HIGH SCHOOL MATHEMATICS TEACHERS' USE OF BELIEFS AND

KNOWLEDGE IN HIGH-QUALITY INSTRUCTION

by

KATHRYN E. RHOADS

A dissertation submitted to the

Graduate School-New Brunswick

Rutgers, The State University of New Jersey

In partial fulfillment of the requirements

For the degree of

Doctor of Philosophy

Graduate Program in Education

Written under the direction of

Keith Weber

And approved by

____

____

____

____

____

New Brunswick, New Jersey

October, 2014

ABSTRACT OF THE DISSERTATION

High School Mathematics Teachers' Use of Beliefs and Knowledge in High-Quality Instruction

By KATHRYN E. RHOADS

Dissertation Director:

Keith Weber

High-quality mathematics instruction is important for students' learning, and teachers are a key part of instruction. As they engage in instruction, teachers draw on their beliefs and knowledge. Yet mathematics education still lacks a robust understanding of the specific ways in which beliefs and knowledge contribute to high-quality instruction, particularly at the high school level. The purpose of this dissertation is to explore the mathematical knowledge and beliefs used by high school teachers who facilitate high-quality instruction. Three main research questions guide this dissertation: (a) What is the nature of mathematical knowledge expressed in exemplary high school mathematics teachers' reflections on teaching? (b) What teacher beliefs and knowledge support high-quality responses to students? (c) How can productive teacher beliefs about mathematics and mathematics teaching lead to instruction that is limited in mathematical richness?

To investigate the first research question, I interviewed 11 high school mathematics teachers who were recognized for exemplary instruction. I used grounded analysis to explore the mathematical knowledge for teaching that was expressed through teachers' reflections on their lessons. In response to the second research question, I observed and interviewed 12 high school teachers, five of whom were recognized for exemplary instruction. I used video-based, stimulated-recall interviews to understand the teacher beliefs and knowledge that supported or hindered high-quality responses to students' mathematical questions, claims, and solutions. To address the third research question, I explored the case of one recognized teacher who expressed beliefs and goals aligned with mathematical meaning and sense making, yet his instruction did not exemplify these aspects. I used observations and interviews to understand the teacher's perspectives on his instruction, and I offer explanations for why this instruction was limited in richness.

The findings highlight the depth and complexity of mathematical knowledge and beliefs used in high-quality instruction and challenge the assumption that either teacher beliefs or teacher knowledge can be studied in isolation or outside of the instruction in which they are used.

Acknowledgements

First and foremost, thank you to my advisor, Keith Weber, for your support throughout this process. Thank you for believing in me and agreeing to supervise my work, despite the fact that it was outside the realm of your current research. My sincere thanks for the time and effort that you spent in helping me to understand research and the workings of academia. I am truly grateful for your guidance.

Thank you to the other members of my committee: Dan Battey, Pablo Mejia, Eugenia Etkina, and Eva Thanheiser. I appreciate your critical guidance for this dissertation, and I have learned a great deal from your expertise. Several other faculty members at Rutgers University also helped to shape my work. Thank you to Carolyn Maher, Drew Gitomer, Sharon Ryan, Alisa Belzer, William Firestone, and Catherine Lugg for your support and guidance. Special thanks to Luis Leyva for working with me to code teacher interviews—your insights were very helpful. Thanks also to Nicole Shea and Aron Samkoff for reading and commenting on drafts of this work.

This work would not have been possible without the teachers and schools who graciously agreed to participate in this project. I am truly appreciative for your help and hope you find the results to be useful.

I would like to acknowledge that portions and earlier versions of this work have been published in conference proceedings (Rhoads, 2011; Rhoads, 2013, see Acknowledgement of Previous Publication section). These opportunities helped to shape and refine the research, and I am grateful for the feedback that colleagues provided at those conferences.

Many thanks are due to Amy Cohen-Corwin and the faculty, staff, and teachers in the New Jersey Partnership for Excellence in Middle School Mathematics. My participation in the project has improved my research, teaching, and learning of mathematics and mathematics education, and I am truly grateful for the experience to work with so many knowledgeable colleagues.

I am also grateful to the P.E.O. Scholar Awards for your support in helping me to complete my doctoral program. Thank you for believing in this research.

To my mathematics education graduate student peers (some of whom are no longer students), Cathleen Rossman, Cecilia Arias, Kristen Lew, Aron Samkoff, Lou Pedrick, Luis Leyva, Jennifer Jones and others, thank you for your encouragement along the way. A special thanks to Iuliana Radu: I am very appreciative of your guidance, expertise, and friendship throughout many professional projects and personal experiences. I have learned a great deal from you.

I am especially grateful for the support of my family. Thank you to my parents, grandparents, brothers, and extended family for believing in me and supporting me in all the professional and personal endeavors I have pursued. I would not be where I am today without your love and support.

Finally, I offer many thanks to my husband, John. I cannot express how grateful I am for your support throughout my doctoral studies and otherwise. Thank you.

Dedication

To John, for your unwavering love and support.

Table of Contents

Abstract of the Dissertation ii

Acknowledgements iv

Dedication vi

Table of Contents vii

List of Tables x

List of Figures xi

Chapter 1: Introduction 1

1.1 Purpose and Research Questions 3

1.2 Overview of the Dissertation 3

Chapter 2: Research on Mathematics Teachers' Knowledge and Beliefs 8

2.1 Definitions 8

2.2 Mathematics Teacher Beliefs 10

2.3 Mathematics Teacher Knowledge 13

2.4 Beliefs, Knowledge, and Instruction: A Call for the Dissertation 37

Chapter 3: Exemplary High School Mathematics Teachers' Reflections on Teaching: A Situated Cognition Perspective 39

3.1 Introduction 40

3.2 Background 41

3.3 Methods 48

3.4 Findings 54

3.5 Conclusion 69

Chapter 4: Teacher Thinking behind Responding to Student Mathematical Productions in High School Mathematics Instruction 73

4.1 Introduction.. 74

4.2 Background 76

4.3 Methods.. 84

4.4 Findings... 107

4.5 Conclusion 139

Chapter 5: Barriers to Implementing Mathematical Richness in High School Mathematics Instruction 148

5.1 Introduction... 149

5.2 Background 151

5.3 Methods... 158

5.4 Background and Setting for the Case.. 175

5.5 Findings... 176

5.6 Discussion 212

Chapter 6: Conclusion 221

6.1 Findings and Significance 221

6.2 Limitations 223

6.3 Directions for Research and Teacher Education.............................. 224

Appendix A: Study 1 Interview Protocol... 227

Appendix B: Supplementary Material for Themes of Teachers’ Reflections................. 231

Appendix C: Study 2 Background Questionnaire... 239

Appendix D: Prelesson Interview Protocol... 240

Appendix E: Stimulated-Recall Interview Protocol 242

Appendix F: Dimensions of Mathematical Quality of Instruction 245

Appendix G: Additional Data Illustrating Mr. Taylor's Overarching Goals and Supporting Orientations and Knowledge 246

References 252

Acknowledgement of Previous Publication 266

List of Tables

Table 2.1 A Comparison of Two Theoretical Perspectives for Mathematical Knowledge for Teaching. 22

Table 3.1 Themes of Reflections: Mathematics-Specific Attunements in Teaching 59

Table 4.1 Characteristics of Classes Observed and Number of Teachers 89

Table 4.2 Number of Responses to SMPs According to MQI of Response 96

Table 4.3 Number of SMPs According to Type and MQI 97

Table 4.4 Coding Scheme for Teachers' Reflections: Goals, Orientations, or Knowledge 101

Table 4.5 Categories of Goals in Teachers' Reflections 104

Table 4.6 Categories of Knowledge in Teachers' Reflections 105

Table 4.7 Number of Teachers Expressing Each Theme in Relationship to MQI of Response 108

Table 4.8 Teachers' Goals Supporting Themes behind High-MQI Responses 110

Table 4.9 Teachers' Knowledge Supporting Themes behind High-MQI Responses ... 110

Table 5.1 Percentage of Segments Scored High, Mid, and Low According to Teacher and Dimension 165

Table 5.2 Content Codes Aligning with Elements of Richness of Mathematics 167

Table 5.3 Counts for Interview Passages Aligned with Mathematical Richness 170

Table 5.4 Goals, Orientations, and Resources Expressed in Interviews 173

Table 5.5 Mr. Taylor's Overarching Goals and Supporting Orientations and Knowledge 177

List of Figures

Figure 4.1 Overview of each teacher's observation and interview data collection. 89

Figure 4.2 Overview of data analysis procedure and model of relationships between components. 94

Figure 4.3 Replica of student solution in Mr. Anderson's class.................................. 112

Figure 4.4 Definitions for geometric sequence provided in Ms. Zimmerman's class. . 120

Figure 5.1 Overview of Mr. Taylor's observation and interview data collection......... 160

Figure 5.2 Overview of data analysis procedure and model of relationships between components. 163

Figure 5.3 Relationships among Mr. Taylor's overarching goals................................ 183

Figure 5.4 Derivation of the standard form of the equation for an ellipse.................... 186

Figure 5.5 Eccentricity definition and visual representation. 191

Figure 5.6 A cutaway view of a semiellipsoid.. 197

Figure 5.7 A model of the reflection property of an ellipse... 197

Figure 5.8 Graph of the whispering chamber floor and dome. 202

Figure 5.9 Relationship between axes and foci in an ellipse. 209

Chapter 1: Introduction

In recent decades, primary and secondary students in the United States have performed inadequately in mathematics when compared to their international peers (U.S. Department of Education, 2008), with students in the United States having poor conceptual understanding of the mathematics they use (Stigler & Hiebert, 1999). In particular, it is common for students to perform mathematical procedures or recite facts without understanding the meaning and relationships behind those facts and procedures. Several researchers and organizations have suggested that to build both interest and success in the mathematical sciences, students of all ages should have opportunities to engage in dynamic mathematical activities such as observing mathematical phenomena, making mathematical conjectures, and communicating, justifying, and debating these conjectures (Davis, Maher, & Noddings, 1990; National Council of Teachers of Mathematics [NCTM], 1989, 2000). Researchers have argued that these processes could lead to increased student engagement and conceptual understanding (Kilpatrick, Martin, & Schifter, 2003).

For such flexible student thinking and open-ended problem solving to be realized in classrooms, teachers must be able to facilitate these learning environments, and creating and working within mathematically engaging classrooms can be demanding for teachers. Such environments require that teachers carefully plan and select mathematically rich activities for instruction, and they also require that teachers make in-the-moment decisions to respond to students in ways that will advance their mathematical thinking. For example, teachers must be able to answer unexpected student questions and quickly evaluate a variety of students' problem-solving strategies. In their own schooling, many mathematics teachers in the United States learned mathematics through practicing and memorizing mechanical procedures (Conference Board of the Mathematical Sciences, [CBMS] 2001). For these teachers, dynamic learning environments such as those described above are unfamiliar territory. Hence, the role of teacher education is particularly important in helping teachers to develop the expertise needed to engage in mathematically rich, student-centered instruction.

To provide effective teacher education, it is important to understand how and why teachers do what they do in instruction. Expertise in mathematics instruction includes teachers' fluency in making both planned and in-the-moment decisions for instruction (Silver & Mesa, 2011), and researchers have argued that such decisions are driven by teachers' beliefs and knowledge (Ball, Thames, & Phelps, 2008; Forgasz & Leder, 2008; Schoenfeld, 2011). Wilson and Cooney (2002) emphasized this point, writing, "The evidence is clear that teacher thinking influences what happens in classrooms, what teachers communicate to students, and what students ultimately learn" (p. 144). Hence, to understand how and why teachers make the decisions that they do, it is important to understand teachers' beliefs and knowledge and how beliefs and knowledge are used in high-quality mathematics instruction.

Although there has been a great deal of research on teachers' knowledge and beliefs and how these are used in instruction (much of which is reviewed in Chapter 2 of this dissertation), with notable exceptions (e.g., Thomas & Yoon, 2014; Törner, Rolka, Rösken, Sriraman, 2010) there is far less research focused on understanding the ways in which beliefs and knowledge interact to inform teachers' pedagogical decisions and subsequent actions. As Simon and Tzur (1999) articulated, "we see a teacher's practice as a conglomerate that cannot be understood by looking at parts split off from the whole (i.e., looking only at beliefs or methods of questioning or mathematical knowledge)" (p. 254). In this line of inquiry, this dissertation will use both interviews about and observations of instruction to understand teacher beliefs and knowledge and ways these are used in instruction, particularly high-quality instruction.

1.1 Purpose and Research Questions

The purpose of this dissertation is to explore the mathematical knowledge and beliefs used by high school teachers who facilitate high-quality instruction. Three main research questions guide this dissertation:

1. What is the nature of mathematical knowledge expressed in exemplary high school mathematics teachers' reflections on teaching?
2. What teacher beliefs and knowledge support high-quality responses to students?
3. How can productive teacher beliefs about mathematics and mathematics teaching lead to instruction that is limited in mathematical richness?

Answering these questions can highlight directions for teacher education regarding both what knowledge and beliefs are needed for high-quality instruction and how these knowledge and beliefs are expressed and used in instruction.

1.2 Overview of the Dissertation

This dissertation addresses the research questions, as follows. I begin with Chapter 2, in which I define the terms used in this dissertation, summarize the existing literature on mathematics teacher beliefs and knowledge, and situate this dissertation within the existing literature. Chapters 3-5 each address one of the research questions and can be read independently of one another.

In Chapter 3, I report on a study that sought to understand how knowledge is used to support high-quality mathematics instruction by considering the mathematical knowledge expressed in exemplary teachers' reflections on their teaching. Participants were 11 high school teachers who were recognized for their exemplary instruction in New Jersey through state and national recognition programs. Through individual interviews, teachers reflected on lessons they had taught with consideration of the mathematical knowledge used in teaching. This study was originally conceived using a cognitive perspective on knowledge, looking at the knowledge that these teachers possessed that helped to make them exemplary; however, I found a situated cognition perspective better elucidated the essence of teachers' accounts. Specifically, teachers did not discuss their knowledge abstractly. Instead, their knowledge was embedded in their accounts of the process of teaching. Teachers' reflections illustrated the simultaneous coordination of several ways of knowing and participating in the process of teaching. For these exemplary teachers, content knowledge and pedagogical content knowledge was expressed through their accounts of teaching and was deeply intertwined with their descriptions of actions in teaching. That is, I found that although the exemplary teachers in this study did not specifically reflect on abstract knowledge, they did express their mathematical understandings through their discussions of how they achieved pedagogical goals.

With the results of the first study in mind, I designed a second study to take a different approach to understanding how teachers use beliefs and knowledge in high-quality instruction. I extended the work from the first study in three main ways. First, a limitation of the first study was that the data collection was limited to interviews with teachers. This afforded the opportunity to understand teachers' perspectives on their instruction, but it did not allow me to explore what teachers actually do in practice. Hence, the second study was designed to include classroom observations to obtain a more complete view of instruction. Second, the first study focused on understanding teachers' knowledge and its use, but teachers' beliefs also impact their instruction in important ways. In fact, in the first study, teachers' reflections often included aspects of their beliefs. As a result, the second study took a more broad approach to understand how both teachers' knowledge and their beliefs interact as they are used in instruction. Third, the first study indicated that teachers' understandings of mathematics were expressed through their discussions of how they achieved pedagogical goals. Building on this finding, the second study sought to explore how teachers' beliefs and knowledge guided specific actions in their instruction.

Participants in the second study were 12 high school mathematics teachers who taught in the greater New Jersey area; five of whom had been recognized for their exemplary instruction. Observations, interviews, and written materials were used to explore teachers' instructional decisions. Each teacher was observed and video-recorded for three consecutive lessons, engaged in a prelesson interview before each lesson (three total), and engaged in a stimulated-recall interview after the three observations were complete. In the stimulated-recall interviews, each teacher was asked to watch videos of and reflect on approximately 6 teaching episodes from the three days of instruction. The analysis of this study was split into two chapters in this dissertation.

In Chapter 4, I report on how teachers in the second study used their knowledge and beliefs in responding to students' mathematical questions, claims, and errors, which the Learning Mathematics for Teaching Project (LMT, 2010) called *student mathematical productions* (SMPs). In observing instruction, I identified SMPs and teachers' responses to these. Each teacher watched and reflected on between one and seven of these episodes in the stimulated-recall interview. Teachers' responses to students were characterized using LMT's (2010) mathematical quality of instruction (MQI) rubric as a guide, and teachers' beliefs and knowledge that corresponded to these specific episodes were explored using grounded analysis of the stimulated-recall interviews. In reflecting on responses to SMPs that were coded high in MQI, teachers expressed how they built on student ideas and emphasized mathematical meaning in their responses. By contrast, in reflecting on responses to SMPs that were coded low in MQI, teachers prioritized goals that were not aligned with the SMP or expressed that they lacked knowledge in the moment of responding to the SMP. This chapter highlights the role of beliefs in teachers' use of mathematical knowledge in teaching decisions.

In Chapter 5, I present the case of one teacher from the second study who expressed orientations and goals for instruction aligned with meaning and sense making in mathematics yet had instruction that was limited in mathematical richness in the sense of LMT's MQI (2010). I used grounded analysis of interviews to describe this teacher's overarching goals for instruction, and I then used additional data from interviews and observations to explore the reasons that the instruction was limited in richness. Through this process, I was better able to understand the teacher's perspective on his instruction, and I identified three themes that could account for the limited richness in instruction: (a) conceptions about what constituted meaning in instruction, (b) inattention to precision and clarity in instruction, and (c) beliefs about students' academic abilities. This chapter points to the depth and complexity of teachers' beliefs and knowledge that are needed to engage in instruction that is mathematically rich and highlights how the teacher's beliefs about students' abilities can shape the mathematics offered to them.

In Chapter 6, I provide concluding thoughts on the dissertation. The chapters in this dissertation provide three views of the teacher beliefs and knowledge that are needed to support high-quality mathematics instruction. In addition, this dissertation challenges the assumption that either teacher beliefs or teacher knowledge can be studied in isolation or outside of the instruction in which they are used.

Chapter 2: Research on Mathematics Teachers' Knowledge and Beliefs

Teachers have an unmistakably important role in instructional improvement. For instance, teachers must attend to students' cognitive issues as well as their affective issues, and at the same time, teachers act as representatives of the mathematics community, establishing the norms of mathematics as a discipline (Yackel & Cobb, 1996). Teachers also mediate students' access to mathematics content through their decisions about what content is discussed, how it is discussed, and how learning is assessed.

To manage such a role, teachers draw on both their beliefs and their knowledge about mathematics, teaching, learning, and students. In this chapter, I clarify the definitions of beliefs and knowledge that are used in this dissertation, summarize the research on mathematics teachers' beliefs and knowledge, and situate this dissertation within the existing research.

2.1 Definitions

There have been several definitions of *beliefs* and *knowledge* in the mathematics education literature (Leder, Pehkonen, & Törner, 2002). Many researchers consider these two constructs to be part of the same continuum of concepts that an individual regards as true (e.g., Anderson, White, & Sullivan, 2005; Beswick, 2007; Furinghetti & Pehkonen, 2002) and distinguish between beliefs and knowledge according to "the quality and quantity of evidence upon which they are based" (Beswick, 2007, p. 96). Whereas knowledge is based upon evidence that allows it to be proven or disproven by others, beliefs are considered to be confidence in the truth of something that cannot be proved by others (Furinghetti & Pehkonen, 2002). Beliefs are frequently thought of as judgments about the world and often can be stated using the auxiliary verb *should.* For example, one teacher belief might be "Students should experience mathematics through hands-on activities," whereas one aspect of a teacher's knowledge might be "The graph of $y = 2x + 5$ is a line with slope 2 and y-intercept (0, 5)." In some cases, the distinctions between beliefs and knowledge are not completely clear; however, I contend that classifying a teacher's particular conception as belief or knowledge is not as important as how the conception helps teachers to carry out their work.

To that end, rather than focus statically on what knowledge and beliefs teachers possess outside of the act of teaching, I take a more situated approach to begin with teachers' instruction and consider not only what knowledge and beliefs are used but also how these are used in instruction. In this dissertation, I draw on Schoenfeld's (2011) theory of goal-oriented decision making and use his definitions of orientations and knowledge. Schoenfeld defined *orientations* to include "dispositions, beliefs, values, tastes, and preferences" (Schoenfeld, 2011, p. 29). The importance of orientations in teaching is that they shape what teachers perceive in situations and hence shape the goals that teachers have in those situations. Schoenfeld's conception of orientations is helpful in this regard: Teachers' goals are influenced by beliefs (e.g., students should not be talking during class) as well as other types of orientations, such as their preferences and values (e.g., I prefer that the classroom is student centered; I value students' thinking). I also use Schoenfeld's definition of *knowledge*, which is information that individuals have "potentially available to bring to bear in order to solve problems, achieve goals, or perform other such tasks" (Schoenfeld, 2011, p. 25). In teaching, teachers may draw on information about mathematics, their students, how to teach mathematics, and so on.

In the sections that follow, I summarize related literature on mathematics teachers' beliefs and knowledge. Most literature has focused specifically on teachers' beliefs rather than the more general concept of orientations. Throughout the dissertation, I use both *beliefs* and *orientations* as they apply to the particular situations being discussed.

2.2 Mathematics Teacher Beliefs

Research in mathematics education has shown that teacher beliefs matter in mathematics instruction. What a teacher believes about mathematics as a discipline, teaching mathematics, students, and themselves drive the decisions that they make in teaching (e.g., Thompson, 1984; Wilson & Cooney, 2002). In this section, I present a brief summary of the research on mathematics teacher beliefs.

Much of the research on teachers' beliefs has been tied to efforts to reform mathematics instruction. Such efforts have been prevalent over the last three decades and emphasize the importance of student-centered approaches to instruction that focus on meaning making and reasoning in mathematics (e.g., Davis, Maher, & Noddings, 1990; NCTM, 1989, 2000). Some research has illustrated that instructional reform cannot be realized unless teachers' beliefs about mathematics and mathematics teaching align with these student-centered practices. For example, Lloyd (1999) illustrated that when using curriculum materials designed to promote student-centered practices in the classroom, teachers' conceptions of what student-centered learning is and how it should be implemented shaped the implementation of this curriculum.

Hence, there has been a recent line of research that has focused on identifying what beliefs teachers have, with an eye towards beliefs that align with student-centered instruction. For example, Barkatsas and Malone (2005) surveyed 465 secondary mathematics teachers in Greece regarding their beliefs about mathematics teaching and learning. Using principal components analysis, the researchers characterized beliefs into two broad orientations: a contemporary orientation aligned with student-centered views of teaching and learning and a traditional orientation aligned with a transmission view of teaching and learning. The researchers also found that teachers' views of mathematics were deeply intertwined with their views of mathematics teaching and learning. As such, professional development aimed at changing teachers' beliefs may need to focus on both mathematics and mathematics pedagogy.

Beswick (2007) began by studying teaching to understand beliefs that support practice. Two secondary teachers were identified because they used constructivist teaching and learning in their classrooms, and Beswick used teacher and student surveys, interviews, and classroom observations to identify beliefs that underpinned these teachers' instruction. She identified nine crucial beliefs that influenced these teachers' practices and categorized these beliefs into categories: beliefs about mathematics, beliefs about students' learning, and beliefs about mathematics teaching. Beswick's study helped to identify some of the beliefs that can support student-centered instruction.

Other research has focused on the structure of teachers' beliefs. For example, Cooney, Shealy, and Arvold (1998) described that not all of a teacher's beliefs are regarded equally by the teacher. A teacher has central beliefs that underpin most of what they do as well as beliefs that are more peripheral. Changing teachers' peripheral beliefs may have little impact on their instructional practices, whereas focusing on teachers' centrally held beliefs may have a positive impact on practice. Chapman (2002) echoed this point, illustrating how understanding a teacher's centrally held beliefs about mathematics can help promote change in their instruction. Aguirre and Speer (2000) conceptualized individual beliefs as part of *belief bundles*. That is, the researchers argued that beliefs are connected to one another in ways that influence the formation of pedagogical goals. The researchers explored these belief bundles by considering shifts in teachers' goals during instruction.

Much of the research that seeks to understand teachers' beliefs has characterized these beliefs by surveys or interviews (e.g., Barkatsas & Malone, 2005; Vacc & Bright, 1999; Wood & Sellers, 1997). However, developing surveys that accurately capture teachers' beliefs is a challenging task (see Philipp et al., 2007), and interviews may not uncover specific beliefs that teachers use in practice. As such, other researchers have contended that one cannot understand teachers' beliefs by written or spoken statements alone. For instance, Leatham (2006) argued that teachers' beliefs are not always articulated clearly, and to more fully understand teachers' beliefs, researchers must observe how teachers use their beliefs in the process of making decisions in the classroom. Similarly, Speer (2005) contended that teachers and researchers should work to develop a shared understanding when discussing beliefs, as language does not always clearly articulate what these beliefs are. Wilson and Cooney (2002) also argued this point, writing, "it seems that both observing and interviewing teachers are necessary if one is interested in comprehending how teachers make sense of their worlds" (Wilson & Cooney, 2002, p. 145).

Following Wilson and Cooney's (2002) recommendation and the line of research advocated by Speer (2005) and Leatham (2006), in this dissertation, I explore teachers' beliefs and other orientations through interviews and observations to understand how teachers make sense of their instruction.

2.3 Mathematics Teacher Knowledge

Recognizing teacher knowledge as an important part of instruction, many researchers have focused on understanding and exploring teachers' mathematical knowledge and its role in teaching. In this section, I review theoretical perspectives on subject-matter knowledge for teaching (both mathematical and otherwise) and empirical research on mathematical knowledge for teaching, with attention to the secondary level.

2.3.1 The Importance of Studying Mathematical Knowledge for High School Teaching

At the high school level, many future teachers in the United States and several other Western countries are required to complete an undergraduate degree in mathematics, and this requirement is based on the assumption that a degree in mathematics will provide the content knowledge needed for high school teaching (CBMS, 2001; Stacey, 2008). However, some researchers have challenged this assumption by finding that teachers who complete an undergraduate major in mathematics may still lack a conceptual understanding of the ideas they will teach. For example, Bryan (1999) interviewed nine preservice secondary teachers near the end of their undergraduate degree. These teachers had considerable difficulty explaining fundamental secondary concepts, such as those regarding functions and exponents, and in some cases teachers also had difficulty answering procedural questions about these ideas. In an interview study with inservice teachers in Cyprus, Cankoy (2010) found that practicing secondary teachers had similar difficulties. These studies and others (e.g., Even, 1990; Sánchez & Llinares, 2003) illustrate how secondary math teachers' conceptual understanding of and facility in explaining fundamental secondary math concepts are not necessarily developed through studying mathematics as an undergraduate. Teachers themselves have also expressed that they did not believe their undergraduate preparation in mathematics helped them to teach high school mathematics. Teachers from the United Kingdom in Goulding, Hatch, and Rodd's (2003) study expressed that they found their undergraduate math courses irrelevant to their teaching, and these courses deterred them from studying advanced mathematics in the future.

To explore why a major in mathematics would not necessarily prepare high school teachers with the content knowledge they need for the classroom, Moreira and David (2008) compared the academic mathematics taught in one Brazilian university's mathematics teacher education program with the mathematics those teachers would teach in secondary school. The researchers argued that ideas in undergraduate mathematics are reduced to a simple structure rather than being linked to concrete meaning as they are in K-12 mathematics. For example, undergraduate mathematics approaches number systems from a set of axioms and definitions, whereas number systems in K-12 mathematics are taught through concrete experiences. With a similar point of view, Deng (2007) argued that postsecondary science and mathematics approaches these disciplines from logical standpoints, whereas high school science and mathematics also have psychological, pedagogical, epistemological, and sociocultural dimensions. In other words, when prospective secondary teachers take undergraduate mathematics courses that are disconnected from the teaching of high school mathematics, they experience only one of the five dimensions of knowledge that Deng hypothesized they need.

2.3.2 Content Knowledge for Teaching

Understanding what teachers need to know about a subject to teach it is important not only for high school mathematics but also for every discipline and certification level. As a consequence, many researchers have studied the subject-matter knowledge needed to teach a discipline.

In Shulman's (1986) Presidential Address to the American Educational Research Association, he called the content knowledge needed for teaching the "missing paradigm" (p. 7) in teacher education research. Shulman recognized that content knowledge alone was not sufficient for effective teaching, and he argued that there must be a transformation from teachers' knowledge of their content to a usable form for pedagogical practice. An important piece of this transformed content knowledge, according to Shulman, was *pedagogical content knowledge* (PCK). Shulman defined PCK as the type of knowledge that includes an understanding of powerful explanations of ideas within the content, ways of conveying the content to others, and what aspects of the content make it difficult or easy to learn. Shulman's address sparked a great deal of research in the teacher education community that has focused on understanding content knowledge needed for teaching, particularly PCK.

One influential interpretation of PCK was offered by Gess-Newsome (1999), who argued that PCK can be conceptualized along a continuum. At one end of this continuum, PCK does not exist but rather is the intersection of knowledge of content, knowledge of pedagogy, and knowledge of context. Each of these types of knowledge is clearly developed and drawn upon to activate PCK in the classroom. Gess-Newsome called this end of the PCK continuum the *integrative model*. At the other end of the continuum is the *transformative model*. In this model, PCK is the transformation of other types of knowledge—knowledge of content, knowledge of pedagogy, and knowledge of context—into a unique form that is usable for teaching: PCK. Moreover, in the transformative model, PCK is the *only* form of knowledge that impacts teaching practice. Both ends of this continuum built on Shulman's (1986) notion of PCK as a transformation of subject-matter knowledge, but each in a different way. Gess-Newsome argued that many researchers studying PCK position their ideas regarding PCK between these two extremes.

2.3.3 Theoretical Perspectives on Mathematical Knowledge for Teaching

Drawing from the work of Shulman (1986), Gess-Newsome (1999), and others in the general education community, researchers in mathematics education have been particularly active in trying to understand the content knowledge needed for teaching. There are a wide variety of theoretical perspectives regarding the mathematical knowledge needed for teaching. In this section, I will present two recent perspectives. One of these is generally aligned with Gess-Newsome's integrative model of PCK whereas the other is more aligned with the transformative model of PCK.

Ball and colleagues' mathematical knowledge for teaching. One widely used perspective on mathematical knowledge for teaching is presented by Ball and colleagues (e.g., Ball, Thames, & Phelps, 2008). Ball et al. used the term *mathematical knowledge for teaching* (MKT) to describe the specific type of mathematical knowledge that teachers of mathematics need. The researchers argued that this knowledge is specific to content but is more than simply common content knowledge. To conceptualize what constitutes MKT, Ball (2000) asked, “What are the recurrent core task domains of teachers’ work?” (p. 244). In other words, the researchers considered the tasks involved in high-quality mathematics teaching to determine what knowledge was needed to carry out each of these tasks effectively. Much of this work came from looking at videos of Ball’s own third-grade classroom (Ball et al., 2008).

According to Ball et al. (2008), MKT consists of both *subject-matter knowledge* (knowledge of mathematics) and PCK (knowledge of teaching mathematics). They further subdivided these two types of knowledge into six components, based on their observations and experience with mathematics teaching (see Ball et al., 2008 for a visual representation of MKT).

Within subject-matter knowledge, Ball et al. (2008) included common content knowledge, horizon content knowledge, and specialized content knowledge. *Common content knowledge* refers to mathematical knowledge that is not unique to the work of teaching. For example, in teaching calculus, a teacher would need to understand what a derivative is and how to find the derivative of a polynomial function, but this knowledge is not unique to teaching. *Horizon content knowledge* refers to knowledge of the relationships among mathematical topics that precede and succeed the current topics being taught. For instance, an algebra teacher with strong horizon content knowledge may introduce the concept of square root in such a way that does not compromise students’ future learning of imaginary numbers. *Specialized content knowledge* refers to the mathematical knowledge that is specific to the profession of teaching (Ball et al., 2008; Hill, Ball, & Schilling, 2008). This type of knowledge is distinct from PCK because it is not knowledge about teaching, but rather knowledge about mathematics that is used in teaching. For example, a teacher with strong specialized content knowledge can recognize mathematically correct and incorrect aspects of students' errors and can determine whether nontraditional solution methods are mathematically sound (Ball et al., 2008). Ball et al. (2008) described specialized content knowledge as "an uncanny kind of unpacking of mathematics that is not needed—or even desirable—in settings other than teaching" (p. 401).

Ball et al. (2008) described that PCK includes knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum. *Knowledge of content and students* includes understanding of how students may look at mathematics and anticipation of common student difficulties with mathematics. Knowledge of content and students also includes facility with interpreting and understanding students' mathematical thinking. *Knowledge of content and teaching* includes knowledge of how to present mathematics in ways that highlight its central features. This also includes facility with designing instruction and structuring mathematical tasks in the classroom. Finally, *knowledge of the curriculum* includes an undersatnding of how to use curricular materials to maximally benefit students' learning of mathematics.

Ball et al. (2008) argued that MKT is a specific type of mathematical knowledge unique to work in the classroom, which is reminiscent of the descriptions of PCK provided by Shulman (1986) and Gess-Newsome (1999). These similarities have sometimes led to confusion in the field regarding the differences between Ball and colleagues' MKT and other researchers' notions of PCK. However, Ball et al. (2008) conceptualized PCK as only a piece of MKT, and they limited the subcategories of PCK to knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum.

Ball et al. (2008) argued that in their model of MKT, each component of knowledge is distinct from the other. Hence, this framework generally aligns with Gess-Newsome's (1999) integrative model of knowledge for teaching. That is, it is possible that one teacher may have sufficient common content knowledge but lack knowledge of content and students, for example. Because of this feature, this framework is particularly useful for assessing teachers' MKT, and several such assessments have grown from this work. In addition, several empirical studies have drawn loosely on Ball et al.'s (2008) framework. These will be summarized later in this chapter.

Although it is logical that Ball et al.'s (2008) MKT framework may apply to high school teaching, the development of this framework was based on elementary and middle school classrooms (Ball, Hill, & Bass, 2005). More research is needed to understand the connections between Ball et al.'s (2008) framework and knowledge for high school teaching.

Silverman and Thompson's mathematical knowledge for teaching. An alternative model for mathematical knowledge for teaching was proposed by Silverman and Thompson (2008). These researchers also adopted the acronym *MKT* but described how their framework differed from Ball et al.'s (2008). Silverman and Thompson wanted to understand the mathematical knowledge that helps teachers to act in spontaneous ways with regards to mathematics in the classroom, carry out a cohesive instructional sequence, and provide a foundation for learning new ideas so that students can see the connectedness of mathematical ideas. The researchers specified that they view MKT as a transformative model of PCK, as introduced by Gess-Newsome (1999). In other words, rather than understanding the individual components of MKT as Ball et al. proposed, Silverman and Thompson sought to understand the cohesiveness of MKT as a type of knowledge that is transformed from other types.

For Silverman and Thompson (2008), MKT consists of two levels of understanding. First, teachers must have a thorough and powerful understanding of the mathematics they will be teaching. The researchers built on Simon's (2006) notion of a *key developmental understanding*. Simon noted two characteristics of key developmental understandings: (a) they involve a change in the way that an individual thinks about or perceives a mathematical relationship, and (b) they are not usually acquired as a result of explanation or demonstration. For example, "Understanding that equal partitioning creates specific units of quantity" (Simon, 2006, p. 361) is a key developmental understanding. Silverman and Thompson believed that a teacher's key developmental understanding allows him or her to have knowledge with pedagogical potential. However, a key developmental understanding of mathematics is not enough to teach mathematics effectively. Silverman and Thompson also argued that teachers need to transform their understanding into a *pedagogically powerful understanding.* When teachers have a pedagogically powerful understanding, or MKT, they can recognize actions that need to be taken to help students develop a key developmental understanding of the mathematical ideas they are teaching. Silverman and Thompson (2008) listed characteristics of a teacher who has developed MKT for a particular topic. Such a teacher

> … (a) has developed a key developmental understanding within which that topic exists, (b) has constructed models of the variety of ways students may understand the content (decentering), (c) has an image of how someone else might come to think of the mathematical idea in a similar way, (d) has an image of the kinds of

> activities and conversations about those activities that might support another person's development of a similar understanding of the mathematical idea, (e) has an image of how students who have come to think about the mathematical idea in the specified way are empowered to learn other, related mathematical ideas. (p. 508)

Silverman and Thompson (2008) described their framework as grounded in ideas from mathematics education research and the learning sciences. In particular, they drew heavily on Simon (1995; 2006) and Piaget (1977/ 2001). I am not aware of any research studies that apply Silverman and Thompson's framework. However, the researchers emphasize that their framework could be used to describe teachers' growth and change in MKT (Silverman & Thompson, 2008).

Summary of theoretical literature on mathematical knowledge for teaching. The two theoretical perspectives presented in this section provide lenses that researchers can use to consider the mathematical knowledge needed for teaching. Table 2.1 provides a summary of each perspective and the research and practice that influenced its development.

The research literature provides additional theoretical perspectives on MKT that are not summarized in this section (e.g., Davis & Simmt, 2006; Rowland, 2008). Despite these different perspectives, there are commonalities: Most researchers emphasize that high-quality instruction requires not only knoweldge of mathematics but also knowledge of how to teach mathematics and mathematics-specific knowledge of the students being taught.

Throughout the remainder of this chapter, I will use the term *mathematical knowledge for teaching* and the acronym MKT to describe the specific mathematical knowledge needed for the classroom. MKT will refer to a particular framework only when explicitly stated.

Table 2.1 *A Comparison of Two Theoretical Perspectives for Mathematical Knowledge for Teaching.*

Theoretical Perspective	Description	Influences
Ball et al. (2008) mathematical knowledge for teaching	MKT consists of subject-matter knowledge and PCK, which are subdivided into six components.	Observations of elementary and middle school classrooms, experience, and related research
Silverman and Thompson (2008) mathematical knowledge for teaching	MKT begins with a key developmental understanding of mathematics, which is transformed into a pedagogically powerful understanding.	Mathematics education research and learning science research

2.3.4 Empirical Research on Mathematical Knowledge for Teaching

A great deal of qualitative and quantitative research has been conducted to understand MKT and the relationships between subject-matter knowledge and PCK. I will describe this research in three sections. First, I will summarize the research that has used grounded perspectives to describe MKT. Second, I will describe qualitative research that explored the relationships between subject-matter knowledge and PCK. Third, I will present quantitative measures of MKT that have been developed and subsequently used to explore MKT and the relationships between subject-matter knowledge and PCK.

Features of mathematical knowledge for teaching. Researchers have identified several features of MKT through grounded research and syntheses of the existing literature. For the sake of brevity, I will first describe the approaches to such research, and then I will summarize the findings at the end of this section.

To identify features of MKT, some researchers have conducted interviews with and observations of classroom teachers. Taking a researcher's viewpoint, Rowland (2007) used videos of twelve elementary teachers in the United Kingdom to describe features of MKT. Rowland coded videos for critical incidents or choices that teachers made that required a specialized knowledge of mathematics and then synthesized these codes into a description of the features of MKT. Ma's (1999) seminal work described features of the elementary mathematics knowledge of successful Chinese math teachers, and Ma used interviews to empirically identify these features. In the domain of calculus, Potari, Zachariades, Christou, Kyriazis, and Pitta-Pantazi (2007) used observations and interviews to explore nine secondary teachers' MKT in the area of calculus in Cyprus. The researchers looked for qualities of teachers' knowledge that contributed to effective or ineffective teaching in the classroom.

The studies cited above sought to understand features of MKT from a researcher's perspective. By contrast, some researchers have asked teachers for their views on MKT. For instance, Kajander (2010) interviewed six Canadian elementary teachers to get a sense of what they believed consitituted MKT. Teachers were invited to share their ideas through interviews, focus groups, journals, and email.

A second approach to understanding MKT has been to synthesize existing literature. Kennedy (1998) synthesized prior research to describe features of MKT that could apply in several settings. Ferrini-Mundy, Floden, McCrory, Burrill, and Sandow (2005) also synthesized prior research to describe features of MKT specific to the domain of algebra. Some researchers have synthesized prior research to develop a framework for MKT and subsequently illustrated the framework's applicability to teachers' understanding of a particular concept. For example, Even (1990) developed a framework that included seven facets of MKT for a particular concept. Even then shared cases of preservice secondary teachers to illustrate how the framework could be used to describe teachers' understandings of *function.* Similarly, Chinnappan and Lawson (2005) used prior research to identfy four ways of understanding a particular concept, then illustrated two Australian secondary teachers' ways of understanding the concept of *square* in the domain of geometry.

The approaches to understanding and describing MKT listed above are situated in many contexts. Each research study is specific to either elementary or secondary teaching, some research is specific to particular domains or topics, and each study was conducted in a particular country and cultural setting. Nevertheless, there are many similarities in the features of MKT described in these studies. To more clearly illustrate these smiliarities, I have synthesized the results of these studies into a list of features of MKT. For each feature in the list, I have cited the studies that reported a similar feature. Drawing from this literature, I identified 12 features of MKT:

1. Understanding the <u>central ideas</u> that carry through much of K-12 mathematics, including equations, multiplicative relationships, and functions (Ferrini-Mundy et al., 2005; Kajander, 2010; Kennedy, 1998; Ma, 1999);
2. Understanding the <u>progression of mathematical ideas</u> that are taught (Ferrini-Mundy et al., 2005; Kajander, 2010; Ma, 1999; Rowland, 2007);
3. Knowledge of appropriate <u>examples and nonexamples</u> of each concept (Even, 1990; Ferrini-Mundy et al., 2005; Kennedy, 1998);

4. Familiarity with appropriate real-life contexts for applying mathematical concepts (Chinnappan & Lawson, 2005; Ferrini-Mundy et al., 2005; Potari et al., 2007);
5. Awareness of connections among mathematical concepts (Chinnappan & Lawson, 2005; Even, 1990; Kennedy, 1998; Ma, 1999; Potari et al., 2007; Rowland, 2007);
6. Fluency with approaches to problem solving and awareness of particular contexts where certain approaches are appropriate (Even, 1990; Ferrini-Mundy et al., 2005; Ma, 1999);
7. Knowledge of justifications and proofs that are appropriate for particular mathematical ideas (Ferrini-Mundy et al., 2005; Kennedy, 1998; Ma, 1999; Potari et al. 2007);
8. Knowledge of appropriate langauge and symbols to accurately and effectively express mathematical ideas (Ferrini-Mundy et al., 2005; Potari et al., 2007);
9. Recognition of apppropriate representations and meaningful ways to model each concept (Chinnappan & Lawson, 2005; Even, 1990; Ferrini-Mundy et al., 2005);
10. Understanding of the nature of mathematics, including the construction of the discipline of mathematics and the means by which truth is established (Even, 1990; Ferrini-Mundy et al., 2005; Kennedy, 1998; Rowland, 2007);
11. Understanding student thinking and recognizing student difficulties with mathematics (Kajander, 2010; Kennedy, 1998; Rowland, 2007); and

12. Ability to enact mathematical knowledge in the classroom to recognize and extend mathematical activity in the moment (Kajander, 2010; Potari et al., 2007; Rowland, 2007).

Each of the features in this list may be appropriate in some situations and not others, as none of the studies in this section identified all of these features.

Qualitative explorations of mathematical knowledge for teaching. Whether researchers consider MKT from an integrative or transformative perspective (Gess-Newsome, 1999), many agree that MKT is multifaceted. As a result, there has been interest in understanding how some aspects of MKT affect others. In particular, researchers have sought to understand the relationships between subject-matter knowledge and PCK.

In line with Gess-Newsome's (1999) transformative model of PCK, some researchers have provided examples of PCK as a transformation of subject-matter knowledge that is useful for the classroom. In a study with 11 preservice secondary mathematics teachers, Ebert (1993) provided teachers with vignettes that contained student misconceptions of functions. Ebert asked teachers to discuss these misconceptions and describe how they would respond to the students and found that the ways in which teachers discussed students' misconceptions were closely tied to their own conceptual understandings of functions. In other words, teachers with limited conceptions of functions also were unable to interpret students' misconceptions or provide adequate explanations, whereas those with strong conceptions of functions were able to provide detailed explanations that addressed students' misconceptions. From this, Ebert hypothesized that PCK—in particular, understanding and discussing student misconceptions—was the result of a transformation of the teacher's subject-matter knowledge. In a similar study, Even (1993) argued that secondary teachers' limited conceptions of functions also limited their PCK and prohibited them from providing students with coherent and mathematically robust explanations for functions. Similar findings were reported in Even and Tirosh (1995).

Other researchers have focused on how teachers' subject-matter knowledge can influence their choice and use of curriculum materials. Recall that knowledge of the curriculum was a piece of content knowledge discussed by Shulman (1986), and Ball et al. (2008) categorized knowledge of curriculum as part of PCK. Sánchez and Llinares (2003) interviewed four secondary mathematics teachers from Spain regarding their content knowledge about functions and their ideas about planned presentations of textbook problems related to the concept of functions. The researchers found that teachers' subject-matter knowledge appeared to help them critically interpret and adapt the textbook for their own teaching needs. Similarly, Lloyd and Wilson (1998) presented a case study of one teacher whose strong conceptual ideas of functions allowed him to effectively implement reform-oriented materials in the classroom.

Additionally, researchers have found that subject-matter knowledge can help teachers to recognize critical mathematical moments in the classroom, and many researchers consider this recognition to be a part of MKT (e.g., Ball et al., 2008). Kahan, Cooper, and Bethea (2003) used undergraduate transcripts, content assessments, lesson plans, and classroom observations of 16 preservice secondary mathematics teachers to explore the relationship between content knowledge and teaching. Content knowledge was measured in terms of the content assessment and the number and difficulty of mathematics courses taken as well as teachers' grades in those courses. To consider the quality of teaching, the researchers rated teachers' lesson plans based on the depth of mathematical content and used an observation framework to analyze the implementation of mathematical knowledge in teachers' lessons. Kahan et al. illustrated that teachers with strong content knowledge tended to have stronger lessons in terms of mathematical quality, and those with weak content knowledge tended to have weaker lessons. However, this relationship was not consistent. What did seem consistent to the researchers was that teachers with stronger content knowledge were more able to recognize teachable, mathematical moments in the classroom and act spontaneously, whereas teachers with limited content knowledge were not able to do so.

One possible interpretation of the studies in this section is that understanding student misconceptions, forming coherent explanations, navigating curriculum materials appropriately, and recognizing teachable moments are ways of observing subject-matter knowledge in teaching. That is, teachers who have sufficient mathematics knowledge are able to perform these actions effectively in the classroom. Other researchers argue, however, that subject-matter knowledge alone does not guarantee that teachers will have sufficient PCK. In a case study with one undergraduate differential equations professor, Speer and Wagner (2009) illustrated that subject-matter knowledge is necessary but not sufficient for developing PCK. The researchers told the story of a mathematics professor with well-developed subject-matter knowledge who was unable to interpret students' thinking and provide scaffolding during classroom discussions. However, Speer and Wagner also argued that the professor's subject-matter knowledge helped him to learn from the experiences of trying to understand students so that his PCK could continue to develop. The researchers explained that subject-matter knowledge

> … can be thought of as both supporting teachers as they do the mathematical work specific to teaching and enabling teachers to learn through such work. The product of that learning then has the potential to become knowledge that can serve as PCK. (pp. 559-560)

In a similar study, Johnson and Larsen (2012) illustrated how one university mathematician misinterpreted students' comments about their mathematical difficulties during an inquiry-oriented abstract algebra course. Johnson and Larsen argued that PCK includes knowing not only students' common difficulties but also the consequences of those difficulties for learning and the ways in which those difficulties are situated in the context of the curriculum.

Powell and Hanna (2006) argued that when teachers have well-developed notions of subject-matter knowledge and PCK, these two dimensions can be indistinguishable. In particular, teachers must interpret students' mathematical thinking, assess the mathematical validity of that thinking, and provide coherent explanations spontaneously in the classroom, and these can all appear as one type of knowledge. This is reminiscent of Gess-Newsome's (1999) transformative model of PCK. Powell and Hanna contended that interactions with students are quality spaces to observe teachers' MKT, and they illustrated teachers' use of subject-matter knowledge and PCK through examples from a research-based, after-school mathematics program for middle school students.

Quantitative measures of mathematical knowledge for teaching. To more fully understand MKT, including its characteristics and how it impacts teaching and learning, some researchers have created measures of MKT. The measures described in this section were statistically validated and used in large-scale studies.

The COACTIV study. The Professional Competence of Teachers, Cognitively Activating Instruction, and the Development of Students' Mathematical Literacy (COACTIV) was a large-scale study conducted in Germany that was aimed at measuring MKT (Baumert et al., 2010; Krauss et al., 2008). The researchers sought to understand the appropriate preparation for future German mathematics teachers.

Krauss et al. (2008) described the development and validation of their assessments to measure teachers' subject-matter knowledge and PCK. For these researchers, the subject-matter knowledge assessment was based on "in-depth background knowledge on the contents of the secondary-level mathematics curriculum" (p.719). The PCK assessment was based on teachers' knowledge of mathematical tasks, knowledge of students' conceptual difficulties, and knowledge of mathematics-specific instructional strategies. A sample of 198 tenth-grade mathematics teachers in Germany were given the two assessments and also asked background questions regarding their education, teaching experience, and so on. Psychometric analysis revealed that the constructs of subject-matter knowledge and PCK, as measured by Krauss et al.'s assessments, were statistically distinguishable. In other words, these were not the same type of knowledge but two distinct bodies of knowledge. Teachers in Germany typically complete one of two types of preparation for teaching secondary mathematics: One preparation focuses more on mathematics and less on teaching, whereas the other focuses more on teaching and less on mathematics. One interesting finding was that teachers with more mathematical background outscored other teachers on both assessments, though they had limited preparation in teaching. In addition, for the teachers with more content preparation, subject-matter knowledge and PCK were less distinguishable; that is, these teachers seemed to have more interconnected subject-matter knowledge and PCK. Krauss et al. also found no correlation between the number of years of teaching experience and the two knowledge categories.

In a follow-up study, Baumert et al. (2010) looked at the relationships among teachers' PCK, their quality of instruction, and student achievement. For this study, a sample of 181 ninth-grade mathematics teachers and 4,353 students across Germany were considered. Teachers were given the subject-matter and PCK assessments discussed in Krauss et al. (2008). To measure the quality of teaching, researchers collected all assessments given by teachers during a school year as well as a homework assignment sample from two compulsory topics. These tasks were analyzed according to the cognitive complexity and the learning support provided. Students' standardized assessments were used to measure student achievement. The researchers found that high levels of teachers' PCK seemed to lead to higher quality instruction in terms of task selection and learning support provided. They also found that teachers' subject-matter knowledge had lower predictive power for student achievement than PCK. Putting the results of these two studies together, it seems as if robust subject-matter knowledge is needed for teachers to develop PCK (Krauss et al., 2008), but in the classroom it is teachers' advanced PCK that makes more impact with regards to task selection and student achievement (Baumert et al., 2010).

The LMT project. The Learning Mathematics for Teaching Project (LMT) follows from Ball and colleagues' conceptualization of MKT (e.g., Ball, 2000; Ball et al., 2008). The purpose of this project was to understand and assess teachers' MKT, with the intention of describing the effectiveness of teacher education and development (LMT, 2012).

Hill, Schilling, and Ball (2004) described the development of measures to assess teachers' MKT. Drawing on the conceptual work of Ball (2000), the researchers analyzed the mathematical knowledge that teachers would need to carry out tasks of teaching. Because MKT was primarily based on elementary mathematics, the researchers chose three domains from the elementary and middle school curriculum for their items: number, operations, and patterns and algebra. The items were also specific to three subcategories of teachers' MKT: common content knowledge, specialized content knowledge (a type of subject-matter knowledge) and knowledge of content and students (a type of PCK; see Ball et al., 2008). Hill et al. (2004) administered their assessments to over 1,500 teachers participating in professional development institutes in California. Through this process they found that knowledge of number and knowledge of operations were related and that common content knowledge was related but distinct from specialized content knowledge.

Since these assessments were developed, researchers within and outside of the LMT project have used them to study the relationship between teachers' MKT and other measures of classroom quality. These assessments have been refined (e.g., Hill, Ball, & Schilling, 2008), and they are publicly available and known as the LMT assessments (LMT, 2012).

The first of the studies to employ these assessments sought to compare elementary teachers' MKT to student achievement. Hill, Rowan, and Ball (2005) used the LMT assessments with 334 first-grade teachers and 365 third-grade teachers and measured their students' achievement by standardized assessments. The sample of students included 1,190 first graders and 1,773 third graders. The researchers found that teachers' MKT, as measured by the LMT assessments, was a significant predictor of student achievement at both grade levels.

In another study using the LMT assessments, Hill (2007) considered middle school teachers' MKT as compared to their years of teaching experience. With a sample of 1,000 teachers, Hill found that teachers with high school teaching experience had higher levels of MKT than their colleagues without high school teaching experience. Another interesting finding was that teachers' MKT appeared to increase with years of experience until teachers had reached about 13 years of experience—then the LMT scores seemed to plateau. In a subsequent study, Hill (2010) considered teachers' background characteristics to determine if predictors of elementary teachers' MKT could be identified. A sample of 625 elementary teachers took the LMT content assessments and provided data for characteristics such as college courses taken, leadership activities, grade taught, and perceptions of their own mathematical knowledge (math self-concept). Although all of these background characteristics seemed to be correlated with teachers' MKT, the grades teachers were teaching at the time and their math self-concept were significant predictors.

In a more qualitatively focused application of the LMT assessments, Hill, Blunk, et al. (2008) compared teachers' scores on the LMT assessments to their mathematical quality of instruction (MQI). To conceptualize facets of MQI, the researchers consulted literature on teachers' knowledge and instruction. They identified five aspects of MQI: (a) the presence of mathematical errors, (b) the mathematical appropriateness of responses to students, (c) the connections between classroom practice and mathematics,

(d) the richness of mathematical content, and (e) the appropriate use of mathematical language. The researchers also developed a coding scheme to capture MQI[1]. Ten elementary teachers were given the LMT assessments, videotaped during instruction, and participated in postobservation interviews. Hill, Blunk, et al. (2008) found that teachers' MKT was positively correlated to aspects of their MQI. However, the researchers presented case studies where teachers' assessed levels of MKT did not correspond to their assessed levels of MQI—that is, one score was high and one was low. The researchers observed that teachers' knowledge of curriculum and curriculum use was an important mediator in these cases. In other words, teachers who had strong MKT but a weak knowledge of how to use the curriculum effectively sometimes had low MQI in the classroom, whereas teachers who had weaker MKT without deviation from their curriculum had the possibility of higher MQI (cf. Sánchez & Llinares, 2003).

More recently, other researchers have used the LMT assessments to study teachers' MKT in various contexts. Bell, Wilson, Higgins, and McCoach (2010) used the LMT measures in conjunction with their own measures to understand the impact of a large-scale professional development program, Developing Mathematical Ideas (DMI). The researchers considered 10 sites that were implementing DMI professional development programs and a control group at each site, with 308 teachers in both treatment and control groups. Most participants were elementary teachers. The researchers emphasized that the LMT measures were appropriate for their needs because DMI was a large-scale program that was widely established and consistent across sites. In addition, the goals of DMI closely aligned with the content of the LMT measures.

---

[1] A later version of the coding scheme is used in this dissertation. See NCTE, 2012.

Although the researchers found the LMT measures helpful in this case, they noted that teachers showed greater improvement on the open-ended measures that were added specifically for this professional development program than they did on the LMT measures.

Other measures. Some researchers have worked to develop specialized MKT assessments for teachers. For example, Izsak, Orrill, Cohen, and Brown (2010) described their development of an assessment designed to measure middle school teachers' MKT of rational numbers.

In another line of research, Shechtman, Roschelle, Haertel, and Knudsen (2010) wanted to understand the implementation of a particular middle school curriculum unit with respect to teachers' MKT. To develop an assessment for teachers' MKT, the researchers created a list of six features of MKT that they believed the implementation of the curriculum required. In the style of the LMT assessments (e.g., Hill et al., 2004), the researchers then developed and validated MKT assessments for middle school teachers based on this list of features. They used experimental design with over 200 teachers to determine how teachers' MKT and subsequent student achievement was impacted by the implementation of the curriculum activities and participation in the associated professional development. The researchers found that the teachers who taught the curriculum had increases in MKT, but these were not significantly different from those of the control group. Moreover, teachers' MKT did not have a strong relationship with student achievement. In addition, the researchers found that teachers' MKT was not correlated with decision making in the areas of topic coverage, choice of teaching goals, or use of technology. In other words, teachers with high levels of MKT did not always make the best instructional decisions.

Summary of empirical literature on mathematical knowledge for teaching. With the aim of understanding the nature and facets of MKT, many researchers have engaged in qualitative and quantitative studies across various certification levels and contexts. Qualitative research has indicated several features of MKT; 12 themes from this research were presented earlier in this review. Researchers have also worked to understand the relationships within MKT, especially the relationship between subject-matter knowledge and PCK. Several qualitative studies have concluded that a teacher's strong subject-matter knowledge can enhance aspects of his or her PCK, including facility with interpreting student thinking, explaining mathematical ideas to students, implementing curriculum materials appropriately, and recognizing teachable moments in the classroom. However, some researchers argued that subject-matter knowledge is not sufficient for PCK. Other researchers claimed that when a teacher's subject-matter knowledge and PCK are effective, these two types of knowledge are indistinguishable in the classroom. Using statistically validated measures, quantitative studies have also shown that subject-matter knowledge contributes to aspects of PCK, such as knowledge of choosing tasks for the classroom, understanding students' conceptual difficulties, and applying appropriate teaching strategies. However, quantitative measures have shown that subject-matter knowledge and PCK are statistically distinguishable. In addition, quantitative studies have shown that MKT may contribute to student achievement and teachers' mathematical quality of instruction, but these results are inconsistent across measures.

There is still a great deal about teachers' mathematical knowledge that is not well understood. This may be because instructional decisions mediate the relationship between teachers' mathematical knowledge and their instruction. Hence, more research is needed on how teachers use their knowledge and beliefs to make and carry out such decisions.

2.4 Beliefs, Knowledge, and Instruction: A Call for the Dissertation

The research in the areas of mathematics teacher beliefs and mathematics teacher knowledge makes it clear that each of these impact instruction. However, despite the fact that researchers have noted the importance of both beliefs and knowledge for mathematics instruction (e.g., Sleep & Eskelson, 2012), with notable exceptions (e.g., Thomas & Yoon, 2014; Törner, Rolka, Rösken, & Sriraman, 2010) there is far less research focused on understanding the ways in which beliefs and knowledge interact to inform teachers' pedagogical decisions. Hence, more research is needed to understand the complexities of how knowledge and beliefs are used in the activity of instruction, and this dissertation will contribute to this area of research.

In addition, researchers studying teacher beliefs have argued that to fully understand these beliefs, it is important to draw on not only teachers' written and spoken descriptions of these beliefs but also teachers' actions in practice (e.g., Leatham, 2006; Speer, 2005; Wilson & Cooney, 2002). Although there have been similar arguments made regarding teacher knowledge (e.g., Davis & Simmt, 2006; Rowland, 2008), the majority of research on mathematics teacher knowledge measures teachers' knowledge by written assessments or task-based interviews (e.g., Baumert et al., 2010; Even, 1990; Hill et al., 2005). These measures provide useful information about teachers' mathematical understandings, but the knowledge assessed through written assessments and task-based interviews does not necessarily illuminate the knowledge that teachers use in practice. As Simon and Tzur (1999) articulated, “we see a teacher’s practice as a conglomerate that cannot be understood by looking at parts split off from the whole (i.e., looking only at beliefs or methods of questioning or mathematical knowledge)” (p. 254). These researchers sought to understand teaching by exploring both what teachers do and what they know and believe. In this line of inquiry, this dissertation will use both interviews and observations of instruction to understand teacher beliefs and knowledge and ways these are used in instructional decision making.

Chapter 3: Exemplary High School Mathematics Teachers' Reflections on Teaching: A Situated Cognition Perspective[2]

Abstract

This study explored the mathematical knowledge that can support high-quality mathematics instruction by considering the mathematical knowledge expressed in exemplary teachers' reflections on their teaching. Participants were 11 teachers in New Jersey who were deemed exemplary through state and national recognition programs. Through individual interviews, teachers reflected on lessons they had taught with consideration of the mathematical knowledge used in teaching. This study was originally conceived using a cognitive perspective on knowledge; however, I found a situated cognition perspective better clarified the essence of teachers' accounts. I used grounded analysis to explore themes of situated knowledge that were present in teachers' reflections.

[2] A version of this chapter is currently being prepared for publication and is co-authored by Keith Weber.

3.1 Introduction

To help novice teachers engage in high-quality mathematics instruction, it is important to understand how expert teachers know and use mathematics in their teaching (Li & Kaiser, 2011). Mathematical knowledge needed for teaching has been the topic of a great deal of research in current decades, yet many questions remain about the mathematical knowledge expert teachers use and how this knowledge is used, particularly at the high school level (Petrou & Goulding, 2011; Stacey, 2008). Studying the mathematical knowledge used in high school teaching is important: In many Western countries, prospective high school teachers complete an undergraduate degree in mathematics, yet it is unclear whether and how this preparation helps them to be expert teachers (Goulding, Hatch, & Rodd, 2003; Stacey, 2008; Zazkis & Leikin, 2010).

Many of the frameworks for mathematical knowledge for teaching have been constructed by researchers (e.g., Ball, Thames, & Phelps, 2008; Silverman & Thompson, 2008), and much of the empirical research on mathematical knowledge for teaching has taken a researcher's perspective (e.g., Ma, 1999; Rowland, 2007).By contrast, research that seeks to understand knowledge from the point of view of expert teachers is limited, with several scholars noting that more research on teachers' perspectives would provide a useful viewpoint (e.g., Asikainen, Pehkonen, & Hirvonen, 2013; Clemente & Ramírez, 2008; Kajander, 2010).

Answering the call for more research on teacher knowledge from teachers' perspectives, this study was designed to understand exemplary teachers' perspectives on the mathematical content knowledge used in their teaching. Specifically, I was interested in both what content knowledge teachers perceived that they used in teaching and how they perceived that they used it. I found that teachers had difficulty answering abstract questions about their knowledge used in teaching, yet they expressed rich content knowledge and pedagogical content knowledge through their accounts of their teaching. In this chapter, I illustrate the mathematical knowledge in teachers' reflections by using a situated perspective. With these findings, I argue that teachers' difficulty in expressing abstractly the content knowledge used in teaching should not be interpreted as teachers lacking such knowledge.

3.2 Background

To position the study, I describe two perspectives on content knowledge for teaching. These two perspectives have helped to shape the research; however, these are not the only two perspectives on knowledge, nor are the distinctions between them always clear.

3.2.1 A Cognitive Perspective on Expert Knowledge

Many researchers have taken a cognitive approach to studying content knowledge for teaching. In cognitive views of teaching and learning, individuals are believed to recognize and construct connections among ideas to develop conceptual understanding and productive mental processes, such as reasoning and problem solving (Greeno et al., 1998; Maher & Davis, 1990). An important assumption of the cognitive approach is that an individual's knowledge can be codified and described through taxonomies or schematic representations.

For instance, Shulman's (1986) conceptualization of content knowledge for teaching aligns with a cognitive perspective. Shulman distinguished between *content* *knowledge*—knowledge of the subject to be taught, including what is true and why it is considered to be true—and *pedagogical content knowledge* (PCK)—including knowledge of powerful explanations of ideas within the content, ways of conveying the content to others, and what aspects of the content make it difficult or easy to learn. More recently, Cochran-Smith and Lytle (1999) described a *knowledge-for-practice* conception of teacher learning, which included cognitive perspectives on teacher knowledge. The knowledge-for-practice conception was based on the assumption that if teachers acquire productive knowledge, this knowledge will translate into practice and improve their teaching (see also Sfard, 1998).

Research on teacher knowledge that is built from a cognitive perspective has pervaded mathematics education research on teacher knowledge, particularly in the United States (Bednarz & Proulx, 2009; Depaepe, Verschaffel, & Kelchtermans, 2013; Sfard, 1998). For example, several mathematics education researchers have used task-based interviews to explore teachers' mathematical knowledge (e.g., Chinnappan & Lawson, 2005; Even, 1993; Lloyd & Wilson, 1998; Ma, 1999; Sánchez & Llinares, 2003), identifying strengths and weaknesses in teachers' knowledge of mathematics and connections between teachers' knowledge and their pedagogical actions.

Building on the work of Shulman (1986), Ball and colleagues (e.g., Ball & Bass, 2002; Ball et al., 2008) analyzed the tasks of teaching in the United States and explicated the mathematics-specific knowledge that was needed to carry out these tasks effectively. The result was a framework for mathematics teachers' content knowledge and PCK: *mathematical knowledge for teaching*. Ball and colleagues also developed written assessments of mathematical knowledge for teaching, and these have been used to explore the extent to which specific types of teacher knowledge are related to measures of instructional quality (Hill, Rowan, & Ball, 2005; Hill, Blunk, et al., 2008). Researchers in Germany have taken a similar approach of using written assessments to identify the specific mathematical knowledge that impacts instruction (Baumert et al., 2010).

Recently, some researchers in the cognitive paradigm have begun seeking teachers' perspectives on mathematical knowledge needed for teaching (e.g., Asikainen et al., 2013; Zazkis & Leikin, 2010). These studies were notably different from those that use task-based interviews or written assessments because they sought to understand teaching knowledge from teachers' perspectives; however, they were still cognitively oriented, as they assumed teachers were able to discuss knowledge abstractly and aimed to identify knowledge that teachers possessed. In these studies, teachers sometimes had difficulty speaking abstractly about the knowledge used in teaching. For instance, through interviews with secondary mathematics teachers in Finland, Asikainen et al. (2013) found that teachers valued many types of knowledge emphasized by researchers, including PCK, but that teachers "may lack the concepts needed to discuss teacher knowledge, even if they are expert in demonstrating effective teaching" (p. 88). In another study Zazkis and Leikin (2010) interviewed high school teachers about how they used advanced (university-level) mathematical knowledge in their teaching. Although teachers mentioned general ways their knowledge was used, they were unable to provide many specific examples. In both cases, the research teams suggested that teachers' inability to describe their knowledge and its use may imply that they lacked mathematical knowledge (Asikainen et al., 2013) or did not use it in their practice (Zazkis & Leikin, 2010). However, it is possible teachers did have and use this knowledge but had difficulty expressing it in ways that were captured by the researchers' cognitive perspectives.

3.2.2 A Situated Cognition Perspective on Expert Knowledge

A second view of expert knowledge for teaching is aligned with *situated cognition* discussed by Greeno (1991) and Brown and colleagues (e.g., Brown, Collins, & Duguid, 1989; Collins, Brown, & Newman, 1989). This theory is built on the assumption that knowledge is situated in the contextual environment where it is used (Greeno, 1991). Hence, knowledge and concepts cannot be fully understood without also understanding the activity in which they are applied (Brown et al., 1989). From this perspective, understanding the abstract knowledge that an individual holds is of limited value; it is important to further understand how knowledge is expressed in meaningful environments where it is used.

Rather than describe knowledge that individuals acquire, Greeno et al. (1998) described learning as developing *attunements* to constraints and affordances within a particular context. Attunements were defined as "regular patterns of an individual's participation" (Greeno et al., 1998, p. 9) and can be conceptualized as an individual's ways of knowing in a particular situation. In the case of mathematics teaching, a teacher's knowledge is shaped by and expressed through his or her activity and practice with students in the classroom. From this perspective, codifying expert content knowledge, pedagogical knowledge, or even PCK (e.g., Ball et al., 2008) is not as useful as understanding how teachers make sense of and participate in the activity of teaching.

Cochran-Smith and Lytle's (1999) *knowledge-in-practice* conception of teacher learning relates to the situated cognition perspective. This conception was built on the assumption that teachers' knowledge is expressed through their teaching or in their accounts of their teaching; that is, "the knowledge teachers need to teach well is embedded in the exemplary practice of experienced teachers" (p. 263). Both what teachers know and how they know it exist in the artistry of teaching and making decisions in the classroom; therefore, there is little separation between what teachers know and what teachers do in the classroom.

Some theoretical frameworks in mathematics education have been built from a situated perspective. For example, Brown and Coles (2011) argued that a teacher's expertise is not captured by a list of knowledge or actions; rather, expertise lies in the appropriate choice of purpose in the classroom and the use of knowledge and actions to meet goals related to that purpose. Hence, Brown and Coles (2011) helped teachers to identify purposes as guiding questions that they ask themselves when teaching. For example, teachers may ask, "'How will I know what [students] know?' … 'How can I share their responses?'" (p. 862). With a similar viewpoint, Davis & Simmt (Davis & Simmt, 2006; Simmt, 2011) proposed that teachers' knowledge of mathematics is enacted through interactions with students, and teachers' understandings of students are at the core of these interactions. Hence, teaching expertise lies in the complex process of "negotiating" (Simmt, 2011, p. 152) between mathematics and students as learners of mathematics.

Research that is more oriented towards the situated cognition perspective is built on the view that teachers use many intertwined ways of knowing as they engage in the complex activity of teaching. In North America, researchers have observed the complexity of teachers' intertwined use of content knowledge and PCK in the classroom (e.g., Powell & Hanna, 2006) and in professional development settings (e.g., Proulx, 2008). In the United Kingdom, Rowland (2008) developed a framework for classroom observation from the perspective of teacher knowledge: *the knowledge quartet*. Rather than separating content knowledge from PCK, Rowland's framework focused on the actions of teaching (e.g., choice of representations, responding to student ideas) that required mathematical knowledge. To study a mathematics teacher's PCK growth, Seymour and Lehrer (2006) explored how the teacher orchestrated students' understandings of mathematics through discourse patterns in the context of the classroom.

In disciplines outside of mathematics education, some researchers have sought teachers' perspectives to understand knowledge in teaching from a situated perspective. These researchers assume that teachers' knowledge and understandings may not be made explicit through interviews and instead are embedded in teachers' stories of practice. For instance, Clemente & Ramírez (2008) asked primary school teachers to narrate the process of teaching reading, and the researchers illustrated how teachers' knowledge "emerges from the action" (p. 1245) in their narratives. In other words, teachers' understandings were expressed through their discussions of the actions of teaching. I am aware of only one similar study in mathematics education: Oslund (2012) analyzed stories of practice from two experienced elementary teachers to illustrate how mathematical knowledge may be expressed through different linguistic patterns. This approach to research is uncommon in mathematics education, and I am not aware of any such studies with high school mathematics teachers.

3.2.3 The Study

The purpose of this study was to understand exemplary teachers' perspectives on the mathematical (content) knowledge that they used in their teaching and how this knowledge was used, and the research began with a cognitive lens. Other researchers using this lens have reported that teachers had difficulty speaking abstractly about the knowledge they used in teaching; however, past research asked teachers to reflect on their knowledge broadly across all aspects of their teaching (Asikainen et al., 2013; Zazkis & Leikin, 2010). Zazkis and Leikin recommended that future research focus on specific classroom scenarios to help teachers to articulate their knowledge use. Taking this recommendation, I asked teachers to reflect on specific lessons and speak about the mathematical knowledge used in these lessons.

Yet, in the present study, exemplary teachers also had difficulty describing the mathematical knowledge used in their practice. Despite this difficulty, teachers expressed rich content knowledge and PCK through their accounts of their teaching. With a situated cognition perspective, I was able to capture knowledge expressed through teachers' reflections that was limited by a cognitive perspective. Specifically, the findings illustrate teachers' mathematics-specific attunements, that is, regularities in ways of knowing and participating, that teachers share through reflection on their teaching. These findings challenge the assumption that teachers' difficulty in abstracting and describing the mathematics used in teaching is problematic. Participants were recognized for exemplary teaching and shared many ways of knowing and using mathematics, despite their difficulty in describing this knowledge abstractly.

3.3 Methods

3.3.1 Participants

My goal in this study was to understand the perspectives of expert teachers. Other researchers have used several ways of identifying exemplary (often termed *expert*) teachers (Li & Kaiser, 2011); in this study, I invited teachers who had been recognized for their teaching through standardized programs. I focused on high school (9th- through 12th-grade) mathematics teachers who had been recognized in New Jersey during the 10 years leading up to this study in at least one of three ways: Each teacher was (a) a state finalist or national recipient of the Presidential Award for Excellence in Math and Science Teaching (National Science Foundation, 2009), (b) named Teacher of the Year in their county (Council of Chief State School Officers, 2012), or (c) a National Board Certified Teacher in mathematics for adolescence and early adulthood (ages 14 to 18+; National Board for Professional Teaching Standards [NBPTS], 2014). These awards require that teachers demonstrate several exemplary qualities, including mastery of the content they teach, the use effective strategies for student engagement and learning, a reflective nature about their practice, and exceptional interpersonal skills.

Twenty-six teachers met the criteria. Six of these could not be located; the remaining 20 teachers were invited to participate. The invitation explained that I was interested in the mathematical knowledge used in teaching and that I valued the opinions and experiences of recognized teachers (in the style of Brown & McIntyre, 1993). Eleven teachers agreed to participate in the research.

Eight females and three males participated with teaching experience ranging from 10 to 32 years. Three participants were recognized in more than one of the ways listed.

All but one of the participants had earned a graduate-level degree. Participants also had various other leadership roles, awards, and honors. At the time of the interview, two participants taught at private schools and nine participants taught at public schools that varied widely in terms of the overall school qualities and socioeconomic status of the students.

By using standardized recognition to identify exemplary teachers, I avoided bringing personal biases into the selection. However, using these criteria is not without its limitations. For instance, these recognitions are culturally specific to the United States, and aspects of teaching valued in this culture may not be valued in others (see Li & Kaiser, 2011). Moreover, professionals within the United States may disagree about recognized teachers' quality of teaching. Also, because of their recognitions, teachers may have (not necessarily intentionally) aligned their interview discussions with the philosophy, rhetoric, and expectations of the recognition. On a positive note, through the process of applying for these awards, participants had practice in articulating the thinking behind their teaching. Keeping these limitations in mind, I believe that this group of participants was indeed an exemplary group but not necessarily representative of all exemplary teachers.

3.3.2 Data Collection

Data included one individual interview and one written lesson plan for each teacher. Interviews were used to capture teachers' perspectives, and lesson plans were used to prompt for examples from teaching situations (Seidman, 2006). Each lesson plan focused on a single topic or problem that ranged from one to five class periods of instruction. Lesson plans were obtained before the interviews so that clarifying interview questions could be added as needed. Interviews were semistructured[3], audio-recorded, and lasted approximately one hour each. The interview protocol (see Appendix A) was created using guidelines from qualitative researchers in social sciences, and each interview proceeded as follows.

Initially, the teacher was reminded that the focus of the interview was the mathematical thinking and knowledge used in teaching. To identify experiences (e.g., graduate school, teacher leadership) on which the teacher may be drawing, the teacher was asked to describe their background in mathematics education (Seidman, 2006). Next, the question "Why did you choose to share this lesson with me?" helped identify lesson features that were significant to the participant (Brenner, 2006). To understand the background for the lesson, the teacher was asked to describe how they created the lesson plan. Throughout the interview, when the teacher discussed ideas that related to mathematical knowledge, follow-up questions were used (Seidman, 2006). For instance, one teacher said he shared the lesson because it connected ideas from geometry and algebra. The teacher was asked to speak more specifically about those connections.

Next, following Merriam (1998), the teacher was asked to elaborate on the intended audience and purpose of the written plan (e.g., to apply for an award, for their own purpose, as part of a professional development project). Additional questions focused on several contextual features of the lesson, such as the topics that were taught before and after the lesson, the number of times the teacher taught the lesson, and the typicality of the lesson when compared to others in the teacher's repertoire.

---

[3] That is, I followed an interview protocol, but teachers were also asked probing questions as necessary.

In the next portion of the interview, the goal was to reconstruct the details of the teaching experience (Seidman, 2006). The teacher was asked the following hypothetical question (Merriam, 1998): “If I were to watch you teach this lesson, what would I see in the lesson that is not included in this plan?”

Once the context of the lesson and the details of the teaching experience were established, the teacher was prompted to reflect specifically on mathematical knowledge. Following Seidman’s (2006) recommended interview structure, it was expected that the earlier, concrete discussion about the lesson would facilitate answers to more abstract questions about mathematical knowledge. To discuss the specific mathematical knowledge used in the lesson, the teacher was asked to simulate a mentoring experience (Seidman, 2006): “If you were going to mentor another teacher who was about to teach this lesson, what would that teacher need to know about mathematics to teach the lesson well?” The teacher was also asked, “How did you use your knowledge of mathematics in this lesson?” The question intentionally presupposes that teachers use mathematical knowledge in teaching so that they can focus on the ways they do so (Patton, 1990). Finally, the teacher was asked to speak more generally about mathematical knowledge and its role in teaching, with the question “In general, how has your knowledge of mathematics influenced your teaching?”

The interview concluded with questions that focused on the teacher’s perceptions about how he or she developed knowledge for mathematics teaching.

Because the focus was on mathematical knowledge *used* in teaching, teachers were not asked to describe their knowledge of mathematics outside of the context of their lesson. For example, a teacher sharing a lesson on exponential functions was not prompted, “Tell me everything you know about exponential functions.” Instead, all questions that prompted teachers to discuss their mathematical knowledge abstractly were related to the lessons that they shared.

3.3.3 Data Analysis

All interviews were fully transcribed. To analyze the data, I used a constructivist grounded theory approach (Charmaz, 2011), as follows. I read the transcripts several times to get a general sense of teachers’ views of their knowledge (Creswell, 2007). Other educational researchers also read through the blinded transcripts, and we discussed our initial reactions. Through this process, I noticed that teachers gave rich accounts of their lessons and their teaching. However, teachers’ responses to questions about mathematical knowledge and its use in teaching were surprising. Responses generally fell into three categories: (a) teachers expressed difficulty in discussing mathematical knowledge abstractly, (b) teachers suggested the concept of *mathematical knowledge* did not sufficiently capture how they were thinking in the classroom, and (c) teachers described other aspects of teaching (e.g., students’ mathematical knowledge, pedagogical actions used to improve students’ mathematical understandings) rather than their abstract mathematical knowledge.

These responses indicated that teachers were de-emphasizing the discussion of their abstract mathematical knowledge. To further explore the first two types of responses, I used techniques from discourse analysis (Chimombo & Roseberry, 1998). In these cases, transcripts were supplemented with notes indicating paralingual moves, such as pauses in discourse. Both verbal cues, such as hedges (e.g., “Aren’t there,” “I suppose,” and “I mean”) and nonverbal cues, such as pauses and lengths of utterances, were considered for implicatures—that is, inferences about information not stated in the text (Chimombo & Roseberry, 1998).

The third type of response that I observed provided a rich foundation for further analysis. It was striking that teachers often answered questions about their mathematical knowledge with descriptions of (a) students' mathematical knowledge or (b) pedagogical actions that teachers used to develop students' mathematical understandings. After reading through the data several times, I felt as though I said, "Tell me about the mathematical knowledge you had to use for this lesson," and teachers were essentially answering, "Let me tell you *how* I met the mathematical needs of students in this lesson." In other words, although I had framed the study from a cognitive perspective, the essence of teachers' accounts seemed to suggest a more situated perspective on knowledge. Hence, I shifted from a cognitive perspective to one more aligned with situated cognition. With this shift, I acknowledged that teachers' mathematical knowledge was not necessarily expressed abstractly but was embedded in their accounts (Cochran-Smith & Lytle, 1999). As a result, I explored the ways of knowing and patterns of participation with mathematics—in particular, mathematics-specific attunements—that teachers' reflections captured.

I then coded teachers' reflections line by line using the "sensitizing concept" (Charmaz, 2002, p. 683) of attunements and with the following question in mind: "What ways of knowing and participating with mathematics do teachers' reflections illustrate?" The result was a list of initial codes. By the using comparative analysis (Charmaz, 2011), codes were grouped into categories. Next, details from the lesson plan were used to supplement each teacher's verbal account. Finally, teachers' reflections were considered holistically, and I searched for confirming and disconfirming evidence (Creswell, 2007) to refine the analysis.

Because the perspective on knowledge shifted during analysis, I was especially interested in member checking to ensure that my interpretations accurately captured teachers' reflections (Charmaz, 2002). All teachers were invited to participate in member-checking interviews, and seven teachers agreed. Each participant was sent a concise summary of the findings by email and then completed an audio-recorded phone interview lasting between 15 and 30 minutes. Teachers were asked to what extent the findings resonated with their own experience in teaching mathematics. They were also given the opportunity to clarify or extend any of the findings. All teachers claimed to relate to the situated nature of their knowledge and overwhelmingly appreciated the themes that were expressed. For instance, Mr. Fisher said, "You got it. You heard. … If I could be clear and write, that's what I would write. I'm very happy with the findings." Ms. Schneider said, "I applaud you for pulling that out," and Ms. Yates said, "You nailed it in what you have there." In addition, the data from these interviews was quite helpful in refining analysis.

3.4 Findings

The findings are organized into three main sections. In the first section, I illustrate how teachers de-emphasized the discussion of the abstract mathematical knowledge used in their teaching. In the second section, I introduce three themes (teachers' attunements) that were rich in teachers' reflections on their teaching. For each theme, I provide a focused account from one teacher that exemplifies several subcategories within that theme[4]. The third section provides an extended description of one teacher's lesson to illustrate how teachers' attunements were coordinated in reflections on teaching.

3.4.1 Teachers' De-emphasis of Abstract Mathematical Knowledge

In the initial interviews, three interview questions directly asked teachers to discuss the mathematical knowledge that they used in teaching. In responding to these questions, teachers de-emphasized discussions of abstract mathematical knowledge. Teachers did so in three ways.

First, three teachers expressed difficulty in answering questions about the mathematical knowledge used in their teaching. For example, in his interview, Mr. Fisher richly described his lesson, and he provided several details about the actions that he took in teaching. Then, when asked what another teacher should know about mathematics to teach the lesson well, Mr. Fisher[5] responded[6], "Mmm [*pause*]. Um, [*pause*] wow. I haven't thought about that." This response was surprising to me, given Mr. Fisher's rich lesson description. But for Mr. Fisher and others, mathematical knowledge used in this lesson was not natural to discuss abstractly.

Reflecting on their knowledge use abstractly is not something that teachers are required to do in their daily work. This could be why researchers such as Zazkis and Leikin (2010) found that teachers provided few specific examples of how they used their advanced mathematical knowledge. However, teachers' difficulty in talking about the knowledge they used does not imply that teachers lack mathematical knowledge. Rather,

[4]Brief examples from other teachers can be found in Appendix B.
[5]All names are pseudonyms.
[6]In this section, *[*pause*]* indicates there is a pause in the discourse. Elsewhere, to increase readability, ellipses indicate the removal of short passages, brackets indicate words that were inserted or changed to clarify meaning, and pauses were not indicated. I do not believe the edits changed the meaning of the text.

I found that teachers' understandings of the mathematics used in teaching were embedded in the actions of teaching. In the member-checking interview, Ms. Yates elaborated on this point:

> *Ms. Y.*: I think that it's hard to talk about because … you don't think about your mathematical knowledge specifically. … You think about what did the students learn before that they can use to apply to this lesson, and what are they going to have to do? … And to me, *that's the mathematical knowledge*: I'm taking prior knowledge, what do they already know, what do I want them to know, and how am I going to create that bridge for them to get from what they know to what they need to know? … So I don't know if that's mathematical, … *but to me that's what you're using*. [Italics are my emphasis.]

Ms. Yates's point here is an important one. She believed that in teaching, she actively thinks about meeting the mathematical needs of students rather than abstract mathematical knowledge. Instead of describing specific mathematical difficulties that students have, Ms. Yates described the *process* of understanding and meeting the needs of students in a specific situation, saying, "I'm taking prior knowledge, what do they already know, what do I want them to know, and how am I going to create that bridge for them to get from what they know to what they need to know?" This quote illustrates Ms. Yates's attunements to students' thinking (elaborated in the next section), and the questions that she asked herself relate closely to researchers' conceptions of mathematical knowledge that is situated in teaching (e.g., Brown & Coles, 2000; Davis & Simmt, 2006).

Second, six teachers indicated that an abstract discussion of mathematical knowledge was missing the essence of expertise in teaching. For instance, Ms. Lombardi shared a lesson on the triangle inequality, eagerly and thoroughly describing her teaching. She generally responded to interview questions confidently and without pauses or hedges. Later, when asked how she used her knowledge of mathematics in the lesson, Ms.

Lombardi responded, “Um [*pause*], I would say just [*pause*], I mean, it's more of a high school level of mathematics, [*pause*] but for the extension activity where they use calculus.” In this excerpt, Ms. Lombardi paused three times, indicating that the response was not straightforward for her. She also added the hedges “just” and “I mean,” suggesting she was distancing herself from the statement about her knowledge (Chimombo & Roseberry, 1998). The short nature of Ms. Lombardi’s response, in contrast to her earlier, lengthier responses, also suggested that abstract mathematical knowledge was not of central importance in Ms. Lombardi’s reflection. Rather, she expressed her knowledge as part of the rich actions of teaching that she described. In the member-checking interview, Ms. Schneider echoed this point:

> *Ms. S.*: I think the mathematical knowledge is what [we] probably went to teacher school for. … That's more or less the science of teaching. … Where knowing your students and being able to find ways of reaching them, that's the art part of it. … I think [what you have described in the themes] is *the* part of math teaching. I think you were able to bring everything together. [Italics were Ms. Schneider’s emphasis.]

Ms Schneider described that the “art” of expertise in mathematics teaching is in “knowing your students and being able to find ways of reaching them,” whereas abstract knowledge learned in “teacher school” does not capture the meaning of what mathematics teachers do. Ms. Schneider emphasized that teachers’ ways of knowing and participating with mathematics go beyond “the science of teaching” and instead are embedded in the process of teaching.

Third and most telling, in response to direct questions about their mathematical knowledge, eight teachers briefly described *how* they met the mathematical needs of their students through their understanding of students as learners or pedagogical actions that they used. In other words, teachers were not abstracting their knowledge, despite direct prompts from the interviewer leading the discussion in this direction, but teachers were sharing their accounts of the *process* of teaching. In these discussions, teachers illustrated various mathematics-specific ways of knowing and participating, or attunements. These attunements were elaborated, expanded, and exemplified in teachers' longer descriptions of their lessons, and I present these themes in the following section.

3.4.2 Themes of Teachers' Reflections: Mathematics-specific Attunements in Teaching

In interviews, teachers discussed three themes that illustrated their mathematics-specific attunements in teaching: (a) knowing students as learners of mathematics, (b) developing mathematical ideas, and (c) promoting students' mathematical activity. Within these three themes were several subcategories. An overview of themes and subcategories is presented in Table 3.1.

Table 3.1 *Themes of Reflections: Mathematics-Specific Attunements in Teaching*

Themes	Number of Teachers (N = 11)
Knowing students as learners	**11**
• Students' mathematical thinking	11
• Students' interests	10
• Students' background knowledge	9
• Students' learning styles	6
Developing mathematical ideas	**11**
• Connections to applications outside of mathematics	11
• Interconnectivity of mathematics	10
• Multiple representations of the content	10
• Technology	8
• Key examples of the content	5
• Mathematical generalizations	5
Promoting students' mathematical activity	**11**
• Active participation from students	10
• Explanations from students	6
• Problem solving activities	4

Knowing students as learners. Through their reflections, participants indicated the importance of knowing their students as learners of mathematics. This theme included (a) understanding typical patterns in students' mathematical thinking and common difficulty with certain topics, (b) knowing what can capture student interest, (c) knowing students' mathematical background, and (d) being aware of students' different learning styles. To illustrate this theme, I present excerpts from Mr. Meyer's interview. I chose to share Mr. Meyer's account because it concisely illustrates several subthemes within this theme. (Similar choices are made for the subsequent sections.) Mr. Meyer shared the following problem:

> Suppose you wanted to make an ice cream cone that would hold as much ice cream as possible. In this activity, you will solve that problem.
>
> 1. Cut a wedge from a circle and remove it. Form the remaining piece of the circle into a cone. Find the angle of the wedge that produces the cone with the greatest volume.
> 2. Make a second cone from the removed wedge. Find a formula for the volume of this second cone in terms of θ, the angle of the wedge.
> 3. Find the wedge angle that produces the maximum total volume of the two cones.

Mr. Meyer described choosing this problem for the lesson: "I thought [this problem] was something that the kids could do with a little bit of guidance. Something that was feasible for them to accomplish. And I thought it would be something that they would find interesting." Mr. Meyer went on to describe the teaching of this problem:

> *Mr. M.*: A lot of [teaching this problem] is figuring out other ways of presenting it, because not every kid is going to be able to come up with this, or not every kid is going to understand how we get here. So it's now coming up with a second approach to get this formula or third approach. And then that comes with every lesson. Just having multiple ways of presenting the same topic to help with the different learning styles. I guess that's where our expertise comes in. … It's just knowing the kids and what their personal strengths are.

Mr. Meyer also described a common student difficulty with this problem:

> *Mr. M.:* It's funny because a lot of these kids had geometry in middle school or in ninth grade, and they have a hard time with the concept of fractional part of a circumference. They don't get it. And I say, okay, what if theta is 90 degrees? They know to multiply it by three fourths, but they don't know where the three fourths came from. They're getting 270 over 360, and that's their three fourths. They don't get that. It's really a weird stumbling block for them.

In Mr. Meyer's description of his teaching (as exemplified here but also throughout his longer account), his attunement to students as learners is paramount, and all four subcategories of this theme are exemplified in Mr. Meyer's account. To choose the cone problem, Mr. Meyer claimed that he considered *students' mathematical* *background* and *what students would find interesting*. Mr. Meyer's *awareness of students' learning styles* motivated his decision to emphasize multiple approaches to the problem. Mr. Meyer also shared his detailed *understanding of students' difficulty* in finding "a fractional part of a circumference."

The mathematics-specific attunements present in this account are related to what Ball et al. (2008) called *knowledge of content and students*, a type of PCK that includes teachers' knowledge of common patterns of students' mathematical thinking as well as common student difficulties with the content. What is different about this example is that Mr. Meyer did not describe de-contexualized PCK. For instance, Mr. Meyer did not describe abstract difficulties that students had with fractions or circles. Rather, as Mr. Meyer described his lesson and its context, his awareness of students as learners of mathematics arose in his reflection without prompting, and Mr. Meyer indicated that part of this awareness led him to choose the problem. Although Mr. Meyer does not codify the knowledge he used in teaching, his lesson reflection reveals his mathematics-specific attunements embedded in his account of his practice (Cochran-Smith & Lytle, 1999).

Teachers were not asked directly about their knowledge of students during the interviews, but the theme of knowing the students as learners of mathematics was emphasized heavily in teachers' accounts. This theme was also described as grounding other mathematical choices, as will be illustrated in the following sections.

Developing mathematical ideas. The second theme in teachers' reflections was developing mathematical ideas in teaching. This was illustrated in six ways. (a) All 11[7]

[7] If a teacher did not mention a theme, this does *not* imply that he or she does not think the theme was important. It is certainly possible that teachers would agree with the theme if prompted (all teachers who

teachers described making connections to applications outside of mathematics. (b) Ten teachers described emphasizing the interconnectivity of mathematics. That is, teachers discussed how they illustrated relationships among mathematical ideas, including the continuity of mathematical topics and relationships among different areas of mathematics. (c) Ten teachers described including multiple representations of the content. Representations included hands-on representations such as manipulatives, visual representations such as graphs, and abstract representations such as equations. (d) Eight teachers described using technology to develop the content because it helped convey meaning, engage the students, or solve problems that had typical solution methods that were beyond the scope of the course. (e) Five teachers described choosing meaningful examples and nonexamples of the content. Teachers ensured that the examples gave a rich picture of the concept, were accessible to students, and included special cases if they existed. (f) Five teachers described developing mathematical ideas through generalizations. That is, the teacher or students made mathematical observations and then developed a general statement about these observations, such as a formal rule. As with the last theme, teachers expressed these attunements *through* their action-centered accounts of practice rather than abstracting their knowledge from the accounts.

For instance, in teaching a unit on logarithms, Ms. Kruger's reflection illustrated how she developed mathematical ideas. This development was motivated by her understanding of students, as she described in the following excerpts:

> *Ms. K.*: By the time kids get to my class, they've seen logs before, but ninety percent of them have no idea why they do it. … I can show them why it makes

---

participated in member-checking interviews agreed with the themes) or the theme would arise if he or she was discussing another lesson.

> sense and show them the math behind it, and that's why I like math—the way it builds on itself and the relationships between different things.

In her lesson, Ms. Kruger asked her students to justify properties of logarithms, such as ln(xy) = ln(x) + ln(y), in terms of properties of exponents (in this case, $e^a e^b = e^{a+b}$):

> *Ms. K.*: I tie [properties of logs] into the properties of exponents so that [students] can see, "Oh, well when you multiply *a* to the *x* and *a* to the *y*, you get *a* to the *x* plus *y*. Well with logs, when you add them, you multiply. Why does that make sense based on what we know about how the functions are related?" ... And I found the more of those building blocks that I can help [students] create, the more likely they are to remember the properties later.

Ms. Kruger's description illustrated her attunement to *emphasizing the interconnectivity of exponents and logarithms*, and she claimed that these connections were motivated by her *awareness of students' mathematical background*, specifically students' lack of exposure to reasons why logarithms make sense and students' prior experience with the rules of exponentiation. Ms. Kruger also claimed the decision to emphasize this interconnectivity was partially motivated by her *awareness of students' thinking*—that they will remember the mathematics better when these connections are made.

To further develop the mathematical ideas in her lesson on logarithms, Ms. Kruger gave the following task to students: "Find a pair of real numbers x and y such that $xy = 6$ but it is not true that $\ln 6 = \ln x + \ln y$." Ms. Kruger commented on the difficulty that students have with this example:

> *Ms. K.*: Half of my kids every year say it's not possible. Half of my kids say, "There's no numbers. That's a trick question. Why did you do that to us?" Or, they skip it, and I have to make them go back and do it.

This excerpt illustrates Ms. Kruger's attunement to *choosing meaningful examples* to develop the content. Ms. Kruger recognized the *student difficulties* with this example, but in spite of students’ difficulty, she believed this example addressed an important mathematical point, namely that logarithms of negative values are undefined.

Ms. Kruger also had students apply logarithms when using an exponential model of the number of people infected by a given virus with respect to time, as she explained:

> *Ms. K.*: My kids get excited about [applications]. Because they're so into biology and the idea of a virus spreading, they love that. The flu, that's what gets them going. So being able to do that with this lesson is a nice thing that they can see it actually working.

Ms. Kruger expressed *awareness of students’ interests*, which she claimed motivated the choice to *make connections to applications outside of mathematics*, and she believed that she did so in a way that coordinated the development of the concept and satisfying the interests of the students.

The development of mathematical ideas in teaching is supported by what Ball et al. (2008) called *knowledge of content and teaching,* another type of PCK. This includes teachers’ knowledge of examples, representations, and connections that can help to explain a concept. What is striking about teachers’ accounts was that they did not explicate their knowledge of content and teaching as abstracted from their teaching. Rather, they described *how* they developed content with the students in context. For instance, in Ms. Kruger’s reflection, she described the process of teaching, and her mathematics-specific attunements were embodied in her account. In addition, teachers described the development of content in conjunction with their understanding of their students as learners of mathematics. For instance, Ms. Kruger claimed that she justified the properties of logs in terms of properties of exponents—at least in part—*because* she recognized this gap in students’ background and *because* she believed it would help students remember the mathematics. In other words, Ms. Kruger’s attunement to students as learners seemed to be coordinated with her attunement to the development of mathematical ideas.

In general, participants discussed the ways in which they developed mathematical ideas in teaching, not in terms of the acquired knowledge that supported this development, but in coordination with their attunement to students as learners of mathematics in the process of meeting the mathematical needs of their students. In fact, in response to interview questions that directly asked about the mathematical knowledge used in their lessons, 10 of 11 teachers described how they developed mathematical ideas, and six of the 10 described developing mathematical ideas in conjunction with knowing students as learners of mathematics. In other words, developing ideas with students seems to be of central importance to these teachers, and their use of mathematical knowledge was embedded within these accounts.

Promoting students' mathematical activity. The third theme in teachers' reflections was promoting students' mathematical activity. That is, in discussions of their lessons, teachers described (a) encouraging active participation from students, (b) emphasizing students' mathematical explanations, and (c) engaging students in problem solving activities.

For example, Ms. Lombardi shared a lesson on the triangle inequality. At the end of this lesson, students were given the following problem: "Randomly cut a stick into three pieces. What is the probability that the three pieces form a triangle if the lengths of the cuts are integers? If the lengths of the cuts are any real number?" Ms. Lombardi described that she chose this problem because of the importance in promoting students' mathematical activity:

> *Ms. L.*: One of the things that I do, pretty much every unit that I teach, ... I try to have something that requires the students to think and communicate, whether it's in written form like this or sometimes even in a presentation format. I think the students are so used to, "Okay, I can quickly get the answer. It's immediate; I don't have to think so much." And I want them to realize that in mathematics, there are times where you're going to have to think and not get the answer. Come back to a problem and look at it again, then take time out from it. And that it's okay to have a problem truly be a problem for a while. … Also, not only did [students] just have a little bit of work, can they explain it? Can [students] explain [their work] using tables and graphs and using the correct vocabulary?

Here Ms. Lombardi discussed her intention for students to *engage in problem solving activities* and *provide explanations* for their work. Ms. Lombardi also described how she considered *students' thinking* when she refined the wording for this problem. She explained, "If I don't specify integers or real numbers, the students tend to focus on integers for some reason. They don't think 'Oh yeah, a length can be two point four,' or whatever."

In Ms. Lombardi's example and in all teachers' reflections, attunements to students' mathematical activity were expressed through a discussion of the process of teaching. Moreover, teachers discussed the importance of students' mathematical activity as a way to meet the mathematical needs of the students, and teachers coordinated students' activity with their awareness of students as learners of mathematics. In fact, teachers' accounts often illustrated the coordination of several attunements, and this will be exemplified in the next section.

3.4.3 Coordinating Mathematics-specific Attunements

In describing the three themes in the previous section, I do not wish to portray that teachers expressed their knowledge in a static way. Arguing for a more situated view of PCK, Mason (2008) cautioned that "if the term PCK is used as a checklist of qualities, quantities, and dimensions, it will only serve to obscure what is essential and central" (p.

305). Teachers in this study expressed a similar view, and Ms. Schneider elaborated on this point in the member-checking interview when discussing the theme of developing mathematical ideas:

> *Ms. S.*: [Novice teachers] come to me with book learning, and they don't see *how* this "developing ideas" works every day. So that's the next thing they need to do. … They can list, they can memorize these six things, but to be able to use it, to be able to see it, and to have them come up with their own means of doing it [is most important]. [Italics were Ms. Schneider's emphasis.]

For Ms. Schneider and the other teachers in this study, the essence of mathematics teaching is in the *activity* of mathematics teaching rather than abstracted knowledge used in this activity. Moreover, the themes described in teachers' reflections were not discussed in isolated ways. In this section, I present one extended account to illustrate how teachers' reflections depicted several attunements for the same episode.

Ms. Johnson shared an Algebra I lesson on functions. She described her choice to also teach students about one-to-one functions, an idea typically taught in Algebra II:

> *Ms. J.*: My rough lesson plan was functions and the vertical line test. A student asked, "Is there a horizontal line test?" There were two choices I could have made at that point. I could have said, "Yes, you'll learn about it in Algebra Two," or, "Yes, and here's why there's a horizontal line test." I chose the second one. … I think it's very important to respect students' questions and respond to them if possible, which is why I chose to diverge from what I planned for that lesson [and go] into, "Yeah, there's a horizontal line test, and it's associated with one-to-one functions." We discussed it that day for about the last 20 minutes of class, and then came back the next class period. By then I had thought more about how [the teaching] was going and I said, "Okay, let me pull out the patty paper [i.e., translucent paper used for folding in geometry] and show them the reflection about *y* equals *x* and why the horizontal line test turns into a vertical line test for the inverse." So we actually got into quite a bit of discussion about the horizontal line test and inverse of a function.

> *Ms. J.*: And I think they followed [the discussion] because instead of just talking about [one-to-one functions] in the abstract, I made sure that I kept the discussion centered in the graphic implications, including pulling out the paddy paper the next class period, and having them actually physically fold [and] reflect across *y* equals *x* and see how it becomes a function or it doesn't become a function. So I think keeping in mind the mathematical sophistication of the students in selecting

> the activities or even the approach I was going to take helped them at least get a useful preview of one-to-one functions.
>
> *Ms. J.*: Because I was responding to a question from one of [the students], ... they had some ownership in what was being discussed in class. … I think their enthusiasm was increased by the fact that they knew [the lesson topic] was coming from them, and they had some voice in the direction the class was going to take. … [Also,] I actively got them involved in exploring ... so that they could see what was going on. Algebra One students are in ninth grade. They're 14 or 15 years old. So they have to be active learners, and I think that doesn't quite come through in the [lesson plan]. … It was keeping the learning active.

Ms. Johnson illustrated all three themes in her account of teaching about one-to-one functions. First, Ms. Johnson’s reflection *emphasized the interconnectivity of mathematical ideas* when she described how she connected the ideas of the vertical line test, horizontal line test, one-to-one functions, and inverse functions with students. She described how she *included multiple representations* when she illustrated these relationships through a hands-on activity: Students reflected injective functions and noninjective functions over the line $y = x$ to observe the relationship between invertible functions and injective functions.

Second, Ms. Johnson’s reflection illustrated *knowing students’ mathematical background*, as she said she was “keeping in mind the mathematical sophistication of the students in selecting the activities or even the approach.” In Ms. Johnson’s reflection, she expressed *knowing what would interest her students*, recognizing that by respecting and responding to students’ questions, they would have more enthusiasm and ownership in the lesson. Third, Ms. Johnson described that she *encouraged active participation from the students* because she believed this was an important aspect of learning for adolescents.

What is striking about Ms. Johnson’s account is, although she was prompted to discuss her mathematical knowledge, she does not discuss her content knowledge of inverse functions and one-to-one functions explicitly. Nor does she describe her PCK, such as her knowledge of content and students or knowledge of content and teaching (Ball et al., 2008). Rather, Ms. Johnson described the process of teaching for the purpose of meeting the mathematical needs of the students. Ms. Johnson expressed how she developed mathematical ideas based on her understanding of students as learners (recognizing that the concept of one-to-one functions may be difficult for the students) and her belief that students should be actively engaged in learning. Another way to consider the coordination of these three themes is to say that in Ms. Johnson's reflection, she is attuned to the students, the methods for developing the mathematics, and the mathematical processes for doing so. As she reflected on her teaching, Ms. Johnson's content knowledge and PCK was situated in her account of teaching.

The three general attunements described in this chapter were pervasive themes in teachers' reflections. However, teachers did not describe the knowledge used in teaching abstractly. Rather, it was contextualized in their accounts of teaching practice. In this way, analyzing the data with a situated cognition perspective allowed me to better capture the rich essence of teachers' accounts. In subsequent member-checking interviews, participants agreed with the findings.

3.5 Conclusion

In this chapter, I illustrated the mathematics-specific attunements in exemplary high school teachers' reflections on teaching by using a situated perspective on knowledge. I originally intended to use a cognitive perspective to describe exemplary mathematics teachers' perspectives on the content knowledge used in their lessons, but despite the focus on specific lessons during interviews, teachers had difficulty in answering direct questions about this knowledge. Rather, teachers illustrated their mathematics-specific attunements through their rich descriptions of teaching.

3.5.1 Limitations

There are some limitations to this study to be noted. The first concerns the sample. I interviewed only 11 participants from a specific region of the United States. The results are therefore specific to these exemplary teachers in this particular cultural context. Other exemplary teachers in other settings may focus on different ideas in their reflections on teaching. Second, the interview method is intriguing because it affords the opportunity to understand teachers' perspectives; however, it is also limiting because I do not have data about what teachers actually do in the classroom setting. In the following two chapters in this dissertation, I report on a study that used observations to explore teachers' use of knowledge during instruction. Third, the findings might not be specific to exemplary teachers. Teachers who are not recognized for their teaching might reflect on their teaching in similar ways. Nonetheless, this study provides an important glimpse at how these particular high school mathematics teachers portray the mathematical knowledge used in teaching in their reflections, and understanding teachers' reflections on teaching can be helpful in structuring teacher education initiatives.

3.5.2 Discussion

Using a cognitive perspective, some researchers have described the content knowledge and PCK that teachers need to teach effectively (e.g., Ball et al., 2008). I *did* find similar themes in this data, but noteworthy is that this content knowledge and PCK was expressed through exemplary teachers' accounts of teaching and was deeply intertwined with their descriptions of actions in teaching. Teachers' reflections illustrated the simultaneous coordination of several attunements in the process of teaching. In addition, teachers found it difficult to discuss the content knowledge used in their lessons in a de-contextualized way.

There are many plausible reasons why the teachers in this study had difficulty in discussing the content knowledge used in their lessons in abstract ways. First, teachers may lack experience in describing their knowledge abstractly. Ms. Yates elaborated on this point in the first section of the findings, and other researchers have also noted this point (e.g., Brown & McIntyre, 1993). Second, interviews with teachers in this study suggested that teachers did not believe that discussing abstract content knowledge used in their lessons captured the essence of the practice of teaching, also described in the first section of the findings. Third, it is possible that a different interview protocol might better capture teachers' perspectives on the abstract mathematical knowledge used in teaching. Regardless of the reason for teachers' difficulty, what was important was that they did express a great deal of content knowledge and PCK in their reflections. That is, although the exemplary teachers in this study did not specifically discuss abstract mathematical ideas in their reflections, they did express their mathematical understandings through their discussions of how they achieved pedagogical goals (cf. Brown & Coles, 2011), and these understandings were explored by using a situated perspective on teachers' knowledge.

The present study challenges the assumption that teachers' difficulty in abstracting and describing their mathematical knowledge implies they are deficient in such knowledge. Participants in this study had been recognized for their exemplary teaching and still had difficulties in answering direct questions about their abstract mathematical knowledge. Moreover, though their knowledge was not captured by the direct interview questions in this study, these teachers shared many ways of knowing and using mathematics that were embedded in their accounts of teaching.

Other researchers have argued that it is important that teachers are able to describe their abstract knowledge because it can be helpful in (a) providing a basis for their reflections on practice (e.g., Clemente & Ramírez, 2008) and (b) mentoring novices—particularly bridging the gap between theoretical teacher preparation and the practicalities of the classroom (e.g., Asikainen et al., 2013). Although I agree that the articulation of abstract knowledge can be helpful in these cases, this study cautions against assuming that when teachers have difficulty abstracting knowledge from their practice that they are deficient in such knowledge. In addition, to bridge the gap between abstract knowledge described by researchers (e.g., Ball et al., 2008) and ways of knowing described by teachers (this study), some researchers have suggested that teachers study researchers' constructions of knowledge for teaching (e.g., Asikainen et al., 2013). A less common suggestion is that researchers build and refine theories from teachers' reflections on practice (e.g., Clemente & Ramírez, 2008). Oslund (2012) echoed this point, saying, "At worst, narrow conceptions of mathematics-for-teaching may ignore the complex and sophisticated sets of competencies teachers bring to the profession and promote (even unknowingly) deficit views of teacher knowledge" (p. 307). In accordance with Clemente and Ramírez (2008) and Oslund (2012), I suggest that researchers critically consider the affordances rather than the deficits of teachers' ways of knowing as they are expressed in reflections on teaching.

Chapter 4: Teacher Thinking behind Responding to Student Mathematical Productions in High School Mathematics Instruction

Abstract

The purpose of this investigation was to explore the teacher orientations and knowledge that support high-quality responses to students in instruction, as measured by the mathematical quality of instruction framework (LMT, 2010). I observed and interviewed 12 high school teachers, five of whom were recognized for exemplary instruction. Video-based, stimulated-recall interviews were used to understand teacher thinking behind specific responses to students' mathematical questions, claims, and solutions. Two themes guided high-quality responses to students: (a) building on the students' mathematical ideas and (b) taking the opportunity to emphasize meaning and sense making. These themes were closely related to teachers' goals, a reflection of teachers' orientations, and supported by various types of teachers' knowledge. By contrast, in lower quality responses, (a) the teacher's goals did not align with the student's production or (b) the teacher lacked knowledge in the moment of responding to the production.

4.1 Introduction

Recent efforts in mathematics education have emphasized the importance of building mathematics instruction around students' mathematical ideas, errors, and confusions (e.g., Fennema et al., 1996; Franke, Kazemi, & Battey, 2007; NCTM, 2000). By contrast, teacher-centered mathematics instruction, in which contributions from students are limited to students' procedural questions and answers, is common in the United States (Stigler & Hiebert, 1999). Hence, for many teachers, working with students' thinking in extended and detailed ways would require a major shift from current practices. Important first steps to this shift are noticing student mathematical productions and responding to them in mathematically-appropriate ways, and these are investigated in the present study.

Drawing on terminology from the Learning Mathematics for Teaching Project (LMT, 2010), I use the term *student mathematical productions* (SMPs) to refer to students' "questions, claims, explanations, solution methods … etc. that contain substantial mathematical ideas" (LMT, 2010, p. 11). SMPs can contain rich mathematical ideas and often provide opportunities to extend, clarify, or enhance the mathematics in instruction (e.g., Peterson & Leatham, 2009; Stockero & Van Zoest, 2013). However, because these productions are generated by students, they can be unusual and unexpected, so working with these productions in ways that advance instruction can be a challenging task for teachers (e.g., Peterson & Leatham, 2009). In particular, teachers must attend to, interpret, and decide how to respond to these SMPs, and then they must draw on their expertise to facilitate a response of high mathematical quality (Jacobs, Lamb, & Philipp, 2010). This work is especially difficult because it is often done in the moment of instruction (Jacobs, Lamb, Philipp, & Schappelle, 2011).

Researchers argue that to engage in any aspect of high-quality instruction, including working with SMPs, teachers must have productive orientations, which guide them to form appropriate instructional goals (Philipp, 2007), and strong mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008). However, not much research has explored both teachers' orientations and teachers' knowledge behind the specific work of responding to SMPs in instruction, and this is particularly true at the high school level.

The aim of the present study is to understand the orientations and knowledge that were used in different types of responses to SMPs. The approach I take in this study is to begin with instruction: I first characterize teachers' responses to SMPs according to their mathematical quality. For each response to an SMP, I seek to understand the teacher's goals, orientations, and knowledge that led the teacher to respond in the way that they did. Specifically, the following research questions guided this study:

1. What goals, orientations, and knowledge do teachers express as they reflect on their decisions to respond to SMPs in instruction?
2. How do teachers' goals, orientations, and knowledge support or hinder high-quality responses to SMPs?

Understanding the goals, orientations, and knowledge that support high-quality responses as well as those that lead to lower-quality responses can help to inform teacher educators about key areas for development. This study also provides a better understanding of the specific ways that teachers' knowledge and orientations are used in the work of responding to SMPs. In addition, this study highlights the integral roles of both orientations and knowledge in deciding how to respond to the SMP. As such, I argue that researchers studying the relationship between mathematical knowledge for teaching and instruction need to consider orientations to fully understand this complex relationship, particularly when studying teachers' in-the-moment decisions.

4.2 Background

This study draws on three areas of research in teacher education: teacher noticing of students' mathematical thinking, teacher thinking in instructional decision making, and the mathematical quality of instruction.

4.2.1 Teacher Noticing of Students' Mathematical Thinking

Research has documented that student-centered practices in mathematics instruction have a positive relationship with student achievement (e.g., Fennema et al., 1996; Silver & Stein, 1996) and are also rich sites for teacher learning (e.g., Leikin & Zazkis, 2010; Weber & Rhoads, 2011). Student-centered instruction requires that teachers engage in noticing and responding to SMPs.

According to the LMT Project (2010), SMPs include students' spoken or written work that includes pertinent mathematical ideas. These may come in the form of questions, solution methods, explanations, and so on, and they may also be student errors or articulated confusion. SMPs must illustrate students' substantial mathematical thinking; they are not "simply answers to problems or pointed questions where [the] teacher has sought a specific, bounded piece of information" (p. 12). In addition, it is not necessary that SMPs be correct or complete to be named a production; indeed, SMPs may include student errors or difficulties that "offer opportunities for discussing and addressing pertinent mathematical ideas" (p. 11). The importance of SMPs is that they afford the teacher an opportunity to respond and clarify, extend, or enhance the mathematics being discussed in ways that build directly from students' thinking (see also Peterson & Leatham, 2009; Stockero & Van Zoest, 2013).

Using SMPs effectively in instruction requires the work of *teacher noticing*. Building from Jacobs et al. (2010), I define teacher noticing of SMPs to include (a) attending to SMPs, (b) interpreting SMPs, and (c) deciding how to respond to SMPs. Each of these aspects of teacher noticing requires unique expertise. For example, simply recognizing an SMP as such does not ensure that a teacher will be able to interpret the SMP effectively or make a decision to respond in a way that advances the mathematics. At the same time, these three aspects of noticing are interrelated and occur almost simultaneously in instruction (Jacobs et al., 2011). Once a teacher has attended to the SMP, interpreted it, and decided how to respond, they must then engage in facilitating the instructional response, and this requires additional expertise.

The first aspect of teacher noticing, attending to student thinking, requires teachers to recognize moments—generated by students—that offer the opportunity to advance mathematical ideas in instruction. Stockero and Van Zoest (2013) called these pivotal teaching moments and described the characteristics of such moments. One of the goals of Stockero and Van Zoest's study was to understand the relationships between the characteristics of the pivotal teaching moment and the response offered to students. Stockero and Van Zoest contributed a useful framework for teacher education, but their study was not designed to explore the teacher thinking behind the second two aspects of teacher noticing: interpreting students' thinking and deciding how to respond to students.

In fact, much of the research on teacher noticing has focused on the aspect of attending to student thinking (e.g., Goldsmith & Seago, 2011; Star, Lynch, & Perova, 2011; Van Es, 2011). Little is known about high school mathematics teachers' thinking behind the in-the-moment work of interpreting SMPs and deciding how to respond to them. The present study will explore these two aspects of teacher noticing, along with instructional responses to SMPs.

4.2.2 Teacher Thinking behind Work with Student Productions

Recent research in mathematics education has illuminated the fact that teachers' knowledge is critically important in all areas of instruction, including working with SMPs. The knowledge that mathematics teaching requires is more than content knowledge of mathematics concepts and procedures. Mathematics teaching also requires what Shulman (1986) called *pedagogical content knowledge* (PCK). That is, teachers must understand powerful explanations of mathematics, know how to convey mathematics to students, and know what aspects of mathematics make it difficult or easy to learn. In mathematics education, Shulman's conceptualization has been extended by Ball and colleagues to a framework for *mathematical knowledge for teaching* (MKT), which includes both content knowledge and PCK (e.g., Ball & Bass, 2002; Ball et al., 2008; Hill, Ball, & Schilling, 2008). Recent research has suggested ways in which MKT and orientations may be used in different aspects of teacher noticing, and this research is described below.

Interpreting SMPs. The work of interpreting SMPs is complex; students' ideas are often incomplete and expressed with language that students commonly use. Hence, researchers have illustrated that working with students requires teachers to listen to students and interpret their mathematical thinking (Johnson & Larsen, 2012; Peterson & Leatham, 2009; Speer & Wagner, 2009). This work requires knowledge that Ball et al. (2008) described this as *knowledge of content and students* (KCS), a type of PCK that includes both "mathematical understanding and a familiarity with students and their mathematical thinking" (p. 401). Researchers studying the instruction of university professors have illustrated that KCS is distinct from mathematics content knowledge, and they illustrated that a lack of this knowledge hindered work with students' ideas (Johnson & Larsen, 2012; Speer & Wagner, 2009).

Deciding how to respond to SMPs. Researchers have also argued that strong MKT is necessary to determine how to use SMPs in instruction. For example, in Davies and Walker's (2005) study, groups of elementary teachers discussed classroom episodes in which the teacher made a decision around student thinking. The researchers found that teachers sometimes had difficulties in deciding how they would use students' thinking in instruction, and Davies and Walker argued that this difficulty was due to the teachers' lack of knowledge of mathematics, mathematics teaching, or student learning. Peterson and Leatham (2009) made a similar point in a study with middle school teachers.

Other researchers have argued that teachers draw on more than only their knowledge when making decisions in the classroom. For example, Herbst and Chazan (2012) explored the justifications behind teachers' pedagogical decision making. They argued that teachers are expected to not only uphold the integrity of mathematics as a discipline but also understand and care for their students while meeting a variety of professional requirements in their schools and districts. Teachers have obligations to the discipline of mathematics, individual students, the collective class of students, and the school or institution (Herbst & Chazan, 2012). Each of these must be balanced in teachers' decisions, and at times, one obligation may be somewhat sacrificed to fulfill another. Herbst and Chazan (2003) defined this as *practical rationality in mathematics teaching*. When making instructional decisions, teachers rely on "a network of differentially prioritized dispositions" (Herbst & Chazan, 2003, p. 13), and Herbst and Chazan argued that it is important to understand the practical rationality behind instructional decisions to improve teacher education.

Researchers have also illustrated how teachers can successfully navigate the in-the-moment decisions they must make in instruction. Brown and Coles (2000) used the concept of *purposes* to describe how teachers navigate such decisions. The researchers describe a purpose as an "idea kept before the mind" (p. 167) in the process of teaching. As teachers make decisions, they attend to several purposes at once. At the beginning of a lesson, a teacher's purpose might be to understand what students already know about the topic. Brown and Coles argued different pedagogical strategies may be effective in different situations and for different teachers; hence, it is the teacher's purpose, rather than other elements of teaching, that ultimately leads to effective instruction. A similar observation was made by Watson and De Geest (2005) who described how teachers' principles and intended directions for teaching—rather than particular teaching strategies or actions—impacted students' learning.

Developing a framework to further explore the complexity of teacher decision making, Schoenfeld (2011) described the thinking behind instructional decisions in terms of goals, orientations, and resources. According to Schoenfeld, teachers make the decisions that they do because of their goals. A *goal* is defined as "something that an individual wants to achieve" (p. 20), and goals may be short term (e.g., responding to a student in a particular way) or long term (e.g., helping students develop understanding across the school year). Teachers set goals based on what they believe to be most important in their work, and Schoenfeld defined these beliefs to be part of the teacher's orientations. Specifically, *orientations* include a teacher's "dispositions, beliefs, values, tastes, and preferences" (p. 29). A teacher's orientations prioritize the goals that they have, and the teacher draws on resources to fulfill those goals. *Resources* include a teacher's intellectual, material, and social resources. Intellectual resources include knowledge, which Schoenfeld defined as "the information [one] has potentially available to bring to bear in order to solve problems, achieve goals, or perform other such tasks" (p. 25). Material resources may include the curriculum, tools for teaching, and so on, whereas social resources include a teacher's social status in the situation (e.g., teachers may establish the classroom rules because they are in a leadership position). In the case of responding to SMPs, Schoenfeld's theory proposes that a teacher forms a goal to respond based on their orientations and carries out the response based on their resources.

Responding to the SMP. After teachers have interpreted the SMP and determined how they will respond, they then engage in the work of responding. Peterson and Leatham (2009) illustrated that intentions to work with student thinking in instruction do not guarantee that teachers will be able to carry out high-quality responses to students. They argued that doing so requires several types of knowledge, including *knowledge of content and teaching*, a type of PCK described by Ball et al. (2008). This knowledge includes knowledge of teaching and knowledge of mathematics needed to do work such as sequence content in instruction, choose examples, and evaluate representations. That is, teachers must have knowledge of how to address students' thinking in instruction.

In the research described in this section, all but one of the studies (i.e., Peterson & Leatham, 2009) explored only one aspect of teacher noticing: either interpreting students' thinking or deciding how to respond to students' thinking. However, as Jacobs et al. (2011) described, noticing and responding to students' thinking often happens almost simultaneously in the classroom. Hence, it is important to explore these aspects in conjunction with one another, as the present study does. Moreover, a key reason for researching teachers' work with SMPs is to understand how to support teachers in carrying out high-quality instruction. As such, the present study will relate teachers' thinking behind responding to SMPs to the mathematical quality of the responses.

4.2.3 Mathematical Quality of Instruction

One goal of the present study is to identify and understand the teacher thinking that supports high-quality responses to SMPs. There are many lenses by which researchers can characterize the quality of responses to SMPs (e.g., Stockero & Van Zoest, 2013). One such lens is the mathematical quality of instruction (MQI; LMT, 2010). This instrument captures whether the response emphasizes mathematical ideas, and it is a widely used, reliable way to describe instruction (see National Center for Teacher Effectiveness [NCTE], 2012). The MQI rubric measures only the mathematical nature of the instruction that is enacted in the classroom, regardless of other factors, such as the style of instruction or teacher intentions. The mathematical nature of instruction is scored along the following dimensions:

1. *Richness of the Mathematics* captures whether and how the mathematics in instruction focuses on meaning or mathematical practices.
2. *Working with Students and Mathematics* indicates whether teachers understand SMPs and respond appropriately.
3. *Errors and Imprecision* assesses the teacher's mathematical errors, imprecision, or lack of clarity.
4. *Student Participation in Meaning-Making and Reasoning* captures whether and how students are engaged with mathematics through questioning, reasoning, and meaning making.
5. *Classroom Work is Connected to Mathematics* indicates whether instruction is focused on mathematics content.

(See NCTE, 2012, for a more complete description of the dimensions.)

The MQI rubric has been used by other researchers as a tool for exploring teaching. For example, Hill, Blunk, et al. (2008) explored how teachers' MKT contributes to MQI. More recently, one issue of the *Journal of Curriculum Studies* (Charalambous & Hill, 2012) was devoted to exploring how both teachers' MKT and curriculum materials contribute to MQI. The results of these studies indicated that teachers' mathematical knowledge helps teachers to implement instruction that exemplifies the elements of MQI. However, researchers also hypothesized that orientations towards mathematics and mathematics teaching may contribute to MQI (Hill, Blunk, et al., 2008; Sleep & Eskelson, 2012).

In the studies by Hill, Blunk, et al. (2008) and Sleep and Eskelson (2012), MKT was measured by written assessments (LMT, 2012), and instruction was considered holistically. To explore the joint roles of orientations and knowledge in instruction, the present study takes a different approach. Specifically, I explore the specific moments of responding to SMPs in instruction, and I am interested in the MKT that teachers use in these moments. As such, I do not assess teachers' knowledge outside of these situations. The rationale for this approach is that a detailed view of teachers' decision making in their instruction will help to provide a sense of the complex ways in which teachers' knowledge and orientations may be contributing to MQI.

4.2.4 Aims and Significance of the Study

To help teachers develop fluency in working with SMPs in ways that emphasize mathematical ideas, it is important to understand the goals, orientations, and knowledge behind such work. With this purpose in mind, the present study has two main goals. First, I aim to understand and explain teachers' responses to SMPs in instruction by studying the resources, orientations, and goals that teachers consider in these instances. Second, I explore how teachers' resources, orientations, and goals support or hinder the MQI of responses to SMPs.

This study highlights the importance of both knowledge and orientations in shaping responses to SMPs. A great deal of research in mathematics education has sought to understand the relationship between MKT and the quality of instruction. I argue that to fully understand this relationship, researchers must consider the role of orientations and the related goals that teachers develop according to these orientations.

4.3 Methods

I explored teachers' thinking behind responses to SMPs by drawing on data from

12 high school mathematics teachers. Each teacher was observed and video-recorded in one class period for three consecutive days. Before each lesson, the teacher was interviewed about lesson planning and anticipated events. After the three observations, the teacher participated in a stimulated-recall interview, watching and reflecting on specific episodes of their teaching. Responses to SMPs were characterized using the MQI rubric (LMT, 2010), and I used grounded analysis to explore how teachers' thinking supported or hindered their responses to SMPs. The study proceeded as follows.

4.3.1 Recruitment

Recruitment was focused on high school (9^{th}- through 12^{th}-grade) teachers from the greater New Jersey area. The aims of recruitment were to observe (a) instruction that scored high on the MQI rubric and (a) a variety of instructional strengths and challenges.

In light of the first aim, I identified potential participants by the following methods. First, I looked at public records to obtain a list of all current high school mathematics teachers in the recruitment area who were (a) awarded the Presidential Award for Mathematics and Science Teaching between 2000 and 2012 (National Science Foundation, 2009), or (b) current (2012) National Board Certified Teachers (NBCTs) in Adolescent and Young Adulthood Mathematics (NBPTS, 2014). Each of these awards requires that teachers demonstrate a commitment to students, knowledge of mathematics and how to teach mathematics, and a reflective nature about their practice. Second, I used public records to compile a list of schools in New Jersey where student achievement in mathematics (according to state assessments) was high with respect to other schools in the state that had similar demographics (New Jersey Monthly, 2012). Third, I asked experts in mathematics education to recommend teachers or schools where they had seen or experienced mathematically-rich teaching. Using a list of potential participants and schools formed by these methods, invitations were sent to 30 individual teachers and 10 additional mathematics departments (to apply to all teachers within the department).

In light of the second aim of recruitment, I invited additional teachers at the same schools that did not necessarily meet the criteria above. That is, when recognized teachers agreed to participate, all teachers teaching the same course at the school were also invited to participate.

4.3.2 Participants

As a result of the recruitment process, 12 teachers (four males and eight females) from five schools agreed to participate in the research. Five teachers were considered experts or had been recognized for their teaching, and two of these teachers participated in the first study reported in this dissertation. Teachers' classroom experience ranged from 1 to 36 years, with a mean of approximately 15 years. Teachers taught at public or private schools in New Jersey or Pennsylvania, and the schools ranged from average to high performing (as determined by state test scores; e.g., State of New Jersey Department of Education, 2013; Pennsylvania Department of Education, 2014). The socioeconomic status of the students at the schools ranged from mid-low SES to high SES (as determined by public records such as the State of New Jersey Department of Education, 2004).

4.3.3 Data Collection

Three types of data were collected for this study: classroom observations, individual interviews, and written teaching materials. For classification purposes, I also collected a background questionnaire from participants.

Background questionnaire. Each participant was sent a background questionnaire, and they were asked to complete the questionnaire at their convenience prior to the final interview. The purpose of this questionnaire was to understand the professional experiences of the teacher. During the interviews, teachers were given the opportunity to elaborate on their responses to the background questionnaire. The full questionnaire is provided in Appendix C.

Written teaching materials. Each teacher also provided copies of their written lesson plans, prepared worksheets, homework, and handouts for all the lessons that were observed. For teachers that used their textbook as a resource, I also made copies of textbook pages that were relevant to the lessons being taught. These materials were used as reference during the prelesson interviews and as data to more fully understand the teacher's decisions around the mathematical topics.

Prelesson interviews. Each teacher participated in three prelesson interviews, each lasting between 15 and 30 minutes. Each prelesson interview was conducted before and close to each lesson observation. Many times, the prelesson interviews occurred before school or during a teacher's free period on the day of the observation. Occasionally, prelesson interviews were conducted by phone the night before the lesson observation.

During this interview, the teacher was asked to describe the lesson they were about to teach and to discuss any anticipated student questions, confusion, ideas, or reactions. A full interview protocol is given in Appendix D. The teacher was sent the interview questions prior to the interview so that they could consider them as they were planning the lesson. This also reduced the time needed to conduct the prelesson interviews. All interviews were audio-recorded, and I took detailed written notes on teachers' discussions. In this study, teachers' prelesson interviews were used to better understand the context for the lessons and the SMPs.

Classroom observations. Each teacher was observed and video-recorded in one class period for three consecutive days. Classes observed ranged from 8 to 22 students, with a median of 18 students[8]. Lessons lasted between 40 and 90 minutes, depending on the schedule at the school. Preference was given to courses and topics that could be observed across multiple teachers. However, due to logistical circumstances, the courses and topics that were observed varied. An overview of the classes observed is provided in Table 4.1.

I observed each lesson from the back of the classroom and recorded the lesson using a tripod-mounted video recorder. The camera was focused on the teacher for the duration of the class. Although the focus was on the teacher, student questions and comments made during whole-class discussion were generally audible, whereas student comments and questions made during group work were not audible. I took detailed written notes during the observations, with attention to SMPs and responses to those productions.

At the end of each lesson, the teacher was asked for their brief reaction to the lesson, and they were invited to mention classroom events or moments that they wanted to discuss in the stimulated-recall interview. Many of these events included SMPs.

---

[8] This is below average in the U.S.: The average class in U.S. high schools is approximately 23 students (U.S. Department of Education, 2007).

Table 4.1 *Characteristics of Classes Observed and Number of Teachers*

Courses	**Number of Teachers (N = 12)**
Type	
College-Preparatory	7
Honors	2
Other (special education, elective)	3
Content	
Precalculus	4
Algebra II	4
Algebra I	2
Geometry	2

Stimulated-recall interview. Each teacher participated in one video-based, stimulated-recall (SR) interview after the three observations were complete. This interview occurred within one week of the last observation and lasted approximately two hours. An overview of the sequence of interviews and observations is provided in Figure 4.1.

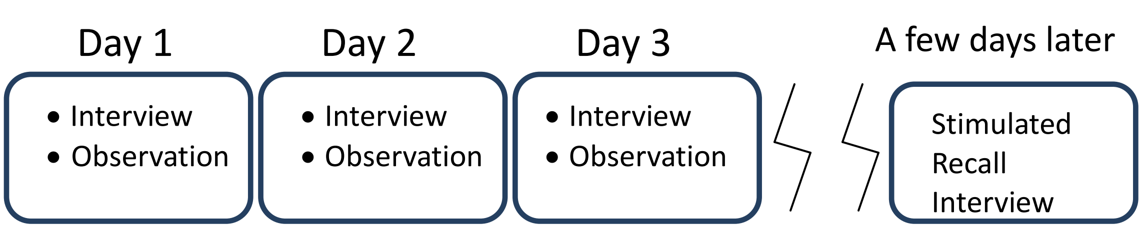


Figure 4.1 Overview of each teacher's observation and interview data collection.

The purpose of this interview was to understand the teacher's thinking behind their responses to SMPs. Several researchers have argued that this SR method is useful for holistically understanding teaching, including the teacher's actions, beliefs, knowledge, and goals (e.g., Dunkin, Welch, Merritt, Phillips, & Craven, 1998; Simon & Tzur, 1999; Speer, 2005). In particular, the SR method offers the opportunity for teachers to reflect on their in-the-moment thinking. For example, Ethell and McMeniman (2000) explained "Video recordings of the classroom practice and related stimulated-recall interviews [can allow] the expert teacher to reflect on the thinking underlying his classroom practice to make explicit the typically tacit cognitive and metacognitive processes that guide his teaching practice" (p. 90). SR methods can also be effective at uncovering teachers' beliefs that are tied to specific examples of instruction (Speer, 2005).

For the interview, the teacher was asked to reflect on the three lessons as a whole then watch approximately six 5-minute video segments of their teaching and reflect on their thinking and decision making in the segments.

Video selection. Because each SR interview was limited to two hours, choices had to be made about the video segments to discuss (similar to Dunkin et al., 1998). SMPs and associated instructional responses were one focus. Additional segments were also chosen, and some of which were analyzed and reported in the following chapter in this dissertation. I used the following procedure for choosing video segments.

To begin the selection process, I watched the videos of the three lessons in their entirety to holistically consider classroom events (Lesh & Lehrer, 2000). During this process, I made notes of events that could be more fully understood with the teacher's explanation (in the style of Powell, Francisco, & Maher, 2003). I compared these notes to my written observation notes to choose segments, and the following five types of segments were given priority: (a) segments that the teacher mentioned as important (in their reflections immediately following instruction), (b) SMPs and associated instructional responses that were pertinent to the lesson, (c) segments that illustrated the teacher's choice to modify the curriculum or deviate from what they had discussed in the prelesson interview, (d) instructional segments that were mathematically rich according to the MQI rubric, and (e) instruction that illustrated teacher error or imprecision, to give the teacher an opportunity to explain the error and the reasoning behind his or her decision.

There was not enough time in the SR interview to include every video segment that met one of these criteria, so preference was given to segments that included more than one of the criteria, segments where the mathematics being discussed was particularly pertinent to the mathematical ideas being studied or to higher mathematical ideas, and segments where it was important to understand the teacher's nonobservable perspective on their decision making. As a result of this process, each teacher watched between five and seven video segments in the SR interview, and these segments contained between one and seven SMPs per teacher. (See Table 4.2 in the Data Analysis section.)

Selecting video segments for discussion imposes some limitations on this study. As with most qualitative studies, the essence of this study is in "particularity rather than generalizability" (Greene & Caracelli, 1997, as cited in Creswell, 2007, p. 193). I am not claiming that a particular video segment is representative of the teacher's instruction across the three days of observation. Rather, I seek to understand the intricacies of teachers' thinking in the specific instances of responding to SMPs. The unit of analysis is the segment of instruction rather than the teacher. In addition, in choosing segments that were high in MQI as well as those that included some error and imprecision, the responses to SMPs are somewhat bipolar. This split was not necessarily problematic, as it allowed me to contrast responses that were characterized in different ways (further described in the Data Analysis section).

A note about the instruction I observed is also relevant here. Although several segments of instruction were mathematically rich, none of the teaching I observed embraced student-centered practices in deep and sustained ways, such as those documented by mathematics educators. (For example, see the instruction documented by Maher and colleagues in Maher & Martino, 1996, and Martino & Maher, 1999, as well as instruction described by Ball, 1993). In the present study, because of the dominance of teacher-centered instruction, SMPs often stood out and created moments that were "interruption(s) in the flow of the lesson" (Stockero & Van Zoest, 2013, p. 127).

Content of videos. Each video segment containing an SMP was clipped around a single mathematical idea. That is, each segment included approximately five minutes of instruction that surrounded the SMP, including the instruction that led to the SMP and the full response to the SMP. Instruction was considered a *response* to an SMP if it followed and explicitly addressed the production. In this data, all responses to SMPs were orchestrated by the teacher; in fact, responses were almost always solely performed by the teacher. Often, the teacher spoke directly to the student making the production or referred to the student by name in the response.

In a few instances, responses to SMPs did not immediately follow the production. For example, in response to a student question, a teacher might say, “That is a great question. I am actually going to address that in a couple of minutes.” In these cases, shorter video segments were pieced together from different points during the three days of observations to fully capture responses.

Interview protocol. The SR interview protocol is provided in Appendix E and was designed following interview procedures recommended by Kvale and Brinkman (2009) and Seidman (2006). Interviews were semistructured to allow for probing questions when necessary. The teacher was first asked introductory questions, such as their opinions on the overall quality and success of the lessons. Following these questions, the teacher and I watched the video segments that were chosen. After watching each episode, the interview questions focused on understanding what the teacher was thinking as they made decisions during the episode. After all video clips were viewed, the teacher was asked to give concluding thoughts about the sequence of lessons. All interviews were audio-recorded, and I took detailed written notes on teachers’ discussions.

4.3.4 Data Analysis

Three main efforts guided the data analysis. First, I characterized responses to SMPs that were discussed in SR interviews in terms of their MQI. Specifically, responses were grouped into three categories: high MQI, mid MQI, and low MQI. Second, I described teachers’ thinking behind their responses to SMPs in terms of goals, orientations, and knowledge. Third, I explored how teachers’ thinking supported or hindered the MQI of their responses to SMPs. The procedure for data analysis and the relationships between components are modeled in Figure 4.2.

Procedure:

1. Characterize responses to SMP using MQI framework
2. Understand teachers' thinking behind responses using grounded analysis
3. Explore relationships between teacher thinking and response to the SMP

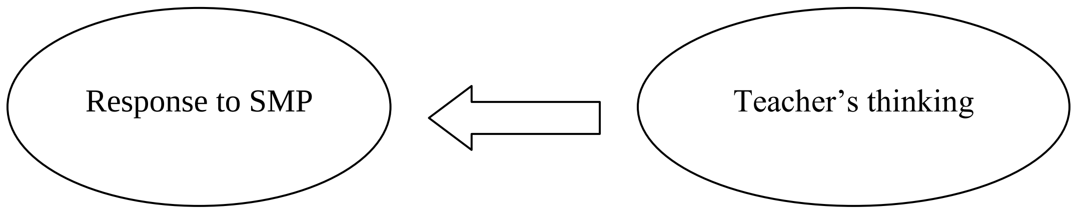


Figure 4.2 Overview of data analysis procedure and model of relationships between components.

Responses to SMPs. For this chapter, I focused exclusively on SMPs that occurred during the video segments discussed in the SR interviews. Using the definitions of SMPs provided in the MQI framework (LMT, 2010), I identified 44 SMPs in the SR video segments. SMPs included 25 instances where students asked mathematically-motivated questions (e.g., "Why is that true?"; "What's the difference between those two ideas?") and 19 instances where students offered solutions or mathematical claims (e.g., "I did it this way, and I got a different answer.").

I then characterized teachers' responses to SMPs using the MQI rubric as a guide. First, I took the MQI certification course and became a certified MQI rater. Second, for each SMP, the instructional response was coded according to MQI rubric in the five dimensions of mathematical quality: classroom work is connected to mathematics, richness of mathematics, working with students and mathematics, errors and imprecision, and student participation in meaning-making and reasoning. Only the response to the SMP was coded. In some cases, the response to the SMP lasted several minutes, whereas in other cases, the response to the SMP lasted a few seconds. Regardless of the length of the response, it was coded according to the MQI rubric.

As previously discussed, the instruction that I observed was generally teacher centered, and this was true of responses to SMPs as well. Nonetheless, there were important differences in the mathematical quality of responses to students, which I explored next. I sorted responses according to their dimensional scores in the following order: (a) classroom work is connected to mathematics (*yes*/1 to *no*/0), (b) richness of the mathematics (*high*/3 to *low*/1), (c) working with students and mathematics (*high*/3 to *low*/1), (c) errors and imprecision (*low*/1 to *high*/3), and (d) student participation in meaning-making and reasoning (*high*/3 to *low*/1). Upon sorting, I noticed that all 44 segments scored *yes*/1 in the dimension of classroom work is connected to mathematics; hence, this dimension was ignored in further classification.

Next, the responses were split into three groups based on their scores in the MQI rubric. I assigned each response one score by calculating the sum of the scores in the dimensions of richness of mathematics and working with students and mathematics and subtracting half of the score for error and imprecision. This method assigned each response a holistic score between 0.5 (1 in richness of mathematics, 1 in working with students and mathematics, and 3 in error and imprecision, which gives 1 + 1 – 1.5 = 0.5) and 5.5 (3 in richness of mathematics, 3 in working with students and mathematics, and 1 in error and imprecision, which gives 3 + 3 – 0.5 = 5.5). Using this calculation, responses were considered to be *high MQI* if they received a score of 4 or greater, *mid MQI* if they received a score between 2.5 and 3.5, and *low MQI* if they received a score of 2 or less.

Of the 44 responses to SMPs, 20 were coded as high MQI, 14 were coded as mid MQI, and 10 were coded as low MQI. Table 4.2 shows the MQI of the response to the SMP according to teacher, and Table 4.3 shows the MQI of the response to the SMP according to SMP type.

Table 4.2 *Number of Responses to SMPs According to MQI of Response*

Teacher	High MQI	Mid MQI	Low MQI	Total
Teacher 1*	6	1	0	**7**
Mr. Anderson*	4	2	0	**6**
Ms. Zimmerman*	2	3	1	**6**
Teacher 2*	2	2	0	**4**
Teacher 3	2	1	0	**3**
Mr. Dillon	1	2	3	**6**
Teacher 4*	1	0	0	**1**
Teacher 5	1	0	0	**1**
Teacher 6	1	0	1	**2**
Teacher 7	0	2	1	**3**
Ms. Carter	0	1	2	**3**
Teacher 8	0	0	2	**2**
Total	**20**	**14**	**10**	**44**

Note: An asterisk (*) denotes that the teacher was previously recognized for exemplary instruction. Pseudonyms are included in this table for the teachers that have episodes discussed in the findings. Other teachers are denoted by number to maintain confidentiality.

Table 4.3 *Number of SMPs According to Type and MQI*

Type of SMP	High MQI	Mid MQI	Low MQI	Total
Student Question	11	6	8	**25**
Student Solution/ Claim	9	8	2	**19**
Total	**20**	**14**	**10**	**44**

Discussion of the lens. Characterizing MQI with the language *high*, *mid*, and *low* is consistent with the MQI framework (LMT, 2010). However, the calculation I used to assign holistic scores to responses to SMPs was my own, and I offer some justifications for using the MQI rubric in the ways that I did. Specifically, I chose to focus on three dimensions: working with students and mathematics, richness of the mathematics, and error and imprecision.

First, working with students and mathematics was a logical choice. The dimension captures "whether teachers can understand and respond to students' mathematically substantive productions (utterances or written work) or mathematical errors" (LMT, 2010, p. 11). This is precisely the type of instruction that I am studying. Second, the richness of the mathematics dimension captures "the depth of the mathematics offered to students … either (a) focused on the meaning of facts or procedures or (b) focused on key mathematical practices" (LMT, 2010, p. 4). These elements of richness are aligned with the types of opportunities that SMPs can afford during instruction: Teachers can build on students' ideas to further develop mathematical richness.

Third, the choice to subtract half of the error and imprecision dimension in calculating a holistic score was done to acknowledge that error and imprecision takes away from the overall MQI of the response (hence subtracting the score), yet responses can still be rich and effective without being perfect (hence halving the score). Note that an error was counted as such only if it was never corrected in the duration of the three days of observation[9].

Fourth, I note that the student participation in meaning-making and reasoning dimension was neglected for this grouping. Student participation in meaning-making and reasoning captures “evidence of students’ involvement in tasks that ask them to ‘do’ mathematics and the extent to which students participate in and contribute to meaning-making and reasoning” (LMT, 2010, p. 17). According to this definition, there is student participation in meaning-making and reasoning inherent in the SMPs themselves, but not necessarily the responses to SMPs. For the purposes of this study, responses could be categorized as high MQI without evidence of student participation in meaning-making and reasoning. This choice was made largely because relatively few instances of student meaning-making and reasoning occurred within the responses to SMPs in my data. As discussed earlier in this chapter, responding to students in ways captured in other dimensions of the MQI rubric is an important first step in working towards student-centered instruction that embraces student participation in meaning-making and reasoning.

[9] Teachers sometimes choose to do sustained work with students’ errors before correcting them (see Ball, 1993). Such a practice did not occur in the data for the present study.

In addition, I do not wish to imply that low-MQI responses were low quality in general. Rather, these responses did not exemplify the categories of MQI used for this coding, but these responses may be considered to be good quality by a different lens.

Teachers' thinking in responding to SMPs. To describe teachers' thinking in responding to SMPs, I analyzed the SR interview using a constructivist approach to grounded theory (Charmaz, 2002). In describing teachers' thinking, my goal was to "explain the teacher's perspective from the researcher perspective" (Simon & Tzur, 1999, p. 254). In other words, in the style of Simon and Tzur (1999), my goal was to understand what teachers perceive, value, and consider in responding to SMPs, but I did so from a researcher's lens, making links to existing mathematics education research.

All SR interviews were fully transcribed. First, I read through teachers' reflections on the SMPs and reduced these reflections to specific thoughts when teaching (see also Schepens, Aelterman, & Keer, 2007). For instance, if a teacher said something like, "When I was responding to the student, I wanted to emphasize the definition of real numbers," this is a reflection on thoughts that the teacher had in the moment of teaching, so it was considered for this analysis. By contrast, if a teacher said something like, "I noticed in watching the video that I am always looking around the room," this would not be considered a reflection on in-the-moment thinking because this thought came in response to watching the video.

With the data reduced to teachers' reflections on their thoughts when teaching, I coded teachers' reflections line by line for the nature of the thought expressed: goal, orientation, or knowledge. To determine whether a teacher's thought was a goal, orientation, or knowledge, I first looked for phrases signaling a particular thought.

However, teachers' language did not always clearly indicate a goal, orientation, or knowledge, so I also used guiding questions in helping to identify these. Table 4.4 gives an overview of the types of phrases and guiding questions that led to the coding of goal, orientation, or knowledge in the reflection.

Next, I coded teachers' thoughts for their content. A short phrase was assigned to each thought that described the content. For instance, the goal "I want what they say to be said accurately" was given the code of *goal for accurate student language* to describe that the teacher was interested in the students' use of language in the classroom. This process yielded a list of content codes that contained topics such as *knowledge of common student errors, orientation for sense making,* and *goal for student engagement.* Existing codes were used as appropriate to code new data. At the end of this process, I revisited this list, modifying and refining the codes, searching for similarities and differences in the data.

Table 4.4 *Coding Scheme for Teachers' Reflections: Goals, Orientations, or Knowledge*

Type of thought	Example phrases	Guiding Question	Examples
<u>Goal:</u> *What the individual wants to achieve in responding to the SMP.* --*May have several goals for an individual response.* --*Goals may pertain to the whole lesson but, in this data, often pertain to a short segment of time.*	"My intention here was to …" "I wanted students to …" "I was trying to emphasize …"	What was the teacher trying to do?	"I wanted to stress than an ordered pair in a real-life problem represents something that's happening that can be put to a sentence." "I was trying to make clear that that [expression] wasn't the same as the previous ones." "I wanted to see what they were thinking and if they were recognizing the differences between linear functions and quadratic functions."
<u>Orientation:</u> *An individual's dispositions, beliefs, values, tastes, and preferences that pertain to the response to the SMP.*	"It's important that …" "I believe that …" "I value …"	What is this teacher's view of this situation? What matters to this teacher?	"Math has to make sense." "I think it's important to follow up on student answers, whether they are right or wrong."
<u>Knowledge:</u> *Information that the individual brings to bear in responding to the SMP.* --*May include knowledge about mathematics, knowledge of students, knowledge of curriculum.*	"I was using knowledge of …" "I know students usually …" "I heard the student say …" "My experience has been that …" "This concept connects to …" "This student understands …"	What information does the teacher recognize in this situation?	"Students always want to make the reference angle the acute angle and the *y*-axis instead of the *x*-axis." "That student was connecting the stuff on the board to variation." "The graph of inverse sine *x* is not very easy to generate."

In the process of coding, I recognized that goals and orientations often had similar content. In looking more carefully at teachers' statements of goals and orientations, it seemed artificial to separate the two. As teachers reflected on their decisions, they professed goals that were an expression of their orientations (Cobb, 1986; Schoenfeld, 2011). For example, if a teacher said, "My goal is for students to share what they are thinking," I did not find that to be fundamentally different from the statement "I think it's really important that students share what they are thinking," because both of these statements are being used in ways that explain the teacher's decision in the segment. Hannah, Stewart, and Thomas (2011) encountered a similar issue with their analysis of teachers' goals and orientations. Indeed, the similarities between goals and orientations may be a result of the fact that teachers were reflecting on and explaining specific actions in their instruction. As a result, goals and orientations were considered together for the remaining analysis but are interpreted separately in the findings when appropriate.

Similar codes were grouped into categories using constant comparisons (Strauss & Corbin, 1990). For instance, the codes of *orientation for sense making, goal to make connections*, and *goal to offer explanations* are related in that they all capture teachers' goals and orientations for *meaning*. Similar categories were made for knowledge, guided by Ball et al.'s (2008) MKT framework and Schoenfeld's (2011) notion of *lesson image*.

How thinking guides responses to SMPs. The ultimate goal in this study was to understand how teachers' thinking supported or hindered the MQI of their responses to SMPs. I did this using a confirming and disconfirming approach to analysis (Creswell, 2007), as follows.

Beginning with the codes for goals and orientations, I compared codes from reflections on high-MQI responses to those from reflections on low-MQI responses. This approach did not overlook any codes, as no codes were present in only mid-MQI responses. I first focused on codes that were more heavily aligned with either high-MQI responses or low-MQI responses, I then reread the transcripts associated with each group, searching for disconfirming evidence for the categories and modifying the definitions of the categories as needed. This led to a list of codes of goals and orientations; each of these codes was present uniquely in either the high-MQI responses or the low-MQI responses, and these codes also categorized the mid-MQI responses. This list of codes is described in Table 4.5.

In addition, some codes were densely present across both high and low-MQI responses. For example, the code *goal for student engagement* was found to be present across all types of responses. Because my intent was to understand the thinking that supports or hinders MQI, these goals were not considered as themes. However, the findings will illustrate how additional goals such as these interacted with the goals that were the result of the previous analysis.

I also compared the knowledge associated with high-MQI responses to knowledge associated with low-MQI responses, yet I found few differences between these. Instead, knowledge played different roles in teachers' decisions, depending on their goals and orientations. Hence, the knowledge codes were not associated with a particular type of response, and the knowledge categories are described in Table 4.6. The relationships between knowledge and goals/ orientations were explored and will be expanded in the Findings.

Table 4.5 *Categories of Goals in Teachers' Reflections*

Categories of goals	**Description**	**Examples**
Build from student ideas	Goals or orientations relating to building instruction around student ideas. May be expressed as using what students say to proceed with the lesson, working to understand and build on students' thinking, etc.	"I'm glad that they feel comfortable giving me answers because I use what they say to kind of move on with the lesson." "Where do I see some things in their thinking that is not correct, and where am I going to have to focus and really emphasize the different things in the lesson?"
Avoid student confusion	Goals or orientations relating to keeping the mathematical ideas simple to avoid student confusion. May be expressed as the teacher changing the mathematical ideas a bit to fit the needs of their students. This code is used when the teacher's goal or orientation takes away some of the mathematical rigor of the task/ opportunity.	"For the sake of ease, I just said the base is the bottom. … I didn't want to open up a can of worms. … In this case, base is bottom, … for their ease." "For now this strategy's going to work every single time. When they get to Algebra Two, they're going to have to think about it more."
Meaning and sense making	Goals or orientations relating to meaning and sense making of mathematics. May be expressed as emphasizing definitions or central ideas, the meaning of notation, how new ideas relate to previous ones, why mathematical ideas are true, clear language use, or making meaningful generalizations. May also be expressed as making mathematics meaningful for students or wanting students to "understand" when "understand" implies meaning and sense making (rather than only procedures).	"I want to try and tie it back into that idea of parent functions" "I want them to see that exponential growth is going to beat linear growth over time." "It's not the answer that's important, it's the definition of what this thing actually us. And then we can use that to help us come across ways of manipulating it."

Emphasizing procedures	Goals or orientations relating to mathematical procedures. May be expressed as the teacher wanting students to recognize a procedure or develop an understanding that does not necessarily emphasize the meaning of the mathematics.	"I want students just to look at a problem right away and know what they should do. I want them to see, okay, there's a sine, so multiply by *r*. It's cosine, so square it."

Table 4.6 *Categories of Knowledge in Teachers' Reflections*

Type of Knowledge	Description	Examples
Knowledge of mathematics	Teacher expresses knowledge about the mathematics they are teaching.	"The graph of inverse sine *x* is not very easy to generate, it has all kinds of implications with domain and range." "I think part of the issue there is the notation. The negative one means two different things."
Knowledge of students	Teacher expresses knowledge of students, including students' learning preferences and characteristics of particular students.	"Sometimes [students] need that bodily, kinesthetic [approach] to actually be able to see it physically move to work." "Those two students probably should have been in the honors group."
Knowledge of content and students	Teacher expresses knowledge of how students work with mathematics, including their difficulties with the content or what methods or explanations are meaningful to them. This may be expressed as knowledge or understanding	"Students always want to make the reference angle the acute angle and the *y*-axis instead of the *x*-axis." "And that's when she finally understood the relationship between *a* sub *n* and *a* sub *n* minus one."

	of the student that the teacher has in the moment, as part of the interaction with the student.	
Knowledge of content and teaching	Teacher expresses knowledge of different representations of concepts, approaches, explanations, etc. that will resonate with students.	"When she didn't get it … on the number line in general terms, I went to numbers. I broke it down to, okay if n is ten and this is the tenth term, then what's a sub n minus one going to represent?"
Knowledge of curriculum	Teacher expresses knowledge of what students have learned in previous courses, what students will learn in subsequent courses, how the textbook presents material, etc.	"The textbook starts with the standard form of the equation." "I know students see induction in Precalculus. I don't know if they see it in any other course."
Lesson image	Teacher expresses how they envisioned the class going before they began teaching. This is their "lesson image" of what they thought would happen (Schoenfeld, 2011)	"I thought we would talk about the ellipse and we would move on." "I did not expect students to have such a difficult time with that concept."
Lacks relevant knowledge	Teacher expresses that they lack relevant knowledge in responding to the production. May be expressed as lack of knowledge about the mathematics, the students' thinking, the curriculum, etc. This code is often applied in addition to one of the previous codes.	"I didn't understand what the student was asking. Now I understand better." "I didn't want to pursue that question because I didn't know the answer." "I don't know any other ways to explain this idea."

Codes and categories that were developed during analysis were cross-checked by an advanced mathematics education graduate student for four of the responses to SMPs, and these responses represented all of the code categories. These four instances were coded separately by each of us then compared and discussed to refine the coding scheme. I revisited the remaining episodes and recoded them according to our revised scheme.

In the findings, I focus on describing thinking behind high-MQI responses and low-MQI responses, as these illustrate teachers' contrasting ways of thinking. Mid-MQI responses will also be mentioned briefly in the Discussion section.

4.4 Findings

In the findings, I illustrate how teachers' orientations, knowledge, and goals contributed to their responses to SMPs. In each video segment chosen for the SR interview, there was evidence that the teacher attended to (heard and provided some response to) the SMP. As such, my purpose in the findings is not to describe what teachers did and did not attend to in the classroom. Instead, the findings highlight teachers' thinking behind the noticing aspects of interpreting the SMP and deciding how to respond to the SMP as well as the action of responding to the SMP.

In reflecting on responses to SMPs that were scored as high MQI, teachers described two goals that were not present in the reflections on low-MQI responses: (a) build on the students' mathematical ideas and (b) take the opportunity to emphasize meaning and sense making. Teachers also expressed knowledge that they used in carrying out these goals. By contrast, in reflecting on responses to SMPs that were scored as low MQI, teachers expressed (a) they had goals that did not build on the SMP or (b) they lacked knowledge in the moment of responding to the SMP. These findings are summarized in Table 4.7.

Table 4.7 *Number of Teachers Expressing Each Theme in Relationship to MQI of Response*

	High MQI		**Mid MQI**		**Low MQI**	
Theme	Responses (out of 20)	Teachers (n = 9)	Responses (out of 14)	Teachers (n = 8)	Responses (out of 10)	Teachers (n = 6)
Build on students' thinking	14	6	3	3	0	0
Emphasize mathematical meaning	18	8	5	3	0	0
Goals not aligned with SMP	0	0	7	4	7	5
Lack of knowledge	0	0	3	3	4	3

The findings are presented in two sections. In the first section, I illustrate how teachers' goals, orientations, and knowledge supported high-MQI responses to SMPs. In the second section, I share teachers' thinking that led to low-MQI responses to SMPs.

4.4.1 Teacher Thinking behind High-MQI Responses to SMPs

Of the 44 responses to SMPs, 20 were coded as high MQI. According to the methods used to categorize responses, high-MQI responses were those in which (a) the teacher understood students' questions, solutions, or claims, and built on students' ideas in instruction, (b) rich mathematics was discussed that focused on meaning or mathematical practices, and (c) teacher errors and imprecision were absent, minor, brief, or not serious enough to distort the content (LMT, 2010).

Two main themes uniquely characterized teachers' reflections on high-MQI responses to SMPs: Teachers expressed that their intentions were to (a) build on the students' mathematical ideas and (b) take the opportunity to emphasize meaning and sense making. As discussed in the Data Analysis section, these goals closely align with teachers' orientations. These themes were present in some reflections on mid-MQI responses, but they were not present in teachers' reflections on low-MQI responses, as illustrated in Table 4.7.

Table 4.8 lists the specific goals that teachers expressed that were associated with each theme and Table 4.9 lists the types of knowledge that supported teachers in carrying out the themes. Whereas the goals listed in Table 4.8 were unique to the high-MQI responses, the knowledge listed in Table 4.9 was not. In addition, all of the knowledge types in Table 4.9 were discussed in conjunction with both themes behind high-MQI responses. Though it was not necessarily unique to high-MQI responses, this knowledge was used in important ways to support teachers' responses, as will be illustrated in the following examples.

Table 4.8 *Teachers' Goals Supporting Themes behind High-MQI Responses*

Build on Students' Mathematical Thinking

Goals:

- Have students share their thinking so that it can guide instruction.
- Watch for and follow up on students' incorrect thinking.
- Build on student solutions and questions when they are offered.
- Stay with students' thinking until they understand the concept fully.
- Let students' level of understanding guide the topics discussed in class.
- Choose the next action in instruction based on the student's thinking.
- Discuss students' incorrect thinking in class.
- Work with the student solutions that are offered.
- Value students' contributions by building on them.

Take the Opportunity to Emphasize Mathematical Meaning

Goals:

- Make connections among mathematical ideas.
- Focus on justification, explanation, and proof.
- Illustrate how math makes sense in the real world.
- Emphasize meaning through definitions.
- Make sense of new ideas by building on previous ideas.
- Emphasize concepts in mathematics.
- Connect representations.
- Compare mathematical ideas to illustrate similarities and differences.

Table 4.9 *Teachers' Knowledge Supporting Themes behind High-MQI Responses*

Knowledge:

- Knowledge of content and students
  - Awareness of students' understandings
  - Recognize student confusion or understanding in the moment
  - Interpret students' thinking in the moment
  - Common student difficulties and ways of thinking
  - Recognize explanations that resonate with students
- Knowledge of content and teaching
  - How to explain concepts in ways that resonate with students
- Knowledge of students
  - Characteristics of particular student
  - Students' learning styles
- Knowledge of mathematics
  - Meaning of the concepts being discussed
  - Different mathematical approaches to a concept
- Knowledge of curriculum
  - Topics students have previously covered
  - Topics students will cover in the future

In this section, I present two examples of high-MQI responses and teachers' thinking behind these responses. Each of these examples highlights both of the themes in the findings.

Mr. Anderson builds on a student solution. This example occurred in Mr. Anderson's Honors Algebra II class of eight students at a private high school. Mr. Anderson had multiple graduate degrees in mathematics education, nearly 30 years of teaching experience, and had been recognized for his excellent instruction. As such, I considered him to be an expert teacher. I will first describe the classroom episode and point to why it was scored as high MQI, and then I will describe Mr. Anderson's reflection on this episode, highlighting the knowledge, orientations, and goals that contributed to this high-MQI response. (A similar approach is taken with subsequent examples.)

Classroom episode. When this episode occurred, the class was studying sequences and series. They had established the formula for the sum of a finite geometric series in a previous class. That is, the sum of the first n terms of a geometric series with common ratio $r \neq 1$ was given by

$$S_n = \frac{t_1(1 - r^n)}{1 - r}$$

where t_1 denotes the first term in the series. In a homework problem, students were asked to show the following:

$$\sum_{k=1}^{n} 2^{k-1} = 2^n - 1$$

The SMP was an incomplete solution to this problem. Figure 4.3 illustrates a replica of the student's solution.

$$\sum_{k=1}^{n} 2^{k-1} = 2^n - 1$$

k = 1	k = 2	k = 3	k = 4	n = 1	n = 2	n = 3	n = 4
1	2	4	8	1	3	7	15

$$1 + 2 + 4 + 8 = 1 + 2 + 4 + 8$$

Figure 4.3 Replica of student solution in Mr. Anderson's class.

The student had written this solution on the classroom board, and he was asked to explain his written work to the class. In this episode, Mr. Anderson interpreted the student's written and spoken explanations, clarified the meaning of the mathematics being discussed, and extended this student's thinking to discuss the nature of proof in mathematics. For these reasons, the segment was coded as high MQI, as I will highlight throughout the transcript:

> *Mr. A.*[10]*:* We're trying to show that this is true, so we're not going to assume that they're true, but we're going to show that they're indeed true.
>
> *Student 1*: Alright, so for this side of the expression [sic], I solved for *t* one, two, three, four.
>
> *Mr. A.*: Okay, what I would say is you evaluated, right, and showed the first four terms. Okay so show me what you have.
>
> *Student 1*: So I have the first four terms as one, two, four, and eight.
>
> *Mr. A.*: Alright, so from when *k* is equal to one, we get one, when *k* is equal to two, you said it's two, when *k* is three. Do we agree with that?
>
> *Students*: Yeah.

[10] Quotations in this chapter are lightly edited in this following ways. Ellipses (…) indicate omissions, which were made for efficiency of expression. Brackets contain text that clarify meaning or replace identifying information. These edits were made with careful attention so that the meaning of the text was preserved.

There is already some richness of mathematics present in this segment, as Mr. Anderson pushed for the accurate student language of “evaluate.” He also revoiced the student’s thinking, and as the segment progressed, Mr. Anderson continued to work to understand the student’s thinking and emphasize mathematical ideas:

> *Student 1*: And then, I did the same for the other side.
>
> *Mr. A.*: Okay, so on the other side … So if n is equal to one, we agree. If n is equal to two, do we agree? If n is equal to three do we agree? … And if n is equal to four, we get fifteen. Now, … to show that these are consistent, we have to make sure that we understand what's going on here. We're saying, when n is equal to one, what we have on the right hand side is one. What we have on the left would be the sum from one to one. … From one to one it's just the first term. When n is equal to two, what are we talking about here?
>
> *Students*: The sum of the—
>
> *Mr. A.*: It's the sum of the first two. Right? So the sum of the first two would be three, which agrees. Are you with me? … If n is equal to three, that means it's the sum of the first three. So that should be seven. Are you with me over there? And if n is equal to four, alright, it's the sum of the first four, which is fifteen. Are you with me here? If I were running for president and that was the support I got, I'd go home, quick.
>
> *Students*: [laughing]
>
> *Mr. A.*: Do you agree?
>
> *Students*: Yes.
>
> *Mr. A.*: Alright. Does that make sense? You do understand that these are sums, right?
>
> *Students*: Yes.

In this excerpt, Mr. Anderson again articulated the student’s thinking then emphasized the meaning behind the summation notation. In the following turn, Mr. Anderson seemed to recognize that the student’s thinking was related to an important mathematical idea, but he gave the student an opening to share any other thinking that he might have.

> *Mr. A.*: Okay. Now, this raises a very interesting question, alright. Because [Student 1], the evidence of your argument--…Go on and explain your reasoning here.

Student 1: What do you mean?

Mr. A.: How did you sort of wrap this up to prove this result?

Student 1: Well I didn't, um.

Mr. A.: So do you feel you want to retract this?

Student 1: Yes.

Mr. A.: Okay. are we okay so far? Alright. This is an important piece, and I love that you approached it this way, what you did. … I need you to be engaged now, okay. Make sure you're with me because … we're looking at the big picture in mathematical thinking. What did [Student 1] do? He's got this statement here. He said, okay, I'm going to try, if n is equal to one, this is true. I'm going to try, if n is equal to two, this is true. I'm going to try if n is equal to three, this is true. Agree? And then he said, and we even did n equal to four. Okay? Now, and then on the basis, I think [Student 1], am I right, that you're saying, well okay it seems to work for these cases, so it would be logical that it worked generally?

Student 1: [Nods yes.]

Mr. A.: Alright. That's sort of a very important idea in mathematics, alright, that you're looking at specific cases. It's sort of analogous to statistics, right? If I said, you all love chicken nuggets, right? You all said, we love chicken nuggets. Then, can I assume then that everybody in the school loves chicken nuggets?

Students: No.

Mr. A.: It's kind of the same thing, right? We've only established that the people in this room love chicken nuggets, right?

Recognizing that the student was trying to generalize the relationship after looking at a few examples, Mr. Anderson emphasized the fact that this argument was not complete in mathematics, and he drew students' attention to what he called the "big picture in mathematical thinking." Continuing this discussion, Mr. Anderson went on to introduce students to the concept of mathematical induction:

Mr. A.: So the analogy here, it's something you're going to learn … next year. There's a method of proof that's called mathematical induction, which is a very powerful method of proof. Alright. And again, I don't want to spoil the party, okay, but it's a fairly sophisticated method, and the analogy is sort of to making a line of dominoes, alright. So you start out, if you have a whole stack of dominoes. [Teacher is setting up white board markers like dominoes.] It's harder with markers. … So here's the idea. There's a whole line of markers, right? So this is the case n equal to one. It holds, agreed? This is the case n equal to two, it holds, this is the case n equal to three, it holds. This is n equal to four it holds, right? So the critical piece in a proof by induction is, … if you can show that any one case

> then implies the next case holds, right, it sort of knocks it over, if you have a line of dominoes. So the idea here would be that if I know that it's true for *n* equal to one, it's true for *n* equal to two, it's true for *n* equal to three, the assumption is that then let's say that it's true for any given one. What I want to try and show is that if it's true for this, then it's true for the next case [pushes one marker over to hit the next]. And once I know that, then I can prove all of them. Okay, but you'll see that obviously in more detail, but your thinking of going case by case and then showing that one case implies the other allows us to do this more generally. [Teacher goes on to show a solution to the problem using direct substitution and the formula that the class has established.]

In this excerpt, Mr. Anderson extended the student's thinking at length and also introduced a more sophisticated idea that related to the student's thinking. The discussion about generalization was also reinforced in response to a different problem later in the class:

> *Mr. A.:* Now the key idea to this really very nice piece of mathematics is we get into what we had discussed earlier with [Student 1], alright. That is, could I just stop here [after checking the first few cases] and say, "Yay, I've proven it"? Would that be enough? No because I've only shown it for these cases, right?

Mr. Anderson's extended response to Student 1's incomplete solution was coded as high MQI. As illustrated through the transcript, Mr. Anderson used the student's written work and spoken explanation to understand the student thinking, articulated the meaning of ideas, and extended the student's inductive thinking by introducing mathematical induction and discussing what is needed to prove a result in mathematics. In other words, as captured in the MQI rubric, Student 1's thinking is woven into the development of the lesson (LMT, 2010), and meaning is emphasized.

Mr. Anderson's reflection. In reflecting on this episode in the SR interview, Mr. Anderson gave a sense of the orientations, goals, and knowledge that supported his decision to respond in the way that he did.

One goal that Mr. Anderson expressed was to have students' ideas drive classroom discussions, and a related goal was to extend the mathematics when he saw the opportunity:

> *Mr. A:* That was more of a situation where I leave it to the kids to decide what questions they want to talk about. If there are opportunities, and at that time I'm really focused, then it's an opportunity for extension.

This goal allowed him to take the opportunity to explore a mathematical idea that had not been in his original lesson agenda. The main goal driving Mr. Anderson's response to this segment was his intention to emphasize the nature of mathematical proof; he wanted students to understand that, in mathematics, one cannot generalize after looking at only a few cases:

> *Mr. A.:* I think central to the idea is not wanting the students to generalize after they only look at some cases. … I think it's not focused on a specific skill, it's focused more on the discipline and the thinking in the discipline. My emphasis was not so much on how do we do this problem. My emphasis was on, if you're thinking about trying to establish a mathematical result, I think that was the bigger idea that I was trying to convey.

In addition, Mr. Anderson's reflection suggests that one of his orientations towards teaching is that it is important to engage students (particularly honors-level students) in mathematically rigorous activities, as he explained:

> *Mr. A.*: [My colleagues and I] have had discussions about—at the upper levels—about what kind of experiences do the kids need to have—this is amongst our very strongest kids—to be prepared to be mathematics majors. And certainly in my own experience, it was the level of rigor, abstraction, formality I think that was really, that was a huge jump from high school to university. … It was almost like a different language. So that I was making a decision about studying mathematics on the basis of something that really wasn't a representative sample of what you would be studying as an undergraduate. I would put induction as one of those topics. … That is a very small example of a topic that is at a higher level of abstraction, rigor, and formality compared to a lot of mathematics.

For Mr. Anderson, introducing the idea of induction was appropriate because it would help prepare these honors students for future mathematics.

Mr. Anderson’s reflection also reveals some of the knowledge that he used to motivate his goals for responding to the SMP. Much of this was Mr. Anderson’s knowledge of the mathematics curriculum, both within and beyond high school. As illustrated above, part of what motivated Mr. Anderson to respond in the way that he did was that he was aware of the type of mathematics that students would encounter in a university setting. He was also aware that induction is taught in Precalculus Honors:

> *Mr. A.:* I think [I was] using knowledge of mathematics that they will see in the future. And I guess you could consider mathematical induction to be knowledge in your mathematical base. I think what's interesting is that I don't think that mathematical induction gets a lot of treatment in general. … I do know that they do see it in Precalculus Honors. I don't know that they see it in any other course. … And when you think of induction, … that's sort of a pretty standard kind of question. Find a pattern and prove it by induction. So I think what I'm using there is, okay I've taught induction in the past, this is a nice connection to that topic.

Mr. Anderson recognized the problems they were working on as “standard” induction problems. In addition, Mr. Anderson was aware that students had probably not had much experience with induction in the past and that they would see mathematics at a much higher level of rigor in the future. This knowledge further motivated Mr. Anderson’s decision to respond to the SMP in the way that he did:

> *Mr. A.:* What I notice is for many students, their previous experiences have not involved this kind of engagement, so it's a big challenge for kids, half of whom are ninth graders and half of whom are tenth graders.

Whereas introducing an advanced mathematical topic to prepare students for the future was unique to Mr. Anderson’s response, this reflection also includes the two themes that supported high-MQI responses in general (Table 4.8). Mr. Anderson described that he allowed students to share their thinking so that it could guide instruction (build on students’ mathematical ideas), and he recognized and took the student’s incomplete solution as an opportunity to emphasize the nature of mathematical proof (take the opportunity to emphasize mathematical meaning). These goals were motivated by Mr. Anderson's awareness that students do not encounter discussions about mathematical proof often in the high school setting, and these goals were supported by his knowledge of the mathematics curriculum, in both high school and university-level mathematics.

Mr. Anderson did not discuss the work of interpreting the student's solution or his thinking as he was carrying out the response, but both of these aspects of noticing and responding to the student contributed to the high-MQI response. For instance, it appears that Mr. Anderson was able to understand the mathematical essence of the student's solution method, even though the student does not write it clearly nor verbalize it completely. This is an example of Mr. Anderson's use of KCS (Ball et al., 2008). To carry out the response, Mr. Anderson drew on his content knowledge of induction and his understanding of what counts as proof in mathematics. Further, Mr. Anderson had a sense of how to discuss mathematical ideas with students, using a line of markers set up as dominoes to explain the concept of mathematical induction, an example of knowledge of content and teaching (Ball et al., 2008). Hence, the goals and knowledge that Mr. Anderson discussed are not sufficient for the response that he gave. The knowledge needed to interpret students' thinking and carry out a response to the student may not be conscious to a teacher until he or she realizes that they do not have such knowledge, as will be illustrated in a later section.

Connections to other high-MQI responses. This example from Mr. Anderson's class is unique in that it was the only one of the 44 responses to SMPs in this data in which the teacher recognized and took the opportunity to introduce a new mathematical idea in light of the SMP. Doing so requires the teacher's knowledge of how the SMP may connect to other mathematical ideas, the SMP itself to afford such an opportunity, and the situation to allow for such discussion (e.g., ample time left in class, relevance of the new topic, etc.). In this sense, this example provides a rich look at how goals, orientations, and knowledge can support a teacher in offering a high-MQI response to an SMP that also extends the current content being discussed.

At the same time, this example has several similarities to others coded as high MQI. Even without the discussion of induction, in this segment Mr. Anderson worked to understand the student thinking, made a decision to respond based on the incorrect thinking, and addressed that thinking with meaning, emphasizing that looking at only a few cases does not prove a result. These were driven by his goals to build on students' mathematical thinking and take the opportunity to emphasize meaning. These themes were present in both responses to students' thinking (such as Mr. Anderson's example) and responses to students' questions, as will be illustrated in the next example.

Ms. Zimmerman responds to a student question. In this section, I share an additional example of how a teacher's thinking supported a high-MQI response to an SMP. This example comes from Ms. Zimmerman's Honors Precalculus class, which was comprised of 22 students. Ms. Zimmerman taught at a public school that was high performing on state and national assessments. She was a NBCT, had over 35 years of teaching experience, and had been recognized for her excellent instruction. As such, I considered Ms. Zimmerman to be an expert teacher.

Classroom episode. When I observed Ms. Zimmerman, her class was studying sequences and had been using the notation of n as the term number in the sequence. They typically represented the terms in a sequence with the notation $A_1, A_2, A_3, \ldots$ Ms.

Zimmerman had given both an explicit definition and a recursive definition for a geometric sequence, as illustrated in Figure 4.4.

Geometric Sequences

- Sequences generated by multiplying a constant to get from term to term
- The constant is called the "common ratio" and is denoted by r

Explicit rule for a geometric sequence: $A_n = A_1 r^{n-1}$

Recursive rule for a geometric sequence: $A_n = A_{n-1} r$ for $n > 1$

Figure 4.4 Definitions for geometric sequence provided in Ms. Zimmerman's class.

Although the idea of geometric sequences was new to them, students had previously worked with explicit and recursive definitions for sequences. In this episode, a student asked about the difference between explicit and recursive definitions. Ms. Zimmerman posed a series of questions that led the student to understand the meaning behind the notation being discussed:

Ms. Z.: So once again, we have an explicit rule, and we have a recursive rule.

Student 1: What's the difference?

Ms. Z.: What's the difference?

Student 1: Like what is the difference between explicit and recursive?

Ms. Z.: Alright. Tell me what variables you see different in them.

Student 1: [Silence.]

Ms. Z.: Are they all using the same variables? They both have *A* sub *n* on the left. What about the right hand side?

Student 1: One is *A* sub *n* minus one times *r* and one is *A* times *r* to the *n* minus one.

Ms. Z.: So what's different?

Student 1: One is like sub of *n* minus one and one is like *r* to the *n* minus one. So one's taking it to the *n*th power and one is, I don't know how. I don't know.

Rather than telling the student the difference between recursive and explicit definitions, Ms. Zimmerman began to ask a series of questions to check for the student's understanding and guide the student towards seeing the differences between the two. When Ms. Zimmerman recognized that the student was still confused, she focused specifically on the meaning of the terms in the definitions:

> *Ms. Z.*: What do you have to know to be able to use this formula? [Pointing to explicit rule.] On the right hand side, what things do you have to know to plug in here?
>
> *Student 1*: *A* sub one and *r*
>
> *Ms. Z.*: You have to know *A* sub one and *r*, and then you have to know the position, right? You're finding the tenth term, the whatever. So you have to know the first term, *A* sub one, and you have to know the common ratio. Do you have to know the first term here? [Pointing to recursive rule.]
>
> *Student 1*: No.
>
> *Ms. Z.*: Do you have to know the common ratio?
>
> *Student 1*: Yes.
>
> *Ms. Z.*: What besides the common ratio do you need to know?
>
> *Student 1*: What you're replacing.
>
> *Ms. Z.*: What is that? [Pointing to A_{n-1}.]
>
> *Student 1*: The term, the *n*th term or whatever?

When Student 1 responds that A_{n-1} is the "*n*th term or whatever," Ms. Zimmerman seemed to recognize that Student 1's difficulties went beyond understanding the difference between explicit and recursive definitions: The student did not seem to understand the meaning of the notation. Recognizing this, Ms. Zimmerman worked to get Student 1 to consider the meaning of A_{n-1}.

> *Ms. Z.*: What's the relationship between *A* sub *n* minus one [A_{n-1}] and *A* sub *n*?
>
> *Student 1*: It could be one greater, and that would give you the ratio. Maybe. I'm not sure, I'm so lost, I'm so sorry.
>
> *Ms. Z.*: If I were here [writes A_n], where would I put *A* sub *n* minus one [A_{n-1}]? If I had commas on either side [writes comma before and comma after A_n] and I was, and *A* sub one's down here [writes A_1 far to the left], where would *A* sub *n* minus one [A_{n-1}] go?
>
> *Student 1*: Further to the left?
>
> *Ms. Z.*: Okay. Further, like way down? How far to the left?

Student 1: I'm going to say in the middle. No.

Ms. Z.: In the middle?

Student 1: No, it would be all the way to the left.

Ms. Z.: On the other side of *A* sub one?

Student 1: I think.

In this excerpt, Ms. Zimmerman tried using a visual representation of the terms in a sequence to help the student understand the meaning of the notation. But when Student 1 said that A_{n-1} would be "all the way to the left," she implied that a sequence would read $A_{n-1}, A_1, \ldots A_n, \ldots$ Ms. Zimmerman again recognized the student's confusion and tried a different approach:

Ms. Z.: Okay, [Student 1].

Student 1: I'm so sorry.

Ms. Z.: No, no, no. It's fine. I'd rather you understand this. Suppose *n* is ten. So this is the tenth term [pointing to A_n].

Student 1: Oh, okay.

Ms. Z.: Where's *A* sub *n* minus one [A_{n-1}]? What is *A* sub *n* minus one [A_{n-1}]? If *n* is ten?

Student 1: *A* sub nine.

Ms. Z.: *A* sub nine. Where's the ninth term in relationship to the tenth term?

Student 1: One to the left.

Ms. Z.: One to the left. Okay. So this [underlines A_{n-1}] is the term right before this [underlines A_n].

Student 1: Yes.

At this point, Ms. Zimmerman recognized that Student 1 finally understood the meaning behind the notation, so she went on to explain the recursive and explicit definitions in terms of this meaning:

Ms. Z.: This definition [recursive rule] says if you want to find the tenth term, you have to know the ninth term and multiply it by *r*. If you want to find the *n*th term, you have to know the term right directly before it to multiply it by *r*. So if you want to find the hundredth term, it's the ninety ninth term times *r*. If you want to know the fifty first term, it's the fiftieth term times *r*. See the difference? For the recursive, it's based on a term before it.

> *Student 1*: Okay.
>
> *Ms. Z.*: The definition says you have to use the term before to find the current term. This definition [explicit rule] says you have to know the first term, you have to know the common ratio, and you have to know the term you're looking for.
>
> *Student 1*: Okay, so the explicit is almost more widely used because
>
> *Ms. Z.*: Correct. Correct, because you don't always know the term right before it. Exactly.
>
> *Student 1*: Okay.

This lengthy segment began with the student question “What is the difference between explicit and recursive?” Ms. Zimmerman asked the student several questions, and as she came to understand the student’s thinking, she responded to and remediated this thinking at length, choosing her subsequent questions based on what she had established the student understood. In addition, Ms. Zimmerman’s response focused on both the meaning of the notation and making comparisons between the two definitions. For these reasons, this response was coded as high MQI.

Ms. Zimmerman’s reflection. In reflecting on this segment, Ms. Zimmerman described that a central goal was that she wanted to understand the student’s thinking and build from that thinking to help the student grasp the meaning of the notation:

> *Ms. Z.:* I like the line of questioning. And when she didn't get it, even on the number line in general terms, I went to numbers. I broke it down to, okay if *n* is ten and this is the tenth term, then what's *A* sub *n* minus one going to represent? And that's when she finally understood the relationship between *A* sub *n* and *A* sub *n* minus one. So if I had to do that again, I would have done it exactly the same way, based on her questions and how I had to proceed with her questions.

For Ms. Zimmerman, it was important to ask questions that helped to elicit the student’s understanding, and she had specific strategies for targeting the student’s misunderstanding. In particular, she emphasized the meaning of the notation by using numbers instead of variables. This points to Ms. Zimmerman’s knowledge of content and teaching (Ball et al., 2008). That is, she is aware of how to explain concepts to students in ways that resonate with them. Ms. Zimmerman also discussed additional knowledge that she used in the moment of instruction, as she was trying to interpret and understand the student's thinking:

> *Ms. Z.:* I was just waiting for her to get it. Like, was that enough of an explanation? … That is something that happens on your feet, when you hear a question and you try to make an answer and you're just not getting through. … You have to hear the question and you have to figure out where the misconception is and the misunderstanding. And then you have to think, "How am I going to get this girl to understand what I'm trying to get through? Okay, let me try this." And if that doesn't work, I have to have another backup plan.

The type of thinking that Ms. Zimmerman described here seems to be KCS (Ball et al., 2008). Importantly, this knowledge was used in the moment of instruction, as Ms. Zimmerman had to work to interpret and understand the student's thinking, then determine how she was going to build from that thinking. Indeed, Ms. Zimmerman drew on her awareness of the student's understanding to determine where to go next in instruction, and she carefully chose the questions that she would ask, based on the student's thinking.

Ms. Zimmerman's reflection highlights both themes that characterized high-MQI responses to students, but particularly the theme of building on students' mathematical thinking. Ms. Zimmerman also described how interpreting the student's thinking, deciding how to respond on the basis of that thinking, and responding to the student were nearly simultaneous. Her in-the-moment PCK (KCS and knowledge of content and teaching, Ball et al., 2008) supported her in aspects of noticing—that is, interpreting the thinking and deciding how to respond—and carrying out her goals.

Another point is noteworthy in this example. Although Ms. Zimmerman ultimately decided to respond to this student at length, she also described other considerations in her decision. Ms. Zimmerman recognized that other students in the class may have already understood the ideas she was emphasizing, and she had not planned to spend so much time exploring the differences between recursive and explicit definitions, as she explained:

> *Ms. Z.:* I liked my development of the questions that [I asked], because I stayed with her. Did it stop the class dead in the water for the people that already understood this? Yes. Were there people there that were probably doing some “Aha” moments in their head, “Oh that's what she was talking about yesterday?” Probably. …
>
> *Ms. Z.:* [Student 1] was just going to let me stay with it until she got it, which I was glad for. As much as I didn't plan that amount of time on that, that's where you lose time. You don't plan on having to stop with that.

In spite her considerations of the other students in the class and her lesson agenda, Ms. Zimmerman still made the decision to follow the student’s line of thinking. In other words, Ms. Zimmerman’s goals and orientations towards understanding and building on the student’s thinking seemed to prevail over her consideration of the other students in the class and her orientation not to “stop the class dead in the water.”

Connections to other high-MQI responses. The example from Ms. Zimmerman’s class includes a lengthy response to the student’s question. This was also true in the example shared from Mr. Anderson’s class, but it was not necessarily true of all high-MQI responses. Teachers were also able to offer shorter responses to students that both built on their thinking and emphasized meaning in ways scored as high MQI. That is, the length of a teacher’s response did not determine the MQI of the response. What was similar across all high-MQI responses was the teacher’s attention to building on student ideas or (often *and*) the teacher’s recognition of and willingness to take an opportunity to emphasize mathematical meaning in light of the SMP.

However, it cannot be ignored that teachers’ knowledge supported them in responding to SMPs in ways that were coded as high MQI. This becomes especially evident when contrasting these instances with low-MQI responses in which teachers' lack of in-the-moment knowledge seemed to hinder their responses to students.

4.4.2 Teacher Thinking behind Low-MQI Responses to SMPs

To fully understand how teachers' thinking can contribute to the MQI of their responses to SMPs, I also sought to understand the goals, orientations, and knowledge behind low-MQI responses. Ten responses to SMPs discussed in SR interviews were coded as low MQI. In this data, low-MQI responses were those in which teachers (a) did not build on student ideas in instruction, (b) built on students' ideas in ways that were incorrect or unclear, or (c) did not focus on mathematical meaning or mathematical practices (LMT, 2010). In reflecting on low-MQI responses, teachers indicated their responses were affected by (a) a lack of knowledge and (b) goals that were not aligned with the SMP. These themes also captured some of the thinking behind mid-MQI responses, but they were not present in teachers' thinking behind high-MQI responses (see Table 4.7). As discussed in the Methods section, I emphasize that low-MQI responses are not necessarily low quality by every metric. Instead, responses are characterized by their mathematical features according to the MQI rubric.

In the previous section, I used a detailed approach to articulate teachers' knowledge, orientations, and goals that supported their high-MQI responses. In this section, I offer examples that are briefer to target and illustrate the characteristics of teachers' thinking that appeared to be hindering them from providing high-MQI responses. This section is organized according to the themes that arose in analysis.

Lack of knowledge. In reflecting on four low-MQI responses, teachers indicated a lack of knowledge that hindered their response to the SMP. In three of these instances, teachers recognized that they lacked knowledge, whereas in one instance, the teacher offered an incorrect response to a student and did not recognize (in class or in the SR interview) that the response was incorrect. Teachers' lack of knowledge included two types: (a) a misinterpretation of the SMP or (b) a lack of knowledge about the mathematics related to the SMP.

Misinterpretation of the SMP. When a student asks a mathematically-related question or poses a mathematical claim in class, these ideas are not always fully developed or articulated in a clear manner (as can be seen in previous examples shared in this chapter). In reflections on two responses to SMPs coded as low MQI, teachers expressed during the SR interview that they misunderstood the SMP in the moment of instruction. I share an example below to illustrate how such a misinterpretation can guide the teacher's response.

Classroom episode. This example comes from Ms. Zimmerman's Honors Precalculus class. In this example, the class was discussing geometric sequences. Recall that the class had typically been representing sequences with the notation $A_1, A_2, A_3, \ldots A_n, \ldots$ Ms. Zimmerman gave the students several sequences and asked them to determine whether each sequence was geometric. If the sequence was geometric, students were asked to state the common ratio, r, between the terms. One of the sequences the teacher provided was

$$\frac{1}{x}, \frac{2}{x^2}, \frac{3}{x^3}, \frac{4}{x^4}, \ldots$$

In the following episode, a student asked whether the common ratio could be dependent on the term number in a sequence. Ms. Zimmerman responded in a misleading way:

Ms. Z.: What about [this] one [pointing to the sequence]? I'm hearing yes and a no. [Student 1]?

Student 1: Can *r* have an *n* in it?

Ms. Z.: Can what?

Student 1: Can *r* have an *n* in it?

Ms. Z.: Can *r*

Student 1: Like can it be *n* over *x*?

Ms. Z.: Sure, can any number, can *n* be a number?

Student 1: No, like, or *A* sub *n* over *x*? Like the term over *x*, do you know what I mean?

Student 2: Like whatever term it is, that will give you

Student 1: Like the first term is one over *x*, the second term is two over *x*.

Although the student's language is unclear in this segment, based on the class's previous definitions, it appears that the student was asking whether a common ratio could be something such as $\frac{n}{x}$, where *n* is the term number in the sequence. Ms. Zimmerman appeared to be having difficulty understanding the student in this segment, as she asked, "Can what?" "Can r…?" "Can *n* be a number?" She then allowed other students to try and explain Student 1's thinking. If *n* represents the term number in the sequence, then $\frac{n}{x}$ would not fit the definition of a common ratio. Assuming that this was the meaning that the student intended with the question, then Ms. Zimmerman's first response is a true statement but does not follow the student's line of thinking:

Ms. Z.: So can the common ratio be a variable? Absolutely.

Student 1: Yeah, can it be *n, A* sub *n* over *x*?

Student 2: But you're not multiplying by *A* sub *n* over *x*.

Student 1: *n* over x

Ms. Z.: It would just be *n* over *x*. Or *n* over *x* squared.

In this segment, Student 1 seemed to be working to clarify her question. Ms. Zimmerman responded by saying, "It would just be *n* over *x*," implying that the term number could be part of the common ratio. This implication is incorrect if the student intended for *n* to be a term number in the sequence. The episode continued, and the class determined that the sequence was not geometric, which did add some clarity, but the first student was left with the incorrect idea that a common ratio in a geometric sequence could be dependent on the term number of the sequence. Because of this incorrect mathematical idea, this response was scored as low MQI.

Ms. Zimmerman's reflection. As Ms. Zimmerman reflected on this segment in the SR interview, she expressed that she had misinterpreted the student's question. Consider the following excerpt from her interview:

> *Ms. Z.*: [Watching video segment] No, no, no, no, no. No. [Finishes watching segment.]
>
> *Ms. Z.:* I'm yelling at myself, yeah. Because I didn't understand, again, can *n*, can you multiply by *n* over *x*, you can. But you can't let *n* change each time, which is what I think she was asking. I think she was using *n* as a position, like one for the first term, two for the second term, three, I think that's what she was saying, but that's not what I was hearing, I'm thinking yeah you can, you can multiply by *n* over *x* every time, but then, that's not what's happening here [referring to example the students were working on].
>
> *Kathryn:* So you were thinking *n* as just another variable, and she was thinking *n* as specific to the
>
> *Ms. Z.:* A position. Right. And now I don't know if she understood that.
>
> …
>
> *Ms. Z.:* Can I teach that again? [laughing]

Ms. Zimmerman's interview makes it clear that she was unsatisfied with her response to the SMP. She explained how, in the moment of teaching, she interpreted the student's question in one way, but, upon reflection, she believed that the student intended a different meaning. Specifically, in the SR interview, Ms. Zimmerman interpreted the student's question in the same way that I explained above.

The type of knowledge that Ms. Zimmerman seemed to be lacking in the segment is an in-the-moment use of KCS (Ball et al., 2008). The reason why this segment was scored as low MQI was because of the incorrect concept of the common ratio that was left with the student. Ms. Zimmerman's misinterpretation of the SMP was a large contributor to this incorrect idea.

Unaware of mathematics related to the SMP. When posed with an SMP, a teacher must also bring their knowledge of mathematics content to mind in the moment. In reflecting on two low-MQI responses, teachers indicated that they did not understand the mathematical content in the SMP, and this lack of understanding seemed to hinder them from providing a response scored higher in MQI.

Classroom episode. To illustrate this theme, I share an example from Ms. Carter's College-Preparatory Precalculus class. Ms. Carter taught at a high-performing, public high school, and the class that I observed had 18 students. Ms. Carter had over 15 years of teaching experience, many of which teaching precalculus. The class was studying the polar coordinate system when I observed them.

Students had discussed how to represent points in the plane with polar coordinates, and they had practiced converting points from rectangular coordinates to polar coordinates. Specifically, if (x, y) are the rectangular coordinates of a point P in the plane, polar coordinates (r, θ) of P are given by $r = \sqrt{x^2 + y^2}$, $x = r\cos\theta$, and $y = r\sin\theta$. Every point in the plane has infinitely many polar coordinates. For the point P with polar coordinates (r, θ), the additional polar coordinates of P are $(r, \theta + 2\pi k)$ for any integer k and $(-r, \theta + (2k + 1)\pi)$ for any integer k.

In this example, the class was graphing rose curves. The class established that these curves came from equations of the form $r = a \cos n\theta$ and $r = a \sin n\theta$ ($n \in \mathbb{N}$, $n \geq 2$). The class used the following procedure to graph rose curves:

1) Determine the length and number of petals in the graph by looking at the equation. (i.e., $|a|$ is the length of each petal, and there are *n* petals if *n* is odd and 2*n* petals if *n* is even.)
2) Use the graphing calculator to find a graph of the equation.
3) Looking at the graph, choose a petal on which to find three points.
4) Use the symmetry of the graph to hypothesize the value for θ—call it θ_1—that gives the maximum value for *r* on that petal. Choose two values of θ in proximity of θ_1.
5) Use the calculator to find the values of *r* that correspond to these three values of θ. (Verify your hypothesis.) This will provide three points on one petal.
6) Use symmetry to complete the graph.

In using this approach to graph, students have a visual sense of the graph before they find exact points on the graph. Because of this, it is possible to choose values for θ for which one is expecting a positive value for *r*, but the value for *r* corresponding to θ is actually negative. Ms. Carter was familiar with this phenomenon, and she anticipated it in her instruction. The class had just graphed $r = 3\cos 2\theta$ by first determining that a maximum value for *r* could be found at θ = 0. Then, the class found the points (3, 0), $\left(2.6, \frac{\pi}{12}\right)$, and $\left(2.6, \frac{23\pi}{12}\right)$ and graphed one petal. From there, they used symmetry to graph the three remaining petals. Afterwards, Ms. Carter drew students' attention to the fact that *r* may be negative when students are not expecting it to be:

Ms. C.: Alright. Let's do this. Watch what happens when I pick pi over two. I don't need to, but let's say I was starting this from the beginning. … Watch what happens when I put in pi over two in my table. It gives me negative three. That's this point down here [points to (-3,$\frac{\pi}{2}$) in polar coordinates]. It's still a point on the graph. So really, if you choose pi over two, it doesn't give you three, it gives you negative three. But does it really matter that that happens? No, because every point can be written other ways. So if you choose these two points on either side of pi over two, which in this case are five pi over twelve and seven pi over twelve, watch what happens there. Five pi over twelve and seven pi over twelve. That is going to send you to this point and to this point. [Pointing to the points $\left(-2.6, \frac{5\pi}{12}\right)$ and $\left(-2.6, \frac{7\pi}{12}\right)$.] Now I don't know why that happens, to be honest with you. But you may choose a petal to work with, and that may happen. But it doesn't matter. You should still be able to know where to plot them. I started thinking I was going to get this petal, and I ended up getting one that was through the circle. Because for some reason, the calculator gave me negatives. Like I said, I'm not sure why that happens, but if it does, it's alright.

In this excerpt, Ms. Carter pointed out to students that they may get negative values for *r* when using the calculator, and she also told students "I don't know why that happens, to be honest with you," and later emphasized, "I'm not sure why that happens."

In response the discussion about the negative values for *r*, a student asked a question about a different type of rose curve, and Ms. Carter did not directly answer this student's question:

Student 1: Would it be the same thing in the horizontal for sine?

Ms. C.: No because I picked zero and it gave me three.

Student 1: No but I mean for sine, like

Ms. C.: You mean like, are you thinking it just would do it for cosine?

Student 1: Yeah.

Ms. C.: Well. No, it may. It may. It could, I don't see why it couldn't, because if I, see but the way sine is, and you'll see in the next example, if this problem was with sine, it would still have four petals, but they would be here. [Pointing to the angle $\frac{\pi}{4}$.] They're not on the axes. But it doesn't mean if you pick pi over four that you're not going to get down here. You still would, you would, it could happen.

Student 1: So it doesn't affect the graph?

Ms. C.: No. Okay, but you'll notice that. Cosine you'll see that the petals are more on what you know is the *x* and *y* axes, but sine is shifted the other way.

In this excerpt, it seems that the student considers that, for the equation $r = 3\cos 2\theta$, the value of r is negative when $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. On the graph, these values appear in the “vertical” petals. Because this graph came from an equation of the form $r = a\cos n\theta$, the student wondered whether the graph of $r = a\sin n\theta$ would have a similar property. Specifically, for the case of $r = a\sin n\theta$, the student wondered whether the value of r is negative when $\theta = 0$ and $\theta = \pi$. He seemed to be thinking of these petals as the “horizontal” petals in the graph of the equation.

In this case, the response to the SMP was scored as low MQI. Ms. Carter appeared to understand the student’s question. She also clarified that graphs of equations in the form $r = a\sin n\theta$ do not have “horizontal” petals (that is, petals that are centered on the angles $\theta = 0$ and $\theta = \pi$). However, Ms. Carter did not answer the student’s question. She said, “It may. … I don’t see why it couldn’t. … It could happen.” She did not rule out the possibility that for the equation $r = a\sin n\theta$, there may be values of θ for which r is negative; however, the question remained unanswered. In addition, Ms. Carter did not include any richness of mathematics in her response.

Ms. Carter’s reflection. During the SR interview, Ms. Carter reflected on this episode. She emphasized the fact that she was aware of this phenomenon and she wanted to draw students’ attention to it, but she wanted it to be introduced to students after they had seen simpler examples:

> *Ms. C.*: Well, [a different] period when I did it, I said, “Okay let's just focus on one petal.” And I remember, I said, “Which petal do you want to focus on?” and he said, “The top one,” I was like okay, and I went, “Oh God!” And I think I said, “You know what, we'll come back to that.” Because I wanted to talk about it, but … on the first example doing this different shape, I didn't want to go there. Even though we talked about that would happen, but I liked it better to show after I did it and explained how I wanted them to use the symmetry to do it. How it wouldn't matter, you would still get somewhere on there. …

When reflecting directly on the student's question, Ms. Carter explained that she considered doing another example, but if she did so, she wanted it to illustrate the student's point:

> *Ms. C.*: Maybe like going back to the student, if I put a sine example up there and trying to make it happen. You know, picking angles to see if it did happen. But then again, I don't know that I would have wanted to pick angles and it didn't happen. But if, which I'm sure it does. I mean, I don't see why it wouldn't, just like what I said, I don't see why it wouldn't. …

In her reflection, Ms. Carter emphasized the fact that she was still unsure whether she would obtain negative values for *r* on the graph of an equation of $r = a \sin n\theta$. Ms. Carter explained that she could have answered the student's question with an example; however, she clarified, "I don't know that I would have wanted to pick angles and it didn't happen." Ms. Carter went on to explain that based on previous experience, she was prepared to show students that they may get negative values for *r* when graphing rose curves:

> *Ms. C.*: That's just something I had been remembering from last year or the year before. I knew it was going to come up. And that's just something over time you learn how to handle. Because I would love for somebody to show me a videotape of when that happened my first year teaching this. I probably would have been like--you know, but you just get more comfortable with this stuff as it goes on and then you're able to just explain it better.

From Ms. Carter's perspective, she believed that she was better able to address this phenomenon because of her experience with it.

This example was chosen to illustrate the point that Ms. Carter's lack of content knowledge about whether this phenomenon happened for different rose curves affected how she responded to the student. However, this was not the only reason that she responded to the student in the way that she did. That is, because one does not have knowledge of a particular mathematical idea does not mean it cannot be explored; one might argue that it is appropriate for teachers to explore mathematical ideas with students, even if they are unsure of the answer. Ms. Carter's reflection suggests that her orientations towards mathematics and towards teaching also guided her response.

For instance, Ms. Carter's main goal in teaching this segment seemed to be focused on procedures: She wanted students to use symmetry to obtain the points on the graph and see that "it wouldn't matter" whether the *r*-values were negative or not. In addition, Ms. Carter explained that she believed she was better prepared to "explain" this issue because she had encountered it before. This suggests that for Ms. Carter, explaining ideas might mean being prepared with the right example to show students. This also aligns with her statement that she didn't want to pick petals on an unfamiliar graph that did not illustrate the point she was trying to make.

Although the student asked *whether* this phenomenon would happen in other cases, a closely related question is *why* this phenomenon happens at all. Ms. Carter readily admitted to students and me that she did not know why it happened. In fact, if she had recognized why this happened, she likely would have been better equipped to answer the student's question. It is curious that Ms. Carter was prepared to illustrate this phenomenon for students but did not seem motivated by previous experiences to try and understand why it happens. This suggests that Ms. Carter's orientations towards mathematics and teaching may be closely intertwined with her content knowledge of mathematics: Ms. Carter may be motivated to understand mathematics in a way that is going to be useful for her in the classroom, to explain the concepts in the ways that she values.

Alternative goals. In reflecting on seven responses to SMPs scored as low MQI, five teachers explained their decisions according to goals that did not align with the SMP.

Specifically, teachers expressed goals to (a) avoid student confusion (three responses across three teachers), (b) emphasize mathematical procedures (two responses across two teachers), or (c) emphasize a mathematical idea that did not build on the SMP (two responses across two teachers).

To illustrate this theme, I share an example of a teacher who had a goal to avoid student confusion. Teachers described this theme by explaining that they intentionally withheld some mathematical ideas in their responses to students because they did not want to overwhelm students. This was not expressed as teachers trying to present abstract ideas in a more understandable way; rather, teachers expressed that their intention was to take away from or withhold the mathematical complexity and rigor in their response.

Classroom episode. The example that I share here comes from Mr. Dillon's Algebra II class at a private high school. This class was comprised of eight students. Mr. Dillon was a first year teacher when I observed him, and he had a college degree in mathematics. When I observed Mr. Dillon, the class was studying rational functions.

Specifically, the class had previously graphed rational functions and they were performing operations with rational expressions. In the episode below, the class was finding the following product:

$$\frac{3x^2+6x}{x^2+3x+2}\cdot\frac{x^2+5x+4}{3x}$$

The teacher worked through this example by factoring and simplifying to obtain the answer of $x+4$. Following this, a student asked about the graph of $y=\frac{3x^2+6x}{x^2+3x+2}\cdot\frac{x^2+5x+4}{3x}$, and Mr. Dillon answered this question in a misleading way:

> *Mr. D.:* And this just becomes *x* plus four. So when I multiply these two things together, instead of multiplying them all out, getting a quartic over a cubic and

then using long division, you'll get the same answer if you do that, you can save yourself a lot of work by just factoring them all out right away.

Student 1: So you're saying those graphs will be the same?

Mr. D.: If you multiply these two things together?

Student 1: Yeah.

Mr. D.: As far as we know, yes. There will be two or three small exceptions, but we're not going to be responsible for that material.

Student 1: Okay.

Mr. D.: But visually, yes. They will look exactly the same. Just domain issues and stuff. Like you still can't plug in zero in here.

This response to the student's question was scored as low MQI. Mr. Dillon appeared to understand the student's question, which seems to be, essentially, "Will the graph of $y = \frac{3x^2+6x}{x^2+3x+2} \cdot \frac{x^2+5x+4}{3x}$ be the same as the graph of $y = x + 4$?" Mr. Dillon then answered the question without attention to meaning in his response. In fact, Mr. Dillon's comments are slightly misleading to students, particularly since the class had already discussed graphing rational functions. He dismisses the "domain issues and stuff" as "two or three small exceptions," whereas most mathematicians would agree that the domain is an important part of the function $y = \frac{3x^2+6x}{x^2+3x+2} \cdot \frac{x^2+5x+4}{3x}$.

Mr. Dillon's reflection. Mr. Dillon reflected on this episode in the SR interview:

Mr. D.: I didn't even need to answer that question. I could have just said, "We're not worried about graphing these." But I think if kids are curious and they want to know, I would say, "Yeah actually it would. It would be the same exact graph." Because I think that emphasizes the fact that it's, that's one value. … The flavor of this chapter [and] this section is to make life easier for yourself, so it actually is still an easy graph is what I was trying to emphasize.

This quote illustrates that Mr. Dillon's goal in this segment was to "make life easier for yourself." He considered dismissing the student's question because he wanted students to understand that the goal of the work they were doing was to recognize the simplicity in mathematics rather than its complexity. The above quote also illustrates another important point: Mr. Dillon valued the student's curiosity and wanted to respond appropriately. For Mr. Dillon, the appropriate response was essentially to say, "It would be the same exact graph." Mr. Dillon also explained that he considered an alternative response to the student:

> *Mr. D.:* I would have drawn in empty circles or I would have drawn out the domain, but I think in my head, I was like, "Well, they're not responsible for it, so I don't want them to be confused with writing it out." … If they were like, "Isn't that sometimes it doesn't always have to be an asymptote?" So I didn't want them to associate writing the domain with that. But it would have been nice to do an example. So I don't have an answer whether that would have been good or not.
>
> *Kathryn:* But that was something that you were considering, and you kind of had to make a decision?
>
> *Mr. D.*: Right, and then I was like, alright, I'm already too far into the topic. If they're interested, they'll talk to me afterwards.

In the quote above, Mr. Dillon indicated that his justification for withholding some of the mathematical ideas was that he wanted to avoid student confusion. Specifically, he did not want students to confuse holes with asymptotes, which they had spent a great deal of time studying. Part of this was due to the fact that Mr. Dillon put value in how students would be assessed, and the concept of holes was not going to appear on students' assessments. Mr. Dillon also expressed that he did not want to deviate too much from his agenda, saying, "I'm already too far into the topic. If they're interested, they'll talk to me afterwards."

Mr. Dillon's goals were motivated by his knowledge of the curriculum. Mr. Dillon knew that the students were not responsible for holes on their department-wide exam. Mr. Dillon also drew on knowledge of students' difficulties with rational functions, anticipating that students might confuse functions that have asymptotes with those that have holes. This knowledge is used in support of his goal to keep the mathematics simple for students. Notably, Mr. Dillon's reflection indicates additional content knowledge that he considered in deciding how to respond to the student. Specifically, Mr. Dillon realized that the graph of $y = \frac{3x^2+6x}{x^2+3x+2} \cdot \frac{x^2+5x+4}{3x}$ would have holes, and he considered using this knowledge to respond to the student's question by drawing holes in the graph or explicitly writing out the domain. However, in this moment, avoiding student confusion was more important to Mr. Dillon than introducing the concept of holes in a graph.

4.5 Conclusion

This study provides a glimpse at the teacher thinking that can support high-MQI responses to SMPs as well as the teacher thinking that can hinder high-MQI responses to SMPs. To build on students' thinking in instruction, it is important that teachers attend to SMPs, interpret them, decide how to respond to them, and facilitate responses that prioritize mathematics (Jacobs et al., 2010; Stockero & Van Zoest, 2013). The themes and examples presented in this chapter provide a glimpse at the cognitive complexity of this work that must be done in the moment of instruction, and understanding teachers' thinking during this process can help teacher educators to better prepare teachers to take advantage of these mathematical opportunities in instruction.

4.5.1 Limitations

There are some limitations to this study. First, I do not claim to capture every aspect of teachers' decision making. The video-based, SR interviews help to elicit aspects of teacher thinking that are not articulated in other ways, but teachers may not be fully aware of their considerations as they make decisions. Second, this study focuses on specific in-the-moment decisions that teachers must make: decisions in responding to

SMPs. The conceptions that teachers bring to bear in these decisions may differ from those that teachers use in making more planned decisions. Third, it was not feasible to have teachers reflect on the complete videos of their teaching, so not all SMPs were used in the SR interviews. Nonetheless, the data shows an intriguing glimpse at how teachers make decisions in responding to SMPs in instruction.

Fourth, although teachers' thinking is an important part of their decision making, I do not claim that if teachers simply develop certain knowledge, orientations, and goals, they will be able to perform well in any classroom situation. Indeed, in this data, when reflecting on some mid-MQI responses, teachers expressed a desire, intention, and orientation to respond in a way that might be expected to be coded as high MQI, but the response was not executed at a high level. Hence, teaching in ways aligned with the MQI framework requires expertise beyond what is described in this chapter.

Finally, in this chapter, I looked at the responses that teachers offered to students through the lens of particular dimensions of MQI. As discussed earlier in this chapter, responses that were coded as high MQI did not take into account student participation in meaning-making and reasoning. In a vision of student-centered mathematics teaching, student participation is equally as important as mathematical richness or any other dimension of the MQI rubric. In addition, researchers have argued that there are additional aspects of teacher-student interactions that mediate students' access to mathematics (e.g., Battey, 2013), and these were not explored. Nonetheless, responses coded as high MQI in this chapter capture important mathematical aspects of instruction. At the same time, responses coded as low MQI are not necessarily low quality by every metric; by a different lens, these responses may be considered to be good quality. Yet low-MQI responses did not exemplify certain mathematical aspects captured by the MQI rubric.

4.5.2 Discussion

Teachers who offered responses to SMPs that were coded as high MQI expressed goals and orientations to build on student ideas and emphasize mathematical meaning in their responses. These goals guided their responses, but teachers' MKT also supported them in interpreting students' thinking and carrying out these goals. In many instances, interpreting the SMP, deciding how to respond to the SMP, and responding to the SMP were almost simultaneous, as illustrated in the high MQI example of Ms. Zimmerman. The themes that guided teachers' actions and their use of knowledge are similar to what Brown and Coles (2000) called purposes in mathematics teaching. Because these themes were not expressed in reflections on low-MQI responses, following Brown and Coles's (2000; 2011) recommendations, these constructs could be explored in future professional development.

The knowledge that supported teachers in carrying out high-MQI responses was made more evident by contrasting these with low-MQI responses. Similar to previous research (e.g., Peterson & Leatham, 2009; Davies & Walker, 2005), the examples in this chapter illustrated how a lack of KCS (Ball et al., 2008) or a lack of mathematical content knowledge hindered teachers from providing high-MQI responses to students.

Noteworthy is that in the present study, KCS was specific to each situation. In the high-MQI response offered by Ms. Zimmerman, KCS supported her in interpreting students' thinking in the moment of instruction, and this work was prolonged and detailed. However, Ms. Zimmerman also lacked KCS in a different instance that was scored as low MQI. In fact, her misinterpretation of the student's question guided the misleading response to the student. Moreover, in the SR interview, Ms. Zimmerman recognized that she misunderstood the student, and this reinforces the notion that what teachers know in one setting may not come to mind in another (cf. Mason & Spence, 1999). As such, a single observation, assessment, or evaluation may not capture the affordances of a teacher's knowledge. If Ms. Zimmerman was evaluated on the basis of the low-MQI response that she offered the student, one interpretation might be that she lacked KCS in general. However, that was not likely the case, as Ms. Zimmerman was able to offer several high and mid-MQI responses to students in this data and she is an NBCT with several additional recognitions for her teaching.

The role of teachers' content knowledge of mathematics was also highlighted in this study. Content knowledge supported high-MQI responses to students, as illustrated in both the high-MQI examples, but particularly in Mr. Anderson's example with his knowledge of mathematics as a discipline and mathematical induction. By contrast, responses to students were sometimes limited by the teacher's lack of content knowledge, as was illustrated with the example of Ms. Carter and polar coordinates. Without an awareness of the mathematics related to the SMP, it is difficult to respond in a way coded as high MQI.

The example of Ms. Carter also illustrated that MKT and orientations may be inter-related. Although Ms. Carter recognized prior to the lesson that she lacked mathematical content knowledge related to the student's question, she was not motivated to seek out this knowledge because it seemed tangential to her procedural goals for the lesson. A teacher's view of what mathematics is, what a mathematics lesson should look like, and what mathematics is necessary or appropriate for students shape not only how they interact with students in instruction but also how they develop new knowledge. A teacher is not necessarily motivated to work to develop new knowledge when they recognize that they lack such knowledge (cf. Leikin & Zazkis, 2010; Weber & Rhoads, 2011).

At the same time, although content knowledge appeared to be necessary to support high-MQI responses to SMPs, it did not necessarily motivate decisions or goals that allowed for productive use of this knowledge. In the case of Mr. Dillon, he understood the mathematics behind holes in the graphs of rational functions, but he chose not to use that knowledge in his response to the student. Ultimately, the response was scored as low MQI because of the misleading way that holes were addressed. Mr. Dillon's case points to the importance of orientations in deciding how to respond to the SMP, and this decision contributes to the overall mathematical quality of the response. In this study, if a teacher did not have orientations that led them to use their content knowledge productively in responding to the student, then a high-MQI response was not realized. This was true for PCK as well: In many low-MQI responses, teachers were able to correctly interpret students' thinking, but this did not support high-MQI responses if teachers did not have goals to build on and work with this thinking in instruction.

The role of orientations and related goals was also highlighted by teachers' dilemmas as they determined how to respond to SMPs. In many cases, teachers' decisions to respond to students were not straightforward; teachers reflected on several considerations that were weighed in the decisions that they made. For example, in responding at length to a student's question about recursive and explicit definitions of sequences, Ms. Zimmerman considered the fact that other students may already understand the content being discussed, and she also considered the great amount of time that it took to build on the student's thinking. However, Ms. Zimmerman's ultimate decision to pursue the student's question reflects what was most important to her at that moment. Similarly, in responding to a student's question comparing two graphs of rational functions, Mr. Dillon considered drawing holes in the graph to illustrate the differences, but he ultimately decided that it was more important to omit those mathematical details to avoid student confusion. Teachers did not necessarily approach these teaching situations with blinders, focused on one goal. Rather, teachers acted rationally and weighed several options in deciding how to respond to students (cf. Herbst & Chazan, 2003; Schoenfeld, 2011). The goals that teachers prioritized in their responses were focused on achieving what they believed was most important for the moment.

As such, it is important to note that the goals that sometimes led to low-MQI responses were not necessarily unproductive. For instance, the goal to avoid student confusion is a reasonable and necessary goal for teaching mathematics. That is, it would not be appropriate to present students with a long list of facts from advanced mathematics that are disconnected from their current mathematical studies. At the same time, several mathematics educators have advocated that student's struggles with mathematics are important aspects of doing mathematics. DeBellis and Goldin (2006) described that when students encounter difficulty with mathematics and are later able to resolve their cognitive struggles, this process can lead not only to students' mathematical learning but also to students' positive feelings about mathematics. What may be problematic is when teachers consistently withhold mathematical ideas that are central to understanding key concepts or rarely allow students the opportunity to struggle with new ideas.

Similarly, the other goals that led to low-MQI responses, emphasizing procedures and focusing on a different mathematical idea, are not necessarily goals that are inappropriate. Many mathematics educators argue that understanding concepts are critically important in mathematics, but this is not to say that students should not obtain procedural fluency (e.g., NCTM, 2000; National Research Council, 2001). In addition, teachers may have carefully-planned mathematical agendas, and weaving students' ideas into these agendas may mean that teachers occasionally choose not to pursue a particular idea. Nonetheless, to achieve a vision of student-centered mathematics, it is important that teachers regularly give voice to students' ideas, especially when these ideas contain substantial mathematical content, as is the case with SMPs. Teacher education can work to help teachers recognize the balance between fulfilling their reasonable goals for the lesson and prioritizing work with students' thinking.

4.5.3 Significance

Responding to SMPs in instruction is critical to the work of student-centered mathematics teaching. The MQI instrument provides a useful lens to describe what it means to respond to SMPs with high mathematical quality. But to help teachers reach this level of quality in their instruction, it is important to understand the teacher thinking needed to carry out this instruction, and that was the purpose of this study.

Other researchers have proposed the types of knowledge that in-the-moment work of responding to SMPs requires (e.g., Davies & Walker, 2005; Johnson & Larsen, 2012; Peterson & Leatham, 2009). The findings of the present study provided specific examples of that knowledge and its role in teachers' responses. The results supported the assumption that MKT is necessary for high-MQI responses to SMPs.

However, the present study also illustrated that MKT was not sufficient for high-MQI responses. A great deal of recent research has focused on the relationship between teachers' knowledge and the quality of their instruction (e.g., Charalambous & Hill, 2012; Hill, Blunk, et al., 2008; Hill, Sleep, Lewis, & Ball, 2007; Kahan, Cooper, & Bethea, 2003). By focusing on the decisions that teachers make in responding to SMPs, the present study highlights that, although MKT is critical for high-quality teaching, it is difficult to fully understand how this MKT contributes to the quality of instruction without understanding the teacher's orientations. Teachers' orientations guide their decisions and filter their use of MKT in their instruction, and the data also suggested that orientations guided how teachers developed new MKT. Hence, with evidence from a specific task of teaching (responding to SMPs), the present study supports the hypothesis that orientations play an important role in mediating MKT and MQI (e.g., Hill, Blunk, et al., 2008; Sleep & Eskelson, 2012). That is, it is important for teachers to develop MKT to work with SMPs in instruction, but one cannot assume that if a teacher does not offer a high-MQI response to an SMP that they lack MKT.

Recognizing the central role of orientations as teachers use their knowledge in instruction, it may be useful for researchers to broaden their views of MKT. For instance, an earlier framework for the knowledge used in mathematics teaching was presented by Fennema and Franke (1992) who argued that beliefs play an important role in the implementation of knowledge in the classroom. These researchers also argued that teachers' knowledge was situated, and to fully understand teachers' knowledge, researchers should also consider the context for teaching. As such, their framework included both beliefs and context as critical aspects of the mathematical knowledge used in teaching. In science education, Magnusson, Krajcik, and Borko's (1999) framework for PCK included both knowledge and beliefs, with orientations to teaching overlaying the implementation of these. Magnusson et al. argued that both knowledge and beliefs are integral to the process of teaching and are driven by teachers' orientations to teaching. The success of a physics teacher education program built around this framework has been documented by Etkina (2010). The present study relates to these frameworks and provides further evidence that to fully understand the quality of mathematics instruction, researchers must consider more than teachers' knowledge alone.

Chapter 5: Barriers to Implementing Mathematical Richness in High School Mathematics Instruction

Abstract

In this chapter, I share the example of one teacher who expressed orientations and goals for instruction aligned with mathematical meaning and sense making yet had instruction that was limited in mathematical richness, according to the mathematical quality of instruction framework (LMT, 2010). I use grounded analysis from interviews to describe this teacher's overarching goals for instruction. Additional data from interviews and observations was used to explore the reasons for the limited richness in instruction, and I identified three of these: (a) conceptions of what constituted meaning in instruction, (b) inattention to precision and clarity in instruction, and (c) beliefs about students' academic abilities. This chapter points to the depth and complexity of teachers' beliefs and knowledge that are needed to engage in instruction that is mathematically rich in the sense of the mathematical quality of instruction framework.

5.1 Introduction

In the United States, several recent efforts have sought to improve mathematics education, including developing and implementing rigorous academic standards for K-12 mathematics, writing and adopting innovative mathematics curricula, and increasing classroom time for students to learn mathematics. However, the benefits that students receive from each of these efforts are mediated by instruction. Cohen and Ball (2000) described instruction as "interactions among teachers, students, and content in environments" (p. 3). These interactions are complex and dynamic, and understanding this complexity is important because it is in these interactions that much of students' learning takes place. Yet there is much about mathematics instruction that is not well understood.

There are several ways in which instruction can be analyzed (e.g., Danielson Group, 2013; McDonald, Kazemi, & Kavanagh, 2013). With one perspective, the Learning Mathematics for Teaching Project (LMT, 2010; 2011) characterized mathematics instruction according to its mathematical features and described the collection of these features as the *mathematical quality of instruction* (MQI). Several aspects of mathematical quality are included in MQI, such as the richness of mathematical ideas, the nature of teachers' responses to students' ideas, and students' participation in mathematical reasoning. Although each of these is an important aspect of mathematics teaching, this chapter focuses on one of these: the richness of the mathematics. The dimension of richness of the mathematics describes the extent to which instruction emphasizes mathematical meaning and mathematical practices. Both of these aspects of instruction are considered by mathematicians and mathematics educators to be central to mathematics as a discipline (Ball & Bass, 2002; NCTM, 2000).

Teachers play an important role in instruction, as they are the ones that plan lessons and guide the events in the classroom. In addition, teachers are representatives of the mathematics community: One of their roles is to help students come to understand what mathematics is and what is mathematically appropriate. Hence, for mathematical richness to be realized in instruction, teachers need to be able to facilitate this richness. Specifically, teachers must make decisions about how to develop mathematical content, and these decisions rely on teachers' orientations and knowledge.

Schoenfeld (2011) proposed that the decisions made around instruction are a result of the teacher's goals, orientations, and resources, including knowledge. In particular, teachers' orientations and resources guide them to form the goals they have for instruction, and teachers also rely on additional resources to carry out their goals. Because teachers' orientations and knowledge are central to their decision making, one role of teacher education is to develop teachers' orientations and knowledge. Hence, it is important to understand how orientations and knowledge may or may not lead to instruction that is mathematically rich.

Recently, researchers have found a positive relationship between teachers' mathematical knowledge for teaching and MQI (Hill, Blunk, et al., 2008; Hill & Charalambous, 2012), but research has also illustrated how MQI can be limited by teachers' decisions around curriculum materials (Hill, Blunk, et al., 2008) and teachers' beliefs about mathematics and mathematics teaching (Hill, Blunk, et al., 2008; Sleep &

Eskelson, 2012). In this chapter, I illustrate another factor that can limit MQI: the teacher's beliefs about students' academic abilities.

To illustrate this point, I share an example of one high school mathematics teacher, Mr. Taylor, who believed that mathematics instruction should be focused on concepts and connections. He expressed goals for instruction that were aligned with richness in the sense of MQI, and these goals also aligned with visions of mathematics teaching advocated by mathematics educators (e.g., NCTM, 2000). Mr. Taylor also articulated content knowledge related to these goals. Yet, surprisingly, his instruction was not coded as high MQI in the richness of the mathematics. In the findings, I present three reasons that the instruction was limited in richness. One of these has not been discussed in the MQI literature: the teacher's beliefs about students' academic abilities. This finding sheds light on the complexity of the ways in which teachers' orientations, goals, and knowledge interact in teaching high school mathematics.

5.2 Background

This chapter draws on literature related to MQI and the teacher thinking behind instructional decisions.

5.2.1 Mathematical Quality of Instruction

Recognizing that instruction impacts students' learning, researchers have used a variety of ways to analyze it (e.g., Brophy & Good, 1986; Danielson Group, 2013). Many evaluations of teaching have focused on observable behaviors in the classroom, such as establishing a clear objective or agenda for students, effectively managing transitions between tasks, assessing students' understanding throughout the lesson, and questioning students frequently. Attention to these features can help teachers to target and improve specific areas of their instruction. However, the LMT Project (2011) argued that such characterizations of practice do not focus on the *mathematical* aspects of mathematics instruction. These researchers presented the MQI framework and corresponding instrument that was designed specifically to describe mathematics instruction in terms of its mathematical characteristics. Further, the LMT Project contended that the quality of mathematical content offered to students may be independent of specific instructional strategies or pedagogical styles. Hence, the MQI instrument focuses on the mathematical nature of instruction, regardless of specific teacher moves, such as questioning techniques or formative assessment.

Drawing from the existing literature on mathematics teaching and learning, the LMT Project (2010) identified five dimensions of MQI. (See Appendix F for a more complete description.) For the purposes of this chapter, I focus on one of these dimensions: richness of the mathematics. This dimension captures both mathematical meaning and mathematical practices that are present in instruction. In the MQI instrument, meaning is characterized by the links made among representations (visual, concrete, verbal, and symbolic), the connections made among mathematical ideas, the meaning of mathematical ideas, and the explanations and justifications behind facts and procedures. The mathematical practices that MQI captures include multiple procedures or solution methods, generalizations, and mathematical language used in instruction.

Both mathematical meaning and practices are central to the work of mathematics at all levels (cf. Mejia-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2012). Mathematics educators argue that if students are given only opportunities to memorize facts and practice procedures in mathematics class, they are not learning mathematics at all (Ball & Bass, 2003). Instead, the discipline of mathematics involves observing mathematical phenomena, making mathematical conjectures, and justifying these conjectures (Davis, Maher, & Noddings, 1990; NCTM, 1989, 2000). Hence, reasoning and sense making are the processes by which one comes to know mathematics (Thompson, 1996). With this view that mathematics involves more than memorizing and applying facts and procedures, explanations and connections (part of richness of mathematics as described by LMT, 2010) are integral pieces to learning mathematics.

In addition to explanations, connections, and sense making, mathematical language is the vehicle by which students learn mathematics (Hill, Blunk, et al., 2008). Hence, the richness of the mathematics in instruction can be enhanced by a fluent use of mathematical language or compromised by an incorrect or imprecise use of language. In a study with 17 preservice elementary teachers, Sleep (2012) found that emphasizing definitions and using intentional redundancy helped teachers to open up and emphasize key mathematical ideas in instruction, whereas repeating imprecise language and providing imprecise or confusing explanations detracted from the key mathematical ideas in the lesson.

The MQI instrument has the potential to be widely useful for the field of mathematics education; however, because the MQI instrument is relatively new, there is a limited body of research that has used this instrument. In an exploratory study, Hill, Blunk, et al. (2008) investigated the relationship between elementary teachers' mathematical knowledge for teaching (MKT; Ball et al., 2008; LMT, 2012) and MQI. These researchers found that, in general, MKT seemed to support MQI; however, they presented four case studies that illustrated that MKT and MQI were mediated by teachers' use of curriculum materials. That is, sometimes teachers with high MKT made decisions to use curriculum materials in ways that did not enhance the meaning of the mathematics, whereas strong curriculum materials sometimes afforded teachers with lower MKT more opportunities to enhance the meaning of the mathematics in the classroom. The researchers also found that teachers' beliefs that mathematics should be fun for students sometimes motivated instructional choices that led to a lower MQI.

As a follow-up to this study, one issue of the *Journal of Curriculum Studies* (Charalambous & Hill, 2012) contained case studies exploring how MKT and curriculum materials uniquely and jointly impact MQI. In one of the articles for the issue, Sleep and Eskelson (2012) compared a teacher with limited MKT and a teacher with strong MKT enacting the same lesson from the same curriculum. They found that the teacher with weaker MKT had a stronger lesson than predicted in terms of MQI, and the researchers hypothesized that this was because the teachers' orientations towards mathematics aligned with the goals of the curriculum materials. By contrast, MKT and student-centered curriculum materials were not sufficient for high MQI: The teacher with strong MKT had an orientation that mathematics consisted of facts and procedures, and the researchers illustrated how these views limited MQI.

The present study extends previous work with the MQI instrument to highlight why content knowledge and beliefs aligned with richness of the mathematics in the sense of MQI would not necessarily lead to instruction that is coded as high in the richness of the mathematics. Specifically, the teacher described in this chapter expressed content knowledge and goals that were focused on mathematics and aligned with mathematical richness, yet his beliefs about students' academic abilities limited his MQI.

5.2.2 Knowledge, Beliefs, and Instructional Decision Making

The teacher has a key role in instruction, as he or she must guide instruction so that students can learn in productive ways. This guidance requires that teachers make choices about what content to teach, how to teach it, what routines will guide classroom activity, and so on. Schoenfeld and the Teacher Model Group at the University of California at Berkley sought to explore how and why teachers make the choices that they do in the classroom (e.g., Schoenfeld, 1999, 2011; Schoenfeld, Minstrell, & Van Zee, 2000; Zimmerlin & Nelson, 2000). Schoenfeld (2011) proposed that teachers' actions in instruction are based upon their goals, orientations, and resources.

Schoenfeld (2011) defined a *goal* to be "something that an individual wants to achieve, even if simply in the service of other goals" (p. 20). Goals for instruction may apply broadly (e.g., I want to make sure that the content is taught correctly) or specifically (e.g., I want that student to stop talking right now). *Orientations* are defined by Schoenfeld to be "dispositions, beliefs, values, tastes, and preferences" (Schoenfeld, 2011, p. 29). A teacher's orientations might include their belief that students should learn mathematics through inquiry or their preference towards regularly assessing student progress. Schoenfeld categorized *resources* as intellectual, material, and social. In teaching, material resources include the physical entities available for use, such as technology, manipulatives, or the textbook. Social resources would include the teacher's position within the classroom and school culture. For example, a veteran teacher may be able to take risks in instruction that a first year teacher cannot, so the veteran has different social resources at her disposal. Intellectual resources include knowledge. Schoenfeld (2011) defined knowledge as “the information that [one] has potentially available to bring to bear in order to solve problems, achieve goals, or perform such other tasks” (p. 25). A teacher’s knowledge may include knowledge about the mathematics he or she is teaching, knowledge about the students, or knowledge about particular strategies for teaching. The central assertion in Schoenfeld’s (2011) book was that teachers’ behavior is goal oriented, and goals are shaped by orientations and carried out by drawing on knowledge.

Although orientations and knowledge both have an impact on instruction, a great deal of research in mathematics education has focused on the knowledge that mathematics teaching requires (see Chapter 2 of this dissertation). Ball et al. (2008) argued that teachers need both knowledge of mathematics and pedagogical content knowledge (PCK) to effectively carry out instruction in the classroom. That is, teachers need to understand more than the mathematical concepts they are teaching. They also need to know ways of explaining and representing the content and have an understanding of the students they are teaching, including common difficulties that students may have with the mathematics.

To be sure, mathematics teachers cannot carry out their work effectively without well-developed knowledge. However, Schoenfeld’s (2011) framework also highlights the importance of orientations in shaping instruction. Recognizing this importance, some mathematics education research has focused on understanding how teachers’ beliefs impact instruction. As described in Chapter 2 of this dissertation, Cooney, Shealy, and Arvold (1998) proposed that not all of a teacher’s beliefs are regarded equally by the teacher. A teacher has central beliefs that underpin most of what they do as well as beliefs that are more peripheral. Exploring how a teacher's beliefs interact in instruction, Aguirre and Speer (2000) conceptualized individual beliefs as part of *belief bundles*. That is, the researchers argued that beliefs are connected to one another in ways that influence the formation of pedagogical goals.

Some research on teachers' beliefs has pointed out inconsistencies between teachers' professed beliefs and their practices. Researchers have explained these by differentiating between teachers' central beliefs and peripheral beliefs (e.g., Raymond, 1997) or by considering teachers' perspectives on practice (Skott, 2001; Sztajn, 2003; see also Philipp, 2007). Leatham (2006) argued that teachers' beliefs are not always articulated clearly, and to more fully understand teachers' beliefs, researchers must seek to understand how teachers use their beliefs in the process of making decisions in the classroom. Leatham proposed that teachers' beliefs are *sensible systems*. With this view, "teachers are seen as complex, sensible people who have reasons for the many decisions they make" (Leatham, 2006, p. 100). Rather than simply point out inconsistencies between teachers' beliefs and their practices, Leatham's approach was to understand these. Similarly, Speer (2005) contended that teachers and researchers should work to develop a shared understanding when discussing beliefs, as language does not always clearly articulate what these beliefs are. Wilson and Cooney (2002) also made a similar point: "it seems that both observing and interviewing teachers are necessary if one is interested in comprehending how teachers make sense of their worlds" (Wilson & Cooney, 2002, p. 145).

5.2.3 Purpose and Significance

Recognizing that teachers' beliefs and knowledge play important roles in the decisions that they make both prior to and during instruction, the larger study reported in this chapter sought to understand how teachers' beliefs and knowledge are used in high-quality instruction. In this chapter, I share the case of Mr. Taylor. In his interviews, Mr. Taylor expressed goals—supported by orientations and knowledge—that aligned with the richness of mathematics dimension of the MQI framework. These goals also aligned with visions of mathematics teaching advocated by mathematics educators (e.g., NCTM, 2000). In fact, Mr. Taylor spent more time in his interviews discussing goals related to mathematical richness than any other teacher in the larger study. However, Mr. Taylor's instruction was not coded as high in richness in the sense of MQI. To explore why this was the case, I draw on specific instances of instruction and Mr. Taylor's reflections on those instances, in the style of Leatham (2006) and Speer (2005).

A central consideration behind Mr. Taylor's instruction was that he believed the richness in his instruction was appropriate for the students in the track he was teaching. This finding contributes to the literature on the knowledge and beliefs that affect MQI (cf. Hill, Blunk, et al., 2008; Sleep & Eskelson, 2012). Specifically, this chapter illustrates how teachers' beliefs about students' abilities can shape the mathematics offered to students.

5.3 Methods

The teacher of focus in this chapter, Mr. Taylor, was a participant in a larger study of 12 high school (9^{th}- through 12^{th}-grade) teachers in the New Jersey area. I will describe the methods in terms of Mr. Taylor. The data collection for the larger study was carried out in the same way and is also described in Chapter 4 of this dissertation; many of the descriptions in this section are adapted from Chapter 4.

5.3.1 Data Collection

Three types of data were collected for this study: classroom observations, individual interviews, and written teaching materials. For classification purposes, I also collected a background questionnaire.

Background questionnaire. Mr. Taylor completed the written background questionnaire, which helped me to understand his professional experiences. During the final interview, Mr. Taylor was given the opportunity to elaborate on his responses to this questionnaire. The full questionnaire is provided in Appendix C.

Written teaching materials. I collected copies of the presentation slides that Mr. Taylor created for class, and I also made copies of textbook pages that were relevant to the lessons being taught. These materials were used as reference during the prelesson interviews and as data to more fully understand Mr. Taylor's decisions around the mathematical topics.

Classroom observations. Mr. Taylor was observed and video-recorded in one precalculus class for three consecutive days. Each of his lessons lasted approximately 40 minutes. During each observation, I sat in the back of the classroom and video-recorded the lesson from a tripod. The camera was focused on the teacher for the duration of the class. Although the focus was on the teacher, student questions and comments made during whole-class discussion were generally audible, whereas student comments and questions made during group work were not audible. I took detailed written notes during the observations, with attention to Mr. Taylor's mathematical choices.

At the end of each lesson, Mr. Taylor was invited to give his overall reaction to the lesson and specify classroom events or moments that he wanted to discuss in the stimulated-recall interview.

Prelesson interviews. I conducted three prelesson interviews, each lasting between 20 and 30 minutes. Each prelesson interview was conducted during the class period before a lesson observation. During this interview, Mr. Taylor was asked to describe the lesson he was about to teach and to discuss any anticipated student questions, confusion, ideas, or reactions. A full interview protocol is given in Appendix D. I sent the interview questions by email prior to our interview, and Mr. Taylor sent written responses prior to our scheduled meeting. This allowed me to form follow-up questions in advance and reduced the time needed to conduct the prelesson interviews. All interviews were audio-recorded, and I took detailed written notes on our discussions.

Stimulated-recall interview. After the observations were complete, Mr. Taylor participated in one video-based, stimulated-recall (SR) interview. This interview occurred six days after the last observation and lasted approximately two hours. An overview of sequence of interviews and observations is provided in Figure 5.1.

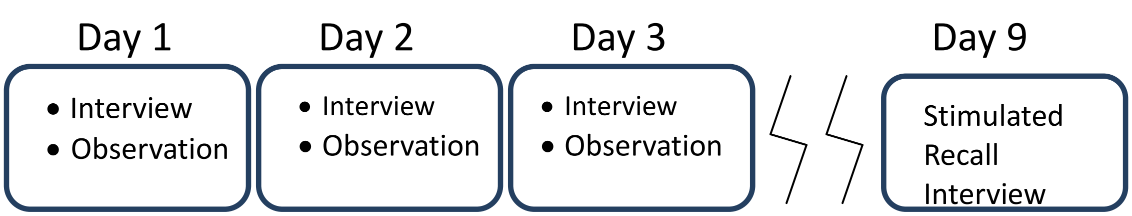


Figure 5.1 Overview of Mr. Taylor's observation and interview data collection. This figure is adapted from Chapter 4 in this dissertation.

This interview was especially important for further understanding Mr. Taylor's goals, orientations, and knowledge: As other researchers have recommended (e.g., Philipp et al., 2007; Speer, 2005) I could explore Mr. Taylor's conceptions, particularly his beliefs, as linked to his classroom actions. In addition, the SR method offered the opportunity for Mr. Taylor to reflect on his in-the-moment thinking. As Ethell and McMeniman (2000) explained, "Video recordings of the classroom practice and related stimulated-recall interviews [can allow] the expert teacher to reflect on the thinking underlying his classroom practice to make explicit the typically tacit cognitive and metacognitive processes that guide his teaching practice" (p. 90).

Video segments. As part of the interview, Mr. Taylor watched six video segments of his teaching, each approximately five minutes in length. Because the SR interview was limited to two hours, choices had to be made about which video segments to discuss (similar to Dunkin, Welch, Merritt, Phillips, & Craven, 1998). The topics in the segments that Mr. Taylor watched included the following:

1. A derivation of the equation for an ellipse centered at (0, 0) on the coordinate plane.
2. A discussion about the location of the foci of an ellipse in a real-world example.
3. A discussion about eccentricity in ellipses.
4. A discussion about the translation of ellipses in the coordinate plane.
5. Practice with drawing diagrams of ellipses given the equations.
6. An instance where students pressed for meaning behind a calculation they were performing.

Among the six segments, all dimensions of MQI were represented at least at the mid level. In addition, each of these segments included both elements of mathematical richness and some imprecision and lack of clarity. This provided Mr. Taylor the opportunity to explain his thinking behind both these characteristics of his instruction. These segments also spanned the three days of instruction. Mr. Taylor requested to discuss the final segment in the above list, and I also flagged this segment as an important one to discuss.

Interview protocol. The SR interview protocol is provided in Appendix E. Following interview procedures recommended by Kvale and Brinkman (2009) and Seidman (2006), the interview was semistructured and proceeded as follows. I first asked Mr. Taylor introductory questions, such as his opinions on the overall quality and success of the lessons. Following these questions, Mr. Taylor and I watched the video segments that were chosen. After we watched each segment, I asked questions that focused on understanding what Mr. Taylor was thinking as he made decisions during the segment. After all video clips were viewed, Mr. Taylor was asked to give concluding thoughts about the sequence of lessons. As in other portions of the study, the interview was audio-recorded, and I took detailed written notes on our discussions.

5.3.2 Data Analysis

Three main efforts guided the data analysis. First, I used the MQI rubric to code lessons. Second, I used qualitative analysis of Mr. Taylor's interviews to capture the orientations, goals, and resources that he expressed. Third, I explored the relationships between Mr. Taylor's thinking and his MQI scores. The procedure for data analysis and the relationships between components are modeled in Figure 5.2.

<u>Procedure</u>:

1. Code lessons according to the MQI framework
2. Understand teachers' goals, orientations, and knowledge using grounded analysis of interviews
3. Explore relationships between teacher thinking and MQI scores

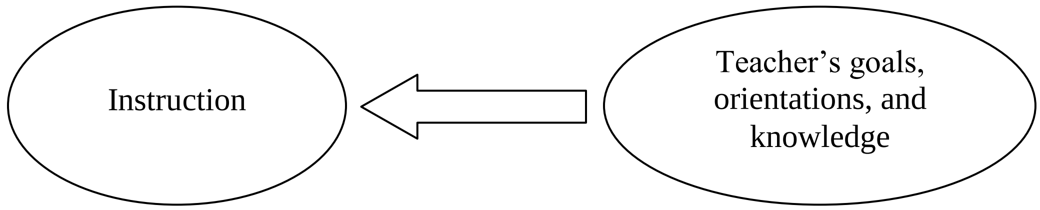


Figure 5.2 Overview of data analysis procedure and model of relationships between components. This figure is adapted from one in Chapter 4 of this dissertation.

Mathematical quality of instruction. To use the MQI rubric, I completed the MQI certification course to become a certified MQI rater. I then coded each lesson in its entirety for MQI, following the procedure outlined in the MQI training (NCTE, 2012). To do so, I began by fully transcribing all videos. Then, each video was broken into segments that were approximately five to seven minutes in length. Each of these segments focused on a single mathematical idea or instructional sequence, and the segments that were used in the SR interviews were kept intact for this portion of the analysis. Finally, I rewatched each segment and then coded the segment along the five dimensions (and 13 subdimensions) of MQI. According to the MQI rubric (LMT, 2010), the dimension of Classroom Work is Connected to Mathematics was given a score of 1 (*yes*) or 0 (*no*), and each of the remaining four dimensions (13 subdimensions) was given a score of 1 (*low*), 2 (*mid*), or 3 (*high*).

Upon completion of this process, Mr. Taylor had 22 lesson segments coded for MQI. To consider MQI holistically, I looked at the percentage of Mr. Taylor's segments that scored high, mid, and low in each of the dimensions of MQI. For example, for the richness of the mathematics, Mr. Taylor had 10 of 22 segments, or 45%, that were coded as mid in richness. This process was completed for all teachers in the larger study, and these percentages for all teachers are provided in Table 5.1. A score of high is desirable in every dimension except error and imprecision, in which a score of low is desirable.

Table 5.1 *Percentage of Segments Scored High, Mid, and Low According to Teacher and Dimension*

	Classroom work is connected to Mathematics		Richness of the Mathematics			Working with Students and Mathematics			Student Participation in Meaning-Making and Reasoning			*Error and Imprecision*		
	Yes	No	High	Mid	Low	High	Mid	Low	High	Mid	Low	*High*	*Mid*	*Low*
Ms. Zimmerman*	87	13	22	35	43	9	52	39	17	48	35	*4*	*74*	*22*
Mr. Anderson*	100	0	21	68	11	8	38	54	4	46	50	*0*	*67*	*33*
Teacher A	96	4	17	43	40	0	35	65	0	39	61	*4*	*70*	*26*
Teacher B*	100	0	10	33	57	10	43	47	0	19	81	*10*	*76*	*14*
Teacher C*	96	4	7	35	58	9	39	52	4	33	63	*7*	*52*	*41*
Teacher D	95	5	0	80	20	14	62	24	0	71	29	*0*	*86*	*14*
Teacher E	100	0	0	59	41	33	41	26	11	44	45	*15*	*78*	*7*
Teacher F	86	14	0	55	45	0	32	68	0	32	68	*5*	*95*	*0*
Mr. Taylor*	100	0	0	45	55	9	32	59	5	55	40	*14*	*82*	*4*
Teacher G	96	4	0	35	65	4	43	53	0	65	35	*4*	*70*	*26*
Teacher H	86	14	0	33	67	0	29	71	0	19	81	*0*	*71*	*29*
Teacher I	87	13	0	26	74	0	22	78	0	9	91	*22*	*74*	*4*

Note. Teachers are ranked by the percentage of segments scored high (then mid) in richness of the mathematics. A score of *yes/ high* is desirable in all dimensions except for error and imprecision, in which a score of *low* is desirable; to highlight this difference, the error and imprecision column is italicized. An asterisk (*) denotes the teacher had been previously recognized for exemplary instruction. Pseudonyms are used for teachers mentioned in this chapter. Other teachers are denoted by letter to maintain confidentiality.

Goals, orientations, and knowledge. To understand Mr. Taylor's goals, orientations, and knowledge for instruction, I used a constructivist approach to grounded theory (Charmaz, 2002). My goal was to "explain the teacher's perspective from the researcher perspective" (Simon & Tzur, 1999, p. 254). In other words, in the style of Simon and Tzur (1999), I wanted to understand how Mr. Taylor described his thinking, but I then took a researcher's lens to analyze those perspectives and make links to existing mathematics education research.

Content of Mr. Taylor's discussions. Analysis of Mr. Taylor's thinking began concurrent with data collection. After each interview, I took detailed written notes about Mr. Taylor's discussion of his lesson. Once data collection was complete, all interviews were fully transcribed. I began by reading through all interviews to get a sense of the three lessons from Mr. Taylor's point of view and wrote reflective memos throughout this process. After reading through the transcripts from all teachers' interviews, I coded interview passages (single units of meaning) inductively according to the content of what was discussed. Codes for this portion of analysis included tags such as *explanations* and *formative assessment.*

These codes were then categorized according to the dimensions of MQI with which they aligned to explore initial trends. For instance, this categorization allowed me to explore whether and how teachers' attention to mathematical meaning and sense making supported instruction that was mathematically rich. To make these categorizations, I referred to the descriptions of the dimensions of MQI (LMT, 2010; see also Appendix F). Codes and descriptions that aligned with mathematical richness are provided in Table 5.2.

Table 5.2 *Content Codes Aligning with Elements of Richness of Mathematics*

Elements of Richness of Mathematics in MQI (LMT, 2010)	Content codes related to element of MQI	Description of code
Linking and Connections Instruction makes explicit connections among representations, among mathematical ideas, across representations and mathematical ideas	Compare mathematical ideas	Make comparisons between two or more mathematical ideas e.g., exponential equation v. linear, polar coordinates v. rectangular
	Connect mathematical ideas	Make connections between ideas -The relationships between new ideas -Relationships between different domains of math -Connections in the development of new ideas -Relationships across all of mathematics (e.g., rational number, functions) Does not include connections that do not enhance the meaning (e.g., apply skills)
	Manipulatives	Use manipulative and hands-on resources for students to enhance the meaning of math. Must discuss some type of meaning.
	Representation	Emphasize representations -Connecting a table to a graph -Connecting a graph to an equation -Representing a concept in different ways
Explanations Instruction gives mathematical meaning to ideas or procedures, meaning of steps, solution methods, including why facts are true, why procedures work,	Concepts	Emphasize concepts. Usually discussed as compared to memorization or procedures.
	Definitions	Bring meaning to concepts in terms of their underlying definitions. Teachers want to make sure students understand how concepts are defined.

what a solution means in the context of the problem	Derivations	Emphasize where ideas come from, how formulas and concepts come about in terms of previous ideas.
	Justifications	Emphasize the reasons why things are so. Does not count when teachers say they want to "explain" but mean to give detailed steps for a procedure.
	Meaning	Emphasize what equations and variables mean, what concepts mean, etc.
	Contextualized problems	Apply math in the real world. This is not the same as emphasizing the history of a topic or relating vocabulary to everyday terms.
<u>Multiple procedures or solution methods</u> Instruction illustrates multiple methods for a single problem, including shortcuts, or multiple procedures for a problem type	Multiple approaches	Show multiple ways to approach a problem
	Shortcuts	Illustrate shortcuts for the problems
<u>Developing mathematical generalizations</u> Instruction develops mathematical generalizations by building up from examples	Generalizations	Use specific examples and make generalizations from those
<u>Mathematical language</u> Instruction characterized by fluent use of mathematical language	Language	Attend to language, either personally or for the students.

The purpose of this classification was exploratory, and these codes were high-level codes in that they did not capture nuances within or relationships among broad concepts. Nonetheless, in this process, Mr. Taylor's case stood out. Among the 12 teachers, Mr. Taylor had the highest frequency of both (a) interview passages that aligned with the richness of the mathematics and (b) richness codes represented in the interviews. Table 5.3 provides the counts of such codes for the 12 teachers in the larger study.

Despite the extensive focus on mathematical meaning and sense making in his interviews, Mr. Taylor's instruction was not coded as high in richness of the mathematics for any segment, and only (45%) of Mr. Taylor's segments were coded as mid in richness. This was surprising.

Table 5.3 *Counts for Interview Passages Aligned with Mathematical Richness*

	Teacher												
Code	Taylor	C	Anderson	A	F	D	Zimmerman	G	B	E	H	I	TOTAL
Context	40	14	6	20	1	2	2	10	5	0	0	0	100
Connect	30	3	19	13	7	4	11	7	21	7	1	1	124
Concepts	25	7	3	0	0	0	2	0	0	9	0	0	46
Meaning	11	25	0	9	12	15	4	10	2	0	1	0	89
Explanation	7	5	8	6	0	1	1	1	2	3	6	1	41
Definition	6	11	4	2	1	0	6	0	0	0	0	0	30
Representation	6	6	5	11	17	1	0	0	0	1	1	0	48
Compare	5	4	0	2	0	0	1	4	2	1	1	0	20
Derivation	4	0	32	0	0	3	5	2	0	0	0	0	46
Manipulatives	2	6	0	3	17	11	0	0	0	0	0	0	39
Language	1	15	0	1	8	1	3	0	0	0	0	0	29
Mult. Aprch.	0	3	0	4	5	6	0	0	0	0	0	0	18
Generalization	0	0	0	1	0	0	9	0	0	0	0	0	10
Shortcuts	0	0	0	0	0	0	0	2	2	3	2	0	9
TOTAL	**137**	**99**	**77**	**72**	**68**	**44**	**44**	**36**	**34**	**24**	**12**	**2**	

How content was discussed. To further capture Mr. Taylor's thinking, I explored how the content of Mr. Taylor's discussions was expressed: as orientations, knowledge, or goals. I looked across all interviews for this evidence, as follows.

I first identified goals, orientations, and knowledge from explicit statements. For instance, teachers would often express their goals by saying, "My goal today is …," "My goal in that segment was …," or "I really wanted to …" Orientations were often expressed with statements such as "I think it's really important that …," or "My philosophy on teaching is …" Teachers expressed knowledge through statements such as, "I know that students usually …," "That student was saying ….," "This concept connects to another one that we did earlier this year," and so on. However, teachers also expressed goals, orientations, and knowledge in less direct language. Hence, after coding for these explicit statements, I revisited interviews and asked guiding questions that helped me identify goals, orientations, and knowledge.

Goals could be identified by answering the question, *What was the teacher trying to do*? Knowledge was identified by answering the question, *What information does the teacher recognize in this situation?,* and orientations were identified by answering the questions, *What is the teacher's view of this situation?* and *What matters to this teacher?*

In looking across the coded data, I noticed that the goals that teachers discussed were not all of the same type. Some goals were overarching across the three lessons, whereas others were specific to a certain situation. Hence, goals were split into two categories to differentiate overarching goals for instruction from specific goals. In addition, knowledge was separated into two categories: knowledge considered when forming goals for instruction and knowledge considered when carrying out goals for instruction (the second type was discussed when reflecting on specific segments in the SR interview). In addition, teachers occasionally discussed material or social resources, so a code was used that separated these from knowledge. The coding scheme for goals, orientations, and resources is summarized in Table 5.4.

In practice, teachers' goals are expressions of their orientations (Cobb, 1986), so it can be difficult to distinguish goals from orientations in interviews. Hannah, Stewart, and Thomas (2011) reported this observation in their analysis of a college professor's orientations and goals: "In performing the analysis, it was often difficult to separate an orientation from a goal, since one usually wants to attain what one values or sees as important" (p. 977). Törner, Rolka, Rösken, & Sriraman (2010) reported a similar finding: "The comments of the teacher in the interview show clearly that goals and beliefs can hardly be separated. That is, when reading through the interview one can identify statements, which can be regarded either as goals or as beliefs" (p. 409).

Recognizing this difficulty, I identified a goal when there was evidence that it was the teacher's intention in instruction, and I identified an orientation when there was evidence that it was the teacher's belief or preference. This means that some goals presented in this chapter are also the teacher's orientations and vice versa. Rather than worry about this overlap, I recognized that a network of teachers' orientations (sometimes goals) supported their overarching goals (sometimes orientations) for instruction. I found the network to be more central to analysis than distinguishing between orientations or goals.

Table 5.4 *Goals, Orientations, and Resources Expressed in Interviews*

Type of thought	**Phrases signaling thought**	**Guiding question**
Goal: What the individual wants to achieve in instruction. May be coded as *overarching* or *specific*.	"My intention here was to …" "I wanted students to …" "I was trying to emphasize …"	What was the teacher trying to do?
Orientation: An individual's dispositions, beliefs, values, tastes, and preferences for instruction.	"It's important that …" "I believe that …" "I value …"	What is this teacher's view of this situation? What matters to this teacher?
Knowledge: Information that the individual brings to bear in instruction. Includes knowledge about mathematics, knowledge of students, knowledge of curriculum, etc. Also includes in-the-moment knowledge of understanding a student. May be coded as considered to *form* goals or considered to *carry out* goals.	"I was using knowledge of …" "I know students usually …" "I was hearing the student say …" "In mathematics, this concept …" "My experience has been that …" "There is a connection to this concept …" "This student understands …"	What information does the teacher recognize in this situation?
Resource: Tool (other than knowledge) that the individual brings to instruction. May be coded as *material* or *social*.	"I am going to use the Smartboard to …" "I have the luxury of complete freedom in my classroom."	What resources, external to the teacher, did the teacher discuss or capitalize on?

To further articulate this network for Mr. Taylor, I listed the overarching goals that he had for instruction, and then I made links to the supporting orientations and knowledge for these overarching goals, as evidenced by his interviews. These networks are presented in the findings and supported by additional evidence in Appendix G.

Relationships between goal networks and instruction. With an understanding of Mr. Taylor's overarching goals and the supporting orientations and knowledge, the next focus in analysis became understanding how and why Mr. Taylor's goals did not lead to high levels of richness in mathematics instruction. To do so, I used an explanation building approach (Yin, 2009), focusing on episodes of instruction. First, I revisited Mr. Taylor's videos to identify episodes that had the potential for rich mathematics that was not realized. These included episodes in which Mr. Taylor made an attempt to make a connection, offer an explanation, or emphasize meaning, but these were either incomplete or imprecise. These also included episodes where Mr. Taylor was implementing a task or activity that he had discussed in a prelesson interview as one that he intended to be mathematically meaningful. Episodes such as those where students were working silently were not considered because silent work could not illustrate evidence of rich mathematics with the methods employed in this study.

Second, I revisited Mr. Taylor's interviews to explore whether his thinking could help to explain why richness was limited. Many of the episodes that were identified were used in the SR interview, and data from the SR interview provided the opportunity to understand Mr. Taylor's thinking behind specific instructional moves. For all episodes that were identified, including those that were not used in the SR interviews, I searched the prelesson and SR interviews for discussions relating to the content of the episodes. To characterize Mr. Taylor's thinking, I coded these interview passages for themes of meaning and formed propositions about why Mr. Taylor's instruction did not achieve high levels of richness. As described by Yin (2009), I used an iterative approach to refine my propositions, revisiting the data and interrogating it for rival explanations. After several iterations, the result was three themes that can account for Mr. Taylor's instruction, and these are presented in the Findings section.

5.4 Background and Setting for the Case

The example in this chapter is an intriguing case because of Mr. Taylor's excellent qualifications and goals that aligned with mathematics educators' visions for mathematics in instruction (e.g., NCTM, 2000). Mr. Taylor was an experienced teacher who had previously received many honors and held several leadership roles. He had nearly 20 years of teaching experience, and he had taught every course in high school mathematics. He also helped to write mathematics curriculum, both for his school and for a large national initiative. In addition, Mr. Taylor regularly facilitated professional development for other teachers and had served as a mentor to several student-teachers.

Mr. Taylor also was also a National Board Certified Teacher in Adolescent and Young Adulthood Mathematics (NBPTS, 2014). This is a noteworthy achievement for teachers in the United States; to become National Board certified, teachers must demonstrate several aspects of excellent teaching, including a commitment to students, knowledge of mathematics and how to teach mathematics, and a reflective nature about their practice. To do so, teachers complete several mathematics exams and submit video-recordings, supporting documents, and written reflections of their teaching. As such, by the standards of National Board Certification, Mr. Taylor was considered to be an excellent teacher.

I observed Mr. Taylor near the end of the school year when he was teaching ellipses as conic sections, and he told me that it had been about 10 years since he had taught this topic. Mr. Taylor taught at a public school that was high performing in the state, according to student standardized test scores, and most students attending the school were from upper-middle-class families. The class was comprised of mostly 11th grade students, and the course was a college-preparatory track, also referred to as an academic track. This track was designed to prepare students to attend university after high school, and the academic level was considered one track below the honors level, as the honors level was intended for students who planned to study mathematics or science in their careers.

5.5 Findings

Below I explore why Mr. Taylor's instruction was limited in mathematical richness despite his goals that prioritized meaning and sense making. The findings are divided into two sections. First, I share Mr. Taylor's overarching goals for instruction and the orientations and knowledge that supported these goals. Second, I draw on specific examples from instruction and data from interviews to (a) illustrate the characteristics of Mr. Taylor's instruction that were limited in mathematical richness in the sense of MQI and (b) propose reasons why Mr. Taylor's goals did not lead to high mathematical richness.

5.5.1 Mr. Taylor's Overarching Goals

To more fully understand Mr. Taylor's case, it is important to first understand Mr. Taylor's intentions for instruction. As illustrated in Table 5.3 in the Data Analysis section, Mr. Taylor's interview passages aligned with mathematical richness more than any other teacher in the larger study. In his interviews, Mr. Taylor shared his overarching goals for instruction, expressed the orientations that gave rise to these goals, and articulated knowledge that supported his goals. The aspects of meaning and sense making that he discussed are aspects of mathematics teaching advocated by mathematics educators (e.g., NCTM, 2000) and align with the richness of the mathematics described in the MQI rubric. These goals are summarized in Table 5.5.

Table 5.5 *Mr. Taylor's Overarching Goals and Supporting Orientations and Knowledge*

Goal 1: **Emphasize concepts (over procedures)**

Orientations:

- Concepts are more critical than procedures when doing mathematics.
- Students should not rely on modeled procedures without understanding.
- Students should consider definitions and concepts when they try to solve problems.
- There are multiple ways that students may solve math problems.
- Students should be able to reason about concepts when they are solving problems.

Knowledge:

- Definitions of concepts: Ellipse, eccentricity
- How the textbook presents material and the ideas and processes that are emphasized in the text
- Students' orientations towards mathematics: Apply procedures to provide correct answers

Goal 2: **Teach concepts through real-world examples**

Orientations:

- Math is a way of explaining the universe.
- Real-world examples should be used to introduce new concepts.
- Students remember mathematics better with real-world examples.
- Real-world examples should not be oversimplified for students.
- Students should solve real mathematical problems that do not have clear solution paths.
- Real-world examples are a meaningful way to apply mathematical facts and procedures.
- Real-world examples are motivating for students.

Knowledge:

- How the textbook present mathematical concepts and procedures
- How mathematics is used in the real world: Whispering chambers, elliptical orbits

Goal 3: **Make connections among mathematical ideas**

Orientations:

- The teacher's role is to make connections to students' previous content and mathematics they will see in the future.

Knowledge:

- Knowledge of previous content and future content
- Connections between mathematical ideas: Ellipse and circle definitions, ellipse and circle equations, transformations of graphs in the coordinate plane
- Representations that illuminate connections between ideas

Goal 4: **Illustrate why mathematical facts are true**

Orientations:

- Students should understand why mathematical facts are true.
- If students understand where mathematical facts come from, it will be easier for them to remember those facts.

Knowledge:

- How to derive mathematical facts: The standard form of the equation for an ellipse
- Student strengths and difficulties with the content
- Students' orientations towards mathematics: New facts appear "magically"

In this section, I will provide some brief quotes from Mr. Taylor's interviews that illustrate his goals and the associated orientations and knowledge. Additional evidence for these goals, orientations, and knowledge can be found in Appendix G.

Emphasize concepts over mathematical procedures. A central goal for Mr. Taylor was to emphasize concepts over mathematical procedures, as he explained:

> *Mr. T*[11]*.:* Down the road, say two years from now, whatever, they're going to forget how to graph [an ellipse] from the standard form. They're going to. But they're not going to forget what an ellipse is. They're going to remember that it has something to do with the foci. In fact, that's the key piece. So when you talk about the big understanding, the deep understanding, it's not the ability to start with a general form of an equation and graph it from that. Because they can always plot points, they can always throw it into some type of graphing calculator to actually use an equation, an algebra equation. But, that's never going to tell them what the foci mean. And if they can pull away from that, … [and get] the real concept of what an ellipse is and how it works and the relationship between the ellipse and the foci, … [those] are the big understandings.

In this excerpt, Mr. Taylor reinforced that concepts, not procedures, are critical to doing mathematics. His goal was for students to understand "the real concept of what an ellipse is and how it works" rather than remember a procedure for graphing an ellipse from the equation.

In addition, Mr. Taylor expressed awareness of his students' orientations towards mathematics. In particular, he was aware that students often want to apply procedures to obtain correct answers rather than understand concepts, and in part, this further motivated him to emphasize concepts:

> *Mr. T.:* I think that's one of the transitions we need to start making in math, as we move towards the future, we have to get kids to stop looking at problems as the ends and think more along the lines of the concepts and the definitions and the theorems are the ends.
>
> *Mr. T.:* Because I know how they do math. And this is a fight that I have with students. … They have gotten to the point where they just want to know "How do I solve the problems in the book? What are the steps?" So in the back of their minds, they have an agenda. It's counter to my agenda. My agenda is to get them to understand the concept. Their agenda is to solve the problems on the homework, to figure out what problems I'm going to ask on the test, and to figure

[11] Quotations in this chapter are lightly edited to increase readability. Ellipses are used in place of omitted words, and brackets are used around words that were added to clarify meaning or replace identifying information. When editing, I paid careful attention to maintain meaning.

> out exactly how to answer those questions. And they're looking for that. They're looking for the bottom line so to speak, without all of the concept piece.

Mr. Taylor recognized that his orientations about mathematics ran counter to most of his students' orientations, and he wanted to change students' views of mathematics. In particular, Mr. Taylor expressed his belief that students should not view "problems"—meaning procedural exercises from the textbook—as the purpose of mathematics; instead, he believed that students should view the understanding of concepts, definitions, and theorems as the purpose.

Teach concepts through real-world examples. A second overarching goal for Mr. Taylor was to teach mathematics through real-world examples. This goal was tied to his belief about mathematics; namely that mathematics explains the universe:

> *Mr. T.:* One of the thoughts I have in general about math is that, it goes back to Galileo, where we use math to explain our universe. So the universe existed first, and math is here to explain it. We don't apply the universe to math. We apply math to the universe. So you can't really talk about a real-world example backwards, this doesn't make any sense. So it makes more sense to talk about, this is the real-world example. Let's see what math we can use to explain it. So then at that point, it gives the real connection to this is, this is real world.

In the discussion above, Mr. Taylor also indicated an orientation about teaching mathematics: Problems from real-world situations should be provided when new mathematical topics are introduced rather than after they are introduced.

Mr. Taylor went on to explain that he believed that students should solve real mathematical problems that do not have clear solution paths, and real-world examples gave the opportunity for students to do so:

> *Mr. T.:* You can boil this all down to a series of steps and formulas, [and the students] are not really learning anything. They can spit back the answers, but if I ask them the question in a slightly different way, they won't be able to answer it. They'll be done. So having this real-world application where it's not so cut and dry, … you're really talking about solving a real problem. And you're approaching math completely differently than you would just to solve a textbook algebra

> problem. You're solving math by taking the knowledge that you know and trying to explain the world in front of you.

Mr. Taylor recognized that another approach to teaching ellipses would be to emphasize "steps and formulas," but he explained that he chose to include a real-world example because he believed that it was important for students to think flexibly.

In addition, Mr. Taylor believed that real-world examples were important so that students could understand the applications of mathematical facts and procedures.

> *Mr. T.:* [Students] are not just following an algebraic process and pulling out numbers and giving me answers based on some algebraic process. So instead of just being a collection of procedures and steps, it's a thing. It's a real life thing that they can connect to something that's in the real world.

For Mr. Taylor, it was important that students are able to connect the mathematics they are learning to mathematical phenomena in the real world.

Make connections among mathematical ideas. A third goal for Mr. Taylor was emphasizing connections among mathematical ideas, including ideas that were not a part of the course he was teaching. This goal was driven by Mr. Taylor's belief that these connections were an important part of teaching mathematics, as he explained:

> *Mr. T.:* If you don't really understand [the content] inside and out and see all the connections to where they're going and where they've been, then you're not really doing [very] much better than the textbook is. And they could just read it from the book. Your job as a teacher is to help them see the content but also make connections to things that they've seen in the past and allude to things that they're going to see in the future and position them in a trajectory so they can hit that.

Mr. Taylor believed that an important part of his "job" was to help students see connections between mathematical ideas and help "position" students along a trajectory so they could make connections among the ideas they were learning.

Mr. Taylor also offered specific examples of connections between ideas that he planned to emphasize within the unit. Specifically, he wanted students to understand the relationships between circles and more general ellipses, and he described the connections in terms of several representations: equations, graphs, and definitions. Mr. Taylor was also thinking about hyperbolas as he was teaching ellipses, and he wanted to emphasize certain concepts, such as eccentricity, because they illustrated connections among several conic sections.

Illustrate why mathematical facts are true. A fourth goal for Mr. Taylor was to illustrate why mathematical facts are true. Mr. Taylor believed that if students understand where mathematical facts come from, it will be easier for them to remember those facts. He was also keenly aware that students did not usually have such an orientation to mathematics, as he explained:

> *Mr. T.:* A lot of times [students] look at this, they're like, "Oh, okay. So there it is, it's magic." And then they … just memorize steps as opposed to trying to understand where these things come from. If they have that anchor of where it comes from, it will be easier for them to remember.

Mr. Taylor also explained that he believed students should have "evidence" for new mathematical ideas and that they should be asking, "Why is this true?" For students to ask such questions, Mr. Taylor believed that it was important for him to model how truth is established in mathematics. Namely, he wanted students to see that new mathematical ideas should make sense and follow from previous ideas[12].

Summary and discussion. Taken together, these four overarching goals for Mr. Taylor's instruction focus on the meaning aspects of mathematical richness as defined by the MQI rubric (LMT, 2010). Mr. Taylor had goals to illustrate connections among mathematical ideas, offer explanations to students, and bring meaning to mathematical concepts. These goals also align with visions of meaningful mathematics in instruction advocated by mathematics educators (e.g., NCTM, 2000).

[12] All of these goals will be elaborated in section 5.5.2, and further examples can be found in Appendix G.

In addition, these goals are consistent with, related to, and support one another. Mr. Taylor's goal to emphasize concepts over procedures seemed to be the most central to his practice. In his interviews, Mr. Taylor explained how using real-world examples, making connections, and illustrating why mathematical facts are true were means to help him achieve his goal of emphasizing concepts. In addition, Mr. Taylor described that illustrating why mathematical facts are true helped him to make connections among mathematical ideas. Figure 5.3 illustrates these relationships, as Mr. Taylor described them.

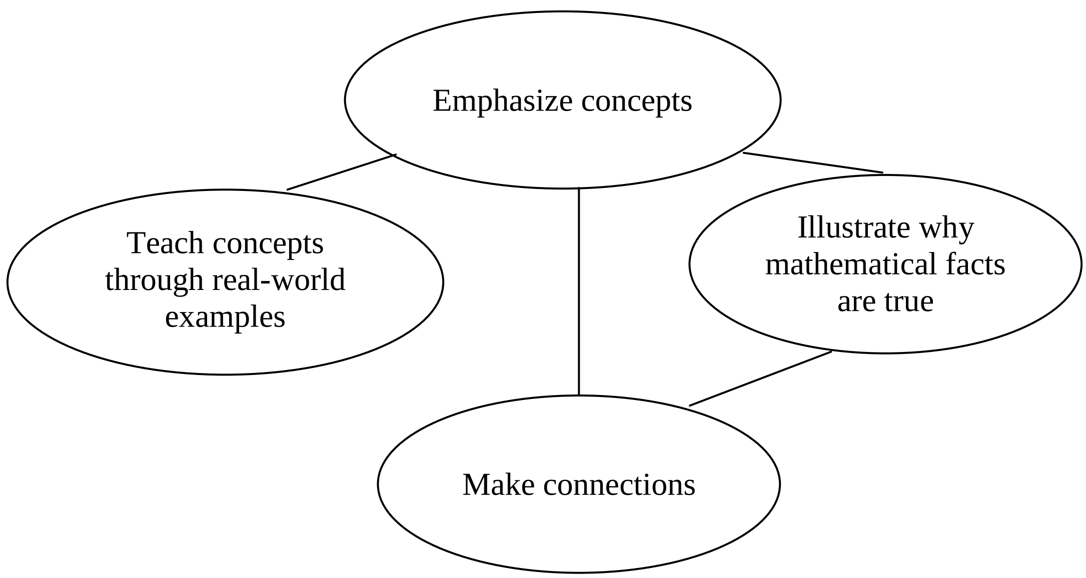


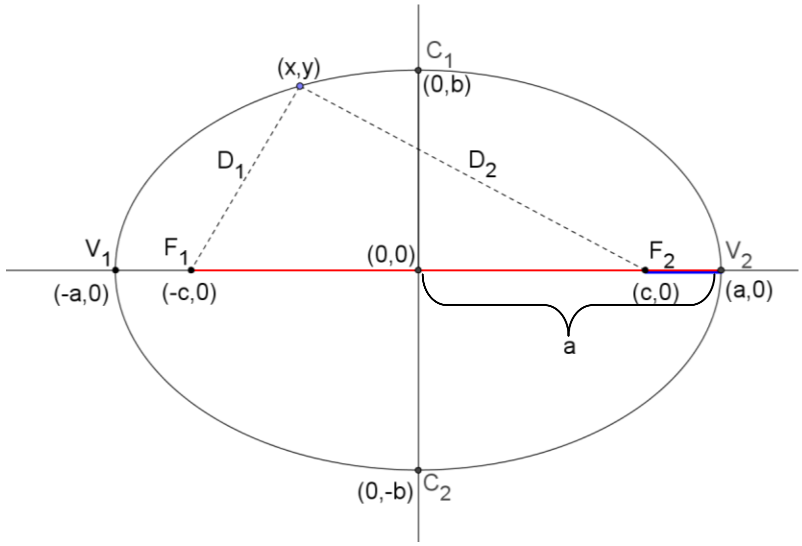
Figure 5.3 Relationships among Mr. Taylor's overarching goals. Relationships between goals are indicated by solid segments.

5.5.2 Examples from Instruction

Despite the fact that Mr. Taylor's goals aligned with meaning and sense making, Mr. Taylor's instruction did not achieve high richness in the sense of MQI. In the examples that follow, I illustrate this point, and I explore the reasons for this by drawing on Mr. Taylor's reflections on these examples. Specifically I present three plausible explanations why Mr. Taylor's instruction was limited in mathematical richness. First, Mr. Taylor's enacted goals did not include the explicit details that could have led to a high score in richness of the mathematics. That is, although Mr. Taylor valued meaning and sense making, his instruction illustrated how he preferred to provide high-level meaning for students, illustrating connections broadly or providing some general sense of why concepts were true, but his instruction was not explicit and detailed enough to be coded high in the MQI rubric. Second, Mr. Taylor did not seem to attend to precision and clarity in the ways called for by the MQI rubric, and this was true both during instruction and in the SR interview. Third, Mr. Taylor believed that the level of richness that he was providing was sufficient and appropriate for the students he was teaching.

Derivation for the equation of an ellipse. One example from instruction where mathematical richness was limited was when Mr. Taylor derived the equation of an ellipse.

Background. Figure 5.4 provides the reader with the derivation for the equation of an ellipse, although this derivation is written out more formally than the approach that Mr. Taylor had planned to take.



- An ellipse is defined to be the set of points in the plane for which the sum of the distances, D_1 and D_2, to two fixed points, F_1 and F_2, called the foci, is constant (see diagram above).
- Without loss of generality, assume that an ellipse is centered at $(0,0)$ on the coordinate plane and oriented so that the major axis lies along the *x*-axis. Then F_1 and F_2 lie on the *x*-axis at points $(-c, 0)$ and $(c, 0)$.
- The major vertices, V_1 and V_2 also lie along the x-axis at points $(a, 0)$ and $(-a, 0)$, $|a| > |c|$.
- So, (Distance from F_1 to the major vertex V_2) = $a + c$ and (Distance from F_2 to V_2) = $a – c$, as illustrated by the red and blue segments in the diagram above. Hence,
  (Distance from F_1 to the major vertex V_2) + (Distance from F_2 to V_2) = $a + c + a – c = 2a$.
- Then, by the definition of ellipse, $D_1 + D_2 = 2a$.
- Let (x, y) be any point on the ellipse. Then by the distance formula,
$$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+\ y^2}=2a$$
- This equation can be rewritten as follows:
$$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+\ y^2}$$
$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$
$$(x+c)^2-\ (x-c)^2-4a^2=-4a\sqrt{(x-c)^2+\ y^2}$$
$$x^2+2xc+c^2-x^2+2xc-c^2-4a^2=-4a\sqrt{(x-c)^2+y^2}$$
$$-xc+a^2=a\sqrt{(x-c)^2+y^2}$$
$$x^2c^2-2a^2xc+a^4=a^2((x-c)^2+y^2)$$
$$x^2c^2-2a^2xc+a^4=a^2(x^2-\ 2xc+c^2+y^2)$$
$$x^2c^2-2a^2xc+a^4=a^2x^2-2a^2xc+a^2c^2+a^2y^2$$
$$x^2c^2-x^2a^2=-a^4+a^2c^2+a^2y^2$$
$$x^2(c^2-a^2)=a^2(c^2-a^2)+a^2y^2$$
$$x^2=a^2+\frac{a^2y^2}{(c^2-\ a^2)}$$
$$x^2-\ \frac{a^2y^2}{(c^2-\ a^2)}=a^2$$
$$\frac{x^2}{a^2}-\frac{y^2}{(c^2-a^2)}=1$$

$$\frac{x^2}{a^2} - \frac{y^2}{(c^2 - a^2)} = 1$$

- The semiminor vertex is at the point (0, b), and a, b, and c are related by the Pythagorean Theorem, as illustrated in the diagram below. Hence, the equation above can be written

$$\frac{x^2}{a^2} - \frac{y^2}{(c^2 - (b^2 + c^2))} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is the standard form for the equation of an ellipse centered at (0,0) with foci on the x-axis, a the length of the semimajor axis, and b the length of the semiminor axis.

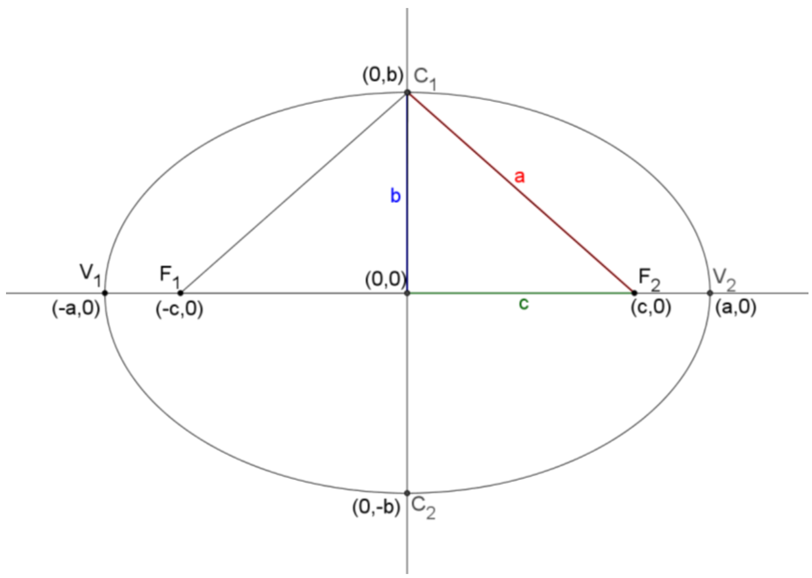


Figure 5.4 Derivation of the standard form of the equation for an ellipse

Mr. Taylor's plan for the segment was to first introduce the definition of an ellipse as the set of points in the plane for which the sum of the distances, D_1 and D_2, to two fixed points, called the foci, is constant (see Figure 5.4). He then planned to consider a particular point on the ellipse, the vertex of the major axis (V_2 in Figure 5.4), to illustrate that the sum $D_1 + D_2$ was equal to twice the length of the semimajor axis (2a where the length of the semimajor axis is a). Using this information, Mr. Taylor then considered another particular point on the ellipse, the vertex of the minor axis (C_1 in

Figure 5.4), to illustrate a relationship among the length of the semimajor axis (a), the length of the semiminor axis (b), and the distance from the center to one focus (c), which was $a = \sqrt{b^2 + c^2}$. With these three facts, the standard form of the equation for an ellipse centered at (0,0) with the major axis along the *x*-axis can be derived, as illustrated in Figure 5.4.

Showing this derivation to students followed from Mr. Taylor's goal to illustrate why mathematical ideas were true (goal 4 in Table 5.5). In the prelesson interview, Mr. Taylor explained how this choice was motivated by his belief that students should understand why mathematical facts are true, but he was also aware that the algebraic manipulation in the derivation may be difficult for students:

> *Mr. T.:* In the academic class, a lot of their algebra skills … are weak. So this algebraic manipulation, for a lot of them, will be beyond them. But seeing that you can do that is really all that I want to get out of that. That you can use algebra to take those pieces and manipulate it to get the general form.

Instruction. In his lesson, Mr. Taylor first illustrated the three assumptions that he needed to carry out the derivation. That is, he defined an ellipse, illustrated how the sum of the two distances from a point on the ellipse to the foci is equal to twice the length of the semimajor axis ($D_1 + D_2 = 2a$), and illustrated the relationship among the semimajor axis, the semiminor axis, and the distance from the center to one focus ($a = \sqrt{b^2 + c^2}$). Then, Mr. Taylor projected 20 lines of the algebraic derivation on the classroom board and explained this algebra as follows:

> *Mr. T.*: Are you ready for the fun part? Alright, so here's the fun part. …We take the distance formula sum, we take the constant two *a*, and we take the Pythagorean Theorem that we used, and we do some lovely algebra. And when we're all done simplifying it, we end up with the standard form for the equation of an ellipse. And to make your awful face go away, I'm going to tell you that you don't need to memorize this. You don't need to reproduce it, all you have to understand is that we're using the distance formula to come up with a standard form for the equation of an ellipse. This right here is the piece that you need to

> make sure is in your notes. [Highlights the standard form of an ellipse equation.] That is the standard form for an ellipse.

In this excerpt, Mr. Taylor illustrated the facts that are used in the derivation without illustrating the (algebraic) justifications for why those facts lead to the standard form of the equation for an ellipse. Students could see the algebra projected on the board and they were provided with the tools that they needed to complete the derivation (i.e., the definition of an ellipse, $D_1 + D_2 = 2a$, and $a = \sqrt{b^2 + c^2}$), but they were not given the opportunity to understand the details of the derivation in class. Because there are no explicit connections or explanations made in this derivation, high richness in the sense of MQI is not realized.

In this segment, Mr. Taylor implied that the algebraic justifications for the derivation were not as important as the facts that led to the derivation: Mr. Taylor briefly showed students the prewritten algebra and said, “you don't need to memorize this. You don't need to reproduce it. All you have to understand is that we're using the distance formula to come up with a standard form for the equation of an ellipse.”

Mr. Taylor’s reflection. During the SR interview, Mr. Taylor watched this segment and had the opportunity to reflect on it. Mr. Taylor explained that, from his perspective, he had achieved his goal:

> *Mr. T.:* I think I got them to the concept that I wanted to get them to. I went through and showed them the algebra, [but] we didn't actually do the algebra. Heaven forbid I actually did all that by hand on the board; they all would have been asleep. But I showed them that they can use the distance formula, and the definition of an ellipse to get to the standard form of an ellipse. Which was the overall goal of that, without having to do a ton of algebra that would lose them.
>
> …
>
> *Kathryn:* Is there anything you would have done differently?
>
> *Mr. T.*: I don't think so in an academic class. I think it's nice that they can see the algebra, but a lot of them don't have the algebra manipulation skills to get through

> that derivation. You'd end up spending, if you actually were to go through that step by step and do it live in front of them so to speak, you'd spend probably the better part of the class trying to derive that. For what goal? What's the goal you're trying to accomplish? And my goal … was not to get them to derive it themselves. [It] was to show them that it was derived. That it did come from somewhere, that it's not that somebody decided to make up this magical equation that works. It has a history. It has a connection to something that they already learned, and they can see that it has an important algebra derivation. Not like they need to know that. They don't need to know the derivation; they just need to know one exists. … In general, I like to show them where things came from. I guess the core idea that they should be asking is "Why? Why is this true?" And if I can't at least give them some evidence that what I'm doing is true, there's no reason they should believe me. … If I can show them where all of my steps came from, then there's good evidence for why what I'm saying is true. So I'm modeling a behavior for them. I'm modeling that the way you do math is you need to demonstrate. Not just, "Oh it's because it is." You need to demonstrate where things come from.

Mr. Taylor's reflection helped to clarify and explain his actions. In particular, Mr. Taylor believed that his goal for the derivation was met, saying, "my goal … was not to get them to derive it themselves. [It] was to show them that it was derived." Mr. Taylor's reflection indicates that he literally intended to "show" students the derivation. That is, he projected the algebraic derivation for the students to see. Although it was important to Mr. Taylor that he illustrate why mathematical facts were true, it was not important to Mr. Taylor that students themselves were able to carry out the derivation or even think through the details of the derivation during class. Mr. Taylor wanted to provide "*some* evidence" for truth and some "history" for the equation, not necessarily all of the rigorous evidence. In fact, Mr. Taylor did illustrate "where things came from," just not *how* the new ideas came to be.

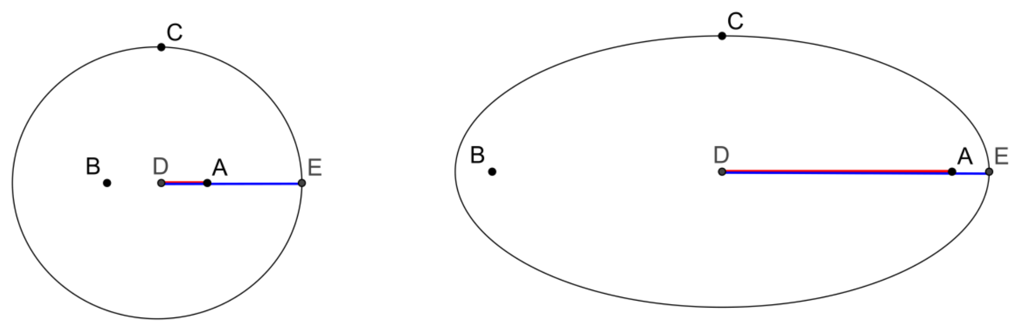
In addition, Mr. Taylor believed that it was important to keep students' attention during the lesson, as he explained, "Heaven forbid I actually did all [the algebra] by hand on the board; they all would have been asleep." In addition, Mr. Taylor's choice to omit the algebraic justifications was driven by his beliefs about the students he was teaching, especially their mathematical abilities. When asked if he would do anything differently if he could, Mr. Taylor explained that he would not because in the academic class “a lot of them don’t have the algebra manipulation skills to get through that derivation.” Ultimately, Mr. Taylor made the choice to leave out the justifications in the derivation because he believed that an overview of the big ideas was sufficient, particularly for the students in his academic-level class.

Eccentricity. Another example where Mr. Taylor began to emphasize the meaning of mathematics but did not make ideas explicit was when he introduced the concept of eccentricity to students.

Background. Eccentricity of an ellipse is denoted e and defined $e = \frac{c}{a}$, where c is the distance from the center to one focus and a is the length of the semimajor axis. This ratio is always between zero and one for an ellipse ($0 \leq e < 1$), and it is a measure of how elongated an ellipse is. That is, if the ratio is equal to zero, then the distance from the center to the focus is equal to zero, so the ellipse is a circle. If this ratio is close to one, then the distance from the center to the focus is almost the length of the semimajor axis, in which case the ellipse is elongated. This explanation is provided in Figure 5.5.

Eccentricity of an ellipse, denoted e, is defined to be $= \frac{c}{a}$, where c is the length from the center to one focus and a is the length of the semimajor axis.
Note that $0 \leq c < a$. Hence,

$$0 \leq \frac{c}{a} < 1$$
$$0 \leq e < 1$$



In the ellipse on the left, D is the center, A is one focus, and E is a vertex on the major axis. Eccentricity is defined as $\frac{DA}{DE}$, and $e \approx \frac{1}{3}$.
In the ellipse on the right, D is the center, A is one focus, and E is a vertex on the major axis. Eccentricity is defined as $\frac{DA}{DE}$, and $e \approx 0.86$.

Figure 5.5 Eccentricity definition and visual representation.

In the interviews, Mr. Taylor expressed how the topic of eccentricity helped him to meet several of his goals. Mr. Taylor planned to relate eccentricity to elliptical orbits of planets in the solar system, and this was an example of his to goal to teach concepts through real-world examples (goal 2 in Table 5.5). In addition, Mr. Taylor believed discussing eccentricity helped him to meet his goals of connecting mathematical ideas and emphasizing concepts (goals 1 and 3 in Table 5.5):

> *Mr. T.:* So the eccentricity when you're talking about an ellipse is between zero and one. The eccentricity for a circle would be zero. The eccentricity for a hyperbola would be a value that's greater than one. So it almost kind of takes an ellipse, pulls it apart, flips it out. So if you're making that, if you know that connection, as you go through, you can really talk about how that standard form relates to the other standard forms, but more importantly, what effect the foci have on the shapes and how moving the foci from being on top of each other, as a

> circle, pulling them apart to make an ellipse and pulling them further apart to make a hyperbola. How that connection works is really important.
>
> *Mr. T.:* There's two ways to talk about an ellipse. You can talk about an ellipse defined upon its eccentricity and the ratio of *c* [distance from center to focus] over *a* [length of semimajor axis]. … But you can also then talk about an ellipse as defined by a sum of distances. … Essentially, an ellipse is still, when it boils down to it, is a set of points that are related to a focal length. You can talk about that focal length by eccentricity, or you could talk about that focal length being a sum of distances.

In addition to expressing Mr. Taylor's orientations towards teaching eccentricity, these quotes illustrate Mr. Taylor's knowledge about eccentricity. Namely, Mr. Taylor understood how circles, ellipses, and hyperbolas were connected through the concept of eccentricity, both conceptually and in terms of their equations. He also recognized that defining an ellipse in terms of its eccentricity is an alternative way of approaching the study of the topic.

Instruction. When the topic of eccentricity arose in the second lesson, students had finished graphing an ellipse where the foci were relatively close to the center of the ellipse (eccentricity of 0.33). The following discussion ensued:

> *Mr. T.:* That almost looks like a what?
>
> *Student 1*: Circle.
>
> *Mr. T.*: Yeah, that almost looks like a circle. In fact, the focal points are significantly closer than when the shape was more elliptical. The focal points when the shape was more elliptical were way, way, way, closer to the edge of the ellipse. But as this is becoming more circular, those focal points are moving closer and closer to each other. What do you think would happen if I move the focal points right on top of each other? What's the shape we would get?
>
> *Student 2*: A line.
>
> *Student 3*: A circle.
>
> *Mr. T.*: That's it. Perfect circle. There's our relationship. There's our connection back to a circle. It has to do with where those focal points are. All of this is dependent upon our focal points. If the focal points are right on top of each other, we have a circle. If we pull those focal points apart, we start getting elliptical. The further we pull those points apart, the more elliptical we get. That has a special word, it's called eccentricity.

In the prelesson interview, Mr. Taylor described that his choice to discuss the topic of eccentricity was because of the connections to circles and other conic sections. Some broad connections were indeed made in this segment: Mr. Taylor discussed that eccentricity has to do with the proximity of the foci to each other, and he also explained that if the foci are "on top of each other," then a circle is formed. There is also a general sense of meaning; that is, eccentricity gives some indication of how elongated an ellipse is. However, eccentricity was not clearly defined in this segment. As such, the connections and meaning that Mr. Taylor emphasized were not justified in detail. Because of the lack of detail and explicitness, this portion was not scored as high MQI.

Following this exchange, Mr. Taylor explained how orbits of planets relate to eccentricity:

> *Mr. T.:* See, planets follow an elliptical path. And if you are on a planet where you have an eccentricity that is close to one, that's a good planet. Because what it means is your shape is more circular, so your seasons are going to be fairly regular. The sun, in our universe, is at one of the focal points. So for part of the year, you're really close to the sun, and then as you move around, you're farther away from the sun, so you get colder. So you can still have sun, but you're getting less radiation because there's less heat. Well, the more elliptical that is, the further away from the sun you go. So if your summer would be you close to the sun, you'd have a very short summer, but then as you moved away from the sun, you'd have a very long winter. Because you'd have to go way out to the edge of the ellipse and then all the way back to get to the sun. So it's actually really, really good that the shape of the ellipse for the earth is really close to a circle. The path around the sun is almost circular for us. Not quite, but almost.

To follow this discussion, Mr. Taylor presented a table that provided students with the eccentricity of the orbits for each of the planets in the solar system. The class discussed that they would rather live on planets such as Earth, for which the eccentricity of its orbit is close to zero, whereas they would not want to live on Mercury because the eccentricity of its orbit is closer to one.

The discussion of elliptical orbits provided students with an application for eccentricity, and it further emphasized the fact that eccentricity has to do with the elongation of an ellipse. There is a minor mathematical error in Mr. Taylor's description, when he said, "an eccentricity that is close to one, that's a good planet. Because what it means is your shape is more circular." Given the discussions that followed this statement, I do not think that students were confused by this minor error. Regardless, this discussion was not scored as high MQI because it did not make explicit connections between circles and ellipses nor did it provide a detailed explanation of why ellipses with greater eccentricity would be more elongated.

In addition, Mr. Taylor's explanation of winters and summers by highly-eccentric, elliptical orbits is misleading because eccentricity does not explain what causes seasons on Earth. In fact, in the Northern Hemisphere, where these students live, Earth is closest to the Sun in January, which is a winter month. Instead, seasons on Earth are caused primarily by the tilt of Earth's axis with respect to the plane that contains its (nearly circular) elliptical path around the Sun. Specifically, the angle formed between Earth's rotational axis and the plane containing its orbit is not a right angle. If it were a right angle, there would be no seasons the way humans know now. For these reasons, this segment was not scored as high in richness of the mathematics.

Mr. Taylor's reflection. In the SR interview, Mr. Taylor watched this segment and explained his teaching choices:

> *Mr. T.:* One of the important things is that I'm going back to the definition of the ellipse being central to foci. And that is the key concept there. I think that gets lost if you're just focusing on the algebra. … By really focusing on eccentricity and talking about how that changes the shape of the ellipse, that really gives them a strong idea of what is really creating the ellipse. The foci are that critical piece.

> And then tying it back to a circle where we went back to a center was exactly where I wanted it to go.
>
> *Mr. T.:* I didn't want to just introduce the ratio saying, "This is the ratio," because it's a decimal. What the heck does that mean? … [It's important that] they can actually see how that's related and talk about how that changes the shape.

By Mr. Taylor's assessment, he met his goals for making connections between mathematical ideas and emphasizing concepts over procedures. For Mr. Taylor, the connection that needed to be made was that the center of a circle is like two foci of an ellipse on top of each other. In terms of the meaning of eccentricity, Mr. Taylor believed that it was sufficient that students see *that* eccentricity changes the shape of an ellipse. Mr. Taylor further explained that he did not want to explicitly talk about how the ratio representing eccentricity is formed because he was concerned that students would not see the how the numerical value affects the visual representation. In other words, Mr. Taylor wanted students to focus broadly on concepts, not on specific details.

Mr. Taylor also reflected on the example of elliptical orbits, and he recognized that there may be inaccuracies in the description that he gave students:

> *Mr. T.:* One of the other things I was thinking when I was watching that was there's probably an astrophysicist who's someday going to see this, who's going to freak out, going, "No, don't explain it that way!" But, to me, it made sense. … So I might not have the explanation of the astrophysics in there exactly right, but I think I have enough in there that it makes sense with the application. … If they go into astrophysics down the road and they're thinking about doing really complicated math, and they're like, "Wow, he was actually wrong." … So sometimes it's oversimplified, but for the academic level, it gets the point across.

Although Mr. Taylor recognized that his explanation of elliptical orbits may be somewhat incorrect, he was not overly concerned because to him, it was most important to emphasize the general concepts rather than correct details. This orientation is also coupled with his view that the students in his class did not need extensive details, as he noted, “sometimes it’s oversimplified, but for the academic level, it gets the point across.”

A whispering chamber. To achieve his goal of using real-world examples to teach concepts (goal 2 in Table 5.5), Mr. Taylor used an example of a whispering chamber to teach the majority of the concepts that I observed. Although this example offered the opportunity to enhance the richness of the mathematics, segments focused on this example were not scored as high in richness because justifications for the mathematical phenomena were not provided to students and there was some lack of clarity in the presentation of tasks.

Background. Many whispering chambers work as follows. Consider a three-dimensional ellipsoid where exactly two of the three axes are equal in length. If a room is shaped as a semiellipsoid of this type (e.g., the height from the floor to the ceiling is equal to half of the width of the room; see Figure 5.6), a person standing at one focus of the ellipse on the floor can whisper and be heard by another person standing at the other focus several feet away. This is due to the reflection property of an ellipse: For any two-dimensional cross section of the ellipsoid cut by a plane that passes through the two foci, sound waves travelling from one focus of the ellipse will meet a point on the ellipse, reflect off that point, and travel to the other focus of the ellipse. A model of the reflection property in a two-dimensional cross section of such an ellipsoid is provided in Figure 5.7.

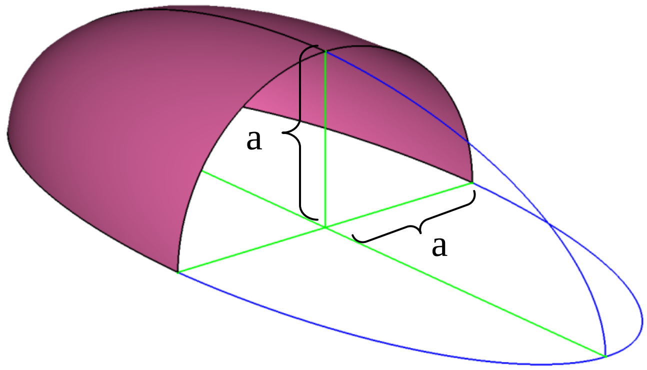


Figure 5.6 A cutaway view of a semiellipsoid. The semiellipsoid in the diagram comes from an ellipsoid in which exactly two of the axes are equal in length.

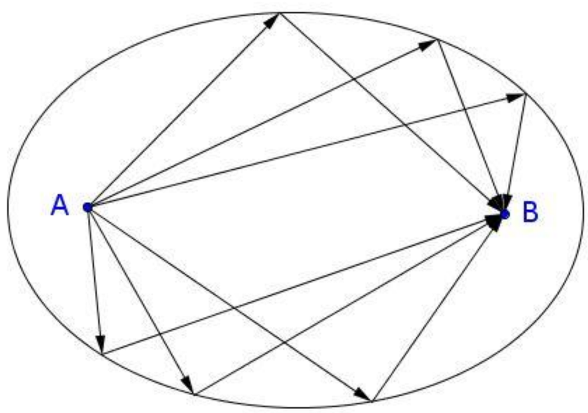


Figure 5.7 A model of the reflection property of an ellipse. This ellipse represents a two-dimensional cross section of an ellipsoid. A and B are foci of the ellipse. Sound can be modeled as a ray that reflects off of the ellipse and travels to the other focus, as illustrated in the diagram.

The textbook for Mr. Taylor's course included a problem that provided students with the length, width, and ceiling height of a particular whispering chamber located in the United States. In the textbook, the ceiling height provided was equal to half of the width of the room. That is, the room was assumed to be a semiellipsoid such as the one in Figure 5.6. In the problem, students were asked to (a) write an equation modeling the shape of the room (i.e., an equation modeling a two-dimensional cross section of the ellipsoid), (b) find the locations in room where individuals would need to stand to observe the whispering phenomenon (i.e., the foci), and (c) find the distance between the individuals who are participating in the whispering phenomenon (i.e., the distance between the foci).

Because Mr. Taylor had visited the whispering chamber in this problem, he did not believe that the textbook captured the phenomenon of that particular whispering chamber correctly. The foci for the ellipse that the textbook used were not where Mr. Taylor recalled standing in the room to observe the whispering phenomenon. When conducting internet research, Mr. Taylor found a value for the ceiling height of the whispering chamber that was more than twice the height listed in the textbook. Hence, Mr. Taylor decided to change the textbook problem because he believed that it was important to portray math in realistic contexts, as he explained:

> *Mr. T.:* [In] the textbook, … they mention the whispering chamber, but it wasn't enough. It was one problem. And they don't really talk about how the chamber is designed. They just talk about the fact that the floor is an ellipse, which is a different ellipse than the one on the ceiling.
>
> *Mr. T.:* [The textbook authors] make it almost seem like that if [students] were to do this math they could go there and stand six feet away from the wall and it would work, but it doesn't because it's not the right spot. So I kind of felt that it wasn't doing it justice.

The whispering chamber problem that Mr. Taylor used in class took the majority of the three lessons that I observed, and it was implemented as described below.

Instruction.

Introducing the problem. Prior to the lesson that I observed, Mr. Taylor introduced the whispering chamber by showing students a video of a tour guide standing at a particular location in the room. This tour guide could speak softly and be heard clearly by the tour group standing several feet away. In the first lesson that I observed, Mr. Taylor introduced the definition of an ellipse and the standard form of the equation for an ellipse. Mr. Taylor then told students that the floor of the whispering chamber was shaped as an ellipse. He provided students with the dimensions of the floor and asked them to create a diagram of the floor. Students then were asked to find the foci in the room. Mr. Taylor informed students that these points are where individuals would need to stand to take advantage of the sound effects in the whispering chamber.

Mr. Taylor's intentions with the whispering camber were to illustrate the meaning of ellipses in a real-world context, which is why students watched the video of the whispering chamber phenomenon prior to this task. However, they had not discussed the mathematical reason for this phenomenon (the reflection property of an ellipse). Hence, despite the connections to the real-world context, this exercise was not high in richness in the sense of MQI because the class did not discuss *why* the foci would be the appropriate places to stand in the room, and students were essentially applying procedures (i.e., graph an ellipse given the lengths of its major axis and minor axis, then find the foci of the ellipse).

Identifying an issue. Once the foci were found, Mr. Taylor drew students' attention to the fact that these were located within six feet of the vertices on the major axis. In the video that students had watched, the tour guide was standing more than six feet from the wall, and Mr. Taylor suggested that perhaps their calculations had not correctly captured the phenomenon:

> *Mr. T.:* We got an issue. … When you look at the room [projects a picture taken from the video that students watched] … You see where this woman is standing? That's the focal point, right? Remember, we saw the video of the tour guide

> walking and she did the demonstration where she stood. Does she look like she's only five point eight feet away from the wall?
>
> *Student*: No.
>
> *Mr. T.*: So what's going on? … Are we wrong? … We found the ellipse on the floor. What did we miss?
>
> *Student*: The ellipse on the ceiling.
>
> *Mr. T.*: Yeah. What if they're not the same? … So all we did was find the ellipse on the floor. We've got to think about how to do the ceiling. So we'll work on that tomorrow to try to find the actual focal point for the sound that deals with the ceiling in the room, okay?

In this segment, Mr. Taylor prompted students to use what they knew from the real-world scenario (i.e., what they observed in watching the video) to determine whether their calculations made sense. Upon determining that their calculations did not match what they observed in the video, the class decided to consider a different ellipse through the ceiling of the room as one that may be causing the sound effect. This segment helped students to determine the reasonableness of their solution, but it did not emphasize the meaning of ellipses as captured in MQI. In particular, the class did not discuss why the ceiling having a different shape than the floor would affect where one should stand in a whispering chamber.

In addition, the presentation of this problem is slightly misleading. Students were not the ones to make the assumption that the ellipse on the floor is what causes the sound effect in the room. Rather, students were told to make a diagram of the floor and then made aware of the fact that the foci in this diagram are not the ones that are used in the whispering phenomenon (according their visual interpretation of the video they had watched). However, because the class had not discussed a reason for why the sound effect was occurring, they did not have a basis for why one two-dimensional ellipse (floor) or another (ceiling) would appropriately capture this phenomenon.

Redirecting attention. The following day, Mr. Taylor began the class by revisiting the problem. In this episode, Mr. Taylor discussed sound reflecting off the surfaces in the whispering chamber:

> *Mr. T.:* We talked about, very briefly at the end [of class yesterday], … the floor can't be where the sound is coming from. Which makes sense. The floor really is just what they're standing on. The sound is reflecting off the ceiling. And that's how the sound is traveling to the other person. It's not reflecting off the side walls. The only way it could get the sound to go from one focal point to the other focal point in the ellipse of the floor would be if the walls are reflecting the sound. That's not what's happening.

The explanation that Mr. Taylor provides above is somewhat incorrect. Sound waves propagate in three dimensions with a spherical wave front. Therefore, when the wave encounters any elliptical surface (be it the partial ellipse formed by the walls or the partial ellipse formed by the ceiling), waves simultaneously reflect. Mr. Taylor’s explanation implies that sound does not reflect off elliptical walls, which is incorrect. Regardless, this explanation does not give an indication of why it would make more sense for sound to reflect off the ceiling as opposed to the walls, so it does not enhance meaning in the sense of MQI.

Representing the problem. In the following class, Mr. Taylor graphed the ellipse that represented the floor of the whispering chamber. He then drew a graph of the dome on the same set of axes. A reproduction of the graph is provided in Figure 5.8.

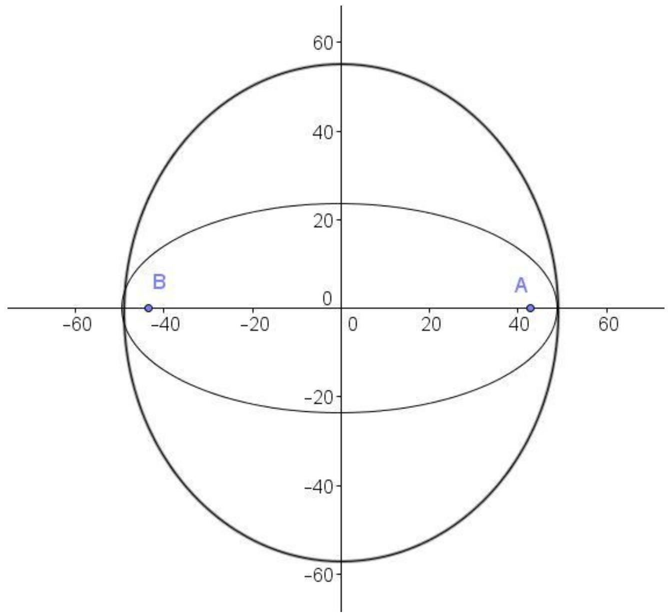


Figure 5.8 Graph of the whispering chamber floor and dome. In the diagram above, the labels on the axes represent feet.

In the discussion that followed, Mr. Taylor's intent was to help students realize that the graph of the ellipse that represented the ceiling of the chamber could not be correct because the major axis would be the vertical axis. This means the foci would not be located in places where individuals could stand. To move towards this point, Mr. Taylor posed the following question for students:

> *Mr. T.:* So that's a representation of our dome. Let me ask you the following question. What axis would the focus be on? …That's the first question. What axis, *x* or *y*, would the focus be on for our dome? …Let's have some argument.

The first student to speak suggested that the foci of the larger ellipse are the intersection points of the two ellipses (the vertices that lie on the *x*-axis). Before Mr. Taylor could respond to this incorrect suggestion, another student asked a question:

> *Student 2*: What are focal points?
>
> *Mr. T.*: Where the sound reflects, so we talk about talking in one spot. Then the sound will reflect off the ellipse and come back to the other focal point. So if we're putting the focal points right here [on the vertices], like [Student 1] suggests, I don't know if that will work, but let's argue about it and see.

Student 2: If you're saying that one point, doesn't it like not matter where you're speaking to, like it could be anywhere on the ellipse, you know what I mean?

Mr. T.: Right, the point could be anywhere on the ellipse, correct.

Student 2: So how do you

Mr. T.: But the focal points will be some spot in the room. So let's try [again].

In Mr. Taylor's response to the student, he focused on the real-world context of the whispering chamber. However, this exchange may have been confusing for students. The representation that Mr. Taylor chose to use might be confusing because he graphed two ellipses on the same coordinate plane, but these ellipses appear in perpendicular planes in the real-world context. In addition, the class had not yet established the definition of major axis or made an explicit observation that the foci always lie along the longer axis of the ellipse, so the question that Mr. Taylor posed ("which axis, x or y, would the focus be on?") was probably confusing for students.

Also in this segment, Student 2 asked what focal points are. Mr. Taylor answered in terms of the real-world example, but the language he used was misleading, as he said, "where the sound reflects." Student 2 seemed to interpret this statement literally, believing that focal points are points *on* the ellipse, where sound is reflecting. The student asked, "If you're saying that one point, doesn't it like not matter where you're speaking to?" Student 2 appears to be thinking that the Student 1's suggestion of focal points at the vertices of the major axis are reasonable, given the fact that sound may reflect off any point on the ellipse. In this exchange, Mr. Taylor did not seem to notice the lack of clarity in his explanation that led to this confusion. (Mr. Taylor also watched this episode in the SR interview, which is explained later.)

Wrapping up the problem. Upon determining that the graph of the dome did not correctly represent the situation, Mr. Taylor's instructions for students were to find a correct graph of the dome, given (a) the height of the dome and (b) the distance between individuals taking advantage of the sound effect. The problem ended with students creating a diagram of such an ellipse.

Although this final task allowed the students to practice useful calculations with ellipses, meaning was not enhanced in the sense of MQI, as this task did not help students to make connections among mathematical ideas or justify the mathematics they were applying. Moreover, this task did not build logically on previous ones. Students' first task was to determine where they needed to stand in the whispering chamber to take advantage of the sound effect. Upon realizing their calculation was not realistic, their task shifted from locating the foci in the room to creating a graph of the dome *using* the information of where individuals stand in the room for the sound effect to occur. Hence, this problem did not follow a logical line of inquiry.

In addition, the culmination of this problem implies that the ellipse that lies on the perimeter of the floor does not capture the sound phenomenon, but an ellipse through apex of the ceiling does. However, with the dimensions that the class used, the room is not a semiellipsoid in which exactly two of the axes are equal in length (e.g., Figure 5.6). Hence, the three-dimensional problem that the students face is much more complex than one that can be explained by a single ellipse, whether that ellipse lies on the floor or passes through the apex of the ceiling.

In fact, in many high school math textbooks, explanations for whispering chambers such as the one provided in Figure 5.7 are provided. These explanations are indeed simplified. However, the dimensions of rooms used in textbook problems are often (or perhaps always) ideal so that the rooms are ellipsoids where exactly two of the axes are equal in length (such as the one pictured in Figure 5.6). Once a more complicated, irregular ellipsoid is introduced, explaining the whispering chamber phenomenon through two-dimensional ellipses is not so straightforward.

Summary. The whispering chamber problem provided the opportunity for students to make connections among a graphical representation, numerical information, and a meaningful context in which ellipses explain a scientific phenomenon. However, the implementation of this problem did not achieve high levels of richness in the sense of MQI because (a) a justification for the whispering phenomenon was not provided, (b) tasks did not follow a logical trajectory, (c) there was lack of clarity in definitions and representations, and (d) the scientific phenomenon was misrepresented.

Mr. Taylor's reflection. In the interviews, Mr. Taylor reflected on the whispering chamber problem as a whole. These reflections illustrate some reasons why Mr. Taylor's instruction was limited in the richness of the mathematics.

As illustrated in other examples in this chapter, Mr. Taylor's intention for meaning did not include explicit connections and justifications. Mr. Taylor believed that using the real-world example helped students to make sense of the mathematics, as he explained:

> *Mr. T.*: In the book, they just say, "the major axis is where the focal point is." Now they're getting with the shape of the room having an effect on where the major axis is. So it makes sense that the shape of the room and where the focal point is and how sound actually works in the room brings them all back to an abstract concept that they can see concretely when they think about this room.

For Mr. Taylor, using the whispering chamber to make sense of ellipses was not about providing details for the connections between the mathematical ideas and their context or providing explicit justifications for the ideas that were being discussed. Rather, it was about visualizing abstract ideas in a concrete way. This visualization also seems to be what Mr. Taylor meant by his contention that students could “make sense” of the major axis in an ellipse through this problem. That is, he believed that visualizing foci on a major axis in terms of individuals standing in a room made an abstract concept concrete.

Mr. Taylor further explained that he chose to deviate from the textbook problem because he thought the assumptions that students make in a real-world problem should be realistic:

> *Mr. T.:* You can go through and do all the calculations correct. You can make a calculation for this ellipse, but if you are basing it off of false assumptions, then it doesn't matter whether your calculations are right or not, it doesn't make sense in the real world. So it's, the whole thing's a mess. It's all false.
>
> *Mr. T.*: As we mentioned before in the textbook, the information's wrong. So going through and actually trying to find the information for this example is really tough. But then again, it makes it a real problem. … It's not that simple. It's a real problem that needs to be solved.

In this quote, Mr. Taylor clarified that he believed it was important that real-world examples be “real” in the sense that they contain numbers that actually come from the context and capture the situation as it is reflected in reality. Although he believed that his approach highlighted the actual numbers and locations involved in the particular whispering chamber example, it is unclear whether Mr. Taylor believed that his approach captured the mathematical and scientific complexity of the chamber.

In addition to reflecting on the problem as a whole, Mr. Taylor also watched and reflected on the segment in which he introduced the graph containing both the ellipse on the perimeter of the floor and the ellipse through the apex of the dome. In reflecting on this segment, Mr. Taylor explained his choice to use a two-dimensional representation instead of a three-dimensional one. Specifically, he was concerned that his students would have perceptual difficulties with a three-dimensional representation necessarily shown in two dimensions on the classroom board: “I thought if I turn the ellipse so that it's a three-dimensional perspective, they might lose the shape. … If I had turned it and made it more three dimensional, I think it would have caused more problems than it solved." In other words, Mr. Taylor's choice of representation was due to his considerations of his students' potential difficulties, and avoiding such difficulties was more important than a more accurate representation of the mathematics.

Mr. Taylor also reflected on his response to the student's question regarding definition of foci:

> *Mr. T.:* I was thinking about [as I was watching that clip] the way we were defining an ellipse [was] focused on the foci, which is critical to what an ellipse actually is. … You can know how to do a problem and know the properties of something without ever knowing its definition. And there's a definition of an ellipse, and then there's how to manipulate an ellipse. And they're two very different things. And a lot of times, what people will miss is the definition part. Because they're only interested in the manipulation. How do you get to the answer? And it's not the answer that's important, it's the definition of what this thing actually is. And then we can use that to help us come across ways of manipulating it. But you always have to keep that definition of what you're talking about in your head.

Mr. Taylor valued the definition of an ellipse in terms of its foci, and in fact, he believed that the definition of an ellipse in terms of its foci was the most important part of what students were doing in the unit. However, in his reflection, Mr. Taylor did not discuss the imprecise way in which he described the foci as "where the sound reflects." For Mr. Taylor, visualizing foci in terms of this real-world example seems to be more important than precision in the definition of foci.

Finally, in reflecting on the whispering chamber problem as a whole, Mr. Taylor did not discuss the trajectory of tasks relating to this problem. For Mr. Taylor, this problem provided a concrete way for students to visualize the mathematics that was being discussed.

Opportunities that afforded meaning. In the segments described thus far, the mathematical richness in Mr. Taylor's instruction was limited, in part, by his beliefs about what would confuse students as well as the depth of meaning that was sufficient for students. However, Mr. Taylor's beliefs about particular students also seemed to support the meaning in his instruction in two episodes. I will briefly describe one of these.

Background. Near the end of the third day of instruction, students were practicing graphing ellipses given an equation in standard form (i.e., $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for an ellipse centered at (0,0)). One way to identify the foci in a graph of an ellipse is to use a visual representation. That is, for a given ellipse with the length of the semimajor axis denoted *a*, the length of the semiminor axis denoted *b*, and the distance from the center to one focus denoted *c*, then, by the definition of an ellipse, *b* and *c* are the lengths of the legs of a right triangle with hypotenuse length equal to *a* (see Figure 5.9). Hence, *a*, *b*, and *c* are related by the Pythagorean Theorem (with variables different than the conventional definition): $b^2 + c^2 = a^2$. Mr. Taylor had illustrated the formula for *a*, *b*, and *c* by showing students a visual representation of this right triangle that is present in an ellipse. (In fact, this representation was used in the derivation of the standard form of the equation for an ellipse and so was also illustrated on the first day of instruction.) However, Mr. Taylor expected that students would mostly rely on the formula ($b^2 + c^2 = a^2$) to solve subsequent problems.

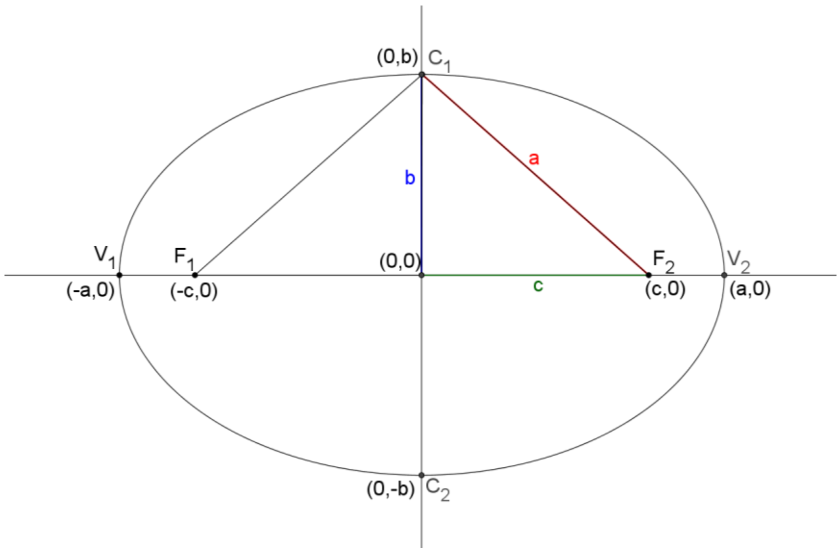


Figure 5.9 Relationship between axes and foci in an ellipse.

Instruction. During instruction, the class was working through an example where they began with the equation of an ellipse (which includes a^2 and b^2) and were asked to graph the ellipse and find the value of *c*. Mr. Taylor explained to students that they should use "the square root of *a* squared minus *b* squared" and asked students to tell him the value of that calculation. Once the class determined the result, one student asked a question:

> *Student 1*: Wait, so *a* is the hypotenuse once—is *c* the hypotenuse? Like what letter represents the hypotenuse?

In response to this question, Mr. Taylor returned to a previous illustration (similar to the one in Figure 5.9) that showed how *c* related to *a* and *b* through a visual representation. The student continued to try and make meaning of this explanation:

> *Student 1*: So you take like, you take the distance, like the, one half of the ellipse and you just like put it to the other corner, and basically you draw that line, I don't know, it's kind of
>
> *Mr. T.*: I see what you're getting. There's two ways to think about how to find *c*. One way is using this formula. Which is fine, this is how a lot of people do it. They memorize this formula. The other way is what you two are trying to do, is go back to the original right triangle, and you have to remember how that right

> triangle's made. If you remember how that right triangle's made, you'll always be able to figure out *a, b* or *c*, depending upon which information you're given. So it's based upon the Pythagorean Theorem. *c* is always going to be from the center out to the focal point. The focal point is always going to be on the major axis. So we know that. *a* is going to be the distance from the minor axis vertex to the focal point.
>
> *Student 1*: So *a* will always be the hypotenuse?
>
> *Mr. T.*: Yes, exactly.
>
> *Student 2*: And *a* is also four [half the length of the major axis] in this case, right?
>
> *Mr. T.*: Right. Does that make sense? So then we can work backwards and find *c*. Using, solving the Pythagorean Theorem. So this is important to understand where this comes from, which is fantastic that you got that, because it's actually a lot easier in the future to remember this and understand how it works. Not everybody's going to be able to retain that. The piece that you *have* to retain is this equation. Does that make sense? Very good. [Italics are the teacher's emphasis.]

The larger segment surrounding this explanation was not scored high in richness because of error and imprecision in language. However, this short explanation that is offered to students is high in richness. It illustrates explicit connections between the visual representation and the formula that Mr. Taylor is using.

A noteworthy feature of this segment is that Mr. Taylor had not originally intended to emphasize the points in this segment that scored high in richness; that is, he expected students to rely on a formula. On the surface, this expectation seems to contradict Mr. Taylor's overarching goals for instruction. However, because Mr. Taylor had explained the meaning behind this formula on the first day of instruction (as background for the derivation for the equation of an ellipse) and the episode described here occurred near the end of the third day, Mr. Taylor had already met his goals for concepts and meaning. In other words, it seems that Mr. Taylor's goals for meaning and sense making were enacted in introductory activities and meant to guide the discussion of new topics but did not replace later tasks that emphasized fluency with procedures.

Regardless, the students in this segment pressed Mr. Taylor for meaning, and there were meaningful connections made in this segment.

Mr. Taylor's reflection. Mr. Taylor explained his perspective on this episode in the SR interview:

> *Mr. T.*: You have these students who are making this huge connection that, wow, this is where *c* comes from because of this right triangle, and that's how they're trying to remember it. So not only are they making the connection, they're now saying it's easier to remember the concept than it is to memorize the formula. … It was like, wow, they got it. They got the concept, and they're using that to derive the formula, so that's where we wanted them to go. But you don't see that a whole lot in academic classes. A lot of times it will be more, you got the general concept and when we got the general concept, here's the pieces, guys, here's the tools, and this is the part that you need to know. …
>
> *Mr. T.*: Those two students probably should have been in the honors group. … But what's so interesting about teaching academic [level] is you have a very interesting dynamic within the same room. … If your makeup of students is all honors, not only do you want them to be able to graph the equation of a circle, you want them to be able to explain to you where it comes from based on the concept. Fantastic. My academic kids, I'm happy if they can explain to me how the foci gives me the ellipse, at least in general. And if they can go further and graph it, that's even better.

Mr. Taylor's explanation of this segment was telling. In particular, this explanation clarifies that Mr. Taylor's goals for meaning and sense making are not necessarily equivalent to what he expects from all students. In reflecting on his response to the two students pressing for meaning in this segment, Mr. Taylor explained, "those two students probably should have been in the honors group." Mr. Taylor believed that it would be important for honors students to explain the origins of the graph of an ellipse, but he does not have the same expectations of his academic-level students. In fact, he claims that "I'm happy if they can explain to me how the foci gives me the ellipse, at least in general." The meaning that Mr. Taylor expects his academic-level students to make remains at what he would describe as a "general" level. This reflects both Mr.

Taylor's conceptions of how meaning should look in the classroom and his beliefs about academic-level students.

5.6 Discussion

Throughout his interviews, Mr. Taylor expressed orientations that privileged mathematical meaning. In fact, Mr. Taylor spent more time discussing goals related to richness in the sense of MQI than any other teacher in the larger study, and these goals also aligned with mathematics educators' visions for high-quality mathematics teaching (e.g., NCTM, 2000). Yet, Mr. Taylor's instruction was not high in richness in the sense of MQI. In this chapter, I discussed three reasons that can explain the limited richness in his instruction.

One reason for the limited richness in Mr. Taylor's instruction was that his goals for mathematical meaning were more general than the richness in instruction that MQI captures. As other researchers have advocated (e.g., Philipp, 2007; Speer, 2005), by seeking Mr. Taylor's perspective on his practice through the SR interview, his understandings of his goals for instruction were made more clear. In the example where he derived the equation of the ellipse, Mr. Taylor wanted students to see *where* the equation came from—that is, the facts used in the derivation—but not *why* the equation followed from those facts. In the example of eccentricity, Mr. Taylor wanted students to understand *what* eccentricity did in a visual sense but not *why* changes in the visual representation made sense. In the example of the whispering chamber, Mr. Taylor wanted students to see that foci appear in the real world, but not why or how the whispering chamber works. In each of these cases, Mr. Taylor believed that he had met his goals for making connections and providing meaning in instruction. In other words, for Mr. Taylor, mathematical meaning in his instruction involved presenting high-level ideas rather than detailed justifications behind those ideas.

Another reason for the limited richness was that Mr. Taylor's instruction lacked clarity at times. Some of this lack of clarity was due to imprecise language. One such example was shared in this chapter: Mr. Taylor's response to the student who asked about the definition of foci. Similar instances occurred throughout Mr. Taylor's instruction. I do not believe that this imprecision in language was due to a lack of content knowledge, however, as Mr. Taylor seemed to understand the *mathematics* behind most of what he was teaching.

Lack of clarity was also present in the ways that Mr. Taylor chose to extend the curriculum. For example, in the whispering chamber problem, there was some lack of clarity in the sequence of tasks, the reasons for why calculations were being made, and the descriptions of the scientific phenomenon. There was also lack of clarity in the eccentricity episode, particularly in the discussion about seasons due to elliptical orbits. Whether due to language or task implementation, lack of clarity in instruction can detract from the meaningful points that teachers try to make (LMT, 2011).

In the SR interview, Mr. Taylor watched many episodes that lacked in clarity, but he did not comment on clarity. That is, there was no evidence that Mr. Taylor noticed the lack of clarity in his instruction. Researchers have argued that teachers must first notice important elements of instruction to improve them (e.g., Star, Lynch, & Perova, 2011; Van Es, 2011). Hence, the lack of clarity in Mr. Taylor's instruction could be due to the fact that he does not attend to this aspect of his instruction as much as others.

At this point, some mathematics educators might speculate that Mr. Taylor lacked MKT needed to carry out high-quality richness. This point deserves some discussion. There was evidence that Mr. Taylor had the *content knowledge* needed to support connections and justifications in his instruction. For instance, Mr. Taylor had prepared a correct algebraic derivation of the equation of an ellipse prior to class, so there was evidence that he understood the derivation. Mr. Taylor also explained in the interviews what eccentricity was—including the ratio definition—and how it affected the elongation of an ellipse. Although it is unclear whether he fully understood the three-dimensional whispering chamber phenomenon, he did understand the reflective property of an ellipse, as he also explained during interviews. Hence, the fact that these mathematical points were not explicit in instruction should not imply that Mr. Taylor does not understand them.

However, there was some evidence that Mr. Taylor's content knowledge of science was limited. For example, in the whispering chamber problem, because Mr. Taylor discussed the appropriateness of the ellipse on the floor versus the ellipse through the apex of the ceiling, it is necessary that he have a working scientific knowledge of how whispering chambers work. However, Mr. Taylor told students that the sound was "not reflecting off the side walls," when in fact, sound does reflect off the walls. Similarly, Mr. Taylor chose to use the example of elliptical orbits, but his explanation of seasons in these orbits was misleading.

In addition, because he made significant changes to the whispering chamber problem stated in the text, Mr. Taylor had to make choices about how to sequence the whispering chamber tasks he used, how to pose questions to students, and upon which information to draw. But these choices did not always lead to instruction that was mathematically rich. Hence, Mr. Taylor may have lacked what Ball et al. (2008) would call *knowledge of content and teaching*, a part of PCK. That is, Mr. Taylor may have lacked some understanding of how to implement the whispering chamber problem in a meaningful way.

Other researchers have argued that a teacher's choice to supplement the curriculum can be particularly taxing on their knowledge and ultimately limit MQI (Hill & Charalambous, 2012), and this is true even for teachers who have demonstrated strong MKT by other measures (Hill, Blunk, et al., 2008). In other words, Mr. Taylor's choice to supplement the curriculum may have accentuated gaps in his knowledge of science and his PCK that would not have been noticeable otherwise.

Arguably the most revealing reason for the limited richness in Mr. Taylor's instruction was his beliefs about students' abilities. Mr. Taylor explained that the limited meaning in his instruction was implemented as he intended, particularly because he was teaching academic-level students. Mr. Taylor also used his beliefs about students to justify some of the lack of clarity in his instruction. For instance, Mr. Taylor recognized that his description of elliptical orbits may be incorrect, and he also recognized that the graph he used to represent the whispering chamber was not completely clear. However, Mr. Taylor did not seem to be concerned about a lack of clarity in either instance because he believed that these were appropriate for his students. In other words, it seemed that Mr. Taylor's orientations for meaning and sense making were filtered by his beliefs about his students, and the goals that Mr. Taylor set for instruction were limited, in part, because of what he believed would be appropriate for his students. That is, Mr. Taylor's beliefs about mathematics and his beliefs about students interacted in ways that influenced the specific details of his goals for instruction (cf. Aguirre & Speer, 2000).

Although Mr. Taylor's beliefs about students limited the richness in his instruction in many cases, beliefs about particular students afforded opportunities for meaning. Two episodes did achieve brief richness, and in these episodes, students pressed Mr. Taylor for meaning. Because he privileged meaning and sense making, he was able to recognize students' quest for meaning and sense making in these instances. However, he attributed this quest for meaning and sense making to the fact that the particular students in the episode "should have been in the honors group." For the majority of the students in his class, he did not have such high expectations.

The discussion of students' academic levels came up in several interviews with teachers in the larger study. I observed only two teachers who taught honors courses: Ms. Zimmerman and Mr. Anderson. Both of these teachers had segments of their instruction that were scored high in richness, and Mr. Anderson's instruction consistently scored either high or mid in richness (89% of segments). These teachers expressed orientations towards mathematical meaning, as Mr. Taylor did, but they also explicitly discussed that they wanted to provide explanations and make connections for students *because* their students were honors students and needed a high level of rigor. Like Mr. Taylor, other teachers in the larger study sometimes discussed that they made choices in limiting the mathematics in their instruction because their students were not honors students, but these teachers did not express orientations towards the meaning of the mathematics like Mr. Taylor did. That is, Mr. Taylor was unique in that he expressed the orientations towards meaning making and richness similar to those that Mr. Anderson and Ms.

Zimmerman expressed, yet he also believed that brief explanations and connections were sufficient because he was teaching academic-level students.

High school teachers' beliefs about students' abilities according to their academic track has been documented by other researchers. For instance, Raudenbush, Rowen, and Cheong (1993) found that high school teachers in the United States reported their goals for higher order thinking in mathematics varied according to the track of students they were teaching. Similarly, Zohar, Degani, and Vaaknin (2001) reported that many Israeli high school mathematics teachers believed that higher order thinking is inappropriate for low-achieving students. Horn (2007) illustrated how US teachers' conceptions of students' abilities are negotiated with colleagues and affect how reform initiatives are accommodated within their schools. That is, schools—both structurally and socially—may perpetuate such beliefs about students' abilities.

The findings reported in this chapter are in accordance with the literature on teachers' beliefs about students according to their academic track. However, the research cited here used teacher self reports (Raudenbush et al., 1993; Zohar et al., 2001) or observations of planning outside of instruction (Horn, 2007). Hence, the present chapter extends this literature by illustrating specific examples of how beliefs about students shaped the mathematics that was offered to students in instruction. In addition, other researchers have noted that teachers' orientations regarding mathematics as an ordered and instrumental body of knowledge can contribute to their beliefs about students according to their academic track (e.g., Horn, 2007; Ruthven, 1987). Yet Mr. Taylor was a unique case in that he valued meaning and sense making in mathematics, but his beliefs about students' abilities seemed to limit his instruction in this regard.

The findings reported in this chapter also contribute to an understanding of the knowledge and beliefs that are necessary for engaging in high-quality instruction as measured by MQI. Researchers have identified that a lack of knowledge or beliefs about mathematics and mathematics teaching can limit MQI (e.g., Charalambous & Hill, 2012; Hill, Blunk, et al., 2008; Sleep & Eskelson, 2012). To add to this literature, the present chapter illustrates that the teachers' beliefs about students' abilities can affect MQI, particularly at the high school level. Mr. Taylor explained that the level of mathematical richness in his instruction was sufficient for the academic-track students he was teaching.

5.6.1 Limitations

This study has some limitations that should be noted. First, my observations of Mr. Taylor were limited to three days of instruction. I do not claim that this instruction is representative of Mr. Taylor's practice as a whole or that the patterns I observed in Mr. Taylor's thinking would apply in other instances of his instruction. Second, using interviews in conjunction with observations provided rich data on Mr. Taylor's perspectives, but there are also limitations to data collection. It was possible to reflect on only six episodes of instruction in Mr. Taylor's SR interview. Although I drew on additional interviews to understand Mr. Taylor's perceptions of instruction, I cannot say how he was thinking in every instance that was not coded as high in richness of the mathematics. Nonetheless, this chapter explores the depth and complexity of Mr. Taylor's orientations and knowledge and how these may have affected his instruction in specific ways. Third, the view of instruction in this chapter is limited to the MQI lens. Mr. Taylor made several other notable pedagogical choices that were not captured by this lens. For example, Mr. Taylor frequently used formative assessment in his instruction, and he also used technology in ways that allowed for dynamic representations of the content. In other words, Mr. Taylor was an excellent teacher in several regards that were not explored in this chapter.

5.6.2 Further Thoughts and Recommendations

In this chapter, I do not intend to imply that Mr. Taylor was not a "good" teacher. On the contrary, Mr. Taylor's admirable goals for instruction and his extended use of a real-world problem made this case intriguing. As such, it is important to give Mr. Taylor credit for the successes in his teaching.

Specifically, Mr. Taylor had orientations and goals to implement meaningful mathematics in his classroom, despite practical and logistical challenges that make it difficult for teachers to do so (cf. Raymond, 1997). In addition, Mr. Taylor was aware that many of his goals ran counter to students' orientations of mathematics, and this further motivated him to implement mathematical meaning in his instruction. Further, the goals that Mr. Taylor discussed are many that mathematics educators advocate as beneficial for students' learning (e.g., NCTM, 2000).

Mr. Taylor also used a complex, real-world problem in his instruction to help students visualize the concepts associated with ellipses. No other teacher I observed for this study used a real-world example in the extensive and independent way that Mr. Taylor did. Indeed, Mr. Taylor engaged in a difficult challenge in trying to extend the curriculum in meaningful ways, and such work is challenging for both novice and experienced teachers. In addition, Mr. Taylor had courage and confidence to try and implement meaningful problems rather than presenting mathematics in a mechanical way. This courage and confidence, coupled with Mr. Taylor's views of mathematics, are important foundations for engaging in instruction that is mathematically rich. Like all teachers, Mr. Taylor could work to improve the richness in his instruction with critical self-reflection.

Specifically, to better implement richness in the ways specified by the MQI rubric, Mr. Taylor and other teachers may benefit from studying and understanding the MQI instrument through video or observations of classroom teaching. For instance, studying MQI may help Mr. Taylor to deepen his conceptions of the level of explicitness and detail that is appropriate for meaning and sense making in instruction. Understanding MQI may also help him to draw attention to certain features of his instruction, such as language and clarity.

Challenging Mr. Taylor's beliefs about students' abilities would likely require more than a detailed study of the MQI rubric. Other researchers have documented teachers' surprise when they witness their students engaging in rich and rigorous mathematics (see Philipp, 2007). Mr. Taylor and all teachers may benefit from engaging in professional development that pushes them to create opportunities for meaningful student thinking and reasoning in their classrooms (Leikin & Zazkis, 2010; Weber & Rhoads, 2011). Seeing the capabilities of students when they are engaged in critical thinking may work to change teachers' beliefs about students' abilities.

Chapter 6: Conclusion

Classroom instruction matters in students' learning, and teachers' decision making is central to mathematics instruction. Teachers make many decisions about instruction, including what content to develop with students and how to develop it. Yet mathematics education still lacks a robust understanding of the beliefs and knowledge supporting decisions that lead to high-quality instruction, particularly at the high school level. Although there is a great deal of research that has focused on teachers' beliefs and knowledge, these are often studied independently from one another (e.g., Hill et al., 2005; Leder, Pehkonen, & Törner, 2002) or assessed outside of classroom instruction (e.g., Hill et al., 2005; Vacc & Bright, 1999). By contrast, in this dissertation, I considered the roles of both knowledge and beliefs as they were grounded in teachers' instructional decisions. Hence, the main contribution of this dissertation is that it provides both an encompassing and an authentic view of the knowledge and beliefs that are used in high-quality instruction.

6.1 Findings and Significance

The findings of this dissertation highlight the depth and complexity of mathematical knowledge and beliefs used in high-quality instruction. In Chapter 3, I reported that exemplary teachers expressed many intertwined ways of knowing about mathematics and mathematics teaching in their reflections; moreover, this knowledge was not expressed abstractly but through teachers' discussions of how they achieved pedagogical goals. These findings challenge the assumption that teachers' difficulty in abstracting and describing their mathematical knowledge implies they are deficient in such knowledge. Rather than seeing teachers' inability to explicate their mathematical knowledge for teaching as a barrier to their development and use of this knowledge, teacher education may want to consider the affordances of using teachers' reflections on their instruction for further understanding and development of the knowledge needed for teaching.

Chapter 4 reported that responses to student mathematical productions coded as high in mathematical quality of instruction (MQI) were supported by goals to build on students' ideas and emphasize meaning and sense making, whereas in low-MQI responses to students, teachers either had alternative instructional goals or lacked knowledge that could have helped them in carrying out the response. Further, this chapter highlighted that although knowledge of students, pedagogy, and mathematics was needed to carry out instructional decisions, teachers' orientations ultimately drove their goals for instruction and the subsequent decisions that they made. This finding implies that developing teachers' knowledge alone may have a limited impact on instruction. It is also necessary to understand and develop teachers' orientations towards mathematics and mathematics teaching.

Chapter 5 illustrated how one teacher's goals for meaning and sense making did not guarantee rich mathematics instruction in the classroom. This chapter highlighted the importance of examining teachers' beliefs as they are grounded in instruction and as they relate to other beliefs and knowledge. The example presented in Chapter 5 was not a case of a mismatch between beliefs and instruction; rather, this teacher's instruction was consistent with what he believed was appropriate for the students he was teaching. Beliefs about mathematics, mathematics teaching, and students all contributed to the level of richness in instruction. In particular, the teacher's beliefs about students' abilities shaped the mathematics that was offered to them.

Taken together, the studies in this dissertation highlight that assumptions about mathematical knowledge for teaching (MKT) should be made cautiously. Teachers may have knowledge that is not expressed in ways that researchers expect (Chapter 3), MKT may be specific to the situation (Chapter 4), and instruction is not a direct consequence of a teacher's MKT (Chapters 4 and 5).

In addition, this dissertation illustrates the interrelated nature of teachers' beliefs and knowledge in making and carrying out instructional decisions. The findings may help to explain why the relationships between teachers' MKT and instruction are inconsistent across studies (see Chapter 2 of this dissertation). Teachers' beliefs and goals filter their use of knowledge in important ways.

In summary, these findings challenge the assumptions that either teacher beliefs or teacher knowledge can be studied in isolation or outside of the instruction in which they are used.

6.2 Limitations

The findings of this dissertation can inform the field of teacher education, but there are limitations to be noted. First, the sample size in each of these studies was appropriate for the qualitative research approach that I took, but it should also be noted when considering the findings of this dissertation. Second, the setting of this dissertation was limited to the New Jersey area of the United States. As such, the findings should be considered in context and may not be representative of other cultural settings. Third, a large part of my understanding of teachers' beliefs and knowledge was built through interviews with teachers. Although this method was informative, it may not capture the full realm of how teachers think about their actions.

Fourth, the time spent with participants in each of these studies was necessarily limited. Although the study reported in Chapters 4 and 5 offered the opportunity to observe teachers across multiple days of instruction, the instruction that was observed may not necessarily be representative of teachers' instruction as a whole. Finally, the notion of high-quality instruction was operationalized by considering the recognition that teachers had received for their teaching or characterizing instruction according to the MQI instrument (LMT, 2010). These are only two of many lenses by which to describe high-quality instruction. In particular, I do not wish to imply that instruction that is considered to be high MQI is definitively high in quality across all lenses.

6.3 Directions for Research and Teacher Education

Chapter 3 found three themes of teachers' attunements to mathematics in teaching, and Chapter 4 found two themes that captured teachers' thinking as they were responding to student mathematical productions. These findings were related to Brown and Coles's (2003, 2011) notion of *purposes* in mathematics teaching. Teachers may benefit from discussing these themes as key ideas to keep in mind during instruction, and this could be explored in future research. In addition, the extent to which a larger sample of exemplary teachers express and integrate these attunements during instruction could also be explored in future research.

Given the important role of teachers' orientations in their instructional decisions (see Chapters 4 and 5), it may be helpful for teachers to critically consider how their beliefs and orientations drive their decisions. Teachers sometimes had several considerations in deciding how to respond to students' mathematical productions or how to present ideas in a lesson. Future research could explore whether explicit discussions about these considerations, with colleagues or mathematics educators, could lead to improved instruction. That is, colleagues may provide ideas about different possible directions for the dilemmas that teachers face. These discussions may broaden teachers' views about what is possible in instruction, thus helping teachers to make both planned and in-the-moment decisions that promote student reasoning and uphold the integrity of the mathematics.

At the same time, teachers' reasons for the decisions that they make are logical and not necessarily always in need of change. For example, Chapter 4 illustrated that a teacher may not pursue a student production because they have a different, yet worthwhile, agenda for the lesson. Future research could explore how the frequency and nature of such decisions affect students' learning of mathematics. Such research could inform the field about the complex and simultaneous work of responding to students, emphasizing mathematics, and navigating a lesson agenda.

Because the MQI instrument (LMT, 2010) focuses specifically on the mathematical characteristics of instruction, teachers may benefit from using this as a tool in professional development. I have reported elsewhere (Rhoads, 2011) that exemplary teachers in the first study of this dissertation believed that mathematics-specific professional development strongly contributed to their expertise, yet teachers also reported that this professional development was rare. In Chapters 4 and 5, teachers' decisions did not always privilege mathematical richness or responding to students in ways that were captured by the rubric. Future research could explore whether and how teachers can (a) deepen their beliefs and knowledge and (b) enhance the mathematics in their instruction by studying and understanding the kind of mathematical work that MQI captures.

Finally, I would like to highlight the many affordances of experienced teachers' beliefs and knowledge that were illustrated in this dissertation. Teachers had well-developed knowledge that they expressed through their discussions of instructional practice, rationally approached responses to students and weighed worthwhile alternative decisions, and carried out goals as they intended them for instruction. The challenge moving forward is for teachers and teacher educators to continue to work together and build on teachers' instructional strengths in ways that resonate with their views of practice.

Appendix A: Study 1 Interview Protocol

Teacher:

School:

Date:

Time Start:

Time End:

Description of setting:

Introduction: Thank you so much for taking the time to meet with me today! I contacted you for this study because you are National Board certified. Because you are an exemplary teacher, you may be able to help others understand what is required for successful mathematics teaching. My main goal is to learn more about the mathematics that you use in your teaching.

I am going to ask you some questions and also look at the lesson plan that you have submitted to me. This process will take about one hour. If you do not feel comfortable answering a question, you do not need to answer it. If at any time you feel uncomfortable with the audio-taping, I can turn it off. We can stop the interview at any time if you become distressed. Feel free to stop me or ask questions along the way.

Also, I would just like to remind you that all data from this interview is confidential. In the interview notes, you will be identified by your name and school, but only I will have access to this information. The notes and audio-recordings from the interview will be kept in a locked file cabinet and shredded after three years. Any reports that are developed from this data will use pseudonyms.

Do you have any questions before we begin?

[State participant number for recording]

I would first like to get a sense of who you are as an educator and what general experiences you have had.

1. Please describe your background in mathematics and mathematics education[13]. (Follow-up questions as needed):
   a. How did you get into teaching?
   b. What training did you have to become a math teacher?
   c. Did you study to be math teacher in college?
   d. Were you alternatively certified?
2. Tell me about your teaching history.
   a. How many years have you been teaching?
   b. What have you taught?
   c. What grades have you taught?
   d. Where have you taught?
   e. What courses have you taught in the past?
   f. When did you teach these courses?
   g. Where did you teach these courses?
   h. What courses are you currently teaching?
3. What led you to teach math?
4. Congratulations on [award]! Why do you think you were chosen to receive this award?
   a. How did it feel to get the award?

[13] On interview protocols used for this study, ample blank space followed each question to allow for written notes.

b. Have you received any other awards for your teaching?

Now I have a sense of your background. These next few questions are about the lesson plan that you shared with me.

5. Why did you choose to share this lesson with me?
6. Describe how you created this lesson plan.
    a. When was it created?
    b. For whom was this lesson plan created?
        i. (Follow-up questions if needed):
        ii. Did you write this for yourself?
        iii. Did you write it for your principal?
        iv. Did you write this for an award?
7. What topics come before and after this lesson?

    <u>[Insert specific questions about lesson here.]</u>
8. If I were to watch you teach this lesson, what would I see in the lesson that is not included in this plan?
9. Describe how you assess if students have learned the material at the end of the lesson.
10. How many times have you taught this lesson?
    a. How would you say it has changed over time?
11. In what areas of this lesson did you have to apply your mathematics content knowledge?

a. What specific content knowledge was used to develop and/ or teach this lesson?

12. If you were going to mentor a new teacher who was about to teach this topic, what would that teacher need to know about the content in order to teach the lesson well?
13. In what ways is this lesson typical of other lessons that you teach on a daily basis? In what ways is it different?

Obviously, one lesson plan is not completely reflective of your practice, so these next few questions are about your practice in general.

14. Describe how your knowledge of mathematics has influenced your teaching.
    a. In what specific areas of teaching have you used this knowledge?
    b. Can you give an example?
15. Where do you think you gained the mathematical knowledge needed to teach high school?
    a. When was this in the scope of your career?
16. What ideas from college mathematics have helped you to teach high school?
    a. What specific aspects of these topics have been important?
    b. Can you give an example?

Okay, that is all the questions I have for today. Is there anything else that you would like to share with me about the mathematical knowledge that you use in teaching?

Appendix B: Supplementary Material for Themes of Teachers' Reflections

Knowing Students as Learners

Students' Background Knowledge

> *Ms. Lombardi:* I was confident my students understood what a compound inequality meant in the way I expected them to write it. … So, I would say, in terms of the expectations that I had for [the students], you have to know what they have already learned because the prior knowledge that they have can determine how something should be taught, or what the expectations are for how they should represent something mathematically. … In another school where I taught this lesson, and they hadn't been through Algebra Two, they could list a range of numbers. Like in words, they could say, "The third side could be any number between three and 11." But since my students have already had Algebra Two, I'd expect them to be able to communicate that symbolically.

> *Ms. Kruger:* Going through [the lesson] like that helps me think about what the kids are going to know and not know. … For the higher level sections, I don't spend a lot of time on that because they already get that they're inverse operations. … But when you get to exponents, two to the third and three squared are two different things. And my lower sections usually don't even realize that. They don't have a strong enough math background that they can look and immediately say, "Well these two numbers are in a different order and it gives me a different answer." … So I spend a lot more time on that. … I wanted to go back and build on things they knew, so I had to figure out what things they knew [and what] they needed to do this lesson, or this unit.

> *Ms. Allen:* I had to go back and look at this logarithm problem several times. … Some of them, you've got to either use change of base formula, which I had not taught them yet, or you're going to have to know your exponents enough to be able to work backwards in your head to find the answer. Which is what I wanted them to do with it at this point, because there was nothing that I felt like they couldn't do. Like, I would think they could figure out what number to the negative four power is 16. They got to think about some things like that.

Students' Mathematical Thinking

> *Ms. Orlando:* What I also think had a lot to do with being a better teacher was the one-on-one tutoring. … Being able to sit with a kid one-on-one and see how that kid was learning. … Having that one-on-one time, you got to see what they were thinking or how they were thinking wrong, and then when you tutored another kid in this same subject, like, "Wow, that kid's thinking the same wrong way." So I think the one-on-one kind of really started to get me [thinking], "Oh well if this

person's having this kind of trouble, maybe if I said it this way in class." Getting the feedback from the student was very helpful one-on-one.

Mr. Travers: It came from seeing kids struggle with any kind of word problem. You know, because they don't see it. And it's, at every level, it's advanced kids, it's kids who are average or above, even a little above average. … I find that's true about a lot of problems where the result is some type of a three dimensional object, we might be talking about surface area of cube, or whatever. They just don't picture it. … That's another piece of why I make them do this chart, because they might say, "Okay, … this was ninety six, this was ninety six," then if the next one is like 40, they might say, "120, that's it." But what about, it could be between 1 and 2, it could be between 2 and 3, you don't know where it peaked. That's why, another reason, you know, you've got to do more than just the superficial.

Ms. Schneider: A lot of what I'm saying when I told you that I translated into student, when I was studying it the first time, that's how I got it into my brain. So, I guess as I study mathematics, and I knew where the pitfalls were and how I overcame them, I passed that along to my students.

<u>Students' Learning Styles</u>

Ms. Hong: We often do, as math teachers, expect [students] to be logically-mathematically inclined. And then, maybe we don't understand sometimes why they don't pick things up so quickly, because maybe we're just not tapping into those strengths that they have. The artist in the classroom is starved to draw. So, why not let them express the lesson using diagrams or pictures or come up to the board and do it. You know, while somebody else in the class can reinforce that task that is algebraically inclined and start developing the steps alongside that person. Or have somebody else design a poster to show the concept in a different way. Have a musician come up and show how the melodies are mathematical. … They represent the topics in different ways, and sometimes I'm amazed by what they come up with as far as—especially musically—to show, maybe not so much geometric sequences, but arithmetic sequences and how you write a melody. It's just incredible. … Any topic that I feel lends itself to this, I break down into different skills to address learning styles … I'm very aware of skills that just pop out in the classroom. Oh you're an artist? Well, come up. You're a musician? Tell me about some of the arithmetic sequences that you've come across in your melodies, in writing melodies.

Ms. Yates: I have a student who, she's brilliant, but she is perceptually impaired, and whenever I have them do any kind of writing, explanation, her spelling, it's just, she gets twice as much time on her tests. .. [It's important to] understand how [students] learn and what they need. And then you take the math, and how can I take the math and apply it to how they learn--this group of kids, this time, these learners in my classroom, this perceptually impaired student, these three boys in

> the back with ADHD? How can I make this lesson of math important to them so that they can all understand it?
>
> *Ms. Schneider:* In my Calculus class, we've actually sung theorems, and one of them I put the hand motions to. But I've had students email me back that they're doing postgraduate work and they'll say, "I still remember how to do partial derivatives because of that silly song in my head." It's never gotten out of their head. … What worked with some kids doesn't always work with other kids. When you have things in the bag, you tweak things, you do things. … Whatever it takes to get the kid to learn.

<u>Students' Interests</u>

> *Ms. Allen:* I do try to find videos that will support what we're doing in class. Like in this one, I showed just a minute and a half news clip from the Haiti disaster, and we talked about the Rictor scale. … I used even the Challenger disaster and then we'll talk about how to solve that type of problem. Just as a way to have them interested and understand that what we are doing does impact their world today.
>
> *Ms. Yates:* When I put these up, the interesting thing about this lesson was when I put these up, they were more interested in, because I didn't have equations, "How did you do those?" … And this one was the one I think that grabbed their interest the most. … That was interesting. The fact that they were so intrigued by this: "Wow, you made those? *You* made those? How do you get it to do certain things?"
>
> *Ms. Hong:* Because I thought the students would be excited when the picture would come out as a flower on the screen. Because you know, they think of how to write words on the calculator and words that are spelled backwards and all that silly stuff. You know, but to see a flower appear, they really didn't know how to do that, so that was one way to inspire them with technology.

<u>Developing Mathematical Ideas</u>

<u>Interconnectivity of Mathematics</u>

> *Mr. Travers:* One example is binomial theorem. Where you're expanding *x* plus *y* to the fifth. And you go through all the theory, and the combinations and all of that, and then you can say, "Alright, well let me just show you a little separate problem. If you have five books and you take three at a time, how many different, well that's the same kind of problem as getting the coefficients for the binomial expansion." And, it might seem like that's not the time to do that, but if it comes up or if you need to reinforce where those numbers were coming from, then it might lend itself to that. So, you know the more knowledge you have, the more

you can tie things in, I guess it really comes down to that. The more knowledge the math teacher has, the more that they can make connections, they can bring several different ideas in. Maybe something that the kids learned in Geometry. You know, if you're teaching algebra and you know a lot about geometry, you can take something from geometry and use it.

Ms. Orlando: The most important thing is the explanation of why … just where it comes from. … [The students ask], “Well who invented [the quadratic formula]?” [Like] it just came out of the sky and someone just put these letters down. … Completing the square is one way to factor, and it’s not really that popular to teach it anymore because you can always use the quadratic formula. … You take that *ax* squared plus *bx* plus *c* equals zero, and you use your algebra on it, and it becomes the quadratic formula. I always show the kids. This just didn’t magically appear one day under the square roots. It came from something very simple that you know.

Mr. Fisher: I want to try to make this consistent to what they already know. … From the teaching end of it. … Everything goes back, and it's really not different. … Don't make it something totally foreign. Make it something connecting back to what they've already done. Because the more you can do that, the stronger it makes it, easier it makes it, the more likely they'll remember it. If you're trying to ask them single pieces of information without reinforcement, without connection, it will fall apart. Whereas, if I can connect it and show it's really not much different, it's just another wrinkle or refinement of a process they've already learned, it makes it better.

Key Examples of the Content

Ms. Orlando: You need to have a thorough understanding of being able to draw, or getting the concept of how [the parabola] opens up, opens down, all the different possibilities of parabolas for the students to visualize what they're coming up with as the answer.

Ms. Yates: So, when I created the graphs, I wanted to make sure that I had representatives of all of the different cases that were going to come up, and even within here, we were headed after this lesson, to damped oscillations. That was coming up the next day. And so, they didn't know what damped oscillation was, but I wanted to expose them to some damped oscillations in this lesson, so that we could come back to them and say, “Remember those graphs from yesterday? You know, we had some of the sinusoids that had a damping factor on them.”

Ms. Lombardi: I would just ask if these lengths [points to several examples, each with three side lengths written] can work to be a triangle. The one that students have the most trouble with is the one where the two shorter sides sum to exactly equal to the third side.

Multiple Representations of the Content

Mr. Travers: As the student, you control the problem more if you can represent it. … Everything stems from being able to understand the problem, represent the problem, and maybe get an equation. Alright. Now, kids don't do a lot of that. … So, that's why I … have a little lesson in "How do you represent a situation, a problem?" … If you can get an equation, you own it. You understand it. The graph is good too, being able to get a graph for it, but the equation itself is crucial.

Ms. Schneider: My teaching has changed tremendously, tremendously. For the better. … We went from chalk and talk and then memorization to the hands-on to the making it relevant, and to the graphing calculator where it's like, "I would never be able to do this by paper and pencil. So that's what that looks like, huh?"

Ms. Yates: I learned that there are students in my classroom who words aren't good enough. They need pictures, they need diagrams, they need that visualization of a model. We did conic sections and we did rotating around an axis. I learned to take in things like wedding bells. You know the crepe wedding bells that you can open up and it becomes a three-dimensional wedding bell that you can hang as a decoration? Well if you start with that on the blackboard and you spin it around, they get the idea of the visualization of the rotation of that. And then again the website was wonderful having access to that. Because the three-dimensional models are there for students to see. Because you can't always draw it the way you want it to look and the way it would look in real life. And then you have tactile kids. And I learned I had to develop projects and I had to come up with activities in the classroom that allowed for manipulatives, even in Calculus and Honors Precalc.

Connections to Applications Outside of Mathematics

Mr. Travers: The other thing is this shuttle problem is a good real-world application. I mean, that's the other thing. The common denominator is, a real-world application is going to be heads and shoulders above any other lesson. There might be other lessons that are good, but if it has that real-world connection, it's going to be a cut above.

Ms. Orlando: As [students] move up in the math, everything becomes so abstract to them. If you can relate it to … something in real life. I'm sure you've seen the show Numbers. … What I liked about that so much is that they could see what some of this really crazy abstract thing has to do with [real life].

Mr. Meyer: I try to find something that's relevant to what we're learning and try to put a realistic spin on [the content] to just give [students] understanding that [math] is used in the real world, and there is a reason that we're learning it.

Mathematical Generalizations

> *Ms. Lombardi:* I might have students do something on their graphing calculator. I have tables in my room right now, so there's five different tables, … and I'll assign different tables [different problems]. … "You guys are doing equation A, and you're going to investigate these properties. You're doing equation B, and then you're doing C. Then you're going to write your result on the Smartboard, and then we're going to take a look at what's going on."

> *Ms. Schneider:* "What do you know about the number of faces and the number of edges?" So we looked through all of [the polyhedra], all of their work for all of them. "What do you think?" Conjecture. … A lot of my approach to mathematics is always patterns. It always makes sense; you've just got to find it.

> *Ms. Hong:* Many of them didn't have time to come up with a rule in class. They had time to discuss it, which was great, because then they could go home with the idea[s from class]: … "How many numbers do the steps increase by? Are we talking about a power of two?" So the discussion happened in the class, but the rule didn't actually come out in the class. The rule came out afterwards.

Technology

> *Mr. Travers:* Being able to use the graphing calculator. That's big. … [The answer's] not going to fall right on two or one or three. Getting the exact decimal, and that's where the technology comes in. So, knowing how to use the graphing calculator is really where the knowledge comes in.

> *Ms. Yates:* I think the technology's helped [my teaching]. When you can put up an emulator that, you know is huge for the kids to see and the things that you can do with the technology, I think it makes it, it pops for the kids. It makes it come more alive.

> *Ms. Hong:* The Texas Instrument, the TI-81 at that time, was new technology, and many of my colleagues were afraid to pick it up. [Laughing] And I was just learning about it … how to use the tool, and I wanted to try it in my classroom. So, my lesson … was a lesson on parametric equations using the TI-81. Because I thought the students would be excited when the picture would come out as a flower, you know, on the screen. … To see a flower appear, they really didn't know how to do that, so that was one way to inspire them with technology. And the TI-81 wasn't really the easiest calculator to use at the time because there were so many menus that they weren't used to accessing. And, I needed to encourage them to not be afraid and try and see what all of these menus do. And, basically the lesson was about that.

Promoting Students' Mathematical Activity

Active Participation from Students

> *Ms. Yates:* I don't like to be the one lecturing all the time. I like to ask questions. I want them to answer, and I want them to participate because that's the only way I know whether to move on, whether they're getting it, or whether I need to stay where I am. … I still love the interaction with the kids.
>
> *Mr. Meyer:* With the lower levels, it's a little bit of lecture and then they try some on their own. And they present what they did and then a little bit more lecture. You have to break it up a little bit more as their attention span and their ability to focus on one task in that span of time is a lot less. So it's more broken up for the lower level.
>
> *Mr. Fisher:* The examples are on Smartboards. Smart had a smart responder. … [The students] all have clickers. I show them the example. We do some problems. We work through the example, now okay you try it. … Engaging students. Now they're active in it. It changes it. [That's] the difference between that and a power point then. A power point is a very passive activity. … This becomes a little more active.

Problem Solving Activities

> *Mr. Travers:* I try to get at least a little bit of problem-solving incorporated into any good lesson. And that might not be the whole lesson is problem-solving, but some aspect of it. [The students] have to think, and it's not just plug in a number and you get the answer. … Think about what it would look like and think about what the outcome would be.
>
> *Mr. Meyer:* The whole thing behind [these types of problems] is [the students] doing a little bit more research, rather than us spoon feeding, "Here's step one. Here's step two of the project, step three of the project." It's just like, "Here's your task. Go and do. Then come back and present what you found."
>
> *Ms. Allen:* There is more than one way to approach a problem. So you've got to give kids time to explore and problem-solve and figure out. And not just solve ten problems and give me the answer. It's got to be, give them a situation and let them use the math that they've learned to reach an answer. And they're not all going to approach it the same way, and so it's helping them use the skills you've given them in a variety of ways.

Explanations from Students

> *Mr. Meyer:* When the kids are presenting, I just stand in the back of the room and let them go. … If somebody doesn't volunteer, … I'll [say], "You had something

that was interesting. Why don't you go up there and show us what you did?" … I'll say, "Let's go up and let's look at this." And we dissect it piece by piece, and we'll point out the good stuff and where they're going. And then say, "Alright, so maybe we have a problem here. What's the problem?" And I let other people say what the issue is, or I'll say, "We have an issue at this point right here, what could we do to fix it, or what is that issue?" … [There are] kids that are like, "I don't know if I got this one right. I got half." And I'm like, "Go up there. Show me what you know." I'm not able to correct something unless I know where they're making their mistakes. So I want them to go up there and I want them to get stuck a little bit.

Mr. Travers: Students were so into it, like, "Okay, we've got to get our facts straight. We've got to. Because we're going to have to go up there and explain." … They were talking about it.

Ms. Schneider: [Students have to] give me the justification. You show me the work based on the justification you gave me. … [Students] always have to have the three paragraph summary. That they know. … It gives you insight. …[It's] in this [summary] that you really know how well they know it.

Appendix C: Study 2 Background Questionnaire

Teacher:

School:

Date:

1. Please describe your background in mathematics and mathematics education, including but not limited to:
    a. How you got into teaching.
    b. The number of years you have taught
    c. The subjects/ domains/ ages/ courses you have taught and how long you taught each of these.
    d. Leadership positions or service in mathematics education (e.g., teacher mentoring, conference presentations, book writing, etc.)
2. What led you to teach math?
3. Have you received any awards or honors for your teaching?
4. Where and how do you think you learned the mathematics that helps to make you an effective teacher?
5. Is there anything else you think I should know about your professional background?

Appendix D: Prelesson Interview Protocol

Hello! I am really looking forward to my upcoming observation! Here are the questions we are going to discuss during the prelesson interview. You are welcome to type or write your responses ahead of time, but it is not necessary.

Also, please bring a copy of materials you used for planning (for example, written lesson plan and/ or worksheets.)

Teacher:

School:

Date lesson will be taught:

1. What are your goals for this lesson?

    a. Why do you have these particular goals?

2. Please describe how you created this lesson plan.

3. How does this lesson draw on the previous day's lesson, if at all?

4. How many times have you taught this lesson?

    a. How would you say it has changed over time?

    b. Why have these changes been made?

5. In creating this lesson, what mathematical ideas did you have to consider?

    a. What underlying ideas were considered?

    b. What ideas beyond the scope of this lesson were considered?

6. In preparing to teach this lesson, what mathematically-related reactions (questions, confusions, excitement, etc.) do you anticipate from students?

a. Can you give specific examples?

7. If another teacher asked for your advice about how to teach this particular topic, what would you tell that teacher?

8. In what ways is this lesson typical of other lessons that you teach on a daily basis? In what ways is it different?

9. Is there anything else I should know about the lesson I am about to watch?

Appendix E: Stimulated-Recall Interview Protocol

Teacher:

School:

Date:

Time Start:

Time End:

Setting:

Introduction: Thank you so much for taking the time to meet with me today! I have two goals for this interview. First, I want to understand your reactions to the clips that I chose, and second I want to understand your perspective on your decision making as you were teaching. I am most interested in your thinking and decisions around content as opposed to classroom management or other aspects of teaching.

First, we will talk about the sequence of lessons in general. Then, I have chosen some clips of specific moments in the classroom. There were many wonderful moments, so I tried to choose those that we haven't already discussed in detail during the prelesson interviews, such as times when you gave a particular explanation or when a student asked an interesting question. Each clip is about 6 minutes long. We will watch the clips together, and I will ask you some follow-up questions. I am most interested in your perspectives on these lessons and the clips that we will watch.

Some teachers have found it helpful to keep notes as they watch the clips in case they see important things that they want to discuss after the clip is over.

In a few months, after I have a report of this interview, you will have a chance to read my report and make comments on whether or not you feel that my interpretations are accurate.

Also, I would just like to remind you that all data from this interview is confidential. Any reports that are developed from this data will use pseudonyms.

Do you have any questions before we begin? Would it be okay if I audio-record this interview?

General questions about the three days:

1. Overall, how do you feel like the teaching of this topic went?

    a. Why do you think so?

2. Can you recall any memorable moments in this sequence of lessons?

    a. Why are these moments memorable?

3. Was there anything that was unexpected during this sequence?

4. Was there any place during this sequence that you modified your original plans for teaching?

    a. If so, how?

    b. Why were these choices made?

5. During this sequence, were there any additional topics or ideas that you considered addressing but chose not to? Why not?

6. During this sequence, were there any alternative approaches to teaching or explaining that you considered? Why did you decide not to use these?

7. Teachers are often said to have a “special” knowledge of mathematics. That is, they understand math in a way that allows them to teach it to others. Can you give describe how you needed to understand this topic in a specific way in order to convey it to the students?

Questions for each clip:

8. What are your overall reactions to that clip?

9. Try to put yourself back into the moment of when you were teaching this. Can you describe what you were thinking as you were teaching this?
    a. What thoughts about mathematics did you as a teacher have to consider as you were teaching this? Perhaps something that you were not saying to the students?

10. Is there anything you would have done differently during this episode? Anything that you would have done different mathematically?

11. Is there anything else you would like to add about this clip?

After all clips:

12. After watching these clips, what do you believe are some of the most important considerations that teachers should have when teaching these topics?

13. Do you have any concluding thoughts about this sequence of lessons?

Is there anything else that you would like to share with me about this sequence of lessons, the notes you made for yourself, or the decisions that you made as you were teaching?

Appendix F: Dimensions of Mathematical Quality of Instruction

1. *Richness of the Mathematics* captures whether and how the mathematics in instruction focuses on meaning or mathematical practices.

   a. Meaning includes (a) linking and connections among representations of a mathematical idea or among different mathematical ideas and (b) explanations, including instruction that provides meaning or explains why a procedure works, what a solution means in the context of a problem, and so on.

   b. Mathematical practices include multiple procedures, developing mathematical generalizations, and fluency in mathematical language

2. *Working with Students and Mathematics* indicates whether teachers understand students' productions and difficulties with the content and respond appropriately.

3. *Errors and Imprecision* assesses the teacher's mathematical errors, imprecision, or lack of clarity.

4. *Student Participation in Meaning-Making and Reasoning* captures whether and how students are engaged with mathematics through questioning, reasoning, and meaning-making.

5. *Classroom Work is Connected to Mathematics* indicates whether instruction is focused on mathematics content.

A more complete description of MQI can be found online (see NCTE, 2012).

Appendix G: Additional Data Illustrating Mr. Taylor's Overarching Goals and Supporting Orientations and Knowledge

Goal 1: **Emphasize concepts (over procedures)**

Orientations:

- Concepts are more critical than procedures when doing mathematics.
- Students should not rely on modeled procedures without understanding.
- Students should consider definitions and concepts when they try to solve problems.
- There are multiple ways that students may solve math problems.
- Students should be able to reason about concepts when they are solving problems.

Knowledge:

- Definitions of concepts: Ellipse, eccentricity
- How the textbook presents material and the ideas and processes that are emphasized in the text
- Students' orientations towards mathematics: Apply procedures to provide correct answers

Mr. T.: One of the important things is that I'm going back to the definition of the ellipse being [based on the] foci. And that is the key concept there that I think gets lost if you're just focusing on the algebra. Because you can graph an ellipse without ever really talking about foci, if you're just using the algebra equation *x* squared over *a* squared plus *y* squared over *b* squared equals one. You never need to talk about foci to graph it, but that's not what an ellipse is. … The foci are that critical piece.

Mr. T.: Down the road, say two years from now, whatever, they're going to forget how to graph [an ellipse] from the standard form. They're going to. But they're not going to forget what an ellipse is. They're going to remember that it has something to do with the foci. In fact, that's the key piece. So when you talk about the big understanding, the deep understanding, it's not the ability to start with a general form of an equation and graph it from that. Because they can always plot points, they can always throw it into some type of graphing calculator to actually use an equation, an algebra equation. But, that's never going to tell them what the foci mean. And if they can pull away from that, … [and get] the real concept of what an ellipse is and how it works and the relationship between the ellipse and the foci, … [those] are the big understandings. If they can't plot—two years from now—an equation of an ellipse, it doesn't bother me. Whereas the textbook problems, that's all the problems are. "Here's the general form, plot the equation of an ellipse on a piece of graph paper. Or write out the points." So it misses a lot.

Mr. T.: When you think about planning lessons and you think about planning units, you have to think about what is the big idea that you want to get them to? You need to think about moving towards that. As opposed to thinking about what problems you want them to solve, in terms of algebra problems in the textbook. And a lot of times, if you're just teaching the examples rather than teaching the concept, you're missing something. And the way textbooks are laid out a lot of times, they're laid out by example. So you have example one applies to questions one through ten. Example two applies to questions fifteen through twenty. It's so wrong, just the way that's done. It should be, here's the content, here's a whole bunch of problems, go figure it out.

Mr. T.: A lot of times in the book—which I despise, I hate it when they do this--… they'll list questions in the book, [like] problem number fifteen, and right next to it, they'll say, "See example three." Really? So they're essentially telling the student, "You can learn how to do problem fifteen by copying example three." They should not be doing math that way. They should be doing math by, … problem fifteen … should say, "See the definition of—in order to solve the problem."

Mr. T.: Well, ten years ago, I was a different teacher. … I had only been teaching for a couple of years, and it was, "Let's get the content done and teach them how to do the problems." So that was the perspective I had. So it was very much centered on the textbook, "Here's how to do the problems, these are the formulas, these are the points they need. So you have to know *a* is found this way, and you can do step one, step two, step three." I don't teach like that anymore at all. Now I teach from the perspective, "There's multiple ways of approaching a problem, and you want to come up with the mathematical clues that will help you solve that problem." Like if you're graphing a parabola, you could come up with a very procedural step one, two, three, four, five, six, seven, eight, nine, ten, eleven, to graph a parabola. Or you could say, "Let's look at this parabola and see what information we can discover. And using the tools that we already know, how can that help us draw a picture of it?" So we start with what do you think. …We get all this information on the board. Now let's draw it. And then we find out, "Wait, our vertex doesn't work." Now if you're following a procedure, step by step, I've seen a lot of kids will just try to make that work anyway. Because it's in the procedure, and now they don't know what to do because it didn't work. Whereas, if you're not following a procedure and it's just one of the clues, if one of the clues doesn't work and all the other clues lead towards this shape, it seems like your vertex is probably wrong. Which is really doing math. You're thinking about it more in terms of an overall picture rather than just procedural steps.

Mr. T.: If you're just doing practicing problems, what you're doing is you're memorizing a pattern. So as soon as they change the problem a little bit, you're done. It doesn't look anything like you've done before, so [students] get upset. Like "You didn't teach this," and "How dare you." Well, I did teach it, but I didn't give you that problem before. And I think that's one of the transitions we need to start making in math, as we move towards the future, we have to get kids to stop looking at problems as the ends and think more along the lines of the concepts and the definitions and the theorems are the ends, and the problems are really just a way of getting there and providing a more variety of problems and challenging them to think about these things in different ways.

Mr. T.: Because I know how they do math. And this is a fight that I have with students. … They have gotten to the point where they just want to know "How do I solve the problems in the book? What are the steps?" So in the back of their minds, they have an agenda. It's counter to my agenda. My agenda is to get them to understand the concept. Their agenda is to solve the problems on the homework, to figure out what problems I'm going to ask on the test, and to figure out exactly how to answer those questions. And they're looking for that. They're looking for the bottom line so to speak, without all of the concept piece.

Mr. T.: So when I start a lesson with a concrete example, a real-world example, that's where I'm getting my concept in. Because as soon as I introduce the algebra, I lose them. They completely forget about the concept, all that goes out the window, all they're interested in is, “Can I get the answer?” So had I started with that, they would have already reached their agenda. And whatever real-life quote unquote example I gave them at that point would not have mattered to them at all. They wouldn't have cared. And it would have been completely lost on them, because they already got their agenda fulfilled. They figured out how to solve the problems. So it didn't matter to them. It's all lovely, he can talk and chat all he wants, I'm going to sit here and know that I just need to plug this number into here and I get my answer and that's all it's going to be. Well, that's a loss at that point. You lost something.

Goal 2: **Teach concepts through real-world examples**

Orientations:

- Math is a way of explaining the universe.
- Real-world examples should be used to introduce new concepts.
- Students remember mathematics better with real-world examples.
- Real-world examples should not be over-simplified for students.
- Students should solve real mathematical problems that do not have clear solution paths.
- Real-world examples are a meaningful way to apply mathematical facts and procedures.
- Real-world examples are motivating for students.

Knowledge:

- How the textbook present mathematical concepts and procedures
- How mathematics is used in the real world: Whispering chambers, elliptical orbits

---

Mr. T.: One of the thoughts I have in general about math is that, it goes back to Galileo, where we use math to explain our universe. So the universe existed first, and math is here to explain it. We don't apply the universe to math. We apply math to the universe. So you can't really talk about a real-world example backwards, this doesn't make any sense. So it makes more sense to talk about, this is the real-world example. Let's see what math we can use to explain it. So then at that point, it gives the real connection to this is, this is real world.

Mr. T.: So if you can find things [real-world examples] that [students] can really do math with and learn through this process, it's going to stick with them longer. Because now they have an anchor, they have a memory anchor to tie this to. … Whereas otherwise, it's just another word problem.

Mr. T.: I like to call them fake real-world problems—where they put them in the book and go, “This is real-world.” And I'm looking at the problems, [and] they have stripped so much of the real math out of the problem to make it doable for that level that they missed the whole point. It's just fake. It doesn't make any sense.

Mr. T.: The more modern books have started getting better, where they have the application, but a lot of times they're still just really basic or they're out-dated or they're uninteresting, or they're over-simplified.

Mr. T.: You can boil this all down to a series of steps and formulas, [and the students] are not really learning anything. They can spit back the answers, but if I ask them the question in a slightly different way, they won't be able to answer it. They'll be done. So having this real-world application where it's not so cut and dry, … you're really talking about solving a real problem. And you're approaching math completely differently than you would just to solve a textbook algebra problem. You're solving math by taking the knowledge that you know and trying to explain the world in front of you.

Mr. T.: [Using real-world examples] has a big payoff in the end, because as I go through these different [topics], I can always go back to those examples. … So we can really get into a lot of the math, but then we can always go back and refer to the real-world examples and say, “Oh yeah, that's right, that's what we're talking about. That's why this works.” Otherwise, it's just graphs and procedures and it doesn't make a whole lot of sense.

Mr. T.: [Students] are not just following an algebraic process and pulling out numbers and giving me answers based on some algebraic process. So instead of just being a collection of procedures and steps, it's a thing. It's a real life thing that they can connect to something that's in the real world, and we continued that as we got into hyperbolas this week. We started again with another real life application, actually we started talking about Comet Ison which is coming in November, which is a great application for hyperbolas.

Mr. T.: So the [whispering chamber] … had this interesting effect where you have the sound at one point reflects off the [dome], and comes down to another point. … What's really interesting about the video is when [the tour guide is] standing next to [the tourists] and giving the talk, it's hard to hear her because the room is so loud and there're so many people in there. But when she moves to that point forty feet away, it's actually a little bit clearer to hear her. So it kind of really drives home that there's something going on there. And it has to do with the way the ellipses work where sound travels from one focus to another focus.

Goal 3: **Make connections among mathematical ideas**

Orientations:

- The teacher's role is to make connections to students' previous content and mathematics they will see in the future.

Knowledge:

- Knowledge of previous content and future content
- Connections between mathematical ideas: e.g., ellipse and circle definitions, ellipse and circle equations, transformations of graphs in the coordinate plane
- Representations which illuminate connections between ideas

Mr. T.: If you don't really understand [the content] inside and out and see all the connections to where they're going and where they've been, then you're not really doing any much better than the textbook is. And they could just read it from the book. Your job as a teacher is to help them see the content but also make connections to things that they've seen in the past and allude to things that they're going to see in the future and position them in a trajectory so they can hit that.

Mr. T.: [In] the first couple of years [of teaching], you're only looking at Algebra One, maybe Algebra Two, and you miss the connections all the way through. So one of the things I'm thinking about is in the future, how does this connect to precalculus? How does this connect to polar coordinates? How is this connecting back to Algebra Two and back to Algebra One?

Mr. T.: So, if you think about the equations of circles, when you're graphing a circle, a circle has a center and the circle is all the points that are equidistant from that center. One of the things we've been talking about is eccentricity. So if you lay the groundwork that when you look at a circle that has a center, then you make that connection to ellipses, that ellipses really kind of have two centers called foci, you can tie those together using the term of eccentricity, because then that follows through to hyperbolas. So the eccentricity when you're talking about for an ellipse is between zero and one. The eccentricity for a circle would be zero. The eccentricity for a hyperbola would be a value that's great than one. So it almost kind of takes an ellipse, pulls it apart, flips it out. So if you're making that, if you know that connection, as you go through, you can really talk about how that standard form relates to the other standard forms, but more importantly, what effect the foci have on the shapes and how moving the foci from being on top of each other, as a circle, pulling them apart to make an ellipse and pulling them further apart to make a hyperbola. How that connection works is really important.

Kathryn: Okay, so you, so the reason why you kind of really want to emphasize [eccentricity] is because of that connection back to circles?

Mr. T.: Yep, and the greater concept of conics in general. Because I think they need that connection all the way through. So from the center out to *c*, that's your focal length, and it's the ratio of *c* over *a*, and that's how you're getting the eccentricity. And then, you always say that you know it has to be a value between zero and one for an ellipse and then greater than one when we get to hyperbolas, so we did explicitly talk about the ratio. But, … I didn't want to just introduce the ratio saying, "This is the ratio," because it's a decimal. What the heck does that mean? … They can actually see how that's related and talk about how that changes the shape.

Mr. T.: [To explain the connection to students], I could take x squared plus y squared equals r squared, divide both sides by r squared so it looked like the standard form of an ellipse. … Which in reality, it is because it just has the same a and b: The major axis and the minor axis are the same value, and the foci are at zero. So being able to understand that myself allowed me to … make that connection for them.

Goal 4: **Illustrate why mathematical facts are true**

Orientations:

- Students should understand why mathematical facts are true.
- If students understand where mathematical facts come from, it will be easier for them to remember those facts.

Knowledge:

- How to derive mathematical facts: The standard form of the equation for an ellipse
- Student strengths and difficulties with the content
- Students' orientations towards mathematics: New facts appear "magically"

Mr. T.: You can use algebra to take those pieces and manipulate it to get the general form. It's not magic. Because a lot of times they look at this, they're like, "Oh, okay. So there it is, it's magic." And then they get to the just memorize steps as opposed to trying to understand where these things come from. If they have that anchor of where it comes from, it will be easier for them to remember that general form.

Mr. T.: I showed them that they can use the distance formula, and the definition of an ellipse to get to the standard form of an ellipse. Which was the overall goal of that.

Mr. T.: [My goal] was to show them that it was derived. That it did come from somewhere, that it's not that somebody decided to make up this magical equation that works. It has a history. It has a connection to something that they already learned, and they can see that it has an important algebra derivation.

Mr. T.: In general, I like to show them where things came from. I guess the core idea that they should be asking, "Why? Why is this true?" And if I can't at least give them some evidence that what I'm doing is true, there's no reason they should believe me. I could be making it all up. I could be wrong. It's possible. It's happened before, where somebody has explained something incorrectly, I'm sure. But if I can show them where all of my steps came from, then there's good evidence for why what I'm saying is true. So I'm modeling a behavior for them. I'm modeling that the way you do math is you need to demonstrate. Not just, oh it's because it is. You need to demonstrate where things come from.

References

Aguirre, J., & Speer, N. M. (2000). Examining the relationship between beliefs and goals in teacher practice. *Journal of Mathematical Behavior, 18,* 327-356. http://dx.doi.org/10.1016/S0732-3123(99)00034-6

Anderson, J., White, P., & Sullivan, P. (2005). Using a schematic model to represent influences on, and relationships between, teachers' problem-solving beliefs and practices. *Mathematics Education Research Journal, 17,* 9-38. doi:10.1007/BF03217414

Asikainen, M. A., Pehkonen, E., & Hirvonen, P. E. (2013). Finnish mentor mathematics teachers' views of the teacher knowledge required for teaching mathematics. *Higher Education Studies, 3*, 79-91. doi:10.5539/hes.v3n1p79

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal, 93*, 373-397. http://dx.doi.org/10.1086/461730

Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education, 51*, 241-247. doi:10.1177/0022487100051003013

Ball, D. L., & Bass, H. (2002). Toward a practice-based theory of mathematical knowledge for teaching. In E. Simmt & B. Davis (Eds.), *Proceedings of the 2002 annual meeting of the Canadian Mathematics Education Study Group* (pp. 3-14). Edmonton, Alberta.

Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 27-44). Reston, VA: National Council of Teachers of Mathematics.

Ball, D. L., Hill, H. C., & Bass, H. (2005, Fall). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 14-22; 43-46.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*, 389-407. doi:10.1177/0022487108324554

Barkatsas, A., & Malone, J. (2005). A typology of mathematics teachers' beliefs about teaching and learning mathematics and instructional practices. *Mathematics Education Research Journal, 17,* 69-90. doi:10.1007/BF03217416

Battey, D. (2013). "Good" mathematics teaching for students of color and those in poverty: The importance of relational interactions in instruction. *Educational Studies in Mathematics, 82*, 125-144. http://dx.doi.org/10.1007/s10649-012-9412-z

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., ... Tsai, Y.-M. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal*, *47*, 133-180. doi:10.3102/0002831209345157

Bednarz, N., & Proulx, J. (2009). Knowing and using mathematics in teaching: Conceptual and epistemological clarifications. *For the Learning of Mathematics, 29*(3), 11-17.

Bell, C. A., Wilson, S. M., Higgins, T., & McCoach, D. B. (2010). Measuring the effects of professional development: The case of developing mathematical ideas. *Journal for Research in Mathematics Education*, *41*, 479-512.

Beswick, K. (2007). Teachers' beliefs that matter in secondary mathematics classrooms. *Educational Studies in Mathematics, 65,* 95-120. doi:10.1007/s10649-006-9035-3

Brenner, M. E. (2006). Interviewing in educational research. In J. L. Green, G. Camilli, & P. B. Elmore (Eds.), *Handbook of complementary methods in education research* (pp. 357-370). Mahwah, NJ: Lawrence Erlbaum Associates.

Brophy, J. E., & Good, T. L. (1986). Teacher behaviour and student achievement. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (pp. 328–375). New York: MacMillan.

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher, 18,* 32-42. doi:10.2307/1176008

Brown, L., & Coles, A. (2000). Complex decision making in the classroom: The teacher as an intuitive practitioner. In T. Atkinson & G. Claxton (Eds.) *The intuitive practitioner: On the value of not always knowing what one is doing.* (pp. 165-181). Open University Press: Buckingham, England.

Brown, L., & Coles, A. (2011). Developing expertise: How enactivism re-frames mathematics teacher development. *ZDM Mathematics Education*, *43*, 861-873. doi:10.1007/s11858-011-0343-4

Brown, S., & McIntyre, D. (1993). *Making sense of teaching.* Buckingham, UK: Open University Press.

Bryan, T. J. (1999). The conceptual knowledge of preservice secondary mathematics teachers: How well do they know the subject matter they will teach? *Issues in the Undergraduate Mathematics Preparation of School Teachers: The Journal*, *1*, 1–12.

Cankoy, O. (2010). Mathematics teachers' topic-specific pedagogical content knowledge in the context of teaching a^0, 0! and a/0. *Educational Sciences: Theory & Practice*, *10*, 749-769.

Chapman, O. (2002). Belief structure and inservice high school mathematics teacher growth. In G. C. Leder, E. Pehkonen, & G. Törner (Eds), *Beliefs: A hidden variable in mathematics education?* (pp. 177-193). Dordrecht, The Netherlands: Kluwer Academic.

Charalambous, C. Y., & Hill, H. C. (Eds.). (2012). *Journal of Curriculum Studies*, *44*(4).

Charmaz, K. (2002). Qualitative interviewing and grounded theory analysis. In J. F. Gubrium & J. A. Holstein (Eds.), *Handbook of interview research: Context and method* (pp. 675-694). Thousand Oaks, CA: Sage.

Charmaz, K. (2011). A constructivist grounded theory analysis of losing and regaining a valued self. In J. Wertz, K. Charmaz, L. M. McMullen, R. Josselson, R. Anderson, & E. McSpadden (Eds.), *Five ways of doing qualitative analysis* (pp. 165-204). New York, NY: The Guilford Press.

Chimombo, M. P. F., & Roseberry, R. I. (1998). *The power of discourse: An introduction to discourse analysis.* Mahwah, NJ: Lawrence Erlbaum Associates.

Chinnappan, M., & Lawson, M. J. (2005). A framework for analysis of teachers' geometric content knowledge and geometric knowledge for teaching. *Journal of Mathematics Teacher Education*, *8*, 197-221. doi:10.1007/s10857-005-0852-6

Clemente, M., & Ramírez, E. (2008). How teachers express their knowledge through narrative. *Teaching and Teacher Education, 24,* 1244-1258. doi:10.1016/j.tate.2007.10.002

Cobb, P. (1986). Contexts, goals, beliefs, and learning mathematics. *For the Learning of Mathematics, 6,* 2-9.

Cochran-Smith, M., & Lytle, S. L. (1999). Relationships of knowledge and practice: Teacher learning in communities. *Review of Research in Education, 24,* 249-305. doi:10.2307/1167272

Cohen, D. K., & Ball, D. L. (2000). *Instruction and innovation: Reconsidering the story*. Working paper, Consortium for Policy Research in Education, Study of Instructional Improvement. Ann Arbor, MI: University of Michigan.

Collins, A., Brown, J. S., & Newman, S. E. (1989). Cognitive apprenticeship: Teaching the crafts of reading, writing, and matheamtics. In L. B. Resnick (Ed.), *Knowing, learning, and instruction: Essays in honor of Robert Glaser* (pp. 453-494). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Conference Board of the Mathematical Sciences. (2001). *The mathematical education of teachers.* Providence RI and Washington DC: American Mathematical Society and Mathematical Association of America.

Cooney, T. J., Shealy, B. E., & Arvold, B. (1998). Conceptualizing belief sturctures of preservice secondary mathematics teachers. *Journal for Research in Mathematics Education*, *29*, 306-333.

Council of Chief State School Officers. (2012). National teacher of the year program. http://www.ccsso.org/ntoy.html

Creswell, J. W. (2007). *Qualitative inquiry and research design: Choosing among five approaches*. Thousand Oaks, CA: Sage.

Danielson Group (2013). The Danielson Group Framework. Retrieved from: http://danielsongroup.org/framework/

Davies, N., & Walker, K. (2005, July). Learning to notice: One aspect of teachers' content knowledge in the numeracy classrooms. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Theory research and practice. Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australia,* pp. 273-280. Sydney: MERGA.

Davis, B., & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the mathematics that teachers (need to) know. *Educational Studies in Mathematics, 61,* 293-319. doi:10.1007/s10649-006-2372-4

Davis, R. B., Maher, C. A., & Noddings, N. (1990). Introduction: Constructivist views on the teaching and learning of mathematics. In *Journal for Research in Mathematics Education Monograph number 4: Constructivist views on the teaching and learning of mathematics* (pp. 1-3). Reston, VA: National Council of Teachers of Mathematics.

DeBellis, V. A., & Goldin, G. A. (2006). Affect and meta-affect in mathematical problem solving: A representational perspective. *Educational Studies in Mathematics, 63,* 131-147. doi:10.1007/s10649-006-9026-4

Deng, Z. (2007). Knowing the subject matter of a secondary-school science subject. *Journal of Curriculum Studies, 39,* 503-535. doi:10.1080/00220270701305362

Depaepe, F., Verschaffel, L., & Kelchtermans, G. (2013). Pedagogical content knowledge: A systematic review of the way in which the concept has pervaded mathematics educational research. *Teaching and Teacher Education, 34,* 12-25. doi:10/1016/j.tate.2013.03.001

Dunkin, M. J., Welch, A., Merritt, A., Phillips, R., & Craven, R. (1998). Teachers' explanations of classroom events: Knowledge and beliefs about teaching civics and citizenship. *Teaching and Teacher Education, 14,* 141-151. doi:10.1016/S0742-051X(97)00033-4

Ebert, C. L. (1993, April). *An assessment of prospective secondary teachers' pedagogical content knowledge about functions and graphs.* Paper presented at the annual meeting of the American Educational Research Association, Atlanta, GA.

Ethell, R. G., & McMeniman, M. M. (2000). Unlocking the knowledge in action of an expert practitioner. *Journal of Teacher Education, 51,* 87-101. doi:10.1177/002248710005100203

Etkina, E. (2010). Pedagogical content knowledge and preparation of high school physics teachers. *Physical Review Special Topics Physics Education Research, 6,* 020110.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics, 21,* 521–544. doi:10.1007/BF00315943

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education, 24,* 94–116. doi:10.2307/749215

Even, R., & Tirosh, D. (1995). Subject-matter knowledge and knowledge about students as sources of teacher presentations of the subject-matter. *Educational Studies in Mathematics*, *29*, 1–20. doi:10.1007/BF01273897

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V., R., & Epson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27,* 403-434. http://dx.doi.org/10.2307/749875

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York: National Council of Teachers of Mathematics.

Ferrini-Mundy, J., Floden, R., McCrory, R., Burrill, G., & Sandow, D. (2005, April). *Knowledge for teaching school algebra: Challenges in developing an analytic framework.* Paper presented at the annual meeting of the American Education Research Association, Montreal, Quebec, Canada.

Forgasz, H. J., & Leder, G. C. (2008). Beliefs about mathematics and mathematics teaching. In P. Sullivan & T. Wood (Eds.), *The international handbook of mathematics teacher education: Vol 1. Knowledge and beliefs in mathematics teaching and teaching development* (pp. 173-192). Rotterdam, the Netherlands: Sense.

Franke, M. L., Kazemi, E., & Battey, D. (2007). Understanding teaching and classroom practice in mathematics. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 225-256). Charlotte, NC: Information Age Publishing.

Furinghetti, F., & Pehkonen, E. (2002). Rethinking characterizations of beliefs. In G. C. Leder, E. Pehkonen, & G. Törner (Eds), *Beliefs: A hidden variable in mathematics education?* (pp. 39-57). Dordrecht, The Netherlands: Kluwer Academic.

Gess-Newsome, J. (1999). Pedagogical content knowledge: An introduction and orientation. In J. Gess-Newsome, & N. G. Lederman (Eds.), *Examining pedagogical content knowledge: The construct and its implications for science education* (pp. 3-17). Hingham, MA: Kluwer Academic Publishers.

Goldsmith, L. T., & Seago, N. (2011). Using classroom artifacts to focus teacher' noticing: Affordances and opportunities. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 169-187). New York, NY: Routledge.

Goulding, M., Hatch, G., & Rodd, M. (2003). Undergraduate mathematics experience: its significance in secondary mathematics teacher preparation. *Journal of Mathematics Teacher Education*, 6, 361–393.

Greeno, J. G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education, 22,* 170-218. http://dx.doi.org/10.2307/749074

Greeno, J. G., & The Middle School Mathematics Through Applications Project Group (1998). The situativity of knowing, learning, and research. *American Psychologist, 53*, 5-26. doi:10.1037//0003-066X.53.1.5

Hannah, J., Stewart, S., & Thomas, M. (2011). Analysing lecturer practice: The role of orientations and goals. *International Journal of Mathematical Education in Science and Technology, 42,* 975-984. http://dx.doi.org/10.1080/0020739X.2011.610008

Herbst, P., & Chazan, D. (2003). Exploring the practical rationality of mathematics teaching through conversations about videotaped episodes: The case of engaging students in proving. *For the Learning of Mathematics, 23,* 2-14.

Herbst, P., & Chazan, D. (2012). On the instructional triangle and sources of justification for actions in mathematics teaching. *ZDM Mathematics Education*, *44*, 601-612. http://dx.doi.org/10.1007/s11858-012-0438-6

Hill, H. C. (2007). Mathematical knowledge of middle school teachers: Implications for the No Child Left Behind policy initiative. *Educational Evaluation and Policy Analysis*, *29*, 95-114. doi:10.3102/0162373707301711

Hill, H. C. (2010). The nature and predictors of elementary teachers' mathematical knowledge for teaching. *Journal for Research in Mathematics Education*, *41*, 513-545.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39,* 372-400.

Hill, H. C., Blunk, M. L., Charalambos, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and mathematical quality of instruction: An exploratory study. *Cognition and Instruction*, *26*, 430-511. doi:10.1080/07370000802177235

Hill, H. C., & Charalambous, C. Y. (2012). Teacher knowledge, curriculum materials, and quality of instruction: Lessons learned and open issues. *Journal of Curriculum Studies*, *44,* 559-576. http://dx.doi.org/10.1080/00220272.2012.716978

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, *42,* 371-406. doi:10.3102/00028312042002371

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal*, *105*, 11-30.

Hill, H. C., Sleep, L., Lewis, J. M., & Ball, D. L. (2007). Assessing teachers' mathematical knowledge: What knowledge matters and what evidence counts? In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 111-155). Charlotte NC: Information Age Publishing.

Horn, I. S. (2007). Fast kids, slow kids, lazy kids: Framing the mismatch problem in mathematics teachers' conversations. *Journal of the Learning Sciences, 16*, 37-79. doi:10.1080/10508400709336942

Hoyles, C. (1992). Mathematics teaching and mathematics teachers: A meta-case study. *For the Learning of Mathematics, 12,* 32-44.

Izsak, A., Orrill, C. H., Cohen, A. S., & Brown, R. E. (2010). Measuring middle grades teachers' understanding of rational numbers with the mixture Rasch model. *The Elementary School Journal*, *110*, 279-300. doi:10.1086/648979

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*, 169-202.

Jacobs, V. R., Lamb, L. L. C., Philipp, R. A., & Schappelle, B. P. (2011). Deciding how to respond on the basis of children's understandings. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 97-116). New York, NY: Routledge.

Johnson, E. M. S., & Larsen, S. P. (2012). Teacher listening: The role of knowledge of content and students. *Journal of Mathematical Behavior, 31,* 117-129. http://dx.doi.org/10.1016/j.jmathb.2011.07.003

Kahan, J. A., Cooper, D. A., & Bethea, K. A. (2003). The role of mathematics teachers' content knowledge in their teaching: A framework for research applied to a study of student teachers. *Journal of Mathematics Teacher Education, 6*, 223–252. doi:10.1023/A:1025175812582

Kajander, A. (2010). Teachers constructing concepts of mathematics for teaching and learning: "It's like the roots beneath the surface, not a bigger garden." *Canadian Journal of Science, Mathematics and Technology Education, 10*, 87-102. doi:10.1080/14926151003778274

Kennedy, M. M. (1998). Education reform and subject matter knowledge. *Journal of Research in Science Teaching, 35*, 249-263. doi:10.1002/(SICI)1098-2736(199803)35:3<249::AID-TEA2>3.3.CO;2-T

Kilpatrick, J., Martin, W. G., & Schifter, D. (Eds.). (2003). *A research companion to Principles and Standard for School Mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Krauss, S., Brunner, M., Kunter, M., Baumert, J., Blum, W., Neubrand, M., & Jordan, A. (2008). Pedagogical content knowledge and content knowledge of secondary mathematics teachers. *Journal of Educational Psychology, 100*, 716-725. doi:10.1037/0022-0663.100.3.716

Kvale, S., & Brinkmann, S. (2009). *Interviews: Learning the craft of qualitative research interviewing*. Thousand Oaks, CA: Sage.

Learning Mathematics for Teaching (2010). Mathematical quality of instruction [Measurement instrument]. Retrieved from http://isites.harvard.edu/icb/icb.do?keyword=mqi_training

Learning Mathematics for Teaching (2011). Measuring the mathematical quality of instruction. *Journal of Mathematics Teacher Education, 14,* 25-47. doi:10.1007/s10857-010-9140-1

Learning Mathematics for Teaching (2012). Learning mathematics for teaching (LMT) project. http://sitemaker.umich.edu/lmt/home

Leatham, K. (2006). Viewing mathematics teachers' beliefs as sensible systems. *Journal of Mathematics Teacher Education, 9,* 91-102. doi:10.1007/s10857-006-9006-8

Leder, G. C., Pehkonen, E., & Törner, G. (Eds). (2002). *Beliefs: A hidden variable in mathematics education?* Dordrecht, The Netherlands: Kluwer Academic.

Leikin, R., & Zazkis, R. (Eds.) (2010). *Learning through teaching mathematics: Development of teachers' knowledge and expertise in practice.* Dordrecht, The Netherlands: Springer.

Lesh, R., & Lehrer, R. (2000). Iterative refinement cycles for videotape analyses of conceptual change. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 665-708). Mahwah, NJ: Lawrence Erlbaum Associates.

Li, Y., & Kaiser, G. (Eds.). (2011). *Expertise in mathematics instruction: An international perspective*. New York: Springer. doi:10.1007/978-1-4419-7707-6

Lloyd, G. M. (1999). Two teachers' conceptions of a reform-oriented curriculum: Implications for mathematics teacher development. *Journal of Mathematics Teacher Education, 2,* 227-252. doi:10.1023/A:1009965804662

Lloyd, G. M., & Wilson, M. (1998). Supporting innovation: The impact of a teacher's conceptions of functions on his implementation of a reform curriculum. *Journal for Research in Mathematics Education*, *29,* 248-274. doi:10.2307/749790

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Magnusson, S., Krajcik, J., & Borko, H. (1999). Nature, sources and development of pedagogical content knowledge for science teaching. In J. Gess-Newsome & N. G. Lederman (Eds.), *Examining pedagogical content knowledge: The construct and its implications for science education* (pp.95-132). Dordrecht, The Netherlands: Kluwer Academic.

Maher, C. A., & Davis, R. B. (1990). Teachers' learning: Building representations of children's meanings. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Journal for Research in Mathematics Education Monograph number 4: Constructivist views on the teaching and learning of mathematics* (pp. 79-90). Reston, VA: National Council of Teachers of Mathematics.

Maher, C. A., & Martino, A. M. (1996). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education, 27,* 194-214.

Martino, A. M. & Maher, C. A. (1999). Teacher questioning to promote justification and generalization in mathematics: What research practice has taught us. *Journal of Mathematical Behavior, 18*, 53-78. doi:10.1016/S0732-3123(99)00017-6

Mason, J. (2008). PCK and beyond. In P. Sullivan & T. Wood (Eds.), *The international handbook of mathematics teacher education: Vol 1. Knowledge and beliefs in mathematics teaching and teaching development* (pp. 301-322). Rotterdam, the Netherlands: Sense.

Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing to act in the moment. *Educational Studies in Mathematics, 38*, 135-161. doi:10.1023/A:1003622804002

McDonald, M., Kazemi, E., & Kavanagh, S. S. (2013). Core practices and pedagogies of teacher education: A call for a common language and collaborative activity. *Journal of Teacher Education, 64*, 378-386. doi: 10.1177/0022487113493807

Mejia-Ramos, J. P., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. *Educational Studies in Mathematics, 79*, 3-18. doi: 10.1007/s10649-011-9349-7

Merriam, S. B. (1998). *Qualitative research and case study applications in education*. San Francisco, CA: Jossey-Bass.

Moreira, P. C., & David, M. M. (2008). Academic mathematics and mathematical knowledge needed in school teaching practice: Some conflicting elements. *Journal of Mathematics Teacher Education, 11*, 23-40. doi:10.1007/s10857-007-9057-5

National Board for Professional Teaching Standards. (2014). Mathematics/ adolescence and young adulthood. http://www.nbpts.org/mathematics-aya

National Center for Teacher Effectiveness. (2012). Mathematical quality of instruction. http://isites.harvard.edu/icb/icb.do?keyword=mqi_training

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics* [Online version]. Retrieved from http://www.nctm.org/standards/content.aspx?id=26792

National Science Foundation. (2009). Presidential awards for excellence in mathematics and science teaching. http://www.paemst.org/controllers/home.cfc?method=view

New Jersey Monthly. (2012, August). The top New Jersey high schools: District factor group. *New Jersey Monthly*. Retrieved from http://njmonthly.com/articles/towns_and_schools/the-top-new-jersey-high-schools-2012.html

Oslund, J. A. (2012). Mathematics-for-teaching: What can be learned from the ethnopoetics of teachers' stories? *Educational Studies in Mathematics, 79,* 293-309. doi:10.1007/s10649-011-9348-8

Patton, M. Q. (1990). *Qualitative evaluation and research methods.* Newbury Park, CA: Sage.

Pennsylvania Department of Education. (2014). Keystone exams. http://www.portal.state.pa.us/portal/server.pt/community/state_assessment_system/20965/keystone_exams/1190529

Peterson, B. E., & Leatham, K. R. (2009). Learning to use students' mathematical thinking to orchestrate a class discussion. In L. Knott (Ed.), *The role of mathematics discourse in producing leaders of discourse* (pp. 99-128). Charlotte, NC: Information Age Publishing.

Petrou, M., & Goulding, M. (2011). Conceptualising teachers' mathematical knowledge in teaching. In T. Rowland & K. Ruthven (Eds.), *Mathematical Knowledge in Teaching* (pp. 9-26). Dordrecht, The Netherlands: Springer.

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 257-315). Charlotte, NC: Information Age Publishing.

Philipp, R. A., Ambrose, R., Lamb, L. L. C., Sowder, J. T., Schappelle, B. P., Sowder, L., Thanheiser, E., & Chavout, J. (2007). Effects of early field experiences on the mathematical content knowledge and beliefs of prospective elementary school teachers: An experimental study. *Journal for Research in Mathematics Education, 38*, 438-476.

Piaget, J. (1977/2001). *Studies in reflecting abstraction* (R. L. Campbell, Trans.). Sussex, UK: Psychology Press.

Potari, D., Zachariades, T., Christou, K., Kyriazis, G., & Pitta-Pantazi, D. (2007). Teachers' mathematical knowledge and pedagogical practices in the teaching of derivative. *Proceedings of the fifth congress of the European Society for Research in Mathematics Education*, 1955-1964. Retrieved from http://ermeweb.free.fr/Cerme5.pdf

Powell, A. B., Francisco, J. M., & Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *Journal of Mathematical Behavior*, *22,* 405-435. doi:10.1016/j.jmathb.2003.09.002

Powell, A. B., & Hanna, E. (2006). Researching teachers' knowledge for teaching mathematics. In S. Alatorre, J. L. Cortina, M. Saiz, & A. Mendez (Eds.), *Proceedings of the 28th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 377-383). Merida, Mexico: Universidad Pedagogica Nacional.

Proulx, J. (2008). Exploring school mathematics as a source for pedagogic reflections in teacher education. *Canadian Journal of Science, Mathematics, and Technology Education, 8,* 331-354. doi:10.1080/14926150802506266

Raudenbush, S. W., Rowan, B., & Cheong, Y. F. (1993). Higher order instructional goals in secondary schools: Class, teacher, and school influences. *American Educational Research Journal*, *30*, 523-553. doi:10.3102/00028312030003523

Raymond, A. M. (1997). Inconsistency between a beginning elementary school teacher's mathematics beliefs and teaching practice. *Journal for Research in Mathematics Education, 28,* 550-576. http://dx.doi.org/10.2307/749691

Rhoads, K. (2011). Mathematical knowledge for teaching: Exemplary high school teachers' views. In S. Brown, S. Larsen, K. Marrongelle, and M. Oehrtman (Eds.), *Proceedings of the 14th Annual Conference on Research in Undergraduate Mathematics Education, Vol. 2* (pp. 381-398). Portland, Oregon.

Rowland, T. (2007). Developing knowledge for mathematics teaching: A theoretical loop. In S. Close, D. Corcoran, & T. Dooley (Eds.), *Proceedings of the second national conference on research in mathematics education MEI 2* (pp. 14-27). Dublin: Saint Patrick's College of Education.

Rowland, T. (2008). Researching teachers' mathematics disciplinary knowledge. In P. Sullivan & T. Wood (Eds.), *The international handbook of mathematics teacher education: Vol 1. Knowledge and beliefs in mathematics teaching and teaching development* (pp. 273-298). Rotterdam, the Netherlands: Sense.

Ruthven, K. (1987). Ability stereotyping in mathematics. *Educational Studies in Mathematics, 18,* 243-253. http://dx.doi.org/10.1007/BF00386197

Sánchez, V., & Llinares, S. (2003). Four student teachers' pedagogical reasoning on functions. *Journal of Mathematics Teacher Education, 6,* 5–25. doi:10.1023/A:1022123615355

Schepens, A., Aelterman, A., & Keer, H. V. (2007). Studying learning processes of student teachers with stimulated recall interviews through changes in interactive cognitions. *Teaching and Teacher Education, 23,* 457-472. doi:10.1016/j.tate.2006.12.014

Schoenfeld, A. H. (1999). Toward a theory of teaching-in-context. *Issues in Education, 4,* 1-94. doi:10.1016/S1080-9724(99)80076-7

Schoenfeld, A. H. (2011). *How we think: A theory of goal-oriented decision making and its educational applications*. New York, NY: Routledge.

Schoenfeld, A. H., Minstrell, J., & Van Zee, E. (2000). The detailed analysis of an established teacher's non-traditional lesson. *Journal of Mathematical Behavior, 18,* 281-325. doi:10.1016/S0732-3123(99)00035-8

Seidman, I. (2006). *Interviewing as qualitative research: A guide for researchers in education and the social sciences*. New York, NY: Teachers College Press.

Seymour, J. R., & Lehrer, R. (2006). Tracing the evolution of pedagogical content knowledge as the development of interanimated discourses. *Journal of the Learning Sciences, 15,* 549-582. doi:10.1207/s15327809jls1504_5

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher, 27,* 4-13. http://dx.doi.org/10.2307/1176193

Shechtman, N., Roschelle, J., Haertel, G., & Knudsen, J. (2010). Investigating links from teacher knowledge, to classroom practice, to student learning in the instructional system of the middle-school mathematics classroom. *Cognition and Instruction, 28,* 317-359. doi:10.1080/07370008.2010.487961

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15,* 4-14. doi:10.3102/0013189X015002004

Silver, E. A., & Mesa, V. (2011). Coordinating characterization of high quality mathematics teaching: Probing the intersection. In Y. Li, & G. Kaiser (Eds.), *Expertise in mathematics instruction: An international perspective* (pp. 63-84). New York, NY: Springer. doi: 10.1007/978-1-4419-7707-6_4

Silver, E. A., & Stein, M. K. (1996). The QUASAR Project: The "revolution of the possible" in mathematics instructional reform in urban middle schools. *Urban Education, 30,* 476-521.

Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education*, *11*, 499-511. doi:10.1007/s10857-008-9089-5

Simmt, E. (2011). Teacher expertise explored as matheamtics for teaching. In Y. Li & G. Kaiser (Eds.), *Expertise in Mathematics Instruction: An international perspective* (pp. 151-164). New York: Springer. doi:10.1007/978-1-4419-7707-6_8

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*, 114–145. doi:10.2307/749205.

Simon, M. A. (2006). Key developmental understanding in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning*, *8*, 359-371. doi:10.1207/s15327833mtl0804_1

Simon, M. A., & Tzur, R. (1999). Explicating the teacher's perspective from the researchers' perspectives: Generating accounts of mathematics teachers' practice. *Journal for Research in Mathematics Education, 30,* 252-264.

Skott, J. (2001). The emerging practices of a novice teacher: The roles of his school mathematics images. *Journal of Mathematics Teacher Education, 4,* 3-28. doi:10.1023/A:1009978831627

Sleep, L. (2012). The work of steering instruction toward the mathematical point: A decomposition of teaching practice. *American Educational Research Journal, 49*, 935-970. doi:10.3102/0002831212448095

Sleep, L., & Eskelson, S. L. (2012). MKT and curriculum materials are only part of the story: Insights from a lesson on fractions. *Journal of Curriculum Studies, 44,* 537-558. http://dx.doi.org/10.1080/00220272.2012.716977

Speer, N. M. (2005). Issues of methods and theory in the study of mathematics teachers' professed and attributed beliefs. *Educational Studies in Mathematics, 58*, 361-391. doi:10.1007/s10649-005-2745-0

Speer, N. M., & Wagner, J. F. (2009). Knowledge needed by a teacher to provide analytic scaffolding during undergraduate mathematics classroom discussions. *Journal for Research in Mathematics Education, 40,* 530-562.

Stacey, K. (2008). Mathematics for secondary teaching: Four components of discipline knowledge for a changing teacher workforce. In P. Sullivan & T. Wood (Eds.), *The international handbook of mathematics teacher education: Vol. 1 Knowledge and beliefs in mathematics teaching and teaching development* (pp. 87-113). Rotterdam, The Netherlands: Sense.

Star, J. R., Lynch, K., & Perova, N. (2011). Using video to improve preservice mathematics teachers' abilities to attend to classroom features: A replication study. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 117-133). New York, NY: Routledge.

State of New Jersey Department of Education. (2004). *District factor groups (DFG) for school districts.* Retrieved from http://www.state.nj.us/education/finance/rda/dfg.shtml

State of New Jersey Department of Education. (2013). High school proficiency assessment. http://www.state.nj.us/education/schools/achievement/13/hspa/

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap.* New York: The free press.

Stockero, S., & Van Zoest, L. (2013). Characterizing pivotal teaching moments in beginning mathematics teachers' practice. *Journal of Mathematics Teacher Eudcation, 16,* 125-147. doi:10.1007/s10857-012-9222-3

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research.* Newbury Park, CA: Sage.

Sztajn, P. (2003). Adapting reform ideas in different mathematics classrooms: Beliefs beyond mathematics. *Journal of Mathematics Teacher Education, 6,* 53-75.

Thomas, M., & Yoon, C. (2014). The impact of conflicting goals on mathematics teaching decisions. *Journal of Mathematics Teacher Education, 17,* 227-243. doi:10.1007/s10857-013-9241-8

Thompson, A. G. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics, 15,* 105-127.

Thompson, P. W. (1996). Imagery and the development of mathematical reasoning. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp.267-283). Mahwah, NJ: Erlbaum.

Törner, G., Rolka, K., Rösken, B., & Sriraman, B. (2010). Understanding a teacher's actions in the classroom by applying Schoenfeld's theory *teaching-in-context:* Reflecting on goals and beliefs. In B. Sriraman, & L. English (Eds.), *Theories of mathematics education: Advances in mathematics education* (pp. 401-420). Berlin: Springer.

U.S. Department of Education. (2007). *Teacher trends.* Retrieved from National Center for Education Statistics website: http://nces.ed.gov/fastfacts/display.asp?id=28

U.S. Department of Education. (2008). *Trends in international mathematics and science study*. Retrieved from National Center for Education Statistics website: http://nces.ed.gov/timss/index.asp

Vacc, N. B., & Bright, G. W. (1999). Elementary preservice teachers' changing beliefs and instructional use of children's mathematical thinking. *Journal for Research in Mathematics Education*, *30,* 89-110.

Van Es, E. A. (2011). A framework for learning to notice student thinking. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 134-151). New York, NY: Routledge.

Watson, A., & De Geest, E. (2005). Principled teaching for deep progress: Improving mathematical learning beyond methods and materials. *Educational Studies in Mathematics, 58,* 209-234. http://dx.doi.org/10.1007/s10649-005-2756-x

Weber, K., & Rhoads, K. (2011). A review of Learning through teaching mathematics: Development of teachers' knowledge and expertise in practice, [book by R. Leikin & R. Zazkis (Eds.)]. *Journal for Research in Mathematics Education, 42*, 521-527.

Wilson, M., & Cooney, T. J. (2002). Mathematics teacher change and development: The role of beliefs. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 127-147). Dordrecht, The Netherlands: Kluwer Academic.

Wood, T., & Sellers, P. (1997). Deepening the analysis: Logitudinal assessment of a problem-centered mathematics program. *Journal for Research in Mathematics Education*, *28*, 163-186.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27,* 458-477.

Yin, R. K. (2009). *Case study research: Design and methods.* Thousand Oaks, CA: Sage.

Zazkis, R., & Leikin, R. (2010). Advanced mathematical knowledge in teaching practice: Perceptions of secondary mathematics teachers. *Mathematical Thinking and Learning*, *12*, 263-281. doi:10.1080/10986061003786349

Zimmerlin, D., & Nelson, M. (2000). A detailed analysis of a beginning teacher carrying out a traditional lesson. *Journal of Mathematical Behavior, 18,* 263-279. http://dx.doi.org/10.1016/S0732-3123(99)00032-2

Zohar, A., Degani, A., & Vaaknin, E. (2001). Teachers' beliefs about low-achieving students and higher order thinking. *Teaching and Teacher Education, 17*, 469-485. doi:10.1007/978-1-4020-1854-1_10

Acknowledgement of Previous Publication

Rhoads, K. (2011). Mathematical knowledge for teaching: Exemplary high school teachers' views. In S. Brown, S. Larsen, K. Marrongelle, and M. Oehrtman (Eds.), *Proceedings of the 14th Annual Conference on Research in Undergraduate Mathematics Education, Vol. 2* (pp. 381-398). Portland, Oregon.

Rhoads, K. (2013). High school mathematics teachers' reflections on videos of their teaching: Exploring teacher knowledge and instruction. In M. V. Martinez & A. Castro Superfine (Eds.), *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (p. 957). Chicago, IL: University of Illinois at Chicago.