ON ERDŐS-KO-RADO FOR RANDOM HYPERGRAPHS

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ABSTRACT OF THE DISSERTATION

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Denote by $\mathcal{H}_k(n,p)$ the random k-graph in which each k-subset of $\{1,\ldots,n\}$ is present with probability p, independent of other choices. This dissertation addresses the question: for which p_0 will $\mathcal{H}_k(n,p)$ satisfy the "Erdős-Ko-Rado property" provided that $p > p_0$? This question was first studied by Balogh, Bohman, and Mubayi where they dealt mainly with $k < n^{\frac{1}{2}-\gamma}$ (for some $\gamma > 0$). Our first main result gives the desired p_0 when $k < \sqrt{cn\log(n)}$ (for $c < \frac{1}{4}$) and indeed contains the main results of Balogh *et al.* concerning when $\mathcal{H}_k(n,p)$ satisfies EKR a.s. (that is, with probability tending to 1 as $n \to \infty$). Additionally, more or less answering a question of Balogh *et al.*, we show: there is a fixed $\varepsilon > 0$ such that if $n = 2k+1$ and $p > 1-\varepsilon$, then $\mathcal{H}_k(n,p)$ has the EKR property a.s.

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Chapter 1

Introduction

One of the most interesting combinatorial trends of the last couple decades has been the investigation of “sparse random” versions of some of the classical theorems of the subject—that is, of the extent to which such results hold in a random setting. This issue has been the subject some spectacular successes, particularly those related to the theorems of Ramsey [23], Turán [29] and Szemerédi [28]; see [12, 2, 24, 19] for origins and, e.g., [9, 27, 10] (or the survey [25]) for a few of the most recent developments.

In this thesis, we are interested in the analogous question for the Erdős-Ko-Rado Theorem [8], another cornerstone of extremal combinatorics. This natural problem has already been considered by Balogh, Bohman and Mubayi [4], and we first quickly recall a few of the notions from that paper.

In what follows k and n are always positive integers with $n > 2k$. As usual we write $[n]$ for $\{1, \ldots, n\}$ and $\binom{V}{k}$ for the collection of k-subsets of a set V. A *k-graph* (or *k-uniform hypergraph*) on V is a multisubset, say $\mathcal{H}$, of $\binom{V}{k}$. Members of V and $\mathcal{H}$ are called *vertices* and *edges* respectively. We use $\mathcal{H}_x$ for the set of edges containing x ($\in V$), called the *star* of x in $\mathcal{H}$. For this thesis we take $V = [n]$ and write $\mathcal{K}$ for $\binom{V}{k}$.

A collection of sets is *intersecting*, or a *clique*, if no two of its members are disjoint. The Erdős-Ko-Rado Theorem says that for any n and k as above, the maximum size of an intersecting k-graph on V is $\binom{n-1}{k-1}$ and this bound is achieved only by the stars.

Following [4] we say $\mathcal{H} \subseteq \mathcal{K}$ satisfies *(strong) EKR* if every largest clique of $\mathcal{H}$ is a star; thus the Erdős-Ko-Rado Theorem says $\mathcal{K}$ satisfies EKR.

For the rest of this thesis we use $\mathcal{H} = \mathcal{H}_k(n, p)$ for the random k-graph on V in which members of $\mathcal{K}$ are present independently, each with probability p.

As suggested above, we are interested in understanding when EKR holds for $\mathcal{H}$; a little more formally:

Question 1.0.1. *For what* $p_0 = p_0(n,k)$ *is it true that* $\mathcal{H}$ *satisfies EKR a.s. provided* $p > p_0$*?*

(As usual, an event—really a sequence of events parameterized by n—holds *almost surely* (a.s.) if its probability tends to 1 as $n \to \infty$.)

Notice that EKR is not an increasing property (that is, it is not invariant under addition of edges) and that, for given n and k,

$$f_{n,k}(p) := \Pr(\mathcal{H}_k(n,p) \text{ satisfies EKR})$$

is not increasing in p. For instance, for sufficiently tiny p (depending on n and k) it will usually be the case that *every* clique is contained in a star. In view of this non-monotonicity, it is natural to define a *threshold* for the property EKR to be the least $p_0 = p_0(n,k)$ satisfying

$$f_{n,k}(p) \geq 1/2 \quad \forall p \geq p_0. \tag{1.1}$$

(This follows the usage in [16] (e.g.), which takes the "threshold" for an *increasing* property Q to be the unique p for which the "p-measure" of Q is 1/2.)

In general—though we will sometimes do better—we tend to regard determination of this threshold to within a constant factor as a satisfactory answer to Question 1.0.1.

For the most part we will not review the contents of [4]. The focus there is mainly on small k; roughly speaking, the authors give fairly complete results for $k = o(n^{1/3})$ and more limited information for k up to $n^{1/2-\varepsilon}$ with $\varepsilon > 0$ fixed (about which we will say a little more in Chapter 3).

The nature of the problem changes around $k = n^{1/2}$, since for k smaller than this, two random k-sets are typically disjoint, while the opposite is true for larger k. Heuristically we may say that the problem becomes more interesting/challenging as k grows and the potential violations of EKR proliferate (though increasing k does narrow the range of p for which we *expect* EKR to hold). At any rate, as noted in [4], very little has been known up to now for k larger than $\sqrt{n}$ (or, indeed, $k > n^{1/2-\varepsilon}$). In this thesis we more or less settle the problem for k in the "smaller" range and a little beyond, and make some progress for k at the very top of its range.

In Chapter 3 we will work, not directly with p, but with $\varphi := p\binom{n-1}{k-1}$, the expected degree of a vertex (as in [4] where they used ρ); this seems more natural as we are most interested in situations where p is tiny while the value of φ is more reasonable. Throughout this thesis we take $\mathsf{m} = \mathsf{E}|\mathcal{H}| = \varphi n/k$, $\Delta = \Delta_{\mathcal{H}}$ (the maximum degree in $\mathcal{H}$) and

$$\mathsf{q} = \Pr(A \cap B \neq \emptyset), \tag{1.2}$$

where A and B chosen uniformly and independently from $\mathcal{K}$. The following is the main result of Chapter 3.

Theorem 1.0.2. *For any fixed $c < 1/4$, if*

$$k < \sqrt{cn \log n} \tag{1.3}$$

and φ is such that

$$\binom{\mathsf{m}}{\Delta}\mathsf{q}^{\binom{\Delta}{2}} < o(1) \quad a.s., \tag{1.4}$$

then $\mathcal{H}$ satisfies EKR a.s.

(Recall $\binom{a}{b} = (a)_b/b! := a(a-1)\cdots(a-b+1)/b!$ for $a \in \mathbb{R}$ and $b \in \mathbb{N}$.) We will say more about the meaning and necessity of (1.4) in Chapter 3.

It is not hard to read off threshold information from Theorem 1.0.2 (with "threshold" as in (1.1), here translated to the corresponding φ_0); for example, for $k = \sqrt{\zeta n} \gg \sqrt{n}$ (satisfying (1.3)), we have $\varphi_0 < (1 + o(1))e^{\zeta} \log n$ (and $\varphi_0 \sim e^{\zeta} \log n$ according to the result of Section 3.10). Other special cases include the main positive results on EKR given in [4] (those in parts (i), (ii) and (iv) of their Theorem 1.1).

We believe Theorem 1.0.2 is true with "$c < 1/4$" replaced by "$c < 1/2$." It is *not* true for $c > 1/2$, roughly because: for $k = \sqrt{cn \log n}$ (with $c > 1/2$), (1.4) first occurs at $\varphi \approx \log n/\log(1/\mathsf{q}) \sim n^c \log n$, where it will be the case that (typically) all degrees are close to φ and for each vertex x the number of edges of $\mathcal{H} \setminus \mathcal{H}_x$ meeting all edges of $\mathcal{H}_x$ is about $\varphi(n/k)\mathsf{q}^{\varphi} \approx n^{c+1/2-1} = n^{\Omega(1)}$, meaning stars are unlikely even to be maxi*mal* cliques.

This is, of course, reminiscent of the Hilton-Milner Theorem [14], which says that the largest *nontrivial* cliques in $\mathcal{K}$ are those of the form $\{A\} \cup \{B \in \mathcal{K}_x : B \cap A \neq \emptyset\}$

(with $A \in \mathcal{K}$ and $x \in V \setminus A$). It seems not impossible that "generic" and "HM" cliques are the main obstructions to EKR in general:

Question 1.0.3. *Is it true that (for any k) if φ satisfies* (1.4) *and*

$$\textit{a.s. every } \mathcal{F}_x \textit{ is a maximal clique in } \mathcal{F},$$

then $\mathcal{F}$ satisfies EKR a.s.?

In Chapter 4 (using methods completely different from those used to prove Theorem 1.0.2) we jump to the other end of the spectrum, taking k to be as large as possible:

Theorem 1.0.4. *There is a fixed $\varepsilon > 0$ such that if $n = 2k + 1$ and $p > 1 - \varepsilon$, then $\mathcal{H}$ satisfies EKR a.s.*

This was prompted by Question 1.4 of [4], *viz.*

Question 1.0.5. *Is it true that for $k \in (n/2 - \sqrt{n}, n/2)$ and $p = .99$, EKR (or weak EKR) holds a.s. for $\mathcal{H}$?*

Theorem 1.0.4 could presumably be extended to the full range of k covered by Question 1.0.5, but this appears to be far short of the truth if $n \geq 2k + 2$, so seems of less interest; we will say more about this in Chapter 4.5.

Chapter 2

Preliminaries

2.1 Usage

Throughout the thesis we take $V = [n]$, $\mathcal{K} = \binom{V}{k}$, and $\mathcal{H} = \mathcal{H}_k(n,p)$ (as already noted) and we let $M = \binom{n-1}{k-1}$ (so $\varphi = Mp$) and $m = |\mathcal{H}|$ (a random variable with mean m). We use v, w, x, y, z for members of V. For a hypergraph $\mathcal{G}$, we let $\mathcal{G}_{\overline{x}} = \mathcal{G} \setminus \mathcal{G}_x$ (recall $\mathcal{G}_x = \{A \in \mathcal{G} : x \in A\}$).

We use $d_{\mathcal{G}}(x)$ for the degree of x in $\mathcal{G}$, and $d_{\mathcal{G}}(x,y)$ for the codegree of x and y in $\mathcal{G}$, and, where not otherwise specified, take d to mean $d_{\mathcal{H}}$. (As already stated, we use Δ for $\Delta_{\mathcal{H}}$.)

We use $B(m,\alpha)$ for a random variable with the binomial distribution $\mathrm{Bin}(m,\alpha)$, log for ln and $\binom{a}{\leq b}$ for $\sum_{i \leq b} \binom{a}{i}$. We use standard asymptotic notation ("big Oh" etc.), but will also sometimes use $a \asymp b$ for $a = \Theta(b)$ and $a \ll b$ for $a = o(b)$. We assume throughout that n is large enough to support our arguments. Following a standard abuse we usually pretend large numbers are integers.

2.2 Negative Association and Large Deviations

Some parts of the analysis in Chapter 3 seem most conveniently handled using the theory of negative association, regarding which we just recall what little we need, in particular confining ourselves to $\{0,1\}$-valued r.v.'s; see e.g. [22, 7] for further background.

Recall that events $\mathcal{A}, \mathcal{B}$ in a probability space are *negatively correlated* (denoted $\mathcal{A} \downarrow \mathcal{B}$) if $\Pr(\mathcal{A}\mathcal{B}) \leq \Pr(\mathcal{A})\Pr(\mathcal{B})$. Given a set S, set $\Omega = \Omega_S = \{0,1\}^S$ and recall that $\mathcal{A} \subseteq \Omega$ is *increasing* if $x \geq y \in \mathcal{A} \ \Rightarrow x \in \mathcal{A}$ (where "$\geq$" is product order on Ω). Say $i \in S$ *affects* $\mathcal{A} \subseteq \Omega$ if there are $\eta \in \mathcal{A}$ and $\nu \in \Omega \setminus \mathcal{A}$ with $\eta_j = \nu_j \ \forall j \neq i$, and write

$\mathcal{A} \perp \mathcal{B}$ if no $i \in S$ affects both $\mathcal{A}$ and $\mathcal{B}$.

Now suppose $(X_i : i \in S)$ is drawn from some probability distribution on Ω. The X_i's are said to be *negatively associated* (NA) if $\mathcal{A} \downarrow \mathcal{B}$ whenever $\mathcal{A}, \mathcal{B}$ are increasing and $\mathcal{A} \perp \mathcal{B}$. If Q_i are events whose indicators are NA then we also say that the Q_i's themselves are NA.

The following observation is surely not news, but as we don't know a reference we give the easy proof.

Proposition 2.2.1. *Suppose that for some $V_1, \ldots, V_s \subseteq V$ and $\ell_1, \ldots, \ell_s$, $A_1, \ldots, A_s$ are chosen independently with A_j uniform from $\binom{V_j}{\ell_j}$. Then the r.v.'s $X_{vj} = \mathbf{1}_{\{v \in A_j\}}$ ($v \in V$, $j \in [s]$) are negatively associated.*

Proof. (*Cf.* [7, Prop. 12].) For each j the vector $(X_{vj} : v \in V)$ is chosen uniformly from the strings of weight ℓ_j in $\{0,1\}^{V_j}$, implying that the r.v.'s X_{vj} ($v \in V$) are NA. (This is standard and easy, though we couldn't find it in writing. A stronger and far more interesting statement is the main result of [5].) We may thus apply [7, Proposition 8], which says that if the collections $\{X_{vj} : v \in V\}$ ($j \in [s]$) are mutually independent and each is NA, then the entire collection $\{X_{vj}\}$ is also NA.

∎

We will use Proposition 2.2.1 in conjunction with the following trivial observations.

Proposition 2.2.2. *If the r.v.'s $X_1, \ldots, X_m$ are NA, $I_1, \ldots, I_r$ are disjoint subsets of $[m]$, and Q_j is an increasing event determined by $\{X_i : i \in I_j\}$, then $Q_1, \ldots, Q_r$ are NA.*

Proposition 2.2.3. *If the events Q_i are NA, then $\Pr(\cap Q_i) \leq \prod \Pr(Q_i)$.*

One virtue of negative association lies in the fact that "Chernoff-type" large deviation bounds for random variables $X = \sum X_i$, where $X_1, \ldots$ are independent Bernoullis, remain valid under the (weaker) assumption that the X_i's are negatively associated. As far as we know, this was first observed by Dubhashi and Ranjan [7, Proposition 7]; it is gotten via the usual argument (Markov's inequality applied to $\exp[tX]$; see e.g. [15, pp.

26-28]), with the identity $\mathsf{E}e^{tX} = \prod \mathsf{E}e^{tX_i}$ replaced by the inequality $\mathsf{E}e^{tX} \le \prod \mathsf{E}e^{tX_i}$. In particular this gives the following bounds (see for example [15, Theorem 2.1 and Corollary 2.4]).

Theorem 2.2.4. *Suppose $X_1, \ldots, X_m$ are either negatively associated or independent* Ber(p) *r.v.'s, $X = \sum X_i$, and $\mu = \mathsf{E}X$. Then for any $\lambda \ge 0$,*

$$\begin{aligned} \Pr(X > \mu + \lambda) &< \exp[-\tfrac{\lambda^2}{2(\mu+\lambda/3)}], \\ \Pr(X < \mu - \lambda) &< \exp[-\tfrac{\lambda^2}{2\mu}], \end{aligned} \tag{2.1}$$

and for any $K > 1$,

$$\Pr(X > K\mu) < [e^{K-1}K^{-K}]^{\mu}. \tag{2.2}$$

Corollary 2.2.5. *The inequality* (2.2) *still holds if instead of $\mathsf{E}X = \mu$ (in Theorem 2.2.4) we assume only $\varrho := \mathsf{E}X \le \mu$.*

Proof. We have (using (2.2) for the inequality)

$$\begin{aligned} \Pr(X > K\mu) &= \Pr(X > (K\mu/\varrho)\varrho) \\ &< [e^{K\mu/\varrho - 1}(K\mu/\varrho)^{-K\mu/\varrho}]^{\varrho} = e^{K\mu-\varrho}K^{-K\mu}(\mu/\varrho)^{-K\mu}. \end{aligned}$$

The last expression is equal to the bound in (2.2) when $\mu = \varrho$ and is easily seen to be decreasing in $\mu \ge \varrho$ (provided $K \ge 1$).

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Chapter 3

Small k

3.1 Main Result

In this chapter we will prove Theorem 1.0.2, which, for ease of reading, we repeat here.

Theorem 3.1.1. *For any fixed $c < 1/4$, if $k < \sqrt{cn \log n}$ and φ is such that*

$$\binom{\mathsf{m}}{\Delta}\mathsf{q}^{\binom{\Delta}{2}} < o(1) \ a.s.$$

, then $\mathcal{H}$ satisfies EKR a.s.

3.2 Remarks

1. When proving Theorem 1.0.2 we may assume $\mathsf{m} = \omega(1)$; for if $\mathsf{m} = o(1)$ then (1.4) fails (the l.h.s. is a.s. 1; actually in this case $\mathcal{H}$ is a.s. empty and does satisfy EKR), while if $\mathsf{m} = \Theta(1)$ then with probability $\Omega(1)$ we have $\Delta = |\mathcal{H}| = 1$ and the expression in (1.4) is m (so (1.4) does not hold).

2. The meaning of (1.4) is as follows. We think of $q^{\binom{t}{2}}$ as the ideal value of the probability that random (independent) k-sets $A_1, \ldots, A_t$ form a clique (it would be the true value if the events $\{A_i \cap A_j \neq \emptyset\}$ were independent). Thus, since $|\mathcal{H}|$ is usually close to m, the left side of (1.4) may be thought of as the expected number of "generic" Δ-cliques in $\mathcal{H}$, and we should perhaps not expect EKR to hold if this number is not small.

At least for k as in (3.1), this intuition turns out to be correct in that, with a minor caveat involving instances with $\Delta = 2$, (1.4) is *necessary* for the conclusion of Theorem 1.0.2; the formal statement and a sketch of the (surprisingly nontrivial) proof are given in Section 3.10.

The rest of this chapter is organized as follows. The problem is most interesting when

$$k > n^{1/2-o(1)}. \tag{3.1}$$

The bulk of our discussion (Sections 3.3 and 3.5-3.8 will deal exclusively with this range, while Section 3.9 handles smaller k).

In proving Theorem 1.0.2 for k as in (3.1) we will find it better to deal first with φ not too far above the "threshold"—this regime will account for most of our work—and then treat larger φ mostly by a reduction to what we've established for smaller. We thus begin in Section 3.3 with an outline of the argument for small φ, in particular deriving Theorem 1.0.2 in this range from three main assertions, Lemmas 3.3.1-3.3.3. These are proved in Sections 3.5-3.7 following a bit of preparation in Section 3.4. Section 3.8 then gives the extension to large φ and, as noted above, Section 3.9 deals with small k. Section 3.10 treats the aforementioned necessity of (1.4).

3.3 Main Points

For the rest of this chapter we fix $c = 1/4 - \varepsilon$ in Theorem 1.0.2. Also, as noted above, the present section assumes k satisfies (3.1) (as well as (1.3)).

As noted earlier, most of our work will deal with φ fairly near the "threshold." Though the problem should become easier as φ grows, some parts of the main argument below break down for larger φ; this could perhaps be remedied, but we have found it easier to first deal directly with smaller φ and then use what we've learned to handle larger values. (A disadvantage of this approach is that it necessarily gives much weaker bounds on the probability that EKR fails than one might *hope* to establish using a more direct argument.)

We thus begin in this section with an outline of where we are headed in the "small φ" regime. As we will see, the "threshold" φ_0 ($:= Mp_0$) is around $\log n/\log(1/\mathsf{q})$. For the remainder of the thesis we define

$$\varphi^* = \frac{\log^3 n}{\log(1/\mathsf{q})}. \tag{3.2}$$

This will serve as a cutoff for "small".

We assume in this section (and again in parts of Section 3.4 and all of Sections 3.5-3.7) that $\varphi \le \varphi^*$ (a restriction which could be relaxed considerably without invalidating the present argument). Thus we want to show

$$\text{for } \varphi \le \varphi^* \text{ satisfying (1.4), } \mathcal{H} \text{ satisfies EKR a.s.} \tag{3.3}$$

(It *is* true that in this regime the problem is most delicate when φ is more or less at the "threshold"; in particular it is only here—see the proof of Lemma 3.3.3—that we must make precise use of (1.4).)

We will make frequent reference to the function in (1.4), so, having specified φ (and therefore m), give it a name:

$$\Lambda(t) = \Lambda_\varphi(t) = \binom{\mathsf{m}}{t}\mathsf{q}^{\binom{t}{2}} \tag{3.4}$$

(with the argument always assumed to lie in $\mathbb{N}$).

Call a clique *trivial* if it is contained in a star. We will show below that there are integers $\alpha = \alpha(n, \varphi) \le \beta = \beta(n, \varphi)$ satisfying, *inter alia*,

$$\Delta \in [\alpha, \beta] \quad \text{a.s.} \tag{3.5}$$

and

$$\Lambda(\alpha) = o(1). \tag{3.6}$$

Thus Theorem 1.0.2 would follow if we could show that $\mathcal{H}$ a.s. does not contain a nontrivial clique of size α, but this is not quite true; for example, if $d_x = \Delta$ is significantly larger than α—say closer to β than α—then an $A \in \mathcal{H} \setminus \mathcal{H}_x$ *typically* misses fewer than $\Delta - \alpha$ edges of $\mathcal{H}_x$, in which case $\{A\} \cup \{B \in \mathcal{H}_x : B \cap A \neq \emptyset\}$ is a nontrivial clique of size greater than α.

A natural way to address this is to compare each clique possessing a sufficiently high degree vertex, say x, directly with the star $\mathcal{H}_x$. This idea is implemented in the first of the following three lemmas; these lemmas will easily yield (3.3) and will also do most of the work when we come to larger φ. (To be clear, the lemmas will depend on further properties of α and β to be established below.)

Set

$$\gamma = \min\{\alpha, \varphi^*/3\}, \tag{3.7}$$

$$\tau = (1-\varepsilon)\gamma \tag{3.8}$$

and

$$\lambda = \max\left\{\frac{\sqrt{\log n}}{\log(1/\mathsf{q})}, 2\sqrt{\frac{\log n}{\log(1/\mathsf{q})}}\right\}. \tag{3.9}$$

(The actual values are not needed in this section. One should think of $\gamma = \alpha$; the technical $\varphi^*/3$ will be needed for the reduction in Section 3.8.)

Lemma 3.3.1. *A.s. there do not exist (in $\mathcal{H}$) a nontrivial clique $\mathcal{C}$ and vertex x such that $|\mathcal{C}| \geq d(x)$, $d_{\mathcal{C}}(x) \geq \tau$, and either $|\mathcal{C}| \geq \alpha$ or $|\mathcal{C}_{\overline{x}}| \geq 2/\varepsilon$.*

Lemma 3.3.2. *A.s. $\mathcal{H}$ does not contain a nontrivial clique with two vertices of degree at least λ.*

Lemma 3.3.3. *A.s. $\mathcal{H}$ does not contain a clique of size γ with at most one vertex of degree greater than λ and all vertices of degree less than τ.*

(For perspective we remark that Lemmas 3.3.1 and 3.3.3 are the main points; Lemma 3.3.2 just makes our lives a little easier when we come to Lemma 3.3.3.)

Lemmas 3.3.1-3.3.3 easily imply (3.3), as follows. Since $\Pr(\Delta < \alpha) = o(1)$ (see (3.5)), it is enough to show that $\mathcal{H}$ a.s. does not contain a nontrivial clique $\mathcal{C}$ with $|\mathcal{C}| \geq \Delta \geq \alpha$. But if $\Delta \geq \alpha$ and $\mathcal{H}$ does contain such a $\mathcal{C}$, then at least one of the following occurs.

(a) There is an x with $d_{\mathcal{C}}(x) \geq \tau$ (and $|\mathcal{C}| \geq \Delta \geq \max\{\alpha, d(x)\}$), so $x, \mathcal{C}$ are as in Lemma 3.3.1.

(b) There are two vertices with degree at least λ in $\mathcal{C}$.

(c) There is at most one vertex x with $d_{\mathcal{C}}(x) \geq \lambda$ and none with $d_{\mathcal{C}}(x) \geq \tau$, so (since $\alpha \geq \gamma$) $\mathcal{C}$ is as in Lemma 3.3.3.

But according to Lemmas 3.3.1-3.3.3, each of (a)-(c) occurs with probability $o(1)$, so we have (3.3).

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As the reader may have noticed, this derivation would remain valid if we dropped the alternative "$|\mathcal{C}_{\overline{x}}| \geq 2/\varepsilon$" in Lemma 3.3.1 and replaced γ by α in Lemma 3.3.3; the stated versions of these lemmas will be needed for dealing with larger φ in Section 3.8.

3.4 Generics

This section establishes basic properties of some of the parameters we will be dealing with, in particular showing that $\mathcal{H}$ a.s. satisfies a few general properties whose failure can then be more or less ignored in what follows.

To begin, we should say something about the intersection probability q (defined in (1.2)). We have $\mathsf{q} = 1 - \vartheta$ with

$$\vartheta = \tfrac{(n-k)_k}{(n)_k} \sim e^{-k^2/n}. \tag{3.10}$$

(The "$\sim$" is valid provided $k = o(n^{2/3})$.) This gives the asymptotics of q for $k = \Omega(\sqrt{n})$; in particular for $k \gg \sqrt{n}$ we have

$$\log(1/\mathsf{q}) \sim e^{-k^2/n}. \tag{3.11}$$

For $k \ll \sqrt{n}$ we instead have

$$\mathsf{q} \sim k^2/n \tag{3.12}$$

(since, with $X_v = \mathbf{1}_{\{v \in A \cap B\}}$,

$$k^2/n = \sum \mathsf{E} X_v \geq \mathsf{q} \geq \sum \mathsf{E} X_v - \sum \mathsf{E} X_v X_w > k^2/n - \tbinom{n}{2}(k/n)^4).$$

Note that in any case we have

$$\varphi^* < n^{1/4-\varepsilon+o(1)}. \tag{3.13}$$

We will usually be dealing with situations in which q is slightly perturbed by information on how relevant k-sets meet some small subset of V. This negligible effect is handled by the next observation.

Proposition 3.4.1. *Fix $W \subseteq V$ of size at most $\mathsf{w} = o(n/\log n)$ and $B \in \binom{V}{k}$, and let A be uniform from $\binom{V}{k}$. Then conditioned on any value of $A \cap W$,*

$$\Pr(A \cap (B \setminus W) \neq \emptyset) < (1 + 2k^2\mathsf{w}/(\mathsf{q}n^2))\mathsf{q}.$$

Proof. The probability is largest when $|W| = \mathsf{w}$ and $B \cap W = A \cap W = \emptyset$, in which case its value is $q = 1 - \varsigma$, with $\varsigma = \frac{(n-\mathsf{w}-k)_k}{(n-\mathsf{w})_k}$. We have

$$\begin{aligned}\frac{\vartheta}{\varsigma} &= \frac{(n-k)_k(n-\mathsf{w})_k}{(n)_k(n-\mathsf{w}-k)_k} \\ &= \prod_{i=0}^{k-1}\left(1 + \frac{k\mathsf{w}}{(n-i)(n-\mathsf{w}-k-i)}\right) = 1 + (1+o(1))\frac{k^2\mathsf{w}}{n^2}\ ;\end{aligned}$$

that is, $\vartheta/\varsigma - 1 \sim k^2\mathsf{w}/n^2$ ($= o(1)$ because of the bound on w). Thus

$$\begin{aligned}\frac{q}{\mathsf{q}} - 1 &= \frac{\vartheta - \varsigma}{1-\vartheta} = \frac{1}{1-\vartheta}\left(\frac{\vartheta}{\varsigma} - 1\right)\varsigma \\ &\sim \frac{k^2\mathsf{w}\varsigma}{(1-\vartheta)n^2} \sim \frac{k^2\mathsf{w}}{(1-\vartheta)n^2}e^{-k^2/n}.\end{aligned}$$

The lemma follows.

∎

In all that follows we assume φ satisfies (1.4). At some (indicated) points in this section, and again throughout Sections 3.5-3.7, we will also stipulate that $\varphi \le \varphi^*$. From now until the "coda" at the end of this section we further assume that

$$\varphi > n^{-o(1)}. \tag{3.14}$$

As we will see in the coda, this is implied by (1.4) if we assume (3.1). Recall (see following the statement of Theorem 1.0.4) we also assume $\mathsf{m} = \omega(1)$ and note that in this section we do *not* assume (3.1).

Recall $m = |\mathcal{H}|$. Let $\psi = \psi(n)$ be some slowly growing function of n (say $\psi = \log n$). Theorem 2.2.4 (for independent Bernoullis) says that a.s.

$$m \in (\mathsf{m} - \psi\sqrt{\mathsf{m}}, \mathsf{m} + \psi\sqrt{\mathsf{m}}). \tag{3.15}$$

From now on we write m_0 for $\mathsf{m} + \psi\sqrt{\mathsf{m}}$.

We next need to say something about the behavior of φ and Δ. Recall that our default for degrees is $\mathcal{H}$; thus, in addition to $\Delta = \Delta_{\mathcal{H}}$, we take $d_x = d(x) = d_{\mathcal{H}}(x)$ and $d(x, y) = d_{\mathcal{H}}(x, y)$. The properties we need will be given in Proposition 3.4.2 once we have introduced the parameters α and β mentioned earlier.

Let α_1 and β be, respectively, the largest integer with $\Pr(d_v \geq \alpha_1) \geq \psi/n$ and the smallest integer with $\Pr(d_v > \beta) < 1/(n\psi)$.

Next, notice that $\Lambda(0) = 1$ and (since $\Lambda(t)/\Lambda(t-1) = ((\mathsf{m} - t + 1)/t)\mathsf{q}^{t-1}$ is decreasing in t) there is some t_0 such that $\Lambda(t)$ is increasing up to t_0 and decreasing thereafter. Thus (1.4) says that there are $\varsigma = \varsigma(n)$ and $\upsilon = \upsilon(n)$, both $o(1)$, such that $\Pr(\Lambda(\Delta) > \varsigma) < \upsilon$. Set $\alpha_2 := \min\{t : \Lambda(t) \leq \varsigma\}$ and $\alpha = \max\{\alpha_1, \alpha_2\}$.

The promised Proposition 3.4.2 now collects properties of these parameters that we will use repeatedly in what follows, often without explicit mention.

Proposition 3.4.2. *For α, β as above:*

$$\alpha \leq \beta; \tag{3.16}$$

$$\Lambda(\alpha) = o(1); \tag{3.17}$$

$$\Delta \leq \beta \text{ a.s.; if } \varphi \leq \varphi^* \text{ then } \Delta \geq \alpha \text{ a.s.}; \tag{3.18}$$

$$\beta/\varphi < n^{o(1)}; \tag{3.19}$$

$$\alpha > (1 - o(1)) \log n / \log(1/\mathsf{q}); \tag{3.20}$$

$$\text{if } \varphi \leq \varphi^* \text{ then } \beta < (1 + o(1))\varphi^* \ (< n^{1/4-\varepsilon+o(1)}). \tag{3.21}$$

(It is not hard to see that in fact $\alpha \sim \beta$ in all cases and $\beta \sim \varphi$ if and only if $\varphi \gg \log n$. What we actually use for the second part of (3.18) is $\alpha_1 k/n \ll 1$.)

For the rest of this chapter we set $\mathcal{P} = \{m \text{ satisfies } (3.15)\} \wedge \{\Delta \leq \beta\}$, noting that (3.18) and our earlier observation that (3.15) holds a.s. give

$$\Pr(\mathcal{P}) = 1 - o(1). \tag{3.22}$$

Proof of Proposition 3.4.2. The first assertion in (3.18) is immediate from the definition of β. From the definition of α_2 we have $\Lambda(\alpha_2) = o(1)$ (namely $\Lambda(\alpha_2) \leq \varsigma$) and $\Delta \geq \alpha_2$

a.s. (since $\Pr(\Delta < \alpha_2) = \Pr(\Lambda(\Delta) > \varsigma) < \upsilon$), implying $\alpha_2 \le \beta$. This gives (3.16) and (3.17).

Let $\beta^* = \lceil \varphi + \eta \rceil$, with η the positive root of $x = \sqrt{2(\varphi + x/3)(\log n + \log \psi)}$. Then Theorem 2.2.4 gives (for any v)

$$\Pr(d_v > \beta^*) < \exp[-\eta^2/(2(\varphi + \eta/3))] = (n\psi)^{-1}, \tag{3.23}$$

whence $\beta \le \beta^*$. (The bound is very crude for smaller values of φ, but we have lots of room in such cases.) In particular, since $\eta = O(\max\{\sqrt{\varphi \log n}, \log n\})$, (3.14) now implies both (3.19) and (3.21) (and $\beta \sim \varphi$ if $\varphi \gg \log n$, but we don't need this).

For (3.20) we have

$$\begin{aligned} \Lambda(\alpha_2) \quad &> \quad \exp[\alpha_2(\log(\mathsf{m}/\alpha_2) - \tfrac{\alpha_2 - 1}{2}\log(1/\mathsf{q}))] \\ &> \quad \exp\left[\tfrac{\alpha_2}{2}((1 - o(1))\log n - \alpha_2 \log(1/\mathsf{q}))\right] \end{aligned}$$

(since $\log(\mathsf{m}/\alpha_2) > (1/2 - o(1))\log n$, as follows from $\mathsf{m} = \varphi n/k$, $\alpha_2 \le \beta$ and (3.19)), and combining this with (3.17) gives $\alpha_2 > (1 - o(1))\log n/\log(1/\mathsf{q})$.

Finally, the second assertion in (3.18) is given by the following more general statement, which we will need again in Section 3.9. Here we assume nothing about n, k, φ ($= Mp$) and $\theta \in \mathbb{N}$ beyond the very minor $p = o(1)$ and $\theta = o(M)$.

Proposition 3.4.3. *If $\Pr(d_v \ge \theta) = \omega(1/n)$ and $\theta k/n = o(1)$ then $\Delta \ge \theta$ a.s.*

(For (3.18)—note we already know $\Delta \ge \alpha_2$ a.s.—the hypothesis $\alpha_1 k/n = o(1)$ follows from $\varphi k/n < n^{-1/4}$ and $\alpha_1/\varphi \le \beta/\varphi < n^{o(1)}$; see (3.13) and (3.19). For $k < n^{1/2 - \Omega(1)}$ and a fixed θ, Proposition 3.4.3 is [4, Lemma 3.6].)

Proof of Proposition 3.4.3. Let $X_v = \mathbf{1}_{\{d_v \ge \theta\}}$ and $X = \sum X_v$. We are assuming $\mathsf{E}X = \omega(1)$, so to finish via the second moment method just need

$$\mathsf{E}X_v X_w \sim \mathsf{E}^2 X_v \tag{3.24}$$

(for $v \ne w$). Letting $Z = d(v, w)$ we have

$$\mathsf{E}X_v X_w < \textstyle\sum_{l \ge 0} \Pr(Z = l) \Pr^2(d_v \ge \theta - l). \tag{3.25}$$

(For equality we would replace d_v by $d(v,\overline{w}) := |\mathcal{H}_v \setminus \mathcal{H}_w|$.)

Now, Z is binomial with $\mathsf{E}Z < \varphi k/n$, so

$$\Pr(Z = l) \ \ (\leq \Pr(Z \geq l)) \ \ < (\varphi k/n)^l. \tag{3.26}$$

On the other hand, since $d_v \sim \mathrm{B}(M,p)$, we have, for each $t \leq \theta$,

$$\frac{\Pr(d_v = t-1)}{\Pr(d_v = t)} \ = \ \frac{t(1-p)}{(M-t+1)p} \sim t/\varphi, \tag{3.27}$$

implying $\Pr(d_v \geq t-1) < (1+\theta/\varphi)\Pr(d_v \geq t)$. Thus (since $\theta k/n = o(1)$) the sum in (3.25) is asymptotic to its zeroth term, and we have (3.24).

(We pickily add—to make sure that $\varphi k/n = o(1)$—that we may assume $\theta \geq \varphi$: there is nothing to prove if $\theta = 0$, and $\Delta \geq \varphi$ is easy if $\varphi \geq 1$ (and $k = o(1)$, which follows from $\theta > 0$ and $\theta k/n = o(1)$).)

∎

We will also eventually (in Section 3.7) need the easy

$$\tbinom{m_0}{\alpha} \sim \tbinom{\mathsf{m}}{\alpha} \tag{3.28}$$

(The ratio of the left- and right-hand sides is

$$\tfrac{(m_0)_\alpha}{(\mathsf{m})_\alpha} < (\tfrac{m_0-\alpha+1}{\mathsf{m}-\alpha+1})^\alpha < \exp[O(\psi\alpha/\sqrt{\mathsf{m}})]$$

and $\psi\alpha/\sqrt{\mathsf{m}} \leq \psi\beta/\sqrt{\mathsf{m}} < n^{-\varepsilon+o(1)}$ (using $\mathsf{m} = \varphi n/k$, (3.14) and (3.21)).)

For $x \in V$, let $W_x = \{y : d(x,y) \geq 2\}$ (a random set determined by $\mathcal{H}_x$). Let $\mathcal{R}$ be the intersection of $\mathcal{P}$ and the events $\{\Delta \geq \alpha\}$,

$$\{d(x,y) \leq 8 \ \ \forall x,y\}, \tag{3.29}$$

and

$$\{|W_x| < \max\{\varphi^2 k^2/n, 6\log n\} \ \ \forall x\}. \tag{3.30}$$

Though defined here in general, $\mathcal{R}$ is only of interest when φ is small:

Proposition 3.4.4. *If $\varphi \leq \varphi^*$, then* $\Pr(\mathcal{R}) = 1 - o(1)$.

Proof. We have already seen (in (3.22) and (3.18)) that $\mathcal{P}$ and $\{\Delta \geq \alpha\}$ hold a.s. That (3.29) does as well follows (*via* the union bound) from the fact, already noted in (3.26), that $\Pr(d(x,y) \geq l) \ll n^{-l/4}$. To deal with (3.30), it is enough to show

Claim. If m satisfies (3.15) and $A_1, \ldots, A_m$ are chosen *independently* (and uniformly) from $\mathcal{K}$, then (3.30) holds a.s.

(Since (3.15) holds a.s. it is enough to show that (3.30) holds a.s. given any m $(= |\mathcal{H}|)$ satisfying (3.15) (equivalently, given $m = m_0$); but for such an m, (i) $A_1, \ldots, A_m$ as in the claim are a.s. distinct and (ii) conditioned on this, the law of $\{A_1, \ldots, A_m\}$ is the same as that of $\mathcal{H}$ given $|\mathcal{H}| = m$.)

Proof of Claim. For a given x we have, for each $y \neq x$, $\Pr(y \in W_x) < \binom{m}{2}(k/n)^4 < (1/2 + o(1))(\varphi k/n)^2$ (using $m \sim \mathsf{m} = \varphi n/k$), implying $\mathsf{E}|W_x| < (1 + o(1))\varphi^2 k^2/(2n)$. On the other hand, the events $\{y \in W_x\}$ are NA (by Propositions 2.2.1 and 2.2.2), and a little calculation, with Corollary 2.2.5, bounds the probability that a particular x violates (3.30) by $o(1/n)$. (In more detail: if $\mu := \varphi^2 k^2/(2n) \geq 3 \log n$, then (2.1) bounds the probability by $\exp[-(9/8) \log n]$; otherwise $K := 6 \log n/\mu > 2$, and (2.2) bounds the probability by $(e^{K-1}K^{-K})^\mu = (e^{1-1/K}K^{-1})^{K\mu} \leq (\sqrt{e}/2)^{6 \log n} = o(1/n)$.)

∎

Coda. Finally, we say why the combination of (1.4) and (3.1) implies (3.14). Suppose instead that the first two conditions hold but $\varphi < n^{-\Omega(1)}$. Then $\Delta < O(1)$ a.s. But if $\Delta = O(1)$, then $\mathsf{q} > n^{-o(1)}$ (see (3.12)) implies $\Lambda(\Delta) = \Omega(\mathsf{m}^\Delta)n^{-o(1)}$, so that (1.4) implies $\mathsf{m} < n^{o(1)}$ (note $\Delta \geq 1$ a.s. since we assume $\mathsf{m} = \omega(1)$). But then (since $\mathsf{m} = \varphi n/k$ and we assume (3.1)) $\varphi < n^{-1/2+o(1)}$, implying that in fact $\Delta \leq 2$ a.s.

Now suppose $\Lambda(2) = o(1)$. Then $k \ll \sqrt{n}$ (otherwise $\mathsf{q} = \Omega(1)$ and $\mathsf{m} = o(1)$, contrary to assumption), and $\Lambda(2) \asymp (\varphi n/k)^2(k^2/n) = \varphi^2 n$, implying $\varphi \ll n^{-1/2}$ and $\Delta = 1$ a.s. But $\Lambda(1) = \mathsf{m}$, so we contradict (1.4) .

∎

3.5 Proof of Lemma 3.3.1

Here and in the next section we take

$$\mathsf{w} = \max\{\varphi^2 k^2/n, 6\log n\} \tag{3.31}$$

and

$$q = (1 + 2k^2\mathsf{w}/(\mathsf{q}n^2))\mathsf{q}; \tag{3.32}$$

thus w is the bound on the $|W_x|$'s in (3.30) (we will use it to bound a related quantity in Section 3.7) and q is the probability bound in Proposition 3.4.1. We will need to say that q is close to q; here and in Section 3.6 we could get by with, for example, $\log(1/q) \sim \log(1/\mathsf{q})$, but for the more delicate situation in Section 3.6 will need

$$q^{\binom{\alpha}{2}} \sim \mathsf{q}^{\binom{\alpha}{2}} \tag{3.33}$$

(that is, $k^2\mathsf{w}\alpha^2/(\mathsf{q}n^2) = o(1)$; in fact, $k^2\mathsf{w}\alpha^2/(\mathsf{q}n^2) < n^{-4\varepsilon+o(1)}$ since $\alpha < n^{1/4-\varepsilon+o(1)}$ (see (3.21)), $\mathsf{w} < n^{1/2-2\varepsilon+o(1)}$ (see (3.13)) and $k^2/(\mathsf{q}n) < 1 + o(1)$.)

We will use (a) of the following observation in the present section and the variant (b) in Section 3.6.

Proposition 3.5.1. (a) *Suppose* $\mathcal{A} = \{A_1, \ldots, A_d\} \subseteq \mathcal{K}_x$ *satisfies*

$$d_{\mathcal{A}}(z) \leq 8 \ \forall z \in V \setminus \{x\} \quad \textit{and} \quad |\{z \in V \setminus \{x\} : d_{\mathcal{A}}(z) \geq 2\}| < \mathsf{w}. \tag{3.34}$$

Then for B *uniform from* $\mathcal{K}_{\overline{x}}$,

$$\Pr(B \cap A_i \neq \emptyset \ \forall i \in [d]) < (1 + o(1))q^d.$$

(b) *The same conclusion holds if* $\mathcal{A} \subseteq \{A \in \mathcal{K}_x : y \notin A\}$ *satisfies* (3.34) *and* B *is uniform from* $\{A \in \mathcal{K}_y : x \notin A\}$.

(Of course the "8" in (3.34) is just the value we happen to have below.)

Proof. The proofs of (a) and (b) are essentially identical and we just give the former. Set $W = \{z \in V \setminus \{x\} : d_{\mathcal{A}}(z) \geq 2\}$. Since the events $\{z \in B\}$ $(z \in V \setminus \{x\})$ are negatively associated (see Proposition 2.2.1), Proposition 2.2.3 and the second condition in (3.34) give

$$\Pr(|B \cap W| = s) \ \leq \ \binom{\mathsf{w}}{s}(k/n)^s \ < \ (\mathsf{w}k/n)^s \ < \ n^{-(2\varepsilon - o(1))s}. \tag{3.35}$$

On the other hand we assert that, with $\mathcal{Q} = \{B \cap A_i \neq \emptyset \ \forall i \in [d]\}$, we have

$$\Pr(\mathcal{Q} | |B \cap W| = s) < q^{d-8s}. \tag{3.36}$$

To see this, condition on the value, Z, of $B \cap W$ (with $|Z| = s$), and let

$$I = \{i \in [d] : B \cap A_i \cap W = \emptyset\}.$$

Then $|I| \geq d - 8s$ (by the first condition in (3.34)) and B must meet the members of $\{A_i : i \in I\}$ in $V \setminus W$, where they are pairwise disjoint. By Proposition 3.4.1, $\Pr(B \cap (A_i \setminus W) \neq \emptyset | B \cap W = Z) < q$ for each i. But, given $\mathcal{R}_Z := \{B \cap W = Z\}$, $B \setminus Z$ is a uniformly chosen $(k-s)$-subset of $V \setminus W$, so by Propositions 2.2.1 and 2.2.2 the events $Q_i = \{B \cap (A_i \setminus W) \neq \emptyset\}$ are conditionally NA given $\mathcal{R}_Z$ (with $\mathcal{Q} = \cap_{i \in I} \mathcal{Q}_i$); thus Proposition 2.2.3 gives

$$\Pr(\mathcal{Q} | \mathcal{R}_Z) < q^{|I|} \leq q^{d-8s},$$

which implies (3.36).

Finally, combining (3.35) and (3.36), we have

$$\begin{aligned} \Pr(\mathcal{Q}) &= \sum_{s \geq 0} \Pr(|B \cap W| = s) \Pr(\mathcal{Q} | |B \cap W| = s) \\ &< \sum_{s \geq 0} n^{-(2\varepsilon - o(1))s} q^{d-8s} \\ &= q^d \sum_{s \geq 0} (n^{-(2\varepsilon - o(1))} q^{-8})^s \sim q^d \end{aligned}$$

∎

Corollary 3.5.2. *Suppose either $\mathcal{A}$ is as in (a) of Proposition 3.5.1 and $\mathcal{B}$ is chosen uniformly from the b-subsets of $\mathcal{K}_{\overline{x}}$, or $\mathcal{A}$ is as in (b) of the proposition and $\mathcal{B}$ is chosen uniformly from the b-subsets of $\{A \in \mathcal{K} : y \in A, x \notin A\}$. Then*

$$\Pr(B \cap A_i \neq \emptyset \ \forall B \in \mathcal{B}, i \in [d]) < (1 + o(1))^b q^{db}.$$

Terminology. Recall that $\mathcal{A}$, $\mathcal{B}$ (two families of sets) are *cross-intersecting* if $A \cap B \neq \emptyset \ \forall A \in \mathcal{A}, B \in \mathcal{B}$.

Proof. Again we just discuss the first case. We may take $\mathcal{B} = \{B_1, \ldots, B_b\}$ with B_i uniform from $\mathcal{K}_{\overline{x}} \setminus \{B_1, \ldots, B_{i-1}\}$. Then, with $\mathcal{Q}_i = \{B_i \cap A_j \neq \emptyset \ \forall j \in [d]\}$, we have

$$\Pr(\cap \mathcal{Q}_i) \leq \prod \Pr(\mathcal{Q}_i) < (1 + o(1))^b q^{db},$$

with the second inequality given by Proposition 3.5.1. (The first is obvious: since the B_i's are drawn without replacement, the probability that all are drawn from those members of $\mathcal{K}_{\overline{x}}$ that meet all A_j's is less than it would be if they were drawn independently.)

∎

Let $\mathcal{Q}(x, r)$ be the event that there is some $\mathcal{C}$ as in Lemma 3.3.1, with $|\mathcal{C}_{\overline{x}}|$ ($= |\mathcal{C}| - d_{\mathcal{C}}(x)$) $= r$, and let $\mathcal{Q}(x) = \cup_{r \geq 1} \mathcal{Q}(x, r)$. By Proposition 3.4.4 it is enough to show that (for any x)

$$\Pr(\mathcal{Q}(x) \wedge \mathcal{R}) = o(1/n). \tag{3.37}$$

(Recall $\mathcal{R}$ was defined in the paragraph containing (3.29) and (3.30).) Let

$$\mathcal{R}_x = \{m \leq m_0;\ d(x) \leq \beta;\ d(x, z) \leq 8 \ \forall z \in V \setminus \{x\};\ |W_x| \leq \mathsf{w}\}.$$

Then $\mathcal{R}_x \supseteq \mathcal{R}$, so for (3.37) it will be enough to bound

$$\Pr(\mathcal{Q}(x) \wedge \mathcal{R}_x) \leq \sum_{r \geq 1} \Pr(\mathcal{Q}(x, r) \wedge \mathcal{R}_x).$$

Set

$$\mathcal{S}(x, r) = \begin{cases} \{d(x) \geq \tau\} & \text{if } r \geq 2/\varepsilon, \\ \{d(x) \geq \alpha - r\} & \text{if } r < 2/\varepsilon, \end{cases}$$

and notice that $\mathcal{S}(x, r) \supseteq \mathcal{Q}(x, r)$. (For $r \geq 2/\varepsilon$ this is contained in the definition of $\mathcal{Q}(x, r)$ (which promises $d_{\mathcal{C}}(x) \geq \tau$), and for smaller r it is given by $d(x) \geq d_{\mathcal{C}}(x) = |\mathcal{C}| - r \geq \alpha - r$.) Thus we have

$$\begin{aligned} \Pr(\mathcal{Q}(x, r) \wedge \mathcal{R}_x) &= \Pr(\mathcal{Q}(x, r) \wedge \mathcal{S}(x, r) \wedge \mathcal{R}_x) \\ &\leq \Pr(\mathcal{S}(x, r)) \Pr(\mathcal{Q}(x, r) | \mathcal{R}_x \wedge \mathcal{S}(x, r)). \end{aligned} \tag{3.38}$$

For all but quite small r, a bound on the second factor in (3.38) will suffice for our purposes. To bound this factor, we condition on values $\mathcal{H}_x = \{A_1, \ldots, A_d\}$ and

$|\mathcal{H}_{\overline{x}}| = t$ satisfying $\mathcal{S}(x,r) \wedge \mathcal{R}_x$ (in particular $d \le \beta$ and $t \le m_0$); thus $\mathcal{H}_{\overline{x}}$ is a uniform t-subset, say $\{B_1, \ldots, B_t\}$, of $\mathcal{K}_{\overline{x}}$. If $\mathcal{Q}(x,r)$ holds under this conditioning, then there are $I \subseteq [d]$ of size at least τ and $J \subseteq [t]$ of size r such that the families $\{A_i : i \in I\}$ and $\{B_j : j \in J\}$ are cross-intersecting (namely, each of the r members of $\mathcal{C}_{\overline{x}}$ meets each of the $d_{\mathcal{C}}(x) \ge \tau$ members of $\mathcal{C}_x$).

The probability that this happens for a fixed I and J as above (note the remaining randomization is in the choice of B_j's) is, by Corollary 3.5.2, less than $(1+o(1))^r q^{\tau r}$, and it follows that the probability of $\mathcal{Q}(x,r)$ under the present conditioning—so also under conditioning on $\mathcal{S}(x,r) \wedge \mathcal{R}_x$—is less than

$$\begin{aligned} \binom{d}{\le r}\binom{m_0}{r}(1+o(1))^r q^{\tau r} &< \left[(1+o(1))\beta m_0 n^{-(1-\varepsilon)}\right]^r \\ &< n^{-(\varepsilon - o(1))r}. \end{aligned} \tag{3.39}$$

Here the first factor on the left bounds the number of possibilities for the $d - d_{\mathcal{C}}(x) \le r$ members of $[d] \setminus I$; the first inequality uses $d \le \beta$; and the second uses $\beta m_0 < (1 + o(1))(\varphi^*)^2 n/k < n^{1-2\varepsilon+o(1)}$ (see (3.13)).

Thus, as suggested above, the second factor on the r.h.s. of (3.38) is enough for us unless r is very small; namely,

$$\sum_{r > 2/\varepsilon} \Pr(\mathcal{Q}(x,r) | \mathcal{R}_x \wedge \mathcal{S}(x,r)) = o(1/n). \tag{3.40}$$

For smaller r we must use the factor $\Pr(\mathcal{S}(x,r))$ from (3.38) (together with (3.39)). Here (3.27) gives $\Pr(d_v = t)/\Pr(d_v = t+1) < n^{o(1)}$ for $t \in [\alpha - r, \alpha]$, which, since $r < O(1)$, implies

$$\Pr(\mathcal{S}(x,r)) < n^{o(1)} \Pr(d_x \ge \alpha + 1) < n^{-1+o(1)}.$$

Finally, recalling (3.39), we find that (for $r \le 2/\varepsilon$) the r.h.s. of (3.38) is less than $n^{-1+o(1)} n^{-(\varepsilon - o(1))r} = n^{-(1+r\varepsilon - o(1))}$, yielding

$$\sum_{r=1}^{\lfloor 2/\varepsilon \rfloor} \Pr(\mathcal{Q}(x,r) \wedge \mathcal{R}_x) < \sum_{r \ge 1} n^{-(1+r\varepsilon-o(1))} = o(1/n),$$

and combining this with (3.40) gives (3.37).

∎

3.6 Proof of Lemma 3.3.2

We prove Lemma 3.3.2 in the following equivalent form.

Lemma 3.6.1. *A.s. there do not exist* $x, y \in V$ *and* $\mathcal{F} \subseteq \mathcal{H}_x$, $\mathcal{G} \subseteq \mathcal{H}_y$ *with* $|\mathcal{F}| = |\mathcal{G}| = \lambda$ *and* $\mathcal{F}$, $\mathcal{G}$ *cross-intersecting.*

Proof. Let $\mathcal{Q}(x, y)$ be the event described in Lemma 3.6.1 and $\mathcal{Q} = \cup \mathcal{Q}(x, y)$. We want $\Pr(\mathcal{Q}) = o(1)$, for which it is enough to show that (for any x, y)

$$\Pr(\mathcal{Q}(x, y) \wedge \mathcal{R}) < o(n^{-2}). \tag{3.41}$$

For the proof of (3.41) we condition on values of: m satisfying (3.15) (so may think of $\mathcal{H}$ as $\{A_i : i \in [m]\}$);

$$I_x := \{i \in [m] : x \in A_i\}, \quad I_y := \{i \in [m] : y \in A_i\}$$

with $|I_x|, |I_y| \leq \beta$ and $|I_x \cap I_y| \leq 8$ (see (3.29)); and a value of $(A_i : i \in I_x)$ for which $|\{z \in V \setminus \{x\} : |\{i : v \in A_i\}| \geq 2\}| < \mathsf{w}$ (see (3.30)). If $\mathcal{Q}(x, y)$ holds (under this conditioning), then there are $J_x \subseteq I_x \setminus I_y$ and $J_y \subseteq I_y \setminus I_x$, each of size $\lambda - 8$, with the families $\{A_i : i \in J_x\}$ and $\{A_j : j \in J_y\}$ cross-intersecting.

The probability that this happens for a given J_x, $(A_i : i \in J_x)$ and J_y is, by Corollary 3.5.2, at most $[(1 + o(1))q^{\lambda-8}]^{\lambda-8} = q^{(1-o(1))\lambda^2}$, whence

$$\begin{aligned} \Pr(\mathcal{Q}(x, y) \wedge \mathcal{R}) &< \tbinom{\beta}{\lambda}^2 q^{(1-o(1))\lambda^2} \\ &< \exp[\lambda(2\log(e\beta/\lambda) - (1 - o(1))\lambda \log(1/q))] \\ &< \exp[(1 - o(1))\lambda^2 \log(1/q)] < o(n^{-3}), \end{aligned} \tag{3.42}$$

where the third inequality uses $\beta < (1 + o(1))\varphi^*$ and $\lambda \geq \sqrt{\log n}/\log(1/q)$ to say $\log(e\beta/\lambda) = O(\log\log n)$ and the last uses $\lambda \geq 2\sqrt{\log n/\log(1/q)}$.

∎

3.7 Proof of Lemma 3.3.3

Of our main points, Lemma 3.3.3 is the only one requiring the full power of the assumption (1.4), as well as the one requiring the most work: there are several ways to handle it, but we (so far) don't see anything very compact.

We again (as in the proof of Proposition 3.4.4) condition on a value of m satisfying (3.15) (so $\mathcal{H}$ is chosen uniformly from the m-subsets of $\mathcal{K}$), and then, rather than dealing directly with $\mathcal{H}$, find it easier to work with sets chosen *independently* from $\mathcal{K}$, which makes essentially no difference since m is so small compared to $|\mathcal{K}|$. Precisely, if m satisfies (3.15), $B_1, \ldots, B_m$ is a uniform m-subset of $\mathcal{K}$, $A_1, \ldots, A_m$ are chosen uniformly and independently from $\mathcal{K}$, and we set $\mathcal{D} = \{A_1, \ldots, A_m \text{ are distinct}\}$, then for any event $\mathcal{B}$ we have

$$\begin{aligned} \Pr(A_1, \ldots, A_m \models \mathcal{B}) &\geq \Pr(\mathcal{D}) \Pr(A_1, \ldots, A_m \models \mathcal{B} | \mathcal{D}) \\ &= \Pr(\mathcal{D}) \Pr(B_1, \ldots, B_m \models \mathcal{B}), \end{aligned}$$

whence

$$\begin{aligned} \Pr(B_1, \ldots, B_m \models \mathcal{B}) &\leq \Pr(A_1, \ldots, A_m \models \mathcal{B}) / \Pr(\mathcal{D}) \\ &\leq [1 - m^2/\tbinom{n}{k}]^{-1} \Pr(A_1, \ldots, A_m \models \mathcal{B}) \\ &= (1 + o(1)) \Pr(A_1, \ldots, A_m \models \mathcal{B}). \end{aligned}$$

It is thus enough to prove the following statement.

Lemma 3.7.1. *Suppose $A_1, \ldots, A_\gamma$ are drawn uniformly and independently from $\mathcal{K}$, and let $\mathcal{Q}$ be the event that $\{A_1, \ldots, A_\gamma\}$ is a clique with at most one vertex of degree greater than λ and none of degree at least τ. Then*

$$\Pr(\mathcal{Q}) = o\left(\tbinom{m}{\gamma}^{-1}\right).$$

Given $A = (A_1, \ldots, A_\gamma) \in \mathcal{K}^\gamma$ we define several related quantities. Write $d_i(v)$ for the degree of v in the multiset $\{A_1, \ldots, A_i\}$ and set $d_v = d_\gamma(v)$. (We no longer default to $d_v = d_{\mathcal{H}}(v)$, since $\mathcal{H}$ plays no further role in this section.) Note that we regard A as given and sometimes (not always) suppress it in our notation; for example $d_i(v)$ could also be written (say) $d_{A,i}(v)$.

We will need to distinguish two possibilities, depending on whether there is or is not an x with $d_A(x) > \lambda$. We treat these in parallel, the analysis in the second case eventually being more or less contained in that for the first. To this end we let $V' = V \setminus \{x\}$ if we have specified such a high-degree x and $V' = V$ otherwise.

Set $W_i = \{v \in V' : d_i(v) = 2\}$, $Z_i = \{v \in V' : d_i(v) \geq 3\}$, $U_i = W_i \cup Z_i$, $W = W_\gamma$, $Z = Z_\gamma$ and $U = U_\gamma$ $(= W \cup Z)$. In addition—now, for reasons which will appear below (see (3.49)-(3.52)), retaining A in the notation—set

$$s_i(A) = |A_i \cap W_{i-1}|, \quad r_i(A) = |A_i \cap Z_{i-1}| \quad \text{for } i \in [\gamma]$$

(with $W_0 = Z_0 = \emptyset$), $\sigma(A) = (s_1(A), \ldots, s_\gamma(A))$, $\rho(A) = (r_1(A), \ldots, r_\gamma(A))$, $s(A) = \sum s_i(A)$ and $r(A) = \sum r_i(A)$. Notice that

$$s(A) = |Z| \text{ and } r(A) = \sum_{v \in Z} (d_v - 3). \tag{3.43}$$

Finally, set

$$\Psi = \sum_{v \in Z} \left[\binom{d_v}{2} - 1 \right] \tag{3.44}$$

and notice that

$$\Psi \;=\; 2|Z| + \sum_{v \in Z} \left[\binom{d_v}{2} - 3 \right] \;=\; 2|Z| + \tfrac{1}{2} \sum_{v \in Z} (d_v - 3)(d_v + 2).$$

We will only use this when $d_v \leq \lambda$ for all $v \in V'$, in which case, in view of (3.43), we have

$$\Psi \leq 2s(A) + (\lambda + 2) r(A)/2. \tag{3.45}$$

From this point we take $A = (A_1, \ldots, A_\gamma)$ with the A_i's as in Lemma 3.7.1 (so chosen uniformly and independently from $\mathcal{K}$); thus the quantities defined above ($d_i(v)$ through Ψ) become random variables determined by A.

Proposition 3.7.2. *With probability* $1 - o\left(\binom{m}{\gamma}^{-1}\right)$,

(a) $|U| < \max\{\varphi^2 k^2/n, \log^6 n\} =: \mathsf{w}$ *and*

(b) $|Z| < \gamma/\varepsilon \ =: \mathsf{z}$.

Proof. Notice first that

$$\binom{m}{\gamma} \;<\; \exp[\gamma \log(em/\gamma)] \;<\; \exp[(1/2 + o(1))\gamma \log n], \tag{3.46}$$

since $m/\gamma \leq 3m/\varphi \sim 3n/k < n^{1/2+o(1)}$.

Since each d_v has the binomial distribution $\mathrm{B}(\gamma, k/n)$, we have (for all v, ℓ) $\Pr(d_v \geq \ell) < (k\gamma/n)^\ell/\ell!$, whence $\mathsf{E}|U| < k^2\gamma^2/(2n)$ and $\mathsf{E}|Z| < (k\gamma)^3/(6n^2) < n^{-(2\varepsilon - o(1))}\gamma$.

On the other hand, by Propositions 2.2.1 and 2.2.2, the events $\{d_v \geq \ell\}$ are negatively associated for any ℓ; so the probabilities in question may be bounded using Corollary 2.2.5. For (b), we have

$$\Pr(|Z| > \mathsf{z}) < n^{-(2\varepsilon - o(1))\gamma/\varepsilon} < n^{-\gamma} = o\left(\binom{m}{\gamma}^{-1}\right).$$

The calculations for (a) are more annoying. Here we set $k = \sqrt{\zeta n}$ and $\mu = k^2\gamma^2/(2n) = \zeta\gamma^2/2$ (our upper bound on $\mathsf{E}|U|$). The desired inequality is

$$\Pr(|U| \geq \mathsf{w}) = o\left(\binom{m}{\gamma}^{-1}\right).$$

We first observe that this is true provided

$$\gamma > 3\log n/\zeta, \tag{3.47}$$

since then (using (2.1) with $\lambda = \mu$) we have (*cf.* (3.46))

$$\Pr(|U| \geq \mathsf{w}) \leq \Pr(|U| \geq 2\mu) < \exp[-\tfrac{3\mu}{8}] = \exp[-\tfrac{3\zeta\gamma^2}{16}] < \exp[-\tfrac{9}{16}\gamma\log n].$$

In particular (3.47) holds if (e.g.) $\zeta \geq 2$, since then (according to (3.20)) we have $\gamma > (1 - o(1))\frac{\log n}{-\log(1-e^{-\zeta})} > 3\log n/\zeta$. So we may assume

$$\gamma \leq 3\log n/\zeta \quad \text{and} \quad \zeta \leq 2.$$

We then have $\log^6 n > 2\mu$, since $\log^6 n \leq 2\mu = \zeta\gamma^2 \leq 9\log^2 n/\zeta$ implies $\zeta < o(1)$, yielding $\log(1/q) = \omega(1)$ and $2\mu = \zeta\gamma^2 = o((\varphi^*)^2) = o(\log^6 n)$, a contradiction. Thus, again using Corollary 2.2.5, we have

$$\Pr(|U| > \mathsf{w}) \leq \Pr(|U| > \log^6 n) < \exp[-\Omega(\log^6 n)] < o\left(\binom{m}{\gamma}^{-1}\right)$$

(the last inequality holding since $\zeta \leq 2$ implies γ $(< \varphi^*) = O(\log^3 n)$).

∎

Set

$$\mathcal{S} = \{|W| \le \mathsf{w}, |Z| \le \mathsf{z}\}.$$

By Proposition 3.7.2, Lemma 3.7.1 will follow from

$$\Pr(\mathcal{Q} \wedge \mathcal{S}) = o\left(\tbinom{m}{\gamma}^{-1}\right). \tag{3.48}$$

For the proof of (3.48) we will bound the probabilities of various events whose union contains $\mathcal{Q}$. Set $\theta = \lfloor (n^{\varepsilon} \log(1/q))^{-1} \rfloor$ and

$$\mathcal{A} = \{\{A_1, \ldots, A_\gamma\} \text{ is a clique}\}.$$

(Note θ need not be large—e.g. it will be zero for k less than about $\sqrt{\varepsilon n \log n}$—so for once we do need the floor symbols. The parts of the following argument involving θ could be avoided when θ is small, but there seems no point in treating this separately.)

For $x \in V$, $d \in (\lambda, \tau]$, and $\sigma, \rho \in \mathbf{N}^\gamma$, let

$$\mathcal{A}(x, d, \rho, \sigma) = \mathcal{A} \wedge \{d_x = d;\ d_v \le \lambda\ \forall v \ne x;\ \rho(A) = \rho;\ \sigma(A) = \sigma\}, \tag{3.49}$$

$$\mathcal{A}(x, d, \rho) = \mathcal{A} \wedge \{d_x = d;\ d_v \le \lambda\ \forall v \ne x;\ \rho(A) = \rho;\ s(A) \le \theta\}, \tag{3.50}$$

$$\mathcal{A}(\rho, \sigma) = \mathcal{A} \wedge \{d_v \le \lambda\ \forall v;\ \rho(A) = \rho;\ \sigma(A) = \sigma\} \tag{3.51}$$

and

$$\mathcal{A}(\rho) = \mathcal{A} \wedge \{d_v \le \lambda\ \forall v;\ \rho(A) = \rho;\ s(A) \le \theta\}. \tag{3.52}$$

For $r, s \in \mathbf{N}$, let $X(r, s) = (\lambda+2)r/2+2s$ (the value in (3.45)), and, for $\varrho = (\varrho_1, \ldots, \varrho_\gamma)$, set $|\varrho| = \sum \varrho_i$.

Lemma 3.7.3. *For any x, d, ρ, σ as above with $|\rho| = r$ and $|\sigma| = s$,*

$$\Pr(\mathcal{A}(x, d, \rho, \sigma) \wedge \mathcal{S}) < \binom{\gamma}{d} \left(\frac{k}{n}\right)^d \left(\frac{\mathsf{z}k}{n}\right)^r \left(\frac{\mathsf{w}k}{n}\right)^s q^{\binom{\gamma}{2} - \binom{d}{2} - X(r,s)} \tag{3.53}$$

and

$$\Pr(\mathcal{A}(\rho, \sigma) \wedge \mathcal{S}) < \left(\frac{\mathsf{z}k}{n}\right)^r \left(\frac{\mathsf{w}k}{n}\right)^s q^{\binom{\gamma}{2} - X(r,s)}. \tag{3.54}$$

For any x, d, ρ as above with $|\rho| = r$,

$$\Pr(\mathcal{A}(x,d,\rho)\wedge\mathcal{S}) < \binom{\gamma}{d}\left(\frac{k}{n}\right)^d\left(\frac{\mathsf{z}k}{n}\right)^r q^{\binom{\gamma}{2}-\binom{d}{2}-X(r,\theta)} \tag{3.55}$$

and

$$\Pr(\mathcal{A}(\rho)\wedge\mathcal{S}) < \left(\frac{\mathsf{z}k}{n}\right)^r q^{\binom{\gamma}{2}-X(r,\theta)}. \tag{3.56}$$

(We will only use (3.53) and (3.54) with $s > \theta$.)

Before proving Lemma 3.7.3 we show that it implies (3.48). Notice that $\mathcal{Q}$ is the (disjoint) union of the events

$$\mathcal{A}(x,d,\rho,\sigma),\quad \mathcal{A}(x,d,\rho),\quad \mathcal{A}(\rho,\sigma)\ \text{ and }\ \mathcal{A}(\rho), \tag{3.57}$$

where $x \in V$, $d \in (\lambda,\tau]$, $\rho \in \mathbb{N}^\gamma$ and $\sigma \in \{(s_1,\ldots,s_\gamma) \in \mathbf{N}^\gamma : \sum s_i > \theta\}$. Thus

$$\Pr(\mathcal{Q}\wedge\mathcal{S}) \le \sum \Pr(\mathcal{E}\wedge\mathcal{S}), \tag{3.58}$$

where $\mathcal{E}$ ranges over the events in (3.57).

It's now convenient to separate the contributions involving x, ρ and σ. Set

$$f(d) = n\binom{\gamma}{d}\left(\frac{k}{n}\right)^d q^{-\binom{d}{2}},$$

$$g(r) = \binom{\gamma+r-1}{r}\left(\frac{\mathsf{z}k}{n}\right)^r q^{-(\lambda+2)r/2},$$

$$h(s) = \binom{\gamma+s-1}{s}\left(\frac{\mathsf{w}k}{n}\right)^s q^{-2s}$$

and

$$h^* = q^{-2\theta}.$$

Then, noting that (e.g.) $|\{\rho \in \mathbf{N}^\gamma : |\rho| = r\}| = \binom{\gamma+r-1}{r}$ and using (3.53)-(3.56), we find that $\Pr(\mathcal{Q}\wedge\mathcal{S})$ (or the r.h.s. of (3.58)) is less than

$$q^{\binom{\gamma}{2}}\left[\sum_{d,r,s} f(d)g(r)h(s) + h^*\sum_{d,r} f(d)g(r) + \sum_{r,s} g(r)h(s) + h^*\sum_{r} g(r)\right],$$

where d, r and s range over $(\lambda,\tau]$, $\mathbf{N}$ and (θ,∞) respectively. Thus, since $q^{\binom{\gamma}{2}} = o\left(\binom{m}{\gamma}^{-1}\right)$ (by (3.17), (3.28) and (3.33) if $\gamma = \alpha$, and with plenty of room if $\gamma = \varphi^*/3$), it is enough to show that each of

$$\textstyle\sum_{r\ge 0} g(r),\quad \sum_{s>\theta} h(s)\quad \text{and}\quad h^*$$

is $O(1)$ and that, with $F = \sum_{d\in(\lambda,\tau]} f(d)$,

$$q^{\binom{\gamma}{2}} F \binom{m}{\gamma}^{-1} \quad (= \Lambda(\gamma)F) \quad = o(1). \tag{3.59}$$

These are all easy calculations, as follows.

First,

$$g(r) \le \left[e\gamma(\mathsf{z}k/n)n^{o(1)}\right]^r < \left[\gamma^2 n^{-1/2+o(1)}\right]^r < n^{-(2\varepsilon-o(1))r}$$

(where the first inequality uses $k > n^{1/2-o(1)} \Rightarrow q > n^{-o(1)} \Rightarrow \log(1/q) = o(\log n) \Rightarrow \lambda\log(1/q) = o(\log n)$), implying $\sum_{r\ge 0} g(r) = 1 + o(1)$.

Second, since

$$\binom{\gamma+s-1}{s}^{1/s} < \frac{e(\gamma+s)}{s} < \frac{e(\gamma+\theta)}{\theta} < n^{\varepsilon+o(1)}$$

(for $s > \theta$), $\frac{\mathsf{w}k}{n} < n^{-2\varepsilon+o(1)}$ and $q = 1 - o(1)$, we have

$$\sum_{s>\theta} h(s) < \sum_{s>\theta} n^{-(\varepsilon-o(1))s} = o(1).$$

Third, $h^* = o(1)$ is immediate from our choice of θ.

The fourth calculation requires a little more care. Notice first that

$$\begin{aligned} f(d) &< n((e\gamma/d)(k/n))^d\, q^{-\binom{d}{2}} \\ &< n\cdot n^{-(1/2-o(1))d} q^{-\binom{d}{2}} < n\cdot\left[n^{-(1-o(1))}q^{-d}\right]^{d/2} \end{aligned} \tag{3.60}$$

(where the second inequality uses $\gamma/d < (1+o(1))\varphi^*/\lambda < n^{o(1)}$). Here we may confine ourselves to

$$d > (1-o(1))\log n/\log(1/q), \tag{3.61}$$

since for smaller d the expression in square brackets in (3.60) is less than $n^{-\Omega(1)}$ (and the exponent $d/2$ is at least $\lambda/2 = \omega(1)$), so that the contribution of such d to F is $o(1)$.

For d as in (3.61) the bound in (3.60) is (rapidly) increasing in d (passing from d to $d+1$ multiplies it by roughly $\sqrt{n}$); so the contribution of such d to $\Lambda(\gamma)F$ is dominated by that of $d = \tau$. For this term we have $\gamma = (1-\varepsilon)^{-1}\tau = (1-\varepsilon)^{-1}d > (1+\varepsilon)\log n/\log(1/q)$ and

$$\begin{aligned} \Lambda(\gamma)f(\tau) &< n^{-(1/2-o(1))\tau+\gamma/2}q^{(\gamma-\tau)(\gamma+\tau-1)/2} \\ &< n^{[\varepsilon/2-(1+\varepsilon)\varepsilon(1-\varepsilon/2)+o(1)]\gamma} \\ &= n^{-(\varepsilon-\varepsilon^2/2-o(1))\gamma}. \end{aligned}$$

Thus we have (3.59).

∎

For the proof of Lemma 3.7.3, we need the following easy observation.

Proposition 3.7.4. *Let $Y_1, \ldots, Y_\ell$ be r.v.'s (not necessarily real-valued) and write y_i for a possible value of Y_i. Let $\mathcal{Z}$ be a set of ("bad") prefixes $(y_1, \ldots, y_i)$ closed under extension (i.e. $i < \ell$ and $(y_1, \ldots, y_i) \in \mathcal{Z}$ imply $(y_1, \ldots, y_i, y_{i+1}) \in \mathcal{Z}$ for every choice of y_{i+1}). Set*

$$\Pr((Y_1, \ldots, Y_i) \in \mathcal{Z} | y_1, \ldots, y_{i-1}) = 1 - \xi(y_1, \ldots, y_{i-1}),$$

where the conditioning has the obvious meaning and when $i = 1$ the l.h.s. is $\Pr((Y_1) \in \mathcal{Z})$. Then

$$\Pr((Y_1, \ldots, Y_\ell) \notin Z) \leq \max_{(y_1,\ldots,y_\ell)\notin\mathcal{Z}} \prod_{i=1}^{\ell} \xi(y_1, \ldots, y_{i-1}) =: \xi.$$

Proof. Define an auxiliary sequence $(X_0, \ldots, X_\ell)$ with $X_0 \equiv 1$ and, for $i \in [\ell]$,

$$X_i = \begin{cases} 0 & \text{if } (Y_1, \ldots, Y_i) \in \mathcal{Z}, \\ \xi(Y_1, \ldots, Y_{i-1})^{-1} X_{i-1} & \text{otherwise.} \end{cases}$$

Then $\mathsf{E} X_\ell = X_0 = 1$ (since $(X_0, \ldots, X_\ell)$ is a martingale), while $X_\ell \geq \xi^{-1}$ whenever $(Y_1, \ldots, Y_\ell) \notin \mathcal{Z}$ (using the fact that $\mathcal{Z}$ is closed under extensions). The conclusion follows.

∎

We now turn to the proof of Lemma 3.7.3, beginning with the simpler (3.54) and (3.56); the arguments for (3.53) and (3.55) are similar, and when we come to these we will mainly just point out the necessary modifications.

For both (3.54) and (3.56) we will apply Proposition 3.7.4 to the sequence $(Y_1, \ldots, Y_{2\gamma})$, where

$$Y_{2j-1} = A_j \cap U_{j-1} \text{ and } Y_{2j} = A_j \setminus U_{j-1}. \tag{3.62}$$

We first prove (3.54) and then discuss the changes needed for (3.56).

Proof of (3.54). Here we say $(Y_1, \ldots, Y_i) \in \mathcal{Z}$ (recall this is the set of "bad" prefixes) if the associated A_j's (or parts of A_j's) satisfy at least one of:

$$\{A_1, \ldots, A_{\lfloor i/2 \rfloor}\} \text{ is not a clique;} \tag{3.63}$$

$$\text{for some } j \le \lceil i/2 \rceil, \ |A_j \cap Z_{j-1}| \ne r_j \text{ or } |A_j \cap W_{j-1}| \ne s_j; \tag{3.64}$$

$$|Z_{\lceil i/2 \rceil}| > \mathsf{z}, \ |W_{\lceil i/2 \rceil}| > \mathsf{w} \text{ or } d_{\lceil i/2 \rceil}(v) > \lambda \text{ for some } v. \tag{3.65}$$

Then $\mathcal{A}(\rho, \sigma) \wedge \mathcal{S} = \{(Y_1, \ldots, Y_{2\gamma}) \notin \mathcal{Z}\}$.

We next need to say something about the quantities

$$\xi(y_1, \ldots, y_{i-1}) = \Pr(Y_1, \ldots, Y_i \notin \mathcal{Z} | y_1, \ldots, y_{i-1})$$

appearing in Proposition 3.7.4, where (we may assume) $(y_1, \ldots, y_{i-1}) \notin \mathcal{Z}$.

If $i = 2j - 1$ then

$$\begin{aligned} \xi(y_1, \ldots, y_{i-1}) &\le \Pr(|A_i \cap Z_{i-1}| \ge r_i, |A_i \cap W_{i-1}| \ge s_i | y_1, \ldots, y_{i-1}) \\ &\le (\mathsf{z}k/n)^{r_i} (\mathsf{w}k/n)^{s_i}. \end{aligned} \tag{3.66}$$

Here we again use Propositions 2.2.1 and 2.2.3, which, since $(y_1, \ldots, y_{i-1}) \notin \mathcal{Z}$ implies $|Z_{i-1}| \le \mathsf{z}$ and $|W_{i-1}| \le \mathsf{w}$, bound the probability in (3.66) by

$$\tbinom{\mathsf{z}}{r_i} \tbinom{\mathsf{w}}{s_i} (k/n)^{r_i} (k/n)^{s_i}.$$

The case $i = 2j$ is more interesting. Here, conditioning on the event $\{(Y_1, \ldots, Y_{i-1}) = (y_1, \ldots, y_{i-1})\}$, we set

$$\beta_j = \sum \{d_{j-1}(v) : v \in A_j \cap U_{j-1}\}. \tag{3.67}$$

(Notice that this is determined by $(y_1,\ldots,y_{i-1})$, which includes specification of $Y_{2j-1} = A_j \cap U_{j-1}$.) We will show

$$\xi(y_1,\ldots,y_{i-1}) \le q^{j-1-\beta_j}. \tag{3.68}$$

Here we only consider (3.63); that is, we we ignore the requirements in (3.65) (those in (3.64) are not affected by Y_i) and show that (given our conditioning) the r.h.s. of (3.68) bounds the probability that A_j meets all of $A_1,\ldots,A_{j-1}$. Now A_j meets at most β_j members of $\{A_1,\ldots,A_{j-1}\}$ in U_{j-1}, so to avoid (3.63) must meet the $j-1-\beta_j$ or more remaining members—say those indexed by I—in $V \setminus U_{j-1}$, where they are pairwise disjoint. This gives (3.68) since the events $Q_h = \{A_j \cap (A_h \setminus U_{j-1}) \neq \emptyset\}$ $(h \in I)$ satisfy $\Pr(Q_h) < q$ (by Proposition 3.4.1) and are NA (by Propositions 2.2.1 and 2.2.2), so by Proposition 2.2.3 we have

$$\Pr(\cap_{h\in I} Q_h) \le \textstyle\prod_{h\in I} \Pr(Q_h) < q^{j-1-\beta_j}.$$

The last thing to notice here is that, provided $d_\gamma(v) \le \lambda\ \forall v$—which in particular is true whenever $(Y_1,\ldots,Y_{2\gamma}) \notin \mathcal{Z}$; see (3.65))—we have

$$\sum \beta_j = \Psi \le X(r,s) \tag{3.69}$$

(see (3.44) for Ψ and (3.45) for the inequality). Finally, combining (3.66), (3.68) and (3.69) (and $\sum_{j\in[\gamma]}(j-1) = \binom{\gamma}{2}$) and applying Proposition 3.7.4 gives (3.54).

∎

Proof of (3.56). We now take $(Y_1,\ldots,Y_i) \in \mathcal{Z}$ if the associated A_j's satisfy at least one of:

$$\{A_1,\ldots,A_{\lfloor i/2\rfloor}\} \text{ is not a clique;} \tag{3.70}$$

$$\textstyle\sum_{j\le\lceil i/2\rceil} s_j(A) > \theta \text{ , or for some } j \le \lceil i/2\rceil,\ \ |A_j \cap Z_{j-1}| \neq r_j; \tag{3.71}$$

$$|Z_{\lceil i/2\rceil}| > \mathsf{z},\ \ |W_{\lceil i/2\rceil}| > \mathsf{w} \text{ or } d_{\lceil i/2\rceil}(v) > \lambda \text{ for some } v. \tag{3.72}$$

Then $\mathcal{A}(\rho) \wedge \mathcal{S} \subseteq \{(Y_1,\ldots,Y_{2\gamma}) \notin \mathcal{Z}\}$.

The arguments bounding the quantities

$$\xi(y_1,\ldots,y_{i-1}) = \Pr(Y_1,\ldots,Y_i \notin \mathcal{Z}|y_1,\ldots,y_{i-1})$$

(again, for $(y_1,\ldots,y_{i-1}) \notin \mathcal{Z}$) are essentially identical to those above. For $i = 2j-1$ the bound

$$\xi(y_1,\ldots,y_{i-1}) \leq \Pr(|A_i \cap Z_{i-1}| \geq r_i|y_1,\ldots,y_{i-1}) \leq (\mathsf{z}k/n)^{r_i} \tag{3.73}$$

is justified in the same way as (3.66). For $i = 2j$ we again define β_j as in (3.67), and (3.68) follows as before. (Note that our only reason for retaining the constraint on $|W_{\lceil i/2 \rceil}|$ in (3.72) is to enforce $\Pr(A_j \cap (A_h \setminus U_{j-1}) \neq \emptyset) < q$ in the proof of (3.68).

Finally, (3.69) again holds provided $(Y_1,\ldots,Y_{2\gamma}) \notin \mathcal{Z}$ (this is where we use the first condition in (3.71)), and combining this with (3.73) and (3.68) we obtain (3.56) *via* Proposition 3.7.4.

∎

We now turn to the parts of Lemma 3.7.3 involving x. For $D \in \binom{[\gamma]}{d}$ let

$$\mathcal{A}(x,D,\rho,\sigma) = \mathcal{A} \wedge \{x \in A_i \Leftrightarrow i \in D;\ d_v \leq \lambda\ \forall v \neq x;\ \rho(A) = \rho;\ \sigma(A) = \sigma\},$$

$$\mathcal{A}(x,D,\rho) = \mathcal{A} \wedge \{x \in A_i \Leftrightarrow i \in D;\ d_v \leq \lambda\ \forall v \neq x;\ \rho(A) = \rho;\ s(A) \leq \theta\}.$$

Since $\Pr(\mathcal{A}(x,d,\rho,\sigma)$ is the sum of the $\Pr(\mathcal{A}(x,D,\rho,\sigma))$'s (and similarly for $\Pr(\mathcal{A}(x,d,\rho))$, (3.53) and (3.55) will follow from (respectively)

$$\Pr(\mathcal{A}(x,D,\rho,\sigma)) < \left(\frac{k}{n}\right)^d \left(\frac{\mathsf{z}k}{n}\right)^r \left(\frac{\mathsf{w}k}{n}\right)^s q^{\binom{t}{2}-\binom{d}{2}-X(r,s)} \tag{3.74}$$

and

$$\Pr(\mathcal{A}(x,D,\rho)) < \left(\frac{k}{n}\right)^d \left(\frac{\mathsf{z}k}{n}\right)^r q^{\binom{t}{2}-\binom{d}{2}-X(r,\theta)}. \tag{3.75}$$

As the proofs of these closely track those of (3.54) and (3.56) (respectively), with exactly the same modifications, we confine ourselves to indicating what changes to the proof of (3.54) are needed for (3.74).

We again apply Proposition 3.7.4, in this case to the sequence $(Y_1, \ldots, Y_{2\gamma})$ given by

$$Y_{2j-1} = A_j \cap (U_{j-1} \cup \{x\}) \text{ and } Y_{2j} = A_j \setminus (U_{j-1} \cup \{x\})$$

(which differs from (3.62) in the addition of $\{x\}$ to the U_{j-1}'s). We say $(Y_1, \ldots, Y_i) \in \mathcal{Z}$ if the associated A_j's satisfy at least one of (3.63), (3.64) (we recall for ease of reading that these were

$$\{A_1, \ldots, A_{\lfloor i/2 \rfloor}\} \text{ is not a clique}$$

and

$$\text{for some } j \leq \lceil i/2 \rceil, \ |A_j \cap Z_{j-1}| \neq r_j \text{ or } |A_j \cap W_{j-1}| \neq s_j),$$

$$|Z_{\lceil i/2 \rceil} > \mathsf{z}, \ |W_{\lceil i/2 \rceil}| > \mathsf{w} \text{ or } d_{\lceil i/2 \rceil}(v) > \lambda \text{ for some } v \neq x \tag{3.76}$$

(which differs from (3.65) in the stipulation $v \neq x$) and

$$\text{for some } j \leq \lceil i/2 \rceil \text{ either } j \in D \text{ and } x \notin A_j \text{ or } j \notin D \text{ and } x \in A_j. \tag{3.77}$$

Then $\mathcal{A}(x, D, \rho, \sigma) \wedge \mathcal{S} = \{(Y_1, \ldots, Y_{2\gamma}) \notin \mathcal{Z}\}$.

The bounds on the quantities

$$\xi(y_1, \ldots, y_{i-1}) = \Pr(Y_1, \ldots, Y_i \notin \mathcal{Z} | y_1, \ldots, y_{i-1})$$

(again, for $(y_1, \ldots, y_{i-1}) \notin \mathcal{Z}$) are modified as follows. For $i = 2j - 1$ we use

$$\xi(y_1, \ldots, y_{i-1}) \leq \begin{cases} (k/n)(\mathsf{z}k/n)^{r_i}(\mathsf{w}k/n)^{s_i} & \text{if } j \in D, \\ (\mathsf{z}k/n)^{r_i}(\mathsf{w}k/n)^{s_i} & \text{otherwise;} \end{cases} \tag{3.78}$$

this is justified (*via* Propositions 2.2.1 and 2.2.3) in the same way as (3.66).

For $i = 2j$ we define β_j as before ($\beta_j = \sum\{d_{j-1}(v) : v \in A_j \cap U_{j-1}\}$) and set $\mathsf{c}_j = [j-1] \setminus D$ (again, a function of $(y_1, \ldots, y_{i-1})$). We then have

$$\xi(y_1, \ldots, y_{i-1}) \leq \begin{cases} q^{\mathsf{c}_j - \beta_j} & \text{if } j \in D, \\ q^{j-1-\beta_j} & \text{otherwise.} \end{cases} \tag{3.79}$$

The proof is essentially the same as that for (3.68), the only difference being that when $j \in D$, there is no requirement that A_j meet those earlier A_l's for which $l \in D$. (On the other hand, the second bound in (3.79) uses the fact that $x \notin A_j$ (for $j \notin D$), which follows from $(y_1, \ldots, y_{i-1}) \notin \mathcal{Z}$; see (3.77).)

Finally, applying Proposition 3.7.4 with the combination of (3.78), (3.79) and $\sum \beta_j = \Psi \le X(r,s)$ (noted earlier in (3.69)) gives (3.74), once we observe that

$$\sum_{j \notin D} (j-1) + \sum_{j \in D} \mathsf{c}_j = \sum_j (j-1) - \sum_{j \in D} |[j-1] \cap D| = \binom{\gamma}{2} - \binom{d}{2}.$$

∎

3.8 Large φ

Here we complete the proof of Theorem 1.0.2 by showing

$$\text{for } \varphi > \varphi^*,\ \mathcal{H} \text{ satisfies EKR a.s.} \tag{3.80}$$

As already mentioned, this is mostly a matter of reducing to φ^* and applying Lemmas 3.3.1-3.3.3. (While there ought to be other ways to handle this, our main argument runs into difficulties when φ is large, since the sets W_x, W, Z used in the proofs of Lemmas 3.3.1-3.3.3 are no longer small.)

For the rest of this section we take $\varphi > \varphi^*$. We use the following natural reduction (coupling). Setting $\rho = \varphi^*/\varphi$, we let $\mathcal{G}$ be the random subhypergraph of $\mathcal{H}$ gotten by retaining edges independently, each with probability ρ; thus $\mathcal{G} \sim \mathcal{H}_k(n, p^*)$, with $p^* = \varphi^*/M$.

We would *like* to say that if EKR fails for $\mathcal{H}$, say at the nontrivial clique $\mathcal{C}$, then there is a decent chance that the clique $\mathcal{D} := \mathcal{C} \cap \mathcal{G}$ satisfies one of the unlikely scenarios described in Lemmas 3.3.1-3.3.3; but this is not always true, since if $\mathcal{C}$ is too close to a star, then $\mathcal{D}$ is likely to actually *be* a star. This special situation is handled by Lemma 3.8.1, and in other cases the desired reduction is given by the routine Proposition 3.8.2.

Set $r_0 = \xi\varphi$ with $\xi = \log(1/q)/(2\log n)$ (as elsewhere, just a convenient value).

Lemma 3.8.1. *A.s. there do not exist (in $\mathcal{H}$) a nontrivial clique $\mathcal{C}$ and vertex x such that $|\mathcal{C}| \ge \max\{\varphi/2, d(x)\}$ and $|\mathcal{C}_{\overline{x}}| \le r_0$.*

Proposition 3.8.2. *Suppose $\mathcal{C}$ is a nontrivial clique of $\mathcal{H}$ with $|\mathcal{C}| \geq \Delta \geq \varphi/2$ and $\Delta_{\mathcal{C}} \leq |\mathcal{C}| - r_0$, and let x be a maximum degree vertex of $\mathcal{C}$. Then with probability at least $1/2 - o(1)$, $\mathcal{D} := \mathcal{C} \cap \mathcal{G}$ satisfies:*

(a) $|\mathcal{D}| \geq \max\{d_{\mathcal{G}}(x), \gamma\}$;

(b) $|\mathcal{D}_{\overline{x}}| > 2/\varepsilon$;

(c) *either* $\Delta_{\mathcal{D}} < \tau$ *or* $d_{\mathcal{D}}(x) > \lambda$.

(Note that γ—which was defined in (3.7)—is now just $\varphi^*/3$.)

Before proving these assertions we show that they (with Lemmas 3.3.1-3.3.3) give (3.80). Since $\Delta \geq \varphi/2$ a.s. (really, $d_v \geq \varphi/2$ $\forall v$ a.s. by Theorem 2.2.4), Lemma 3.8.1 says it is enough to show that $\mathcal{H}$ is unlikely to contain a nontrivial clique $\mathcal{C}$ with $|\mathcal{C}| \geq \Delta \geq \varphi/2$ and $\Delta_{\mathcal{C}} < |\mathcal{C}| - r_0$. So we suppose this does happen, let x be some maximum degree vertex of $\mathcal{C}$, and observe that $\mathcal{D}$ and x are then fairly likely (that is, with probability at least $1/2-o(1)$) to exhibit one of the improbable behaviors described in Lemmas 3.3.1-3.3.3; namely this is true if the conclusions of Proposition 3.8.2 hold:

(i) if $\mathcal{D}$ has at least two vertices of degree at least λ, then Lemma 3.3.2 applies; otherwise:

(ii) if $\Delta_{\mathcal{D}} < \tau$ then we are in the situation of Lemma 3.3.3 (since (a) of Proposition 3.8.2 gives $|\mathcal{D}| \geq \gamma$ and we assume $\mathcal{D}$ has at most one vertex of degree at least λ);

(iii) if $\Delta_{\mathcal{D}} \geq \tau$ then in fact $d_{\mathcal{D}}(x) \geq \tau$ (by (c), since we assume $\mathcal{D}$ has at most one vertex of degree at least λ $(< \tau)$); so in view of (a) and (b) we are in the situation of Lemma 3.3.1.

∎

Proof of Lemma 3.8.1. We need one preliminary observation. For given x and $\mathcal{B} \subseteq \mathcal{K}_x$, let $g(\mathcal{B})$ be the probability that A chosen uniformly from $\mathcal{K}_{\overline{x}}$ meets all members of $\mathcal{B}$.

Suppose that, for some s, $\mathcal{B}$ is a uniform s-subset of $\mathcal{K}_x$ and A is uniform from $\mathcal{K}_{\overline{x}}$ (these choices made independently). Then

$$\mathsf{E}g(\mathcal{B}) = \Pr(A \cap B \neq \emptyset \ \forall B \in \mathcal{B}) < \mathsf{q}^s, \tag{3.81}$$

the inequality holding because (i) $\Pr(A \cap B \neq \emptyset) < \mathsf{q}$ for A and B uniform from $\mathcal{K}_{\overline{x}}$ and $\mathcal{K}_x$ respectively, and (ii) the probability in (3.81) is no more than it would be if the members of $\mathcal{B}$ were chosen independently. Markov's Inequality thus gives (for any $a \leq s$)

$$\Pr(g(\mathcal{B}) > \mathsf{q}^a) < \mathsf{q}^{s-a}.$$

Now let $\mathcal{S} = \mathcal{P} \wedge \{d(x) \geq \varphi/2 \ \forall x\}$ (recall $\mathcal{P}$ was defined in the paragraph containing (3.22)), noting that (by (3.22) and Theorem 2.2.4) $\Pr(\overline{\mathcal{S}}) = o(1)$. Let

$$\mathcal{Q}(x) = \{\exists \mathcal{B} \subseteq \mathcal{H}_x : \ |\mathcal{B}| = d(x) - r_0 \text{ and } g(\mathcal{B}) > \mathsf{q}^{\varphi/4}\}$$

and $\mathcal{Q} = \cup \mathcal{Q}(x)$. Then

$$\begin{aligned} \Pr(\mathcal{Q} \wedge \mathcal{S}) &< n\tbinom{\beta}{r_0}\mathsf{q}^{\varphi/4 - r_0} \\ &< n \exp[r_0 \log(e\beta/r_0) - (\varphi/4 - r_0)\log(1/\mathsf{q})]. \end{aligned} \tag{3.82}$$

Recalling that $\varphi \sim \beta$ we have

$$r_0 \log(e\beta/r_0) \sim \xi\varphi \log(1/\xi) < (1/8)\varphi \log(1/\mathsf{q})$$

(since $\log(1/q) > n^{-1/4+\Omega(1)}$ implies $\log(1/\xi) < (1/4)\log n$); so, noting that $q > n^{-o(1)}$ implies $r_0 = o(\varphi)$ and recalling that $\varphi > \varphi^*$, we find that the r.h.s. of (3.82) is $o(1)$.

Thus, with $\mathcal{T}$ the event in Lemma 3.8.1, the lemma will follow from

$$\Pr(\mathcal{T} \wedge \overline{\mathcal{Q}} \wedge \mathcal{S}) = o(1). \tag{3.83}$$

We show

$$\Pr(\mathcal{T} \wedge \overline{\mathcal{Q}} \wedge \mathcal{S}) \leq n \sum_{r=1}^{r_0} (\beta m_0 \mathsf{q}^{\varphi/4})^r \tag{3.84}$$

(and then observe that the r.h.s. is small).

Proof of (3.84) *and* (3.83) . We consider occurrence of $\mathcal{T}$ at a given x, writing $\mathcal{T}(x)$ for this event. Since

$$\Pr(\mathcal{T} \wedge \overline{\mathcal{Q}} \wedge \mathcal{S}) \leq \Pr(\mathcal{T} | \overline{\mathcal{Q}} \wedge \{d(x) \leq \beta, m \leq m_0\})$$

(the conditioning event contains $\overline{\mathcal{Q}} \wedge \mathcal{S}$), it is enough to show

$$\Pr^*(\mathcal{T}(x)) < (\beta m_0 \mathsf{q}^{\varphi/4})^r,$$

where $\Pr^*$ denotes probability under conditioning on some $\mathcal{H}_x$ of size at most β satisfying $\overline{\mathcal{Q}}(x)$, together with a value $m \le m_0$ of $|\mathcal{H}|$.

If, under this conditioning, $\mathcal{T}(x)$ occurs at $\mathcal{C}$ with $|\mathcal{C} \setminus \mathcal{H}_x| = r$ $(\in [1, r_0])$, then, since $d_{\mathcal{C}}(x) = |\mathcal{C}| - r \ge \varphi/2 - r$ and $|\mathcal{H}_x \setminus \mathcal{C}| \le r$, there are $\mathcal{B} \subseteq \mathcal{H}_x$ and $\mathcal{D} \subseteq \mathcal{H}_{\overline{x}}$ (namely $\mathcal{B} = \mathcal{C}_x$, $\mathcal{D} = \mathcal{C}_{\overline{x}}$) with

$$|\mathcal{B}| = |\mathcal{C}| - r \ge \max\{d(x) - r, \varphi/2 - r\},$$

$|\mathcal{D}| = r$, $\mathcal{B}$ and $\mathcal{D}$ cross-intersecting, and $g(\mathcal{B}) \le \mathsf{q}^{\varphi/4}$ (the last property implied by $\overline{\mathcal{Q}}(x)$; of course if $|\mathcal{B}| \ge d(x) - r$ and $g(\mathcal{B}) > \mathsf{q}^{\varphi/4}$, then $g(\mathcal{B}') > \mathsf{q}^{\varphi/4}$ for any $(d(x) - r_0)$-subset $\mathcal{B}'$ of $\mathcal{B}$). But the probability that this occurs given $\mathcal{H}_x$ and m as above is at most $\binom{d(x)}{\le r}\binom{m-d(x)}{r}\mathsf{q}^{\varphi r/4} < (\beta m_0 \mathsf{q}^{\varphi/4})^r$ (which gives (3.84)).

Finally (now for (3.83)), we have $\beta m_0 \mathsf{q}^{\varphi/4} < \varphi^2 n^{1/2+o(1)} \mathsf{q}^{\varphi/4} = o(1/n)$, where the first inequality uses $\beta \sim \varphi$ and $m_0 \sim \varphi n/k$, and the second holds because $\varphi^2 \mathsf{q}^{\varphi/4}$ is decreasing in $\varphi > \varphi^*$ and is $o(1/n)$ when $\varphi = \varphi^*$.

∎

Proof of Proposition 3.8.2. Of course $\Pr(|\mathcal{D}| \ge d_{\mathcal{G}}(x)) \ge 1/2$, so it's enough to show that each of the other requirements (namely, $|\mathcal{D}| \ge \gamma$ and those in (b), (c)) holds a.s. These are all routine applications of Theorem 2.2.4 (or Corollary 2.2.5): First, $|\mathcal{D}|$ is binomial with mean $|\mathcal{C}|\rho \ge (\varphi/2)\rho = \varphi^*/2 = 3\gamma/2$, implying $\Pr(|\mathcal{D}| < \gamma) < \exp[-\Omega(\gamma)]$. Second, $\mathsf{E}|\mathcal{D}_{\overline{x}}| \ge r_0\rho = \xi\varphi^* = \omega(1)$, so $\Pr(|\mathcal{D}_{\overline{x}}| < 2/\varepsilon) < \exp[-\omega(1)]$. Third, since $\tau \gg \lambda$ we have either $\Delta_{\mathcal{C}}\rho$ $(= \mathsf{E} d_{\mathcal{D}}(x))) > 2\lambda$, implying $\Pr(d_{\mathcal{D}}(x) < \lambda) = o(1)$, or $\Delta_{\mathcal{C}}\rho < \tau/2$, implying $\Pr(\Delta_{\mathcal{D}} \ge \tau) < n\exp[-\Omega(\tau)] = o(1)$; thus (c) also holds a.s.

∎

3.9 Small k

Finally, we turn to the proof of Theorem 1.0.2 for $k < n^{1/2-\Omega(1)}$, say

$$k \le n^{1/2-\varepsilon} \tag{3.85}$$

with $\varepsilon > 0$ fixed. This is, as noted earlier, easier than what we've already done, one reason being the absence of the issue discussed following (3.6): there will now always be an α such that $\Delta \geq \alpha$ a.s. and there is a.s. no nontrivial clique of size at least α. This will mean that here we only need Proposition 3.4.3 (which for k as in (3.85) and *fixed* α was proved in [4]) and a simpler version of Lemma 3.3.3. Since most of this consists of simpler versions of earlier arguments, parts of the discussion will be a bit sketchy.

It will be helpful to think of three regimes: (i) $\varphi < n^{-\Omega(1)}$; (ii) $n^{-o(1)} < \varphi \ll 1$; and (iii) $\varphi = \Omega(1)$. The last of these is treated in [4, Theorem 1.1(iv)], so we may concentrate on the first two.

We first need to specify α. If we are in regime (ii) then we take α as in Section 3.4 (recall this assumed $\varphi > n^{-o(1)}$ but not (3.1), noting that, in addition to $\Delta \geq \alpha$ a.s. (see (3.18)) and $\Lambda(\alpha) = o(1)$ (see (3.17)), we have $\alpha = \omega(1)$. (Remark: here $\alpha = \alpha_1$.)

If we are in regime (i) then α is the least integer satisfying $\varphi \gg n^{-1/\alpha}$, for which (since $\Pr(d_v \geq \alpha) \asymp \varphi^\alpha$) Proposition 3.4.3 gives $\Delta \geq \alpha$ a.s. Note that here too we have $\Lambda(\alpha) = o(1)$, which is given by (1.4) once we observe that, by Harris' Inequality [13],

$$\begin{aligned} \Pr(\Delta \leq \alpha) &= \Pr(d_v \leq \alpha \ \forall v) \\ &\geq \prod_v \Pr(d_v \leq \alpha) = (1 - O(1/n))^n = \Omega(1). \end{aligned} \tag{3.86}$$

(Of course if $\varphi \ll n^{-1/(\alpha+1)}$, then $\Delta = \alpha$ a.s., and it is not hard to see that if $\varphi \asymp n^{-1/(\alpha+1)}$, then $\Delta \in \{\alpha, \alpha+1\}$ a.s. and each possibility occurs with probability $\Omega(1)$.)

For regime (i) we will usually use c in place of α to remind ourselves that the value is a constant. Note that we may assume $c \geq 3$, since if $c \leq 2$ then $\varphi^2 n \asymp \Lambda(2) = o(1)$ gives $\varphi \ll 1$ and $\Delta_{\mathcal{H}} \leq 1$ a.s.

In either case we need to show that $\mathcal{H}$ is unlikely to contain a nontrivial α-clique. The arguments for the two regimes are similar and we treat them in parallel. In each case we will avoid some complications by first disposing of $\mathcal{C}$'s with very large degrees (*cf.* Lemma 3.3.1).

If $\mathcal{H}$ contains a nontrivial clique of maximum degree at least d then it contains a

"Hilton-Milner" family of size $d+1$, that is, $A_0,\ldots,A_d$ such that $\cap_{i=1}^d A_i\setminus A_0\neq\emptyset$ and $A_i\cap A_0\neq 0\ \forall i\in[d]$. The probability that this occurs is less than

$$\binom{\mathsf{m}}{d+1}(d+1)n(k/n)^d\mathsf{q}^d<\varphi^{d+1}k^{2d-1}n^{-(d-1)} \tag{3.87}$$

(where the factor n on the l.h.s. is for a choice of $x\in\cap_{i=1}^d A_i\setminus A_0$ and the inequality uses $\mathsf{m}=\varphi n/k$). We then need to show that the r.h.s. of (3.87) is $o(1)$ for suitable d.

For regime (i) we take $d=c-1$. We have

$$\Lambda(c)\asymp(\varphi n/k)^c(k^2/n)^{\binom{c}{2}}=\left[\varphi k^{c-2}n^{-(c-3)/2}\right]^{c/2},$$

so $\Lambda(c)=o(1)$ implies $k^{c-2}\ll n^{(c-3)/2}/\varphi$. Thus (for typographical reasons considering the $(c-2)^{\mathrm{nd}}$ power of the r.h.s. of (3.87))

$$\begin{aligned}\left[\varphi^c k^{2c-3}n^{-(c-3)}\right]^{c-2} &\ll \frac{\varphi^{c(c-2)}n^{(2c-3)(c-3)/2}}{n^{(c-2)(c-3)}\varphi^{2c-3}} = \left[\varphi^{c-1}n^{1/2}\right]^{c-3}\\ &= O(n^{-(c-1)/(c+1)+1/2})^{c-3}\\ &= O(n^{-(c-3)^2/(2(c+1))}) = o(1),\end{aligned}$$

where (in the third step) we used $\varphi=O(n^{-1/(c+1)})$. Thus the r.h.s. of (3.87) is $o(1)$.

For regime (ii) we take $d=\lfloor\alpha/2\rfloor$ (say) and find that, since $k<n^{1/2-\varepsilon}$, the r.h.s. of (3.87) is less than $n^{-\Omega(\alpha)}$.

So (in either case) we just need to show that $\mathcal{H}$ is unlikely to contain a nontrivial α-clique with maximum degree at most $d-1$ (d as above). The reduction to independent A_i's preceding Lemma 3.7.1 of course remains valid here, so the following analogue of Lemma 3.3.3 finishes it.

Lemma 3.9.1. *Let α be as above, suppose $A_1,\ldots,A_c$ are drawn uniformly and independently from $\mathcal{K}$, and let $\mathcal{Q}$ be the event that the multiset $\mathcal{C}:=\{A_1,\ldots,A_c\}$ is a nontrivial clique with $\Delta_{\mathcal{C}}\leq d-1$. Then* $\Pr(\mathcal{Q})=o\left(\binom{m}{\alpha}^{-1}\right)$.

Proof. This is a (much) simpler version of the proof of Lemma 4.4.4. We retain the definitions of $d_i(v)$ and d_v from that argument, but now set $W_i=\{v: d_i(v)\geq 2\}$, $W=W_c$, $s_i(A)=|A_i\cap W_{i-1}|$,

$$s(A)=\sum s_i(A)=\sum_{v\in W}(d_v-2)$$

and

$$\Psi = \sum_{v \in Z} \left[\binom{d_v}{2} - 1 \right] = \tfrac{1}{2} \sum_{v \in Z} (d_v + 1)(d_v - 2),$$

noting that if all d_v's are at most d then

$$\Psi \le (d+1)s(A)/2. \tag{3.88}$$

For a counterpart of Proposition 3.7.2, with $\mathsf{w} = (\alpha/\varepsilon)$ (ε as in (3.85)) and $\mathcal{S} = \{|W| \le \mathsf{w}\}$, we have

$$\Pr(\overline{\mathcal{S}}) = o(\mathsf{m}^{-\alpha})$$

(since $\mathsf{E}|W| < (\varphi k)^2/n < n^{-2\varepsilon}$ implies $\Pr(|W| \ge \mathsf{w}) < n^{-2\varepsilon \mathsf{w}} < n^{-2\alpha}$, while $\mathsf{m}^\alpha < n^\alpha$). So we need $\Pr(\mathcal{Q} \wedge \mathcal{S}) = o\left(\binom{m}{\alpha}^{-1} \right)$.

We again let $\mathcal{A} = \{\mathcal{C} \text{ is a clique}\}$ and for $\sigma = (s_1, \ldots, s_\alpha) \in \mathbb{N}^\alpha$ set

$$\mathcal{A}(\sigma) = \mathcal{A} \wedge \{\Delta_{\mathcal{C}} \le d-1\} \wedge \mathcal{S} \wedge \{\sigma(A) = \sigma\}.$$

We have $\mathcal{Q} \wedge \mathcal{S} = \cup_\sigma \mathcal{A}(\sigma)$ so, finally, just need to show

$$\textstyle\sum_\sigma \Pr(\mathcal{A}(\sigma)) < o\left(\binom{\mathsf{m}}{\alpha}^{-1} \right); \tag{3.89}$$

with q as in (3.32) (with the present w), this will follow from

Lemma 3.9.2. *For any σ as above with $\sum s_i = s$,*

$$\Pr(\mathcal{A}(\sigma)) \le \min\{(\mathsf{w}k/n)^s q^{\binom{\alpha}{2} - (d+1)s/2}, (\mathsf{w}k/n)^s\}. \tag{3.90}$$

Before sketching the proof of this, we show that it implies (3.89), beginning with regime (i) (so $\alpha = c$ and $d = c-2$ and $\binom{\mathsf{m}}{\alpha} \asymp \mathsf{m}^c$). We use the first bound in (3.90) for $s := |\sigma| < c$ and the second for $s \ge c$. For the latter we find that the contribution to $\mathsf{m}^c \sum_{|\sigma| \ge c} \Pr(\mathcal{A}(\sigma))$ is at most

$$\sum_{s \ge c} \binom{s+c-1}{c-1} (\varphi n/k)^c (\mathsf{w}k/n)^s < \sum_{s \ge c} ((s+c)\varphi \mathsf{w})^c (\mathsf{w}k/n)^{s-c} = o(1).$$

For the former, the product of m^c and the first bound in (3.90) is

$$\begin{aligned} (\varphi n/k)^c (\mathsf{w}k/n)^s q^{(c-1)(c-s)/2} &\sim (\varphi n/k)^c (\mathsf{w}k/n)^s (k^2/n)^{(c-1)(c-s)/2} \\ &= \varphi^c \mathsf{w}^s \left[(n/k)(k^2/n)^{(c-1)/2} \right]^{c-s}. \end{aligned}$$

If the expression in brackets is at most 1, then we have

$$\mathsf{m}^c \sum_{s<c} \Pr(\mathcal{A}(\sigma)) = O(\varphi^c) \tag{3.91}$$

(since w and $\binom{s+c-1}{c-1}$ are $O(1)$, as is the number of terms in the sum), and otherwise the sum in (3.91) is on the order of

$$\varphi^c \left[(n/k)(k^2/n)^{(c-1)/2}\right]^c \asymp \Lambda(c) = o(1).$$

For regime (ii), we use the second bound in (3.90) for $s \geq 3\alpha/2$, yielding

$$\begin{aligned} \binom{\mathsf{m}}{\alpha} \sum_{|\sigma| \geq 3\alpha/2c} \Pr(\mathcal{A}(\sigma)) &< \sum_{s \geq 3\alpha/2} \binom{s+\alpha-1}{\alpha-1} (\varphi n/k)^\alpha (\mathsf{w}k/n)^s \\ &< \sum_{s \geq 3\alpha/2} ((s+\alpha)\varphi \mathsf{w})^\alpha (\mathsf{w}k/n)^{s-\alpha} = o(1) \end{aligned}$$

(since $\alpha \leq \varphi^* < n^{o(1)}$; see (3.21), (3.2)). On the other hand, the first bound in (3.90) gives (with d as above)

$$\binom{\mathsf{m}}{\alpha} \sum_{|\sigma| < 3\alpha/2} \Pr(\mathcal{A}(\sigma)) < \binom{\mathsf{m}}{\alpha} \sum_{s \geq 3\alpha/2} (\mathsf{w}k/n)^s q^{\binom{\alpha}{2} - (d+1)s/2} = o(1)$$

(because: $\binom{\mathsf{m}}{\alpha}$ and the number of terms in the sum are each at most $\exp[O(\alpha \log n)]$, the q-term is less than $\exp[-\Omega(\alpha^2 \log n)]$ (since $q < n^{-\Omega(1)}$; see (3.12)), and, as noted above, $\alpha = \omega(1)$.)

∎

Proof of Lemma 3.9.2. This is similar to the proof of Lemma 4.4.4 and we just indicate the little changes. For the first bound in (3.90) we follow the proof of (3.54) (beginning with the paragraph containing (3.62)), with changes: replace the γ's by c's and the U's by W's; in (3.64) and (3.65) omit the condition involving Z and replace λ by $d-1$ in (3.65); omit the first factor in (3.66) (the proof doesn't change); and replace $X(r,s)$ in (3.69) by $(d+1)s(A)/2$ (see (3.88)).

For the second bound we use the same modifications and simply sacrifice the contributions of the terms with $i = 2j$ (so for these we can just say $\xi(y_1, \ldots, y_{i-1}) \leq 1$; thus the clique condition (3.63) could be omitted here).

∎

3.10 Necessity

In combination with Theorem 1.0.2, the next result says that (for k as in (1.3)) (1.4) actually *characterizes* the situations in which EKR holds a.s. To say this properly we should remove $\Delta = 2$ from the discussion. (Since there is no such thing as a nontrivial clique of size 2, failure of (1.4) at $\Delta = 2$ should not suggest failure of EKR.) Thus (given φ) we define $\Lambda' = \Lambda'_\varphi$ by $\Lambda'(2) = 0$ and $\Lambda'(t) = \Lambda(t)$ if $t \neq 2$.

Theorem 3.10.1. *For any fixed $\delta > 0$, if k is as in Theorem 1.0.2 and*

$$\Pr(\Lambda'(\Delta) > \delta) > \delta, \tag{3.92}$$

then with probability $\Omega(1)$, $\mathcal{H}$ does not satisfy EKR.

(It is easy to see that Theorem 1.0.2 remains true with (1.4) replaced by "$\Lambda'(\Delta) < o(1)$ a.s." Note that, while the switch to Λ'—or some substitute—is needed to make Theorem 3.10.1 correct, the change is irrelevant in the situations that usually interest us, where at least $\Delta = \omega(1)$ a.s.; in particular, as this will be true in what follows, we continue to write Λ rather than Λ'.)

We believe Theorem 3.10.1 does not require the restriction on k, but our proof doesn't give this.

Notes on the proof of Theorem 3.10.1. This seems not entirely straightforward. It becomes easier (still not immediate) if we retreat to, say, $k = O(\sqrt{n})$. At any rate we give only a sketch of the argument, restricting to k as in (3.1) to avoid some annoyances, with details—such as they are—mostly restricted to the more interesting points. (Some instances of failure of EKR for smaller k are given in [4].)

Set $\alpha = \max\{t \in \mathbb{N} : \Lambda(t) > \delta\}$ and $\mathcal{A} = \{\Delta_{\mathcal{H}} \leq \alpha\}$; thus (3.92) is

$$\Pr(\mathcal{A}) > \delta. \tag{3.93}$$

It is easy to check (*cf.* (3.20)) that

$$\alpha \sim \frac{\log n}{\log(1/\mathsf{q})} \quad (< n^{1/4-\varepsilon+o(1)}), \tag{3.94}$$

and we observe that (for any v)

$$\Pr(d_v > \alpha) = O(1/n), \tag{3.95}$$

since otherwise Proposition 3.4.3 gives $\Delta > \alpha$ a.s., contradicting (3.93).

Here we do (finally) need some concrete notion of a "generic" clique: taking $\mathsf{z} = \alpha/\varepsilon$ (*cf.* Proposition 3.7.2(b)), say a clique—possibly with repeated edges—is *generic* if it has maximum degree at most 3 and at most z vertices with degree equal to 3. Then with

$$\mathcal{B} = \{\mathcal{H} \text{ contains a generic clique of size } \alpha\},$$

Theorem 3.10.1 will follow if we show that (assuming (3.93))

$$\Pr(\mathcal{AB}) = \Omega(1). \tag{3.96}$$

(The negative results of [4] are achieved by showing (probable) existence of Δ-cliques of maximum degree 2.)

Here, as in some earlier instances, we first observe that it is enough to prove (3.96) with $\mathcal{H} = \mathcal{H}_k(n,p)$ replaced by $\mathcal{H}$ consisting of m independent A_i's for suitable m; specifically, Theorem 3.10.1 will follow from:

Lemma 3.10.2. *For any m satisfying* (3.15) *and $\mathcal{H} = \{A_1, \ldots, A_m\}$, with the A_i's chosen uniformly and independently from $\mathcal{K}$,*

$$\mathbb{P}(\mathcal{AB}) = \Omega(1). \tag{3.97}$$

(So we are using "$\mathbb{P}$" for probabilities in this model. Note $\mathcal{H}$ may now—in principle, though in reality essentially never—have repeated edges.)

We first assert that

$$\mathbb{P}(\mathcal{A}) = \Omega(1). \tag{3.98}$$

This actually requires a little argument, but we just point out the difficulty. The combination of (3.93) and Proposition 3.4.3 give $\Pr(d_v > \alpha) = O(1/n)$, which easily implies the corresponding statement for $\mathbb{P}$, the change in the distribution of d_v from $\text{Bin}(M,p)$ to $\text{Bin}(m,k/n)$ having almost no effect. But getting from this to (3.98)—an implication which for $\mathcal{H}_k(n,p)$ is given by Harris' Inequality; see (3.86)—is no longer immediate, since negative association now works against us.

One way to handle this is to compare the present $\mathcal{H}$ with $\mathcal{H}' = \mathcal{H}_k(n,p')$, with $p' > p$ chosen so that, writing $\Pr'$ for the corresponding probabilities, we have $\Pr'(d_v > \alpha) = O(1/n)$ and $|\mathcal{H}'| \geq m$ a.s. (We can then couple so that $\mathcal{H}' \supset \mathcal{H}$ a.s.—note $\mathcal{H}$ a.s. avoids repeats—yielding $\mathbb{P}(\Delta \leq \alpha) > \Pr'(\Delta \leq \alpha) - o(1) = \Omega(1)$. Of course one must show there is such a p', but we omit this easy arithmetic.)

For the proof of (3.97) we use the second moment method. Set $M = [m]$ and $\mathcal{S} = \binom{M}{\alpha}$. We now use $\mathcal{G}$ for the set of generic α-cliques (again, with repeated edges allowed). For $S \subseteq M$ write A_S for the multiset $\{A_i : i \in S\}$ and Δ_S for Δ_{A_S} (so $\Delta = \Delta_M$). In addition, set $\mathcal{B}_S = \{A_S \in \mathcal{G}\}$ and $X_S = \mathbf{1}_{\mathcal{B}_S}$ (these are only of interest if $S \in \mathcal{S}$) and $X = \sum_{S \in \mathcal{S}} X_S$.

We actually need estimates for the quantities $\mathsf{E}X_S$ and $\mathsf{E}X_S X_T$ (for $S, T \in \mathcal{S}$) *conditioned on* $\mathcal{A}$, but will get these by first dealing with the unconditional versions and then showing that the conditioning has little effect. Thus we show (for any S,T)

$$\mathsf{E}X_S \sim \mathsf{q}^{\binom{\alpha}{2}}; \tag{3.99}$$

$$\mathsf{E}X_S X_T < (1+o(1))\mathsf{q}^{2\binom{\alpha}{2} - \binom{|S \cap T|}{2}}; \tag{3.100}$$

$$\mathsf{E}[X_S|\mathcal{A}] \sim \mathsf{E}X_S \quad \text{and} \quad \mathsf{E}[X_S X_T|\mathcal{A}] \sim \mathsf{E}X_S X_T. \tag{3.101}$$

We will say a little about the proofs of these main points below. Once they are established we have, setting $\tilde{\mathsf{E}}[\cdot] = \mathsf{E}[\cdot|\mathcal{A}]$,

$$\mu := \tilde{\mathsf{E}}X \sim \tbinom{m}{\alpha}\mathsf{q}^{\binom{\alpha}{2}} \sim \Lambda(\alpha) = \Omega(1)$$

(using (3.15) for "$\sim$") and an easy calculation gives

$$\begin{aligned} \tilde{\mathsf{E}}X^2 &= \sum_S \sum_T \tilde{\mathsf{E}}X_S X_T \\ &< (1+o(1))\tbinom{m}{\alpha}\mathsf{q}^{2\binom{\alpha}{2}} \sum_{i=0}^{\alpha} \tbinom{\alpha}{i}\tbinom{m-\alpha}{\alpha-i}\mathsf{q}^{-\binom{i}{2}} \sim \mu^2 + \mu, \end{aligned}$$

whence

$$\mathbb{P}(X \neq 0) \geq \mu^2/\tilde{\mathsf{E}}X^2 = \Omega(1),$$

which is what we want.

The proofs of (3.100) and $\mathsf{E}X_S < (1+o(1))\mathsf{q}^{\binom{\alpha}{2}}$ (for (3.99)) are like that of Lemma 3.7.1 and we will not pursue them here.

The proof of the reverse inequality in (3.99) is similar in spirit, if less so in details. We again think of choosing $A_1, \ldots, A_\alpha$ in order and use d_i for degrees in $\{A_1, \ldots, A_i\}$. Set $Z_i = \{v : d_i(v) \geq 3\}$, $\mathcal{Q}_i = \{|Z_i| \leq \mathsf{z}\}$, $\mathcal{R}_i = \{A_i \cap Z_{i-1} = \emptyset\}$, $\mathcal{T}_i = \{A_i \cap A_j \neq \emptyset\ \forall j \in [i-1]\}$ and

$$\mathcal{B}_i = \{\{A_1, \ldots, A_i\} \text{ is a generic clique}\}.$$

Then $\mathcal{B}_i = \mathcal{B}_{i-1}\mathcal{R}_i\mathcal{T}_i\mathcal{Q}_i$ and

$$\mathbb{P}(\mathcal{B}_i) \geq \mathbb{P}(B_{i-1})\mathbb{P}(\mathcal{R}_i\mathcal{T}_i|B_{i-1}) - \mathbb{P}(\overline{\mathcal{Q}}_i). \tag{3.102}$$

Setting $\delta_i = in^{-1/4}$, we show by induction on i (with $i = 1$ trivial)

$$\mathbb{P}(\mathcal{B}_i) \geq (1-\delta_i)\mathsf{q}^{\binom{i}{2}} \tag{3.103}$$

(which suffices because of (3.94)).

The relevant probabilities are bounded as follows. First, the proof of Proposition 3.7.2(b) gives

$$\mathbb{P}(\overline{\mathcal{Q}}_i) < \eta \tag{3.104}$$

for some $\eta < \binom{m}{\alpha}^{-2+o(1)}$. Second, trivially,

$$\mathbb{P}(\mathcal{R}_i|\mathcal{B}_{i-1}) \geq 1 - \mathsf{z}k/n \tag{3.105}$$

(this just uses $\mathcal{B}_{i-1} \subseteq \mathcal{Q}_{i-1}$). Third,

$$\mathbb{P}(\mathcal{T}_i|\mathcal{R}_i\mathcal{B}_{i-1}) \geq ((1-2\xi)\mathsf{q})^{i-1}, \tag{3.106}$$

where $\xi = 2k^2\mathsf{z}/(\mathsf{q}n^2)$; this is mostly given by the next two observations, the first of which is an easy calculation along the lines of Proposition 3.4.1. (Here "mostly" refers to a small detail we're omitting: in (3.106) we use, for example,

$$\mathbb{P}(\mathcal{T}_i|\mathcal{R}_i\mathcal{B}_{i-1}) > (1-\xi)^{i-1}\prod_{j<i}\mathbb{P}(A_i \cap A_j \neq \emptyset|\mathcal{R}_i\mathcal{B}_{i-1}),$$

which requires a proof since the events in question are (slightly) negatively correlated.)

Proposition 3.10.3. *For $Z \subseteq V$ of size at most z, $B \in \mathcal{K}$ and A uniform from $\binom{V\setminus Z}{k}$,*

$$\Pr(A \cap B) > (1-\xi)\mathsf{q}.$$

Proposition 3.10.4. *If $C_1, \ldots, C_s, D_1, \ldots, D_s$ are subsets of V with $|C_i| = |D_i|\ \forall i$ and the D_i's pairwise disjoint, and A is uniform from $\binom{V}{k}$, then*

$$\Pr(A \cap C_i \neq \emptyset\ \forall i) \geq \Pr(A \cap D_i \neq \emptyset\ \forall i). \tag{3.107}$$

This follows *via* induction from the fact—an easy coupling argument—that (3.107) holds when $x \in C_i \cap C_j$ $(i \neq j)$, $D_i = C_i \setminus \{x\} \cup \{y\}$ for some $y \in V \setminus \cup C_\ell$, and $D_\ell = C_\ell$ for $\ell \neq i$.

By (3.104)-(3.106) and (3.103) for $i-1$, the r.h.s. of (3.102) is at least

$$\begin{aligned}(1-\delta_{i-1})\mathsf{q}^{\binom{i-1}{2}}&[(1-\mathsf{z}k/n)(1-\xi)^{i-1}-\eta']\mathsf{q}^{i-1}\\ &> (1-\delta_{i-1})[1-\{\mathsf{z}k/n+(i-1)\xi+\eta'\}]\mathsf{q}^{\binom{i}{2}},\end{aligned}$$

where we set $\eta' = \eta[(1-\delta_i)\mathsf{q}^{\binom{i}{2}}]^{-1}$. This gives (3.103) since the expression in $\{\ \}$'s (whose dominant term is $\mathsf{z}k/n$) is easily seen to be less than $n^{-1/4}$.

Finally we turn to (3.101), for which we need the following observation.

Proposition 3.10.5. *Let $s \in [m]$ and $t = m - s$. Suppose $S \in \binom{M}{s}$ and $\mathcal{D}$ is an s-multisubset of $\mathcal{K}$ with $\Delta_{\mathcal{D}} \leq C$. If $\mathbb{P}(B(t,k/n) \geq \alpha - C) = \rho/n$ then*

$$|\mathbb{P}(\Delta \leq \alpha | A_S = \mathcal{D}) - \mathbb{P}(\Delta \leq \alpha)| \leq st\rho/n. \tag{3.108}$$

Proof. With $T = M \setminus S$, the assertion is given by

$$\begin{aligned}\mathbb{P}(\Delta_T \leq \alpha) &\geq \mathbb{P}(\Delta \leq \alpha | A_S = \mathcal{D})\\ &\geq \mathbb{P}(\Delta_T \leq \alpha) - sk\rho/n \ \geq\ \mathbb{P}(\Delta \leq \alpha) - sk\rho/n.\end{aligned}$$

The first and third inequalities are trivial. For the second, setting $V(\mathcal{D}) = \{v : d_{\mathcal{D}}(v) > 0\}$ and using our assumption on $\mathcal{D}$, we find that on $\{A_S = \mathcal{D}\}$,

$$\{\Delta_T \leq \alpha\} \setminus \{\Delta \leq \alpha\} \ \subseteq\ \{\exists v \in V(\mathcal{D}), d_T(v) \geq \alpha - C\}.$$

But the probability of the latter event is at most $|V(\mathcal{D})|\rho/n \leq sk\rho/n$.

$\blacksquare$

The arguments for the two statements in (3.101) are similar and we only discuss the first. This is equivalent to $\mathbb{P}(\mathcal{A}|\mathcal{B}_S) \sim \mathbb{P}(\mathcal{A})$ or, in view of (3.98), $\mathbb{P}(\mathcal{A}|\mathcal{B}_S) = \mathbb{P}(\mathcal{A}) \pm o(1)$, which will follow if we show that, for any generic α-clique $\mathcal{D}$,

$$\mathbb{P}(\mathcal{A}|A_S = \mathcal{D}) = \mathbb{P}(\mathcal{A}) \pm o(1).$$

This is, of course, an instance of Proposition 3.10.5, for which we just have to make sure that, with $s = \alpha$, $C = 3$ and ρ as in the proposition, the bound in (3.108) is $o(1)$; but this follows from (3.94), $t \leq m < (1 + o(1))\alpha n/k$ and $\alpha/\varphi < n^{o(1)}$, the latter given by (3.27) and (3.19).

Chapter 4

$n = 2k + 1$

4.1 Main Result

We again recall the statement to be proved:

Theorem 4.1.1. *There is a fixed $\varepsilon > 0$ such that if $n = 2k + 1$ and $p > 1 - \varepsilon$, then $\mathcal{H}$ satisfies EKR a.s.*

Note that for n, k as in Theorem 1.0.4, EKR is unlikely unless p is large, since a simple calculation shows that for p less than about $3/4$ stars are unlikely even to be maxi*mal* cliques. We will elaborate on this in Section 4.5.

We haven't thought very hard about whether the ε in Theorem 1.0.4 could be pushed to .01, since this seems somewhat beside the point (and since it seems not wildly unethical to regard ".99" as really meaning "$1 - \varepsilon$ for some fixed $\varepsilon > 0$").

4.2 Preliminaries

4.2.1 Usage

Recall that we use $M = \binom{2k}{k-1}$ and let $N = \binom{2k}{k}$. As usual, 2^S is the power set of S.

For graphs, xy is an edge joining vertices x and y; $N(x)$ is, as usual, the neighborhood of x (and $N(X) = \cup_{x \in X} N(x)$); and $\nabla(X, Y)$ is the set of edges joining the disjoint vertex sets X, Y.

In this chapter we take $\mathcal{F}$ to be a member of $\mathcal{M}$, the collection of *nontrivial* maximal intersecting families in $\mathcal{K}$. We now take $p = 1 - \varepsilon$, with $\varepsilon > 0$ fixed but small enough to support our arguments. (We make no attempt to optimize.)

4.2.2 Isoperimetry and Degree

For $A \subseteq \binom{[2k]}{k}$ let $\delta(A) = (|\nabla A| - |A|)/|A|$, where $\nabla A = \{y \in \binom{[2k]}{k+1} : \exists x \in A, y \supset x\}$ (the *upper shadow* of A). We will use the following consequence of the Kruskal-Katona Theorem ([20], [17] or e.g. [6]).

Proposition 4.2.1. *For $A \subseteq \binom{[2k]}{k}$ with $|A| \le N/2$,*

$$\delta(A) \ge \tfrac{\log 2}{k} \; \log_2 \left(\tfrac{N}{2|A|}\right). \tag{4.1}$$

(Notice that $N/2 = \binom{2k-1}{k}$. The $\log 2$ in (4.1) can probably be replaced by 1, but cannot be replaced by $k/(k-1)$.)

Proof. We use Lovász' version [21, Problem 13.31] of Kruskal-Katona, which in the present situation says that if $|A| = \binom{x}{k}$, then $|\nabla(A)| \ge \binom{x}{k-1}$. (This is ordinarily stated for the *lower* shadow, which is equivalent here since our universe is of size $2k$.)

Let $|A| = \binom{2k-t}{k}$, noting that $|A| \le N/2$ implies $t \ge 1$, and $\psi = k^{-1}\log 2$. Then $\frac{N}{2|A|} = \frac{(2k)_k}{2(2k-t)_k}$ and, from Kruskal-Katona (Lovász),

$$\delta(A) \ge \binom{2k-t}{k-1}/\binom{2k-t}{k} - 1 = \frac{t-1}{k-t+1}.$$

Thus (4.1) will follow from

$$f(t) := \frac{t-1}{k-t+1} - \psi \log_2 \left[\frac{(2k)_k}{2(2k-t)_k}\right] \ge 0 \qquad \text{for } t \ge 1,$$

so (since $f(1) = 0$) from $f'(t) \ge 0$. But, recalling the value of ψ, we have

$$f'(t) = \frac{k}{(k-t+1)^2} - \frac{1}{k}\sum_{i=0}^{k-1} \frac{1}{2k-t-i} \ge \frac{k}{(k-t+1)^2} - \frac{1}{k-t+1} \ge 0.$$

∎

The following result of P. Frankl [11] will also be helpful in getting things started. (We give the result for general k, n and i, but will only use it with $n = 2k+1$ and $i = 3$.) Given k and $n > 2k$, set, for each $i \in \{3, \ldots, k+1\}$,

$$\mathcal{G}_i = \{A \in \mathcal{K} : 1 \in A, A \cap \{2, \ldots, i\} \ne \emptyset\} \cup \{A \in \mathcal{K} : A \supseteq \{2, \ldots, i\}\}. \tag{4.2}$$

Theorem 4.2.2 ([11])**.** *For any k, n and i as above, if $\mathcal{G} \subseteq \mathcal{K}$ is a clique with $|\mathcal{G}| > |\mathcal{G}_i|$, then $\Delta_{\mathcal{G}} > \Delta_{\mathcal{G}_i}$.*

4.2.3 Graphs

Two special graph-theoretic notions will be relevant in what follows. First, for a bigraph Σ with bipartition $\Gamma_1 \cup \Gamma_2$, say $X \subseteq \Gamma_i$ is said to be *closed* if $N(x) \subseteq N(X) \Rightarrow x \in X$. Second, for a (general) graph Σ and positive integer j, $W \subseteq V(\Sigma)$ is j*-linked* if for all $u, v \in W$ there are $u = u_0, u_1, , \ldots, , u_l = v$ with $u_i \in W$ $(\forall i)$ and $\rho(u_{i-1}, u_i) \leq j$ for $i \in [l]$, where ρ is graph-theoretic distance. We will eventually need the following observation from [26].

Proposition 4.2.3. *Let Σ be a graph and suppose A and T are subsets of $V(\Sigma)$ with $T \subseteq N(A)$, $A \subseteq N(T)$ and A j-linked. Then T is $(j+2)$-linked.*

Proof. Given $u, v \in T$, choose $x, y \in A$ with $x \sim u$, $y \sim v$, and then $x = x_0, \ldots, x_\ell = y$ with $x_i \in A$ and $\rho(x_{i-1}, x_i) \leq j$ $(i \in [\ell])$. If we now let $u_0 = u$, $u_\ell = v$ and $x_i \sim u_i \in T$ for $i \in [\ell - 1]$, then $\rho(u_{i-1}, u_i) \leq 1 + \rho(x_{i-1}, x_i) + 1 \leq j + 2$ (for $i \in [\ell]$). The proposition follows.

∎

We also find some use for the following standard bound.

Proposition 4.2.4. *In any graph with all degrees at most d, the number of trees of size u rooted at some specified vertex is at most $(ed)^{u-1}$.*

Proof. This follows easily from the fact (see e.g. [18, p.396, Ex.11]) that the infinite d-branching rooted tree contains precisely $\frac{1}{(d-1)u+1}\binom{Du}{u} \leq (ed)^{u-1}$ rooted subtrees of size u.

∎

4.2.4 Etc.

We make repeated use of the fact that for positive integers a, b with $a \leq b/2$,

$$\binom{b}{\leq a} \leq \exp[a \log(eb/a)]. \tag{4.3}$$

4.3 Setting Up

The statement we are to prove is

$$\max_{\mathcal{F}\in\mathcal{M}} |\mathcal{H}\cap\mathcal{F}| < \max_x |\mathcal{H}\cap\mathcal{K}_x| \text{ a.s.,} \tag{4.4}$$

but we will find it better to work with a variant, (4.7) below. This requires a little preparation.

For $x\in[n]$ and $0\le\ell\le n-1$, let Γ_ℓ^x denote the collection of ℓ-subsets of $[n]\setminus\{x\}$. Let Σ^x be the usual bigraph on $\Gamma_k^x\cup\Gamma_{k+1}^x$ (that is, with adjacency given by set containment), and write N^x for neighborhood in Σ^x. For $A\subseteq\Gamma_k^x$ set $\delta_x(A)=(|N^x(A)|-|A|)/|A|$ (so $N^x(A)$ is the upper shadow of A in $2^{[n]\setminus\{x\}}$ and our usage here follows that in Proposition 4.2.1).

For $\mathcal{F}\in\mathcal{M}$ (and $x\in[n]$), let $A^x(\mathcal{F})=\mathcal{F}\setminus\mathcal{K}_x$, $J^x(\mathcal{F})=\mathcal{K}_x\setminus\mathcal{F}$ and $G^x(\mathcal{F})=N^x(A^x(\mathcal{F}))$; thus $A^x(\mathcal{F})$ and $G^x(\mathcal{F})$ are subsets of Γ_k^x and Γ_{k+1}^x respectively. Note that

$$|\mathcal{H}\cap A^x(\mathcal{F})|-|\mathcal{H}\cap J^x(\mathcal{F})|=|\mathcal{H}\cap\mathcal{F}|-|\mathcal{H}\cap\mathcal{K}_x|. \tag{4.5}$$

For $\mathcal{B}\subseteq 2^{[n]}$ set $\mathcal{B}^c=\{[n]\setminus T : T\in\mathcal{B}\}$. It is easy to see that maximality of $\mathcal{F}$ implies that (for any x) $A^x(\mathcal{F})$ is closed in Σ^x and $G^x(\mathcal{F})=J^x(\mathcal{F})^c$. The converse is also true (and similarly easy): if $x\in[n]$ and A is a nonempty closed subset of Γ_k^x (in Σ^x), then $(\mathcal{K}_x\setminus N^x(A)^c)\cup A\in\mathcal{M}$. (If $A=\emptyset$ then $(\mathcal{K}_x\setminus N^x(A)^c)\cup A=\mathcal{K}_x$ is maximal intersecting but not in $\mathcal{M}$, which does not contain stars.)

Let $\mathcal{Q}$ be the event that there are $\mathcal{F}\in\mathcal{M}$ and $x\in[n]$ for which $A^x(\mathcal{F})$ is 2-linked (in Σ^x),

$$\delta_x(A^x(\mathcal{F}))>1/(3k), \tag{4.6}$$

and $|\mathcal{H}\cap\mathcal{F}|\ge|\mathcal{H}\cap\mathcal{K}_x|$. Our main point, the aforementioned variant of (4.4), is

$$\Pr(\mathcal{Q})=o(1). \tag{4.7}$$

Before proving this (in Section 4.4), we show that it implies (4.4), by showing that failure of (4.4) implies $\mathcal{Q}$. Supposing (4.4) fails, choose $\mathcal{F}\in\mathcal{M}$ with $|\mathcal{H}\cap\mathcal{F}|$ maximum and fix x with $d_{\mathcal{F}}(x) = \Delta(\mathcal{F})$. Let $A = A^x(\mathcal{F})$ and $J = J^x(\mathcal{F})$. By (4.5) (and our assumption that $|\mathcal{H} \cap \mathcal{F}| \geq |\mathcal{H} \cap \mathcal{K}_y|\ \forall y$) we have

$$|\mathcal{H} \cap A| \geq |\mathcal{H} \cap J|. \tag{4.8}$$

Note also that

$$|A| \leq (k+1)|\mathcal{F}|/n < (k+1)M/n, \tag{4.9}$$

since $(|\mathcal{F} \setminus A| =)\ \Delta(\mathcal{F}) \geq k|\mathcal{F}|/n$ (as is true for any $\mathcal{F} \subseteq \mathcal{K}$).

Suppose first that A is 2-linked in Σ^x. In this case we claim that $(\mathcal{F}, x)$ itself satisfies $\mathcal{Q}$, i.e. that (4.6) holds. Take $\mathcal{G}_3$ as in (4.2). If $|\mathcal{F}| > |\mathcal{G}_3|$, then Theorem 4.2.2 gives $\Delta(\mathcal{F}) > \Delta(\mathcal{G}_3) \sim 3M/4$, whence $|A| < (1+o(1))M/4$ and (4.6) (actually a little more) is given by (4.1). If, on the other hand, $|\mathcal{F}| \leq |\mathcal{G}_3|$, then, noting that $M - |\mathcal{G}_3| \sim M/(4k)$, we have, using (4.5) and (4.9),

$$\delta_x(A) = (M - |\mathcal{F}|)/|A| > (2 - o(1))(M - |\mathcal{G}_3|)/M \sim 1/(2k).$$

Now suppose A is not 2-linked. Let $A_1, \ldots, A_s$ be the 2-linked components (defined in the obvious way) of A, and $J_i = N^x(A_i)^c$. Then $J_i \cup \cdots \cup J_s$ is a partition of J. Moreover, each A_i is closed, so that (see the paragraph following (4.5)) $\mathcal{F}_i := (\mathcal{K}_x \setminus J_i) \cup A_i \in \mathcal{M}$ for each i. Suppose w.l.o.g. that $|A_1| = \max_i |A_i|$. Then for $i \geq 2$ we have $|A_i| \leq |A/2| < (1/4 + o(1))M$, implying (again using (4.1)) $\delta_x(A_i) > (\log 2 - o(1))/k$. So we have $\mathcal{Q}$ if $|\mathcal{H} \cap A_i| \geq |\mathcal{H} \cap J_i|$ for some $i \geq 2$; but if this is not the case then (again using (4.5))

$$\begin{aligned} |\mathcal{H} \cap \mathcal{F}_1| - |\mathcal{H} \cap \mathcal{K}_x| &= |\mathcal{H} \cap A_1| - |\mathcal{H} \cap J_1| \\ &= |\mathcal{H} \cap A| - |\mathcal{H} \cap J| - \sum_{i \geq 2}(|\mathcal{H} \cap A_i| - |\mathcal{H} \cap J_i|) \\ &> |\mathcal{H} \cap A| - |\mathcal{H} \cap J| = |\mathcal{H} \cap \mathcal{F}| - |\mathcal{H} \cap \mathcal{K}_x|, \end{aligned}$$

contradicting the assumed maximality of $|\mathcal{H} \cap \mathcal{F}|$.

∎

4.4 Main Point

4.4.1 More Set-Up

For the remainder of our discussion we work with a fixed $x \in [n]$ and drop the super- and subscripts x from our notation; so to begin, we set $\Sigma^x = \Sigma$ and $\Gamma^x_\ell = \Gamma_\ell$. We will use G_A for the neighborhood of $A \subseteq \Gamma_k$ in Σ and

$$\delta(A) = \frac{|G_A|}{|A|} - 1 \quad (= \delta_x(A)).$$

We extend $\mathcal{H}$ to Γ_{k+1} by declaring that $T \in \mathcal{H}$ if and only if $[n] \setminus T \in \mathcal{H}$ (so here T is a $(k+1)$-set off x and $[n] \setminus T$ is a k-set on x); we may then forget about $J(\mathcal{F})$ $(= J^x(\mathcal{F}))$ and regard $\mathcal{H}$ as a subset of $\Gamma_k \cup \Gamma_{k+1}$. Note that (*cf.* (4.5)) "$|\mathcal{H} \cap \mathcal{F}| \geq |\mathcal{H} \cap \mathcal{K}_x|$" in the definition of $\mathcal{Q}$ is then the same as "$|\mathcal{H} \cap G_A| \geq |\mathcal{H} \cap A|$" with $A = \mathcal{F} \setminus \mathcal{K}_x$ and (thus) $G_A = J^x(\mathcal{F})^c$.

For the proof of (4.7) we will bound the probability that $\mathcal{Q}$ occurs at our given x with specified sizes of A and G_A, and then sum over possibilities for these sizes. (Of course we need a bound $o(1/n)$ since we must eventually sum over x.) Thus we assume throughout that we have fixed a, g with

$$\delta := \frac{g-a}{a} > \frac{1}{3k}, \tag{4.10}$$

and write $\mathcal{A} = \mathcal{A}(a, g)$ for the set of A's satisfying

$$A \text{ is closed and 2-linked, } |A| = a \text{ and } |G_A| = g. \tag{4.11}$$

Notice that for $A \in \mathcal{A}$ we have

$$\begin{aligned} |\nabla(G_A, \Gamma_k \setminus A)| &= (k+1)g - ka \\ &= (k+1)(1+\delta)a - ka = (1 + (k+1)\delta)a. \end{aligned}$$

Let $\mathcal{Q}(a, g)$ $(= \mathcal{Q}_x(a, g))$ be the event that there is some $A \in \mathcal{A}(a, g)$ with

$$|\mathcal{H} \cap G_A| \leq |\mathcal{H} \cap A|. \tag{4.12}$$

We show

$$\sum_{a,g} \Pr(\mathcal{Q}(a,g)) = o(1/n), \tag{4.13}$$

which, since the union of the $\mathcal{Q}(a,g)$'s is occurrence of $\mathcal{Q}$ at x, gives (4.7).

The bound (4.13) is (of course) the heart of the matter, and the rest of our discussion is devoted to its proof. This turns out to be rather delicate, and a rough indication of where we are headed may be helpful.

For $A \in \mathcal{A}$ we have

$$\mathsf{E}|\mathcal{H} \cap G_A| - \mathsf{E}|\mathcal{H} \cap A| = \delta ap,$$

so can rule out (4.12) if we can say that the quantities $|\mathcal{H} \cap G_A|$ and $|\mathcal{H} \cap A|$ are close to their expectations, where "close" means somewhat small relative to δap ($\approx \delta a$). The problem (of course) is that though each of these *individual* events is likely, there are too many of them to allow a simple union bound.

Our remedy for this is to exploit similarities among the A's (and similarly G_A's, but for this very rough description we stick to A's) to avoid paying repeatedly for the same unlikely events. To do this we specify each $A \in \mathcal{A}$ *via* several "approximations," beginning with a set S_A for which $A \Delta S_A$ is *fairly* small, and then adding and subtracting lesser pieces. It will then follow that $|\mathcal{H} \cap A|$ is close to its expectation provided this is true of $|\mathcal{H} \cap B|$ for each of the relevant pieces B.

Thus we will want to say that, with B ranging over some to-be-specified collection of subsets of Γ_k, it is likely that all $|\mathcal{H} \cap B|$'s are close to their expectations. Of course the probability that this fails for a particular B grows with $|B|$ (since the benchmark δap does not change), so we would like to arrange that the larger B's are not too numerous. For example, the aforementioned S_A's will necessarily be large (of size roughly a), but there will be relatively few of them, reflecting the fact that a single S will typically be S_A for many A's. We may think of $\mathcal{A}$ as consisting of a large number of variations on a relatively small number of themes, though as we will see, controlling these themes and variations turns out to be not very straightforward.

Our approach has its roots in the beautiful ideas of A.A. Sapozhenko [26], which were originally developed to deal with "Dedekind's Problem" and related questions in asymptotic enumeration.

Proof of (4.13). As our fixed x plays no further role in what follows, we will feel free to recycle and use "x" (along with u, v, y, z) to denote a general member of our ground set, which we may now think of as $[2k]$.

We divide the proof of (4.13) into two cases, large and small δ, beginning with the second, which is by far the more interesting. (Our treatment of this case can be adapted to work in general—actually with most of the contortions below becoming unnecessary and/or vacuous—but this seems pointless given how much simpler the proof is for large δ.)

4.4.2 Small δ

Assume then that $\delta \leq 1$ (say), and note that in this case (4.1) gives $a > (4/e)^k$ (which is pretty far from the truth but we have plenty of room here). Before dealing with $\mathcal{H}$ we will spend some time developing the aforementioned approximations to A and G.

For $R \subseteq V := V(\Sigma)$, let $N^i(R) = \{u \in V : \rho(u, R) \leq i\}$ (where, recall, ρ is graph-theoretic distance). For $A \in \mathcal{A}$ $(= \mathcal{A}(a, g))$, say a path is A-*good* if it is of the form vx_1yx_2 with $x_1, x_2 \in A$ (so in particular has length 3), and for $v \in \Gamma_{k+1}$, let $f(v, A)$ denote the number of A-good paths beginning with v. Fix a small $\zeta > 0$, and set $\vartheta = \zeta/2$ and

$$G_A^0 = \{v \in G_A : f(v, A) \geq (1/4)k^{3-\zeta}\}.$$

For $T \subseteq \Gamma_k$ set $W_T = N^3(T) \cap \Gamma_{k+1}$ and

$$S_T = \{x : d_{W_T}(x) \geq k/2\} \quad (\subseteq \Gamma_k). \tag{4.14}$$

For $T \subseteq A \in \mathcal{A}$, let $F_{A,T} = \nabla(N(T), \Gamma_k \setminus A)$ and $Z_{A,T} = N(N^2(T) \cap A) \subseteq W_T$. Notice that $w \in Z_{A,T}$ if and only if either $w \in N(T)$ or there is a path $xyzw$ with $x \in T$ and $yz \notin F_{A,T}$ (equivalently an A-good path from w to T); in particular $Z_{A,T}$ *is determined by* T *and* $F_{A,T}$.

Lemma 4.4.1. *There is a fixed* K *such that for each* $A \in \mathcal{A}$ *there is a* $T \subseteq A$ *satisfying*

(T1) $|T| \leq Kak^{-3+\zeta}\log k$,

(T2) $|F_{A,T}| \leq K\delta ak^{-1+\zeta}\log k$,

(T3) $|G_A^0 \setminus Z_{A,T}| \leq Kak^{-2}$,

(T4) $|W_T \setminus G_A| < K\delta ak^{\zeta}\log k$, *and*

(T5) $|A \setminus S_T| < K\delta ak^{-\vartheta}$.

The following auxiliary definitions and lemma will be helpful in the proof of Lemma 4.4.1 and again later in the proof of Lemma 4.4.4. Fix $A \in \mathcal{A}$, set $G_A = G$ and $G_A^0 = G^0$, and define

$$H = \{y \in G : d_A(y) < k^{1-\vartheta}\},$$

$$B = \{x \in A : d_H(x) > k/2\},$$

$$I = \{y \in G \setminus H : d_{A\setminus B}(y) < k^{1-\vartheta}/2\}$$

and

$$C = \{x \in A \setminus B : d_{H\cup I}(x) > k/4\}.$$

Lemma 4.4.2. *With the above definitions, $|H \cup I| < O(\delta a)$, $|C| < O(\delta ak^{-\vartheta})$ and $G \setminus G^0 \subseteq H \cup I$.*

Proof. We have

$$(k+1-k^{1-\vartheta})|H| \leq |\nabla(H, \Gamma_k \setminus A)| \leq |\nabla(G, \Gamma_k \setminus A)| = (1+(k+1)\delta)a,$$

$$(k/2)|B| < |\nabla(B, H)| < k^{1-\vartheta}|H|,$$

$$(k^{1-\vartheta}/2)|I| < |\nabla(I, B)| < k|B|/2$$

and

$$(k/4)|C| < |\nabla(C, H \cup I)| < |H \cup I|k^{1-\vartheta},$$

implying $|H| < (4+o(1))\delta a$ (using (4.6)), $|B| < (8+o(1))\delta ak^{-\vartheta}$, $|I| < (8+o(1))\delta a$ and $|C| < (48+o(1))\delta ak^{-\vartheta}$. This gives the first two assertions in the lemma. The third is given by the observation that for $y \in G \setminus (H \cup I)$ the number of paths $ywzx$ with $(w, z, x) \in (A \setminus B) \times (G \setminus H) \times A$ is at least $(k^{1-\vartheta}/2)(k/2)k^{1-\vartheta}$.

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Proof of Lemma 4.4.1. Here we will find it more convenient to use "big Oh" notation; that is, we will prove the lemma with each of the bounds $K \cdot X$ appearing in (T1)-(T5) replaced by $O(X)$. We first show existence of T satisfying (T1)-(T3) and then observe that any such T also satisfies (T4) and (T5).

Let $q = 16k^{-3+\zeta} \log k$ and $\mathbf{T} = A_q$. To show that there is a T satisfying (T1)-(T3), it is enough to show that the stated bounds (again, in their "big Oh" forms) hold for the *expectations* of the set sizes in question, since Markov's Inequality then implies existence of a T for which each of these quantities is at most three times its expectation. This is of course true for $\mathsf{E}|\mathbf{T}| = aq$. For (T2) we have

$$\begin{aligned} \mathsf{E}|F_{A,\mathbf{T}}| &= \sum_{x \in G} \Pr(x \in N(\mathbf{T})) d_{\Gamma_k \setminus A}(x) \\ &\leq q \sum_{x \in G} d_A(x) d_{\Gamma_k \setminus A}(x) \\ &\leq qk|\nabla(G, \Gamma_k \setminus A)| < O(\delta a k^{-1+\zeta} \log k). \end{aligned}$$

To bound the expectation for (T3), notice that for $v \in G^0$, there are at least $(1/8)k^{3-\zeta}$ vertices $x \in A$ for which $x \in T$ implies $v \in Z_{A,T}$. (This is true of any x for which there is an A-good path from v to x and, since two vertices at distance 3 are connected by exactly two paths of length 3 in Σ, the number of such x's is at least $f(v, A)/2$.) The probability that such a v does not belong to $Z_{A,\mathbf{T}}$ is thus at most $(1-q)^{(1/8)k^{3-\zeta}} < k^{-2}$, so that $\mathsf{E}|G^0 \setminus Z_{A,\mathbf{T}}| < gk^{-2}$ (which gives the bound in (T3) since we assume $g = O(a)$; of course the assumption isn't really needed here, as we could instead have arranged $\mathsf{E}|G^0 \setminus Z_{A,\mathbf{T}}| < gk^{-3}$).

This completes the discussion of (T1)-(T3) and we turn to the last two properties requested of T. We first observe that (T4) follows from (T2), since in fact

$$|W_T \setminus G| \leq k|F_{A,T}|.$$

To see this just notice that if $w \in W_T \setminus G$, then (since $w \in W_T$) there is a path $xyzw$ with $x \in T$ and (therefore) $y \in N(T)$, but $z \notin A$ (since $w \notin G$), so that $yz \in F_{A,T}$ (and each such yz gives rise to at most k such w's).

For (T5), note that (according to the definition of S_T in (4.14)) any $x \in A \setminus S_T$ has at least $k/4$ neighbors in one of $G \setminus G^0$, $G^0 \setminus W_T$. By Lemma 4.4.2, x's of the first type belong to $B \cup C$ and number at most $O(\delta a k^{-\vartheta})$. On the other hand, by (T3) (and (4.6)), the number of the second type is at most

$$(4/k)|G^0 \setminus W_T|(k+1) < O(ak^{-2}) < o(\delta a k^{-\vartheta}).$$

∎

We think of W_T in Lemma 4.4.1 as a first approximation to G_A, and $Z_{A,T}$ as a second approximation satisfying

$$Z_{A,T} \subseteq W_T \cap G_A \tag{4.15}$$

that discards vertices that got into W_T on spurious grounds. Similarly, the next lemma prunes our first approximation, S_T, of A to get a better second approximation.

Lemma 4.4.3. *There is a fixed K such that for any $A \in \mathcal{A}$ and $T \subseteq A$ satisfying (T4), there is some $U \subseteq W_T \setminus G_A$ with*

(U1) $|U| \leq K\delta a k^{-1+\zeta} \log^2 k$ *and*

(U2) $|(S_T \setminus A) \setminus N(U)| \leq K\delta a$.

The second approximation mentioned above is then $S_T \setminus N(U)$, which in particular satisfies

$$S_T \supseteq S_T \setminus N(U) \supseteq S_T \cap A. \tag{4.16}$$

Proof of Lemma 4.4.3. Here we again (as in the proof of Lemma 4.4.1) switch to "big Oh" notation. Set $G = G_A$, $W = W_T$ and $S = S_T$. Let $q = 4k^{-1}\log k$ and $\mathbf{U} = (W \setminus G)_q$. By the definition of $S = S_T$, each $x \in S \setminus A$ has at least $k/4$ neighbors in one of $W \setminus G$, G. Let

$$L = \{x \in S \setminus A : d_{W \setminus G}(x) \geq k/4\}.$$

Then $|L| \leq (4/k)|W \setminus G|(k+1) = O(\delta a k^{\zeta} \log k)$ (by (T4)). On the other hand, for $x \in L$ we have $\Pr(x \notin N(\mathbf{U})) \leq (1-q)^{k/4} < k^{-1}$, so there is some U with

$$|L \setminus N(U)| \leq \mathsf{E}|L \setminus N(\mathbf{U})| < |L|/k = o(\delta a).$$

Finally, since $x \in (S \setminus A) \setminus L$ implies $d_G(x) > k/4$, we have

$$|(S \setminus A) \setminus L| \leq (4/k)|\nabla(G, \Gamma_k \setminus A)| = 4(1 + (k+1)\delta)a/k = O(\delta a).$$

The lemma follows.

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Now write K for the larger of the constants appearing in Lemmas 4.4.1 and 4.4.3. For each $A \in \mathcal{A}$ fix some $T = T_A \subseteq A$ satisfying (T1)-(T5) and then some $U = U_A \subseteq W_T \setminus G_A$ satisfying (U1)-(U2), and set: $W_A = W_T$, $S_A = S_T$, $F_A = F_{A,T}$, $Z_A = Z_{A,T}$, $S'_A = S_T \setminus N(U)$ and $R_A = R(A) = (T_A, F_A, U_A)$. (We prefer R_A but will use $R(A)$ to avoid double subscripts.) We may think of T_A, F_A, U_A as "primary" objects, which we will need to specify, and W_A, S_A, Z_A, S'_A as "secondary" objects, which are functions of the primary objects.

Let $\mathcal{R} = \{R_A : A \in \mathcal{A}\}$. For each $R \in \mathcal{R}$ fix some $A^* = A^*_R \in \mathcal{A}$ with $R_{A^*} = R$, and let $G^*_R = G_{A^*}$. If $R = R_A$ then we also set $W_R = W_A$ (which is the same for all A with $R_A = R$), and similarly for the other objects subscripted by A in the preceding paragraph. Now suppose $A \in \mathcal{A}$, $G = G_A$, $R = R_A$, $A^* = A^*_R$ and $G^* = G^*_R$. Notice that, given A^* and G^*,

$$A \textit{ is determined by } A \setminus A^* \textit{ and } G \cap G^*. \tag{4.17}$$

Actually A is determined by B, G_B, $A \setminus B$ and $G \cap G_B$ *whenever* $A, B \subseteq \Gamma_k$ are closed with $G = G_A$, since

$$A \cap B = \{x \in B : N(x) \subseteq G \cap G_B\}$$

(namely, $x \in A$ if and only if $N(x) \subseteq G$, which for $x \in B$ is the same as $N(x) \subseteq G \cap G_B$).

We now turn to $\mathcal{H}$. Note that in what follows we assume the constant ε $(= 1 - p)$ is small enough to support our argument, making no attempt to optimize.

For $\eta > 0$ and $B \subseteq \Gamma_k$, set

$$E_{B,\eta} = \{||\mathcal{H} \cap B| - |B|p| > \eta\delta a p\}. \tag{4.18}$$

(The second p on the right-hand side is unnecessary but we keep it as a reminder of where we are: if p were smaller, then this factor *would* be relevant.) Say a collection of sets, $\mathcal{B}$, is *η-nice* if

$$\Pr(\cup_{B\in\mathcal{B}} E_{B,\eta}) < \exp[-\Omega(ak^{-2})]. \tag{4.19}$$

Fix a smallish η; for concreteness, say $\eta = 0.08$ (we need $6\eta < 0.5$). The next, regrettably (but as far as we can see unavoidably) elaborate statement is most of the story.

Lemma 4.4.4. *The following collections are η-nice:*

(a) $\{W_R : R \in \mathcal{R}\}$;

(b) $\{S_R : R \in \mathcal{R}\}$;

(c) $\{W_R \setminus Z_R : R \in \mathcal{R}\}$;

(d) $\{S_R \setminus S'_R : R \in \mathcal{R}\}$;

(e) $\{S'_R \setminus A^*_R : R \in \mathcal{R}\}$;

(f) $\{A^*_R \setminus S'_R : R \in \mathcal{R}\}$;

(g) $\{G^*_R \setminus Z_R : R \in \mathcal{R}\}$;

(h) $\{A \setminus A^*_{R(A)} : A \in \mathcal{A}\}$;

(i) $\{A^*_{R(A)} \setminus A : A \in \mathcal{A}\}$;

(j) $\{G_A \setminus G^*_{R(A)} : A \in \mathcal{A}\}$;

(k) $\{G_A \cap (G^*_{R(A)} \setminus Z_{R(A)}) : A \in \mathcal{A}\}$.

Before proving this, we show that it supports (4.13):

Corollary 4.4.5. *The collections $\mathcal{A}$ and $\{G_A : A \in \mathcal{A}\}$ are (6η)-nice.*

Of course this gives the relevant portion of (4.13), since $\mathcal{Q}(a,g)$ implies that for some $A \in \mathcal{A}$ either $|\mathcal{H} \cap A| \geq |A|p + \delta ap/2$ or $|\mathcal{H} \cap G_A| \leq |G_A|p - \delta ap/2$, each of which (by Corollary 4.4.5) occurs with probability $\exp[-\Omega(ak^{-2})]$ and

$$\sum_{a>(4/e)^k} \sum_{g\leq 2a} \exp[-\Omega(ak^{-2})] = o(1/n).$$

Proof of Corollary 4.4.5. This is just a matter of building the relevant sets, starting from the collections in Lemma 4.4.4 and applying the (trivial) observations:

if $\{K_B : B \in \mathcal{B}\}$ is α-nice, $\{L_B : B \in \mathcal{B}\}$ is β-nice and $K_B \cap L_B = \emptyset \quad \forall B \in \mathcal{B}$, then $\{K_B \cup L_B : B \in \mathcal{B}\}$ is $(\alpha + \beta)$-nice;

if $\{K_B : B \in \mathcal{B}\}$ is α-nice, $\{L_B : B \in \mathcal{B}\}$ is β-nice and $K_B \supseteq L_B \quad \forall B \in \mathcal{B}$, then $\{K_B \setminus L_B : B \in \mathcal{B}\}$ is $(\alpha + \beta)$-nice.

Using these (in combination with Lemma 4.4.4), we find that:

$$\{Z_R = W_R \setminus (W_R \setminus Z_R) : R \in \mathcal{R}\} \text{ is } (2\eta)\text{-nice;}$$

$$\{S'_R = S_R \setminus (S_R \setminus S'_R) : R \in \mathcal{R}\} \text{ is } (2\eta)\text{-nice;}$$

$$\{A^*_R = (S'_R \setminus (S'_R \setminus A^*_R)) \cup (A^*_R \setminus S'_R) : R \in \mathcal{R}\} \text{ is } (4\eta)\text{-nice;}$$

$$\{A = (A \setminus A^*_{R(A)}) \cup (A^*_{R(A)} \setminus (A^*_{R(A)} \setminus A)) : A \in \mathcal{A}\} = \mathcal{A} \text{ is } (6\eta)\text{-nice;}$$

$$\begin{aligned}\{G_A = (G_A \setminus G^*_{R(A)}) \cup (G_A \cap (G^*_{R(A)} \setminus Z_{R(A)})) \cup Z_{R(A)}) : A \in \mathcal{A}\} \\ = \{G_A : A \in \mathcal{A}\} \text{ is } (4\eta)\text{-nice.}\end{aligned}$$

∎

Proof of Lemma 4.4.4. For the rest of this discussion we write E_B for $E_{B,\eta}$. We want to show that (4.19) holds for each of the collections $\mathcal{B}$ appearing in (a)-(k). This is all based on the union bound: in each case we bound the size of the $\mathcal{B}$ in question and show, using what we know about the sizes of members of $\mathcal{B}$, that $\Pr(E_B)$ is much smaller than $|\mathcal{B}|^{-1}$ for each $B \in \mathcal{B}$.

We are interested in bounding probabilities of the type

$$\Pr(||\mathcal{H} \cap B| - |B|p| > \eta\delta p)$$

using Theorem 2.2.4; but, since $p = 1 - \varepsilon \approx 1$, we can do a little better by applying these theorems with $\xi = |B \setminus \mathcal{H}|$ (which has the distribution $\mathrm{B}(|B|, \varepsilon)$), using the trivial observation that, for any $\lambda > 0$ (always equal to $\eta\delta ap$ in what follows),

$$\Pr(||\mathcal{H} \cap B| - |B|p| > \lambda) = \Pr(||B \setminus \mathcal{H}| - |B|\varepsilon| > \lambda). \tag{4.20}$$

(For most of the argument this change will make little difference, but it will be crucial when we come to items (h)-(k).)

Items (a) and (b). To make things easier to read, set $\beta = Kak^{-3+\zeta}\log k$ (the bound in (T1) of Lemma 4.4.1). The number of possibilities for each of W_R, S_R is bounded by the number of possible T_R's, which (by (4.3)) is at most

$$\exp[\beta\log(eN/\beta)] < \exp[\beta\cdot 4k] \tag{4.21}$$

(recall $N = \binom{2k}{k}$). On the other hand (T4) and $|S_T| \le 2(k+1)|W_T|/k$ (see (4.14)) imply that, for any T,

$$|W_T|, |S_T| < O(\delta ak^{\zeta}\log k) + g = O(\delta ak^{\zeta}\log k + a), \tag{4.22}$$

so that Theorem 2.2.4 gives (for any T)

$$\max\{\Pr(E_{W_T}), \Pr(E_{S_T})\} < \exp[-\Omega(\eta^2\delta^2 a/\max\{\delta k^{\zeta}\log k, 1\})]. \tag{4.23}$$

(In a little more detail: we apply the theorem—using (4.20) if desired though, as noted above, it is not really needed here—with $\lambda = \eta\delta ap$ and $m = O(\delta ak^{\zeta}\log k + a)$ to bound the left side of (4.23) by $\exp[-\Omega(\lambda^2/\max\{m\varepsilon, \lambda\})]$, and observe that $\lambda^2/\max\{m\varepsilon,\lambda\} = \Omega(\eta^2\delta^2 a/\max\{\delta k^{\zeta}\log k, 1\})$.)

That the collections in (a) and (b) are η-nice now follows upon multiplying the bounds in (4.21) and (4.23) (and using (4.6)).

Item (c). Since each of Z_R, W_R is determined by T_R and F_R, the number of possibilities for $W_R \setminus Z_R$ is at most the product of the bound in (4.21) (which will be negligible here) and the number of possibilities for F_R given $T = T_R$. The latter is at most the number of subsets of $\nabla(N(T), \Gamma_k)$ of size less than $\mathsf{c} := K\delta ak^{-1+\zeta}\log k$ (the bound in (T2)), which, since

$$|\nabla(N(T), \Gamma_k)| \le k^2|T| < Kak^{-1+\zeta}\log k =: \lambda \tag{4.24}$$

(see (T1)), is less than

$$\begin{aligned}\exp[\mathsf{c}\log(e\lambda/\mathsf{c})] &= \exp[O(\delta ak^{-1+\zeta}\log k\log(e/\delta))]\\ &= \exp[O(\delta ak^{-1+\zeta}\log^2 k)].\end{aligned} \tag{4.25}$$

(Here we again use (4.3) (for the initial bound) and (4.6) (for the second line). Strictly speaking, the application of (4.3) is only justified when $\delta \le 1/2$; but for larger δ we can just use the trivial bound 2^λ, which for such δ is smaller than the expression in (4.25).)

On the other hand, using (T2), we have

$$|W_R \setminus Z_R| \le k|F_R| = O(\delta a k^\zeta \log k).$$

(For the first inequality, fix A with $R(A) = R$ and note that for any $w \in W_R \setminus Z_R$ $(= W_A \setminus Z_A)$ there is a path $xyzw$ with $x \in T$ $(= T_A)$—such a path exists since $w \in W_A$—and $yz \in F_A$ (since otherwise $y \in A$ and $w \in Z_A$).)

Thus (for any R)

$$\begin{aligned}\Pr(E_{W_R \setminus Z_R}) &< \exp[-\Omega(\eta^2\delta^2 a^2/(\delta a k^\zeta \log k))] \\ &= \exp[-\Omega(\eta^2 \delta a/(k^\zeta \log k))],\end{aligned}$$

which, combined with the (relatively insignificant) bounds in (4.21) and (4.25), gives

$$\textstyle\sum_R \Pr(E_{W_R \setminus Z_R}) = \exp[-\Omega(\eta^2\delta a/(k^\zeta \log k))].$$

Items (d)-(g). For each of these the number of sets in question is $|\mathcal{R}|$, the number of possibilities for (T_R, F_R, U_R). As already observed, the number of (T_R, F_R)'s is at most the product of the bounds in (4.21) and (4.25). On the other hand, with $\mathsf{c} = K\delta a k^{-1+\zeta} \log^2 k$ (the bound on $|U|$ in (U1)) and $\lambda = K\delta a k^\zeta \log k$ (the bound on $|W_T \setminus G_A|$ in (T4))—so c and λ have changed from what they were above—the number of possibilities for U_R given T_R is at most

$$\exp[\mathsf{c}\log(e\lambda/\mathsf{c})] = \exp[O(\delta a k^{-1+\zeta} \log^3 k)] \tag{4.26}$$

(which dominates the bounds from (4.21) and (4.25)).

We next need to bound the sizes of the various sets under discussion. We have

$$|S_R \setminus S'_R| \le (k+1)\mathsf{c} = O(\delta a k^\zeta \log^2 k) \tag{4.27}$$

(again, from (U1))

$$|S'_R \setminus A^*_R| = O(\delta a) \tag{4.28}$$

(using (U2));

$$|A_R^* \setminus S_R'| = O(\delta a k^{-\zeta}) \tag{4.29}$$

(using (T5) and the fact—see (4.16)—that $A_R^* \setminus S_R' = A_R^* \setminus S_R$); and

$$|G_R^* \setminus Z_R| \le |(G_R^*)^0 \setminus Z_R| + |G_R^* \setminus (G_R^*)^0| = O(ak^{-2} + \delta a) = O(\delta a) \tag{4.30}$$

(using (T3), Lemma 4.4.2 and (4.6); note that this bound actually applies to $|G_A \setminus Z_R|$ for any A with $R(A) = R$).

The largest of the preceding bounds is the $O(\delta a k^{\zeta} \log^2 k)$ in (4.27); so for each of the sets B appearing in (d)-(g) (i.e. $B = S_R \setminus S_R'$ in (d) and so on), we have

$$\begin{aligned} \Pr(E_B) &< \exp[-\Omega(\eta^2\delta^2 a^2/(\delta a k^{\zeta} \log^2 k)] \\ &= \exp[-\Omega(\eta^2 \delta a/(k^{\zeta} \log^2 k))]; \end{aligned} \tag{4.31}$$

and, since $\eta^2\delta a/(k^{\zeta}\log^2 k)$ (from the exponent in (4.31)) is much larger than the exponent in (4.26), it follows that each of the collections in (d)-(g) is nice.

Items (h)-(k). Here we first dispose of the sizes of the individual sets, before turning to the more interesting problem of bounding the sizes of the collections in question.

For (h) and (i), notice that for any $A, A' \in \mathcal{A}$ with $R(A) = R(A')$ we have

$$|A \setminus A'| \le |A \cap (S_R' \setminus A')| + |A \setminus S_R'| = O(\delta a + \delta a k^{-\zeta}) = O(\delta a)$$

(using (U2) and (T5), as earlier in (4.28) and (4.29)); in particular this bounds the sizes of the sets in (h), (i) (namely $|A \setminus A_R^*|$ and $|A_R^* \setminus A|$ where $R = R(A)$) by $O(\delta a)$. For (j) and (k), a similar bound—that is,

$$\max\{|G_A \setminus G_R^*|, |G_A \cap (G_R^* \setminus Z_R)|\} = O(\delta a)$$

(again, for A with $R(A) = R$) follows from (4.30) (which, as noted there, is valid with G_R^* replaced by any G_A with $R(A) = R$) and the fact (see (4.15)) that $G_A \supseteq Z_R$ whenever $R(A) = R$.

We now turn to the sizes of the collections in (h)-(k), each of which is at most $|\mathcal{A}|$. We will show

$$|\mathcal{A}| < \exp[O(\delta a)]. \tag{4.32}$$

Before doing so we observe that this is enough to show that the collections in (h)-(k) are η-nice; namely, for B belonging to any of these collections (so $|B| = O(\delta a)$) and small enough ε, Theorem 2.2.4 (applied with $m = O(\delta a)$ and $q = \varepsilon$—and now really using (4.20)—gives

$$\Pr(E_B) < \exp[-\Omega(\eta\delta a \log(\eta/\varepsilon)].$$

(Here $|\mathcal{H} \setminus B| < |B|\varepsilon - \eta\delta ap$ is simply impossible, so we are just using

$$\Pr(|\mathcal{H} \setminus B| > |B|\varepsilon + \eta\delta ap) < \Pr(|\mathcal{H} \setminus B| > \eta\delta ap) < \exp[-\eta\delta ap \log(\tfrac{\eta\delta ap}{em\varepsilon})].)$$

Proof of (4.32). According to (4.17), we may bound $|\mathcal{A}|$ by the number of possibilities for the pair $(A\setminus A^*_R, G_A\cap G^*_R)$, so by our earlier bound on $|\mathcal{R}|$—essentially (4.26)—times the number of possibilities for $(A\setminus A^*_R, G_A\cap G^*_R)$ given R. So it is enough to show that, once we know R—and therefore A^*_R and G^*_R—the number of choices for each of $A\setminus A^*_R$, $G_A\cap G^*_R$ is less than $\exp[O(\delta a)]$.

The second of these is easy: since (by (4.15)) each of G_A, G^*_R contains Z_R (which is determined by R), the number of possibilities for $G\cap G^*_R$ given R (and therefore G^*_R) is at most $\exp_2[|G^*_R \setminus Z_R|]$, and we have already seen in (4.30) that $|G^*_R \setminus Z_R| = O(\delta a)$.

The case of $A \setminus A^*_R$ is more interesting. Here we may decompose

$$A \setminus A^*_R = (A \cap (S'_R \setminus A^*_R)) \cup (A \setminus (S'_R \cup A^*_R))$$

and consider the two terms on the right-hand side separately. The number of possibilities for the first term is at most $\exp_2[|S'_R \setminus A^*_R|]$ (again, given R, which determines S'_R and A^*_R), while (U2) (or (4.28)) gives $|S'_R \setminus A^*_R| = O(\delta a)$.

So it is enough to show that the number of possibilities for $A\setminus(S'_R\cup A^*_R)$ is $\exp[o(\delta a)]$. In fact, it is enough to prove such a bound on the number of possibilities for $A \setminus S'_R$ which determines $A \setminus (S'_R \cup A^*_R)$ since we know A^*_R. Here we recall that (4.16) gives $A\setminus S'_R = A\setminus S_R$ (so we may use these interchangeably, and similarly for $A\cap S'_R = A\cap S_R$), and that—crucially—(T5) gives

$$|A \setminus S_R| = O(\delta a k^{-\vartheta}). \tag{4.33}$$

Note that this final point differs from its earlier counterparts in the present argument in that we now have less control over the size of the universe from which the set in question (i.e. $A \setminus S_R$) is being drawn (in contrast to, for example, F_R in (c), which was drawn from $\nabla(N(T), \Gamma_k \setminus T)$, whose size was bounded in (4.24), or, in the present case, $A \cap (S'_R \setminus A^*_R)$, which is drawn from the quite small $S'_R \setminus A^*_R$). Thus, for example, if we try to apply (4.3) with α the bound in (4.33) and $\beta = N$ $(= \binom{2k}{k})$, then we can only say that the number of possibilities for $A \setminus S_R$ is less than $\exp[O(\delta a k^{-\vartheta}) \log(eN/\delta a k^{-\vartheta})]$, which for somewhat small a may be far larger than the desired $\exp[O(\delta a)]$. This little difficulty will be handled by Proposition 4.2.4.

Write t $(= O(\delta a k^{-\zeta}))$ for the bound on $|A \setminus S_R|$ given in (4.33). Denote by Λ the ("Johnson") graph on Γ_k in which two vertices (a.k.a. k-sets) are adjacent if they are at distance 2 in Σ, and set $d = k^2$ (so Λ is d-regular). Since our A's induce connected subgraphs of Λ (another way of saying they are 2-linked), there is, for each A under discussion, a rooted forest with roots in $S_R \cap A = S'_R \cap A$, set of non-roots equal to $A \setminus S_R$, at least one non-root in each component, and at most t vertices overall; thus we just need to bound the number of such forests.

(Note that existence of said forest requires $S_R \cap A \neq \emptyset$, which, since we assume δ is not too large, holds because the bound in (T5) is less than a. If $S_R \cap A = \emptyset$—as can happen for large δ—then the forest has a single root, the number of possibilities for which we can only bound by N (in place of the bound for (ii) below). This change would cause trouble in the present regime, but not for large δ, where, as will appear below, our probability bounds improve.)

For the desired bound we may think of specifying a forest as above by specifying:

(i) the number, say $q \leq t$ (or $q \leq t/2$, but this doesn't matter), of roots;

(ii) the actual roots, $x_1, \ldots, x_q \in S'_R \cap A$;

(iii) for each $i \in [q]$, the size, say α_i, of the component (tree) rooted at x_i; and

(iv) the components themselves.

We may bound the numbers of possibilities in (ii), (iii) and (iv) by $\binom{(1+O(\delta))a}{q}$, $\binom{t}{q}$

and $(ed)^t$ (respectively). The first of these derives from (U2), according to which we have $|S'_R| < a + O(\delta a)$; the second is the number of sequences $(\alpha_1, \ldots, \alpha_q)$ of positive integers summing to at most t (a slight overcount since our α_i's are all at least 2); and the third follows from Proposition 4.2.4. Thus the number of forests is at most

$$\textstyle\sum_{q \le t} \binom{(1+O(\delta))a}{q} \binom{t}{q} (ed)^t = \exp[O(t \log k)] = \exp[o(\delta a)]. \tag{4.34}$$

∎

4.4.3 Big δ

Finally we turn to the case of large δ ($\delta > 1$), showing (for any a, g with $\delta = (g-a)/a > 1$)

$$\Pr(\mathcal{Q}(a, g)) < \varepsilon^{g/3}, \tag{4.35}$$

which, with the trivial $g \ge k$, bounds the contribution to (4.13) of the terms under discussion by

$$\sum_{g \ge k} \sum_{a < g} \varepsilon^{g/3} = o(1/n).$$

To begin, notice that in the present situation Theorem 2.2.4 bounds the probability of (4.12) (for a given $A \in \mathcal{A}(a, g)$) by

$$\Pr(|G_A \setminus \mathcal{H}| > g/2) < (2e\varepsilon)^{g/2}. \tag{4.36}$$

On the other hand, to bound the number of possibilities for A (i.e. the size of $\mathcal{A}(a, g)$), we may think of specifying A *via* the following steps.

(i) Choose $T \subseteq G = G_A$ of size $C(g/k) \log k$ such that, with

$$S = S_T = \{x \in \Gamma_k : d_T(x) > (C/2) \log k\},$$

we have

$$|A \setminus S| < k^{-2} a \tag{4.37}$$

and, with $Z = Z(G) = \{x : d_G(x) \ge k/4\}$

$$|S \setminus Z| < k^{-1} g \tag{4.38}$$

(The proof of the existence of such a T is similar to—easier than—the proof of Lemma 4.4.1, and we omit the details, just noting that, since $S \setminus Z \subseteq N(G)$, fewer than gk vertices are relevant to (4.38).)

Notice that by (4.38) (and the definition of Z), we have

$$|S| \leq (4/k)g(k+1) + g/k = O(g). \tag{4.39}$$

(ii) For each $x \in A \setminus S$, choose some neighbor of x (necessarily in G) and let T' be the collection of these neighbors; thus (recalling (4.37)) we have $|T'| \leq |A \setminus S| < k^{-2}a$. Notice also that $T \cup T'$ is 4-linked (by Proposition 4.2.3, using the fact that A is 2-linked).

(iii) Finally, choose A from $S \cup N(T')$.

We should then bound the number of ways in which these steps can be carried out:

(i) Since $T \cup T'$ is 4-linked Proposition 4.2.4) (applied to the graph on Γ_k with adjacency meaning distance in Σ at most 4, a regular graph of degree $d < k^4$) bounds the number of choices for $T \cup T'$ by

$$N \exp[O((g/k) \log k) \log d] < N \exp[C(g/k) \log^2 k]$$

(where the N is for a choice of *some* vertex of $T \cup T'$).

(ii) The number of choices for $T' \setminus T$ given $T \cup T'$ is $\exp[O(k^{-2}a \log(gk/a)]$. Note that once we know $T \cup T'$ and $T \setminus T'$, we also know T and thus S.

(iii) Given S, there are $\exp[O(a \log(g/a)]$ choices for $A \subseteq S \cup N(T')$.

Of course for sufficiently (not very) small ε, all of these bounds are dominated by the one in (4.36), so we have (4.35).

∎

4.5 $p > 3/4$

Here we justify the assertion following Theorem 1.0.4, showing that for fixed $p < 3/4$ (say $p = 3/4 - \varepsilon$), a.s. no $\mathcal{H}_x$ is a maximal clique (so $\mathcal{H}$ is not EKR). This is a simple application of the second moment method, as follows.

Fix x and for $A \in \mathcal{K}_{\overline{x}}$ set $E(A) = \{\mathcal{H}_x \cup \{A\}$ is a clique of $\mathcal{H}\}$ and $X_A = \mathbf{1}_{E(A)}$. Set $X = \sum\{X_A : A \in \mathcal{K}_{\overline{x}}\}$. Then

$$\mu := \mathsf{E}X = \binom{2k}{k} p(1-p)^k > (1 + 4\varepsilon - o(1))^k.$$

On the other hand, writing $A \sim B$ if $|A \cap B| = k-1$, we have

$$\mathsf{E}X_A X_B = \begin{cases} p^2(1-p)^{2k-1} & \text{if } A \sim B, \\ \mathsf{E}X_A \mathsf{E}X_B & \text{if } A \not\sim B \neq A, \end{cases}$$

which, with A, B running over $\mathcal{K}_{\overline{x}}$, yields

$$\begin{aligned} \mathrm{Var}(X) &= \sum_A \sum_B \mathrm{cov}(X_A, X_B) \\ &\leq \sum_A \mathsf{E}X_A + \sum_A \sum_{B \sim A} \mathsf{E}X_A X_B \\ &= \mu[1 + k^2 p(1-p)^{k-1}]. \end{aligned}$$

Thus

$$\begin{aligned} \Pr(\mathcal{H}_x \text{ is a maximal clique of } \mathcal{H}) &= \Pr(X \neq 0) \\ &\leq \frac{\mathrm{Var}(X)}{\mu^2} < (1 + 4\varepsilon - o(1))^{-k} \end{aligned}$$

and (now letting x vary)

$$\Pr(\text{some } \mathcal{H}_x \text{ is a maximal clique of } \mathcal{H}) < n(1 + 4\varepsilon - o(1))^{-k} = o(1).$$

∎

A similar discussion explains the comment following Theorem 1.0.4 (that a positive answer to Question 1.0.5 is not very satisfactory for $n \geq 2k+2$). For example for $n = 2k+2$, arguing as above—now with special treatment for $A, B \in \mathcal{K}_{\overline{x}}$ with $|A \cap B| \in \{k-1, k-2\}$—shows that a.s. no $\mathcal{H}_x$ is a maximal clique if $p < C/k$ for a suitable constant C.

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