therefore satisfying the time dependent Schrödinger equation in order t^0.

First order perturbation theory

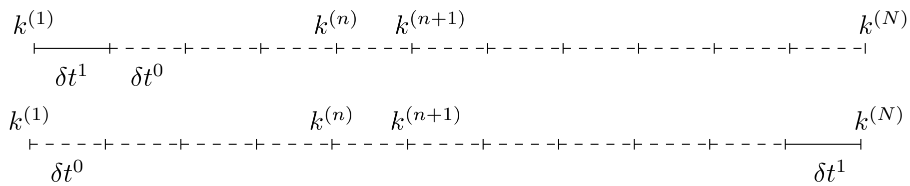
As stated above however an expansion of the exponential in (2.52) is not valid for order t^4 or higher. Therefore the exponentials in (2.52) have to be kept, the products of the overlaps can be expanded in δt again. Analogous to above one gets for $U(k^{(1)},k^{(N)})$:

$$
\begin{aligned}
U(k^{(1)},k^{(N)};t)=\int_{k^{(2)}}\cdots\int_{k^{(N-1)}}\prod_{n=1}^{N-1}&\left[\left(1-\frac{ic\delta t}{k_1^{(n)}-k_2^{(n)}}\right)\frac{1}{2}\sum_P\delta\left(k_1^{(n)}-k_{P1}^{(n+1)}\right)\delta\left(k_2^{(n)}-k_{P2}^{(n+1)}\right)\right.\\
&\left.-\frac{c\delta t}{2\pi}\frac{\delta(k_1^{(n)}+k_2^{(n)}-k_1^{(n+1)}-k_2^{(n+1)})}{(k_1^{(n)}-k_1^{(n+1)}+i\varepsilon)(k_1^{(n)}-k_2^{(n+1)}+i\varepsilon)}\right]e^{-i\left(k_1^{(1)\,2}+k_2^{(1)\,2}+\cdots+k_1^{(N)\,2}+k_2^{(N)\,2}\right)\delta t}
\end{aligned}
\tag{2.71}
$$

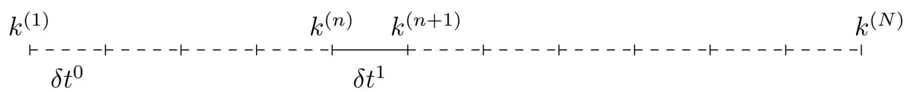
As every δt in the product of overlaps comes in a product with the slope of the interaction strength c every insertion of a δt from the overlaps is equivalent to perturbation theory in c. Therefore one gets for order c^0:

$$\mathcal{O}(c^0): \qquad U(k^{(1)},k^{(N)}) = e^{-i\left(k_1^{(1)\,2}+k_2^{(1)\,2}\right)t}\frac{1}{2}\sum_P \delta\left(k_1^{(1)}-k_{P1}^{(N)}\right)\delta\left(k_2^{(1)}-k_{P2}^{(N)}\right) \tag{2.72}$$

For order c^1 one has one insertion of the summands in the product proportional to c in (2.71). Those insertions can be at the edge of the set of possible $k^{(n)}$ variables, that means they contain $k^{(1)}$ or $k^{(N)}$



or in the middle of this set, only containing $k^{(2)},\ldots,k^{(N-1)}$:



As there are $N-3$ possibilities for insertions in the middle, for $N\to\infty$ the edge terms are suppressed by $\frac{1}{N}$. Therefore one gets:

$$\begin{aligned}
U(k^{(1)},k^{(N)};t) &= \sum_{n=2}^{N-2}\int_{k^{(n)}}\int_{k^{(n+1)}}\frac{1}{2}\sum_P \delta\left(k_1^{(1)}-k_{P1}^{(n)}\right)\delta\left(k_2^{(1)}-k_{P2}^{(n)}\right)\\
&\left[-\frac{ic\delta t}{k_1^{(n)}-k_2^{(n)}}\frac{1}{2}\sum_P \delta\left(k_1^{(n)}-k_{P1}^{(n+1)}\right)\delta\left(k_2^{(n)}-k_{P2}^{(n+1)}\right)\right.\\
&\left.-\frac{c\delta t}{2\pi}\frac{\delta(k_1^{(n)}+k_2^{(n)}-k_1^{(n+1)}-k_2^{(n+1)})}{(k_1^{(n)}-k_1^{(n+1)}+i\varepsilon)(k_1^{(n)}-k_2^{(n+1)}+i\varepsilon)}\right]\frac{1}{2}\sum_P \delta\left(k_1^{(n+1)}-k_{P1}^{(N)}\right)\delta\left(k_2^{(n+1)}-k_{P2}^{(N)}\right)\\
&e^{-in\left(k_1^{(1)\,2}+k_2^{(1)\,2}\right)\delta t}e^{-i(N-n)\left(k_1^{(N)\,2}+k_2^{(N)\,2}\right)\delta t}\\
&= \sum_{n=2}^{N-2}\int_{k^{(n)}}\int_{k^{(n+1)}}\left[-\frac{ic\delta t}{k_1^{(n)}-k_2^{(n)}}\frac{1}{2}\sum_P \delta\left(k_1^{(1)}-k_{P1}^{(N)}\right)\delta\left(k_2^{(1)}-k_{P2}^{(N)}\right)\right.\\
&\left.-\frac{c\delta t}{2\pi}\frac{\delta(k_1^{(1)}+k_2^{(1)}-k_1^{(N)}-k_2^{(N)})}{(k_1^{(1)}-k_1^{(N)}+i\varepsilon)(k_1^{(1)}-k_2^{(N)}+i\varepsilon)}\right]e^{-in\left(k_1^{(1)\,2}+k_2^{(1)\,2}\right)\delta t}e^{-i(N-n)\left(k_1^{(N)\,2}+k_2^{(N)\,2}\right)\delta t}
\end{aligned}$$

and therefore for order c^1:

$$\mathcal{O}(c^1): \qquad U(k^{(1)},k^{(N)};t) = -\frac{ict}{k_1^{(1)}-k_2^{(1)}} e^{-i\left(k_1^{(1)\,2}+k_2^{(1)\,2}\right)t}\,\frac{1}{2}\sum_P \delta\left(k_1^{(1)}-k_{P1}^{(N)}\right) \delta\left(k_2^{(1)}-k_{P2}^{(N)}\right) - \frac{ic}{2\pi}\frac{e^{-i\left(k_1^{(1)\,2}+k_2^{(1)\,2}\right)t}-e^{-i\left(k_1^{(N)\,2}+k_2^{(N)\,2}\right)t}}{k_1^{(1)\,2}+k_2^{(1)\,2}-k_1^{(N)\,2}-k_2^{(N)\,2}} \frac{\delta\left(k_1^{(1)}+k_2^{(N)}-k_1^{(N)}-k_2^{(N)}\right)}{\left(k_1^{(1)}-k_1^{(N)}+i\varepsilon\right)\left(k_1^{(1)}-k_2^{(N)}+i\varepsilon\right)} \tag{2.73}$$

where again the limit $N\to\infty$ and $\delta t\to 0$ was taken. The same result can be obtained by means of time dependent perturbation theory in the parameter c:

Starting e.g. from the state $|k_1k_2\rangle_0$ with $c=0$ at $t=0$ the time evolution operator is obtained with first order perturbation theory to be:

$$U(t) = e^{-iH_0t} - e^{-iH_0t}\left(-i\int_0^t \mathrm{d}t'\; V_I(t')\right) \tag{2.74}$$

where $H_0 = \int_{\mathbb{R}} \mathrm{d}x\;\partial_t b^\dagger(x)\partial_t b(x)$ and $V_I(t) = e^{iH_0t}\int_{\mathbb{R}}\mathrm{d}x\;c\cdot t\;b^\dagger(x)b^\dagger(x)b(x)b(x)e^{-iH_0t}$. After inserting complete sets of states and Fourier transforming the operators as well as inserting $e^{-\varepsilon t}$ for convergence (adiabatic) one gets:

$$\hat{U}(t) = \int \mathrm{d}^2k \int \mathrm{d}^2q\;\; |q_1q_2\rangle_N\;{}_0\langle k_1k_2|\;\; U(k,q) \tag{2.75}$$

with $U(k,q)$ to order c^1 one gets after some calculation:

$$\begin{aligned} U(k,q) =& e^{-i(k_1^2+k_2^2)t}\;{}_N\langle q_1q_2 \mid k_1k_2\rangle_0 - {}_N\langle q_1q_2|\, e^{-iH_0t} ic\int_0^t \mathrm{d}t'\; t' e^{iH_0t'} \\ & \int \frac{\mathrm{d}^4p}{(2\pi)^3}\delta(p_1+p_2-p_3-p_4)b^\dagger(p_1)b^\dagger(p_2)b(p_3)b(p_4)e^{-i(k_1^2+k_2^2)t'}e^{-\varepsilon t'}\;|k_1k_2\rangle_0 \\ =& e^{-i(k_1^2+k_2^2)t}\;{}_N\langle q_1q_2 \mid k_1k_2\rangle_0 - \int \mathrm{d}^2p\;\delta(p_1+p_2-k_1-k_2)\;{}_N\langle q_1q_2 \mid p_1p_2\rangle_0\,\frac{ic}{2\pi} \\ & \left[\frac{t\;e^{-i(k_1^2+k_2^2)t}}{-i(k_1^2+k_2^2-p_1^2-p_2^2)-\varepsilon} - \frac{e^{-i(k_1^2+k_2^2)t}-e^{-i(p_1^2+p_2^2)t}}{\left(-i(k_1^2+k_2^2-p_1^2-p_2^2)-\varepsilon\right)^2}\right] \\ =& \left(1-\frac{ict}{k_1-k_2}\right)e^{-i(k_1^2+k_2^2)t}\frac{1}{2}\sum_P \delta(k_1-q_{P1})\delta(k_2-q_{P2}) \\ & -\frac{ic}{2\pi}\frac{\delta(q_1+q_2-k_1-k_2)}{(k_1-q_1+i\varepsilon)(k_1-q_2+i\varepsilon)}\frac{e^{-i(k_1^2+k_2^2)t}-e^{-i(q_1^2+q_2^2)t}}{k_1^2+k_2^2-q_1^2-q_2^2} \end{aligned} \tag{2.76}$$

and therefore the same result.

The further steps in this problem is trying to get a closed expression for large times and after that generalisation to many particles. A further interesting problem would be to have the transition from negative to positive interaction strength $c(t)$ or the other way around to see if a phase transition from states with bound states to no bound states can be seen.

2.2.2 Time-Periodic interaction strength

Being able to produce a generic time dependent interaction strength in experiments as mentioned above, a further interest lies in changing the interaction strength periodically. With the overlaps developed above starting from the general form of the time evolution operator (2.52) it is also possible to not take an infinitessimal time slice δt but finite times t_n for the interaction c_n. Therefore a time periodic interaction can be described with e.g. $t_n = t_{n \bmod 2}$ and $\gamma_n = ic_n = ic_{n \bmod 2}$ in

$$U(k^{(N)}, k^{(1)}) = \int_{k^{(2)}} \cdots \int_{k^{(N-1)}} \prod_{n=1}^{N-1} \left[\frac{\left(1 + \frac{\gamma_n}{k_1^{(n)} - k_2^{(n)}}\right)\left(1 - \frac{\gamma_{n+1}}{k_1^{(n)} - k_2^{(n)}}\right)}{\sqrt{G_{n,n}(k) G_{n+1,n+1}(k)}} \right.$$
$$\times \frac{1}{2} \sum_P \delta(k_1^{(n)} - k_{P1}^{(n+1)}) \delta(k_2^{(n)} - k_{P2}^{(n+1)}) - \frac{i}{2\pi} \delta(k_1^{(n)} + k_2^{(n)} - k_1^{(n+1)} - k_2^{(n+1)})$$
$$\left. \times \frac{\gamma_n - \gamma_{n+1}}{\sqrt{G_{n,n}(k^{(n)}) G_{n+1,n+1}(k^{(n+1)})}} \frac{1}{(k_1^{(n)} - k_1^{(n+1)} + i\varepsilon)(k_1^{(n)} - k_2^{(n+1)} + i\varepsilon)} \right]$$
$$\times e^{-i(k_1^{(1)\,2} + k_2^{(1)\,2}) t_1} \cdots e^{-i(k_1^{(N)\,2} + k_2^{(N)\,2}) t_N} \tag{2.77}$$

which corresponds to a interaction as depicted in Figure 2.1.

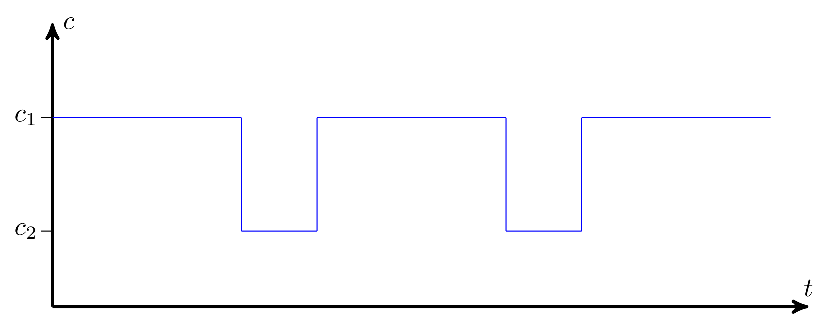


Figure 2.1: time periodic interaction c over time t;

Here it would be interesting to study the behaviour at large times t which is equivalent to large N. It would be worthwhile to see for example how the effective interaction strength compares to the mean interaction strength or if the result corresponds to an infinite temperature independently of the initial state as described in [15]. However no method of approximation for large times or large number of cycles in the periodic interaction could be found yet for this expression of $U(k^{(N)}, k^{(1)})$. Also integrating over $k^{(2)}$ to $k^{(N)}$ and trying to see a series behaviour in N seems to be very hard, as the gaussian factors in the integral makes it more difficult to close the contour and already after the first integration one has to deal with special functions like complementary error functions or Meijer G functions.

Therefore it could be easier to take the Yudson representation and see if any approximations can be done there. In the following the same conventions for the wave function as in [12] are taken. That means the eigenstates of the Lieb-Liniger model are different from the eigenstates above by a phase:

$$|\lambda\rangle = \int_x \prod_{i<j} Z_{ij}^c(x_i, x_j) \prod_j e^{i\lambda_j x_j} b^\dagger(x_j) |0\rangle \tag{2.78}$$

with

$$Z_{ij}^c(x_i, x_j) = \frac{\lambda_i - \lambda_j - ic\,\mathrm{sgn}\,(x_i - x_j)}{\lambda_i - \lambda_j - ic} \tag{2.79}$$

Again for $c > 0$ and two particles one can insert complete states of the form (2.31). Therefore the time evolution operator can again be written as:

$$\begin{aligned}\hat{U}(t) = &\int_{\mu^{(1)}} \cdots \int_{\mu^{(N)}} \left(\mu_1^{(N)} \mu_2^{(N)}\right| e^{-iH(c_{N-1})t_{N-1}} \left|\mu_1^{(N-1)} \mu_2^{(N-1)}\right\rangle_{N-1} \cdots \\ &\left(\mu_1^{(2)} \mu_2^{(2)}\right| e^{-iH(c_1)t_1} \left|\mu_1^{(1)} \mu_2^{(1)}\right\rangle_1 \; e^{-iH(c_N)t} \left|\mu_1^{(N)} \mu_2^{(N)}\right\rangle_N \left(\mu_1^{(1)} \mu_2^{(1)}\right| \\ = &\int_{\mu^{(1)}} \int_{\mu^{(N)}} e^{-iH(c_N)t} \left|\mu_1^{(N)} \mu_2^{(N)}\right\rangle_N \left(\mu_1^{(1)} \mu_2^{(1)}\right| \; U_Y(\mu^{(1)}, \mu^{(N)})\end{aligned} \tag{2.80}$$

Then using (2.78) and the Yudson state (2.32) one can either integrate out the spacial or the rapidity dependence in the matrix elements. Integrating out the rapidities again leads to complementary error functions right from the first integral. Integrating out the spatial dependency one gets for the matrix elements:

$$
\begin{aligned}
(\mu_1\mu_2|e^{-iH(c)t}\,|\lambda_1\lambda_2\rangle_c = e^{-i(\lambda_1^2+\lambda_2^2)t}\int_x \Big[&e^{-i(\lambda_1-\mu_1)x_1+i(\lambda_2-\mu_2)x_2}\theta(x_1-x_2)+\\
&\frac{\lambda_1-\lambda_2+ic}{\lambda_1-\lambda_2-ic}e^{-i(\lambda_1-\mu_2)x_1+i(\lambda_2-\mu_1)x_2}\theta(x_2-x_1)\Big]\\
=&e^{-i(\lambda_1^2+\lambda_2^2)t}2\pi i\;\delta(\lambda_1+\lambda_2-\mu_1-\mu_2)\left[\frac{1}{\lambda_1-\mu_1+i\varepsilon}+\frac{1}{\lambda_2-\mu_1+i\varepsilon}\frac{\lambda_1-\lambda_2+ic}{\lambda_1-\lambda_2-ic}\right]
\end{aligned}
\tag{2.81}
$$

Inserting into (2.80) and integrating out the δ-functions one arrives at the expression:

$$
\begin{aligned}
U_Y(\mu^{(1)},\mu^{(N)}) = &\int \mathrm{d}\mu_1^{(2)}\;\ldots\int \mathrm{d}\mu_1^{(N-1)}\;2\pi i^{N-1}\delta\left(\mu_1^{(N)}+\mu_2^{(N)}-\mu_1^{(1)}-\mu_2^{(1)}\right)\\
&\exp\left(-i\sum_{j=1}^{N-1}\left(\mu_1^{(j)}\right)^2 t_j - i\sum_{j=1}^{N-1}\left(\mu_1^{(j)}-\mu_1^{(N)}-\mu_2^{(N)}\right)^2 t_j\right)\\
&\prod_{i=1}^{N-1}\left[\frac{1}{\mu_1^{(i)}-\mu_1^{(i+1)}+i\varepsilon}+\frac{1}{\mu_1^{(N)}+\mu_2^{(N)}-\mu_1^{(i)}-\mu_1^{(i+1)}+i\varepsilon}\frac{2\mu_1^{(i)}-\mu_1^{(N)}-\mu_2^{(N)}+ic_i}{2\mu_1^{(i)}-\mu_1^{(N)}-\mu_2^{(N)}-ic_i}\right]
\end{aligned}
\tag{2.82}
$$

However this expression doesn't seem to be easier for large t approximations. Evaluating one of the integrals again leads to the special functions.

The goal for the periodic interaction is finding an approximation for large N. After that a generalisation to many particles, periodicity between only negative or between positive and negative interaction strengths would be of interest. Having found an expression for that especially the characteristic function $G(u)$ and a survey of the fluctuation theorems in this problem would be of interest (cf. [16])

2.3 Lieb-Liniger model with external homogenous field

2.3.1 Wave function in external homogenous field

Ultracold atomic experiments are usually conducted in the presence of the gravitational field of the earth. An effort has to be made to cancel it out. However it is also interesting to include this field e.g. in the Lieb-Liniger model as an external linear field. Another interesting application for this model could be bosons interacting with an external homogenous electric field. The first quantized

N-particle Hamiltonian for this model is

$$H = -\sum_i \frac{\partial^2}{\partial x_i^2} + 2c \sum_{1 \le i < j \le N} \delta(x_i - x_j) + \alpha \sum_i x_i \tag{2.83}$$

where $c > 0$ is again the interaction strength and $\alpha > 0$ is the constant force potential. For one particle the delta function is not present and the solution is the usual solution to a particle in a constant force field, which is:

$$\psi_1(x) = \alpha^{-1/3} \mathrm{Ai} \left(\alpha^{1/3} x - \alpha^{-2/3} E \right) \tag{2.84}$$

where E is the energy of the solutions and only the wave function decaying to 0 for $|x| \to \infty$ was taken.

The solution for two and three particles are constructed in [17] by separating into center of mass and relative coordinates for two particles and for three particles separating into different zones, effectively reducing it to two particle problems in those regions.

However a more elegant solution was found in [18] by means of the Gaudin operator introduced in chapter 2.1.2. Utilizing again the bosonic symmetry of the wave function the problem can be considered only in the domain $D : x_1 < \cdots < x_N$. The Schrödinger equation in this domain is

$$E\psi = -\sum_{i=1}^{N} \frac{\partial^2 \psi}{\partial x_i^2} + \alpha \sum_i x_i \psi \tag{2.85}$$

with the boundary condition (cusp condition) analogous to chapter 2.1.2

$$\left[1 - \frac{1}{c} \left(\frac{\partial}{\partial x_{j+1}} - \frac{\partial}{\partial x_j} \right) \right]_{x_{j+1} = x_j} \psi = 0 \tag{2.86}$$

As in chapter 2.1.2 the solution of the Schrödinger equation with Hamilton operator (2.83) is then given by

$$\psi = N_c O_c \psi_F \tag{2.87}$$

where N is the normalization. The Gaudin operator O_c

$$O_c = \prod_{1 \le i < j \le N} \left[\mathrm{sgn}\,(x_j - x_i) + \frac{1}{c} \left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_i} \right) \right] \tag{2.88}$$

is acting on the antisymmetric wave function ψ_F, which is the solution to the Schrödinger equation

$$E\psi_F = -\sum_{i=1}^{N} \frac{\partial^2 \psi_F}{\partial x_i^2} + \alpha \sum_{i=1}^{N} x_i \psi_F \tag{2.89}$$

which can be written in the form of a Slater determinant

$$\psi_F = \alpha^{-N/6} \frac{1}{\sqrt{N!}} \det \left(\mathrm{Ai} \left(\alpha^{1/3} x_j - \alpha^{-2/3} E_i \right) \right) \tag{2.90}$$

with $E = \sum_i E_i$.

The wave function (2.87) can be shown to be the eigenfunction to (2.83) in the same way as in chapter 2.1.2. The proof that it obeys the cusp condition is analogous to chapter 2.1.2, as the operator O_c is equivalent and the wave function ψ_F is also antisymmetric. It furthermore has to be shown that it commutes with the Hamilton operator in the domain D:

$$H_D = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \alpha \sum_{i=1}^{N} x_i \tag{2.91}$$

Having in mind that O_c is acting on an antisymmetric ψ_F the commutator of O_c with $\frac{\partial^2}{\partial x_i^2}$ is easily seen to be satisfied. The commutator

$$\left[\sum_i x_i, O_c \right] = 0 \tag{2.92}$$

is shown in [18]. Therefore the wave function (2.87) satisfies in the same fashion as in chapter 2.1.2 the Schrödinger equation with Hamilton operator (2.83).

It should be mentioned here however that the eigenstates with total energy E are degenerate, as only the total energy E is conserved, not however the singe particle energies E_i. Therefore it is for example not easily possible to get the S-matrix and wave function by means of the usual Bethe Ansatz technique.

2.3.2 Time periodic external homogenous field

Again following [18] the time dependent solutions of the Lieb-Liniger model in an external linear potential can be constructed in the following way. Starting from the wave function at $\alpha = 0$, which is the time dependent solution of the Lieb-Liniger model

$$i \frac{\partial \psi_{\alpha=0}}{\partial t} = -\sum_i \frac{\partial^2 \psi_{\alpha=0}}{\partial x_i^2} + 2c \sum_{1 \le i < j \le N} \delta(x_i - x_j) \psi_{\alpha=0} \tag{2.93}$$

one can get the wave function for $\alpha \neq 0$ by:

$$\psi(x_1, \dots, x_n; t) = e^{-i\alpha t \sum_{i=1}^{N}(x_i + at^2/3)} \psi_{\alpha=0}(x_1 + \alpha t^2, \dots, x_N + \alpha t^2; t) \tag{2.94}$$

This can be easily seen to be the right wave function by plugging it into the time dependent Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = -\sum_i \frac{\partial^2 \psi}{\partial x_i^2} + 2c \sum_{1 \leq i < j \leq N} \delta(x_i - x_j)\psi + \alpha \sum_i x_i \psi \tag{2.95}$$

At $t = 0$ the initial conditions coincide. The solution of the Schrödinger equation with $\alpha = 0$ can be calculated as in [12] and one gets:

$$\psi_{\alpha=0} = \int \frac{\mathrm{d}^N \lambda}{(2\pi)^N} G(\lambda_1, \dots, \lambda_N) e^{i\sum_{i=1}^{N}(\lambda_i x_i - \lambda_i^2 t)} \tag{2.96}$$

where $G(\lambda_1, \dots, \lambda_N)$ can be found by comparing with the expression from [12]:

$$\begin{aligned} |\psi_{\alpha=0}(t)\rangle =& e^{-iHt} \int_y \phi(y) \prod_i b^\dagger(y_i) \, |0\rangle \\ =& N! \int_y \theta(\mathbf{y})\phi(y) \int_\lambda e^{-i\sum_j \lambda_j^2 t} \prod_j e^{-i\lambda_j y_j} \int_x \prod_{i<j} Z_{ij}^x(\lambda_i - \lambda_j) \prod_j e^{i\lambda_j x_j} b^\dagger(x_j) \, |0\rangle \\ =& \int_x \int_\lambda \left[\theta(\mathbf{y})\phi(y) N! \prod_k e^{-i\lambda_k y_k} \prod_{i<j} Z_{ij}^x(\lambda_i - \lambda_j) \right] e^{i\sum_{i=1}^{N}(\lambda_i x_i - \lambda_i^2 t)} \prod_j b^\dagger(x_j) \end{aligned} \tag{2.97}$$

with $\phi(y)$ being the initial condtions and $\theta(\mathbf{y}) = \theta(y_1 < y_2 < \cdots < y_N)$. Therefore one gets:

$$G(\lambda_1, \dots, \lambda_N) = \theta(\mathbf{y})\phi(y) N! \prod_k e^{-i\lambda_k y_k} \prod_{i<j} Z_{ij}^x(\lambda_i - \lambda_j) \tag{2.98}$$

Because of (2.94) it is now sufficient for the time evolution with $\alpha \neq 0$ to make the substitution

$$\exp\left(i \sum_{i=1}^{N} (\lambda_i x_i - \lambda_i^2 t) \right) \quad \rightarrow \quad \exp\left(-i \sum_{j=1}^{N} \left((\lambda_j - \alpha t)x_j + \frac{(\lambda_j - \alpha t)^3 - \lambda_j^3}{3\alpha} \right) \right) \tag{2.99}$$

in (2.96) and therefore get for the time evolution with $\alpha \neq 0$ of the state $|x_1, \dots, x_N\rangle$:

$$\begin{aligned} e^{-H_\alpha t} \, |x_1, \dots, x_N\rangle = \int_y \theta(\mathbf{x}) \int_\lambda \prod_k e^{-i\lambda_k x_k} \prod_{i<j} Z_{ij}^y(\lambda_i - \lambda_j) \\ \exp\left(-i \sum_{j=1}^{N} \left((\lambda_j - \alpha t)x_j + \frac{(\lambda_j - \alpha t)^3 - \lambda_j^3}{3\alpha} \right) \right) \prod_j b^\dagger(y_j) \, |0\rangle \end{aligned} \tag{2.100}$$

For a time periodic evolution one has to apply different time evolution operators $\exp(-iH_{\alpha_1} t_1)$ and $\exp(-iH_{\alpha_2} t_2)$ consecutively. For the following time evolution the current time evolution can be seen as initial conditions and therefore one gets for example for the second time evolution:

$$e^{-iH_{\alpha_2}t_2}e^{-iH_{\alpha_1}t_1}|x_1,\dots,x_N\rangle = \int_z\int_\mu\int_y \theta(\mathbf{y})\theta(\mathbf{x})\prod_k e^{-i(\lambda_j x_j+\mu_j y_j)}\mathcal{S}_y\prod_{i<j}Z^y_{ij}(\lambda_i-\lambda_j)Z^z_{ij}(\mu_i-\mu_j)$$
$$\exp\left(-i\sum_{j=1}^N\left((\lambda_j-\alpha_1t_1)y_j+\frac{(\lambda_j-\alpha_1t_1)^3-\lambda_j^3}{3\alpha_1}\right)\right)$$
$$\exp\left(-i\sum_{j=1}^N\left((\mu_j-\alpha_2t_2)z_j+\frac{(\mu_j-\alpha_2t_2)^3-\mu_j^3}{3\alpha_2}\right)\right)\prod_j b^\dagger(z_j)|0\rangle \quad (2.101)$$

with $\mathcal{S}_y$ being the symmetrizer in $y_1,\dots,y_N$. For two particles in particular this can be written after integrating out the y dependence:

$$e^{-iH_{\alpha_2}t_2}e^{-iH_{\alpha_1}t_1}|x_1x_2\rangle = \int_\mu\int_\lambda\int_z \theta(\mathbf{x})\prod_k e^{-i\lambda_k x_k}\delta(\mu_1+\mu_2-\lambda_1-\lambda_2+2\alpha_1t_1)\ 2\pi i$$
$$\left(\frac{1}{\lambda_1-\alpha_1t_1-\mu_1+i\varepsilon}+\frac{1}{\lambda_2-\alpha_1t_1-\mu_1+i\varepsilon}\frac{\lambda_1-\lambda_2+ic}{\lambda_1-\lambda_2-ic}\right)$$
$$\exp\left(i\sum_j\frac{(\lambda_j-\alpha_1t_1)^3-\lambda_j^3}{3\alpha_1}\right)\ \exp\left(-i\sum_{j=1}^N\left((\mu_j-\alpha_2t_2)z_j+\frac{(\mu_j-\alpha_2t_2)^3-\mu_j^3}{3\alpha_2}\right)\right)$$
$$Z^z_{12}(\mu_1-\mu_2)\prod_k b^\dagger(z_k)|0\rangle \quad (2.102)$$

In the delta function the change of total momentum by the accerating constant field can be seen.

The goal for this model would also to get an approximation for large times for two and N particles for arbitrary interaction strength c to see for example the amount of energy put into or taken out of the system or to see the behaviour for large t, e.g. if the result corresponds to an infinite temperature independently of the initial state as described in [15] for periodically driven systems.

Chapter 3

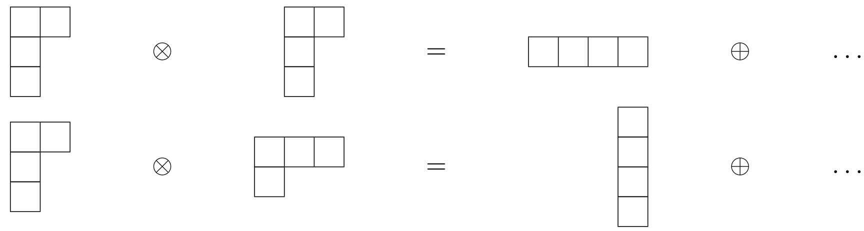
Gaudin-Yang Model

3.1 Solution of the Gaudin-Yang Model

When looking at the Lieb-Liniger model the goal was to find a totally symmetric wave function for indistinguishable bosons with no further degree of freedom. This is however a huge restriction on the generality, as there is a great interest in higher spin bosons and especially Fermi gases in one dimension interacting via a two-body delta-function potential (cf. [1]). In order to describe that, the Hamiltonian in first quantization is exactly the same as defined in Section 2.1.2 as there is no spin interaction:

$$H_N = -\sum_i \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<j} \delta(x_i - x_j) \tag{3.1}$$

with $c > 0$. The space of wave function however changed by introducing inner degrees of freedom. The total wave function now can be either totally antisymmetric (fermionic) or totally symmetric (bosonic). The orbital wave function however can have a generic symmetry and can be classified according to a certain young tableau $\mathcal{T}$ depending on the other degrees of freedom. For example in the case of the internal degrees being spin degrees of freedom, a bosonic wave function totally symmetric in the N particle indices, with a certain total spin, can be written as a sum over products of a spin function in representation $\mathcal{R}$ with a spatial wave function in the same representation $\mathcal{R}$ of the symmetric group S_N. A fermionic wave function totally antisymmetric in the N particle indices can be written as a sum over products of a spin function in representation $\mathcal{R}$ with a orbital wave function in the conjugate representation $\bar{\mathcal{R}}$ of S_N (cf. [19]). For example for four particles and a specific young tableau:



(3.2)

(3.3)

For the case of spin being the internal degree of freedom this can also be seen by considering the second quantized hamiltonian (cf. [20])

$$H = \int_{\mathbb{R}} \mathrm{d}x \sum_a \left(\partial_x b_a^\dagger(x) \partial_x b_a(x) + c \sum_{a,b} b_a^\dagger(x) b_b^\dagger(x) b_a(x) b_b(x) \right) \tag{3.4}$$

with the Ansatz for the wave function

$$|\psi(\{k\})\rangle = \frac{1}{\sqrt{N!}} \sum_{\{a\}} \int_{\mathbb{R}^N} \mathrm{d}^N x \; \Psi_{a_1,\ldots,a_N}(\{k\}|\{x\}) b_{a_1}^\dagger(x_1) \ldots b_{a_N}^\dagger(x_N) \, |0\rangle \tag{3.5}$$

The wave function $\Psi_{\{a\}}(\{k\}|\{x\})$ can then again be seen as the sum of products of spin and orbital wave functions or their conjugates over irreducible representations of S_N as described above with the Hamiltonian (3.1) acting on the orbital part of the wave function. Therefore it is sufficient to find the orbital wave function which is either in the same or conjugate representation as the spin wave function depending on if it is a boson or a fermion. The problem was solved for spin $1/2$ by Yang in 1967 ([4]) and for a general symmetry of the wave function by Sutherland ([6]) in 1968.

3.1.1 Conditions for Wave function

As the Hamiltonian is the same as in 2.1.2 the Schrödinger equation can again be reexpressed as as the free wave equation in the sector $D_Q : x_{Q1} < \cdots < x_{QN}$ with continuity and jump-equation at the boundaries of the domain D_Q (cf.[10, 21]). Assuming the exchange of particles x_{Qa} and $x_{Q(a+1)}$ one gets:

continuity condition:

$$\psi|_{x_{Q(a+1)} - x_{Qa} = +0} \equiv \psi|_{x_{Q(a+1)} - x_{Qa} = -0} \tag{3.6}$$

cusp condition:

$$\left(\frac{\partial\psi}{\partial x_{Q(a+1)}} - \frac{\partial\psi}{\partial x_{Qa}}\right)\Bigg|_{x_{Q(a+1)}-x_{Qa}=+0} - \left(\frac{\partial\psi}{\partial x_{Q(a+1)}} - \frac{\partial\psi}{\partial x_{Qa}}\right)\Bigg|_{x_{Q(a+1)}-x_{Qa}=-0} \equiv 2c\ \psi|_{x_{Q(a+1)}-x_{Qa}=\pm 0} \tag{3.7}$$

To find a wave function satisfying this, a Bethe Ansatz is made, assuming that in every sector D_Q the wave function is a superposition of plane wave solutions (cf. [10]):

$$\psi|_{\{x\}\in D_Q} = \sum_{P\in S_N} \langle Q||P\rangle e^{i(\bar{P}k,\bar{Q}x)} \tag{3.8}$$

with $(\bar{P}k, \bar{Q}x) = \sum_j k_{Pj} x_{Q_j}$ and $\bar{P} = P^{-1}$.

The full wave function can then be written as:

$$\psi(\{x\}) = \sum_{Q\in S_N} \theta(\mathbf{x}_Q)\ \psi|_{\{x\}\in D_Q} \tag{3.9}$$

where $\theta(\mathbf{x}_Q)$ indicates the ordering of the domain $D_Q : x_{Q1} < \cdots < x_{QN}$.

Applying the conditions (3.6) and (3.7) to the wave function (3.8) in the domain D_Q one gets:

continuity condition:

$$\langle Q||P\rangle + \langle Q||P(a\ a+1)\rangle = \langle Q(a\ a+1)||P\rangle + \langle Q(a\ a+1)||P(a\ a+1)\rangle \tag{3.10}$$

cusp condition:

$$i(k_{P(a+1)} - k_{Pa})\left(\langle Q||P\rangle - \langle Q||P(a\ a+1)\rangle + \langle Q(a\ a+1)||P\rangle - \langle Q(a\ a+1)||P(a\ a+1)\rangle\right) = 2c(\langle Q||P\rangle + \langle Q||P(a\ a+1)\rangle) \tag{3.11}$$

where $P(a\ a+1)$ indicates that in the permutation P the ath and $(a+1)$st element have been transposed. From those two conditions, a relation for passing from sector Q to $Q(a\ a+1)$ can be inferred (cf. [10]):

$$\langle Q(a\ a+1)||P\rangle = x_{Pa,P(a+1)}\langle Q||P\rangle + (1 + x_{Pa,p(a+1)})\langle Q||P(a\ a+1)\rangle \tag{3.12}$$

with

$$x_{ij} = \frac{ic}{k_i - k_j} \tag{3.13}$$

which can directly be verified by plugging it into (3.10) and (3.11). As $\langle Q||P\rangle$ has $N!^2$ elements for the different permutations there now exist $(N-1)(N!)^2$ equations (3.12). It can be shown, that they are mutually consistent (cf.[4, 10]) for unequal $\{k\}$. This can be done by proving that the coefficients $\langle Q||P\rangle$ for every sector P can be uniquely constructed from the $N!$ coefficients in some chosen initial sector by means of the relation (3.12). For a sketch of the proof define the ring element:

$$\left|\bar{P}\right\rangle = \sum_{Q\in S_N} \bar{Q}\langle Q||P\rangle \tag{3.14}$$

Relation (3.12) can be written in the so called Yang representation for S_N (cf. [4, 10]) as:

$$|\overline{P(a\ a+1)}\rangle = Y_{a\,a+1}\left|\bar{P}\right\rangle \tag{3.15}$$

with the operator

$$Y_{a\,a+1} = \frac{(a\ a+1) - x_{Pa,P(a+1)}}{1 + x_{Pa,P(a+1)}} \tag{3.16}$$

where $(a\,a+1)$ acting on the basis Q again stands for the transposition from the sector where $x_{Q1} < \cdots < x_{Qa} < x_{Q(a+1)} < \ldots x_{QN}$ to the sector $x_{Q1} < \cdots < x_{Q(a+1)} < x_{Qa} < \ldots x_{QN}$. Writing the vector $\left|\bar{P}\right\rangle$ in the form (cf. [4, 10])

$$\left|\bar{P}\right\rangle \equiv |P_1, P_2, \ldots, P_N\rangle \equiv |ijk\ldots\rangle \tag{3.17}$$

one gets the operator as used in [4]:

$$Y_{12}^{ij}\,|ijk\ldots\rangle = |jik\ldots\rangle \qquad \text{with} \qquad Y_{12}^{ij} = \frac{(12) - x_{ij}}{1 + x_{ij}} \tag{3.18}$$

This operator Y_{aa+1}^{ij} can now be used to construct every vector $\left|\bar{P}\right\rangle$ starting from some arbitrary initial vector $\left|\bar{I}\right\rangle$. The coherence condition for that is that every path from one permutation to another has to give the same result. It is sufficient to show that:

$$Y_{12}^{ij}Y_{12}^{ij} = 1 \tag{3.19}$$

$$Y_{12}^{jk}Y_{23}^{ik}Y_{12}^{ij} = Y_{23}^{ij}Y_{12}^{ik}Y_{23}^{jk} \tag{3.20}$$

which can directly be verified [10]. It corresponds to the Permutation

$$(1\ 3) = (1\ 2)(2\ 3)(1\ 2) = (2\ 3)(1\ 2)(2\ 3) \tag{3.21}$$

of the elements ijk.

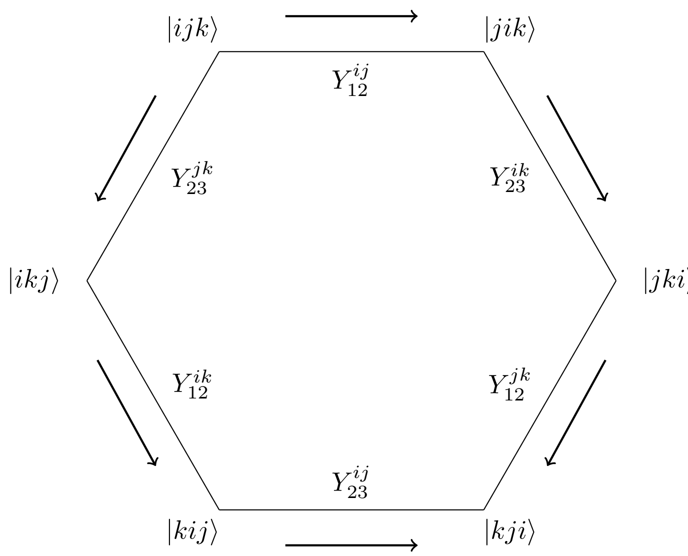
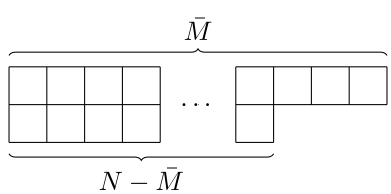


Figure 3.1: Consistency condition on the Y-operators. [22]

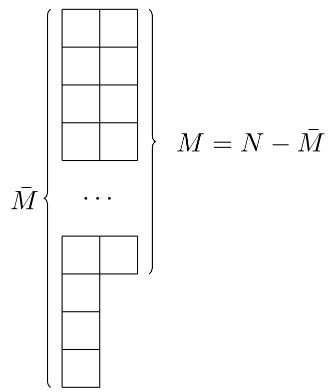
Therefore the wave function is entirely determined from the $N!$ parameters in one initial sector and there is one wave function for every young tableau.

Furthermore there are symmetry conditions on the wave function depending on the representation the wave function is in. In the following the case spin 1/2 is going to be discussed. The higher spin fermion and boson cases essentially boil down to constructing a suitable irreducible representation of the symmetric group (cf. [6, 10]).

The symmetry condition for a system with N spin 1/2 fermions with total spin $S = \frac{N}{2} - M = \bar{M} - \frac{N}{2}$ is of interest here. As each fermion is antisymmetric in the two spin variables the symmetry type of the spin wave function has to be $[\bar{M}, N - \bar{M}]$:



As the total wave function is antisymmetric the orbital wave function is in the conjugate representation and therefore in the $\bar{\mathcal{R}} = [2^M, 1^{N-2M}]$ representation.



Therefore the wave function cannot be antisymmetrized in $\bar{M}+1$ variables and it is separately antisymmetric in $\bar{M}$ and $N-\bar{M}$ variables. Assuming all particles in the subset U of $\{1,\ldots,N\}$ of order $\bar{M}$ are downspin, this can be written in terms of the amplitude in (3.8) as (cf. [10]):

$$\langle Q(i,j)||P\rangle = -\langle Q||P\rangle \qquad \forall (i,j\in U)\vee(i,j\in N\setminus U) \tag{3.22}$$

This condition can be expressed in the ring defined in (3.14):

$$(i,j)\left|\bar{P}\right\rangle = -\left|\bar{P}\right\rangle \qquad (i,j\in U)\vee(i,j\in N\setminus U) \tag{3.23}$$

It simply states, that the element $\left|\bar{P}\right\rangle$ belongs to the representation $\bar{\mathcal{R}}$. A representation of $\bar{\mathcal{R}}$ is obtained by going to the conjugate basis

$$|\tilde{P}\rangle = \sum_{Q} \mathrm{sgn}\,(Q)\,\bar{Q}\langle Q||P\rangle \tag{3.24}$$

with the permutation operator

$$(ij) = -\frac{1}{2}(1+\vec{\sigma}_i\vec{\sigma}_j) \tag{3.25}$$

(cf. [10]) acting on the space V^N with $V=\mathbb{C}^2$, where $\vec{\sigma}_i$ is the Pauli-matrices acting on the i-th spinor.

3.1.2 Periodic boundary conditions

The next step to solve the model is to put the model in a finite volume and impose boundary conditions. By doing so one introduces a volume cutoff and therefore regulates the infrared behaviour.

Furthermore it is the most convenient way to get the full wave function and is needed if one wants to study the thermodynamics of the model. In the following periodic boundary conditions of length L will be imposed. This is equivalent to putting the system on a ring of length L. It will be seen, that those boundary conditions are compatible with the Bethe-Ansatz type solution.

For the periodic system the domain D_Q now has a new condition (cf. [10]):

$$D_Q: \quad x_{Q1} < x_{Q2} < \cdots < x_{QN} \quad , \quad x_{QN} - x_{Q1} < L \tag{3.26}$$

As the wave function should be periodic one gets the condition for the domain D_Q (cf. [10])

$$\psi(x_{Q1}, \ldots, x_{QN}) \equiv \psi(x_{Q2}, \ldots, x_{QN}, x_{Q1} + L) \tag{3.27}$$

Introducing the Permutation C which is a cycle of length N, the right side of (3.27) is defined in the domain QC. This again can be stated as conditions on the amplitude of (3.8) (cf. [10]):

$$\langle Q||P\rangle = \langle QC||PC\rangle e^{ik_{P1}L} \tag{3.28}$$

This can be written in the defined ring as (cf. [10])

$$\left|\bar{P}\right\rangle e^{-ik_{P1}L} = C\left|\overline{PC}\right\rangle = CY_{N-1\,N}\ldots Y_{23}Y_{12}\left|\bar{P}\right\rangle \tag{3.29}$$

where the relation (3.15) was used. Due to the fact, that equation (3.29) holds for every P (cf. [21]) one can choose some initial permutation I_j and momentum k_j and get the eigenvalue equation (cf. [10, 21]):

$$Z_j\left|I_j\right\rangle \equiv X_{j+1\,j}X_{j+2\,j}\ldots X_{j-1\,j}\left|I_j\right\rangle = e^{-ik_jL}\left|I_j\right\rangle \tag{3.30}$$

where the operators X_{ij} were obtained by applying the cycle C on the Y operators (cf. [10, 21]). They are defined as (cf. [4, 10, 21]):

$$X_{ij} = (ij)Y_{ij}^{ij} = \frac{1-(ij)x_{ij}}{1+x_{ij}} \tag{3.31}$$

and as the Y operators satisfy conditions (3.19) and (3.20), which are now called Yang-Baxter relations.

It is important here to note that the Z_j commute among themselves, which can be inferred from the fact that the X_{ij} satisfy the Yang-Baxter equation (cf. [10]). Therefore the eigenvectors in (3.30) can be chosen independently of j (cf. [10]).

Therefore with (3.30) one has an eigenvalue equation for some chosen initial sector of the permutation P. Consequently by solving this equation in the representation discussed in section 3.1.1 one can get all the amplitudes $\langle Q||P\rangle$ by applying the operators Y or X and imposing the symmetry conditions. The problem now is to solve this eigenvalue equation in the chosen representation. This will be done in the next section with the algebraic Bethe Ansatz technique or Quantum Inverse Scattering developed by Baxter [23] for the eight-vertex lattice model and extended by Faddeev and Takhtajan [24].

3.1.3 Algebraic Bethe Ansatz

In the represenation chosen in 3.1.1 one gets the operator Z_j by choosing a basis (cf. [22]):

$$(Z_j)^{b_1\dots b_N}_{a_1\dots a_N} = (X_{jj-1}\dots X_{j1}X_{jN}\dots X_{jj+1})^{b_1\dots b_N}_{a_1\dots a_N} \tag{3.32}$$

with the permutation operator $(ij)^{ab}_{cd} = -\frac{1}{2}\left(\delta^a_c\delta^b_d + (\vec{\sigma}_i)^a_c(\vec{\sigma}_j)^b_d\right)$. It will in the following be useful to introduce the continous variable κ instead of $k_i - k_j$ in X_{ij}:

$$X_{ij}(\kappa) = \frac{\kappa - ic(ij)}{\kappa + ic} \tag{3.33}$$

Futhermore introducing an auxiliary space V_A (cf. [22]) the operator X can be defined acting on the space $V_i \times V_A$ (cf. [22]):

$$(X^{jA}(\kappa))^{b,v}_{a,u} = \frac{\kappa\ 1^{b,v}_{a,u} - ic\ (ij)^{b,v}_{a,u}}{\kappa + ic} \tag{3.34}$$

Inspired by the six-vertex model define futhermore the monodromy matrix $T(\kappa)$ by (cf. [22]):

$$T(\kappa)^{b_1\dots b_N,v}_{a_1\dots a_N,u} = \sum_{\{s\}}(X_{1A}(\kappa-k_1))^{b_1,s_1}_{a_1,u}(X_{2A}(\kappa-k_2))^{b_2,s_2}_{a_2,s_1}\dots(X_{NA}(\kappa-k_N))^{b_N,v}_{a_N,s_{N-1}} \tag{3.35}$$

Introducing also the transfer matrix τ by taking the trace over the auxiliary variables $\tau(\kappa) = \mathrm{tr}_A\left(T(\kappa)\right)$ one gets back the Z_j by $Z_j = \tau(\kappa = k_j)$ (cf. [22]).

Observing now with the aid of the Yang-Baxter equation for X, wich is also satisfied for the continous parameter with an appropiate shift (cf. [22])

$$X_{kj}(\kappa-\lambda)X_{ki}(\kappa)X_{ij}(\lambda)=X_{ij}(\lambda)X_{ki}(\kappa)X_{kj}(\kappa-\lambda) \tag{3.36}$$

that by multiplying with (kj) and extending this relation into auxiliary space denoted by the indeces a and b one gets the relation (cf. [25]):

$$Y_{ab}(\kappa-\lambda)X_{ai}(\kappa)X_{ib}(\lambda)=X_{ib}(\lambda)X_{ai}(\kappa)Y_{ab}(\kappa-\lambda) \tag{3.37}$$

where $Y_{ab}(\kappa-\lambda)$ acts totally on the auxiliary space. In the following this Y operator acting only on the auxiliary space will be called R-matrix:

$$R^{uv}_{st}(\kappa-\lambda)=(Y_{ab})^{uv}_{st}(\kappa-\lambda) \tag{3.38}$$

Therefore, with the fact, that the X operators with different indices commute this can be iterated and one gets as the so called Yang-Baxter algebra for the monodromy matrix:

$$R(\kappa-\lambda)(T(\kappa)\otimes T(\lambda))=(T(\lambda)\otimes T(\kappa))R(\kappa-\lambda) \tag{3.39}$$

Remark

Multiplying with $R^{-1}(\kappa-\lambda)$ from the left and taking the trace in the auxiliary spin space one gets a continous version of the commutation relation of the Z_j:

$$[Z(\kappa),Z(\lambda)]=0 \tag{3.40}$$

Expanding the $Z(\kappa)$ in the continous parameters gives an infinete set of charges commuting with $Z(\kappa)$ and therefore making the problem integrable.

The R-matrix and the operator X_{Aj} can be written as matrices in the auxiliary spin space (cf.

[22, 25]):

$$R(\kappa) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\kappa) & a(\kappa) & 0 \\ 0 & a(\kappa) & b(\kappa) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.41}$$

$$X_{Aj}(\kappa) = \begin{pmatrix} (a(\kappa) + b(\kappa)/2) + b(\kappa)/2\sigma_j^z & b(\kappa)\sigma_j^- \\ b(\kappa)\sigma_j^+ & (a(\kappa) + b(\kappa)/2) - b(\kappa)/2\sigma_j^z \end{pmatrix} \tag{3.42}$$

with $a(\kappa) = \frac{\kappa}{\kappa+ic}$ and $b(x) = \frac{ic}{\kappa+ic}$. The monodromy matrix T can also be written as a matrix in the auxiliary space:

$$T(\kappa) = \begin{pmatrix} A(\kappa) & B(\kappa) \\ C(\kappa) & D(\kappa) \end{pmatrix} \tag{3.43}$$

where the operators A, B, C, D act on the space V^N and $Z(\kappa) = A(\kappa) + D(\kappa)$. With the help of the Yang-Baxter Algebra (3.39) one can infer commutation relations for the A, B, C, D (cf. [21, 22, 25]):

$$B(\kappa)B(\lambda) = B(\lambda)B(\kappa) \tag{3.44}$$

$$A(\kappa)B(\lambda) = \frac{1}{a(\lambda-\kappa)}B(\lambda)A(\kappa) - \frac{b(\lambda-\kappa)}{a(\lambda-\kappa)}B(\kappa)A(\lambda) \tag{3.45}$$

$$D(\kappa)B(\lambda) = \frac{1}{a(\kappa-\lambda)}B(\lambda)D(\kappa) - \frac{b(\kappa-\lambda)}{a(\kappa-\lambda)}B(\kappa)D(\lambda) \tag{3.46}$$

Eigenstates of $Z(\kappa)$ can now be constructed by starting of with the state $|\omega\rangle = |\uparrow \dots \uparrow\rangle = \binom{1}{0}^{\otimes N}$ and applying products of the operator $B(\kappa)$ on it, playing the role of creation operators of downspins (cf. [22, 25]). For the state with M down spins, which is the desired eigenstate of $Z(\kappa)$, one acts M times with B (cf. [22]):

$$|I\rangle = B(\lambda_1) \dots B(\lambda_M) |\omega\rangle = \sum_{j_1,\dots,j_M} \langle j_1 \dots j_M || I\rangle \; \sigma_{j_1}^- \dots \sigma_{j_M}^- \, |\omega\rangle \tag{3.47}$$

where in $\langle j_1 \dots j_M || I\rangle$ the positions of the down spins are specified. These amplitudes are the ones needed to get every other amplitude by application of Y or by imposing the symmetry conditions.

It can be shown, that the state $|\omega\rangle$ is an eigenstate to $A(\kappa)$ and $D(\kappa)$ with the eigenvalues 1 and $d(\kappa) = \prod_{j=1}^{N} \frac{\kappa-k_j}{\kappa-k_j+ic}$ respectively (cf. [22, 25]) by applying the operators X in (3.42) according to (3.35) on this state. With this and the commutation relations of A, B and D one can apply the operators A and D to the state $|I\rangle$ and one gets (cf. [22, 25]):

$$A(\kappa)\,|I\rangle = \prod_{j=1}^{M} \frac{1}{a(\lambda_j-\kappa)}\,|I\rangle + B(\kappa)\sum_{j=1}^{M} F_j(\lambda)\prod_{\substack{k=1\\k\neq j}}^{M} B(\lambda_k)\,|0\rangle \tag{3.48}$$

$$D(\kappa)\,|I\rangle = d(\kappa)\prod_{j=1}^{M} \frac{1}{a(\kappa-\lambda_j)}\,|I\rangle + B(\kappa)\sum_{j=1}^{M} \tilde{F}_j(\kappa)\prod_{\substack{k=1\\k\neq j}}^{M} B(\lambda_k)\,|0\rangle \tag{3.49}$$

with the functions F_j and $\tilde{F}_j$ determined by the permutation relations of A and D with B (cf. [22, 25]):

$$F_j(\kappa) = -\frac{b(\lambda_j-\kappa)}{a(\lambda_j-\kappa)}\prod_{\substack{k=1\\k\neq j}}^{M}\frac{1}{a(\lambda_k-\lambda_j)} \tag{3.50}$$

$$\tilde{F}_j(\kappa) = -\frac{b(\kappa-\lambda_j)d(\lambda_j)}{a(\kappa-\lambda_j)}\prod_{\substack{k=1\\k\neq j}}^{M}\frac{1}{a(\lambda_j-\lambda_k)} \tag{3.51}$$

Therefore for $|I\rangle$ to be an eigenstate of $Z(\kappa) = A(\kappa) + B(\kappa)$ a sufficient condition is, that $F_j(\kappa) + \tilde{F}_j(\kappa) = 0$ for all $j = 1, \ldots, M$. This is equivalent to (cf. [22]):

$$d(\lambda_j) = \prod_{l=1}^{N}\frac{\lambda_j - k_l}{\lambda_j - k_l + ic} \equiv \prod_{\substack{k=1\\k\neq j}}^{M}\frac{\lambda_k-\lambda_j+ic}{\lambda_k-\lambda_j-ic} \tag{3.52}$$

With this condition imposed, the state $|I\rangle$ is an eigenstate of $Z(\kappa)$ to the eigenvalue $z(\kappa)$ (cf. [22]):

$$z(\kappa) = \prod_{j=1}^{M}\frac{1}{a(\lambda_j-\kappa)} + d(\kappa)\prod_{j=1}^{M}\frac{1}{a(\kappa-\lambda_j)} \tag{3.53}$$

Going back to the initial problem for the eigenvalues and eigenfunctions of Z_j in (3.30) one gets by inserting $\kappa = k_j$ and with the definition $\Lambda_j = \lambda_j + \frac{ic}{2}$ the final conditions on the parameters $\{\Lambda\}$ and $\{k\}$ coming from the periodic boundary conditions (cf. [22]):

$$e^{ik_jL} = \prod_{l=1}^{M}\frac{\Lambda_k - k_j - ic/2}{\Lambda_k - k_j + ic/2} \tag{3.54}$$

$$\prod_{\substack{j=1\\j\neq l}}\frac{\Lambda_j-\Lambda_l+ic}{\Lambda_j-\Lambda_l-ic} = \prod_{i=1}^{N}\frac{\Lambda_l - k_i - ic/2}{\Lambda_l - k_i + ic/2} \tag{3.55}$$

Therefore the model is solved completely.

3.1.4 Remark: Wave function without periodic boundary conditions

The wave function constructed in 3.1.3 and 3.1.2 is also a valid wave function for the spin 1/2 case without periodic boundary conditions for arbitrary nonequal k and Λ which one would get in the $L \to \infty$ case. This can be seen as the wave function satisfies the Schrödinger equation and has the wanted symmetry. Alternatively it can be constructed as a solution to the conditions on the wave function derived in 3.1.1 as well as the symmetry of the wave function depending on the irreducible representation (cf. [26]).

3.2 Gaudin-Yang with external homogenous field

Analogous to the Lieb-Liniger model with linear force potential a method to construct the wave function of the non integrable model with the Hamiltonian:

$$H = -\sum_i \frac{\partial^2}{\partial x_i^2} + 2c \sum_{1 \le i < j \le N} \delta(x_i - x_j) + \alpha \sum_i x_i \tag{3.56}$$

could be by studying the Gaudin operator O_c. The operator again has to commute with the Hamiltonian. Furthermore, as the symmetry of the orbital wave function now can be determined by an arbitrary Young tableau, or in the case of spin 1/2 by a Young tableau of the form $[2^M, 1^{N-2M}]$ as for the Gaudin-Yang model without the external field, the operator has to construct the wanted symmetry of the orbital wave function. Moreover the wave function constructed by application of the Gaudin operator O_c has to obey the cusp condition from (3.7). This condition can be also written as:

$$\left[\left(\frac{\partial}{\partial x_{Q(j+1)}} - \frac{\partial}{\partial x_{Qj}}\right)\frac{1}{2}(1 + P_{jj+1}) - c\right]_{x_{Q(j+1)} - x_{Qj} = +0} \psi_{a_1 \ldots a_N}(x_1, \ldots, x_N) = 0 \tag{3.57}$$

where the operator P_{ij} flips the coordinates on positions i and j. Alternatively this could be written as

$$\left[\left(\frac{\partial}{\partial x_{Q(j+1)}} - \frac{\partial}{\partial x_{Qj}}\right)\frac{1}{2}\left(1 + \xi P^s_{Q(j+1)Qj}\right) - c\right]_{x_{Q(j+1)} - x_{Qj} = +0} \psi_{a_1 \ldots a_N}(x_1, \ldots, x_N) = 0 \tag{3.58}$$

where $\xi = -1$ for fermions and $\xi = 1$ for bosons and P^s_{ij} exchanging a_i and a_j. The operator for the totally symmetric wave function of the Lieb Liniger model can be written in the way:

$$O_c = \prod_{i>j} \left[\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \frac{1}{2} \left(1 + P_{ij}\right) + c \ \left(\mathrm{sgn}(x_i - x_j)\right)^{\frac{1}{2}(1+P_{ij})} \right] \tag{3.59}$$

with P_{ij} being in the totally symmetric representation $[N]$ of the symmetric group, therefore $P_{ij} = 1$. This operator acting on a totally antisymmetric wave function constructed by taking the slater determinant of the single particle solutions gives the wave function of the interacting model, as seen in chapter 2.1.2 and 2.3.1.

Furthermore when taking the P_{ij} in the totally antisymmetric representation $[1^N]$ with $P_{ij} = -1$ the operator becomes

$$O_c = c^{\frac{N(N-1)}{2}} \tag{3.60}$$

and therefore the wave function totally antisymmetric in the coordinates $\{x\}$ can also be obtained by applying the operator O_c in the appropriate representation on the totally antisymmetric wave function obtained by taking the slater determinant of the single particle wave functions.

The idea for an arbitrary symmetry of the spatial wave function would be now to take the P_{ij} in the appropriate representation or the operator P^s_{ij} acting on the spin wave function in the conjugate representation and again let the operator O_c act on the slater determinant of the single particle wave functions. This however is not yet proven.

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