STRUCTURE AND DYNAMICS OF NONCOMMUTATIVE SOLITONS:

SPECTRAL THEORY AND DISPERSIVE ESTIMATES

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A dissertation submitted to the

Graduate School-New Brunswick

Rutgers, The State University of New Jersey

In partial fulfillment of the requirements

For the degree of

Doctor of Philosophy

Graduate Program in Physics

Written under the direction of

Avraham Soffer

And approved by

New Brunswick, New Jersey

October 2014

ABSTRACT OF THE DISSERTATION

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We consider the Schrödinger equation with a Hamiltonian given by a second order difference operator with nonconstant growing coefficients, on the half one dimensional lattice. This operator appeared first naturally in the construction and dynamics of noncommutative solitons in the context of noncommutative field theory. We prove pointwise in time decay estimates, with the optimal decay rate $t^{-1}\log^{-2} t$ generically. We use a novel technique involving generating functions of orthogonal polynomials to achieve these estimates. We construct a ground state soliton for this equation and analyze its properties. In particular we arrive at ℓ^∞ and ℓ^1 estimates as well as a quasi-exponential spatial decay rate. We completely determine the spectrum of the associated linearized Hamiltonian and prove the optimal decay rate of $t^{-1}\log^{-2} t$ for the associated time decay estimate. These results are to appear in forthcoming papers.

DEDICATION

This work is dedicated to my mother, without whom nothing would be possible.

ACKNOWLEDGEMENT

I am in gratitude to my advisor, Avy Soffer, for his wisdom and respect, Andrew Comech for his guidance and introducing me to the field nonlinear PDE, and Marius Beceanu, for his patience and esteemed comments. I thank the members of my committee, Joel Lebowitz, Shelly Goldstein, and Joel Shapiro for their honesty and support. I thank Michael Kiessling and A. Shadi Tahvildar-Zadeh for introducing me to many important concepts in mathematical physics. I am in gratitude to Roderich Tumulka for his enthusiastic support in all respects. This work was supported in part by the NSF grant DMR 1104500.

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Part 1

Introduction and Background

The notion of noncommutative soliton arises when one considers the nonlinear Klein-Gordon equation (NLKG) for a field which is dependent on, for example, two "noncommutative coordinates", x_1, x_2, whose coordinate functions satisfy $[X_1, X_2] = i\epsilon$. By going to a representation of the above canonical commutation relation, one can reduce the dynamics of the problem to an equation for the coefficients of an expansion in the Hilbert space representation of the above CCR, see e.g. [15]. By restricting to rotationally symmetric functions the nocommutative deformation of the Laplacian reduces to a second order finite difference operator, which is symmetric, and with variable coefficient growing like n, the lattice coordinate, at infinity. Therefore, this operator is unbounded, and in fact has continuous spectrum $[0, \infty)$. These preliminary analytical results, as well as additional numerical results, were obtained by Chen, Fröhlich, and Walcher. [5]. The dynamics and scattering of the (perturbed) soliton can then be inferred from the NLKG with such a discrete operator as the linear part. We will be interested in studying the dynamics of discrete NLKG and discrete NLS equations with these hamiltonians.

We follow the presentation of these methods in the manner of [5]. A soliton, or solitary wave, is a localized stationary solution to a nonlinear PDE, an example of which is the 2D real nonlinear Klein-Gordon equation: $-\partial_t^2\Phi = -\Delta\Phi + V(\Phi)$, where $\Phi : \mathbb{R}_t \times \mathbb{R}^2_{(x_1,x_2)} \to \mathbb{R}$, $V : \mathbb{R} \to \mathbb{R}$. A noncommutative soliton is a solitary wave solution to a PDE which has been deformed by enhancing the pointwise algebra of functions to one with a nontrivial spatial commutation relation. Here this is to say that $[X_1, X_2] = 0 \mapsto [X_1, X_2] = i\epsilon$, where X_1, X_2 are the coordinate functions respectively for the Cartesian spatial coordinates x_1, x_2. By demanding that $\partial_x X = 1$ one may arrive at $\partial_{x_1} \mapsto \frac{i}{\epsilon}[X_2, \cdot]$ and $\partial_{x_2} \mapsto \frac{i}{\epsilon}[X_1, \cdot]$. Finally, by considering formal power series expansions of functions on space one may deform the pointwise product $\Phi_1\Phi_2(x_1, x_2)$ to the Moyal star product

$$\Phi_1 \star \Phi_2(x_1, x_2) = e^{i(\epsilon/2)(\partial_{a_1}\partial_{b_2} - \partial_{a_2}\partial_{b_1})}\Phi_1(a_1, b_1)\Phi_2(a_2, b_2)\lfloor_{(a_j,b_j)=(x_1,x_2)}.$$

By analogy to $[Q,P]=i\epsilon \Rightarrow [A^\dagger,A]=1$ it is typical to consider the associated harmonic oscillator representation

$$A := (2\epsilon)^{-1/2}(X_1+iX_2), \quad A^\dagger := (2\epsilon)^{-1/2}(X_1-iX_2),$$

$$-\partial_t^2\Phi = -\Delta\Phi + V(\Phi) \quad \mapsto \quad -\partial_t^2\Phi = (2/\epsilon)[A,[A^\dagger,\Phi]] + V(\Phi).$$

Let Π_n be the projection onto the n-th excited state of the harmonic oscillator system: $\Pi_n v = v(n)\chi_n$. One may observe that

$$(\chi_n, R^2\chi_n) = (\chi_n, (X_1^2+X_2^2)\chi_n) = 2\epsilon(n-1/2),$$

$$[A,[A^\dagger,\Pi_n]] = \begin{cases} -(n+1)\Pi_{n+1} + (2n+1)\Pi_n - n\Pi_{n-1} & , \quad n>0 \\ -\Pi_1+\Pi_0 & , \quad n=0. \end{cases}$$

so that spherically symmetric functions are deformed into operators which are diagonal in the harmonic oscillator basis. Therefore if one restricts the original PDE to spherically symmetric solutions then the noncommutative PDE becomes a finite difference equation on the half-lattice of excited state level indices of a harmonic oscillator system: $\Phi(t,r)\mapsto\Phi(t,n)$.

As an alternative to noncommutative deformation one may consider the following. If one performs the change of variables $r^2=2\epsilon n_*$, then the radial 2D laplacian takes the form $r^{-1}\partial_r r\partial_r = (2/\epsilon)\partial_{n_*}n_*\partial_{n_*}$. Upon a careful choice of direct discretization one finds $(2/\epsilon)\partial_{n_*}n_*\partial_{n_*} \mapsto (2/\epsilon)D_+ nD_-$, which has the same action on the χ_n as $-\Delta \mapsto (2/\epsilon)[A,[A^\dagger,\cdot]]$ has on the Π_n, where D_+, D_- are respectively the forward and backward finite difference operators.

In this sense the resulting real radial 2D nonlinear Klein-Gordon equation can be considered as either a noncommutative version or a discrete version of the original commutative differential one. The result of the discretization procedure is perhaps a bit unexpected as the harmonic oscillator representation needn't look anything like the the original commutative spatial representation. Nevertheless, the original equation can be recovered with the limit $\epsilon \searrow 0$ as either the commutative limit of a noncommutative system (like $\hbar \searrow 0$) or as the continuum limit of a lattice system. We will work with the finite difference representation exclusively. Henceforth $n \mapsto x \in \mathbb{Z}_+$, since the lattice will be our space, and we set $\epsilon = 2$.

The principle of replacing the usual space with a so-called noncommutative space (or space-time) has found extensive use for model building in physics and in particular for allowing easier construction of localized solutions, see e.g. [2][22] for surveys. An example of the usefulness of this approach is that it may provide a robust procedure for circumventing classical nonexistence theorems for solitons, e.g. that of Derrick [8]. The NLKG variant of the equation we study here first appeared in the context of string theory and associated effective actions in the presence of background D-brane configurations, see e.g. [15]. We have decided to look in a completely different direction. The NLS variant and its solitons can in principle be materialized experimentally with optical devices, suitably etched, see e.g. [6]. Thus the dynamics of NLS with such solitons may offer new and potentially useful coherent states for optical devices. Furthermore, we believe the NLS solitons to have special properties, in particular asymptotic stability as opposed to the conjectured asymptotic metastability of the NLKG solitons conjectured in [5].

We will be following a procedure for the proof of asymptotic stability which has become standard within the study of nonlinear PDE. Crucial aspects of the theory and associated results were established by Buslaev and Perelman [3], Buslaev and Sulem [4], and Gang and Sigal [14]. Important elements of these methods are the dispersive estimates. Various such estimates have been found in the context of 1D lattice systems, for example see the work of A.I. Komech, E.A. Kopylova, and M. Kunze [20] and of I. Egorova, E. Kopylova, G. Teschl [12], as well as the continuum 2D problem to which our system bears many resemblances, see e.g. the work of E. A. Kopylova and A.I. Komech [21]. Extensive results have been found on the asymptotic stability on solitons of 1D nonlinear lattice Schrödinger equations by F. Palmero et al. [24] and P.G. Kevrekidis, D.E. Pelinovsky, and A. Stefanov [18]. Important aspects of the application of these models to optical nonlinear waveguide arrays has been established by H.S. Eisenberg et al. [**13**].

This work is part of a series of forthcoming papers devoted to the construction, scattering, and asymptotic stability of these noncommutative solitons. The first three chapters are devoted to separate aspects of the problem in order of necessity. The fourth chapter addresses further work on this subject matter that we would like to pursue. The organization of this work is as follows.

In Chapter 1 we focus on a key estimate that is needed for scattering and stability, namely the decay in time of the solution, at the optimal rate. Fortunately, in the generic case, we find it is integrable, given by $t^{-1}\log^{-2} t$. The proof of this result is rather direct, and employs the generating functions of the corresponding generalized eigenfunctions, to explicitly represent and estimate the resolvent of the hamiltonian at all energies. We also conclude the absence of positive eigenvalues and singular continuous spectrum.

Preliminary results for the scattering theory of the associated noncommutative waves and solitons were found by Durhuus and Gayral [**9**]. In particular they find local decay estimates for the associated noncommutative NLS. We utilize alternative methods and find local decay for both the free Schrödinger operator as well as a class of rank one perturbations thereof. An important element of this analysis is the study of the spectral properties of the free and perturbed Schrödinger operator. We extend the analysis of Chen, Frölich, and Walcher [**5**] and reproduce some of their results with alternative techniques.

In Chapter 2 we address the construction and properties of a family of ground state solitons. These stationary states satisfy a nonlinear eigenvalue equation, are positive, monotonically decaying and sharply peaked for large spectral parameter. The proof of this result follows directly from our spectral results in Chapter 1 by iteration for small data and root finding for large data. The existence and many properties of solutions for a similar nonlinear eigenvalue equation were found by Durhuus, Jonssen, and Nest

[10][11]. We utilize a simple power law nonlinearity for which their existence proofs do not apply. We additionally find estimates for the peak height, spatial decay rate, norm bounds, and parameter dependence.

In Chapter 3 we focus on deriving a decay rate estimate for the Hamiltonian which results from linearizing the original NLS around the soliton constructed in Chapter 1. We determine the full spectrum of this operator, which is the union of a multiplicity 2 null eigenvalue and a real absolutely continuous spectrum. This establishes a well-defined set of modulation equations [31] and points toward the asymptotic stability of the soliton.

In Chapter 4 we describe how the results from Chapter 3 can be applied to prove stability of the soliton we constructed in Chapter 2. The issue of asymptotic stability of NLS solitons has been sufficiently well-studied in such a broad context that the proof thereof is often considered as following straightforwardly from the appropriate spectral and decay estimates, of the kind found in Chapter 3. We sketch how the theory of modulation equations established by Soffer and Weinstein [31] can be used to prove asymptotic stability. Chen, Fröhlich, and Walcher conjectured that in the NLKG case the corresponding solitons are unstable but with exponential long decay: the so-called metastability property, see e.g. [5]. There is a great deal of evidence to suggest that this is in fact the case but a proof has yet to be provided. This will be the subject of future work.

Part 2

Main

Notation

Let $\mathbb{Z}_+$ and $\mathbb{R}_+$ respectively be the nonnegative integers and nonnegative reals and $\mathscr{H} = \ell^2(\mathbb{Z}_+, \mathbb{C})$ the Hilbert space of square integrable complex functions, e.g. $v : \mathbb{Z}_+ \ni x \mapsto v(x) \in \mathbb{C}$, on the 1D half-lattice with inner product $(\cdot, \cdot)$, which is conjugate-linear in the first argument and linear in the second argument, and the associated norm $||\cdot||$, where $||v|| = (v, v)^{1/2}$, $\forall v \in \mathscr{H}$. Where the distinction is clear from context $||\cdot|| \equiv ||\cdot||_{\mathrm{op}}$ will also represent the norm for operators on $\mathscr{H}$ given by $||A||_{\mathrm{op}} = \sup_{v\in\mathscr{H}} ||v||^{-1}||Av||$, for all bounded A on $\mathscr{H}$. Denote the lattice ℓ^1 norm by $||\cdot||_1$ where $||v||_1 = \sum_{x=0}^{\infty} |v(x)|$, $\forall v \in \ell^1(\mathbb{Z}_+, \mathbb{C})$.

We denote by $\otimes$ the tensor product and by $z \mapsto \overline{z}$ complex conjugation for all $z \in \mathbb{C}$. We write $\mathscr{H}^*$ for the space of linear functionals on $\mathscr{H}$: the dual space of $\mathscr{H}$. For every $v \in \mathscr{H}$ one has that $v^* \in \mathscr{H}^*$ is its dual satisfying $v^*(w) = (v, w)$ for all $v, w \in \mathscr{H}$. For every operator A on $\mathscr{H}$ we take $\mathcal{D}(A)$ as standing for the domain of A. For each operator A on $\mathscr{H}$ define A^* on $\mathscr{H}^*$ to be its dual and $A^\dagger$ on $\mathscr{H}$ its adjoint such that $v^*(Aw) = A^*v^*(w) = (A^\dagger v, w)$ for all $v \in \mathcal{D}(A^\dagger)$ and all $w \in \mathcal{D}(A)$. Let $\{\chi_x\}_{x=0}^{\infty}$ be the orthonormal set of vectors such that $\chi_x(x) = 1$ and $\chi_{x_1}(x_2) = 0$ for all $x_2 \neq x_1$. We write $\Pi_x = \chi_x \otimes \chi_x^*$ for the orthogonal projection onto the space spanned by χ_x.

We define $\mathscr{T}$ to be the topological vector space of all complex sequences on $\mathbb{Z}_+$ endowed with topology of pointwise convergence, $\mathcal{B}(\mathscr{H})$ to be the space of bounded linear operators on $\mathscr{H}$, and $\mathcal{L}(\mathscr{T})$ to be the space of linear operators on $\mathscr{T}$, endowed with the pointwise topology induced by that of $\mathscr{T}$. When an operator A on $\mathscr{H}$ can be given by an explicit formula through $A(x_1, x_2) = (\chi_{x_1}, A\chi_{x_2}) < \infty$ for all $x_1, x_2 \in \mathbb{Z}_+$ one may make the natural inclusion of A into $\mathcal{L}(\mathscr{T})$, the image of which will also be denoted by A. We consider $\mathscr{T}$ to be endowed with pointwise multiplication, i.e. the product uv is specified by $(uv)(x) = u(x)v(x)$ for all $u, v \in \mathscr{T}$.

We represent the *spectrum* of each A on $\mathscr{H}$ by $\sigma(A)$. We term each element $\lambda \in \sigma(A)$ a *spectral value*. We write $\sigma_{\mathrm{d}}(A)$ for the *discrete spectrum*, $\sigma_{\mathrm{e}}(A)$ for the *essential spectrum*, $\sigma_{\mathrm{p}}(A)$ for the *point spectrum*, $\sigma_{\mathrm{ac}}(A)$ for the *absolutely continuous spectrum*, and $\sigma_{\mathrm{sc}}(A)$ for the *singularly continuous spectrum*. Should an operator satisfy the spectral theorem one may implement spectral projections. For each operator A, these will be written as Π^A_{d} and the like for each of the distinguished subsets of the spectral decomposition of A. Define $R^A_{\cdot} : \rho(A) \to \mathcal{B}(\mathscr{H})$, the resolvent of A, to be specified by $R^A_z := (A - z)^{-1}$, where $\rho(A) := \mathbb{C} \setminus \sigma(A)$ is the resolvent set of A and where by abuse of notation $zI \equiv z \in \mathcal{B}(\mathscr{H})$ here.

Allow an *eigenvector* of A to be a vector $v \in \mathscr{H}$ for which $Av = \lambda v$ for some $\lambda \in \mathbb{C}$. Should A admit inclusion into $\mathcal{L}(\mathscr{T})$, we define a *generalized eigenvector* of A be a vector $\phi \in \mathscr{T} \setminus \mathscr{H}$ which satisfies $A\phi = \lambda\phi$ for some $\lambda \in \mathbb{C}$ such that $\phi(x)$ is polynomially bounded, which is to say that there exists a $p \geq 0$ such that $\lim_{x \nearrow \infty} (x + 1)^{-p} \phi(x) = 0$. We define a *spectral vector* of A to be a vector which is either an eigenvector or generalized eigenvector of A. We define the subspace of spectral vectors associated to the set $\Sigma \subseteq \sigma(A)$ to be the *spectral space over* Σ.

We write $\partial_z \equiv \frac{\partial}{\partial z}$ and $\mathrm{d}_z \equiv \frac{\mathrm{d}}{\mathrm{d}z}$ respectively for formal partial and total derivative operators with respect to a parameter $z \in \mathbb{R}, \mathbb{C}$.

CHAPTER 1

The linear scalar problem

1. Results

Definition 1.1. *Define L_0 to be the operator on $\mathscr{H}$ with action*

$$L_0 v(x) = \begin{cases} -(x+1)v(x+1) + (2x+1)v(x) - xv(x-1) & , \quad x > 0 \\ -v(1) + v(0) & , \quad x = 0. \end{cases} \tag{1}$$

and domain $\mathcal{D}(L_0) := \{v \in \mathscr{H} \mid ||Mv|| < \infty\}$, where M is the multiplication operator with action $Mv(x) = xv(x)$ $\forall v \in \mathscr{T}$.

Consider the linear Schrödinger equation

$$i\partial_t u = L_0 u + Vu \tag{2}$$

where $u : \mathbb{R}_t \times \mathbb{Z}_+ \to \mathbb{C}$ and V is a potential (energy) multiplication operator on $\mathscr{H}$. To find solutions to Equation (2) it is sufficient to analyze the spectral measure of $L_0 + V$.

Definition 1.2. *For A an essentially self adjoint operator on $\mathscr{H}$, such that each $\lambda \in \sigma(A)$ has multiplicity 1, we denote the set of spectral vectors of A by $\{\phi^A_\lambda\}_{\lambda \in \sigma(A)}$. Let a choice of* normalization vector *for A be a fixed $v_{\mathrm{N}} \in \mathcal{H}$ such that $(v_{\mathrm{N}}, \phi^A_\lambda) = 1$ for all $\lambda \in \sigma(A)$. Furthermore let the* spectral integral weight *w^A_λ be given by $w^A_\lambda := (v_{\mathrm{N}}, \delta^A_\lambda v_{\mathrm{N}})$ for all $\lambda \in \sigma(A)$. We define the* resolvent vector *$\psi^A_\cdot : \rho(A) \to \mathscr{H}$ by $\psi^A_z := R^A_z v_{\mathrm{N}}$ and the* auxilliary resolvent vector *$\xi^A_\cdot : \rho(A) \to \mathscr{H}$ by $\xi^A_z := \psi^A_z - v^*_{\mathrm{N}}(\psi^A_z)\phi^A_z$, where $\phi^A_z \in \mathscr{T}$ is the analytic continuation $\sigma(A) \ni \lambda \to z \in \mathbb{C}$ of $\phi^A_\lambda \in \mathscr{T}$. Let the* resolvent function *$f^A_\cdot : \rho(A) \to \mathbb{C}$ be specified by $(v_{\mathrm{N}}, R^A_z v_{\mathrm{N}})$.*

The above permits the useful representation $\psi_z^A = f_z^A \phi_z^A + \xi_z^A$. ϕ_z^A and ψ_z^A are connected to the Stieltjes transform theory of orthogonal polynomials, in that ϕ_z^A is termed a *primary polynomial* and ψ_z^A the corresponding *secondary polynomial* [33].

PROPOSITION 1. *The operator L_0 has the following properties.*

(1) *L_0 is essentially self-adjoint.*

(2) *The spectral space over each $\lambda \in \sigma(L_0)$ has multiplicity 1.*

(3) *The spectrum of L_0 is absolutely continuous, $\sigma(L_0) = \sigma_{\mathrm{ac}}(L_0) = [0, \infty)$, and for choice of normalization $\phi_\lambda^{L_0}(0) = 1$, its generalized eigenfunctions are the Laguerre polynomials $\phi_\lambda^{L_0} = \sum_{k=0}^{x} \frac{(-\lambda)^k}{k!} \binom{x}{k}$.*

Chen, Frohlich, and Walcher determined the above properties for L_0 in [5] via methods which are different from ours with the apparent exception of a proof of essential self-adjointness.

DEFINITION 1.3. *Define L to be the operator on $\mathscr{H}$ with domain $\mathcal{D}(L) = \mathcal{D}(L_0)$ and specified by $L := L_0 - q\Pi_0$ where $q \geq 0$ is a fixed constant.*

THEOREM 1. *Let ϕ_λ^L, $\lambda \in \sigma(L)$, denote spectral vectors of L chosen to satisfy the normalization condition $(\chi_0, \phi_\lambda^L) = \phi_\lambda^L(0) = 1$, $\forall \lambda \in \sigma(L)$. L has the following properties.*

(1) *$\sigma_{\mathrm{d}}(L) = \sigma_{\mathrm{p}}(L) = \{\lambda_0\}$, where $\lambda_0 < 0$ uniquely satisfies $1 = q\psi_{\lambda_0}^{L_0}(0)$ and the unique eigenfunction over λ_0 is $\psi_{\lambda_0}^L = q\psi_{\lambda_0}^{L_0}$.*

(2) *$\sigma_{\mathrm{e}}(L) = \sigma_{\mathrm{ac}}(L) = [0, \infty)$.*

(3) *$\mathrm{d}\mu^L(\lambda)\Pi_{\mathrm{e}}^{L_0} = w_\lambda^L \phi_\lambda^L \otimes \phi_\lambda^{L,*} \, \mathrm{d}\lambda \; \Pi_{\mathrm{e}}^{L_0}$, where $w_\lambda^L = \{[1 - qe^{-\lambda}\mathcal{P}E_1(-\lambda)]^2 + [\pi q e^{-\lambda}]^2\}^{-1} e^{-\lambda}$, $\mathrm{d}\lambda$ is the Lebesgue measure on $[0, \infty)$, and $\mathcal{P}E_1(-\lambda) = -\mathrm{Ei}(\lambda) := -\int_{-\lambda}^{\infty} \mathrm{d}u \; u^{-1} e^{-u}$, $\lambda > 0$, is the exponential integral. The generalized eigenfunctions of L are given by $\phi_\lambda^L = \phi_\lambda^{L_0} + q\xi_\lambda^{L_0}$, $\lambda \in \sigma_{\mathrm{ac}}(L)$.*

We are ultimately interested in studying the solutions of a nonlinear equation so it is important to acquire decay estimates for dispersive "scattering states".

Definition 1.4. *Let* $W_{\kappa,\tau}$ *be the multiplication operator weight specified by* $W_{\kappa,\tau}v(x) = (x+\kappa)^{\tau}v(x)$, $\forall v \in \mathscr{T}$, *where* $0 < \kappa \in \mathbb{R}$, $\tau \in \mathbb{R}$.

Theorem 2. *For all* $-3 \geq \tau \in \mathbb{R}$, $t > 0$, $v \in \ell^1$, *there exists a constant* $c > 0$ *and a* $1 < \kappa \in \mathbb{R}$ *such that*

$$||W_{\kappa,\tau}e^{-itL_0}W_{\kappa,\tau}v||_\infty < ct^{-1}||v||_1 \tag{3}$$

Theorem 3. *For all* $-3 \geq \tau \in \mathbb{R}$, $v \in \ell^1$, *there exists a* $1 < \kappa \in \mathbb{R}$ *such that*

$$||W_{\kappa,\tau}e^{-itL}\Pi_{\mathrm{e}}^{L}W_{\kappa,\tau}v||_\infty = \mathcal{O}(t^{-1}\log^{-2}t), \quad t \nearrow \infty \tag{4}$$

Our proof of these estimates will rely heavily on the generating functions of the generalized eigenvalues. This approach draws upon known properties of certain special functions. Hereby the problem of sequences on a lattice will be transformed into a problem of analytic functions in the complex plane.

2. Spectral Properties of L_0

LEMMA 2.1. *Any vector, v, the set of whose components, $\{v(x)\}_{x=0}^{\infty}$, have finitely many nonzero elements is a semi-analytic vector for L_0, which is to say that $||L_0^k v|| \leq c_v(2k)!$ where c_v depends on v alone.*

PROOF. Define $x_v := \sup_x\{x : v(x) \neq 0\}$.

$$||L_0 v||_2^2 = \sum_{x=0}^{\infty} |-(x+1)v(x+1) + (2x+1)v(x) - xv(x-1)|^2 \tag{5}$$

$$\leq \sum_{x=0}^{\infty}[(x+1)|v(x+1)| + (2x+1)|v(x)| + x|v(x-1)|]^2 \tag{6}$$

$$\leq \sum_{x=0}^{\infty}\{[(x_v-1)+1]|v(x+1)| + (2x_v+1)|v(x)| + (x_v+1)|v(x-1)|\}^2 \tag{7}$$

$$= \sum_{x=0}^{\infty}\{[x_v|v(x+1)|]^2 + 2x_v(2x_v+1)|v(x+1)||v(x)| \tag{8}$$

$$+ 2x_v(x_v+1)|v(x)||v(x-1)| + [(2x_v+1)|v(x)|]^2 \tag{9}$$

$$+ 2(2x_v+1)(x_v+1)|v(x)||v(x-1)| + [(x_v+1)|v(x-1)|]^2\} \tag{10}$$

$$\leq \sum_{x=0}^{\infty}\{[x_v|v(x)|]^2 + 2x_v(2x_v+1)|v(x+1)||v(x)| \tag{11}$$

$$+ 2x_v(x_v+1)|v(x+1)||v(x)| + [(2x_v+1)|v(x)|]^2 \tag{12}$$

$$+ 2(2x_v+1)(x_v+1)|v(x+1)||v(x)| + [(x_v+1)|v(x)|]^2\} \tag{13}$$

$$\leq 16(x_v+1)^2||v||_1^2 \tag{14}$$

$$||L_0 v||_1 = \sum_{x=0}^{\infty} |-(x+1)v(x+1) + (2x+1)v(x) - xv(x-1)| \tag{15}$$

$$\leq \sum_{x=0}^{\infty} |(x+1)v(x+1) + (2x+1)v(x) + xv(x-1)| \tag{16}$$

$$\leq \sum_{x=0}^{\infty}[(x+1)|v(x+1)| + (2x+1)|v(x)| + x|v(x-1)|] \tag{17}$$

$$\leq \sum_{x=0}^{\infty}\{[(x_v-1)+1]|v(x+1)|+(2x_v+1)|v(x)|+(x_v+1)|v(x-1)|\} \tag{18}$$

$$\leq \sum_{x=0}^{\infty}[x_v|v(x+1)|+(2x_v+1)|v(x)|+(x_v+1)|v(x-1)|] \tag{19}$$

$$\leq 4(x_v+1)||v||_1. \tag{20}$$

Let $a_v := 4(x_v+1)$. We have then that $||L_0v||_1^2, ||L_0v||_2^2 \leq a_v||v||_1$. One may observe that $x_{L_0v} = x_v+1 \Rightarrow a_{L_0v} = a_v+4$. One then has that

$$||L_0^kv||_2 \leq a_{L_0^{k-1}v}||L_0^{k-1}v||_1 \leq a_{L_0^{k-1}v}a_{L_0^{k-2}v}||L_0^{k-2}v||_1 \leq \dots \tag{21}$$

$$\leq \prod_{j=1}^{k} a_{L_0^{k-j}v}||v||_1 = 4^k\prod_{j=1}^{k}(j+x_v)||v||_1 = 4^k(x_v!)^{-1}(k+x_v)!||v||_1. \tag{22}$$

Since $4^k(x_v!)^{-1}(k+x_v)!||v||_1$ is monotonically increasing in $k \in \mathbb{Z}_+$ we may, without loss of generality, take that $k > x_v$ and $k > 4$ to bound this expression. One may show through the monotonicity in k of $\binom{x}{k}$ for $0 \leq k \leq \lfloor x/2 \rfloor$ that $\binom{x}{k} \leq (\lfloor x/2 \rfloor!)^{-2}(x!)$, where $\lfloor a \rfloor = \sup_{a\geq n\in\mathbb{Z}} n$, for all $a \in \mathbb{R}$, is the floor function. One then has

$$4^k(x_v!)^{-1}(k+x_v)!||v||_1 = 4^kk!\binom{k+x_v}{k}||v||_1 \leq 4^kk!\binom{2k}{k}||v||_1 \tag{23}$$

$$= 4^k(k!)^{-1}(2k)!||v||_1 \leq 4^4(4!)^{-1}(2k)!||v||_1 < c_v(2k)!, \tag{24}$$

where $c_v = 11||v||_1$. □

Proof of Proposition (1) Part (1). The set of vectors, v, with finitely many nonzero components is dense in $\mathscr{H}$. This dense set is semi-analytic for L_0. By the extension of Nelson's analytic vector theorem to semi-analytic vectors, see e.g. [**28**], it is therefore the case that L_0 is essentially self-adjoint. □

Proof of Proposition (1) Part (2). One has that $L_0v(x) = \lambda v(x)$, $v \in \mathscr{T}$, specifies a countable family of coupled elementary algebraic equations. A unique solution may be found for each λ by specifying v_0 and solving inductively in increasing

$x \in \mathbb{Z}_+$. A choice of normalization will fix $v(0)$. Therefore each solution is, up to normalization, uniquely specified by λ. $\square$

Proof of Proposition (1) Part (3). L_0 is an essentially self-adjoint, second order, finite difference operator or *Jacobi operator*. It is well known that the theory of Jacobi operators is intimately connected with that of orthogonal polynomials. In particular spectral equations for operators extended to formal sequence spaces may be viewed as recursion formulas for families, indexed by lattice site, of orthogonal polynomials defined on the spectrum of the operator in question, see e.g. [**27**]. For $L_0 \in \mathcal{L}(\mathscr{T})$ it is the case that $L_0 v(x) = \lambda v(x)$ takes the form of the recursion formula for the Laguerre polynomials. By part (2) of the proposition these solutions are unique. $\square$

The Laguerre polynomials, $\phi_\lambda^{L_0}(x) \equiv \phi_\lambda(x)$, have known completeness and orthogonality relations whose roles will be reversed here:

$$\delta_{x_1,x_2} = \int_0^\infty \mathrm{d}\lambda\, e^{-\lambda}\phi_\lambda(x_1)\phi_\lambda(x_2), \quad \delta(\lambda_1 - \lambda_2) = e^{-(\lambda_1+\lambda_2)/2}\sum_{n=0}^{\infty}\phi_{\lambda_1}(x)\phi_{\lambda_2}(x), \tag{25}$$

where here $\delta(\cdot)$ is Dirac's delta distribution supported on $\sigma(L_0)$. The RHS of these equations converge in the distributional sense respectively on $\ell^2(\mathbb{Z}_+)$ and $L^2(\mathbb{R}_+)$. The former equation expresses components of the spectral measure of L_0 and in particular $w_\lambda^{L_0} = e^{-\lambda}$. In what follows we will often suppress the dependence of a quantity on L_0, e.g. $w_\lambda^{L_0} \equiv w_\lambda$. We will use χ_0 a normalization vector for L_0.

3. Spectral Properties of L

Consider that A_0 is an essentially self adjoint operator on $\mathscr{H}$ and has spectrum $\sigma(A_0) = \sigma_{\mathrm{ac}}(A_0) = [0, \infty)$. Furthermore consider that A on $\mathcal{H}$ has domain $\mathcal{D}(A) = \mathcal{D}(A_0)$ and is specified by $A := A_0 - q\Pi$, where $q \in \mathbb{C}$ is a fixed constant and Π is a rank-1 orthogonal projection operator. There exists a $\upsilon_{\mathrm{P}} \in \mathscr{H}$ of unit norm for which $\Pi = \upsilon_{\mathrm{P}} \otimes \upsilon_{\mathrm{P}}^*$. Define $f_z^A := (\upsilon_{\mathrm{P}}, R_z^A \upsilon_{\mathrm{P}})$. By Weyl's theorem, the perturbation $-q\Pi$ cannot change the essential spectrum of A_0 but it can introduce eigenvalues. Let $u \in \mathscr{H}$ satisfy $\upsilon_{\mathrm{P}}^*(u) \neq 0$, then

$$Au = zu \quad \Rightarrow \quad 1 = qf_z^A. \tag{26}$$

A will then have as many eigenvalues as $qf_z^A - 1$ has zeroes. The corresponding eigenfunctions are then given by

$$Au = \lambda u \quad \Rightarrow \quad u = q\upsilon_{\mathrm{P}}^*(u) R_\lambda^{A_0} \upsilon_{\mathrm{P}}. \tag{27}$$

Definition 3.1. *Let $\mathcal{T} \in \mathcal{L}(\mathscr{T})$ be the* binomial transform*, see e.g.* [19]*, defined by*

$$\mathcal{T}v(k) = \sum_{x=0}^{\infty} \mathcal{T}(k,x)v(x) = \sum_{x=0}^{\infty} (-1)^k \binom{k}{x} v(x), \quad \forall v \in \mathscr{T}. \tag{28}$$

$\mathcal{T}$ is involutive in the sense that $\mathcal{T}^2 = I$. One has that $\mathcal{T}v(0) = v(0)$ and the useful representation $\chi_0(x) = \sum_{k=0}^{\infty} (-1)^x \binom{x}{k}$. We take the conventions that $x!, \binom{x}{k}, \sum_{x=0}^{k} v(x) = 0$ for $k, x < 0$ and $k < x$ for all $v \in \mathscr{T}$.

Lemma 3.1. *One may check by direct computation that*

$$\mathcal{T}L_0 v(k) = (k+1)\mathcal{T}v(k+1), \quad \forall v \in \mathscr{T} \tag{29}$$

and may recover the usual definition of the Laguerre polynomials hereby.

PROOF. Consider

$$L_0\phi_\lambda(x) = \lambda\phi_\lambda(x) \Rightarrow (k+1)\mathcal{T}\phi_\lambda(k+1) = \lambda\mathcal{T}\phi_\lambda(k). \tag{30}$$

Choosing $\mathcal{T}\phi_\lambda(0) = \phi_\lambda(0) = 1$ one has by induction that $\mathcal{T}\phi_\lambda(k) = \frac{\lambda^k}{k!}$. One may then apply the binomial transform again to arrive at

$$\phi_\lambda(x) = \sum_{k=0}^{x} \frac{(-\lambda)^k}{k!}\binom{x}{k}. \tag{31}$$

□

LEMMA 3.2. *One has the representation*

$$\psi_z(x) = e^{-z}\sum_{k=0}^{x}(-1)^k\binom{x}{k}E_{k+1}(-z), \tag{32}$$

where

$$E_p(z) := \int_1^\infty \mathrm{d}t\ e^{-zt}t^{-p}, \qquad p \in \mathbb{C}, z \in \mathbb{C}\setminus(-\infty, 0] \tag{33}$$

are the generalized exponential integrals *for which we take the principal branch with standard branch cut* $\Sigma = (-\infty, 0]$.

PROOF. Consider the $(L_0 - z)\psi_z = \chi_0$, where $\psi_z, \chi_0 \in \mathscr{T}$, $L_0 \in \mathcal{L}(\mathscr{T})$. By binomial transform of this equation one finds

$$(k+1)\mathcal{T}\psi_z(k+1) = z\mathcal{T}\psi_z(k) + 1. \tag{34}$$

$\mathcal{T}\psi_z(k) = e^{-z}E_{k+1}(-z)$ satisfies this recursion formula. □

DEFINITION 3.2. *For any single-valued or multi-valued function* $f : \mathbb{C} \to \mathbb{C}$, *an element of a set of linear functionals on some suitable Banach space with norm given through integration over* λ, *and with poles, branch points, and branch cuts found in* *the subset* $\Sigma \subseteq \mathbb{R}$ *let* $\mathcal{P}f : \Sigma \to \mathbb{C}$ *be the* principal part of f *defined by the weak limit*

$$\mathcal{P}f(\lambda) := \frac{1}{2} \operatorname*{w-lim}_{\epsilon \searrow 0} \left[f(\lambda + i\epsilon) + f(\lambda - i\epsilon) \right], \quad \lambda \in \Sigma, \tag{35}$$

which converges in the distributional sense. We analogously define the δ-part *of* f *to be*

$$\delta f(\lambda) := \frac{1}{2\pi i} \operatorname*{w-lim}_{\epsilon \searrow 0} \left[f(\lambda + i\epsilon) - f(\lambda - i\epsilon) \right], \quad \lambda \in \Sigma. \tag{36}$$

We have kept vague the specification of the sense in which the above definitions converge weakly for the purposes of generality. The details of such convergence in our work will be clear from context. One may extend the domain of $\mathcal{P}f$ to the complex plane and produce a single valued function, which we will also denote f, through

$$\mathcal{P}f(z) := \begin{cases} f(z), & z \in \mathbb{C} \setminus \Sigma \\ \mathcal{P}f(z), & z \in \Sigma. \end{cases} \tag{37}$$

One may observe that the analogous extension of $\delta f(\lambda)$ vanishes away from $\Sigma \subseteq \mathbb{R}$. This prescription extends to weak limits in $z \in \mathbb{C}$ of complex sequences $v_z \in \mathscr{T}$ whose components depend upon z.

The generalized exponential integrals have the convergent series expansion [34]

$$E_{n+1}(z) = -\frac{(-z)^n}{n!} \log(z) + \frac{e^{-z}}{n!} \sum_{k=1}^{n} (-z)^{k-1} (n-k)! \tag{38}$$

$$+ \frac{e^{-z}(-z)^n}{n!} \sum_{k=0}^{\infty} \frac{z^k}{n!} F(k+1), \tag{39}$$

where $F(x) := \mathrm{d}_x \log \Gamma(x)$ is the digamma function. One may therefore observe that

$$\operatorname*{w-lim}_{\epsilon \searrow 0} E_{n+1}(-x \pm i\epsilon) = \mathcal{P}E_{n+1}(-x) \mp i\pi \frac{(x)^n}{n!}, \quad x > 0, \tag{40}$$

where for the sake of generality the limit is weak with respect to $L^2([a,\infty),\mathbb{C})$, $a>0$. In particular one often writes $\mathcal{P}E_1(-x) = -\mathrm{Ei}(x)$ where

$$\mathrm{Ei}(x) := -\int_{-x}^{\infty} \mathrm{d}u\; u^{-1}e^{-u}, \qquad x>0 \tag{41}$$

is *the exponential integral.*

Proof of Theorem (1) Part (1). Eigenvalues are be given by zeros of $qe^{-z}E_1(-z)-1$. L is essentially self-adjoint so $\sigma(L)\subseteq\mathbb{R}$. Analytic continuation of $E_1(-z)$ to $z>0$ from above or below will result in the sum of a real function and an imaginary constant

$$\lim_{\epsilon\searrow 0} E_1(-x\pm i\epsilon) = -\mathrm{Ei}(x) \mp i\pi, \quad x>0 \tag{42}$$

so there can be no positive eigenvalues. $qe^{-z}E_1(-z)$ diverges for $z\to 0$ so $z=0$ cannot be an eigenvalue. All eigenvalues must be negative. Let $z=-a<0$. It is the case that $e^aE_1(a)$ is monotonically decreasing for increasing $a\in(0,\infty]$

$$\mathrm{d}_a[e^aE_1(a)] = -\int_0^{\infty}\mathrm{d}x\; e^{-x}(x+a)^{-2} < 0, \tag{43}$$

where we have used manifest dominated convergence of the integral to pass the derivative through the integral. Furthermore since

$$\lim_{a\searrow 0}\int_0^{\infty}\mathrm{d}t\; e^{-t}(t+a)^{-1} = \infty, \quad \lim_{a\nearrow\infty}\int_0^{\infty}\mathrm{d}t\; e^{-t}(t+a)^{-1} = 0, \tag{44}$$

it follows that $e^aE_1(a)$ takes each on the interval $[0,\infty)$ exactly once, where we have used manifest uniform convergence of the integrand to pass the limit through the integral. Therefore $qe^{-z}E_1(-z)-1$ has exactly one root for each fixed $q>0$. □

Proof of Theorem (1) Part (2). By the argument of the Proof of Theorem (1) Part (1), there can be no embedded eigenvalues. By Weyl's critereon the perturbation of $L_0\mapsto L$ leaves the essential spectrum unchanged. The argument for the proof of $\sigma(L_0)=\sigma_{\mathrm{ac}}(L_0)$ follows without change for the spectrum of L. □

Definition 3.3. *Let A be an operator on $\mathscr{H}$ which is self-adjoint on its domain $\mathcal{D}(A)$ and λ an element of the discrete spectrum of A. Define $\mathcal{P}^A_\lambda \equiv \mathcal{P}(A-\lambda)^{-1}$, $\lambda \in \sigma(A)$ to be the principal part of the resolvent of A given by the strong limit*

$$\mathcal{P}^A_\lambda := \frac{1}{2} \operatorname*{s-lim}_{\epsilon \searrow 0} \left[R^A_{\lambda+i\epsilon} + R^A_{\lambda-i\epsilon} \right]. \tag{45}$$

Denote by $\delta^A_\lambda \equiv \delta(A-\lambda) \equiv \Pi^A_\lambda$, $\lambda \in \sigma(A)$ the spectral projection defined by the strong limit

$$\delta^A_\lambda := \frac{1}{2\pi i} \operatorname*{s-lim}_{\epsilon \searrow 0} \left[R^A_{\lambda+i\epsilon} - R^A_{\lambda-i\epsilon} \right]. \tag{46}$$

If λ is instead an element of the essential spectrum of A one has that $\mathcal{P}^A_\lambda, \delta^A_\lambda$ are defined by weak limits. If and only if the essential spectrum of A is absolutely continuous it is the case that $\mathrm{d}\mu^A_{\mathrm{e}}(\lambda) = \delta^A_\lambda \, \mathrm{d}\lambda$, where $\mathrm{d}\mu^A_{\mathrm{e}}(\lambda)$ is the essential spectral measure of A and $\mathrm{d}\lambda$ is the Lebesgue measure on $\sigma_{\mathrm{e}}(A)$.

The above definition permits the useful representation $\delta^A_\lambda = w^A_\lambda \phi^A_\lambda \otimes \phi^{A,*}_\lambda$. One may typically observe through the spectral representation of R^A_z that $\mathcal{P}\psi^A_\lambda = \mathcal{P}^A_\lambda v_{\mathrm{N}}$ and that $\mathcal{P}\xi^A_\lambda = \mathcal{P}\psi^A_\lambda - v^*_{\mathrm{N}}(\mathcal{P}\psi^A_\lambda)\phi^A_\lambda = \xi^A_\lambda$, $\forall \lambda \in \sigma(A)$.

We recall the method of spectral shifts as applied to rank-1 perturbations, see e.g. [29], for operators of the form specified by the $A_0, \Pi, A = A_0 - q\Pi$ considered above. Through the resolvent formula it follows that

$$R^A_z = R^{A_0}_z + R^{A_0}_z q \Pi R^A_z \quad \Rightarrow \quad \Pi R^A_z = \Pi R^{A_0}_z + f^{A_0}_z q \Pi R^A_z \tag{47}$$

$$\Rightarrow \quad \Pi R^A_z = (1 - q f^{A_0}_z)^{-1} \Pi R^{A_0}_z \quad \Rightarrow \quad R^A_z = R^{A_0}_z + (1 - q f^{A_0}_z)^{-1} R^{A_0}_z q \Pi R^{A_0}_z. \tag{48}$$

For A essentially self-adjoint one may apply the definitions of $\mathcal{P}^A_\lambda$ and δ^A_λ and find the corresponding shifts to $\mathcal{P}^{A_0}_\lambda$ and $\delta^{A_0}_\lambda$. For $\lambda \in \sigma(A)$ it follows that

$$\mathcal{P}^A_\lambda = \mathcal{P}^{A_0}_\lambda + g^{A_0}_\lambda [(1 - q\mathcal{P} f^{A_0}_\lambda)(\mathcal{P}^{A_0}_\lambda q \Pi \mathcal{P}^{A_0}_\lambda - \pi^2 \delta^{A_0}_\lambda q \Pi \delta^{A_0}_\lambda) \tag{49}$$

$$- \pi^2 q \delta f^{A_0}_\lambda (\mathcal{P}^{A_0}_\lambda q \Pi \delta^{A_0}_\lambda + \delta^{A_0}_\lambda q \Pi \mathcal{P}^{A_0}_\lambda)] \tag{50}$$

$$\delta_\lambda^A = \delta_\lambda^{A_0} + g_\lambda^{A_0}[(1 - q\mathcal{P}f_\lambda^{A_0})(\mathcal{P}_\lambda^{A_0} q\Pi\delta_\lambda^{A_0} + \delta_\lambda^{A_0} q\Pi\mathcal{P}_\lambda^{A_0}) \tag{51}$$

$$+ q\delta f_\lambda^{A_0}(\mathcal{P}_\lambda^{A_0} q\Pi\mathcal{P}_\lambda^{A_0} - \pi^2\delta_\lambda^{A_0} q\Pi\delta_\lambda^{A_0})], \tag{52}$$

where

$$g_\lambda^{A_0} := [(1 - q\mathcal{P}f_\lambda^{A_0})^2 + (q\pi\delta f_\lambda^{A_0})^2]^{-1}. \tag{53}$$

If $v_{\mathrm{P}} = v_{\mathrm{N}}$ is a normalizing vector for A_0 one may greatly simplify the expression of the shift of the spectral projection. Since one has $f_z^A = (v_{\mathrm{N}}, R_z^A v_{\mathrm{N}})$ it follows that $\mathcal{P}f_\lambda^A = v_{\mathrm{N}}^*(\mathcal{P}\psi_\lambda^A) = \mathcal{P}v_{\mathrm{N}}^*(\psi_\lambda^A)$ and $\delta f_\lambda^A = w_\lambda^A$. One may find that

$$R_z^A\alpha = \psi_z^A = (1 - qf_z^{A_0})^{-1}\psi_z^{A_0} \Rightarrow f_z^A = (1 - qf_z^{A_0})^{-1} f_z^{A_0}, \tag{54}$$

$$\mathcal{P}f_z^A = g_\lambda^{A_0}[\mathcal{P}f_\lambda^{A_0} - q(\mathcal{P}f_\lambda^{A_0})^2 - q(\pi w_\lambda^{A_0})^2], \quad \delta f_\lambda^A = w_\lambda^A = g_\lambda^{A_0} w_\lambda^{A_0}, \tag{55}$$

$$\mathcal{P}\psi_\lambda^A = g_\lambda^{A_0}[\mathcal{P}f_\lambda^{A_0}\phi_\lambda^{A_0} - q\mathcal{P}f_\lambda^{A_0}\xi_\lambda^{A_0} + \xi_\lambda^{A_0} - q(\mathcal{P}f_\lambda^{A_0})^2\phi_\lambda^{A_0} - q(\pi w_\lambda^{A_0})^2\phi_\lambda^{A_0}], \tag{56}$$

$$\delta\psi_\lambda^A = w_\lambda^A\phi_\lambda^A = g_\lambda^{A_0} w_\lambda^{A_0}(\phi_\lambda^{A_0} + q\xi_\lambda^{A_0}) \Rightarrow \phi_\lambda^A = \phi_\lambda^{A_0} + q\xi_\lambda^{A_0} \tag{57}$$

$$\delta_\lambda^A = w_\lambda^A\phi_\lambda^A \otimes \phi_\lambda^{A,*} = g_\lambda^{A_0} w_\lambda^{A_0}(\phi_\lambda^{A_0} + q\xi_\lambda^A) \otimes (\phi_\lambda^{A_0,*} + q\xi_\lambda^{A_0,*}) \tag{58}$$

$$g_\lambda^{A_0} = [(1 - q\mathcal{P}f_\lambda^{A_0})^2 + (q\pi w_\lambda^{A_0})^2]^{-1} \tag{59}$$

If $\delta_\lambda^{A_0}$ is regular at the threshold of $\sigma(A_0)$ then an analysis of the threshold behavior of δ_λ^A is strongly controlled by the threshold behavior of $g_\lambda^{A_0}$.

Proof of Theorem (1) Part (3). One may straightforwardly take the presented techniques for rank-1 spectral shifts with the assignments $A_0 = L_0$, $A = L$, $v_{\mathrm{N}} = v_{\mathrm{P}} = \chi_0$. □

Of particular importance in our analysis will be the function

$$g_\lambda := \{[1 - qe^{-\lambda}\mathcal{P}E_1(-\lambda)]^2 + [\pi qe^{-\lambda}]^2\}^{-1}, \tag{60}$$

which satisfies $w_\lambda^L = g_\lambda w_\lambda$. This function strongly controls the behavior of the spectral measure near the threshold due to the logarithmic divergence of $\mathcal{P}E_1(-\lambda)$ there.

4. Decay Estimates for L_0 and L

The Mourre estimate, see e.g. [23], has been proven for L_0 by Chen, Fröhlich, and Walcher [5], in order to prove its the spectrum is absolutely continuous and equal to $[0,\infty)$. We want to study pointwise decay estimates in time, which requires knowledge of the aysmptotic properties of the resolvent at thresholds. The Mourre estimates do not apply at thresholds so we will need to use alternative methods.

Local decay estimates for L_0 have been found by Durhuus and Gayral [9] in the context of more general noncommutative solitons (where L_0 corresponds their "diagonal case with 2 noncommuting spatial coordinates"). They found an unweighted estimate of the form $||e^{-itL_0}v||_\infty \leq c|t|^{-1}(1+\log|t|)||v||_1$ for $|t|\geq 1$. We present an alternative approach, in the context of Jacobi operators, which enhances the local decay estimate for the free Schrödinger operator and provides integrable decay for rank one boundary perturbations thereof. We find weighted estimates

$$||W_{\kappa,\tau}e^{-itL_0}W_{\kappa,\tau}v||_\infty < ct^{-1}||v||_1, \quad ||W_{\kappa,\tau}e^{-itL}\Pi_{\mathrm{e}}^{L}W_{\kappa,\tau}v||_\infty = \mathcal{O}(t^{-1}\log^{-2}t)$$

for $t\nearrow\infty$.

4.1. Weighted estimates for spectral vectors.

Definition 4.1. *Let $\mathbb{S}_r,\mathbb{D}_r\subset\mathbb{C}$ be respectively the circle and the disc of radius $r>0$ centered at the origin and $u\in\mathscr{T}$ a formal sequence for which there exist constants $r,c>0$ for which $|u(x)|r^{-x}<c$ for all $x\in\mathbb{Z}_+$. The* generating function *of u is the function $\zeta(u,\cdot):\mathbb{S}_{r'}\to\mathbb{C}$ defined by $\zeta(u,s):=\sum_{x=0}^\infty u(x)s^x$, where $r'<r$. This permits the presentation of u via*

$$u(x)=\oint_\gamma \mathrm{d}s\ (2\pi i s)^{-1}s^{-x}\zeta(u,s), \tag{61}$$

where γ is any positively oriented simple closed curve in $\mathbb{D}_r$ which encloses and does not pass through the origin.

The Laguerre polynomials have the well-known generating function [35]

$$\zeta(\phi_\lambda, s) := \sum_{x=0}^{\infty} \phi_\lambda(x) s^x = (1-s)^{-1} \exp[-(1-s)^{-1} s\lambda], \quad |s| < 1. \tag{62}$$

We will also employ the notion of a reduced generating function.

Definition 4.2. *For a given generating function $\zeta(v, s)$ of a vector v let the* reduced generating function *be the function $\widehat{\zeta}(v, s) := (1-s)\zeta(v, s)$.*

For example we have that $\widehat{\zeta}(\phi_\lambda, s) = \exp[-(1-s)^{-1}s\lambda]$.

Definition 4.3. *For $s \in \mathbb{C}$ we let*

$$r := |s|, \quad \theta := \arg(s), \quad \widehat{s} := (1-s)^{-1}s, \tag{63}$$

$$\epsilon := 1 - r^2, \quad \widehat{\epsilon} := -2^{-1} + 16^{-1}\epsilon. \tag{64}$$

Should many variables s_j be present we define $\epsilon_j := 1 - |s_j|^2$ correspondingly for each.

Lemma 4.1. *One has the representation*

$$\xi_z(x) = \int_0^\infty \mathrm{d}\lambda\; e^{-\lambda}(\lambda - z)^{-1}[\phi_\lambda(x) - \phi_z(x)], \tag{65}$$

where

$$K(\eta, z, s) := (\eta - z)^{-1}\left[\exp(-\widehat{s}\eta) - \exp(-\widehat{s}z)\right] \tag{66}$$

for $\eta \in \mathbb{R}_+$, $z \in \mathbb{C}$, $s \in \mathbb{S}_r$.

Proof. By the spectral representation of $R_z^{L_0}$ it is the case that $\psi_z(x) = \int_0^\infty \mathrm{d}\lambda\; e^{-\lambda}(\lambda - z)^{-1}\phi_\lambda(x)$, where we have used the normalization condition $\phi_\lambda(0) = 1$, $\forall \lambda \in \sigma(L_0)$. □

We are primarily concerned with estimates of operators in generating function presentation. In such forms one finds line integrations over dummy complex variables,

s_j, with *a priori* separate sums for each $x_j \in \mathbb{Z}_+$. One is therefore permitted to make the associated contours dependent on x_j. We then will hereafter take

$$r := |s_1| \overset{\text{set}}{=} 1 - (x+\kappa)^{-1} < 1 \tag{67}$$

$$\Rightarrow \quad \epsilon = 1 - [1-(x+\kappa)^{-1}]^2 = 2(x+\kappa)^{-1} + \mathcal{O}((x+\kappa)^{-2}) \tag{68}$$

where dependence on $x \in \mathbb{Z}_+$ and $1 < \kappa \in \mathbb{R}$ will always be suppressed. One may observe that for sufficiently large κ one may make ϵ as small as one would like and therefore will be treated as a small parameter. One may also observe that

$$|(1-s)^{-1}| \leq x+\kappa, \quad |\widehat{s}| \leq x+\kappa-1 < x+\kappa \tag{69}$$

as well as the crucial estimate

$$|s^{-x}| < e < 3. \tag{70}$$

One is then permitted to work with polynomially weighted spaces instead of exponentially weighted ones.

LEMMA 4.2. *One has that* $|\exp(-\widehat{s}\lambda)| \leq \exp(-\widehat{\epsilon}\lambda)$ *for sufficiently large* κ*.*

PROOF. We recall that we have universally chosen that $|s| \overset{\text{set}}{=} 1-(x+\kappa)^{-1}$. Let $l := \Re\widehat{s}$, $m := 1-\cos\theta$, and $n := r\cos\theta - r^2$. Let $\mathbb{H}_{\mathrm{L}} := \{z \in \mathbb{C} : \Re z < 0\}$ be the left half plane. With regard to $\widehat{s}$ we need only consider $l := \Re\widehat{s}$. By inspection of $\widehat{s}$ one can see that for $s \in \mathbb{S}$ the supremum of $|\exp(-\widehat{s}\lambda)|$ should be found in $\mathbb{S}_r \cap \mathbb{H}_{\mathrm{L}}$. Furthermore, although for unrestricted s one has $0 \leq m \leq 2$ it is the case that for restricted s one has $1 \leq m \leq 2$ and thereby $m = \mathcal{O}(1)$. For $s \in \mathbb{S}_r \cap \mathbb{H}_{\mathrm{L}}$ it follows that

$$r = (1-\epsilon)^{1/2} = 1 - \frac{1}{2}\epsilon + \mathcal{O}(\epsilon^2) \tag{71}$$

$$n = r\cos\theta - r^2 = -m - \frac{1}{2}(1-m)\epsilon + \mathcal{O}(\epsilon^2) \tag{72}$$

$$l = n(\epsilon - 2n)^{-1} \tag{73}$$

$$= -m - \frac{1}{2}(1-m)\epsilon + \mathcal{O}(\epsilon^2)\left[\epsilon - 2 - m - \frac{1}{2}(1-m)\epsilon + \mathcal{O}(\epsilon^2)\right]^{-1} \tag{74}$$

$$= -2^{-1} + (4m)^{-1}\epsilon + \widehat{l}(\epsilon), \tag{75}$$

where $\widehat{l}(\epsilon) = \mathcal{O}(\epsilon^2)$. One then has

$$|\exp(-\widehat{s}\lambda)| \leq \sup_{\theta\in[\pi/2,3\pi/2]} \exp[-l\lambda] \tag{76}$$

$$= \sup_{\theta\in[\pi/2,3\pi/2]} \exp\{-[-2^{-1} + (4m)^{-1}\epsilon + \widehat{l}(\epsilon)]\lambda\} \tag{77}$$

$$= \sup_{\theta\in[\pi/2,3\pi/2]} \exp\{-[-2^{-1} + 8^{-1}\epsilon + \widehat{l}(\epsilon)]\lambda\}. \tag{78}$$

κ is a free parameter and as it increases ϵ will decrease monotonically. Therefore there must exist a large enough κ so that $|\widehat{l}(\epsilon)| \leq (16)^{-1}\epsilon$, $\forall\theta$. One then has that

$$\sup_{\theta_1\in[\pi/2,3\pi/2]} \exp\{-[-2^{-1} + 8^{-1}\epsilon + \widehat{l}(\epsilon)]\lambda\} \leq \exp\{-[-2^{-1} + 8^{-1}\epsilon - 16^{-1}\epsilon]\lambda\} \tag{79}$$

$$= \exp[-(-2^{-1} + 16^{-1}\epsilon)\lambda]. \tag{80}$$

□

Lemma 4.3. *One has that* $(\epsilon_1 + \epsilon_2)^{-1} < 4^{-1}(x_1 + \kappa)(x_2 + \kappa)$.

Proof.

$$\epsilon = 1 - r^2 = [2 - (x+\kappa)^{-1}](x+\kappa)^{-1} \tag{81}$$

$$(\epsilon_1 + \epsilon_2)^{-1} = (x_1+\kappa)(x_2+\kappa)[4 - (x_1+\kappa)^{-1} - (x_2+\kappa)^{-1}] \tag{82}$$

$$< 4^{-1}(x_1+\kappa)(x_2+\kappa). \tag{83}$$

□

LEMMA 4.4. *One has the generating function representation*

$$\xi_z(x) = \oint_{\mathbb{S}_r} \mathrm{d}s\ (2\pi i s)^{-1} s^{-x} \zeta(\xi_z, s), \quad \forall z \in \mathbb{C} \tag{84}$$

where $\zeta(\xi_z, s) := (1-s)^{-1} \int_0^\infty \mathrm{d}\eta\ e^{-\eta} K(\eta, z, s)$.

PROOF. First, let $z \in \mathbb{C} \setminus \mathbb{R}_+ =: \Sigma$ and $l := \Re \widehat{s}$. Since $|s| < 1$ there exists a $c > 0$ such that $|\widehat{s}| < c$. It follows that

$$|K(\eta, z, s)| \leq \left|(\eta - z)^{-1}\right| [|\exp(-\widehat{s}\eta)| + |\exp(-\widehat{s}z)|] \tag{85}$$

$$\leq [\mathrm{dist}(\Sigma, z)]^{-1} [\exp(-l\eta) + \exp(-c|z|)] < \infty. \tag{86}$$

Second, let $z \equiv \lambda \in \mathbb{R}_+$. By mean value theorem one has

$$K(\eta, \lambda, s) = (\eta - \lambda)^{-1} [\Re \exp(-\widehat{s}\eta) - \Re \exp(-\widehat{s}z)] \tag{87}$$

$$+ i(\eta - \lambda)^{-1} [\Im \exp(-\widehat{s}\eta) - \Im \exp(-\widehat{s}z)] \tag{88}$$

$$= \mathrm{d}_\eta \Re \exp(-\widehat{s}\eta)\lfloor_{\eta=\mu_1} + i \mathrm{d}_\eta \Im \exp(-\widehat{s}\eta)\lfloor_{\eta=\mu_2} \tag{89}$$

$$= \frac{1}{2}\Big[(-\widehat{s}) \exp(-\widehat{s}\mu_1) + (-\overline{\widehat{s}}) \exp(-\overline{\widehat{s}}\mu_1) \tag{90}$$

$$+ (-\widehat{s}) \exp(-\widehat{s}\mu_2) - (-\overline{\widehat{s}}) \exp(-\overline{\widehat{s}}\mu_2)\Big], \tag{91}$$

where $\mu_j \equiv \mu_j(r, \theta, \eta, \lambda) \in [\min(\eta, \lambda), \max(\eta, \lambda)]$. Then

$$|K(\eta, \lambda, s)| \leq \frac{1}{2} [|\widehat{s}| |\exp(-\widehat{s}\mu_1)| + |\widehat{s}| |\exp(-\widehat{s}\mu_1)| \tag{92}$$

$$+ |\widehat{s}| |\exp(-\widehat{s}\mu_2)| + |\widehat{s}| |\exp(-\widehat{s}\mu_2)|] \tag{93}$$

$$= |\widehat{s}| [\exp(-l\mu_1) + \exp(-l\mu_2)] \tag{94}$$

$$\leq 2|\widehat{s}| \exp[-\widehat{\epsilon}(\eta + \lambda)] < \infty. \tag{95}$$

One may observe that

$$\int_0^\infty \mathrm{d}\eta\ e^{-\eta}|K(\eta,z,s)| \le \int_0^\infty \mathrm{d}\eta\ e^{-\eta}2|\widehat{s}|\exp[-\widehat{\epsilon}(\eta+\lambda)] \tag{96}$$

$$= 2|\widehat{s}|\exp(-\widehat{\epsilon}\lambda)\int_0^\infty \mathrm{d}\eta\ \exp[-(1+\widehat{\epsilon})\eta] \tag{97}$$

$$= 2|\widehat{s}|\exp(-\widehat{\epsilon}\lambda)(1+\widehat{\epsilon})^{-1} < \infty \tag{98}$$

The multi-integral of the generating function representation of $\xi_z(x)$ converges absolutely and thereby Fubini's theorom permits

$$\xi_z(x) = \int_0^\infty \mathrm{d}\eta\ e^{-\eta}(\eta-z)^{-1}[\phi_\eta(x)-\phi_z(x)] \tag{99}$$

$$= \int_0^\infty \mathrm{d}\eta\ e^{-\eta}\oint_{\mathbb{S}_r} \mathrm{d}s\ (2\pi i s)^{-1}s^{-x}(1-s)^{-1}K(\eta,z,s) \tag{100}$$

$$= \oint_{\mathbb{S}_r} \mathrm{d}s\ (2\pi i s)^{-1}s^{-x}(1-s)^{-1}\int_0^\infty \mathrm{d}\eta\ e^{-\eta}K(\eta,z,s) \tag{101}$$

for all $z \in \mathbb{C}$. □

Lemma 4.5. *It is the case that*

$$|\mathrm{d}_\lambda^n\widehat{\zeta}(\phi_\lambda,s)| < \widehat{c}(\phi_\lambda,n)\exp(-\widehat{\epsilon}\lambda), \quad n\in\mathbb{Z}_+, \tag{102}$$

where $\widehat{c}(\phi_\lambda,n) := (x+\kappa)^n$ *and*

$$|\mathrm{d}_\lambda^n\widehat{\zeta}(\xi_\lambda,s)| < \widehat{c}(\xi_\lambda,n)\exp(-\widehat{\epsilon}\lambda), \quad n=0,1,2, \tag{103}$$

where

$$\widehat{c}(\xi_\lambda,0) := 4(x+\kappa), \quad \widehat{c}(\xi_\lambda,1) := 6(x+\kappa), \quad \widehat{c}(\xi_\lambda,2) := 8(x+\kappa)^2. \tag{104}$$

Proof. For ϕ_λ:

$$|\mathrm{d}_\lambda^n\widehat{\zeta}(\phi_\lambda,s)| = |\mathrm{d}_\lambda^n\exp(-\widehat{s}\lambda)| = |\widehat{s}^n\exp(-\widehat{s}\lambda)| \tag{105}$$

$$\le |\widehat{s}|^n|\exp(-\widehat{s}\lambda)| \le |\widehat{s}|^n\exp(-\widehat{\epsilon}\lambda) < (x+\kappa)^n\exp(-\widehat{\epsilon}\lambda) \tag{106}$$

For ξ_λ: One may observe that $K(\eta,\lambda,s) = K(\lambda,\eta,s)$. Then, by integration by parts, one has

$$\mathrm{d}_\lambda\widehat{\zeta}(\xi_\lambda,s) = \int_0^\infty \mathrm{d}\eta\; e^{-\eta}\mathrm{d}_\lambda K(\eta,\lambda,s) = \int_0^\infty \mathrm{d}\eta\; e^{-\eta}\mathrm{d}_\eta K(\eta,\lambda,s) \tag{107}$$

$$= -K(0,\lambda,s) + \widehat{\zeta}(\xi_\lambda,s) \tag{108}$$

and thereby

$$\mathrm{d}^n_\lambda\widehat{\zeta}(\xi_\lambda,s) = -\sum_{k=0}^{n-1}\mathrm{d}_\lambda K(0,\lambda,s) + \widehat{\zeta}(\xi_\lambda,s), \tag{109}$$

where the sum is defined to vanish when the upper bound is negative. It follows that

$$|K(\eta,\lambda,s)| \leq 2|\widehat{s}|\exp[-\widehat{\epsilon}(\eta+\lambda)], \quad |\widehat{\zeta}(\xi_\lambda,s)| \leq 2|\widehat{s}|(1+\widehat{\epsilon})^{-1}\exp(-\widehat{\epsilon}\lambda). \tag{110}$$

Consider an arbitrary $f \in C^2(\mathbb{R},\mathbb{R})$ and let f_* be its Newton quotient so that

$$f_*(a_0,a) := (a_0-a)^{-1}[f(a_0)-f(a)]. \tag{111}$$

One has by mean value theorem

$$\mathrm{d}_a f_*(a_0,a) = (a_0-a)^{-2}[f(a_0)-f(a)-(a_0-a)\mathrm{d}_a f(a)] \tag{112}$$

$$= (a_0-a)^{-1}[\mathrm{d}_{a_1}f(a_1)-\mathrm{d}_a f(a)], \quad a_1 \in [\min(a_0,a),\max(a_0,a)] \tag{113}$$

$$= (a_0-a)^{-1}(a_1-a)\mathrm{d}^2_{a_2}f(a_2), \quad a_2 \in [\min(a_1,a),\max(a_1,a)] \tag{114}$$

$$\Rightarrow \quad |\mathrm{d}_a f_*(a_0,a)| \leq |(a_0-a)^{-1}||a_1-a||\mathrm{d}^2_{a_2}f(a_2)| \leq |\mathrm{d}^2_{a_2}f(a_2)|. \tag{115}$$

Let $(\Re,\Im)z$ be a presentation for the real and imaginary parts of $z \in \mathbb{C}$ whose ordering in compatible with the respective ordering of $\pm$. Let $i_+ := 1$, $i_- := i$ and

$\mu_\pm \in [\min(\eta,\lambda), \max(\eta,\lambda)]$. It follows that

$$|\mathrm{d}_\lambda(\Re,\Im)K(\eta,\lambda,s)| \leq |\mathrm{d}^2_\lambda(\Re,\Im)\exp(-\widehat{s}\lambda)|\lfloor_{\lambda=\mu_\pm} \tag{116}$$

$$= |\mathrm{d}^2_\lambda(2i_\pm)^{-1}[\exp(-\widehat{s}\lambda) \pm \exp(-\overline{\widehat{s}}\lambda)]|\lfloor_{\lambda=\mu_\pm} \tag{117}$$

$$\leq |\widehat{s}|^2 \exp[-\widehat{\epsilon}(\eta+\lambda)] \tag{118}$$

$$\Rightarrow \quad |\mathrm{d}_\lambda K(\eta,\lambda,s)| \leq 2|\widehat{s}|^2 \exp[-\widehat{\epsilon}(\eta+\lambda)]. \tag{119}$$

Then

$$|\widehat{\zeta}(\xi_\lambda,s)| \leq 2|\widehat{s}|(1+\widehat{\epsilon})^{-1}\exp(-\widehat{\epsilon}\lambda) < 4(x+\kappa)\exp(-\widehat{\epsilon}\lambda) \tag{120}$$

$$|\mathrm{d}_\lambda\widehat{\zeta}(\xi_\lambda,s)| \leq 2|\widehat{s}|[(1+\widehat{\epsilon})^{-1}+1]\exp(-\widehat{\epsilon}\lambda) < 6(x+\kappa)\exp(-\widehat{\epsilon}\lambda) \tag{121}$$

$$|\mathrm{d}^2_\lambda\widehat{\zeta}(\xi_\lambda,s)| \leq 2|\widehat{s}|[(1+\widehat{\epsilon})^{-1}+1+|\widehat{s}|]\exp(-\widehat{\epsilon}\lambda) < 8(x+\kappa)^2\exp(-\widehat{\epsilon}\lambda). \tag{122}$$

$\square$

Remark. *If estimates of* $\mathrm{d}^n_\lambda\widehat{\zeta}(\xi_\lambda,s)$ *for* $2 < n \in \mathbb{Z}$ *were required the above method would not follow so straightforwardly due to the inapplicability of the mean value theorem for yet higher derivatives.*

Corollary 4.1. *One has that*

$$\left|\mathrm{d}^n_\lambda\left[w_\lambda^{1/2}\phi_\lambda(x)\right]\right| < c(\phi_\lambda,n)\exp(-16^{-1}\epsilon\lambda), \quad n=0,1,2, \tag{123}$$

where $c(\phi_\lambda,n) := 3^{n+1}(x+\kappa)^{n+1}$ *and*

$$\left|\mathrm{d}^n_\lambda\left[w_\lambda^{1/2}\xi_\lambda(x)\right]\right| < c(\xi_\lambda,n)\exp(-16^{-1}\epsilon\lambda), \quad n=0,1,2, \tag{124}$$

where

$$c(\xi_\lambda,0) := 12(x+\kappa)^2, \quad c(\xi_\lambda,1) := 24(x+\kappa)^2, \quad c(\xi_\lambda,2) := 36(x+\kappa)^3 \tag{125}$$

Proof. For ϕ_λ:

$$\left|\mathrm{d}_\lambda^n\left[w_\lambda^{1/2}\phi_\lambda(x)\right]\right| = \left|\oint_{\mathbb{S}_r}\mathrm{d}s\ (2\pi i s)^{-1}s^{-x}(1-s)^{-1}\mathrm{d}_\lambda^n\left[w_\lambda^{1/2}\widehat{\zeta}(\phi_\lambda, s)\right]\right|, \tag{126}$$

$$= \left|\oint_{\mathbb{S}_r}\mathrm{d}s\ (2\pi i s)^{-1}s^{-x}(1-s)^{-1}\right. \tag{127}$$

$$\left.\times\sum_{k=0}^{n}\binom{n}{k}\mathrm{d}_\lambda^{n-k}w_\lambda^{1/2}\mathrm{d}_\lambda^k\widehat{\zeta}(\phi_\lambda, s)\right|, \tag{128}$$

$$\leq \oint_{\mathbb{S}_r}\left|\mathrm{d}s\ (2\pi i s)^{-1}\right|\left|s^{-x}\right|\left|(1-s)^{-1}\right| \tag{129}$$

$$\times\sum_{k=0}^{n}\binom{n}{k}\left|\mathrm{d}_\lambda^{n-k}w_\lambda^{1/2}\right|\left|\mathrm{d}_\lambda^k\widehat{\zeta}(\phi_\lambda, s)\right|, \tag{130}$$

$$< (1)(3)(x+\kappa)\sum_{k=0}^{n}\binom{n}{k}2^{-(n-k)}w_\lambda^{1/2}\widehat{c}(\phi_\lambda, k)\exp(-\widehat{\epsilon}\lambda) \tag{131}$$

$$= 3(x+\kappa)\sum_{k=0}^{n}\binom{n}{k}2^{-(n-k)}\widehat{c}(\phi_\lambda, k)\exp(-16^{-1}\epsilon\lambda). \tag{132}$$

For $n = 0$:

$$\left|w_\lambda^{1/2}\phi_\lambda(x)\right| < 3(x+\kappa)\widehat{c}(\phi_\lambda, 0)\exp(-16^{-1}\epsilon\lambda) = 3(x+\kappa)\exp(-16^{-1}\epsilon\lambda). \tag{133}$$

For $n = 1$:

$$\left|\mathrm{d}_\lambda\left[w_\lambda^{1/2}\phi_\lambda(x)\right]\right| < 3(x+\kappa)\left[2^{-1}\widehat{c}(\phi_\lambda, 0) + \widehat{c}(\phi_\lambda, 1)\right]\exp(-16^{-1}\epsilon\lambda) \tag{134}$$

$$= 3(x+\kappa)\left[2^{-1} + (x+\kappa)\right]\exp(-16^{-1}\epsilon\lambda) \tag{135}$$

$$< 6(x+\kappa)^2\exp(-16^{-1}\epsilon\lambda). \tag{136}$$

For $n = 2$:

$$\left|\mathrm{d}_\lambda^2 \left[w_\lambda^{1/2}\phi_\lambda(x)\right]\right| < 3(x+\kappa)\left[2^{-2}\widehat{c}(\phi_\lambda,0) + 2^{-1}\widehat{c}(\phi_\lambda,1) + \widehat{c}(\phi_\lambda,2)\right] \tag{137}$$

$$\times \exp(-16^{-1}\epsilon\lambda) \tag{138}$$

$$= 3(x+\kappa)\left[2^{-2} + 2^{-1}(x+\kappa) + (x+\kappa)^2\right] \tag{139}$$

$$\times \exp(-16^{-1}\epsilon\lambda) \tag{140}$$

$$< 9(x+\kappa)^3 \exp(-16^{-1}\epsilon\lambda). \tag{141}$$

<u>For ξ_λ:</u>

$$\left|w_\lambda^{1/2}\xi_\lambda(x)\right| < 3(x+\kappa)\sum_{k=0}^{n}\binom{n}{k}2^{-(n-k)}\widehat{c}(\xi_\lambda,k)\exp(-16^{-1}\epsilon\lambda) \tag{142}$$

For $n = 0$:

$$\left|w_\lambda^{1/2}\xi_\lambda(x)\right| < 3(x+\kappa)\widehat{c}(\xi_\lambda,0)\exp(-16^{-1}\epsilon\lambda) = 12(x+\kappa)^2\exp(-16^{-1}\epsilon\lambda). \tag{143}$$

For $n = 1$:

$$\left|\mathrm{d}_\lambda \left[w_\lambda^{1/2}\xi_\lambda(x)\right]\right| < 3(x+\kappa)\left[2^{-1}\widehat{c}(\xi_\lambda,0) + \widehat{c}(\xi_\lambda,1)\right]\exp(-16^{-1}\epsilon\lambda) \tag{144}$$

$$= 3(x+\kappa)\left[2(x+\kappa) + 6(x+\kappa)\right]\exp(-16^{-1}\epsilon\lambda) \tag{145}$$

$$= 24(x+\kappa)^2\exp(-16^{-1}\epsilon\lambda). \tag{146}$$

For $n = 2$:

$$\left|\mathrm{d}_\lambda^2 \left[w_\lambda^{1/2}\xi_\lambda(x)\right]\right| < 3(x+\kappa)\left[2^{-2}\widehat{c}(\xi_\lambda,0) + 2^{-1}\widehat{c}(\xi_\lambda,1) + \widehat{c}(\xi_\lambda,2)\right] \tag{147}$$

$$\times \exp(-16^{-1}\epsilon\lambda) \tag{148}$$

$$= 3(x+\kappa)\left[(x+\kappa) + 3(x+\kappa) + 8(x+\kappa)^2\right] \tag{149}$$

$$\times \exp(-16^{-1}\epsilon\lambda) \tag{150}$$

$$< 36(x+\kappa)^3\exp(-16^{-1}\epsilon\lambda). \tag{151}$$

□

4.2. Local time decay for L_0.

Proof of Theorem (2). Let $t>0$, $|s_j| \equiv r_j \overset{\text{set}}{=} 1-(x_j+\kappa)^{-1}$, $j=1,2$, for $1<\kappa\in\mathbb{R}$. It is the case that

$$\left|e^{-itL_0}(x_1,x_2)\right| = \left|\int_0^\infty \mathrm{d}\lambda\ e^{-it\lambda} w_\lambda \phi_\lambda(x_1)\phi_\lambda(x_2)\right| \tag{152}$$

$$= \left|\int_0^\infty \mathrm{d}\lambda\ (-it)^{-1}\mathrm{d}_\lambda e^{-it\lambda}\left[w_\lambda^{1/2}\phi_\lambda(x_1)\right]\left[w_\lambda^{1/2}\phi_\lambda(x_2)\right]\right| \tag{153}$$

$$\leq \left|-(-it)^{-1} - \int_0^\infty \mathrm{d}\lambda\ (-it)^{-1}e^{-it\lambda}\left\{\mathrm{d}_\lambda\left[w_\lambda^{1/2}\phi_\lambda(x_1)\right]\right.\right. \tag{154}$$

$$\left.\left.\times\left[w_\lambda^{1/2}\phi_\lambda(x_2)\right] + \left[w_\lambda^{1/2}\phi_\lambda(x_1)\right]\mathrm{d}_\lambda\left[w_\lambda^{1/2}\phi_\lambda(x_2)\right]\right\}\right| \tag{155}$$

$$\leq t^{-1}\left(1+\int_0^\infty \mathrm{d}\lambda\ \left\{\left|\mathrm{d}_\lambda\left[w_\lambda^{1/2}\phi_\lambda(x_1)\right]\right|\ \left|w_\lambda^{1/2}\phi_\lambda(x_2)\right|\right.\right. \tag{156}$$

$$\left.\left.+\left|w_\lambda^{1/2}\phi_\lambda(x_1)\right|\ \left|\mathrm{d}_\lambda\left[w_\lambda^{1/2}\phi_\lambda(x_2)\right]\right|\right\}\right) \tag{157}$$

$$< t^{-1}\left(1+\int_0^\infty \mathrm{d}\lambda\ \left\{(6)(x_1+\kappa)^2\exp(-16^{-1}\epsilon_1\lambda)(3)(x_2+\kappa)\exp(-16^{-1}\epsilon_2\lambda)\right.\right. \tag{158}$$

$$\left.\left.+(3)(x_1+\kappa)\exp(-16^{-1}\epsilon_1\lambda)(6)(x_2+\kappa)\exp(-16^{-1}\epsilon_2\lambda)\right\}\right) \tag{159}$$

$$\leq t^{-1}\left\{1+18(x_1+\kappa)^2(x_2+\kappa)^2\int_0^\infty \mathrm{d}\lambda\ \exp[-16^{-1}(\epsilon_1+\epsilon_2)\lambda]\right\} \tag{160}$$

$$= t^{-1}\left[1+288(x_1+\kappa)^2(x_2+\kappa)^2(\epsilon_1+\epsilon_2)^{-1}\right] \tag{161}$$

$$< t^{-1}\left[1+72(x_1+\kappa)^3(x_2+\kappa)^3\right] \tag{162}$$

$$\leq 73(x_1+\kappa)^3(x_2+\kappa)^3 t^{-1}. \tag{163}$$

□

4.3. Local time decay for L. We recall without proof Lemma 3.12 from [**21**]:

Lemma. *Let $\mathscr{B}$ be a Banach space and $\lambda_+>\lambda_-$ be real constants. If $F(\lambda)$ has the properties*

(1) $F \in C(\lambda_-, \lambda_+; \mathscr{B})$

(2) $F(\lambda_-) = F(\lambda) = 0, \quad \lambda > \lambda_+$

(3) $\mathrm{d}_\lambda F \in L^1(\lambda_- + \delta, \lambda_+; \mathscr{B}), \quad \forall \delta > 0$

(4) $\mathrm{d}_\lambda F(\lambda) = \mathcal{O}([\lambda - \lambda_-]^{-1} \log^{-3}[\lambda - \lambda_-]), \quad \lambda \searrow \lambda_-$

(5) $\mathrm{d}^2_\lambda F(\lambda) = \mathcal{O}([\lambda - \lambda_-]^{-2} \log^{-2}[\lambda - \lambda_-]), \quad \lambda \searrow \lambda_-$

then

$$\int_{\lambda_-}^{\infty} \mathrm{d}\lambda \; e^{-it\lambda} F(\lambda) = \mathcal{O}(t^{-1} \log^{-2} t), \quad t \nearrow \infty \tag{164}$$

in the norm of $\mathscr{B}$.

Proof of Theorem (3). Let $\mathscr{B} = \{A \in \mathcal{L}(\mathscr{T}) : ||A||_{\mathscr{B}} < \infty\}$ be the Banach space complete in the norm

$$||A||_{\mathscr{B}} := \sup_{v \in \ell^1} \frac{||W_{\kappa,\tau} A W_{\kappa,\tau} v||_1}{||v||_1}. \tag{165}$$

Let $F(\lambda) = \delta^L_\lambda$. We will verify the appropriate properties of $F(\lambda)$ for $\lambda_- = 0$ and $\lambda_+ = \infty$.

We recall that

$$F(\lambda, x_1, x_2) = w^L_\lambda \phi^L_\lambda(x_1) \phi^L_\lambda(x_2) = g_\lambda w_\lambda [\phi_\lambda(x_1) + q\xi_\lambda(x_1)][\phi_\lambda(x_2 + q\xi_\lambda(x_2)]. \tag{166}$$

One may observe that

$$|\mathrm{d}^n_\lambda [w^{1/2}_\lambda \phi^L_\lambda(x)]| \leq |\mathrm{d}^n_\lambda [w^{1/2}_\lambda \phi_\lambda(x)]| + q|\mathrm{d}^n_\lambda [w^{1/2}_\lambda \xi_\lambda(x)]| \tag{167}$$

$$< [c(\phi_\lambda, n) + qc(\xi_\lambda, n)] \exp(-16^{-1} \epsilon \lambda), \tag{168}$$

$$< c(\phi^L_\lambda, n) \exp(-16^{-1} \epsilon \lambda), \quad n = 0, 1, 2 \tag{169}$$

where here we choose

$$c(\phi^L_\lambda, 0) := 3(1+4q)(x+\kappa)^2, \tag{170}$$

$$c(\phi^L_\lambda, 1) := 6(1+4q)(x+\kappa)^2, \tag{171}$$

$$c(\phi^L_\lambda, 2) := 9(1+4q)(x+\kappa)^3. \tag{172}$$

One may see by inspection that $g_\lambda := \{[1 - qe^{-\lambda}\mathcal{P}E_1(-\lambda)]^2 + [\pi q e^{-\lambda}]^2\}^{-1}$ has the properties:

$$g_\lambda = |g_\lambda| \le \widehat{g}_0(q) < \infty, \quad \forall \lambda \in [0,\infty) \tag{173}$$

$$|\mathrm{d}_\lambda g_\lambda| \le \widehat{g}_0(q)\widehat{g}_1(q,\delta) < \infty, \quad \forall \lambda \in [\delta,\infty) \tag{174}$$

$$g_0 = g_\infty = 0, \tag{175}$$

$$\mathrm{d}_\lambda g_\lambda = \mathcal{O}(\lambda^{-1}\log^{-1}\lambda), \quad \lambda \searrow 0 \tag{176}$$

$$\mathrm{d}^2_\lambda g_\lambda = \mathcal{O}(\lambda^{-2}\log^{-3}\lambda), \quad \lambda \searrow 0 \tag{177}$$

$$\subset \mathcal{O}(\lambda^{-2}\log^{-2}\lambda) \tag{178}$$

where $0 < \widehat{g}_0(q), \widehat{g}_1(q,\delta) < \infty$ are constants whose other properties are not needed here. g_λ is the only function of λ involved in the definition of $F(\lambda)$ whose derivatives are unbounded in the neighborhood of the threshold $\lambda = 0$ and thereby the derivatives of g_λ are dominant in determining the properties of the derivatives of $F(\lambda)$.

<u>Properties (1), (2):</u> One may observe that the properties follow by inspection.

Property (3): For $\lambda \in [\delta, \infty)$ one has

$$
\begin{aligned}
(179)\qquad |\mathrm{d}_\lambda F(\lambda, x_1, x_2)| &= |\mathrm{d}_\lambda \{g_\lambda [w_\lambda^{1/2} \phi_\lambda^L(x_1)][w_\lambda^{1/2} \phi_\lambda^L(x_2)]\}| \\
(180)\qquad &\leq |\mathrm{d}_\lambda g_\lambda||[w_\lambda^{1/2} \phi_\lambda^L(x_1)]||[w_\lambda^{1/2} \phi_\lambda^L(x_2)]| \\
(181)\qquad &\quad + |g_\lambda||\mathrm{d}_\lambda [w_\lambda^{1/2} \phi_\lambda^L(x_1)]||[w_\lambda^{1/2} \phi_\lambda^L(x_2)]| \\
(182)\qquad &\quad + |g_\lambda||[w_\lambda^{1/2} \phi_\lambda^L(x_1)]||\mathrm{d}_\lambda [w_\lambda^{1/2} \phi_\lambda^L(x_2)]| \\
(183)\qquad &< \widehat{g}_0(q)\widehat{g}_1(q, \delta)(3)(1 + 4q)(x_1 + \kappa)^2 \exp(-16^{-1}\epsilon_1 \lambda) \\
(184)\qquad &\quad \times (3)(1 + 4q)(x_2 + \kappa)^2 \exp(-16^{-1}\epsilon_1 \lambda) \\
(185)\qquad &\quad + \widehat{g}_0(q)(6)(1 + 4q)(x_1 + \kappa)^2 \exp(-16^{-1}\epsilon_1 \lambda) \\
(186)\qquad &\quad \times (3)(1 + 4q)(x_2 + \kappa)^2 \exp(-16^{-1}\epsilon_2 \lambda) \\
(187)\qquad &\quad + \widehat{g}_0(q)(3)(1 + 4q)(x_1 + \kappa)^2 \exp(-16^{-1}\epsilon_1 \lambda) \\
(188)\qquad &\quad \times (6)(1 + 4q)(x_2 + \kappa)^2 \exp(-16^{-1}\epsilon_2 \lambda) \\
(189)\qquad &= c_0(q, \delta)(x_1 + \kappa)^2 (x_2 + \kappa)^2 \exp[-16^{-1}(\epsilon_1 + \epsilon_2)\lambda],
\end{aligned}
$$

where $c_0(q, \delta)$ is a constant.

Property (4): For $\lambda \searrow 0$ one has

$$|\mathrm{d}_\lambda F(\lambda,x_1,x_2)| = |\mathrm{d}_\lambda\{g_\lambda[w_\lambda^{1/2}\phi_\lambda^L(x_1)][w_\lambda^{1/2}\phi_\lambda^L(x_2)]\}| \tag{190}$$

$$\leq |\mathrm{d}_\lambda g_\lambda||[w_\lambda^{1/2}\phi_\lambda^L(x_1)]||[w_\lambda^{1/2}\phi_\lambda^L(x_2)]| \tag{191}$$

$$+ |g_\lambda||\mathrm{d}_\lambda[w_\lambda^{1/2}\phi_\lambda^L(x_1)]||[w_\lambda^{1/2}\phi_\lambda^L(x_2)]| \tag{192}$$

$$+ |g_\lambda||[w_\lambda^{1/2}\phi_\lambda^L(x_1)]||\mathrm{d}_\lambda[w_\lambda^{1/2}\phi_\lambda^L(x_2)]| \tag{193}$$

$$|\mathrm{d}_\lambda F(\lambda,x_1,x_2)| = |\mathrm{d}_\lambda\{g_\lambda[w_\lambda^{1/2}\phi_\lambda^L(x_1)][w_\lambda^{1/2}\phi_\lambda^L(x_2)]\}| \tag{194}$$

$$< |\mathrm{d}_\lambda g_\lambda|(3)(1+4q)(x_1+\kappa)^2\exp(-16^{-1}\epsilon_1\lambda) \tag{195}$$

$$\times (3)(1+4q)(x_2+\kappa)^2\exp(-16^{-1}\epsilon_1\lambda) \tag{196}$$

$$+ \widehat{g}_0(q)(6)(1+4q)(x_1+\kappa)^2\exp(-16^{-1}\epsilon_1\lambda) \tag{197}$$

$$\times (3)(1+4q)(x_2+\kappa)^2\exp(-16^{-1}\epsilon_2\lambda) \tag{198}$$

$$+ \widehat{g}_0(q)(3)(1+4q)(x_1+\kappa)^2\exp(-16^{-1}\epsilon_1\lambda) \tag{199}$$

$$\times (6)(1+4q)(x_2+\kappa)^2\exp(-16^{-1}\epsilon_2\lambda) \tag{200}$$

$$\leq c_1(q,\delta)(x_1+\kappa)^2(x_2+\kappa)^2\exp[-16^{-1}(\epsilon_1+\epsilon_2)\lambda]|\mathrm{d}_\lambda g_\lambda| \tag{201}$$

$$= \mathcal{O}(\lambda^{-1}\log^{-1}\lambda) \tag{202}$$

in the norm of $\mathscr{B}$, where $c_1(q,\delta)$ is a constant.

Property (5): For $\lambda \searrow 0$ one has

$$|\mathrm{d}_\lambda^2 F(\lambda,x_1,x_2)| = |\mathrm{d}_\lambda^2\{g_\lambda[w_\lambda^{1/2}\phi_\lambda^L(x_1)][w_\lambda^{1/2}\phi_\lambda^L(x_2)]\}| \tag{203}$$

$$\leq c_2(q,\delta)(x_1+\kappa)^3(x_2+\kappa)^3\exp[-16^{-1}(\epsilon_1+\epsilon_2)\lambda]|\mathrm{d}_\lambda^2 g_\lambda| \tag{204}$$

$$= \mathcal{O}(\lambda^{-2}\log^{-2}\lambda) \tag{205}$$

in the norm of $\mathscr{B}$, where $c_2(q,\delta)$ is a constant. $\square$

CHAPTER 2

The nonlinear stationary problem

1. Results

Consider the discrete NLS

$$i\partial_t w = L_0 w - |w|^{2\sigma} w, \quad 1 \leq \sigma \in \mathbb{Z} \tag{206}$$

where $w : \mathbb{R}_t \times \mathbb{Z}_+ \to \mathbb{C}$. The existence of a $u : \mathbb{Z}_+ \to \mathbb{C}$ which satisfies the nonlinear finite difference equation

$$L_0 u = \zeta u + |u|^{2\sigma} u, \tag{207}$$

furnishes a stationary state of the discrete NLS of the form $w(t) = e^{-i\zeta t} u$. One expects that, due to the attractive nature of the nonlinearity, a negative "nonlinear eigenvalue", $\zeta = -a < 0$, will allow the existence of a sharply peaked, monotonically decaying "ground state soliton". We will therefore exclusively look for solutions to

$$L_0 u = -au + u^{2\sigma+1}, \tag{208}$$

where $u : \mathbb{Z}_+ \to \mathbb{R}_+$ and $a > 0$. Solutions with these characteristics are self-focusing and tend to be sharply localized. They are therefore termed solitary waves or *solitons* generally.

THEOREM 4. *There exists a $\mu_* > 0$ such that for each $\mu > \mu_*$ there exists a solution to Equation (208) with $\zeta = -\mu < 0$ and $u = \alpha_\mu$, which is:*

(1) *positive: $\alpha_\mu(x) > 0$ for all $x \in \mathbb{Z}_+$*
(2) *monotonically decaying: $\alpha_\mu(x+1) - \alpha_\mu(x) < 0$ for all $x \in \mathbb{Z}_+$*
(3) *absolutely integrable: $\alpha_\mu \in \ell^1$*

DEFINITION 1.1. *We define* $\Pi := I - \Pi_0$ *and write* $\widehat{v} \equiv \Pi v$ *for all* $v \in \mathscr{H}$ *and write* $\widehat{A} \equiv \Pi A$ *for all* $A \in \mathcal{B}(\mathscr{H})$.

The proof of Theorem (4) will proceed as follows:

(1) Consider Equation (208) with $\zeta = -a < 0$, where a is a free parameter, $u : \mathbb{Z}_+ \to \mathbb{R}_+$, and $p := 2\sigma + 1$: $L_0 u = -au + u^p$. Split this equation into a boundary piece and a tail piece by applying Π_0 and Π respectively.

(2) We take $b := u(0)$ to be a fixed constant and iterate the tail piece of Equation (208) via

$$u_{n+1}(a, b) = \widehat{\psi}_{-a} b^p + \widehat{R}^{L_0}_{-a} u^p_n(a, b), \tag{209}$$

$u_n(a, b) \in \mathscr{H}$ for all n. We show that for large enough a this iteration converges pointwise monotonically and that $||u_n(a, b)||_1 \leq s_n(a)$ is bounded as $n \nearrow \infty$, for a sequence of constants $\{s_n(a)\}_{n=0}^{\infty}$. We define the limit of this iteration to be $u_*(a, b) \equiv \lim_{n \nearrow \infty} u_n(a, b)$.

(3) The construction of $u_*(a, b)$ sets $u(1) = u_*(a, b; 1) \equiv q(a, b)$. We substitute this into the boundary piece of Equation (208) which then takes the form

$$0 = b^p - (a + 1)b + q(a, b). \tag{210}$$

We will now take b to be a variable. If the solution, u, is positive and monotonically decaying then one must have

$$0 < u(1) < u(0) \quad \Rightarrow \quad a^{(p-1)^{-1}} < b < (a + 1)^{(p-1)^{-1}}. \tag{211}$$

We show that for all a sufficiently large there is a unique $b = b_*(a) \in (a^{(p-1)^{-1}}, (a + 1)^{(p-1)^{-1}})$ which solves the boundary equation.

(4) We define the solution we desire, α_μ, by

$$\alpha_\mu(x) := \begin{cases} b_*(\mu) & , \ x = 0 \\ u_*(\mu, b_*(\mu); x) & , \ x > 0 \end{cases} . \tag{212}$$

(5) The three properties (i.e. positivity, monotonicity, ℓ^1) of the solution will then be verified in turn.

Typically one can arrive at the existence of a soliton with such properties via variational or rearrangement arguments. We will use much more elementary techniques which yield yet other properties due to the dependence on explicit constructions. One such result which will be of use later on is

Proposition 2.

(213)

$$||\widehat{\alpha}_\mu||_1 \leq s(\mu) = \mu^{-1}(\mu+1)^{(p-1)^{-1}} + \mu s^p(\mu) = \mu^{-(p-1)^{-1}(p-2)} + \mathcal{O}(\mu^{-(p-1)^{-1}(2p-3)}).$$

2. Existence of α_μ

2.1. Away from the boundary. Consider two forms of Equation (208):

$$L_0 u = -au + u^p \tag{I}$$

$$u = R^{L_0}_{-a} u^p. \tag{II}$$

One may project the equation of form (II) to a "tail" piece by applying Π:

$$u = R^{L_0}_{-a}(\Pi_0 + \Pi)u^p \quad \Rightarrow \quad \widehat{u} = \widehat{\psi}_{-a} u^p(0) + \widehat{R}^{L_0}_{-a} \widehat{u}^p. \tag{214}$$

We will fix $u(0) \equiv b$ and iterate by substituting the LHS into the RHS. We will show the conditions under which this converges and

Definition 2.1. *Let $u(0) \equiv b$ be a fixed parameter which satisfies $a^{(p-1)^{-1}} < b < (a+1)^{(p-1)^{-1}}$. Let $\{u_n(a,b)\}_{n=0}^{\infty}$ be a sequence of vectors in $\mathscr{H}$ defined by a fixed $u_0(a,b)$ and inductively by*

$$u_{n+1}(a,b) = \widehat{\psi}_{-a} b^p + \widehat{R}^{L_0}_{-a} u^p_n(a,b), \tag{215}$$

such that $u_n(a,b) = \widehat{u}_n(a,b)$.

The requirement that $a^{(p-1)^{-1}} < b < (a+1)^{(p-1)^{-1}}$ follows from $0 < u(1) < u(0)$ such that $u(0) = b$ and $u(x)$ in monotonically decreasing for increasing x.

Lemma 2.1. *One has that $||\psi_{-a}||_1 = a^{-1}$ for all $a > 0$.*

Proof. $\psi_{-a}(x) > 0$ for all $a \in \mathbb{R}_+$ and $x \in \mathbb{Z}_+$. Therefore

$$||\psi_{-a}||_1 = \sum_{x=0}^{\infty} \psi_{-a}(x) \tag{216}$$

$$= \sum_{x=0}^{\infty} e^a \sum_{k=0}^{x} (-1)^k \binom{x}{k} E_{k+1}(a) = \sum_{x=0}^{\infty} e^a \int_1^\infty \mathrm{d}t e^{-at} t^{-1} (1-t^{-1})^x \tag{217}$$

$$= e^a \int_1^\infty \mathrm{d}t e^{-at} t^{-1} \sum_{x=0}^{\infty} (1-t^{-1})^x = \int_0^\infty \mathrm{d}t e^{-at} = a^{-1}. \tag{218}$$

□

Lemma 2.2. *Let $\{s_n(a)\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers defined by a fixed $s_0(a)$ and inductively by*

$$s_{n+1}(a) = a^{-1} r(a) + a^{-1} s_n^p(a), \tag{219}$$

where

$$r(a) := (a+1)^{(p-1)^{-1}}. \tag{220}$$

If $||u_j(a,b)||_1 \le s_j(a)$ for some j then $||u_k(a,b)||_1 < s_k(a)$ for all $j < k$.

Proof. Consider the known bound [1]

$$(z+n)^{-1} < e^z E_n(z) \le (z+n-1)^{-1}, \quad 0 < z \in \mathbb{R} \tag{221}$$

Since $\psi_{-a}(0) = e^a E_1(a)$, one then has that

$$\left|\left|\widehat{\psi}_{-a}\right|\right|_1 = ||\psi_{-a}||_1 - \psi_{-a}(0) < a^{-1}(a+1)^{-1}. \tag{222}$$

Then, since $b < (a+1)^{(p-1)^{-1}}$ one has

$$\left|\left|\widehat{\psi}_{-a} b^p\right|\right|_1 < a^{-1} r(a). \tag{223}$$

One may then observe that

$$\left|\left|u_{j+1}\right|\right|_1 < a^{-1}r(a) + a^{-1}\left|\left|u_j^p\right|\right|_1 \leq a^{-1}r(a) + a^{-1}s_j^p(a) = s_{j+1}(a). \tag{224}$$

□

LEMMA 2.3. *Let* $g : \mathbb{R}_+^2 \to \mathbb{R}$ *be specified by*

$$g(a, s) := a^{-1}r(a) + a^{-1}s^p - s. \tag{225}$$

and $s_{\min}(a) := (a/p)^{(p-1)^{-1}}$. *For sufficiently large* $a > 0$ *it is the case that* $g(a, s)$ *has exactly two roots in* s*:* $s_-(a)$ *which satisfies* $0 < s_-(a) < s_{\min}(a)$ *and* $s_+(a)$ *which satisfies* $s_{\min}(a) < s_+(a)$.

PROOF. One may observe that $g(a, s)$ has a global minimum at $s = s_{\min}(a)$. It is the case that

$$g(a, s_{\min}(a)) = a^{-1}r(a) + a^{-1}(a/p)^{p(p-1)^{-1}} - a^{(p-1)^{-1}}p^{-(p-1)^{-1}} \tag{226}$$

$$= a^{-1}(a+1)^{(p-1)^{-1}} - a^{(p-1)^{-1}}p^{-(p-1)^{-1}}(1 - p^{-1}) \tag{227}$$

$$\mathrm{d}_a g(a, s_{\min}(a)) = -a^{-1}(a+1)^{(p-1)^{-1}}\left[a^{-1} - (a+1)^{-1}(p-1)^{-1}\right] \tag{228}$$

$$- a^{-(p-2)(p-1)^{-1}}p^{-(p-1)^{-1}}(p-1)^{-1}(1 - p^{-1}) < 0, \quad \forall a > 0 \tag{229}$$

since

$$a^{-1} - (a+1)^{-1}(p-1)^{-1} > 0, \quad \forall a > 0. \tag{230}$$

One has that

$$\lim_{a \nearrow \infty} g(a, s_{\min}(a)) = -\infty \tag{231}$$

and

$$g(a,0) = a^{-1}r(a) > 0, \quad \forall a > 0 \tag{232}$$

$$\lim_{a \nearrow \infty} g(a,0) = \infty \tag{233}$$

$$\lim_{s \nearrow \infty} g(a,s) = \infty, \quad \forall a > 0 \tag{234}$$

By intermediate value theorem there must be at least one root. By Descartes rule of signs, $g(a,s)$ has either 0 or 2 positive roots in s. Therefore $g(a,s)$ has exactly 2 roots for all sufficiently large $a > 0$. $\square$

Definition 2.2. *Let $a_0 > 0$ be the unique value of a such that $g(a,s)$ has two distinct roots in s for all $a > a_0$.*

$a_0 > 0$ is the unique value of a for which that $g(a,s)$ has one root in s of multiplicity 2.

Lemma 2.4. *Let $h : \mathbb{R}_+^2 \to \mathbb{R}_+$ be defined by*

$$h(a,s) := a^{-1}r(a) + a^{-1}s^p. \tag{235}$$

The map $h(a,\cdot) : s \mapsto h(a,s)$ is contractive on the domain $0 \le s < s_{\min}(a)$ for all $a > a_0$.

Proof.

$$\partial_s g(a,s)|_{s=0} = \; -1 \tag{236}$$

$$\partial_s g(a,s)|_{s=s_{\min}(a)} = 0 \tag{237}$$

$$\partial_s^2 g(a,s) > 0, \quad \forall s > 0 \tag{238}$$

$$\Rightarrow \quad 0 \le \partial_s h(a,s) < 1, \quad \forall s : 0 \le s < s_{\min}(a) \tag{239}$$

By mean value theorem, for any $s, s_1 \in [0, s_{\min})$ there exists an $s_2 < |s - s_1|$ such that

$$\frac{h(a,s) - h(a,s_1)}{s - s_1} = \partial_s h(a,s)|_{s=s_2} < 1, \tag{240}$$

which completes the proof. $\square$

LEMMA 2.5. *For all $a > a_0$ and for sufficiently small $||u_0(a,b)||_1 \geq 0$ one has that $\lim_{n\nearrow\infty} ||u_n(a,b)||_1 < \infty$.*

PROOF. Let $a > a_0$. Given the iteration $s_{n+1}(a) = h(a, s_n(a))$, the choice of any $s_0(a)$ which satisfies $0 \leq s_0(a) \leq s_-(a)$ gives a sequence $\{s_n(a)\}_{n=0}^{\infty}$ which converges to $\lim_{n\nearrow\infty} s_n(a) = s_-(a) < \infty$ monotonically from below as $n \nearrow \infty$ for all $a > a_0$. For $0 \leq ||u_0(a,b)||_1 \leq s_0(a)$ it is the case that $\lim_{n\nearrow\infty} ||u_n(a,b)||_1 \leq \lim_{n\nearrow\infty} s_n(a) = s_-(a) < \infty$. $\square$

LEMMA 2.6. *Let $s_*(a) := \lim_{n\nearrow\infty} s_n(a) = s_-(a)$ for all $a > a_0$. One has that $s_*(a) \searrow 0$ monotonically as $a \nearrow \infty$ for all $a > a_0$.*

PROOF. Let $a > a_0$. Consider the graph of the map $s \mapsto h(a,s) = a^{-1}r(a) + a^{-1}s^p$. It is the case that $\partial_a h(a,s) < 0$ and $h(a,s) \geq s$ for $0 \leq s \leq s_*(a)$, with equality for $h(a, s_*(a)) = s_*(a)$, which completes the proof. $\square$

LEMMA 2.7. *We define*

$$\theta_1(a,b) := \widehat{\psi}_{-a} b^p, \tag{241}$$

$$\theta_k(a,b) := \widehat{R}_{-a}^{L_0} \sum_{m_1=1}^{p} \binom{p}{m_1} \theta_1^{p-m_1}(a,b) \cdots \sum_{m_i=1}^{m_{j-1}} \binom{m_{j-1}}{m_j} \theta_j^{m_j - m_{j-1}}(a,b) \cdots \tag{242}$$

$$\times \sum_{m_{k-2}=1}^{m_{k-3}} \binom{m_{k-3}}{m_{k-2}} \theta_{k-2}^{m_{k-3}-m_{k-2}}(a,b) \theta_{k-1}^{m_{k-2}}(a,b). \tag{243}$$

If $u_0(a,b)=0$ *then one has*

$$u_n(a,b)=\sum_{k=1}^{n}\theta_k(a,b). \tag{244}$$

PROOF. One may verify by induction:

$$u_1(a,b)=\theta_1(a,b) \tag{245}$$

(246)

$$
\begin{aligned}
u_{n+1}(a,b) &= \theta_1(a,b)+\widehat{R}_{-a}^{L_0}u_n^p(a,b) \\
&= \theta_1(a,b)+\widehat{R}_{-a}^{L_0}\left[\sum_{k=1}^{n}\theta_k(a,b)\right]^p && (247)\\
&= \theta_1(a,b)+\widehat{R}_{-a}^{L_0}\left\{\theta_1^p(a,b)+\sum_{m_1=1}^{p}\binom{p}{m_1}\theta_1^{p-m_1}(a,b)\left[\sum_{k=2}^{n}\theta_k(a,b)\right]^{m_1}\right\} && (248)\\
&= \theta_1(a,b)+\widehat{R}_{-a}^{L_0}\Bigg[\theta_1^p(a,b)+\sum_{m_1=1}^{p}\binom{p}{m_1}\theta_1^{p-m_1}(a,b)\theta_2^{m_1}(a,b)+\cdots && (249)\\
&\qquad +\sum_{m_1=1}^{p}\binom{p}{m_1}\theta_1^{p-m_1}(a,b)\cdots\sum_{m_j=1}^{m_{j-1}}\binom{m_{j-1}}{m_j}\theta_j^{m_j-m_{j-1}}(a,b)\cdots && (250)\\
&\qquad \times\sum_{m_{k-2}=1}^{m_{k-3}}\binom{m_{k-3}}{m_{k-2}}\theta_{k-2}^{m_{k-3}-m_{k-2}}(a,b)\theta_{k-1}^{m_{k-2}}(a,b) && (251)\\
&\qquad +\sum_{m_1=1}^{p}\binom{p}{m_1}\theta_1^{p-m_1}(a,b)\cdots\sum_{m_j=1}^{m_{j-1}}\binom{m_{j-1}}{m_j}\theta_j^{m_i-m_{j-1}}(a,b)\cdots && (252)\\
&\qquad \times\sum_{m_{n-1}=1}^{m_{n-2}}\binom{m_{n-2}}{m_{n-1}}\theta_{n-1}^{m_{n-2}-m_{n-1}}(a,b)\theta_n^{m_{n-1}}(a,b)\Bigg] && (253)\\
&= \theta_1(a,b)+\theta_2(a,b)+\cdots+\theta_j(a,b)+\cdots+\theta_{n+1}(a,b) && (254)\\
&= \sum_{k=1}^{n+1}\theta_k(a,b). && (255)
\end{aligned}
$$

$\square$

Lemma 2.8. *For $u_0(a,b) = 0$ one has that $u_{n+1}(a,b;x) > u_n(a,b;x)$ for all $n \geq 0, x > 0$ (strict pointwise increase in n for all $x > 0$) for all $a > a_0$.*

Proof. Since $\theta_n(a,b;x) > 0$, $\forall x > 0$, and $u_{n+1}(a,b) - u_n(a,b) = \theta_{n+1}(a,b)$ it must be the case that the sequence $\{u_n(a,b;x)\}_{n=0}^{\infty}$ is strictly increasing in n for all $x > 0$ and for all $a > a_0$. $\square$

Lemma 2.9. *For $u_0(a,b) = 0$ and all $a > a_0$ one has that $\lim_{n\nearrow\infty} u_n(a,b) = \sum_{n=1}^{\infty} \theta_n(a,b) \in \ell^1$.*

Proof. Let $a > a_0$. One may observe that that $\lim_{n\nearrow\infty} ||u_n(a,b)||_1 \leq s_*(a) < \infty \Rightarrow \lim_{n\nearrow\infty} u_n(a,b;x) < s_*(a) < \infty$ for all $x \in \mathbb{Z}_+$. Then by monotonic increase of $u_n(a,b;x)$ in n for all $x > 0$, it follows that $u_n(a,b;x) = \sum_{k=1}^{n} \theta_k(a,b;x)$ exists for each n, x and uniquely determines $\lim_{n\nearrow\infty} u_n(a,b)$. $\square$

Lemma 2.10. *Let $u_*(a,b) := \sum_{n=1}^{\infty} \theta_n(a,b) = \lim_{n\nearrow\infty} u_n(a,b)$ for all $a > a_0$. One has that $u_*(a,b;x)$ is a monotonically increasing function in $b \geq 0$ for all $x > 0$ and for all $a > a_0$.*

Proof. $u_*(a,b;x) := \sum_{n=1}^{\infty} \theta_n(a,b;x)$ can be represented as a power series in b for all $x > 0$ with only positive powers and positive coefficients. $\square$

2.2. At the boundary. We now consider the equation of form (I). We can project it onto a "boundary" piece by applying Π_0:

$$0 = (\Pi_0 + \Pi)(-L_0 u - au + u^p) \quad \Rightarrow \quad 0 = u^p(0) - (a+1)u(0) + u(1) \tag{256}$$

Given $u_*(a,b) = \sum_{k=1}^{\infty} \theta_k(a,b)$ we will substitute $b = u(0)$ and $q(a,b) \equiv u_*(a,b;1) = u(1)$ in the boundary equation and thereby consider

$$0 = b^p - (a+1)b + q(a,b) \tag{257}$$

on the interval $a^{(p-1)^{-1}} \leq b \leq (a+1)^{(p-1)^{-1}}$.

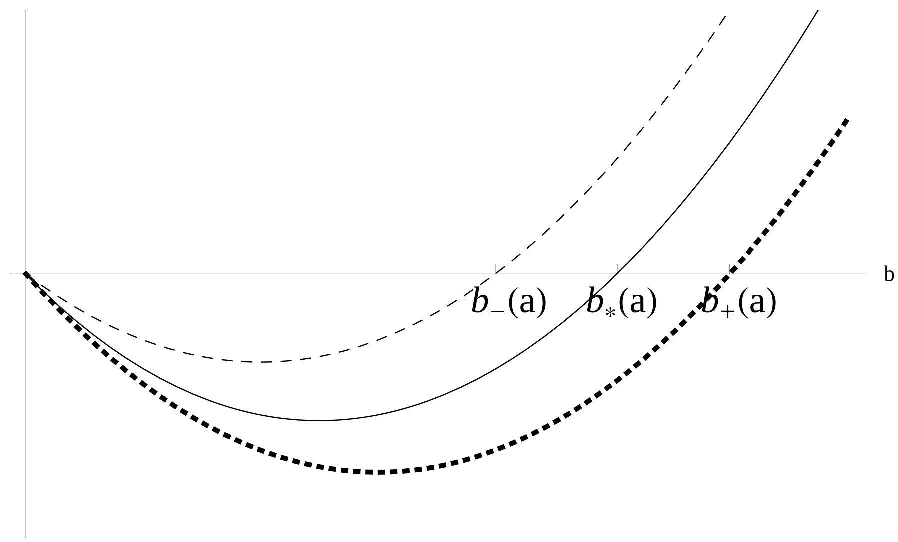

FIGURE 1. The dashed line is the graph of $f_+(a,b)$, the solid is of $f(a,b)$, and the dotted is of $f_-(a,b)$.

DEFINITION 2.3.

$$b_-(a) := a^{(p-1)^{-1}}, \quad b_+(a) := (a+1)^{(p-1)^{-1}} \tag{258}$$

$$\Sigma_a := \{b \in \mathbb{R} : b_-(a) \leq b \leq b_+(a)\} \tag{259}$$

$$q_-(a,b) := 0, \quad q(a,b) := u_*(a,b;1), \quad q_+(a,b) := b \tag{260}$$

$$f(a,b,q) := b^p - (a+1)b + q \tag{261}$$

$$f_-(a,b) := f(a,b,q_-(a,b)), \quad f_*(a,b) := f(a,b,q(a,b)), \tag{262}$$

$$f_+(a,b) := f(a,b,q_+(a,b)) \tag{263}$$

LEMMA 2.11. *$q(a,b) > q_-(a,b)$ for all $b \in \Sigma_a$ and for all $a > a_0$.*

PROOF. Let $a > a_0$.

$$q(a,b) = \sum_{n=1}^{\infty} \theta_n(a,b;1) > \theta_1(a,b;1) = \widehat{\psi}_{-a}(1) b^p \tag{264}$$

$$= e^a \left[E_1(a) - E_2(a)\right] b^p > 0 = q_-(a,b) \tag{265}$$

since $E_2(a) < E_1(a), \ \forall a > 0$ [1]. □

LEMMA 2.12. *There exists an* $a_1 \geq a_0$ *such that* $q(a,b) < q_+(a,b)$ *for all* $b \in \Sigma_a$ *and for all* $a > a_1$.

PROOF. Let $a > a_0$. It is the case that $\lim_{a \nearrow \infty} s_*(a) = 0$ monotonically and $\lim_{a\to\infty} b_-(a) = \infty$ monotonically so there exists an $a_1 \geq a_0$ such that $s_*(a) < b_-(0)$, $\forall a > a_1$. Clearly $q(a,b) < s_*(a)$. Then $q(a,b) < s_*(a) < b_-(a) \leq b = q_+(a,b)$, $\forall a > a_1$ and $\forall b \in \Sigma_a$. $\square$

LEMMA 2.13. *Let* $a_1 > 0$ *be the smallest value for which* $a_1 \geq a_0$ *and for which* $q(a,b) < q_+(a,b)$ *for all* $b \in \Sigma_a$ *and for all* $a > a_1$. *Define* $a_2 := (p-1)^{-1}$, $a_3 := \max\{a_1, a_2\}$. *One has that:*

- $f_-(a,b)$ *is negative and monotonically increasing for all* $b \in \Sigma_a \setminus b_+(a)$ *and for all* $a > a_3$.
- $f_+(a,b)$ *is positive and monotonically increasing in* b *for all* $b \in \Sigma_a \setminus b_-(a)$ *and for all* $a > a_3$.

PROOF. One may observe that $f_-(a,b) < f_*(a,b) < f_+(a,b)$ for all $a > a_1$ and all $b \in \Sigma_a$. $f_-(a,b_+(a)) = 0$ and $f_+(a,b_-(a)) = 0$ for all $a \geq 0$.
For $f_-(a,b)$: Let $a > a_3$. It follows that

$$\partial_b f_-(a,b) = pb^{p-1} - (a+1) > pb_-^{p-1}(a) - (a+1) = a(p-1) - 1. \tag{266}$$

Then $\partial_b f_-(a,b) > 0$, $\forall b \in \Sigma_a$ and for all $a > a_3$. Since $f_-(a,b_+(a)) = 0$ one has that $f_-(a,b) < 0$ for all $b \in \Sigma_a \setminus b_+(a)$.
For $f_+(a,b)$: Let $a > a_3$.

$$\partial_b f_+(a,b) = pb^{p-1} - a \geq pb_-^{p-1}(a) - a = a(p-1) > 0, \ \forall a > 0. \tag{267}$$

Then since $f_+(a,b_-(a)) = 0$ one finds that $f_+(a,b) > 0$ $\forall b \in \Sigma_a \setminus b_-(a)$. $\square$

LEMMA 2.14. $f_*(a,b)$ *is monotonically increasing in* b *and has exactly one root in* b *for all* $b \in \Sigma_a$ *and for all* $a > a_3$.

Proof. Let $a > a_3$. $f_-(a,b)$ is monotonically increasing in b for all $b \in \Sigma_a$. $q(a,b)$ is monotonically increasing in b for all $b > 0$. $f_*(a,b) = f_-(a,b) + q(a,b)$ so $f_*(a,b)$ must be monotonically increasing in b for all $b \in \Sigma_a$.

Since $f_*(a, b_-(a)) < 0 = f_+(b_-(a)) = f_-(a, b_+(a)) < f_*(a, b_+(a))$, by intermediate value theorem there must be at least one $b = b_*(a)$ for which $f_*(a,b) = 0$ and since $f_*(a,b)$ is monotonically increasing on this interval, there must be exactly one such $b = b_*(a)$. $\square$

Definition 2.4. *Let $b_*(a)$ be the unique value of $b \in \Sigma_a$ for which $f_*(a,b) = 0$ for all $a > a_3$ and define $\alpha_a \in \mathscr{H}$ via*

$$\alpha_a(x) := \begin{cases} b_*(a) & , \ x = 0 \\ u_*(a, b_*(a)) & , \ x > 0 \end{cases} , \tag{268}$$

for all $a > a_3$.

3. Properties of α_μ

3.1. Monotonicity. Consider

$$-L_0 u = V(u), \tag{269}$$

where $V(\cdot) : \mathbb{R} \to \mathbb{R}$ satisfies $V(r) = 0$, $V(r_0) = 0$, and $V(r) > 0$ for all $r \in (0, r_0)$ where $r_0 > 0$. It was shown in [**10**] that for any solution of this equation for which $\lim_{x \nearrow \infty} u(x) = 0$, there exists an x_* such that $u(x)$ is a monotonically decaying for increasing $x \geq x_*$. We will apply their argument, section 3 of [**10**], to show that for sufficiently large a one must have that $\alpha_a(x)$ is monotonically decaying in x for all $x \in \mathbb{Z}_+$. For our case we have that $V(r) = ar - r^p$, $r \in \mathbb{R}$, which satisfies the desired criteria.

Equation (269) may be summed to give an alternative finite difference equation. One has the two forms:

$$(x+1)[u(x+1) - u(x)] - x[u(x) - u(x-1)] = V(u(x)), \tag{270}$$

$$u(x+1) - u(x) = (x+1)^{-1} \sum_{y=0}^{x} V(u_y). \tag{271}$$

The argument proceeds as follows.

Lemma 3.1. *There exists an $a_4 \geq a_3$ such that $s_*(a) < b_-(a)$ for all $a > a_4$.*

Proof. Let $a > a_3$. It is the case that $s_*(a) \searrow 0$ monotonically as $a \nearrow \infty$. It is clear that $b_-(a) \nearrow \infty$ monotonically as $a \nearrow \infty$. $\square$

Lemma 3.2. *Let $u \in \mathscr{H}$ solve Equation (269). Let c_- and c_+ be constants which satisfy $0 < c_- < u(0)$ and $0 < ||\Pi u|| < c_+ < \infty$. If additionally $c_+ < c_-$ then $u(x)$ is monotonically decreasing in x for all $x \in \mathbb{Z}_+$.*

Proof. Let u solve Equation (269). Let c_- and c_+ be constants which satisfy $0 < c_- < u(0)$ and $0 < ||\Pi u|| < c_+ < \infty$. Furthermore let $c_+ < c_-$. One has that

$u(x) < u(0)$ for all $x > 0$ and in particular $u(1) < u(0)$. Assume that there exists an $0 < x_0 \in \mathbb{Z}_+$ such that $u(x_0+1) - u(x_0) = 0$ for $u(x_0),\ u(x_0+1) \in (0, c_-)$. One then has $u(x_0+2) - u(x_0+1) > 0$ since $(x_0+2)[u(x_0+2) - u(x_0+1)] = V(u(x_0+1)) > 0$. Then assume generally that there exists an $0 < x_1 \in \mathbb{Z}_+$ such that $u(x_1+1) - u(x_1) > 0$. This gives that $\sum_{y=0}^{x_1} V(u(y)) > 0$ since $u(x_1+1) - u(x_1) = (x_1+1)^{-1} \sum_{y=0}^{x_1} V(u(y)) > 0$. One then has that

$$u(x_1+2) - u(x_1+1) = (x_1+2)^{-1} \sum_{y=0}^{x_1+1} V(u(y)) \tag{272}$$

$$= (x_1+2)^{-1} V(u(x_1+1)) + (x_1+2)^{-1} \sum_{y=0}^{x_1} V(u(y)) > 0 \tag{273}$$

since $V(u(x_1+1))$ and $\sum_{y=0}^{x_1} V(u(y))$ are both positive. If $u(x_1+2) \geq c_-$ then one has a contradiction. On the other hand if $u(x_1+2) < c_-$ then one may repeat the above process with x_1 replaced with x_1+1. Therefore if there exists an $x_1 > 0$ such that $u(x_1+1) - u(x_1) > 0$, the $u(x)$ for the subsequent $x > x_1+1$ will continue to rise at least until $u(x_3) \geq c_-$, which is the point greater than which $V(x)$ remains negative, for some $x_3 > x_2 + 1$. One therefore has a contradiction if $u(x)$ fails to be monotonically decreasing as $x \nearrow \infty$ for all $x \in \mathbb{Z}_+$. □

Lemma 3.3. *Let $\mu_* > 0$ be the smallest value such that $\mu_* \geq a_3$ and such that $s_*(\mu) < b_-(\mu)$ for all $\mu > \mu_*$. One has that $\alpha_\mu(x)$ decreases monotonically as $x \nearrow \infty$ for all $x \in \mathbb{Z}_+$ and for all $\mu > \mu_*$*

Proof. Consider Lemma 4.1 and let $u = \alpha_\mu$, $c_- = b_-(\mu)$, and $c_+ = s_*(\mu)$ for all $\mu > \mu_*$. □

3.2. Asymptotic behavior. In [5] the asymptotic behavior of solutions of the finite difference nonsingular Sturm-Liouville problem $\Pi(L_0 - \lambda)\Pi u = 0$ were studied with various boundary conditions. One solution, ϕ_λ, has the well-known asymptotic behavior

$$\phi_\lambda(x) \sim e^{\lambda/2} J_0(2\sqrt{\lambda x}) \quad \text{as} \quad x \nearrow \infty, \tag{274}$$

where $J_0(z)$ is the Bessel function of the first kind of degree 0. They studied a particular solution, which they call Ψ_λ, which is a linear combination of ϕ_λ and ψ_λ. It satisfies the boundary conditions $\Psi_\lambda(0) = 0$ and $\Psi_\lambda(1) = 1$. Ψ_λ was shown to have the asymptotic behavior

$$\Psi_\lambda(x) \sim \pi e^{-\lambda/2} Y_0(2\sqrt{\lambda x}) + e^{\lambda/2} \mathcal{P} E_1(-\lambda) J_0(2\sqrt{\lambda x}) \quad \text{as} \quad x \nearrow \infty, \tag{275}$$

where $Y_0(z)$ is the Bessel function of the second kind of degree 0. One can determine the asymptotic behavior of $\psi_{-a}(x)$ as $x \nearrow \infty$ by finding the appropriate linear combination of Bessel functions such that upon analytic continuation, the real part is of the appropriate linear combination of the asymptotic forms of $\phi_\lambda(x)$ and $\Psi_\lambda(x)$ is monotonically decreasing as $x \nearrow \infty$. This combination must be monotonically decaying for $\lambda = -a < 0$ one can straightforwardly determine that the asymptotic behavior must be of the form

$$\psi_{-a}(x) \sim 2e^{a/2} K_0(2\sqrt{ax}) \sim e^{a/2} \pi^{1/2} (ax)^{-1/4} e^{-2\sqrt{ax}} \quad \text{as} \quad x \nearrow \infty, \tag{276}$$

where $K_0(z)$ is the modified Bessel function of the second kind of degree 0.

Lemma 3.4. *$\phi_{-a}(x), \psi_{-a}(x) > 0$ for all $x \in \mathbb{Z}_+$ and $a > 0$. $\phi_{-a}(x)$ is monotonically increasing and $\psi_{-a}(x)$ is monotonically decreasing in increasing x for all $a > 0$.*

Proof. One can see that $\phi_{-a}(x)$ is positive and monotonically increasing by inspection of $\phi_{-a}(x) = \sum_{k=0}^{x} \frac{a^k}{k!}\binom{x}{k}$. One can observe that $\psi_{-a}(x)$ is positive for all x, a

$$\psi_{-a}(x) = e^a \sum_{k=0}^{x} (-1)^k \binom{x}{k} \mathrm{E}_{k+1}(a) = e^a \int_1^\infty \mathrm{d}t \; e^{-at} t^{-1} \left(1 - t^{-1}\right)^x > 0 \tag{277}$$

as well as monotonically decreasing

$$\psi_{-a}(x+1) - \psi_{-a}(x) = -e^{a} \int_1^\infty \mathrm{d}t \; e^{-at} t^{-2} \left(1 - t^{-1}\right)^x < 0. \tag{278}$$

$\square$

Lemma 3.5. *The resolvent $R_z^{L_0}$ has the Sturm-Liouville (SL) representation*

$$R_z^{L_0}(x_1, x_2) = \begin{cases} \phi_z(x_1)\psi_z(x_2), & x_1 \le x_2 \\ \phi_z(x_2)\psi_z(x_1), & x_1 \ge x_2 \end{cases} . \tag{279}$$

Proof. The operator L_0 on $\mathscr{H}$ is a singular, second order, finite difference Sturm-Liouville operator. This is made manifest when put into SL form, $L_0 = D_+ M D_-$, where D_+, D_- are the respectively the usual forward and backward finite difference operators

$$D_+ v(x) = v(x+1) - v(x) \tag{280}$$

$$D_- v(x) = \begin{cases} v(x) - v(x-1) & , \; x > 0 \\ v(x) & , \; x = 0, \end{cases} \tag{281}$$

and $M$is lattice index multiplication operator $Mv(x) = xv(x)$ for all $v \in \mathscr{T}$.

L_0 is singular at the boundary point $x = 0$. When its domain and range are restricted to functions only on lattice points for $x > 0$ it is the case that L_0 is a non-singular operator. This restricted operator, $\Pi L_0 \Pi$, is second order and nonsingular therefore $\Pi L_0 \Pi u = zu$, $u \in \mathscr{T}$, admits two linearly independent solutions which satisfy linearly independent boundary conditions.

The finite difference Wronskian, also known as the Cassoratian, of two vectors $u, v \in \mathscr{T}$ is given by (see e.g. [**17**])

$$W[u,v](x) = u(x)v(x+1) - u(x+1)v(x). \tag{282}$$

A Jacobi operator $A \in \mathcal{L}(\mathscr{T})$ can be brought into the form

$$Av(x) = \eta(x)v(x+1) + \omega(x)v(x) + \eta(x-1)v(x-1), \quad \eta(x), \omega(x) \in \mathbb{R}. \tag{283}$$

As is specified by the finite difference Sturm-Liouville theory, if A is a Sturm-Liouville operator, u_z, v_z are linearly independent solutions of $Au = zu$, and $\eta(x)W[u_z, v_z](x) = 1\ \forall x$ then

$$R_z^A(x_1, x_2) = \begin{cases} u_z(x_1)v_z(x_2), & x_1 \le x_2 \\ u_z(x_2)v_z(x_1), & x_1 \ge x_2 \end{cases}. \tag{284}$$

This construction follows for $A = \Pi L_0 \Pi$, $u_z = \Pi\phi_z$, $v_z = \Pi\psi_z$. Since L_0 is singular at $x = 0$ one cannot adjust boundary conditions any further than fixing a scale factor for spectral solutions. One can simulate a second boundary condition with the introduction of either a linear perturbation or of an inhomogenous source supported at the singular point. This is to say that one can respectively consider the equations

$$(L_0 - q\Pi_0)u = zu \quad \text{or} \quad L_0 u = zu + q\chi_0 \tag{285}$$

where $q \in \mathbb{C}$ is a parameter, the tuning of which simulates the tuning of a second boundary condition. By taking the latter form with $q = 1$ one may arrive at ψ_z for the second solution. It therefore must be the case that $\Pi R_z^{L_0}\Pi = \Pi R_z^{\Pi L_0 \Pi}\Pi$ with the above prescription and $R_z^{L_0} = R_z^{\Pi L_0 \Pi}$ may be checked for $x = 0$ directly at boundary values. $\square$

One may observe that one has $R_{-a}^{L_0}(x_1, x_2) > 0$ for all $a > 0$ and $0 \le x_1, x_2 \in \mathbb{Z}$.

Lemma 3.6. *Consider the equation*

$$(L_0 + a)u = V(u)u, \tag{286}$$

where $V(\cdot) : \mathbb{R} \to \mathbb{R}$ *is continuous, locally bounded, and satisfies* $\lim_{r \searrow 0} V(r)r = 0$*. If* u *is a solution of this equation which is positive for all* x *and for which* $\lim_{x \nearrow \infty} u(x) =$

0 then $u(x) \sim c_0 x^{-1/2} e^{-c_1\sqrt{x}}$ as $x \nearrow \infty$ for some $0 < c_0, c_1 < \infty$ and for each fixed $a > 0$.

PROOF. Consider $a > 0$ a fixed constant. Let $\Pi_{\le x_*} := \sum_{x=0}^{x_*} \Pi_x$ and $\Pi_{>x_*} := I - \Pi_{\le x_*}$ for some $x_* \in \mathbb{Z}_+$. Let $q := ||\Pi_{>x_*} V(u)||_{\text{op}} = ||\Pi_{>x_*} V(u)||_\infty$. Furthermore let x_* satisfy $0 < q < a$ for all $a > 0$. One may find

$$(L_0 + a)u = V(u)u \tag{287}$$

$$\Rightarrow \quad u = [L_0 + a - \Pi_{>x_*(a)} V(u)]^{-1} \Pi_{\le x_*(a)} u \tag{288}$$

$$= R^{L_0}_{-a} [1 - \Pi_{>x_*} V(u) R^{L_0}_{-a}]^{-1} \Pi_{\le x_*} u \tag{289}$$

$$= R^{L_0}_{-a} \sum_{n=0}^{\infty} [\Pi_{>x_*} V(u) R^{L_0}_{-a}]^n \Pi_{\le x_*} u \tag{290}$$

where the sum converges absolutely. One has from the Sturm-Liouville form of the resolvent

$$R^{L_0}_{-a} \chi_x \le \phi_{-a}(x) R^{L_0}_{-a} \chi_0 = \phi_{-a}(x) \psi_{-a}, \tag{291}$$

for all $a > 0$ and all $x \in \mathbb{Z}_+$. One then has, for all $a > 0$ and all $x \in \mathbb{Z}_+$

$$u(x) \le R^{L_0}_{-a} \sum_{n=0}^{\infty} \left[q(u, x_*(a)) R^{L_0}_{-a} \right]^n \Pi_{\le x_*(a)} u(x) \tag{292}$$

$$\le \left[R^{L_0}_{-a+q} \sum_{y=0}^{x_*} u(y) \chi_y \right](x) \tag{293}$$

$$\le \sum_{y=0}^{x_*} u(y) \phi_{-a+q}(y) \psi_{-a+q}(x) \tag{294}$$

One may therefore conclude

$$u(x) \le c \psi_{-a+q}(x), \tag{295}$$

where $c = \sum_{y=0}^{x_*} u(y) \phi_{-a+q}(y) < \infty$. One has that $\psi_{-a+q}(x) \sim c' x^{-1/2} e^{-2\sqrt{(a-q)x}}$ as $x \nearrow \infty$ with a fixed, where c' is a constant that depends on $-a+q$ alone. By inspection the appropriate constants $0 < c_0, c_1 < \infty$ may be found for each fixed $a > 0$. $\square$

LEMMA 3.7. *$\alpha_\mu(x) \sim c_0 x^{-1/2} e^{-c_1\sqrt{x}}$ as $x \nearrow \infty$ for all $x \in \mathbb{Z}_+$ and for all $\mu > \mu_*$.*

PROOF. For $\mu > \mu_*$ it must be the case that $\alpha_\mu(x)$ exists as defined, be monotonically decreasing in x, and since it solves Equation (208), which is of the form given by Equation (269) it must have the desired asymptotic behavior. $\square$

PROOF OF THEOREM (4). Existence and Property (1) are given by Definition (2.4) and arguments on which the definition depends. Property (2) is given by Lemma (3.3). Property (3) is given by Lemma (3.7). $\square$

Since we now have existence and the desired properties, we can estimate an upper bound on norms of the tail of α_μ. We may use the convergence for the iteration of the $s_n(\mu)$ functions for large values of μ to verify Proposition (2) directly.

CHAPTER 3

The linearized matrix problem

1. Results

We would like to study the evolution of solutions which, at least at an initial time $t=0$, are close to the stationary soliton solution $\widehat{u}(t)=e^{-i(-\mu t+\nu)}\alpha_\mu$, where $\nu\in\mathbb{R}$ is an arbitrary phase factor. We then consider the ansatz $u=e^{-i\theta}(\alpha_{\mu_1}+\beta)$ where

$$\theta(t):=-\int_0^t \mu_1(s)\mathrm{d}s+\nu_1(t), \tag{296}$$

$$\mu_1(t)=\mu+\widehat{\mu}(t),\quad \mu_1(0)=\mu,\quad \nu_1(t)=\nu+\widehat{\nu}(t),\quad \nu_1(0)=\nu, \tag{297}$$

and $\beta:\mathbb{R}_t\times\mathbb{Z}_+\to\mathbb{C}$ has the property that it and $\mathrm{d}_t\beta$ are small in norm at $t=0$. If one finds that $u(t)\to e^{-i(-\mu_\infty t+\nu_\infty)}\alpha_{\mu_*}$ in norm as $t\nearrow\infty$ for some μ_∞, ν_∞, for all μ, ν and all sufficiently small β then one calls $\widehat{u}(t)$ *asymptotically stable*. The most important element of the proof of this analysis is the study of the spectral measure of the operator one obtains by linearizing around α_μ. One then considers the associated linearized NLS.

If $u=e^{-i\theta}(\alpha_{\mu_1}+\beta)$ satisfies the NLS then β satisfies the linearized NLS (LNLS)

$$i\mathrm{d}_t\vec{\beta}=H\vec{\beta}+\vec{\gamma}, \tag{298}$$

where

$$H:=(L_0+\mu)D-(\sigma+1)\alpha^{2\sigma}D-\sigma\alpha^{2\sigma}J, \tag{299}$$

$$D:=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad J:=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},\quad \vec{\beta}:=\begin{bmatrix}\beta\\ \overline{\beta}\end{bmatrix},\quad \vec{\gamma}:=\begin{bmatrix}\gamma\\ -\overline{\gamma}\end{bmatrix}, \tag{300}$$

$$\alpha_{\mu_1} = \alpha + \widehat{\alpha}, \quad \alpha_\mu = \alpha, \quad \gamma = \gamma_0 + \gamma_1, \quad \gamma_0 := -\mathrm{d}_t\nu_1\alpha_{\mu_1} + i\partial_{\mu_1}\alpha_{\mu_1}\mathrm{d}_t\mu_1, \tag{301}$$

$$\gamma_1 := \widehat{\mu}\beta - [(\sigma+1)\beta + \sigma\overline{\beta}]\sum_{j=1}^{2\sigma}\binom{2\sigma}{j}\widehat{\alpha}^j\alpha^{2\sigma-j} - \mathrm{d}_t\nu_1\beta \tag{302}$$

$$+ \sum_{(j,k)}{}' \binom{\sigma}{j}\overline{\beta}^j\alpha_{\mu_1}^{\sigma-j}\binom{\sigma+1}{k}\beta^k\alpha_{\mu_1}^{\sigma+1-k}, \tag{303}$$

where $\sum'_{(j,k)}$ sums over all $(j,k) \in \mathbb{Z}\times\mathbb{Z}$ for $0 \le j \le \sigma$ and $0 \le k \le \sigma+1$ with the exclusion of $(0,0)$, $(0,1)$ and $(1,0)$.

We will arrive at the properties of H by studying a sequence of simpler operators.

Definition 1.1.

$$H_0 := (L_0 + \mu)D, \quad H_1 := H_0 - q_1\Pi_0 D, \quad H_2 := H_1 - q_2\Pi_0 J, \tag{304}$$

$$\rho := \alpha(0), \quad q_1 := (\sigma+1)\rho^{2\sigma}, \quad q_2 := \sigma\rho^{2\sigma}, \tag{305}$$

$$L := L_0 - q_1\Pi_0, \quad U := H - H_2. \tag{306}$$

Although H_0 is self-adjoint, it is the case that H_2 and H are not. This property of H is typical of linearized operators and makes the analysis very difficult for most systems.

Theorem 5. *The spectrum of H_2 has the following properties.*

(1) $\sigma_{\mathrm{d}}(H_2) = \sigma_{\mathrm{p}}(H_2) = \{(-1)^j i(2\sigma)^{1/2}\mu^{-\sigma}[1+\mathcal{O}(\mu^{-1})]\}_{j=0}^1$.

(2) $\sigma_{\mathrm{e}}(H_2) = \sigma_{\mathrm{c}}(H_2) = \sigma_{\mathrm{ac}}(H_2) = (-\infty,-\mu]\cup[\mu,\infty)$.

Theorem 6. *The spectrum of H has the following properties.*

(1) $\sigma_{\mathrm{d}}(H) = \sigma_{\mathrm{p}}(H) = \{0\}$*, with multiplicity 2.*

(2) $\sigma_{\mathrm{e}}(H) = \sigma_{\mathrm{c}}(H) = \sigma_{\mathrm{ac}}(H) = (-\infty,-\mu]\cup[\mu,\infty)$.

Theorem 7. *For all $-3 \ge \tau \in \mathbb{R}$, $v \in \ell^1$, there exists a constant $1 < \kappa \in \mathbb{R}$ such that*

$$||W_{\kappa,\tau}e^{-itH_2}\Pi_{\mathrm{e}}^{H_2}W_{\kappa,\tau}v||_\infty = \mathcal{O}(t^{-1}\log^{-2}t), \quad t\nearrow\infty \tag{307}$$

THEOREM 8. *For all* $-3 \geq \tau \in \mathbb{R}$, $W_{\kappa,\tau}^{-1} v \in \ell^1$, *there exists a constant* $1 < \kappa \in \mathbb{R}$ *such that*

$$||W_{\kappa,\tau} e^{-itH} \Pi_{\mathrm{e}}^H W_{\kappa,\tau} v||_\infty = \mathcal{O}(t^{-1} \log^{-2} t), \quad t \nearrow \infty \tag{308}$$

One may extract from $E_1(z) := \int_1^\infty \mathrm{d}t\, e^{-zt} t^{-1}$ the well-known asymptotic expansion

$$E_1(z) := e^{-z} z^{-1} \sum_{k=1}^{n-1} \frac{k!}{(-z)^k} + \mathcal{O}(n! z^{-n}), \tag{309}$$

which is valid for large values of $\Re z$. This in turn gives

$$\psi_z(0) := z^{-1} \sum_{k=1}^{n-1} \frac{k!}{(-z)^k} + \mathcal{O}(n! z^{-n}), \tag{310}$$

which will be a crucial tool in our analysis.

2. Spectral Properties of H_2

Consider that A is an essentially self-adjoint operator on $\mathscr{H}$ and $B \in \mathcal{B}(\mathscr{H})$. If A and B commute strongly on their common domain then R_z^A commutes strongly with B on $\mathscr{H}$ for all $z \in \rho(A)$. Furthermore consider $H = AD$ where D is the diagonal matrix defined above, it is the cas that $R_z^H = \begin{bmatrix} R_z^A & 0 \\ 0 & -R_{-z}^A \end{bmatrix}$ since

$$H - z = \begin{bmatrix} A - z & 0 \\ 0 & -A - z \end{bmatrix} = \begin{bmatrix} A - z & 0 \\ 0 & -(A + z) \end{bmatrix} \tag{311}$$

$$\Rightarrow (H - z)^{-1} = \begin{bmatrix} (A - z)^{-1} & 0 \\ 0 & -(A + z)^{-1} \end{bmatrix}. \tag{312}$$

One may write $R_z^{H_1} = \begin{bmatrix} R_{z_1}^L & 0 \\ 0 & -R_{z_2}^L \end{bmatrix}$, where here $L = L_0 - q_1\Pi_0$ and where q_1, z_1, z_2 are defined as given above. Since H_2^2 and H^2 are self-adjoint, it follows that $\sigma(H_2), \sigma(H) \subseteq \mathbb{R} \cup i\mathbb{R}$, see e.g. [25].

The first part of our analysis will be dedicated to proving that the point spectrum of H_2 consists of a conjugate pair of complex eigenvalues which are very close to the origin for large μ.

Lemma 2.1. *The eigenvalues of H_2 are given by the roots of $h(z) := q_2^2 f_{z_1}^L f_{z_2}^L - 1$, where $z_1 := z - \mu$ and $z_2 := -z - \mu$.*

Proof. Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and take $v_1(0) \neq 0$.

$$H_2\vec{v} = z\vec{v} \tag{313}$$

$$(H_1 - q_2\Pi_0 J)\vec{v} = \tag{314}$$

$$\Rightarrow \quad \vec{v} = R_z^{H_1} q_2 \Pi_0 J \vec{v} \tag{315}$$

$$\Rightarrow \quad (\chi_0, \vec{v}) = (\chi_0, R_z^{H_1} q_2 \Pi_0 J \vec{v}) \tag{316}$$

$$\vec{v}(0) = q_2 f_z^{H_1} J \vec{v}(0) \tag{317}$$

$$\begin{bmatrix} v_1(0) \\ v_2(0) \end{bmatrix} = q_2 \begin{bmatrix} 0 & f_{z_1}^L \\ f_{z_2}^L & 0 \end{bmatrix} \begin{bmatrix} v_1(0) \\ v_2(0) \end{bmatrix} \tag{318}$$

$$\Rightarrow \quad v_1(0) = q_2^2 f_{z_1}^L f_{z_2}^L v_1(0) \tag{319}$$

$$\Rightarrow \quad 1 = q_2^2 f_{z_1}^L f_{z_2}^L. \tag{320}$$

□

Next we find some preliminary estimates which are asymptotic in $\mu \nearrow \infty$.

LEMMA 2.2. *Let* $\widehat{\epsilon} := (\psi_{-\mu}, (I - \Pi_0)\alpha^{2\sigma+1})$. *It is the case that* $\rho^{-1}\widehat{\epsilon} = \mu^{-(2\sigma+2)} + \mathcal{O}(a^{-(2\sigma+3)})$ *for all* $\sigma \in \mathbb{Z}_+$.

PROOF. One has that $(L_0 + \mu)\alpha = \alpha^{2\sigma+1} \Rightarrow \alpha = R_{-\mu}^{L_0}\alpha^{2\sigma+1} \Rightarrow 1 = \rho^{2\sigma} f_{-\mu} + \rho^{-1}\widehat{\epsilon}$. Let $\epsilon_x := \alpha(x)/\alpha(x-1)$, for all $0 < x \in \mathbb{Z}$. One may observe that

$$-\mu\alpha + \alpha^{2\sigma+1} = L_0\alpha \tag{321}$$

$$\Rightarrow \quad -\mu\alpha(x) + \alpha^{2\sigma+1}(x) = -(x+1)\alpha(x+1) \tag{322}$$

$$+ (2x+1)\alpha(x) - x\alpha(x-1) \tag{323}$$

$$\Rightarrow \quad x\epsilon_1 \dots \epsilon_{x-1} + (\rho\epsilon_1 \dots \epsilon_x)^{2\sigma+1} = -(x+1)\rho\epsilon_1 \dots \epsilon_{x+1} \tag{324}$$

$$+ (2x+1+\mu)\rho\epsilon_1 \dots \epsilon_x \tag{325}$$

$$\Rightarrow \quad x + (\rho\epsilon_1 \dots \epsilon_{x-1})^{2\sigma}\epsilon_x^{2\sigma+1} = -(x+1)\epsilon_x\epsilon_{x+1} + (2x+1+\mu)\epsilon_x \tag{326}$$

$$\Rightarrow \quad \epsilon_x = \left[\mu + 1 + 2x - (x+1)\epsilon_{x+1}\right]^{-1} \left[x + (\rho\epsilon_1 \dots \epsilon_{x-1})^{2\sigma}\epsilon_x^{2\sigma+1}\right] \tag{327}$$

The analogous equation for $x = 0$ is

$$\rho^{2\sigma} = \mu + 1 - \epsilon_1. \tag{328}$$

We recall that $\sum_{x=0}^{\infty}\psi_{-\mu}(x) = \mu^{-1}$. Therefore

$$\sum_{x=1}^{\infty}\psi_{-\mu}(x) = \sum_{x=0}^{\infty}\psi_{-\mu}(x) - \psi_{-\mu}(0) = \mu^{-1} - [\mu^{-1} - \mu^{-2} + \mathcal{O}(\mu^{-3})] \tag{329}$$

$$= \mu^{2} + \mathcal{O}(\mu^{-3}), \tag{330}$$

$$\rho^{-1}\widehat{\epsilon} = \rho^{-1}\sum_{x=1}^{\infty}\psi_{-\mu}(x)\alpha^{2\sigma+1}(x) < \rho^{-1}\sum_{x=1}^{\infty}\psi_{-\mu}(x)\alpha^{2\sigma+1}(1) \tag{331}$$

$$= \rho^{2\sigma}\epsilon_1^{2\sigma+1}\sum_{x=1}^{\infty}\psi_{-\mu}(x) < (\mu+1)[\mu^{-1} + \mathcal{O}(\mu^{-2})]^{2\sigma+1}[\mu^{-2} + \mathcal{O}(\mu^{-3})] \tag{332}$$

$$< \mu^{-(2\sigma+2)} + \mathcal{O}(\mu^{-(2\sigma+3)}). \tag{333}$$

Since the lowest order of this bound has a power which depends explicitly on σ and should hold for all $0 < \sigma \in \mathbb{Z}$ it follows that the full asymptotic expansion, as well as the error term, must consist of orders which are powers of μ^{-1} that depend explicitly on σ.

$$\psi_{-\mu}(1) = (1+\mu)\psi_{-\mu}(0) - 1 \tag{334}$$

$$= (1+\mu)[\mu^{-1} - \mu^{-2} + 2\mu^{-3} + \mathcal{O}(\mu^{-4})] - 1 = \mu^{-2} + \mathcal{O}(\mu^{-3}), \tag{335}$$

$$\rho^{-1}\widehat{\epsilon} = \rho^{-1}\sum_{x=1}^{\infty}\psi_{-\mu}(x)\alpha^{2\sigma+1}(x) \tag{336}$$

$$= \psi_{-\mu}(1)\rho^{2\sigma}\epsilon_1^{2\sigma+1} + \rho^{-1}\sum_{x=2}^{\infty}\psi_{-\mu}(x)\alpha^{2\sigma+1}(x) \tag{337}$$

$$= [\mu^{-2} + \mathcal{O}(\mu^{-3})][\mu + 1 + \mathcal{O}(\mu^{-1})][\mu^{-1} + \mathcal{O}(\mu^{-2})]^{2\sigma+1} \tag{338}$$

$$+ \rho^{-1}\sum_{x=2}^{\infty}\psi_{-\mu}(x)\alpha^{2\sigma+1}(x) \tag{339}$$

$$= \mu^{-(2\sigma+2)} + \mathcal{O}(\mu^{-(2\sigma+3)}) + \rho^{-1}\sum_{x=2}^{\infty}\psi_{-\mu}(x)\alpha^{2\sigma+1}(x) \tag{340}$$

$$\Rightarrow \quad \rho^{-1}\sum_{x=2}^{\infty}\psi_{-\mu}(x)\alpha^{2\sigma+1}(x) = \mathcal{O}(\mu^{-(2\sigma+3)}) \tag{341}$$

$$\Rightarrow \quad \rho^{-1}\widehat{\epsilon} = \mu^{-(2\sigma+2)} + \mathcal{O}(\mu^{-(2\sigma+3)}) \tag{342}$$

□

Now we show that there are no real eigenvalues through a series of lemmas.

LEMMA 2.3. *It is the case that* $h(0) = 2\sigma^{-1}\mu^{-(2\sigma+2)} + \mathcal{O}(\mu^{-(2\sigma+3)}) > 0$.

PROOF. One may observe that

$$\rho^{2\sigma}f_{-\mu} = 1 - \mu^{-(2\sigma+2)} + \mathcal{O}(\mu^{-(2\sigma+3)}) \tag{343}$$

$$f_{-\mu}^{-1} = \mu + 1 - \mu^{-1} + 3\mu^{-2} - 16\mu^{-3} + \mathcal{O}(\mu^{-4}) \tag{344}$$

$$\rho^{2\sigma} = \mu + 1 - \mu^{-1} + 3\mu^{-2} - 16\mu^{-3} + \mathcal{O}(\mu^{-4}) \tag{345}$$

$$- \mu^{-(2\sigma+1)} - \mu^{-(2\sigma+2)} + \mathcal{O}(\mu^{-(2\sigma+3)}) \tag{346}$$

$$\rho^{2\sigma}f_{-2\mu} = 2^{-1}[1 + 2^{-1}\mu^{-1} - \mu^{-2} + \mathcal{O}(\mu^{-3})] \tag{347}$$

$$q_1 f_{-2\mu} - 1 = 2^{-1}(\sigma - 1) + 4^{-1}(\sigma + 1)\mu^{-1} - 4^{-1}(\sigma + 1)\mu^{-2} + \mathcal{O}(\mu^{-3}) \tag{348}$$

$$h(0) = [(q_1 f_{-\mu} - 1)^{-1}q_2 f_{-\mu}]^2 - 1 \tag{349}$$

$$= \{\sigma^{-1}[1 - (1 + \sigma^{-1})\mu^{-(2\sigma+2)} + \mathcal{O}(\mu^{-(2\sigma+3)})]^{-1}\sigma\rho^{2\sigma}f_{-\mu}\}^2 - 1 \tag{350}$$

$$= \{[1 + (1 + \sigma^{-1})\mu^{-(2\sigma+2)} + \mathcal{O}(\mu^{-(2\sigma+3)})] \tag{351}$$

$$\times [1 - \mu^{-(2\sigma+2)} + \mathcal{O}(\mu^{-(2\sigma+3)})]\}^2 - 1 \tag{352}$$

$$= [1 + \sigma^{-1}\mu^{-(2\sigma+2)} + \mathcal{O}(\mu^{-(2\sigma+3)})]^2 - 1 \tag{353}$$

$$= 2\sigma^{-1}\mu^{-(2\sigma+2)} + \mathcal{O}(\mu^{-(2\sigma+3)}) > 0. \tag{354}$$

□

Lemma 2.4. *For $z = a \in [-\mu, \mu]$, we define $a_1 := a - \mu$, $a_2 := -a - \mu$. One has for $\sigma = 1$*

$$\lim_{a \nearrow \mu} h(a) = 2^{-1}\mu - 4^{-1} + \mathcal{O}(\mu^{-1}) > 0 \tag{355}$$

and for $1 < \sigma \in \mathbb{Z}_+$

$$\lim_{a \nearrow \mu} h(a) = (\sigma^2 - 1)^{-1} + (\sigma + 1)^{-1}(\sigma - 1)^{-2}\sigma^2\mu^{-1} \tag{356}$$

$$+ 4^{-1}(\sigma + 1)^{-1}(\sigma - 1)^{-3}(3\sigma^2 + 10\sigma - 9)\mu^{-2} + \mathcal{O}(\mu^{-3}) > 0. \tag{357}$$

Proof. Let $s_1 := -a_1/a$, $s_1 \in [0, 2]$. For $a_1 \gg 1$ one has

$$(\sigma + 1)^{-1}s_1(q_1 f_{a_1} - 1) = [1 - (\sigma + 1)^{-1}s_1] + (1 - s_1^{-1})\mu^{-1} \tag{358}$$

$$- (1 + s_1^{-1} - 2s_1^{-2})\mu^{-2} + (16 + 3s_1^{-1} + 2s_1^{-2} + 6s_1^{-3} - 24s_1^{-4})\mu^{-4} \tag{359}$$

$$+ \mathcal{O}(\mu^{-5}) - \mu^{-(2\sigma+2)} - (1 - s_1^{-1})\mu^{-(2\sigma+3)} + \mathcal{O}(\mu^{-(2\sigma+4)}). \tag{360}$$

We recall that the generalized exponential integrals have the representation

$$E_{n+1}(z) = -\frac{(-z)^n}{n!}\log z + \frac{e^{-z}}{n!}\sum_{k=1}^{n}(-z)^{k-1}(n-k)! \tag{361}$$

$$+ \frac{e^{-z}(-z)^n}{n!}\sum_{k=0}^{\infty}\frac{z^k}{k!}\digamma(k+1). \tag{362}$$

This permits one to write

$$f_{-a} = e^a E_1(a) = -e^a \log a + \sum_{k=0}^{\infty}\frac{a^k}{k!}\digamma(k+1) \tag{363}$$

and hereby one may observe that near $a = 0$ the above expression for f_{-a} is dominated by logarithmic behavior. Therefore $\lim_{a \nearrow \mu}(-f_{a_1}^L) = \lim_{a \nearrow \mu}(q_1 f_{a_1} - 1)^{-1} f_{a_1} = q_1^{-1}$.

$$\lim_{a \nearrow \mu} h(a) = \lim_{a \nearrow \mu} (q_1 f_{z_1} - 1)^{-1} q_2 f_{z_1} (q_1 f_{z_2} - 1)^{-1} q_2 f_{z_2} - 1 \tag{364}$$

$$= q_1^{-1} q_2^2 (q_1 f_{-2\mu} - 1)^{-1} f_{-2\mu} - 1 \tag{365}$$

$$= (\sigma + 1)^{-1} \sigma^2 [(\sigma + 1)\rho^{2\sigma} f_{-2\mu} - 1]^{-1} \rho^{2\sigma} f_{-2\mu} - 1 \tag{366}$$

$$= 2^{-1} (\sigma + 1)^{-1} \sigma^2 [2^{-1}(\sigma - 1) + 4^{-1}(\sigma + 1)\mu^{-1} \tag{367}$$

$$- 4^{-1}(\sigma + 1)\mu^{-2} + \mathcal{O}(\mu^{-3})]^{-1} [1 + 2^{-1}\mu^{-1} - \mu^{-2} + \mathcal{O}(\mu^{-3})] \tag{368}$$

<u>For $\sigma = 1$</u>:

(369)

$$\lim_{a \nearrow \mu} h(a) = 4^{-1} [2^{-1}\mu^{-1} - 2^{-1}\mu^{-2} + \mathcal{O}(\mu^{-3})]^{-1} [1 + 2^{-1}\mu^{-1} - \mu^{-2} + \mathcal{O}(\mu^{-3})] - 1$$

$$= 2^{-1}\mu [1 - \mu^{-1} + \mathcal{O}(\mu^{-2})]^{-1} [1 + 2^{-1}\mu^{-1} - \mu^{-2} + \mathcal{O}(\mu^{-3})] - 1 \tag{370}$$

$$= 2^{-1}\mu [1 + \mu^{-1} + \mathcal{O}(\mu^{-2})] [1 + 2^{-1}\mu^{-1} - \mu^{-2} + \mathcal{O}(\mu^{-3})] - 1 \tag{371}$$

$$= 2^{-1}\mu - 4^{-1} + \mathcal{O}(\mu^{-1}) > 0 \tag{372}$$

<u>For $\sigma > 1$</u>:

(373) $$\lim_{a\nearrow\mu} h(a) = (\sigma^2-1)^{-1}\sigma^2\{1 - 2^{-1}(\sigma-1)^{-1}(\sigma+1)[-\mu^{-1}+\mu^{-2}+\mathcal{O}(\mu^{-3})]\}^{-1}$$

(374) $$\times [1 + 2^{-1}\mu^{-1} - \mu^{-2} + \mathcal{O}(\mu^{-3})] - 1$$

(375) $$= (\sigma^2-1)^{-1}\sigma^2\{1 + 2^{-1}(\sigma-1)^{-1}(\sigma+1)[-\mu^{-1}+\mu^{-2}+\mathcal{O}(\mu^{-3})$$

(376) $$+ [2^{-1}(\sigma-1)^{-1}(\sigma+1)]^2[-\mu^{-1}+\mathcal{O}(\mu^{-2})]^2 + \mathcal{O}(\mu^{-3})\}$$

(377) $$\times [1 + 2^{-1}\mu^{-1} - \mu^{-2} + \mathcal{O}(\mu^{-3})] - 1$$

(378) $$= (\sigma^2-1)^{-1}\sigma^2[1 - 2^{-1}(\sigma-1)^{-1}(\sigma+1)\mu^{-1}$$

(379) $$+ 4^{-1}(\sigma-1)^2(\sigma+1)(3\sigma-1)\mu^{-2} + \mathcal{O}(\mu^{-3})]$$

(380) $$\times [1 + 2^{-1}\mu^{-1} - \mu^{-2} + \mathcal{O}(\mu^{-3})] - 1$$

(381) $$= (\sigma^2-1)^{-1}\sigma^2[1 - (\sigma-1)^{-1}\mu^{-1}$$

(382) $$+ 4^{-1}(\sigma-1)^2(3\sigma^2 10\sigma - 9)\mu^{-2} + \mathcal{O}(\mu^{-3})] - 1$$

(383) $$= (\sigma^2-1)^{-1} - (\sigma+1)^{-1}(\sigma-1)^{-2}\sigma^2\mu^{-1}$$

(384) $$+ 4^{-1}(\sigma+1)^{-1}(\sigma-1)^3(3\sigma^2+10\sigma-9)\mu^{-2} + \mathcal{O}(\mu^{-3}) > 0$$

$\square$

LEMMA 2.5. *Let $h_0(z) := (q_1 f_{z_1} - 1)^{-1}(q_1 f_{z_2} - 1)^{-1}(2\sigma+1)\rho^{4\sigma}$. It is the case that $h_0(z) > 0$ for all $z = a \in [-\mu, \mu]$.*

PROOF. Let $c_0 := (2\sigma+1)^{-1}\rho^{-4\sigma}$, $c_1 := (2\sigma+1)^{-1}(\sigma+1)\rho^{-2\sigma}$, $c_2 := c_1^2 - c_0 = (2\sigma+1)^{-2}\rho^{-4\sigma}\sigma^2$. One then has $h(z) = h_0(z)[c_2 - (f_{z_1} - c_1)(f_{z_2} - c_1)]$. For all $\sigma \in \mathbb{Z}_+$ and $a \in [-\mu, \mu]$ it is the case that $q_1 f_{a_1} = q_1 f_{a-\mu}$ and $q_1 f_{a_2} = q_1 f_{-a-\mu}$ vary monotonically between

(385) $$q_1 f_{-2\mu} = 2^{-1}(\sigma+1) + 4^{-1}(\sigma+1)\mu^{-1} - 4^{-1}(\sigma+1)\mu^{-2} + \mathcal{O}(\mu^{-3}) > 1$$

and

$$\lim_{a \nearrow 0} q_1 f_{-a} = \infty > 1. \tag{386}$$

Therefore $(q_1 f_{a_i} - 1)^{-1} > 0$ which concludes the proof. $\square$

LEMMA 2.6. *For* $1 \ll (2\sigma + 1)^{-1}\mu = \mathcal{O}(\mu)$ *it is the case that* $(f_{a_j} - c_1)$, $j = 1, 2$, *have unique roots,* $a = r_j := (-1)^j(\sigma + 1)^{-1}(\mu + \sigma) + \mathcal{O}(\mu^{-1})$.

PROOF. Consider $c_1 = f_{a_j}$. Let $a_1 = -s_1\mu$, $s_1 \in [0, 2]$, and assume that $1 \ll s_1\mu = \mathcal{O}(\mu)$ so that the asymptotic expansion of f_{a_1} is valid.

$$c_1 = f_{a_1} = (s_1\mu)^{-1} - (s_1\mu)^{-2} + \mathcal{O}(\mu^{-3}) \tag{387}$$

$$= (s_1\mu)^{-1}[1 - (s_1\mu)^{-1} + \mathcal{O}(\mu^{-2})] \tag{388}$$

$$\Rightarrow \quad s_1 = \mu^{-1}c_1^{-1}[1 - (s_1\mu)^{-1} + \mathcal{O}(\mu^{-2})] \tag{389}$$

$$= (\sigma + 1)^{-1}(2\sigma + 1)[1 + \mu^{-1} + \mathcal{O}(\mu^{-2})] \tag{390}$$

$$\times [1 - (s_1\mu)^{-1} + \mathcal{O}(\mu^{-2})] \tag{391}$$

$$= (\sigma + 1)^{-1}(2\sigma + 1)[1 + (1 - s^{-1})\mu^{-1} + \mathcal{O}(\mu^{-2})] \tag{392}$$

$$\Rightarrow \quad s_1 = (\sigma + 1)^{-1}(2\sigma + 1) \tag{393}$$

$$\times \{1 + [1 - (2\sigma + 1)^{-1}(\sigma + 1)]\mu^{-1} + \mathcal{O}(\mu^{-2})\} \tag{394}$$

$$= (\sigma + 1)^{-1}(2\sigma + 1) + (\sigma + 1)^{-1}\sigma\mu^{-1} + \mathcal{O}(\mu^{-2}). \tag{395}$$

This result satisfies the assumptions and the root must be unique, therefore the above value of s_1 specifies the unique root. In terms of a one has

$$a_1 = a - \mu = -s_1\mu \tag{396}$$

$$\Rightarrow \quad a = (1 - s_1)\mu = \{1 - [(\sigma + 1)^{-1}(2\sigma + 1) + (\sigma + 1)^{-1}\sigma\mu^{-1} + \mathcal{O}(\mu^{-2})]\}\mu \tag{397}$$

$$= -(\sigma + 1)^{-1}(\mu + \sigma) + \mathcal{O}(\mu^{-1}). \tag{398}$$

Due to symmetry between the a_j there are two roots $a = r_j = (-1)^j(\sigma+1)^{-1}(\mu+\sigma) + \mathcal{O}(\mu^{-1})$. □

LEMMA 2.7. *For $1 \ll r_j = \mathcal{O}(\mu)$ one has that $h(z) > 0$ for all $z = a \in [-\mu, \mu]$.*

PROOF. Let $h_1(z) := (f_{z_1} - c_1)(f_{z_2} - c_1)$. One can observe that $h_1(0) = (f_{-\mu} - c_1)^2 > 0$ and $\lim_{a \nearrow \mu}(f_{a_1} - c_1) = \infty$. Then since $(f_{a_j} - c_1)$ are monotonic in a and have unique roots r_i, it is the case that $h(a) > 0$ for $a \in (r_1, r_2)$ and $h(a) < 0$ for $a \in [-\mu, \mu] \setminus (r_1, r_2)$. We have shown that $h(0) > 0$ and $h(\mu) = h(-\mu) > 0$. Since $h(a) > 0$ for $a \in (r_1, r_2)$ and $h(a) < 0$ for $a \in [-\mu, \mu] \setminus (r_1, r_2)$ it must be the case that $h(a) > 0$ for $a \in [-\mu, \mu] \setminus (r_1, r_2)$. It remains to consider $a \in (r_1, r_2)$. By assumption $1 \ll a = \mathcal{O}(\mu)$ and therefore the asymptotic expansion of the f_{a_j} is valid.

$$h_1(a) = (f_{a_1} - c_1)(f_{a_2} - c_1) = f_{a_1} f_{a_2} - c_1(f_{a_1} + f_{a_2}) + c_1^2 \tag{399}$$

$$= [(\mu - a)^{-1} + \mathcal{O}(\mu^{-2})][(\mu + a)^{-1} + \mathcal{O}(\mu^{-2})] \tag{400}$$

$$- c_1\{[(\mu - a)^{-1} + \mathcal{O}(\mu^{-2})] + [(\mu + a)^{-1} + \mathcal{O}(\mu^{-2})]\} + c_1^2 \tag{401}$$

$$= (\mu^2 - a^2)^{-1} - (\mu^2 - a^2)^{-1}(2\sigma + 1)^{-1}(\sigma + 1) \tag{402}$$

$$\times [1 + \mathcal{O}(\mu^{-1})][2 + \mathcal{O}(\mu^{-1})] + c_1^2 + \mathcal{O}(\mu^{-3}) \tag{403}$$

$$= -(\mu^2 - a^2)^{-1}(2\sigma + 1)^{-1} + c_1^2 + \mathcal{O}(\mu^{-3}). \tag{404}$$

Therefore $-h(a)$ decays monotonically as $|a| \nearrow r_2$ for sufficiently large μ. This guarantees that $h(a) > 0$ for all $a \in (r_1, r_2)$. □

LEMMA 2.8. *It is the case that $h(z)$ has no roots for $z \in (-\infty, -\mu] \cup [\mu, \infty)$.*

PROOF. This follows by the same principle which permits L from having embedded eigenvalues. □

Now we prove the existence and location (asymptotically) of the imaginary roots.

LEMMA 2.9. *$h(z)$ has exactly two roots, $\lambda_\pm = z_\pm := \pm i(2\sigma)^{1/2}\mu^{-\sigma}[1 + \mathcal{O}(\mu^{-1})]$, for $z \in i\mathbb{R}$.*

Proof. One may observe that

$$f_z = (-z)^{-1} - (-z)^{-2} + 2(-z)^{-3} - 6(-z)^{-4} + \mathcal{O}(z^{-5}), \tag{405}$$

$$f_z^L = (1 - q_1 f_z)^{-1} f_z, \quad \partial_z f_z^L = (1 - q_1 f_z)^{-2} \partial_z f_z \tag{406}$$

$$\partial_z f_z = (-1)(f_z + z^{-1}), \quad \partial_z^2 f_z = 2q_1(1 - q_1 f_z)^{-3}(\partial_z f_z)^2 + (1 - q_1 f_z)^{-2} \partial_z^2 f_z, \tag{407}$$

$$f'_{-\mu} = \mu^{-2} - 2\mu^{-3} + 6\mu^{-4} + \mathcal{O}(\mu^{-5}), \quad f''_{-\mu} = 2\mu^{-3} + 6\mu^{-4} + \mathcal{O}(\mu^{-5}) \tag{408}$$

and that for $z \in i\mathbb{R}$ one has that $h(z) = |q_2 f^L_{\mu - z}|^2 - 1 \in \mathbb{R}$. Furthermore, for all $z \in i\mathbb{R}$ the asymptotic expansion of f_{z_j} is valid since $\Re z_j = a \gg 1$.

First consider $|z| \ll 1$. One finds

$$\partial_z^2 h(z) = q_2^2(\partial_z^2 f_{z_1}^L f_{z_2}^L + 2\partial_z f_{z_1}^L \partial_z f_{z_2}^L + f_{z_1}^L \partial_z^2 f_{z_2}^L) \tag{409}$$

$$= q_2^2(\partial_{z_1}^2 f_{z_1}^L f_{z_2}^L - 2\partial_{z_1} f_{z_1}^L \partial_{z_2} f_{z_2}^L + f_{z_1}^L \partial_{z_2}^2 f_{z_2}^L) \tag{410}$$

$$= q_2^2\{[2q_1(1 - q_1 f_{z_1})^{-3}(\partial_{z_1} f_{z_1})^2 + (1 - q_1 f_{z_1})^{-2} \partial_{z_1}^2 f_{z_1}]\}(1 - q_1 f_{z_2})^{-1} f_{z_2} \tag{411}$$

$$- 2(1 - q_1 f_{z_1})^{-2} \partial_{z_1} f_{z_1} (1 - q_1 f_{z_2})^{-2} \partial_{z_2} f_{z_2} \tag{412}$$

$$+ (1 - q_1 f_{z_1})^{-1} f_{z_1} [2q_1(1 - q_1 f_{z_2})^{-3}(\partial_{z_2} f_{z_2})^2 + (1 - q_1 f_{z_2})^{-2} \partial_{z_2}^2 f_{z_2}]\} \tag{413}$$

$$h''(0) = q_2^2\{[2q_1(1 - q_1 f_{-\mu})^{-3}(f'_{-\mu})^2 + (1 - q_1 f_{-\mu})^{-2} f''_{-\mu}]\}(1 - q_1 f_{-\mu})^{-1} f_{-\mu} \tag{414}$$

$$- 2(1 - q_1 f_{-\mu})^{-2} f'_{-\mu} (1 - q_1 f_{-\mu})^{-2} f'_{-\mu} \tag{415}$$

$$+ (1 - q_1 f_{-\mu})^{-1} f_{-\mu} [2q_1(1 - q_1 f_{-\mu})^{-3}(f'_{-\mu})^2 + (1 - q_1 f_{-\mu})^{-2} f''_{-\mu}]\} \tag{416}$$

$$(417) \qquad = q_2^2[2(q_1 f_{-\mu} - 1)^{-4}(f'_{-\mu})^2(2q_1 f_{-\mu} - 1) - 2(q_1 f_{-\mu} - 1)^{-3} f''_{-\mu} f_{-\mu}]$$

$$(418) \qquad = q_2^2(2\{\sigma[1 + (1 + \sigma^{-1})\mu^{-(2\sigma+2)} + \mathcal{O}(\mu^{-(2\sigma+3)})]\}^{-4}$$

$$(419) \qquad \times [\mu^{-2} - 2\mu^{-3} + 6\mu^{-4} + \mathcal{O}(\mu^{-6})]^2$$

$$(420) \qquad \times \{2[(\sigma + 1) - (\sigma + 1)\mu^{-(2\sigma+2)} + \mathcal{O}(\mu^{-(2\sigma+3)})] - 1\}$$

$$(421) \qquad - 2\{\sigma[1 + (1 + \sigma^{-1})\mu^{-(2\sigma+2)} + \mathcal{O}(\mu^{-(2\sigma+3)})]\}^{-3}$$

$$(422) \qquad \times [2\mu^{-3} - 6\mu^{-4} + \mathcal{O}(\mu^{-5})]$$

$$(423) \qquad \times [\mu^{-1} - \mu^{-2} + 2\mu^{-3} - 6\mu^{-4} + \mathcal{O}(\mu^{-5})])$$

$$(424) \qquad = 2\sigma^{-2}\mu^{-2}[1 - 2\mu^{-1} + \mathcal{O}(\mu^{-2})].$$

Assume that $h(z) = h(0) + 2^{-1}h''(0)z^2 + \epsilon$ where $|\epsilon| \le \mathcal{O}(\mu^{-4})$. One finds

$$(425) \quad h(z) = 0$$

$$(426)$$

$$\Rightarrow z^2 = -2[h(0) + \epsilon][h''(0)]^{-1}$$

$$(427) \qquad = -(2)[2\sigma^{-1}\mu^{-(2\sigma+2)} + \mathcal{O}(\mu^{-(2\sigma+3)}) + \epsilon](2^{-1}\sigma^2\mu^2)[1 - 2\mu^{-1} + \mathcal{O}(\mu^{-2})]^{-1}$$

$$(428) \qquad = -2\sigma\mu^{2\sigma}[1 + \mathcal{O}(\mu^{-1}) + \epsilon].$$

This result is compatible with the assumptions, thus there are at least the two imaginary roots given by $z_\pm = \pm i(2\sigma)^{1/2}\mu^{-\sigma}[1 + \mathcal{O}(\mu^{-1})]$.

It remains to be shown that there are no other imaginary roots. It is sufficient to prove that $h(z)$ has nonpositive curvature for all $z \in i\mathbb{R}$. Let $z_j = (-1)^{j+1}ib - \mu$, where $b \in \mathbb{R}$.

$$(429) \quad h(z) = q_2^2(q_1 - f_{z_1}^{-1})(q_1 - f_{z_2}^{-1}) - 1$$

$$(430) \qquad = q_2^2[(\sigma + 1)\mu - (\mu - ib) + \mathcal{O}(1)]^{-1}[(\sigma + 1)\mu - (\mu + ib) + \mathcal{O}(1)]^{-1} - 1$$

$$(431) \qquad = q_2^2(\sigma^2\mu^2 + b^2)^{-1}[1 + \mathcal{O}(\mu^{-1})] - 1.$$

Therefore $h(ib)$ decays monotonically as $|b| \nearrow \infty$ and there are only two imaginary roots. $\square$

Proof of Theorem (5) Part (1). We have exhaustively shown that

$$\lambda_{\pm} := \pm i(2\sigma)^{1/2}\mu^{-\sigma}[1 + \mathcal{O}(\mu^{-1})] \tag{432}$$

are the only roots of $h(z)$ for $z \in \mathbb{R} \cup i\mathbb{R}$. The absence of embedded eigenvalues follows from arguments similar to those for $\sigma(L)$. $\square$

Proof of Theorem (5) Part (2). By Weyl's critereon it is the case that $\sigma_{\mathrm{e}}(H_2) = \sigma_{\mathrm{e}}(H_0) = (-\infty, -\mu] \cup [\mu, \infty)$. It is clear that there exists a well-defined absolutely continuous spectral measure on $\sigma_{\mathrm{e}}(H_2)$. The representation of $H_2 v = zv$ as a coupled series of algebraic equations guarantees that each $\lambda \in \sigma_{\mathrm{e}}(H_2)$ has multiplicity 1. Therefore one must have that $\sigma_{\mathrm{e}}(H_2) = \sigma_{\mathrm{ac}}(H_0)$. $\square$

3. Spectral Properties of H

We recall Proposition (2) from Chapter 2:

$$||(I-\Pi_0)\alpha_\mu||_1 \leq \mu^{-(2\sigma)^{-1}(2\sigma-1)} + \mathcal{O}(\mu^{-(2\sigma)^{-1}(4\sigma-1)}). \tag{433}$$

This gives

$$||U|| \leq 2(2\sigma+1)||(I-\Pi_0)\alpha^{2\sigma}||_1 \leq 2(2\sigma+1)||(I-\Pi_0)\alpha||_1 \tag{434}$$

$$\leq 2(2\sigma+1)\mu^{-(2\sigma)^{-1}(2\sigma-1)} + \mathcal{O}(\mu^{-(2\sigma)^{-1}(4\sigma-1)}) =: m(\mu). \tag{435}$$

We recall without proof a statement of Kato [**16**] regarding norm resolvent convergence.

Proposition. *For A a closed operator and $\{A_n\}_{n=0}^\infty$ a sequence of closed operators, if $R_z^{A_n}$ converges in norm to R_z^A for some $z \in \rho(A)$ then the convergence holds for every $z \in \rho(A)$.*

Proof of Theorem (6) Part (1). The discrete spectrum of H can be at most $||U|| \leq m(\mu)$ away from that of H_2. We therefore only need to consider the shift of the eigenvalues of H_2 and possible production of eigenvalues from the thresholds of H_2.

Consider the eigenvalues near the origin. By standard arguments, see e.g. [**25**], the kernel of H is spanned by linear combinations of matrix vectors composed the set $\{T_j\alpha\}_{j=1}^n$ where $\{T_j\}_{j=1}^n$ is the set of generators of symmetries of the soliton manifold, in which α lies. In our case there are only two symmetries: phase rotation and energy translation. The kernel is then spanned by matrix linear combinations of α and $\partial_\mu\alpha$ and thereby there exists an eigenvalue of multiplicity 2 at the origin. These eigenvalues must be result of the shift of the eigenvalues of H_2 to the origin.

Now consider the possibility of eigenvalues near the threshold. Consider that by the resolvent identity, one has $R_z^H - R_z^{H_2} = R_z^{H_2} U R_z^H$. Without loss of generality, let z be chosen so that $||R_z^{H_2}|| \leq |z+\epsilon(\mu)|^{-1}$ and $|z+\epsilon(\mu)|^{-1}||U|| < 1$ for some

$\epsilon(\mu) = \mathcal{O}(\mu^{-\sigma})$, i.e. $\epsilon(\mu)$ is due to the presence of the eigenvalues of H_2. Then

$$\lim_{\mu \nearrow \infty} ||R_z^{H_2}|| \leq \lim_{\mu \nearrow \infty} |z + \epsilon(\mu)|^{-1} = |z|^{-1} \tag{436}$$

and

$$||R_z^H|| = ||(1 - R_z^{H_2}U)^{-1} R_z^{H_2}|| \leq (1 - ||R_z^{H_2}||\ ||U||)^{-1} ||R_z^{H_2}|| \tag{437}$$

$$\leq [1 - |z + \epsilon(\mu)|^{-1} m(\mu)]^{-1} |z + \epsilon(\mu)|^{-1} \tag{438}$$

$$\lim_{\mu \nearrow \infty} ||R_z^H|| \leq \lim_{\mu \nearrow \infty} [1 - |z + \epsilon(\mu)|^{-1} m(\mu)]^{-1} |z + \epsilon(\mu)|^{-1} = |z|^{-1}. \tag{439}$$

One may then find

$$\lim_{\mu \nearrow \infty} ||R_z^H - R_z^{H_2}|| = \lim_{\mu \nearrow \infty} ||R_z^{H_2} U R_z^H|| \leq \lim_{\mu \nearrow \infty} ||R_z^{H_2}||\ ||U||\ ||R_z^H|| \tag{440}$$

$$\leq \lim_{\mu \nearrow \infty} |z|^{-2} m(\mu) = 0. \tag{441}$$

It is the case that A is closed if R_z^A exists and is bounded for at least one $z \in \mathbb{C}$. This is clearly the case for both H_2, H. Therefore by the principle of norm resolvent convergence it is the case that $(-\mu, 0) \cup (0, \mu) \subset \rho(H)$. □

Proof of Theorem (6) Part (2). By Weyl's critereon $\sigma_{\mathrm{e}}(H) = \sigma_{\mathrm{e}}(H_2)$. One may explicitly construct an absolutely continuous spectral measure by expanding $R_z^H = (1 - R_z^{H_2}U)^{-1} R_z^{H_2}$ as a convergent series in U, taking a limit $z \to \lambda \in \sigma_{\mathrm{e}}(H_2)$, and collecting the imaginary terms. The representation of $Hv = zv$ as a coupled series of algebraic equations guarantees that each $\lambda \in \sigma_{\mathrm{e}}(H)$ has multiplicity 1. Therefore one must have that $\sigma_{\mathrm{e}}(H) = \sigma_{\mathrm{ac}}(H_2)$. □

4. Decay Estimates for H_2 and H

We recall without proof Lemma 3.12 from [21]:

LEMMA. *Let $\mathscr{B}$ be a Banach space and $\lambda_+ > \lambda_-$ be real constants. If $F(\lambda)$ has the properties*

(1) $F \in C(\lambda_-, \lambda_+; \mathscr{B})$

(2) $F(\lambda_-) = F(\lambda) = 0, \quad \lambda > \lambda_+$

(3) $\mathrm{d}_\lambda F \in L^1(\lambda_- + \delta, \lambda_+; \mathscr{B}), \quad \forall \delta > 0$

(4) $\mathrm{d}_\lambda F(\lambda) = \mathcal{O}([\lambda - \lambda_-]^{-1} \log^{-3}[\lambda - \lambda_-]), \quad \lambda \searrow \lambda_-$

(5) $\mathrm{d}_\lambda^2 F(\lambda) = \mathcal{O}([\lambda - \lambda_-]^{-2} \log^{-2}[\lambda - \lambda_-]), \quad \lambda \searrow \lambda_-$

then

$$\int_{\lambda_-}^{\infty} \mathrm{d}\lambda \; e^{-it\lambda} F(\lambda) = \mathcal{O}(t^{-1} \log^{-2} t), \quad t \nearrow \infty \tag{442}$$

in the norm of $\mathscr{B}$.

We will verify that $F(\lambda) = \delta_\lambda^{H_2}$ satisfies the desired properties for both $\lambda \geq \mu$ and $\lambda \leq \mu$.

PROOF OF THEOREM (7). Let $\mathscr{B} = \left\{ A \in \mathcal{L}(\vec{\mathscr{T}}) : ||A||_{\mathscr{B}} < \infty \right\}$ be the Banach space complete in the norm

$$||A||_{\mathscr{B}} := \sup_{v \in \ell^1} \frac{||W_{\kappa,\tau} A W_{\kappa,\tau} v||_1}{||v||_1}, \tag{443}$$

where $\vec{\mathscr{T}}$ is the natural extension of $\mathscr{T}$ to the matrix system. Let $F(\lambda) = \delta_\lambda^{H_2}$. We will verify the appropriate properties of $F(\lambda)$ for $\lambda_- = 0$ and $\lambda_+ = \infty$.

Let $X_1 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $X_2 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $g_\lambda := \left[(1 - q_1 \mathcal{P} f_\lambda)^2 + (q_1 \pi w_\lambda)^2 \right]^{-1}$, and

$$\widehat{g}_{1,\lambda} := \left[\left(1 - q_2^2 \mathcal{P} f_{\lambda_1}^L f_{\lambda_2}^L \right)^2 + \left(\pi w_{\lambda_1}^L q_2^2 f_{\lambda_2}^L \right)^2 \right]^{-1}, \tag{444}$$

$$\widehat{g}_{2,\lambda} := \left[\left(1 - q_2^2 f_{\lambda_1}^L \mathcal{P} f_{\lambda_2}^L \right)^2 + \left(\pi w_{\lambda_2}^L q_2^2 f_{\lambda_1}^L \right)^2 \right]^{-1}. \tag{445}$$

Since $\psi_z^L = (1-q_1 f_z)^{-1}\psi_z$, $f_z^L = (1-q_1 f_z)^{-1} f_z$, and $\psi_z = f_z\phi_z + \xi_z$, for $\lambda \geq 0$ one has

$$\mathcal{P}\psi_\lambda^L = g_\lambda\left[\mathcal{P}f_\lambda\phi_\lambda - q_1\mathcal{P}f_\lambda\xi_\lambda + \xi_\lambda - q_1\left(\mathcal{P}f_\lambda\right)^2\phi_\lambda - q_1(\pi w_\lambda)^2\phi_\lambda\right], \tag{446}$$

$$\mathcal{P}f_\lambda^L = g_\lambda\left[\mathcal{P}f_\lambda - q_1\left(\mathcal{P}f_\lambda\right)^2 - q_1(\pi w_\lambda)^2\right], \tag{447}$$

$$\phi_\lambda^L = \phi_\lambda + q_1\xi_\lambda, \quad w_\lambda^L = g_\lambda w_\lambda, \tag{448}$$

and for $\lambda < 0$ one has

$$\psi^L = (1-q_1 f_\lambda)^{-1}\psi_\lambda, \quad f^L = (1-q_1 f_\lambda)^{-1} f_\lambda. \tag{449}$$

By the method of spectral shifts one has

$$R_z^{H_2} = R_z^{H_1} + R_z^{H_1} q_2 \Pi_0 J R_z^{H_2} \tag{450}$$

$$= R_z^{H_1} + R_z^{H_1} q_2 \Pi_0 J\left(1 - f_z^{H_1} q_2 J\right)^{-1} R_z^{H_1} \tag{451}$$

$$= R_z^{H_1} + R_z^{H_1} q_2 \Pi_0 J\left(1 - q_2^2 f_{z_1}^L f_{z_2}^L\right)^{-1}\left(1 + f_z^{H_1} q_2 J\right) R_z^{H_1} \tag{452}$$

$$= R_z^{H_1} + q_2\left|1 - q_2^2 f_{z_1}^L f_{z_2}^L\right|^{-2}\left(1 - q_2^2 \overline{f}_{z_1}^L \overline{f}_{z_2}^L\right) R_z^{H_1}\left(J + q_2 J f_z^{H_1} J\right)\Pi_0 R_z^{H_1} \tag{453}$$

$$= \left(R_{z_1}^L X_1 - q_2 R_{z_2}^L X_2\right) + \left|1 - q_2^2 f_{z_1}^L f_{z_2}^L\right|^{-2}\left(1 - q_2^2 \overline{f}_{z_1}^L \overline{f}_{z_2}^L\right) \tag{454}$$

$$\times\left(R_{z_1}^L X_1 - R_{z_2}^L X_2\right)\left[J - q_2\left(f_{z_1}^L X_2 - f_{z_2}^L X_1\right)\right]\Pi_0\left(R_{z_1}^L X_1 - R_{z_2}^L X_2\right) \tag{455}$$

from which one finds for $\lambda \geq \mu$:

$$\lim_{\epsilon\searrow 0} R_{\lambda\pm i\epsilon}^{H_2} = \left[\left(\mathcal{P}_{\lambda_1}^L X_1 - R_{\lambda_2}^L X_2\right) \pm i\pi\delta_{\lambda_1}^L X_1\right] \tag{456}$$

$$+ q_2\widehat{g}_{1,\lambda}\left[\left(1 - q_2^2\mathcal{P}f_{\lambda_1}^L f_{\lambda_2}^L\right) \pm i\pi w_{\lambda_1}^L q_2^2 f_{\lambda_2}^L\right] \tag{457}$$

$$\times\left[\left(\mathcal{P}_{z_1}^L X_1 - R_{z_2}^L X_2\right) \pm i\pi\delta_{\lambda_1}^L X_1\right] \tag{458}$$

$$\times\left[\left(J - q_2\mathcal{P}f_{\lambda_1}^L X_2 + q_2 f_{\lambda_2}^L X_1\right) \mp i\pi w_{\lambda_1}^L q_2 X_2\right] \tag{459}$$

$$\times\,\Pi_0\left[\left(\mathcal{P}_{\lambda_1}^L X_1 - R_{\lambda_2}^L X_2\right) \pm i\pi\delta_{\lambda_1}^L X_1\right] \tag{460}$$

$$= \Big[\big(\mathcal{P}^L_{\lambda_1} X_1 - R^L_{\lambda_2} X_2\big) \pm i\pi w^L_{\lambda_1} \phi^L_{\lambda_1} \otimes \phi^{L,*}_{\lambda_1} X_1\Big] \tag{461}$$

$$+ q_2 \widehat{g}_{1,\lambda} \left[\left(1 - q_2^2 \mathcal{P} f^L_{\lambda_1} f^L_{\lambda_2}\right) \pm i\pi w^L_{\lambda_1} q_2^2 f^L_{\lambda_2}\right] \tag{462}$$

$$\times \left[\left(\mathcal{P}\psi^L_{\lambda_1} X_1 - \psi^L_{\lambda_2} X_2\right) \pm i\pi w^L_{\lambda_1} \phi^L_{\lambda_1} X_1\right] \tag{463}$$

$$\left[\left(J - q_2 \mathcal{P} f^L_{\lambda_1} X_2 + q_2 f^L_{\lambda_2} X_1\right) \mp i\pi w^L_{\lambda_1} q_2 X_2\right] \tag{464}$$

$$\otimes \left[\left(\mathcal{P}\psi^{L,*}_{\lambda_1} X_1 - \psi^{L,*}_{\lambda_2} X_2\right) \pm i\pi w^L_{\lambda_1} \phi^{L,*}_{\lambda_1} X_1\right] \tag{465}$$

and for $\lambda \leq \mu$:

$$\lim_{\epsilon \searrow 0} R^{H_2}_{\lambda \pm i\epsilon} = \left[\left(R^L_{\lambda_1} X_1 - \mathcal{P}^L_{\lambda_2} X_2\right) \mp i\pi \delta^L_{\lambda_2} X_2\right] \tag{466}$$

$$+ q_2 \widehat{g}_{2,\lambda} \left[\left(1 - q_2^2 f^L_{\lambda_1} \mathcal{P} f^L_{\lambda_2}\right) \mp i\pi w^L_{\lambda_2} q_2^2 f^L_{\lambda_1}\right] \tag{467}$$

$$\times \left[\left(R^L_{\lambda_1} X_1 - \mathcal{P}^L_{\lambda_2} X_2\right) \mp i\pi \delta^L_{\lambda_2} X_2\right] \tag{468}$$

$$\left[\left(J - q_2 f^L_{\lambda_1} X_2 + q_2 \mathcal{P} f^L_{\lambda_2} X_1\right) \pm i\pi w^L_{\lambda_2} q_2 X_1\right] \tag{469}$$

$$\times \Pi_0 \left[\left(R^L_{\lambda_1} X_1 - \mathcal{P}^L_{\lambda_2} X_2\right) \mp i\pi \delta^L_{\lambda_2} X_2\right] \tag{470}$$

$$= \left[\left(R^L_{\lambda_1} X_1 - \mathcal{P}^L_{\lambda_2} X_2\right) \mp i\pi w^L_{\lambda_2} \phi^L_{\lambda_2} \otimes \phi^{L,*}_{\lambda_2} X_2\right] \tag{471}$$

$$+ q_2 \widehat{g}_{2,\lambda} \left[\left(1 - q_2^2 f^L_{\lambda_1} \mathcal{P} f^L_{\lambda_2}\right) \mp i\pi w^L_{\lambda_2} q_2^2 f^L_{\lambda_1}\right] \tag{472}$$

$$\times \left[\left(\psi^L_{\lambda_1} X_1 - \mathcal{P}\psi^L_{\lambda_2} X_2\right) \mp i\pi w^L_{\lambda_2} \phi^L_{\lambda_2} X_2\right] \tag{473}$$

$$\left[\left(J - q_2 f^L_{\lambda_1} X_2 + q_2 \mathcal{P} f^L_{\lambda_2} X_1\right) \pm i\pi w^L_{\lambda_2} q_2 X_1\right] \tag{474}$$

$$\otimes \left[\left(\psi^L_{\lambda_1} X_1 - \mathcal{P}\psi^L_{\lambda_2} X_2\right) \mp i\pi w^L_{\lambda_2} \phi^{L,*}_{\lambda_2} X_2\right]. \tag{475}$$

One may expand the above expressions and look for the resulting imaginary piece to find $\delta^{H_2}_\lambda$ for $\lambda \geq \mu$ or $\lambda \leq \mu$. We will not do so and will analyze its properties from the unexpanded forms for simplicity instead.

We will once again use the definitions for $W_{\kappa,\tau}$, ϵ, and the like from Chapter 1. Furthermore we will employ the spectral decay estimates of Corollary (4.1) as well as the quasi-exponential decay estimates of Theorem (4) of Chapter 02. From the latter one can see that for $\lambda > \mu$ it is the case that $\mathrm{d}_\lambda^n \left[w_{\lambda_1}^{1/2} \psi_{\lambda_2}(x) \right] \in \ell^1$, as a function of (x, λ), is $\ell^1(\mathbb{Z}_+ \times [\mu, \infty), \mathbb{R})$.

By considering the many definitions, there is one crucial function which strongly determines our estimates: $\mathcal{P} f_a = e^a E_1(a) \sim -\log(a)$ as $0 < a \searrow 0$. Only powers of $\mathcal{P} f_a$ can be nonanalytic or unbounded. We will therefore proceed to prove the desired properties of $F(\lambda)$ by addressing the powers of $\mathcal{P} f_a$ alone.

We recall that by inspection one may observe that $g_\lambda := \{[1 - q_1 e^{-\lambda} \mathcal{P} E_1(-\lambda)]^2 + [\pi q_1 e^{-\lambda}]^2\}^{-1}$ has the properties:

$$g_\lambda = |g_\lambda| \leq \widehat{g}_0(q_1) < \infty, \quad \forall \lambda \in [0, \infty) \tag{476}$$

$$|\mathrm{d}_\lambda g_\lambda| \leq \widehat{g}_0(q_1) \widehat{g}_1(q_1, \delta) < \infty, \quad \forall \lambda \in [\delta, \infty) \tag{477}$$

$$g_0 = g_\infty = 0, \tag{478}$$

$$\mathrm{d}_\lambda g_\lambda = \mathcal{O}(\lambda^{-1} \log^{-1} \lambda), \quad \lambda \searrow 0 \tag{479}$$

$$\mathrm{d}_\lambda^2 g_\lambda = \mathcal{O}(\lambda^{-2} \log^{-3} \lambda), \quad \lambda \searrow 0 \tag{480}$$

$$\subset \mathcal{O}(\lambda^{-2} \log^{-2} \lambda) \tag{481}$$

where $0 < \widehat{g}_0(q_1), \widehat{g}_1(q_1, \delta) < \infty$ are constants whose other properties are not needed here. g_λ is the only function of λ involved in the definition of $F(\lambda)$ whose derivatives are unbounded in the neighborhood of the threshold $\lambda = 0$ and thereby the derivatives of g_λ and positive powers of $\mathcal{P} f_\lambda$ and its derivatives are dominant in determining the properties of the derivatives of $F(\lambda)$. We will therefore only consider the dominant factors with respect to these quantities.

Consider the contributions to the imaginary part of $\lim_{\epsilon \searrow 0} R_{\lambda \pm i\epsilon}^{H_2}$ for either $\lambda \geq \mu$ or $\lambda \leq \mu$. Due to the symmetry between these two ranges of λ it is sufficient to analyze the case of $\lambda \geq \mu$ alone and we will do so exclusively in the following.

Properties (1), (2): By considering the control that the factors of $w_{\lambda_1}^L = g_{\lambda_1} w_{\lambda_1}$ impose, one may see that all possible contributions are bounded in λ_1 and x.

Property (3): The bounds for the derivatives of ϕ_{λ_1}, ξ_{λ_1}, and ψ_{λ_2} cannot present a problem with the chosen norm on $\mathscr{B}$. Exponential decay as $\lambda_1 \nearrow \infty$ ensures that the upper bound of integration cannot be a problem. The only remaining potential issue comes from the behavior at the threshold, which is not relevant.

Property (4): The bounds for the derivatives of ϕ_{λ_1}, ξ_{λ_1}, and ψ_{λ_2} cannot present a problem with the chosen norm on $\mathscr{B}$.There will be two dominant factors. One of is $\mathrm{d}_\lambda g_{\lambda_1} = \mathcal{O}(\lambda_1^{-1} \log^{-1} \lambda_1)$ as $\lambda_1 \searrow 0$. The other dominant factor is of the form $g_{\lambda_1}^2 \mathrm{d}_{\lambda_1} (\mathcal{P} f_{\lambda_1})^2 = \mathcal{O}(\lambda_1^{-1} \log^{-1} \lambda_1)$ as $\lambda_1 \searrow 0$.

Property (5): The bounds for the derivatives of ϕ_{λ_1}, ξ_{λ_1}, and ψ_{λ_2} cannot present a problem with the chosen norm on $\mathscr{B}$. There will be one dominant factor, which is $g_{\lambda_1}[\mathrm{d}_{\lambda_1}(\mathcal{P} f_{\lambda_1})^2]^2 = \mathcal{O}(\lambda^{-2} \log^{-2} \lambda)$.

$\square$

Proof of Theorem (8). By the Duhamel formula it is the case that

$$-i\mathrm{d}_t u = Hu = H_2 u + Uu \tag{482}$$

$$\Rightarrow \quad u(t) = e^{-itH_2} u_0 - i \int_0^t \mathrm{d}t_1 \; e^{-i(t-t_1)H_2} U u(t) \tag{483}$$

$$\Rightarrow \quad W_{\kappa,\tau} u(t) = W_{\kappa,\tau} e^{-itH_2} u_0 - i \int_0^t \mathrm{d}t_1 \; W_{\kappa,\tau} e^{-i(t-t_1)H_2} U u(t_1) \tag{484}$$

$$||W_{\kappa,\tau} u(t)||_1 \leq ||W_{\kappa,\tau} e^{-itH_2} u_0||_1 + \int_0^t \mathrm{d}t_1 \; ||W_{\kappa,\tau} e^{-i(t-t_1)H_2} U u(t_1)||_1 \tag{485}$$

$$\leq c_0 ||W_{\kappa,\tau} e^{-itH_2} W_{\kappa,\tau} W_{\kappa,\tau}^{-1} u_0||_\infty \tag{486}$$

$$+ c_1 \int_0^t \mathrm{d}t_1 \; ||W_{\kappa,\tau} e^{-i(t-t_1)H_2} W_{\kappa,\tau} W_{\kappa,\tau}^{-1} U W_{\kappa,\tau}^{-1} W_{\kappa,\tau} u(t_1)||_\infty \tag{487}$$

$$\leq c_0 ||W_{\kappa,\tau} e^{-itH_2} W_{\kappa,\tau}|| \; ||W_{\kappa,\tau}^{-1} u_0||_1 \tag{488}$$

$$+ c_1 \int_0^t \mathrm{d}t_1 \; ||W_{\kappa,\tau} e^{-i(t-t_1)H_2} W_{\kappa,\tau}|| \; ||W_{\kappa,\tau}^{-2} U|| \; ||W_{\kappa,\tau} u(t_1)||_1 \tag{489}$$

$$\leq c_2 [(t + c_3) \log^2 (t + c_3)]^{-1} \tag{490}$$

$$+ c_4 \int_0^t \mathrm{d}t_1 \; [(t - t_1 + c_3) \log^2 (t - t_1 + c_3)]^{-1} ||W_{\kappa,\tau} u(t_1)||_1 \tag{491}$$

Let $f(t) := ||W_{\kappa,\tau}u(t)||_1$ and $g(t) := [(t+c_3)\log^2(t+c_3)]^{-1}$. By Gronwall's Lemma one has

$$f(t) \le c_2 g(t) + c_4 \int_0^t \mathrm{d}t_1\ g(t-t_1) f(t_1) \tag{492}$$

$$\Rightarrow \quad f(t) \le c_2 g(t) + c_2 c_4 \int_0^t \mathrm{d}t_1\ g(t_1) g(t-t_1) \exp\left[c_4 \int_{t_1}^t \mathrm{d}t_2\ g(t-t_2)\right] \tag{493}$$

Furthermore

$$\exp\left[c_4 \int_{t_1}^t \mathrm{d}t_2\ g(t-t_2)\right] = \exp\left[c_4 \int_0^{t-t_1} \mathrm{d}t_2\ g(t_2)\right] \tag{494}$$

$$\le \exp\left[c_4 \int_0^{\infty} \mathrm{d}t_2\ g(t_2)\right] \le c_5, \tag{495}$$

and

$$\int_0^t \mathrm{d}t_1\ g(t_1) g(t-t_1) = 2\int_0^{t/2} \mathrm{d}t_1\ g(t_1) g(t-t_1) \le 2\int_0^{t/2} \mathrm{d}t_1\ g(t_1) g(t/2) \tag{496}$$

$$\le c_6[(t+c_7)\log^2(t+c_7)]^{-1}. \tag{497}$$

One therefore has

$$f(t) \le c_2 g(t) + c_2 c_4 c_5 c_6 [(t+c_7)\log^2(t+c_7)]^{-1} \tag{498}$$

$$\le c_8[(t+c_9)\log^2(t+c_9)]^{-1} = \mathcal{O}(t^{-1}\log^{-2} t), \tag{499}$$

as $t \nearrow \infty$. In the above c_j, $j = 0, \ldots, 9$, are constants, the properties of which are not important. $\square$

CHAPTER 4

Conclusions and Conjectures

The goal for the work at this stage has been ultimately to arrive at the proof of Theorem (8). This is the most important element of the proof of asymptotic stability. One can see from the proof itself that as long as the decay estimates are integrable, and the perturbation decays faster than the square of the inverse of the weights, one can control the evolution rather easily. This will also be the case in the context of the LNLS (298) where one employs a Duhamel formula of the form

$$\vec{\beta}(t) = e^{-itH}\vec{\beta}_0 - i\int_0^t \mathrm{d}t_1\; e^{-i(t-t_1)H}\vec{\gamma}(t), \tag{500}$$

where $\vec{\beta}_0 = \vec{\beta}(t=0)$, to specify the evolution of β. This alone, however, is not enough to determine the evolution of β as the parameters μ_1 and ν_1 are time dependent. One must include separate evolution equations for these as well.

First one assumes that $(\vec{\alpha}, \vec{\beta}(0))_{\vec{\mathscr{H}}} = (\vec{\alpha}, \mathrm{d}_t\vec{\beta}(0))_{\vec{\mathscr{H}}} = 0$, where $(\cdot,\cdot)_{\vec{\mathscr{H}}}$ is natural the extension of the inner product of $\mathscr{H}$ to the matrix system and $\vec{\alpha} = \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix}$. This condition ensures that $\vec{\beta}$ remains in the span of the generalized eigenvectors of H. Then one takes the inner product of $\vec{\alpha}$ with both sides of the LNLS (298) to arrive at

$$0 = (\alpha, \gamma) \quad \Rightarrow \quad \mathrm{d}_t\nu_1 = (\alpha, \alpha_{\mu_1})^{-1}(\alpha, \Re\gamma_1), \quad \mathrm{d}_t\mu_1 = i(\alpha, \partial_{\mu_1}\alpha_{\mu_1})^{-1}(\alpha, \Im\gamma_1). \tag{501}$$

Equations (500) and (501) together constitute the *modulation equations* for the NLS (206), where μ_1 and ν_1 are the *modulation parameters* [**31**].

CONJECTURE 1. *The soliton manifold specified by the coordinates (μ, ν) with respect to $u(t) = e^{-i(-\mu t+\nu)}\alpha_\mu$ is asymptotically stable under perturbed evolution via the discrete NLS (206).*

It is reasonable to assume that such a claim is true and it should be the case that one can prove it with an application of bootstrapping estimates. The case of the real Nonlinear Klein-Gordon equation is likely to be much harder to address.

Consider the discrete real Nonlinear Klein-Gordon equation (rNLKG) specified by

$$-\partial_t^2 u = L_0 u - u^p, \quad 1 < p \in \mathbb{Z}. \tag{502}$$

It was this equation that was first studied in the context of noncommutative field theory, see e.g. [**15**], and the mathematical analysis of [**5**], [**10**], and [**11**]. There is an approach to the rNLKG which is similar to that of the methods of linearization taken with the method of modulation equations, but it is of a different character. There, the analogue of the stationary solution will be of the form $u(t) = \cos(\mu t + \nu)\alpha_\mu$. Due to the presence of nonlinearity this will not be a stationary solution in general. For the NLS one could interpret the stationary state as a nonlinear variant of the evolution of an eigenfunction with an associated isolated eigenvalue. For the rNLKG one is typically lead to interpret the "quasi-stationary state" as the nonlinear variant of the evolution of a resonance function with an associated embedded eigenvalue. The coupling of the "radiation" β to the soliton will introduce an instability and lead to a resonance with a decay time. This picture was introduced and elaborated upon in the work of Soffer and Weinstein on nonlinear resonances and the nonlinear Fermi-Golden Rule [**32**].

Chen, Frölich, and Walcher in [**5**] conjecture that Equation (502) has localized metastable solutions. This leads us to the following conjecture.

Conjecture 2. *Solutions of the discrete rNLKG* (502) *which begin close to* $u(t) = \cos(\mu t + \nu)\alpha_\mu$ *are metastable resonance functions.*

We seek to address compare and contrast the proofs of these conjectures in future work.

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