A CLASS OF INPUT/OUTPUT RANDOM SYSTEMS: MONOTONICITY AND A SMALL-GAIN THEOREM

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ABSTRACT OF THE DISSERTATION

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We expand upon the theory of *random dynamical systems* (RDS) of L. Arnold, developing a theory of *random dynamical systems with inputs and outputs* (RDSIO)—an abstract framework for the treatment of noise-driven systems subject to stochastic inputs and yielding random outputs. This development allows for one to study both autonomous RDS and proper RDSIO as the feedback interconnection of smaller random input/output modules. As "proof of concept," we prove a small-gain theorem for autonomous RDS which can be realized as the feedback interconnection of monotone RDSI with monotone or anti-monotone outputs. This result gives sufficient conditions for autonomous RDS to possess unique, globally attracting equilibria—the RDS in question need not be itself monotone.

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My family has always been there for me despite my not being there for them nearly as much. I can barely believe the unconditional love and support I get across all layers, from immediate family to distant relatives—thank you!

Dedication

This work is dedicated to my grandmother "Vovó" Maria, and to the memory of my late grandparents, "Vovô" Fernando, "Vovó" Lygia, and "Vovô" Marcondes, without whose love and support I would not have come this far.

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Chapter 1

Introduction

The study of the long-term behavior of systems evolving over time started long ago. It has been about three and a half centuries since its formal mathematical treatment first began, along with the development of Newtonian mechanics, while some of its philosophical underpinnings—for instance, the idea of causality—go at least as far back as Aristotle. Ever since, the body of theory often known simply as *dynamical systems* has become an increasingly active area of research. It has motivated the development of a great deal of modern mathematics in areas such as differential geometry, algebraic topology, and ergodic theory, in addition to providing for better descriptions and understanding of natural phenomena.

Over the past several decades, randomness or uncertainty have been gradually, and systematically, incorporated into the study of the dynamic behavior of systems evolving by both discrete or continuous time increments—as well as "mixed" time-evolution regimes with continuous intervals interspersed by discrete "jumps." This has motivated the development of several conceptual approaches to describing "random dynamics" in a unified way for large classes of systems such as, for instance, *stochastic analysis* [46, 32], which describes differential equations driven by semimartingales rather than just Gaussian white noise, *random dynamical systems* [4, 8], which unifies discrete- and continuous-time systems driven by stationary noise processes, and, more recently, *stochastic dynamic equations*, which describes systems evolving over time-parameter sets not necessarily equipped with the structure of a semigroup [49, 23]. This is the tradition within which lies this work.

To motivate the discussion, consider a simple biochemical circuit in which three *species*—say, *lacI*, *tetR* and *cI*, as in the *repressilator* [18]—interact with one another as

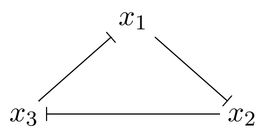

Figure 1.1: Biochemical Circuit

illustrated in Figure 1.1. The "arrow" in '$x \dashv y$' indicates that species 'x' represses the production of species 'y.' The concentrations of the species in this biochemical circuit may then be modeled, in continuous time, by a system of differential equations such as

$$\begin{cases} \dot{\xi}_1 &= a_1\xi_1 + h_1(\xi_3) \\ \dot{\xi}_2 &= a_2\xi_2 + h_2(\xi_1) \\ \dot{\xi}_3 &= a_3\xi_3 + h_3(\xi_2) \end{cases},$$

where $a_1, a_2, a_3 < 0$ are the rates of degradation, and h_1, h_2, h_3 are nonincreasing functions of their arguments. This scenario and many variations, such as the *Goodwin model* of gene expression [22, 21, 42, 28], are studied in the context of molecular biology.

It is natural that the rates of degradation, as well as the strength of the interactions between the species, may depend on environmental factors such as temperature or pressure, as well as the concentrations of other enzymes and such. If this is the case, then a more realistic model would have been a system of parametrized differential equations of the form

$$\begin{cases} \dot{\xi}_1 &= a_1(\lambda)\xi_1 + h_1(\lambda, \xi_3) \\ \dot{\xi}_2 &= a_2(\lambda)\xi_2 + h_2(\lambda, \xi_1) \\ \dot{\xi}_3 &= a_3(\lambda)\xi_3 + h_3(\lambda, \xi_2) \end{cases},$$

where the parameter λ lives in a parameter space Λ, and may be an ordered tuple $\lambda = (\lambda_1, \ldots, \lambda_m)$ encoding all relevant external factors upon which the dynamics of the circuit depends.

The next step is to add noise to the values of the parameter λ. As noted above, many different approaches to how this noise could be modeled have been considered. One such approach is to introduce a stochastic process $(\lambda_t)_{t \geqslant 0}$ evolving on Λ, and consider the system of "random differential equations" ("RDE")

$$\begin{cases} \dot{\xi}_1 &=& a_1(\lambda_t(\omega))\xi_1 + h_1(\lambda_t(\omega), \xi_3) \\ \dot{\xi}_2 &=& a_2(\lambda_t(\omega))\xi_2 + h_2(\lambda_t(\omega), \xi_1) \\ \dot{\xi}_3 &=& a_3(\lambda_t(\omega))\xi_3 + h_3(\lambda_t(\omega), \xi_2) \end{cases} , \tag{1.1}$$

now effectively parametrized by the random outcome ω in the probability space modeling the underlying uncertainty.

In this work we will approach systems such as (1.1) from the point of view of "global convergence to a unique equilibrium."

1.0.1 Random Dynamics

In the late 1980's, Ludwig Arnold conceived a deep and elegant approach to the foundations of random dynamics. His paradigm of a random dynamical system (RDS) is based on an ultimately simple idea: to view it as consisting of two ingredients, a stochastic (but autonomous) noise process combined with a classical dynamical system that is driven by this process. The noise process is itself described by a *measure-preserving dynamical system* (*MPDS*) $\theta \colon \mathcal{T} \times \Omega \to \Omega$, evolving on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in discrete or continuous time-increments encoded in $\mathcal{T}$. This MPDS may represent random environmental perturbations, internal variability, randomly fluctuating parameters, model uncertainty, or measurement errors. But the formalism allows for deterministically periodic, or almost-periodic, driving processes as well. The resulting theory, developed in a series of papers by many authors through the 1990's, provides a seamless integration of classical ergodic theory with modern dynamical systems, giving a theoretical framework parallel to classical smooth and topological dynamics—stability, attractors, bifurcation theory, and so forth—, while allowing one to treat in a unified way the most important classes of dynamical systems evolving subject to randomness—random difference or differential equations,

$$x^+ = f(\theta_n\omega, x)\,, \quad \dot{x} = f(\theta_t\omega, x)\,,$$

as well as stochastic differential equations [4, 12, 8].

1.0.2 Systems with Inputs and Outputs

Our motivation for studying RDS with inputs and outputs arises from the need to provide foundations for a constructive theory of interconnections and feedback for stochastic systems, one that will eventually generalize successful and widely applied deterministic approaches to the analysis and design of dynamic networks [35, 30, 31]. To explain this need, we recall the basic paradigm of (deterministic) control theory. The objects of study are systems with inputs and outputs. One may think, for concreteness, of a system of ordinary differential equations

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & f_1(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_k(t)) \\ & \vdots & \\ \dot{x}_n & = & f_n(\underbrace{x_1(t), \ldots, x_n(t)}_{\text{states}}, \underbrace{u_1(t), \ldots, u_k(t)}_{\text{inputs}}) \end{array} \right. \tag{1.2}$$

supplemented by a set of output variables $y_1, \ldots, y_p$,

$$y_j(t) \;=\; h_j(x(t)) \,, \quad j = 1, \ldots, p \,,$$

which are indeed functions of the state vector x. The inputs $u_1, \ldots, u_k$ may be viewed as controls, forcing functions, external signals, or stimuli, depending on the context. Under suitable hypotheses on the inputs and the righthand side f, the system of differential equations (1.2) will generate a (deterministic) flow $\varphi(t, x, u)$, giving the state of the system at time t when it started at x and is subject to the input $u = (u_1, \ldots, u_k)$ [52]. The outputs $y_1, \ldots, y_p$ represent responses, typically a partial readout of the system state vector $(x_1, \ldots, x_n)$.

Such formalism, which originated in the analysis of engineering systems, is also natural in biology. Cells are not autonomous systems; they process external information, provided by physical (radiation, mechanical, temperature) or chemical (drugs, growth factors, hormones, nutrients) inputs. Cells also produce signals which we may view as outputs, such as chemical signals sent to other cells, commands to motors that move flagella or pseudopods, or the internal activation of transcription factors which may be monitored by measurement technologies. Thus the control theory formalism, in

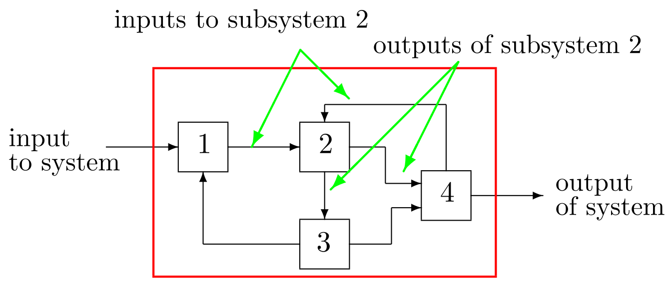

Figure 1.2: A system viewed as an interconnection of subsystems with inputs and outputs

contrast to dynamical systems theory—which deals with isolated systems—is not only reasonable, but also natural in biology.

There is also a different, and not as intuitive, reason for considering systems with inputs and outputs. Cells can be seen as composed of a large number of subsystems; "networks" of proteins, RNA, DNA, and metabolites involved in various processes, such as cell growth, maintenance, division, and death. Indeed, one of the important themes in current molecular biology is that of understanding cell behavior in terms of cascades and feedback interconnections of elementary "modules" [25, 37, 17]. The idea is that one should be able to decompose large systems into smaller subsystems, and then study the dynamics of the larger system in light of the (hopefully simpler) dynamics of the smaller modules, and how these modules interact with each other through the feedback interconnections. As illustrated in Figure 1.2, one might represent this situation as an overall system composed of four subsystems. Although the figure also shows inputs and outputs for the overall system, even if the entire system were autonomous (no arrows into or out of the large box), one must necessarily consider subsystems that admit time-dependent input signals and produce output signals, in order to be able to define such interconnections. Thus, when using a decomposition-based approach, the control-theoretic formalism is a necessity, even in the analysis of autonomous systems.

1.0.3 Thesis Outline

The main objective of this work is to carry the successful decomposition-based, feedback interconnections approach to deterministic systems, outlined above, over to the RDS theory of Arnold. This entails,

(1) carefully extending Arnold's axiomatic framework to accommodate stochastic inputs, addressing all emerging algebraic and measurability technicalities,

(2) giving a treatment of cascades and feedback interconnections of random systems with stochastic inputs and noisy outputs, including the search for an appropriate mode of convergence which interacts well with desirable regularity assumptions on the outputs, and transfers over from subsystem to subsystem, and

(3) the application of this theory to the analysis of examples not encompassed by the previously existing theory of RDS.

We define a *random dynamical system with inputs* (*RDSI*) to be an ordered triple $(\theta, \varphi, \mathcal{U})$ in which θ is an MPDS, as described in Section 2.1, $\mathcal{U}$ is a class of admissible stochastic inputs (Definition 3.14), and $\varphi(t, \omega, x, u)$ is a semiflow driven by θ, giving the state of the system at time t when starting from initial state x and subject to random outcome ω and input u (Definition 3.16). This notion extends the concept of RDS of Arnold in the sense that an RDSI evolving subject to a stationary input reduces to an RDS (Lemma 3.29 and Proposition 3.30). A *random dynamical system with inputs and outputs* (*RDSIO*), in turn, is defined as an ordered quadruple $(\theta, \varphi, \mathcal{U}, h)$ in which $(\theta, \varphi, \mathcal{U})$ is an RDSI and h is an *output function*—a readout of the current state of the system, and which also depends on the random outcome ω.

The main result of this work is Theorem 4.28, the *Small-Gain Theorem* (*SGT*). The key ingredients of this result are,

(1) the idea of monotonicity [8], which we extend for RDSI,

(2) the notion of *temperedness*, a growth condition (along orbits of the underlying MPDS) for random variables from which we derive concepts of convergence and continuity, and

(3) a *converging input to converging state* (*CICS*) result, namely, Theorems 4.11 and 4.12.

To facilitate the discussion throughout the remainder of this introduction, we provide informal statements of Theorems 4.11 and 4.28 as "Theorems" 1.1 and 1.2, respectively. These will be dropped in favor of the formal statements in Chapter 4 once we have developed the needed language and notation.

Theorem 1.1 (CICS Redux)**.** *Suppose that*

(*i*) *X, U are separable Banach spaces, partially ordered by solid, normal, minihedral cones, and*

(*ii*) *$(\theta, \varphi, \mathcal{U})$ is a tempered, monotone RDSI with a*

(*iii*) *continuous input to state characteristic $\mathcal{K}$.*

If u is a tempered input with asymptotically precompact (pullback) tails, and which converges (in the tempered sense) to a stationary input u_∞, then $\varphi(t, \omega, x, u)$ converges to $\mathcal{K}(u_\infty)$ for every tempered initial state x.

The *input to state characteristic* $\mathcal{K}$ in the assumptions of the theorem above is a mapping associating, to each (tempered) stationary input u fed into the system, a unique, globally attracting equilibrium $\mathcal{K}(u)$ to which the system converges (in the tempered sense) for every (tempered) initial state x. For an RDSIO, the output can be composed with the i/s characteristic, yielding the *input to output characteristic* $\mathcal{K}^Y$ of the system. For an RDSIO arising from "opening the loop" in an RDS, this i/o characteristic is an operator on the space of tempered inputs. If this operator has a unique, globally attracting fixed point, then we say that the system satisfies the *small-gain condition* (*SGC*). The small-gain condition combined with estimates from monotonicity reduces the behavior of closed-loop trajectories of the system to the circumstances in Theorem 1.1, yielding the global convergence in the SGT,

Theorem 1.2 (SGT Redux)**.** *Suppose that*

(*i*) *X, U are separable Banach spaces, partially ordered by solid, normal, minihedral, and*

(*ii*) *$(\theta, \varphi, \mathcal{U}, h)$ is a tempered, monotone RDSIO with a* *(iii)* *continuous input/state characteristic, and a*

(iv) *order-preserving or order-reversing output which preserves temperedness.*

If the i/o characteristic of the RDSIO $(\theta, \varphi, \mathcal{U}, h)$ *satisfies the SGC, then the closed-loop of* $(\theta, \varphi, \mathcal{U}, h)$ *has a unique, globally attracting equilibrium in the universe of tempered initial states.*

To illustrate how Theorem 1.2 may be applied, lets go back to our toy example, the system generated by (1.1) modeling the biochemical circuit illustrated in Figure 1.1. If the noise process $(\lambda_t)_{t\geqslant 0}$ is *stationary*, meaning that

$$\mathbb{P}(\lambda_{t_1+s} \in A_1, \ldots, \lambda_{t_k+s} \in A_k) \equiv \mathbb{P}(\lambda_{t_1} \in A_1, \ldots, \lambda_{t_k} \in A_k)\,,$$

then (1.1) can be rewritten in the "canonical" form

$$\left\{\begin{array}{lcl} \dot{\xi}_1 & = & a_1(\theta_t\omega)\xi_1 + h_1(\theta_t\omega, \xi_3) \\ \dot{\xi}_2 & = & a_2(\theta_t\omega)\xi_2 + h_2(\theta_t\omega, \xi_1) \\ \dot{\xi}_3 & = & a_3(\theta_t\omega)\xi_3 + h_3(\theta_t\omega, \xi_2) \end{array}\right. , \tag{1.3}$$

in terms of an MPDS $\theta\colon \mathbb{R}_{\geqslant 0} \times \Omega \to \Omega$, for appropriately redefined Ω, a_1, a_2, a_3, h_1, h_2 and h_3.

Under suitable hypotheses on the a_j's and h_j's, this system of random differential equations generates an autonomous RDS. We now describe how our theory of RDSIO can be used to study the asymptotic behavior of this RDS.

First, one opens up the feedback loop, rewriting the model as a system of *random differential equations with inputs (RDEI)*

$$\left\{\begin{array}{lcl} \dot{\xi}_1 & = & a_1(\theta_t\omega)\xi_1 + u_t^{(1)}(\omega) \\ \dot{\xi}_2 & = & a_2(\theta_t\omega)\xi_2 + u_t^{(2)}(\omega) \\ \dot{\xi}_3 & = & a_3(\theta_t\omega)\xi_3 + u_t^{(3)}(\omega) \end{array}\right. , \tag{1.4}$$

together with a set of outputs

$$\left\{\begin{array}{lclcl} u_t^{(2)}(\omega) & = & y_t^{(1)}(\omega) & = & h_2(\theta_t\omega, \xi_1) \\ u_t^{(3)}(\omega) & = & y_t^{(2)}(\omega) & = & h_3(\theta_t\omega, \xi_2) \\ u_t^{(1)}(\omega) & = & y_t^{(3)}(\omega) & = & h_1(\theta_t\omega, \xi_3) \end{array}\right. . \tag{1.5}$$

Now the RDEI in (1.4) is not only linear, but also monotone (with respect to the partial order induced by the positive orthant). It generates an RDSI which, under the hypotheses that θ is ergodic and the a_j's are "negative on average," possesses a well-defined, tempered continuous i/s characteristic $\mathcal{K}$, that is, a map associating a globally attracting equilibrium $\mathcal{K}(u)$ to each (tempered) stationary input u. Composing this i/s characteristic with the output of the system, (1.5), we obtain its i/o characteristic $\mathcal{K}^Y$.

It remains to check whether the i/o characteristic satisfies the SGC. This will be the case, for instance, if the h's are of the form

$$h(\omega, x) = \frac{\alpha(\omega)}{\beta(\omega) + g(x)} \, ,$$

where α and β are uniformly bounded away from zero and infinity along each orbit of θ, and g is a continuous, bounded, order-preserving, sublinear map. It can be shown, using the Thompson metric induced by the underlying partial order (Appendix D), that the i/o characteristic has a unique, globally attracting fixed point u_∞. It then follows from the SGT that $\mathcal{K}(u_\infty)$, the state characteristic corresponding to u_∞, is a unique, globally attracting equilibrium of the original, closed-loop system,

$$\left\{ \begin{array}{rcl} \dot{\xi}_1 & = & a_1(\theta_t\omega)\xi_1 + \dfrac{\alpha_1(\theta_t\omega)}{\beta_1(\theta_t\omega) + g_1(\xi_3)} \\ \dot{\xi}_2 & = & a_2(\theta_t\omega)\xi_2 + \dfrac{\alpha_2(\theta_t\omega)}{\beta_2(\theta_t\omega) + g_2(\xi_1)} \\ \dot{\xi}_3 & = & a_3(\theta_t\omega)\xi_3 + \dfrac{\alpha_3(\theta_t\omega)}{\beta_3(\theta_t\omega) + g_3(\xi_2)} \end{array} \right. . \qquad (1.6)$$

1.1 How This Work Is Organized

We end this introduction with a more thorough description of the content of each chapter.

1.1.1 Preliminary Notions

Chapter 2 lays down the foundations upon which our theory rests.

In Section 2.1, we review the concept of measure preserving dynamical systems, introducing much of the notation and terminology for our random analogues of points, paths and constant trajectories in the deterministic theory, as well as key conventions we shall follow in dealing with them.

Sections 2.2 and 2.3 are self-contained reviews of all the concepts we shall need from the theory of random sets and functional analysis.

In Section 2.4 we develop the notions of convergence with respect to which we will study asymptotic behavior. This includes an extension of the concepts of $\liminf$ and $\limsup$ to stochastic processes evolving in certain partially ordered spaces (possibly infinite-dimensional).

1.1.2 Random Dynamical Systems with Inputs and Outputs

In this chapter we develop the axiomatic foundations of our theory of random dynamical systems with inputs and outputs.

Section 3.1 is a brief review of the necessary elements from Arnold's RDS theory, setting the stage for the introduction of RDSIO in Section 3.2, where they are motivated and defined.

The appropriate stochastic analogue of constant inputs in the deterministic theory are further discussed in Section 3.3, followed by the definition of input to state characteristics for RDSI.

Although some examples are already given in Sections 3.1–3.3 along with the development of the theory, we close the chapter in Section 3.4 with a few more examples and simulations in discrete time, plus a thorough description of sufficient conditions for random differential equations with inputs to generate RDSI.

1.1.3 Monotone RDSIO and a Small-Gain Theorem

This is the main chapter of this work.

We start by extending the concept of monotonicity to RDSI in Section 4.1. This is applied in Section 4.2 to derive two converging input to converging state results for monotone RDSI.

Output functions, which had been considered in the previous chapter mostly from an algebraic perspective, are reconsidered in Section 4.3, where we discuss their measurability, regularity, growth and monotonicity properties.

The Small-Gain Theorem, the main result of this work, is proved in Section 4.4, after having introduced closed-loop trajectories and the Small-Gain Condition.

Section 4.5 is devoted to applications of the SGT to establish unique, globally attracting equilibria for some classes of discrete and continuous RDS.

1.1.4 Future Work

We started this work with the clear goal of extending the SGT for RDS. Thus many questions which, although important, might have distracted us from this objective, were set aside as we looked for the path to the SGT. In the last chapter of this work we sketch a few possible directions for future research.

1.1.5 The Appendices

We wanted for this work to be as self-contained as possible. We do not claim originality over any of the content in the appendices. However, it was often difficult to find classical results presented exactly the way we needed them. In other occasions, they had to be patched together from various sources. Finally, there were a few situations in which it was convenient to refer to a technique or notation in the proof of a classical result, in which case it was desirable to have this result readily available for reference. This was collected into Appendices A–D for the reader's convenience.

Chapter 2

Preliminary Notions

We build upon the "random dynamical systems" framework of Arnold [4]. The foundation for this framework is the concept of "measure preserving dynamical system," recalled in Definition 2.1 below. Throughout this chapter, we expand upon this concept, connecting it to stochastic processes and notions of "temperedness" and "long-term behavior." Along the way we introduce some notation and terminology not found in [4] to facilitate the discussion. This will set the stage for a brief review of random dynamical systems in the beginning of next chapter, then followed by the introduction of our new concept of "random dynamical systems with inputs."

2.1 Measure Preserving Dynamical Systems

Definition 2.1 (Measure-Preserving Dynamical System)**.** A *measure - preserving dynamical system*[1] (*MPDS*) is an ordered quadruple

$$\theta = (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathcal{T}})$$

consisting of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a topological group $(\mathcal{T}, +)$, and a *measurable flow* $(\theta_t)_{t\in\mathcal{T}}$ of invertible, measure-preserving maps $\Omega \to \Omega$—in other words,

$$\theta \colon \mathcal{T} \times \Omega \longrightarrow \Omega$$

is a $(\mathcal{B}(\mathcal{T}) \otimes \mathcal{F})$-measurable group action[2] with the property that $\mathbb{P} \circ \theta_t = \mathbb{P}$ for each $t \in \mathcal{T}$. △

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[1] Arnold [4, page 635] and Chueshov [8, Definition 1.1.1 on page 10] refer to such an object primarily as a 'metric dynamical system.' We find 'measure-preserving,' which Arnold also uses as a synonym, less confusing and more informative.

[2] $\theta_{t+s}\omega = \theta_t\theta_s\omega$ for every $s, t \in \mathcal{T}$, for every $\omega \in \Omega$.

Example 2.2 (Bernoulli Shifts). Consider a probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$. Let

$$\Omega := (\Omega_0)^{\mathbb{Z}}$$

be the family of all two-sided sequences $k \mapsto \omega_k \in \Omega_0$, $k \in \mathbb{Z}$. As usual, we denote such sequences as $\omega = (\omega_k)_{k\in\mathbb{Z}}$. Recall that a *cylinder subset* of Ω is a set of the form

$$\{\omega \in \Omega\,;\ \omega_{k_j} \in E_j\,,\ j = 1, \ldots, m\}$$

for some $m \in \mathbb{N}$, $E_j \in \mathcal{F}_0$, $k_j \in \mathbb{Z}$, $j = 1, \ldots, m$. Let $\mathcal{C}$ denote the family of all cylinder subsets of Ω and let

$$\mathcal{F} := \sigma(\mathcal{C})$$

be the σ-algebra generated by $\mathcal{C}$. Define $\mathbb{P}'\colon \mathcal{C} \to [0,1]$ by

$$\mathbb{P}'(C) := \prod_{j=1}^{m} \mathbb{P}_0(E_j) \tag{2.1}$$

for every

$$C = \{\omega \in \Omega\,;\ \omega_{k_j} \in E_j\,,\ j = 1, \ldots, m\} \in \mathcal{C}\,. \tag{2.2}$$

In particular, $\mathbb{P}'(\varnothing) = 0$ and $\mathbb{P}'(\Omega) = 1$. Well-established measure theory results can be applied to uniquely extend $\mathbb{P}'$ to a probability measure

$$\mathbb{P}\colon \mathcal{F} \to [0,1]$$

(see constructions developed in Sections 1.4 and 2.5 of [20]). This yields the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Now set $\mathcal{T} := \mathbb{Z}$ and let $\theta\colon \mathbb{Z} \times \Omega \to \Omega$ be the map defined by

$$\theta_n\omega := (\omega_{k+n})_{k\in\mathbb{Z}}\,, \quad (n, \omega) \in \mathbb{Z} \times \Omega \tag{2.3}$$

—in other terms, one shifts the original sequence, $(\omega_k)_{k\in\mathbb{Z}}$, n slots to the left. Note that θ_n is invertible, with $\theta_n^{-1} = \theta_n$ for each $n \in \mathbb{Z}$ and $\theta_0 = id_\Omega$. For any cylinder set $C \in \mathcal{C}$ as in (2.2) and any $n \in \mathbb{Z}$, we have

$$(\theta_n)^{-1}(C) = \{\omega \in \Omega\,;\ \omega_{k_j - n} \in E_j\,,\ j = 1, \ldots, k\} \in \mathcal{C}\,, \tag{2.4}$$

hence

$$\theta^{-1}(C) = \bigcup_{n\in\mathbb{Z}} \big(\{n\} \times (\theta_n)^{-1}(C)\big) \in \mathcal{B}(\mathbb{Z}) \otimes \mathcal{F}\,.$$

Since $C \in \mathcal{C}$ was chosen arbitrarily, this shows that θ is $(\mathcal{B}(\mathbb{Z}) \otimes \mathcal{F})$-measurable. It follows straight from the definition in (2.3) that θ is a group action of $\mathbb{Z}$ on Ω, for

$$\begin{aligned}\theta_{n_1+n_2}\omega &= (\omega_{k+n_1+n_2})_{k\in\mathbb{Z}}\\ &= \theta_{n_1}(\omega_{k+n_2})_{k\in\mathbb{Z}}\\ &= \theta_{n_1}\theta_{n_2}\omega\,, \quad \forall\omega\in\Omega\,,\ \forall n_1,n_2\in\mathbb{Z}\,.\end{aligned}$$

Furthermore, we have $\mathbb{P}'\circ\theta_n = \mathbb{P}'$ for every $n\in\mathbb{Z}$ by (2.1) and (2.4). Therefore we must indeed have $\mathbb{P}\circ\theta_n = \mathbb{P}$ for every $n\in\mathbb{Z}$ also, otherwise the uniqueness of the extension of $\mathbb{P}'$ to $\mathbb{P}$ would be violated. This completes the construction of the (discrete) MPDS

$$\theta = (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathcal{T}})\,.$$

In what follows, whenever we say 'let θ be the *Bernoulli shift* of $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$,' we shall mean the MPDS constructed as just described above. ◊

In this work we will be dealing with MPDS's mainly in the abstract setting. For several other concrete examples see Chueshov [8, Section 1.1].

The "time group" $\mathcal{T}$ will always refer to either $\mathbb{R}$ or $\mathbb{Z}$, depending on whether one is talking about continuous or discrete time, respectively. In either case, $\mathcal{T}$ will be equipped with the usual[3] order and $\mathcal{T}_{\geqslant 0}$ will denote the nonnegative elements of $\mathcal{T}$. Therefore the notations '$t\in\mathcal{T}_{\geqslant 0}$' and '$t\geqslant 0$' will be used interchangeably.

We will occasionally need to make measure-theoretic considerations about $\mathcal{T}$, or Borel subsets of it. If $\mathcal{T}=\mathbb{R}$, that is, in continuous time, then we tacitly equip any Borel subset of $\mathcal{T}$ with the measure induced by the Lebesgue measure on $\mathbb{R}$. If $\mathcal{T}=\mathbb{Z}$, discrete time, then we think of the counting measure on $\mathbb{Z}$.

When $\mathcal{T}=\mathbb{Z}$, it follows from the group action that θ is completely determined by θ_1. In that case we will abuse notation and use the same θ to denote both the underlying MPDS and θ_1.

[3] As far as the abstract theory goes, $(\mathcal{T},+)$ could have been an arbitrary topological group, and $(\mathcal{T},\geqslant)$ could have been any directed set such that the partial order, topology and group operation were compatible in some sense. We will not discuss any applications in such general context though.

The symbols 'θ,' 'Ω,' '$\mathcal{F}$,' '$\mathbb{P}$' and '$\mathcal{T}$' are reserved throughout this entire work to have the meanings and perform the functions assigned to them in Definition 2.1. Moreover, whenever we talk about an 'MPDS θ,' it is tacitly understood that $\theta = (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathcal{T}})$, unless otherwise specified.

Definition 2.3 (θ-Invariant Sets)**.** Given an MPDS θ, a set $B \in \mathcal{F}$ is said to be *θ-invariant* if $\theta_t(B) = B$ for all $t \in \mathcal{T}$. △

When MPDS are involved, it is often the case that a condition depending on $\omega \in \Omega$ is stated to be satisfied for all $\omega \in \widetilde{\Omega}$, for some θ-invariant $\widetilde{\Omega} \subseteq \Omega$ of full measure[4]. It is often not necessary to specify said $\widetilde{\Omega}$ though. So we say *for θ-almost all* $\omega \in \Omega$ or write

$$\widetilde{\forall}\omega \in \Omega$$

to mean 'for all $\omega \in \widetilde{\Omega}$, for some θ-invariant set $\widetilde{\Omega} \subseteq \Omega$ of full measure.' See also Subsection 2.1.1 below.

The next two definitions will not be needed until much later. We state them here so they are easier to find.

Definition 2.4 (Ergodic MPDS)**.** An MPDS θ is said to be *ergodic* if, whenever $B \in \mathcal{F}$ is θ-invariant, then we have either $\mathbb{P}(B) = 0$ or $\mathbb{P}(B) = 1$. △

Definition 2.5 (Periodic MPDS)**.** An MPDS θ is said to be *periodic* with *period* $P > 0$ or *P-periodic* if $\theta_{t+P} = \theta_t$ for every $t \in \mathcal{T}$. △

The next example will be needed for the construction carried out in Example 2.10 further down.

Example 2.6 (Factoring of MPDS). Given an MPDS θ with $\mathcal{T} = \mathbb{Z}$ or $\mathbb{R}$, and given $k \in \mathcal{T}_{\geqslant 0}$, let

$$\begin{aligned} \hat{\theta}\colon \mathcal{T}\times\Omega &\longrightarrow \Omega \\ (t,\omega) &\longmapsto \theta_{kt}\omega \end{aligned}.$$

Then $\hat{\theta}$ is also an *MPDS*. To see this, first note that $\hat{\theta}$ is the composition of θ with the mapping

$$(t,\omega) \longmapsto (kt,\omega) \in \mathcal{T}\times\Omega\,, \quad (t,\omega) \in \mathcal{T}\times\Omega\,. \tag{2.5}$$

[4]In other words, $\theta_t\widetilde{\Omega} = \widetilde{\Omega}$ for all $t \in \mathcal{T}$, and $\mathbb{P}(\widetilde{\Omega}) = 1$.

Now θ is $(\mathcal{B}(T)\otimes\mathcal{F})$-measurable by the definition of MPDS, while (2.5) is continuous in the first coordinate and constant in the second, and thus also $(\mathcal{B}(T)\otimes\mathcal{F})$-measurable. Therefore $\hat{\theta}$ is $(\mathcal{B}(T)\otimes\mathcal{F})$-measurable. The other properties of an MPDS are inherited by $\hat{\theta}$ directly from θ. $\Diamond$

2.1.1 θ-Almost Everywhere Equal Maps

Random variables or θ-stochastic processes which agree on a θ-invariant subset of full measure of Ω will be identified in the most natural way. We briefly describe this identification in an abstract setting which will comprise all situations we shall encounter in this work.

Let G and H be any nonempty sets and let $H^{G\times\Omega}$ be the family of all maps $G\times\Omega\to H$. Then

$$a,b\in H^{G\times\Omega}\,,\ a\sim b \quad\Longleftrightarrow\quad a(g,\omega)=b(g,\omega)\,,\quad \forall g\in G\,,\ \widetilde{\forall}\omega\in\Omega\,,$$

is an equivalence relation in $H^{G\times\Omega}$. Indeed, it is immediate from the condition above that $\sim$ is reflexive and symmetric. And since the intersection of two θ-invariant subsets of full measure is also a θ-invariant subset of full measure, we see that $\sim$ is also transitive. As usual, we denote the family $H^{G\times\Omega}$ modulo this equivalence relation by $H^{G\times\Omega}/\sim$ and the equivalence class of an element $a\in H^{G\times\Omega}$ by $[a]$.

Now let G, H_1, H_2, H_3 be any nonempty sets and suppose that there is an operation $*\colon H_1\times H_2\to H_3$ linking H_1 and H_2 to H_3. This naturally induces an operation

$$*\colon H_1^{G\times\Omega}\times H_2^{G\times\Omega}\longrightarrow H_3^{G\times\Omega}$$

linking $H_1^{G\times\Omega}$ and $H_2^{G\times\Omega}$ to $H_3^{G\times\Omega}$, defined by

$$(a*b)(g,\omega):=a(g,\omega)*b(g,\omega)\,,\quad (g,\omega)\in G\times\Omega\,. \tag{2.6}$$

This operation can now be projected onto

$$\begin{aligned} *\colon (H_1^{G\times\Omega}/\sim)\times(H_2^{G\times\Omega}/\sim) \quad&\longrightarrow\quad H_3^{G\times\Omega}/\sim \\ ([a],[b]) \quad&\longmapsto\quad [a*b] \end{aligned}\,,$$

in the sense that it is well-defined irrespective of the representative. This will follow from (2.6) together with, once again, the fact that the intersection of two θ-invariant subsets of full measure are also a θ-invariant subset of full measure.

Thus in what follows, whenever pertinent, we tacitly identify $H^{G\times\Omega}$ with $H^{G\times\Omega}/\sim$, drop the brackets and so write $a = b$ to mean $a \sim b$. This abstract construction applies to both θ-stochastic processes—upon replacing G by the appropriate discrete or continuous time group $\mathcal{T}$—and random variables—upon taking G to be an arbitrary singleton. The operation $*$ will typically be addition of vectors or scalars, or multiplication of a vector by a scalar.

2.1.2 θ-Stationary Processes

Let X be a topological space and consider the measurable space $(X, \mathcal{B}(X))$ consisting of X equipped with its Borel σ-algebra. In the context of random dynamical systems, the analogue of a point in the state space X for a deterministic system is a *random variable* $\Omega \to X$—that is, a Borel-measurable map $\Omega \to X$. In this work we use the terms 'random variable' and 'Borel-measurable map' interchangeably (see also Definition 2.11 and observation right after it). We denote the set of all random variables into a topological space X by $X^{\Omega}_{\mathcal{B}}$.

Similarly, the analogue of paths $\mathcal{T}_{\geqslant 0} \to X$ in the deterministic case will be *θ-stochastic processes*[5] $\mathcal{T}_{\geqslant 0} \times \Omega \to X$—in other words, $(\mathcal{B}(\mathcal{T}_{\geqslant 0}) \otimes \mathcal{F})$-measurable maps $\mathcal{T}_{\geqslant 0} \times \Omega \to X$. Given any such map q, we denote $q_t := q(t, \cdot)\colon \Omega \to X$ for each $t \geqslant 0$. In particular, $q_t \in X^{\Omega}_{\mathcal{B}}$ for every $t \geqslant 0$ by Lemma C.3(b). The set of all θ-stochastic processes on a topological space X is denoted by $\mathcal{S}^X_\theta$.

We discuss next an analogue, in the stochastic setting, of constant paths in the deterministic setting. We start by defining a "translation" or "shift" operator in $\mathcal{S}^X_\theta$.

[5] *A 'θ-stochastic process' is indeed a stochastic process in the traditional sense. We use the prefix 'θ-' to emphasize the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and time semigroup $\mathcal{T}_{\geqslant 0}$ specified by the given MPDS.*

For each $s \in \mathcal{T}_{\geqslant 0}$, define[6]

$$\begin{aligned} \rho_s : \mathcal{S}_\theta^X &\longrightarrow \mathcal{S}_\theta^X \\ q &\longmapsto \rho_s(q) \end{aligned} \tag{2.7}$$

by

$$[\rho_s(q)]_t(\omega) := q_{t+s}(\theta_{-s}\omega)\,, \quad (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,. \tag{2.8}$$

The symbol ρ is reserved in this work to denote this operator. So even if we are working with two θ-stochastic processes $q_1 \in \mathcal{S}_\theta^{X_1}$ and $q_2 \in \mathcal{S}_\theta^{X_2}$ evolving on (possibly different) topological spaces X_1, X_2, we shall use the same ρ to denote their shifts. In other words, for any $s_1, s_2 \geqslant 0$, the maps $\rho_{s_1}(q_1)\colon \mathcal{T}_{\geqslant 0} \times \Omega \to X_1$ and $\rho_{s_2}(q_2)\colon \mathcal{T}_{\geqslant 0} \times \Omega \to X_2$ are simply the θ-stochastic process defined on the appropriate topological space by (2.8).

Definition 2.7 (θ-Stationary Processes)**.** A θ-stochastic process $\bar{q} \in \mathcal{S}_\theta^X$ is said to be *θ-stationary* if

$$\rho_s(\bar{q}) = \bar{q}\,, \quad \forall s \geqslant 0\,,$$

in the sense of the identification in the previous subsection—that is,

$$[\rho_s(\bar{q})]_t(\omega) = \bar{q}_t(\omega)$$

for all $s, t \in \mathcal{T}_{\geqslant 0}$, for θ-almost all $\omega \in \Omega$. △

Contrary to the convention in the terminology for 'θ-stochastic processes' above, the prefix 'θ-' is really relevant in the definition of θ-stationary processes. We will see in Proposition 2.9 that a θ-stationary θ-stochastic process $\bar{q}$ is indeed *stationary* in the traditional stochastic processes sense—that is,

$$\mathbb{P}(\bar{q}_{t_1} \in A_1, \ldots, \bar{q}_{t_k} \in A_k) = \mathbb{P}(\bar{q}_{t_1+h} \in A_1, \ldots, \bar{q}_{t_k+h} \in A_k)$$

for all $k \in \mathbb{N}$, for any $A_1, \ldots, A_k \in \mathcal{F}$, and any $t_1, \ldots t_k, h \geqslant 0$ (see, for instance, [40, Section 1.3]). The converse, however, is not true, as illustrated in Example 2.10. In fact, this same example shows that a θ-stochastic process may be θ-stationary with respect to an MPDS $\theta = (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathcal{T}})$, while not being $\hat{\theta}$-stationary with respect to another MPDS $\hat{\theta} = (\Omega, \mathcal{F}, \mathbb{P}, (\hat{\theta}_t)_{t\in\mathcal{T}})$ defined over the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$

[6] We prove in Lemma C.11 that ρ_s is well-defined for every $s \geqslant 0$.

and time group $(\mathcal{T}, +)$. But before we can discuss that any further, we first need the characterization of θ-stationary processes given in the next result.

Lemma 2.8. *A θ-stochastic process $\bar{q} \in \mathcal{S}_\theta^X$ is θ-stationary if, and only if there exists a random variable $q \in X_\mathcal{B}^\Omega$ such that*

$$\bar{q}_t(\omega) = q(\theta_t \omega)\,, \quad \forall t \in \mathcal{T}_{\geqslant 0}\,,\ \widetilde{\forall}\omega \in \Omega\,. \tag{2.9}$$

Proof. (Sufficiency) Suppose that (2.9) holds for some $q \in X_\mathcal{B}^\Omega$ and let $\widetilde{\Omega} \subseteq \Omega$ be a corresponding θ-invariant set of full measure. Pick any $s \in \mathcal{T}_{\geqslant 0}$. Then

$$[\rho_s(\bar{q})]_t(\omega) = \bar{q}_{t+s}(\theta_{-s}\omega) = q(\theta_{t+s}\theta_{-s}\omega) = q(\theta_t\omega) = \bar{q}_t(\omega)\,.$$

for all $t \in \mathcal{T}_{\geqslant 0}$, for all $\omega \in \widetilde{\Omega}$. So $\bar{q}$ is θ-stationary.

(Necessity) Suppose that $\bar{q} \in \mathcal{S}_\theta^X$ is θ-stationary and define $q \in X_\mathcal{B}^\Omega$ by

$$q(\omega) := \bar{q}_0(\omega)\,, \quad \omega \in \Omega\,. \tag{2.10}$$

We have

$$\bar{q}_{t+s}(\theta_{-s}\omega) = [\rho_s(\bar{q})]_t(\omega) = \bar{q}_t(\omega)\,, \quad \forall s, t \in \mathcal{T}_{\geqslant 0}\,,\ \widetilde{\forall}\omega \in \Omega\,.$$

Setting $t = 0$ and renaming s as t, we then have

$$\bar{q}_t(\theta_{-t}\hat{\omega}) = \bar{q}_0(\hat{\omega}) = q(\hat{\omega})\,, \quad \forall t \in \mathcal{T}_{\geqslant 0}\,,\ \widetilde{\forall}\hat{\omega} \in \Omega\,.$$

Let $\widetilde{\Omega} \subseteq \Omega$ be a corresponding θ-invariant set of full measure. Given any $\omega \in \widetilde{\Omega}$ and any $t \in \mathcal{T}_{\geqslant 0}$, we may apply this property with $\hat{\omega} = \theta_t\omega$ due to the θ-invariance of $\widetilde{\Omega}$, thus obtaining

$$\bar{q}_t(\omega) = q(\theta_t\omega)\,.$$

This shows that (2.9) holds. $\square$

Note that the random variable q associated to $\bar{q}$ is unique up to a θ-invariant set of measure zero. Indeed, it is determined θ-almost everywhere by Equation (2.10). We will always use an overbar to denote the θ-stationary θ-stochastic process $\bar{q}$ associated with a given random variable q.

Proposition 2.9. *A θ-stationary stochastic process $\bar{q} \in \mathcal{S}_\theta^X$ is stationary in the sense that*

$$\mathbb{P}(\bar{q}_{t_1} \in A_1, \ldots, \bar{q}_{t_k} \in A_k) = \mathbb{P}(\bar{q}_{t_1+h} \in A_1, \ldots, \bar{q}_{t_k+h} \in A_k) \tag{2.11}$$

for all $k \in \mathbb{N}$, for any $A_1, \ldots, A_k \in \mathcal{F}$, and any $t_1, \ldots t_k, h \geqslant 0$.

Proof. Let $q := \bar{q}_0$ be the generator of $\bar{q}$ given by Lemma 2.8. In particular, $\bar{q}_t$ and $q(\theta_t \cdot)$ differ only on the same (θ-invariant) subset of probability zero for all $t \geqslant 0$. Therefore

$$\mathbb{P}(\bar{q}_{t_1} \in A_1, \ldots, \bar{q}_{t_k} \in A_k) = \mathbb{P}(q(\theta_{t_1} \cdot) \in A_1, \ldots, q(\theta_{t_k} \cdot) \in A_k)\,,$$

$$\mathbb{P}(\bar{q}_{t_1+h} \in A_1, \ldots, \bar{q}_{t_k+h} \in A_k) = \mathbb{P}(q(\theta_{t_1+h} \cdot) \in A_1, \ldots, q(\theta_{t_k+h} \cdot) \in A_k)\,,$$

and so we conclude that (2.11) is equivalent to

$$\mathbb{P}(q(\theta_{t_1} \cdot) \in A_1, \ldots, q(\theta_{t_k} \cdot) \in A_k) = \mathbb{P}(q(\theta_{t_1+h} \cdot) \in A_1, \ldots, q(\theta_{t_k+h} \cdot) \in A_k)\,,$$

which we now proceed to prove.

First note that

$$\begin{aligned} \{\omega \in \Omega\,;\; q(\theta_t \omega) \in A\} &= \theta_{-t}(\{\omega \in \Omega\,;\; q(\omega) \in A\}) \\ &= \theta_{-t}\big(q^{-1}(A)\big)\,, \quad \forall t \geqslant 0\,,\ \forall A \in \mathcal{F}\,. \end{aligned}$$

Denote

$$A'_j := q^{-1}(A_j)\,, \quad j = 1, \ldots, k\,.$$

Because θ_{-h} is invertible, and in virtue of the group action property, we also have

$$\begin{aligned} \theta_{-h}\big(\theta_{-t_1}(A'_1) \cap \cdots \cap \theta_{-t_k}(A'_k)\big) &= \big(\theta_{-h}\theta_{-t_1}(A'_1) \cap \cdots \cap \theta_{-h}\theta_{-t_k}(A'_k)\big) \\ &= \theta_{-h-t_1}(A'_1) \cap \cdots \cap \theta_{-h-t_k}(A'_k)\,. \end{aligned}$$

Furthermore,

$$\mathbb{P}\big(\theta_{-h}\big(\theta_{-t_1}(A'_1) \cap \cdots \cap \theta_{-t_k}(A'_k)\big)\big) = \mathbb{P}\big(\theta_{-t_1}(A'_1) \cap \cdots \cap \theta_{-t_k}(A'_k)\big)$$

by the measure-preserving property. Thus

$$\begin{aligned} \mathbb{P}(q(\theta_{t_1} \cdot) \in A_1, \ldots, q(\theta_{t_k} \cdot) \in A_k) &= \mathbb{P}(\theta_{-t_1}(A'_1) \cap \cdots \cap \theta_{-t_k}(A'_k)) \\ &= \mathbb{P}(\theta_{-h-t_1}(A'_1) \cap \cdots \cap \theta_{-h-t_k}(A'_k)) \\ &= \mathbb{P}(q(\theta_{t_1+h} \cdot) \in A_1, \ldots, q(\theta_{t_k+h} \cdot) \in A_k)\,. \end{aligned}$$

Since $k \in \mathbb{N}$, $A_1, \dots, A_k \in \mathcal{F}$ and $t_1, \dots t_k, h \geqslant 0$ were chosen arbitrarily, this completes the proof. $\square$

Example 2.10 (Stationarity Does Not Imply θ-Stationarity). We observe that the converse of Proposition 2.9 does not hold, as illustrated by the following simple example. Let θ be the Bernoulli shift of $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$, where $\Omega_0 := \{0, 1\}$, $\mathcal{F}_0 := 2^{\Omega_0}$, and $\mathbb{P}_0 \colon \mathcal{F}_0 \to [0, 1]$ is determined by

$$\mathbb{P}_0(\{0\}) = \mathbb{P}_0(\{1\}) = \frac{1}{2}\,.$$

Let $q \colon \Omega \to [0, 1]$ be the random variable defined by

$$q(\omega) := \omega_0\,, \quad \omega = (\omega_k)_{k \in \mathbb{N}} \in \Omega\,,$$

and let $\bar{q}$ be the θ-stationary θ-stochastic process generated by q via Lemma 2.8, defined by

$$\bar{q}_n(\omega) := q(\theta_n \omega)\,, \quad (n, \omega) \in \mathbb{Z} \times \Omega\,. \tag{2.12}$$

By Proposition 2.9, $\bar{q}$ is a stationary process.

Now let $\hat{\theta}$ be the MPDS over $(\Omega, \mathcal{F}, \mathbb{P})$ defined as in Example 2.6 with, say, $k = 2$. We will show that $\bar{q}$ is not $\hat{\theta}$-stationary. Suppose on the contrary that it is. Then

$$\bar{q}_n(\omega) = \bar{q}_0(\hat{\theta}_n \omega) = q(\hat{\theta}_n \omega)\,, \quad \forall n \in \mathbb{Z}\,, \quad \forall \omega \in \hat{\Omega}\,, \tag{2.13}$$

where $\hat{\Omega}$ is a $\hat{\theta}$-invariant subset of full-measure of Ω. The first equality follows from Lemma 2.8, and the second one follows from the construction of $\bar{q}$. Combining (2.12) and (2.13) with $n = 1$, we obtain

$$q(\hat{\theta}_1 \omega) = q(\theta_2 \omega) = \omega_2 = \omega_1 = q(\theta_1 \omega)\,, \quad \forall \omega \in \hat{\Omega}\,.$$

Thus $\hat{\Omega}$ is contained in

$$\begin{aligned} D \;&:=\; \{\omega \in \Omega\,;\ \omega_2 = \omega_1\} \\ &=\; \bigcup_{i=1}^{2} \left[\left(\prod_{j=-\infty}^{0} \Omega_0 \right) \times \{i\} \times \{i\} \times \left(\prod_{j=3}^{\infty} \Omega_0 \right) \right]. \end{aligned}$$

Since $\mathbb{P}(\hat{\Omega}) = 1$, we must then have $\mathbb{P}(D) = 1$ also. This is a contradiction with the fact that

$$\mathbb{P}(D) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} < 1$$

by construction. Therefore q cannot be $\hat{\theta}$-stationary. $\lozenge$

2.2 Random Sets

Definition 2.11 (Random Set)**.** Given a topological space X, a multifunction $D\colon \Omega \to 2^X$ is said to be a *random set* (or simply *measurable*) if

$$D^{-1}(U) := \{\omega \in \Omega\,;\ D(\omega) \cap U \neq \varnothing\} \in \mathcal{F}$$

for every open set $U \subseteq X$. $\triangle$

Note that

$$\omega \longmapsto \{v(\omega)\}\,, \quad \omega \in \Omega\,,$$

is a random set for any random variable $v \in X^{\Omega}_{\mathcal{B}}$. Conversely, if D is a random set such that $D(\omega)$ is a singleton $\{v_\omega\}$ for each $\omega \in \Omega$, then the mapping

$$\omega \longmapsto v_\omega\,, \quad \omega \in \Omega\,,$$

is a random variable. Thus we will sometimes refer to a random variable v as a *random singleton*, if for some reason framing said random variable as a random set is pertinent.

Proposition 2.12. *Given a topological space X, a multifunction $D\colon \Omega \to 2^X$ is measurable if, and only if its* closure $\overline{D}\colon \Omega \to 2^X$, *defined by*

$$\overline{D}(\omega) := \overline{D(\omega)}\,, \quad \omega \in \Omega\,,$$

is also measurable.

Proof. Indeed, for any open subset $U \subseteq X$ and any subset $A \subseteq X$, we have $A \cap U \neq \varnothing$ if, and only if $\overline{A} \cap U \neq \varnothing$. Thus

$$\{\omega \in \Omega\,;\ D(\omega) \cap U \neq \varnothing\} = \{\omega \in \Omega\,;\ \overline{D(\omega)} \cap U \neq \varnothing\}$$

for every open subset $U \subseteq X$. This proves the equivalence. $\square$

Lemma 2.13. *Let X be a topological space. If $D, E\colon \Omega \to 2^X$ are measurable multifunctions, then their union $D \cup E\colon \Omega \to 2^X$,*

$$(D \cup E)(\omega) := D(\omega) \cup E(\omega)\,, \quad \omega \in \Omega\,,$$

is also measurable.

Proof. Fix arbitrarily an open subset $U \subseteq X$. Then

$$(D \cup E)^{-1}(U) = D^{-1}(U) \cup E^{-1}(U)\,,$$

which belongs to $\mathcal{F}$. $\square$

Example 2.14 (Random Open/Closed Balls)*.* This example generalizes Example 1.3.1 in [8]. Let (X, d) be a metric space. Given any random variables $a\colon \Omega \to X$ and $r\colon \Omega \to \mathbb{R}_{\geqslant 0}$, let $B\colon \Omega \to 2^X$ be the multifunction defined by

$$B(\omega) := B_{r(\omega)}(a(\omega)) = \{x \in X\,;\ d(x, a(\omega)) < r(\omega)\}\,, \quad \omega \in \Omega\,.$$

Then B is a random set, referred to as the *random open ball of radius r and centered at a*. Indeed, let U be any open subset of X. First note that

$$\begin{aligned} B^{-1}(U) &= \{\omega \in \Omega\,;\ B(\omega) \cap U \neq \varnothing\} \\ &= \{\omega \in \Omega\,;\ a(\omega) \in U_{r(\omega)}\}\,, \end{aligned}$$

where we denote

$$S_\delta := \bigcup_{x \in S} B_\delta(x)\,, \quad S \subseteq X\,, \quad \delta \geqslant 0\,.$$

Now let $(s_n)_{n\in\mathbb{N}}$ be any enumeration of the nonnegative rational numbers. Then

$$\{\omega \in \Omega\,;\ a(\omega) \in U_{r(\omega)}\} = \bigcup_{n=1}^{\infty} \left([r \geqslant s_n] \cap [a \in U_{s_n}]\right)\,, \tag{2.14}$$

where we denote

$$[r \geqslant s_n] := \{\omega \in \Omega\,;\ r(\omega) \geqslant s_n\}\,, \quad n \in \mathbb{N}\,,$$

and, likewise,

$$[a \in U_{s_n}] := \{\omega \in \Omega\,;\ a(\omega) \in U_{s_n}\}\,, \quad n \in \mathbb{N}\,.$$

The righthand side in (2.14) is clearly $\mathcal{F}$-measurable, since a and r are assumed to be random variables. Thus $B^{-1}(U)$ is $\mathcal{F}$-measurable. Since the open subset $U \subseteq X$ was chosen arbitrarily, this proves that B is a random set.

Now consider the multifunction $\overline{B}\colon \Omega \to 2^X$ defined by

$$\overline{B}(\omega) := \overline{B}_{r(\omega)}(a(\omega))\,, \quad \omega \in \Omega\,.$$

Then

$$\overline{B}(\omega) = \overline{B(\omega)}\,, \quad \omega \in \Omega\,.$$

So it follows from Proposition 2.12 that $\overline{B}$ is also a random set. We shall refer to $\overline{B}$ as the *random closed ball of radius r and centered at a.* $\Diamond$

2.2.1 Polish Spaces

The concept of random set is much easier to deal with when the underlying topological space is separable and metrizable by a complete metric. Whenever an abbreviation may be convenient, we shall follow the well-established tradition of referring to those as *Polish spaces*. In Polish spaces there are several properties equivalent to that of being a random set, providing us with more tools to check for measurability. We list some of these in the propositions below.

Proposition 2.15. *Suppose that (X, d) is a separable metric space. A multifunction $D\colon \Omega \to 2^X$ is a random set if, and only if*

$$\omega \longmapsto \operatorname{dist}(x, D(\omega)) := \inf_{y \in D(\omega)} d(x, y)\,, \quad \omega \in \Omega\,,$$

defines a Borel-measurable[7] *map $\Omega \to \overline{\mathbb{R}}_{\geqslant 0}$ for each $x \in X$.*

Proof. See [27, Proposition 1.4 on page 142]. $\square$

The two standard pieces of terminology below will not be used very often throughout the rest of this work. They will, however, make it easier to parse the statement of Proposition 2.18 below.

[7] *Our convention is that* $\inf \varnothing := +\infty$.

Definition 2.16 (Representation of Multifunctions)**.** Let Ω, X, Y be nonempty sets, $D\colon \Omega \to 2^X$ and $g\colon \Omega \times Y \to X$. We say that the pair (Y, g) *represents* the multifunction D if

$$g(\omega, Y) := \{g(\omega, y)\,;\ y \in Y\} = D(\omega)$$

for each $\omega \in \Omega$. $\triangle$

Definition 2.17 (Carathéodory Maps)**.** Let $(\Omega, \mathcal{F})$ be a measurable space, X and Y be topological spaces. A map $g\colon \Omega \times Y \to X$ is said to be *Carathéodory* if

$$g(\omega, \cdot)\colon Y \longrightarrow X$$

is continuous for every $\omega \in \Omega$ and

$$g(\cdot, y)\colon \Omega \longrightarrow X$$

is $\mathcal{F}$-measurable for every $y \in Y$. $\triangle$

If Y is a separable[8] metric space and X is a metric space, then a Carathéodory map $g\colon \Omega \times Y \to X$ is, in fact, $(\mathcal{F} \otimes \mathcal{B}(Y))$-measurable, sometimes expressed simply as *jointly measurable* (see [27, Proposition 1.6 on page 142]). As we noted before, in this work X and Y will often have at least the topological structure of a Polish space, in which case Carathéodory maps will then be jointly measurable.

Proposition 2.18. *Suppose that $D\colon \Omega \to 2^X \backslash \{\varnothing\}$ is a closed random set in a Polish space X. Then there exist a Polish space Y and a Carathéodory map $g\colon \Omega \times Y \to X$ such that (Y, g) represents D. Furthermore, for each metric d_X in X, a metric d_Y in Y can be chosen[9] such that*

$$d_X(g(\omega, y_1), g(\omega, y_2)) \leqslant d_Y(y_1, y_2)(1 + d_X(g(\omega, y_1), g(\omega, y_2)))$$

for all $\omega \in \Omega$ and all $y_1, y_2 \in Y$.

Proof. See [29, Theorem 1 and Corollary 1.1 on pages 134–135]. $\square$

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[8]If Y is not separable, this might fail; [2] refers to [15] for a counterexample.

[9]*It is to be tacitly understood that d_X and d_Y generate the topologies in X and Y, respectively, and that the metric spaces (X, d_X) and (Y, d_Y) are complete.*

Proposition 2.19 (Measurable Selection Theorem)**.** *Suppose that X is a Polish space and let $D\colon \Omega \to 2^X \backslash \{\varnothing\}$ be a multifunction such that $D(\omega)$ is closed for each $\omega \in \Omega$. Then D is a (closed) random set if, and only if there exists a sequence $(v_n)_{n\in\mathbb{N}}$ of measurable maps $\Omega \to X$ such that*

$$v_n \in D\,, \quad \forall n \in \mathbb{N}\,,$$

and

$$D(\omega) = \overline{\{v_n(\omega)\,;\; n \in \mathbb{N}\}}\,, \quad \forall \omega \in \Omega\,.$$

In particular, any closed random set $D\colon \Omega \to 2^X \backslash \{\varnothing\}$ has a measurable selection*—that is, a measurable map $v\colon \Omega \to X$ such that $v(\omega) \in D(\omega)$ for each $\omega \in \Omega$.*

Proof. ($\Rightarrow$) Suppose D is a closed random set. By Proposition 2.18, there exist a Polish space Y and a Carathéodory map $g\colon \Omega \times Y \to X$ such that (Y, g) represents D. Let $(y_n)_{n\in\mathbb{N}}$ be any dense sequence in Y. It then follows from continuity with respect to the second variable that

$$D(\omega) = g(\omega, Y) = \overline{\{g(\omega, y_n)\,;\; n \in \mathbb{N}\}}\,, \quad \forall \omega \in \Omega\,.$$

Since $\omega \mapsto g(\cdot, y_n)$, $\omega \in \Omega$, is measurable for each $n \in \mathbb{N}$, the result then follows with $v_n := g(\cdot, y_n)$, $n \in \mathbb{N}$.

($\Leftarrow$) Now suppose that $(v_n)_{n\in\mathbb{N}}$ is a sequence of measurable maps $\Omega \to X$ such that

$$D(\omega) = \overline{\{v_n(\omega)\,;\; n \in \mathbb{N}\}}\,, \quad \forall \omega \in \Omega\,.$$

Fix arbitrarily an open subset U of X. Then

$$\begin{aligned}\{\omega \in \Omega\,;\; D(\omega) \cap U \neq \varnothing\} &= \{\omega \in \Omega\,;\; \{v_n(\omega)\,;\; n \in \mathbb{N}\} \cap U \neq \varnothing\} \\ &= \bigcup_{n\in\mathbb{N}} \{\omega \in \Omega\,;\; v_n(\omega) \in U\}\,,\end{aligned}$$

which is the countable union of $\mathcal{F}$-measurable subsets of Ω. Since U open in X was chosen arbitrarily, this shows D is measurable.

For the second conclusion of the proposition, we may simply take v to be any of the v_n's. $\square$

2.2.2 Universally Measurable Sets

Given a measurable space $(\Omega, \mathcal{F})$ and a probability measure ν on this space, we denote by $\bar{\mathcal{F}}^{\nu}$ the *completion of* $\mathcal{F}$ *with respect to* ν (or the ν*-completion of* $\mathcal{F}$)—that is, the σ-algebra consisting of all subsets $A \subseteq \Omega$ such that $B \subseteq A \subseteq C$ for some $B, C \in \mathcal{F}$ with $\nu(B) = \nu(C)$. Denote by $N(\Omega, \mathcal{F})$ the collection of all probability measures on $(\Omega, \mathcal{F})$.

Definition 2.20 (Universal σ-Algebra)**.** The *universal* σ*-algebra* of a measurable space $(\Omega, \mathcal{F})$ is defined to be the σ-algebra

$$\mathcal{F}^u := \bigcap_{\nu \in N(\Omega, \mathcal{F})} \bar{\mathcal{F}}^{\nu}$$

of subsets of Ω. The sets in $\mathcal{F}^u$ are called the *universally measurable* sets associated with $(\Omega, \mathcal{F})$. $\triangle$

Since $\mathcal{F} \subseteq \bar{\mathcal{F}}^{\nu}$ for any $\nu \in N(\Omega, \mathcal{F})$, it follows straight from the definition that $\mathcal{F} \subseteq \mathcal{F}^u$. Furthermore, we have $\mathcal{F}^u \subseteq \mathcal{F}$ whenever $(\Omega, \mathcal{F}, \nu)$ is a complete probability space for some probability measure ν on $(\Omega, \mathcal{F})$. So, in this case, we have indeed $\mathcal{F}^u = \mathcal{F}$.

Proposition 2.21 (Measurable Projection Theorem)**.** *Let X be a Polish space, and let $(\Omega, \mathcal{F})$ be a measurable space. If $M \subseteq \Omega \times X$ is $(\mathcal{F} \otimes \mathcal{B}(X))$-measurable, then the projection*

$$\operatorname{proj}_{\Omega} M := \{\omega \in \Omega\,;\ (\omega, x) \in M \textit{ for some } x \in X\}$$

is universally measurable; that is, $\operatorname{proj}_{\Omega} M \in \mathcal{F}^u$*. In particular, if $(\Omega, \mathcal{F}, \nu)$ is complete for some probability measure ν, then* $\operatorname{proj}_{\Omega} M$ *is $\mathcal{F}$-measurable.*

Proof. See [11, Proposition 8.4.4 on page 281]. $\square$

2.2.3 Tempered Random Sets

In what follows we suppose given an MPDS $\theta = (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathcal{T}})$.

Definition 2.22 (Real-Valued Tempered Random Variables)**.** A real-valued function $r \colon \Omega \to \mathbb{R}$ is said to be a *tempered random variable* (*with respect to the underlying MPDS* θ) if it is Borel-measurable, and, for every $\gamma > 0$,

$$\sup_{s \in \mathcal{T}} |r(\theta_s \omega)|\, \mathrm{e}^{-\gamma |s|} < \infty\,, \quad \widetilde{\forall} \omega \in \Omega\,. \tag{2.15}$$

We denote the family of real-valued, tempered random variables (with respect to the MPDS θ) by $\mathbb{R}_\theta^\Omega$. $\triangle$

Equation (2.15) is equivalent to

$$|r(\theta_s\omega)| \leqslant K_{\gamma,\omega}\, \mathrm{e}^{\gamma|s|}\,, \quad \forall s \in \mathcal{T}\,, \quad \widetilde{\forall}\omega \in \Omega\,,$$

for constants $K_{\gamma,\omega} \geqslant 0$ depending on $\gamma > 0$ and $\omega \in \Omega$. Thus tempered random variables can also be interpreted simply as random variables with sub-exponential growth along θ-almost every orbit of θ.

Remark 2.23. (1) Any Borel-measurable function which is bounded along θ-almost every orbit of θ is tempered. In particular, any θ-almost everywhere constant random variable is tempered.

(2) If $r_1 \in \mathbb{R}_\theta^\Omega$ and $|r_2(\omega)| \leqslant |r_1(\omega)|$ for θ-almost every $\omega \in \Omega$, then r_2 is also tempered.

(3) A θ-invariant subset $\widetilde{\Omega} \subseteq \Omega$ of full measure on which (2.15) holds can be chosen to be the same for every $\gamma > 0$. Indeed, let $(\gamma_k)_{k\in\mathbb{N}}$ be a sequence convergent to zero from above. For each $k \in \mathbb{N}$, let $\widetilde{\Omega}_k \subseteq \Omega$ be a θ-invariant subset of full measure such that (2.15) holds for every $\omega \in \widetilde{\Omega}_k$ with $\gamma = \gamma_k$. Observe that

$$\sup_{s\in\mathcal{T}} |r(\theta_s\omega)|\, \mathrm{e}^{-\delta|s|} \leqslant \sup_{s\in\mathcal{T}} |r(\theta_s\omega)|\, \mathrm{e}^{-\gamma|s|}$$

whenever $0 < \gamma < \delta$. Thus

$$\widetilde{\Omega} := \bigcap_{k=1}^{\infty} \widetilde{\Omega}_k$$

is a θ-invariant subset of full measure on which (2.15) holds for every $\gamma > 0$.

(4) We do not require the bound in Definition 2.22 to be independent of $\omega \in \Omega$. In fact, if this were the case, then r would have been essentially bounded. For suppose that, for some $\gamma > 0$, there exists a $K_\gamma \geqslant 0$ such that

$$\sup_{s\in\mathcal{T}} |r(\theta_s\omega)|\, \mathrm{e}^{-\gamma|s|} \leqslant K_\gamma\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

Then

$$|r(\omega)| \leqslant \sup_{s\in\mathcal{T}} |r(\theta_s\omega)|\, \mathrm{e}^{-\gamma|s|} \leqslant K_\gamma\,, \quad \widetilde{\forall}\omega \in \Omega\,,$$

showing that r is essentially bounded. $\square$

Lemma 2.24. *For any $r_1, r_2 \in \mathbb{R}^{\Omega}_{\theta}$, r_1+r_2 and $r_1 r_2$ are also tempered random variables. In particular, $\mathbb{R}^{\Omega}_{\theta}$ is a commutative ring with the operations of pointwise addition and multiplication.*

Proof. Let $\widetilde{\Omega}$ be a θ-invariant subset of full measure of Ω over which (2.15) holds for both r_1 and r_2, for any $\gamma > 0$ (see Remark 2.23(3) above). Fix any such γ arbitrarily. Then

$$\begin{aligned}\sup_{s\in\mathcal{T}} |(r_1+r_2)(\theta_s\omega)| e^{-\gamma|s|} &\leqslant \sup_{s\in\mathcal{T}} |r_1(\theta_s\omega)| e^{-\gamma|s|} + \sup_{s\in\mathcal{T}} |r_2(\theta_s\omega)| e^{-\gamma|s|} \\ &< \infty\,, \quad \forall \omega \in \widetilde{\Omega}\,.\end{aligned}$$

Since $\gamma > 0$ was chosen arbitrarily, this shows that $r_1 + r_2$ is tempered.

Similarly, for any $\gamma > 0$,

$$\begin{aligned}\sup_{s\in\mathcal{T}} |r_1(\theta_s\omega) r_2(\theta_s\omega)| e^{-\gamma|s|} &= \sup_{s\in\mathcal{T}} |r_1(\theta_s\omega)| e^{-\frac{\gamma}{2}|s|} |r_2(\theta_s\omega)| e^{-\frac{\gamma}{2}|s|} \\ &\leqslant \left(\sup_{s\in\mathcal{T}} |r_1(\theta_s\omega)| e^{-\frac{\gamma}{2}|s|}\right)\left(\sup_{s\in\mathcal{T}} |r_2(\theta_s\omega)| e^{-\frac{\gamma}{2}|s|}\right) \\ &< \infty\,, \quad \forall \omega \in \widetilde{\Omega}\,.\end{aligned}$$

This shows that $r_1 r_2$ is also tempered. □

Corollary 2.25. *For any $r_1, r_2 \in (\mathbb{R})^{\Omega}_{\theta}$, the random variables $r_1 \wedge r_2 \colon \Omega \to \mathbb{R}$ and $r_1 \vee r_2 \colon \Omega \to \mathbb{R}$ defined, respectively, by*

$$(r_1 \wedge r_2)(\omega) := \min\{r_1(\omega), r_2(\omega)\}\,, \quad \omega \in \Omega\,,$$

and

$$(r_1 \vee r_2)(\omega) := \max\{r_1(\omega), r_2(\omega)\}\,, \quad \omega \in \Omega\,,$$

are tempered.

Proof. Indeed,

$$r_1 \wedge r_2 = \frac{r_1+r_2}{2} - \frac{|r_1-r_2|}{2}$$

and

$$r_1 \vee r_2 = \frac{r_1+r_2}{2} + \frac{|r_1-r_2|}{2}\,.$$

Now it follows from Remark 2.23(2) and Lemma 2.24 that $|r_1 - r_2|$ is tempered. Hence $r_1 \wedge r_2$ and $r_1 \vee r_2$ are also tempered by the same lemma. □

Definition 2.26 (Tempered Random Sets)**.** Let (X, d) be a metric space. A random set $D\colon \Omega \to 2^X$ is said to be *tempered* (*with respect to* θ) if there exist $x_0 \in X$ and a nonnegative tempered random variable $r\colon \Omega \to \mathbb{R}_{\geqslant 0}$ such that

$$D(\omega) \subseteq \overline{B}_{r(\omega)}(x_0)\,, \quad \widetilde{\forall}\omega \in \Omega\,. \tag{2.16}$$

A Borel-measurable map $v\colon \Omega \to X$ is said to be a *tempered random variable* (*with respect to* θ) if the random singleton defined by $\omega \mapsto \{v(\omega)\}$, $\omega \in \Omega$, is a tempered random set.

We denote the family of tempered random sets (with respect to θ) by $(2^X)^\Omega_\theta$, and the family of tempered random variables (with respect to θ) is denoted by X^Ω_θ. △

Example 2.27 (Tempered Random Balls). Let (X, d) be a metric space. If

$$a\colon \Omega \longrightarrow X \quad \text{and} \quad r\colon \Omega \to \mathbb{R}_{\geqslant 0}$$

are tempered random variables, then

$$B(\cdot) := B_{r(\cdot)}(a(\cdot))\colon \Omega \longrightarrow 2^X \backslash \{\varnothing\}$$

is a tempered random set.

It was shown in Example 2.14 that B is a random set, so it remains to show that it is tempered. Let $x_0 \in X$ and r_0 be a nonnegative tempered random variable such that

$$\{a(\omega)\} \subseteq \overline{B}_{r_0(\omega)}(x_0)\,, \quad \widetilde{\forall}\omega \in \Omega\,;$$

in other words,

$$d(a(\omega), x_0) \leqslant r_0(\omega)\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

Then

$$d(x, x_0) \leqslant d(x, a(\omega)) + d(a(\omega), x_0) \leqslant r(\omega) + r_0(\omega)\,, \quad \forall x \in B(\omega)\,, \quad \widetilde{\forall}\omega \in \Omega\,;$$

in other terms,

$$B(\omega) \subseteq B_{(r+r_0)(\omega)}(x_0)\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

Since $r + r_0$ is tempered by Lemma 2.24, we conclude that B is tempered.

The same construction and estimates show that the closed random ball

$$\overline{B}(\cdot) := \overline{B}_{r(\cdot)}(a(\cdot))$$

is also tempered, provided that its center a and its radius r are also tempered. $\Diamond$

Remark 2.28. (1) Note that if there exist $x_0 \in X$ and a nonnegative tempered random variable r such that (2.16) holds, then for every $x \in X$, there exists a nonnegative random variable r_x such that (2.16) holds with x and r_x in place of x_0 and r, respectively. Indeed, one can simply take $r_x(\cdot) := r(\cdot)+d(x_0, x)$. This is particularly convenient when the underlying metric space is a normed vector space, in which case we may always choose the reference point to be 0.

(2) Definitions 2.22 and 2.26 agree for random variables when $X = \mathbb{R}$. Indeed, if r is tempered in the sense of Definition 2.22, then it can be seen to be also tempered in the sense of Definition 2.26 with $x_0 = 0$. Conversely, if r is tempered in the sense of Definition 2.26, then

$$|r(\omega)| \leqslant |r(\omega) - x_0| + |x_0| \leqslant r_0(\omega) + |x_0| \,, \quad \widetilde{\forall}\omega \in \Omega \,,$$

for some $x_0 \in \mathbb{R}$ and some nonnegative tempered random variable r_0 (in the sense of Definition 2.22). Thus it follows by Lemma 2.24 and Remark 2.23(2) that r is also tempered in the sense of Definition 2.22. $\square$

Most of the time the underlying MPDS θ will be clear from the context. Therefore we shall say simply ‘tempered’ to mean ‘tempered with respect to θ,’ unless there is any risk of confusion.

Lemma 2.29. *Let (X, d) be a metric space. If $D, E \colon \Omega \to 2^X$ are tempered random sets, then their union $D \cup E$ is also a tempered random set.*

Proof. It follows from Lemma 2.13 that $D \cup E$ is measurable. Now let $r, s \in (\mathbb{R}_{\geqslant 0})^{\Omega}_{\theta}$ be tempered random variables such that

$$D \subseteq \overline{B}_r(0) \quad \text{and} \quad E \subseteq \overline{B}_s(0) \,.$$

Therefore

$$D \cup E \subseteq \overline{B}_{r \vee s}(0) \,.$$

It follows from Corollary 2.25 that $r \vee s$ is tempered, showing that $D \cup E$ is tempered. $\square$

It is also worth noting that the concept of temperedness is independent of the norm in finite-dimensional spaces. Thus the analysis in each of the finite-dimensional examples we shall discuss in the next chapter remains valid regardless of the underlying norm. We parse this into the following two results.

Lemma 2.30. *Suppose that $(X, \|\cdot\|)$ is a real normed space. Then a random variable $R\colon \Omega \to X$ is tempered if, and only if $\|R(\cdot)\|\colon \Omega \to \mathbb{R}_{\geqslant 0}$ is a tempered random variable.*

Proof. This is a rehash of Remark 2.28(1) for normed spaces. $\square$

Lemma 2.31. *Suppose that $(X, \|\cdot\|)$ is a finite-dimensional, real normed space. Then R is a tempered random variable if, and only if R is tempered with respect to any norm $\|\cdot\|_*\colon X \to \mathbb{R}_{\geqslant 0}$; in other words, temperedness is independent of the norm in finite-dimensional spaces.*

Proof. Since all norms in a finite dimensional space are equivalent, they all generate the same topology. Therefore measurability does not depend on the norm. Furthermore, if $\|\cdot\|_* \leqslant \beta\|\cdot\|$ for some $\beta \geqslant 0$ and $\|R\|$ is tempered, then $\|R\|_*$ is also tempered. It follows from Lemma 2.30 that R is tempered with respect to any norm in X. $\square$

The next result generalizes Lemma 2.24.

Lemma 2.32. *Suppose that $(X, \|\cdot\|)$ is a real normed space. Then $R_1 + R_2$ and rR_1 are tempered random variables $\Omega \to X$ for any $R_1, R_2 \in X^\Omega_\theta$ and any $r \in \mathbb{R}^\Omega_\theta$. In particular, X^Ω_θ is a module over the ring of real-valued tempered random variables.*

Proof. Since $\|R_1 + R_2\| \leqslant \|R_1\| + \|R_2\|$ and $\|rR_1\| = |r|\|R_1\|$, this follows straight from Lemma 2.24, plus Corollary 2.30. $\square$

Temperedness often implies other useful properties. The results below will allow us to avoid making redundant assumptions without having to digress into proving that such extra hypotheses are actually not needed.

Lemma 2.33. *If $b\colon \Omega \to \mathbb{R}$ is a tempered random variable, then*

$$t \longmapsto b(\theta_t\omega)\,, \quad t \in \mathbb{R}\,,$$

is locally essentially bounded for θ-almost all $\omega \in \Omega$. In particular,

$$b(\theta.\omega)\colon \mathbb{R} \longrightarrow \mathbb{R}$$

is locally integrable for θ-almost all $\omega \in \Omega$.

Proof. It follows from Corollary C.9 that $t \mapsto b(\theta_t\omega)$, $t \in \mathbb{R}$, is Borel-measurable for each $\omega \in \Omega$. Now let $\widetilde{\Omega}_1$ be the set of all $\omega \in \Omega$ such that

$$K_{1,\omega} := \sup_{s\in\mathbb{R}} b(\theta_s\omega)\,\mathrm{e}^{-|s|} < \infty\,.$$

Because b is assumed to be tempered, $\widetilde{\Omega}_1$ can be chosen to be a θ-invariant subset of full measure of Ω. Fix arbitrarily $\omega \in \widetilde{\Omega}_1$. For each finite interval $[\alpha, \beta] \subseteq \mathbb{R}$, we have

$$0 \leqslant b(\theta_t\omega) \leqslant K_{1,\omega} \max\{\mathrm{e}^{|\alpha|}, \mathrm{e}^{|\beta|}\}\,, \quad \alpha \leqslant t \leqslant \beta\,.$$

So $t \mapsto b(\theta_t\omega)$, $t \in \mathbb{R}$, is locally bounded—and thus locally integrable. □

2.3 Partially Ordered Spaces

Definition 2.34 (Partially Ordered Topological Spaces)**.** A *partial ordered topological space* is an ordered pair $(X, \leqslant)$ in which X is a topological space and $\leqslant$ is a closed partial order (in the product space $X \times X$); that is,

(1) $x \leqslant x$ for every $x \in X$ ($\leqslant$ is *reflexive*),

(2) $x \leqslant y$ and $y \leqslant x$ imply $x = y$ ($\leqslant$ is *antisymmetric*),

(3) $x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$ ($\leqslant$ is *transitive*), and

(4) if $(x_\alpha)_{\alpha\in A}$ and $(y_\alpha)_{\alpha\in A}$ are nets in X convergent to x_∞, y_∞, respectively, and $x_\alpha \leqslant y_\alpha$ for every $\alpha \in A$, then $x_\infty \leqslant y_\infty$ ($\leqslant$ is closed in $X \times X$). △

In this work we shall deal exclusively with partially ordered sets which are also equipped with a topology which is *compatible* with the partial order in the sense of (4) in the above definition. Therefore we will often refer to partially ordered topological spaces simply as *partially ordered spaces*.

Definition 2.35 (Intervals and Extremes)**.** Let X be a partially ordered topological space.

(1) For any $a, b \in X$, the *order-interval* $[a, b]$ is defined by

$$[a, b] := \{x \in X\,;\; a \leqslant x \leqslant b\}\,.$$

We will often refer to these as simply *intervals*.

(2) An element $v \in X$ is said to be an *upper* (*lower*) *bound* of a subset $B \subseteq X$ if $x \leqslant v$ ($x \geqslant v$) for every $x \in B$.

(3) An upper (lower) bound $v_0 \in X$ of a subset $B \subseteq X$ is said to be the[10] *supremum* (*infimum*), denoted $\sup B$ ($\inf B$), if any other upper (lower) bound satisfies $v \geqslant v_0$ ($v \leqslant v_0$). The supremum (infimum) is also referred to as the *least upper bound* (*greatest lower bound*).

(4) An element $v \in B \subseteq X$ is said to be *maximal* (*minimal*) *in* B if $x \geqslant v$ ($x \leqslant v$) for some $x \in B$ implies $x = v$.

(5) A subset $B \subseteq X$ is said to be *order-bounded* if $B \subseteq [a, b]$ for some $a, b \in X$; in other terms, if it is contained in some order-interval. $\triangle$

Remark 2.36. (1) Simple examples in $\mathbb{R}^2$ show that $\sup B$ or $\inf B$, even if they exist, need not be in the closure $\overline{B}$ of B in X. (See *cone-induced partial orders* below. See also Lemma 2.38.)

(2) When the supremum (infimum) of B exists and belongs to B, it is a maximal (minimal) element of the set.

[10] Provided existence, uniqueness of the supremum (infimum) follows from the antissymmetric property of $\leqslant$.

(3) A maximal (minimal) element need not be an upper (lower) bound. Indeed, the definition of maximal (minimal) element does not require that the element be related to every other member of the set. □

Lemma 2.37. *Let A, B be subsets of a partially ordered topological space $(X, \leqslant)$. If* $\sup A$ *and* $\sup B$ *exist, and if for every $a \in A$ there exists an element $b \in B$ such that $a \leqslant b$, then* $\sup A \leqslant \sup B$*. The same is true if for every $b \in B$ there exists an $a \in A$ such that $a \leqslant b$.*

Similarly, if $\inf A$ *and* $\inf B$ *exist, and if for every $a \in A$ there exists an element $b \in B$ such that $a \leqslant b$, then* $\inf A \leqslant \inf B$*. The same is also true if for every $b \in B$ there exists an $a \in A$ such that $a \leqslant b$.*

Proof. Suppose $\inf A$ and $\inf B$ exist and that for every $b \in B$ there exists an $a \in A$ such that $a \leqslant b$. We have $\inf A \leqslant a \leqslant b$. So indeed $\inf A \leqslant b$ for every $b \in B$. This proves $\inf A \leqslant \inf B$.

All other cases can be proved using the exact same argument. □

Lemma 2.38. *Let B be a subset of a partially ordered topological space $(X, \leqslant)$. Then* $\sup B$ *exists if, and only if* $\sup \overline{B}$ *exists. In this case,* $\sup B = \sup \overline{B}$*. Analogously,* $\inf B$ *exists if, and only if* $\inf \overline{B}$ *exists, in which case* $\inf B = \inf \overline{B}$*.*

Proof. Suppose $\sup B$ exists. Given any $x \in \overline{B}$, let $(x_\alpha)_{\alpha \in A}$ be a net in B converging to x. We have

$$x_\alpha \leqslant \sup B\,, \quad \forall \alpha \in A\,,$$

hence $x \leqslant \sup B$ by Definition 2.34(4). So $\sup B$ is an upper bound for $\overline{B}$. Now let $v \in X$ be any upper bound of $\overline{B}$. Then v is an upper bound of B, so $v \geqslant \sup B$. This shows that $\sup B$ is the least upper bound of $\overline{B}$. In other words, $\sup \overline{B}$ exists and is equal to $\sup B$.

Conversely, if $\sup \overline{B}$ exists, then it is clearly an upper bound for B. Now any upper bound of B is also an upper bound of $\overline{B}$ by the argument above. Thus $\sup \overline{B}$ is the least upper bound of B as well.

The proof for inf's is entirely analogous. □

2.3.1 Partially Ordered Vector Spaces

The core of our theory will be developed on subsets of separable, real Banach spaces. In those cases, the underlying partial order will be usually what we call a *cone-induced order.*

Definition 2.39 (Cone)**.** Let V be a real topological vector space. A *cone* in V is a subset $V_+ \subseteq V$ such that

(1) V_+ is closed,

(2) $V_+ + V_+ \subseteq V_+$,

(3) $\alpha V_+ \subseteq V_+$ for every $\alpha \geqslant 0$, and

(4) $V_+ \cap (-V_+) = \{0\}$. △

Definition 2.40 (Cone-Induced Partial Order)**.** Given a subset X of real topological vector space V and a cone $V_+ \subseteq V$, the binary relation $\leqslant_{V_+}$ on X defined by

$$x \leqslant_{V_+} y \quad \Longleftrightarrow \quad y - x \in V_+$$

is a partial order with closed diagonal

$$\{(x, y) \in X \times X\, ;\; x \leqslant_{V_+} y\}\,,$$

called the *partial order in X induced by V_+.*

As usual, we write $x <_{V_+} y$ when $x \leqslant_{V_+} y$ and $x \neq y$. Naturally, $x \geqslant_{V_+} y$ means that $y \leqslant_{V_+}$ and $x >_{V_+} y$ means that $y <_{V_+} x$. If $\operatorname{int} V_+ \neq \varnothing$, then we write $x \ll_{V_+} y$ or $y \gg_{V_+} x$ whenever $y - x \in \operatorname{int} V_+$. △

Whenever the underlying cone is clear from the context we shall drop the index 'V_+' in '$\leqslant_{V_+}$' and write just '$\leqslant$.' The same convention applies to '$<_{V_+}$,' '$\ll_{V_+}$,' '$\geqslant_{V_+}$,' '$>_{V_+}$,' and '$\gg_{V_+}$.'

The subset $X \subseteq V$ in Definition 2.40 will often be taken to be V itself, V_+ or $\operatorname{int} V_+$—when it is nonempty. In any case, it will always be a Borel subset of V. However we seem to neither lose anything from making the definition as general as we did, nor to gain anything from making it more specific.

Definition 2.41 (Solid and Minihedral Cones)**.** Let $V_+ \subseteq V$ be a cone in a real topological vector space V.

(1) If the interior $\operatorname{int} V_+$ of V_+ is nonempty, then V_+ is said to be a *solid* cone. In this case, any subset $X \subseteq V$ is said to be *strongly ordered by* V_+. We then write $x \ll y$ (or $y \gg x$) if $y - x \in \operatorname{int} V_+$.

(2) If every order-bounded, finite subset $M = \{v_1, \dots, v_k\} \subseteq V$ has a supremum (not necessarily in M), then V_+ is said to be *minihedral*. If every order-bounded subset $M \subseteq V$ has a supremum, then V_+ is said to be *strongly minihedral*. △

Definition 2.42 (Normal and Regular Cones)**.** Let $V_+ \subseteq V$ be a cone in a real normed vector space V.

(1) V_+ is said to be *normal* if there exists a constant $C_V \geqslant 0$ such that $0 \leqslant x \leqslant y$ implies $\|x\| \leqslant C_V \|y\|$.

(2) V_+ is said to be *regular* if every order-bounded monotone sequence converges in norm; that is,

$$x_1 \leqslant x_2 \leqslant x_3 \leqslant \cdots \leqslant x_n \leqslant \cdots \leqslant u$$

for some $u \in V$ implies that the sequence $(x_n)_{n \in \mathbb{N}}$ converges in norm to some $x_\infty \in V$. △

We follow up with a review of several results concerning normed spaces partially ordered by cones having one or more of the properties above. Proofs which are not readily found in the standard literature are provided.

Lemma 2.43. *Let V be a real normed space, partially ordered by a solid cone $V_+ \subseteq V$. Then*

(1) *the order interval $[-u, u]$ has nonempty interior for any $u \in \operatorname{int} V_+$; in particular, there exists $u \gg 0$ such that $[-u, u]$ contains the unit ball.*

(2) $V_+ = \overline{\operatorname{int} V_+}$*; in other words, a solid cone is the closure of its interior.*

Proof. (1) Pick any $u \gg 0$ and let $\delta > 0$ be such that $B_\delta(u) \subseteq V_+$. We show that $B_\delta(0) \subseteq [-u, u]$. Indeed, $B_\delta(u) \subseteq V_+$ is equivalent to

$$u + x \geqslant 0\,, \quad \forall x \in B_\delta(0)\,.$$

Thus

$$-x \leqslant u\,, \quad \forall x \in B_\delta(0)\,,$$

and

$$x \geqslant -u\,, \quad \forall x \in B_\delta(0)\,.$$

This amounts to

$$-\,u \leqslant x \leqslant u\,, \quad \forall x \in B_\delta(0)\,, \tag{2.17}$$

verifying the claim.

To prove the second part, multiply each term in (2.17) by $1/\delta$. This shows that $B_1(0) \subseteq [-(1/\delta)u, (1/\delta)u]$, where $B_1((1/\delta)u) \subseteq V_+$, and so $(1/\delta)u \in \operatorname{int} V_+$. This establishes (1).

(2) The inclusion $\overline{\operatorname{int} V_+} \subseteq V_+$ is just a topological fact, since V_+ is closed and $\operatorname{int} V_+ \subseteq V_+$. To prove the reciprocal inclusion, pick any $v \geqslant 0$, any $u \gg 0$, and let $u_n := u/n$, $n \in \mathbb{N}$. In particular, $u_n \gg 0$ for each $n \in \mathbb{N}$, and $u_n \to 0$ as $n \to \infty$. Hence $v + u_n \gg 0$ for each $n \in \mathbb{N}$ and $v + u_n \to v$ as $n \to \infty$. This shows that $v \in \overline{\operatorname{int} V_+}$. $\square$

Lemma 2.44. *Suppose that $(x_\alpha)_{\alpha \in A}$ is a net in a normed space X, partially ordered by a solid, normal cone $X_+ \subseteq X$. Suppose, in addition, that the net converges to an element $x_\infty \in X$, and that the infima and suprema*

$$x_\alpha^- := \inf\{x_{\alpha'}\,;\; \alpha' \geqslant \alpha\}$$

and

$$x_\alpha^+ := \sup\{x_{\alpha'}\,;\; \alpha' \geqslant \alpha\}$$

exist for every $\alpha \in A$. Then the nets $(x_\alpha^-)_{\alpha \in A}$ and $(x_\alpha^+)_{\alpha \in A}$ so defined also converge to x_∞.

Proof. Fix arbitrarily $x_0 \in \operatorname{int} X_+$. Then $rx_0 \in \operatorname{int} X_+$ for every scalar $r > 0$, and so the order interval $[-rx_0, rx_0]$ is a neighborhood of the origin by Lemma 2.43. It then follows that $[x_\infty - rx_0, x_\infty + rx_0]$ is a neighborhood of the limit x_∞ for each $r > 0$. Thus from the hypothesis of convergence, for every $r > 0$, there exists an $\alpha_r \in A$ such that

$$x_\alpha \in [x_\infty - rx_0, x_\infty + rx_0]\,, \quad \forall \alpha \geqslant \alpha_r\,.$$

Now

$$x_\alpha^- = \inf\{x_{\alpha'}\,;\ \alpha' \geqslant \alpha\} \geqslant x_\infty - rx_0\,, \quad \forall \alpha \geqslant \alpha_r\,, \quad \forall r > 0\,,$$

and, similarly,

$$x_\alpha^+ = \sup\{x_{\alpha'}\,;\ \alpha' \geqslant \alpha\} \leqslant x_\infty + rx_0\,, \quad \forall \alpha \geqslant \alpha_r\,, \quad \forall r > 0\,;$$

that is,

$$x_\infty - rx_0 \leqslant x_\alpha^- \leqslant x_\alpha^+ \leqslant x_\infty + rx_0\,, \quad \forall \alpha \geqslant \alpha_r\,, \quad \forall r > 0\,.$$

Let $C_{X_+} \geqslant 0$ be the normality constant of X_+. Then

$$\begin{aligned} \|x_\alpha^- - x_\infty\| &\leqslant \|x_\alpha^- - (x_\infty - rx_0)\| + \|rx_0\| \\ &\leqslant C_{X_+}\|(x_\infty + rx_0) - (x_\infty - rx_0)\| + \|rx_0\| \\ &= (2C_{X_+} + 1)\|x_0\|r\,, \end{aligned}$$

for every $\alpha \geqslant \alpha_r$ and any $r > 0$. Since

$$(2C_{X_+} + 1)\|x_0\|r \longrightarrow 0 \quad \text{as} \quad r \to 0\,,$$

we conclude that $\|x_\alpha^- - x_\infty\| \to 0$ as $\alpha \to \infty$. The proof that $\|x_\alpha^+ - x_\infty\| \to 0$ as $\alpha \to \infty$ as well is entirely analogous. $\square$

The hypothesis that the cone be solid is not necessary, as illustrated in Remark 2.45 below. Nevertheless, the conclusion of Lemma 2.44 may still fail if it is not satisfied (see Remark 2.46). We do not know the extent to which normality is a necessary condition for the conclusions of the lemma to hold.

Remark 2.45. The hypothesis that the cone be solid is not necessary. Let $X := \mathbb{R}^2$ and $X_+ := \mathbb{R}_{\geqslant 0} \times \{0\}$, and let $(x_\alpha, y_\alpha)_{\alpha \in A}$ be a net in X which is convergent in the Euclidean norm to an $(x_\infty, y_\infty) \in X$, and such that

$$(x_\alpha^-, y_\alpha^-) := \inf_{\alpha' \geqslant \alpha} (x_{\alpha'}, y_{\alpha'}) \quad \text{and} \quad (x_\alpha^+, y_\alpha^+) := \sup_{\alpha' \geqslant \alpha} (x_{\alpha'}, y_{\alpha'})$$

are well-defined for each $\alpha \in A$.

We first note that $y_{\alpha_1} = y_{\alpha_2}$ for any $\alpha_1, \alpha_2 \in A$. Indeed, let $\beta \in A$ be such that $\alpha_1, \alpha_2 \leqslant \beta$. We have

$$(x_{\alpha_1}^-, y_{\alpha_1}^-) \leqslant (x_{\alpha'}, y_{\alpha'}) \,, \quad \forall \alpha' \geqslant \alpha_1 \,,$$

therefore

$$x_{\alpha'} \geqslant x_{\alpha_1}^- \quad \text{and} \quad y_{\alpha'} = y_{\alpha_1}^- \,, \quad \forall \alpha' \geqslant \alpha_1 \,.$$

Similarly,

$$y_{\alpha'} = y_{\alpha_2}^- \,, \quad \forall \alpha' \geqslant \alpha_2 \,, \quad \text{and} \quad y_{\alpha'} = y_\beta^- \,, \quad \forall \alpha' \geqslant \beta \,.$$

Putting these together we obtain

$$y_\beta = y_\beta^- = y_{\alpha_1}^- = y_{\alpha_2}^- = y_{\alpha_1} = y_{\alpha_2} \,.$$

Now

$$y_\alpha = y_\infty \,, \quad \forall \alpha \in A \,,$$

and so, indeed,

$$(x_\alpha^-, y_\alpha^-) = \left(\inf_{\alpha' \geqslant \alpha} x_{\alpha'}, y_\infty \right) \quad \text{and} \quad (x_\alpha^+, y_\alpha^+) = \left(\sup_{\alpha' \geqslant \alpha} x_{\alpha'}, y_\infty \right) \,, \quad \forall \alpha \in A \,.$$

From the hypothesis that $x_\alpha \to x_\infty$, it follows—from Lemma 2.44 with $X := \mathbb{R}$ and $X_+ := \mathbb{R}_{\geqslant 0}$, if you must—that

$$x_\alpha^- \longrightarrow x_\infty \quad \text{and} \quad x_\alpha^+ \longrightarrow x_\infty \,,$$

from which

$$(x_\alpha^-, y_\alpha^-) \longrightarrow (x_\infty, y_\infty) \quad \text{and} \quad (x_\alpha^+, y_\alpha^+) \longrightarrow (x_\infty, y_\infty) \,,$$

then immediately follows. $\square$

Remark 2.46. The conclusion of Lemma 2.44 may fail if the cone is not solid. Consider, for instance, the normed space $L^1([0,1],\mathbb{R})$ of real-valued, Lebesgue-integrable functions on $[0,1]$, equipped with the partial order induced by the cone of nonnegative functions $L^1_{\geqslant 0}([0,1],\mathbb{R})$. This is a normal cone (the normality constant is 1) which is not solid (given an arbitrary $f \in L^1_{\geqslant 0}([0,1],\mathbb{R})$, it is not difficult to construct a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1([0,1],\mathbb{R})\backslash L^1_{\geqslant 0}([0,1],\mathbb{R})$ such that $\|f_n - f\|_{L^1([0,1],\mathbb{R})} \to 0$ as $n \to \infty$). We will construct a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1([0,1],\mathbb{R})$ satisfying the hypotheses of Lemma 2.44 for which the conclusion does not hold.

For each $k \in \mathbb{Z}_{\geqslant 0}$, let

$$N_k := \sum_{j=0}^{k} j\,.$$

Define $f_n \colon [0,1] \to \mathbb{R}$ by

$$f_n(x) := \chi_{\left[\frac{j-1}{k},\frac{j}{k}\right]}(x)\,, \quad x \in [0,1]\,,$$

for each $n = N_{k-1} + j$, where j ranges over $\{1,\ldots,k\}$ and k runs through the positive integers. Then

$$\sup_{n\geqslant n_0} f_n(x) = 1\,, \quad \forall x \in [0,1]\,, \quad \forall n_0 \in \mathbb{N}\,,$$

even though $f_n \to 0$ as $n \to \infty$. Indeed,

$$\|f_n\|_{L^1([0,1],\mathbb{R})} \leqslant \frac{1}{k}$$

for every $n \geqslant N_{k-1}+1$, for each positive integer k. □

Proposition 2.47. *Suppose that V is a real Banach space, partially ordered by a solid, normal, minihedral cone $V_+ \subseteq V$. Then every precompact subset $B \subseteq V$ has a supremum and an infimum.*

Proof. For compact subsets $B \subseteq V$, see [34, Theorem 6.5, page 62]. It follows for precompact sets in view of Lemma 2.38: if B is precompact, then $\overline{B}$ is compact and so $\sup B = \sup \overline{B}$ and $\inf B = \inf \overline{B}$ by the lemma. □

Corollary 2.48. *If V is a finite-dimensional, real normed space, partially ordered by a solid, normal, minihedral cone $V_+ \subseteq V$, then every bounded subset $B \subseteq V$ has a supremum and an infimum.*

Proof. Every finite-dimensional, real normed space is automatically complete (Banach). Moreover, every bounded subset is precompact. □

Proposition 2.47 and its corollary—or, more precisely, their extension to random sets in Subsection 2.3.2 below—are a key tool in this work. Therefore it is convenient to have a shorthand terminology to refer to Banach spaces which are partially ordered by a solid, normal and minihedral cone.

Definition 2.49 (RTA Spaces)**.** For brevity, we shall refer to a real Banach space V, partially ordered by a solid, normal, minihedral cone $V_+ \subseteq V$, as an *RTA space.* △

Definition 2.50 (Shell)**.** For any compact subset K of an RTA space, the set

$$\operatorname{shell}(K) := \{\sup E\,;\ E \text{ is a precompact subset of } K\}$$

will be referred to as the *shell* of K. △

Proposition 2.47 guarantees that the shell of a compact subset of an RTA space is well-defined.

Theorem 2.51 (Compact Shells)**.** *The shell of a compact subset of an RTA space is compact.*

Proof. Let X be an arbitrary compact subset of an arbitrary RTA space V. By Lemma 2.38, we then have

$$\operatorname{shell}(X) = \{\sup E\,;\ E \in F(X)\}\,.$$

Now $(F(X), d_H)$ is a compact metric space by Proposition A.5, and $\operatorname{shell}(X)$ is the image of $F(X)$ under the map

$$E \longmapsto \sup E\,. \quad E \in F(X)\,, \tag{2.18}$$

Therefore it suffices to show that this map is continuous.

Since the cone V_+ is solid, we may choose an u in the interior of V_+ such that the order interval

$$[-u, u] := \{x \in V\,;\ -u \leqslant x \leqslant u\}$$

contains the unit ball $B_1(0)$. Below we will denote the normality constant of V_+ by K.

Fix arbitrarily $\delta > 0$ and $A, C \in F(X)$ such that $d_H(A, C) < \delta$. By Proposition A.1, we have

$$A \subseteq C_\delta = C + B_\epsilon(0) \subseteq C + [-\delta u, \delta u]\,,$$

hence

$$\sup A \leqslant \sup C + \delta u\,.$$

Likewise,

$$\sup C \leqslant \sup A + \delta u\,.$$

Combining these two inequalities, we obtain

$$-\delta u \leqslant \sup A - \sup C \leqslant \delta u\,,$$

so that

$$0 \leqslant \sup A - \sup C + \delta u \leqslant 2\delta u\,.$$

Now, by normality,

$$\begin{aligned} \|\sup A - \sup C\| &\leqslant \|\sup A - \sup C + \delta u\| + \delta\|u\| \\ &\leqslant (2K+1)\|u\|\delta\,. \end{aligned}$$

This shows that (2.18) is uniformly continuous on $F(X)$, thus completing the proof. $\square$

2.3.2 Random Sets in Partially Ordered Spaces

The results below were proven in Chueshov [8]. We restate them here using our notation mostly for ease of reference. However mild generalizations are provided in a few of these results.

Suppose that $(X, \leqslant)$ is a partially ordered space. For any $a, b \in X^\Omega_\mathcal{B}$, we write $a \leqslant b$ to mean that $a(\omega) \leqslant b(\omega)$ for θ-almost all $\omega \in \Omega$. Similarly, for any $p, q \in \mathcal{S}^X_\theta$, we write $p \leqslant q$ to mean that $p(t, \omega) \leqslant q(t, \omega)$ for all $t \geqslant 0$, for θ-almost all $\omega \in \Omega$. Taking into account the identification of θ-almost everywhere equal maps discussed in Section 2.1.1, this convention induces partial orders in $X^\Omega_\mathcal{B}$ and $\mathcal{S}^X_\theta$.

Proposition 2.52. *Suppose that $a, b \colon \Omega \to V$ are random variables in a real normed space V, partially ordered by a cone $V_+ \subseteq V$. Then the multivalued map $[a, b] \colon \Omega \to 2^X$, defined by*

$$[a, b](\omega) := [a(\omega), b(\omega)]\,, \quad \omega \in \Omega\,,$$

is a random closed set provided that at least one of the three conditions below is satisfied.

(1) *V_+ is solid and $a \ll b$.*

(2) *V_+ is solid, normal and minihedral, and $a \leqslant b$.*

(3) *V is finite-dimensional and $a \leqslant b$.*

Proof. See [8, Proposition 3.2.1, page 88]. □

Proposition 2.53. *Suppose that $V_+ \subseteq V$ is a solid, normal cone. Then a random set $D \in (2^X)^\Omega_{\mathcal{B}}$ is bounded (tempered) if, and only if there exists a random (tempered random) variable $v \colon \Omega \to \operatorname{int} V_+$ such that*

$$D(\omega) \subseteq [-v(\omega), v(\omega)]\,, \quad \forall \omega \in \Omega\,.$$

Proof. See [8, Proposition 3.2.2, page 89]. □

Theorem 2.54 (Suprema and Infima of Random Sets)**.** *Suppose that V is a separable, real Banach space, partially ordered by a solid, normal, minihedral cone $V_+ \subseteq V$. If $D \in (2^X)^\Omega_{\mathcal{B}}$ is a precompact random set, then $\sup D, \inf D \colon \Omega \to V$ are random variables in V. Moreover, if D is tempered, then $\sup D$ and $\inf D$ are also tempered.*

Proof. For compact random sets, see Chueshov [8, Theorem 3.2.1, page 90]. From Theorem 2.47, $\sup D$ and $\inf D$ are well-defined. From Proposition 2.12, $\overline{D}$ is a compact random set. From Lemma 2.38, $\sup D = \sup \overline{D}$ and $\inf D = \inf \overline{D}$, which are then Borel-measurable. For the second statement, observe that temperedness of random sets is preserved by closure. □

Corollary 2.55. *If V is a finite-dimensional, real normed space, partially ordered by a solid, normal, minihedral cone $V_+ \subseteq V$, then $\sup D$ and $\inf D$ are random variables* *for every bounded random set $D \in (2^X)^{\Omega}_{\mathcal{B}}$. In particular,* $\sup D$ *and* $\inf D$ *are tempered random variables for every tempered random set $D \in (2^X)^{\Omega}_{\theta}$.*

Proof. Finite-dimensional, real normed spaces are automatically separable and complete, so the hypotheses of Theorem 2.54 are satisfied. Moreover, bounded sets are precompact in finite-dimensional normed spaces.

For the second statement, we need only observe that tempered random sets are bounded (random sets). □

2.4 Asymptotic Behavior Concepts

In this work asymptotic behavior is studied in the "pullback" sense.

Definition 2.56 (Pullback)**.** The *pullback* of a θ-stochastic process $\xi \in \mathcal{S}^X_{\theta}$ is the θ-stochastic process[11] $\check{\xi} \in \mathcal{S}^X_{\theta}$ defined by

$$\check{\xi}_t(\omega) := \xi_t(\theta_{-t}\omega)\,, \quad (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,.$$

If emphasis or contrast is needed, we shall refer to the original θ-stochastic process ξ as the *forward* one. △

Intuitively, in the forward θ-stochastic process one keeps seeing random dynamical fluctuations along the way, while its pullback gives "photographs" of what the "present" state would look like if it had started to evolve a long time ago [14]. We shall always use the check mark ˇ to indicate the pullback of the θ-stochastic process being accented. Note that, while the concepts of random variable and θ-stochastic process defined at the beginning of Subsection 2.1.2 do not depend explicitly on the measurable flow $(\theta_t)_{t\in\mathcal{T}}$ component of the underlying MPDS θ, the concept of pullback does.

In the context of random dynamical systems, pullback trajectories seem to be the mathematically natural object to study, as we will discuss in more detail later on, after we formally introduce RDS in Definition 3.1. We can advance that it follows from the measure-preserving property that

$$\mathbb{P}(\xi_t \in A) = \mathbb{P}(\theta_t[\xi_t \in A]) = \mathbb{P}(\check{\xi}_t \in A)\,, \quad \forall t \geqslant 0\,, \quad \forall A \in \mathcal{F}\,. \tag{2.19}$$

[11] $\check{\xi}$ is indeed a θ-stochastic process by Lemma C.10.

Thus, at least as far as practical matters are concerned, the asymptotic behavior of forward and pullback trajectories are equivalent in the sense of probability.

Definition 2.57 (Tail)**.** The *tail* (*from moment* τ) of the pullback trajectories of a θ-stochastic process $\xi \in \mathcal{S}_\theta^X$ is the multifunction $\beta_\xi^\tau \colon \Omega \to 2^X \backslash \{\varnothing\}$, defined by

$$\beta_\xi^\tau(\omega) := \{\xi_t(\theta_{-t}\omega)\,;\ t \geqslant \tau\}\,, \quad \omega \in \Omega\,,$$

for each $\tau \geqslant 0$. △

Lemma 2.58. *If* (X, d) *is a Polish space and* $(\Omega, \mathcal{F}, \mathbb{P})$ *is complete, then* β_ξ^τ *is a random set for each* $\tau \geqslant 0$*, for any* θ*-stochastic process* $\xi \in \mathcal{S}_\theta^X$*.*

Proof. Fix $\xi \in \mathcal{S}_\theta^X$, $\tau \geqslant 0$ and $x \in X$ arbitrarily. We will show that

$$\omega \longmapsto \operatorname{dist}(x, \beta_\xi^\tau(\omega))\,, \quad \omega \in \Omega\,, \tag{2.20}$$

is measurable. Let $g \colon \mathcal{T}_{\geqslant 0} \times \Omega \to \mathbb{R}_{\geqslant 0}$ be the map defined by

$$g(t, \omega) := d(x, \xi_t(\theta_{-t}\omega))\,, \quad (t, \omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,.$$

As a composition of measurable functions, g is measurable. It follows straight from the definition of the tails of the pullback trajectories of ξ that

$$\operatorname{dist}(x, \beta_\xi^\tau(\omega)) = \inf_{t \geqslant \tau} g(x, \omega)\,, \quad \forall \omega \in \Omega\,.$$

Fix $a \geqslant 0$ arbitrarily. Then

$$\begin{aligned}
E_a &:= \{\omega \in \Omega\,;\ \operatorname{dist}(x, \beta_\xi^\tau(\omega)) < a\} \\
&= \{\omega \in \Omega\,;\ \inf_{t \geqslant \tau} g(t, \omega) < a\} \\
&= \operatorname{proj}_\Omega \{(t, \omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,;\ g(t, \omega) < a \text{ and } t \geqslant \tau\} \\
&= \operatorname{proj}_\Omega \left(g^{-1}([0, a)) \cap [\tau, \infty) \times \Omega\right)\,.
\end{aligned}$$

From the assumptions that (X, d) is a Polish space and $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, it follows by Proposition 2.21 that E_a is $\mathcal{F}$-measurable. Since $a \geqslant 0$ was chosen arbitrarily, this shows that map defined in (2.20) is measurable. Since X is separable and $x \in X$ was also chosen arbitrarily, it follows from Proposition 2.15 that β_ξ^τ is a random set. Finally, $\xi \in \mathcal{S}_\theta^X$ and $\tau \geqslant 0$ were chosen arbitrarily as well, therefore the argument above proves the lemma. □

2.4.1 Tempered Convergence and Continuity

Definition 2.59 (Tempered Convergence)**.** Suppose θ is an MPDS and (X, d) is a metric space. We say that a net $(\xi_\alpha)_{\alpha\in A}$ in $X^\Omega_\mathcal{B}$ converges in the *tempered sense* to a random variable $\xi_\infty \in X^\Omega_\mathcal{B}$ if there exists a nonnegative, tempered random variable $r\colon \Omega \to \mathbb{R}_{\geqslant 0}$ and an $\alpha_0 \in A$ such that

(1) $\xi_\alpha(\omega) \longrightarrow \xi_\infty(\omega)$ as $\alpha \to \infty$ for θ-almost all $\omega \in \Omega$, and

(2) $d(\xi_\alpha(\omega), \xi_\infty(\omega)) \leqslant r(\omega)$ for all $\alpha \geqslant \alpha_0$, for θ-almost all $\omega \in \Omega$.

In this case we write $\xi_\alpha \to_\theta \xi_\infty$ (as $\alpha \to \infty$). △

Definition 2.60 (Tempered Continuity)**.** Suppose θ is an MPDS and X, U are metric spaces. A map $\mathcal{K}\colon \mathcal{U} \subseteq U^\Omega_\mathcal{B} \to X^\Omega_\mathcal{B}$ is said to be *tempered continuous* if $\mathcal{K}(u_\alpha) \to_\theta \mathcal{K}(u_\infty)$ for every net $(u_\alpha)_{\alpha\in A}$ in $\mathcal{U}$ such that $u_\alpha \to_\theta u_\infty$ for some $u_\infty \in \mathcal{U}$. △

2.4.2 Tempered and Eventually Precompact Trajectories

Definition 2.61 (Eventually Precompact Trajectories)**.** We say that a θ-stochastic process $\xi \in \mathcal{S}^X_\theta$ is *eventually precompact* if there exists a $\tau_\xi \geqslant 0$ such that $\beta^{\tau_\xi}_\xi(\omega)$ is precompact for θ-almost all $\omega \in \Omega$. We denote the subset of all eventually precompact θ-stochastic processes $\xi \in \mathcal{S}^X_\theta$ by $\mathcal{K}^X_\theta$. △

Note that $\beta^{\tau_1}_\xi(\omega) \subseteq \beta^{\tau_2}_\xi(\omega)$ whenever $\tau_1 \geqslant \tau_2$. So, in this definition, it is indeed true that $\beta^\tau_\xi(\omega)$ is precompact for every $\tau \geqslant \tau_\xi$, for θ-almost every $\omega \in \Omega$.

Definition 2.62 (Tempered Trajectories)**.** A θ-stochastic process $\xi \in \mathcal{S}^X_\theta$ is said to be *tempered* if there exists a nonempty, tempered random set $D \in (2^X)^\Omega_\theta$ such that

$$\beta^0_\xi(\omega) \subseteq D(\omega)\,, \quad \widetilde{\forall}\omega \in \Omega\,; \tag{2.21}$$

in other words,

$$\check{\xi}_t(\omega) = \xi_t(\theta_{-t}\omega) \in D(\omega)\,, \quad \forall t \geqslant 0\,, \quad \widetilde{\forall}\omega \in \Omega\,. \tag{2.22}$$

Any $D \in (2^X)^\Omega_\theta$ for which the inclusions above hold is called a *rest set*. The subset of all tempered θ-stochastic processes $\xi \in \mathcal{S}^X_\theta$ is denoted by $\mathcal{V}^X_\theta$. △

Observe that, by virtue of θ-invariance, condition (2.22) is equivalent to

$$\xi_t(\omega) \in D(\theta_t\omega)\,, \quad \forall t \geqslant 0\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

Lemmas 2.63–2.65 below further motivate the concept of tempered θ-stochastic processes just introduced. The idea is to have a term to talk about θ-stochastic processes which, as far as their oscillatory behavior is concerned, look somewhat like a θ-stationary process generated by a tempered random variable (Lemma 2.63). Since this pertains to long-term behavior, this property should be preserved by shifting (Lemma 2.64) or "concatenating" (Lemma 2.65) tempered θ-stochastic processes.

Lemma 2.63. *If $\xi\colon \Omega \to X$ is a tempered random variable, then the θ-stationary process $\bar{\xi}\colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$ generated by ξ is a tempered θ-stochastic process.*

Proof. Consider the random singleton $D := \{\xi\}$. Then

$$\bar{\xi}_t(\theta_{-t}\omega) = \xi(\theta_t\theta_{-t}\omega) = \xi(\omega) \in D(\omega)\,, \quad \forall t \geqslant 0\,, \quad \forall\omega \in \Omega\,.$$

Since ξ is tempered by hypothesis, D is also tempered, and so $\bar{\xi}$ is tempered with rest set D. □

Lemma 2.64. *If $\xi\colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$ is a tempered θ-stochastic process, then the shift $\rho_s(\xi)$ is also tempered for any $s \geqslant 0$.*

Proof. Let D be a rest set for ξ. Fix $s \geqslant 0$ arbitrarily. We have

$$[\rho_s(\xi)]_t(\theta_{-t}\omega) = \xi_{s+t}(\theta_{-(s+t)}\omega) \in D(\omega)\,, \quad \forall t \geqslant 0\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

Thus D is also a rest set for $\rho_s(\xi)$. Since $s \geqslant 0$ was chosen arbitrarily, this completes the proof. □

For each $s \geqslant 0$, we define an operator $\lozenge_s\colon \mathcal{S}_\theta^X \times \mathcal{S}_\theta^X \to \mathcal{S}_\theta^X$ as follows. Given $\xi, \zeta \in \mathcal{S}_\theta^X$, $\xi \lozenge_s \zeta$ consists of the truncation of ξ at time s, "continued" by ζ from then onwards. More precisely, we define $u \lozenge_s v\colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$ by

$$(\xi \lozenge_s \zeta)_t(\omega) = \begin{cases} \xi_t(\omega)\,, & 0 \leqslant t < s \\ \zeta_{t-s}(\theta_s\omega)\,, & s \leqslant t \end{cases}\,, \quad (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,.$$

When Ω is a singleton, this construction reduces to the standard deterministic way of concatenating paths.

Lemma 2.65. *Let $\xi, \zeta \colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$ be tempered θ-stochastic processes. Then for any $s \geqslant 0$, the concatenation $\xi \Diamond_s \zeta$ is also tempered.*

Proof. Let $D, E \in (2^X)^\Omega_\theta$ be rest sets for ξ, ζ, respectively, and let $\widetilde{\Omega} \subseteq \Omega$ be a θ-invariant subset of full measure such that

$$\xi_t(\theta_{-t}\omega) \in D(\omega) \quad \text{and} \quad \zeta_t(\theta_{-t}\omega) \in E(\omega)\,, \quad \forall t \geqslant 0\,, \quad \forall \omega \in \widetilde{\Omega}\,.$$

Fix $s \geqslant 0$ arbitrarily. For any $t \in [0, s)$, we have

$$(\xi \Diamond_s \zeta)_t(\theta_{-t}\omega) = \xi_t(\theta_{-t}\omega) \in D(\omega)\,, \quad \forall \omega \in \widetilde{\Omega}\,.$$

Similarly, for any $t \in [s, \infty)$, we have

$$(\xi \Diamond_s \zeta)_t(\theta_{-t}\omega) = \zeta_{t-s}(\theta_{-(t-s)}\omega) \in E(\omega)\,, \quad \forall \omega \in \widetilde{\Omega}\,.$$

This shows that

$$(\xi \Diamond_s \zeta)_t(\theta_{-t}\omega) \in (D \cup E)(\omega)\,, \quad \forall t \geqslant 0\,, \quad \forall \omega \in \widetilde{\Omega}\,.$$

Now $D \cup E$ is a tempered random set by Lemma 2.29, thus completing the proof. $\square$

Lemma 2.66. *Let $\xi \colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$ be a tempered θ-stochastic process in a real normed space X. If X is finite-dimensional, then ξ is eventually precompact; in other terms, $\mathcal{V}^X_\theta \subseteq \mathcal{K}^X_\theta$.*

Proof. Indeed, let D be a rest set for ξ. In virtue of temperedness, $D(\omega)$ is, in particular, bounded for θ-almost all $\omega \in \Omega$. Under the assumption that X is finite-dimensional, each such bounded $D(\omega)$ is precompact. It follows from (2.21) that $\beta^0_\xi(\omega)$ is precompact for θ-almost all $\omega \in \Omega$, which means that ξ is eventually precompact (with $\tau_\xi = 0$). $\square$

Proposition 2.67. *Suppose that (X, d) is a metric space. If $\xi \colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$ is a tempered θ-stochastic process and $\xi_\infty \colon \Omega \to X$ is a map such that*

$$\xi_t(\theta_{-t}\omega) \longrightarrow \xi_\infty(\omega)\,, \quad \text{as} \quad t \to \infty\,, \quad \widetilde{\forall}\omega \in \Omega\,, \tag{2.23}$$

then ξ_∞ is a tempered random variable and the convergence occurs in the tempered sense. Furthermore, ξ is, in this case, eventually precompact with $\tau_\xi = 0$.

Proof. It follows from [36, Chapter 11, §1, Property M7 on page 248] that ξ_∞ is measurable. (If $\mathcal{T} = \mathbb{R}$, then think of ξ_∞ as the limit along a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathcal{T}_{\geqslant 0}$ going to infinity.) Let D be a rest set for ξ, and let $x_0 \in X$ and $r \in (\mathbb{R}_{\geqslant 0})^\Omega_\theta$ be such that $D(\cdot) \subseteq \overline{B}_{r(\cdot)}(x_0)$. Then, by continuity,

$$d(\xi_\infty(\omega), x_0) = \lim_{t\to\infty} d(\xi_t(\theta_{-t}\omega), x_0) \leqslant r(\omega)\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

We conclude that ξ_∞ is tempered. Furthermore,

$$d(\xi_t(\theta_{-t}\omega), \xi_\infty(\omega)) \leqslant d(\xi_t(\theta_{-t}\omega), x_0) + d(x_0, \xi_\infty(\omega)) \leqslant 2r(\omega)\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

Thus convergence occurs in the tempered sense.

The second statement follows straight from Lemma 2.66. □

Recall that an RTA space is a Banach space X which is partially ordered by a solid, normal, minihedral cone $X_+ \subseteq X$. It follows from Theorem 2.54 that the infima and suprema in the definition below are well-defined.

Definition 2.68 (Lower and Upper Tails)**.** Let X be a separable RTA space. Given $\xi \in \mathcal{K}^X_\theta$ and $\tau_\xi \geqslant 0$ such that $\beta^{\tau_\xi}_\xi(\omega)$ is precompact for θ-almost all $\omega \in \Omega$, the net $(a_\tau)_{\tau\geqslant\tau_\xi}$ of random variables $\Omega \to X$ defined by

$$a_\tau(\omega) := \inf \beta^\tau_\xi(\omega) = \inf_{t\geqslant\tau} \xi_t(\theta_{-t}\omega)\,, \quad \omega \in \Omega\,, \quad \tau \geqslant \tau_\xi\,,$$

is referred to as a *lower tail* (*of the pullback trajectories*) of ξ. Similarly, the net $(b_\tau)_{\tau\geqslant\tau_\xi}$ of random variables $\Omega \to X$ defined by

$$b_\tau(\omega) := \sup \beta^\tau_\xi(\omega) = \sup_{t\geqslant\tau} \xi_t(\theta_{-t}\omega)\,, \quad \omega \in \Omega\,, \quad \tau \geqslant \tau_\xi\,,$$

is referred to as an *upper tail* (*of the pullback trajectories*) of ξ. △

Lower and upper tails of pullback trajectories are a prevalent concept in this work.

Since we shall be interested solely in the asymptotic behavior of lower and upper tails, the nonuniqueness implicit in the definition above will be harmless. Indeed, if $(a_\tau)_{\tau\geqslant\tau_\xi}$ and $(a'_\tau)_{\tau\geqslant\tau'_\xi}$ are lower tails of ξ, then $a_\tau = a'_\tau$ for every $\tau \geqslant \max\{\tau_\xi, \tau'_\xi\}$. Naturally, the same is true of upper tails.

2.4.3 θ-Limits

Eventually precompact θ-stochastic process evolving on a separable RTA space are particularly well-behaved. In this section we shall develop notions of lim inf and lim sup for such processes. These concepts will be the backbones of the constructions leading up to the small-gain theorem in the next chapter.

Proposition 2.69. *Let X be a separable RTA space, $\xi\colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$ be any tempered, eventually precompact θ-stochastic process, and let $(a_\tau)_{\tau \geqslant \tau_\xi}$ and $(b_\tau)_{\tau \geqslant \tau_\xi}$ be, respectively, a lower and an upper tail of the pullback trajectories of ξ. Then $(a_\tau)_{\tau \geqslant \tau_\xi}$ and $(b_\tau)_{\tau \geqslant \tau_\xi}$ both converge in the tempered sense. Moreover, the limits*

$$a_\infty := \lim_{\tau \to \infty} a_\tau$$

and

$$b_\infty := \lim_{\tau \to \infty} b_\tau$$

are tempered random variables such that

$$a_\sigma \leqslant a_\tau \leqslant a_\infty \leqslant b_\infty \leqslant b_\tau \leqslant b_\sigma\,, \quad \forall \tau \geqslant \sigma \geqslant \tau_\xi\,. \tag{2.24}$$

Proof. Fix $\omega \in \Omega$ arbitrarily such that $\beta_\xi^{\tau_\xi}(\omega)$ is precompact. We shall show that every sequence $(\tau_n)_{n \in \mathbb{N}}$ in $\mathcal{T}_{\geqslant \tau_\xi}$ going to infinity has a subsequence along which $(a_\tau(\omega))_{\tau \geqslant \tau_\xi}$ converges to the same value $a_\infty(\omega)$. Thus $(a_\tau(\omega))_{\tau \geqslant \tau_\xi}$ itself converges to $a_\infty(\omega)$.

Since $\beta_\xi^\tau \subseteq \beta_\xi^\sigma$ whenever $\tau \geqslant \sigma \geqslant \tau_\xi$, we have

$$a_\sigma \leqslant a_\tau \leqslant b_\tau \leqslant b_\sigma\,, \quad \forall \tau \geqslant \sigma \geqslant \tau_\xi\,. \tag{2.25}$$

Let $(\tau_n)_{n \in \mathbb{N}}$ be any sequence going to infinity in $\mathcal{T}_{\geqslant \tau_\xi}$. By passing to a subsequence, if necessary, we may assume that $(\tau_n)_{n \in \mathbb{N}}$ is nondecreasing. So

$$a_{\tau_1}(\omega) \leqslant a_{\tau_2}(\omega) \leqslant a_{\tau_3}(\omega) \leqslant \cdots \leqslant a_{\tau_n}(\omega) \leqslant \cdots\,.$$

Now

$$a_{\tau_n}(\omega) \in \operatorname{shell}(\beta_\xi^{\tau_\xi}(\omega))\,, \quad \forall n \in \mathbb{N}\,,$$

and, since $\beta_\xi^{\tau_\xi}(\omega)$ is precompact, we know from Theorem 2.51 that $\operatorname{shell}(\beta_\xi^{\tau_\xi}(\omega))$ is compact. So, $a_{\tau_n}(\omega) \longrightarrow a_\infty(\omega)$ as $n \to \infty$ for some $a_\infty(\omega) \in X$ ([50, Lemma 1.2

on page 3]). Using the same argument, we can show that if $(\sigma_n)_{n\in\mathbb{N}}$ is any monotone sequence in $\mathcal{T}_{\geqslant\tau_\xi}$ going to infinity, then there exists $\widetilde{a}_\infty(\omega) \in X$ such that

$$\widetilde{a}_\infty(\omega) := \lim_{n\to\infty} a_{\sigma_n}(\omega)\,.$$

Now there are subsequences $(k_n)_{n\in\mathbb{N}}$ and $(l_n)_{n\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ such that

$$\tau_n \leqslant \sigma_{k_n} \quad \text{and} \quad \tau_{l_n} \leqslant \sigma_n\,,$$

and so

$$a_{\tau_n}(\omega) \leqslant a_{\sigma_{k_n}}(\omega) \quad \text{and} \quad a_{\tau_{l_n}}(\omega) \leqslant a_{\sigma_n}(\omega)\,, \qquad \forall n \in \mathbb{N}\,.$$

Passing the limit as n goes to infinity we get $a_\infty(\omega) \leqslant \widetilde{a}_\infty(\omega)$ and $a_\infty(\omega) \geqslant \widetilde{a}_\infty(\omega)$, showing that in fact $a_\infty(\omega) = \widetilde{a}_\infty(\omega)$.

Since $\beta_\xi^{\tau_\xi}(\omega)$ is precompact for θ-almost all $\omega \in \Omega$, a map $a_\infty\colon \Omega \to X$ is well-defined θ-almost everywhere by

$$a_\infty(\omega) := \lim_{\tau\to\infty} a_\tau(\omega)\,, \quad \omega \in \Omega\,.$$

In particular,

$$a_\infty := \lim_{n\to\infty} a_{\tau_n}$$

for any sequence $(\tau_n)_{n\in\mathbb{N}}$ in $\mathcal{T}_{\geqslant\tau_\xi}$ going to infinity. So, measurability follows from [36, Chapter 11, §1, Property M7 on page 248].

The proof that $(b_\tau)_{\tau\geqslant\tau_\xi}$ converges to a random variable $b_\infty\colon \Omega \to X$ goes along the same lines.

We obtain (2.24) by fixing $\sigma \geqslant \tau_\xi$ arbitrarily and taking the limit as τ goes to infinity in (2.25).

By Theorem 2.54,

$$a_{\tau_\xi} = \inf \beta_\xi^{\tau_\xi} \quad \text{and} \quad b_{\tau_\xi} = \sup \beta_\xi^{\tau_\xi}$$

are tempered. It follows from (2.24) and normality that

$$\begin{aligned}
\|a_\infty(\omega)\| &\leqslant \|a_{\tau_\xi}(\omega)\| + \|a_{\tau_\xi}(\omega) - a_\infty(\omega)\| \\
&\leqslant \|a_{\tau_\xi}(\omega)\| + C_{X_+}\|b_{\tau_\xi}(\omega) - a_{\tau_\xi}(\omega)\| \\
&\leqslant (1 + C_{X_+})\|a_{\tau_\xi}(\omega)\| + \|b_{\tau_\xi}(\omega)\|\,, \quad \widetilde{\forall}\omega \in \Omega\,,
\end{aligned}$$

where C_{X_+} is the normality constant of the underlying cone X_+. This shows that a_∞ is tempered. Furthermore,

$$\|a_\tau(\omega) - a_\infty(\omega)\| \leqslant C_{X_+}(\|a_{\tau_\xi}(\omega)\| + \|b_{\tau_\xi}(\omega)\|)\,, \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall \tau \geqslant \tau_\xi\,.$$

Therefore $a_\tau \to_\theta a_\infty$, as we wanted to show. The proof that b_∞ is tempered and $b_\tau \to_\theta b_\infty$ goes along the same lines. □

Remark 2.70. The key step in the proof of the proposition above was the observation that $\mathrm{shell}(\beta_\xi^{\tau_\xi}(\omega))$ is compact. A simpler proof is possible in finite-dimensional spaces.

One first notes that a normal cone in a finite-dimensional space is automatically regular. Indeed, normality implies that order-bounded sequences are also norm-bounded, and thus precompact. By [50, Lemma 1.2 on page 3], monotone sequences in partially ordered, compact spaces converge. This shows that every monotone, order-bounded sequence must be convergent in a finite-dimensional space which is partially ordered by a normal cone, thus establishing regularity.

Now, going back to the proof of the proposition, we have

$$a_{\tau_1}(\omega) \leqslant a_{\tau_2}(\omega) \leqslant a_{\tau_3}(\omega) \leqslant \cdots \leqslant a_{\tau_n}(\omega) \cdots \leqslant b_{\tau_\xi}(\omega)\,.$$

Thus, if the X is finite-dimensional, then the sequence $(a_{\tau_n}(\omega))_{n\in\mathbb{N}})$ converges by regularity. □

The proposition above motivates a definition of a "tempered lim sup" and a "tempered lim inf" in separable RTA spaces.

Definition 2.71 (Tempered lim inf and lim sup)**.** Given a separable RTA space X and a tempered, eventually precompact θ-stochastic process $\xi\colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$, we define

$$\text{“}\,\theta\text{-}\underline{\lim}\,\xi\,\text{”}$$

to be the (tempered) random variable $\Omega \to X$ given by

$$[\theta\text{-}\underline{\lim}\,\xi](\omega) := \varliminf_{t\to\infty} \xi_t(\theta_{-t}\omega) := \sup_{\tau\geqslant\tau_\xi} \inf_{t\geqslant\tau} \xi_t(\theta_{-t}\omega)\,, \quad \omega \in \Omega\,,$$

where $\tau_\xi \geqslant 0$ is chosen arbitrarily so that $\beta_\xi^{\tau_\xi}(\omega)$ is precompact for θ-almost all $\omega \in \Omega$. Similarly, we define

$$\text{“}\,\theta\text{-}\overline{\lim}\,\xi\,\text{”}$$

to be the (tempered) random variable $\Omega \to X$ given by

$$[\theta\text{-}\overline{\lim}\,\xi](\omega) := \overline{\lim_{t\to\infty}}\,\xi_t(\theta_{-t}\omega) := \inf_{\tau\geqslant\tau_\xi}\sup_{t\geqslant\tau}\xi_t(\theta_{-t}\omega)\,, \quad \omega\in\Omega\,.$$

Conversely, when we write

$$\theta\text{-}\underline{\lim}\,\xi \quad \text{or} \quad \theta\text{-}\overline{\lim}\,\xi$$

for some θ-stochastic process $\xi\colon \mathcal{T}_{\geqslant 0}\times\Omega\to X$, it is to be tacitly understood that ξ is[12] tempered and eventually precompact, and that the symbols represent the random variables defined above. △

It follows straight from the definition above that

$$\theta\text{-}\underline{\lim}\,\xi \leqslant \theta\text{-}\overline{\lim}\,\xi$$

for any tempered, eventually precompact θ-stochastic process $\xi\colon \mathcal{T}\times\Omega\to X$. Moreover, we will have equality if, and only if ξ converges in the tempered sense.

Lemma 2.72. *Suppose that $\xi\colon \mathcal{T}_{\geqslant 0}\times\Omega\to X$ is a tempered, eventually precompact θ-stochastic process on a separable RTA space X. Then*

$$\check{\xi}_t \longrightarrow_\theta \xi_\infty \quad \textit{as} \quad t\to\infty \tag{2.26}$$

for some $\xi\in X_\theta^\Omega$ if, and only if

$$\theta\text{-}\underline{\lim}\,\xi = \theta\text{-}\overline{\lim} = \xi_\infty\,. \tag{2.27}$$

Proof. ($\Leftarrow$) Suppose that (2.27) holds for some $\xi\in X_\theta^\Omega$. Let $(a_\tau)_{\tau\geqslant\tau_\xi}$ and $(b_\tau)_{\tau\geqslant\tau_\xi}$ be a lower and an upper tail of the pullback trajectories of ξ, respectively. By definition, we have

$$a_\tau(\omega)\leqslant\check{\xi}_\tau(\omega)\leqslant b_\tau(\omega)\,,\quad \forall\tau\geqslant\tau_\xi\,,\quad \widetilde{\forall}\omega\in\Omega\,.$$

By Proposition 2.69, we have

$$a_\tau(\omega)\leqslant\xi_\infty(\omega)\leqslant b_\tau(\omega)\,,\quad \forall\tau\geqslant\tau_\xi\,,\quad \widetilde{\forall}\omega\in\Omega\,.$$

[12] Whether temperedness and eventual precompactness are being assumed or have been proved to hold will always be clear from the context.

Thus by normality

$$\|\check{\xi}_\tau(\omega) - \xi_\infty(\omega)\| \leqslant C_{X_+} \|b_\tau(\omega) - a_\tau(\omega)\|\,, \quad \forall \tau \geqslant 0\,, \quad \widetilde{\forall}\omega \in \Omega\,,$$

where $C_{X_+} \geqslant 0$ is the normality constant of the underlying cone $X_+ \subseteq X$. Again by Proposition 2.69 (together with the hypothesis that $\theta\text{-}\underline{\lim}\,\xi = \theta\text{-}\overline{\lim}\,\xi$), $b_\tau - a_\tau \to_\theta 0$. Combining this with the inequality we obtain (2.26).

($\Rightarrow$) Now suppose (2.26) holds. Fix arbitrarily $\omega \in \Omega$ such that

$$\check{\xi}_t(\omega) \longrightarrow \xi_\infty(\omega) \quad \text{as} \quad t \to \infty\,. \tag{2.28}$$

Then it follows from Lemma 2.44 that

$$a_\tau(\omega) = \inf_{t \geqslant \tau} \check{\xi}_t(\omega) \longrightarrow \xi_\infty(\omega)$$

and

$$b_\tau(\omega) = \sup_{t \geqslant \tau} \check{\xi}_t(\omega) \longrightarrow \xi_\infty(\omega)$$

as $\tau \to \infty$. Since (2.28) holds for θ-almost all $\omega \in \Omega$, we conclude that (2.27) also holds. $\square$

Naturally, inequalities are also preserved by tempered lim inf and tempered lim sup.

Lemma 2.73. *Suppose that $\xi_1, \xi_2 \colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$ are tempered, eventually precompact θ-stochastic processes on a separable RTA space X. If $\xi_1 \leqslant \xi_2$, then*

$$\theta\text{-}\underline{\lim}\,\xi_1 \leqslant \theta\text{-}\underline{\lim}\,\xi_2 \quad \textit{and} \quad \theta\text{-}\overline{\lim}\,\xi_1 \leqslant \theta\text{-}\overline{\lim}\,\xi_2\,.$$

Proof. We will carry out the details for

$$\theta\text{-}\overline{\lim}\,\xi_1 \leqslant \theta\text{-}\overline{\lim}\,\xi_2\,.$$

The other inequality can be proved in the exact same way. Let $\tau_\xi \geqslant 0$ be such that $\beta_{\xi_1}^{\tau_\xi}(\omega)$ and $\beta_{\xi_2}^{\tau_\xi}(\omega)$ are precompact for θ-almost every $\omega \in \Omega$, and let $(b_\tau^{(1)})_{\tau \geqslant \tau_\xi}$ and $(b_\tau^{(2)})_{\tau \geqslant \tau_\xi}$ be upper tails of the pullback trajectories of ξ_1 and ξ_2, respectively. Since

$$(\xi_1)_t(\omega) \leqslant (\xi_2)_t(\omega)\,, \quad \forall t \geqslant 0\,, \quad \widetilde{\forall}\omega \in \Omega\,,$$

it follows by θ-invariance that

$$(\xi_1)_t(\theta_{-t}\omega) \leqslant (\xi_2)_t(\theta_{-t}\omega)\,, \quad \forall t \geqslant 0\,, \quad \widetilde{\forall}\omega \in \Omega\,,$$

Hence

$$\begin{aligned} b_\tau^{(1)}(\omega) &= \sup\{(\xi_1)_t(\theta_{-t}\omega)\,;\ t \geqslant \tau\} \\ &\leqslant \sup\{(\xi_2)_t(\theta_{-t}\omega)\,;\ t \geqslant \tau\} \\ &= b_\tau^{(2)}(\omega)\,, \quad \forall \tau \geqslant \tau_\xi\,, \quad \widetilde{\forall}\omega \in \Omega\,, \end{aligned}$$

by Lemma 2.37. By taking the limits as $\tau \to \infty$ in the inequality above, it follows from Proposition 2.69 that

$$\theta\text{-}\overline{\lim}\,\xi_1 = \lim_{\tau\to\infty}(\xi_1)_\tau \leqslant \lim_{\tau\to\infty}(\xi_2)_\tau = \theta\text{-}\overline{\lim}\,\xi_2\,.$$

This completes the proof. $\square$

Chapter 3

Random Dynamical Systems with Inputs and Outputs

In this chapter, unless otherwise specified, X, U and Y will be assumed to be Polish spaces, equipped with their respective Borel σ-algebras whenever measure-theoretic considerations are being made. We shall also assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ constituting the underlying MPDS θ be complete. Much of what we will do would still make sense in a slightly more general setting. However our most important definitions and results will depend on one or more features of this infrastructure. Thus assuming that these conditions are all satisfied from the beginning will allow for a more unified treatment of the subject. We will draw remarks discussing what might have happened under weaker assumptions whenever pertinent.

3.1 Random Dynamical Systems

We begin by reviewing the "random dynamical systems" framework of L. Arnold [4]. We take the opportunity to work out in the detail the linear example, which shall serve as a point of reference and scaffold for examples discussed throughout the rest of the work.

Definition 3.1 (Random Dynamical Systems). A (*continuous*) *random dynamical system* (*RDS*) *on* X is an ordered pair (θ, φ) in which θ is an MPDS and

$$\varphi : \mathcal{T}_{\geqslant 0} \times \Omega \times X \longrightarrow X$$

is a (*continuous*) *cocycle over* θ—that is, a $(\mathcal{B}(\mathcal{T}_{\geqslant 0}) \otimes \mathcal{F} \otimes \mathcal{B}(X))$-measurable map such that[1]

[1]For any nonempty set S, $id_S \colon S \to S$ is the identity map on S, defined by $id_S(s) := s$ for each $s \in S$.

(S1) $\varphi(t,\omega) := \varphi(t,\omega,\cdot) : X \to X$ is continuous for each $(t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega$,

(S2) $\varphi(0,\omega) = id_X$ for each $\omega \in \Omega$, and

$$\varphi(t+s,\omega) = \varphi(t,\theta_s\omega) \circ \varphi(s,\omega)\,, \quad \forall \omega \in \Omega\,, \quad \forall s,t \geqslant 0$$

(*cocycle property*). △

The cocycle property generalizes the semigroup property of deterministic dynamical systems. Indeed, RDS include deterministic dynamical systems as the special case in which Ω is a singleton.

As with the notation for MPDS in the previous chapter, the symbols 'φ' and 'X' will also be reserved throughout the rest of this work to carry the meanings and perform the functions assigned in Definition 3.1. Therefore when we refer to an RDS (θ,φ), or to a cocycle φ over an MPDS θ, we tacitly assume φ to have state space X. In particular, X will always have at least the structure of a Polish space,[2] as stated in the beginning of the chapter.

Example 3.2 (RDS Generated by Random Linear Differential Equations)*.* Suppose

$$\mathcal{T} = \mathbb{R}\,,$$

and let $A\colon \Omega \to M_{n\times n}(\mathbb{R})$ be a random $n \times n$ real matrix such that, for each $\omega \in \Omega$,

$$t \longmapsto \|A(\theta_t\omega)\|\,, \quad t \in \mathbb{R}\,,$$

is locally integrable. For each $\omega \in \Omega$, let

$$\Xi(\cdot,\cdot,\omega)\colon \mathbb{R} \times \mathbb{R} \longrightarrow M_{n\times n}(\mathbb{R})$$

be the *fundamental matrix solution* of the linear differential equation

$$\dot{\xi} = A(\theta_t\omega)\xi\,, \quad t \in \mathbb{R} \tag{3.1}$$

(see [52, Section C.4 on pages 487–491])—that is, for each fixed $s \in \mathbb{R}$,

$$\Xi(s,\cdot,\omega)\colon \mathbb{R} \longrightarrow M_{n\times n}(\mathbb{R})$$

[2]The definition of RDS would still make sense if X were only assumed to be a topological space. However, as pointed out in the beginning of the chapter, we shall not need to deal with the concept in such generality.

is the unique absolutely continuous $\mathbb{R} \to M_{n\times n}(\mathbb{R})$ map such that

$$\Xi(s,s,\omega) = I_n := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

and

$$\frac{d}{dt}\Xi(s,t,\omega) = A(\theta_t\omega)\Xi(s,t,\omega)$$

for Lebesgue-almost all $t \in \mathbb{R}$. We discuss further properties of Ξ in Lemma 3.3 below. For the immediate let

$$\begin{aligned} \Phi\colon \mathbb{R}_{\geqslant 0} \times \Omega \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (t,\omega,x) &\longmapsto \Xi(0,t,\omega)\cdot x \end{aligned} .$$

Then clearly $\Phi(t,\omega,\cdot) = \Xi(0,t,\omega)\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuous for each fixed $(t,\omega) \in \mathbb{R}_{\geqslant 0} \times \Omega$. Moreover, $\Phi(0,\omega,x) = x$ for every $(\omega,x) \in \Omega \times \mathbb{R}^n$, and

$$\frac{d}{dt}\Phi(t,\omega,x) = A(\theta_t\omega)\Phi(t,\omega,x)$$

for Lebesgue-almost all $t \geqslant 0$, for each $(\omega,x) \in \Omega \times \mathbb{R}^n$. It can then be shown using uniqueness of solutions for (3.1) that Φ has the cocycle property,

$$\Phi(t+s,\omega,x) = \Phi(t,\theta_s\omega,\Phi(t,\omega,x))\,, \quad \forall s,t \geqslant 0\,, \quad \forall \omega \in \Omega\,, \quad \forall x \in \mathbb{R}^n\,.$$

(The argument goes along the lines of the proof of Lemma 3.3, invoking Lemma B.5 from Appendix B.) It can also be shown that Φ is indeed $(\mathcal{B}(\mathbb{R}_{\geqslant 0})\otimes\mathcal{F}\otimes\mathcal{B}(X))$-measurable (see Remark 3.43 and Example 3.44 at the end of Subsection 3.4.2). Thus (θ,Φ) constitutes an RDS, henceforth referred to as the RDS *generated* by the (*linear*) *random differential equation* (*RDE*) (3.1). $\Diamond$

We will build upon this example, referring to it several times throughout the rest of the work. Thus it will be convenient to have the symbols 'Ξ' and 'Φ' locked as well. Thus whenever we use them, the assumptions and the construction in Example 3.2 are to be tacitly understood.

Lemma 3.3 (Properties of the Fundamental Matrix Solution)**.** *Assume the same hypotheses as in* Example 3.2. *Then*

(1) $\Xi(0,t,\omega)\cdot\big(\Xi(0,s,\omega)\big)^{-1}\equiv\Xi(s,t,\omega)$, *and*

(2) $\Xi(s,t,\theta_\sigma\omega)\equiv\Xi(\sigma+s,\sigma+t,\omega)$.

Proof. In each case the proof comes down to showing that, for each arbitrarily fixed $s,\sigma\in\mathbb{R}$ and $\omega\in\Omega$, the function of $t\in\mathbb{R}$ defined by each side of the equation satisfies the same initial value problem. So equality follows by uniqueness of solutions.

(1) We have

$$\frac{d}{dt}\Xi(s,t,\omega)=A(\theta_t\omega)\Xi(s,t,\omega)$$

and, likewise,

$$\begin{aligned}\frac{d}{dt}\left(\Xi(0,t,\omega)\cdot\big(\Xi(0,s,\omega)\big)^{-1}\right) &= \left(\frac{d}{dt}\Xi(0,t,\omega)\right)\cdot\big(\Xi(0,s,\omega)\big)^{-1}\\ &= A(\theta_t\omega)\left(\Xi(0,t,\omega)\cdot\big(\Xi(0,s,\omega)\big)^{-1}\right)\end{aligned}$$

for Lebesgue-almost all $t\in\mathbb{R}$. Moreover,

$$\begin{aligned}\big(\Xi(s,t,\omega)\big)_{t=s} &= \Xi(s,s,\omega)\\ &= I_n\\ &= \Xi(0,s,\omega)\cdot\big(\Xi(0,s,\omega)\big)^{-1}\\ &= \left(\Xi(0,t,\omega)\cdot\big(\Xi(0,s,\omega)\big)^{-1}\right)_{t=s}.\end{aligned}$$

Thus both

$$t\longmapsto\Xi(s,t,\omega)\,,\quad t\in\mathbb{R}\,,$$

and

$$t\longmapsto\Xi(0,t,\omega)\cdot\big(\Xi(0,s,\omega)\big)^{-1}\,,\quad t\in\mathbb{R}\,,$$

are solutions of the initial value problem

$$\dot{\xi}=A(\theta_t\omega)\xi\,,\quad \xi(s)=I_n\,,\quad t\in\mathbb{R}\,.$$

The equality then follows by the uniqueness of solutions for said initial value problem (see Lemma B.5).

(2) The argument is essentially the same. We have

$$\frac{d}{dt}\Xi(s,t,\theta_\sigma\omega) = A(\theta_t\theta_\sigma\omega)\Xi(s,t,\theta_\sigma\omega) = A(\theta_{\sigma+t}\omega)\Xi(s,t,\theta_\sigma\omega)$$

and, by the Chain Rule,

$$\begin{aligned}\frac{d}{dt}\Xi(\sigma+s,\sigma+t,\omega) &= 1\cdot\left(\frac{d}{d\tau}\Xi(\sigma+s,\tau,\omega)\right)_{\tau=\sigma+t}\\ &= \big(A(\theta_\tau\omega)\Xi(\sigma+s,\tau,\omega)\big)_{\tau=\sigma+t}\\ &= A(\theta_{\sigma+t}\omega)\Xi(\sigma+s,\sigma+t,\omega)\end{aligned}$$

for Lebesgue-almost all $t\in\mathbb{R}$. So both sides of (2), as functions of $t\in\mathbb{R}$, satisfy the differential equation

$$\dot{\xi} = A(\theta_{\sigma+t}\omega)\xi\,,\quad t\in\mathbb{R}\,.$$

Since they also agree at $t=s$, where we have

$$\Xi(s,s,\theta_\sigma\omega) = I_n = \Xi(\sigma+s,\sigma+s,\omega)\,,$$

equality follows once again from uniqueness of solutions for the corresponding initial value problem. □

3.1.1 Trajectories and Equilibria

We now review some basic RDS concepts, introducing a few pieces of notation not found in Arnold [4] or Chueshov [8].

Let (θ,φ) be an RDS. Given $x\in X^\Omega_{\mathcal{B}}$, we define the (*forward*) *trajectory starting at* x to be the θ-stochastic process $\xi^x\in\mathcal{S}^X_\theta$ defined by

$$\xi^x_t(\omega) := \varphi(t,\omega,x(\omega))\,,\quad (t,\omega)\in\mathcal{T}_{\geqslant 0}\times\Omega\,. \tag{3.2}$$

The *pullback trajectory starting at* x is, in turn, defined to be the θ-stochastic process $\check{\xi}^x:\mathcal{T}_{\geqslant 0}\times\Omega\to X$ defined by

$$\check{\xi}^x_t(\omega) := \xi^x_t(\theta_{-t}\omega) = \varphi(t,\theta_{-t}\omega,x(\theta_{-t}\omega))\,,\quad (t,\omega)\in\mathcal{T}_{\geqslant 0}\times\Omega\,. \tag{3.3}$$

So, the pullback trajectory starting at x of an RDS is simply the pullback (in the sense of Definition 2.56) of the forward trajectory starting at x of the same RDS. Recall that we will always use the check mark ˇ to indicate the pullback of the θ-stochastic process being accented.

We slightly modify the standard notion of equilibrium for RDS (see, for instance, [8, Definition 1.7.1 on page 38]) to allow for the defining property to hold only θ-almost everywhere, as opposed to everywhere.

Definition 3.4 (Equilibrium)**.** An *equilibrium*[3] of an RDS (θ, φ) is a random variable $x \in X_{\mathcal{B}}^{\Omega}$ such that

$$\xi_t^x(\omega) = \varphi(t, \omega, x(\omega)) = x(\theta_t \omega)$$

for all $t \geqslant 0$, for θ-almost all $\omega \in \Omega$. △

In view of the notion of pullback convergence with which we will be working (see Subsection 3.1.2), it is more natural to think of the concept of equilibrium in terms of pullback trajectories. Observe that a random variable $x \in X_{\mathcal{B}}^{\Omega}$ is an equilibrium of the RDS (θ, φ) if, and only if

$$\check{\xi}_t^x(\omega) = \varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega)) = x(\omega)\,, \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall t \geqslant 0\,.$$

For deterministic systems, a point in the state space is an equilibrium if, and only if the trajectory of the system starting at that point is constant. The natural analog with 'θ-stationary processes' in place of 'constant trajectories' holds for RDS.

Proposition 3.5. *Given an RDS (θ, φ) and a random state $x \in X_{\mathcal{B}}^{\Omega}$, the following three properties are equivalent.*

(1) *x is an equilibrium.*

(2) *The trajectory ξ^x, as defined in* Equation (3.2)*, is θ-stationary.*

(3) *The map $t \mapsto \check{\xi}_t^x \in X_{\mathcal{B}}^{\Omega}$, $t \in \mathcal{T}_{\geqslant 0}$, is constant.*

Proof. (1) $\Leftrightarrow$ (2). If x is an equilibrium, then ξ^x must be θ-stationary by Definition 3.4 and Lemma 2.8. Conversely, if ξ^x is θ-stationary, then $x = \xi_0^x$ is an equilibrium by the same lemma.

[3] Sometimes referred to as a *random invariant point*, or *random fixed point*, or *stationary solution*. See [45, Definition 2 on page 584], and also Proposition 3.5 below.

(1) $\Leftrightarrow$ (3). If x is an equilibrium, then $\check{\xi}^x_t = x$ for all $t \geqslant 0$ as observed above. Conversely, if (3) holds, then

$$\xi^x_t(\omega) = \check{\xi}^x_t(\theta_t\omega) = \check{\xi}^x_0(\theta_t\omega) = x(\theta_t\omega)$$

for all $t \geqslant 0$, for θ-almost all $\omega \in \Omega$. Thus x is an equilibrium. □

3.1.2 Pullback Convergence

In this work, we follow the tradition developed and canonized in the literature on nonautonomous dynamical systems of considering convergence in the pullback sense [4, 8, 33, 13].

Definition 3.6. (Pullback Convergence) A θ-stochastic process $\xi \in \mathcal{S}^X_\theta$ is said to *converge to a random variable* $\xi_\infty \in X^\Omega_{\mathcal{B}}$ *in the pullback sense* if

$$\check{\xi}_t(\omega) = \xi_t(\theta_{-t}\omega) \longrightarrow \xi_\infty(\omega) \quad \text{as} \quad t \to \infty\,,$$

for θ-almost all $\omega \in \Omega$. △

One may argue that forward convergence would have been a more natural object to consider when thinking about applications. Recall that almost sure convergence implies convergence in probability. It then follows from (2.19) in Section 2.4 that a θ-stochastic process which converges in the pullback sense will also converge, in probability, in the forward sense. Therefore as far as applications go, pullbacks are still useful.

Mathematically, there are a couple other reasons pullbacks seem to be the most natural sense in which to study asymptotic behavior of RDS.

First note that

$$(t, x) \longmapsto \check{\Phi}_t x := \check{\xi}^x_t\,, \quad (t, x) \in \mathcal{T}_{\geqslant 0} \times X^\Omega_{\mathcal{B}}\,,$$

defines a (*skew-product*) *semiflow* $\check{\Phi}\colon \mathcal{T}_{\geqslant 0} \times X^\Omega_{\mathcal{B}} \to X^\Omega_{\mathcal{B}}$ on $X^\Omega_{\mathcal{B}}$—that is,

$$\check{\Phi}_0 x = x\,, \quad \forall x \in X^\Omega_{\mathcal{B}}\,,$$

and

$$\check{\Phi}_{s+t} x = \check{\Phi}_s \check{\Phi}_t x\,, \quad \forall s, t \in \mathcal{T}_{\geqslant 0}\,,\ \forall x \in X^\Omega_{\mathcal{B}}\,.$$

In other words, pullback trajectories of the RDS (θ, φ) are forward paths of the dynamical system $\check{\Phi}$, which could then be studied in light of the classical theory provided that $X_{\mathcal{B}}^{\Omega}$ is equipped with a suitable topology. However, one of the points of the RDS approach of Arnold lies precisely on the benefits of an explicit separation of the deterministic and random components of a system evolving subject to random perturbations, in particular, dropping the requirement that the space of random outcomes has any topological structure at all.

Pullbacks have also been noted as a more appropriate sense in which to formulate notions of "invariance" for nonautonomous systems. This is true even for deterministic systems [33]. Proposition 3.7 and Remark 3.8 below illustrate this point.

Proposition 3.7. *Let (θ, φ) be an RDS. Suppose that there exist a random initial state $x \in X_{\mathcal{B}}^{\Omega}$ and a map $x_\infty : \Omega \to X$ such that*

$$\check{\xi}_t^x(\omega) \longrightarrow x_\infty(\omega) \quad \text{as} \quad t \to \infty\,, \quad \widetilde{\forall}\omega \in \Omega\,. \tag{3.4}$$

Then x_∞ is an equilibrium.

Proof. For each $t \in \mathcal{T}_{\geqslant 0}$, the map

$$\omega \longmapsto \check{\xi}_t(\omega) = \varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega))\,, \quad \omega \in \Omega\,,$$

is measurable. So it follows from [36, Chapter 11, §1, Property M7 on page 248] that x_∞ is measurable. (If $\mathcal{T} = \mathbb{R}$, just pick a subsequence $(t_n)_{n\in\mathbb{N}}$ going to infinity in $[0, \infty)$.)

Now for each $\omega \in \Omega$ such that the limit in (3.4) exists, and each $\tau \in \mathcal{T}_{\geqslant 0}$, we also have

$$\lim_{t\to\infty} \varphi(t-\tau, \theta_{\tau-t}\omega, x(\theta_{\tau-t}\omega)) = x_\infty(\omega)\,.$$

By θ-invariance, the limit in (3.4) exists for $\theta_\tau\omega$ as well. Hence

$$\begin{aligned} x_\infty(\theta_\tau\omega) &= \lim_{t\to\infty} \varphi(t, \theta_{-t}\theta_\tau\omega, x(\theta_{-t}\theta_\tau\omega)) \\ &= \lim_{t\to\infty} \varphi(\tau + t - \tau, \theta_{-(t-\tau)}\omega, x(\theta_{-(t-\tau)}\omega)) \\ &= \lim_{t\to\infty} \varphi\big(\tau, \theta_{t-\tau}\theta_{-(t-\tau)}\omega, \varphi(t-\tau, \theta_{\tau-t}\omega, x(\theta_{\tau-t}\omega))\big) \\ &= \varphi(\tau, \omega, x_\infty(\omega)) \end{aligned}$$

by the cocycle property in (S2) and continuity property in (S1). $\square$

The next example illustrates the kind of constraint imposed upon an equilibrium of an RDS by its being a forward limit.

Remark 3.8 (Forward Limits and Equilibria). Given an RDS (θ, φ), assume that $x_\infty \in X_{\mathcal{B}}^{\Omega}$ is an equilibrium of the RDS, and suppose that x_∞ is the pointwise limit of the forward trajectory of (θ, φ) starting at x_∞—in other words,

$$\varphi(t, \omega, x_\infty(\omega)) = x_\infty(\theta_t \omega)\,, \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall t \geqslant 0\,,$$

and

$$\lim_{t\to\infty} \varphi(t, \omega, x_\infty(\omega)) = x_\infty(\omega)\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

Thus

$$\lim_{t\to\infty} x_\infty(\theta_t \omega) = x_\infty(\omega)\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

Now

$$x_\infty(\theta_\tau \omega) = \lim_{t\to\infty} x_\infty(\theta_t \theta_\tau \omega) = x_\infty(\omega)\,, \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall \tau \in \mathbb{R}\,,$$

meaning that x_∞ must be constant along each of its orbits. □

As pointed out above, a θ-stochastic process converges in probability in the forward sense if, and only if it also converges in probability in the pullback sense, in which case the limits are the same except on a θ-invariant subset of probability zero of Ω. The next example shows that this equivalence is not true in general for pointwise convergence.

Example 3.9 (Forward versus Pullback Convergence). Convergence in the pullback sense does not imply convergence in the forward sense. Consider the MPDS

$$\theta = (\Omega, \mathcal{F}, \mathbb{P}, (\theta_n)_{n\in\mathbb{Z}})\,,$$

where $\Omega := \{-1, 1\}$, $\mathcal{F} := 2^{\Omega}$, $\mathbb{P}$ is determined by

$$\mathbb{P}(\{-1\}) = \mathbb{P}(\{1\}) = \frac{1}{2}\,,$$

and θ is the measure preserving dynamical system generated by

$$\begin{aligned} \theta\colon \Omega &\longrightarrow \Omega \\ \omega &\longmapsto -\omega \end{aligned}\,.$$

Thus

$$\theta_n\omega = (-1)^n\omega\,, \quad \forall n \in \mathbb{Z}\,, \quad \forall\omega \in \Omega\,.$$

Let $X := \{-1, 1\}$ also, and consider the random variable

$$\begin{aligned} x_0 \colon \Omega &\longmapsto \Omega \\ \omega &\longmapsto \omega \end{aligned}$$

and the θ-stochastic process $\xi \in \mathcal{S}_\theta^X$ defined by

$$\xi_n(\omega) := (-1)^n\omega\,, \quad (n,\omega) \in \mathbb{Z}_{\geqslant 0} \times \Omega\,.$$

Then ξ converges to x_0 in the pullback sense,

$$\xi_n(\theta_{-n}\omega) = (-1)^n\theta_{-n}\omega = (-1)^0\omega = \omega\,, \quad \forall n \geqslant 0\,, \quad \forall\omega \in \Omega\,.$$

However ξ clearly does not converge in the forward sense.

A similar construction yields a counterexample to the hypothesis that convergence in the forward sense implies convergence in the pullback sense. Let $\tilde{\xi} \in \mathcal{S}_\theta^X$ be the θ-stochastic process defined by

$$\tilde{\xi}_n(\omega) := \omega\,, \quad (n,\omega) \in \mathbb{Z}_{\geqslant 0} \times \Omega\,.$$

Then $\tilde{\xi}$ converges to x_0 in the forward sense. However,

$$\tilde{\xi}_n(\theta_{-n}\omega) = \theta_{-n}\omega = (-1)^{-n}\omega\,, \quad \forall n \geqslant 0\,, \quad \forall\omega \in \Omega\,.$$

So $\tilde{\xi}$ does not converge in the pullback sense. $\Diamond$

3.1.3 Perfection of Crude Cocycles

We briefly review the concept of perfection of crude cocycles discussed in Arnold's [4, Section 1.2]. It is customary for the definition of an RDS to require that the cocycle property of the flow φ in (S2) holds for every $s, t \in \mathcal{T}_{\geqslant 0}$ and **every** $\omega \in \Omega$. If we want to emphasize this fact we shall say that φ is a *perfect cocycle* (over the underlying MPDS θ).

Definition 3.10 (Crude Cocycle)**.** We say that $\varphi\colon \mathcal{T}_{\geqslant 0} \times \Omega \times X \to X$ is a *crude cocycle* (*over* θ) if it is a $(\mathcal{B}(\mathcal{T}_{\geqslant 0}) \otimes \mathcal{F} \otimes \mathcal{B})$-measurable map satisfying (S1) and

(S2′) $\varphi(0,\omega) = id_X$ for each $\omega \in \Omega$ and, for every $s \geqslant 0$, there exists a subset $\Omega_s \subseteq \Omega$ of full measure such that

$$\varphi(t+s,\omega) = \varphi(t,\theta_s\omega) \circ \varphi(s,\omega)\,, \quad \forall t \geqslant 0\,, \quad \forall \omega \in \Omega_s\,.$$

(The Ω_s's need not be θ-invariant.) △

In particular, a perfect cocycle is a crude cocycle with $\Omega_s = \Omega$ for each $s \geqslant 0$.

As Arnold points out, there are circumstances in which this flexibility in the requirements for a cocycle is desirable. For instance, the flow of a stochastic differential equation is only guaranteed to be a crude cocycle [4, Section 2.3]. Another example will come up below after we introduce RDS with inputs. For consider (deterministic) controlled dynamical systems. Such systems yield a (deterministic) dynamical system when restricted to a constant input. One would then expect a sensible extension of the concept to random dynamical systems to have an analogous property. However, as we shall see in the proof of Lemma 3.29 below, the restriction of the flow of an RDS with inputs to a θ-stationary input is not necessarily a perfect cocycle.

Definition 3.11 (Indistinguishable Crude Cocycles)**.** Let θ be an MPDS and

$$\varphi, \psi \colon \mathcal{T}_{\geqslant 0} \times \Omega \times X \to X$$

be crude cocycles over θ. If there exists a subset $N \in \mathcal{F}$ such that $\mathbb{P}(N) = 0$ and

$$\{\omega \in \Omega\,;\ \varphi(t,\omega) \neq \psi(t,\omega)\,,\ \text{for some } t \geqslant 0\} \subseteq N\,,$$

then φ and ψ are said to be *indistinguishable.* △

Definition 3.12 (Perfection of Crude Cocycles)**.** Let $\varphi, \psi \colon \mathcal{T}_{\geqslant 0} \times \Omega \times X \to X$ be crude cocycles over an MPDS θ. We say that ψ is a *perfection* of φ if ψ is a perfect cocycle, and φ and ψ are indistinguishable. In this case we may also say that φ is *perfected by* ψ. △

Arnold's theory of perfection of crude cocycles will not be needed in this work. The Ω_s's of the crude cocycles we shall have to deal with will be the same for every $s \geqslant 0$, and actually θ-invariant. So, it will be enough to simply redefine the flow on a

θ-invariant subset of measure zero of Ω. (See Subsection 3.3.1 below.) We nevertheless state the proposition below for the sake of completeness.

Proposition 3.13. *Let $\theta = (\mathcal{F}, \Omega, \mathbb{P}, (\theta_t)_{t\in\mathcal{T}})$ be an MPDS with $\mathcal{T} = \mathbb{Z}$ or $\mathcal{T} = \mathbb{R}$. Suppose $\varphi\colon \mathcal{T}_{\geqslant 0} \times \Omega \times X \to X$ is a crude cocycle over θ evolving on a locally compact, locally connected, Hausdorff topological space X. Then φ can be perfected; in other words, there exists a perfect cocycle $\psi\colon \mathcal{T}_{\geqslant 0} \times \Omega \times X \to X$ such that φ and ψ are indistinguishable.*

Proof. See Arnold [4, Theorem 1.2.1] for the discrete case, which actually holds with weaker hypotheses and yields stronger conclusions. For the continuous case, see Arnold [4, Theorem 1.2.2 and Corollary 1.2.4]. □

3.2 RDS with Inputs and Outputs

We now introduce our new concept of "random dynamical systems with inputs." It extends the notion of RDS to systems in which there is a stochastic external input, or forcing function. A contribution of this work is the precise formulation of this concept, particularly the way in which the argument of the input is shifted in the semigroup (cocycle) property.

Recall the θ-shift operator introduced in Subsection 2.1.2. Given any $u \in \mathcal{S}_\theta^U$ and any $s \geqslant 0$, their θ-shift $\rho_s(u)\colon \mathcal{T}_{\geqslant 0} \times \Omega \to U$ is given by

$$[\rho_s(u)]_t(\omega) := u_{t+s}(\theta_{-s}\omega)\,, \quad (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,.$$

Recall, also, the operation of θ-concatenation of θ-stochastic processes introduced in Section 2.4. Given $u, v \in \mathcal{S}_\theta^U$ and $s \in \mathcal{T}_{\geqslant 0}$, their θ-concatenation was defined to be the θ-stochastic process $u\Diamond_s v\colon \mathcal{T}_{\geqslant 0} \times \Omega \to U$, given by

$$(u\Diamond_s v)_\tau(\omega) := \begin{cases} u_\tau(\omega)\,, & 0 \leqslant \tau < s \\ v_{\tau-s}(\theta_s\omega)\,, & s \leqslant \tau \end{cases}\,, \quad (\tau,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,.$$

Definition 3.14 (θ-Inputs)**.** We say that a subset $\mathcal{U} \subseteq \mathcal{S}_\theta^U$ is a *class of θ-inputs* if it has the following closure properties.

(J1) $\rho_s(u) \in \mathcal{U}$ for any $u \in \mathcal{U}$ and any $s \geqslant 0$, and

(J2) $u \Diamond_s v \in \mathcal{U}$ for any $u, v \in \mathcal{U}$ and any $s \geqslant 0$. $\triangle$

Thus, in other words, a class of θ-inputs is a subset of $\mathcal{S}_\theta^U$ which is closed under θ-shifts and θ-concatenation.

Example 3.15 (θ-Inputs). It follows from Lemmas 2.64 and 2.65 that the family $\mathcal{V}_\theta^U$ of tempered θ-stochastic processes $\mathcal{T}_{\geqslant 0} \times \Omega \to U$ is a class of θ-inputs. Moreover, it follows from Lemma 2.63 that $U_\theta^\Omega \subseteq \mathcal{V}_\theta^U$, where we identify[4] U_θ^Ω with the subset of $\mathcal{S}_\theta^U$ consisting of the θ-stationary θ-stochastic processes $\mathcal{T}_{\geqslant 0} \times \Omega \to U$ generated (via Lemma 2.8) by tempered random variables $\Omega \to U$.

It is not difficult to see that the family $\mathcal{K}_\theta^U$ of eventually precompact θ-stochastic processes $\mathcal{T}_{\geqslant 0} \times \Omega \to U$ also satisfies (J1) and (J2), thus constituting a class of θ-inputs as well. Note that it is not necessarily true, in general, that $U_\theta^\Omega \subseteq \mathcal{K}_\theta^U$.

We introduce a third notable class of θ-inputs. Let $\mathcal{S}_\infty^U$ be the family consisting of all θ-stochastic processes $u \in \mathcal{S}_\theta^U$ such that

$$t \longmapsto |u_t(\omega)|\,, \quad t \geqslant 0\,,$$

is locally essentially bounded for each $\omega \in \Omega$. To see that $\mathcal{S}_\infty^U$ is indeed a class of θ-inputs, fix arbitrarily $s \geqslant 0$, $u, v \in \mathcal{S}_\infty^U$, and $\omega \in \Omega$. Then

$$t \longmapsto |[\rho_s(u)]_t(\omega)| = u_{s+t}(\theta_{-s}\omega)\,, \quad t \geqslant 0\,,$$

is locally essentially bounded. We can readily see from the definition above that

$$t \longmapsto |(u \Diamond_s v)_t(\omega)|\,, \quad t \geqslant 0\,,$$

is also locally essentially bounded. This shows $\mathcal{S}_\infty^U$ satisfies (J1) and (J2). It follows from Lemma 2.33 that $U_\theta^\Omega \subseteq \mathcal{S}_\infty^U$.

Finally, note that the intersection of classes of θ-inputs is a class of θ-inputs. In particular, $\mathcal{V}_\theta^U \cap \mathcal{K}_\theta^U$, $\mathcal{V}_\theta^U \cap \mathcal{S}_\infty^U$, $\mathcal{K}_\theta^U \cap \mathcal{V}_\theta^U$ and $\mathcal{V}_\theta^U \cap \mathcal{K}_\theta^U \cap \mathcal{S}_\infty^U$ are classes of θ-inputs. Furthermore, $U_\theta^\Omega \subseteq \mathcal{V}_\theta^U \cap \mathcal{S}_\infty^U$. $\Diamond$

[4] This identification will be henceforth tacitly understood.

Definition 3.16 (Random Dynamical Systems with Inputs)**.** A *random dynamical system with inputs* (*RDSI*) is an ordered triple $(\theta, \varphi, \mathcal{U})$ consisting of an MPDS $\theta = (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathcal{T}})$, a class of θ-inputs $\mathcal{U} \subseteq \mathcal{S}_\theta^U$, and a map

$$\varphi : \mathcal{T}_{\geqslant 0} \times \Omega \times X \times \mathcal{U} \to X$$

satisfying,

(I1) $\varphi_u := \varphi(\cdot,\cdot,\cdot,u)\colon \mathcal{T}_{\geqslant 0} \times \Omega \times X \to X$ is $(\mathcal{B}(\mathcal{T}_{\geqslant 0}) \otimes \mathcal{F} \otimes \mathcal{B})$-measurable for each fixed $u \in \mathcal{U}$;

(I2) $\varphi(t, \omega, \cdot, u) : X \to X$ is continuous for each fixed $(t, \omega, u) \in \mathcal{T}_{\geqslant 0} \times \Omega \times \mathcal{U}$;

(I3) $\varphi(0, \omega, x, u) = x$ for each $(\omega, x, u) \in \Omega \times X \times \mathcal{U}$;

(I4) for any $s, t \geqslant 0$, $\omega \in \Omega$, $x \in X$, and $u, v \in \mathcal{U}$, if

$$\varphi(s, \omega, x, u) = y$$

and

$$\varphi(t, \theta_s\omega, y, v) = z\,,$$

then

$$z = \varphi(s + t, \omega, x, u\Diamond_s v)\,; and$$

(I5) given any $t \geqslant 0$, $\omega \in \Omega$, $x \in X$, and $u, v \in \mathcal{U}$, if $u_\tau(\omega) = v_\tau(\omega)$ for Lebesgue-almost all $\tau \in [0, t)$, then $\varphi(t, \omega, x, u) = \varphi(t, \omega, x, v)$. △

As with MPDS and RDS, whenever we talk about an RDSI $(\theta, \varphi, \mathcal{U})$, we tacitly assume the notation laid above, unless otherwise specified. (I1) and (I2) are regularity conditions. (I3) means that "nothing has happened if one is still at time $t = 0$." (I4) generalizes the cocycle property of RDS (see also Remark 3.17 below), and (I5) states that the evolution of an RDS subject to an input u is, so to speak, "independent of irrelevant random input values."

Remark 3.17. Notice that for each arbitrarily fixed $s, t \geqslant 0$, $x \in X$, and $\omega \in \Omega$,

$$\varphi(t + s, \omega, x, u) \;=\; \varphi(t, \theta_s\omega, \varphi(s, \omega, x, u), \rho_s(u))\,, \quad \forall u \in \mathcal{U}\,.$$

This follows straight from (I4) with $v = \rho_s(u)$, which then yields $u\Diamond_s v = u$. □

The shift operator ρ_s has a physical interpretation. The right-hand side in

$$[\rho_s(u)]_t(\omega) = u_{t+s}(\theta_{-s}\omega)$$

is the input as interpreted by an observer of the RDSI φ who started at time $t_1 = 0$, while the left-hand side is how someone who started observing the system at time $t_2 = s$ would describe it t units of time later—that is, at time $t+s$ from the perspective of the first observer. Following this interpretation, a θ-stationary input would then be an input which is observed to look the same, regardless of when one started observing it.

Example 3.18. (RDSI Generated by Random Differential Linear Equations with Inputs) This generalizes Example 3.2. Given an MPDS θ, suppose that $A\colon \Omega \to M_{n\times n}(\mathbb{R})$ and $B\colon \Omega \to M_{n\times k}(\mathbb{R})$ are random real matrices such that, for each $\omega \in \Omega$,

$$t \longmapsto \|A(\theta_t\omega)\|\,, \quad t \geqslant 0\,,$$

is locally integrable and

$$t \longmapsto \|B(\theta_t\omega)\|\,, \quad t \geqslant 0\,,$$

is locally essentially bounded. Let $U := \mathbb{R}^k$ and let $\mathcal{S}_\infty^U \subseteq \mathcal{S}_\theta^U$ be the class of θ-inputs from Example 3.15. We consider the *random differential equation with inputs* (*RDEI*)

$$\dot{\xi} = A(\theta_t\omega)\xi + B(\theta_t\omega)u_t(\omega)\,, \quad t \geqslant 0\,, \quad \omega \in \Omega\,, \quad u \in \mathcal{S}_\infty^U\,. \tag{3.5}$$

Let $\Xi\colon \mathbb{R}\times\mathbb{R}\times\Omega \to M_{n\times n}(\mathbb{R})$ be the fundamental matrix solution of the homogeneous, linear RDE

$$\dot{\xi} = A(\theta_t\omega)\xi\,, \quad t \geqslant 0\,,$$

and let (θ, Φ) be the RDS generated by the same equation (refer to Example 3.2). For each fixed $(\omega, u) \in \Omega \times \mathcal{S}_\infty^U$, define $\Psi(\cdot,\omega,u)\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}^n$ by

$$\Psi(t,\omega,u) := \int_0^t \Xi(\sigma,t,\omega)B(\theta_\sigma\omega)u_\sigma(\omega)\,d\sigma\,, \quad t \geqslant 0\,.$$

Finally, define

$$\begin{array}{rcl} \varphi\colon \mathbb{R}_{\geqslant 0}\times\Omega\times\mathbb{R}^n\times\mathcal{S}_\infty^U & \longrightarrow & \mathbb{R}^n \\ (t,\omega,x,u) & \longmapsto & \Phi(t,\omega,x)+\Psi(t,\omega,u) \end{array}\,.$$

Then for any arbitrarily fixed (ω, x, u), we have

$$\varphi(0, \omega, x, u) = x\,,$$

and, by the Chain Rule,

$$\begin{aligned}
\frac{d}{dt}\varphi(t,\omega,x,u) &= \frac{d}{dt}\Phi(t,\omega,x) + \frac{d}{dt}\Psi(t,\omega,u) \\
&= A(\theta_t\omega)\Phi(t,\omega,x) \\
&\quad + \Xi(t,t,\omega)B(\theta_t\omega)u_t(\omega) + \int_0^t A(\theta_t\omega)\Xi(\sigma,t,\omega)B(\theta_\sigma\omega)u_\sigma(\omega)\,d\sigma \\
&= A(\theta_t\omega)\left(\Phi(t,\omega,x) + \int_0^t \Xi(\sigma,t,\omega)B(\theta_\sigma\omega)u_\sigma(\omega)\,d\sigma\right) \\
&\quad + B(\theta_t\omega)u_t(\omega) \\
&= A(\theta_t)\varphi(t,\omega,x,u) + B(\theta_t\omega)u_t(\omega)
\end{aligned}$$

for Lebesgue-almost every $t \geqslant 0$. Thus φ indeed satisfies the RDEI (3.29).

Furthermore, $(\theta, \varphi, \mathcal{S}_\infty^U)$ is an RDSI. Indeed, (I2) and (I3) follow directly from the analogous properties of Φ. Properties (I4) and (I5) follow from uniqueness of solutions applied for each fixed $\omega \in \Omega$—one basically has to check that both sides of the identities in (I4) and (I5), when looked at as functions of t for arbitrarily fixed values of the other variables, define solutions of the same differential equation with the same initial condition. The measurability requirement in (I1) seems to be the trickiest property to check. For this, as well as the details of how to check (I2)–(I5), we refer the reader to Theorem 3.42 and Example 3.44 further down.

We shall refer to the RDSI $(\theta, \varphi, \mathcal{S}_\infty^U)$ constructed above as the RDSI *generated* by the RDEI (3.29). $\Diamond$

Recall the concepts of tempered and precompact trajectories from Section 2.4. In what follows, we shall often require that the RDSI in question preserves these properties when subject to inputs having them. The next two definitions make this idea precise.

Definition 3.19 (Tempered RDSI)**.** An RDSI $(\theta, \varphi, \mathcal{U})$ is said to be *tempered* if the θ-stochastic processes

$$(t,\omega) \longmapsto \varphi(t,\omega,x,u)\,, \quad (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,,$$

are tempered for each tempered random initial state $x \in X_\theta^\Omega$ and each tempered input $u \in \mathcal{U}$. $\triangle$

Given any tempered initial state $x \in X_\theta^\Omega$ and any tempered input $u \in \mathcal{S}_\theta^U$, let $D\colon \Omega \to 2^X \backslash \{\varnothing\}$ be a rest set. Then

$$\varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega), u) \in D(\omega)\,, \quad \forall t \geqslant 0\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

Thus

$$\omega \longmapsto \varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega, u))\,, \quad \omega \in \Omega\,,$$

is a tempered random variable for each $t \geqslant 0$.

Definition 3.20 (Compact RDSI)**.** An RDSI $(\theta, \varphi, \mathcal{U})$ is said to be *compact* if the θ-stochastic processes

$$(t, \omega) \longmapsto \varphi(t, \omega, x, u)\,, \quad (t, \omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,,$$

are eventually precompact for every tempered initial state $x \in X_\theta^\Omega$ and every eventually precompact input $u \in \mathcal{U}$. $\triangle$

Although the context here is somewhat different, this definition is related to the concept of *compact* RDS given in [8, Definition 1.4.3 on page 30]. Chueshov does not require the "entering time" $t_0(\omega)$ to be uniform in ω, while we do require the entering time τ_ξ in Definition 2.61 to be the same for θ-almost every $\omega \in \Omega$. On the other hand, Chueshov requires the "absorbing set" to be the same for every initial state, while we allow for it to depend on $x \in X_\theta^\Omega$.

Finally, we introduce a concept of outputs.

Definition 3.21 (Output Functions)**.** An output function is a $(\mathcal{F} \otimes \mathcal{B}(X))$-measurable map $h\colon \Omega \times X \to Y$ into a Polish space Y such that $h(\omega, \cdot)\colon X \to Y$ is continuous for each $\omega \in \Omega$. In this context Y is called an *output space*. $\triangle$

Definition 3.22 (Random Dynamical Systems with Inputs and Outputs)**.** A *random dynamical system with inputs and outputs* (*RDSIO*) is a quadruple $(\theta, \varphi, \mathcal{U}, h)$, such that $(\theta, \varphi, \mathcal{U})$ is an RDSI, and h is an output function. $\triangle$

It may sometimes be useful to refer to a *random dynamical system with outputs* (*RDSO*) only, by which we mean an ordered triple (θ, φ, h) where (θ, φ) is an RDS and h is an output function.

The Ω-component in the domain of output functions is important. It allows for the concept to model uncertainties in the readout as well.

We will discuss systems with outputs in greater depth in the next subsection. We will be concerned, at first, with issues of algebraic nature; in other words, how inputs, outputs and pullbacks interact. We will consider regularity properties of output functions in the next chapter, in the context of setting up the stage for the Small-Gain Theorem (Theorem 4.28).

3.2.1 Pullback trajectories

Let $(\theta, \varphi, \mathcal{U}, h)$ be an RDSIO. Given $x \in X^{\Omega}_{\mathcal{B}}$ and $u \in \mathcal{U}$, we define the (*forward*) *trajectory starting at* x *and subject to* u to be the θ-stochastic process $\xi^{x,u} \in \mathcal{S}^X_\theta$ defined by

$$\xi^{x,u}_t(\omega) := \varphi(t, \omega, x(\omega), u)\,, \quad (t, \omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,.$$

We then define the *pullback trajectory starting at* x *and subject to* u to be the θ-stochastic process $\check{\xi}^{x,u} \in \mathcal{S}^X_\theta$ defined by

$$\check{\xi}^{x,u}_t(\omega) := \xi^{x,u}_t(\theta_{-t}\omega) = \varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega), u)\,, \quad (t, \omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,.$$

The (*forward*) *output trajectory corresponding to initial state* x *and input* u is defined to be the θ-stochastic process $\eta^{x,u} \in \mathcal{S}^Y_\theta$, where

$$\eta^{x,u}_t(\omega) := h(\theta_t\omega, \xi^{x,u}_t(\omega)) = h(\theta_t\omega, \varphi(t, \omega, x(\omega), u))\,, \quad (t, \omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,,$$

while the *pullback output trajectory corresponding to initial state* x *and input* u is analogously defined to be the θ-stochastic process $\check{\eta}^{x,u} \in \mathcal{S}^Y_\theta$, where

$$\check{\eta}^{x,u}_t(\omega) := \eta^{x,u}_t(\theta_{-t}\omega) = h(\omega, \varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega), u))\,, \quad (t, \omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,.$$

Note that

$$\check{\eta}^{x,u}_t \equiv h(\omega, \check{\xi}^{x,u}_t(\omega))\,.$$

For RDSI the definitions of ‘forward’ and ‘pullback’ trajectories are the same, and we also use the notations $\xi^{x,u}$ and $\check{\xi}^{x,u}$.

For RDSO the definitions are analogous, except that they do not depend on any inputs. So forward and pullback trajectories are defined as for RDS (refer to (3.2) and (3.3)), and we also use the notations ξ^x and $\check{\xi}^x$, respectively. Forward and pullback output trajectories are defined analogously. We define the (*forward*) *output trajectory corresponding to initial state* x to be the θ-stochastic process $\eta^x \in \mathcal{S}_\theta^Y$ defined by

$$\eta_t^x(\omega) := h(\theta_t\omega, \xi_t^x(\omega)) = h(\theta_t\omega, \varphi(t, \omega, x(\omega)))\,, \quad (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,,$$

and the *pullback output trajectory corresponding to initial state* x to be the θ-stochastic process $\check{\eta}^x \in \mathcal{S}_\theta^Y$ defined by

$$\check{\eta}_t^x(\omega) := h(\omega, \check{\xi}_t^x(\omega)) = h(\omega, \varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega)))\,, \quad (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,.$$

Observe that the input u is not shifted in the argument of φ in the pullback, while at first one might intuitively think it should have been. There are several reasons this is so. First notice that this

$$\check{\xi}_t^{x,u}(\omega) = \xi_t^{x,u}(\theta_{-t}\omega)\,, \quad \forall (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,.$$

So $\check{\xi}^{x,u}$ is just the pullback, in the sense of Definition 2.56, of the θ-stochastic process $\xi^{x,u}$. However we are more concerned with what happens in the context of “cascades” and “feedback interconnections” of RDSIO. So, this issue asks for further scrutiny. But before we get deep into it, we first discuss discrete RDSIO. This will further motivate axioms (I1)–(I5) in the definition of an RDSI, produce—and completely characterize—a whole class of examples, and provide the framework for said discussion of pullback trajectories and cascades.

We say that an RDSI (or RDSIO) is *discrete* when $\mathcal{T} = \mathbb{Z}$. A notable class of discrete RDSI is the class of RDSI generated by “random difference equations with inputs,” the discrete-time object analogous to RDEI which we now describe.

Definition 3.23 ((Random) Transition Maps)**.** A $(\mathcal{F} \otimes \mathcal{B} \otimes \mathcal{B}(U))$-measurable map $f\colon \Omega \times X \times U \to X$ will be said to be a (*random*) *transition map* if

(T) $f(\omega,\cdot,\tilde{u})\colon X \longrightarrow X$ is continuous for each $(\omega,\tilde{u}) \in \Omega \times U$. $\triangle$

Given a transition map f and a class of θ-inputs $\mathcal{U}$, we say that a map $\varphi\colon \mathcal{T}_{\geqslant 0} \times \Omega \times X \times U \to X$ is a *solution* of the *random difference equation with inputs* (*RdEI*)

$$\xi^+ = f(\theta_n\omega, \xi, u_n(\omega))\,, \quad n \geqslant 0\,, \quad \omega \in \Omega\,, \quad u \in \mathcal{U}\,,$$

if

$$\varphi(0,\omega,x,u) = x\,, \quad \forall(\omega,x,u) \in \Omega \times X \times \mathcal{U}\,,$$

and

$$\varphi(n+1,\omega,x,u) = f(\theta_n\omega, \varphi(n,\omega,x,u), u_n(\omega))$$

for every $(n,\omega,x,u) \in \mathcal{T}_{\geqslant 0} \times \Omega \times X \times \mathcal{U}$.

Theorem 3.24 (Random Difference Equations with Inputs)**.** *Let θ be an MPDS, $\mathcal{U}$ be a class of θ-inputs, and f a transition map. Then the RdEI*

$$\xi^+ = f(\theta_n\omega, \xi, u_n(\omega))\,, \quad n \geqslant 0\,, \quad \omega \in \Omega\,, \quad u \in \mathcal{U}\,, \tag{3.6}$$

has a unique solution $\varphi\colon \mathcal{T}_{\geqslant 0} \times \Omega \times X \times \mathcal{U} \to X$. Furthermore, the ordered triple $(\theta,\varphi,\mathcal{U})$ is an RDSI. (In this case we refer to the map f as the generator *of the RDSI $(\theta,\varphi,\mathcal{U})$.)*

Proof. Define $\varphi\colon \mathcal{T}_{\geqslant 0} \times \Omega \times X \times \mathcal{U} \to X$ recursively by

$$\varphi(0,\omega,x,u) := x\,, \quad (\omega,x,u) \in \Omega \times X \times \mathcal{U}\,, \tag{3.7}$$

and

$$\varphi(n+1,\omega,x,u) := f(\theta_n\omega, \varphi(n,\omega,x,u), u_n(\omega))\,, \quad (n,\omega,x,u) \in \mathcal{T}_{\geqslant 0} \times \Omega \times X \times \mathcal{U}\,. \tag{3.8}$$

Thus φ is a solution of (3.6). It is uniquely determined by its state at $n = 0$ in (3.7) and the inductive process in (3.8).

It remains to show that $(\theta,\varphi,\mathcal{U})$ is an RDSI. We shall check (I1), (I2), (I3), (I5) and (I4) in this order.

(I1) Fix arbitrarily $u \in \mathcal{U}$. One first shows, using induction on $n \geqslant 1$, that

$$\varphi(n,\cdot,\cdot,u) = f(\theta_{n-1}\cdot, \varphi(n-1,\cdot,\cdot,u), u_{n-1}(\cdot)) \tag{3.9}$$

is $(\mathcal{F} \otimes \mathcal{B})$-measurable for each $n \geqslant 1$. Indeed, at $n = 1$ we have

$$\varphi(1, \cdot, \cdot, u) = f(\theta_{1-1} \cdot, \varphi(1-1, \cdot, \cdot, u), u_{1-1}(\cdot)) = f(\cdot, \cdot, u_0(\cdot)),$$

which is $(\mathcal{F} \otimes \mathcal{B})$-measurable, since f is $(\mathcal{F} \otimes \mathcal{B} \otimes \mathcal{B}(U))$-measurable and u_0 is $\mathcal{F}$-measurable. The inductive step follows straight from (3.9), since the righthand side is a composition of measurable functions and, hence, itself measurable.

Now pick any $A \in \mathcal{B}$. We then have

$$\varphi(\cdot, \cdot, \cdot, u)^{-1}(A) = \bigcup_{n=0}^{\infty} \{n\} \times \varphi(n, \cdot, \cdot, u)^{-1}(A) \in 2^{\mathbb{Z}_{\geqslant 0}} \otimes \mathcal{F} \otimes \mathcal{B},$$

since it is a countable union of $(2^{\mathbb{Z}_{\geqslant 0}} \otimes \mathcal{F} \otimes \mathcal{B})$-measurable sets. Since $A \in \mathcal{B}$ and $u \in \mathcal{U}$ were chosen arbitrarily, this proves that (I1) holds.

(I2) This follows from (T), (3.7) and (3.8), again by induction. Fix arbitrarily $\omega \in \Omega$ and $u \in \mathcal{U}$. At $n = 0$, it follows straight from (3.7) that $\varphi(0, \omega, \cdot, u)$ is continuous. So, once (I2) has been proved for a certain value of $n \geqslant 0$, we conclude from (T) and (3.8) that

$$\varphi(n+1, \omega, \cdot, u) = f(\theta_n \omega, \varphi(n, \omega, \cdot, u), u_n(\omega))$$

is also continuous. This completes the induction, proving (I2).

(I3) This is the same as (3.7).

(I5) This is shown, once again, by induction. Fix $\omega \in \Omega$ and $x \in X$ arbitrarily. The base of the induction is given by (3.7). Now assume (I5) holds for a certain value of $t = n \geqslant 0$. If $u, v \in \mathcal{U}$ are such that

$$u_j(\omega) = v_j(\omega), \quad \forall j = 0, 1, \ldots, n,$$

then

$$\varphi(n, \omega, x, u) = \varphi(n, \omega, x, v)$$

by the induction hypothesis. So, it follows from (3.8) that

$$\begin{aligned} \varphi(n+1, \omega, x, u) &= f(\theta_n \omega, \varphi(n, \omega, x, u), u_n(\omega)) \\ &= f(\theta_n \omega, \varphi(n, \omega, x, v), v_n(\omega)) \\ &= \varphi(n+1, \omega, x, v). \end{aligned}$$

This proves (I5).

(I4) Fix $\omega \in \Omega$, $x \in X$, $u, v \in \mathcal{U}$ and $p \geqslant 0$ arbitrarily. We use induction on n to show that

$$\varphi(n+p, \omega, x, u \Diamond_p v) = \varphi(n, \theta_p \omega, \varphi(p, \omega, x, u), v)\,, \quad \forall n \geqslant 0\,. \tag{3.10}$$

For $n = 0$, (3.10) holds in virtue of (I3) and (I5). Indeed, we have

$$(u \Diamond_p v)_j(\omega) = u_j(\omega)\,, \quad \forall j = 0, \ldots, p-1\,,$$

therefore

$$\varphi(0+p, \omega, x, u \Diamond_p v) = \varphi(p, \omega, x, u) = \varphi(0, \theta_p \omega, \varphi(p, \omega, x, u), v)\,.$$

Now suppose (3.10) holds for some $n \geqslant 0$. Set $y := \varphi(p, \omega, x, u)$. By the induction hypothesis,

$$\varphi(n, \theta_p \omega, y, v) = \varphi(n+p, \omega, x, u \Diamond_p v)\,.$$

Hence

$$\begin{aligned}
\varphi(n+1, \theta_p \omega, y, v) &= f(\theta_n \theta_p \omega, \varphi(n, \theta_p \omega, y, v), v_n(\theta_p \omega)) \\
&= f(\theta_{n+p} \omega, \varphi(n+p, \omega, x, u \Diamond_p v), (u \Diamond_p v)_{n+p}(\omega)) \\
&= \varphi(n+p+1, \omega, x, u \Diamond_p v)\,.
\end{aligned}$$

(3.10) then follows by induction. Since $\omega \in \Omega$, $x \in X$, $u, v \in \mathcal{U}$ and $p \geqslant 0$ were chosen arbitrarily, this establishes (I4), completing the proof that $(\theta, \varphi, \mathcal{U})$ is an RDSI. □

From the construction of a discrete RDSI $(\theta, \varphi, \mathcal{U})$ from a transition map f above, it is clear how the value of the flow φ at time $n+1$, when subject to ω, depends on the input u through its value $u_n(\omega)$ at time n. So, when one shifts the ω-argument of φ in the pullback trajectory to $\theta_{-n}\omega$, there is no need to modify the input; the value of $\varphi(n, \theta_{-n}\omega, x(\theta_{-n}\omega), u)$ will automatically depend on $u_n(\theta_{-n}\omega)$ already. This is the second reason we defined the pullback trajectories of RDSI as we did.

We now discuss the third and most important reason this is the mathematically sensible way of defining pullback trajectories for RDSI.

Consider the topological product $Z := X_1 \times X_2$ of two Polish spaces X_1 and X_2, let $g\colon \Omega \times Z \longrightarrow Z$ be a transition map, and let (θ, ψ) be the discrete RDS generated by g. Now suppose g can be written as

$$g(\omega, (x_1, x_2)) \equiv \begin{pmatrix} f_1(\omega, x_1) \\ f_2(\omega, x_2, h_1(\omega, x_1)) \end{pmatrix}, \tag{3.11}$$

where $f_1\colon \Omega \times X_1 \to X_1$ is the generator of some RDS (θ, φ_1), $h_1\colon \Omega \times X_1 \to Y_1$ is an output function, thus yielding an RDSO (θ, φ_1, h_1), and $f_2\colon \Omega \times X_2 \times U_2 \to X_2$ is the generator of some RDSI $(\theta, \varphi_2, \mathcal{U}_2)$ with input space $U_2 = Y_1$. We use η_1 to denote the output trajectories of (θ, φ_1, h_1), ξ for the state trajectories of ψ, and ξ_2 for the state trajectories of $(\theta, \varphi_2, \mathcal{U}_2)$. Finally, let $\pi_2\colon X_1 \times X_2 \to X_2$ be the projection onto the second coordinate.

Proposition 3.25. *For any random initial state*

$$z = (x_1, x_2) \in Z^{\Omega}_{\mathcal{B}(Z)} = (X_1)^{\Omega}_{\mathcal{B}(X_1)} \times (X_2)^{\Omega}_{\mathcal{B}(X_2)},$$

the following two identities hold.

(1) $\psi(n, \omega, z(\omega)) \equiv \begin{pmatrix} \varphi_1(n, \omega, x_1(\omega)) \\ \varphi_2(n, \omega, x_2(\omega), (\eta_1)^{x_1}) \end{pmatrix}$, *and*

(2) $\pi_2(\check{\xi}^z_n(\omega)) \equiv (\check{\xi}_2)^{x_2, (\eta_1)^{x_1}}_n(\omega)$.

Proof. (1) For each arbitrarily fixed $\omega \in \Omega$ and $z = (x_1, x_2) \in Z^{\Omega}_{\mathcal{B}(Z)}$, we use induction on $n \geqslant 0$. At $n = 0$, it follows from (S2)/(I3) that

$$\psi(0, \omega, z(\omega)) = z(\omega) = \begin{pmatrix} x_1(\omega) \\ x_2(\omega) \end{pmatrix} = \begin{pmatrix} \varphi_1(0, \omega, x_1(\omega)) \\ \varphi_2(0, \omega, x_2(\omega), (\eta_1)^{x_1}) \end{pmatrix}.$$

Now suppose that (1) holds for some $n \geqslant 0$. Since

$$(\eta_1)^{x_1}_n(\omega) = h_1\big(\theta_n \omega, \varphi_1(n, \omega, x_1(\omega))\big)$$

by definition, it follows that

$$\begin{aligned}
\psi(n+1,\omega,z(\omega)) &= g\big(\theta_n\omega,\psi(n,\omega,z(\omega))\big) \\
&= \begin{pmatrix} f_1\big(\theta_n\omega,\varphi_1(n,\omega,x_1(\omega))\big) \\ f_2(\theta_n\omega,\varphi_2(n,\omega,x_2(\omega),(\eta_1)^{x_1}),(\eta_1)^{x_1}_n(\omega)) \end{pmatrix} \\
&= \begin{pmatrix} \varphi_1(n+1,\omega,x_1(\omega)) \\ \varphi_2(n+1,\omega,x_2(\omega),(\eta_1)^{x_1}) \end{pmatrix}.
\end{aligned}$$

This completes the induction.

(2) We prove by induction that (2) holds, for each $n \geqslant 0$, for all random initial states $z = (x_1, x_2) \in Z^{\Omega}_{\mathcal{B}(Z)}$, and all $\omega \in \Omega$. At $n = 0$, we have

$$\begin{aligned}
\pi_2(\check{\xi}^z_0(\omega)) &= \pi_2\big(\psi(0,\omega,z(\omega))\big) \\
&= \pi_2(x_1(\omega),x_2(\omega)) \\
&= x_2(\omega) \\
&= \varphi_2(0,\omega,x_2(\omega),(\eta_1)^{x_1}) \\
&= (\check{\xi}_2)^{x_2,(\eta_1)^{x_1}}_0
\end{aligned}$$

for any $z = (x_1, x_2) \in Z^{\Omega}_{\mathcal{B}(Z)}$ and any $\omega \in \Omega$.

Now assume (2) has been shown to hold for all nonnegative integer values of n up to some $n_0 \geqslant 0$, for all random initial states $z = (x_1, x_2) \in Z^{\Omega}_{\mathcal{B}(Z)}$ and all $\omega \in \Omega$. Given $z = (x_1, x_2) \in Z^{\Omega}_{\mathcal{B}(Z)}$, define $\hat{z} = (\hat{x}_1, \hat{x}_2) \in Z^{\Omega}_{\mathcal{B}(Z)}$ by

$$\begin{aligned}
\hat{z}(\omega) &:= g(\theta_{-1}\omega, z(\theta_{-1}\omega)) \\
&= \begin{pmatrix} f_1(\theta_{-1}\omega, x_1(\theta_{-1}\omega)) \\ f_2\big(\theta_{-1}\omega, x_2(\theta_{-1}\omega), h_1(\theta_{-1}\omega, x_1(\theta_{-1}\omega))\big) \end{pmatrix}, \quad \omega \in \Omega\,.
\end{aligned} \tag{3.12}$$

Fix $\omega \in \Omega$ arbitrarily and denote $\hat{\omega} := \theta_{-(n_0+1)}\omega$. Then

$$\begin{aligned}
\pi_2(\check{\xi}^z_{n_0+1}(\omega)) &= \pi_2\big(\psi(n_0+1, \hat{\omega}, z(\hat{\omega}))\big) \\
&= \pi_2\Big(\psi\big(n_0, \theta_{-n_0}\omega, \psi(1, \hat{\omega}, z(\hat{\omega}))\big)\Big) \\
&= \pi_2\Big(\psi\big(n_0, \theta_{-n_0}\omega, g(\hat{\omega}, z(\hat{\omega}))\big)\Big) \\
&= \pi_2\big(\psi(n_0, \theta_{-n_0}\omega, \hat{z}(\theta_{-n_0}\omega))\big) \\
&= \pi_2(\check{\xi}^{\hat{z}}_{n_0}(\omega)) \\
&= (\check{\xi}_2)^{\hat{x}_2,(\eta_1)^{\hat{x}_1}}_{n_0}(\omega)
\end{aligned}$$

by the induction hypothesis. Now

$$h_1(\hat{\omega}, x_1(\hat{\omega})) = (\eta_1)^{x_1}_0(\hat{\omega})\,,$$

and

$$(\eta_1)^{\hat{x}_1} = \rho_1((\eta_1)^{x_1})$$

by Lemma 3.26 below, thus

$$\begin{aligned}
(\check{\xi}_2)^{\hat{x}_2,(\eta_1)^{\hat{x}_1}}_{n_0}(\omega) &= \varphi_2(n_0, \theta_{-n_0}\omega, \hat{x}_2(\theta_{-n_0}\omega), (\eta_1)^{\hat{x}_1}) \\
&= \varphi_2(n_0, \theta_{-n_0}\omega, f_2(\hat{\omega}, x_2(\hat{\omega}), (\eta_1)^{x_1}_0(\hat{\omega})), (\eta_1)^{\hat{x}_1}) \\
&= \varphi_2(n_0, \theta_{-n_0}\omega, \varphi_2(1, \hat{\omega}, x_2(\hat{\omega}), (\eta_1)^{x_1}), \rho_1((\eta_1)^{x_1})) \\
&= \varphi_2(n_0+1, \theta_{-(n_0+1)}\omega, x_2(\theta_{-(n_0+1)}\omega), (\eta_1)^{x_1}) \\
&= (\check{\xi}_2)^{x_2,(\eta_1)^{x_1}}_{n_0+1}(\omega)\,.
\end{aligned}$$

So,

$$\pi_2(\check{\xi}^z_{n_0+1}(\omega)) = (\check{\xi}_2)^{x_2,(\eta_1)^{x_1}}_{n_0+1}(\omega)\,.$$

Since $z = (x_1, x_2) \in Z^{\Omega}_{\mathcal{B}(Z)}$ and $\omega \in \Omega$ were arbitrary, this completes the inductive step, thus proving (2). $\square$

The lefthand side of (2) in the proposition above is the projection over the second coordinate of the pullback trajectory starting at $z = (x_1, x_2)$ of the RDS (θ, ψ). The righthand side is the pullback trajectory of the RDSI $(\theta, \varphi_2, \mathcal{U}_2)$ starting at x_2 and subject to the input $(\eta_1)^{x_1}$, the output trajectory of (θ, φ_1, h_1) starting at x_1. Proposition 3.25 then says that they coincide.

An analogous result holds in continuous time for systems generated by RDEI. The same principle also applies to more complicated RDS or RDSI which can be decomposed as the cascade/feedback interconnection of two or more coordinate RDS or RDSI.

We now state and prove the technical lemma referred to in the proof of item (2) in Proposition 3.25.

Lemma 3.26. *Let $f\colon \Omega \times X \to X$ be the generator of a discrete RDS (θ, φ) and $h\colon \Omega \times X \to Y$ an output function—thus yielding an RDSO (θ, φ, h). Given $x \in X^{\Omega}_{\mathcal{B}}$, define $\hat{x} \in X^{\Omega}_{\mathcal{B}}$ by*

$$\hat{x}(\omega) := f(\theta_{-1}\omega, x(\theta_{-1}\omega))\,, \quad \omega \in \Omega\,.$$

Then $\eta^{\hat{x}} = \rho_1(\eta^x)$.

Proof. Indeed, we have

$$\begin{aligned}
\eta^{\hat{x}}_n(\omega) &= h\big(\theta_n\omega, \varphi(n, \omega, \hat{x}(\omega))\big) \\
&= h\Big(\theta_n\omega, \varphi\big(n, \omega, f(\theta_{-1}\omega, x(\theta_{-1}\omega))\big)\Big) \\
&= h\Big(\theta_n\omega, \varphi\big(n, \omega, \varphi(1, \theta_{-1}\omega, x(\theta_{-1}\omega))\big)\Big) \\
&= h\big(\theta_{n+1}\theta_{-1}\omega, \varphi(n+1, \theta_{-1}\omega, x(\theta_{-1}\omega))\big) \\
&= \eta^x_{n+1}(\theta_{-1}\omega) \\
&= (\rho_1(\eta^x))_n(\omega)
\end{aligned}$$

for every $n \geqslant 0$ and every $\omega \in \Omega$. $\square$

3.2.2 Measurability Properties

Definition 3.27 (Tails)**.** Let $(\theta, \varphi, \mathcal{U})$ be an RDSI. Given a random initial state $x \in X^{\Omega}_{\mathcal{B}}$, an input $u \in \mathcal{U}$ and a starting time $\tau \geqslant 0$, we define the *tail (from moment τ) of the pullback trajectory starting at x and subject to u* to be the multivalued function

$$\begin{aligned}
\gamma^{\tau}_{x,u}\colon \Omega &\longrightarrow 2^X \backslash \{\varnothing\} \\
\omega &\longmapsto \{\varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega), u)\,;\ t \geqslant \tau\}\,.
\end{aligned}$$

We refer to $\gamma^0_{x,u}$ as the *pullback orbit starting from x and subject to u.* $\triangle$

Proposition 3.28. *If $(\theta, \varphi, \mathcal{U})$ is an RDSI evolving on a Polish space (X, d) and the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of θ is complete, then $\gamma^{\tau}_{x,u}$ is a random set for every $\tau \geqslant 0$, for any random initial state $x \in X^{\Omega}_{\mathcal{B}}$ and any input $u \in \mathcal{U}$.*

Proof. The proof is essentially the same as the proof of Proposition 1.5.1 in [8]. Fix $x \in X^{\Omega}_{\mathcal{B}}$, $u \in \mathcal{U}$, $\tau \geqslant 0$ and $y \in X$ arbitrarily. We want to show that

$$\omega \longmapsto \operatorname{dist}(y, \gamma^{\tau}_{x,u}(\omega)), \quad \omega \in \Omega,$$

is Borel-measurable. Note that

$$\operatorname{dist}(y, \gamma^{\tau}_{x,u}(\omega)) = \inf_{t \geqslant \tau} g(t, \omega), \quad \forall \omega \in \Omega,$$

where $g \colon \Omega \times \mathcal{T}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is defined by

$$g(t, \omega) := d(y, \varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega), u)), \quad (t, \omega) \in \mathcal{T}_{\geqslant 0} \times \Omega.$$

Observe that g is $(\mathcal{B}(\mathcal{T}_{\geqslant 0}) \otimes \mathcal{F})$-measurable, since it is a composition of measurable functions. So

$$g^{-1}([0, a)) \in \mathcal{F}, \quad \forall a \geqslant 0.$$

Then

$$\begin{aligned} \{\omega \in \Omega\,;\; \operatorname{dist}(y, \gamma^{\tau}_{x,u}(\omega)) < a\} &= \left\{\omega \in \Omega\,;\; \inf_{t \geqslant \tau} g(t, \omega) < a\right\} \\ &= \operatorname{proj}_{\Omega}\{(t, \omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,;\; g(t, \omega) < a \text{ and } t \geqslant \tau\} \\ &= \operatorname{proj}_{\Omega}\left(g^{-1}([0, a)) \cap [\tau, \infty) \times \Omega\right) \end{aligned}$$

is measurable by the Measurable Projection Theorem (Proposition 2.21) for any $a \geqslant 0$. This proves the result. □

We emphasize the need that X be a Polish space and $(\Omega, \mathcal{F}, \mathbb{P})$ be complete so we can apply the Measurable Projection Theorem. Of course if we did not assume $(\Omega, \mathcal{F}, \mathbb{P})$ to be complete it would still have been true that $\gamma^{\tau}_{x,u}$ is a $(\Omega, \mathcal{F}^{u})$-random set.

3.3 Input to State Characteristics

It turns out that, in a sense we shall make precise in the next subsection, an RDSI evolving subject to a θ-stationary input looks like an RDS. This section is devoted to the study of this situation. We will be particularly interested in RDSI with the property that, for each θ-stationary input, the corresponding RDS has a unique, globally attracting equilibrium.

3.3.1 θ-Stationary Inputs

The concept of RDSI subsumes that of an RDS, as we shall see below. Denote the subset of $\mathcal{S}_\theta^U$ consisting of θ-stationary inputs by $\bar{\mathcal{S}}_\theta^U$. We identify $\bar{\mathcal{S}}_\theta^U$ and $U_{\mathcal{B}}^\Omega$ via Lemma 2.8.

Let $(\theta, \varphi, \mathcal{U})$ be a RDSI, and suppose that $\bar{u} \in \mathcal{U} \cap \bar{\mathcal{S}}_\theta^U$ is some θ-stationary input. Consistent with the convention that an overbar is used to indicate the θ-stationary process associated with a given random variable, we remove the bar to denote the random variable associated with a given θ-stationary process. So, we denote by u the random variable in $U_{\mathcal{B}}^\Omega$ associated, via Lemma 2.8, with $\bar{u}$. We then define

$$\varphi_u := \varphi(\cdot, \cdot, \cdot, \bar{u}) \colon \mathcal{T}_{\geqslant 0} \times \Omega \times X \longrightarrow X\,.$$

Lemma 3.29. *φ_u is a crude cocycle.*

Proof. It follows from (I1) that φ_u is $(\mathcal{B}(\mathcal{T}_{\geqslant 0}) \otimes \mathcal{F} \otimes \mathcal{B})$-measurable. From (I2), $\varphi_u(t, \omega, \cdot)$ is continuous for each $(t, \omega) \in \mathcal{T}_{\geqslant 0} \times \Omega$, yielding (S1). From (I3), we know that $\varphi_u(0, \omega, \cdot) = id_X$ for every $\omega \in \Omega$. So, to verify (S2′) it remains to prove that φ_u satisfies the "crude cocycle property." Let $\widetilde{\Omega} \subseteq \Omega$ be a θ-invariant subset of full measure such that

$$[\rho_s(\bar{u})]_t(\omega) = \bar{u}_t(\omega)\,, \quad \forall \omega \in \widetilde{\Omega}\,, \quad \forall s, t \geqslant 0\,. \tag{3.13}$$

Fix arbitrarily $\omega \in \widetilde{\Omega}$. For any $s, t \geqslant 0$, we have $\theta_s\omega \in \widetilde{\Omega}$ by θ-invariance, and so it follows from (3.13) and (I5) that

$$\varphi(t, \theta_s\omega, \varphi_u(s, \omega, x), \rho_s(\bar{u})) = \varphi(t, \theta_s\omega, \varphi_u(s, \omega, x), \bar{u})\,.$$

It then follows from (I4)—see Remark 3.17—that

$$\begin{aligned}\varphi_u(t+s,\omega,x) &= \varphi(t+s,\omega,x,\bar{u})\\ &= \varphi(t,\theta_s\omega,\varphi(s,\omega,x,\bar{u}),\rho_s(\bar{u}))\\ &= \varphi(t,\theta_s\omega,\varphi_u(s,\omega,x),\bar{u})\\ &= \varphi_u(t,\theta_s\omega,\varphi_u(s,\omega,x))\,.\end{aligned}$$

We conclude that (S2′) is satisfied with $\Omega_s := \widetilde{\Omega}$ for every $s \geqslant 0$. □

Proposition 3.30. *φ_u can be perfected.*

Proof. Let $\widetilde{\Omega}$ be the θ-invariant subset of full measure of Ω from the proof of Lemma 3.29, and denote $N := \Omega \backslash \widetilde{\Omega}$. Thus N is θ-invariant and $\mathbb{P}(N) = 0$. Define $\psi_u \colon \mathcal{T}_{\geqslant 0} \times \Omega \times X \to X$ by

$$\psi_u(t,\omega,x) := \begin{cases} \varphi_u(t,\omega,x)\,, & \text{if } \omega \in \widetilde{\Omega}\,,\\ x\,, & \text{if } \omega \in N\,. \end{cases} \tag{3.14}$$

We will show that ψ_u is a perfection of φ_u.

To verify measurability, pick any open subset $A \subseteq X$. Then

$$\psi_u^{-1}(A) = \big(\varphi_u^{-1}(A)\backslash(\mathcal{T}_{\geqslant 0} \times N \times X)\big) \cup \mathcal{T}_{\geqslant 0} \times N \times A\,.$$

Thus $\psi_u^{-1}(A) \in \mathcal{B}(\mathcal{T}_{\geqslant 0}) \otimes \mathcal{F} \otimes \mathcal{B}$, proving that ψ_u is $(\mathcal{B}(\mathcal{T}_{\geqslant 0}) \otimes \mathcal{F} \otimes \mathcal{B})$-measurable.

If $\omega \in \widetilde{\Omega}$, then $\psi_u(t,\omega,\cdot) = \varphi_u(t,\omega,\cdot) = \varphi(t,\omega,\cdot,u)$, which is continuous for any $t \geqslant 0$ by (I2). And if $\omega \in N$, then $\psi_u(t,\omega,\cdot) = id_X$, and thus also continuous for any $t \geqslant 0$. This shows ψ_u satisfies (S1).

It is clear from (3.14) and (I3) that $\psi_u(0,\omega,\cdot) = id_X$ for any $\omega \in \Omega$. We already know from the proof of Lemma 3.29 that ψ_u satisfies the cocycle property for every $\omega \in \widetilde{\Omega}$. For $\omega \in N$, it follows from θ-invariance and (3.14) that

$$\begin{aligned}\varphi(t+s,\omega,x) &= x\\ &= \varphi(t,\theta_s\omega,x)\\ &= \varphi(t,\theta_s\omega,\varphi(s,\omega,x))\,, \quad \forall t,s \geqslant 0\,, \quad \forall x \in X\,.\end{aligned}$$

This completes the proof of (S2), thus completing the proof that (θ,ψ_u) is an RDS.

Finally, since

$$\{\omega \in \Omega\,;\ \varphi_u(t,\omega) \neq \psi_u(t,\omega)\,,\ \text{for some } t \geqslant 0\} \subseteq N$$

and $\mathbb{P}(N) = 0$, we conclude that φ_u and ψ_u are indistinguishable (see Definition 3.11). This completes the proof that ψ_u is a perfection of φ_u. □

Remark 3.31. We emphasize that the perfection of φ_u constructed in Proposition 3.30 (1) does not depend on whether $\mathcal{T}$ is discrete or continuous and (2) does not require additional topological properties for the state space X (contrast with Proposition 3.13). In particular, Proposition 3.30 would still hold in continuous time even if X was an infinite-dimensional space. As noted before, this ad hoc construction was only possible in virtue of the fact that the subset $N_s := \Omega\backslash\Omega_s$ on which the cocycle property does not hold is contained in a θ-invariant subset N of probability zero of Ω. □

Given an RDSI $(\theta, \varphi, \mathcal{U})$ and a θ-stationary input $u \in \mathcal{U}$, we shall always assume that φ_u has already been replaced by an indistinguishable perfection, and then refer to the resulting RDS (θ, φ_u).

3.3.2 Input to State Characteristics

Let $(\theta, \varphi, \mathcal{U})$ be an RDSI and suppose that $\bar{u} \in \mathcal{U}$ is a θ-stationary process, with generating random variable u (refer to Lemma 2.8). Any equilibrium ξ of the RDS (θ, φ_u) will be referred to as an *equilibrium associated to* $\bar{u}$ (or *to* u). The set of all equilibria associated to $\bar{u}$ (or to u) is denoted as $\mathcal{E}(\bar{u})$ (we may also write $\mathcal{E}(u)$). Thus, in other words, an element $\xi \in \mathcal{E}(\bar{u})$ is a random variable $\Omega \to X$ such that

$$\varphi_u(t, \theta_{-t}\omega, \xi(\theta_{-t}\omega)) = \xi(\omega)\,, \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall t \geqslant 0\,. \tag{3.15}$$

For deterministic systems—when Ω is a singleton and we may identify the set of θ-inputs $\mathcal{U}$ with the input space U—, if the set $\mathcal{E}(\bar{u})$ consists of a single, globally attracting equilibrium, then the mapping $u \mapsto \mathcal{E}(\bar{u})$, $u \in U$, is the object known as the "input to state characteristic" in the literature on input/output systems. For systems with outputs, composition with the output map h provides the "input to output characteristic" [3]. One of the contributions of this work is the extention of these concepts to RDSI and RDSIO.

In this section we introduce the notion of input to state characteristics for RDSI and discuss a class of examples. Systems with outputs will be considered in greater detail in the next chapter.

For reasons which will be illustrated in Example 3.34 and become clearer in the proof of Theorem 4.11 ("converging input to converging state"), further conditions on the convergence to a globally attracting equilibrium are needed.

Definition 3.32. (I/S Characteristic) An RDSI $(\theta, \varphi, \mathcal{U})$ is said to have an *input to state (i/s) characteristic* $\mathcal{K}\colon U_\theta^\Omega \to X_\theta^\Omega$ if

$$U_\theta^\Omega \subseteq \mathcal{U}$$

and

$$\breve{\xi}_t^{x,u} \longrightarrow_\theta \mathcal{K}(u) \quad \text{as} \quad t \to \infty\,,$$

for every $x \in X_\theta^\Omega$, for every $u \in U_\theta^\Omega$. $\triangle$

Example 3.34 below illustrates the concepts of tempered RDSI (Definition 3.19) and i/s characteristics (Definition 3.32 above). Temperedness features in said example will be a special case (with $p = 1$ or $p = \infty$) of the general result below.

Proposition 3.33. *Suppose $r\colon \Omega \to \mathbb{R}_{\geqslant 0}$ is a tempered random variable. For each $\gamma > 0$ and each $p \in [1, \infty]$, the map*

$$\omega \longmapsto \| r(\theta.\omega)\, \mathrm{e}^{-\gamma|\cdot|} \|_{L^p(\mathbb{R})}\,, \quad \omega \in \Omega\,,$$

is a tempered random variable. Moreover, temperedness bounds are uniform in $p \in [1, \infty]$; that is, for each $\gamma > 0$ and each $\delta > 0$,

$$\sup_{p\in[1,\infty]} \sup_{s\in\mathbb{R}} \| r(\theta.\theta_s\omega)\, \mathrm{e}^{-\gamma|\cdot|} \|_{L^p(\mathbb{R})}\, \mathrm{e}^{-\delta|s|} < \infty\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

Proof. For each $\gamma > 0$, set

$$K_{\gamma,\omega} := \sup_{s\in\mathbb{R}} r(\theta_s\omega)\, \mathrm{e}^{-\gamma|s|}$$

for every $\omega \in \Omega$ such that the supremum above is finite. Since r is tempered by assumption, this will hold true for θ-almost all $\omega \in \Omega$.

Fix arbitrarily $\gamma > 0$ and choose any $\delta > 0$. We consider two different cases.

(Case $1 \leqslant p < \infty$) Setting $m := \min\{\gamma, \delta\} > 0$ and using the triangle inequality, we obtain

$$\begin{aligned}
\|r(\theta.\theta_s\omega)\,\mathrm{e}^{\gamma|\cdot|}\,\|_{L^p(\mathbb{R})}\,\mathrm{e}^{-\delta|s|} &= \left(\int_{-\infty}^{\infty}[r(\theta_{t+s}\omega)\,\mathrm{e}^{-\gamma|t|-\delta|s|}]^p\,dt\right)^{1/p} \\
&\leqslant \left(\int_{-\infty}^{\infty}[r(\theta_{t+s}\omega)\,\mathrm{e}^{-m|t+s|}]^p\,dt\right)^{1/p} \\
&\leqslant \left(\int_{-\infty}^{\infty}[r(\theta_{t+s}\omega)\,\mathrm{e}^{-\frac{m}{2}|t+s|}]^p\,\mathrm{e}^{-\frac{pm}{2}|t+s|}\;dt\right)^{1/p} \\
&\leqslant K_{\frac{m}{2},\omega}\left(\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{pm}{2}|t+s|}\;dt\right)^{1/p} \\
&= K_{\frac{m}{2},\omega}\left(\tfrac{4}{pm}\right)^{1/p} \\
&< \infty
\end{aligned}$$

for all $s \in \mathbb{R}$, for θ-almost all $\omega \in \Omega$. In fact, since the map

$$p \longmapsto K_{\frac{m}{2},\omega}\left(\frac{4}{pm}\right)^{1/p}, \quad 1 \leqslant p < \infty\,, \tag{3.16}$$

is continuous in p and

$$\lim_{p\to\infty} K_{\frac{m}{2},\omega}\left(\frac{4}{pm}\right)^{1/p} = K_{\frac{m}{2},\omega}\,,$$

we then conclude that the map in (3.16) is bounded. Thus

$$M_{\gamma,\delta,\omega} := \sup_{p\in[1,\infty)}\sup_{s\in\mathbb{R}}\|r(\theta.\theta_s\omega)\,\mathrm{e}^{-\gamma|\cdot|}\,\|_{L^p(\mathbb{R})}\,\mathrm{e}^{-\delta|s|} < \infty\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

(Case $p = \infty$) The trick is basically the same as before. We have

$$\begin{aligned}
\|r(\theta.\theta_s\omega)\,\mathrm{e}^{\gamma|\cdot|}\,\|_{L^\infty(\mathbb{R})}\,\mathrm{e}^{-\delta|s|} &= \sup_{t\in\mathbb{R}} r(\theta_{t+s}\omega)\,\mathrm{e}^{-\gamma|t|-\delta|s|} \\
&\leqslant \sup_{t\in\mathbb{R}} r(\theta_{t+s}\omega)\,\mathrm{e}^{-m|t+s|} \\
&= K_{m,\omega}\,,
\end{aligned}$$

which is finite for all $s \in \mathbb{R}$, for θ-almost all $\omega \in \Omega$.

Combining both cases we conclude that

$$\sup_{p\in[1,\infty]}\sup_{s\in\mathbb{R}}\|r(\theta.\theta_s\omega)\,\mathrm{e}^{-\gamma|\cdot|}\,\|_{L^p(\mathbb{R})}\,\mathrm{e}^{-\delta|s|} = \max\{M_{\gamma,\delta,\omega}, K_{m,\omega}\}\,,$$

which is finite for θ-almost all $\omega \in \Omega$. Since $\gamma, \delta > 0$ were chosen arbitrarily, this completes the proof. $\square$

Example 3.34 (I/S Characteristics for RDSI Generated by Linear RDEI). Consider the RDSI $(\theta, \varphi, \mathcal{S}_\infty^U)$ from Example 3.18, generated by the RDEI

$$\dot{\xi} = A(\theta_t\omega)\xi + B(\theta_t\omega)u_t(\omega)\,, \quad t \geqslant 0\,, \quad u \in \mathcal{S}_\infty^U\,, \tag{3.17}$$

where $X = \mathbb{R}^n$, $U = \mathbb{R}^k$, and $A\colon \Omega \to M_{n\times n}(\mathbb{R})$ and $B\colon \Omega \to M_{n\times k}(\mathbb{R})$ are random matrices such that

$$t \longmapsto \|A(\theta_t\omega)\|\,, \quad t \geqslant 0\,,$$

is locally integrable and

$$t \longmapsto \|B(\theta_t\omega)\|\,, \quad t \geqslant 0\,,$$

is locally essentially bounded for every $\omega \in \Omega$. Now suppose in addition that A, B are such that

(L1) B is tempered, and

(L2) there exist a $\lambda > 0$ and a nonnegative, tempered random variable $\gamma \in (\mathbb{R}_\geqslant)_\theta^\Omega$ such that the fundamental matrix solution Ξ of the homogeneous part of (3.17) satisfies

$$\|\Xi(s, s+r, \omega)\| \leqslant \gamma(\theta_s\omega)\,\mathrm{e}^{-\lambda r}\,, \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall s \in \mathbb{R}\,, \quad \forall r \geqslant 0\,.$$

Then $(\theta, \varphi, \mathcal{S}_\infty^U)$ is tempered (in the sense of Definition 3.19) and has a continuous i/s characteristic $\mathcal{K}\colon U_\theta^\Omega \to X_\theta^\Omega$ (Definition 3.32). We will prove this in several steps, indicated below.

Construction of $\mathcal{K}\colon U_\theta^\Omega \to X_\theta^\Omega$. We first claim that the limit

$$\lim_{t\to\infty} \check{\xi}_t^{x,\bar{u}}(\omega) = \int_{-\infty}^0 \Xi(\sigma, 0, \omega)B(\theta_\sigma\omega)u(\theta_\sigma\omega)\,d\sigma \tag{3.18}$$

exists, for each $x \in X_\theta^\Omega$ and each $u \in U_\theta^\Omega$, for θ-almost $\omega \in \Omega$. Let Φ and Ψ be as in Example 3.18, so that we may write

$$\varphi(t, \omega, x, u) \equiv \Phi(t, \omega, x) + \Psi(t, \omega, u)\,.$$

Then

$$\check{\xi}_t^{x,\bar{u}}(\omega) \equiv \Phi(t, \theta_{-t}\omega, x(\theta_{-t}\omega)) + \Psi(t, \theta_{-t}\omega, u)\,.$$

So, it is enough to show that

$$\lim_{t\to\infty} \Phi(t, \theta_{-t}\omega, x(\theta_{-t}\omega)) = 0\,, \quad \forall x \in X_\theta^\Omega\,, \quad \widetilde{\forall}\omega \in \Omega\,, \tag{3.19}$$

and that

$$\lim_{t\to\infty} \Psi(t, \theta_{-t}\omega, \bar{u}) = \int_{-\infty}^{0} \Xi(\sigma, 0, \omega) B(\theta_\sigma\omega) u(\theta_\sigma\omega)\, d\sigma\,, \quad \forall u \in U_\theta^\Omega\,, \quad \widetilde{\forall}\omega \in \Omega\,. \tag{3.20}$$

Fix arbitrarily $x \in X_\theta^\Omega$ and let $\omega \in \Omega$ be such that

$$K_{\omega,\frac{\lambda}{2},x} := \sup_{s\in\mathbb{R}} \gamma(\theta_s\omega)|x(\theta_s\omega)|\,\mathrm{e}^{-\frac{\lambda}{2}|s|} < \infty\,, \tag{3.21}$$

where $\lambda > 0$ and γ nonnegative and tempered are given by (L2). Combining (L2) and (3.21), we obtain

$$\begin{aligned}
|\Phi(t, \theta_{-t}\omega, x(\theta_{-t}\omega))| &= |\Xi(0, t, \theta_{-t}\omega) \cdot x(\theta_{-t}\omega)| \\
&\leqslant \gamma(\theta_{-t}\omega)\,\mathrm{e}^{-\lambda t}\,|x(\theta_{-t}\omega)| \\
&= \left(\gamma(\theta_{-t}\omega)|x(\theta_{-t}\omega)|\,\mathrm{e}^{-\frac{\lambda}{2}|-t|}\right)\mathrm{e}^{-\frac{\lambda}{2}t} \\
&\leqslant K_{\omega,\frac{\lambda}{2},x}\,\mathrm{e}^{-\frac{\lambda}{2}t}\,, \quad \forall t \geqslant 0\,.
\end{aligned}$$

Hence

$$|\Phi(t, \theta_{-t}\omega, x(\theta_{-t}\omega))| \longrightarrow 0 \quad \text{as} \quad t \to \infty\,.$$

Since $K_{\omega,\frac{\lambda}{2},x}$ is finite for θ-almost all $\omega \in \Omega$—recall that, by Lemma 2.32(3), the product of two tempered random variables is tempered—, this holds θ-almost everywhere. So since $x \in X_\theta^\Omega$ was chosen arbitrarily, this proves (3.19).

Now fix arbitrarily $u \in U_\theta^\Omega$. Then by Lemma 3.3(2) and a change of variables,

$$\begin{aligned}
\Psi(t, \theta_{-t}\omega, \bar{u}) &= \int_0^t \Xi(\sigma, t, \theta_{-t}\omega) B(\theta_{\sigma-t}\omega) u(\theta_{\sigma-t}\omega)\, d\sigma \\
&= \int_0^t \Xi(\sigma - t, 0, \omega) B(\theta_{\sigma-t}\omega) u(\theta_{\sigma-t}\omega)\, d\sigma \\
&= \int_{-t}^0 \Xi(\sigma, 0, \omega) B(\theta_\sigma\omega) u(\theta_\sigma\omega)\, d\sigma\,, \quad \forall t \geqslant 0\,, \quad \forall\omega \in \Omega\,.
\end{aligned}$$

In virtue of (L2), for each $\omega \in \Omega$ such that

$$L_{\omega,\frac{\lambda}{2},u} := \sup_{s\in\mathbb{R}} \gamma(\theta_s\omega)\|B(\theta_s\omega)\| \cdot |u(\theta_s\omega)| \, \mathrm{e}^{-\frac{\lambda}{2}|s|} < \infty \,, \tag{3.22}$$

we have

$$\begin{aligned} |\Xi(\sigma,0,\omega)B(\theta_\sigma\omega)u(\theta_\sigma\omega)| &\leqslant \gamma(\theta_\sigma\omega)\,\mathrm{e}^{-\lambda|\sigma|}\,\|B(\theta_\sigma\omega)\| \cdot |u(\theta_\sigma\omega)| \\ &\leqslant L_{\omega,\frac{\lambda}{2},u}\,\mathrm{e}^{-\frac{\lambda}{2}|\sigma|}\,, \quad \forall \sigma \leqslant 0\,. \end{aligned}$$

Thus, since

$$\sigma \longmapsto L_{\omega,\frac{\lambda}{2},u}\,\mathrm{e}^{-\frac{\lambda}{2}|\sigma|}\,, \quad \sigma \leqslant 0\,,$$

is integrable on $(-\infty,0]$, so is

$$\sigma \longmapsto \Xi(\sigma,0,\omega)B(\theta_\sigma\omega)u(\theta_\sigma\omega)\,, \quad \sigma \leqslant 0\,.$$

In particular, it follows from dominated convergence that the limit

$$\begin{aligned} \lim_{t\to\infty} \Psi(t,\theta_{-t}\omega,\bar{u}) &= \lim_{t\to\infty} \int_{-t}^{0} \Xi(\sigma,0,\omega)B(\theta_\sigma\omega)u(\theta_\sigma\omega)\,d\sigma \\ &= \int_{-\infty}^{0} \Xi(\sigma,0,\omega)B(\theta_\sigma\omega)u(\theta_\sigma\omega)\,d\sigma \end{aligned}$$

exists. Finally, observe that, for each $u \in U_\theta^\Omega$, $L_{\omega,\frac{\lambda}{2},u}$ as defined in (3.22) is finite for θ-almost all $\omega \in \Omega$. This establishes (3.20).

We have then proved that (3.18) holds, for each $x \in X_\theta^\Omega$ and each $u \in U_\theta^\Omega$, for θ-almost all $\omega \in \Omega$.

Define $\mathcal{K}\colon U_\theta^\Omega \to X_\mathcal{B}^\Omega$ by

$$[\mathcal{K}(u)](\omega) := \int_{-\infty}^{0} \Xi(\sigma,0,\omega)B(\theta_\sigma\omega)u(\theta_\sigma\omega)\,d\sigma\,, \quad \omega \in \Omega\,.$$

It remains to show that $\mathcal{K}(U_\theta^\Omega) \subseteq X_\theta^\Omega$.

Indeed, fix $u \in U_\theta^\Omega$ arbitrarily. It follows from the estimate

$$|\Xi(\sigma,0,\omega)B(\theta_\sigma\omega)u(\theta_\sigma\omega)| \leqslant \gamma(\theta_\sigma\omega)\|B(\theta_\sigma\omega)\| \cdot |u(\theta_\sigma\omega)|\,\mathrm{e}^{-\lambda|\sigma|}\,, \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall \sigma \leqslant 0\,, \tag{3.23}$$

shown above, that

$$\begin{aligned} |[\mathcal{K}(u)](\omega)| &\leqslant \int_{-\infty}^{0} \gamma(\theta_\sigma\omega)\|B(\theta_\sigma\omega)\| \cdot |u(\theta_\sigma\omega)|\,\mathrm{e}^{-\lambda|\sigma|}\;d\sigma \\ &\leqslant \int_{-\infty}^{\infty} \gamma(\theta_\sigma\omega)\|B(\theta_\sigma\omega)\| \cdot |u(\theta_\sigma\omega)|\,\mathrm{e}^{-\lambda|\sigma|}\;d\sigma \\ &= \|(\gamma\|B\| \cdot |u|)(\theta.\omega)\,\mathrm{e}^{-\lambda|\cdot|}\,\|_{L^1(\mathbb{R})}\,, \quad \widetilde{\forall}\omega \in \Omega\,. \end{aligned}$$

From Proposition 3.33,

$$\omega \longmapsto \|(\gamma\|B\| \cdot |u|)(\theta.\omega)\,\mathrm{e}^{-\lambda|\cdot|}\,\|_{L^1(\mathbb{R})}\,, \quad \omega \in \Omega\,,$$

is tempered. Thus $\mathcal{K}(u)\colon \Omega \to X$ is also tempered.

$\mathcal{K}$ **is an i/s characteristic.** To show that $\mathcal{K}$ is an i/s characteristic, it remains to show that the convergence in both (3.19) and (3.20) is tempered.

Fix $x \in X_\theta^\Omega$ arbitrarily. From the estimates above, we have

$$\begin{aligned} |\Phi(t, \theta_{-t}\omega, x(\theta_{-t}\omega))| &\leqslant \gamma(\theta_{-t}\omega)|x(\theta_{-t}\omega)|\,\mathrm{e}^{-\lambda t} \\ &\leqslant \sup_{s\in\mathbb{R}} \gamma(\theta_s\omega)|x(\theta_s\omega)|\,\mathrm{e}^{-\lambda|s|} \\ &= \|(\gamma|x|)(\theta.\omega)\,\mathrm{e}^{-\lambda|\cdot|}\,\|_{L^\infty(\mathbb{R})}\,, \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall t \geqslant 0\,. \end{aligned}$$

It follows from Proposition 3.33 (applied with $p = \infty$) that

$$\omega \longmapsto \|(\gamma|x|)(\theta.\omega)\,\mathrm{e}^{-\lambda|\cdot|}\,\|_{L^\infty(\mathbb{R})}\,, \quad \omega \in \Omega\,,$$

is tempered. We conclude that convergence in (3.19) is tempered.

Similarly, for any arbitrarily fixed $u \in U_\theta^\Omega$, we have

$$\begin{aligned} |\Psi(t, \theta_{-t}\omega, \bar{u}) - [\mathcal{K}(u)](\omega)| &= \left| \int_{-\infty}^{-t} \Xi(\sigma, 0, \omega)B(\theta_\sigma\omega) \cdot u(\theta_\sigma\omega)\,d\sigma \right| \\ &\leqslant \int_{-\infty}^{\infty} \gamma(\theta_\sigma\omega)\|B(\theta_\sigma\omega)\| \cdot |u(\theta_\sigma\omega)|\,\mathrm{e}^{-\lambda|\sigma|}\,d\sigma \\ &= \|(\gamma\|B\| \cdot |u|)(\theta.\omega)\,\mathrm{e}^{-\lambda|\cdot|}\,\|_{L^1(\mathbb{R})} \end{aligned}$$

for all $t \geqslant 0$, for θ-almost all $\omega \in \Omega$. As we saw above, the rightmost term in these inequalities is a tempered random variable. So the convergence in (3.20) is also tempered.

$\mathcal{K}$ **is continuous.** Suppose that $u_\alpha \to_\theta u_\infty \in U_\theta^\Omega$ for some net $(u_\alpha)_{\alpha\in A}$ in U_θ^Ω. Let $\alpha_0 \in A$ and $r \in (\mathbb{R}_{\geqslant 0})_\theta^\Omega$ be such that

$$|u_\alpha(\omega) - u_\infty(\omega)| \leqslant r(\omega)\,, \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall \alpha \geqslant \alpha_0\,. \tag{3.24}$$

Then it follows from (3.24) and (3.23)—with '$|u|$' replaced by 'r,' that

$$\begin{aligned} |[\mathcal{K}(u_\alpha)](\omega) - [\mathcal{K}(u_\infty)](\omega)| &= \left| \int_{-\infty}^{0} \Xi(\sigma, 0, \omega) B(\theta_\sigma \omega) \cdot (u_\alpha(\theta_\sigma \omega) - u_\infty(\theta_\sigma \omega))\, d\sigma \right| \\ &\leqslant \int_{-\infty}^{\infty} \gamma(\theta_\sigma \omega) \|B(\theta_\sigma \omega)\| r(\theta_\sigma \omega)\, \mathrm{e}^{-\lambda|\sigma|}\, d\sigma \\ &= \|(\gamma \|B\| r)(\theta.\omega)\, \mathrm{e}^{-\lambda|\cdot|}\, \|_{L^1(\mathbb{R})} \end{aligned}$$

for every $\alpha \geqslant \alpha_0$, for θ-almost all $\omega \in \Omega$. As above,

$$\omega \longmapsto \|(\gamma \|B\| r)(\theta.\omega)\, \mathrm{e}^{-\lambda|\cdot|}\, \|_{L^1(\mathbb{R})}\,, \quad \omega \in \Omega\,,$$

is tempered. Since

$$|u_\alpha(\omega) - u_\infty(\omega)| \longrightarrow 0 \quad \text{as} \quad \alpha \to \infty\,, \quad \widetilde{\forall} \omega \in \Omega\,,$$

it follows from θ-invariance that

$$|u_\alpha(\theta_\sigma \omega) - u_\infty(\theta_\sigma \omega)| \longrightarrow 0 \quad \text{as} \quad \alpha \to \infty\,, \quad \widetilde{\forall} \omega \in \Omega\,, \quad \forall \sigma \leqslant 0\,.$$

It then follows from dominated convergence, as in the proof of (3.20), that

$$|(\mathcal{K}(u_\alpha))(\omega) - (\mathcal{K}(u_\infty))(\omega)| \longrightarrow 0 \quad \text{as} \quad \alpha \to \infty\,, \quad \widetilde{\forall} \omega \in \Omega\,,$$

as well. This shows that $\mathcal{K}(u_\alpha) \to_\theta \mathcal{K}(u_\infty)$. Since $u_\infty \in U_\theta^\Omega$ and the net $(u_\alpha)_{\alpha \in A}$ converging to u_∞ were arbitrary, this proves $\mathcal{K}$ is (tempered) continuous.

φ **is tempered.** The argument here goes along the same lines of what we have been doing throughout this example. Fix arbitrarily any tempered input $u \in \mathcal{S}_\infty^U$ and any tempered initial state $x \in X_\theta^\Omega$. When we were showing that $\mathcal{K}$ is an i/s characteristic, we saw that

$$|\Phi(t, \theta_{-t}\omega, x(\theta_{-t}\omega))| \leqslant r_1(\omega)\,, \quad \widetilde{\forall} \omega \in \Omega\,, \quad \forall t \geqslant 0\,,$$

where $r_1 \colon \Omega \to \mathbb{R}_{\geqslant 0}$ is a tempered random variable defined by

$$r_1(\omega) := \|(\gamma |x|)(\theta.\omega)\, \mathrm{e}^{-\lambda|\cdot|}\, \|_{L^\infty(\mathbb{R})}\,, \quad \omega \in \Omega\,.$$

Now let $D \in (2^U)_\theta^\Omega$ be a (tempered) rest set for u, and let $r \in (\mathbb{R}_{\geqslant 0})_\theta^\Omega$ be such that

$$D(\omega) \subseteq \{u \in U\,;\ \|u\| \leqslant r(\omega)\}\,, \quad \widetilde{\forall} \omega \in \Omega\,.$$

Thus, indeed,

$$\|u_t(\theta_{-t}\omega)\| \leqslant r(\omega)\,, \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall t \geqslant 0\,.$$

Therefore

$$\begin{aligned} |\Psi(t,\theta_{-t}\omega,u)| &= \left|\int_0^t \Xi(\sigma-t,0,\omega)B(\theta_{\sigma-t}\omega)u_\sigma(\theta_{-\sigma}\theta_{\sigma-t}\omega)\,d\sigma\right| \\ &\leqslant \int_0^t \|\Xi(\sigma-t,0,\omega)B(\theta_{\sigma-t}\omega)\|\cdot r(\theta_{\sigma-t}\omega)\,d\sigma \\ &= \int_{-t}^0 \|\Xi(\sigma,0,\omega)B(\theta_\sigma\omega)\|r(\theta_\sigma\omega)\,d\sigma \\ &\leqslant \int_{-\infty}^{\infty} \gamma(\theta_\sigma\omega)\|B(\theta_\sigma\omega)\|r(\theta_\sigma\omega)\,\mathrm{e}^{-\lambda|\sigma|}\,d\sigma \\ &= \|(\gamma\|B\|r)(\theta.\omega)\,\mathrm{e}^{-\lambda|\cdot|}\,\|_{L^1(\mathbb{R})}\,, \quad \widetilde{\forall}\omega\in\Omega\,, \quad \forall t \geqslant 0\,. \end{aligned}$$

Denote

$$r_2(\omega) := \|(\gamma\|B\|r)(\theta.\omega)\,\mathrm{e}^{-\lambda|\cdot|}\,\|_{L^1(\mathbb{R})}\,, \quad \omega \in \Omega\,.$$

The map $r_2\colon \Omega \to \mathbb{R}_{\geqslant 0}$ so-defined is tempered. Now r_1+r_2 is tempered, and furthermore

$$|\check{\xi}_t^{x,u}(\omega)| = |\varphi(t,\theta_{-t},x(\theta_{-t}\omega),u)| \leqslant r_1(\omega)+r_2(\omega)\,, \quad \widetilde{\forall}\omega\in\Omega\,, \quad \forall t\geqslant 0\,.$$

In other terms, $\xi^{x,u}$ is tempered. Since the tempered θ-input u and the tempered initial state x were chosen arbitrarily, this completes the proof that φ is indeed a tempered cocycle. $\Diamond$

Remark 3.35. If $\|A(\cdot)\| \in L^1(\Omega,\mathcal{F},\mathbb{P})$, the largest eigenvalue $\overline{\lambda}(\cdot)$ of the Hermitian part of $A(\cdot)$ is such that

$$\mathbb{E}\overline{\lambda} := \int_\Omega \overline{\lambda}(\omega)\,d\mathbb{P}(\omega) < 0\,,$$

and the underlying MPDS θ is ergodic, then it follows from [8, Theorem 2.1.2, page 60] that (L2) holds with $\lambda := -(\mathbb{E}\overline{\lambda}+\epsilon)$ for any choice of $\epsilon \in (0,-\mathbb{E}\overline{\lambda})$. $\square$

Remark 3.36. We showed in the example above that

$$|\Phi(t,\theta_{-t}\omega,x(\theta_{-t}\omega))| \longrightarrow 0 \quad \text{as} \quad t\to\infty\,, \quad \widetilde{\forall}\omega\in\Omega\,,$$

for every tempered initial state $x \in (\mathbb{R}^n)^\Omega_\theta$. To further illustrate the role of temperedness in the above convergence, we consider the one-dimensional scenario below.

Suppose that $a, b\colon \Omega \to \mathbb{R}$ are random variables such that $t \mapsto b(\theta_t\omega)$, $t \in \mathbb{R}$, is absolutely continuous, and

$$a(\theta_t\omega) = \frac{d}{dt}\left[b(\theta_t\omega)\right]$$

for Lebesgue-almost all $t \in \mathbb{R}$, for every $\omega \in \Omega$. In this case, the RDS (θ, Φ) generated by the linear RDE

$$\dot{\xi} = a(\theta_t\omega)\xi\,, \quad t \geqslant 0\,,$$

is given by

$$\Phi(t, \omega, x) = x\,\mathrm{e}^{\int_0^t a(\theta_\tau\omega)\,d\tau} = x\,\mathrm{e}^{b(\theta_t\omega)-b(\omega)}\,, \quad (t, \omega, x) \in \mathbb{R}_{\geqslant 0} \times \Omega \times \mathbb{R}\,.$$

Now for each $c \in \mathbb{R}$, the random variable $x_c\colon \Omega \to \mathbb{R}$ defined by

$$x_c(\omega) := c\,\mathrm{e}^{b(\omega)}\,, \quad \omega \in \Omega\,,$$

is an equilibrium of (θ, Φ). Indeed,

$$\begin{aligned}
\Phi(t, \theta_{-t}\omega, x_c(\theta_{-t}\omega)) &= x_c(\theta_{-t}\omega)\,\mathrm{e}^{b(\theta_t\theta_{-t}\omega)-b(\theta_{-t}\omega)} \\
&= c\,\mathrm{e}^{b(\theta_{-t}\omega)}\,\mathrm{e}^{b(\omega)-b(\theta_{-t}\omega)} \\
&= c\,\mathrm{e}^{b(\omega)} \\
&= x_c(\omega)\,, \quad \forall \omega \in \Omega\,.
\end{aligned}$$

In particular, (θ, Φ) has multiple equilibria. In virtue of the example above, if a belongs to $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{E}[a] < 0$, then $x_c \equiv 0$ is the only such equilibria which is also tempered. □

3.4 More Examples

In this last section, we present a few more examples of RDSI.

In the first subsection, we discuss the discrete-time analogues of Examples 3.2, 3.18 and 3.34. This is followed by some explicit examples and a few numerical simulations.

In Subsection 3.4.2 we give sufficient conditions for an RDEI to generate an RDSI—a continuous-time analogue of Theorem 3.24.

3.4.1 Discrete Time

We start by outlining the discrete-time analogue of the program developed for RDEI in Examples 3.2, 3.18 and 3.34.

Example 3.37 (I/S Characteristics for RDSI Generated by Linear RdEI). Fix any discrete MPDS θ (that is, $\mathcal{T} = \mathbb{Z}$). Let $X := \mathbb{R}^n$, $U := \mathbb{R}^k$, $\mathcal{U} := \mathcal{S}_\theta^U$, and suppose that $A\colon \Omega \to M_{n\times n}(\mathbb{R})$ and $B\colon \Omega \to M_{n\times k}(\mathbb{R})$ are Borel-measurable. Applying Theorem 3.24 with

$$\begin{array}{rcl} f\colon \Omega \times X \times U & \longrightarrow & X \\ (\omega, x, u) & \longmapsto & A(\omega)x + B(\omega)u \end{array},$$

we conclude that the random difference equation with inputs

$$\xi^+ = A(\theta_n\omega)\xi + B(\theta_n\omega)u_n(\omega)\,, \quad n \geqslant 0\,, \quad \omega \in \Omega\,, \quad u \in \mathcal{S}_\theta^U\,, \tag{3.25}$$

generates an RDSI $(\theta, \varphi, \mathcal{U})$. Indeed, we can show by induction that

$$\varphi(n, \omega, x, u) \equiv \left(\prod_{j=0}^{n-1} A(\theta_j\omega)\right) x + \sum_{j=0}^{n-1}\left(\prod_{k=j+1}^{n-1} A(\theta_k\omega)\right) B(\theta_j\omega)u_j(\omega)\,, \tag{3.26}$$

and that the pullback trajectories are given by

$$\begin{aligned} \varphi(n, \theta_{-n}\omega, x(\theta_{-n}\omega), u) \;\equiv\;& \left(\prod_{j=-n}^{-1} A(\theta_j\omega)\right) x(\theta_{-n}\omega) \\ &+ \sum_{j=-n}^{-1}\left(\prod_{k=j+1}^{-1} A(\theta_k\omega)\right) B(\theta_j\omega)u_{j+n}(\theta_{-n}\omega)\,. \end{aligned}$$

We denote

$$\Xi(s, s+r, \omega) := \prod_{j=s}^{s+r-1} A(\theta_j\omega)\,, \quad s \in \mathbb{Z}\,, \quad r \geqslant 0\,, \quad \omega \in \Omega\,,$$

following the convention that, when $r = 0$, the "empty product" from s to $s-1$ evaluates to

$$\Xi(s, s, \omega) = \prod_{j=s}^{s-1} A(\theta_j\omega) := I_n\,, \quad s \in \mathbb{Z}\,.$$

Note that we are using the same notation we used for the fundamental solution of linear RDE (Example 3.2) for the "fundamental solution" of the linear part of (3.25). But since we shall not consider "mixed-time systems" in this work, there is not risk of confusion. It should be always clear from the context what we mean by 'Ξ.' Using this notation, we may rewrite the forward and pullback flow as

$$\varphi(n,\omega,x,u) \equiv \Xi(0,n,\omega)x + \sum_{j=0}^{n-1} \Xi(j+1,n,\omega)B(\theta_j\omega)u_j(\omega)$$

and

$$\varphi(n,\theta_{-n}\omega,x(\theta_{-n}\omega),u) \equiv \Xi(-n,0,\omega)x(\theta_{-n}\omega) + \sum_{j=-n}^{-1} \Xi(j+1,0,\omega)B(\theta_j\omega)u_{j+n}(\theta_{-n}\omega)$$

Now suppose that A and B have, in addition, properties

($l1$) B is tempered, and

($l2$) there exist a $\lambda \in (0,1)$ and a nonnegative, tempered random variable $\gamma\colon \Omega \to \mathbb{R}_{\geqslant 0}$ such that

$$\left\| \prod_{j=s}^{s+r-1} A(\theta_j\omega) \right\| \leqslant \gamma(\theta_s\omega)\lambda^r\,, \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall s \in \mathbb{Z}\,, \quad \forall r \geqslant 0\,.$$

Then the RDSI $(\theta,\varphi,\mathcal{U})$ constructed above is tempered, and has a continuous i/s characteristic $\mathcal{K}\colon U_\theta^\Omega \to X_\theta^\Omega$,

$$[\mathcal{K}(u)](\omega) = \sum_{j=-\infty}^{-1} \Xi(j+1,0,\omega)B(\theta_j\omega)u(\theta_j\omega)\,, \quad \forall u \in U_\theta^\Omega\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

This follows from ($l1$) and ($l2$), along the same lines as in Example 3.34, and so we omit the details. ♢

We now give a few explicit examples and numerical simulations fitting within this setting.

Example 3.38 (Cantor Set). Let θ be the Bernoulli shift of the probability space

$$(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)\,,$$

where $\Omega_0 := \{0,1\}$, $\mathcal{F}_0 := 2^{\Omega_0}$, and $\mathbb{P}_0\colon \mathcal{F}_0 \to [0,1]$ is defined by

$$\mathbb{P}_0(\{0\}) := \frac{1}{2}\,, \quad \mathbb{P}_0(\{1\}) := \frac{1}{2}$$

(refer to Example 2.2). Take $X := \mathbb{R}_{\geqslant 0}$, $U := [0,1]$, and consider the discrete RDSI $(\theta, \varphi, \mathcal{S}_\theta^U)$ generated by the RdEI

$$\xi^+ = \frac{1}{3}\xi + \frac{2}{3}u_n(\omega)\,, \quad n \geqslant 0\,, \quad \omega \in \Omega\,, \quad u \in \mathcal{S}_\theta^U\,.$$

Note that the coefficients satisfy ($l1$) and ($l2$), so it follows from Example 3.37 that φ has an i/s characteristic $\mathcal{K}\colon U_\theta^\Omega \to X_\theta^\Omega$ given by

$$[\mathcal{K}(u)](\omega) = \sum_{j=-\infty}^{-1} \frac{2u(\theta_j\omega)}{3^{-j}} = \sum_{j=1}^{\infty} \frac{2u(\theta_{-j}\omega)}{3^j}\,, \quad \omega \in \Omega\,, \quad u \in U_\theta^\Omega\,.$$

Now consider the input $v \in \mathcal{S}_\theta^U$ defined by

$$v(\omega) = v((\omega_k)_{k\in\mathbb{Z}}) := \omega_0\,, \quad \omega \in \Omega\,.$$

We have

$$[\mathcal{K}(v)](\omega) = \sum_{j=1}^{\infty} \frac{2\omega_{-j}}{3^j}\,, \quad \forall \omega \in \Omega\,,$$

and so $\mathcal{K}(v)$ is "uniformly distributed over the Cantor set" in the following sense: *the probability that $\mathcal{K}(v)$ belongs to an interval which was not removed in the n^{th} step of the construction of the Cantor set is 2^{-n} for each $n \geqslant 0$.* Indeed, for any nonnegative integer n, any interval which was not removed in the n^{th} step of the construction of the Cantor set has the form

$$I_{a_1,\ldots,a_n} = \left[\sum_{j=1}^{n} \frac{a_j}{3^j}\,,\ \sum_{j=1}^{n} \frac{a_j}{3^j} + \frac{1}{3^n}\right] = \left[\sum_{j=1}^{n} \frac{a_j}{3^j}\,,\ \sum_{j=1}^{n} \frac{a_j}{3^j} + \sum_{j=n+1}^{\infty} \frac{2}{3^j}\right]$$

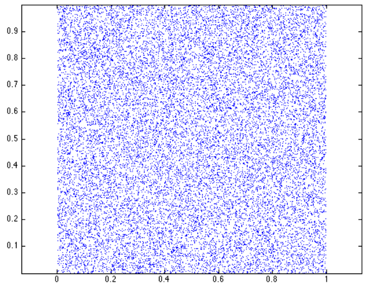
for some $a_1, \ldots, a_n \in \{0, 2\}$. Therefore $[\mathcal{K}(v)](\omega)$ belongs to $I_{a_1,\ldots,a_n}$ if, and only if $2\omega_{-j} = a_j$ for $j = 1, \ldots, n$. Now by construction

$$\mathbb{P}(\{\omega \in \Omega\,;\ \omega_{-j} = a_j/2\,,\ j = 1, \ldots, n\}) = \left(\frac{1}{2}\right)^n$$

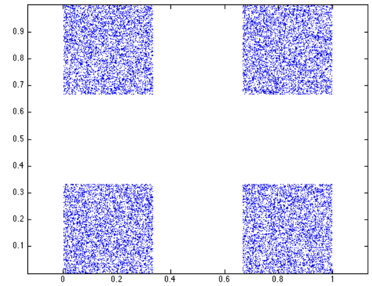
(refer once again to Example 2.2). $\Diamond$

The construction in the example above can be extended to arbitrary finite dimensions, thus yielding *Cantor dusts*. Figure 3.1 illustrates the pullback convergence to a 2-dimensional Cantor dust of a random variable which is uniformly distributed on the unit square. Figure 3.2[5] illustrates the 3-dimensional Cantor dust.

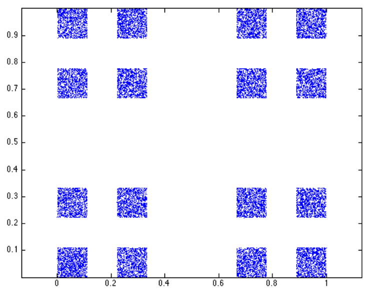
[5]Negative of `http://commons.wikimedia.org/wiki/File%3ACantors_cube.jpg`.



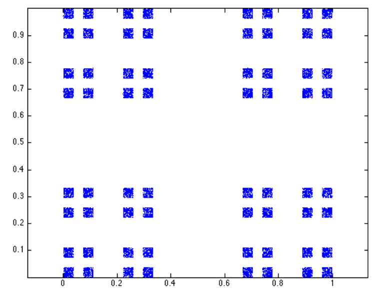
(a) $n = 0$



(b) $n = 1$



(c) $n = 2$



(d) $n = 3$

Figure 3.1: Pullback convergence of uniform distribution over the unit square to "uniform distribution" over the 2-dimensional Cantor dust.

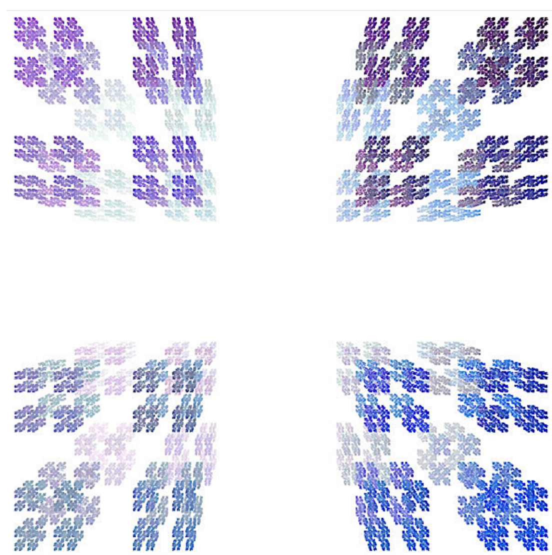


Figure 3.2: 3-dimensional Cantor dust.

Example 3.39 (Barnsley Fern). *Iterated function systems* (*IFS*), in the sense of [6, Definition 1 on page 82], can be interpreted as RDS or RDSI. We use the classical example of the *Barnsley fern* [6, Table 3.8.3 on page 87, and Figure 3.8.3 on page 92],

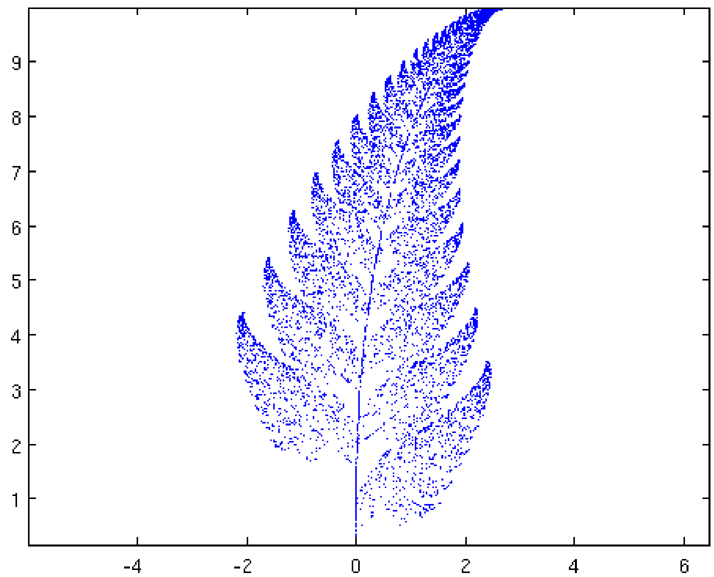


Figure 3.3: Barnsley Fern

also illustrated here in Figure 3.3 below, to show how the *Random Iteration Algorithm* [6, Program 3.8.2 on page 91] can be described as an RDSI. (See also [5].)

Let θ be the Bernoulli shift of the probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$, where $\Omega_0 := \{1,2,3,4\}$, $\mathcal{F}_0 := 2^{\Omega_0}$ and $\mathbb{P}_0 \colon \mathcal{F}_0 \to [0,1]$ is defined by

$$\mathbb{P}_0(\{1\}) := 0.01\,, \quad \mathbb{P}_0(\{2\}) := 0.85\,, \quad \mathbb{P}_0(\{3\}) := 0.07\,, \quad \mathbb{P}_0(\{4\}) := 0.07\,.$$

Let $X := \mathbb{R}^2$ and $U := [0,1] \times [0,1]$, and consider the discrete RDSI $(\theta, \varphi, \mathcal{S}_\theta^U)$ generated by the RdEI

$$\xi^+ = A(\theta_n\omega)\xi + u_n(\omega)\,, \quad n \geqslant 0\,, \quad u \in \mathcal{S}_\theta^U\,,$$

where $A \colon \Omega \to M_{2\times 2}(\mathbb{R})$ is defined as follows. First define $A_0 \colon \Omega_0 \to M_{2\times 2}(\mathbb{R})$ by

$$A_0(1) := \begin{bmatrix} 0 & 0 \\ 0 & 0.16 \end{bmatrix}\,,$$

$$A_0(2) := \begin{bmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{bmatrix}\,,$$

$$A_0(3) := \begin{bmatrix} 0.2 & -0.26 \\ 0.23 & 0.22 \end{bmatrix}$$

and

$$A_0(4) := \begin{bmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{bmatrix} .$$

Then define A by setting

$$A(\omega) := A_0(\omega_0)\,, \quad \omega \in \Omega\,.$$

The largest singular values of $A_0(1)$, $A_0(2)$, $A_0(3)$ and $A_0(4)$ can be numerically estimated to be, respectively,

$$\sigma_0^{max}(1) = 0.16\,,$$

$$\sigma_0^{max}(2) \approx 0.8509\,,$$

$$\sigma_0^{max}(3) \approx 0.3407\,,$$

and

$$\sigma_0^{max}(4) \approx 0.3792\,.$$

Now $A(\omega) = A_0(\omega_0)$ and so the largest singular value of $A(\omega)$ is

$$\sigma^{max}(\omega) = \sigma_0^{\max}(\omega_0)$$

for each $\omega \in \Omega$. Thus

$$\left\| \prod_{j=s}^{s+r-1} A(\theta_j \omega) \right\| \leqslant \gamma \lambda^r\,, \quad \forall \omega \in \Omega\,, \quad \forall s \in \mathbb{Z}\,, \quad \forall r \geqslant 0\,,$$

where

$$\lambda := \sigma_0^{max}(2) \in (0,1)\,,$$

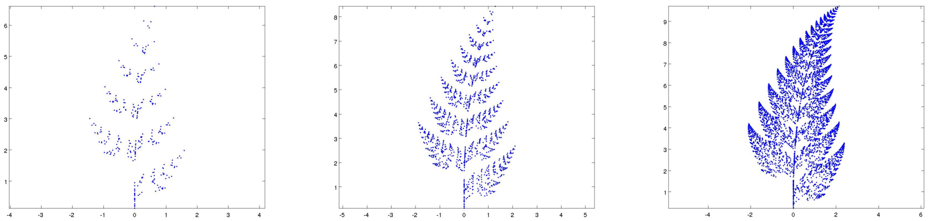
and γ is a nonnegative constant depending only on the matrix norm[6] $\|\cdot\|$. Thus A satisfies ($l2$). So, it follows as in Example 3.37, with

$$B \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the observation that all θ-inputs in $\mathcal{S}_\theta^U$ are uniformly bounded, that φ has a continuous i/s characteristic $\mathcal{K}\colon U_\theta^\Omega \to X_\theta^\Omega$ given by

$$[\mathcal{K}(u)](\omega) = \sum_{j=-\infty}^{-1} \left(\prod_{k=j+1}^{-1} A(\theta_k \omega) \right) u(\theta_j \omega)\,, \quad \forall u \in U_\theta^\Omega\,, \quad \widetilde{\forall} \omega \in \Omega\,.$$

[6]For instance, if $\|\cdot\|$ is the operator norm induced by the Euclidean norm in $\mathbb{R}^n$, then $\gamma = 1$.



(a) $t = 6$ (b) $t = 10$ (c) $t = 18$

Figure 3.4: Simulation of tempered, pullback convergence of φ to $\mathcal{K}(u)$, starting at $x = 0$.

Now consider the θ-stationary input $u \in U_\theta^\Omega$ defined as follows. First define

$$u_0(1) := \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$u_0(2) := \begin{bmatrix} 0 \\ 1.6 \end{bmatrix},$$

$$u_0(3) := \begin{bmatrix} 0 \\ 1.6 \end{bmatrix},$$

and

$$u_0(4) := \begin{bmatrix} 0 \\ 0.44 \end{bmatrix}.$$

Then set

$$u(\omega) := u_0(\omega_0)\,, \quad \omega \in \Omega\,.$$

The support of the image of $\mathcal{K}(u)$ is the Barnsley fern. Figure 3.4 shows the results after steps $n = 6$, $n = 10$ and $n = 18$ of a simulation of the pullback trajectories of φ starting at $x = 0$, and subject to the input u defined above. $\Diamond$

Example 3.40 (Barnsley Fern to Maple Leaf)*.* Let θ be the same MPDS as in Example 3.39, except for having instead

$$\mathbb{P}_0(\{1\}) := 0.10\,, \quad \mathbb{P}_0(\{2\}) := 0.35\,, \quad \mathbb{P}_0(\{3\}) := 0.35\,, \quad \mathbb{P}_0(\{4\}) := 0.20\,.$$

Set

$$C_0(1) := \begin{bmatrix} 0.14 & 0.01 \\ 0 & 0.51 \end{bmatrix},$$

$$C_0(2) := \begin{bmatrix} 0.43 & 0.52 \\ -0.45 & 0.5 \end{bmatrix},$$

$$C_0(3) := \begin{bmatrix} 0.45 & -0.49 \\ 0.47 & -1.62 \end{bmatrix}$$

and

$$C_0(4) := \begin{bmatrix} 0.49 & 0 \\ 0 & 0.51 \end{bmatrix},$$

then define $C \colon \Omega \to M_{2\times 2}(\mathbb{R})$ by

$$C(\omega) := C_0(\omega_0)\,, \quad \omega \in \Omega\,,$$

and consider the (discrete) RDSI $(\theta, \varphi, \mathcal{S}_\theta^U)$ generated by the RdEI

$$\xi^+ = C(\theta_n\omega)\xi + u_n(\omega)\,, \quad n \geqslant 0\,, \quad u \in \mathcal{S}_\theta^U\,.$$

As in Example 3.39, the largest singular values of $C_0(1)$, $C_0(2)$, $C_0(3)$ and $C_0(4)$ can be estimated to be, respectively,

$$\sigma_0^{max}(1) \approx 0.5101\,,$$

$$\sigma_0^{max}(2) \approx 0.7214\,,$$

$$\sigma_0^{max}(3) \approx 0.9461$$

and

$$\sigma_0^{max}(4) = 0.51\,.$$

So, it can be shown as in the previous example that φ has a continuous i/s characteristic.

The state characteristic corresponding to the θ-stationary input $u \in U_\theta^\Omega$ defined by

$$u(\omega) := u_0(\omega_0)\,, \quad \omega \in \Omega\,,$$

where

$$u_0(1) := \begin{bmatrix} -0.08 \\ -1.31 \end{bmatrix},$$

$$u_0(2) := \begin{bmatrix} 1.49 \\ -0.75 \end{bmatrix},$$

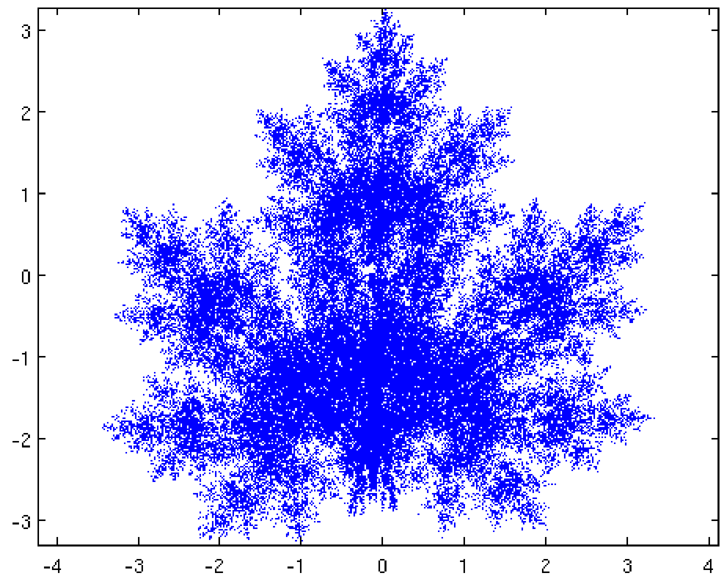


Figure 3.5: Maple Leaf

$$u_0(3) := \begin{bmatrix} -1.62 \\ -0.74 \end{bmatrix} ,$$

and

$$u_0(4) := \begin{bmatrix} 0.02 \\ 1.62 \end{bmatrix} ,$$

is a distribution over the "maple leaf," as illustrated by the numerical simulation in Figure 3.5.

Now let $x_\infty \colon \Omega \to \mathbb{R}^2$ be the distribution over the Barnsley fern obtained as the (pullback) limit of the RDSI in Example 3.39—in other words, the state characteristic corresponding to the input specified in the example. Since x_∞ is tempered (in fact, it is bounded), the pullback trajectory of φ starting at x_∞ and subject to u also converges to the distribution over the maple leaf. Figure 3.6 illustrates the transition. $\Diamond$

3.4.2 Random Differential Equations with Inputs

In this section we give sufficient conditions for an RDEI to generate an RDSI. To a large extent, this amounts to solving each ODE in a family parametrized by ω, as it was the case in [4, 8] for RDS generated by RDE. The greatest technical challenge

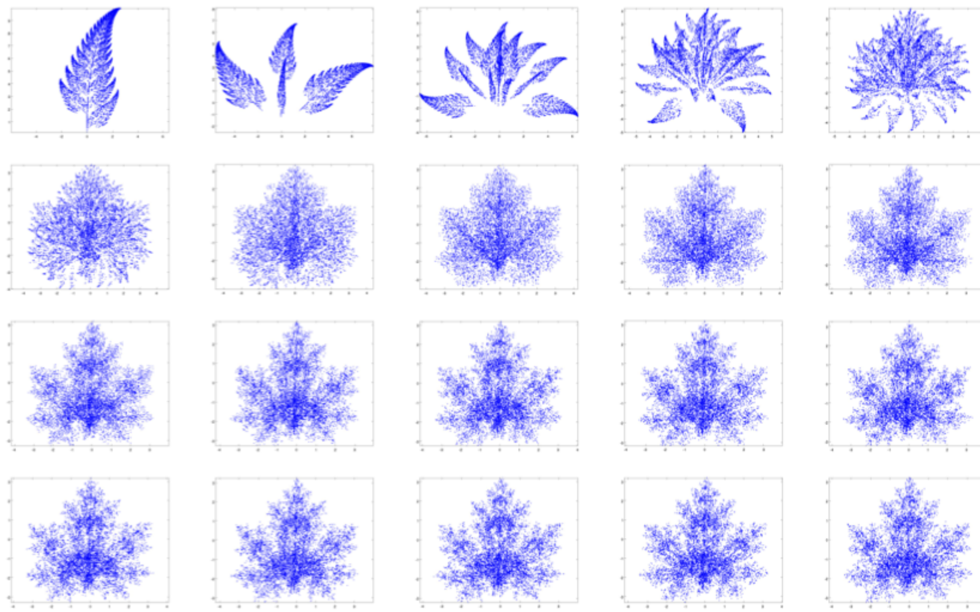


Figure 3.6: Barnsley Fern into Maple Leaf

here is establishing measurability properties—more specifically, axiom (I1) in Definition 3.16. The analogous measurability issue for RDS is not discussed in detail in either of Arnold's or Chueshov's monographs, and so we take the opportunity to give a thorough and self-contained proof.

For the reader's convenience, we give a brief review of the theory of existence and uniqueness for (deterministic) ordinary differential equations, introducing all the notation and terminology we shall need, in Appendix B.

Given a map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and an $X \subseteq \mathbb{R}^n$, denote

$$\|f\|_X := \sup_{x\in X} |f(x)| + \sup_{\substack{x,y\in X\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|}\,. \tag{3.27}$$

We say that f is *locally Lipschitz* if

$$\sup_{\substack{x,y\in K\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|} < \infty$$

for every compact $K \subseteq \mathbb{R}^n$. In this case, f is also continuous, and so $\|f\|_K < \infty$ for every such K. Note that the family $\mathcal{C}^{0,1}_{loc}(\mathbb{R}^n;\mathbb{R}^n)$ of locally Lipschitz maps $\mathbb{R}^n \to \mathbb{R}^n$ with multiplication by a real scalar constitutes a vector space on which the map

$$\|\cdot\|_K\colon \mathcal{C}^{0,1}_{loc}(\mathbb{R}^n;\mathbb{R}^n) \longrightarrow \mathbb{R}_{\geqslant 0}$$

defined as above constitutes a pseudonorm for each compact $K \subseteq \mathbb{R}^n$.

For any map $f \colon \mathbb{R}^n \to \mathbb{R}^n$, we call

$$\operatorname{supp} f := \{x \in \mathbb{R}^n \,;\; f(x) \neq 0\}$$

the *support of* f. This same map is said to be *compactly supported* if $\overline{\operatorname{supp} f}$ is compact.

For any $f \colon \mathbb{R}^n \to \mathbb{R}^n$, it follows straight from (3.27) that $\|f\|_{X_1} \leqslant \|f\|_{X_2}$ whenever $X_1 \subseteq X_2 \subseteq \mathbb{R}^n$. For a locally Lipschitz, compactly supported f, the maximum of $\|f\|_K$ is attained at $K := \overline{\operatorname{supp} f}$ (see Lemma B.1).

In Appendix B, we introduce a working notion of admissible (deterministic) "righthand sides" f for a nonautonomous ODE

$$\dot{\xi} = f(t, \xi)\,, \quad t \geqslant 0\,,$$

and give sufficient growth conditions of f yielding globally defined solutions (Proposition B.7). We now proceed to extend this to RDEI/RDSI. We begin by extending the concept of righthand side.

Definition 3.41 (θ-Righthand Side)**.** Let U be a Borel subset of $\mathbb{R}^k$ and $\mathcal{U}$ be a set of θ-inputs $\mathbb{R}_{\geqslant 0} \times \Omega \to U$. A $(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(U))$-measurable map $f \colon \Omega \times \mathbb{R}^n \times U \to \mathbb{R}^n$ is said to be a *θ-righthand side* (*with respect to* $\mathcal{U}$) if

(R1) $f(\omega, \cdot, u) \colon \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz for every $\omega \in \Omega$ and every $u \in U$, and

(R2) for each $\omega \in \Omega$, every $u \in \mathcal{U}$ and any $b > a \geqslant 0$,

$$\int_a^b \|f(\theta_t\omega, \cdot, u_t(\omega))\|_K \, dt < \infty$$

for every compact $K \subseteq \mathbb{R}^n$. △

For each positive integer k, let $H_k \colon \mathbb{R}^n \to \mathbb{R}$ be a smooth[7] "bump function" such that

$$H_k(x) = 1\,, \quad \forall x \in \overline{B}_k(0)\,,$$

$$0 \leqslant H_k(x) \leqslant 1\,, \quad \forall x \in B_{k+1}(0) \backslash \overline{B}_k(0)\,,$$

[7] With continuous derivatives of all orders.

and

$$H_k(x) = 0\,, \quad \forall x \in \mathbb{R}^n \backslash \overline{B}_{k+1}(0)\,.$$

(For the construction of smooth bump functions, refer to [39, Lemma 2.22 on page 42].) In particular, the first order partial derivatives of H_k are continuous and compactly supported. Therefore H_k is globally Lipschitz. Denote

$$L_k := \sup_{\substack{x,y \in \mathbb{R}^n \\ x \neq y}} \frac{|H_k(x) - H_k(y)|}{|x-y|} < \infty\,.$$

If $f\colon \Omega \times \mathbb{R}^n \times U \to \mathbb{R}^n$ is a θ-righthand side with respect to a set of θ-inputs $\mathcal{U}$, then

$$(\omega, x, u) \longmapsto H_k(x) f(\omega, x, u)\,, \quad (\omega, x, u) \in \Omega \times \mathbb{R}^n \times U$$

is also a θ-righthand side with respect to $\mathcal{U}$ for any positive integer k. In particular,

$$\|H_k(\cdot) f(\omega, \cdot, u)\|_{\mathbb{R}^n} = \|H_k(\cdot) f(\omega, \cdot, u)\|_{\overline{B}_k(0)}\,, \quad \forall \omega \in \Omega\,, \quad \forall u \in U\,.$$

(See Lemma B.4.)

We are now ready to show how RDSI can be obtained from RDEI.

Theorem 3.42 (RDSI Generated by RDEI)**.** *Suppose that $f\colon \Omega \times \mathbb{R}^n \times U \to \mathbb{R}^n$ is a θ-righthand side with respect to $\mathcal{S}_\infty^U$. Suppose that f satisfies the growth condition*

$$|f(\omega, x, u)| \leqslant \alpha(\omega)|x| + \beta(\omega) + c(u)\,, \quad \forall \omega \in \Omega\,, \quad \forall (x, u) \in \mathbb{R}^n \times U\,, \tag{3.28}$$

for some tempered random variables $\alpha, \beta\colon \Omega \to \mathbb{R}_{\geqslant 0}$, and some continuous function $c\colon U \to \mathbb{R}_{\geqslant 0}$. Then the RDEI

$$\dot{\xi} = f(\theta_t\omega, \xi, u_t(\omega))\,, \quad t \geqslant 0\,, \quad \omega \in \Omega\,, \quad u \in \mathcal{S}_\infty^U\,, \tag{3.29}$$

generates an RDSI $(\theta, \varphi, \mathcal{S}_\infty^U)$, uniquely determined by the properties that

$$\varphi(0, \omega, x, u) = x\,, \quad \forall (\omega, x, u) \in \Omega \times \mathbb{R}^n \times \mathcal{S}_\infty^U\,,$$

and

$$\frac{d}{dt}\varphi(t, \omega, x, u) = f(\theta_t\omega, \varphi(t, \omega, x, u), u_t(\omega))\,, \tag{3.30}$$

for each $(\omega, x, u) \in \Omega \times \mathbb{R}^n \times \mathcal{S}_\infty^U$, for Lebesgue-almost every $t \geqslant 0$.

Proof. The proof consists of two main steps. The first step is to construct the "flow" $\varphi\colon \mathbb{R}_{\geqslant 0} \times \Omega \times \mathbb{R}^n \times \mathcal{S}_\infty^U \to \mathbb{R}^n$ of (3.29). We then show that it satisfies axioms (I1)–(I5) in Definition 3.16.

Fix arbitrarily $\omega \in \Omega$, $u \in \mathcal{S}_\infty^U$, and define $g_{\omega,u}\colon \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$g_{\omega,u}(t,x) := f(\theta_t\omega, x, u_t(\omega)), \quad (t,x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n .$$

Since $g_{\omega,u}$ is the composition of measurable maps, it is itself $(\mathcal{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathcal{B}(\mathbb{R}^n))$ - measurable. By (R1), $g_{\omega,u}(t,\cdot) = f(\theta_t\omega, \cdot, u_t(\omega))$ is locally Lipschitz for each $t \geqslant 0$, hence $g_{\omega,u}$ satisfies (Q1). By (R2),

$$\int_a^b \|g_{\omega,u}(t,\cdot)\|_K \, dt = \int_a^b \|f(\theta_t\omega, \cdot, u_t(\omega))\|_K \, dt < \infty$$

for any $b > a \geqslant 0$ and any compact $K \subseteq \mathbb{R}^n$, hence $g_{\omega,u}$ satisfies (Q2) also. We conclude that $g_{\omega,u}$ is a (deterministic) righthand side.

Now

$$\begin{aligned} |g_{\omega,u}(t,x)| &= |f(\theta_t\omega, x, u_t(\omega))| \\ &\leqslant \alpha(\theta_t\omega)|x| + \big(\beta(\theta_t\omega) + c(u_t(\omega))\big), \quad \forall t \geqslant 0, \quad \forall x \in \mathbb{R}^n . \end{aligned}$$

The hypotheses that α and β are tempered, $u \in \mathcal{S}_\infty^U$ and c is continuous guarantee that the functions

$$t \longmapsto \alpha(\theta_t\omega), \quad t \geqslant 0, \quad t \longmapsto \beta(\theta_t\omega), \quad t \geqslant 0,$$

and

$$t \longmapsto c(u_t(\omega)), \quad t \geqslant 0,$$

are locally integrable. Thus $g_{\omega,u}$ satisfies growth condition (B.3). It then follows from Proposition B.7 that the ODE

$$\dot{\xi} = f(\theta_t\omega, \xi, u_t(\omega)) = g_{\omega,u}(t,\xi), \quad t \geqslant 0,$$

generates a continuous global flow $\varphi_{\omega,u}\colon \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \mathbb{R}^n$, uniquely determined by the properties that

$$\varphi_{\omega,u}(0,x) = x, \quad \forall x \in \mathbb{R}^n ,$$

and

$$\frac{d}{dt}\varphi_{\omega,u}(t,x) = f(\theta_t\omega, \varphi_{\omega,u}(t,x), u_t(\omega)) = g_{\omega,u}(t, \varphi_{\omega,u}(t,x)) \tag{3.31}$$

for all $x \in \mathbb{R}^n$, for Lebesgue-almost every $t \geqslant 0$.

Define $\varphi\colon \mathbb{R}_{\geqslant 0} \times \Omega \times \mathbb{R}^n \times \mathcal{S}_\infty^U \to \mathbb{R}^n$ by

$$\varphi(t,\omega,x,u) := \varphi_{\omega,u}(t,x)\,, \quad (t,\omega,x,u) \in \mathbb{R}_{\geqslant 0} \times \Omega \times \mathbb{R}^n \times \mathcal{S}_\infty^U\,.$$

We proceed to show that $(\theta, \varphi, \mathcal{S}_\infty^U)$ satisfies the axioms of an RDSI.

(I1) Fix arbitrarily $u \in \mathcal{S}_\infty^U$ and denote

$$\varphi_u(t,\omega,x) := \varphi(t,\omega,x,u)\,, \quad \forall (t,\omega,x) \in \mathbb{R}_{\geqslant 0} \times \Omega \times \mathbb{R}^n\,.$$

We shall show that

$$\varphi_u(t,\omega,x) = \lim_{m\to\infty} \varphi_u^{(m)}(t,\omega,x)\,, \quad \forall (t,\omega,x) \in \mathbb{R}_{\geqslant 0} \times \Omega \times \mathbb{R}^n\,, \tag{3.32}$$

where, for each positive integer m,

$$\varphi_u^{(m)}(t,\omega,x) := \lim_{i\to\infty} \varphi_u^{(m,i)}(t,\omega,x)\,, \quad (t,\omega,x) \in \mathbb{R}_{\geqslant 0} \times \Omega \times \mathbb{R}^n\,, \tag{3.33}$$

and the $\varphi_u^{(m,i)}$ are defined recursively by

$$\varphi_u^{(m,0)}(t,\omega,x) := x\,, \quad (t,\omega,x) \in \mathbb{R}_{\geqslant 0} \times \Omega \times \mathbb{R}^n\,,$$

$$\varphi_u^{(m,i)}(t,\omega,x) := x + \int_0^t H_m(\varphi_u^{(m,i-1)}(s,\omega,x)) f(\theta_s\omega, \varphi_u^{(m,i-1)}(s,\omega,x), u_s(\omega))\,ds\,,$$

$$(t,\omega,x) \in \mathbb{R}_{\geqslant 0} \times \Omega \times \mathbb{R}^n\,, \quad i = 1,2,3,\ldots\,.$$

Before we get to that, assume that we have shown that the limits in (3.33) exist and (3.32) holds. If we can show that each $\varphi_u^{(m,i)}$ above is $(\mathcal{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^n))$-measurable, then it follows from the limit in (3.33) that $\varphi_u^{(m)}$ is $(\mathcal{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^n))$-measurable for each positive integer m, and then from the limit in (3.32) that φ_u is itself $(\mathcal{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^n))$-measurable.

Throughout the rest of the proof of (I1), we shall say simply ‘measurable’ to mean ‘$(\mathcal{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^n))$-measurable,’ for short. Fix arbitrarily any positive integer m.

Clearly, $\varphi_u^{(m,0)}$ is globally defined and measurable. Now suppose it has been established that $\varphi_u^{(m,i-1)}$ is globally defined and measurable for some $i \geqslant 1$. Then the integrand

$$(t,\omega,x) \longmapsto H_m(\varphi_u^{(m,i-1)}(t,\omega,x)) f(\theta_t\omega, \varphi_u^{(m,i-1)}(t,\omega,x), u_t(\omega)) \,, \qquad (t,\omega,x) \in \mathbb{R}_{\geqslant 0} \times \Omega \times \mathbb{R}^n \,,$$

is measurable, since it is the composition measurable maps. Furthermore, for any arbitrarily fixed $\omega \in \Omega$ and $x \in \mathbb{R}^n$, the function

$$\begin{aligned} t \quad \mapsto \quad & |H_m(\varphi_u^{(m,i-1)}(t,\omega,x)) f(\theta_t\omega, \varphi_u^{(m,i-1)}(t,\omega,x), u_t(\omega))| \\ \leqslant \quad & \|H_m(\cdot) f(\theta_t\omega, \cdot, u_t(\omega))\|_{\mathbb{R}^n} \\ \leqslant \quad & \|H_m(\cdot) f(\theta_t\omega, \cdot, u_t(\omega))\|_{\overline{B}_m(0)} \,, \quad t \geqslant 0 \,, \end{aligned}$$

is locally integrable by the last statement in Lemma B.4. Thus $\varphi_u^{(m,i)}$ is globally defined. In particular, it follows from Proposition C.7, applied with '$\Omega \times \mathbb{R}^n$' for '$X$' and '$\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^n)$'for '$\mathcal{F}$,' that $\varphi_u^{(m,i)}$ is measurable. This completes the induction step, thus establishing the measurability of $\varphi_u^{(m,i)}$ for all $i \geqslant 0$. Since the positive integer m was chosen arbitrarily, this is true for every such m.

It follows as in Theorem B.8 that the limit in (3.33) exists. Furthermore, for each positive integer m, it follows from the same lemma, applied for each $\omega \in \Omega$, that $\varphi_u^{(m)}$ is the unique global solution of the RDE

$$\dot{\xi} = H_m(\xi) f(\theta_t\omega, \xi, u_t(\omega)) \,, \quad t \geqslant 0 \,, \quad \omega \in \Omega \,.$$

It remains to show that (3.32) holds. Fix arbitrarily $\omega \in \Omega$, $x \in \mathbb{R}^n$ and $T \geqslant 0$, then chose a positive integer m_0 such that

$$\varphi_u(t,\omega,x) \in \overline{B}_{m_0}(0) \,, \quad \forall t \in [0,T] \,.$$

Then

$$\begin{aligned} \frac{d}{dt}\varphi_u(t,\omega,x) \quad &= \quad f(\theta_t\omega, \varphi_u(t,\omega,x), u_t(\omega)) \\ &= \quad H_m(\varphi_u(t,\omega,x)) f(\theta_t\omega, \varphi_u(t,\omega,x), u_t(\omega)) \,, \quad \forall m \geqslant m_0 \,, \end{aligned}$$

for Lebesgue-almost every $t \in [0,T]$. By uniqueness, we must then have

$$\varphi_u^{(m)}(t,\omega,x) = \varphi_u(t,\omega,x) \,, \quad \forall t \in [0,T] \,, \quad \forall m \geqslant m_0 \,.$$

This shows that

$$\lim_{m\to\infty} \varphi_u^{(m)}(t,\omega,x) = \varphi_u(t,\omega,x)\,, \quad \forall t \in [0,T]\,.$$

Since $\omega \in \Omega$, $x \in \mathbb{R}^n$ and $T \geqslant 0$ were chosen arbitrarily, this establishes (3.32).

Since $u \in \mathcal{S}_\infty^U$ was chosen arbitrarily, this shows that $\varphi_u = \varphi(\cdot,\cdot,\cdot,u)$ is $(\mathcal{F}\otimes\mathcal{B}(\mathbb{R}_{\geqslant 0})\otimes\mathcal{B}(\mathbb{R}^n))$-measurable for each such u. This establishes (I1).

(I2) For each arbitrarily fixed $(t,\omega,u) \in \mathbb{R}_{\geqslant 0}\times\Omega\times\mathcal{S}_\infty^U$,

$$\varphi(t,\omega,\cdot,u) = \varphi_{\omega,u}(t,\cdot)$$

is continuous from the construction at the beginning of the proof. This establishes (I2).

(I3) For any $(\omega,x,u) \in \Omega\times\mathbb{R}^n\times\mathcal{S}_\infty^U$, it also follows straight from the construction at the beginning of the proof that

$$\varphi(0,\omega,x,u) = \varphi_{\omega,u}(0,x) = x\,.$$

This establishes (I3).

(I4) Fix $s \in \mathbb{R}_{\geqslant 0}$, $\omega \in \Omega$, $x \in X$ and $u, v \in S_\theta^U$ arbitrarily. Set

$$y := \varphi(s,\omega,x,u)\,.$$

We want to show that

$$\varphi(t,\theta_s\omega,y,v) = \varphi(s+t,\omega,x,u\Diamond_s v)\,, \quad \forall t \geqslant 0\,.$$

We have

$$\begin{aligned}
\varphi(t,\theta_s\omega,y,v) &= y + \int_0^t f(\theta_{s+\tau}\omega, \varphi(\tau,\theta_s\omega,y,v), v_\tau(\theta_s\omega))\,d\tau \\
&= \varphi(s,\omega,x,u) + \int_0^t f(\theta_{s+\tau}\omega, \varphi(\tau,\theta_s\omega,y,v), v_\tau(\theta_s\omega))\,d\tau \\
&= x + \int_0^s f(\theta_\sigma\omega, \varphi(\sigma,\omega,x,u), u_\sigma(\omega))\,d\sigma \\
&\qquad + \int_0^t f(\theta_{s+\tau}\omega, \varphi(\tau,\theta_s\omega,y,v), v_\tau(\theta_s\omega))\,d\tau \\
&= x + \int_0^s f(\theta_\sigma\omega, \varphi(\sigma,\omega,x,u), u_\sigma(\omega))\,d\sigma \\
&\qquad + \int_s^{s+t} f(\theta_{\tilde\tau}\omega, \varphi(\tilde\tau - s,\theta_s\omega,y,v), v_{\tilde\tau-s}(\theta_s\omega))\,d\tilde\tau\,,
\end{aligned}$$

hence

$$\varphi(t, \theta_s\omega, y, v) = x + \int_0^{s+t} f(\theta_\tau\omega, \psi_{s,\omega,x,u,v}(\tau), (u\Diamond_s v)_\tau(\omega))\, d\tau\,, \quad \forall t \geqslant 0\,, \tag{3.34}$$

where $\psi_{s,\omega,x,u,v} \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}^n$ is defined by

$$\psi_{s,\omega,x,u,v}(\tau) := \begin{cases} \varphi(\tau, \omega, x, u)\,, & 0 \leqslant \tau < s \\ \varphi(\tau - s, \theta_s\omega, y, v)\,, & s \leqslant \tau \end{cases}\,. \tag{3.35}$$

It follows straight from the definition of $\psi := \psi_{s,\omega,x,u,v}$ that

$$\begin{aligned} \psi(\tau) &= \varphi(\tau, \omega, x, u) \\ &= x + \int_0^\tau f(\theta_\sigma\omega, \varphi(\sigma, \omega, x, u), u_\sigma(\omega))\, d\sigma \\ &= x + \int_0^\tau f(\theta_\sigma\omega, \psi(\sigma), (u\Diamond_s v)_\sigma(\omega))\, d\sigma \end{aligned}$$

for $0 \leqslant \tau < s$. And combining (3.35) with (3.34), we obtain

$$\begin{aligned} \psi(\tau) &= \varphi(\tau - s, \theta_s\omega, y, v) \\ &= x + \int_0^\tau f(\theta_\sigma\omega, \psi(\sigma), (u\Diamond_s v)_\sigma(\omega))\, d\sigma \end{aligned}$$

for $\tau \geqslant s$. In summary,

$$\psi(\tau) = x + \int_0^\tau f(\theta_\tau\omega, \psi(\tau), (u\Diamond_s v)_\tau(\omega))\, d\tau\,, \quad \forall \tau \geqslant 0\,.$$

Now

$$\varphi(\tau, \omega, x, u\Diamond_s v) = x + \int_0^\tau f(\theta_\sigma\omega, \varphi(\sigma, \omega, x, u\Diamond_s v), (u\Diamond_s v)_\sigma(\omega))\, d\sigma\,, \quad \forall \tau \geqslant 0\,,$$

while

$$\psi(0) = x = \varphi(0, \omega, x, u\Diamond_s v)\,.$$

Therefore it follows by uniqueness that

$$\varphi(\tau, \omega, x, u\Diamond_s v) = \psi(\tau)\,, \quad \forall \tau \geqslant 0\,.$$

In particular,

$$\varphi(s + t, \omega, x, u\Diamond_s v) = \psi(s + t) = \varphi(t, \theta_s\omega, y, v)\,, \quad \forall t \geqslant 0\,.$$

(I5) Finally, given $\tau \geqslant 0$, $\omega \in \Omega$, $x \in \mathbb{R}^n$ and $u, v \in \mathcal{S}_\infty^U$ such that $u_t(\omega) = v_t(\omega)$ for Lebesgue-almost all $t \in [0, \tau)$, we then have

$$\begin{aligned} \frac{d}{dt}\varphi(t,\omega,x,u) &= f(\theta_t\omega, \varphi(t,\omega,x,u), u_t(\omega)) \\ &= g_{\omega,u}(t, \varphi(t,\omega,x,u)) \\ &= g_{\omega,v}(t, \varphi(t,\omega,x,u)) \\ &= f(\theta_t\omega, \varphi(t,\omega,x,u), v_t(\omega)) \end{aligned}$$

for Lebesgue-almost every $t \in [0, \tau)$. It then follows from Lemma B.5 that

$$\varphi(t,\omega,x,u) = \varphi_{\omega,u}(t,x) = \varphi_{\omega,v}(t,x) = \varphi(t,\omega,x,v)\,, \quad \forall t \in [0,\tau)\,.$$

Taking the limits as t approaches τ from the left, we obtain

$$\begin{aligned} \varphi(\tau,\omega,x,u) &= \varphi_{\omega,u}(\tau,x) \\ &= \lim_{t\to\tau^-} = \varphi_{\omega,u}(t,x) \\ &= \lim_{t\to\tau^-} = \varphi_{\omega,v}(t,x) \\ &= \varphi_{\omega,v}(\tau,x) \\ &= \varphi(\tau,\omega,x,v)\,. \end{aligned}$$

This establishes (I5).

Finally, suppose that $(\theta, \tilde{\varphi}, \mathcal{S}_\infty^U)$ is an RDSI satisfying (3.30) for each $(\omega, x, u) \in \Omega \times \mathbb{R}^n \times \mathcal{S}_\infty^U$, for Lebesgue-almost every $t \geqslant 0$. Then for each arbitrarily fixed $\omega \in \Omega$ and $u \in \mathcal{S}_\infty^U$, the map $\tilde{\varphi}(\cdot, \omega, \cdot, u)$ satisfies (3.31) for all $x \in \mathbb{R}^n$, for Lebesgue-almost every $t \geqslant 0$. Furthermore,

$$\tilde{\varphi}(0,\omega,x,u) = x\,, \quad \forall x \in \mathbb{R}^n\,,$$

by (I3). Therefore $\tilde{\varphi}(\cdot, \omega, \cdot, u) = \varphi_{\omega,u}$. Since $\omega \in \Omega$ and $u \in \mathcal{S}_\infty^U$ were chosen arbitrarily, this shows that $\tilde{\varphi} = \varphi$, thus completing the proof that $(\theta, \varphi, \mathcal{S}_\infty^U)$ is an RDSI. $\square$

Remark 3.43. Essentially the same argument used above to check (I1) can be used to settle measurability for RDS generated by RDE under Arnold's hypotheses. (See [4, Theorem 2.2.2 and Remark 2.2.3(iii) on pages 60–61].) $\square$

Example 3.44 (Linear RDS/RDSI). The "righthand side"

$$\begin{aligned} f\colon \Omega\times\mathbb{R}^n &\longmapsto \mathbb{R}^r \\ (\omega,x) &\longrightarrow A(\omega)x \end{aligned}$$

of the linear RDE (3.1) in Example 3.2 satisfies the hypotheses of [4, Theorem 2.2.2/Remark 2.2.3(iii) on pages 60–61]. Thus the "RDS" (θ,Φ) constructed in the example is indeed an RDS.

Likewise, the "righthand side"

$$\begin{aligned} f\colon \Omega\times\mathbb{R}^n\times\mathbb{R}^k &\longmapsto \mathbb{R}^r \\ (\omega,x,u) &\longrightarrow A(\omega)x+B(\omega)u \end{aligned}$$

of the linear RDEI (3.29) in Example 3.18 satisfies the hypotheses of Theorem 3.42—it is a θ-righthand side satisfying growth condition (3.28). Thus the "RDSI" $(\theta,\varphi,\mathcal{S}_\infty^U)$ constructed in the example is indeed an RDSI. $\Diamond$

As noted earlier, the natural continuous-time analogue of Proposition 3.25 also holds for RDS or RDSI generated by RDEI which can be interpreted as a cascade or feedback interconnection of lower-dimensional systems.

Chapter 4

Monotone RDSIO and a Small-Gain Theorem

In this chapter we shall again be concerned with order relations. Unless otherwise noted, we will tacitly assume that X and U are closed order-intervals of separable RTA spaces—not necessarily the same underlying space for both X and U. In particular, X and U will be convex, and the underlying cones will have nonempty interior.

4.1 Monotone RDSI

Given a partially ordered space $(X, \leqslant)$, recall the partial orders induced in $X_{\mathcal{B}}^{\Omega}$ and $\mathcal{S}_{\theta}^{X}$ as discussed in Subsection 2.3.2.

Definition 4.1 (Monotone RDSI)**.** An RDSI $(\theta, \varphi, \mathcal{U})$ is said to be *monotone* if the underlying state and input spaces are partially ordered spaces $(X, \leqslant_X)$, $(U, \leqslant_U)$, and

$$\varphi(\cdot, \cdot, x(\cdot), u) \leqslant_X \varphi(\cdot, \cdot, z(\cdot), v)$$

whenever $x, z \in X_{\mathcal{B}}^{\Omega}$ and $u, v \in \mathcal{U}$ are such that $x \leqslant_X z$ and $u \leqslant_U v$. △

Remark 4.2. Of course that if the inequality above holds pointwise in X, that is, if

$$\varphi(t, \omega, x, u) \leqslant_X \varphi(t, \omega, z, v)$$

holds for every $t \geqslant 0$, every $\omega \in \Omega$, and every $x, z \in X$ and $u, v \in \mathcal{U}$ such that $x \leqslant_X z$ and $u \leqslant_U v$, then $(\theta, \varphi, \mathcal{U})$ is monotone in the sense of the definition above. This is how we shall typically check for monotonicity. Proposition 4.4 below should further motivate our choice for a looser definition. □

4.1.1 Monotone Characteristics

Definition 4.3 (Monotone Characteristics)**.** Suppose $(X, \leqslant_X)$ and $(U, \leqslant_U)$ are partially ordered spaces. A map $\mathcal{M}\colon E \subseteq U^{\Omega}_{\mathcal{B}} \to X^{\Omega}_{\mathcal{B}}$ is said to be *monotone* or *order-preserving* if $\mathcal{M}(u) \leqslant_X \mathcal{M}(v)$ whenever $u, v \in E$ satisfy $u \leqslant_U v$. Analogously, if $\mathcal{M}(u) \geqslant_X \mathcal{M}(v)$ whenever $u \leqslant_U v$, then $\mathcal{M}$ is said to be *anti-monotone* or *order-reversing.* △

Most of the time, the underlying partially ordered space will be clear from the context. So, unless there is any risk of confusion, we shall again drop the indices in '$\leqslant_X$' and '$\leqslant_U$,' and write simply '$\leqslant$.'

Proposition 4.4. *If an RDSI $(\theta, \varphi, \mathcal{U})$ is monotone and has an i/s characteristic $\mathcal{K}\colon U^{\Omega}_{\theta} \to X^{\Omega}_{\theta}$, then $\mathcal{K}$ is order-preserving; in other words, if $u, v \in U^{\Omega}_{\theta}$ and $u \leqslant v$, then $\mathcal{K}(u) \leqslant \mathcal{K}(v)$.*

Proof. The proof is straightforward, and we emphasize its main purpose of pointing out a subtlety in Definition 4.1 which might have otherwise gone overlooked (see Remark 4.5 below). Pick any $u, v \in U^{\Omega}_{\theta}$ such that $u \leqslant v$, and fix $x \in X^{\Omega}_{\theta}$ arbitrarily. Then $x \leqslant x$ and $\bar{u} \leqslant \bar{v}$. By Definition 4.1, there exists a θ-invariant subset of full-measure $\widetilde{\Omega} \subseteq \Omega$ such that

$$\varphi(t, \omega, x(\omega), \bar{u}) \leqslant \varphi(t, \omega, x(\omega), \bar{v}), \quad \forall t \geqslant 0, \quad \forall \omega \in \widetilde{\Omega}. \tag{4.1}$$

Thus

$$\varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega), \bar{u}) \leqslant \varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega), \bar{v}), \quad \forall t \geqslant 0, \quad \forall \omega \in \widetilde{\Omega},$$

in view of the θ-invariance of $\widetilde{\Omega}$. The result then follows by taking the limit as $t \to \infty$ on both sides of the inequality above for each fixed $\omega \in \widetilde{\Omega}$. □

Remark 4.5. Note that we do not need for (4.1) to hold for $\omega \in \Omega \backslash \widetilde{\Omega}$. □

4.1.2 Infinitesimal Characterization of Monotonicity

Let U be a Borel subset of $\mathbb{R}^k$ and $f\colon \Omega \times \mathbb{R}^n \times U \to \mathbb{R}^n$ be a θ-righthand side, with respect to $\mathcal{S}_\infty^U$, satisfying growth condition (3.28). Then the RDEI

$$\dot{\xi} = f(\theta_t\omega, \xi, u_t(\omega))\,, \quad t \geqslant 0\,, \quad \omega \in \Omega\,, \quad u \in \mathcal{S}_\infty^U\,, \tag{4.2}$$

generates an RDSI $(\theta, \varphi, \mathcal{S}_\infty^U)$, as we saw in Theorem 3.42. We now discuss sufficient conditions for this RDSI to be monotone.

Recall what it means for RDEI (4.2) to generate the RDSI $(\theta, \varphi, \mathcal{S}_\infty^U)$. This is to say that

$$\varphi(0, \omega, x, u) = x\,, \quad \forall (\omega, x, u) \in \Omega \times \mathbb{R}^n \times \mathcal{S}_\infty^U\,,$$

and

$$\frac{d}{dt}\varphi(t, \omega, x, u) = f(\theta_t\omega, \varphi(t, \omega, x, u), u_t(\omega))\,,$$

for each $(\omega, x, u) \in \Omega \times \mathbb{R}^n \times \mathcal{S}_\infty^U$, for Lebesgue-almost all $t \geqslant 0$. Putting this in perspective against our definition of monotonicity for RDSI—Definition 4.1 above—, we see that monotonicity properties for RDSI generated by RDEI can be obtained by applying known results from the deterministic theory ω-wise.

We follow the framework of Angeli and Sontag [3].

Definition 4.6 (Tangent Cone)**.** Given a nonempty $S \subseteq \mathbb{R}^n$ and a $p \in S$, we define the *tangent cone to S at p* to be the collection of all points of the form

$$\lim_{k\to\infty} \frac{1}{t_k}(p_k - p)$$

for some sequences $(p_k)_{k\in\mathbb{N}}$ in S and $(t_k)_{k\in\mathbb{N}}$ in $(0, \infty)$ such that $p_k \longrightarrow p$ and $t_k \longrightarrow 0$ as $k \to \infty$. $\triangle$

Theorem 4.7 (Monotone RDSI by RDEI)**.** *Assume $\mathbb{R}^n$ and $\mathbb{R}^k$ to be partially ordered by cones $K_{\mathbb{R}^n}$ and $K_{\mathbb{R}^k}$, respectively, and suppose $U \subseteq \mathbb{R}^k$ is closed and order-convex. Let $f\colon \Omega \times \mathbb{R}^n \times U \to \mathbb{R}^n$ be a θ-righthand side with respect to $\mathcal{S}_\infty^U$ which satisfies growth condition* (3.28)*. Then the RDSI $(\theta, \varphi, \mathcal{S}_\infty^U)$ generated by* (4.2) *is monotone if, and only if, for θ-almost every $\omega \in \Omega$,*

$$x \leqslant z\,,\ u \leqslant v \quad \Rightarrow \quad f(\omega, z, v) - f(\omega, x, u) \in T_{z-x}K_{\mathbb{R}^n}\,.$$

Proof. This follows from [3, Theorem 1 on page 1686], applied for each $\omega \in \Omega$ for which the condition holds. $\square$

Proposition 4.8 (Kamke Conditions)**.** *Assume the same hypotheses as in* Theorem 4.7*, except for assuming that $\mathbb{R}^n$ and $\mathbb{R}^k$ are partially ordered specifically by their respective positive orthant cones, that U has nonempty interior, and that $f(\omega, \cdot, \cdot)\colon \mathbb{R}^n \times U \to \mathbb{R}^n$ is continuously differentiable for θ-almost all $\omega \in \Omega$. Then the RDSI $(\theta, \varphi, \mathcal{S}_\infty^U)$ generated by* (4.2) *is monotone if, and only if, for θ-almost every ω,*

(K1) $\dfrac{\partial f_i}{\partial x_j}(x, u) \geqslant 0$*, for every $x \in \mathbb{R}^n$, every $u \in \operatorname{int} U$, and every $i, j \in \{1, \ldots, n\}$ such that $i \neq j$, and*

(K2) $\dfrac{\partial f_i}{\partial u_j}(x, u) \geqslant 0$*, for every $x \in \mathbb{R}^n$, every $u \in \operatorname{int} U$, every $i \in \{1, \ldots, n\}$, and every $j \in \{1, \ldots, k\}$.*

Proof. This follows from [3, Proposition III.2 on page 1687], applied for each $\omega \in \Omega$ such that the conditions hold. $\square$

Example 4.9 (Monotone, Linear RDSI)*.* In Example 3.18, equip $\mathbb{R}^n$ and $\mathbb{R}^k$ with their respective positive orthant cones, and suppose that each of the off-diagonal entries of A are nonnegative θ-almost everywhere, and that each of the entries of B are nonnegative θ-almost everywhere. Then the RDSI in the example is monotone. Indeed, if this is the case, then it is not difficult to see that

$$\begin{aligned} f\colon \Omega \times \mathbb{R}^n \times \mathbb{R}^k &\longrightarrow \mathbb{R}^n \\ (\omega, x, u) &\longmapsto A(\omega)x + B(\omega)u \end{aligned}$$

satisfies (K1) and (K2). $\Diamond$

4.2 Converging Input to Converging State Property

Example 4.10 (CICS Property for Linear RDSI)*.* Recall the RDSI $(\theta, \varphi, \mathcal{S}_\infty^U)$ from Example 3.34, generated by the RDEI

$$\dot{\xi} = A(\theta_t\omega)\xi + B(\theta_t\omega)u_t(\omega)\,, \quad t \geqslant 0\,, \quad \omega \in \Omega\,, \quad u \in \mathcal{S}_\infty^U\,,$$

evolving on the state space $X = \mathbb{R}^n$, with input space $U = \mathbb{R}^k$, and with $A\colon \Omega \to M_{n\times n}(\mathbb{R})$ and $B\colon \Omega \to M_{n\times k}(\mathbb{R})$ being random matrices satisfying the integrability and growth conditions specified in the example. We saw that $(\theta, \varphi, \mathcal{S}_\infty^U)$ has a continuous i/s characteristic $\mathcal{K}\colon U_\theta^\Omega \to X_\theta^\Omega$.

Now consider the following, slightly modified situation. Suppose that the pullback of a tempered θ-input $u \in \mathcal{S}_\infty^U$ converges (in the tempered sense) to a $u_\infty \in U_\theta^\Omega$. One may expect that the continuity of φ (on the state variable) and $\mathcal{K}$ would imply that

$$\breve{\xi}_t^{x,u}(\omega) \longrightarrow_\theta [\mathcal{K}(u_\infty)](\omega)\,, \quad \text{as} \quad t \to \infty\,, \quad \forall x \in X_\theta^\Omega\,. \tag{4.3}$$

This is indeed true of this particular example, as we will proceed to show. We note, however, that this is not true in general. In fact, this "converging input to converging state" property might fail even in the deterministic case, as illustrated by the counterexample in [48].

Fix arbitrarily a tempered initial state $x \in X_\theta^\Omega$. As we saw in Example 3.34, φ is tempered. Therefore $\xi^{x,u}$ is tempered. So if we can prove θ-almost sure pointwise convergence in (4.3), then it follows from Proposition 2.67 that convergence is also tempered.

From Example 3.34, we have

$$\varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega), u) = \Phi(t, \theta_{-t}\omega, x(\theta_{-t}\omega)) + \Psi(t, \theta_{-t}\omega, u)\,, \quad \forall (t,\omega) \in \mathbb{R}_{\geqslant 0} \times \Omega\,.$$

In the same example, we showed that

$$\Phi(t, \theta_{-t}\omega, x(\theta_{-t}\omega)) \longrightarrow 0 \quad \text{as} \quad t \to \infty\,, \quad \widetilde{\forall}\omega \in \Omega\,. \tag{4.4}$$

So it remains to show that

$$\Psi(t, \theta_{-t}\omega, u) \longrightarrow [\mathcal{K}(u_\infty)](\omega) \quad \text{as} \quad t \to \infty\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

From a change of variables combined with splitting the integral defining $[\mathcal{K}(u_\infty)](\omega)$ into an integral from $-\infty$ to $-t$ plus another one from $-t$ to 0, we obtain

$$\begin{aligned}
&|\Psi(t, \theta_{-t}\omega, u) - [\mathcal{K}(u_\infty)](\omega)| \\
&\qquad \leqslant \int_{-\infty}^{-t} |\Xi(\sigma, 0, \omega) B(\theta_\sigma\omega) u_\infty(\theta_\sigma\omega)|\, d\sigma \\
&\qquad\quad + \int_{-t}^{0} \|\Xi(\sigma, 0, \omega) B(\theta_\sigma\omega)\| \cdot |u_{\sigma+t}(\theta_{-t}\omega) - u_\infty(\theta_\sigma\omega)|\, d\sigma
\end{aligned}$$

for θ-almost every $\omega \in \Omega$, for every $t \geqslant 0$. Since the integral

$$\int_{-\infty}^{0} |\Xi(\sigma, 0, \omega) B(\theta_\sigma \omega) u_\infty(\theta_\sigma \omega)| \, d\sigma$$

converges for θ-almost all $\omega \in \Omega$ (refer to the estimates and computations in Example 3.34), it follows from dominated convergence that

$$\int_{-\infty}^{-t} |\Xi(\sigma, 0, \omega) B(\theta_\sigma \omega) u_\infty(\theta_\sigma \omega)| \, d\sigma \longrightarrow 0 \quad \text{as} \quad t \to \infty\,, \quad \widetilde{\forall} \omega \in \Omega\,.$$

It remains to show that the second integral in the inequality above also goes to zero θ-almost surely.

Since u is tempered by hypothesis, there exists a tempered random variable $r \colon \Omega \to \mathbb{R}_{\geqslant 0}$ such that

$$|u_t(\theta_{-t}\omega)| + |u_\infty(\omega)| \leqslant r(\omega)\,, \quad \widetilde{\forall} \omega \in \Omega\,.$$

Now[1]

$$\begin{aligned}
\int_{-t}^{0} & \|\Xi(\sigma, 0, \omega) B(\theta_\sigma \omega)\| \cdot |u_{\sigma+t}(\theta_{-t}\omega) - u_\infty(\theta_\sigma \omega)| \, d\sigma \\
&= \int_{-\infty}^{0} \|\Xi(\sigma, 0, \omega) B(\theta_\sigma \omega)\| \cdot |u_{\sigma+t}(\theta_{-t}\omega) - u_\infty(\theta_\sigma \omega)| \cdot \chi_{[-t,0]}(\sigma) \, d\sigma \\
&\leqslant \int_{-\infty}^{0} \|\Xi(\sigma, 0, \omega) B(\theta_\sigma \omega)\| r(\theta_\sigma \omega) \, d\sigma\,, \quad \forall t \geqslant 0\,,
\end{aligned}$$

the last of the integrals being convergent just as above. From the hypotheses that u is tempered and $u_t \longrightarrow_\theta u_\infty$ as $t \to \infty$, we have

$$|u_{\sigma+t}(\theta_{-(\sigma+t)}\theta_\sigma \omega) - u_\infty(\theta_\sigma \omega)| \longrightarrow 0 \quad \text{as} \quad t \to \infty\,, \quad \widetilde{\forall} \omega \in \Omega\,, \quad \forall \sigma \leqslant 0\,.$$

So, it follows once again by dominated convergence that

$$\int_{-t}^{0} \|\Xi(\sigma, 0, \omega) B(\theta_\sigma \omega)\| \cdot |u_{\sigma+s}(\theta_{-t}\omega) - u_\infty(\theta_\sigma \omega)| \, d\sigma \longrightarrow 0$$

as $t \to \infty$, for θ-almost all $\omega \in \Omega$. Since $x \in X_\theta^\Omega$ was chosen arbitrarily, this proves (4.3). $\Diamond$

[1]Here $\chi_{[-t,0]} \colon \mathbb{R} \to \mathbb{R}$ is the characteristic function of the interval $[-t, 0]$, defined by $\chi(\sigma) := 1$ for $\sigma \in [-t, 0]$, and $\chi(\sigma) := 0$ for $\sigma \notin [-t, 0]$. Therefore the abuse of notation in the second integral is harmless—the integrand is 0 between $-\infty$ and $-t$.

4.2.1 Random CICS

The 'converging input to converging state' result below was first stated and proved for deterministic and finite-dimensional "monotone control systems" by Angeli and Sontag [3, Proposition V.5(2)]. In [19, Theorem 1], Enciso and Sontag explore normality to extend the result to infinite-dimensional systems. Replacing the geometric properties in [19] by minihedrality, it is possible to extend this result further to monotone RDSI.

Theorem 4.11 (Random CICS)**.** *Suppose that X and U are separable RTA spaces. Let $(\theta, \varphi, \mathcal{U})$ be a tempered, monotone RDSI with state space X and input space U, and suppose that φ has a continuous i/s characteristic $\mathcal{K}\colon U_\theta^\Omega \to X_\theta^\Omega$. If $u \in \mathcal{U}$ and $u_\infty \in U_\theta^\Omega$ are such that*

(i) *u is tempered and*

(ii) *$\check{u}_t \longrightarrow_\theta u_\infty$ as $t \to \infty$,*

then

$$\check{\xi}_t^{x,u} \longrightarrow_\theta \mathcal{K}(u_\infty) \quad \textit{as} \quad t \to \infty\,, \quad \forall x \in X_\theta^\Omega\,. \tag{4.5}$$

Proof. Fix arbitrarily $x \in X_\theta^\Omega$. Since φ is assumed to be tempered, the hypothesis (i) that u is tempered implies that the θ-stochastic process $\xi^{x,u}$ is also tempered (refer to Definition 3.19). Thus it remains to prove the pointwise convergence in (4.5), the temperedness bit then following straight from Proposition 2.67; more precisely, it remains to show that

$$\check{\xi}_t^{x,u}(\omega) \longrightarrow [\mathcal{K}(u_\infty)](\omega) \quad \text{as} \quad t \to \infty\,, \quad \widetilde{\forall}\omega \in \Omega\,, \tag{4.6}$$

and it will then follow directly from the aforementioned proposition that convergence is tempered.

This will require a little setting up.

In virtue of the pointwise convergence implied in (ii), it will follow that $\beta_u^0(\omega)$ is precompact for θ-almost all $\omega \in \Omega$. Let $(a_\tau)_{\tau \geqslant 0}$ and $(b_\tau)_{\tau \geqslant 0}$ be, respectively, lower and upper tails of the pullback trajectories of u (refer to Definition 2.68). Observe that $a_\tau, b_\tau \in U_\theta^\Omega$ for each $\tau \geqslant 0$, and that

$$a_\tau, b_\tau \longrightarrow_\theta u_\infty \quad \text{as} \quad \tau \to \infty \tag{4.7}$$

(refer to Proposition 2.69 and Lemma 2.72).

For each $\tau \geqslant 0$, let $\bar{a}_\tau$ and $\bar{b}_\tau$ be the θ-stationary processes generated by a_τ and b_τ, respectively. Then

$$(\bar{a}_\tau)_s(\omega) = a_\tau(\theta_s\omega) = \inf_{t\geqslant\tau} u_t(\theta_{-t}\theta_s\omega) \leqslant u_{\tau+s}(\theta_{-(\tau+s)}\theta_s\omega) = [\rho_\tau(u)]_s(\omega)$$

for θ-almost every $\omega \in \Omega$, for every $\tau, s \geqslant 0$, and, similarly,

$$[\rho_\tau(u)]_s(\omega) \leqslant (\bar{b}_\tau)_s(\omega)\,, \quad \widetilde{\forall}\omega \in \Omega\,, \quad \forall \tau, s \geqslant 0\,.$$

Thus

$$\bar{a}_\tau \leqslant \rho_\tau(u) \leqslant \bar{b}_\tau\,, \quad \forall \tau \geqslant 0\,. \tag{4.8}$$

We now return to (4.6). Using the cocycle property, we may rewrite

$$\begin{aligned}
\check{\xi}_t^{x,u}(\omega) &= \varphi(t-\tau, \theta_{-(t-\tau)}\omega, \varphi(\tau, \theta_{-t}\omega, x(\theta_{-t}\omega), u), \rho_\tau(u)) \\
&= \varphi(t-\tau, \theta_{-(t-\tau)}\omega, x_\tau(\theta_{-(t-\tau)}\omega), \rho_\tau(u)) \\
&= \check{\xi}_{t-\tau}^{x_\tau,\rho_\tau(u)}(\omega)\,, \quad \forall \omega \in \Omega\,, \quad \forall t \geqslant \tau \geqslant 0\,,
\end{aligned}$$

where $x_\tau \in X_\theta^\Omega$ is defined by $x_\tau := \check{\xi}_\tau^{x,u}$. Therefore

$$\|\check{\xi}_{\tau+s}^{x,u}(\omega) - [\mathcal{K}(u_\infty)](\omega)\| = \|\check{\xi}_s^{x_\tau,\rho_\tau(u)}(\omega) - [\mathcal{K}(u_\infty)](\omega)\|$$

for θ-almos every $\omega \in \Omega$, for all $\tau, s \geqslant 0$. For any such ω, s, τ, we have

$$\begin{aligned}
\|\check{\xi}_s^{x_\tau,\rho_\tau(u)}(\omega) - [\mathcal{K}(u_\infty)](\omega)\| \;\leqslant\; & \|\check{\xi}_s^{x_\tau,\rho_\tau(u)}(\omega) - \check{\xi}_s^{x_\tau,\bar{a}_\tau}(\omega)\| \\
& + \|\check{\xi}_s^{x_\tau,\bar{a}_\tau}(\omega) - [\mathcal{K}(a_\tau)](\omega)\| \\
& + \|[\mathcal{K}(a_\tau)](\omega) - [\mathcal{K}(u_\infty)](\omega)\|\,.
\end{aligned}$$

From (4.7) and the continuity of $\mathcal{K}$, there exist θ-invariant subsets $\widetilde{\Omega}_a$ and $\widetilde{\Omega}_b$ of full measure of Ω such that

$$\|[\mathcal{K}(a_\tau)](\omega) - [\mathcal{K}(u_\infty)](\omega)\| \longrightarrow 0 \quad \text{as} \quad \tau \to \infty\,, \quad \forall \omega \in \widetilde{\Omega}_a\,,$$

and

$$\|[\mathcal{K}(b_\tau)](\omega) - [\mathcal{K}(u_\infty)](\omega)\| \longrightarrow 0 \quad \text{as} \quad \tau \to \infty\,, \quad \forall \omega \in \widetilde{\Omega}_b\,.$$

Similarly, from the definition of i/s characteristic, for any integer $n \geqslant 0$, there exist θ-invariant subsets $\widetilde{\Omega}_{a,n}$ and $\widetilde{\Omega}_{b,n}$ of full measure of Ω such that

$$\|\check{\xi}_s^{x_n,\bar{a}_n}(\omega) - [\mathcal{K}(a_n)](\omega)\| \longrightarrow 0 \quad \text{as} \quad s \to \infty \,, \quad \forall \omega \in \widetilde{\Omega}_{a,n} \,,$$

and

$$\|\check{\xi}_s^{x_n,\bar{b}_n}(\omega) - [\mathcal{K}(b_n)](\omega)\| \longrightarrow 0 \quad \text{as} \quad s \to \infty \,, \quad \forall \omega \in \widetilde{\Omega}_{b,n} \,.$$

Now by (4.8) and monotonicity, for each integer $n \geqslant 0$, there exists a θ-invariant subset of full measure $\widetilde{\Omega}_{\leqslant,n} \subseteq \Omega$ such that

$$\check{\xi}_s^{x_n,\bar{a}_n}(\omega) \leqslant \check{\xi}_s^{x_n,\rho_n(u)}(\omega) \leqslant \check{\xi}_s^{x_n,\bar{b}_n}(\omega) \,, \quad \forall s \geqslant 0 \,, \quad \forall \omega \in \widetilde{\Omega}_{\leqslant,n} \,.$$

Let

$$\widetilde{\Omega} := \widetilde{\Omega}_a \cap \widetilde{\Omega}_b \cap \left(\bigcap_{n=0}^{\infty} \widetilde{\Omega}_{a,n} \right) \cap \left(\bigcap_{n=0}^{\infty} \widetilde{\Omega}_{b,n} \right) \cap \left(\bigcap_{n=0}^{\infty} \widetilde{\Omega}_{\leqslant,n} \right) .$$

Thus $\widetilde{\Omega}$ is a countable intersection of θ-invariant subsets of full measure of Ω and, hence, itself a θ-invariant subset of full measure of Ω. We shall show that convergence in (4.6) occurs for every $\omega \in \widetilde{\Omega}$.

Fix arbitrarily an $\omega \in \widetilde{\Omega}$ and a positive integer k. It follows from the construction of $\widetilde{\Omega}$ that there exists an integer $n_k \geqslant 0$ such that

$$\|[\mathcal{K}(a_\tau)](\omega) - [\mathcal{K}(u_\infty)](\omega)\| < 1/k \,, \quad \forall \tau \geqslant n_k \,,$$

and

$$\|[\mathcal{K}(b_\tau)](\omega) - [\mathcal{K}(u_\infty)](\omega)\| < 1/k \,, \quad \forall \tau \geqslant n_k \,.$$

Now we can use the convergence in the definition of i/s characteristic to choose an $s_k \geqslant 0$ such that

$$\|\check{\xi}_s^{x_{n_k},\bar{a}_{n_k}}(\omega) - [\mathcal{K}(a_{n_k})](\omega)\| < 1/k \,, \quad \forall s \geqslant s_k \,,$$

and

$$\|\check{\xi}_s^{x_{n_k},\bar{b}_{n_k}}(\omega) - [\mathcal{K}(b_{n_k})](\omega)\| < 1/k \,, \quad \forall s \geqslant s_k \,.$$

Again from the construction of $\widetilde{\Omega}$, we have

$$\check{\xi}_s^{x_{n_k},\bar{a}_{n_k}}(\omega) \leqslant \check{\xi}_s^{x_{n_k},\rho_{n_k}(u)}(\omega) \leqslant \check{\xi}_s^{x_{n_k},\bar{b}_{n_k}}(\omega) \,, \quad \forall s \geqslant 0 \,.$$

Thus

$$\|\check{\xi}_s^{x_{n_k},\rho_{n_k}(u)}(\omega) - \check{\xi}_s^{x_{n_k},\bar{a}_{n_k}}(\omega)\| \leqslant C\|\check{\xi}_s^{x_{n_k},\bar{b}_{n_k}}(\omega) - \check{\xi}_s^{x_{n_k},\bar{a}_{n_k}}(\omega)\|\,, \quad \forall s \geqslant 0\,,$$

where $C \geqslant 0$ is the normality constant for U_+. Now

$$\begin{aligned}
\|\check{\xi}_s^{x_{n_k},\bar{b}_{n_k}}(\omega) - \check{\xi}_s^{x_{n_k},\bar{a}_{n_k}}(\omega)\| &\leqslant \|\check{\xi}_s^{x_{n_k},\bar{b}_{n_k}}(\omega) - [\mathcal{K}(b_{n_k})](\omega)\| \\
&\quad + \|[\mathcal{K}(b_{n_k})](\omega) - [\mathcal{K}(u_\infty)](\omega)\| \\
&\quad + \|[\mathcal{K}(u_\infty)](\omega) - [\mathcal{K}(a_{n_k})](\omega)\| \\
&\quad + \|[\mathcal{K}(a_{n_k})](\omega) - \check{\xi}_s^{x_{n_k},\bar{a}_{n_k}}(\omega)\| \\
&\leqslant 4/k\,, \quad \forall s \geqslant s_k\,.
\end{aligned}$$

We conclude that

$$\begin{aligned}
\|\check{\xi}_t^{x,u}(\omega) - [\mathcal{K}(u_\infty)](\omega)\| &= \|\check{\xi}_{t-n_k}^{x_{n_k},\rho_{n_k}(u)}(\omega) - [\mathcal{K}(u_\infty)](\omega)\| \\
&\leqslant \|\check{\xi}_{t-n_k}^{x_{n_k},\rho_{n_k}(u)}(\omega) - \check{\xi}_{t-n_k}^{x_{n_k},\bar{a}_{n_k}}(\omega)\| \\
&\quad + \|\check{\xi}_{t-n_k}^{x_{n_k},\bar{a}_{n_k}}(\omega) - [\mathcal{K}(a_{n_k})](\omega)\| \\
&\quad + \|[\mathcal{K}(a_{n_k})](\omega) - [\mathcal{K}(u_\infty)](\omega)\| \\
&< 4C/k + 1/k + 1/k \\
&= (4C+2)/k\,, \quad \forall t \geqslant n_k + s_k\,.
\end{aligned}$$

Since $\omega \in \widetilde{\Omega}$ and the positive integer k were chosen arbitrarily, this completes the proof. $\square$

4.2.2 Compact RDSI

The hypothesis (ii) that the θ-input u converges served two key purposes in the proof of Theorem 4.11. It was first used to show that u was indeed eventually precompact. It was then used to establish (4.7). If we know a priori that u is eventually precompact, then we may still construct lower and upper tails for u and compute its θ-limits,

$$\theta\text{-}\underline{\lim}\, u \quad \text{as} \quad \theta\text{-}\overline{\lim}\, u\,.$$

If the θ-limits

$$\theta\text{-}\underline{\lim}\, \xi^{x,u} \quad \text{as} \quad \theta\text{-}\overline{\lim}\, \xi^{x,u}$$

also exist, then a natural question would be how these θ-limits may compare with

$$\mathcal{K}(\theta\text{-}\underline{\lim}\, u) \quad \text{as} \quad \mathcal{K}(\theta\text{-}\overline{\lim}\, u)\,.$$

We address this question in the next result.

Theorem 4.12 (Sub-CICS)**.** *Suppose that X and U are separable RTA spaces. Let $(\theta, \varphi, \mathcal{U})$ be a tempered, compact, monotone RDSI with state space X and input space U, and suppose that φ has a continuous i/s characteristic $\mathcal{K}\colon U_\theta^\Omega \to X_\theta^\Omega$. Then*

$$\mathcal{K}(\theta\text{-}\underline{\lim}\, u) \leqslant \theta\text{-}\underline{\lim}\, \xi^{x,u}$$

and

$$\theta\text{-}\overline{\lim}\, \xi^{x,u} \leqslant \mathcal{K}(\theta\text{-}\overline{\lim}\, u)$$

for every $x \in X_\theta^\Omega$ and every tempered, eventually precompact $u \in \mathcal{U}$.

Proof. We work out the details for the first inequality, the proof of the second one being entirely analogous. Fix arbitrarily a tempered initial state $x \in X_\theta^\Omega$ and a tempered, eventually precompact input $u \in \mathcal{U}$. By Definitions 3.19 and 3.20, the θ-stochastic process $\xi^{x,u}$ is also tempered and eventually precompact. Let $\tau_u \geqslant 0$ be such that $\beta_u^{\tau_u}(\omega)$ and $\beta_{\xi^{x,u}}^{\tau_u}(\omega)$ are precompact for θ-almost every $\omega \in \Omega$, and let $(a_\tau)_{\tau \geqslant \tau_u}$ be a lower tail of the pullback trajectories of u. From Proposition 2.69, both $\theta\text{-}\underline{\lim}\, u$ and $\theta\text{-}\underline{\lim}\, \xi^{x,u}$ exist and define tempered random variables in their respective spaces. Also from Proposition 2.69, we know that

$$a_\tau \longrightarrow_\theta \theta\text{-}\underline{\lim}\, u \quad \text{as} \quad \tau \to \infty\,.$$

Thus

$$\mathcal{K}(a_\tau) \longrightarrow_\theta \mathcal{K}(\theta\text{-}\underline{\lim}\, u) \quad \text{as} \quad \tau \to \infty$$

by continuity. Therefore it is enough to show that

$$\mathcal{K}(a_{\tau_n}) \leqslant \theta\text{-}\underline{\lim}\, \xi^{x,u}\,, \quad \forall n \in \mathbb{N}\,, \tag{4.9}$$

for an arbitrarily fixed sequence $(\tau_n)_{n\in\mathbb{N}}$ going to infinity in $[\tau_u, \infty)$.

Fix arbitrarily $n \in \mathbb{N}$. Let $\bar{a}_{\tau_n}$ be the θ-stationary process generated by a_{τ_n}. As in the proof of Theorem 4.11, we have $\bar{a}_{\tau_n} \leqslant \rho_{\tau_n}(u)$ and $\check{\xi}_t^{x,u} = \check{\xi}_{t-\tau_n}^{\check{\xi}_{\tau_n}^{x,u},\rho_{\tau_n}(u)}$ for all $t \geqslant \tau_n$. Thus, by monotonicity,

$$\xi_s^{\check{\xi}_{\tau_n}^{x,u},\bar{a}_{\tau_n}} \leqslant \xi_s^{\check{\xi}_{\tau_n}^{x,u},\rho_{\tau_n}(u)}, \quad \forall s \geqslant 0 .$$

Since we are assuming φ to be tempered, the pullback state $\check{\xi}_{\tau_n}^{x,u}$ is a tempered random variable. By Lemmas 2.64 and 2.63, the θ-stochastic processes $\rho_{\tau_n}(u)$ and $\bar{a}_{\tau_n}$ are tempered. Thus $\xi_s^{\check{\xi}_{\tau_n}^{x,u},\rho_{\tau_n}(u)}$ and $\xi_s^{\check{\xi}_{\tau_n}^{x,u},\bar{a}_{\tau_n}}$ are eventually tempered. It follows from the existence of the i/s characteristic and Lemmas 2.72 and 2.73 that

$$\mathcal{K}(a_{\tau_n}) = \theta\text{-}\underline{\lim}\, \xi^{\check{\xi}_{\tau_n}^{x,u},\bar{a}_{\tau_n}} \leqslant \theta\text{-}\underline{\lim}\, \xi^{\check{\xi}_{\tau_n}^{x,u},\rho_{\tau_n}(u)} = \theta\text{-}\underline{\lim}\, \xi^{x,u} .$$

Since $n \in \mathbb{N}$ was chosen arbitrarily, this proves (4.9). $\square$

Theorem 4.12 is a key ingredient in the proof of the Small-Gain Theorem RTA spaces (Theorem 4.28 below).

4.3 Output Functions Revisited

We now take a closer look at output functions from the point of view of regularity, growth, and order-preserving/order-reversing properties.

4.3.1 Measurability

We begin with a technical measurability consideration.

Lemma 4.13. *Consider an output function $h\colon \Omega \times X \to Y$. If $D\colon \Omega \to 2^X \backslash \{\varnothing\}$ is a closed random set in X, then*

$$\omega \longrightarrow h(\omega, D(\omega)) := \{h(\omega, x)\,;\; x \in D(\omega)\}, \quad \omega \in \Omega ,$$

is a random set in Y.

Proof. By Proposition 2.18, there exist a Polish space (Z, d_Z) and a Carathéodory map $g\colon \Omega \times Z \to X$ such that $D(\omega) = g(\omega, Z)$ for every $\omega \in \Omega$. Let $(z_k)_{k \in \mathbb{N}}$ be a dense sequence in Z.

Fix $y \in Y$ arbitrarily. We want to show that

$$\omega \longmapsto \mathrm{dist}_Y(y, h(\omega, D(\omega)))\,, \quad \omega \in \Omega\,,$$

is $\mathcal{F}$-measurable. Since $(z_k)_{k\in\mathbb{N}}$ is dense in Z and $g(\omega,\cdot)$ is continuous, $(g(\omega,z_k))_{k\in\mathbb{N}}$ is dense in $D(\omega)$ for each $\omega \in \Omega$. Now $h(\omega,\cdot)$ is also continuous, so $(h(\omega, g(\omega, z_k)))_{k\in\mathbb{N}}$ is dense in $h(\omega, D(\omega))$ for each $\omega \in \Omega$. It follows that

$$\begin{aligned} \mathrm{dist}_Y(y, h(\omega, D(\omega))) &= \inf_{x\in D(\omega)} d_Y(y, h(\omega, x)) \\ &= \inf_{k\in\mathbb{N}} d_Y(y, h(\omega, g(\omega, z_k)))\,, \quad \omega \in \Omega\,. \end{aligned}$$

For each fixed $k \in \mathbb{N}$,

$$g(\cdot, z_k) \colon \Omega \longrightarrow X$$

is $\mathcal{F}$-measurable. Thus

$$(\cdot, g(\cdot, z_k)) \colon \Omega \longrightarrow \Omega \times X$$

is also $\mathcal{F}$-measurable. Since h is $(\mathcal{F} \otimes \mathcal{B}(X))$-measurable and

$$d_Y(y, \cdot) \colon Y \longrightarrow \mathbb{R}_{\geqslant 0}$$

is continuous, it follows that

$$d_Y(y, h(\cdot, g(\cdot, z_k))) \colon \Omega \longrightarrow \mathbb{R}_{\geqslant 0}$$

is $\mathcal{F}$-measuable. We conclude that $\mathrm{dist}_Y(y, h(\cdot, D(\cdot)))$ is $\mathcal{F}$-measurable—it is the infimum of countably many compositions of measurable functions, hence itself measurable. Since $y \in Y$ was taken arbitrarily, this completes the proof. $\square$

Corollary 4.14. *Assume the same hypotheses as in* Lemma 4.13*. If D is compact, then $h(\cdot, D(\cdot))$ is a compact random set.*

Proof. Indeed, $h(\cdot, D(\cdot))$ is a random set by the lemma. Furthermore, $h(\omega, D(\omega))$ is a compact subset of Y for each $\omega \in \Omega$ by the continuity of $h(\omega, \cdot)$. $\square$

Corollary 4.15. *Assume the same hypotheses as in* Lemma 4.13*. If $D \colon \Omega \to 2^X \backslash \{\varnothing\}$ is a random set—not necessarily closed—, then $h(\cdot, D(\cdot))$ is also a random set.*

Proof. By Proposition 2.12, the closure $\overline{D}$ of D is a (closed) random set. By Lemma 4.13, $h(\cdot, \overline{D}(\cdot))$ is a random set. It then follows, again from Proposition 2.12, that $\overline{h(\cdot, \overline{D}(\cdot))}$ is a (closed) random set.

It now follows from continuity that

$$\overline{h(\omega, \overline{D}(\omega))} = \overline{h(\omega, D(\omega))}\,, \quad \forall \omega \in \Omega\,.$$

So $\overline{h(\cdot, D(\cdot))}$ is a random set. It then follows, once again from Proposition 2.12 that $h(\cdot, D(\cdot))$ is a random set. □

4.3.2 Temperedness Preserving Outputs

Given an output function $h\colon \Omega \times X \to Y$ (Definition 3.21), we define its *induced output characteristic* $h_*\colon X^\Omega_{\mathcal{B}} \to Y^\Omega_{\mathcal{B}}$ by

$$[h_*(x)](\omega) := h(\omega, x(\omega))\,, \quad \omega \in \Omega\,,$$

for each $x \in X^\Omega_{\mathcal{B}}$. This is the natural way to map random states $x \in X^\Omega_{\mathcal{B}}$ into random readouts $y \in Y^\Omega_{\mathcal{B}}$, generalizing what is accomplished by the output function $h\colon X \to Y$ itself in the deterministic setting.

In the context of "closed-loop systems," "cascades" and "feedback interconnections," we shall be interested in output funcions h such that $h_*(X^\Omega_\theta) \subseteq Y^\Omega_\theta$.

Definition 4.16 (Temperedness Preserving Outputs)**.** An output function

$$h\colon \Omega \times X \longrightarrow Y$$

is said to *preserve temperedness* if the random set

$$h(\cdot, D(\cdot))\colon \Omega \longrightarrow 2^Y \backslash \{\varnothing\}$$

is tempered for every tempered random set $D\colon \Omega \to 2^X \backslash \{\varnothing\}$. △

In particular,

$$\omega \longmapsto h(\omega, x(\omega))\,, \quad \omega \in \Omega\,,$$

defines a tempered random variable $\Omega \to Y$ whenever $x\colon \Omega \to X$ is also a tempered random variable. Thus the induced output characteristic $h_*\colon X^\Omega_\mathcal{B} \to Y^\Omega_\mathcal{B}$ satisfies $h_*(X^\Omega_\theta) \subseteq Y^\Omega_\theta$.

Temperedness preservation will be typically a consequence of growth conditions on the outputs, which is what one would actually check for in examples and applications.

Definition 4.17 (Growth Conditions for Output Functions)**.** Given an output function $h\colon \Omega \times X \to Y$, we label the following growth conditions for ease of reference.

(G1) There exist $M_0, M_1 \in (\mathbb{R}_{\geqslant 0})^\Omega_\theta$ and $n \in \mathbb{N}$ such that

$$\|h(\omega, x)\| \leqslant M_0(\omega) + M_1(\omega)\|x\|^n\,, \quad \forall x \in X\,, \quad \widetilde{\forall}\omega \in \Omega$$

(*tempered polynomial growth*). △

Proposition 4.18. *Suppose that $h\colon \Omega \times X \to Y$ is an output function satisfying* (G1)*. Then h is temperedness preserving.*

Proof. Given any tempered random set $D \in (2^X\backslash\{\varnothing\})^\Omega_\theta$, let $r \in (\mathbb{R}_{\geqslant 0})^\Omega_\theta$ be such that $D(\cdot) \subseteq B_{r(\cdot)}(0)$. Then

$$h(\cdot, D(\cdot)) \subseteq B_{M_0(\cdot)+M_1(\cdot)(r(\cdot))^n}(0)\,.$$

Now $B_{M_0(\cdot)+M_1(\cdot)(r(\cdot))^n}(0)$ is a tempered random variable in virtue of Example 2.14 and Lemma 2.24, thus completing the proof. □

We end this subsection with a couple of technical properties of temperedness preserving outputs.

Lemma 4.19. *Let $h\colon \Omega \times X \to Y$ be a temperedness preserving output. If $\xi\colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$ is a tempered θ-stochastic process in X, then the θ-stochastic process $\eta^\xi\colon \mathcal{T}_{\geqslant 0} \times \Omega \to Y$ defined by*

$$\eta^\xi_t(\omega) := h(\theta_t\omega, \xi_t(\omega))\,, \quad (t, \omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,,$$

is also tempered.

Proof. Indeed, given a (tempered) rest set D for ξ, it follows straight from Definition 4.16 that $h(\cdot, D(\cdot))$ is a (tempered) rest set for η^ξ. □

Lemma 4.20. *Suppose that $h\colon \Omega \times X \to Y$ is a temperedness preserving output function. Then the restriction $h_*\big|_{X_\theta^\Omega}\colon X_\theta^\Omega \to Y_\theta^\Omega$ of the induced output characteristic to X_θ^Ω is tempered continuous.*

Proof. Let $(x_\alpha)_{\alpha\in A}$ be any net in X_θ^Ω such that $x_\alpha \longrightarrow_\theta x_\infty$ as $\alpha \to \infty$ for some $x_\infty \in X_\theta^\Omega$. Then $x_\alpha(\omega) \longrightarrow x_\infty(\omega)$ as $\alpha \to \infty$ for θ-almost every $\omega \in \Omega$. It then follows from the continuity of h with respect to its second variable that

$$\begin{aligned}
[h_*(x_\alpha)](\omega) &= h(\omega, x_\alpha(\omega)) \\
&\to h(\omega, x_\infty(\omega)) \\
&= [h_*(x_\infty)](\omega) \quad \text{as} \quad \alpha \to \infty\,, \quad \widetilde{\forall}\omega \in \Omega\,.
\end{aligned}$$

It remains to show that convergence is tempered.

Let $\alpha_0 \in A$ and $r \in (\mathbb{R}_{\geqslant 0})_\theta^\Omega$ be such that

$$\|x_\alpha(\omega) - x_\infty(\omega)\| \leqslant r(\omega)\,, \quad \forall \alpha \geqslant \alpha_0\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

In other terms,

$$x_\alpha(\omega) \in B_{r(\omega)}(x_\infty(\omega))\,, \quad \forall \alpha \geqslant \alpha_0\,, \quad \widetilde{\forall}\omega \in \Omega\,,$$

where the random set $B_{r(\cdot)}(x_\infty(\cdot))$ is tempered by Example 2.27. Since h is temperedness preserving by hypothesis, we conclude that $h(\cdot, B_{r(\cdot)}(x_\infty(\cdot)))$ is a tempered random set. Let R be a nonnegative tempered random variable such that

$$h(\omega, B_{r(\omega)}(x_\infty(\omega))) \subseteq B_{R(\omega)}(0)\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

Then

$$[h_*(x_\alpha)](\omega) \in B_{R(\omega)}(0)\,, \quad \forall \alpha \geqslant \alpha_0\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

Therefore

$$\|[h_*(x_\alpha)](\omega) - [h_*(x_\infty)](\omega)\| \leqslant R(\omega) + \|[h_*(x_\infty)](\omega)\|\,, \quad \forall \alpha \geqslant \alpha_0\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

Since $R(\cdot) + \|[h_*(x_\alpha)](\cdot)$ is a tempered random variable, this completes the proof that convergence is tempered. $\square$

Definition 4.21 (I/O Characteristic)**.** Suppose that an RDSIO $(\theta, \varphi, \mathcal{U}, h)$ is such that the underlying RDSI $(\theta, \varphi, \mathcal{U})$ has an i/s characteristic $\mathcal{K}\colon U_\theta^\Omega \to X_\theta^\Omega$, and the output function h preserves temperedness. Then the induced output characteristic $h_*\colon X_\theta^\Omega \to Y_\theta^\Omega$ of $(\theta, \varphi, \mathcal{U}, h)$ is well-defined, and so the map

$$\mathcal{K}^Y := h_* \circ \mathcal{K}\colon U_\theta^\Omega \longrightarrow Y_\theta^\Omega$$

is also well-defined. In this case the system is said to have an *input to output* (*i/o*) *characteristic* and, accordingly, $\mathcal{K}^Y$ is referred to as the *input to output* (*i/o*) *characteristic* of the system. △

In the particular case when $Y = U$, the i/o characteristic is an operator on the space U_θ^Ω of tempered random variables $\Omega \to U$. This operator can be informally interpreted as the "gain" of the system, a measure of how much a θ-stationary "signal" u changes when "processed" by the system.

4.3.3 Monotone and Anti-Monotone Outputs

Definition 4.22 (Monotone and Anti-Monotone Outputs)**.** Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be partially ordered spaces. An output function $h\colon \Omega \times X \to Y$ is said to be *monotone* if

$$\widetilde{\forall}\omega \in \Omega\,, \quad x_1 \leqslant_X x_2 \quad \Rightarrow \quad h(\omega, x_1) \leqslant_Y h(\omega, x_2)\,.$$

Analogously, if

$$\widetilde{\forall}\omega \in \Omega\,, \quad x_1 \leqslant_X x_2 \quad \Rightarrow \quad h(\omega, x_1) \geqslant_Y h(\omega, x_2)\,,$$

then h is said to be *anti-monotone.* △

Most often the underlying partial order will be clear from the context and we shall use simply $\leqslant$ to denote either of $\leqslant_X$ or $\leqslant_Y$. Furthermore, whenever we refer to a 'monotone RDSI,' an 'order-preserving map,' etc, the underlying spaces will be tacitly understood to be partially ordered.

Lemma 4.23. *Suppose that* $h\colon \Omega \times X \to Y$ *is a monotone* (*anti-monotone*) *output function. Then the induced output characteristic* h_* *is order-preserving* (*order-reversing*)*.*

Proof. Suppose first that h is monotone. By Definition 4.22, there exists a θ-invariant subset of full measure $\widetilde{\Omega}_1 \subseteq \Omega$ such that

$$\forall \omega \in \widetilde{\Omega}_1\,, \quad p \leqslant_X q \quad \Rightarrow \quad h(\omega, p) \leqslant_Y h(\omega, q)\,.$$

Now pick any $x_1, x_2 \in X^\Omega_{\mathcal{B}}$ such that $x_1 \leqslant x_2$ and let $\widetilde{\Omega}_2 \subseteq \Omega$ be a θ-invariant subset of full measure such that

$$x_1(\omega) \leqslant_X x_2(\omega)\,, \quad \forall \omega \in \widetilde{\Omega}_2\,.$$

Set $\widetilde{\Omega} := \widetilde{\Omega}_1 \cap \widetilde{\Omega}_2$. Then $\widetilde{\Omega}$ is a θ-invariant subset of full measure and

$$(h_*(x_1))(\omega) = h(\omega, x_1(\omega)) \leqslant_Y h(\omega, x_2(\omega)) = (h_*(x_2))(\omega)\,, \quad \forall \omega \in \widetilde{\Omega}\,.$$

Since $x_1, x_2 \in X^\Omega_{\mathcal{B}}$ with $x_1 \leqslant x_2$ were chosen arbitrarily, this proves h_* is order preserving.

If h is anti-monotone, then the proof that h_* is order-reversing is essentially the same. One needs only replace occurrences of '$\leqslant_Y$' above by '$\geqslant_Y$.' □

4.4 Small-Gain Theorem

Definition 4.24 (Closed-Loop Trajectory)**.** A θ-stochastic process $\xi \in \mathcal{S}^X_\theta$ is said to be a *closed-loop trajectory* of an RDSIO $(\theta, \varphi, \mathcal{U}, h)$ (starting at ξ_0) if

(1) $Y = U$,

(2) the θ-stochastic process $\eta^\xi \colon \mathcal{T}_{\geqslant 0} \times \Omega \longrightarrow U$ defined by

$$\eta^\xi_t(\omega) := h(\theta_t \omega, \xi_t(\omega))\,, \quad t \geqslant 0\,, \quad \omega \in \Omega\,,$$

belongs to $\mathcal{U}$, and

(3) $\xi_t(\omega) = \varphi(t, \omega, \xi_0(\omega), \eta^\xi)$ for all $t \geqslant 0$ and all $\omega \in \Omega$. △

Property (1) is quite natural. It does not make sense to talk about feeding the output of the system back into it, thus "closing the loop," if the output and input spaces do not coincide. The θ-stochastic process η^ξ defined in property (2) is the "readout" of the (θ-stochastic) trajectory ξ on the state space. Naturally, we can only feed this readout as an input to the system if it is itself an admissible θ-input. Property (3) then states that the original trajectory ξ could be recovered by letting the system evolve starting at ξ_0 and subject to the θ-input η^ξ.

Lemma 4.25. *Suppose that X, U are separable RTA spaces. Let $(\theta, \varphi, \mathcal{U}, h)$ be a monotone RDSIO with state space X, input and output spaces U, possessing a continuous i/s characteristic $\mathcal{K}\colon U_\theta^\Omega \to X_\theta^\Omega$, and a monotone or anti-monotone, temperedness preserving output function h. Given a tempered, eventually precompact closed loop trajectory $\xi\colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$ of $(\theta, \varphi, \mathcal{U}, h)$, let $\eta^\xi\colon \mathcal{T}_{\geqslant 0} \times \Omega \to U$ be the corresponding (tempered, eventually precompact) output trajectory along ξ; that is,*

$$\eta_t^\xi(\omega) = h(\theta_t\omega, \xi_t(\omega))\,, \quad \forall (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,.$$

Let $(a_\tau)_{\tau \geqslant \tau_\xi}$ and $(b_\tau)_{\tau \geqslant \tau_\xi}$ be, respectively, lower and upper tails of the pullback trajectories of η^ξ. Then

$$\begin{aligned}(\mathcal{K}^Y)^{2k}(a_\tau) &\leqslant \theta\text{-}\underline{\lim}\,\eta^\xi \\ &\leqslant \theta\text{-}\overline{\lim}\,\eta^\xi \leqslant (\mathcal{K}^Y)^{2k}(b_\tau)\,, \quad \forall k \in \mathbb{N}\,, \quad \forall \tau \geqslant \tau_\xi\,.\end{aligned}$$

Proof. $\mathcal{K}$ is monotone by Proposition 4.4. So, since

$$a_\tau \leqslant \theta\text{-}\underline{\lim}\,\eta^\xi \leqslant \theta\text{-}\overline{\lim}\,\eta^\xi \leqslant b_\tau\,, \quad \forall \tau \geqslant \tau_\xi\,,$$

we have

$$\mathcal{K}(a_\tau) \leqslant \mathcal{K}(\theta\text{-}\underline{\lim}\,\eta^\xi) \leqslant \mathcal{K}(\theta\text{-}\overline{\lim}\,\eta^\xi) \leqslant \mathcal{K}(b_\tau)\,, \quad \forall \tau \geqslant \tau_\xi\,.$$

By Theorem 4.12,

$$\mathcal{K}(\theta\text{-}\underline{\lim}\,\eta^\xi) \leqslant \theta\text{-}\underline{\lim}\,\xi \leqslant \theta\text{-}\overline{\lim}\,\xi \leqslant \mathcal{K}(\theta\text{-}\overline{\lim}\,\eta^\xi)\,. \tag{4.10}$$

Combining these with the previous inequalities, we obtain

$$\mathcal{K}(a_\tau) \leqslant \theta\text{-}\underline{\lim}\,\xi \leqslant \theta\text{-}\overline{\lim}\,\xi \leqslant \mathcal{K}(b_\tau)\,, \quad \forall \tau \geqslant \tau_\xi\,. \tag{4.11}$$

Suppose first that h is monotone. By Lemma 4.23, h_* preserves the inequalities above; that is,

$$\mathcal{K}^Y(a_\tau) \leqslant h_*(\theta\text{-}\underline{\lim}\,\xi) \leqslant h_*(\theta\text{-}\overline{\lim}\,\xi) \leqslant \mathcal{K}^Y(b_\tau)\,, \quad \forall \tau \geqslant \tau_\xi\,.$$

By Lemma 4.26(1) below, we now have

$$\mathcal{K}^Y(a_\tau) \leqslant \theta\text{-}\underline{\lim}\,\eta^\xi \leqslant \theta\text{-}\overline{\lim}\,\eta^\xi \leqslant \mathcal{K}^Y(b_\tau)\,, \quad \forall \tau \geqslant \tau_\xi\,.$$

Now suppose that we have shown that

$$(\mathcal{K}^Y)^k(a_\tau) \leqslant \theta\text{-}\underline{\lim}\,\eta^\xi \leqslant \theta\text{-}\overline{\lim}\,\eta^\xi \leqslant (\mathcal{K}^Y)^k(b_\tau)\,, \quad \forall \tau \geqslant \tau_\xi\,, \tag{4.12}$$

for some $k \in \mathbb{N}$. Then, again, combining the monotonicity of $\mathcal{K}$ and h_*, (4.10) and Lemma 4.26(1), we obtain

$$\begin{aligned} \mathcal{K}\big((\mathcal{K}^Y)^k(a_\tau)\big) \leqslant \mathcal{K}(\theta\text{-}\underline{\lim}\,\eta^\xi) &\leqslant \theta\text{-}\underline{\lim}\,\xi \\ &\leqslant \theta\text{-}\overline{\lim}\,\xi \leqslant \mathcal{K}(\theta\text{-}\overline{\lim}\,\eta^\xi) \leqslant \mathcal{K}\big((\mathcal{K}^Y)^k(b_\tau)\big)\,, \end{aligned}$$

hence

$$\begin{aligned} (\mathcal{K}^Y)^{k+1}(a_\tau) \leqslant h_*(\theta\text{-}\underline{\lim}\,\xi) &\leqslant \theta\text{-}\underline{\lim}\,\eta^\xi \\ &\leqslant \theta\text{-}\overline{\lim}\,\eta^\xi \leqslant h_*(\theta\text{-}\overline{\lim}\,\xi) \leqslant (\mathcal{K}^Y)^{k+1}(b_\tau) \end{aligned}$$

for every $\tau \geqslant \tau_\xi$. It follows by induction that (4.12) holds for every $k \in \mathbb{N}$. In particular, the conclusion of the lemma holds.

If h is anti-monotone, then h_* is order-reversing by Lemma 4.23. Thus applying h_* to each term in the chain of inequalities in (4.11) yields

$$\mathcal{K}^Y(b_\tau) \leqslant h_*(\theta\text{-}\overline{\lim}\,\xi) \leqslant h_*(\theta\text{-}\underline{\lim}\,\xi) \leqslant \mathcal{K}^Y(a_\tau)\,, \quad \forall \tau \geqslant \tau_\xi\,.$$

Applying Lemma 4.26(2) below, we get

$$\mathcal{K}^Y(b_\tau) \leqslant \theta\text{-}\underline{\lim}\,\eta^\xi \leqslant \theta\text{-}\overline{\lim}\,\eta^\xi \leqslant \mathcal{K}^Y(a_\tau)\,, \quad \forall \tau \geqslant \tau_\xi\,. \tag{4.13}$$

Applying $\mathcal{K}$ to each term in (4.13) and using (4.10) once again, we get

$$\mathcal{K}\big(\mathcal{K}^Y(b_\tau)\big) \leqslant \theta\text{-}\underline{\lim}\,\xi \leqslant \theta\text{-}\overline{\lim}\,\xi \leqslant \mathcal{K}\big(\mathcal{K}^Y(a_\tau)\big)\,, \quad \forall \tau \geqslant \tau_\xi\,.$$

Applying h_* to each term in the inequalities above and using Lemma 4.26(2) below once again to simplify, we then get

$$(\mathcal{K}^Y)^2(a_\tau) \leqslant \theta\text{-}\underline{\lim}\,\eta^\xi \leqslant \theta\text{-}\overline{\lim}\,\eta^\xi \leqslant (\mathcal{K}^Y)^2(b_\tau)\,, \quad \forall \tau \geqslant \tau_\xi\,.$$

The argument can now be completed by induction on k just as in the previous case, using the monotonicity of $\mathcal{K}$, the anti-monotonicity of h_*, (4.10) and Lemma 4.26(2) to simplify the two terms in the middle after each application of $\mathcal{K}$ and h_*, respectively. $\square$

Lemma 4.26. *Assume the same hypotheses as in* Lemma 4.25.

(1) *If h is monotone, then*

$$h_*(\theta\text{-}\underline{\lim}\,\xi) \leqslant \theta\text{-}\underline{\lim}\,\eta^\xi \leqslant \theta\text{-}\overline{\lim}\,\eta^\xi \leqslant h_*(\theta\text{-}\overline{\lim}\,\xi)\,.$$

(2) *If h is anti-monotone, then*

$$h_*(\theta\text{-}\overline{\lim}\,\xi) \leqslant \theta\text{-}\underline{\lim}\,\eta^\xi \leqslant \theta\text{-}\overline{\lim}\,\eta^\xi \leqslant h_*(\theta\text{-}\underline{\lim}\,\xi)\,.$$

Proof. Since $\theta\text{-}\underline{\lim}\,\eta^\xi \leqslant \theta\text{-}\overline{\lim}\,\eta^\xi$ always holds so long as both terms are well-defined (refer to Definition 2.71), we essentially have only four inequalities to prove. The argument for each of them goes along the same lines, so we shall provide the details for only one of the inequalities. Namely, we assume that h is anti-monotone, and prove that

$$h_*(\theta\text{-}\overline{\lim}\,\xi) \leqslant \theta\text{-}\underline{\lim}\,\eta^\xi\,.$$

Let $(\alpha_\tau)_{\tau\geqslant\tau_\xi}$ and $(\beta_\tau)_{\tau\geqslant\tau_\xi}$ be, respectively, lower and upper tails of the pullback trajectories of ξ. Since

$$\xi_t(\theta_{-t}\omega) \leqslant \beta_\tau(\omega)\,, \quad \widetilde{\forall}\omega\in\Omega\,, \quad \forall t\geqslant\tau\geqslant\tau_\xi\,,$$

it follows from the anti-monotonicity of h that

$$h(\omega,\xi_t(\theta_{-t}\omega)) \geqslant h(\omega,\beta_\tau(\omega))\,, \quad \widetilde{\forall}\omega\in\Omega\,, \quad \forall t\geqslant\tau\geqslant\tau_\xi\,.$$

Therefore

$$\begin{aligned} a_\tau(\omega) &= \inf_{t\geqslant\tau}\eta^\xi_t(\theta_{-t}\omega) \\ &= \inf_{t\geqslant\tau}h(\omega,\xi_t(\theta_{-t}\omega)) \\ &\geqslant h(\omega,\beta_\tau(\omega)) \\ &= [h_*(\beta_\tau)](\omega)\,, \quad \widetilde{\forall}\omega\in\Omega\,, \quad \forall\tau\geqslant\tau_\xi\,. \end{aligned}$$

We know from Lemma 4.20 that h_* is tempered continuous. So, by letting $\tau\to\infty$ in the chain of equalities and inequalities above, we obtain

$$\theta\text{-}\underline{\lim}\,\eta^\xi = \lim_{\tau\to\infty}a_\tau \geqslant \lim_{\tau\to\infty}h_*(\beta_\tau) = h_*(\theta\text{-}\overline{\lim}\,\xi)\,.$$

As noted above, the proofs of the other inequalities are entirely analogous. $\square$

We are now ready to introduce the small-gain condition, then state and prove the Small-Gain Theorem for RDS.

Definition 4.27 (Small-Gain Condition)**.** We say that an RDSIO $(\theta, \varphi, \mathcal{U}, h)$ satisfying the hypotheses of Lemma 4.25 satisfies the *Small-Gain Condition* if there exists a (necessarily unique) $u_\infty \in U_\theta^\Omega$ such that

$$[(\mathcal{K}^Y)^k(u)](\omega) \longrightarrow u_\infty(\omega)$$

as $k \to \infty$ for θ-almost all $\omega \in \Omega$, for every $u \in U_\theta^\Omega$. △

Observe that we do not ask that convergence in the Small-Gain Condition be tempered.

Theorem 4.28 (Small-Gain Theorem)**.** *Suppose that X, U are separable RTA spaces. Let $(\theta, \varphi, \mathcal{U}, h)$ be a tempered, monotone RDSIO with state space X, input and output spaces U, possessing a continuous i/s characteristic $\mathcal{K} \colon U_\theta^\Omega \to X_\theta^\Omega$, and a monotone or anti-monotone, temperedness preserving output function h. If $(\theta, \varphi, \mathcal{U}, h)$ satisfies the Small-Gain Condition, then*

$$\check{\xi}_t \longrightarrow_\theta \mathcal{K}(u_\infty) \quad \textit{as} \quad t \to \infty$$

for every tempered, eventually precompact closed-loop trajectory $\xi \colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$ of $(\theta, \varphi, \mathcal{U}, h)$; in other words, every tempered, eventually precompact closed loop trajectory of $(\theta, \varphi, \mathcal{U}, h)$ converges (in the pullback, tempered sense) to $\mathcal{K}(u_\infty)$.

Proof. Consider the notation introduced in the statement and proof of Lemma 4.25, which also give us

$$\begin{aligned}(\mathcal{K}^Y)^{2k}(a_\tau) &\leqslant \theta\text{-}\underline{\lim}\, \eta^\xi \\ &\leqslant \theta\text{-}\overline{\lim}\, \eta^\xi \leqslant (\mathcal{K}^Y)^{2k}(b_\tau)\,, \quad \forall k \in \mathbb{N}\,, \quad \forall \tau \geqslant 0\,.\end{aligned}$$

By the Small-Gain Condition,

$$\lim_{k\to\infty} [(\mathcal{K}^Y)^{2k}(a_\tau)](\omega) = \lim_{k\to\infty} [(\mathcal{K}^Y)^{2k}(b_\tau)](\omega) = u_\infty(\omega)\,, \quad \widetilde{\forall} \omega \in \Omega\,,$$

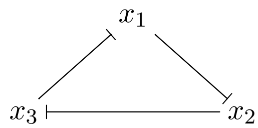


Figure 4.1: Biochemical Circuit

where $u_\infty \in U_\theta^\Omega$ is given in the definition of the Small-Gain Condition (Definition 4.27). Thus

$$u_\infty = \theta\text{-}\underline{\lim}\, \eta^\xi = \theta\text{-}\overline{\lim}\, \eta^\xi = u_\infty\,.$$

So

$$\breve{\eta}_t^\xi \longrightarrow_\theta u_\infty \quad \text{as} \quad t \to \infty$$

by Lemma 2.72. It then follows from Theorem 4.11 that

$$\breve{\xi}_t = \breve{\xi}_t^{\xi_0,\eta^\xi} \longrightarrow_\theta \mathcal{K}(u_\infty) \quad \text{as} \quad t \to \infty\,,$$

as we wanted to show. □

4.5 Applications

In this section we provide several examples and constructions illustrating how the theory developed in this work may be applied.

One may allude to the example in the introduction, namely, a biochemical circuit as illustrated in Figure 4.1, as a prototype for the more general examples discussed in what follows. As outlined in the introduction, this biochemical circuit may be modeled by an RDE

$$\left\{\begin{array}{rcl} \dot{\xi}_1 & = & a_1(\theta_t\omega)\xi_1 + \dfrac{b_1(\theta_t\omega)}{\beta_1(\theta_t\omega) + g_1(\xi_3)} \\ \dot{\xi}_2 & = & a_2(\theta_t\omega)\xi_2 + \dfrac{b_2(\theta_t\omega)}{\beta_2(\theta_t\omega) + g_2(\xi_1)} \\ \dot{\xi}_3 & = & a_3(\theta_t\omega)\xi_3 + \dfrac{b_3(\theta_t\omega)}{\beta_3(\theta_t\omega) + g_3(\xi_2)} \end{array}\right. ,$$

for some nondecreasing functions $g_1, g_2, g_3 \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$. Note that this could be rewritten in matrix notation as

$$\dot{\xi} = A(\theta_t\omega)\xi + B(\theta_t\omega)h(\theta_t\omega, \xi)\,,$$

where

$$A(\omega) \equiv \begin{bmatrix} a_1(\omega) & 0 & 0 \\ 0 & a_2(\omega) & 0 \\ 0 & 0 & a_3(\omega) \end{bmatrix}, \quad B(\omega) \equiv \begin{bmatrix} b_1(\omega) & 0 & 0 \\ 0 & b_2(\omega) & 0 \\ 0 & 0 & b_3(\omega) \end{bmatrix}$$

and

$$h(\omega, \xi) \equiv \begin{bmatrix} \frac{1}{\beta_1(\omega) + g_1(\xi_3)} & \frac{1}{\beta_2(\omega) + g_2(\xi_1)} & \frac{1}{\beta_3(\omega) + g_3(\xi_2)} \end{bmatrix}^T .$$

The same biochemical network could also be modeled, in discrete time, by an RdE

$$\begin{cases} \xi_1^+ &= \xi_1 + a_1(\theta_n\omega)\xi_1 + \dfrac{b_1(\theta_n\omega)}{\beta_1(\theta_n\omega) + g_1(\xi_3)} \\ \xi_2^+ &= \xi_2 + a_2(\theta_n\omega)\xi_2 + \dfrac{b_2(\theta_n\omega)}{\beta_2(\theta_n\omega) + g_2(\xi_1)} \\ \xi_3^+ &= \xi_3 + a_3(\theta_n\omega)\xi_3 + \dfrac{b_3(\theta_n\omega)}{\beta_3(\theta_n\omega) + g_3(\xi_2)} \end{cases} .$$

This could be rewritten as

$$\xi^+ = A(\theta_n\omega)\xi + B(\theta_n\omega)h(\theta_n\omega, \xi) ,$$

where

$$A(\omega) \equiv \begin{bmatrix} 1 + a_1(\omega) & 0 & 0 \\ 0 & 1 + a_2(\omega) & 0 \\ 0 & 0 & 1 + a_3(\omega) \end{bmatrix} ,$$

and B and h are as in the continuous-time example.

We will give a few explicit examples, in continuous and discrete time, of how the Small-Gain Theorem may be directly applied to establish the existence and uniqueness of a globally attracting equilibrium for some classes of non-monotone, nonlinear RDS such as the ones generated by the RDE or RdE above.

4.5.1 Continuous Time

Suppose that $A\colon \Omega \to M_{n\times n}(\mathbb{R})$ and $B\colon \Omega \to M_{n\times k}(\mathbb{R})$ are random matrices satisfying the hypotheses in Examples 3.34 and 4.9; thus

$$t \longmapsto A(\theta_t\omega) , \quad t \geqslant 0 , \quad \text{and} \quad t \longmapsto B(\theta_t\omega) , \quad t \geqslant 0 ,$$

are locally essentially bounded for each $\omega \in \Omega$, conditions (L1) and (L2) from Example 3.34 are satisfied, all off-diagonal entries of $A(\omega)$ are nonnegative for θ-almost every $\omega \in \Omega$, and all entries of $B(\omega)$ are nonnegative for θ-almost every $\omega \in \Omega$. We shall consider the RDE

$$\dot{\xi} = A(\theta_t\omega)\xi + B(\theta_t\omega)h(\theta_t\omega, \xi)\,, \quad t \geqslant 0\,, \quad \omega \in \Omega\,, \tag{4.14}$$

for several classes of nonlinearity $h\colon \Omega \times \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^k_{\geqslant 0}$. In each of Examples 4.29–4.31 below, we will apply the Small-Gain Theorem to show that the RDS generated by (4.14) has a unique, globally attracting, positive equilibrium.

Equip $\mathbb{R}^n$ and $\mathbb{R}^k$ with their respective positive orthant cone-induced partial orders, thus yielding separable RTA spaces. Let $X := \mathbb{R}^n_{\geqslant 0}$ and $U := \mathbb{R}^k_{\geqslant 0}$, which are closed order-intervals. Under the hypotheses on A and B described above, the RDEI

$$\dot{\xi} = A(\theta_t\omega)\xi + B(\theta_t\omega)u_t(\omega)\,, \quad t \geqslant 0\,, \quad \omega \in \Omega\,, \quad u \in \mathcal{S}^U_\infty\,,$$

generates a tempered (Example 3.34), monotone (Example 4.9) RDSI $(\theta, \varphi, \mathcal{S}^U_\infty)$ possessing a continuous i/s characteristic $\mathcal{K}\colon U^\Omega_\theta \to X^\Omega_\theta$ given by

$$[\mathcal{K}(u)](\omega) = \int_{-\infty}^{0} \Xi(\sigma, 0, \omega)B(\theta_\sigma\omega)u(\theta_\sigma\omega)\, d\sigma\,, \quad \forall u \in U^\Omega_\theta\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

Thus the burden of satisfying the hypotheses of the Small-Gain Theorem has now fallen all on h—the RDS generated by (4.14) will have a unique, globally attracting equilibrium whenever h is a monotone or anti-monotone, temperedness preserving output function such that the RDSIO $(\theta, \varphi, \mathcal{U}, h)$ satisfies the Small-Gain Condition.

Example 4.29 (Saturated Readouts). Consider an output function $h\colon \Omega \times X \to U$ defined by

$$h(\omega, x) := \left(\frac{\alpha_j(\omega)}{\beta_j(\omega) + g_j(x)}\right)_{j=1}^{k}\,, \quad (\omega, x) \in \Omega \times X\,,$$

where $\alpha, \beta\colon \Omega \to \mathbb{R}^k_{\geqslant 0}$ and $g\colon \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^k_{\geqslant 0}$ satisfy the following hypotheses:

(P1) α and β are continuous and uniformly bounded away from zero and infinity along θ-almost every orbit; more precisely, for θ-almost every $\omega \in \Omega$,

$$t \longmapsto \alpha(\theta_t\omega) \in \mathbb{R}^k\,, \quad t \in \mathbb{R}\,,$$

and

$$t \longmapsto \beta(\theta_t\omega) \in \mathbb{R}^k\,, \quad t \in \mathbb{R}\,,$$

are continuous, and there exist an $\epsilon = \epsilon(\omega) \gg 0$ and an $M = M(\omega) \geqslant 0$ such that

$$\epsilon \leqslant \alpha(\theta_t\omega), \beta(\theta_t\omega) \leqslant M\,, \quad \forall t \in \mathbb{R}\,,$$

and

(P2) g is continuous, order-preserving, sublinear, and bounded.

It follows straight from the monotonicity of g in (P2) that h is anti-monotone.

It follows from (P1) that

$$0 \leqslant h(\theta_s\omega, x(\theta_s\omega)) \leqslant \frac{M(\omega)}{\epsilon(\omega)}\,, \quad \forall s \in \mathbb{R}\,, \quad \forall \omega \in \Omega\,,$$

for any $x \in X_\theta^\Omega$. In particular, h preserves temperedness.

It remains to check that the i/o characteristic $\mathcal{K}^Y$ of $(\theta, \varphi, \mathcal{S}_\infty^U, h)$ satisfies the Small-Gain Condition.

For each $u \in U_\theta^\Omega$,

$$[\mathcal{K}^Y(u)](\omega) = \left(\frac{\alpha_j(\omega)}{\beta_j(\omega) + g_j\left(\int_{-\infty}^0 \Xi(\sigma, 0, \omega) B(\theta_\sigma\omega) u(\theta_\sigma\omega)\, d\sigma\right)} \right)_{j=1}^k\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

Fix arbitrarily such an u. Fix arbitrarily any $\omega \in \Omega$ for which $[\mathcal{K}(u)](\omega)$ is defined and (P1) holds. For each $t \in \mathbb{R}$, we have

$$[\mathcal{K}^Y(u)](\theta_t\omega) = \left(\frac{\alpha_j(\theta_t\omega)}{\beta_j(\theta_t\omega) + g_j\left(\int_{-\infty}^t \Xi(\sigma, t, \omega) B(\theta_\sigma\omega) u(\theta_\sigma\omega)\, d\sigma\right)} \right)_{j=1}^k$$

by a simple, linear change of variables. Set $A_\omega := A(\theta.\omega)$, $B_\omega := B(\theta.\omega)$, $\alpha_\omega := \alpha(\theta.\omega)$, and $\beta_\omega := \beta(\theta.\omega)$. Thus A_ω and B_ω are locally integrable matrix paths satisfying (L1′), (L2′), (M1′) and (M2′) in Section D.4, plus α_ω, β_ω and g satisfy (i) and (ii) in the hypotheses of Proposition D.23. Consider the (discrete) dynamical system generated by the difference equation

$$\nu^+ = \mathcal{H}_\omega(\nu)\,,$$

where $\mathcal{H}_\omega \colon L^\theta_+(\mathbb{R}^k) \to L^\theta_+(\mathbb{R}^k)$ is defined by

$$[\mathcal{H}_\omega(\nu)](t) := \left(\frac{(\alpha_\omega)_j(t)}{(\beta_\omega)_j(t) + g_j\left(\int_{-\infty}^t \Xi_\omega(\sigma,t) B_\omega(\sigma)\nu(\sigma)\, d\sigma\right)} \right)_{j=1}^k , \quad t \in \mathbb{R} ,$$

for each $\nu \in L^\theta_+(\mathbb{R}^k)$. It follows from Proposition D.23 that this discrete system has a unique, globally attracting fixed point

$$u_\omega \in \operatorname{int} L^\infty_+(\mathbb{R}^k) \subseteq L^\theta_+(\mathbb{R}^k) .$$

Furthermore, as shown in this same proposition, we may choose u_ω to be continuous, and such that convergence occurs pointwise; that is,

$$\lim_{m\to\infty} [\mathcal{H}^m_\omega(\nu)](t) = u_\omega(t) , \quad \forall t \in \mathbb{R} , \quad \forall \nu \in L^\theta_+(\mathbb{R}^k) .$$

We now show that the map $u_\infty \colon \Omega \to U$ defined by

$$u_\infty(\omega) := u_\omega(0) , \quad \omega \in \Omega ,$$

belongs to U^Ω_θ, and is the unique, globally attracting fixed point of $\mathcal{K}^Y$. Fix arbitrarily $u \in U^\Omega_\theta$. Then

$$\lim_{m\to\infty} [(\mathcal{K}^Y)^m(u)](\omega) = \lim_{m\to\infty} [\mathcal{H}^m_\omega(u)](0) = u_\omega(0) = u_\infty(\omega) , \quad \widetilde{\forall} \omega \in \Omega .$$

In particular, u_∞ is the θ-almost sure, pointwise limit of measurable maps

$$\omega \longmapsto [(\mathcal{K}^Y)^m(u)](\omega) , \quad \omega \in \Omega , \quad m = 1, 2, 3, \dots ,$$

hence itself measurable. Fix arbitrarily any $\omega \in \Omega$ for which the limit above holds. By the uniqueness of the continuous representatives u_ω, we have

$$u_\infty(\theta_t\omega) = u_{\theta_t\omega}(0) = u_\omega(t) , \quad \forall t \in \mathbb{R} .$$

Therefore $t \mapsto u_\infty(\theta_t\omega)$, $t \in \mathbb{R}$, is bounded. In particular,

$$\sup_{t\in\mathbb{R}} |u_\infty(\theta_t\omega)|\, \mathrm{e}^{-\gamma|t|} < \infty , \quad \forall \gamma > 0 .$$

We conclude that u_∞ is tempered, and a fixed point of $\mathcal{K}^Y$. Since $u \in U^\Omega_\theta$ was chosen arbitrarily, this also shows that u_∞ is globally attractive. $\Diamond$

Example 4.30 (Unbounded g)*.* Now consider an output function $h\colon \Omega \times X \to U$ defined by

$$h(\omega, x) := \left(\frac{\alpha_j(\omega) + \widetilde{\alpha}_j(\omega) g_j(x)}{\beta_j(\omega) + \widetilde{\beta}_j(\omega) g_j(x)} \right)_{j=1}^{k}, \quad (\omega, x) \in \Omega \times X\,,$$

where $\alpha, \widetilde{\alpha}, \beta, \widetilde{\beta}\colon \Omega \to \mathbb{R}^k_{>0}$ and $g\colon \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^k_{\geqslant 0}$ satisfy

(P1′) α, $\widetilde{\alpha}$, β, and $\widetilde{\beta}$ are continuous and uniformly bounded away from zero along the orbit of ω, and satisfy

$$\frac{\alpha_j(\theta_t\omega)}{\beta_j(\theta_t\omega)} \geqslant \frac{\widetilde{\alpha}_j(\theta_t\omega)}{\widetilde{\beta}_j(\theta_t\omega)}, \quad \forall t \in \mathbb{R}\,, \quad j = 1, \ldots, k\,,$$

for θ-almost every $\omega \in \Omega$, and

(P2′) g is continuous, order-preserving, and sublinear.

Then h is anti-monotone, temperedness preserving, and the i/o characteristic $\mathcal{K}^Y$ of $(\theta, \varphi, \mathcal{S}^U_\infty, h)$ satisfies the Small-Gain Condition. This follows along the same lines of Example 4.29, using Proposition D.24. ♢

Example 4.31 (Periodic θ)*.* In Example 4.29, suppose that the underlying MPDS θ is T-periodic; that is, there exists $T > 0$ such that

$$\theta_{t+T}\omega = \theta_t\omega\,, \quad \forall t \in \mathbb{R}\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

Then g need not be bounded in order for the Small-Gain Condition to be satisfied. This follows along the lines of Example 4.29, via Proposition D.28. ♢

4.5.2 Discrete Time

Each of the continuous-time examples above has a discrete-time counterpart. The starting point is Example 3.37. Suppose that θ is a discrete MPDS—that is, $\mathcal{T} = \mathbb{Z}$—, and suppose that $A\colon \Omega \to M_{n\times n}(\mathbb{R})$ and $B\colon \Omega \to M_{n\times k}(\mathbb{R})$ are Borel-measurable maps with θ-almost everywhere nonnegative entries, and satisfying ($l1$) and ($l2$). We shall consider the RdE

$$\xi^+ = A(\theta_n\omega)\xi + B(\theta_n\omega)h(\theta_n\omega, \xi)\,, \quad n \geqslant 0\,, \quad \omega \in \Omega\,, \tag{4.15}$$

for the same classes as nonlinearity $h\colon \Omega \times \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^k_{\geqslant 0}$ as in the continuous-time examples above, applying the Small-Gain Theorem once again to show that, for each such classes, the RDS generated by (4.15) has a unique, globally attracting, positive equilibrium.

We once again equip $\mathbb{R}^n$ and $\mathbb{R}^k$ with the partial orders induced by their respective positive orthant cones. As we saw in Example 3.37,

$$\xi^+ = A(\theta_n\omega)\xi + B(\theta_n\omega)u_n(\omega)$$

generates a tempered RDSI $(\theta, \varphi, \mathcal{U})$ possessing an i/s characteristic $\mathcal{K}\colon U^\Omega_\theta \to X^\Omega_\theta$ given by

$$[\mathcal{K}(u)](\omega) := \sum_{j=-\infty}^{-1} \left(\prod_{l=j+1}^{-1} A(\theta_l\omega) \right) B(\theta_j\omega)u(\theta_j\omega)\,, \quad \forall u \in U^\Omega_\theta\,, \quad \widetilde{\forall}\omega \in \Omega\,.$$

In view of the assumptions that A and B have θ-almost everywhere nonnegative entries, the RDSI is also monotone, which can be shown straight from (3.26).

Example 4.32 (Saturated Readouts). Consider an output function $h\colon \Omega \times X \to U$ defined by

$$h(\omega, x) := \left(\frac{\alpha_j(\omega)}{\beta_j(\omega) + g_j(x)} \right)_{j=1}^{k}\,, \quad (\omega, x) \in \Omega \times X\,,$$

where $\alpha, \beta\colon \Omega \to \mathbb{R}^k_{\geqslant 0}$ and $g\colon \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^k_{\geqslant 0}$ satisfy the following hypotheses:

(p1) α and β are uniformly bounded away from zero and infinity along θ-almost every orbit; more precisely, for θ-almost every $\omega \in \Omega$, there exist an $\epsilon = \epsilon(\omega) \gg 0$ and an $M = M(\omega) \geqslant 0$ such that

$$\epsilon \leqslant \alpha(\theta_m\omega), \beta(\theta_m\omega) \leqslant M\,, \quad \forall m \in \mathbb{Z}\,,$$

and

(p2) g is continuous, order-preserving, sublinear, and bounded.

It follows as in Example 4.29 that h is anti-monotone and temperedness. One can then show, using Proposition D.33, that the i/o characteristic $\mathcal{K}^Y\colon U^\Omega_\theta \to U^\Omega_\theta$ of $(\theta, \varphi, \mathcal{U}, h)$ satisfies the Small-Gain Condition. $\Diamond$

Example 4.33 (Unbounded g)*.* Consider an output function $h\colon \Omega \times X \to U$ defined by

$$h(\omega, x) := \left(\frac{\alpha_j(\omega) + \widetilde{\alpha}_j(\omega) g_j(x)}{\beta_j(\omega) + \widetilde{\beta}_j(\omega) g_j(x)} \right)_{j=1}^{k}, \quad (\omega, x) \in \Omega \times X\,,$$

where $\alpha, \widetilde{\alpha}, \beta, \widetilde{\beta}\colon \Omega \to \mathbb{R}^k_{>0}$ and $g\colon \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^k_{\geqslant 0}$ satisfy

(p1′) α, $\widetilde{\alpha}$, β, and $\widetilde{\beta}$ are uniformly bounded away from zero along the orbit of ω, and satisfy

$$\frac{\alpha_j(\theta_m \omega)}{\beta_j(\theta_m \omega)} \geqslant \frac{\widetilde{\alpha}_j(\theta_m \omega)}{\widetilde{\beta}_j(\theta_m \omega)}\,, \quad \forall m \in \mathbb{Z}\,, \quad j = 1, \ldots, k\,,$$

for θ-almost every $\omega \in \Omega$, and

(p2′) g is continuous, order-preserving, and sublinear.

Then h is anti-monotone, temperedness preserving, and the i/o characteristic $\mathcal{K}^Y$ of $(\theta, \varphi, \mathcal{S}^U_\infty, h)$ satisfies the Small-Gain Condition. This follows along the same lines of Examples 4.29, using Proposition D.34. ♢

Example 4.34 (Periodic θ)*.* In Example 4.32, suppose that the underlying MPDS θ is T-periodic; that is, there exists $T > 0$ such that

$$\theta_{m+T}\omega = \theta_m \omega\,, \quad \forall m \in \mathbb{Z}\,, \quad \widetilde{\forall} \omega \in \Omega\,.$$

Then g need not be bounded in order for the Small-Gain Condition to be satisfied. This follows along the lines of Example 4.29, via Proposition D.38. ♢

Chapter 5

Future Work

There are a few directions in which this research could be advanced.

Applications to Systems Biology

Although this has been the underlying motivation for this work, no concrete biological examples have been carefully examined yet in light of the theory just developed. Thus there is plenty of room for research on this front. The study of concrete examples would greatly inform which directions the development of this theory should take.

Checking the Small-Gain Condition

As illustrated in Examples 4.29–4.34, the Small-Gain Condition can be somewhat difficult to check directly. The space U_θ^Ω on which the i/o characteristic is defined has no obvious underlying topology, making it very difficult to frame the problem of checking the Small-Gain Condition within the context of fixed-point theorems. Indeed, all the examples explicitly considered in this work rely on the construction carried out in Appendix D using the Thompson metric. Thus many interesting examples not fitting within the Thompson metric framework, or not satisfying the required boundedness conditions for it to be applicable, are left out.

With regards to the Thompson metric, we are confident that the constrains in Appendix D could be relaxed, even if just marginally. The work in the Ph.D thesis of Mircea-Dan Rus [47] might might provide some of the tools with which this could be achieved.

Another possible approach would be to look for other topologies with which we could equip U_θ^Ω. Known results from the analogous deterministic theory, as well as hypotheses from the biological sciences, will inform the choice for other classes of examples worthy of scrutiny.

Input to State Stability

Naturally, many of the axioms and definitions in our theory of RDSIO were motivated by and tailored to fit the monotone systems approach to a Small-Gain Theorem. This begs the question, what else could be done with this theory? Looking at RDSIO with other objectives in mind would be the natural step towards consolidating the definitions presented in this work—or finding out how they should perhaps be modified.

"Input to State Stability" (ISS), in the sense of control theory, seems like a good candidate. Besides having many powerful applications to engineering [35], it is well-developed in the deterministic case [51, 53, 31], which could once again be used as a guide. Could an equally fruitful theory of ISS be developed over the same abstract RDSIO framework? In other words, are the definitions sensible outside the context of monotonicity? If not, then what fails? In this case, could the foundations be re-designed so as to accommodate a unified approach to both monotone systems theory and ISS?

Some of these questions are already work in progress. The challenges are not unrelated to the difficulties described above with expanding the realm of applications of the Small-Gain Theorem. The lack of an obvious norm in the space of tempered random variables, with or without partial orders in the picture, makes the definition of 'input to state stability' from the deterministic theory very difficult to translate to RDSI. Nevertheless, we are optimistic about the prospects. Some preliminary analysis of the linear case suggests that we should be able to make some progress by thinking about "random norms;" in other words, looking for at norms as random variables.

Appendix A

The Hausdorff Distance

Recall that the *Hausdorff distance* between two nonempty, bounded subsets A and C of a metric space (X, d) is defined to be the nonnegative real number

$$d_H(A, C) := \max\left\{\sup_{a\in A} \operatorname{dist}(a, C),\ \sup_{c\in C} \operatorname{dist}(c, A)\right\},$$

where

$$\operatorname{dist}(x, E) := \inf_{y\in E} d(x, y)\,, \quad x \in X\,, \quad \varnothing \neq E \subseteq X\,,$$

is the distance between a point and a nonnempty subset of X.

Given a metric space (X, d), we denote the family of nonempty, bounded, closed subsets of X by $F(X)$. When (X, d) is a compact metric space, the restriction $d_H\big|_{F(X)\times F(X)}$ of the Hausdorff distance to $F(X)$ constitutes a metric with respect to which $F(X)$ is also compact (Proposition A.5). This property of the Hausdorff distance was used in Chapter 2 to show that the shell of a compact subset of an RTA space is also compact (Theorem 2.51).

For the reader's convenience, we work out in detail the properties of the Hausdorff metric leading up to Proposition A.5. The proofs follow the presentation in [26], with a few corrections and simplifications. See also [6, Sections 2.6 and 2.7].

Given a metric space (X, d), a point $x \in X$, and an $\epsilon > 0$, we denote the *ball of radius* ϵ *and centered at* x by $B_\epsilon(x)$; in other terms,

$$B_\epsilon(x) := \{y \in X\,;\ d(y, x) < \epsilon\}\,.$$

For a nonempty subset $A \subseteq X$ and an $\epsilon > 0$, we then denote

$$A_\epsilon := \bigcup_{a\in A} B_\epsilon(a)\,.$$

Proposition A.1. *Let (X,d) be a metric space. For any nonempty, bounded subsets $A, C \subseteq X$, we have*

$$d_H(A,C) = \inf\{\epsilon > 0\,;\ A \subseteq C_\epsilon \text{ and } C \subseteq A_\epsilon\}\,.$$

In particular, the Hausdorff distance between any nonempty, bounded subsets $A, C \subseteq X$ is always finite.

Proof. We show that

$$d_H(A,C) \leqslant \inf\{\epsilon > 0\,;\ A \subseteq C_\epsilon \text{ and } C \subseteq A_\epsilon\} \tag{A.1}$$

and

$$d_H(A,C) \geqslant \inf\{\epsilon > 0\,;\ A \subseteq C_\epsilon \text{ and } C \subseteq A_\epsilon\}\,. \tag{A.2}$$

(A.1). Since A and C are bounded by hypothesis, there exists an $\epsilon > 0$ such that $A \subseteq C_\epsilon$ and $C \subseteq A_\epsilon$. Fix any such an ϵ arbitrarily. Then

$$\operatorname{dist}(a,C), \operatorname{dist}(c,A) < \epsilon\,, \quad \forall a \in A\,, \quad \forall c \in C\,.$$

Therefore

$$d_H(A,C) \leqslant \epsilon\,.$$

Taking the infimum on the righthand side of the inequality above over all $\epsilon > 0$ such that $A \subseteq C_\epsilon$ and $C \subseteq A_\epsilon$, we obtain (A.1). In particular, $d_H(A,C)$ is finite.

(A.2). It follows straight from the definition of $d_H(A,C)$ that

$$\operatorname{dist}(a,C), \operatorname{dist}(c,A) \leqslant d_H(A,C)\,, \quad \forall a \in A\,, \quad \forall c \in C\,.$$

Therefore

$$A \subseteq C_{d_H(A,C)+1/n} \quad \text{and} \quad C \subseteq A_{d_H(A,C)+1/n}\,, \quad \forall n \in \mathbb{N}\,,$$

and so

$$d_H(A,C) + \frac{1}{n} \in \{\epsilon > 0\,;\ A \subseteq C_\epsilon \text{ and } C \subseteq A_\epsilon\}\,, \quad \forall n \in \mathbb{N}\,.$$

This establishes (A.2). $\square$

Proposition A.2. *Let (X,d) be a metric space. Then the restriction $d_H\big|_{F(X)\times F(X)}$ of the Hausdorff distance to $F(X)$ is a metric on $F(X)$.*

Proof. It follows straight from the definition that

$$d_H(A, A) = 0\,, \quad \forall A \in F(X)\,,$$

and

$$d_H(A, C) = d_H(C, A)\,, \quad \forall A, C \in F(X)\,.$$

It remains to show that

$$d_H(A, C) > 0\,, \quad \forall A, C \in F(X)\,:\; A \neq C\,, \tag{A.3}$$

and

$$d_H(A, C) \leqslant d_H(A, D) + d_H(D, C)\,, \quad \forall A, C, D \in F(X)\,. \tag{A.4}$$

To that end, fix arbitrarily $A, C, D \in F(X)$.

(A.3). It follows straight from the definition that $d_H(A, C) \geqslant 0$. So (A.3) is equivalent to

$$d_H(A, C) = 0 \quad \Rightarrow \quad A = C\,.$$

Suppose $d_H(A, C) = 0$. Then

$$\mathrm{dist}(a, C) = \mathrm{dist}(c, A) = 0\,, \quad \forall a \in A\,, \quad \forall c \in C\,.$$

In other terms, every point of A is an accumulation point of C, and vice-versa. Since A and C are assumed to be closed, we then have $A \subseteq \bar{C} = C$ and, likewise, $C \subseteq \bar{A} = A$. This yields $A = C$. Since $A, C \in F(X)$ were chosen arbitrarily, this proves (A.3).

(A.4). Pick any $\epsilon_1, \epsilon_2 > 0$ such that $A \subseteq C_{\epsilon_1}$, $C \subseteq A_{\epsilon_1}$, $C \subseteq D_{\epsilon_2}$ and $D \subseteq C_{\epsilon_2}$. Then

$$A \subseteq C_{\epsilon_1} \subseteq D_{\epsilon_1+\epsilon_2} \quad \text{and} \quad D \subseteq C_{\epsilon_2} \subseteq A_{\epsilon_2+\epsilon_1}\,,$$

and so

$$d_H(A, D) \leqslant \epsilon_1 + \epsilon_2$$

by Proposition A.1. Taking the infimum on the righthand side over all such ϵ_1 and ϵ_2, it follows, again from Proposition A.1, that

$$d_H(A, D) \leqslant d_H(A, C) + d_H(C, D)\,.$$

Since $A, C, D \in F(X)$ were chosen arbitrarily, this establishes (A.4), completing the proof of the lemma. $\square$

Recall that a metric space (X, d) is said to be *totally bounded* if, for every $\epsilon > 0$, there exist finitely many points $x_1, \ldots, x_k \in X$ such that

$$X = \bigcup_{j=1}^{k} B_\epsilon(x_j)\,. \tag{A.5}$$

Lemma A.3. *If (X, d) is a totally bounded metric space, then $(F(X), d_H)$ is also totally bounded.*

Proof. Given any $\epsilon > 0$, let $x_1, \ldots, x_k \in X$ be such that (A.5) holds. Let

$$\{C_1, \ldots, C_{2^k-1}\} := 2^{\{x_1,\ldots,x_k\}} \backslash \{\varnothing\}$$

be the family of all $2^k - 1$ nonempty subsets of $\{x_1, \ldots, x_k\}$—the order in which the subsets are labeled is irrelevant. Note that each C_j is finite, hence closed, and so belongs to $F(X)$. We claim that

$$F(X) = \bigcup_{j=1}^{2^k-1} B^H_{2\epsilon}(C_j)\,,$$

where we use the superscript 'H' in '$B^H_{2\epsilon}(C_j)$' just to emphasize that we are referring to a ball in $(F(X), d_H)$.

Given any $A \in F(X)$, we have

$$\varnothing \neq A \subseteq \bigcup_{j=1}^{k} B_\epsilon(x_j)\,,$$

and so

$$A \subseteq C_\epsilon\,,$$

where

$$C := \{x \in \{x_1, \ldots, x_k\}\,;\ B_\epsilon(x) \cap A \neq \varnothing\} \neq \varnothing\,.$$

Since $B_\epsilon(x) \cap A \neq \varnothing$ for each $x \in C$, we also have

$$C \subseteq A_\epsilon\,.$$

It follows from Proposition A.1 that $d_H(A, C) \leqslant \epsilon$, showing that $A \in B^H_{2\epsilon}(C)$ for some $C \in \{C_1, \ldots, C_{2^k-1}\}$.

Since $A \in F(X)$ was chosen arbitrarily, this proves the claim. And since $\epsilon > 0$ was also chosen arbitrarily, this establishes the lemma. $\square$

Lemma A.4. *If (X,d) is a complete metric space, then $(F(X),d_H)$ is also complete.*

Proof. Let $(A_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(F(X),d_H)$. We will show that $(A_n)_{n\in\mathbb{N}}$ converges to

$$A_\infty := \{x \in X\,;\; x \in \overline{\{a_n\}_{n\in\mathbb{N}}} \text{ and } a_n \in A_n\,,\ \forall n \in \mathbb{N}\}\,.$$

We first observe that A_∞ is indeed in $F(X)$. To show that A_∞ is closed, one may employ a "diagonal argument." Let $(x_k)_{k\in\mathbb{N}}$ in A_∞ and $x_\infty \in X$ be such that

$$\lim_{k\to\infty} x_k = x_\infty\,. \tag{A.6}$$

For each $k \in \mathbb{N}$, we have $x_k \in \overline{\{a_n\}_{n\in\mathbb{N}}}$ for some sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n \in A_n$ for each $n \in \mathbb{N}$. Thus we may construct a strictly increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ and a sequence $(\hat{a}_k)_{k\in\mathbb{N}}$ such that

$$\hat{a}_k \in A_{n_k} \quad \text{and} \quad d(\hat{a}_k, x_k) < 1/k\,, \quad \forall k \in \mathbb{N}\,.$$

Now $\hat{a}_k \longrightarrow x_\infty$ as $k \to \infty$, showing that $x_\infty \in D_\infty$. Since $(x_k)_{k\in\mathbb{N}}$ in A_∞ and $x_\infty \in X$ such that (A.6) holds were chosen arbitrarily, this proves that A_∞ is closed. It will follow from the discussion below that A_∞ is also bounded and nonempty.

Fix arbitrarily $\epsilon > 0$, and let $(N_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers with the property that

$$m,n \geqslant N_k \quad \Rightarrow \quad d_H(A_m,A_n) < \epsilon/2^{k+1}\,, \quad \forall k \in \mathbb{N}.$$

We claim that

$$A_\infty \subseteq (A_n)_\epsilon \quad \text{and} \quad A_n \subseteq (A_\infty)_\epsilon\,, \quad \forall n \geqslant N_1\,. \tag{A.7}$$

In light of Proposition A.1, this implies that

$$d_H(A_n, A_\infty) \leqslant \epsilon\,, \quad \forall n \geqslant N_1\,.$$

Since $\epsilon > 0$ is being chosen arbitrarily, this would show that $(A_n)_{n\in\mathbb{N}}$ converges to A_∞. Thus it remains to prove (A.7).

$A_\infty \subseteq (A_n)_\epsilon$. For any $n \geqslant N_1$, we have

$$d_H(A_m, A_n) \leqslant \epsilon/2^{1+1}\,, \quad \forall m \geqslant N_1\,,$$

hence

$$A_m \subseteq (A_n)_{\epsilon/4}\,, \quad \forall m \geqslant N_1\,.$$

Therefore

$$A_\infty \subseteq \overline{(A_n)_{\epsilon/4}} \subseteq (A_n)_\epsilon\,.$$

$A_n \subseteq (A_\infty)_\epsilon$. Fix arbitrarily $n \geqslant N_1$ and $x \in A_n$. We will recursively construct a sequence $(f_k)_{k\in\mathbb{N}}$ with the properties that

$$f_k \in A_{N_k}\,, \quad \forall k \in \mathbb{N}\,,$$

$$d(x, f_1) < \epsilon/2^2\,, \quad \text{and} \quad d(f_k, f_{k+1}) < \epsilon/2^{k+1}\,, \quad \forall k \in \mathbb{N}\,.$$

Since $A_n \subseteq (A_{N_1})_{\epsilon/2^2}$, there exists an $f_1 \in A_{N_1}$ such that $d(x, f_1) < \epsilon/2^2$. Similarly, $A_{N_1} \subseteq (A_{N_2})_{\epsilon/2^2}$. Therefore there exists an $f_2 \in D_{N_2}$ such that $d(f_1, f_2) < \epsilon/2^2$. Now having chosen $f_1, \ldots, f_k$ for which the properties above hold, note that $A_{N_k} \subseteq (A_{N_{k+1}})_{\epsilon/2^{k+1}}$, and then choose $f_{k+1} \in A_{N_{k+1}}$ such that $d(f_k, f_{k+1}) < \epsilon/2^{k+1}$.

Observe that $(f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence, for

$$\begin{aligned} d(f_k, f_{k+l}) &\leqslant d(f_k, f_{k+1}) + \cdots + d(f_{k+l-1}, f_{k+l}) \\ &\leqslant \epsilon/2^{k+1} + \cdots + \epsilon/2^{k+l} \\ &\leqslant \epsilon/2^k\,, \quad \forall k, l \in \mathbb{N}\,. \end{aligned}$$

We are assuming that (X, d) is complete, therefore there exists an $f_\infty \in X$ to which $(f_k)_{k\in\mathbb{N}}$ converges. In particular, $f_\infty \in A_\infty$, and

$$d(x, f_\infty) \leqslant d(x, f_1) + d(f_1, f_\infty) < \epsilon/2^2 + \epsilon/2 < \epsilon\,.$$

This means that $x \in (A_\infty)_\epsilon$. Since $x \in A_n$ was chosen arbitrarily, we conclude that $A_n \subseteq (A_\infty)_\epsilon$.

Since $n \geqslant N_1$ was chosen arbitrarily in each case, the argument above proves (A.7). This completes the proof of the lemma. □

Proposition A.5. *If (X, d) is a compact metric space, then $(F(X), d_H)$ is also compact.*

Proof. Since a metric space is compact if, and only if it is complete and totally bounded [41, Theorem 45.1 on page 276], this follows straight from Lemmas A.3 and A.4. □

Appendix B

Ordinary Differential Equations

In this appendix we review the notation, terminology, and results from the theory of (deterministic) ODE used in the construction of RDSI via RDEI in Subsection 3.4.2.

B.1 Righthand Sides

Before we introduce our class of admissible "righthand sides" $f\colon I \times X \to \mathbb{R}^n$ for a nonautonomous ODE

$$\dot{\xi} = f(t,\xi)\,, \quad t \in I\,,$$

considered over some interval $I \subseteq \mathbb{R}$, we review some properties of Lipschitz functions. The notation introduced in Subsection 3.4.2 shall be tacitly assumed.

Lemma B.1. *Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz, compactly supported map. Then $\|f\|_{\overline{\operatorname{supp} f}} = \|f\|_{\mathbb{R}^n}$. In particular, $\|f\|_X = \|f\|_{\overline{\operatorname{supp} f}}$ whenever $\overline{\operatorname{supp} f} \subseteq X \subseteq \mathbb{R}^n$.*

Proof. Denote $K := \overline{\operatorname{supp} f}$ for short. Since $f(x) = 0$ for any $x \in \mathbb{R}^n \backslash K$, we can readily see that

$$\sup_{x\in K} |f(x)| = \sup_{x\in\mathbb{R}^n} |f(x)|\,.$$

The inequality

$$\sup_{\substack{x,y\in K\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|} \leqslant \sup_{\substack{x,y\in \mathbb{R}^n\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|}\,.$$

follows straight from the inclusion $K \subseteq \mathbb{R}^n$. Thus it remains to show the converse inequality.

If $x', y' \in \mathbb{R}^n \backslash \operatorname{int} K$, then $|f(x') - f(y')| = 0$. So, if $x' \neq y'$, then

$$\frac{|f(x')-f(y')|}{|x'-y'|} = 0 \leqslant \sup_{\substack{x,y\in K\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|}\,.$$

Now suppose that $x' \in \operatorname{int} K$ and $y' \in \mathbb{R}^n \setminus \operatorname{int} K$. Consider the straight line

$$l := \{(1-s)x' + sy' \,;\ 0 \leqslant s \leqslant 1\}$$

joining x' and y'. Since l is connected, there exists a point $z' \in l \cap \partial K$. In particular, z' belongs to K, f vanishes at z', and $|x'-y'| \geqslant |x'-z'|$. Therefore

$$\frac{|f(x')-f(y')|}{|x'-y'|} = \frac{|f(x')|}{|x'-y'|} \leqslant \frac{|f(x')-f(z')|}{|x'-z'|} \leqslant \sup_{\substack{x,y\in K\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|}\,.$$

We conclude that

$$\sup_{\substack{x,y\in \mathbb{R}^n\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|} \leqslant \sup_{\substack{x,y\in K\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|}\,,$$

completing the proof of the result. □

Definition B.2 (Righthand Side)**.** A $(\mathcal{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathcal{B}(\mathbb{R}^n))$-measurable map

$$f\colon \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \mathbb{R}^n$$

is said to be a *righthand side* if

(Q1) $f(t,\cdot)\colon \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz for every $t \geqslant 0$, and

(Q2) for each compact $K \subseteq \mathbb{R}^n$,

$$\int_a^b \|f(t,\cdot)\|_K \, dt < \infty$$

for every $b > a \geqslant 0$. △

Remark B.3. Any $(\mathcal{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathcal{B}(\mathbb{R}^n))$-measurable map

$$f\colon \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \mathbb{R}^n$$

satisfying (Q1) is Carathéodory (recall Definition 2.17). Thus, in the definition above, the measurability of

$$t \longmapsto \|f(t,\cdot)\|_K\,, \quad t \geqslant 0\,,$$

in (Q2) follows directly from Lemma C.2. In other words, property (Q2) is, in effect, only asking that the integral be finite—the requirement that the integrand is measurable is automatically satisfied by the properties that f be $(\mathcal{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathcal{B}(\mathbb{R}^n))$-measurable and satisfies (Q1). □

Recall the smooth bump functions H_k described in Subsection 3.4.2.

Lemma B.4. *If* $f\colon \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \mathbb{R}^n$ *is a righthand side, then*

$$(t,x) \longmapsto H_k(x) f(t,x)\,, \quad (t,x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$$

is also a righthand side for any positive integer k*. In particular,*

$$\|H_k(\cdot) f(t,\cdot)\|_{\mathbb{R}^n} = \|H_k(\cdot) f(t,\cdot)\|_{\overline{B}_k(0)}\,, \quad \forall t \geqslant 0\,.$$

Proof. Measurability follows from the well-known fact that the product of measurable functions is measurable.

Fix arbitrarily a positive integer k and a compact $K \subseteq \mathbb{R}^n$. For any $x, y \in K$ such that $x \neq y$, we have

$$\begin{aligned} \frac{|H_k(x)f(t,x) - H_k(y)f(t,y)|}{|x-y|} &\leqslant |H_k(x)|\frac{|f(t,x)-f(t,y)|}{|x-y|} \\ &\quad + \frac{|H_k(x)-H_k(y)|}{|x-y|}|f(t,y)| \\ &\leqslant (1+L_k)\|f(t,\cdot)\|_K\,, \end{aligned}$$

therefore $H_k(\cdot)f(t,\cdot)$ is locally Lipschitz for each $t \geqslant 0$. Since $K \subseteq \mathbb{R}^n$ compact was chosen arbitrarily, this establishes (R1).

Taking the supremum over all distinct x and y in an arbitrary compact $K \subseteq \mathbb{R}^n$, we obtain

$$\|H_k(\cdot)f(t,\cdot)\|_K \leqslant (1+L_k)\|f(t,\cdot)\|_K\,, \quad \forall t \geqslant 0\,.$$

Thus

$$t \longmapsto \|H_k(\cdot)f(t,\cdot)\|_K\,, \quad t \geqslant 0\,,$$

is locally integrable by comparison. This holds for any compact subset $K \subseteq \mathbb{R}^n$, thus establishing (R2).

Since the positive integer k was chosen arbitrarily, this proves the lemma.

The last statement follows straight from Lemma B.1. $\square$

B.2 Global Solutions

Lemma B.5 (Local Uniqueness)**.** *Suppose that* $f\colon \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \mathbb{R}^n$ *is a righthand side. Then for each* $x \in \mathbb{R}^n$*, the initial value problem*

$$\dot{\xi} = f(t, \xi)\,, \quad 0 \leqslant t < \tau\,, \quad \xi(0) = x\,, \tag{B.1}$$

has at most one solution.

Proof. This follows along the lines of the Uniqueness Theorem of Giuliano [1, Theorem 3.5.1 on page 143].

Suppose that $\xi, \zeta\colon [0,\tau) \to \mathbb{R}^n$ are solutions of (B.1); in other words, ξ and ζ are absolutely continuous, and

$$\frac{d\xi}{dt}(t) = f(t, \xi(t)) \quad \text{and} \quad \frac{d\zeta}{dt}(t) = f(t, \zeta(t)) \tag{B.2}$$

for Lebesgue-almost every $t \in [0, \tau)$. Fix arbitrarily $\tau' \in [0, \tau)$. We may choose $b \geqslant 0$ such that

$$|\xi(t) - x| \leqslant b \quad \text{and} \quad |\zeta(t) - x| \leqslant b\,, \quad \forall t \in [0, \tau']\,.$$

Set $K := \overline{B}_b(x)$.

Suppose on the contrary that $\xi\big|_{[0,\tau']} \neq \zeta\big|_{[0,\tau']}$. By continuity, the set

$$\{t \in [0, \tau']\,;\; \xi(t) \neq \zeta(t)\}$$

is open, and therefore it can be expressed as the countable union of disjoint open intervals (relative to $[0, \tau']$). Let $a < b$ be the endpoints of any such interval. Thus $\xi(a) = \zeta(a)$ and $\xi(t) \neq \zeta(t)$ for all $t \in (a, b)$. Set $\psi\colon (a, b) \to \mathbb{R}_{>0}$ by

$$\psi(t) := |\xi(t) - \zeta(t)|^2 = \langle \xi(t) - \zeta(t), \xi(t) - \zeta(t)\rangle\,, \quad t \in (a, b)\,.$$

Then ψ is absolutely continuous and $\psi(t) > 0$ for all $t \in (a, b)$. We shall derive a contradiction by showing that $\psi \equiv 0$.

By (B.2) and the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}
\frac{d\psi}{dt}(t) &= 2\langle f(t, \xi(t)) - f(t, \zeta(t)), \xi(t) - \zeta(t)\rangle \\
&\leqslant 2|f(t, \xi(t)) - f(t, \zeta(t))| \cdot |\xi(t) - \zeta(t)| \\
&\leqslant 2\|f(t, \cdot)\|_K \cdot |\xi(t) - \zeta(t)|^2\,,
\end{aligned}$$

hence

$$\frac{1}{\psi(t)}\frac{d\psi}{dt}(t) \leqslant 2\|f(t,\cdot)\|_K$$

for Lebesgue-almost every $t \in (a,b)$. It now follows by a change of variables that

$$\int_{a'}^{b'} \frac{1}{\psi(t)}\frac{d\psi}{dt}(t)\,dt = \int_{\psi(a')}^{\psi(b')} \frac{1}{u}\,du \leqslant \int_{a'}^{b'} 2\|f(t,\cdot)\|_K\,dt$$

whenever $a < a' < b' < b$. Fix arbitrarily $b' \in (a,b)$. We have

$$\lim_{a' \to a^+} \int_{a'}^{b'} 2\|f(t,\cdot)\|_K\,dt = \int_{a}^{b'} 2\|f(t,\cdot)\|_K\,dt < \infty$$

by (Q2), while, on the other hand,

$$\lim_{a' \to a^+} \int_{\psi(a')}^{\psi(b')} \frac{1}{u}\,du = \infty\,,$$

since $\psi(a') \longrightarrow 0$ as $a' \to a^+$. This is a contradiction.

We conclude that $\xi\big|_{[0,\tau']} = \zeta\big|_{[0,\tau']}$. Since $\tau' \in [0,\tau)$ was chosen arbitrarily, this completes the proof that $\xi = \zeta$. $\square$

Lemma B.6 (Gronwall's Inequality)**.** *Suppose that* $\alpha\colon [\tau_1,\tau_2] \to \mathbb{R}_{\geqslant 0}$ *is integrable,* $\mu\colon [\tau_1,\tau_2] \to \mathbb{R}_{\geqslant 0}$ *is continuous,* $c \geqslant 0$*, and*

$$\mu(t) \leqslant c + \int_{\tau_1}^{t} \alpha(s)\mu(s)\,ds\,, \quad \forall t \in [\tau_1,\tau_2]\,.$$

Then

$$\mu(t) \leqslant c\,\mathrm{e}^{\int_{\tau_1}^{t} \alpha(s)\,ds}\,, \quad \forall t \in [\tau_1,\tau_2]\,.$$

Proof. See [52, Lemma C.3.1 on page 475]. $\square$

Proposition B.7 (Global Solutions for ODE)**.** *Suppose that* $f\colon \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \mathbb{R}^n$ *is a righthand side satisfying the growth condition*

$$|f(t,x)| \leqslant \alpha(t)|x| + \beta(t)\,, \quad \forall t \geqslant 0\,, \quad \forall x \in \mathbb{R}^n\,, \tag{B.3}$$

for some locally integrable functions $\alpha, \beta\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$*. Then the ODE*

$$\dot{\xi} = f(t,\xi)\,, \quad t \geqslant 0\,, \tag{B.4}$$

generates a continuous global flow $\varphi\colon \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \mathbb{R}^n$*, uniquely determined by the properties that, for each* $x \in \mathbb{R}^n$*,*

$$\varphi(0, x) = x\,,$$

$\varphi(\cdot, x)\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}^n$ *is absolutely continuous, and*

$$\frac{d}{dt}\varphi(t, x) = f(t, \varphi(t, x))$$

for Lebesgue-almost every $t \geqslant 0$*.*

Proof. Local existence follows from the Existence Theorem of Carathéodory, since

$$|f(t, x)| \leqslant \|f(t, \cdot)\|_K\,, \quad \forall (t, x) \in \mathbb{R}_{\geqslant 0} \times K\,,$$

and

$$t \longmapsto \|f(t, \cdot)\|_K\,, \quad t \geqslant 0\,,$$

is locally integrable (by (Q2)) for every compact $K \subseteq \mathbb{R}^n$ (see, for instance, [10, Theorem 1.1 on page 43]). Local uniqueness follows from Lemma B.5 above.

It follows from growth condition (B.3) and Lemma B.6 that no solution of (B.4) blows up in finite time. Thus maximal solutions are globally defined.

Continuity with respect to t and x can also be patched up from the hypotheses by means of Lemma B.6. Combining (B.3) and Lemma B.6, one can show that φ is bounded on $[0, T] \times K$ for finite $T \geqslant 0$ and compact $K \subseteq \mathbb{R}^n$. Another application of Lemma B.6 to the righthand side of

$$|\varphi(t_1, x_1) - \varphi(t_2, x_2)| \leqslant |\varphi(t_1, x_1) - \varphi(t_1, x_2)| + |\varphi(t_1, x_2) - \varphi(t_2, x_2)|$$

yields local continuity, completing the proof. $\square$

A standard technique for establishing measurability is to realize the function one is trying to show to be measurable as the limit of a sequence of maps which are known to be measurable. The result below provides a "canonical" way of realizing the flow of an ODE as the limit of a sequence defined recursively starting from a constant map, in the case when the righthand side is compactly supported "uniformly in t." This, together with the the measurability properties discussed in the next appendix, is the key ingredient in the verification of property (I1) for RDSI generated by θ-righthand sides in the proof of Theorem 3.42.

Theorem B.8 (Canonical Limit Representation)**.** *Suppose that* $f\colon \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \mathbb{R}^n$ *is a righthand side which is compactly supported uniformly in* $t \in \mathbb{R}_{\geqslant 0}$*; that is, such that*

$$\overline{\operatorname{supp} f(t,\cdot)} \subseteq K\,, \quad \forall t \geqslant 0\,,$$

for some compact $K \subseteq \mathbb{R}^n$*. Let* $\varphi\colon \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \mathbb{R}^n$ *be the global flow generated by the ODE*

$$\dot{\xi} = f(t,\xi)\,, \quad t \geqslant 0\,. \tag{B.5}$$

Then

$$\varphi(t,x) = \lim_{m\to\infty} \varphi_m(t,x)\,, \quad \forall (t,x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n\,, \tag{B.6}$$

where $(\varphi_m)_{m\in\mathbb{Z}_{\geqslant 0}}$ *is the sequence of* $\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \mathbb{R}^n$ *maps defined recursively by*

$$\varphi_0(t,x) := x\,, \quad (t,x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n\,,$$

and

$$\varphi_m(t,x) := x + \int_0^t f(s,\varphi_{m-1}(s,x))\,ds\,, \quad (t,x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n\,, \quad m \geqslant 1\,.$$

Proof. First note that f satisfies growth condition (B.3) with $\alpha = 0$ and $\beta = \|f(\cdot,\cdot)\|_K$. Thus (B.5) indeed generates a global flow φ by Proposition B.7.

Fix arbitrarily $x \in \mathbb{R}^n$. By Lemma B.1,

$$\|f(t,\cdot)\|_{\mathbb{R}^n} = \|f(t,\cdot)\|_K < \infty\,, \quad \forall t \geqslant 0\,.$$

Consider the primitive

$$\begin{array}{rcl} F\colon \mathbb{R}_{\geqslant 0} & \longrightarrow & \mathbb{R}_{\geqslant 0} \\ t & \longmapsto & \displaystyle\int_0^t \|f(s,\cdot)\|_{\mathbb{R}^n}\,ds \end{array}.$$

We will show by induction on $m \in \mathbb{N}$ that

$$|\varphi_m(t,x) - \varphi_{m-1}(t,x)| \leqslant \frac{[F(t)]^m}{m!}\,, \quad \forall t \geqslant 0\,, \quad \forall m \in \mathbb{N}\,. \tag{B.7}$$

For $m = 1$, we have

$$\begin{aligned}
|\varphi_1(t,x) - \varphi_0(t,x)| &\leqslant \int_0^t |f(s,x)|\,ds \\
&\leqslant \int_0^t \|f(s,\cdot)\|_{\mathbb{R}^n}\,ds \\
&= \frac{[F(t)]^1}{1!}\,, \quad \forall t \geqslant 0\,.
\end{aligned}$$

Now suppose (B.7) has been shown to hold for $m = 1, \ldots, k$, for some $k \geqslant 1$. Then

$$\begin{aligned}
|\varphi_{k+1}(t,x) - \varphi_k(t,x)| &\leqslant \int_0^t |f(s,\varphi_k(s,x)) - f(s,\varphi_{k-1}(s,x))|\,ds \\
&\leqslant \int_0^t \|f(s,\cdot)\|_{\mathbb{R}^n} |\varphi_k(s,x) - \varphi_{k-1}(s,x)|\,ds \\
&\leqslant \int_0^t F'(s)\frac{[F(s)]^k}{k!}\,ds \\
&= \frac{[F(t)]^{k+1}}{(k+1)!}\,, \quad \forall t \geqslant 0\,.
\end{aligned}$$

This completes the induction argument, establishing (B.7).

Now fix arbitrarily $T \geqslant 0$. We abuse notation and denote the restrictions $\varphi\big|_{[0,T]\times\{x\}}$ and $\varphi_m\big|_{[0,T]\times\{x\}}$ by φ and φ_m, respectively. Since F is nondecreasing, we have

$$\begin{aligned}
|\varphi_{m+k}(t,x) - \varphi_m(t,x)| &\leqslant \sum_{j=m+1}^{m+k} |\varphi_j(t,x) - \varphi_{j-1}(t,x)| \\
&\leqslant \sum_{j=m+1}^{m+k} \frac{[F(T)]^j}{j!}\,, \quad \forall t \in [0,T]\,, \quad \forall m \geqslant 0\,, \quad \forall k > 0\,.
\end{aligned}$$

Since the series

$$\sum_{j=0}^{\infty} \frac{[F(T)]^j}{j!} = \mathrm{e}^{F(T)}$$

converges, we conclude that $(\varphi_m)_{m\in\mathbb{Z}_{\geqslant 0}}$ is a Cauchy sequence. Let $\tilde{\varphi}\colon [0,T] \to \mathbb{R}^n$ be the limit,

$$\tilde{\varphi}(t) := \lim_{m\to\infty} \varphi_m(t,x)\,, \quad t \in [0,T]\,.$$

By the continuity hypothesis in (Q1), it follows that

$$\lim_{m\to\infty} f(t,\varphi_{m-1}(t,x)) = f(t,\tilde{\varphi}(t))\,, \quad \forall t \in [0,T]\,.$$

Since

$$|f(t,\varphi_{m-1}(t,x))| \leqslant \|f(t,\cdot)\|_{\mathbb{R}^n}\,, \quad \forall t \in [0,T]\,, \quad \forall m \in \mathbb{N}\,,$$

it then follows from Lebesgue Dominated Convergence Theorem that

$$\tilde{\varphi}(t) = \int_0^t f(s, \tilde{\varphi}(t))\, ds\,, \quad \forall t \in [0, T]\,.$$

By Lemma B.5, $\varphi = \tilde{\varphi}$.

Since $x \in \mathbb{R}^n$, and then $T \geqslant 0$ were chosen arbitrarily, this proves (B.6), completing the proof of the theorem. $\square$

Appendix C

Measure and Integration

Though important, interesting and often nontrivial, measurability issues are usually a distraction from the main ideas when one is talking about random dynamical systems. Thus we collect less notable technicalities concerning measurability into this appendix. We refer to the definitions and notational conventions laid down in Chapter 2.

C.1 Carathéodory Functions

Recall the definition of Carathéodory functions—functions which are measurable with respect to the first variable and continuous with respect to the second (Definition 2.17). We derive a couple of properties posessed by such functions.

Lemma C.1. *Let $(T, \mathcal{F})$ be a measurable space, X be a separable topological space, and $f\colon T \times X \to \mathbb{R}$ be a Carathéodory function. Then the function $F\colon T \to \overline{\mathbb{R}}$ defined by*

$$F(t) := \sup_{x \in X} f(t, x)\,, \quad t \in T\,,$$

is $\mathcal{F}$-measurable.

Proof. Let $(x_k)_{k\in\mathbb{N}}$ be a dense sequence in X. Since f is assumed to be Carathéodory, the projection map $f(\cdot, x_k)\colon T \to \mathbb{R}$ is measurable for each $k \in \mathbb{N}$. Therefore it is enough to show that

$$F(t) = \sup_{k\in\mathbb{N}} f(t, x_k)\,, \quad \forall t \in T\,. \tag{C.1}$$

Fix $t \in T$ arbitrarily, and let $(y_m)_{m\in\mathbb{N}}$ be a sequence in X such that

$$F(t) = \lim_{m\to\infty} f(t, y_m)\,.$$

For each $m \in \mathbb{N}$, pick $k_m \in \mathbb{N}$ such that

$$|f(t, x_{k_m}) - f(t, y_m)| < 1/m\,.$$

This is possible since $f(t,\cdot)$ is continuous and $(x_k)_{k\in\mathbb{N}}$ is dense in X. Then

$$\lim_{m\to\infty} f(t,x_{k_m}) = \lim_{m\to\infty} f(t,y_m) = F(t)\,.$$

Since $f(t,x_{k_m}) \leqslant F(t)$ for every $m\in\mathbb{N}$, this proves (C.1). And since $t\in T$ was chosen arbitrarily, this proves the result. □

Lemma C.2. *Suppose that $f\colon \mathbb{R}_{\geqslant 0}\times\mathbb{R}^n \to \mathbb{R}^n$ is Carathéodory, and let X be any nonempty subset of $\mathbb{R}^n$. Then the $\mathbb{R}_{\geqslant 0}\to\overline{\mathbb{R}}$ map defined by*

$$t \longmapsto \|f(t,\cdot)\|_X\,,\quad t\geqslant 0\,,$$

is $\mathcal{B}(\mathbb{R}_{\geqslant 0})$-measurable.

Proof. Since X is separable and $|f|$ is also Carathéodory, it follows straight from Lemma C.1 that

$$t \longmapsto \sup_{x\in X} |f(t,x)|\,,\quad t\geqslant 0\,,$$

is $\mathcal{B}(\mathbb{R}_{\geqslant 0})$-measurable.

Now set

$$C := (X\times X)\backslash\{(x,y)\in X\times X\,;\; x=y\}\,.$$

Then

$$(t,(x,y)) \longmapsto \frac{|f(t,x)-f(t,y)|}{|x-y|}\,,\quad (t,(x,y))\in\mathbb{R}_{\geqslant 0}\times C\,,$$

is also Carathéodory. Furthermore, $C\subseteq\mathbb{R}^{2n}$ is separable. Thus, again from Lemma C.1,

$$t \longmapsto \sup_{\substack{x,y\in X\\ x\neq y}} \frac{|f(t,x)-f(t,y)|}{|x-y|} = \sup_{(x,y)\in C} \frac{|f(t,x)-f(t,y)|}{|x-y|}\,,\quad t\geqslant 0\,,$$

is $\mathcal{B}(\mathbb{R}_{\geqslant 0})$-measurable.

Putting these two together, we conclude that

$$t \longmapsto \|f(t,\cdot)\|_X = \sup_{x\in X}|f(t,x)| + \sup_{\substack{x,y\in X\\ x\neq y}} \frac{|f(t,x)-f(t,y)|}{|x-y|}\,,\quad t\geqslant 0\,,$$

is $\mathcal{B}(\mathbb{R}_{\geqslant 0})$-measurable. □

C.2 Products of Measurable Spaces

This section is devoted to establishing Proposition C.7, which was applied in Subsection 3.4.2 to the study of measurability properties of RDSI generated by RDEI.

The following result is a standard fact from product measures, stated here for the reader's convenience.

Lemma C.3. *Let $(X,\mathcal{F}),(Y,\mathcal{G}),(Z,\mathcal{H})$ be measurable spaces.*

(a) *If $E \in \mathcal{F}\otimes\mathcal{G}$, then*

$$E_x := \{y \in Y\,;\ (x,y) \in E\} \in \mathcal{G}\,, \quad \forall x \in X\,,$$

and likewise

$$E^y := \{x \in X\,;\ (x,y) \in E\} \in \mathcal{F}\,, \quad \forall y \in Y\,.$$

(b) *If $f\colon X\times Y \to Z$ is an $(\mathcal{F}\otimes\mathcal{G})$-measurable map, then the projection maps $f_x\colon Y \to Z$ and $f^y\colon X \to Z$, defined by*

$$f_x(y) := f(x,y) =: f^y(x)\,, \quad (x,y) \in X \times Y\,,$$

are, respectively, $\mathcal{G}$-measurable and $\mathcal{F}$-measurable for all $x \in X$ and all $y \in Y$.

Proof. See [7, Lemma 10.6 on page 116]. □

Lemma C.4. *Suppose that $(X,\mathcal{F})$ is a measurable space, and that $A \in \mathcal{B}(\mathbb{R}_{\geqslant 0})\otimes\mathcal{F}$. Then*

$$\begin{aligned} F\colon \mathbb{R}_{\geqslant 0}\times X &\longrightarrow \mathbb{R}_{\geqslant 0} \\ (t,x) &\longmapsto \int_0^t \chi_A(s,x)\,ds \end{aligned}$$

is $(\mathcal{B}(\mathbb{R}_{\geqslant 0})\otimes\mathcal{F})$-measurable.

Proof. Since $\mathbb{R}_{\geqslant 0}$ is separable, it is enough to show that F is Carathéodory [27, Proposition 1.6 on page 142].

Continuity with respect to $t \in \mathbb{R}_{\geqslant 0}$. Fix arbitrarily $x \in X$. From Lemma C.3,

$$A^x := \{s \in \mathbb{R}_{\geqslant 0}\,;\ (s,x) \in A\} \in \mathcal{B}(\mathbb{R}_{\geqslant 0})\,,$$

therefore

$$s \longmapsto \chi_A(s,x) = \chi_{A^x}(s)\,, \quad s \in \mathbb{R}_{\geqslant 0}\,, \tag{C.2}$$

is measurable. Thus (C.2) is indeed locally integrable, from which it then follows that $F(\cdot, x)$ is continuous.

$\mathcal{F}$-masurability with respect to $x \in X$. Fix arbitrarily $t \in \mathbb{R}_{\geqslant 0}$, and denote the restriction to $[0,t]$ of the Borel measure on $\mathcal{B}(\mathbb{R}_{\geqslant 0})$ by μ_t. Then

$$F(t,x) = \int_0^t \chi_{A^x}(s)\,ds = \mu_t(A^x)\,, \quad \forall x \in X\,.$$

Upon defining an arbitrary finite measure on $(X, \mathcal{F})$—say, the atomic measure on an arbitrarily chosen point of X—, we may then apply [7, Lemma 10.8 on page 117] to conclude that $F(t,\cdot)$ is $\mathcal{F}$-measurable. □

Corollary C.5. *Suppose that $(X, \mathcal{F})$ is a measurable space, and that*

$$f \colon \mathbb{R}_{\geqslant 0} \times X \longrightarrow \mathbb{R}_{\geqslant 0}$$

is a simple function. Then

$$\begin{aligned} F \colon \mathbb{R}_{\geqslant 0} \times X &\longrightarrow \mathbb{R}_{\geqslant 0} \\ (t,x) &\longmapsto \int_0^t f(s,x)\,ds \end{aligned}$$

is also $(\mathcal{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathcal{F})$-measurable.

Proof. This follows straight from Lemma C.4. Upon rewriting

$$f = \sum_{j=1}^{k} a_j \chi_{A_j}$$

for some $a_1, \ldots, a_k \geqslant 0$ and some $A_1, \ldots, A_k \in \mathcal{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathcal{F}$, we obtain

$$F(t,x) = \sum_{j=1}^{k} a_j \int_0^t \chi_{A_j}(s,x)\,ds\,, \quad \forall (t,x) \in \mathbb{R}_{\geqslant 0} \times X\,.$$

So, F is a linear combination of $(\mathcal{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathcal{F})$-measurable functions, and thus itself $(\mathcal{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathcal{F})$-measurable. □

Corollary C.6. *Suppose that* $(X, \mathcal{F})$ *is a measurable space, and that*

$$f \colon \mathbb{R}_{\geqslant 0} \times X \longrightarrow \mathbb{R}_{\geqslant 0}$$

is a $(\mathcal{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathcal{F})$*-measurable function such that* $f(\cdot, x) \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ *is locally integrable for each* $x \in X$*. Then*

$$\begin{aligned} F \colon \mathbb{R}_{\geqslant 0} \times X &\longrightarrow \mathbb{R}_{\geqslant 0} \\ (t, x) &\longmapsto \int_0^t f(s, x)\, ds \end{aligned}$$

is also $(\mathcal{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathcal{F})$*-measurable.*

Proof. Let $(f_m)_{m \in \mathbb{N}}$ be a nondecreasing sequence of simple functions

$$\mathbb{R}_{\geqslant 0} \times X \longrightarrow \mathbb{R}_{\geqslant 0}$$

which converges pointwise to f. Set

$$\begin{aligned} F_m \colon \mathbb{R}_{\geqslant 0} \times X &\longrightarrow \mathbb{R}^n \\ (t, x) &\longmapsto \int_0^t f_m(s, x)\, ds \end{aligned} \quad , \qquad m = 1, 2, 3, \ldots .$$

By Corollary C.5, each F_m is $(\mathcal{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathcal{F})$-measurable. By the Monotone Convergence Theorem,

$$\begin{aligned} F(t, x) &= \int_0^t f(s, x)\, ds \\ &= \lim_{m \to \infty} \int_0^t f_m(s, x)\, ds \\ &= \lim_{m \to \infty} F_m(t, x)\,, \quad \forall (t, x) \in \mathbb{R}_{\geqslant 0} \times X\,. \end{aligned}$$

As the limit of a sequence of $(\mathcal{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathcal{F})$-measurable maps, F is $(\mathcal{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathcal{F})$-measurable. □

Proposition C.7. *Suppose that* $(X, \mathcal{F})$ *is a measurable space, and that*

$$f \colon \mathbb{R}_{\geqslant 0} \times X \longrightarrow \mathbb{R}^n$$

is a $(\mathcal{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathcal{F})$*-measurable function such that* $f(\cdot, x) \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}^n$ *is locally integrable for each* $x \in X$*. Then*

$$\begin{aligned} F \colon \mathbb{R}_{\geqslant 0} \times \Omega &\longrightarrow \mathbb{R}^n \\ (t, x) &\longmapsto \int_0^t f(s, x)\, ds \end{aligned}$$

is also $(\mathcal{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathcal{F})$*-measurable.*

Proof. Indeed, write

$$f = (f_1, \dots, f_n) = (f_1^+ - f_1^-, \dots, f_n^+ - f_n^-)\,,$$

where $f_j^+ := \max\{0, f_j\}$ and $f_j^- := \max\{-f_j, 0\}$ are, respectively, the positive and negative parts of each coordinate f_j of f. Then

$$F(t,x) \equiv \left(\int_0^t f_1^+(s,x)\,ds - \int_0^t f_1^-(s,x)\,ds, \dots, \int_0^t f_n^+(s,x)\,ds - \int_0^t f_n^-(s,x)\,ds \right).$$

Since $f_1^+, f_1^-, \dots, f_n^+, f_n^-$ satisfy the hypotheses of Corollary C.6, this shows that F is $(\mathcal{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathcal{F})$-measurable. □

C.3 θ-Stochastic Processes

Lemma C.8. *Suppose θ is an MPDS and X is a topological space. If $q\colon \Omega \to X$ is Borel-measurable, then*

$$\begin{aligned} \bar{q}\colon \mathcal{T} \times \Omega &\longrightarrow X \\ (t,\omega) &\longmapsto q(\theta_t \omega) \end{aligned}$$

is $(\mathcal{B}(\mathcal{T}) \otimes \mathcal{F})$-measurable.

Proof. Indeed, given any Borel subset $B \subseteq X$, we have

$$(\bar{q})^{-1}(B) = \theta^{-1}(q^{-1}(B))\,.$$

Now $q^{-1}(B) \in \mathcal{F}$ by hypothesis. So $\theta^{-1}(q^{-1}(B)) \in \mathcal{B}(\mathcal{T}) \otimes \mathcal{F}$ by the measurability properties of an MPDS (Definition 2.1). We conclude that $(\bar{q})^{-1}(B) \in \mathcal{B}(\mathcal{T}) \otimes \mathcal{F}$. Since $B \in \mathcal{B}(X)$ was arbitrary, this proves $\bar{q}$ is a θ-stochastic process. □

Corollary C.9. *Under the same assumptions as in* Lemma C.8*,*

$$t \longmapsto q(\theta_t \omega)\,, \quad t \in \mathcal{T}\,,$$

defines a $\mathcal{B}(\mathcal{T})$-measurable map for every $\omega \in \Omega$.

Proof. This follows from combining Lemma C.8 with Lemma C.3(b). □

Lemma C.10. *The pullback of a θ-stochastic process is a θ-stochastic process.*

Proof. Given a θ-stochastic process $q\colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$, we may look at its pullback $\check{q}$ as the composition of q with the map

$$(t,\omega) \longmapsto (t, \theta_{-t}\omega)\,, \quad (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,. \tag{C.3}$$

The projection

$$(t,\omega) \longmapsto \theta_{-t}\omega\,, \quad (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,, \tag{C.4}$$

is the composition of $\theta\colon \mathcal{T} \times \Omega \to \Omega$ with

$$(t,\omega) \longmapsto (-t, \omega)\,, \quad (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,. \tag{C.5}$$

Since θ and (C.5) are measurable, so is (C.4). The first coordinate of (C.3) is readily seen to be measurable. We conclude that (C.3) is measurable.

This shows that $\check{q}$ is the composition of measurable maps. We conclude that $\check{q}$ is measurable. □

Lemma C.11. *The ρ-shift of a θ-stochastic process is a θ-stochastic process.*

Proof. Fix arbitrarily a θ-stochastic process $q\colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$ and an $s \geqslant 0$. Then $\rho_s(q)$ is the composition of q with

$$(t,\omega) \longmapsto (t+s, \theta_s\omega)\,, \quad (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,,$$

both coordinates of which are readily seen to be measurable. This shows that $\rho_s(q)$ is measurable. □

Lemma C.12. *The θ-concatenation of θ-stochastic processes is a θ-stochastic process.*

Proof. Fix arbitrarily θ-stochastic processes $q, r\colon \mathcal{T}_{\geqslant 0} \times \Omega \to X$, and an $s \geqslant 0$. Note that

$$(q \diamondsuit_s r)_t(\omega) = \begin{cases} q_t(\omega)\,, & 0 \leqslant t < s \\ [\rho_s(r)]_t(\omega)\,, & s \leqslant t \end{cases}\,, \quad \forall (t,\omega) \in \mathcal{T}_{\geqslant 0} \times \Omega\,.$$

Fix arbitrarily an $A \in \mathcal{B}$. Since q is measurable by hypothesis and $\rho_s(r)$ is measurable by Lemma C.11, we have

$$q^{-1}(A) \in \mathcal{B}(\mathcal{T}_{\geqslant 0}) \otimes \mathcal{F} \quad \text{and} \quad [\rho_s(r)]^{-1}(A) \in \mathcal{B}(\mathcal{T}_{\geqslant 0}) \otimes \mathcal{F}\,.$$

Thus

$$(q \Diamond_s r)^{-1}(A) = \big(q^{-1}(A) \cap [0,s) \times \Omega\big) \cup \big([s,\infty) \times \Omega\big) \in \mathcal{B}(\mathcal{T}_{\geqslant 0}) \otimes \mathcal{F}\,.$$

Since $A \in \mathcal{B}$ was chosen arbitrarily, this completes the proof. $\square$

Appendix D

The Thompson Metric

In Definition 2.39 we defined cones in the context of topological vector spaces and asked that they be closed with respect to the underlying topology. This requirement guarantees that the partial order induced by the cone is compatible with the topology of the space, in the sense that

$$\lim_{n\to\infty} x_n \leqslant \lim_{n\to\infty} y_n \tag{D.1}$$

whenever $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are sequences such that

$$x_n \leqslant y_n\,, \quad \forall n \in \mathbb{N}\,,$$

and their limits exist. In this appendix we will drop the requirement that the cone be closed. In fact, we will (at first) ignore the underlying topological structure of the vector space altogether, and introduce a metric which depends solely on the relationship between the partial order and the linear structure. This will provide us with a tool to look for fixed points of operators defined on partially ordered vector spaces arising without any obvious underlying topology.

D.1 Basic Definitions

Definition D.1 (Algebraic Cone)**.** Let V be a real vector space. An (*algebraic*) *cone in* V is a nonempty subset $V_+ \subseteq V$ such that

(1) $V_+ + V_+ \subseteq V_+$,

(2) $\alpha V_+ \subseteq V_+$ for every $\alpha > 0$, and

(3) $V_+ \cap (-V_+) \subseteq \{0\}$. △

An algebraic cone induces a partial order in the underlying vector space just like in Definition 2.40. As before, we will abuse the terminology by often referring to any vector in V_+ as *nonnegative.* Although it no longer makes sense to wonder about when (D.1) may hold, it follows from axioms (1) and (2) in Definition D.1 that this partial order is still compatible with the linear structure of the vector space, in the sense that

$$x \leqslant y \quad \text{and} \quad x' \leqslant y' \quad \Leftrightarrow \quad x + x' \leqslant y + y' \,,$$

$$x \leqslant y \quad \Leftrightarrow \quad tx \leqslant ty \,, \quad \forall t > 0 \,,$$

and

$$x \leqslant y \quad \Leftrightarrow \quad tx \geqslant ty \,, \quad \forall t < 0 \,.$$

Throughout this appendix this will be our working definition of 'cone.' Whenever the vector space in question is also equipped with a topology, and the cone is closed with respect to the this topology, we shall explicitly state so.

Let V be a real vector space which is partially ordered by a cone $V_+ \subseteq V$. Then

$$x, y \in V_+ \,, \quad x \sim y \quad \Leftrightarrow \quad \exists c > 0 : \quad c^{-1}x \leqslant y \leqslant cx$$

defines an equivalence relation in V_+. This equivalence relation partitions V_+ into its *parts.* Note that one of the parts is the singleton $C_0 := \{0\}$, consisting of just the origin. We shall refer to all other parts as the *nonzero parts of* V_+, and define a metric on each of them.

Example D.2 (The Parts of $\mathbb{R}^n_{\geqslant 0} \subseteq \mathbb{R}^n$). The parts of the cone $\mathbb{R}_{\geqslant 0} \subseteq \mathbb{R}$ are

$$\{0\} \quad \text{and} \quad \mathbb{R}_{>0} \,.$$

The parts of $\mathbb{R}^2_{\geqslant 0} \subseteq \mathbb{R}^2$ are

$$\{0\} \,, \quad \{0\} \times \mathbb{R}_{>0} \,, \quad \mathbb{R}_{>0} \times \{0\} \,, \quad \text{and} \quad \mathbb{R}_{>0} \times \mathbb{R}_{>0} \,.$$

In general, $\mathbb{R}^n_{\geqslant 0} \subseteq \mathbb{R}^n$ will have 2^n parts, namely, $\{0\}$, $\mathbb{R}^n_{>0}$, and the projections of $\mathbb{R}^n_{>0}$ over each of the lower-dimensional coordinate subspaces. $\Diamond$

Note that, whenever $c^{-1}x \leqslant y \leqslant cx$ for some $x, y \in V_+ \backslash \{0\}$ and $c > 0$, we must indeed have $c \geqslant 1$. For since $x > 0$ and $(c - c^{-1})x \geqslant 0$, we must also have $c - c^{-1} \geqslant 0$. Thus the definition below is well-posed.

Definition D.3 (Thompson Metric)**.** For each nonzero part C of the cone V_+, the map $d_C\colon C \times C \to \mathbb{R}_{\geqslant 0}$, defined by

$$d_C(x,y) := \inf\{\log c\,;\; c^{-1}x \leqslant y \leqslant cx\}\,, \quad x, y \in C\,,$$

is called the *Thompson metric on* C. △

It is not difficult to show that d_C is always a pseudometric [9, Proposition 3.7(1) on page 12]. If V is a topological space and V_+ is closed, then

$$x, y \in C\,, \quad d_C(x,y) = 0 \quad \Leftrightarrow \quad x = y\,,$$

yielding a metric. Indeed, if $d_C(x,y) = 0$, then there exists a sequence $(c_n)_{n\in\mathbb{N}}$ such that

$$c_n^{-1}x \leqslant y \leqslant c_n x\,, \quad \forall n \in \mathbb{N}\,, \quad \text{and} \quad \lim_{n\to\infty} c_n = 1\,.$$

By (D.1), we then have $x \leqslant y \leqslant x$. Thus $x = y$. However it is possible to express necessary and sufficient conditions for d_C to be a metric strictly in algebraic terms.

Definition D.4 (Almost Archimedean Cones)**.** Let V be a real vector space. An (algebraic) cone $V_+ \subseteq V$ will be said to be *algebraically almost Archimedean* if

$$x, y \in V \quad \text{and} \quad -ty \leqslant x \leqslant ty\,,\ \forall t > 0 \quad \Rightarrow \quad x = 0\,.$$

If V is a real normed space, then an (algebraic) cone $V_+ \subseteq V$ will be said to be *topologically almost Archimedean* if $\overline{V_+ \cap W}$ is either empty or an (algebraic) cone for every two-dimensional subspace $W \subseteq V$. △

Lemma D.5. *Suppose V_+ is a closed cone in a normed vector space V. Then V_+ is both algebraically and topologically almost Archimedean.*

Proof. Algebraically almost Archimedean. Suppose $x, y \in V$ are such that

$$-ty \leqslant x \leqslant ty\,, \quad \forall t > 0\,.$$

Letting $t \to 0$, we get $0 \leqslant x \leqslant 0$, which then yields $x = 0$.

Topologically almost Archimedean. Let $W \subseteq V$ be any two-dimensional subspace. Since W is finite-dimensional, it is closed. Thus $V_+ \cap W$ is closed cone. In particular,

$$\overline{V_+ \cap W} = V_+ \cap W$$

is a cone. □

Remark D.6. When V is a real normed space, it can actually be shown that both definitions are equivalent, so we would have only had to check either one in the proof above. But since this fact will not be needed here, we will avoid the detour to prove that. Nevertheless, when talking about almost Archimedean cones, we will drop the qualifiers 'algebraically' or 'topologically' unless we would like to emphasize them. □

Proposition D.7. *Suppose that V_+ is an (algebraic) cone in a real vector space V. For any part $C \subseteq V_+$, the Thompson metric d_C is a metric if, and only if V_+ is almost Archimedean.*

Proof. See [9, Proposition 3.7(2) on pages 12–13]. □

Unless there is any risk of ambiguity, we will omit the index 'C' designating the part, writing simply 'd' for the Thompson metric on any part. We set $d(0,0) = 0$ by convention.

We shall need one last fact about almost Archimedean cones.

Lemma D.8. *Suppose V_+ and W_+ are (algebraic) cones in a real vector space V, and that $W_+ \subseteq V_+$.*

(1) *If V_+ is algebraically almost Archimedean, then W_+ is also algebraically almost Archimedean.*

(2) *If V is a normed space and V_+ is topologically almost Archimedean, then W_+ is also topologically almost Archimedean.*

Proof. (1) Given any $x, y \in V$,

$$\begin{aligned} -ty \leqslant_{W_+} x \leqslant_{W_+} ty\,, \quad \forall t > 0 \qquad &\Rightarrow \qquad -ty \leqslant_{V_+} x \leqslant_{V_+} ty\,, \quad \forall t > 0 \\ &\Rightarrow \qquad x = 0\,, \end{aligned}$$

since V_+ is algebraically almost Archimedean by hypothesis. Thus W_+ is also algebraically almost Archimedean.

(2) For any two-dimensional subspace $U \subseteq V$, we have

$$\overline{W_+ \cap U} \subseteq \overline{V_+ \cap U}\,.$$

Now $\overline{V_+ \cap U}$ is either empty or an (algebraic) cone, since we are assuming V_+ to be topologically almost Archimedean. In particular, $\overline{W_+ \cap U}$ cannot contain any nontrivial subspaces of V. Thus $\overline{W_+ \cap U}$ is itself either empty or an (algebraic) cone. Since the two-dimensional subspace $U \subseteq V$ was chosen arbitrarily, we conclude that W_+ is also topologically almost Archimedean. □

The 'Thompson metric' was introduced by A. C. Thompson in [54], where he showed that, under the assumption that the underlying cone be normal, this metric is complete, and proved a fixed point result for a class of nonlinear operators which are contractive with respect to the metric. The Thompson metric is related to the *Hilbert projective metric*, a thorough account of which is given in [43, 44]. Properties of the Thompson metric were derived solely in terms of the relationship between the underlying partial order and the linear structure of the vector space under consideration—that is, disregarding its topological structure, if any—by Ş. Cobzaş and M.-D. Rus in [9].

D.2 Basic Properties

Lemma D.9. *Given a real vector space V, partially ordered by a cone $V_+ \subseteq V$, and $\beta \in V_+$, let*

$$\begin{aligned} \tau_\beta \colon V_+ &\longrightarrow V_+ \\ x &\longmapsto \beta + x \end{aligned}$$

be the translation of V_+ by β. Then τ_β is nonexpansive with respect to the Thompson metric.

Proof. Indeed, given any $x, y \in V_+$ and $c \geqslant 1$ such that

$$c^{-1}x \leqslant y \leqslant cx\,,$$

we have

$$c^{-1}\beta \leqslant \beta \leqslant c\beta\,.$$

By adding up these inequalities term-wise, we get

$$c^{-1}(\beta + x) = c^{-1}\tau_\beta(x) \leqslant \tau_\beta(y) \leqslant c\tau_\beta(x) = c(x+\beta)\,.$$

This shows that $\tau_\beta(x)$ and $\tau_\beta(y)$ are in the same part of V_+ whenever x and y are. Furthermore, in this case, $d(\tau_\beta(x), \tau_\beta(y)) \leqslant d(x, y)$. □

Order-preserving *sublinear* maps, which come up naturally in many applications, interact particularly well with the Thompson metric. Indeed, they are nonexpansive with respect to d, as we show in Lemma D.14.

Definition D.10 (Sublinear Maps)**.** Let V, W be real vector spaces, partially ordered by cones $V_+ \subseteq V$ and $W_+ \subseteq W$. A map $g\colon V_+ \to W_+$ is said to be *sublinear* if

$$\lambda g(x) \leqslant g(\lambda x) \tag{D.2}$$

for all $\lambda \in [0,1]$, for all $x \in V_+$. △

Equivalently, $g\colon V_+ \to W_+$ will be sublinear if

$$g(\lambda x) \leqslant \lambda g(x)\,, \quad \forall \lambda \geqslant 1\,, \quad \forall x \in V_+\,. \tag{D.3}$$

Remark D.11. Observe that any map $g\colon V_+ \to W_+$ will satisfy (D.2), for any $x \in V_+$, with $\lambda = 0$ or $\lambda = 1$. So we need only check it for $\lambda \in (0,1)$. Similarly, (D.3) always holds for $\lambda = 1$. □

Remark D.12. If $g^*\colon V \to W$ is linear and $g^*(V_+) \subseteq W_+$, then g^* is also order-preserving. Indeed,

$$g^*(y) - g^*(x) = g^*(y - x) \geqslant 0$$

whenever $y \geqslant x \geqslant 0$. Moreover, the restriction

$$g := g^*\big|_{V_+} : V_+ \to W_+$$

is sublinear, for the equality holds in either (D.2) or (D.3). □

Lemma D.13. *Let U, V, and W be real vector spaces, partially ordered by cones U_+, V_+, and W_+, respectively. If the maps $g\colon U_+ \to V_+$ and $h\colon V_+ \to W_+$ are sublinear, and h is order-preserving, then their composition $h \circ g\colon U_+ \to W_+$ is sublinear.*

Proof. Indeed, given any $x \in U_+$ and any $\lambda \in (0,1)$, we have

$$h(g(\lambda x)) \leqslant h(\lambda g(x)) \leqslant \lambda h(g(x)) .$$

The first inequality follows from the sublinearity of g combined with the monotonicity of h. The second inequality follows from the sublinearity of h. □

Lemma D.14. *Suppose that V and W are real vector spaces, partially ordered by cones $V_+ \subseteq V$ and $W_+ \subseteq W$, respectively. If $g\colon V_+ \to W_+$ is order-preserving and sublinear, then g is nonexpansive with respect to the Thompson metric; more precisely, whenever x and y are in the same part of V_+, $g(x)$ and $g(y)$ are also in the same part of W_+, and, in this case,*

$$d(g(x), g(y)) \leqslant d(x, y) .$$

Proof. Pick any x and y which are in the same part of V_+, and any $c \geqslant 1$ such that

$$c^{-1}x \leqslant y \leqslant cx . \tag{D.4}$$

Then

$$c^{-1}y \leqslant x \leqslant cy .$$

Thus by sublinearity and monotonicity,

$$c^{-1}g(x) \leqslant g(c^{-1}x) \leqslant g(y) \leqslant g(cx) \leqslant cg(x) , \tag{D.5}$$

and

$$c^{-1}g(y) \leqslant g(c^{-1}y) \leqslant g(x) \leqslant g(cy) \leqslant cg(y) .$$

These two inequalities combined show that $g(x) = 0$ if, and only if $g(y) = 0$. If this is the case, then $d(g(x), g(y)) = 0 \leqslant d(x, y)$. Otherwise, both $g(x)$ and $g(y)$ are in the same nonzero part of W_+, and it follows from (D.5), plus the arbitrary choice of $c \geqslant 1$ satisfying (D.4) that $d(g(x), g(y)) \leqslant d(x, y)$, completing the proof. □

Lemma D.15. *If V_+ is a solid, closed cone in a normed space V, then* $\operatorname{int} V_+$ *is a part.*

Proof. Given any $u, v \gg 0$, let $\gamma, \delta > 0$ be such that $B_\gamma(u), B_\delta(v) \subseteq V_+$. Then

$$B_\gamma(u) \subseteq B_{\|u\|+\gamma}(0)$$

and (see the proof of Lemma 2.43)

$$B_\delta(0) \subseteq [-v, v] \,.$$

Therefore

$$B_\gamma(u) \subseteq [-Rv, Rv] \,,$$

with

$$R := \frac{\|u\| + \gamma}{\delta} \,.$$

Similarly,

$$B_\delta(v) \subseteq [-Su, Su] \,,$$

with

$$S := \frac{\|v\| + \delta}{\gamma} \,.$$

We conclude that

$$c^{-1} v \leqslant u \leqslant cv \,,$$

where

$$c := \max\{R, S\} \,.$$

This shows that $\operatorname{int} V_+$ is contained in a part C of V_+.

To see that $C \subseteq \operatorname{int} V_+$ also, pick any $u \in C$, any $v \in \operatorname{int} V_+$, and let $c \geqslant 1$ be such that $c^{-1} v \leqslant u \leqslant cv$. In particular,

$$-u \leqslant -c^{-1} v \leqslant c^{-1} v \leqslant u \,,$$

thus

$$[-c^{-1} v, c^{-1} v] \subseteq [-u, u] \,.$$

Let $\delta > 0$ be such that $B_\delta(v) \subseteq V_+$. Then $B_\delta(0) \subseteq [-v, v]$, so indeed

$$B_{c^{-1}\delta}(0) \subseteq [-c^{-1} v, c^{-1} v] \subseteq [-u, u] \,.$$

Now

$$B_{c^{-1}\delta}(u) \subseteq V_+$$

showing that $u \in \operatorname{int} V_+$. This completes the proof that V_+ is a part. $\square$

Observe that if x, y, z are in the interior of V_+, then so are $x + z$ and $y + z$. In particular, $x, y, x + z, y + z$ are all in the same[1] part of V_+.

Proposition D.16. *Suppose that V is a Banach space, partially ordered by a solid, closed cone $V_+ \subseteq V$. For any $\beta \in \operatorname{int} V_+$, the translation*

$$\begin{aligned} \tau_\beta \colon \operatorname{int} V_+ &\longrightarrow \operatorname{int} V_+ \\ x &\longmapsto x + \beta \end{aligned}$$

is nonexpansive with respect to the Thompson metric on $\operatorname{int} V_+$; that is,

$$d(\tau_\beta(x), \tau_\beta(y)) = d(x + \beta, y + \beta) \leqslant d(x, y)\,, \quad \forall x, y \in \operatorname{int} V_+\,.$$

Furthermore, for any $B \in \operatorname{int} V_+$, the restriction of τ_β to $[0, B] \cap \operatorname{int} V_+$ is a strict contraction; that is, there exists an $L = L(\beta, B) \in [0, 1)$ such that

$$d(x + \beta, y + \beta) \leqslant L d(x, y)\,, \quad \forall x, y \in \operatorname{int} V_+ \cap [0, B]\,.$$

Proof. It follows from Lemmas D.5 and D.8 and $\operatorname{int} V_+$ is almost Archimedean. The result then follows from [38, Theorem 2.6 on page 85]. $\square$

In order to apply Banach's Fixed Point Theorem, the only ingredient now missing is completeness. The result below provides necessary and sufficient conditions for it to happen.

Proposition D.17. *Let V be a real Banach space which is partially ordered by a cone $V_+ \subseteq V$. Then each of the parts of V_+ is complete with respect to the Thompson metric if, and only if V_+ is normal.*

---

[1] Note that this is not true in general. The easiest class of examples to construct is to take x, y in any nonzero part of V_+ other than the interior and z in the interior. Thus $x + z$ and $y + z$ are in the same part (they are both in $\operatorname{int} V_+$), but $x, y, x + z, y + z$ are not all in the same part.

Proof. For a detailed conceptual approach, see [9, Theorem 4.20 on pages 34–35]. A direct, elementary proof straight from the definitions can be pieced together using [24, Theorem 2.1.1 on pages 27–30], [16, Definition 19.1 and Proposition 19.1 on pages 219–220], [34, Theorem 4.7 and Inequality (4.12) on pages 42–43], and [9, Theorem 4.14 on pages 30–32]. □

D.3 Cones of Nonnegative Functions

Let T be an arbitrary nonempty set, and consider the space $(\mathbb{R}^k)^T$ of $\mathbb{R}^k$-valued functions on T. The positive orthant cone $\mathbb{R}^k_{\geqslant 0} \subseteq \mathbb{R}^k$ induces the cone $(\mathbb{R}^k)^T_+ := (\mathbb{R}^k_{\geqslant 0})^T$ of nonnegative functions in $(\mathbb{R}^k)^T$.

We introduce a couple of pieces of notation. Given any $\alpha = (\alpha_1, \dots, \alpha_k)$ and $u = (u_1, \dots, u_k)$ in $(\mathbb{R}^k)^T$, we define the *Hadamard product* $\alpha \odot u$ of α and u by

$$(\alpha \odot u)_j(t) := \alpha_j(t) u_j(t)\,, \quad t \in T\,, \quad j = 1, \dots, k\,.$$

Note that the Hadamard product is bilinear. In particular,

$$u \longmapsto \alpha \odot u\,, \quad u \in (\mathbb{R}^k)^T\,,$$

is linear. If $\alpha \geqslant 0$, then this map is also order preserving. For any $\alpha \in (\mathbb{R}^k_{>0})^T$, we also define their *coordinatewise inverse* $\alpha^{-1} \colon T \to \mathbb{R}^k$,

$$\alpha^{-1}(t) := \left(\frac{1}{\alpha_1(t)}, \dots, \frac{1}{\alpha_k(t)} \right)\,, \quad t \in T\,.$$

The following lemma describes how these two operations interact with the Thompson metric.

Lemma D.18. *The Thompson metrics on the parts of* $(\mathbb{R}^k_{\geqslant 0})^T$ *satisfy the following two properties.*

(1) *For any u, v and α in $(\mathbb{R}^k_{\geqslant 0})^T$ such that u and v are in the same part, $\alpha \odot u$ and $\alpha \odot v$ are also in the same part, and*

$$d(\alpha \odot u, \alpha \odot v) \leqslant d(u, v)\,.$$

(2) *For any* $u, v \in (\mathbb{R}^k_{>0})^T$ *which are in the same part of* $(\mathbb{R}^k_{\geqslant 0})^T$*, we also have* u^{-1} *and* v^{-1} *in the same part, and*

$$d(u^{-1}, v^{-1}) = d(u, v) \,.$$

Proof. (1) For any $c \geqslant 1$,

$$\begin{aligned} c^{-1}u \leqslant v \leqslant cu \quad &\Leftrightarrow \quad c^{-1}u_j \leqslant v_j \leqslant cu_j \,, \quad j = 1, \ldots, k \,, \\ &\Rightarrow \quad c^{-1}\alpha_j u_j \leqslant \alpha_j v_j \leqslant c\alpha_j u_j \,, \quad j = 1, \ldots, k \,, \\ &\Leftrightarrow \quad c^{-1}(\alpha \odot u) \leqslant (\alpha \odot v) \leqslant c(\alpha \odot u) \,. \end{aligned}$$

This proves (1).

(2) Similarly, given $c \geqslant 1$, we have

$$\begin{aligned} c^{-1}u \leqslant v \leqslant cu \quad &\Leftrightarrow \quad c^{-1}u_j \leqslant v_j \leqslant cu_j \,, \quad j = 1, \ldots, k \,, \\ &\Leftrightarrow \quad cu_j^{-1} \geqslant v_j^{-1} \geqslant c^{-1}u_j^{-1} \,, \quad j = 1, \ldots, k \,, \\ &\Leftrightarrow \quad c^{-1}u^{-1} \leqslant v^{-1} \leqslant cu^{-1} \,. \end{aligned}$$

This proves (2). $\square$

Observe that u and u^{-1} need not be in the same part. For instance, if $T := \mathbb{R}_{\geqslant 0}$ and $u \in (\mathbb{R}_{\geqslant 0})^T$ is defined by

$$u(t) := 1 + t \,, \quad t \geqslant 0 \,,$$

then

$$u^{-1}(t) = \frac{1}{1+t} \,, \quad \forall t \geqslant 0 \,.$$

Since u^{-1} is uniformly bounded in $t \in \mathbb{R}_{\geqslant 0}$ and

$$c^{-1}(1 + t) \longrightarrow \infty \quad \text{as} \quad t \to \infty \,,$$

for any $c \geqslant 1$, we conclude that there is no such c for which

$$c^{-1}u(t) \leqslant u^{-1}(t) \leqslant cu(t) \,, \quad \forall t \geqslant 0 \,.$$

D.4 Tempered Paths

For the sake of convenience, we will refer to a measurable map $B\colon \mathbb{R} \to M_{n\times k}(\mathbb{R})$ as a *tempered path* if it satisfies the following growth condition,

(L1′) for every $\delta > 0$,

$$K_\delta := \sup_{s\in\mathbb{R}} \|B(s)\|\, \mathrm{e}^{-\delta|s|} < \infty\,.$$

In particular, B is locally essentially bounded. In fact, the natural analogues of all properties of tempered random variables are still true for tempered paths in the sense of (L1′) (refer to Lemma 2.32). So, in particular, the family $L^\theta(M_{n\times k}(\mathbb{R}))$ of tempered paths $\mathbb{R} \to M_{n\times k}(\mathbb{R})$ is a vector space over the real scalars. Of course all of the above can be also said about vector-valued paths $\mathbb{R} \to \mathbb{R}^n$ upon identifying $\mathbb{R}^n$ with $M_{n\times 1}(\mathbb{R})$. We equip $M_{n\times k}(\mathbb{R})$ with the partial order induced by the nonnegative orthant cone; that is, the cone of $n \times k$ real matrices having all their entries nonnegative. We then equip $L^\theta(M_{n\times k}(\mathbb{R}))$ with the partial order induced by the cone $L^\theta_+(M_{n\times k}(\mathbb{R}))$ of (Lebesgue-almost surely) nonnegative paths in $M_{n\times k}(\mathbb{R})$.

Accordingly, we have the natural analogue of property (L2) in Example 3.34 for a locally integrable matrix path $A\colon \mathbb{R} \to M_{n\times n}(\mathbb{R})$,

(L2′) there exist a $\lambda > 0$ and a tempered path $\gamma\colon \mathbb{R} \to \mathbb{R}$ (in the sense of (L1′) above) such that the fundamental matrix solution (see Example 3.2 also)

$$\Xi\colon \mathbb{R}\times\mathbb{R} \to M_{n\times n}(\mathbb{R})$$

of the linear differential equation

$$\dot\xi = A(t)\xi\,, \quad t \in \mathbb{R}\,, \tag{D.6}$$

satisfies

$$\|\Xi(s, s+r)\| \leqslant \gamma(s)\, \mathrm{e}^{-\lambda r}\,, \quad \forall s \in \mathbb{R}\,, \quad \forall r \geqslant 0\,.$$

When we say 'suppose that A satisfies (L2′),' it is to be tacitly understood that Ξ has the meaning described above.

Lemma D.19. *Let* $A\colon \mathbb{R} \to M_{n\times n}(\mathbb{R})$ *and* $B\colon \mathbb{R} \to M_{n\times k}(\mathbb{R})$ *be locally integrable matrix paths satisfying* (L1′) *and* (L2′)*. Suppose, in addition, that*

(M1′) B *is nonnegative; that is,* $B_{ij}(t) \geqslant 0$ *for Lebesgue-almost every* $t \in \mathbb{R}$*, for* $i = 1, \dots, n$*,* $j = 1, \dots, k$*, and*

(M2′) *all off-diagonal entries of* A *are nonnegative; that is,* $A_{ij}(t) \geqslant 0$ *for Lebesgue-almost every* $t \in \mathbb{R}$*, for all* $i, j = 1, \dots, n$ *such that* $i \neq j$*.*

Then

$$[\mathcal{J}^*(u)](t) := \int_{-\infty}^{t} \Xi(\sigma,t)B(\sigma)u(\sigma)\,d\sigma\,, \quad t \in \mathbb{R}\,, \quad u \in L^\theta(\mathbb{R}^k)\,, \tag{D.7}$$

defines an order-preserving, linear operator $\mathcal{J}^*\colon L^\theta(\mathbb{R}^k) \to L^\theta(\mathbb{R}^n)$*. In particular,*

$$\mathcal{J} := \mathcal{J}^*\big|_{L^\theta_+(\mathbb{R}^k)}\colon L^\theta_+(\mathbb{R}^k) \to L^\theta_+(\mathbb{R}^n)$$

is sublinear, and thus nonnexpansive with respect to the Thompson metric.

Proof. We first show that, under (L1′) and (L2′), (D.7) defines a linear operator

$$\mathcal{J}^*\colon L^\theta(\mathbb{R}^k) \longrightarrow L^\theta(\mathbb{R}^n)\,.$$

We then show that, under (M1′) and (M2′), we also have $\mathcal{J}^*(L^\theta_+(\mathbb{R}^k)) \subseteq L^\theta_+(\mathbb{R}^n)$. Hence $\mathcal{J}^*$ is order-preserving, and so $\mathcal{J}$ is well-defined and sublinear (see Remark D.12).

Fix arbitrarily $u \in L^\theta(\mathbb{R}^k)$ and $t \in \mathbb{R}$. Since $\Xi(\cdot,t)$ is continuous and B, u are locally essentially bounded, the integrand in (D.7) is locally integrable. To show that the integral

$$\int_{-\infty}^{t} \Xi(\sigma,t)B(\sigma)u(\sigma)\,d\sigma$$

converges, we apply (L2′), then (L1′). This yields

$$\begin{aligned}
|\Xi(\sigma,t)B(\sigma)u(\sigma)| &= \gamma(\sigma)\|B(\sigma)\||u(\sigma)|\,\mathrm{e}^{-\lambda|t-\sigma|} \\
&\leqslant \gamma(\sigma)\|B(\sigma)\||u(\sigma)|\,\mathrm{e}^{-\lambda|\sigma|}\,\mathrm{e}^{\lambda|t|} \\
&\leqslant K\,\mathrm{e}^{-\frac{\lambda}{3}|\sigma|}\,, \quad \forall \sigma \leqslant 0\,,
\end{aligned}$$

for some constant $K \geqslant 0$ comprising $\mathrm{e}^{\lambda|t|}$ and the temperedness constants for γ and B, with $\delta = \lambda/3$.

To see that the map $\mathcal{J}^*(u)\colon \mathbb{R} \to \mathbb{R}^n$ so defined is a tempered path, pick any $\delta > 0$, and fix $s \in \mathbb{R}$ arbitrarily. Let $m := \min\{\lambda, \delta\}$. Then, for some constant $K \geqslant 0$ constructed as above, we have

$$\begin{aligned}
\mathrm{e}^{-\delta|s|}\,|[\mathcal{J}^*(u)](s)| &= \mathrm{e}^{-\delta|s|}\left|\int_{-\infty}^{s} \Xi(\sigma,s)B(\sigma)u(\sigma)\,d\sigma\right| \\
&\leqslant \int_{-\infty}^{s} \gamma(\sigma)\|B(\sigma)\||u(\sigma)|\,\mathrm{e}^{-\delta|s|-\lambda|s-\sigma|}\;d\sigma \\
&\leqslant \int_{-\infty}^{s} \gamma(\sigma)\|B(\sigma)\||u(\sigma)|\,\mathrm{e}^{-m|\sigma|}\;d\sigma \\
&\leqslant K\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{m}{3}|\sigma|}\;d\sigma\,,
\end{aligned}$$

which is finite and does not depend on s. This shows that $\mathcal{J}^*(u) \in L^\theta(\mathbb{R}^n)$. It follows straight from the linearity of the integral and matrix multiplication that $\mathcal{J}^*$ is also linear.

Now assume that (M1′) and (M2′) hold in addition to (L1′) and (L2′). Then clearly $Bu \in L_+^\theta(\mathbb{R}^n)$ for any $u \in L_+^\theta(\mathbb{R}^k)$. Now (M2′) is equivalent to the Kamke condition for the linear equation (D.6). Therefore the flow of (D.6) is monotone with respect to the positive-orthant cone-induced partial order (see [50, Chapter 3, Proposition 1.1 on pages 32–33]). Thus

$$\Xi(\sigma,t)B(\sigma)u(\sigma) \geqslant 0$$

for Lebesgue-almost all $\sigma \leqslant t$, for every $t \in \mathbb{R}$, for each $u \in L_+^\theta(\mathbb{R}^k)$, and so

$$[\mathcal{J}^*(u)](t) = \int_{-\infty}^{t} \Xi(\sigma,t)B(\sigma)u(\sigma)\,d\sigma \geqslant 0\,, \quad \forall t \in \mathbb{R}\,; \tag{D.8}$$

in other words, $\mathcal{J}^*(L_+^\theta(\mathbb{R}^k)) \subseteq L_+^\theta(\mathbb{R}^n)$. Since $\mathcal{J}^*$ is linear, this implies that it is also order-preserving, as we pointed out in Remark D.12. In particular,

$$\mathcal{J} := \mathcal{J}^*\big|_{L_+^\theta(\mathbb{R}^k)}\colon L_+^\theta(\mathbb{R}^k) \longrightarrow L_+^\theta(\mathbb{R}^n)$$

is sublinear (and order-preserving). It then follows from Lemma D.14 that $\mathcal{J}$ is nonexpansive with respect to the Thompson metric. □

D.5 Conditions for Strict Contractiveness

Now consider the space $L^\infty(\mathbb{R}^n)$ of Borel-measurable, essentially bounded, vector-valued functions $\mathbb{R} \to \mathbb{R}^n$. Once again, we equip $L^\infty(\mathbb{R}^n)$ with the partial order induced by the cone $L^\infty_+(\mathbb{R}^n)$ of nonnegative (Borel-measurable, essentially bounded) functions $\mathbb{R} \to \mathbb{R}^n_{\geqslant 0}$.

Recall that L^∞ is a Banach space when equipped with the norm $\|\cdot\|_{L^\infty}$, defined by

$$\begin{aligned} \|u\|_{L^\infty} &:= \operatorname{ess\,sup}\{|u(t)|\,;\; t \in M\} \\ &:= \inf\{K \geqslant 0\,;\; \mu(|u| > K) = 0\}\,, \quad u \in L^\infty(M)\,. \end{aligned}$$

Moreover, $L^\infty_+(\mathbb{R}^n)$ is a solid, normal cone. The interior $\operatorname{int} L^\infty_+(\mathbb{R}^n)$ of $L^\infty_+(\mathbb{R}^n)$ is the family of functions uniformly (essentially) bounded away from zero; that is, u belongs to $\operatorname{int} L^\infty_+$ if, and only if there exists an $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \gg 0$ such that $\mu(u < \epsilon) = 0$. For any $u = (u_1, \ldots, u_n) \in \operatorname{int} L^\infty_+$, their coordinatewise inverse $u^{-1} = (u_1^{-1}, \ldots, u_k^{-1})$ is well-defined and also belongs to $\operatorname{int} L^\infty_+$, since u is essentially bounded both away from zero and from infinity.

Any path $u \in L^\infty(\mathbb{R}^n)$ has a representative which is bounded everywhere. Assume without loss of generality that u is one such representative. Then indeed $u \in L^\theta(\mathbb{R}^n)$. Having this identification in mind, we may thus write $L^\infty(\mathbb{R}^n) \subseteq L^\theta(\mathbb{R}^n)$.

Lemma D.20. *Suppose that* $A\colon \mathbb{R} \to M_{n\times n}(\mathbb{R})$ *and* $B\colon \mathbb{R} \to M_{n\times k}(\mathbb{R})$ *are locally integrable matrix paths satisfying* (L1′), (L2′), (M1′) *and* (M2′). *Let*

$$\mathcal{H}\colon L^\infty_+(\mathbb{R}^k) \longrightarrow L^\infty_+(\mathbb{R}^k)$$

be defined by

$$[\mathcal{H}(u)](t) := \left(\frac{\alpha_j(t)}{\beta_j(t) + g_j\left(\displaystyle\int_{-\infty}^{t} \Xi(\sigma, t) B(\sigma) u(\sigma)\, d\sigma \right)} \right)_{j=1}^{k}, \quad t \in \mathbb{R}\,,$$

for each $u \in L^\infty_+(\mathbb{R}^k)$, *where*

(i) $\alpha = (\alpha_1, \ldots, \alpha_k)$ *and* $\beta = (\beta_1, \ldots, \beta_k)$ *are in* $\operatorname{int} L^\infty_+(\mathbb{R}^k) \cap C^0(\mathbb{R}^k)$, *and*

(ii) $g = (g_1, \ldots, g_k)\colon \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^k_{\geqslant 0}$ *is continuous, bounded, order-preserving and sub-linear.*

Then $\mathcal{H}(L^\infty_+(\mathbb{R}^k)) \subseteq \operatorname{int} L^\infty_+(\mathbb{R}^k)$. *Furthermore,*

$$\mathcal{I} := \mathcal{H}\big|_{\operatorname{int} L^\infty_+(\mathbb{R}^k)}\colon \operatorname{int} L^\infty_+(\mathbb{R}^k) \longrightarrow \operatorname{int} L^\infty_+(\mathbb{R}^k)$$

is a strict contraction with respect to the Thompson metric on $\operatorname{int} L_+^\infty(\mathbb{R}^k)$*; that is, there exists an* $L \in [0,1)$ *such that*

$$d(\mathcal{I}(u), \mathcal{I}(v)) \leqslant L d(u,v)\,, \quad \forall u, v \in \operatorname{int} L_+^\infty(\mathbb{R}^k)\,. \tag{D.9}$$

Proof. First note that $\mathcal{H}$ is well-defined. Indeed, fix $u \in L_+^\infty(\mathbb{R}^k)$ arbitrarily. In view of (L1′), (L2′), (M1′) and (M2′), it follows from (D.8) in the proof of Lemma D.19 that

$$[\mathcal{J}(u)](t) = \int_{-\infty}^{t} \Xi(\sigma,t) B(\sigma) u(\sigma)\, d\sigma \geqslant 0\,, \quad \forall t \in \mathbb{R}\,.$$

So the g_j term in the denominator is well-defined. It then follows straight from (i) and (ii) that $\mathcal{H}(u)$ is nonnegative and bounded coordinatewise. Moreover,

$$\alpha_j(t) \geqslant \epsilon_j\,,$$

$$\beta_j(t) \leqslant B_j\,,$$

and

$$0 \leqslant g_j \left(\int_{-\infty}^{t} \Xi(\sigma,t) B(\sigma) u(\sigma)\, d\sigma \right) \leqslant M_j$$

for every $t \in \mathbb{R}$, for some $\epsilon_j > 0$, $B_j < \infty$ and $M_j < \infty$, $j \in \{1, \dots, k\}$. Hence

$$\big([\mathcal{H}(u)](t)\big)_j \geqslant \frac{\epsilon_j}{B_j + M_j} > 0\,, \quad \forall t \in \mathbb{R}\,, \quad j \in \{1, \dots, k\}\,.$$

This shows that $\mathcal{H}(u) \in \operatorname{int} L_+^\infty(\mathbb{R}^k)$. Since $u \in L_+^\infty(\mathbb{R}^k)$ was chosen arbitrarily, we have indeed $\mathcal{H}(L_+^\infty(\mathbb{R}^k)) \subseteq \operatorname{int} L_+^\infty(\mathbb{R}^k)$, proving the first part of the result.

We now proceed to establish the strict contractiveness part of the result. Consider the operator $\mathcal{G} \colon L_+^\infty(\mathbb{R}^n) \to L_+^\infty(\mathbb{R}^k)$, defined by

$$[\mathcal{G}(\xi)](t) := g(\xi(t)) = (g_1(\xi(t)), \dots, g_k(\xi(t)))\,, \quad t \in \mathbb{R}\,, \quad \xi \in L_+^\infty(\mathbb{R}^n)\,.$$

Since g is nonnegative, continuous and bounded by hypothesis, $\mathcal{G}$ is well-defined. Note that $\mathcal{G}$ is also sublinear and order-preserving. Observe that, by combining this with the various pieces of notation introduced in the previous section, we may write

$$\mathcal{H}(u) = \alpha \odot \big(\beta + \mathcal{G}\big(\mathcal{J}(u)\big)\big)^{-1}\,, \quad \forall u \in \operatorname{int} L_+^\infty(\mathbb{R}^k)\,.$$

Now fix arbitrarily $u, v \in \operatorname{int} L^\infty_+(\mathbb{R}^k)$. By the observation above and Lemma D.18, we have

$$\begin{aligned} d(\mathcal{I}(u), \mathcal{I}(v)) &= d(\mathcal{H}(u), \mathcal{H}(v)) \\ &= d\left(\alpha \odot \left(\beta + \mathcal{G}\big(\mathcal{J}(u)\big)\right)^{-1}, \alpha \odot \left(\beta + \mathcal{G}\big(\mathcal{J}(v)\big)\right)^{-1}\right) \\ &\leqslant d\left(\left(\beta + \mathcal{G}\big(\mathcal{J}(u)\big)\right)^{-1}, \left(\beta + \mathcal{G}\big(\mathcal{J}(v)\big)\right)^{-1}\right) \\ &= d\left(\beta + \mathcal{G}\big(\mathcal{J}(u)\big), \beta + \mathcal{G}\big(\mathcal{J}(v)\big)\right) . \end{aligned}$$

Define $M \in L^\infty_+(\mathbb{R}^k)$ by

$$M(t) := (M_1, \ldots, M_k)\,, \quad t \in \mathbb{R}\,.$$

By Proposition D.16, there exists an $L := L(\beta/2, \beta/2 + M) \in [0, 1)$ such that

$$d(\beta/2 + x, \beta/2 + y) \leqslant L d(x, y)\,, \quad \forall x, y \in [\beta/2, \beta/2 + M] \cap \operatorname{int} L^\infty_+(\mathbb{R}^k)\,.$$

Since $\mathcal{G}(L^\infty_+(\mathbb{R}^n)) \subseteq [0, M]$, it follows that

$$d\left(\beta + \mathcal{G}\big(\mathcal{J}(u)\big), \beta + \mathcal{G}\big(\mathcal{J}(v)\big)\right) \leqslant L d\left(\beta/2 + \mathcal{G}\big(\mathcal{J}(u)\big), \beta/2 + \mathcal{G}\big(\mathcal{J}(v)\big)\right)$$

Because $\mathcal{G}$, $\mathcal{J}$ and

$$w \longmapsto \beta/2 + w \in L^\infty_+(\mathbb{R}^k)\,, \quad w \in L^\infty_+(\mathbb{R}^k)\,,$$

are order-preserving and sublinear, it follows from Lemmas D.13 and D.14 that

$$d\left(\beta/2 + \mathcal{G}\big(\mathcal{J}(u)\big), \beta/2 + \mathcal{G}\big(\mathcal{J}(v)\big)\right) \leqslant d(u, v)\,.$$

Combining all these inequalities we obtain

$$d(\mathcal{I}(u), \mathcal{I}(v)) \leqslant L d(u, v)\,.$$

Since $u, v \in \operatorname{int} L^\infty_+(\mathbb{R}^k)$ were chosen arbitrarily, this completes the proof that $\mathcal{I}$ is a strict contraction with respect to the Thompson metric. □

Lemma D.21. *Assume the same hypotheses as in* Lemma D.20*, except for replacing* (i) *and* (ii) *in that lemma by* (i′) *$\alpha = (\alpha_1, \dots, \alpha_k)$, $\widetilde{\alpha} = (\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_k)$, $\beta = (\beta_1, \dots, \beta_k)$ and $\widetilde{\beta} = (\widetilde{\beta}_1, \dots, \widetilde{\beta}_k)$ are in $\operatorname{int} L^\infty_+(\mathbb{R}^k) \cap C^0(\mathbb{R}^k)$, and satisfy*

$$\frac{\alpha_j(t)}{\beta_j(t)} \geqslant \frac{\widetilde{\alpha}_j(t)}{\widetilde{\beta}_j(t)}\,, \quad \forall t \in \mathbb{R}\,, \quad j = 1, \dots, k\,, \tag{D.10}$$

and

(ii′) *$g = (g_1, \dots, g_k)\colon \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^k_{\geqslant 0}$ is continuous, order-preserving and sublinear.*

Let $\mathcal{H}\colon L^\infty_+(\mathbb{R}^k) \to L^\infty_+(\mathbb{R}^k)$ be defined by

$$[\mathcal{H}(u)](t) := \left(\frac{\alpha_j(t) + \widetilde{\alpha}_j(t) g_j\left(\displaystyle\int_{-\infty}^t \Xi(\sigma,t)B(\sigma)u(\sigma)\,d\sigma\right)}{\beta_j(t) + \widetilde{\beta}_j(t) g_j\left(\displaystyle\int_{-\infty}^t \Xi(\sigma,t)B(\sigma)u(\sigma)\,d\sigma\right)} \right)_{j=1}^k\,, \quad t \in \mathbb{R}\,,$$

for each $u \in L^\infty_+(\mathbb{R}^k)$. Then the same conclusions as in Lemma D.20 *hold; that is, $\mathcal{H}(L^\infty_+(\mathbb{R}^k)) \subseteq \operatorname{int} L^\infty_+(\mathbb{R}^k)$, and $\mathcal{I} := \mathcal{H}\big|_{\operatorname{int} L^\infty_+(\mathbb{R}^k)}\colon \operatorname{int} L^\infty_+(\mathbb{R}^k) \to \operatorname{int} L^\infty_+(\mathbb{R}^k)$ is a strict contraction with respect to the Thompson metric on $\operatorname{int} L^\infty_+(\mathbb{R}^k)$.*

Proof. Fix arbitrarily $u \in L^\infty_+(\mathbb{R}^k)$. As in Lemma D.20, the hypotheses (L1′), (L2′), (M1′) and (M2′) guarantee that the g_j terms are well-defined. It follows from (i′) and (ii′) that $\mathcal{H}(u)$ is nonnegative coordinatewise. Also by (i′) and (ii′), for each $j \in \{1, \dots, k\}$, there exist $\epsilon_j > 0$, $B_j < \infty$ and $M_j < \infty$ such that

$$\alpha_j(t), \widetilde{\alpha}_j(t), \beta_j(t), \widetilde{\beta}_j(t) \geqslant \epsilon_j\,, \quad \forall t \in \mathbb{R}\,,$$

and

$$\alpha_j(t), \widetilde{\alpha}_j(t), \beta_j(t), \widetilde{\beta}_j(t) \leqslant B_j\,, \quad \forall t \in \mathbb{R}\,.$$

Thus

$$\big([\mathcal{H}(u)](t)\big)_j \leqslant \frac{B_j + B_j g_j([\mathcal{J}(u)](t))}{\epsilon_j + \epsilon_j g_j([\mathcal{J}(u)](t))} \leqslant \max_{r \geqslant 0} \frac{B_j + B_j r}{\epsilon_j + \epsilon_j r} < \infty\,, \quad \forall t \in \mathbb{R}\,.$$

This shows that $\mathcal{H}(u)$ is also bounded coordinatewise. Since u was chosen arbitrarily, this establishes that $\mathcal{H}$ is well-defined. Moreover,

$$\big([\mathcal{H}(u)](t)\big)_j \geqslant \frac{\epsilon_j + \epsilon_j g_j([\mathcal{J}(u)](t))}{B_j + B g_j([\mathcal{J}(u)](t))} \geqslant \min_{r \geqslant 0} \frac{\epsilon_j + \epsilon_j r}{B_j + B_j r} > 0\,, \quad \forall t \in \mathbb{R}\,.$$

So we also have $\mathcal{H}(L^\infty_+(\mathbb{R}^k)) \subseteq \operatorname{int} L^\infty_+(\mathbb{R}^k)$.

For the strict contractiveness part of the result, we first note that we may rewrite

$$\frac{\alpha_j(t) + \widetilde{\alpha}_j(t) g_j([\mathcal{J}(u)](t))}{\beta_j(t) + \widetilde{\beta}_j(t) g_j([\mathcal{J}(u)](t))} \equiv c_j(t) + \frac{e_j(t)}{f_j(t) + g_j([\mathcal{J}(u)](t))}\,,$$

where

$$c_j := \frac{\widetilde{\alpha}_j}{\widetilde{\beta}_j}\,, \quad e_j := \frac{\widetilde{\alpha}_j}{\widetilde{\beta}_j}\left(\frac{\alpha_j}{\widetilde{\alpha}_j} - \frac{\beta_j}{\widetilde{\beta}_j}\right)\,, \quad \text{and} \quad f_j := \frac{\beta_j}{\widetilde{\beta}_j}\,,$$

for each $j \in \{1, \ldots, k\}$. Observe that $c, f \gg 0$, and $e \geqslant 0$. Now setting

$$c := (c_1, \ldots, c_k)\,, \quad e := (e_1, \ldots, e_k)\,, \quad \text{and} \quad f := (f_1, \ldots, f_k)\,,$$

we may rewrite

$$\mathcal{H}(u) = c + e \odot \big(f + \mathcal{G}\big(\mathcal{J}(u)\big)\big)^{-1}\,, \quad \forall u \in \operatorname{int} L^\infty_+(\mathbb{R}^k)\,.$$

So for any $u, v \in \operatorname{int} L^\infty_+(\mathbb{R}^k)$, we have

$$\begin{aligned}
d(\mathcal{I}(u), \mathcal{I}(v)) &= d(\mathcal{H}(u), \mathcal{H}(v)) \\
&\leqslant Ld\left(c/2 + e \odot \big(f + \mathcal{G}\big(\mathcal{J}(u)\big)\big)^{-1}, c/2 + e \odot \big(f + \mathcal{G}\big(\mathcal{J}(v)\big)\big)^{-1}\right) \\
&\leqslant Ld\left(e \odot \big(f + \mathcal{G}\big(\mathcal{J}(u)\big)\big)^{-1}, e \odot \big(f + \mathcal{G}\big(\mathcal{J}(v)\big)\big)^{-1}\right) \\
&\leqslant Ld\left(\big(f + \mathcal{G}\big(\mathcal{J}(u)\big)\big)^{-1}, \big(f + \mathcal{G}\big(\mathcal{J}(v)\big)\big)^{-1}\right) \\
&= Ld\left(f + \mathcal{G}\big(\mathcal{J}(u)\big), f + \mathcal{G}\big(\mathcal{J}(v)\big)\right) \\
&\leqslant Ld\left(\mathcal{G}\big(\mathcal{J}(u)\big), \mathcal{G}\big(\mathcal{J}(v)\big)\right) \\
&\leqslant Ld\,(u, v)\;.
\end{aligned}$$

In the above the first inequality follows from Proposition D.16, with

$$L := L(c/2, c/2 + M') < 1\,,$$

constructed as in the proof of Lemma D.20 with

$$M'(t) := \left(\frac{\operatorname{ess\,sup}_{t\in\mathbb{R}} |e_1(t)|}{\operatorname{ess\,inf}_{t\in\mathbb{R}} |f_1(t)|}, \ldots, \frac{\operatorname{ess\,sup}_{t\in\mathbb{R}} |e_k(t)|}{\operatorname{ess\,inf}_{t\in\mathbb{R}} |f_k(t)|}\right)\,, \quad t \in \mathbb{R}\,,$$

the second and fourth inequalities follow from Lemma D.9. The third inequality and the subsequent equality follow from Lemma D.18. The last inequalities follow from Lemmas D.19, D.13 and D.14. This shows that $\mathcal{I} = \mathcal{H}\big|_{\operatorname{int} L^\infty_+(\mathbb{R}^k)}$ is a strict contraction. $\square$

D.6 Unique, Globally Attracting Equilibria

Lemma D.22. *Under the same hypotheses as in either* Lemma D.20 *or* Lemma D.21*, the discrete dynamical system on* $\operatorname{int} L^\infty_+(\mathbb{R}^k)$ *generated by the difference equation*

$$u^+ = \mathcal{H}(u) \tag{D.11}$$

has a unique, globally attracting equilibrium $u_\infty \in \operatorname{int} L^\infty_+(\mathbb{R}^k)$ *with respect to the Thompson metric on the part* $\operatorname{int} L^\infty_+(\mathbb{R}^k)$*.*

Proof. It follows from Lemma D.15 that $\operatorname{int} L^\infty_+(\mathbb{R}^k)$ is a part. Since $L^\infty(\mathbb{R}^k)$ is a Banach space and $L^\infty_+(\mathbb{R}^k)$ is a normal cone, the Thompson metric on $\operatorname{int} L^\infty_+(\mathbb{R}^k)$ is complete by Proposition D.17.

Under the hypotheses of either Lemma D.20 or Lemma D.21,

$$\mathcal{H}(\operatorname{int} L^\infty(\mathbb{R}^k)) \subseteq \mathcal{H}(L^\infty(\mathbb{R}^k)) \subseteq \operatorname{int} L^\infty(\mathbb{R}^k)\,,$$

and $\mathcal{H}$ is a strict contraction (with respect to the Thompson metric) on $\operatorname{int} L^\infty_+(\mathbb{R}^k)$. Therefore $\mathcal{H}$ has a unique, globally attracting fixed point $u_\infty \in \operatorname{int} L^\infty(\mathbb{R}^k)$ by the Banach Fixed Point Theorem. $\square$

Proposition D.23. *Suppose that* $A\colon \mathbb{R} \to M_{n\times n}(\mathbb{R})$ *and* $B\colon \mathbb{R} \to M_{n\times k}(\mathbb{R})$ *are locally integrable matrix paths satisfying* (L1′), (L2′), (M1′) *and* (M2′)*. Let*

$$\mathcal{H}\colon L^\theta_+(\mathbb{R}^k) \longrightarrow L^\theta_+(\mathbb{R}^k)$$

be defined by

$$[\mathcal{H}(u)](t) := \left(\frac{\alpha_j(t)}{\beta_j(t) + g_j\left(\displaystyle\int_{-\infty}^t \Xi(\sigma,t)B(\sigma)u(\sigma)\,d\sigma \right)} \right)_{j=1}^k, \quad t \in \mathbb{R}\,,$$

for each $u \in L^\theta_+(\mathbb{R}^k)$*, where*

(i) $\alpha = (\alpha_1, \ldots, \alpha_k)$ *and* $\beta = (\beta_1, \ldots, \beta_k)$ *are in* $\operatorname{int} L^\infty_+(\mathbb{R}^k) \cap C^0(\mathbb{R}^k)$*, and*

(ii) $g = (g_1, \ldots, g_k)\colon \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^k_{\geqslant 0}$ *is continuous, bounded, order-preserving and sublinear.*

Then the discrete dynamical system on $L_+^\theta(\mathbb{R}^k)$ generated by the difference equation

$$u^+ = \mathcal{H}(u)$$

has a unique, globally attracting equilibrium $u_\infty \in \operatorname{int} L_+^\infty(\mathbb{R}^k) \subseteq L_+^\theta(\mathbb{R}^k)$ in the sense of almost-everywhere, pointwise convergence. Furthermore, the representative u_∞ can be chosen to be continuous, in which case convergence is actually everywhere; that is,

$$[\mathcal{H}^k(u)](t) \longrightarrow u_\infty(t) \quad \text{as} \quad k \to \infty\,, \quad \forall t \in \mathbb{R}\,, \quad \forall u \in L_+^\theta(\mathbb{R}^k)\,.$$

Proof. As in the proof of Lemma D.20, the assumptions imply that $\mathcal{H}(u)$ is in fact bounded coordinatewise for each $u \in L_+^\theta(\mathbb{R}^k)$. Therefore

$$\mathcal{H}(L_+^\theta(\mathbb{R}^k)) \subseteq L_+^\infty(\mathbb{R}^k) \subseteq L^\theta(\mathbb{R}^k)\,.$$

Again by Lemma D.20, we also have

$$\mathcal{H}(L_+^\infty(\mathbb{R}^k)) \subseteq \operatorname{int} L_+^\infty(\mathbb{R}^k)\,,$$

so indeed

$$\mathcal{H}^2(L_+^\theta(\mathbb{R}^k)) \subseteq \operatorname{int} L_+^\infty(\mathbb{R}^k)\,.$$

By Lemma D.22, $\mathcal{H}\big|_{\operatorname{int} L_+^\infty(\mathbb{R}^k)}$ has a unique, globally attracting fixed point u_∞ with respect to the Thompson metric.

Fix $u \in L_+^\theta(\mathbb{R}^k)$ arbitrarily, and let

$$u_k := \mathcal{H}^k(u)\,, \quad k = 0, 1, 2, \dots\,.$$

Then

$$u_k \in \operatorname{int} L_+^\infty(\mathbb{R}^k)\,, \quad k = 2, 3, 4, \dots\,.$$

Moreover,

$$d(u_k, u_\infty) \longrightarrow 0\,, \quad \text{as} \quad k \to \infty\,,$$

since u_∞ is the unique, globally attracting fixed point of $\mathcal{H}\big|_{\operatorname{int} L_+^\infty(\mathbb{R}^k)}$ with respect to the Thompson metric. Now

$$\mathrm{e}^{-d(u_k,u_\infty)}\, u_\infty \leqslant u_k \leqslant \mathrm{e}^{d(u_k,u_\infty)}\, u_\infty\,, \quad k = 2, 3, 4, \dots,\,.$$

Thus by the triangle inequality, and by normality,

$$\begin{aligned}
\|u_k - u_\infty\|_\infty &\leqslant \|u_k - \mathrm{e}^{-d(u_k,u_\infty)}\, u_\infty\|_\infty + \|\,\mathrm{e}^{-d(u_k,u_\infty)}\, u_\infty - u_\infty\|_\infty \\
&\leqslant 1 \cdot \|(\mathrm{e}^{d(u_k,u_\infty)} - \mathrm{e}^{-d(u_k,u_\infty)})u_\infty\|_\infty + \|(\mathrm{e}^{-d(u_k,u_\infty)} - 1)u_\infty\|_\infty \\
&\leqslant (|\,\mathrm{e}^{d(u_k,u_\infty)} - \mathrm{e}^{-d(u_k,u_\infty)}\,| + |\,\mathrm{e}^{-d(u_k,u_\infty)} - 1|)\|u\|_\infty \\
&\to 0\,, \quad \text{as} \quad k \to \infty\,.
\end{aligned}$$

In particular, $(u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence. Furthermore, u_k is continuous for each $k \in \mathbb{N}$, since α, β, g are continuous by hypothesis and $\mathcal{J}(u)$ is continuous for each $u \in L^\theta_+(\mathbb{R})$. Thus indeed $(u_k)_{k\in\mathbb{N}}$ converges uniformly to a continuous function which is equal to u_∞ in the sense of L^∞. □

Proposition D.24. *Assume the same hypotheses as in* Proposition D.23*, except for replacing* (i) *and* (ii) *in that proposition by*

(i′) $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\widetilde{\alpha} = (\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_k)$, $\beta = (\beta_1, \ldots, \beta_k)$ *and* $\widetilde{\beta} = (\widetilde{\beta}_1, \ldots, \widetilde{\beta}_k)$ *are in* $\operatorname{int} L^\infty_+(\mathbb{R}^k) \cap C^0(\mathbb{R}^k)$, *and satisfy*

$$\frac{\alpha_j(t)}{\beta_j(t)} \geqslant \frac{\widetilde{\alpha}_j(t)}{\widetilde{\beta}_j(t)}\,, \quad \forall t \in \mathbb{R}\,, \quad j = 1, \ldots, k\,, \tag{D.12}$$

and

(ii′) $g = (g_1, \ldots, g_k) \colon \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^k_{\geqslant 0}$ *is continuous, order-preserving and sublinear,*

and let $\mathcal{H} \colon L^\theta_+(\mathbb{R}^k) \to L^\theta_+(\mathbb{R}^k)$ *be defined by*

$$[\mathcal{H}(u)](t) := \left(\frac{\alpha_j(t) + \widetilde{\alpha}_j(t) g_j \left(\int_{-\infty}^t \Xi(\sigma,t) B(\sigma) u(\sigma)\, d\sigma \right)}{\beta_j(t) + \widetilde{\beta}_j(t) g_j \left(\int_{-\infty}^t \Xi(\sigma,t) B(\sigma) u(\sigma)\, d\sigma \right)} \right)_{j=1}^k \,, \quad t \in \mathbb{R}\,,$$

for each $u \in L^\theta_+(\mathbb{R}^k)$. *Then the discrete dynamical system on* $L^\theta_+(\mathbb{R}^k)$ *generated by the difference equation*

$$u^+ = \mathcal{H}(u)$$

has a unique, globally attracting equilibrium $u_\infty \in \operatorname{int} L^\infty_+(\mathbb{R}^k) \subseteq L^\theta_+(\mathbb{R}^k)$ *in the sense of almost-everywhere, pointwise convergence. Furthermore, the representative* u_∞ *can* *be chosen to be continuous, in which case convergence is actually everywhere; that is,*

$$[\mathcal{H}^k(u)](t) \longrightarrow u_\infty(t) \quad \textit{as} \quad k \to \infty\,, \quad \forall t \in \mathbb{R}\,, \quad \forall u \in L_+^\theta(\mathbb{R}^k)\,.$$

Proof. The proof is the same as in Proposition D.23. We have

$$\mathcal{H}^2(L_+^\theta(\mathbb{R}^k)) \subseteq \operatorname{int} L_+^\infty(\mathbb{R}^k)$$

by Lemma D.21, and $\mathcal{H}\big|_{\operatorname{int} L_+^\infty(\mathbb{R}^k)}$ has a unique, globally attracting fixed point u_∞ (with respect to the Thompson metric) by Lemma D.22. It then follows as in the proof of Proposition D.23 that $(\mathcal{H}^k(u))_{k\in\mathbb{N}}$ is a Cauchy sequence in $L_+^\infty(\mathbb{R}^k)$, each term of which is continuous. This establishes the uniform convergence to a continuous function which is almost everywhere equal to u_∞. □

D.7 The Periodic Case

Lemma D.25. *Suppose that $A\colon \mathbb{R} \to M_{n\times n}(\mathbb{R})$ is a locally integrable, T-periodic matrix path satisfying* (L2′)*. Then the map*

$$t \longmapsto \int_{-\infty}^{t} \Xi(\sigma, t)\, d\sigma\,, \quad t \in \mathbb{R}\,, \tag{D.13}$$

is T-periodic. In particular, it is uniformly bounded away from infinity; in other words, there exist $P \geqslant 0$ such that

$$\left\| \int_{-\infty}^{t} \Xi(\sigma, t)\, d\sigma \right\| \leqslant P < \infty\,, \quad \forall t \in \mathbb{R}\,. \tag{D.14}$$

Proof. Convergence of the integral for each $t \in \mathbb{R}$ follows from the same estimates as in Lemma D.19. To establish periodicity, first note that, since A is T-periodic,

$$\begin{aligned} \frac{d}{dt}\Xi(\sigma+T, t+T) &= A(t+T)\Xi(\sigma+T, t+T) \\ &= A(t)\Xi(\sigma+T, t+T)\,, \quad \forall \sigma, t \in \mathbb{R}\,. \end{aligned}$$

Since

$$\Xi(\sigma+T, \sigma+T) = I_n\,,$$

it then follows by uniqueness that

$$\Xi(\sigma+T, t+T) \equiv \Xi(\sigma, t)\,.$$

Now a simple change of variables yields

$$\int_{-\infty}^{t+T} \Xi(\sigma, t+T)\, d\sigma \equiv \int_{-\infty}^{t} \Xi(\sigma+T, t+T)\, d\sigma \equiv \int_{-\infty}^{t} \Xi(\sigma, t)\, d\sigma \,.$$

This shows that the map defined in (D.13) is T-periodic. Since the map is also continuous, the estimate in (D.14) hold on $[0, T]$ for some $P \geqslant 0$. It then holds along the whole line by periodicity. $\square$

Lemma D.26. *Assume the same hypotheses as in* Lemma D.20*, except that g is not necessarily bounded, but with the additional hypotheses that the matrix paths A and B are T-periodic. Let $\mathcal{H}\colon L_+^\infty(\mathbb{R}^k) \to L_+^\infty(\mathbb{R}^k)$ be defined by*

$$[\mathcal{H}(u)](t) := \left(\frac{\alpha_j(t)}{\beta_j(t) + g_j\left(\int_{-\infty}^{t} \Xi(\sigma, t) B(\sigma) u(\sigma)\, d\sigma \right)} \right)_{j=1}^{k}, \quad t \in \mathbb{R}\,,$$

for each $u \in L_+^\infty(\mathbb{R}^k)$. Then $\mathcal{H}(L_+^\infty(\mathbb{R}^k)) \subseteq \operatorname{int} L_+^\infty(\mathbb{R}^k)$, and

$$\mathcal{I} := \mathcal{H}\big|_{\operatorname{int} L_+^\infty(\mathbb{R}^k)} \circ \mathcal{H}\big|_{\operatorname{int} L_+^\infty(\mathbb{R}^k)} \colon \operatorname{int} L_+^\infty(\mathbb{R}^k) \longrightarrow \operatorname{int} L_+^\infty(\mathbb{R}^k)$$

is a strict contraction with respect to the Thompson metric on $\operatorname{int} L_+^\infty(\mathbb{R}^k)$.

Proof. By temperedness, B is locally bounded. It then follows from periodicity that B is globally bounded. Now for any $u \in L_+^\infty(\mathbb{R}^k)$, we have

$$|[\mathcal{J}(u)](t)| = \left| \int_{-\infty}^{t} \Xi(\sigma, t) B(\sigma) u(\sigma)\, d\sigma \right| \leqslant P \|B\|_\infty \|u\|_\infty\,, \quad \forall t \in \mathbb{R}\,,$$

where $P \geqslant 0$ is given by Lemma D.25. Since g_j is continuous, it follows that there exists an $M_{j,u} \geqslant 0$ such that

$$g_j([\mathcal{J}(u)](t)) \leqslant M_{j,u}\,, \quad \forall t \in \mathbb{R}\,, \quad j = 1, \ldots, k\,.$$

Thus

$$\big([\mathcal{H}(u)](t)\big)_j \geqslant \frac{\epsilon_j}{B_j + M_{j,u}} > 0\,, \quad \forall t \in \mathbb{R}\,, \quad j = 1, \ldots, k\,,$$

where the ϵ_j's and B_j's are defined as in the proof of Lemma D.20. This proves that $\mathcal{H}(u) \in \operatorname{int} L_+^\infty(\mathbb{R}^k)$. Since $u \in L_+^\infty(\mathbb{R}^k)$ was chosen arbitrarily, we conclude that in fact $\mathcal{H}(L_+^\infty(\mathbb{R}^k)) \subseteq \operatorname{int} L_+^\infty(\mathbb{R}^k)$.

By decreasing the values of the ϵ_j's and increasing the values of the B_j's, if necessary, we may assume without loss of generality that

$$\alpha_j(t) \leqslant B_j\,, \quad \forall t \in \mathbb{R}\,, \quad j = 1, \ldots, k\,,$$

and

$$\beta_j(t) \geqslant \epsilon_j\,, \quad \forall t \in \mathbb{R}\,, \quad j = 1, \ldots, k\,.$$

Therefore

$$\big([\mathcal{H}(u)](t)\big)_j \leqslant \frac{B_j}{\epsilon_j}\,, \quad \forall t \in \mathbb{R}\,, \quad j = 1, \ldots, k\,, \quad \forall u \in L^\infty_+(\mathbb{R}^k)\,,$$

and so $\mathcal{H}(L^\infty_+(\mathbb{R}^k)) \subseteq [0, M]$, where $M \in \operatorname{int} L^\infty(\mathbb{R}^k)$ is defined by

$$M(t) := \left(\frac{B_1}{\epsilon_1}, \ldots, \frac{B_k}{\epsilon_k}\right)\,, \quad t \in \mathbb{R}\,.$$

Fix arbitrarily $u, v \in \operatorname{int} L^\infty(\mathbb{R}^k)$. It follows as in the computations in the proof of Lemma D.20 that

$$\begin{aligned}
d(\mathcal{I}(u), \mathcal{I}(v)) &= d(\mathcal{H}^2(u), \mathcal{H}^2(v)) \\
&= d\left(\alpha \odot \Big(\beta + \mathcal{G}\big(\mathcal{J}(\mathcal{H}(u))\big)\Big)^{-1}, \alpha \odot \Big(\beta + \mathcal{G}\big(\mathcal{J}(\mathcal{H}(v))\big)\Big)^{-1}\right) \\
&\leqslant d\left(\beta + \mathcal{G}\big(\mathcal{J}(\mathcal{H}(u))\big), \beta + \mathcal{G}\big(\mathcal{J}(\mathcal{H}(v))\big)\right) \\
&\leqslant Ld\left(\beta/2 + \mathcal{G}\big(\mathcal{J}(\mathcal{H}(u))\big), \beta/2 + \mathcal{G}\big(\mathcal{J}(\mathcal{H}(v))\big)\right) \\
&\leqslant Ld\left(\mathcal{H}(u), \mathcal{H}(v)\right) \\
&= Ld\left(\alpha \odot \left(\beta + \mathcal{G}(\mathcal{J}(u))\right)^{-1}, \alpha \odot \left(\beta + \mathcal{G}(\mathcal{J}(v))\right)^{-1}\right) \\
&\leqslant Ld\left(u, v\right)\,,
\end{aligned}$$

where

$$L := L(\beta/2, \beta/2 + M)$$

is given by Proposition D.16. Since $u, v \in \operatorname{int} L^\infty(\mathbb{R}^k)$ were chosen arbitrarily, this completes the proof of the result. $\square$

Corollary D.27. *The discrete dynamical system on* $\operatorname{int} L_+^\infty(\mathbb{R}^k)$ *generated by the difference equation*

$$u^+ = \mathcal{H}(u)$$

has a unique, globally attracting equilibrium $u_\infty \in \operatorname{int} L_+^\infty(\mathbb{R}^k)$ *with respect to the Thompson metric on the part* $\operatorname{int} L_+^\infty(\mathbb{R}^k)$*.*

Proof. For $\mathcal{H}^2$ is a strict contraction on a complete metric space (see proof of Lemma D.22 also). □

Proposition D.28. *Assume the same hypotheses as in* Proposition D.23*, except that* g *is not necessarily bounded, but with the additional hypotheses that the matrix paths* A *and* B *are* T*-periodic. Let* $\mathcal{H}\colon L_+^\theta(\mathbb{R}^k) \to L_+^\theta(\mathbb{R}^k)$ *be defined by*

$$[\mathcal{H}(u)](t) := \left(\frac{\alpha_j(t)}{\beta_j(t) + g_j\left(\int_{-\infty}^t \Xi(\sigma, t) B(\sigma) u(\sigma)\, d\sigma \right)} \right)_{j=1}^k , \quad t \in \mathbb{R} ,$$

for each $u \in L_+^\theta(\mathbb{R}^k)$*. Then the discrete dynamical system on* $L_+^\theta(\mathbb{R}^k)$ *generated by the difference equation*

$$u^+ = \mathcal{H}(u)$$

has a unique, globally attracting equilibrium $u_\infty \in \operatorname{int} L_+^\infty(\mathbb{R}^k) \subseteq L_+^\theta(\mathbb{R}^k)$ *in the sense of almost-everywhere, pointwise convergence. Furthermore, the representative* u_∞ *can be chosen to be continuous, in which case convergence is actually everywhere; that is,*

$$[\mathcal{H}^k(u)](t) \longrightarrow u_\infty(t) \quad as \quad k \to \infty , \quad \forall t \in \mathbb{R} , \quad \forall u \in L_+^\theta(\mathbb{R}^k) .$$

Proof. We have

$$\mathcal{H}(L_+^\theta(\mathbb{R}^k)) \subseteq L_+^\infty(\mathbb{R}^k) ,$$

as in the proof of Proposition D.23, and

$$\mathcal{H}(L_+^\infty(\mathbb{R}^k)) \subseteq \operatorname{int} L_+^\infty(\mathbb{R}^k) ,$$

by Lemma D.26. Therefore

$$\mathcal{H}^2(L_+^\infty(\mathbb{R}^k)) \subseteq \operatorname{int} L_+^\infty(\mathbb{R}^k) .$$

From Corollary D.27, $\mathcal{H}\big|_{\operatorname{int} L^\infty_+(\mathbb{R}^k)}$ has a unique, globally attracting fixed point u_∞ (with respect to the Thompson metric). It then follows as in the proof of Proposition D.23 that $(\mathcal{H}^k(u))_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty_+(\mathbb{R}^k)$, each term of which is a continuous path. This establishes the uniform convergence of $(\mathcal{H}^k(u))_{k\in\mathbb{N}}$ to a continuous function which is equal almost everywhere to u_∞. □

D.8 Discrete Time

The discussion in Sections D.4–D.7 above has a natural discrete-time counterpart. The proofs follow along the same lines, and so we omit most of the details.

D.8.1 Tempered Paths

We begin by establishing the appropriate analogue of Lemma D.19. A map

$$B\colon \mathbb{Z} \longrightarrow M_{n\times k}(\mathbb{R})$$

will be said to be a (*discrete*) *tempered path* if

(l1′) for every $\delta > 0$,

$$K_\delta := \sup_{s\in\mathbb{Z}} \|B(s)\|\, \mathrm{e}^{-\delta|s|} < \infty\,.$$

As long as there is no risk of confusion, we will also denote the family of (discrete) tempered paths $\mathbb{Z} \to M_{n\times k}(\mathbb{R})$ by $L^\theta(M_{n\times k}(\mathbb{R}))$. We continue to equip $M_{n\times k}(\mathbb{R})$ with the partial order induced by the nonnegative orthant cone, and thus equip $L^\theta(M_{n\times k}(\mathbb{R}))$ with the partial order induced by the cone $L^\theta_+(M_{n\times k}(\mathbb{R}))$ of nonnegative tempered paths $\mathbb{Z} \to M_{n\times k}(\mathbb{R}_{\geqslant 0})$.

Given a matrix path $A\colon \mathbb{Z} \to M_{n\times n}(\mathbb{R})$, we define $\Xi\colon \mathbb{D} \to M_{n\times n}(\mathbb{R})$ by

$$\Xi(s, s+r) := \prod_{j=s}^{s+r-1} A(j)\,, \quad (s, s+r) \in \mathbb{D}\,, \tag{D.15}$$

where

$$\mathbb{D} := \{(s, s+r)\,;\ s \in \mathbb{Z} \text{ and } r \in \mathbb{Z}_{\geqslant 0}\}\,,$$

and with the notational convention that

$$\Xi(s,s) = \prod_{j=s}^{s-1} A(j) := I_n\,, \quad s \in \mathbb{Z}\,. \tag{D.16}$$

We will be interested in matrix paths $A\colon \mathbb{Z} \to M_{n\times n}(\mathbb{R})$ for which

(l2′) there exist a $\lambda \in (0,1)$ and a nonnegative tempered function $\gamma\colon \mathbb{Z} \to \mathbb{R}$ (in the sense of (l1′) with $n = k = 1$) such that

$$\|\Xi(s,s+r)\| \leqslant \gamma(s)\lambda^r\,, \quad \forall s \in \mathbb{Z}\,, \quad \forall r \in \mathbb{Z}_{\geqslant 0}\,. \tag{D.17}$$

Of course this is equivalent to say that there exist a $\lambda > 0$ and a nonnegative tempered function $\gamma\colon \mathbb{Z} \to \mathbb{R}$ such that

$$\|\Xi(s,s+r)\| \leqslant \gamma(s)\,\mathrm{e}^{-\lambda r}\,, \quad \forall s \in \mathbb{Z}\,, \quad \forall r \in \mathbb{Z}_{\geqslant 0}\,.$$

Thus (l2′) is indeed the natural discrete-time analogue of (L2′). However it is much more convenient, in this discrete-time context, to carry out the computations using the form (D.17) of the estimate.

Lemma D.29. *Let $A\colon \mathbb{Z} \to M_{n\times n}(\mathbb{R})$ and $B\colon \mathbb{Z} \to M_{n\times k}(\mathbb{R})$ be nonnegative matrix paths satisfying* (l1′) *and* (l2′)*—that is, $A_{ij}(m) \geqslant 0$ for every $m \in \mathbb{Z}$, for $i,j = 1,\dots,n$, and analogously for B. Then*

$$[\mathcal{J}^*(u)](m) := \sum_{j=-\infty}^{m-1} \Xi(j+1,m)B(j)u(j)\,, \quad m \in \mathbb{Z}\,, \quad u \in L^\theta(\mathbb{R}^k)\,,$$

defines an order-preserving, linear operator $\mathcal{J}^*\colon L^\theta(\mathbb{R}^k) \to L^\theta(\mathbb{R}^n)$. In particular,

$$\mathcal{J} := \mathcal{J}^*\big|_{L^\theta_+(\mathbb{R}^k)}\colon L^\theta_+(\mathbb{R}^k) \to L^\theta_+(\mathbb{R}^n)$$

is sublinear, and thus nonexpansive with respect to the Thompson metric.

Proof. Analogous to the proof of Lemma D.19. The series converges by comparison with the geometric series in virtue of (l2′) and temperedness. (See also the proof of Lemma D.35 below.) ∎

D.8.2 Conditions for Strict Contractiveness

Now let $L^\infty(\mathbb{R}^n)$ denote the space of bounded, vector-valued two-sided sequences

$$\mathbb{Z} \longrightarrow \mathbb{R}^n\,.$$

Again, equip $L^\infty(\mathbb{R}^n)$ with the partial order induced by the cone $L^\infty_+(\mathbb{R}^n)$ of nonnegative, bounded two-sided sequences $\mathbb{Z} \to \mathbb{R}^n_{\geqslant 0}$. Recall that $L^\infty(\mathbb{R}^n)$ is a Banach space when equipped with the norm $\|\cdot\|_{L^\infty}$, defined by

$$\|u\|_{L^\infty} := \sup_{s\in\mathbb{Z}} |u(s)|\,, \quad u \in L^\infty(\mathbb{R}^n)\,.$$

Moreover, $L^\infty_+(\mathbb{R}^n)$ is a solid, normal cone. The interior $\operatorname{int} L^\infty_+(\mathbb{R}^n)$ of $L^\infty_+(\mathbb{R}^n)$ is the family of two-sided sequences $\mathbb{Z} \to \mathbb{R}^n_{\geqslant 0}$ which are uniformly bounded away from zero and infinity—that is, $u = (u_1, \ldots, u_n) \in \operatorname{int} L^\infty_+(\mathbb{R}^n)$ if, and only if there exist $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \gg 0$ and $M = (M_1, \ldots, M_n)$ such that $\epsilon_i \leqslant u_i(m) \leqslant M_i$ for every $m \in \mathbb{Z}$, for $i = 1, \ldots, n$. Finally, given $u = (u_1, \ldots, u_n) \in \operatorname{int} L^\infty_+(\mathbb{R}^n)$, their coordinatewise inverse $u^{-1} = (u_1^{-1}, \ldots, u_n^{-1})$ is well-defined and also belongs to $\operatorname{int} L^\infty_+(\mathbb{R}^n)$.

Lemma D.30. *Suppose that $A\colon \mathbb{Z} \to M_{n\times n}(\mathbb{R})$ and $B\colon \mathbb{Z} \to M_{n\times k}(\mathbb{R})$ are nonnegative matrix paths satisfying* (l1′) *and* (l2′)*. Let $\mathcal{H}\colon L^\infty_+(\mathbb{R}^k) \to L^\infty_+(\mathbb{R}^k)$ be defined by*

$$[\mathcal{H}(u)](m) := \left(\frac{\alpha_i(m)}{\beta_i(m) + g_i\left(\displaystyle\sum_{j=-\infty}^{m-1} \Xi(j+1, m)B(j)u(j)\right)}\right)_{i=1}^{k}\,, \quad m \in \mathbb{Z}\,,$$

for each $u \in L^\infty_+(\mathbb{R}^k)$, where

(i) *$\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_k)$ are in $\operatorname{int} L^\infty_+(\mathbb{R}^k)$, and*

(ii) *$g = (g_1, \ldots, g_k)\colon \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^k_{\geqslant 0}$ is continuous, bounded, order-preserving and sub-linear.*

Then $\mathcal{H}(L^\infty_+(\mathbb{R}^k)) \subseteq \operatorname{int} L^\infty_+(\mathbb{R}^k)$. Furthermore,

$$\mathcal{I} := \mathcal{H}\big|_{\operatorname{int} L^\infty_+(\mathbb{R}^k)} \colon \operatorname{int} L^\infty_+(\mathbb{R}^k) \longrightarrow \operatorname{int} L^\infty_+(\mathbb{R}^k)$$

is a strict contraction with respect to the Thompson metric on $\operatorname{int} L^\infty_+(\mathbb{R}^k)$*; that is, there exists an* $L \in [0,1)$ *such that*

$$d(\mathcal{I}(u), \mathcal{I}(v)) \leqslant L d(u,v)\,, \quad \forall u, v \in \operatorname{int} L^\infty_+(\mathbb{R}^k)\,. \tag{D.18}$$

Proof. Analogous to the proof of Lemma D.20. □

Lemma D.31. *Assume the same hypotheses as in* Lemma D.30*, except for replacing* (i) *and* (ii) *in that lemma by*

(i′) $\alpha = (\alpha_1, \ldots, \alpha_k)$*,* $\widetilde{\alpha} = (\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_k)$*,* $\beta = (\beta_1, \ldots, \beta_k)$ *and* $\widetilde{\beta} = (\widetilde{\beta}_1, \ldots, \widetilde{\beta}_k)$ *are in* $\operatorname{int} L^\infty_+(\mathbb{R}^k)$*, and satisfy*

$$\frac{\alpha_i(m)}{\beta_i(m)} \geqslant \frac{\widetilde{\alpha}_i(m)}{\widetilde{\beta}_i(m)}\,, \quad \forall m \in \mathbb{Z}\,, \quad i = 1, \ldots, k\,, \tag{D.19}$$

and

(ii′) $g = (g_1, \ldots, g_k)\colon \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^k_{\geqslant 0}$ *is continuous, order-preserving and sublinear,*

and let $\mathcal{H}\colon L^\infty_+(\mathbb{R}^k) \to L^\infty_+(\mathbb{R}^k)$ *be defined by*

$$[\mathcal{H}(u)](m) := \left(\frac{\alpha_i(m) + \widetilde{\alpha}_i(m) g_i\left(\sum_{j=-\infty}^{m-1} \Xi(j+1, m) B(j) u(j) \right)}{\beta_i(m) + \widetilde{\beta}_i(m) g_i\left(\sum_{j=-\infty}^{m-1} \Xi(j+1, m) B(j) u(j) \right)} \right)_{i=1}^{k}\,, \quad m \in \mathbb{Z}\,,$$

for each $u \in L^\infty_+(\mathbb{R}^k)$*. Then the same conclusions as in* Lemma D.30 *hold; that is,* $\mathcal{H}(L^\infty_+(\mathbb{R}^k)) \subseteq \operatorname{int} L^\infty_+(\mathbb{R}^k)$*, and* $\mathcal{I} := \mathcal{H}\big|_{\operatorname{int} L^\infty_+(\mathbb{R}^k)} \colon \operatorname{int} L^\infty_+(\mathbb{R}^k) \to \operatorname{int} L^\infty_+(\mathbb{R}^k)$ *is a strict contraction with respect to the Thompson metric on* $\operatorname{int} L^\infty_+(\mathbb{R}^k)$*.*

Proof. Analogous to the proof of Lemma D.21. □

D.8.3 Unique, Globally Attracting Equilibria

Lemma D.32. *Under the same hypotheses as in either* Lemma D.30 *or* Lemma D.31*, the discrete dynamical system on* $\operatorname{int} L^\infty_+(\mathbb{R}^k)$ *generated by the difference equation*

$$u^+ = \mathcal{H}(u) \tag{D.20}$$

has a unique, globally attracting equilibrium $u_\infty \in \operatorname{int} L^\infty_+(\mathbb{R}^k)$ *with respect to the Thompson metric on the part* $\operatorname{int} L^\infty_+(\mathbb{R}^k)$*.*

Proof. Analogous to the proof of Lemma D.22. $\square$

Proposition D.33. *Suppose that* $A\colon \mathbb{Z} \to M_{n\times n}(\mathbb{R})$ *and* $B\colon \mathbb{Z} \to M_{n\times k}(\mathbb{R})$ *are non-negative matrix paths satisfying* (l1′) *and* (l2′)*. Let* $\mathcal{H}\colon L^\theta_+(\mathbb{R}^k) \to L^\theta_+(\mathbb{R}^k)$ *be defined by*

$$[\mathcal{H}(u)](m) := \left(\frac{\alpha_i(m)}{\beta_i(m) + g_i\left(\displaystyle\sum_{j=-\infty}^{m-1} \Xi(j+1,m)B(j)u(j)\right)} \right)_{i=1}^{k}, \quad m \in \mathbb{Z},$$

for each $u \in L^\theta_+(\mathbb{R}^k)$*, where*

(i) $\alpha = (\alpha_1,\dots,\alpha_k)$ *and* $\beta = (\beta_1,\dots,\beta_k)$ *are in* $\operatorname{int} L^\infty_+(\mathbb{R}^k)$*, and*

(ii) $g = (g_1,\dots,g_k)\colon \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^k_{\geqslant 0}$ *is continuous, bounded, order-preserving and sub-linear.*

Then the discrete dynamical system on $L^\theta_+(\mathbb{R}^k)$ *generated by the difference equation*

$$u^+ = \mathcal{H}(u)$$

has a unique, globally attracting equilibrium $u_\infty \in \operatorname{int} L^\infty_+(\mathbb{R}^k) \subseteq L^\theta_+(\mathbb{R}^k)$ *in the sense of pointwise convergence.*

Proof. This follows along the same lines as the proof of Proposition D.23. Note that, in the discrete-time case, we need not worry about the continuity[2] of u_∞. Pointwise convergence follows straight from

$$|u_k(m) - u_\infty(m)| \leqslant \|u_k - u_\infty\|_{L^\infty} \longrightarrow 0 \quad \text{as} \quad k \to \infty$$

for each $m \in \mathbb{Z}$. $\square$

[2]More precisely, u_∞ is automatically continuous, since the standard topology on $\mathbb{Z}$ is discrete.

Proposition D.34. *Assume the same hypotheses as in* Proposition D.33*, except for replacing* (i) *and* (ii) *in that proposition by*

(i′) $\alpha = (\alpha_1, \ldots, \alpha_k)$*,* $\widetilde{\alpha} = (\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_k)$*,* $\beta = (\beta_1, \ldots, \beta_k)$ *and* $\widetilde{\beta} = (\widetilde{\beta}_1, \ldots, \widetilde{\beta}_k)$ *are in* $\operatorname{int} L^\infty_+(\mathbb{R}^k)$*, and satisfy*

$$\frac{\alpha_i(m)}{\beta_i(m)} \geqslant \frac{\widetilde{\alpha}_i(m)}{\widetilde{\beta}_i(m)}\,, \quad \forall m \in \mathbb{Z}\,, \quad i = 1, \ldots, k\,, \tag{D.21}$$

and

(ii′) $g = (g_1, \ldots, g_k)\colon \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^k_{\geqslant 0}$ *is continuous, order-preserving and sublinear,*

and let $\mathcal{H}\colon L^\theta_+(\mathbb{R}^k) \to L^\theta_+(\mathbb{R}^k)$ *be defined by*

$$[\mathcal{H}(u)](m) := \left(\frac{\alpha_i(m) + \widetilde{\alpha}_i(m) g_i\left(\displaystyle\sum_{j=-\infty}^{m-1} \Xi(j+1,m)B(j)u(j)\right)}{\beta_i(m) + \widetilde{\beta}_i(m) g_i\left(\displaystyle\sum_{j=-\infty}^{m-1} \Xi(j+1,m)B(j)u(j)\right)} \right)_{i=1}^{k}\,, \quad m \in \mathbb{Z}\,,$$

for each $u \in L^\theta_+(\mathbb{R}^k)$*. Then the discrete dynamical system on* $L^\theta_+(\mathbb{R}^k)$ *generated by the difference equation*

$$u^+ = \mathcal{H}(u)$$

has a unique, globally attracting equilibrium $u_\infty \in \operatorname{int} L^\infty_+(\mathbb{R}^k) \subseteq L^\theta_+(\mathbb{R}^k)$ *in the sense of pointwise convergence.*

Proof. Analogous to the proof of Proposition D.24. □

D.8.4 The Periodic Case

As in the continuous-time case, the boundedness assumption for g in Proposition D.33 is not needed if A and B are T-periodic.

Lemma D.35. *Suppose that* $A\colon \mathbb{Z} \to M_{n\times n}(\mathbb{R})$ *is a* T*-periodic matrix path satisfying* (12′)*. Then the map*

$$m \longmapsto \sum_{j=-\infty}^{m-1} \Xi(j+1,m)\,, \quad m \in \mathbb{Z}\,, \tag{D.22}$$

is T-periodic. In particular, it is uniformly bounded away from infinity; in other words, there exists a $P \geqslant 0$ such that

$$\left\| \sum_{j=-\infty}^{m-1} \Xi(j+1,m) \right\| \leqslant P < \infty\,, \quad \forall m \in \mathbb{Z}\,. \tag{D.23}$$

Proof. The convergence of the series for each fixed $m \in \mathbb{Z}$ follows from the estimate in (12′) and the definition of temperedness in (11′). For some $\lambda \in (0,1)$, we have

$$\begin{aligned}
\left| \sum_{j=-\infty}^{m-1} \Xi(j+1,m) \right| &\leqslant \sum_{j=-\infty}^{m-1} \gamma(j+1)\lambda^{m-j-1} \\
&\leqslant \lambda^{|m|} \sum_{j=-\infty}^{m-1} \gamma(j+1)\lambda^{\frac{1}{2}|j+1|}\lambda^{\frac{1}{2}|j+1|} \\
&\leqslant \lambda^{|m|} K_\delta \sum_{j=-\infty}^{m-1} \left(\lambda^{\frac{1}{2}}\right)^{|j+1|}\,,
\end{aligned}$$

with

$$\delta := -\frac{1}{2}\log\lambda > 0$$

in the temperedness constant of γ. Thus the series in (D.22) converges by comparison with the geometric series.

To establish T-periodicity, recall the definition of Ξ in (D.15) and (D.16). We have

$$\begin{aligned}
\sum_{j=-\infty}^{m+T-1} \Xi(j+1,m+T) &= \sum_{j=-\infty}^{m+T-1} \left(\prod_{k=j+1}^{m+T-1} A(k) \right) \\
&= \sum_{\hat{\jmath}=-\infty}^{m-1} \left(\prod_{k=\hat{\jmath}+T+1}^{m+T-1} A(k) \right) \\
&= \sum_{\hat{\jmath}=-\infty}^{m-1} \left(\prod_{\hat{k}=\hat{\jmath}+1}^{m-1} A(k+T) \right) \\
&= \sum_{j=-\infty}^{m-1} \left(\prod_{k=j+1}^{m-1} A(k) \right) \\
&= \sum_{j=-\infty}^{m-1} \Xi(j+1,m)\,, \quad \forall m \in \mathbb{Z}\,,
\end{aligned}$$

thus establishing that the map defined in (D.22) is T-periodic.

The fact that there exists a $P \geqslant 0$ such that (D.23) holds now follows from the simple observation that the map defined by (D.22) only takes up finitely many values. $\square$

Lemma D.36. *Assume the same hypotheses as in* Lemma D.30*, except that g is not necessarily bounded, but with the additional hypotheses that the matrix paths A and B be T-periodic. Let $\mathcal{H}\colon L^\infty_+(\mathbb{R}^k) \to L^\infty_+(\mathbb{R}^k)$ be defined by*

$$[\mathcal{H}(u)](m) := \left(\frac{\alpha_i(m)}{\beta_i(m) + g_i\left(\displaystyle\sum_{j=-\infty}^{m-1} \Xi(j+1,m)B(j)u(j)\right)} \right)_{i=1}^{k}, \quad m \in \mathbb{Z}\,,$$

for each $u \in L^\infty_+(\mathbb{R}^k)$. Then $\mathcal{H}(L^\infty_+(\mathbb{R}^k)) \subseteq \operatorname{int} L^\infty_+(\mathbb{R}^k)$, and

$$\mathcal{I} := \mathcal{H}\big|_{\operatorname{int} L^\infty_+(\mathbb{R}^k)} \circ \mathcal{H}\big|_{\operatorname{int} L^\infty_+(\mathbb{R}^k)} \colon \operatorname{int} L^\infty_+(\mathbb{R}^k) \longrightarrow \operatorname{int} L^\infty_+(\mathbb{R}^k)$$

is a strict contraction with respect to the Thompson metric on $\operatorname{int} L^\infty_+(\mathbb{R}^k)$*.*

Proof. Analogous to the proof of Lemma D.26. For each arbitrarily fixed $u \in L^\infty_+(\mathbb{R}^k)$ and $i \in \{1,\dots,k\}$, we can use Lemma D.35 and the continuity of g_i to show that

$$m \longmapsto g_i\left(\sum_{j=-\infty}^{m-1} \Xi(j+1,m)B(j)u(j)\right), \quad m \in \mathbb{Z}\,,$$

is bounded. □

Corollary D.37. *The discrete dynamical system on* $\operatorname{int} L^\infty_+(\mathbb{R}^k)$ *generated by the difference equation*

$$u^+ = \mathcal{H}(u)$$

has a unique, globally attracting equilibrium $u_\infty \in \operatorname{int} L^\infty_+(\mathbb{R}^k)$ with respect to the Thompson metric on the part $\operatorname{int} L^\infty_+(\mathbb{R}^k)$*.*

Proof. Analogous to the proof of Lemma D.22. □

Proposition D.38. *Assume the same hypotheses as in* Proposition D.33*, except that g is not necessarily bounded, but with the additional hypotheses that the matrix paths A and B are T-periodic. Let $\mathcal{H}\colon L^\theta_+(\mathbb{R}^k) \to L^\theta_+(\mathbb{R}^k)$ be defined by*

$$[\mathcal{H}(u)](m) := \left(\frac{\alpha_i(m)}{\beta_i(m) + g_i\left(\displaystyle\sum_{j=-\infty}^{m-1} \Xi(j+1,m)B(j)u(j)\right)} \right)_{i=1}^{k}, \quad m \in \mathbb{Z}\,,$$

for each $u \in L_+^\theta(\mathbb{R}^k)$*. Then the discrete dynamical system on* $L_+^\theta(\mathbb{R}^k)$ *generated by the difference equation*

$$u^+ = \mathcal{H}(u)$$

has a unique, globally attracting equilibrium $u_\infty \in \operatorname{int} L_+^\infty(\mathbb{R}^k) \subseteq L_+^\theta(\mathbb{R}^k)$ *in the sense of pointwise convergence.*

Proof. Analogous to the proof of Proposition D.28. $\square$

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