Applications of Weak attraction theory in Out($\mathbb{F}_N$)

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Abstract

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Given a finite rank free group $\mathbb{F}_N$ of rank ≥ 3 and two exponentially growing outer automorphisms ψ and ϕ with dual lamination pairs $\Lambda^{\pm}_{\psi}$ and $\Lambda^{\pm}_{\phi}$ associated to them, which satisfy a notion of independence described in this paper, we will use the pingpong techniques developed by Handel and Mosher [14] to show that there exists an integer $M > 0$, such that for every $m, n \geq M$, the group $G_M = \langle \psi^m, \phi^n \rangle$ will be a free group of rank two and every element of this free group which is not conjugate to a power of the generators will be fully irreducible and hyperbolic. We will also look at a different proof of the theorem of Kapovich and Lustig in [18] which shows that the Cannon-Thurston map for a fully-irreducible hyperbolic automorphism exists and is finite-to-one.

Dedicated to my parents and all my teachers.

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1 Introduction

Let $\mathbb{F}_N$ denote a finite rank free group of rank $N \geq 3$. The Outer automorphism group of $\mathbb{F}_N$ (denoted by $\mathrm{Out}(\mathbb{F}_N)$) is the quotient group $\mathrm{Aut}(\mathbb{F}_N)/\mathrm{Inn}(\mathbb{F}_N)$. Long ago, Nielsen gave us a finite generating set for $\mathrm{Out}(\mathbb{F}_N)$ and Whitehead modified this generating set and used them to produce an algorithm to test when an element of $\mathbb{F}_N$ can be part of some basis of the free group. However the study of $\mathrm{Out}(\mathbb{F}_N)$ grew slowly for the next few decades until Culler and Vogtmann constructed *Outer space* in [7]. Since then, over the last two decades this group has been studied with great interest by Geometric group theorists. Some of the very interesting results proven during the early phase were the *Scott Conjecture*[19] and the *Tit's Alternative for* $\mathrm{Out}(\mathbb{F}_N)$[22], [3]. In the process of these developments the theory of *Train Track maps* evolved. Inspired by Thurston's idea of train tracks and John Stallings' elegant paper about folding maps [26], train track maps have become a key tool for studying outer automorphisms of free groups. Roughly speaking, train track maps are a special type of homotopy equivalence maps on a graph G whose fundamental group is the given Free group. Not all automorphisms admit a train track map, however, "fully-irreducibles" do. For the more complex types of automorphisms we use the notion of a *Relative Train track map*, where the graph G has to be broken down into a filtration of invariant subgraphs and each filtration element is studied individually. Fortunately, the number of levels of filtrations are finite. The theory of train track maps saw further developments in a series of other papers [21], [12], [14] and it became the key tool in understanding the dynamics of elements of $\mathrm{Out}(\mathbb{F}_N)$.

My first goal as a graduate student was to have a detailed understanding of this theory and all the results that were proved using this technology.

Ignoring the base points, conjugacy classes in F_n are represented by closed loops in G. The group $\mathrm{Out}(\mathbb{F}_\mathrm{N})$ acts on the set of all conjugacy classes of F_n and this action is studied via the induced train track map. Conjugacy classes are either fixed or their growth is bounded by some polynomial function or they grow exponentially under iterations by a nontrivial element $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$. Accordingly, if ϕ (nontrivial) makes some conjugacy class grow exponentially, we call it an *exponentially growing* outer automorphism, otherwise if the growth rate of every conjugacy class is bounded by some polynomial function we call it *polynomially growing.* It is to be noted that, as n grows, there exist outer automorphisms of arbitrarily high polynomial growth. A very special class of exponentially growing outer automorphisms are the *fully irreducibles*, no power of which has any invariant free factors. These elements closely resemble the *pseudo Anosov* homeomorphisms of surfaces. There exist two types of fully irreducible elements: *geometric*, which are induced by homeomorphisms of surfaces of appropriate topological type and *nongeometric*, which are not induced by any such surface homeomorphism. In [24] Rivin shows that the most generic type of outer automorphisms (in the sense of random walks) are fully irreducibles. but [10] shows that geometric fully irreducibles are rare, leaving us with nongeometric fully irreducibles as the generic elements of $\mathrm{Out}(\mathbb{F}_\mathrm{N})$. It was shown by Bestvina, Feighn[2] and Brinkmann [4] that an automorphism was *hyperbolic* if and only if it did not have any periodic conjugacy classes. Thus in case of fully irreducibles, nongeometric and hyperbolic are equivalent.

The question that grabbed my interest was: exactly how generic are they from a dynamical viewpoint ?

Farb and Mosher proved the following in [1]:

Given two independent pseudo-Anosov mapping classes f , g of a closed and oriented surface, sufficiently high powers of f and g will generate a free group of rank 2 and every element of this group will be pseudo-Anosov.

The **primary goal of my thesis** now became to prove something similar in the case of $\mathrm{Out}(\mathbb{F}_\mathrm{N})$ and also investigate how easily can one actually produce the nongeometric fully irreducibles (most generic type) from ones that might not be fully irreducible themselves. Along with that, the aim was to prove it in a way that somehow explains the cause of this abundance. I also wanted to see what are all possible types of outer automorphisms which when composed after passing to sufficiently high powers give the generic elements of $\mathrm{Out}(\mathbb{F}_\mathrm{N})$ and why it so happens. This broadly explains the problem of my thesis and also my deep interest in it.

The first result of my thesis is the following:

Theorem *Let $(\phi, \Lambda^{\pm}_{\phi}), (\psi, \Lambda^{\pm}_{\psi})$ be two exponentially growing, pairwise independent elements of $\mathrm{Out}(\mathbb{F}_\mathrm{N})$. Then there exists an $M \geq 0$, such that for all $p, q \geq M$ the group $\langle \psi^p, \phi^q \rangle$ will be free of rank two and every element of this free group, not conjugate to some power of the generators, will be hyperbolic and fully-irreducible.*

Kapovich and Lustig had already proved an analogous version of the aforementioned result of Farb-Mosher for $\mathrm{Out}(\mathbb{F}_\mathrm{N})$ [17], in which they used the theory of *Currents in* F_n [15], [16].

Our problems however, were different. To explain, let us look at an *attracting lamination* Λ^+_ϕ as a special collection of lines, which is closed in the weak topology and it has a certain collection of dense leaves called *generic leaves*. Under iterations of ϕ every conjugacy class either does or does not limit to some generic leaf (called *attracted* or *nonattracted*) in Λ^+_ϕ. Kapovich and Lustig start with fully irreducible, nongeometric elements for which the notion of "independence" was already defined in [22], but the right definition for non fully-irreducibles was not known when I started. One big obstacle was that for hyperbolic fully irreducibles every conjugacy class gets attracted to Λ^+_ϕ, for non fully-irreducibles the situation is much worse. Understanding the details proofs of [1] and [22] revealed the correct definition. What we define as ***independent pairs*** roughly means that the laminations $\Lambda^\pm_\phi$ and $\Lambda^\pm_\psi$ fill the free group, they are pairwise disjoint and every conjugacy class is attracted to one of the laminations under iteration by the appropriate automorphism and the laminations themselves are attracted to each other under iterations by the appropriate automorphism.

Designing the proof in a way that enabled us to see exactly why hyperbolic fully irreducibles are abundant was a challenge. The tool of choice for me was to use train track maps. There is a rich theory of the study of *laminations* for exponentially growing elements of $\mathrm{Out}(\mathbb{F}_\mathrm{N})$. By design, the theory of laminations captured astonishing amount of the dynamical character of ϕ. Bestvina, Feighn and Handel were the first to develop this theory in [22]. Later on, it was studied in more details in [21], [12]. Finally in [13], [14] we get an almost complete understanding of this *Weak Attraction* theory. One technical improvement we introduce is to modify some of the results of [14] so that they can be applied to elements of $\mathrm{Out}(\mathbb{F}_\mathrm{N})$ that are not rotationless

Once this was established, the rest of the proof is divided into two parts. First is to do a pingpong argument on the leaves of the laminations and generate the attracting and repelling lamination pairs for elements of the group $\langle \psi^p, \phi^q \rangle$, and it was here that the reason for abundance of hyperbolic fully irreducibles become apparent. Two laminations which are "sufficiently distinct" from one another mix up very quickly under iterations of ϕ and ψ due to pingpong and possibly fill the entire group to produce a new lamination. This also shows why the elements of the free group will be nongeometric, since all conjugacy classes are weakly attracted to the attracting lamination produced by pingpong. The second part of the proof uses *Stallings graph* to support a contradiction argument to show that the laminations produced by pingpong indeed fill F_n. This completes the proof.

It gave rise to very interesting **applications**, one of which is the main theorem by Kapovich, Lustig in [17] (mentioned above) which follows as a corollary by taking ϕ, ψ to be hyperbolic fully irreducibles. The other corollaries give a complete picture of what happens when we work with arbitrary fully irreducible elements. It turns out that except in the particular case, when both ϕ and ψ are induced by pseudo-Anosov homeomorphisms of the same surface $\mathcal{S}$ (with one boundary component), we almost always end up with hyperbolic fully irreducibles. In the only exceptional case we can can prove that we will end up with a free group of rank two all whose elements are induced by pseudo-Anosov homeomorphisms of the S. Very similar to what Farb, Mosher proved in [1] except that they had closed surface.

2 Preliminaries

2.1 Free group, Free basis and Free factors:

Let $\mathbb{F}$ be a group and $S \subset \mathbb{F}$ be a subset. Define S^{-1} to be the set $\{s^{-1} | s \in S\}$. A *word* over the set $S \cup S^{-1}$ is a finite sequence $s_1, s_2, ..., s_p$ where each $s_i \in S \cup S^{-1}$. This is often abused with the *evaluation* of a word, which is the product of the word using the group operation in $\mathbb{F}$ and written as as a concatenation $s_1 s_2 ..... s_p$. It is possible that there might be cancellations in this product whenever we have consecutive letter pairs of the form ss^{-1} or $s^{-1}s$. If no such cancellations are possible, the word is called a *reduced word*. A reduced word is said to be *cyclically reduced* if it is stable under conjugation by elements of $\mathbb{F}$.

We say that a set $S \subset \mathbb{F}$ is a *free basis* if every word in $\mathbb{F}$ can be written as a unique reduced word over $S \cup S^{-1}$. A group is said to be a *free group* if it has a free basis. The group operation in this case becomes concatenation of reduced words followed by subsequent cancellations to form a reduced word. Given a free group, any of it's free basis always has the same cardinality, usually denoted by *rank*$(\mathbb{F})$. This number is the dimension of the vector space $ab(\mathbb{F}) \otimes_{\mathbb{Z}} \mathbb{R}$. If rank$(\mathbb{F})$ is finite we call $\mathbb{F}$ to be a finite rank free group. For the purpose of this work, we only deal with finite rank free groups such that rank($\mathbb{F}$) ≥ 3.

Notation: For convenience, we will denote $a^{-1} \in \mathbb{F}_N$ by A.

The group operation described above can be realized geometrically. For this, let R be the standard rose with n-petals (a simplicial complex with one vertex and n oriented-edges attached to it.), where $n = rank(\mathbb{F})$. The fundamental group of R, with the unique vertex as a basepoint, is $\mathbb{F}$. There is a bijective correspondence between the edges in R and the elements of the free basis of $\mathbb{F}$. Label each edge by a unique element of the free basis. Every reduced word in $\mathbb{F}$ can be now realized as a loop based at the vertex of R. Cancellation in $\mathbb{F}$ corresponds to ignoring edges in R over which there is backtracking. This process is called *tightening* a loop(or a path in general). The theory of fundamental groups tells us that there is bijective correspondence between reduced words in $\mathbb{F}$ and loops based at the basepoint (homotopy rel basepoint). If we forget the basepoint, this bijection induces a bijection between the conjugacy classes of reduced words in $\mathbb{F}$ and the set of free homotopy classes of *circuits* in R, where a circuit is a continuous, locally injective map $S^1 \to R$. This bijection is of fundamental importance to us.

It is a well known result that a subgroup of a free group is free and has both algebraic proofs and topological proofs which use covering space theory. However, subgroup of a finitely generated free group $\mathbb{F}$ may have higher rank than $\mathbb{F}$, and can be infinite. There are, however, a special class of subgroups of $\mathbb{F}$ called *free factors* which are of more interest to us. A subgroup $H \leq \mathbb{F}$ is called a free factor if $\mathbb{F} = H * B$, where B can be the trivial group. In this case H has rank less than $\mathbb{F}$. Moreover, the free basis of H can be extended to form a free basis of $\mathbb{F}$. Similarly, we define a *free factor system* to be a collection of conjugacy classes of free factors $\mathcal{F} = \{[H_1], [H_2], ..., [H_p]\}$ such that $\mathbb{F} = H_1 * H_2 * .... * H_p * B$, for some B, possibly trivial. (Here $[H]$ denotes the conjugacy class of the subgroup H).

More generally, one can define a *subgroup system* to be a finite collection of conjugacy classes of finitely generated subgroups of $\mathbb{F}_N$. Every free factor system is a subgroup system.

We say that a conjugacy class $[c]$ is *carried by a free factor system* (or a *subgroup system*) $\mathcal{F} = \{[H_1], [H_2], ..., [H_p]\}$ if there exists some i such that $c \in H_i$. It is possible that there does not exist any proper free factor of $\mathbb{F}_N$ which contains c, in which case we say that the conjugacy class *c fills the free group* $\mathbb{F}_N$.

Example 2.1. Let $\mathbb{F}_N = \langle a, b, c\rangle$. Then the subgroups $H_1 = \langle ab, a\rangle$ and $H_2 = \langle c\rangle$ are both free factors of $\mathbb{F}_N$ and $\mathbb{F}_N = H_1 * H_2$.

Note that H_1 is same as the subgroup $\langle a, b\rangle$ with just a different choice of basis.

$\mathcal{F}_1 = \{[H_1]\}$, $\mathcal{F}_2 = \{[\langle a\rangle], [\langle b\rangle]\}$ and $\{[H_1], [H_2]\}$ are examples of free factor systems. The conjugacy class $[abaaba]$ is carried by both $\mathcal{F}_1$ and $\mathcal{F}_3$ but not by $\mathcal{F}_2$. $[abc]$ is an example of a conjugacy class that is not carried by any of the above free factor systems, but it is not a filling conjugacy class.

Example 2.2. With the $\mathbb{F}_N$ as above one can see that $\langle a^2b^2, ab\rangle$ is not a free factor of $\mathbb{F}_N$ since a^2b^2 cannot be extended to a basis of $\mathbb{F}_N$. Thus $\mathcal{A} = \{[\langle a^2b^2, ab\rangle], [H_1]\}$ is a subgroup system of $\mathbb{F}_N$ but not a free factor system.

2.2 $\mathrm{Aut}(\mathbb{F}_\mathrm{N})$ and it's elements:

By $\mathrm{Aut}(\mathbb{F}_\mathrm{N})$ we denote the automorphism group of $\mathbb{F}_N$. When one tries to study $\mathrm{Aut}(\mathbb{F}_\mathrm{N})$, there are some purely algebraic techniques, but we shall employ more geometric methods. The geometric tools that has been developed to study Free groups, it's automorphisms and their dynamics arises from a very important geometric property of finitely generated free groups. A free group $\mathbb{F}_N$, is a 0-hyperbolic group (in the sense of Gromov) and hence, has a well defined Gromov boundary, which we denote by $\partial\mathbb{F}_N$. The Cayley graph of $\mathbb{F}_N$ is a tree and the boundary of this tree is homeomorphic to the boundary of $\mathbb{F}_N$, which is a Cantor set.

The group $\mathrm{Aut}(\mathbb{F}_\mathrm{N})$ acts on $\partial\mathbb{F}_N$. Let $\widehat{\Phi} : \partial\mathbb{F}_N \to \partial\mathbb{F}_N$ denote the action of $\Phi \in \mathrm{Aut}(\mathbb{F}_\mathrm{N})$ and let $\mathrm{Fix}(\widehat{\Phi})$ denote the fix point set of this action. Let $\mathrm{Fix}(\Phi)$ denote the fixed subgroup of Φ, considered as an automorphism of $\mathbb{F}_N$. The solution of Scott conjecture by Bestvina-Handel in [19] tells us that $\mathrm{rank}(\mathrm{Fix}(\Phi)) \leq \mathrm{rank}(\mathbb{F}_N)$. Let $\partial\mathrm{Fix}(\Phi) \subset \partial\mathbb{F}_N$ denote it's boundary. $\partial\mathrm{Fix}(\Phi)$ is trivial when $\mathrm{Fix}(\Phi)$ is trivial, has two points when $\mathrm{Fix}(\Phi)$ has rank 1 or a Cantor set when $\mathrm{Fix}(\Phi)$ has rank ≥ 2. We call an element P of $\mathrm{Fix}(\widehat{\Phi})$ *attracting fixed point* if there exists an open neighborhood $U \subset \partial\mathbb{F}_N$ of P such that we have $\widehat{\Phi}(U) \subset U$ and for every point $Q \in U$ the sequence $\widehat{\Phi}^n(Q)$ converges to P. Let $\mathrm{Fix}_+(\widehat{\Phi})$ denote the set of attracting fixed points of $\mathrm{Fix}(\widehat{\Phi})$. Similarly let $\mathrm{Fix}_-(\widehat{\Phi})$ denote the attracting fixed points of $\mathrm{Fix}(\widehat{\Phi}^{-1})$. The following lemma from [9] gives the relation between these sets.

Lemma 2.3. *For each $\Phi \in \mathrm{Aut}(\mathbb{F}_\mathrm{N})$ the set $Fix(\widehat{\Phi})$ is a union of the sets $Fix(\Phi)$, $Fix_-(\widehat{\Phi})$, $Fix_+(\widehat{\Phi})$. This union is a disjoint union except for the case* *when Fix*(Φ) *has rank 1 and Fix*($\widehat{\Phi}$) $= \partial$*Fix*(Φ) $=\{\xi_-, \xi_+\}$ *where* $\xi_- = Fix_-(\widehat{\Phi})$ *and* $\xi_+ = Fix(\widehat{\Phi})$.

Let $\mathrm{Fix}_N(\widehat{\Phi}) = \mathrm{Fix}(\widehat{\Phi}) - \mathrm{Fix}_-(\widehat{\Phi}) = \partial\mathrm{Fix}(\Phi) \cup \mathrm{Fix}_+(\widehat{\Phi})$ denote the set of non-repelling fixed points. This set carries some vital information about the dynamics of the outer automorphism class of Φ, as we will later see.

2.3 $\mathrm{Out}(\mathbb{F}_\mathrm{N})$ and it's elements:

Let $X_\mathcal{C}$ denote the set of all conjugacy classes of reduced words in $\mathbb{F}_N$, and $X_\mathcal{F}$ denote the set of all conjugacy classes of free factors in $\mathbb{F}_N$. $\mathrm{Out}(\mathbb{F}_\mathrm{N})$ admits an action on the set $X_\mathcal{C}$ and the set $X_\mathcal{F}$.

An outer automorphism of $\mathbb{F}_N$ is said to to be *fully-irreducible* if it does not have any periodic orbits in $X_\mathcal{F}$. If the action of an outer automorphism fixes a point in $X_\mathcal{F}$, it is said to be *reducible*. The other remaining case when it has a periodic orbit of period greater than 1, it is called *irreducible*. Any such irreducible automorphism can be made reducible by passing onto some some power, called *rotationless* power, which we shall explain later in this chapter. It is to be noted that detecting whether an automorphism is fully irreducible is a difficult problem. One tool that helps us in this process is the subject of Chapter 2, *train track maps,* which are some well controlled homotopy equivalences on graphs that helps us study finer properties of individual outer automorphisms.

An automorphism $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ to be **hyperbolic** if there exists some $M > 0$

and $\lambda > 1$ such that

$$\lambda|g| \leq max\{|\phi^M(g)|, |\phi^{-M}(g)|\} \quad \forall g \in \mathbb{F}_N$$

Brinkmann [4] showed that this is equivalent to requiring that ϕ has no periodic conjugacy classes.

Fully-irreducible outer automorphisms are of two types: *Geometric* or *Nongeometric*.

Consider a compact, connected surface $\mathcal{S}$ with one boundary component. Any such surface can be homotoped to it's spine, which is a finite, connected graph, whose fundamental group is a finite rank free group, say $\mathbb{F}_N$. A pseudo-Anosov homeomorphism of $\mathcal{S}$ induces an automorphism of $\mathbb{F}_N$. The outer automorphism class of such an automorphism is called *geometric fully-irreducible* outer automorphism. Note that the homeomorphism fixes the boundary curve and hence the induced outer automorphism will fix the conjugacy class representing the boundary curve. Thus every geometric fully-irreducible outer automorphism fixes a (necessarily unique) conjugacy class. The following result from [19, Theorem 4.1]

Proposition 2.4. *Let $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ be a fully-irreducible element. If there exists some conjugacy class $\mathcal{C}$ such that $\phi(\mathcal{C}) = \mathcal{C}$ or $\bar{\mathcal{C}}$, then ϕ is a geometric fully-irreducible outer automorphism, i.e. it is induced by a pseudo-Anosov homeomorphism of some surface $\mathcal{S}$ with one boundary component. The conjugacy class of the boundary curve in $\mathbb{F}_N$ is $\mathcal{C}$.*

This result tells us that nongeometric fully-irreducible elements of $\mathrm{Out}(\mathbb{F}_\mathrm{N})$

are all hyperbolic.

Example 2.5. Nongeometric fully irreducible example: $\mathbb{F}_N = \langle a, b, c \rangle$. Consider the map:

$a \mapsto ab, b \mapsto ac, c \mapsto a$

reducible example: $\mathbb{F}_N = \langle a, b, c \rangle$. Consider the map:

$a \mapsto a, b \mapsto Abaca, c \mapsto bacaB$

2.4 Bringing graphs to the picture:

For our purposes, graphs are finite 1-dimensional CW-complexes.

A *path* in a finite graph Γ is a locally injective, continuous map $\gamma : J \to \Gamma$ from a closed, connected, nonempty subset $J \subset \mathbb{R}$ to Γ such that if we take any lift $\widetilde{\gamma} : J \to \widetilde{\Gamma}$ of this path to the universal covering tree $\widetilde{\Gamma}$ of Γ , then this lift is proper (inverse image of a compact set is compact). A *circuit* is a continuous, locally injective map of a oriented circle into Γ. A path is said to be *trivial* if J is a point. Two paths are *equivalent* if they are equal up to an orientation preserving homeomorphism between their domains. Every nontrivial path is expressed as a concatenation of edges and partial edges, concatenated at the vertices, in the following sense:

- a *finite path* is a finite concatenation $E_0E_1....E_n$ for $n \geq 0$ where only E_0, E_n are the possible partial edges.

- a *positive ray* is an infinite concatenation $E_0E_1....$, where E_0 is the only possible partial edge.

- a *negative ray* is an infinite concatenation $....E_{-2}E_{-1}E_0$, where E_0 is the only possible partial edge.

- a *line* is a bi-infinite concatenation $...E_{-1}E_0E_1....$, where none of the E_i's are partial.

These expressions are unique upto a translation of indices. By the locally injective property we must have $E_i \neq \overline{E_{i+1}}$, where $\overline{E_j}$ denotes the inverse of E_j.

Notation: Let $\widehat{\mathcal{B}}(\Gamma)$ denote the set of all paths, circuits and lines in Γ. The weak topology on $\widehat{\mathcal{B}}(\Gamma)$ is defined by the basis elements $\widehat{N}(\Gamma, \alpha)$, where α is a finite path in Γ and $\widehat{N}(\Gamma, \alpha)$ is the collection of all paths, circuits and lines that contain α as a subpath. Denote by $\mathcal{B}(\Gamma)(\subset \widehat{\mathcal{B}}(\Gamma))$ the compact subspace consisting of all lines in Γ with basis elements $N(\Gamma, \alpha) = \widehat{N}(\Gamma, \alpha) \cap \mathcal{B}(\Gamma)$.
Consider the continuous function $\gamma : J \to \Gamma$, where J is closed, connected and nonempty. If J is noncompact , each lift of $\widetilde{\gamma}$ is proper and induces an injection from the ends of J to the ends of Γ. If J is a compact interval then either γ is homotopic rel endpoints to a unique nontrivial path denoted by $[\gamma]$ in Γ, or γ is homotopic to a constant path in which case $[\gamma]$ denotes the trivial path. $[\gamma]$ is called the *tightening* of γ. If J is noncompact, then γ is homotopic to a unique path $[\gamma]$ in Γ and this homotopy is proper, meaning that its lifts to the universal cover $\widetilde{\Gamma}$ are all proper; equivalently, γ and the path $[\gamma]$ have lifts to $\widetilde{\Gamma}$ which have the same finite endpoints and the same infinite end.
For $n \geq 2$ let R be the graph which is the wedge of n circles with directed edges labeled $E_1, E_2, ..., E_n$. Then $\pi_1(R)$ of this graph is $\mathbb{F}_N$, the free group of rank n with $E_1, E_2, ...E_n$ as the free basis. A graph R with such labeling is

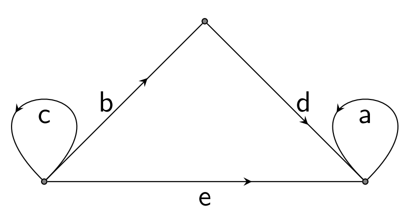
called *base rose*. **Notation:** We shall always denote the base rose of rank n by R.

A *rank n core graph* G is a finite, connected graph with Euler characteristic $\mathcal{X}(G) = 1 - n$. By definition, it does not have any valence 1 vertex.

A *marked graph* is a rank n core graph G with a homotopy equivalence from the base rose $\rho : R \to G$. The homotopy equivalence ρ is called a *marking*. One can assign lengths to each edge and give the graph a path metric structure. Such a graph is called a *marked metric graph*. Recall that every graph is a Eilenberg-Mclane space and thus, a map $f : G \to H$ is a homotopy equivalence if and only if the induced map $f_* : \pi_1(G) \to \pi_1(H)$ is an isomorphism. Hence, we say that a homotopy equivalence $f : G \to G$ *represents* an outer automorphism ϕ if the outer automorphism class of the induced automorphism f_* is ϕ.

A *topological representative* for an outer automorphism ϕ is a marked graph G (with marking ρ) equipped with a homotopy equivalence $f : G \to G$ such that f takes vertices to vertices and edges to edge paths in G and the homotopy equivalence $\bar{\rho} \circ f \circ \rho : R \to R$ represents ϕ.

Example 2.6.



This is a graph G with the marking from the standard rose given by $a \mapsto a, b \mapsto AEbd, c \mapsto Ece$. Under this marking, consider the map $f : G \to G$ on the graph given by:

$$a \mapsto a, b \mapsto bdaEc, c \mapsto bdaEceaDB, d \mapsto ea, e \mapsto ea$$

This marked graph together with the map f represents the outer automorphism given by:

$$a \mapsto a, b \mapsto Abaca, c \mapsto bacaB$$

We will later use this example again to understand several concepts related to train-track theory.

Let $\widetilde{R}$ be the tree which is the universal cover of R. Then it's set of ends is a Cantor set. If $\partial\mathbb{F}_N$ is the Gromov boundary of $\mathbb{F}_N$, then $\partial\mathbb{F}_N$ can be identified with this Cantor set of ends. Under this identification, let

$$\widetilde{\mathcal{B}} = (\partial\mathbb{F}_N \times \partial\mathbb{F}_N - \Delta)/\mathbb{Z}_2$$

where Δ is the diagonal set of the product $\partial\mathbb{F}_N \times \partial\mathbb{F}_N$ and $\mathbb{Z}_2$ acts by interchanging factors. Equip $\widetilde{\mathcal{B}}$ with the *weak topology* induced by the standard cantor topology on $\partial\mathbb{F}_N$. The group $\mathbb{F}_N$ acts on $\widetilde{\mathcal{B}}$ with compact but non-Hausdorff quotient space $\mathcal{B} = \widetilde{\mathcal{B}}/\mathbb{F}_N$. The elements of $\mathcal{B}$ are called *lines*. The induced quotient topology will also be called the *weak topology*. A lift of a line $l \in \mathcal{B}$ is an element $\widetilde{l} \in \widetilde{\mathcal{B}}$ whose quotient is l and the two elements of $\widetilde{l}$ are called it's *endpoints*.

Given any marked graph G, one can naturally identify the two spaces $\mathcal{B}(G)$ and

$\mathcal{B}$ by considering a homeomorphism between the two Cantor sets $\partial\mathbb{F}_N$ and set of ends of G. $\mathrm{Out}(\mathbb{F}_\mathrm{N}) \curvearrowright \mathcal{B}$. The actions comes from the action of $\mathrm{Aut}(\mathbb{F}_\mathrm{N})$on $\partial\mathbb{F}_N$, described earlier. Given any two marked graphs G, G' and a homotopy equivalence $f : G \to G'$ between them, the induced map $f_\# : \widehat{\mathcal{B}}(G) \to \widehat{\mathcal{B}}(G')$ is continuous and the restriction $f_\# : \mathcal{B}(G) \to \mathcal{B}(G')$ is a homeomorphism. With respect to the identification $\mathcal{B}(G) \approx \mathcal{B} \approx \mathcal{B}(G')$, if f preserves the marking then $f_\# : \mathcal{B}(G) \to \mathcal{B}(G')$ is equal to the identity map on $\mathcal{B}$. When $G = G'$, $f_\#$ agree with their homeomorphism $\mathcal{B} \to \mathcal{B}$ induced by the outer automorphism associated to f. A critical lemma while dealing with homotopy equivalences between graphs is the *Bounded Cancellation lemma* due to Cooper [6]:

Lemma 2.7. *For any homotopy equivalence between marked graphs $f : G \to G'$ there exists a constant BCC(f) such that for any lift $\widetilde{f} : \widetilde{G} \to \widetilde{G'}$ to universal covers and any path $\widetilde{\gamma}$ in $\widetilde{G}$, the path $f_\#(\gamma)$ is contained in the BCC(f) neighborhood of the image $f(\gamma)$.*

Proof. see [20] Lemma 2.3.1 . □

A line(path) γ is said to be *weakly attracted* to a line(path) β under the action of $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$, if the $\phi^k(\gamma)$ converges to β in the weak topology. This is same as saying, for any given finite subpath of β, $\phi^k(\gamma)$ contains that subpath for some value of k; similarly if we have a homotopy equivalence $f : G \to G$, a line(path) γ is said to be *weakly attracted* to a line(path) β under the action of $f_\#$ if the $f^k_\#(\gamma)$ converges to β in the weak topology. The *accumulation set* of a ray γ in G is the set of lines $l \in \mathcal{B}(G)$ which are elements of the weak closure of γ; which is same as saying every finite subpath of l occurs infinitely many times as a subpath γ.

A line $l \in \mathcal{B}(G)$ is *birecurrent* if l is in the closure of some(any) positive subray of l and l is in the closure of some(any) negative subray of l. This is equivalent to saying that every finite subpath of l occurs infinitely many times in l in both directions.

2.5 Attracting Laminations:

For any marked graph G, the natural identification $\mathcal{B} \approx \mathcal{B}(G)$ induces a bijection between the closed subsets of $\mathcal{B}$ and the closed subsets of $\mathcal{B}(G)$. A closed subset in any of these two cases is called a *lamination*, denoted by Λ. Given a lamination $\Lambda \subset \mathcal{B}$ we look at the corresponding lamination in $\mathcal{B}(G)$ as the realization of Λ in G. An element $\lambda \in \Lambda$ is called a *leaf* of the lamination.

A lamination Λ is called an *attracting lamination* for ϕ is it is the weak closure of a line l (called the *generic leaf of* Λ) satisfying the following conditions:

- l is birecurrent leaf of Λ.
- l has an *attracting neighborhood* V, in the weak topology, with the property that every line in V is weakly attracted to l.
- no lift $\widetilde{l} \in \mathcal{B}$ of l is the axis of a generator of a rank 1 free factor of F_r .

We know from [20] that with each $\phi \in \mathrm{Out}(\mathbb{F}_{\mathrm{N}})$ we have a finite set of laminations $\mathcal{L}(\phi)$, called the set of *attracting laminations* of ϕ, and the set $\mathcal{L}(\phi)$ is invariant under the action of ϕ. When $\mathcal{L}(\phi)$ has multiple elements, ϕ can permute the elements of $\mathcal{L}(\phi)$. This makes it slightly tricky to study the dynamics of elements that are not fully-irreducible. This motivates the notion of *forward rotationless outer automorphisms.*

2.6 Principal automorphisms and rotationless outer automorphisms:

Given an outer automorphism $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$, we can consider a lift Φ in $\mathrm{Aut}(\mathbb{F}_\mathrm{N})$. We say that $\Phi \in \mathrm{Aut}(\mathbb{F}_\mathrm{N})$ in the outer automorphism class of ϕ is a *principal automorphism* if $\mathrm{Fix}_N(\widehat{\Phi})$ has at least 3 points or $\mathrm{Fix}_N(\widehat{\Phi})$ has exactly two points which are neither the endpoints of an axis of a covering translation, nor the endpoints of a of a generic leaf of the attracting lamination Λ^+_ϕ. The set of all principal automorphisms of ϕ is denoted by $P(\phi)$. Roughly speaking, what such lifts guarantees is the existence of certain lines which are not a part of the attracting lamination but it still fills the free group F_r. Such lines (called *singular lines*) will be a key tool in describing the set of lines which are not attracted to the attracting lamination of ϕ.
We then have the following lemma from [9] and [12]:

Lemma 2.8. *If $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ is fully irreducible and Φ is a principal automorphism representing ϕ, then:*

1. *If $Fix(\Phi)$ is trivial then $Fix_N(\widehat{\Phi})$ is a finite set of attractors.*

2. *If $Fix(\Phi) = \langle\gamma\rangle$ is infinite cyclic, then $Fix_N(\widehat{\Phi})$ is the union of the endpoints of the axis of the covering translation $t^{\pm}_\gamma$ with a finite set of t_γ-orbits of attractors.*

3. *If* $P \in Fix_N(\widehat{\Phi})$ *is an attractor then it is not the end points of an axis of any covering translation* t_γ*.*

Let $\mathrm{Per}(\widehat{\Phi}) = \cup_{k\geq 1}\mathrm{Fix}(\widehat{\Phi}^k)$, $\mathrm{Per}_+(\widehat{\Phi}) = \cup_{k\geq 1}\mathrm{Fix}_+(\widehat{\Phi}^k)$ and similarly define $\mathrm{Per}_-(\widehat{\Phi})$ and $\mathrm{Per}_N(\widehat{\Phi})$.

We say that $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ is rotationless if $\mathrm{Fix}_N(\widehat{\Phi}) = \mathrm{Per}_N(\widehat{\Phi})$ for all $\Phi \in P(\phi)$, and if for each $k \geq 1$ the map $\Phi \to \Phi^k$ induces a bijection between $P(\phi)$ and $P(\phi^k)$.

The following two important facts about rotationless automorphisms are taken from [21]. Whenever we write "pass to a rotationless power "we intend to use this uniform constant K given by the fact.

Lemma 2.9. *[21, Lemma 4.43] There exists a* K *depending only upon the rank of the free group* F_r *such that for every* $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$, ϕ^K *is rotationless.*

Lemma 2.10. *[21] If* $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ *is rotationless then:*

- *Every periodic conjugacy class of* ϕ *is a fixed conjugacy class.*
- *Every free factor system which is periodic under* ϕ *is fixed.*
- *The set* $\mathcal{L}(\phi)$ *is fixed pointwise.*

3 Train track maps

3.1 Topological representatives and Train track maps

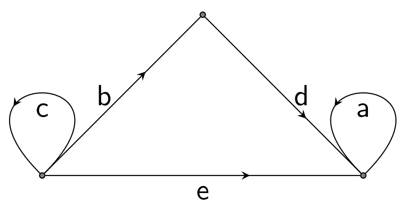
Recall that for any $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ a *topological representative* is a homotopy equivalence $f : G \to G$ such that $\rho : R \to G$ is a marked graph, f takes vertices to vertices and edges to paths and $\overline{\rho} \circ f \circ \rho : R \to R$ represents ϕ. A nontrivial path γ in G is a *periodic Nielsen path* if there exists a k such that $f^k_\#(\gamma) = \gamma$; the minimal such k is called the period and if $k = 1$, we call such a path *Nielsen path.* A periodic Nielsen path is *indivisible* if it cannot be written as a concatenation of two or more nontrivial periodic Nielsen paths.

Notation: Given a subgraph $H \subset G$ let $G \setminus H$ denote the union of edges in G that are not in H.

Filtrations, stratas and train track maps: A strictly increasing sequence of subgraphs $G_0 \subset G_1 \subset \cdots \subset G_k = G$, each with no isolated vertices, is called a *filtration of* G. The subgraph $H_i = G_i \setminus G_{i-1}$ is called a *stratum.* H_i will be called a stratum of *height* i. The stratum H_k is called the *top stratum.* Any path, circuit, ray or line is said to be of *height* s, if the highest strata it crosses is H_s. A topological representative is said to *preserve the filtration* if $f(G_i) \subset G_i$ for all i. Given some increasing sequence $\mathcal{F} = \mathcal{F}_1 \sqsubset \mathcal{F}_2 \cdots \sqsubset \mathcal{F}_N$ of free factor systems, we say that f *realizes* $\mathcal{C}$ if there exists a sequence of filtration elements G_{i_j} $(1 \leq j \leq N)$ such that $\mathcal{F}_j = [G_{i_j}]$

Let $f : G \to G$ be a topological representative of ϕ and $G_0 \subset G_1 \subset \cdots \subset G_k = G$ be a filtration preserved by f. Let H_r be a stratum with edges $\{E_1, E_2, ..., E_s\}$, define the *transition matrix of* H_r to be the square matrix whose ijth entry is equal to the number of times that the edge E_j crosses the path $f(E_i)$, both orientations counted. The transition matrix is *irreducible* if - for each pair i, j there exists some $p > 0$ such that the ijth entry of $p-$th power of the transition matrix is nonzero. If the transition matrix of a stratum H_i is irreducible, then we say that the stratum H_i is *irreducible.* It is known that if a some strata is not irreducible then the filtration can be may be refined so that every strata of the filtration is irreducible. Hence, for the rest of this work, whenever we mention a strata we assume that it is a irreducible strata.

Example 3.1.



Going back to example 2.6 let's call the vertex of the circuit a to be $*$. The map $f : G \to G$ was given by:

$$a \mapsto a, b \mapsto bdaEc, c \mapsto bdaEceaDB, d \mapsto ea, e \mapsto ea$$

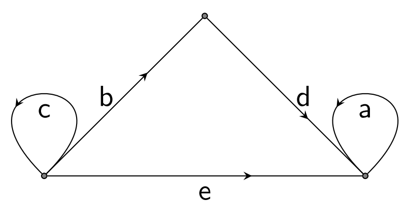
We can see that the graph G has a filtration given by $G_0 = \{*\}, G_1 = \{a\}, G_2 = \{a, e\}, G_3 = \{a, e, d\}, G_4 = \{a, e, d, c, b\}$. The corresponding stratum are $H_1 = \{a\}, H_2 = \{e\}, H_3 = \{d\}, H_4 = \{c, b\}$.

EG and NEG strata : Given some irreducible stratum H_r and a transition matrix T_r corresponding to that strata, the Perron Frobenius theorem [25] tells us that T_r has a unique eigenvalue (called the *Perron-Frobenius eigenvalue*) $\lambda \geq 1$ such that T_r has a positive eigenvector, which is also unique upto scalar multiplication, associated to eigenvalue λ. If $\lambda > 1$ we say that the stratum H_r is *exponentially growing strata or EG strata*. In this case we can write down the finitely many Perron-Frobenius eigenvalues in an increasing sequence and we call this sequence Γ. Note that we can order Γ lexicographically. If $\lambda = 1$, then we say that H_r is an *nonexponentially growing strata or NEG strata*. Given some NEG stratum H_r we can enumerate and orient its edges as $E_1, E_2, ..., E_N$ so that $f_{\#}(E_i) = E_i.u_{i-1}$ where u_{i-1} is a circuit in G_{i-1}. This is a consequence of assuming that the strata is H_r is irreducible, which in turn implies that the transition matrix is the permutation matrix of a permutation with single cycles in its cyclic decomposition. The NEG edge E_i is said to be *linear* if u_{i-1} is a nontrivial periodic Nielsen path. If each edge of a NEG stratum H_r is linear, then we say that H_r is a *linear stratum*. If $N = 1$, then H_r is a *fixed stratum* and E is a *fixed edge*. If each u_i is a trivial path, then we say that H_r is a *periodic* stratum and each edge E in H_r is a *periodic edge*. If the transition matrix is the zero matrix we say that H_r is a *zero stratum*.

A *direction* at some point $x \in G$ is the germ of finte paths with initial vertex x. If x is not a vertex of G, then number of directions at x is 2. A *turn* in a marked graph G is a pair of oriented edges of G originating at a common vertex. A turn is said to be *nondegenerate* if the nonoriented edges defining it are distinct; *degenerate* otherwise. A turn is *contained in the filtration (stratum)* element G_r (H_r) if both edges defining the turn are in G_r (H_r). If $E_1'E_2......E_{k-1}E_k'$ is the edge path associated to a α then we say that α *contains* *the turn* or *crosses the turn* $(E_i, \overline{E}_{i+1})$.

Given a marked graph G and a homotopy equivalence $f : G \to G$ that takes edges to paths, one can define a new map Tf by setting $Tf(E)$ to be the first edge in the edge path associated to $f(E)$; similarly let $Tf(E_i, E_j) = (Tf(E_i), Tf(E_j))$. So Tf is a map that takes turns to turns. We say that a nondegenerate turn is illegal if for some iterate of Tf the turn becomes degenerate; otherwise the turn is legal. A path is said to be legal if it contains only legal turns and it is r-legal if it is of height r and all its illegal turns are in G_{r-1}.

Example 3.2.



From example 3.1 we can see that H_1 is a fixed strata, H_2 is a linear-NEG strata and H_4 is an EG strata with expansion factor approximately 2.414.

Relative train track map. Given $\phi \in \text{Out}(\mathbb{F}_\text{N})$ and a topological representative $f : G \to G$ with a filtration $G_0 \subset G_1 \subset \cdots \subset G_k$ which is preserved by f, we say that f is a train relative train track map if the following conditions are satisfied:

1. f maps r-legal paths to r-legal paths.

2. If γ is a path in G with its endpoints in H_r then $f_\#(\gamma)$ has its end points

in H_r.

3. If E is an edge in H_r then $Tf(E)$ is an edge in H_r

Example 3.3. Example 3.1 is an example of a relative train-track map.

The following result relates the notion of attracting laminations to that of exponentially growing stratas. Note that, an outer automorphism has an attracting lamination if and only if it has an exponentially growing strata.

Lemma 3.4. *If $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ preserves each element of $\mathcal{L}(\phi)$ then for any relative train track map $f : G \to G$ representing ϕ there is a bijective correspondence between $\mathcal{L}(\phi)$ and the set of EG strata $H_r \subset G$, where $\Lambda_r \in \mathcal{L}(\phi)$ corresponds to H_r if and only if the realization of each generic leaf of Λ_r has height r.*

For any topological representative $f : G \to G$ and exponentially growing stratum H_r, let $N(f, r)$ be the number of indivisible Nielsen paths $\rho \subset G$ that intersect the interior of H_r. Let $N(f) = \Sigma_r N(f, r)$. Let N_{min} be the minimum value of $N(f)$ that occurs among the topological representatives with $\Gamma = \Gamma_{min}$. We call a relative train track map stable if $\Gamma = \Gamma_{min}$ and $N(f) = N_{min}$. The following result is Theorem 5.12 in [19] which assures the existence of a stable relative train track map.

Lemma 3.5. *Every $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ has a stable relative train track representative.*

Example 3.6. Example 3.1 is an example of a stable relative train track map.

If $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ is fully irreducible then the above fact implies that there exists a stable train track representative for ϕ.

The reason why stable relative train track maps are important is due to the following fact

Lemma 3.7. *(Theorem 5.15, [19]) If $f : G \to G$ is a stable relative train track representative of $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$, and H_r is an exponentially growing stratum, then there exists at most one indivisible Nielsen path ρ of height r. If such a ρ exists, then the illegal turn of ρ is the only illegal turn in H_r and ρ crosses every edge of H_r.*

Example 3.8. Let's go back to example 3.2 to understand this concept. As we already know that the strata $H_4 = \{b, c\}$ we will do some iteration to get a feel for the attraction to lamination.

$$a \mapsto a, b \mapsto bdaEc, c \mapsto bdaEceaDB, d \mapsto ea, e \mapsto ea$$

$f^5_\#(c) \quad = \quad bdaEceaEbdaEceaDBeaEbdaEceaEbdaEceaDBeaECeADBeaE$
$bdaEceaEbdaEceaDBeaEbdaEceaEbdaEceaDBeaECeADBeaEbdAECeADB$
$eAECeADBeaEbdaEceAEbdAECeADBeAECeADBeAEbd$
$AECeADBeAECeADB$

One can notice a pattern (highlighted out in blue) that begins to appear in both directions of the line repeatedly. These are actually segments of the image of c under f. As one keep iterating this EG edge for higher and higher iterates, one actually ends up converging to a generic leaf. The blue segment grows exponentially under iteration and gives more blue segments in it's images. Now consider the circuit Ebd and lets look at some of it's iterates.

$f^4_\#(Ebd) \quad = \quad AAAAEbdaEceaEbdaEceaDBeaEbdaEceaEbdaEceaDBe$
$aECeADBeaEbdaEceaEbdaEceaDBeaEbdaEceaEbdaEceaDBeaECeA$
$DBeaEbdAECeADBeAECeADBeaaaa$

If one looks closely, one will notice the same patterns emerging in the iteration of the circuit. Eventually, as we keep iterating this circuit gets attracted to the lamination associated to the EG strata H_4. Intuitively, this happens because as we iterate the blue segment, it grows exponentially and there is enough cushion to prevent the edges in the EG strata H_4 from canceling out, which forces it to spit out more blue segments under iteration. This idea will charaterize circuits that get attracted to the lamination associated to H_4. We will make it precise once we have defined CT's, which is our next topic.

Splittings, complete splittings and CT's. Given relative train track map $f : G \to G$, splitting of a line, path or a circuit γ is a decomposition of γ into subpaths $....\gamma_0\gamma_1.....\gamma_k....$ such that for all $i \geq 1$ the path $f^i_\#(\gamma) = ..f^i_\#(\gamma_0)f^i_\#(\gamma_1)...f^i_\#(\gamma_k)...$ The terms γ_i are called the *terms* of the splitting of γ.

Given two linear edges E_1, E_2 and a root-free closed Nielsen path ρ such that $f_\#(E_i) = E_i.\rho^{p_i}$ then we say that E_1, E_2 are said to be in the *same linear family* and any path of the form $E_1\rho^m\overline{E}_2$ for some integer m is called an *exceptional path.*

Complete splittings: A splitting of a path or circuit $\gamma = \gamma_1 \cdot \gamma_2 ...... \cdot \gamma_k$ is called complete splitting if each term γ_i falls into one of the following categories:

- γ_i is an edge in some irreducible stratum.
- γ_i is an indivisible Nielsen path.
- γ_i is an exceptional path.
- γ_i is a maximal subpath of γ in a zero stratum H_r and γ_i is taken.

Completely split improved relative train track maps. A *CT* or a completely split improved relative train track maps are topological representatives with particularly nice properties. But CTs do not exist for all outer automorphisms. Only the rotationless outer automorphisms are guranteed to have a CT representative as has been shown in the following Theorem from [21, Theorem 4.28].

Lemma 3.9. *For each rotationless $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ and each increasing sequence $\mathcal{F}$ of ϕ-invariant free factor systems, there exists a CT $f : G \to G$ that is a topological representative for ϕ and f realizes $\mathcal{F}$.*

The following properties are used to define a CT in [21]. There are actually nine properties. But we will state only the ones we need. The rest are not directly used here but they are all part of the proof of various propositions and lemmas we will be needing and which we have stated here as facts.

1. **(Rotationless)** Each principal vertex is fixed by f and each periodic direction at a principal vertex is fixed by Tf.

2. **(Completely Split)** For each edge E in each irreducibe stratum, the path $f(E)$ is completely split.

3. **(vertices)** The endpoints of all indivisible Nielsen paths are vertices. The terminal endpoint of each nonfixed NEG edge is principal.

4. **(Periodic edges)** Each periodic edge is fixed.

CTs have very nice properties. The reader can look them up [21] for a detailed exposition or [12] for a quick reference. We list below only a few of them that is needed for us.

Lemma 3.10. *[21, Lemma 4.11] A completely split path or circuit has a unique complete splitting.*

Lemma 3.11. *[21] If σ is a finite path or a circuit with endpoint in vertices, then $f^k_\#(\sigma)$ is completely split for all sufficiently large $k \geq 1$.*

Lemma 3.12. *[21] Every periodic Nielsen path is fixed. Also, for each EG stratum H_r there exists at most one indivisible Nielsen path of height r, upto reversal of orientation.*

3.2 Attracting Laminations and their properties under CTs

Our main results in this paper use the properties of generic leaves and their behavior under CTs. This has been first studied in [21] and then again in [[14]] extensively. We will only state some important results in this topic, for a more detailed exposition with proofs the reader is suggested to look into those papers. The following lemma is a collection of results from [20]. Item 2 is used to define the nonattracting subgroup system in the next section. Item 3 is proved in [[14]]

Lemma 3.13. *With the same notations as used in Lemma 3.4, for each EG stratum $H_r \subset G$ we have:*

- *The leaves of Λ_r are characterized as the set of lines to which some (any) edge of H_r is weakly attracted under iterates of f*
- *A path or circuit $\sigma \subset G$ is weakly attracted to Λ_r under iterates of f if and only if there exists $k \geq 0$ and a splitting $f^k_\#(\sigma)$ so that some term in the splitting is an edge in H_r.*
- *No conjugacy class in F_r is weakly attracted to Λ_r under iterates of ϕ^{-1}.*

Lemma 3.14. *With the notational setup of Lemma 3.4, for any EG stratum $H_r \subset G$ corresponding to an attracting lamination Λ_r, the following conditions are equivalent for being a generic leaf λ of Λ_r*

1. *λ is birecurrent and has height r*
2. *Both ends of λ have height r.*

The following facts are important since it is the first step to understanding the properties of lines which are not attracted to an attracting lamination Λ^r under the action of a relative train track map. It follows from corollary 4.2.4 and Lemma 4.2.2 of [20].

Lemma 3.15. *[14] Suppose $f : G \to G$ is a relative train track map and $\Lambda \in \mathcal{L}(\phi)$ is an attracting lamination for ϕ. Then being attracted to Λ under iteration by $f_\#$ is an open condition on paths in $\mathcal{B}(G)$.*

Lemma 3.16. *Let $f : G \to G$ be a CT representing a rotationless $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ and σ be a circuit in G. Let Λ^+_ϕ be an attracting lamination for ϕ which corresponds to the exponentially growing strata H_r in G. Then σ is attracted to Λ^+_ϕ if and only if some term in the complete splitting of σ is an edge in H_r.*

This lemma implies that if we consider any circuit in the filtration G_{r-1} then it will never get attracted to the lamination Λ^+_ϕ. Might be circuits which does not intersect H_r but intersects some higher strata. In that case it may as well happen that under iteration the circuit crosses H_r and gets attracted to the lamination.

Dual lamination pairs. We have already seen that the set of lines carried by a free factor system is a closed set and so, together with the fact that the weak closure of a generic leaf λ of an attracting lamination Λ is the whole lamination Λ tells us that $\mathcal{A}_{supp}(\lambda) = \mathcal{A}_{supp}(\Lambda)$. In particular the free factor support of an attracting lamination Λ is a single free factor. Let $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ be an outer automorphism and Λ^+_ϕ be an attracting lamination of ϕ and Λ^-_ϕ be an attracting lamination of ϕ^{-1}. We say that this lamination pair is a *dual lamination pair* if $\mathcal{A}_{supp}(\Lambda^+_\phi) = \mathcal{A}_{supp}(\Lambda^-_\phi)$. By Lemma 3.2.4 of [20] there is bijection between $\mathcal{L}(\phi)$ and $\mathcal{L}(\phi^{-1})$ induced by this duality relation. The following fact is Lemma 2.35 in [[14]]; it establishes an important property of lamination pairs in terms of inclusion. We will use it in proving duality for the attracting and repelling laminations we produce in Proposition 5.4.

Lemma 3.17. *If $\Lambda^\pm_i, \Lambda^\pm_j$ are two dual lamination pairs for $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ then $\Lambda^+_i \subset \Lambda^+_j$ if and only if $\Lambda^-_i \subset \Lambda^-_j$.*

Lemma 3.18. *[14] Let $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ be rotationless and σ be any conjugacy class. Let $\Lambda^{\pm}_{\phi}$ be a dual lamination pair for ϕ. Then σ is attracted to Λ^{+}_{ϕ} under iterates of ϕ if and only if it is attracted to Λ^{-}_{ϕ} under iterates of ϕ^{-1}.*

Together with the observation in 3.16, we can conclude that if a circuit is supported in the lower filtration element Γ_{r-1}, then it is neither attracted to Λ^{+}_{ϕ} nor to Λ^{-}_{ϕ}. In fact, such a circuit is carried by the *nonattracting subgroup system*, which is our next topic of discussion. However, a circuit may cross the H_r strata but eventually when we take iterates, all the edges in H_r cancel out. This may happen(although not always) for example, when the circuit has height $> r$. This is why we need the complete splitting in the hypothesis of lemma 3.16

3.3 Nonattracting subgroup system:

In this section we will define the subgroup system $\mathcal{A}_{na}\Lambda^{+}_{\phi}$ which contains the data about the conjugacy classes which are not attracted to Λ^{+}_{ϕ} under iterates of ϕ (hence the name). The definition and facts given here are from section 5 of [[14]].

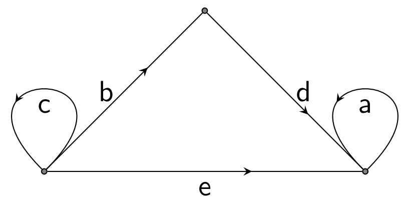
Definition 3.19. Suppose $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ is rotationless and $f : G \to G$ is a CT representing ϕ such that Λ^{+}_{ϕ} is an invariant attracting lamination which corresponds to the EG stratum $H_s \in G$. The *nonattracting subgraph* Z of G is defined as a union of irreducible stratas H_i of G such that no edge in H_i is weakly attracted to Λ^{+}_{ϕ}. This is equivalent to saying that a strata $H_r \subset G \setminus Z$ if and only if there exists $k \geq 0$ some term in the complete splitting of $f^k_{\#}(E_r)$

is an edge in H_s. Define the path $\widehat{\rho}_s$ to be trivial path at any chosen vertex if there does not exist any indivisible Nielsen path of height s, otherwise $\widehat{\rho}_s$ is the unique closed indivisible path of height s (from definition of stable train track maps).

The groupoid $\langle Z, \widehat{\rho}_s \rangle$ - Let $\langle Z, \widehat{\rho}_s \rangle$ be the set of lines, rays, circuits and finite paths in G which can be written as a concatenation of subpaths, each of which is an edge in Z, the path $\widehat{\rho}_s$ or its inverse. Under the operation of tightened concatenation of paths in G, this set forms a groupoid (Lemma 5.6, [[14]]).

Define the graph K by setting $K = Z$ if $\widehat{\rho}_s$ is trivial and let $h : K \to G$ be the inclusion map. Otherwise define an edge E_ρ representing the domain of the Nielsen path $\rho_s : E_\rho \to G_s$, and let K be the disjoint union of Z and E_ρ with the following identification. Given an endpoint $x \in E_\rho$, if $\rho_s(x) \in Z$ then identify $x \sim \rho_s(x)$.Given distinct endpoints $x, y \in E_\rho$, if $\rho_s(x) = \rho_s(y) \notin Z$ then identify $x \sim y$. In this case define $h : K \to G$ to be the inclusion map on K and the map ρ_s on E_ρ. It is not difficult to see that the map h is an immersion. Hence restricting h to each component of K, we get an injection at the level of fundamental groups. The *nonattracting subgroup system* $\mathcal{A}_{na}(\Lambda^+_\phi)$ is defined to be the subgroup system defined by this immersion.

Example 3.20. The lamination associated to the EG strata H_4 in Example 3.2 has a nontrivial non-attracting subgroup system.



$$a \mapsto a, b \mapsto bdaEc, c \mapsto bdaEceaDB, d \mapsto ea, e \mapsto ea$$

One can see that the conjugacy class determined by the circuit a is clearly nonattracted to the lamination and is carried by the nonattracting subgroup system of the attracting lamination. In this particularly nice case the nonattracting subgraph Z is just G_3. Thus $\mathcal{A}_{na}(\Lambda^+) = \{[\langle a \rangle]\}$

Some important properties of $\mathcal{A}_{na}\Lambda^+_\phi$ which we need are stated below. For a more detailed exposition, please refer to section 5 of [[14]].

Lemma 3.21. *([[14]]- Lemma 5.6, 5.7)*

1. *$\langle Z, \widehat{\rho}_s \rangle$ is $f_\#$ invariant.*
2. *The set of lines carried by $\langle Z, \widehat{\rho}_s \rangle$ is same as the set of lines carried by $\mathcal{A}_{na}(\Lambda^+_\phi)$*
3. *The set of circuits carried by $\langle Z, \widehat{\rho}_s \rangle$ is same as the set of circuits carried by $\mathcal{A}_{na}(\Lambda^+_\phi)$*
4. *The set of lines carried by $\langle Z, \widehat{\rho}_s \rangle$ is closed in the weak topology.*
5. *A conjugacy class $[c]$ is not attracted to Λ^+_ϕ if and only if it is carried by $\mathcal{A}_{na}(\Lambda^+_\phi)$.*

6. *$\mathcal{A}_{na}(\Lambda^+_\phi)$ does not depend on the choice of the CT representing ϕ.*

The following fact is corollary 5.10 in [[14]] which tells us the important fact that dual lamination pairs have same nonattracting subgroup systems. Henceforth, for dual lamination pairs we will use the notation $\mathcal{A}_{na}\Lambda^\pm_\phi$

Lemma 3.22. *[12] Given $\phi, \phi^{-1} \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ both rotationless elements and a dual lamination pair $\Lambda^\pm_\phi$ we have $\mathcal{A}_{na}(\Lambda^+_\phi) = \mathcal{A}_{na}(\Lambda^-_\phi)$*

Lemma 3.23. *([[14]] -Proposition 5.5) Given a rotationless ϕ and a CT $f : G \to G$ representing ϕ such that the attracting lamination Λ^+_ϕ corresponds to the EG stratum H_r, then the subgroup system $\mathcal{A}_{na}(\Lambda^+_\phi)$ satisfies the following:*

1. *$\mathcal{A}_{na}(\Lambda^+_\phi)$ is a free factor system if and only if the stratum H_r is not geometric.*

2. *$\mathcal{A}_{na}(\Lambda^+_\phi)$ is malnormal.*

Corollary 3.24. *Let $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ be a fully irreducible geometric element which is induced by a pseudo-Anosov homeomorphism of the surface $\mathcal{S}$ with one boundary component. Let $[c]$ be a conjugacy class representing $\partial\mathcal{S}$. Then $\mathcal{A}_{na}(\Lambda^+_\phi) = [\langle c \rangle]$.*

Proof. The surface homeomorphism leaves $\partial\mathcal{S}$ invariant implies that $[c]$ is ϕ periodic. By passing to a rotationless power we may assume that $[c]$ is fixed by ϕ. ϕ being fully irreducible implies $\mathcal{A}_{na}(\Lambda^+_\phi) = [\langle c \rangle]$. □

4 Singular lines, Extended boundary and Weak attraction theorem

In this section we will look at some results from [14] which analyze and identify the set of lines which are not weakly attracted to an attracting lamination $\Lambda^{\pm}_{\phi}$, given some exponentially growing element in $\mathrm{Out}(\mathbb{F}_{\mathrm{N}})$. Most of the results stated here are in terms of rotationless elements as in the original work. However, we note that being weakly attracted to a lamination Λ_{ϕ} is not dependent on whether the element is rotationless. The first time a special case of this theorem appeared was in the Tit's alternative paper [20]. All facts stated here about rotationless elements also hold for non rotationless elements also, unless otherwise mentioned. This has been pointed out in Remark 5.1 in [14] The main reason for using rotationless elements is to make use of the train track structure from the CT theory. We will use some of the facts to prove lemmas about non rotationless elements which we will need later on.

Denote the set of lines not attracted to Λ^{+}_{ϕ} by $\mathcal{B}_{na}(\Lambda^{+}_{\phi})$. The non-attracting subgroup system carries partial information about such lines as we can see in Lemma 3.21. Other obvious lines which are not attracted are the generic leaves of Λ^{-}_{ϕ}. There is another class of lines, called singular lines, which we define below, which are not weakly attracted to Λ^{+}_{ϕ}.

Define a *singular line* for ϕ to be a line $\gamma \in \mathcal{B}$ if there exists a principal lift Φ of ϕ and a lift $\widetilde{\gamma}$ of γ such that the endpoints of $\widetilde{\gamma}$ are contained in $\mathrm{Fix}_N(\Phi) \subset \partial F_r$. The set of all singular lines of ϕ is denoted by $\mathcal{B}^{sing}(\phi)$. The lemma [Lemma 2.1, [14]] below summarizes this discussion.

Lemma 4.1. *Given a rotationless $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ and an attracting lamination Λ^+_ϕ, any line γ that satisfies one of the following three conditions is in $\mathcal{B}_{na}(\Lambda^+_\phi)$.*

1. *γ is carried by $\mathcal{A}_{na}\Lambda^\pm_\phi$*

2. *γ is a generic leaf of some attracting lamination for ϕ^{-1}*

3. *γ is in $\mathcal{B}^{sing}(\phi)$.*

But these are not all lines that constitute $\mathcal{B}_{na}(\Lambda^+_\phi)$. An important theorem in [Theorem 2.6, [14], stated here as Lemma 4.3, tells us that there is way to concatenate lines from the three classes we mentioned in the above fact which will also result in lines that are not weakly attracted to Λ^+_ϕ. Fortunately, these are all possible types of lines in $\mathcal{B}_{na}(\Lambda^+_\phi)$. A simple explanation of why the concatenation is necessary is, one can construct a line by connecting the base points of two rays, one of which is asymptotic to a singular ray in the forward direction of ϕ and the other is asymptotic to a singular ray in the backward direction of ϕ. This line does not fall into any of the three categories we see in the fact above. The concatenation process described in [14] takes care of such lines. We will not describe the concatenation here, but the reader can look up section 2.2 in [14].

Definition 4.2. Let $A \in \mathcal{A}_{na}\Lambda^\pm_\phi$ and $\Phi \in P(\phi)$, we say that Φ is $A-related$ if $\mathrm{Fix}_N(\widehat{\Phi}) \cap \partial A \neq \emptyset$. Define the extended boundary of A to be

$$\partial^{ext}(A, \phi) = \partial A \cup \left(\bigcup_{\Phi} Fix_N(\widehat{\Phi}) \right)$$

where the union is taken over all A-related $\Phi \in P(\phi)$.

Let $\mathcal{B}^{ext}(A,\phi)$ denote the set of lines which have end points in $\partial^{ext}(A,\phi)$; this set is independent of the choice of A in its conjugacy class. Define

$$\mathcal{B}^{ext}(\Lambda^+_\phi) = \bigcup_{A\in\mathcal{A}_{na}\Lambda^{\pm}_\phi} \mathcal{B}^{ext}(A,\phi)$$

. We can now state the main result about non-attracted lines.

Lemma 4.3. *Suppose $\phi, \psi = \phi^{-1} \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ be rotationless elements and Λ^+_ϕ is an attracting lamination for ϕ. Then any line γ is in $\mathcal{B}_{na}(\Lambda^+_\phi)$ if and only if one of the following conditions hold:*

1. *γ is in $\mathcal{B}^{ext}(\Lambda^+_\phi)$*

2. *γ is in $\mathcal{B}^{sing}(\phi)$*

3. *γ is a generic leaf of some attracting lamination for ψ*

It is worth noting that the sets of lines mentioned in Lemma 4.3 are not necessarily pairwise disjoint.

4.1 Weak attraction theorem

Lemma 4.4 ([14] Corollary 2.17)**.** *Let $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ be a rotationless and exponentially growing. Let $\Lambda^{\pm}_\phi$ be a dual lamination pair for ϕ. Then for any line $\gamma \in \mathcal{B}$ not carried by $\mathcal{A}_{na}(\Lambda^{\pm}_\phi)$ at least one of the following hold:*

1. *γ is attracted to Λ^+_ϕ under iterations of ϕ.*

2. *γ is attracted to Λ^-_ϕ under iterations of ϕ^{-1}.*

Moreover, if V_ϕ^+ *and* V_ϕ^- *are attracting neighborhoods for the laminations* Λ_ϕ^+ *and* Λ_ϕ^- *respectively, there exists an integer* $l \geq 0$ *such that at least one of the following holds:*

- $\gamma \in V_\phi^-$.
- $\phi^l(\gamma) \in V_\phi^+$
- γ *is carried by* $\mathcal{A}_{na}(\Lambda_\phi^\pm)$.

Corollary 4.5. *Let* $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ *be exponentially growing and* $\Lambda_\phi^\pm$ *be geometric dual lamination pair for* ϕ *such that* ϕ *fixes* Λ_ϕ^+ *and* ϕ^{-1} *fixes* Λ_ϕ^-*with attracting neighborhoods* $V_\phi^\pm$. *Then there exists some integer* l *such that for any line* γ *in* $\mathcal{B}$ *one of the following occurs:*

- $\gamma \in V_\phi^-$.
- $\phi^l(\gamma) \in V_\phi^+$.
- γ *is carried by* $\mathcal{A}_{na}(\Lambda_\phi^\pm)$

Proof. Let K be a positive integer such that ϕ^K is rotationless. Then by definition $\mathcal{A}_{na}(\Lambda_\phi^\pm) = \mathcal{A}_{na}(\Lambda_{\phi^K}^\pm)$. Also ϕ fixes Λ_ϕ^+ implies $\Lambda_\phi^+ = \Lambda_{\phi^K}^+$ and the attracting neighborhoods V_ϕ^+ and $V_{\phi^K}^+$ can also be chosen to be the same weak neighborhoods. Then by Lemma 4.4 we know that there exists some positive integer m such that the conclusions of the Weak attraction theorem hold for ϕ^K. Let $l := mK$. This gives us the conclusions of the corollary. Before we end we note that by definition of an attracting neighborhood $\phi(V_\phi^+) \subset V_\phi^+$ which implies that if $\phi^l(\gamma) \in V_\phi^+$, then $\phi^t(\gamma) \in V_\phi^+$ for all $t \geq l$. $\square$

Lemma 4.6. *Suppose $\phi, \psi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ are two exponentially growing automorphisms with attracting laminations Λ^+_ϕ and Λ^+_ψ, respectively. If a generic leaf $\lambda \in \Lambda^+_\phi$ is in $\mathcal{B}_{na}(\Lambda^+_\psi)$ then the whole lamination $\Lambda^+_\phi \subset \mathcal{B}_{na}(\Lambda^+_\psi)$.*

Proof. Recall that a generic leaf is bi-recurrent. Hence, $\lambda \in \mathcal{B}_{na}(\Lambda^+_\psi)$ implies that λ is either carried by $\mathcal{A}_{na}$ or it is a generic leaf of some element of $\mathcal{L}(\psi^{-1})$. First assume that λ is carried by $\mathcal{A}_{na}$. Then using Lemma 3.21 item 4, we can conclude that Λ^+_ϕ is carried by $\mathcal{A}_{na}(\Lambda^+_\psi)$.

Alternatively, if λ is a generic leaf of some element $\Lambda^-_\psi \in \mathcal{L}(\psi^{-1})$, then the weak closure $\overline{\lambda} = \Lambda^+_\phi = \Lambda^-_\phi$ and we know Λ^-_ψ does not get attracted to Λ^+_ψ by Fact 3.1. Hence, $\Lambda^+_\phi \subset \mathcal{B}_{na}(\Lambda^+_\psi)$. □

5 Pingpong argument for exponential growth

Lemma 5.1 ([20] Section 2.3)**.** *If $f : G \longrightarrow G$ is a train track map for an irreducible $\phi \in \mathrm{Out}(\mathbb{F}_{\mathrm{N}})$ and α is a path in some leaf λ of G such that $\alpha = \alpha_1\alpha_2\alpha_3$ is a decomposition into subpaths such that $|\alpha_1|, |\alpha_3| \geq 2C$ where C is the bounded cancellation constant for the map f, then $f^k_{\#}(\alpha_2) \subset f^k_{\#\#}(\alpha)$ for all $k \geq 0$.*

Proof. Let α be any path with a decomposition $\alpha = \alpha_1\alpha_2\alpha_3$. Take lifts to universal cover of G. If $\tilde{\gamma}$ is a path in $\tilde{G}$ that contains $\tilde{\alpha}$, then decompose $\tilde{\gamma} = \tilde{\gamma_1}\tilde{\alpha_2}\tilde{\gamma_3}$ such that $\tilde{\alpha_1}$ is the terminal subpath of $\tilde{\gamma_1}$ and $\tilde{\alpha_3}$ is the initial subpath of $\tilde{\gamma_3}$. Following the proof of [20] if $K = 2C$ then $\tilde{\gamma}$ can be split at the endpoints of $\tilde{\alpha_2}$. Thus, $\tilde{f}^k_{\#}(\tilde{\gamma}) = \tilde{f}^k_{\#}(\tilde{\gamma_1})\tilde{f}^k_{\#}(\tilde{\alpha_2})\tilde{f}^k_{\#}(\tilde{\gamma_3})$. The result now follows from the definition of $f^k_{\#\#}(\alpha)$.

$\square$

Lemma 5.2 ([14] Lemma 1.1)**.** *Let $f : G \to G$ be a homotopy equivalence representing $\phi \in \mathrm{Out}(\mathbb{F}_{\mathrm{N}})$ such that there exists a finite path $\beta \subset G$ having the property that $f_{\#\#}(\beta)$ contains three disjoint copies of β. Then ϕ is exponentially growing and there exists a lamination $\Lambda \in \mathcal{L}(\phi)$ and a generic leaf λ of $\Lambda \in \mathcal{L}(\phi)$ such that Λ is ϕ-invariant and ϕ fixes λ preserving orientation, each generic leaf contains $f^i_{\#\#}(\beta)$ as a subpath for all $i \geq 0$ and $N(G, \beta)$ is an attracting neighborhood for Λ.*

We adapt the following notation for the statement of the next proposition: Let $\epsilon \in \{-, +\}$ and $i \in \{0, 1\}$. Here i will be used to represent ψ(if $i = 0$) or ϕ(if $i = 1$). Together the tuple $\mu_i := (i, \epsilon) \in \{0, 1\} \times \{-, +\}$ will represent

ψ(if $\mu_0 = (0,+)$), ψ^{-1} (if $\mu_0 = (0,-)$) and so on. In this notation we write $\Lambda_0^\epsilon := \Lambda_\psi^\epsilon$ and so on.

We also write notations like $-\epsilon$ where it means $-- = +$ and $-+ = -$ depending on value of epsilon.

Standing assumptions for the rest of this section: ϕ, ψ are exponentially growing elements of $\mathrm{Out}(\mathbb{F}_\mathrm{N})$ such that the following conditions are satisfied:

1. ϕ, ψ are not powers of one another.

2. There exists dual lamination pairs $\Lambda_\psi^\pm$ and $\Lambda_\phi^\pm$ such that $\Lambda_\psi^\pm$ is attracted to Λ_ϕ^ϵ under iterates of ϕ^ϵ and $\Lambda_\phi^\pm$ is attracted to Λ_ψ^ϵ under iterates of ψ^ϵ.

3. ψ^ϵ fixes Λ_ψ^ϵ and ϕ^ϵ fixes Λ_ϕ^ϵ.

4. Both $\Lambda_\psi^\pm$ and $\Lambda_\phi^\pm$ are non-geometric or every lamination pair of every element of $\langle \psi, \phi \rangle$ is geometric.

Remark 5.3. Some remarks regarding the set of hypothesis.

- hypothesis 2 and 3 are needed to play the ping-pong game.

- hypothesis 4 is needed to prove that the attracting and repelling laminations produced out of ping-pong are dual. This, as we will see later in Proposition 5.5 , is not required if ϕ, ψ are fully irreducible.

The lemma that has been proven by Handel and Mosher in Proposition 1.3 in [14] is a weaker version of the following proposition. What they have shown (with slightly weaker conditions than hypothesis 2 above) is that the lemma is true for $k = 1$ and only under positive powers of ψ and ϕ. Strengthening that one part of the hypothesis enables us to extend their result to both positive and negative exponents and also for reduced words with arbitrary k (see statement of 5.4 for description of k). They also have the assumption that ϕ, ψ are both rotationless, which they later on discovered, is not necessary; one can get away with the hypothesis 3 above. The techniques of proof is similar.

Pingpong lemma:

Proposition 5.4. *Let ψ, ϕ be exponentially growing elements of* $\mathrm{Out}(\mathbb{F}_\mathrm{N})$*, which satisfy the hypothesis 5 mentioned above.*
Then there exists some integer $M > 0$ and attracting neighborhoods $V^\pm_\phi$, $V^\pm_\psi$ of $\Lambda^\pm_\phi$ and $\Lambda^\pm_\psi$,respectively, such that for every pair of finite sequences $n_i \geq M$ and $m_i \geq M$ if

$$\xi = \psi^{\epsilon_1 m_1}\phi^{\epsilon'_1 n_1} .... \psi^{\epsilon_k m_k}\phi^{\epsilon'_k n_k}$$

($k \geq 1$) is a cyclically reduced word then w will be exponentially-growing and have a lamination pair $\Lambda^\pm_\xi$ satisfying the following properties:

1. *Every conjugacy class carried by $\mathcal{A}_{na}(\Lambda^\pm_\xi)$ is carried by both $\mathcal{A}_{na}(\Lambda^\pm_\phi)$ and $\mathcal{A}_{na}(\Lambda^\pm_\psi)$*

2. *$\psi^{m_i}(V^\pm_\phi) \subset V^+_\psi$ and $\psi^{-m_i}(V^\pm_\phi) \subset V^-_\psi$.*

3. *$\phi^{n_j}(V^\pm_\psi) \subset V^-_\phi$ and $\phi^{-n_j}(V^\pm_\psi) \subset V^-_\phi$.*

4. *$V^+_\xi :\ = V^{\epsilon_1}_\psi$ is an attracting neighborhood of Λ^+_ξ*

5. *$V^-_\xi :\ = V^{-\epsilon'_k}_\phi$ is an attracting neighborhood of Λ^-_ξ*

6. *(uniformity) Suppose* $U_{\psi}^{\epsilon_1}$ *is an attracting neighborhood of* $\Lambda_{\psi}^{\epsilon_1}$ *and* λ_{ξ}^{+} *is a generic leaf of* Λ_{ξ}^{+}*. Then* $\lambda_{\xi}^{+} \in U_{\psi}^{\epsilon_1}$ *for sufficiently large* M*.*

7. *(uniformity) Suppose* $U_{\phi}^{\epsilon_k'}$ *is an attracting neighborhood of* $\Lambda_{\phi}^{\epsilon_k'}$ *and* λ_{ξ}^{-} *is a generic leaf of* Λ_{ξ}^{-}*. Then* $\lambda_{\xi}^{-} \in U_{\phi}^{\epsilon_k'}$ *for sufficiently large* M*.*

Proof. Let $g_{\mu_i} : G_{\mu_i} \longrightarrow G_{\mu_i}$ be stable relative train train-trak maps and $u_{\mu_j}^{\mu_i} : G_{\mu_i} \longrightarrow G_{\mu_j}$ be the homotopy equivalence between the graphs which preserve the markings, where $i \neq j$.

Let $C_1 > 2\mathrm{BCC}\{g_{\mu_i} | i \in \{0,1\}\}$. Let $C_2 > \mathrm{BCC}\{u_{\mu_j}^{\mu_i} | i,j \in \{0,1\}, i \neq j\}$. Let $C \geq C_1, C_2$.

Let λ_i^{ϵ} be generic leaves of laminations Λ_i^{ϵ}.

STEP 1:

Using the fact that Λ_1^{ϵ} is weakly attracted to $\Lambda_0^{\epsilon'}$, under the action if $\psi^{\epsilon'}$, choose a finite subpath $\alpha_1^{\epsilon} \subset \lambda_1^{\epsilon}$ such that

- $(u_{\mu_0}^{\mu_1})_{\#}(\alpha_1^{\epsilon}) \to \lambda_0^{\epsilon'}$ weakly, where $\mu_0 = (0, \epsilon')$ and $\mu_1 = (1, \epsilon)$.
- α_1^{ϵ} can be broken into three segments: initial segment of C edges, followed by a subpath α^{ϵ} followed by a terminal segment with C edges.

STEP 2:

Now using the fact $\Lambda_0^{\epsilon} \to \Lambda_1^{\epsilon'}$ weakly under iterations of $\phi^{\epsilon'}$, we can find positive integers $p_{\mu_1}^{\epsilon}$ (there are four choices here that will yield four integers) such that $\alpha_1^{\epsilon'} \subset (g_{\mu_1}^{p_{\mu_1}^{\epsilon}} u_{\mu_1}^{\mu_0})_{\#}(\lambda_0^{\epsilon})$, where $\mu_0 = (0, \epsilon), \mu_1 = (1, \epsilon')$.

Let C_3 be greater than $\mathrm{BCC}\{g_{\mu_1}^{p_{\mu_1}^{\epsilon}} u_{\mu_1}^{\mu_0}\}$ (four maps for four integers $p_{\mu_1}^{\epsilon}$) .

STEP 3:

Next, let $\beta_0^\epsilon \subset \lambda_0^\epsilon$ be a finite subpath such that $(g_{\mu_1}^{p_{\mu_1}^\epsilon})_\#(\beta_0^\epsilon)$ contains $\alpha_1^{\epsilon'}$ protected by C_3 edges in both sides, where $\mu_0 = (0,\epsilon)$ and $\mu_1 = (1,\epsilon')$. Also, by increasing β_0^ϵ if necessary, we can assume that $V_\psi^\epsilon = N(G_{\mu_0}, \beta_0^\epsilon)$ is an attracting neighborhood of Λ_0^ϵ .

Let σ be any path containing β_0^ϵ. Then $(g_{\mu_1}^{p_{\mu_1}^\epsilon} u_{\mu_1}^{\mu_0})_\#(\sigma) \supset \alpha_0^\epsilon$. Thus by using Lemma 5.1 we get that $(g_{\mu_1}^{p_{\mu_1}^\epsilon + t} u_{\mu_0}^{\mu_1})_\#(\sigma) = (g_{\mu_1}^t)_\#((g_{\mu_1}^{p_{\mu_1}^\epsilon} u_{\mu_1}^{\mu_0})_\#(\sigma))$ contains $(g_{\mu_1}^t)_\#(\alpha^\epsilon)$ for all $t \geq 0$.

Thus we have $(g_{\mu_1}^{p_{\mu_1}^\epsilon + t} u_{\mu_1}^{\mu_0})_{\#\#}(\beta_0^\epsilon) \supset (g_{\mu_1}^t)_\#(\alpha^\epsilon)$ for all $t \geq 0$.

STEP 4:

Next step is reverse the roles of ϕ and ψ to obtain positive integers $q_{\mu_1}^{\epsilon'}$ and paths $\gamma_1^{\epsilon'} \subset \lambda_{\mu_1}^{\epsilon'}$ such that $(g_{\mu_0}^{q_{\mu_0}^\epsilon + t} u_{\mu_0}^{\mu_1})_{\#\#}(\gamma_1^{\epsilon'}) \supset (g_{\mu_0}^t)_\#(\beta_0^\epsilon)$ for all $t \geq 0$, where $\mu_0 = (0,\epsilon), \mu_1 = (1,\epsilon')$

STEP 5:

Finally, let k be such that $(g_{\mu_1}^k)_\#(\alpha^\epsilon)$ contains three disjoint copies of γ_1^ϵ and that $(g_{\mu_0}^k)_\#(\beta_0^\epsilon)$ contains three disjoint copies of β_0^ϵ for $\epsilon = 0, 1$. Let $p \geq$ max $\{p_{\mu_1}^\epsilon\} + k$ and $q \geq$ max $\{q_{\mu_0}^\epsilon\} + k$.

Let $m_i \geq q$ and $n_i \geq p$.

The map $f_\xi = g_{(0,\epsilon_1)}^{m_1} u_{(0,\epsilon_1)}^{(1,\epsilon_1')} g_{(1,\epsilon_1')}^{n_1} u_{(1,\epsilon_1')}^{(0,\epsilon_2)} \ldots\ldots\ldots g_{(1,\epsilon_k')}^{n_k} u_{(1,\epsilon_k')}^{(0,\epsilon_1)} : G_{(0,\epsilon_1)} \to G_{(0,\epsilon_1)}$ is a topological representative of ξ. With the choices we have made, $g_{(1,\epsilon_k')}^{n_k} u_{(1,\epsilon_k')}^{(0,\epsilon_1)})_{\#\#}(\beta_0^{\epsilon_1})$ contains three disjoint copies of $\gamma_1^{\epsilon_k'}$ and so $(g_{(0,\epsilon_k)}^{m_k} u_{(0,\epsilon_k)}^{(1,\epsilon_k')} g_{(1,\epsilon_k')}^{n_k} u_{(1,\epsilon_k')}^{(0,\epsilon_1)})_{\#\#}(\beta_0^{\epsilon_1})$ will contain three disjoint copies of $\beta_0^{\epsilon_k}$. Continuing in this fashion in the end we get that $(f_\xi)_{\#\#}(\beta_0^{\epsilon_1})$ contains three disjoint copies of $\beta_0^{\epsilon_1}$. Thus by Lemma 5.2 ξ is an exponentially growing element of $\mathrm{Out}(\mathbb{F}_\mathrm{N})$ with an attracting lamination Λ_ξ^+ which has $V_\xi^+ = N(G_{\mu_0}, \beta_0^{\epsilon_1}) = V_\psi^{\epsilon_1}$ as an attracting neighborhood.

Similarly, if we take inverse of ξ and interchange the roles played by ψ, ϕ with ϕ^{-1}, ψ^{-1}, we can produce an attracting lamination Λ_ξ^- for ξ^{-1} with an attracting neighborhood $V_\xi^- = N(G_{(1,-\epsilon'_k)}, \gamma_1^{-\epsilon'_k}) = V_\phi^{-\epsilon'_k}$. which proves property (4) and (5) of the proposition. We shall later show that Λ_ξ^+ and Λ_ξ^- form a dual-lamination pair.

Hence,every reduced word of the group $\langle \phi^n, \psi^m \rangle$ will be exponentially growing if $n \geq p, m \geq q$. Let $M \geq p, q$.

Now, we prove the conclusion related to non-attracting subgroup. By corollary 4.5 there exists l so that if τ is neither an element of $V_\phi^{-\epsilon'_k} = V_\xi^-$ nor is it carried by $\mathcal{A}_{na}\Lambda_\phi^\pm$ then $\phi_\#^{\epsilon'_k t}(\tau) \in V_\phi^{\epsilon'_k}$ for all $t \geq l$. Increase M if necessary so that $M > l$. Under this assumption,

$\xi_\#(\tau) \in \psi^{\epsilon_1 m_1}(V_\phi^{\epsilon'_1}) \subset V_\xi^+$. So τ is weakly attracted to Λ_ξ^+. Hence we conclude that if $\tau \notin V_\xi^-$ and not attracted to Λ_ξ^+, then τ is carried by $\mathcal{A}_{na}\Lambda_\phi^\pm$.

Similarly, if τ is not in V_ξ^+ and not attracted to Λ_ξ^- then τ is carried by $\mathcal{A}_{na}\Lambda_\psi^\pm$.

Next, suppose that τ is a line that is not attracted to any of $\Lambda_\xi^+, \Lambda_\xi^-$. Then τ must be disjoint from V_ξ^+, V_ξ^-. So, is carried by both $\mathcal{A}_{na}\Lambda_\psi^\pm$ and $\mathcal{A}_{na}\Lambda_\phi^\pm$. Restricting our attention to periodic line, we can say that every conjugacy class that is carried by both $\mathcal{A}_{na}\Lambda_\xi^+$ and $\mathcal{A}_{na}\Lambda_\xi^-$ is carried by both $\mathcal{A}_{na}\Lambda_\psi^\pm$ and $\mathcal{A}_{na}\Lambda_\phi^\pm$. That the two non-attracting subgroup systems are mutually malnormal gives us the first conclusion.

The proof in [14] that Λ_ξ^- and Λ_ξ^+ are dual lamination pairs will carry over in this situation and so $\mathcal{A}_{na}\Lambda_\xi^+ = \mathcal{A}_{na}\Lambda_\xi^-$.

Thus we have the proof of the first conclusion of our proposition. □

The following are the main ingredients that will be used in the to show applications of the main theorem of this paper:

Proposition 5.5. *If we assume that ψ, ϕ are fully-irreducible, then we do not need the second and third bulleted item in the hypothesis 5 for the ping-pong proposition.*

Remark 5.6. In the case of fully-irreducible, situation is much simpler

1. Hypothesis 1 implies hypothesis 2.
2. Hypothesis 3 is obvious since the attracting and repelling lamination pairs for fully irreducible elements are unique.
3. Hypothesis 4 is needed to prove that the laminations produced from the ping-pong argument are dual. We will see a direct proof in the lines of Proposition 1.3 [14] if the elements are fully irreducible, without using hypothesis 3 needed for propostion 5.4

Proof. We will show that the laminations produced from pingpong argument are dual. As in the proposition assume

$$\xi = \psi^{\epsilon_1 m_1}\phi^{\epsilon'_1 n_1}........\psi^{\epsilon_k m_k}\phi^{\epsilon'_k n_k}$$

Suppose the laminations Λ^+_ξ and Λ^-_ξ produced using the ping-pong type argument are not dual. Index all the dual lamination pairs of ξ as $\{\Lambda^{\pm}_i\}_{i\in I}$ and assume that

$$\Lambda^+_\xi = \Lambda^+_i \text{ and } \Lambda^-_\xi = \Lambda^-_j \text{ for some } i \neq j$$

Case 1: $\Lambda_i^+ \nsubseteq \Lambda_j^+$. This implies that a generic leaf λ of Λ_j^+ is not attracted to $\Lambda_i^+ = \Lambda_\xi^+$ under iteration by ξ. Also, λ is not attracted to $\Lambda_j^- = \Lambda_\xi^-$ under iteration by ξ^{-1}. In particular, $\lambda \notin V_\xi^-$. By the discussion at the end of the ping-pong argument, this implies that λ is carried by $\mathcal{A}_{na}\Lambda_\phi^\pm$. But ϕ being fully-irreducible, $\mathcal{A}_{na}\Lambda_\phi^\pm$ is either trivial, which gives us that λ does not exist(contradiction), or $\mathcal{A}_{na}\Lambda_\phi^\pm = [c]$, which implies that λ is a circuit. The later is impossible since a generic leaf cannot be a circuit (Lemma 3.1.16, [20]).

Case 2: $\Lambda_i^+ \subset \Lambda_j^+$. By Fact 2.18 this means $\Lambda_i^- \subset \Lambda_j^-$. This implies that some generic leaf λ of Λ_i^- is not attracted to Λ_j^- under iteration by ξ^{-1} (since proper inclusion implies there is a generic leaf of Λ_i whose height is less than the height of the EG stratum corresponding to Λ_j). Also, λ is not attracted to $\Lambda_i^+ = \Lambda_\xi^+$ under iteration by ξ. In particular, $\lambda \notin V_\xi^+$. By discussion at the end of the ping-pong proposition, this implies that λ is carried by $\mathcal{A}_{na}\Lambda_\psi^\pm$. The same arguments as in case 1 works and we get a contradiction.

$\square$

Corollary 5.7. *If in proposition 5.4 if we drop bulleted items 2, 3 in the hypothesis 5 and instead assume that ψ, ϕ are fully-irreducible outer automorphisms such that ϕ is geometric and ψ is hyperbolic(or vice versa), then the resulting laminations Λ_ξ^+ and Λ_ξ^- produced by the ping-pong argument will be dual. Moreover, $\mathcal{A}_{na}\Lambda_\xi^\pm$ will be trivial if ξ is not conjugate to a power of ϕ*

Proof. The result follows from Proposition 5.5 and the conclusion 1 from Proposition 5.4 $\square$

Corollary 5.8. *If in proposition 5.4 if we drop bulleted items 2, 3 in the hypothesis 5 and instead assume that ψ and ϕ are fully-irreducible and geometric* *and fix the same conjugacy class, then the resulting laminations* Λ^+_ξ *and* Λ^-_ξ *produced by the ping-pong argument will be dual and* $\Lambda^\pm_\xi$ *will be geometric and* $\mathcal{A}_{na}(\Lambda^\pm_\xi)$ *will be equal to* $\mathcal{A}_{na}(\Lambda^\pm_\phi)$*. If they don't fix the same conjugacy class, then* $\mathcal{A}_{na}\Lambda^\pm_\xi$ *is trivial when* ξ *is not conjugate to a power of* ϕ *or* ψ

Proof. When both are geometric and fix the same conjugacy class they arise from pseudo-Anosov homeomorphism of the same surface with connected boundary and the conjugacy class corresponding to the boundary, $[c]$ is equal to $\mathcal{A}_{na}\Lambda^\pm_\phi$ and $\mathcal{A}_{na}\Lambda^\pm_\psi$. So, every reduced word ξ in ϕ and ψ will fix $[c]$. We get the conclusion about duality of Λ^+_ξ and Λ^-_ξ by using proposition 5.5 and conclusion 1 of proposition 5.4 tells us that $\mathcal{A}_{na}\Lambda^\pm_\xi = [c]$.

If they are both geometric but they do not fix the same conjugacy class $\mathcal{A}_{na}(\Lambda^\pm_\phi)$ and $\mathcal{A}_{na}(\Lambda^\pm_\psi)$ are conjugacy classes of infinite cyclic subgroups which have generators which are not powers of each other. Proposition 5.5 tells us that the laminations Λ^+_ξ and Λ^-_ξ are dual. Then using the conclusion(1) from pingpong Lemma 5.4 we can conclude that $\mathcal{A}_{na}(\Lambda^\pm_\xi)$ does not carry any conjugacy classes and hence is trivial when ξ is not conjugate to some power of ψ or ϕ. $\square$

6 Proof of main theorem

We begin this section by introducing the concept of Stallings graph associated to a free factor, which will contain the information about lines which are carried by the free factor.

Stallings graph : Consider a triple (Γ, S, p) consisting of a marked graph Γ, a connected subgraph S with no valence one vertices and a homotopy equivalence $p : \Gamma \to R_N$, which is a homotopy inverse of the marking on Γ. p takes vertices to vertices and edges to edge-paths and is a immersion when restricted to S. This enforces that any path in S is mapped to a path in R_N. If in addition we have $[F] = [S]$, we say that the triple (Γ, S, p) is a representative of the free factor F and S is said to be the *Stallings graph* for F. We get a metric on each edge of Γ by pulling back the metric from R_N via p. Under this setting, a line l is carried by $[F]$ if and only if its realization l_Γ in Γ is contained in S, in which case the restriction of p to the line l_Γ is an immersion whose image in R_N is l.

Lemma 6.1. *Every proper free factor F has a realization (Γ, S, p).*

The description of the Stallings graph and the proof of the fact mentioned above can be found in the proof of Theorem I, section 2.4 in [14]. The proof of existence uses the Stallings fold theorem to construct S, hence the name. The following fact is an important tool for the proof of the main theorem. It is used to detect fully-irreducibility when we are given an exponentially growing element.

Lemma 6.2. *Let $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ be a rotationless and exponentially growing element. Then for each attracting lamination Λ^+_ϕ, if the subgroup system $\mathcal{A}_{na}\Lambda^+_\phi$ is trivial and the free factor system $\mathcal{A}_{supp}(\Lambda^+_\phi)$ is not proper then ϕ is fully irreducible.*

It is worth noting that the fully irreducible element we will get from this lemma is hyperbolic, since $\mathcal{A}_{na}\Lambda^+$ trivial implies that there are no periodic conjugacy classes and by [4] it is hyperbolic. In the next lemma we will extend this result to include the geometric case also.

Lemma 6.3. *For each exponentially growing $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ if there exists an attracting lamination Λ^+_ϕ such that $\mathcal{A}_{na}\Lambda^+_\phi = [\langle c\rangle]$, where $\mathcal{A}_{supp}[c]$ and $\mathcal{A}_{supp}(\Lambda^+)$ are not proper then ϕ is fully irreducible and geometric.*

Proof. We follow the footsteps of the proof of the Lemma 6.2. Suppose ϕ is not fully irreducible. Pass on to a rotationless power and assume ϕ is rotationless. Let $[F]$ be the conjugacy class of the proper, non trivial free factor fixed by ϕ. Choose a CT $f : G \to G$ such that F is realized by some filtration element G_r and $[G_r] = [F]$. Since $\mathcal{A}_{supp}(\Lambda^+)$ is not proper, the lamination Λ^+_ϕ corresponds to the highest strata H_s and $r < s$. Next recall that a strata $H_i \subset G \setminus Z$ if and only if there exists some $k \geq 0$ so that some term in the complete splitting of $f^k_\#(E_i)$ (for some edge $E_i \subset H_i$)is an edge in H_s. This implies that $G_{s-1} \subset Z$, since f preserves the filtration. Hence we have $G_r \subset G_{s-1} \subset Z$. This implies that $[F] \precneqq [\langle c\rangle]$. So, $F = \langle c^p\rangle$ for some $p > 1$. But the conjugacy class $[c]$ fills contradicts our assumption that F is a proper non trivial free factor. $\square$

We write down the hypothesis for main theorem followed by some remarks as to how the assumptions are used.

Hypothesis :

Let $\phi, \psi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ be exponentially growing outer automorphisms, which are not conjugate to powers of each other and which do not have a common periodic free factor. Also let ϕ, ψ have dual lamination pairs $\Lambda^{\pm}_{\phi}$ and $\Lambda^{\pm}_{\psi}$ such that the following hold:

- ψ^{ϵ} leaves $\Lambda^{\epsilon}_{\psi}$ invariant and ϕ^{ϵ} leaves $\Lambda^{\epsilon}_{\phi}$ invariant.
- $\Lambda^{\pm}_{\psi}$ is attracted to $\Lambda^{\epsilon}_{\phi}$ under iterates of ϕ^{ϵ} and $\Lambda^{\pm}_{\phi}$ is attracted to $\Lambda^{\epsilon}_{\psi}$ under iterates of ψ^{ϵ}
- $\{\Lambda^{\pm}_{\phi}\} \cup \{\Lambda^{\pm}_{\psi}\}$ fills.
- Both $\Lambda^{\pm}_{\psi}$ and $\Lambda^{\pm}_{\phi}$ are non-geometric or every lamination pair of every element of $\langle \psi, \phi \rangle$ is geometric.
- The non-attracting subgroup systems $\mathcal{A}_{na}(\Lambda^{\pm}_{\phi})$ and $\mathcal{A}_{na}(\Lambda^{\pm}_{\psi})$ are mutually malnormal.

Definition 6.4. Any two pairs $(\phi, \Lambda^{\pm}_{\phi}), (\psi, \Lambda^{\pm}_{\psi})$ which satisfy the hypothesis set 6 above will be called *pairwise independent*.

When we abuse the definition and say that *ϕ, ψ are independent*, it is understood that we also have their dual lamination pairs $\Lambda^{\pm}_{\phi}$ and $\Lambda^{\pm}_{\psi}$ which satisfy the set of hypothesis.

Remark 6.5. Bulleted item 5 is a technical requirement to prove the first conclusion of the Proposition 5.4. The rest are as follows:

1. The first two bullets are required to prove the conclusion about the free group of rank two in the statement of Theorem 6.7 and are also needed to apply Proposition 5.4.

2. The third bullet will be used to deduce a contradiction in the proof of showing that ξ is fully irreducible

3. The fifth bullet implies that there is no line that is carried by both the nonattracting subgroup subgroup systems. This will be used to conclude hyperbolicity.

The following result is same as Lemma 3.4.2 in [20].

Lemma 6.6. *Let $\phi, \psi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ be exponentially growing elements, which are not conjugates to powers of each other, which satisfy the first three bullets of 6 . Then there exists some $M_0 \geq 0$ such that the group $G_{M_0} = \langle \psi^m, \phi^n \rangle$ is free for every $m, n \geq M_0$*

We say that l is a *periodic line* if $l = ...\rho\rho\rho....$ is a bi-infinite iterate of some finite path ρ. In this case we write $l = \rho_\infty$

Theorem 6.7. *Let ϕ, ψ be two exponentially growing, independent elements of $\mathrm{Out}(\mathbb{F}_\mathrm{N})$. Then there exists a $M \geq 0$, such that for all $n, m \geq M$ the group $\langle \psi^m, \phi^n \rangle$ will be free and every element of this free group, not conjugate to some power of the generators, will be hyperbolic and fully-irreducible.*

Proof. We already know that there exists some $M_0 > 0$, such that for all $m, n \geq M_0$ the group $G_{M_0} = \langle \psi^m, \phi^n \rangle$ will be free group of rank two (proposition6.6). It remains to show that , by increasing M_0 if necessary, every reduced word in this group will be fully irreducible. We shall prove it by contradiction.

Suppose that there does not exist any such $M \geq M_0$. This implies that for M large, there exist $m(M), n(M)$, such that the group $\langle \psi^{m(M)}, \phi^{n(M)} \rangle$ contains at least one reduced word ξ_M (not conjugate to some power of the generators) which is either reducible or fully irreducible but not hyperbolic. Using the hypothesis of mutual malnormality of $\mathcal{A}_{na}(\Lambda^{\pm}_{\psi})$ and $\mathcal{A}_{na}(\Lambda^{\pm}_{\phi})$ together with conclusion 1 of Proposition 5.4 we know that for sufficiently large M, $\mathcal{A}_{na}(\Lambda^{\pm}_{\xi_M})$ must be trivial. Hence after passing to a subsequence if necessary, assume that all ξ_M's are reducible.

We also make an assumption that this ξ_M begins with a nonzero power of ψ and ends in some nonzero power of ϕ; if not, then we can conjugate to achieve this. Thus as M increases, we have a sequence of reducible elements $\xi_M \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$. Pass to a subsequence to assume that the ξ_M's begin with a positive power of ψ and end with a positive power of ϕ. If no such subsequence exist, then change the generating set of G_{M_0} by replacing generators with their inverses.

Let $\xi_M = \psi^{m_1}\phi^{\epsilon'_1 n_1} .... \psi^{\epsilon_k m_k}\phi^{n_k}$ where $m_i = m_i(M), n_j = n_j(M)$ and k depend on M.

We note that by our assumptions, the exponents get larger as M increases. From the Ping-Pong lemma we know that there exists attracting neighborhoods $V^{\pm}_{\psi}$ and $V^{\pm}_{\phi}$ for the dual lamination pairs $\Lambda^{\pm}_{\psi}$ and $\Lambda^{\pm}_{\phi}$, respectively, such that if $i \neq 1$

$$\psi^{\epsilon_i m_i(M)}(V^{\pm}_{\phi}) \subset V^{\epsilon_i}_{\psi} \text{ and } \psi^{m_1(M)}(V^{\pm}_{\phi}) \subset V^{+}_{\psi} \subset V^{+}_{\xi_M}$$

where each of ξ_M's are exponentially-growing and equipped with a lamination pair $\Lambda^{\pm}_{\xi_M}$ (with attracting neighborhoods $V^{\pm}_{\xi_M}$) such that $\mathcal{A}_{na}\Lambda^{\pm}_{\xi_M}$ is trivial (using conclusion 1 of proposition 5.4 and bullet 6 in the hypothesis set 6).

Using Lemma 6.2 the automorphisms ξ_M's being reducible implies that $\mathcal{A}_{supp}(\Lambda^{\pm}_{\xi_M}) = [F_{\xi_M}]$ is proper. Fix a marked metric graph $H = R_N$, the standard rose. Denote the stallings graph (discussed at the beginning of this section) associated to F_{ξ_M} by K_M, equipped with the immersion $p_M : K_M \to H$. A *natural vertex* is a vertex with valence greater than two and a *natural edge* is an edge between two natural vertices. We can subdivide every natural edge of K_M into *edgelets*, so that each edgelet is mapped to an edge in H and label the edgelet by its image in H.

Let γ^-_M be a generic leaf of Λ^-_M and γ^+_M be a generic leaf of Λ^+_M. We note that every natural edge in K_M is mapped to an edge path in H, which is crossed by both γ^-_M and γ^+_M.

We claim that the edgelet length of every natural edge in K_M is uniformly bounded above. *Once we have proved the above claim*, it immediately follows that (after passing to a subsequence if necessary) there exists homeomorphisms $h_{M,M'} : K_M \to K_{M'}$ which maps edgelets to edgelets and preserves labels. Hence, we can assume that the sequence of graphs K_M is eventually constant (upto homeomorphism) and $F_{\xi_M} = F$, is independent of M.

Next, observe that, if α is any finite subpath of a generic leaf of Λ^+_ψ, by enlarging α if necessary we can assume that it defines an attracting neighborhood of Λ^+_ψ. By using the uniformity of attracting neighborhoods from the ping-pong lemma (conclusion 4,5) we know that γ^+_M belongs to this neighborhood for sufficiently large M. This means $\alpha \subset \gamma_M^+$ for sufficiently large M, which implies that the realizations of λ_ψ^+ lift to K_M. A similar argument gives the same conclusion about λ_ϕ^-. Thus both λ_ψ^+ and λ_ϕ^- are carried by F , which implies that F carries Λ_ψ^+ and Λ_ϕ^- which contradicts our hypothesis. Hence, F cannot be proper and so the ξ_M's are fully irreducible for all sufficiently large M - contradiction.

proof of claim : Suppose that the edgelet length of the natural edges of K_M is not uniformly bounded. Then there exists a sequence of natural edges $\{E_M\}$ such that their edgelet lengths go to infinity as $M \to \infty$. Let l be a weak limit of some subsequence $\{E_M\}$ and $\sigma \subset l$ be any finite subpath. For sufficiently large M, $\sigma \subset E_M \subset \gamma_M^+$. Hence $l \in L^+ = \{$All weak limits of all subsequences of $\gamma_M^+\}$. Similarly, $l \in L^- = \{$All weak limits of all subsequences of $\gamma_M^-\}$. It remains to show that the intersection of this two sets is empty. Suppose not. Let γ^* be a weak limit of some subsequence of γ_M^-. We claim that γ^* is not attracted to Λ_ϕ^+. If not, then $\phi^p(\gamma^*) \in V_\phi^+$ for some $p \geq 0$. This means that for sufficiently large M, $\phi^p(\gamma_M^-) \in V_\phi^+$, implying that $\xi_M(\gamma_M^-) \in V_{\xi_M}^+$ for sufficiently large M, which is a contradiction to the fact that a generic leaf of $\Lambda_{\xi_M}^-$ is not attracted to $\Lambda_{\xi_M}^+$ under action of ξ_M. By similar arguments we can show that if γ^* is a weak limit of some sequence of γ_M^+, then γ^* is not attracted to Λ_ψ^-.

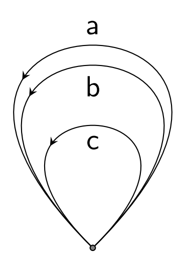
Let $l \in L^+ \cap L^-$. Then $l \in \mathcal{B}_{na}(\Lambda_\phi^+) \cap \mathcal{B}_{na}(\Lambda_\psi^-)$ by above arguments and by lemma 4.3. If l is not carried by $\mathcal{A}_{na}(\Lambda_\psi^\pm)$, then by the weak attraction theorem (Lemma 4.4) l is contained in every attracting neighborhood of the generic leaf λ_ψ^+. This implies that $\lambda_\psi^+ \subset cl(l) \subset \mathcal{B}_{na}(\Lambda_\phi^+)$. But this contradicts our hypothesis that Λ^+_ψ is attracted to Λ^+_ϕ. Hence l must be carried by $\mathcal{A}_{na}(\Lambda^\pm_\psi)$. By a symmetric argument we can show that l must be carried by $\mathcal{A}_{na}(\Lambda^\pm_\phi)$. But this is not possible, since by our assumption these two subgroup systems are mutually malnormal. So $L^+ \cap L^- = \emptyset$

$\square$

Example 6.8. Consider example 3.20 and call the corresponding outer automorphism ϕ.

Let us define $\psi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ by:

$$a \mapsto ba, b \mapsto a, c \mapsto c$$



The stable relative train track map of ϕ can be drawn on the rose itself. It is exponentially growing and the EG strata is $H_2 = \{a, b\}$. There is a closed indivisible Nielsen path $abAB$ crossing the top strata. Together with the lower strata $H_1 = \{c\}$ the two define the nonattracting subgroup system $[\langle abAB, c\rangle]$. Define the map $\xi = \phi^{-2}\psi^2$.

$$a \mapsto aaBACabaaCaba, b \mapsto aBACabaaCaba, c \mapsto ABAcAABAcabCabA$$

This map is fully irreducible and hyperbolic.

It is however, not easy to come up with an exponent that we can verify will always work. However, at least in this particular case the magic number 2 actually works as a lower bound. This has been checked on the train track computing software developed by Thierry Coulbois by trying on large exponents and really complicated words in ϕ and ψ.

7 APPLICATIONS

This section is dedicated to looking at some applications of the Theorem 6.7 . The corollary that is stated below is probably the first time that we see some result on the dynamics of mixed type of fully irreducible automorphisms. There are in fact three corollaries packed under the same hood. The first one is a well known theorem from [17] but the proof in their paper is very different from the technique we use here. The other two items are new.

Corollary 7.1. *If ϕ, ψ are fully-irreducible elements of* $\mathrm{Out}(\mathbb{F}_\mathrm{N})$ *which are not conjugate to powers of each other, then there exists an integer $M \geq 0$ such that for every $m, n \geq M$, $G_M = \langle \psi^m, \phi^n \rangle$ is a free group of rank two, all whose elements are fully-irreducible.*
Moreover, M can be chosen so that

1. *If both ϕ, ψ are hyperbolic, then every element of G is hyperbolic.*
2. *If ψ is hyperbolic and ϕ is geometric, then every element of G not conjugate to a power of ϕ is hyperbolic.*
3. *If both ψ and ϕ are geometric but do not fix the same conjugacy class,*

then every element of G not conjugate to a power of ϕ or ψ is hyperbolic.

Proof. The conclusion about the free group is from Proposition 6.6. It is easy to check that ϕ, ψ satisfy the hypothesis of Theorem 6.7 except bullet four in definition6.4. But Proposition 5.5 tells us that bullet four is not required to draw all the conclusions in pingpong for this special case. Hence we can make the conclusion of the theorem which, along with the fact that the conjugate of any power of a fully-irreducible outer automorphism is also fully-irreducible, gives us that every element of G is fully-irreducible.

We now look at the proofs of the statements in moreover part:

1. follows immediately since conclusion 1 of Proposition5.4 tells us that the $\mathcal{A}_{na}\Lambda^{\pm}_{\xi}$ is trivial which implies that no element of G_M has any periodic conjugacy classes. [4] tells us that they are hyperbolic.

2. follows from corollary 5.7 and Theorem 6.7

3. follows from corollary 5.8 and Theorem 6.7

□

The following result is another interesting application of the main theorem and several other technical lemmas that we have developed along the way. The proof of the result is almost same as Theorem 6.7 same but a small modification is needed in the last part of the proof.

A version of the theorem below, when the surface $\mathcal{S}$ is without boundary is an important result proved in [1] where they develop the theory of convex cocompact subgroups of $\mathcal{MCG}(\mathcal{S})$.

Theorem 7.2. *Let $\mathcal{S}$ be a connected, compact surface (not necessarily oriented) with one boundary component. Let $f, g \in \mathcal{MCG}(\mathcal{S})$ be pseudo-Anosov homeomorphisms of the surface which are not conjugate to powers of each other. Then there exists some integer M such that the group $G_M = \langle f^m, g^n \rangle$ will be free for every $m, n > M$, and every element of this group will be a pseudo-Anosov.*

Proof. Let $\psi, \phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ be the fully irreducible, geometric outer automorphisms induced by f, g respectively, where $F_r = \pi_1(\mathcal{S})$. We will prove the result for $\psi, \phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ which will imply the theorem.

Let $[c]$ be the conjugacy class corresponding to $\partial\mathcal{S}$. Then $[c]$ fills F_r and $\psi([c]) = \phi([c]) = [c]$ and $\mathcal{A}_{na}\Lambda^\pm_\psi = \mathcal{A}_{na}\Lambda^\pm_\phi = [\langle c \rangle]$. Proposition 5.4 along with proposition 5.5 tells us that there exists an integer M such that every cyclically reduced word ξ in the group G_M will be exponentially growing with a dual lamination pair $\Lambda^\pm_\xi$ such that any conjugacy class carried by $\mathcal{A}_{na}\Lambda^\pm_\xi$ will be carried by both $\mathcal{A}_{na}\Lambda^\pm_\psi$ and $\mathcal{A}_{na}\Lambda^\pm_\phi$. Hence we can conlude that $\mathcal{A}_{na}\Lambda^\pm_\xi = [\langle c \rangle]$. We check that we satisfy all the hypothesis required for the main theorem except the last two bullets. Proposition 5.5 voids the need for bullet five. We modify the proof of Theorem 6.7 by using Lemma 6.3 so that we do not need the last bullet. Using Proposition 6.6 we can conclude that by increasing M if necessary, we may assume that G_M is free of rank two.

The proof of being being fully irreducible follows the exact same steps, but in this case we use Lemma 6.3 to start the contradiction argument as in the proof of Theorem 6.7. Namely, assume that there does not exist any M such that every element of G_M is fully irreducible and let $\xi_M \in G_M$ be a reducible element for each M. After passing to a subsequence if necessary assume that $\xi_M = \psi^{m_1}\phi^{\epsilon_1' n_1}........\psi^{\epsilon_k m_k}\phi^{n_k}$ where $m_i = m_i(M) > 0, n_j = n_j(M) > 0$ and k depend on M. From above discussion we have $\mathcal{A}_{na}\Lambda^{\pm}_{\xi_M} = [\langle c\rangle]$, where $\mathcal{A}_{supp}[c]$ is not proper. Using Lemma 6.3, $\mathcal{A}_{supp}\Lambda^{\pm}_{\xi_M}$ must be proper and non trivial for all M. The rest of the argument follows through except that when we look at the part of the proof of Theorem 6.7 separated under "proof of claim ", the proof breaks down. We will just focus on this part and modify the proof to finish our theorem.

The goal of that part of the proof is to show that the edgelet length of the natural edges of K_M is uniformly bounded. Suppose the claim is false. Then there exists a sequence of natural edges E_M whose edgelet length goes to infinity. Let l be some weak limit of this sequence. If we show l is a periodic line $l = \rho_\infty$ then $\rho \subset E_M$ for all large M, which implies that free factor support of ρ is contained in the proper free factor F_{ξ_M}. The contradiction is achieved by showing that $[\rho]$ fills F_r.

Let σ be a weak limit of some subsequence of γ^-_M. We claim that σ is not attracted to Λ^+_ϕ. If not, then $\phi^p(\sigma) \in V^+_\phi$ for some $p \geq 0$. This means that for sufficiently large M, $\phi^p(\gamma^-_M) \in V^+_\phi$, implying that $\xi_M(\gamma^-_M) \in V^+_{\xi_M}$ for sufficiently large M, which is a contradiction to the fact that a generic leaf of $\Lambda^-_{\xi_M}$ is not attracted to $\Lambda^+_{\xi_M}$ under action of ξ_M. By similar arguments we can show that if σ' is a weak limit of some sequence of γ^+_M, then σ' is not attracted to Λ^-_ψ.

Let $l \in cl(\sigma) \cap cl(\sigma')$. Then $l \in \mathcal{B}_{na}(\Lambda^+_\phi) \cap \mathcal{B}_{na}(\Lambda^-_\psi)$ by above arguments and by Lemma 4.3. If l is not carried by $\mathcal{A}_{na}(\Lambda^{\pm}_\psi)$, then by the weak attraction theorem (Lemma 4.4) l is contained in every attracting neighborhood of the generic leaf λ^+_ψ. This implies that $\lambda^+_\psi \subset cl(l) \subset cl(\sigma) \subset \mathcal{B}_{na}(\Lambda^+_\phi)$. But this contradicts our hypothesis that Λ^+_ψ is attracted to Λ^+_ϕ. Hence l must be carried by $\mathcal{A}_{na}(\Lambda^\pm_\psi) = [\langle c \rangle]$. Hence $l = c_\infty$ is a periodic line and $[c]$ fills F_r and we get our contradiction by taking $c = \rho$.

$\square$

8 Ending laminations, Cannon-Thurston maps

In this section we will use the weak attraction theory to give a short and elegant proof that the Cannon-Thurston map from $\partial\mathbb{F}_N$ to the boundary of the mapping torus of a fully-irreducible hyperbolic outer automorphism is a finite-to-one map. This result was already proved by Kapvich and Lustig in [18] by using tools that are very different from the ones we have seen here. It seems that the train track theory proof that we will see here gives a slightly better understanding of this map. We begin with some definitions.

Let Γ be a word-hyperbolic group and $H < \Gamma$ is a word-hyperbolic subgroup. If the inclusion map $i : H \to \Gamma$ extends to a continuous map of the boundaries $\widehat{i} : \partial H \to \partial\Gamma$ then $\widehat{i}$ is called a Cannon-Thurston map. When it does exist, it is an interesting question to know what its properties are. Its precise behavior is captured by the notion of *Ending laminations*, denoted by L_ϕ. We will skip the original definition here but instead use the one that will be more useful for us. The original definition was given in [23] and for Free groups it was later modified and used in [18]. The following definition is needed to state the proposition we will be needing from [18]

Definition 8.1. Given a set $R \subset \partial\mathbb{F}_N$, define the *diagonal closure* of R to be the set

$$\mathrm{Diag}(R) = \{(X, Y) \in \partial^2\mathbb{F}_N | \exists Z_0 = X, Z_1, ..., Z_k = Y, \text{such that} (Z_i, Z_{i+1}) \in R\}$$

Note that taking $k = 1$ implies $R \subseteq \mathrm{Diag}(R)$.

Kapovich and Lustig showed:

Proposition 8.2. *[18, Proposition 4.5] Let* $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ *be a fully-irreducible, hyperbolic element. Then*

$$L_\phi = Diag(\Lambda^+_\phi).$$

.

The Cannon-Thurston map, when it exists, identifies the endpoints of the certain leaves of ending lamination. Our goal here is to understand the class of leaves that get identified by this map by using the theory of laminations and singular lines. Let Γ_ϕ denote the mapping torus for a hyperbolic $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$. The precise statement is given by:

Proposition 8.3. *[23] If* $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ *is a hyperbolic outer automorphism then the Cannon-Thurston map* $\widehat{i} : \partial\mathbb{F}_N \to \partial\Gamma_\phi$ *exists. Moreover,* $\widehat{i}(X) = \widehat{i}(Y)$ *if and only if the line* $l \in \mathcal{B}$ *joining* X *to* Y *is in* $L_\phi \cup L_{\phi^{-1}}$.

We will restrict our attention to $\phi \in \mathrm{Out}(\mathbb{F}_\mathrm{N})$ fully irreducible hyperbolic outer automorphism and Λ^+_ϕ be it's attracting lamination and $\mathcal{S}(\phi)$ be it's set of singular lines. Let $E^+ = \Lambda^+_\phi \cup \mathcal{S}(\phi)$. Similarly define E^- and let $E_\phi = E^+ \cup E^-$.

Proposition 8.4. *Diag*$(\Lambda^+) = E^+$

Proof. First we show that Diag$(\Lambda^+) \subseteq E^+$. For this, let $(X,Y) \in$ Diag(Λ^+). Then there exists a finite sequence of points $Z_0 = X, Z_1, ..., Z_k = Y$ in $\partial\mathbb{F}_N$ such that the line with endpoints (Z_i, Z_{i+1}) is a leaf of Λ^+. If $k = 1$, X, Y are endpoints of a leaf of Λ^+ and we are done.

Suppose $k \neq 1$. Let us denote these leaves by $l_1, l_2, ..., l_k$. By our assumption all l_i's are asymptotic and hence fall in the same equivalence class of asymptotic leaves. By [11, Lemma 3.3] there exists a principal automorphism Φ, representing ϕ that fixes all these leaves. Hence the points $X, Y \in \text{Fix}_+(\widehat{\Phi})$ and they define an element of $\mathcal{S}(\phi)$.

For the reverse inclusion Diag$(\Lambda^+) \supseteq E^+$, suppose $l \in E^+$ with endpoints P, Q. If $l \in \Lambda^+$ we are done. So assume that $l \in \mathcal{S}(\phi)$. This implies there exists a principal automorphism Φ representing ϕ and $P, Q \in \text{Fix}_+(\widehat{\Phi})$. By [11, Lemma 3.1] both P and Q are the endpoints of some leaves l_P and l_Q of Λ^+. But [11, Lemma 3.3] tells us that there is bijection between principal automorphisms representing ϕ and the equivalence classes of asymptotic rays in Λ^+. Thus l_P and l_Q are asymptotic and thus $P, Q \in \text{Fix}_+(\widehat{\Phi})$. Hence the line that joins P, Q is an element of $\mathcal{S}(\phi)$. $\square$

Proposition 8.5. *For a fully-irreducible hyperbolic* $\phi \in \text{Out}(\mathbb{F}_\text{N})$*, the Cannon-Thurston map* $\widehat{i} : \partial\mathbb{F}_N \longrightarrow \partial G_\phi$ *is a finite-to-one map.*

Proof. Suppose $P_1, P_2, P_3......$ be distinct points such that $\widehat{i}(P_1) = \widehat{i}(P_2) = \widehat{i}(P_3)......$ This implies the lines l_i's are leaves of the ending lamination E_ϕ. Without loss of generality assume that l_i's belong to E^+. Then there exists a principal automorphism representing ϕ that fixes all these lines. Therefore, any such sequence of distinct points P_i's will define distinct lines in $\mathcal{S}(\phi)$. Since the number of lines in $\mathcal{S}(\phi)$ is finite, $\widehat{i}^{-1}(X)$ for all $X \in \partial G_\phi$ must be a finite set.

Moreover, we know that the finiteness in $\mathcal{S}(\phi)$ comes from the finiteness of the possible number of isogrediance classes of principal automorphisms of ϕ. From Remark 3.9 [21] we learn that this bound is actually uniform since the number of Nielsen classes with principal vertices are uniformly bounded.

□

Future research questions:

1. Is there a uniform lower bound for the exponents of the groups constructed in 7.1 ? This result is true in the case of mapping class groups of closed surfaces and was shown by Fujiwara in [8]. But the technique he uses (acylindrical actions) is still unknown for $\mathrm{Out}(\mathbb{F}_\mathrm{N})$.

2. Attempt to prove a similar result as 6.7 for polynomially growing outer automorphisms. A partial answer is already known from [5]. Even more generally one can attempt to answer the question for arbitrary outer automorphisms.

3. Is Proposition 8.5 true for hyperbolic outer automorphisms that are not fully irreducible ?

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