MODELING VORTEX-INDUCED FLUID-STRUCTURE INTERACTION USING AN EXTENSION OF JOURDAIN'S PRINCIPLE

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ABSTRACT OF THE DISSERTATION

MODELING VORTEX-INDUCED FLUID-STRUCTURE INTERACTION USING AN EXTENSION OF JOURDAIN'S PRINCIPLE

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A first-principles variational approach is proposed for reduced-order order modeling of fluid-structure interaction systems, specifically vortex-induced vibration. Fluid-structure interaction has to be taken into account in design and analysis of a large portion of engineering applications, yet a comprehensive theoretical development where analytical equations are derived from first principles is nonexistent. Not only does there exist much ambiguity concerning the general behavior of such systems, but the nature of the Lagrangian-Eulerian transformation is yet to be fully understood. Also, a general variational principle that is purely defined in a Eulerian description is nonexistent. Consequently, the use of variational methods for fluid-structure interaction problems has been relatively successful only for simple problems. Moreover, a review of the literature suggests that the Navier-Stokes (N-S) equations could not be obtained using a variational principle. This can be avoided by using Jourdain's principle (JP). Therefore, we have modified Jourdain's principle and obtained the

first purely Eulerian variational formulation. Subsequently, by extending the JP for systems of changing mass, we have shown that the N-S equations can be obtained via a variational approach. Moreover, having shown that conservative terms of the N-S equations do not commute with the Eulerian variational operator, a correction term is obtained that must be added to the classical energy equation in integral form for Newtonian incompressible viscous fluids. Regarding vortex-induced vibration, an elastically supported, inverted pendulum that is immersed in a flow is considered as a study system. The pendulum is allowed to move transversely to the flow direction. This problem has generally been used as a test bed of vortex-induced vibration models, as it provides a simple geometry, yet possesses the nonlinearity of these phenomena. It is shown that the reduced-order modeling can be done without any *ad hoc* assumptions regarding the fluid forcing function. There exists no reduced-order model in the literature that does not make such assumptions. Based on the theoretical results as well as the reduced-order model, we conclude that the first principles development herein is a viable framework for the modeling of complex fluid-structure interaction problems such as vortex-induced oscillations.

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Dedication

To my mother, Meetra Emamy

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Chapter 1
Introduction and Preliminaries

The problem of fluid-structure interaction (FSI) has long been one of the great challenges in engineering. It is a crucial consideration in the design of many engineering systems, such as offshore structures, aircraft and bridges. While the importance of the subject has been understood for over a century, it has been only in the past few decades that efforts have been made to analytically model the general behavior of such systems. Parallel to analytical attempts, many experiments have been devoted to gathering data and interpreting such interactions. Consequently, analytical dynamics modeling of such problems has evolved with coupling to experimental data resulting in various semi-analytical representations. Generally, attempts have been made to model vortex-induced vibration (VIV) problems as few degrees of freedom (DOF) oscillatory models; therefore, they are referred to as *reduced-order* models.

Regarding the experimental studies on VIV, certain types of structural configurations have been preferred in the literature where a rigid solid body with one or two degree(s) of freedom is immersed in a flow. While the experiments have been conducted on variety of solid shapes (and occasionally on flexible bodies), *reduced-order semi-analytical models* have been generally developed for single DOF rigid bluff bodies, specifically for circular cylinders. The most commonly used model, called the *model problem* [7], is a type of inverted pendulum that is immersed in a flow, rests on elastic supports and can only move transversely to the flow direction. A second model is the translating cylinder. Schematic diagrams of elements of two representative configurations of the *model problem* are shown in Figures 1.2 and 1.1.

The *model problem* has been widely used since it possesses a simple geometric configuration, and yet, it exhibits the majority of the nonlinear behaviors of VIV systems. Consequently, the majority of VIV experiments have been conducted on the *model problem*. Both in experimental and analytical studies, the flow is controlled or considered to be two dimensional for all time, as are the shedding vortices.

The purpose of this work is to present our theoretical development that can improve existing models by deriving reduced order models from first principles where assumptions are explicitly stated. Therefore, experimental observations are not the main focus this research work. However, few key features observed in the experimental studies are summarized for those who are not familiar with the subject. An excellent review of the experiments of VIV can be found in [6].

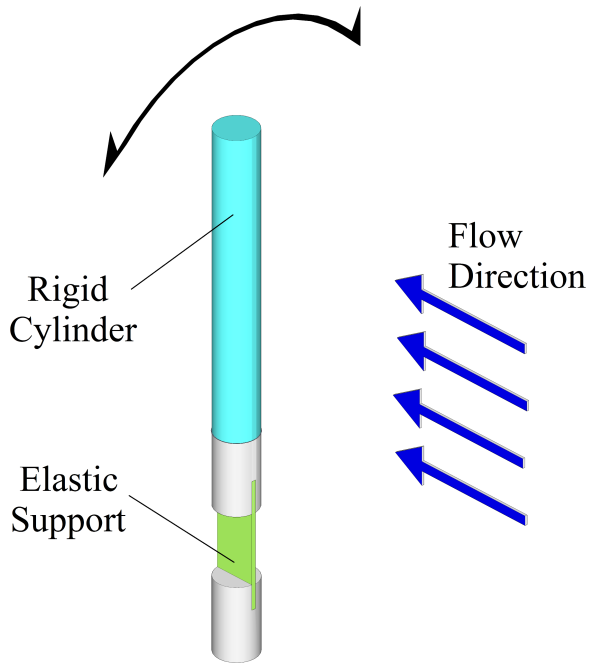

Figure 1.1: A representative configuration of the *model problem:* inverted pendulum.

Starting with the stagnant fluid, if the speed of the flow past the cylinder is

increased, three different behavioral regimes are identified: *pre-synchronization*, *resonant synchronization* and *classical lock-in*. *Pre-synchronization* is the first regime where the structure starts oscillating and vortices are first observed. The amplitude of the structural oscillations are low and the vortices' strength are weak to moderate. Observed in this region is a beating behavior, that is, the amplitude of structural response increases and decreases gradually as the structure oscillates. Moreover, the flow drives the structure in this region.

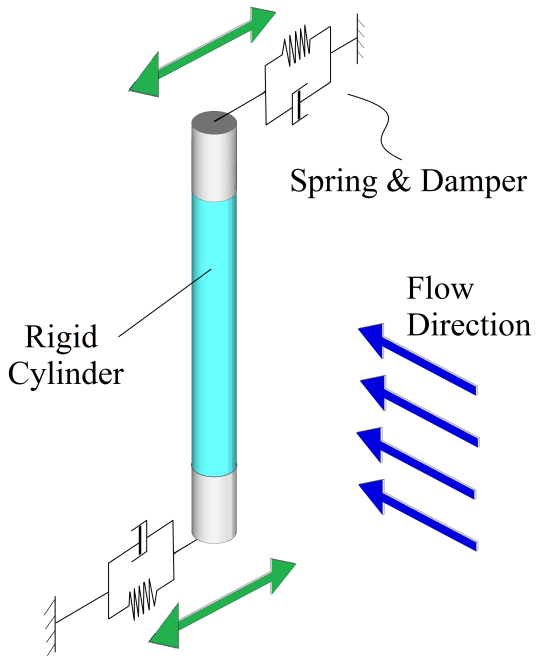

Figure 1.2: A representative configuration of the *model problem:* translating cylinder.

As the average velocity of the flow is increased, vortices become stronger until the frequency of the vortex shedding reaches the natural frequency of the structure, where near-resonant behavior is observed. Thus, the structural response reaches a maximum and it is called the *resonant synchronization* region. Similar to the *pre-synchronization* region, beating behavior is noticeable but weaker, and the structure remains driven by the flow.

If the velocity is increased further, constant structural oscillation amplitude and frequency are observed for a range of flow velocities. This phenomenon is called the *classical lock-in*. Unlike the other two regions, the flow is modulated by the structure and the vortices observed are the least organized. The existence of three distinct regimes in the frequency-amplitude response curves of an inverted pendulum is shown in Figure 1.3.

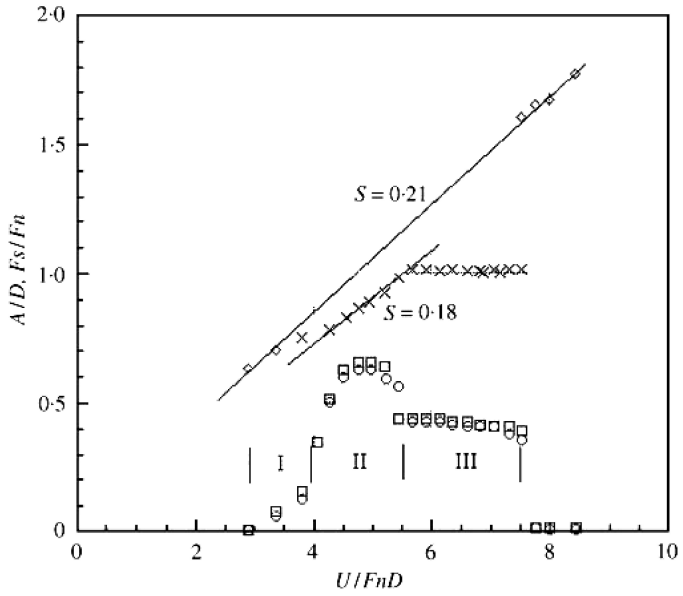

Figure 1.3: The frequency-amplitude response curves of an inverted pendulum, where A is the amplitude of oscillations, D is the diameter of the cylinder, F_s is the frequency of oscillations, F_n is the natural frequency of the cylinder, U represents the fluid velocity. □, ◯, amplitude of oscillation for two independent but identical experimental runs; ×, frequency of ocillation and vortex shedding frequency in which VIV was observed; ◇, frequency of vortex shedding where the cylinder was stationary; I, pre-synchronization; II, resonant synchronization; III, classic lock-in ([1], with permission from Elsevier).

Also observed in many of the experiments is the existence of hysteresis behavior, that is, the maximum amplitude of the oscillations are larger as the velocity is increased than when it is decreased, as shown in Figure 1.4. VIV is a complicated

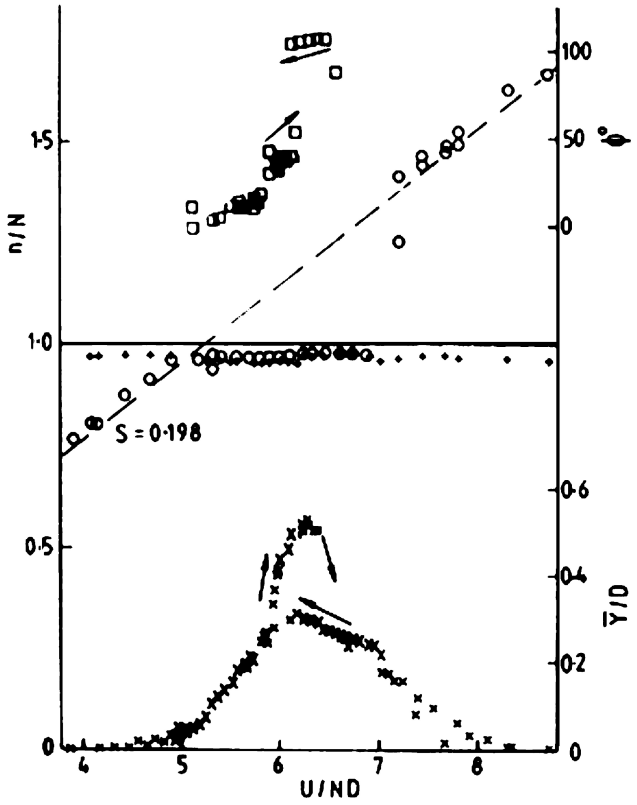

Figure 1.4: Oscillation characteristics of a vibrating circular cylinder wit light damping. n is the frequency of vortex shedding, N is the frequency of oscillations, $\bar{Y}/D$ is the normalized maximum amplitude of oscillation at the reduced velocity U/ND, and ϕ° denotes the phase angle between the fluid force and the cylinder displacemen. $+$, frequency of oscillation; $\bigcirc$, frequency of vortex-shedding; $\square$, phase angle; $\times$, amplitude of oscillation ([2] in [3]).

phenomenon. The structural response depends on many factors, such as shedding frequency, Reynolds number, material damping, structural stiffness, surface roughness, cylinder length, density of the fluid and mass of the cylinder, [6] and [8]. Therefore, *reduced-order* modeling of VIV has evolved in parallel to experiments in order to increase our understanding of this phenomenon.

Efforts to model VIV as reduced-order systems can be divided into two categories: empirical models and first-principles models. Moreover, the empirical models can be divided into two subcategories: wake-oscillator (wake-body) models and experimental

force-coefficient models.

The wake-oscillator models are based on the assumption that an immersed structure in a flow experiences hydrodynamic forces that are the result of nonlinear oscillator like systems. Therefore, the aim is to obtain nonlinear fluid force equations from the experimentally acquired data that can be coupled with the structural equation of motion (EOM). One of the earliest models is the one proposed by Hartlen and Currie [9]. They used a van der Pol type fluid oscillator to model the fluid-structure system,

$$\begin{cases} \ddot{x} + 2\zeta\dot{x} + x = a\omega_0^2 C_L \\ \ddot{C}_L - \alpha\omega_0\dot{C}_L + \dfrac{\gamma}{\omega_0}\dot{C}_L^3 + \omega_0^2 C_L = b\dot{x}, \end{cases} \tag{1.1}$$

where a, ω_0, and ζ are the known structural parameters, and the fluid parameters α, b, and γ are found experimentally.

The experimental force coefficient models are single degree-of-freedom (DOF) models. They only include a single forcing function obtained experimentally. Generally, the empirical models have relative success in capturing the features of VIV. However, these models neglect the dynamics of the flow by only considering the forces as they are seen by the structure. Therefore, they do not provide much understanding of the physics of the problem. These *ad hoc* methods are outside of the scope of this work that is focused on first principles models, specifically, using variational principles. Good reviews of the empirical models can be found in [6], [10] and [11].

While variational principles have been known for well over a century, it was not until 1973 that McIver was among the first researchers to propose the use of variational methods in modeling the fluid-structure interaction problems [12]. Also, the work by Benaroya and Wei in 2000 is one of the earliest attempts to use those methods for VIV problems [1]. Consequently, the literature on the subject is very limited. Therefore, we continue this chapter in the next section by reviewing a broader topic, that is, the application of variational principles for fluid systems.

Having reviewed the literature in Section 1.1, one main challenge identified is the relation of variational principles in the Lagrangian frame of reference to the Eulerian frame of reference. These relations are reviewed in Section 1.2. Then, the relation between the Lagrangian and the Eulerian variational operators are developed in Section 1.3. Afterwards, based on our discussions in Sections 1.2 and 1.3, some of the challenges that have been faced when applying variational methods to fluid systems are explained in Section 1.4. Having reviewed and discussed the literature on the subject, we explain the motivation behind this research work in Section 1.5. Finally, an outline is provided for the remainder of this dissertation in Section 1.6.

1.1 Variational Principles for Fluid Systems

Efforts to model fluid dynamics by using variational principles can be traced back to a work by Millikan in 1929 [13]. He tried to obtain the governing equations of steady motion of a viscous, incompressible fluid by using Lagrange's equations. He started from the balance of energy equation of a control volume. Then, he imposed the continuity equation to the balance of energy equation by means of Lagrangian undetermined multipliers. He concluded that it is impossible to obtain the Navier-Stokes equations for a steady motion of a viscous, incompressible fluid from Lagrange's equations unless certain conditions are met. However, Millikan could not find realistic examples where those conditions are violated. We do not encounter these issues in this dissertation.

Since Millikan's work, tens of attempts have been made to carry out similar studies. Next, we review a few of these that are believed to be good representative works. Later, we use some of these papers to point out the difficulties faced in using variational methods for fluid systems. The challenges in this field are extensive. These papers provide thorough examinations of the subject and present well-explained derivations that, in part, have helped the development of the current study.

Another early attempt in applying Hamilton's principle to a system of particles is the work by Eckart (1938) [14]. His goal was to obtain the equations governing electrodynamic motions. He used Hamilton's principle and imposed Maxwell's equations in the form of electromagnetic potentials (as constraints) by utilizing Lagrange multipliers. The resulting variational equation can be considered for modeling an irrotational ideal fluid system if the electromagnetic potentials are replaced with velocity potentials.

In 1954, Herivel argued that it is hard to imagine how Hamilton's principle would follow the same exact rules in the Eulerian description as it follows in the Lagrangian one [15]. Therefore, one must first transform the Lagrangian function to Eulerian form before imposing the variations. He also explained that Hamilton's principle can only be used in the absence of irreversible processes, that is, it can only be applied to an ideal fluid. Therefore, Herivel used a Jacobian matrix and transformed the Lagrangian coordinates into the Eulerian frame. Similar to Eckart's work, he used Lagrangian multipliers to impose the continuity and the entropy transport equations. Then, he utilized a specific form of Clebsch's transformation to obtain the velocity potentials. The resulting variational equation was an extension of Hamilton's principle, containing Lagrangian variations and Eulerian functions. Compared to the earlier models, Herivel's method has the advantage that it can also be used for rotational motions.

In a later paper, Eckart (1960) discussed that finding a general variation principle in a Eulerian representation is very hard, therefore, the Lagrangian approach is preferable [16]. Considering the conservation law in terms of a Lagrangian energy-moment tensor, he used a Jacobian matrix to backtrack the particles to their initial positions while normalizing the coordinates with respect to density. Then, he obtained the Lagrangian equations for the motion of both incompressible and compressible fluids. Also, he discussed the importance of Clebsch's transformation when integrating the

problems involving vorticity equations. Eckart's model has the disadvantage that it requires the coordinates of each particle to be specified at least at two different points in time.

In a later work, Bretherton (1970) tried to obtain the Eulerian equations of motion from the Lagrangian formulation of Hamilton's principle [17]. His main aim was to clarify some of the inconsistencies in the literature. He considered the fact that there must exist an inverse mapping function of Lagrangian trajectories that can be used to trace particle positions to some earlier configuration. As a result, he assumed that the density at each instance of time is related to the density at some reference time via the Jacobian mapping of Lagrangian trajectories to the inverse mapping function. Having obtained the Eulerian variations as functions of the Lagrangian virtual displacement, he applied the Eulerian variation to the action integral. The results were equations containing the Eulerian functions and the Lagrangian virtual displacements. As an example, Bretherton obtained Kelvin's circulation theorem using his variational approach.

Another effort in applying Hamilton's principle to fluid mechanics problems is the work by Leech (1977) [18]. He argued that Hamilton's principle cannot directly be applied to a control volume as it is defined for a system of particles. Also, he discussed that the variational operator does not commute with the control volume. Rather it commutes with mass integrals as the integration limits are invariant. Moreover, he noted that Hamilton's principle expressed in the form of $\delta \int L dt = 0$ can only be used for conservative systems. For nonconservative systems, the original statement of the principle must be used. Additionally, he assumed that there must exist a function that maps the instantaneous displacements of fluid particles to a reference state. Then, he used the Jacobian matrix to map the integrals back to a reference frame - in the same manner that is usually used to prove Reynolds' transport theorem (RTT). His manipulations resulted in an integral equation containing the variation of the

mapping functions. Then, the author suggested that one may use Hamilton's principle to optimize an assumed solution. Therefore, a class of displacement functions (called *admissible functions*) can be chosen for a problem and their weighting functions can be optimized using his formula. To elaborate on his method, Leech considered a few problems. One such problem was the interaction between an incompressible inviscid flow and a structure, called the d'Alembert's paradox, in which d'Alembert proved that the drag force is zero.

An important approach to modeling fluid dynamics using variational principles is by McIver (1973) [12]. He presented his extended form of Hamilton's principle for problems involving fluid-structure interactions. Having considered Hamilton's principle for a system of continuous particles, McIver utilized Reynolds' transport theorem to modify the principle for a system of changing mass (control volume, CV). For a moving control volume, it is customary to consider the relative velocities of the fluid particles with respect to the control volume. However, McIver considered the velocity of the control volume with respect to the fluid particles, $\boldsymbol{u}_r$, for which he did not provide any justification. Perhaps McIver's aim was to introduce the backtracking concept that has been used by the previously mentioned authors at this stage of his formulation. Also, he assumed that the virtual work performed on a control volume is purely due to the surface tractions at the control surface. Therefore, using the stress dyadic, $\bar{\boldsymbol{\sigma}}$, he considered the virtual work to be

$$\delta W = \int_{CS} \delta \boldsymbol{r} \cdot \bar{\boldsymbol{\sigma}} \cdot \boldsymbol{n} \, dA, \tag{1.2}$$

where $\delta \boldsymbol{r}$ is the virtual displacement, $\boldsymbol{n}$ is the normal vector to the differential surface element dA, and CS is the control surface. For fluid-structure interaction problems, the control volume can be chosen in a manner that some portions of the control surface match the structural surfaces. Denoting the portions of the CS where the

flow cannot pass through by CS_C (closed CS), and representing the rest of the CS by CS_O (open CS), McIver's extension of Hamilton's principle is given by

$$\delta \int_{t_1}^{t_2} (T - \Pi)_{CV} dt + \int_{t_1}^{t_2} \int_{CS_O} \left[\delta \boldsymbol{r} \cdot \bar{\boldsymbol{\sigma}} \cdot \boldsymbol{n} + \rho \left(\boldsymbol{u} \cdot \delta \boldsymbol{r} \right) \left(\boldsymbol{u}_r \cdot \boldsymbol{n} \right) \right] dA dt + \int_{t_1}^{t_2} \int_{CS_C} \delta \boldsymbol{r} \cdot \bar{\boldsymbol{\sigma}} \cdot \boldsymbol{n} dA dt = 0, \quad (1.3)$$

where $\boldsymbol{u}$ is the absolute velocity of the fluid particles, T is the kinetic energy and Π is the potential energy.

Equation 1.3 represents a stationary process if the integrand of the second term always disappears at the CS_O, that is,

$$\bar{\boldsymbol{\sigma}} \ + \rho \boldsymbol{u} \boldsymbol{u}_r = 0, \qquad \text{or} \qquad (\bar{\boldsymbol{\sigma}} \ + \rho \boldsymbol{u} \boldsymbol{u}_r) \cdot \boldsymbol{n} = 0 \quad \text{ at } CS_O. \quad (1.4)$$

Therefore, the applicability of McIver's equation is restricted to the cases where such a control volume can be distinguished from the physics of the problem, where the fluid is bounded by the structure. McIver considered two simple problems as examples, a rocket problem and a flexible pipe problem.

Along similar lines, Xing and Price (2000) modified Hamilton's principle for non-linear ship-water interactions [19]. They considered that imposing virtual displacements cause the particles to be virtually transported across an assumed control volume. They defined a general integral function of interest, say H, as

$$H\left[\phi\right] = \int_{t_1}^{t_2} \int_{CV} F\left(\phi, \frac{\partial \phi}{\partial t}\right) dV dt, \quad (1.5)$$

where ϕ is a continuos differentiable function of displacement, $\boldsymbol{x}$, and time, t. Denoting the local variation (Eulerian) by $\bar{\delta}$ and the material variation (Lagrangian) by δ, they obtained the local variation of the H function to be

$$\bar{\delta}H = \int_{t_1}^{t_2} \left\{ \int_{CV} \bar{\delta}F dV + \int_{CS} F\left(\phi, \frac{\partial \phi}{\partial t}\right)(\delta \boldsymbol{x} \cdot \boldsymbol{n})\, dA \right\} dt. \tag{1.6}$$

Therefore, the variation of H is twofold: the Eulerian variation inside the control volume and the flux of H due to Lagrangian virtual displacements. Then, their model was applied to a rigid ship travelling in calm water and in waves. Xing and Price's method requires further simplifications and assumptions as it contained both Lagrangian and Eulerian variations.

Benaroya and Wei (2000) considered a more general type of FSI problem where the fluid contains the structure [1]. They showed that Hamilton's principle becomes the balance of energy rates when the configuration is not known at any time. Similar to McIver's approach, they used the RTT to relate Hamilton's principle to a control volume. However, unlike the McIver's use of RTT, they chose the conventional form of RTT when the relative velocities, $\boldsymbol{u}_r$, are the relative fluid particle velocities with respect to the control volume. They presented their governing equation as

$$\begin{aligned} \frac{d}{dt}\left(T_{\text{structure}} + \Pi_{\text{structure}}\right)_{CV} &= \int_{CS} \frac{1}{2}\rho u^2 \left(\boldsymbol{u}_r \cdot \boldsymbol{n}\right) dA \\ &\quad + \int_{CS} (-p\boldsymbol{n} + \boldsymbol{\tau}) \cdot \boldsymbol{u} dA - \left(m_{\text{fluid}} u \dot{u}\right)_{CV}, \end{aligned} \tag{1.7}$$

where m_{fluid} is the mass of fluid contained by the CV, p is the pressure, and $\boldsymbol{\tau}$ is the shearing force. They explained that the terms on left-hand side of Equation 1.7 are the structural dynamic terms, and the right-hand side terms can be evaluated experimentally. The result is the acceleration of the structure that can be integrated twice to obtain the structure's displacement.

In parallel with their theoretical development, Benaroya and Wei conducted a series of experiments on the VIV of a circular cylinder in uniform flow. The cylinder was free to vibrate transversely to the flow direction. Having input the experimental data to Equation 1.7, they showed that their model is successful in predicting the

frequencies of the structural oscillation as well as in capturing the beating behavior that is usually observed in VIV. However, the predicted response amplitudes were roughly half of the experimental values. Their results are shown in Figure 1.5. They concluded that these differences are most likely due to the choice of control volume. In their subsequent paper [4], they examined the effect of the choice of the experimental control volume on the predictions of their model. They found that the predictions are indeed influenced by the selection of control volume. Having obtained a CV for which the predictions of the model matched the experimental values (shown in Figure 1.6), they concluded that the control volume must contain both upstream and downstream sections of the flow where downstream control surface is far enough from the structure as to not pass through the vortex formation region, yet not too far as it will not capture the true kinetic energy flux due to the dissipation of energy.

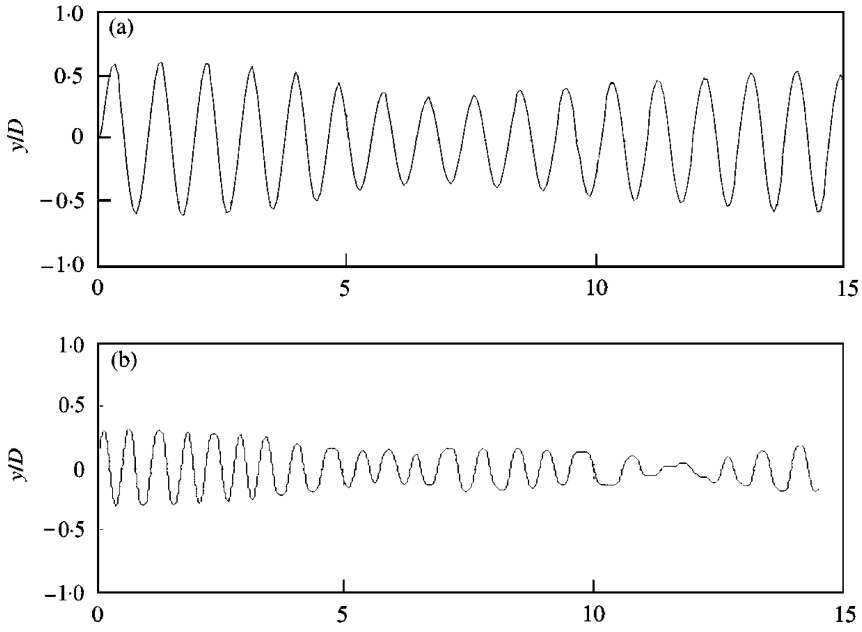

Figure 1.5: Predicted (bottom) vs. Experimental (top) amplitudes. Both plots are in seconds ([1], with permission from Elsevier).

Benaroya and Wei showed that when the configuration is unknown, which is the

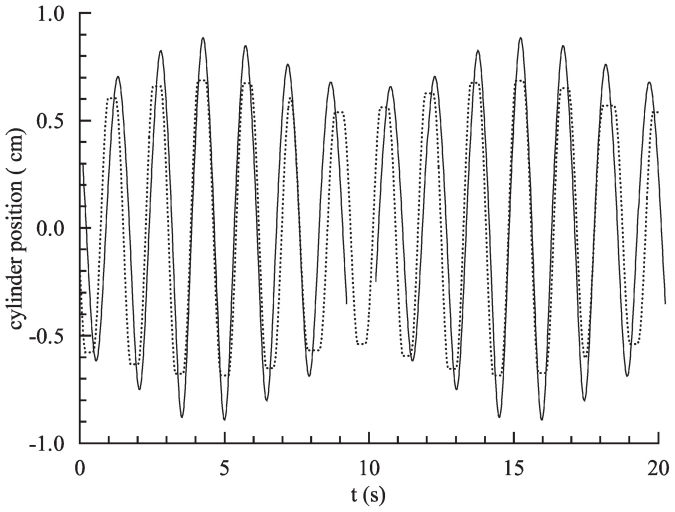

Figure 1.6: Dong *et al.* [4] results obtained for the phase-averaged cylinder position vs. time, where the solid line represents the experimental result and the dotted line is the computed result for a specific control volume (with permission from Elsevier).

case for the majority of fluid and fluid-structure interaction problems, the result of Hamilton's principle is not a variational principle. However, the satisfactory results of their studies motivated the research work by Gabbai and Benaroya to modify the same approach as to obtain a variational method [7]. The experiments show the existence of a formation region (cavity) in vicinity of a cylinder that is immersed in a flow. They assumed that the energy is evenly exchanged between the cylinder and the wake in formation region. Denoting the displacement of this cavity by w, they obtained their variational equation as

$$\int_{t_1}^{t_2} a m_{\text{cavity}} \dot{w} \delta \dot{w} dt + \delta \int_{t_1}^{t_2} \frac{1}{2} m \dot{x}^2 dt - \delta \int_{t_1}^{t_2} \frac{1}{2} k x^2 dt - \int_{t_1}^{t_2} c \dot{x} \delta x dt - \int_{t_1}^{t_2} \delta W \left(\dot{w}, \ddot{w}, x, \dot{x}, \ddot{x}, t \right) dt - \delta \int_{t_1}^{t_2} F(w,t) \delta w dt = 0, \quad (1.8)$$

where the overdot denotes d/dt, m denotes the mass, x is the displacement of the cylinder, k is the structural stiffness, c is the structural damping, δ is the variational

operator, t is time, F is the fluid stiffness, and W represents instantaneous total work done by the transverse hydrodynamic force acting on the cylinder, $F_{fl/st}$, and by the viscous and pressure forces inside the cavity, $F_{\mu/p}$. Therefore,

$$\delta W\left(\dot{w},\ddot{w},x,\dot{x},\ddot{x},t\right)=-F_{fl/st}\left(\dot{w},\ddot{w},\dot{x},\ddot{x},t\right)\delta x+F_{\mu/p}\left(\dot{w},\ddot{w},\dot{x},\ddot{x},t\right)\delta w. \tag{1.9}$$

Then, based on the literature, the authors proposed some general functions of $\dot{w}$, $\ddot{w}$, $\dot{x}$, $\ddot{x}$, t, and lift coefficient for $F_{fl/st}$ and $F_{\mu/p}$. They showed that three of the existing wake-oscillator models are each a specific case of their more general model.

As evident from the literature, the efforts to apply Hamilton's principle and Lagrange's equations to the problems of fluid dynamics have had relative success in certain cases, mainly ideal fluids. There exist no general variational approaches for fluid-structure interactions, nor for fluid dynamics problems. One main challenge arises from relating the variational principles in the Lagrangian frame to the Eulerian frame. Therefore, we next consider the Eulerian and Lagrangian reference frames and review the relations between dynamic properties described in these reference frames.

1.2 Eulerian and Lagrangian Descriptions

In classical mechanics, one has two alternative descriptions to observe and analyze dynamic systems: the Lagrangian description and the Eulerian description. Each of these observation methods has advantages and drawbacks compared to the other one. Generally, one exercises care by considering the physics of the problem in order to understand the convenience of each description on a case by case basis.

The Lagrangian reference frame has long been used in solid mechanics, while the Eulerian reference frame has been preferred in fluid mechanics. The fact that the first principles of mechanics are defined in the Lagrangian reference frame is the main advantage of this descriptions. Then again, it has been shown that the EOM of fluid

systems become less complicated in the Eulerian reference frame.

In the Lagrangian frame, also called the particles description or material coordinate, one observes the trajectories of specific particles during some interval of time. Consequently, a set of coordinates, which are usually the initial positions, are tagged to a set of particles and the resulting EOM are differential equations of the trajectories (paths) of those particles. The Lagrangian trajectories, $\boldsymbol{r}$, can be expressed as

$$\boldsymbol{r} = \boldsymbol{r}\left(\boldsymbol{A}, t\right), \tag{1.10}$$

where $\boldsymbol{A}$ are the initial positions and t is time. Since $\boldsymbol{A}$ are constant coordinates, $\boldsymbol{r}$ are coordinates dependent on t.

The Eulerian frame of reference, also called space-fixed coordinates or configuration frame, describes a system at some fixed points in space. Unlike the Lagrangian coordinates, the Eulerian coordinates are not carried by the particles, thus they remain unaltered by the physics of the problem. Therefore, the EOM represent the dynamics of the particles that occupy certain spatial points at some instants of time, that is, the Eulerian coordinates, $\boldsymbol{x}$, are independent of time, t.

Generally, the transformation of the first principles of mechanics from the Lagrangian frame to the Eulerian description is based on three conditions [5]:

1. The velocity obtained from both frames must be equal to each other at a given time and a given spatial position. This is very clear since an actual flow particle has a unique velocity at any instant.

2. Similar to velocities, the time derivatives of the fluid properties obtained from either one of the representations must match the other one at a given time and spatial position.

3. The time derivative of an integral of a function over a moving material volume

can be related into an integral of that function over an arbitrary control volume using Reynolds' transport theorem.

The majority of the challenges in applying the first principles of mechanics to systems of fluids is due to the difficulties in utilizing the above conditions. These difficulties are explored in more detail in the following sections.

1.2.1 Relating the displacement fields

The displacement field in the Lagrangian frame is defined by the particle trajectories, Equation 1.10. In the field of fluid mechanics, they are often referred to as *pathlines*. The particles trajectories, as they are treated in the Lagrangian frame of reference, do not have an exactly equivalent Eulerian concept, as the Eulerian coordinates are independent of time. The closest concept to pathlines is the existence of a function that maps the current state of the particles to an earlier configuration. For convenience, the earlier state can be chosen to be the initial configuration. Therefore, the Eulerian mapping function can be expressed as

$$\boldsymbol{A} = \boldsymbol{A}\left(\boldsymbol{x}, t\right). \tag{1.11}$$

In order to relate this mapping function to particle trajectories, assumptions must be made that the mapping from $\boldsymbol{A}$ to $\boldsymbol{r}$ is continuous and unique such that two adjacent particles will never be separated and neither particle will coexist at the same position at the same time. These assumptions are the basis of continuum mechanics, and they require the field to be a smooth continuum down to an arbitrarily small spatial scale. If these assumptions hold, then the relation between $\boldsymbol{A}$ and $\boldsymbol{r}$ can be expressed by [5]

$$\boldsymbol{r} = \boldsymbol{r}\left(\boldsymbol{A}, t\right) \qquad \Leftrightarrow \qquad \boldsymbol{A} = \boldsymbol{A}\left(\boldsymbol{r}, t\right). \tag{1.12}$$

The importance of Equation 1.12 will become more clear when considering the velocity transformation example of next section.

1.2.2 Relating the velocity fields

As mentioned earlier, the velocities obtained by using any observation frame must match the real velocity field. Denoting the velocity field in the Lagrangian frame by $\boldsymbol{v}(\boldsymbol{A}, t)$ and in the Eulerian frame by $\boldsymbol{u}(\boldsymbol{x}, t)$, then it must be true that

$$\boldsymbol{v}(\boldsymbol{A}, t) = \boldsymbol{u}(\boldsymbol{x}, t)|_{\boldsymbol{x}=\boldsymbol{r}}, \tag{1.13}$$

or, alternatively that

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{v}[\boldsymbol{A}(\boldsymbol{r}, t), t]|_{\boldsymbol{r}=\boldsymbol{x}}. \tag{1.14}$$

In order to make these relations more clear, we consider the following simple example. We assume that the Eulerian velocity field of a one dimensional flow is known to be

$$u(x, t) = -\alpha x + \beta t, \tag{1.15}$$

where α and β are known constants, and the equation is expressed by scalar variables since the flow is one-dimensional.

We wish to obtain the Lagrangian velocity field. Using Equation 1.13, we can write

$$v(A, t) = u(r, t) \qquad \Rightarrow \qquad v(A, t) = -\alpha r + \beta t. \tag{1.16}$$

Lagrangian coordinates $r = r(A, t)$ are dependent functions of only time, since the initial positions are fixed. Therefore, the velocity $v(A, t)$ is obtained by differentiating r with respect to time, resulting in

$$\frac{dr(A, t)}{dt} = -\alpha r(A, t) + \beta t, \tag{1.17}$$

which is an ordinary differential equation (ODE). Solving this ODE for r and differentiating it with respect to time, the Lagrangian velocity field is obtained to be

$$v\left(A,t\right)=-\alpha\left[\frac{\beta}{\alpha^{2}}+A\right]\exp\left(-\alpha t\right)+\frac{\beta}{\alpha}. \tag{1.18}$$

Now, we wish to obtain the Eulerian velocity, Equation 1.15, from Equation 1.18. In order to utilize Equation 1.14, the mapping function, $\boldsymbol{A}$, is required. Considering Equation 1.12, the particles trajectories must be obtained first. This is done by integrating Equation 1.18 with respect to time and obtaining the constants of integration by imposing the initial conditions. The result is

$$r\left(A,t\right)=\left(\frac{\beta}{\alpha^{2}}+A\right)\exp\left(-\alpha t\right)+\frac{\beta}{\alpha^{2}}\left(\alpha t-1\right). \tag{1.19}$$

The mapping function, A, is the inverse function of Equation 1.19 and it is obtained to be

$$A\left(r,t\right)=\left[-\frac{\beta}{\alpha^{2}}\left(\alpha t-1\right)+r\right]\exp\left(\alpha t\right)-\frac{\beta}{\alpha^{2}}. \tag{1.20}$$

By substituting the mapping function from Equation 1.20 into the Lagrangian velocity field of Equation 1.18 and replacing r with x, the Eulerian velocity field Equation 1.15 is obtained.

In general, for two- or three-dimensional flow these transformations are not as easy as the one-dimensional example here, as the differential equations are generally coupled. Moreover, the velocity field must be known in, at least, one of these configurations, that is, a problem must be first solved. Additionally, the initial positions are not observable for a fluid system, unlike for solids (for instance in theory of elasticity) [5].

1.2.3 Relating the time derivatives of the system properties

Regarding the time derivatives, the following notations have been adopted throughout this paper: d/dt represents the total derivative of a Lagrangian function, D/Dt denotes the material derivative of an Eulerian function, and $\partial/\partial t$ is the partial derivative with respect to time.

The Lagrangian trajectories, $\boldsymbol{r}(\boldsymbol{A},t)$, and velocities, $\boldsymbol{v}(\boldsymbol{A},t)$, are functions of initial positions and time. Therefore, their time derivatives are a simple differentiation with respect to time by holding the initial conditions fixed, that is

$$\boldsymbol{v}(\boldsymbol{A},t)=\frac{d\boldsymbol{r}(\boldsymbol{A},t)}{dt},\qquad \boldsymbol{a}_L(\boldsymbol{A},t)=\frac{d\boldsymbol{v}(\boldsymbol{A},t)}{dt}=\frac{d^2\boldsymbol{r}(\boldsymbol{A},t)}{dt^2}, \tag{1.21}$$

where $\boldsymbol{a}_L(\boldsymbol{A},t)$ are the Lagrangian accelerations.

Regarding the Eulerian description, functions are space-time dependent on the Eulerian velocity field. As these functions are essentially the system properties observed at fixed points in space, the advection of the properties must also be included. Therefore, the total derivative is considered to be the material derivative defined as

$$\frac{D(\quad)}{Dt}=\frac{\partial(\quad)}{\partial t}+\boldsymbol{u}\cdot\nabla(\quad). \tag{1.22}$$

The term material derivative is used since this derivation is aimed to match the total derivative of a Lagrangian function, that is the material-fixed frame. As an example, the Eulerian acceleration, $\boldsymbol{a}$, can be obtained as

$$\boldsymbol{a}(\boldsymbol{x},t)=\frac{D\boldsymbol{u}(\boldsymbol{x},t)}{Dt}=\frac{\partial\boldsymbol{u}(\boldsymbol{x},t)}{\partial t}+\boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla\boldsymbol{u}(\boldsymbol{x},t). \tag{1.23}$$

As mentioned before, a fluid property obtained from any observational frame must match the unique real property of the system. Therefore, for derivatives of a property

function, say Φ, we have

$$\frac{d\boldsymbol{\Phi}_L(\boldsymbol{A},t)}{dt} = \left.\frac{D\boldsymbol{\Phi}_E(\boldsymbol{x},t)}{Dt}\right|_{\boldsymbol{x}=\boldsymbol{r}}, \tag{1.24}$$

where $\boldsymbol{\Phi}_E(\boldsymbol{x},t)$ is the Eulerian representation of the Lagrangian function $\boldsymbol{\Phi}_L(\boldsymbol{A},t)$.

1.2.4 Reynolds' Transport Theorem (RTT)

Reynolds' transport theorem is an effective tool to relate an integral over a material volume (system volume, SV) to an integral over a control volume. It can be thought of as the three dimensional form of the Leibniz integral. The RTT can be applied to a scalar-valued spatial function, say $b(\boldsymbol{x},t)$. For a fixed control volume, the RTT is expressed as [20]

$$\frac{d}{dt}\int_{SV(t)} \rho b(\boldsymbol{x},t)\, dV(t) = \int_{CV}\left\{\frac{D[\rho b(\boldsymbol{x},t)]}{Dt} + \rho b(\boldsymbol{x},t)[\nabla\cdot\boldsymbol{u}]\right\} dV, \tag{1.25}$$

where ρ is the fluid density.

The RTT can be applied to a general moving and deforming control volume that contains a solid system as well as a fluid. The general form of the RTT is given by

$$\frac{d}{dt}\int_{SV(t)} \rho b(\boldsymbol{x},t)\, dV(t) = \int_{CV(t)} \frac{\partial[\rho b(\boldsymbol{x},t)]}{\partial t} dV(t) + \int_{CS(t)} \rho b(\boldsymbol{x},t)[\boldsymbol{u}\cdot\boldsymbol{n}]\, dA(t). \tag{1.26}$$

When using the RTT, the equations can often be simplified by utilizing the Gauss (divergence) theorem, that is,

$$\int_{CV} \nabla\cdot\boldsymbol{F} dV = \int_{CS} \boldsymbol{F}\cdot\boldsymbol{n} dA, \tag{1.27}$$

where $\boldsymbol{F}$ is a vector-valued spatial function. If including a scalar-valued function, say

$h(\boldsymbol{x}, t)$, it can be shown that Gauss' theorem will become

$$\int_{CV} (\boldsymbol{F} \cdot \nabla h + h \nabla \cdot \boldsymbol{F})\, dV = \int_{CS} h \boldsymbol{F} \cdot \boldsymbol{n} dA. \tag{1.28}$$

Considering our discussions in this section, we expect to have a well-defined relation between the Lagrangian variational operator and Eulerian one. However, relating the Lagrangian variation to Eulerian ones faces additional challenges which are discussed next.

1.3 Lagrangian and Eulerian Variations

Gelfand and Fomin (1963) used a family of surface transformations and utilized an interim surface to obtain the relation between the Lagrangian variational operator, δ, and the Eulerian one, $\bar{\delta}$, as [21, p.168]

$$\delta\,(\qquad) = \bar{\delta}\,(\qquad) + \delta x_i \frac{\partial\,(\qquad)}{\partial x_i}. \tag{1.29}$$

While the reader is encouraged to view the mathematically rigorous derivations of Gelfand and Fomin, we will obtain this relation from a graphical point of view so as to clarify the physics of this transformation.

We start by considering a particle i with its known Lagrangian trajectory $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{A}_i, t)$, where $\boldsymbol{A}_i$ is the initial position of the particle i. The Lagrangian path $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{A}_i, t)$ represents the position of the particle i at time t, which was at the position $\boldsymbol{A}_i$ at time $t = 0$. Therefore, $\boldsymbol{r}$ represents the time-varying position of the particle in the same frame in which $\boldsymbol{A}_i$ was originally observed. Consequently, if $\boldsymbol{A}_i$ is Eulerian, $\boldsymbol{r}$ is the Eulerian position at time t.

By applying the virtual displacement $\delta\boldsymbol{r}$, the particle will be transferred to an

imaginary path that differs from the actual path by $\delta \boldsymbol{r}$ for all time. Since $\delta \boldsymbol{r}$ are arbitrary infinitesimal vectors that are compliant with system constraints, the resulting new paths can be thought of as alternative possible trajectories.

As mentioned earlier, the Lagrangian coordinates are time dependent whereas the Eulerian ones are time independent. Moreover, while the virtual displacement, $\delta \boldsymbol{r}$, is imposed by holding time fixed ($\delta t = 0$), yet it gets carried away by the particle. This is due to the fact that it is imposed in a material frame. Therefore, one must include the advection effect while acquiring the Lagrangian-Eulerian variational relation.

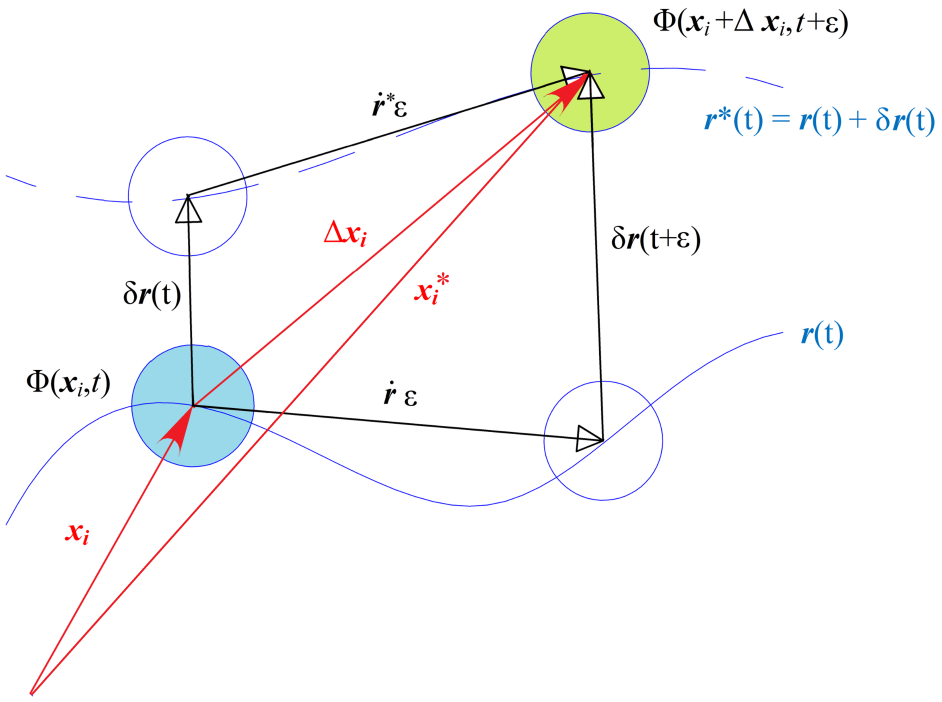

Figure 1.7: Relating the Lagrangian and the Eulerian variational operators.

Considering the particle i and its real path $\boldsymbol{r}$, let $\boldsymbol{r}^*$ be a possible alternative path that results by imposing the virtual displacement $\delta \boldsymbol{r}$, as shown in Figure 1.7. We are interested in obtaining the resulting variation of a spatial function, say $\Phi(\boldsymbol{x}, t)$, due to an imposed $\delta \boldsymbol{r}$. In order to consider advection, we assume that the particle will move on path $\boldsymbol{r}^*$ during an infinitesimal virtual time, say ε. Denoting the difference in value

of Φ between the virtual position, $\boldsymbol{r}^*(\boldsymbol{A}_i, t+\varepsilon)$, and actual position, $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{A}_i, t)$, by $\triangle\Phi$, we have

$$\triangle\Phi = \Phi(\boldsymbol{x}, t)|_{\boldsymbol{r}^*(\boldsymbol{A}_i, t+\varepsilon)} - \Phi(\boldsymbol{x}, t)|_{\boldsymbol{r}=\boldsymbol{r}(\boldsymbol{A}_i, t)}. \tag{1.30}$$

Using an Eulerian position vector, $\boldsymbol{x}$, we denote the actual position of the particle by $\boldsymbol{x}_i$, and the virtual position by $\boldsymbol{x}_i^*$. Therefore, the displacement vector, $\triangle\boldsymbol{x}_i$, is

$$\triangle\boldsymbol{x}_i = \boldsymbol{x}_i^* - \boldsymbol{x}_i. \tag{1.31}$$

Consequently, Equation 1.30 can be modified using Equation 1.31 as

$$\triangle\Phi = \Phi(\boldsymbol{x}_i + \triangle\boldsymbol{x}_i, t+\varepsilon) - \Phi(\boldsymbol{x}_i, t). \tag{1.32}$$

Applying Taylor expansion about $\boldsymbol{x}_i$ and t to the first term on the right-hand side of Equation 1.32, we obtain

$$\triangle\Phi = \Phi(\boldsymbol{x}_i, t) + \varepsilon \left[\frac{\partial\Phi(\boldsymbol{x}', t')}{\partial t'}\right]_{\substack{\boldsymbol{x}'=\boldsymbol{x}_i \\ t'=t}} + \triangle\boldsymbol{x}_i \left[\frac{\partial\Phi(\boldsymbol{x}', t')}{\partial \boldsymbol{x}'}\right]_{\substack{\boldsymbol{x}'=\boldsymbol{x}_i \\ t'=t}} - \Phi(\boldsymbol{x}_i, t) + HOT, \tag{1.33}$$

where HOT stands for higher order terms. By definition of the variational operator

$$\delta\Phi = \lim_{\varepsilon\to 0} \frac{\triangle\Phi}{\varepsilon}, \tag{1.34}$$

therefore,

$$\delta\Phi = \lim_{\varepsilon\to 0} \left\{ \frac{\varepsilon}{\varepsilon} \left[\frac{\partial\Phi(\boldsymbol{x}', t')}{\partial t'}\right]_{\substack{\boldsymbol{x}'=\boldsymbol{x}_i \\ t'=t}} + \frac{\triangle\boldsymbol{x}_i}{\varepsilon} \left[\frac{\partial\Phi(\boldsymbol{x}', t')}{\partial \boldsymbol{x}'}\right]_{\substack{\boldsymbol{x}'=\boldsymbol{x}_i \\ t'=t}} + \frac{HOT}{\varepsilon} \right\}. \tag{1.35}$$

By definition

$$\delta \boldsymbol{x}_i = \lim_{\varepsilon \to 0} \frac{\triangle \boldsymbol{x}_i}{\varepsilon}, \tag{1.36}$$

and

$$\bar{\delta}\Phi = \frac{\partial \Phi(\boldsymbol{x}, t)}{\partial t}, \tag{1.37}$$

since $\bar{\delta}\Phi$ is the local variation of function Φ for which $\delta \boldsymbol{x}_i$ act as velocity. Therefore, by taking the limit of Equation 1.35, $\delta\Phi$ is obtained as

$$\delta\Phi(\boldsymbol{x}, t) = \bar{\delta}\Phi(\boldsymbol{x}, t) + \delta \boldsymbol{x}_i \frac{\partial \Phi(\boldsymbol{x}, t)}{\partial \boldsymbol{x}_i}, \tag{1.38}$$

which is the same as Equation 1.29.

As it is apparent from Equation 1.38, the Lagrangian variation in Eulerian frame becomes mixed Lagrangian-Eulerian variations. This is because in the Eulerian description the coordinates are fixed in space, thus no variation is permitted on the displacement field. Therefore, $\delta \boldsymbol{x}_i$ is an Eulerian representation of a Lagrangian virtual displacement. This introduces major difficulties when modeling fluid systems, as is explained in the next section where some other challenges are also discussed.

1.4 Challenges Faced Using Virtual Displacement

In relating the equations for the solid to the fluid, requiring the relation of the Lagrangian descriptions to the Eulerian one has its own challenges. Initially, these difficulties manifest themselves in relating the Eulerian and the Lagrangian variational operators, as the Lagrangian concept of virtual displacement does not have an Eulerian counterpart. Some main challenges are discussed in this section.

Mentioned in our literature review (Section 1.1) was the existence of backtracking steps required in these type of problems. The reason is clear from the simple example of velocity transformation in Section 1.2.2. As evident from our analysis (Equations

1.13 - 1.20), in order to transform the Lagrangian velocities to the Eulerian one, the mapping function (Equation 1.11) must be known. The mapping function is also required for virtual displacements which are in many ways similar to velocities. However, the initial conditions are not observable in the Eulerian description, nor are they in most fluid dynamics problems. Therefore, one possibility is to guess an *ad hoc* mapping function, as explored by Leech [18].

Alternatively, some have chosen to keep the Lagrangian virtual displacements in the variational formulation and obtain the necessary condition on the control surface by which the action function assumes stationary values inside the control volume. This approach will impose additional constraints on the choice of control volume that generally cannot be made, or at least are not easily distinguished, as was encountered by McIver [12].

Another challenge encountered by keeping the virtual displacement $\delta \boldsymbol{r}$ is when the boundary conditions are expressed in the velocity format, as is generally true for fluid systems. For many problems in solids mechanics, this difficulty has been overcome by utilizing some generalized coordinates, q, as

$$\delta \boldsymbol{r} = \sum_{i=1}^{n} \frac{\partial \boldsymbol{r}}{\partial q_i} \delta q_i, \tag{1.39}$$

and then, by using the relation

$$\frac{\partial \dot{\boldsymbol{r}}}{\partial \dot{q}_i} = \frac{\partial \boldsymbol{r}}{\partial q_i}. \tag{1.40}$$

While Equation 1.40 is true for holonomic systems, requiring the velocity field to be integrable, it is not valid for nonholonomic systems [22]. Thus, it cannot be used for *ideal fluids* and it has limited applicability for viscid incompressible fluids. Leech discussed the fact that *ideal fluids* (inviscid and incompressible) are nonholonomic; while viscous incompressible fluids can be considered holonomic, the boundary conditions might not be so [18].

Moreover, the existence of nonconservative forces (that do perform virtual work) can reduce the generality of stationary principles as variational methods. It is due to the fact that generally the contribution of the nonconservative forces in the equations of motion is not a consequence of the variational operations, but rather it is by the appropriate choice of the equations representing nonconservative forces. In order to keep the generality of the variational approach, Bateman (1931) proposed that there must exist a secondary system that absorbs the energy dissipated by the original system, resulting in a set of complimentary equations [23]. This secondary set must not add any additional restriction to the system, thus, the solution of the complimentary equations must be a function of the solutions of the original system. However, he showed that the complimentary equations do not generally meet this requirement for nonlinear systems, that is, the number of variables required in Lagrange's function cannot be reduced. Concluding his paper, Bateman stated:

> "The researches of Clark Millikan showed, indeed, that there was no prospect of the discovery of a function L depending only on the quantities occurring in the equations of motion and the equation of continuity."

Although this statement leaves very little promise for using stationary principles for fluid systems, Hamilton's principle of varying action as stated by

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt - \sum_{i=1}^{N} \frac{\partial T_i}{\partial \dot{\boldsymbol{r}}} \cdot \delta \boldsymbol{r}_i \bigg|_{t_1}^{t_2} = 0, \tag{1.41}$$

is not a stationary principle, if

$$\sum_{i=1}^{N} \frac{\partial T_i}{\partial \dot{\boldsymbol{r}}} \cdot \delta \boldsymbol{r}_i \bigg|_{t_1}^{t_2} \neq 0. \tag{1.42}$$

However, it requires that the configuration be known at two instances of time. When

modelling fluid systems, the boundary conditions for particle trajectories are generally nonexistent. In the absence of these conditions, Hamilton's principle is not a variational principle, as was shown by Benaroya and Wei [1]. In this dissertation, Jourdain's variational principle is proposed as a possible basis for overcoming some of these difficulties.

As evident from our discussions thus far, there exist many difficulties in modeling FSI system which have motivated the current research work. These are summarized next.

1.5 Motivation Behind This Research

The problem of fluid-structure interaction is a crucial consideration in many applications from designing aircraft to analyzing the dynamics of blood cells. In fact, the literature on the subject comprises a substantial portion of the mechanics literature. Yet, there exists no compelling analytical theory which can explain the nature of this interaction. In solid mechanics, variational principles and methods are one of the most effective tools in modeling the dynamics of a system. However, as evident from our literature review, these methods have had very limited success in modeling fluid dynamics. Therefore, there exists a need for developing variational methods that can be used for modeling fluid systems.

Among the FSI problems, VIV is one of the most challenging and the least understood phenomena, in part due to very complex behavior of the fluid. In order to make the challenges faced more clear we further discuss the model problem defined earlier, best understood when the cylinder is fixed.

For a fixed cylinder at low Reynolds numbers ($\mathrm{Re} \leq 40$), the *boundary layer approximation method* does not have a known exact solution. At about $\mathrm{Re} = 40$, the vortices come to exist, thus the flow becomes unstable, and essentially all our

understanding of the problem is from experimental observations. It has been found experimentally that for a fixed circular cylinder, the frequency of the vortex shedding is proportional to the velocity of the free stream and corresponds to a constant Strouhal number of 0.2 for flow with Reynolds numbers of 300 to 2×10^5 [6]. While the constant Strouhal number reduces the level of difficulty in modeling the fixed cylinder substantially, no analytical solution is known for this problem.

The level of difficulty increases substantially when the structure is free to vibrate. Consequently, the majority of reduced-order models in the literature are mainly obtained by assuming a class of equation, and then by curve fitting to a set of experimental data. However, in order to set up an experiment, at least a crude governing equation is required. Moreover, these *ad hoc* models do not provide any insight into dynamics of the fluid system. Thus they will not help us to understand the nature of VIV. Therefore, our aim is to develop a method which is based on first-principles to overcome this difficulty.

As mentioned, the work of Benaroya and Wei [1] is one of the earliest attempts in modeling the VIV problem using first principles. Also, their energy equation resulted in satisfactory results for a specific control volume. The energy equation or conservation of energy law must be independent of the selection of the control volume. While their justification based on experimental considerations are indeed valid, the theoretical structural response of amplitude one-half the actual response poses some questions that if there exists other characteristics of the system which are not considered by the classical energy equation. The motivation behind the current research is to also consider the reduced-order modeling using conservation of energy.

Last but not least, there exist no analytical models that do not assume some form of fluid force function on solid surface to obtain wake-oscillator type equations. The main motivation for this research work was to overcome this difficulty, and to develop a method which does not require assumptions regarding the lift and/or drag forcing

functions.

1.6 Dissertation Outline

D'Alembert's principle, Lagrange's equations, and Hamilton's principle all utilize the concept of virtual displacements. There are many challenges and roadblocks in applying these principles directly to fluid-dominated systems.

In the following chapters we identify these difficulties and introduce to notion of using Jourdain's variational principle as a basis for such modeling as it is based on the idea of a virtual velocity. This has not been done before. Jourdain's principle is explored and an extension is obtained for systems of changing mass.

Our main objective is to utilize this variational principle in the derivation of physically meaningful reduced-order mathematical models for VIV.

In Chapter 2, as a first step, beginning with the constitutive relation for a Newtonian incompressible viscous fluid, and applying Jourdain's variational principle, we derive the Navier-Stokes equation. While important in its own right, this result verifies our extension of Jourdain's variational principle.

In Chapter 3, Jourdain's variational operator is used to derive an energy rate equation, thus extending the results of Chapter 2. The energy equation results in a correction term that does not exist in the classical energy equation. The reason for this is found to be that the fluid acceleration defined in the Eulerian reference frame is not reversible with respect to velocity.

It is also shown that the energy equation corresponds to a variational energy formulation. Moreover, a modified variational energy equation is obtained which allows the boundary conditions to be implemented explicitly or implicitly.

In Chapter 4, the energy equation is used to include the structure in the variational formulation. The result is a single governing equation which can be coupled with the

experimental data to predict the structural response. The model is compared to that proposed by McIver. Also, the single governing equation of motion is obtained for the model problem and is compared with Benaroya and Wei's model. Moreover, based on the definition of Jourdain's variational operator and Rayleigh's dissipation function, an equation is obtained for cases where the control volume is very close to the structure. The equation obtained may explain the reason why the classical energy equation predicted structural oscillations with amplitudes half of that observed in Benaroya and Wei's experiments.

In Chapter 5, we propose our methodology for obtaining wake-oscillator type equations from the variational energy formulation of Chapter 4. Two coupled non-linear equations are obtained for the model problem. Also, we show how our method can be combined with dimensional and similarity methods via an example to obtain a model similar to that of the classical Hartlen and Currie model [9].

In Chapter 6, we conclude our work and explain what can be added to this research work to help us better understand the nature of VIV.

We include an appendix that outlines the connections between the key variational principles that embody this work.

Chapter 2

An Extension of Jourdain's Principle to Systems of Changing Mass

As mentioned in the previous chapter, one of the challenges in using D'Alembert's principle, Lagrange's equations, and Hamilton's principle is that the virtual displacement used in these methods (which is a Lagrangian concept) does not have a well-defined Eulerian counterpart. In principle, this challenge can to be overcome by utilizing Jourdain's variational principle that assumes the displacement field to be frozen and imposes a set of virtual velocities, instead. However, a review of the literature suggests that this principle has not been used in modeling fluid dynamic systems or fluid-structure interaction systems. Regarding the dynamics of solid systems, Jourdain's principle has attracted a very limited number of researchers when compared with other mentioned methods. Mainly, it has been preferred in the modeling of nonholonomic systems because the velocities are not integrable.

We start this chapter with a short description of Jourdain's principle. The relation between d'Alembert's and Jourdain's principles is reviewed in Section 2.2, and some characteristics of Jourdain's principle are discussed in Section 2.3. Then, the Eulerian representation of Jourdain's principle is obtained in Section 2.4 and it is extended to systems of changing mass in Section 2.5. In Section 2.6, it is shown that the Eulerian representation of Jourdain's principle results in the Navier-Stokes equations for a general control volume of incompressible viscous fluid. The chapter concludes with a comparative discussion on the extended Jourdain's principle.

2.1 Jourdain's Principle

In 1909, Jourdain published his variational principle to explain the gap between d'Alembert's principle and Gauss's principle of least constraint (Gibbs-Appell equations) and the differences in the variational constraints imposed [24]. He considered the variational constraints to be

$$\delta \boldsymbol{r} = \delta t = 0. \tag{2.1}$$

Analogous to the other mentioned variational methods, Jourdain's principle is based on the *dynamic equilibrium* relation, and for a system of N particles is given by the relation

$$\sum_{i=1}^{N} (m_i \ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i) \cdot \delta \dot{\boldsymbol{r}}_i \ = 0, \qquad \text{where} \qquad \delta \boldsymbol{r} = 0,\ \delta t = 0, \tag{2.2}$$

and where m_i is the mass of particle i, vector $\boldsymbol{F}_i$ is the force acting on particle i, and $\delta \dot{\boldsymbol{r}}_i$ is the variation of the velocity of the particle i (called virtual velocity).

Jourdain showed that his method demands less derivations for nonholonomic systems when compared with d'Alembert's and Gauss's principles. (Jourdain concluded that his constraints lead to a new method because the differential quotient appears both before and after the variational operator (δ).) However, he did not explain the physical meaning of his virtual velocities, which do not correspond to any virtual displacements.

Since the terms in Equation 2.2 are power relations, the equation has also been referred to as the *principle of virtual power* (PVP). Repeatedly, Kane's equations are also referred to as PVP. To avoid any confusion, we will use the term Jourdain's principle (JP).

2.2 Deriving the Jourdain's Principle from d'Alembert's Principle

Jourdain's principle can be derived from d'Alembert's principle by direct differentiation [25] or by the means of a Taylor series expansion [26]. We choose direct differentiation as this approach provides the format required when applying the RTT.

D'Alembert's principle is stated as

$$\sum_{i=1}^{N} \left(m_i \ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i\right) \cdot \delta \boldsymbol{r}_i = 0, \qquad \text{where} \qquad \delta t = 0. \tag{2.3}$$

Differentiation of Equation 2.3 with respect to time yields

$$\sum_{i=1}^{N} \left\{ \frac{d}{dt}\left(m_i \ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i\right) \cdot \delta \boldsymbol{r}_i + \left(m_i \ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i\right) \cdot \frac{d}{dt}\left(\delta \boldsymbol{r}_i\right) \right\} = 0, \qquad \text{where} \qquad \delta t = 0. \tag{2.4}$$

Using the *commutation rule,*

$$\frac{d}{dt}\left(\delta \boldsymbol{r}\right) - \delta\left(\frac{d}{dt}\boldsymbol{r}\right) = 0, \tag{2.5}$$

we have

$$\sum_{i=1}^{N} \left\{ \frac{d}{dt}\left(m_i \ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i\right) \cdot \delta \boldsymbol{r}_i + \left(m_i \ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i\right) \cdot \delta \dot{\boldsymbol{r}}_i \right\} = 0, \qquad \text{where} \qquad \delta t = 0. \tag{2.6}$$

Now, we can impose Jourdain's constraints,

$$\delta t = 0, \qquad \delta \boldsymbol{r} = 0, \qquad \text{and} \qquad d\left(\delta \boldsymbol{r}\right) \neq 0, \tag{2.7}$$

and obtain Jourdain's principle,

$$\begin{cases} \sum_{i=1}^{N} (m_i \ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i) \cdot \delta \dot{\boldsymbol{r}}_i = 0 \\ \delta t = 0 \\ \delta \boldsymbol{r} = 0 \\ d(\delta \boldsymbol{r}) \neq 0, \end{cases} \tag{2.8}$$

where $d(\delta \boldsymbol{r})$ is the differential of the virtual displacement $\delta \boldsymbol{r}$. Jourdain's constraints, expressed by Equation 2.7, assume that a system can have an alternatively possible velocity field at a time instance with the same corresponding displacement field.

In order to distinguish between Jourdain's and d'Alembert's variational operators, many authors have used the sign δ_1 to denote Jourdain's variation. We prefer to retain the same notation (δ) with Jourdain's constraints in the remainder of this work where it will be understood from the context.

2.3 Characteristics of Jourdain's Principle

As emphasized by Jourdain, his constraints lead to a different variation principle. Basically, JP does not share the similarities with the other mentioned virtual principles. These differences are noted below.

Kövecses and Clenghorn [27] investigated Jourdain's principle. They proposed that the position vector, $\boldsymbol{r}$, can be represented by using its trajectories in both Lagrangian and Eulerian frames. Based on their hybrid parameterization, they show that unlike the virtual displacements and velocities utilized in Lagrange's equation, d'Alembert principle, and Hamilton's principle, the virtual velocities in JP are not necessarily infinitesimal quantities. Also, they pointed out that JP always results in an alternative possible state, whereas this can only be accomplished by utilizing the above mentioned methods for holonomic systems.

While the majority of the limited literature on JP is focused on its application to nonholonomic systems, an important feature of this principle was revealed by Papastavridis [25]. While examining the principle and comparing it with Lagrange's equations, he showed that Jourdain's principle is independent of the commutation rule (2.5). He discussed that if the commutation rule is not valid, then there exists a vector $\delta^* \boldsymbol{r}$ such that

$$\delta^* \boldsymbol{r} = \frac{d}{dt}(\delta \boldsymbol{r}) - \delta\left(\frac{d}{dt}\boldsymbol{r}\right). \tag{2.9}$$

By setting $\delta^* \boldsymbol{r} \neq 0$, Papastavridis proved that Jourdain's principle results in the correct EOM. While the reader is urged to consider the discussion by Papastavridis, we proceed with an interpretation suited for our purposes that results in a simplified proof of this feature.

Let us consider the relation between Lagrangian variations (δ) and Eulerian variations ($\bar{\delta}$) as defined by Equation 1.38, and impose the Jourdain constraints ($\delta \boldsymbol{x}_i = 0$). The result is

$$\delta\Phi(\boldsymbol{x}, t) = \bar{\delta}\Phi(\boldsymbol{x}, t). \tag{2.10}$$

This is a very important property, as it states that Jourdain's variation of a spatial function is the same in both Lagrangian and Eulerian descriptions. Using this property, some of the problems that have been encountered in extending variational principles for fluid systems can be avoided by using JP.

In order to prove that JP is independent of the commutation rule, we impose Jourdain's variational operator to the total derivative of a function. Using Equation 2.10, we can write

$$\delta\left[\frac{D}{Dt}\Phi(\boldsymbol{x}, t)\right] = \bar{\delta}\left[\frac{D}{Dt}\Phi(\boldsymbol{x}, t)\right]. \tag{2.11}$$

It has been shown that the Lagrangian variation commutes with time differentiation

while the Eulerian one does not [19]; that is,

$$\delta\left[\frac{D}{Dt}\Phi(\boldsymbol{x},t)\right] = \frac{D}{Dt}\left[\delta\Phi(\boldsymbol{x},t)\right] \tag{2.12}$$

$$\bar{\delta}\left[\frac{D}{Dt}\Phi(\boldsymbol{x},t)\right] \neq \frac{D}{Dt}\left[\bar{\delta}\Phi(\boldsymbol{x},t)\right]. \tag{2.13}$$

Using Equations 2.10-2.13, we have

$$\begin{aligned}\bar{\delta}\left[\frac{D}{Dt}\Phi(\boldsymbol{x},t)\right] &= \delta\left[\frac{D}{Dt}\Phi(\boldsymbol{x},t)\right] \\ &= \frac{D}{Dt}\left[\delta\Phi(\boldsymbol{x},t)\right] \\ &= \frac{D}{Dt}\left[\bar{\delta}\Phi(\boldsymbol{x},t)\right].\end{aligned} \tag{2.14}$$

Comparing Equations 2.13 and 2.14, the independence of JP with regard to the commutation rule is proven. In other words, we used Equation 2.12 that implies $\delta\boldsymbol{r}^* = 0$, then by imposing the Jourdain constraints, we obtained $\delta\boldsymbol{r}^* \neq 0$.

2.4 Eulerian-Lagrangian Description of Jourdain's Principle

In Section 2.2, it was shown that JP can be obtained by differentiating d'Alembert's principle, and then imposing Jourdain's constraints. Similarly, in the following, we start by manipulating d'Alembert's principle, and then, apply Jourdain's constraints. Therefore, Jourdain's constraints are kept beside d'Alembert's principle as shown in Equation 2.16, where the curly bracket on the left is to remind us that the same mathematical manipulations must be applied to all those terms. As shown in Section 2.2, all the terms on the right of the bracket together are equivalent to Jourdain's principle.

D'Alembert's principle for a system of N particles can be stated as

$$\sum_{i=1}^{N} \frac{d}{dt} \left(m_i \dot{\boldsymbol{r}}_i \right) \cdot \delta \boldsymbol{r}_i = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i, \tag{2.15}$$

and by differentiating this with respect to time and imposing Jourdain's constraints, JP can be written as

$$\begin{cases} \dfrac{d}{dt} \left[\displaystyle\sum_{i=1}^{N} \frac{d}{dt} \left(m_i \dot{\boldsymbol{r}}_i \right) \cdot \delta \boldsymbol{r}_i \right] = \dfrac{d}{dt} \left[\displaystyle\sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i \right] \\ \delta t = 0 \\ \delta \boldsymbol{r}_i = 0 \\ \dfrac{d}{dt} \left(\delta \boldsymbol{r}_i \right) \neq 0. \end{cases} \tag{2.16}$$

In the above, both sides of the equation are differentiated using the product rule and use is made of the $\delta \boldsymbol{r}_i = 0$ constraint.

As mentioned in Section 1.2, the velocity and resultant force observed at a point in the Eulerian space must be the same as the velocity and forces obtained from the Lagrangian frame for a particle that occupies that Eulerian point, that is,

$$\boldsymbol{v} \left(\boldsymbol{A}_i, t \right) = \boldsymbol{u} \left(\boldsymbol{r}_i, t \right) \tag{2.17}$$

$$\boldsymbol{F}(\boldsymbol{A}_i, t) = \boldsymbol{F}_E(\boldsymbol{r}_i, t), \tag{2.18}$$

where $\boldsymbol{v}$ $(= \dot{\boldsymbol{r}})$ is the Lagrangian velocity and $\boldsymbol{u}$ is the Eulerian velocity (the same as before), $\boldsymbol{F}_E$ is the Eulerian representation of the force $\boldsymbol{F}(\boldsymbol{A}_i, t)$. Note that $\boldsymbol{r}_i$ and $\boldsymbol{r}\left(\boldsymbol{A}_i, t\right)$ are two alternative ways to denote the same parameter.

In continuum mechanics, a set of particles is assumed to be continuous in such a way that two particles do not occupy the same position, and there exist no gaps unless in isolated points. By the assumption that there exists a unique function that maps the Lagrangian reference frame to the Eulerian one, the continuum assumptions are applied to the Eulerian frame. Consequently, the change in the system properties,

as viewed from a fixed position in space, is assumed to be smooth, continuous and differentiable, meaning that two consecutive particles occupying a point in space, say $\boldsymbol{x}$, are allowed to possess infinitesimally different properties.

Also, as discussed in Section 1.2.1, for each particle trajectory there exists an inverse mapping function to the initial position of that particle (Equation 1.12). While the initial position is a fixed variable in the Lagrangian frame, it can be selected to be any position on the path, differing only by the reference times. Let us call any of these points a possible initial position. Alternatively, by mapping these different time references into a specific one, there must exist an instantaneous spatial function whose outcome is the path-history of the particle occupying that space. As a result, these possible initial positions can be considered to be a spatial function. Therefore, in the realm of continuum mechanics, we assume that there must exist an Eulerian smooth, continuous, differentiable function $\boldsymbol{\Lambda}(\boldsymbol{x}, t)$ where

$$\boldsymbol{r} = \boldsymbol{\Lambda}(\boldsymbol{r}, t), \tag{2.19}$$

and

$$\frac{d}{dt}\boldsymbol{r} = \frac{d}{dt}\boldsymbol{\Lambda}(\boldsymbol{r}, t) = \frac{D}{Dt}\boldsymbol{\Lambda}(\boldsymbol{x}, t)\bigg|_{\boldsymbol{x}=\boldsymbol{r}}. \tag{2.20}$$

Note that Equation 2.19 becomes Equation 1.11 for a fixed initial position, thus it can then no longer be differentiated. Also, from Equations 2.17 and 2.20 we have

$$\boldsymbol{u}(\boldsymbol{x}, t) = \frac{D}{Dt}\boldsymbol{\Lambda}(\boldsymbol{x}, t), \tag{2.21}$$

since $\boldsymbol{v}(\boldsymbol{A}, t) = d\boldsymbol{r}/dt$.

By substituting Equations 2.17, 2.18, 2.19 and 2.20 into Equation 2.16, Jourdain's

principle can be stated as

$$\begin{cases} \dfrac{d}{dt}\left\{\displaystyle\sum_{i=1}^{N}\frac{d}{dt}\left[m_i\boldsymbol{u}\left(\boldsymbol{r}_i,t\right)\right]\cdot\delta\boldsymbol{\Lambda}\left(\boldsymbol{r}_i,t\right)\right\} = \dfrac{d}{dt}\left\{\displaystyle\sum_{i=1}^{N}\boldsymbol{F}_E(\boldsymbol{r}_i,t)\cdot\delta\boldsymbol{\Lambda}\left(\boldsymbol{r}_i,t\right)\right\} \\ \delta t = 0 \\ \delta\boldsymbol{\Lambda}\left(\boldsymbol{r}_i,t\right) = 0 \\ \dfrac{d}{dt}\left(\delta\boldsymbol{\Lambda}\left(\boldsymbol{r}_i,t\right)\right) \neq 0. \end{cases} \tag{2.22}$$

Note that $\boldsymbol{u}$, $\boldsymbol{F}_E$, and $\boldsymbol{\Lambda}$ are all Eulerian functions, however, Equation 2.22 is a Lagrangian equation due to the presence of $\boldsymbol{r}_i$ (since we are following the particles). Also, the variational operator δ is Lagrangian.

If the set of particles remains continuous for all time, the summation can be replaced with integration over the material domain. Therefore, Equation 2.22 becomes

$$\begin{cases} \dfrac{d}{dt}\left\{\displaystyle\int_{V_m}\frac{d}{dt}\left[\rho\boldsymbol{u}\left(\boldsymbol{r},t\right)\right]\cdot\delta\boldsymbol{\Lambda}\left(\boldsymbol{r},t\right)\right\}dV_m = \dfrac{d}{dt}\left\{\displaystyle\int_{V_m}\boldsymbol{f}\left(\boldsymbol{r},t\right)\cdot\delta\boldsymbol{\Lambda}\left(\boldsymbol{r},t\right)\right\}dV_m \\ \delta t = 0 \\ \delta\boldsymbol{\Lambda}\left(\boldsymbol{r},t\right) = 0 \\ \dfrac{d}{dt}\left(\delta\boldsymbol{\Lambda}\left(\boldsymbol{r},t\right)\right) \neq 0, \end{cases} \tag{2.23}$$

where V_m is the material volume, ρ is the density, and $\boldsymbol{f}$ is the force density. Since all the functions are Eulerian, only the bounds of integrations are required to evaluate those integrals. If the material volume is known in the Eulerian description for all times, then the effect of the Lagrangian paths $\boldsymbol{r}$ inside the domain becomes irrelevant to the integration and it can be replaced by the Eulerian coordinate $\boldsymbol{x}$.

Based on the work of Xing and Price [19] on the effect of virtual displacement $\delta\boldsymbol{x}$ on an integration over a control volume, we saw that the virtual displacements are present on the surfaces of the control volume and not inside the domain in Equation 1.6. Considering Jourdain's constraints, the virtual velocities have no effect on the control surfaces, confirming what was stated in the previous paragraph.

By mapping the system from the Lagrangian frame of reference to the Eulerian representation, the material volume, V_m, will be mapped to an Eulerian system (material) volume, $V_E(t)$, so that

$$dV_m(\boldsymbol{r},t) = \det\left[\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}}\right] dV_E(\boldsymbol{x},t), \tag{2.24}$$

where we used the determinant of Jacobian for the mapping.

Thus far, the derivations have been kept general, that is, we did not specify the compressibility of the material. We next limit our derivation to incompressible flows, where the incompressibility condition implies that

$$\det\left[\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}}\right] = 1, \tag{2.25}$$

for all times. Therefore, by substituting Equations 2.24 and 2.25 into Equation 2.23, utilizing Equation 1.24, and then, replacing $\boldsymbol{r}$ for the reason explained after Equation 2.23, Jourdain's principle for an incompressible set of continuous particles becomes

$$\begin{cases} \dfrac{d}{dt}\left\{\displaystyle\int_{V_E(t)} \frac{D}{Dt}\left[\rho \boldsymbol{u}(\boldsymbol{x},t)\right] \cdot \delta\boldsymbol{\Lambda}(\boldsymbol{x},t)\, dV_E(t)\right\} \\ \qquad = \dfrac{d}{dt}\left\{\displaystyle\int_{V_E(t)} \boldsymbol{f}(\boldsymbol{x},t) \cdot \delta\boldsymbol{\Lambda}(\boldsymbol{x},t)\, dV_E(t)\right\} \\ \delta t = 0 \\ \delta\boldsymbol{\Lambda}(\boldsymbol{x},t) = 0 \\ \dfrac{D}{Dt}\left(\delta\boldsymbol{\Lambda}(\boldsymbol{x},t)\right) \neq 0, \end{cases} \tag{2.26}$$

where by Equation 2.21 $\boldsymbol{u}(\boldsymbol{x},t) = \frac{D}{Dt}\boldsymbol{\Lambda}(\boldsymbol{x},t)$; and where d/dt is used to emphasize that it is a differentiation of a material volume, and D/Dt is used inside the integrand since the associated function is Eulerian.

Equation 2.26 is still mixed Eulerian-Lagrangian. The variations δ are Lagrangian, and the same set of particles are being followed. In the following section we impose

Jourdain's constraints and convert Equation 2.26 into an equation for a system of changing mass with the Eulerian variational operator.

2.5 Extended JP for General Control Volume

Jourdain's principle for a system of particles was obtained in Equation 2.26, which must be considered together with Equation 2.21. In relating the integrals over the system volume to those over a control volume, an effective tool is Reynolds' transport theorem (Section 1.2.4). In the following, each side of Equation 2.26 is considered separately for simplicity, and is manipulated so as to become applicable to the cases where general control volumes are considered. The resulting equation is completely Eulerian.

2.5.1 Left-hand side of Equation 2.26

We start by considering the left-hand side of Equation 2.26, and apply the RTT for a general control volume as per Equation 1.26,

$$\begin{aligned}\frac{d}{dt}\left\{\int_{V_E(t)}\frac{D}{Dt}\left[\rho\boldsymbol{u}\left(\boldsymbol{x},t\right)\right]\cdot\delta\boldsymbol{\Lambda}\left(\boldsymbol{x},t\right)dV_E\left(t\right)\right\}&\\ =\int_{CV(t)}\frac{\partial}{\partial t}\left\{\frac{D}{Dt}\left[\rho\boldsymbol{u}\left(\boldsymbol{x},t\right)\right]\cdot\delta\boldsymbol{\Lambda}\left(\boldsymbol{x},t\right)\right\}dV\left(t\right)&\\ +\int_{CS(t)}\left\{\frac{D}{Dt}\left[\rho\boldsymbol{u}\left(\boldsymbol{x},t\right)\right]\cdot\delta\boldsymbol{\Lambda}\left(\boldsymbol{x},t\right)\right\}\left[\boldsymbol{u}\left(\boldsymbol{x},t\right)\cdot\boldsymbol{n}\right]dA\left(t\right).&\quad(2.27)\end{aligned}$$

Now by applying the Gauss (divergence) theorem (Equation 1.28) to the right-hand side of Equation 2.27, where we can the combine the two terms on the right-hand

side, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_{V_E(t)} \frac{D\left(\rho \boldsymbol{u}\right)}{Dt}\cdot \delta\boldsymbol{\Lambda} dV_E\left(t\right)\right\}\\
&=\int_{CV(t)}\left[\frac{\partial}{\partial t}\left(\rho\frac{D\boldsymbol{u}}{Dt}\cdot\delta\boldsymbol{\Lambda}\right)+\boldsymbol{u}\cdot\nabla\left(\rho\frac{D\boldsymbol{u}}{Dt}\cdot\delta\boldsymbol{\Lambda}\right)+\left(\rho\frac{D\boldsymbol{u}}{Dt}\cdot\delta\boldsymbol{\Lambda}\right)\left(\nabla\cdot\boldsymbol{u}\right)\right]dV\left(t\right)\\
&\qquad=\int_{CV(t)}\left[\frac{D}{Dt}\left(\rho\frac{D\boldsymbol{u}}{Dt}\cdot\delta\boldsymbol{\Lambda}\right)+\left(\rho\frac{D\boldsymbol{u}}{Dt}\cdot\delta\boldsymbol{\Lambda}\right)\left(\nabla\cdot\boldsymbol{u}\right)\right]dV\left(t\right), \quad (2.28)
\end{aligned}$$

where the argument $(\boldsymbol{x}, t)$ is omitted since we recognize that all the functions are expressed in the Eulerian frame. Moreover, the density ρ has been pulled out of the differentiation since for incompressible flow the density is constant.

By imposing the incompressibility constraint, $\nabla\cdot\boldsymbol{u}=0$, and then differentiating the remaining terms, Equation 2.28 becomes

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_{V_E(t)} \frac{D\left(\rho \boldsymbol{u}\right)}{Dt}\cdot \delta\boldsymbol{\Lambda} dV_E\left(t\right)\right\}\\
&\qquad=\int_{CV(t)}\frac{D}{Dt}\left(\rho\frac{D\boldsymbol{u}}{Dt}\cdot\delta\boldsymbol{\Lambda}\right)dV\left(t\right)\\
&\qquad=\int_{CV(t)}\left[\rho\frac{D^2\boldsymbol{u}}{Dt^2}\cdot\delta\,\boldsymbol{\Lambda}+\rho\frac{D\boldsymbol{u}}{Dt}\cdot\frac{D}{Dt}\left(\delta\boldsymbol{\Lambda}\right)\right]dV\left(t\right). \quad (2.29)
\end{aligned}$$

In deriving Equation 2.29, we have not yet imposed Equation 2.7, Jourdain's constraints. Since δ is the Lagrangian variational operator, the commutation rule still holds,

$$\frac{D\left[\delta\left(\boldsymbol{\Lambda}\right)\right]}{Dt}=\delta\left[\frac{D\left(\boldsymbol{\Lambda}\right)}{Dt}\right]. \quad (2.30)$$

Substituting Equation 2.21 in Equation 2.30, we have

$$\frac{D\left[\delta\left(\boldsymbol{\Lambda}\right)\right]}{Dt}=\delta\boldsymbol{u}. \quad (2.31)$$

Finally, by applying Jourdain's constraints as expressed in Equation 2.26, in particular

$\delta\mathbf{\Lambda} = 0$, and using Equation 2.31, the left-hand side of Equation 2.26 becomes

$$\frac{d}{dt}\left\{\int_{V_E(t)} \frac{D\left(\rho\boldsymbol{u}\right)}{Dt}\cdot\delta\mathbf{\Lambda}dV_E\left(t\right)\right\} = \int_{CV(t)} \rho\frac{D\boldsymbol{u}}{Dt}\cdot\delta\boldsymbol{u}dV\left(t\right), \tag{2.32}$$

where we realize that $\delta\boldsymbol{u}$ is a Eulerian variation of the Eulerian velocity. This is because the Jourdain's variation of a spatial function is the same in both Lagrangian and Eulerian descriptions, as was expressed by Equation 2.10. We will not use the notation $\bar{\delta}$ introduced earlier in the remainder of this dissertation by understanding that the Lagrangian variational operator becomes the Eulerian operator after imposing Jourdain's constraints, $\delta\mathbf{\Lambda} = 0$; because the majority of the equations will be expressed in the Eulerian reference frame.

2.5.2 Right-hand side of Equation 2.26

For the right-hand side of Equation 2.26, the steps are similar to those above, resulting in

$$\begin{aligned}\frac{d}{dt}&\left\{\int_{V_E(t)} \boldsymbol{f}\cdot\delta\mathbf{\Lambda}dV_E\left(t\right)\right\}\\ &= \int_{CV(t)}\left[\frac{\partial}{\partial t}\left(\boldsymbol{f}\cdot\delta\mathbf{\Lambda}\right) + \boldsymbol{u}\cdot\nabla\left(\boldsymbol{f}\cdot\delta\mathbf{\Lambda}\right) + \left(\boldsymbol{f}\cdot\delta\mathbf{\Lambda}\right)\left(\nabla\cdot\boldsymbol{u}\right)\right]dV\left(t\right)\\ &= \int_{CV(t)}\frac{D}{Dt}\left(\boldsymbol{f}\cdot\delta\mathbf{\Lambda}\right)dV\left(t\right) = \int_{CV(t)}\boldsymbol{f}\cdot\delta\boldsymbol{u}dV\left(t\right).\end{aligned} \tag{2.33}$$

2.5.3 New version of Equation 2.26

By substituting Equations 2.32 and 2.33 into Equation 2.26, and by considering Equation 2.10, the extended Jourdain's principle for a system of changing mass in a control volume becomes

$$\int_{CV(t)}\rho\frac{D\boldsymbol{u}\left(\boldsymbol{x},t\right)}{Dt}\cdot\delta\boldsymbol{u}\left(\boldsymbol{x},t\right)dV\left(t\right) = \int_{CV(t)}\boldsymbol{f}\left(\boldsymbol{x},t\right)\cdot\delta\boldsymbol{u}\left(\boldsymbol{x},t\right)dV\left(t\right), \tag{2.34}$$

where all the functions and variations are represented in the Eulerian frame.

We have derived a variational formulation that is expressed purely in the Eulerian frame. To verify our mathematical manipulations, we derive the Navier-Stokes equations using Equation 2.34 in the following section.

2.6 Extended Jourdain's Principle for Viscous Incompressible Fluids

In fluid mechanics, forces acting on a fluid particle are generally divided into three categories: *line forces*, *body forces*, and *surface forces* [28]. *Line forces*, also called *surface tension*, are the forces observed at an interface layer between gas and fluid or two immiscible liquids due to intermolecular attractive forces. These forces do not appear directly in the EOM and they are considered as boundary conditions. Thus, they are not considered in this work.

Body forces are due to an action at a distance, for instance, gravitational, magnetic, and electromagnetic forces. Considering the purpose of the current study, only gravitational forces might be required. *Body forces*, $\boldsymbol{F}_b$, can be denoted in terms of the body force per unit volume (body force density), $\boldsymbol{f}_b$, as

$$\boldsymbol{F}_b = \int_{CV} \boldsymbol{f}_b dV. \tag{2.35}$$

Surface forces, as their name implies, are defined on a surface. Considering the definition of the stress tensor, $\bar{\boldsymbol{\sigma}}$, surface force per unit surface area, $\boldsymbol{f}_s$, can be obtained as

$$\boldsymbol{f}_s = \boldsymbol{n}^T \cdot \bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}} \cdot \boldsymbol{n}, \tag{2.36}$$

where ^{T} denotes transpose and $\boldsymbol{n}$ is the normal vector to the surface of interest. Therefore, the total surface force, $\boldsymbol{F}_s$, for a control volume can be obtained by using

Gauss' theorem (Equation 1.27) as

$$\boldsymbol{F}_s = \int_{CS} \boldsymbol{f}_s dA = \int_{CS} \bar{\boldsymbol{\sigma}} \cdot \boldsymbol{n} dA = \int_{CV} \nabla \cdot \bar{\boldsymbol{\sigma}} \; dV, \tag{2.37}$$

where $\boldsymbol{n}$ is the normal vector to the surface of the control volume.

The total active force in a control volume is obtained by integrating the force density over that control volume,

$$\begin{aligned} \int_{CV} \boldsymbol{f}\left(\boldsymbol{x}, t\right) dV &= \int_{CV} \boldsymbol{f}_b dV + \int_{CS} \boldsymbol{f}_s dA \\ &= \int_{CV} \left(\boldsymbol{f}_b + \nabla \cdot \bar{\boldsymbol{\sigma}}\right) dV. \end{aligned} \tag{2.38}$$

Therefore, the extended Jourdain's principle can be modified by substituting Equation 2.38 into Equation 2.34, resulting in the equation

$$\int_{CV(t)} \left(\rho \frac{D\boldsymbol{u}\left(\boldsymbol{x}, t\right)}{Dt} - \boldsymbol{f}_b - \nabla \cdot \bar{\boldsymbol{\sigma}} \right) \cdot \delta \boldsymbol{u}\left(\boldsymbol{x}, t\right) dV\left(t\right) = 0. \tag{2.39}$$

In order to expand Equation 2.39 further, we consider the constitutive relation for Newtonian incompressible fluids, that is,

$$\bar{\boldsymbol{\sigma}} = -p\bar{\boldsymbol{I}} + 2\mu\bar{\boldsymbol{S}}, \tag{2.40}$$

where p is the thermodynamic pressure, $\bar{\boldsymbol{I}}$ is the identity tensor, μ is the coefficient of dynamic viscosity, and $\bar{\boldsymbol{S}}$ is a symmetric tensor defined as

$$\bar{\boldsymbol{S}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2.41}$$

Therefore, Equation 2.39 becomes

$$\int_{CV(t)} \left(\rho \frac{D\boldsymbol{u}\left(\boldsymbol{x},t\right)}{Dt} - \boldsymbol{f}_{b}\left(\boldsymbol{x},t\right) + \nabla p\left(\boldsymbol{x},t\right) - \mu \nabla^{2}\boldsymbol{u}\left(\boldsymbol{x},t\right) \right) \cdot \delta\boldsymbol{u}\left(\boldsymbol{x},t\right) dV\left(t\right) = 0. \tag{2.42}$$

Since $\bar{\delta}\boldsymbol{u}$ is a nonzero vector, the terms of the integrand inside the parentheses must add to zero. These are the Navier-Stokes equations. Therefore, the governing EOM of an incompressible viscous flow can be obtained from our derived variational formulation.

2.7 Discussion

Our review of the literature states that a purely Eulerian variational method does not exist. Also, the Navier-Stokes equations have not been obtained using a variational method. We have done this here.

Considering the derivations presented here, the main reason for the difficulties encountered in the literature can be traced to the Lagrangian-Eulerian relations. An essential component in relating a Lagrangian function to a Eulerian one is a mapping function of Lagrangian trajectories. In order to show this dependency, using Equations 2.29 and 2.33, we can write

$$\begin{aligned} \int_{CV(t)} \left[\rho \frac{D^{2}\boldsymbol{u}}{Dt^{2}} \cdot \delta\,\boldsymbol{\Lambda} + \rho \frac{D\boldsymbol{u}}{Dt} \cdot \frac{D}{Dt}\left(\delta\boldsymbol{\Lambda}\right) \right] dV\left(t\right) \\ = \int_{CV(t)} \left[\frac{D}{Dt}\boldsymbol{f} \cdot \delta\boldsymbol{\Lambda} + \boldsymbol{f} \cdot \frac{D}{Dt}\left(\delta\boldsymbol{\Lambda}\right) \right] dV\left(t\right), \end{aligned} \tag{2.43}$$

that is, our extension of d'Alembert principle before imposing the Jourdain constraints. As it is evident from this equation, the mapping function $\boldsymbol{\Lambda}$ is required. In general, it is not possible to obtain function $\boldsymbol{\Lambda}$ from the information available for a control volume, because Lagrangian trajectories are defined by initial conditions that

are not observable in a Eulerian frame. However, they are required in d'Alembert's principle and consequently in Lagrange's equations and Hamilton's principle. As we showed, this problem can be avoided by using JP due to Jourdain's constraints.

We have shown that Jourdain's principle can be modified to obtain a purely Eulerian variational method. Also, it was shown that the Navier-Stokes equations can be obtained from the extended JP presented here. In our derivations, we did not make any assumptions, but used Jourdain's principle and continuum mechanics.

The absence of virtual displacements in our variational formulation might lead to a more advantageous approach for modeling viscous fluids. A variational function obtained by imposing virtual displacements is stationary inside an assumed control volume only if the first variation vanishes without any restriction on the second variation. The works of Millikan [13] and Bateman [23], discussed earlier, confirm that a Lagrangian function cannot be found for which the mentioned condition holds. Therefore, the existence of extrema must be investigated. If the required conditions to vanish the first variation result in restrictions on the second variation, the function can have an extremum. In general, it is hard to find an extremum inside the control volume. Therefore, investigating the control surface is preferable. However, the virtual displacements are not reversible on the control surface. Thus the conditions imposed to vanish the first variation no longer hold inside the control volume [29, p. 42-43], forcing one to try to obtain a function that is stationary, again. Since Jourdain's principle does not use the virtual displacement, it does not have the restriction on the boundary surfaces, as is evident from our derivations.

One of the main objectives of the current research work is to utilize a variational principle in reduced-order modeling of VIV problems. In reduced-order modeling, one tries to reduce the number of degrees of freedom in such a way that the few resulting EOM can capture the main characteristics of the nonlinear problem. In a sense, reduced-order modeling can be thought of as averaging over the material

domain over some time span(s).

An interesting yet challenging problem that might arise is the *Stokes drift* type phenomena. Stokes (1847) considered the dynamics of oscillatory waves and showed that the time average of the Eulerian velocities lag the average of the Lagrangian velocities [30]. The term *Stokes drift* was later adopted for these differences in velocities. It is interesting that the information required to obtain Stokes drift is readily available from the Eulerian velocities [5], that is, such challenges can be avoided by solving the problem in either one of the descriptions. Therefore, since the variational approach proposed here is purely Eulerian, unlike the methods available in the literature, it may have some additional advantages regarding *Stokes drift* type phenomena. However, it is too early to make such a conclusion and further investigation is required.

Since the ultimate goal of this research work is reduced-order modeling of the VIV problem, Equation 2.42 in its current format is not beneficial to our goal. In the next chapter such EOMs are derived in an energy framework. This takes us one step closer to a general equation from which we can extract reduced order coupled differential equations governing the fluid and the structure.

Chapter 3

Energy Equation from the Extended Jourdain's Principle

Reduced-order modeling of VIV is the focus of the current work. In this chapter, we look to derive the energy equation for a fluid via Jourdain's principle. Equation 2.42 obtained in the previous chapter provides us with the well-known Navier-Stokes equations, which together with the conservation of mass are four coupled ordinary differential equations. The extended Jourdain's principle is modified further in this chapter by considering the definition of Jourdain's variational operator in order to reduce the number of required equations to one by obtaining a single EOM in an energy format. Here, we consider the variational formulation of a control volume containing only fluid particles. Fluid-structure interaction problems are considered in the next chapter, where a rigid structure is introduced within the fluid flow.

We start the chapter by reviewing the Reynolds' transport theorem beyond our discussion in Section 1.2.4. Then, a few relevant ideas on Jourdain's variational operator are considered. In Sections 3.3 and 3.4, we explain our methodology for finding the energy rate equations directly from Jourdain's principle for conservative systems. Section 3.5 focus on the concept of the Rayleigh dissipation function. In Section 3.6, we derive an energy equation from the extended Jourdain's principle. The chapter is concluded with some final remarks on energy equations.

3.1 Reynolds' Transport Theorem for a Control Volume

In Section 1.2.4, we discussed that the RTT can be utilized to relate the change of a real-valued spatial function of a control system (Lagrangian concept) to that of a control volume (Eulerian concept) occupying the same volume at that instant of time by using Equations 1.25 and 1.26. Recall that the system volume is a volume containing the same particles for all time, while the control volume is an imaginary volume of space and not material.

Similarly, the RTT can be used for evaluating the rate of change of a real-valued spatial function, say $b(\boldsymbol{x}, t)$, of a control volume (not a control system) as [20]

$$\frac{D}{Dt}\int_{CV} b(\boldsymbol{x}, t)\, dV = \int_{CV} \frac{\partial}{\partial t}\left[b(\boldsymbol{x}, t)\right] dV + \int_{CS} \left[b(\boldsymbol{x}, t)\right](\boldsymbol{v}_{CS} \cdot \boldsymbol{n})\, dA, \tag{3.1}$$

where $\boldsymbol{v}_{CS}$ is the velocity of the control surface element, dA.

The RTT can also be expressed in terms of the relative velocity, $\boldsymbol{u}_r$, defined as

$$\boldsymbol{u}_r = \boldsymbol{u} - \boldsymbol{v}_{CS} \qquad \Rightarrow \qquad \boldsymbol{v}_{CS} = \boldsymbol{u} - \boldsymbol{u}_r, \tag{3.2}$$

where $\boldsymbol{u}$ is the fluid velocity. Substituting $\boldsymbol{v}_{CS}$ from Equation 3.2 into Equation 3.1, we obtain

$$\frac{D}{Dt}\int_{CV} b\, dV = \int_{CV} \frac{\partial b}{\partial t}\, dV + \int_{CS} b\ \left[(\boldsymbol{u} - \boldsymbol{u}_r) \cdot \boldsymbol{n}\right] dA, \tag{3.3}$$

which, by applying the divergence theorem, becomes

$$\begin{aligned} \frac{D}{Dt}\int_{CV} b\, dV &= \int_{CV} \left[\frac{\partial b}{\partial t} + \nabla \cdot (b\ \boldsymbol{u})\right] dV - \int_{CS} b\ (\boldsymbol{u}_r \cdot \boldsymbol{n})\, dA \\ &= \int_{CV} \left[\frac{\partial b}{\partial t} + \nabla b \cdot\ \boldsymbol{u} + b(\nabla \cdot \boldsymbol{u})\right] dV - \int_{CS} b\ (\boldsymbol{u}_r \cdot \boldsymbol{n})\, dA. \end{aligned} \tag{3.4}$$

By imposing the incompressibility condition, $\nabla \cdot \boldsymbol{u} = 0$, on Equation 3.4, the RTT

for a general control volume is found to be

$$\frac{D}{Dt}\int\limits_{CV} b\ dV = \int\limits_{CV} \frac{Db}{Dt} dV - \int\limits_{CS} b\ \left(\boldsymbol{u}_r \cdot \boldsymbol{n}\right) dA. \tag{3.5}$$

Moreover, if $b\left(\boldsymbol{x}, t\right)$ is a function of velocity $\boldsymbol{u}$, the RTT can be expressed in terms of $\boldsymbol{u}_r$ and $\boldsymbol{v}_{CS}$, as well. Next, we consider a few notes on Jourdain's variational operator.

3.2 A Few Notes on Jourdain's Variational Operator

As mentioned earlier, Jourdain's variational principle assumes an alternative possible velocity field for the system, while time and displacements are considered to be frozen. Therefore, the variation of a real-valued function of vectors is defined as the resulting change in the function value due to the imposed virtual velocities, while neglecting the terms of order higher than one with respect to the velocity. Thus, Jourdain's variation of a function ψ, say $\delta\psi$, is defined as

$$\delta\psi = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\psi\left(\boldsymbol{u} + \varepsilon\ \delta\boldsymbol{u}\right) - \psi\left(\boldsymbol{u}\right)\right], \tag{3.6}$$

where $\boldsymbol{u}$ is the velocity field and $\delta\boldsymbol{u}$ is the variation of the velocity field.

From the vector calculus, the derivative ($\partial\psi/\partial\boldsymbol{u}$, also referred to as an *abstract derivative*) of a real-valued function ψ of vectors $\boldsymbol{u}$ is defined [31] by the relation

$$\left.\frac{\partial\psi\left(\boldsymbol{u} + \varepsilon\boldsymbol{w}\right)}{\partial\varepsilon}\right|_{\varepsilon=0} \equiv \frac{\partial\psi}{\partial\boldsymbol{u}} \cdot \boldsymbol{w} \tag{3.7}$$

for all $\boldsymbol{w}$. For a specific set of vectors $\boldsymbol{w}$, Equation 3.7 results in the directional derivative, that is, the derivative of ψ in the direction of $\boldsymbol{w}$. Therefore, by setting

$\boldsymbol{w} = \delta\boldsymbol{u}$ in Equation 3.7,

$$\begin{aligned} \frac{\partial\psi}{\partial\boldsymbol{u}} \cdot \delta\boldsymbol{u} &= \left.\frac{\partial\psi\left(\boldsymbol{u} + \varepsilon\ \delta\boldsymbol{u}\right)}{\partial\varepsilon}\right|_{\varepsilon=0} \\ &\equiv \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[\psi\left(\boldsymbol{u} + \varepsilon\ \delta\boldsymbol{u}\right) - \psi\left(\boldsymbol{u}\right)\right]. \end{aligned} \tag{3.8}$$

Comparing Equations 3.6 and 3.8, we obtain

$$\delta\psi = \frac{\partial\psi}{\partial\boldsymbol{u}} \cdot \delta\boldsymbol{u}. \tag{3.9}$$

Therefore, Jourdain's variation of a function is equal to its derivative with respect to velocity in the direction of the virtual velocity. Since $\delta\boldsymbol{u}$ is a set of arbitrary vectors that are compatible with the system constraints, the existence of any constraint on the velocity field limits the direction of the variational operator. Thus, the kinematic constraints do not necessarily impose limitations on the magnitude of $\delta\boldsymbol{u}$. However, the magnitude of $\delta\boldsymbol{u}$ at a point, say $\boldsymbol{x}^*$, must be small enough so that the function ψ be smooth in the neighborhood of radius $|\delta\boldsymbol{u}|$ about $\boldsymbol{x}^*$.

3.3 Obtaining the Energy Rate Equation in the Lagrangian Reference Frame

In Section 2.2, it was shown that Jourdain's principle can be obtained from d'Alembert's principle. Hamilton's principle is also derived from d'Alembert's principle, bringing it to a monogenic[1] form, which is of great theoretical and practical importance. Moreover, for a general system, Hamilton's principle leads to a system of simultaneous

[1] In mathematics, a combinatoric system is a finite set of rules which together with a finite set of promises results in a finite set of conclusions. If a system is such that for each rule there exists a single promise and a single conclusion, the system is called monogenic [32]. Therefore, regarding our discussion here, monogenic means that each force, conservative or non-conservative, corresponds to a single potential function.

differential equations of second order called the *Lagrangian equations of motion* [29, pp.111-119]. In order to demonstrate our methodology for obtaining the energy rate equations from Jourdain's principle, a short review of the derivation of the Lagrangian EOM is essential.

Consider a Lagrangian function L defined as

$$L = L\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right), \qquad \text{for} \qquad i = 1, 2, \cdots, N, \tag{3.10}$$

where N is the number of particles and $\boldsymbol{r}_i$ denote the Lagrangian coordinates of particle i. The function L is defined for a conservative system as

$$L = T - \Pi, \tag{3.11}$$

where T is the kinetic energy and Π is the potential energy. The function L defines the entire dynamics of the system. Therefore, the action integral in the absence of nonconservative forces is defined as

$$\int_{t_1}^{t_2} \delta_d L\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right) = 0, \tag{3.12}$$

where $\delta_d L$ is the variation of function L, and the subscript d is used to differentiate between d'Alembert's and Jourdain's variational operators.

The variation of a Lagrangian function is, by definition, the resulting difference found by imposing d'Alembert's virtual displacements $\delta_d \boldsymbol{r}_i$ on $L\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right)$ and eliminating the terms with order higher than one with respect to $\delta_d \boldsymbol{r}_i$ while holding time frozen, as follows,

$$\int_{t_1}^{t_2} \delta_d L \; dt = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t_1}^{t_2} \left[L\left(\frac{d}{dt}\left(\boldsymbol{r}_i + \varepsilon \; \delta_d \boldsymbol{r}_i\right), \boldsymbol{r}_i + \varepsilon \; \delta_d \boldsymbol{r}_i, t \right) - L\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right) \right] \; dt. \tag{3.13}$$

In the Lagrangian frame of reference, the variational operator and time differentiation commute. Thus, Equation 3.13 can be written as

$$\int_{t_1}^{t_2} \delta_d L \; dt = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t_1}^{t_2} \left[L\left(\dot{\boldsymbol{r}}_i + \varepsilon \; \delta_d \dot{\boldsymbol{r}}_i, \boldsymbol{r}_i + \varepsilon \; \delta_d \boldsymbol{r}_i, t\right) - L\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right) \right] \; dt. \tag{3.14}$$

Applying the Taylor expansion on $L\left(\dot{\boldsymbol{r}}_i + \varepsilon \; \delta_d \dot{\boldsymbol{r}}_i, \boldsymbol{r}_i + \varepsilon \; \delta_d \boldsymbol{r}_i, t\right)$ and neglecting the terms of order higher than one with respect to $\delta_d \boldsymbol{r}_i$, Equation 3.14 becomes

$$\begin{aligned} \int_{t_1}^{t_2} \delta_d L \; dt &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t_1}^{t_2} \left[L\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right) + \varepsilon \; \delta_d \dot{\boldsymbol{r}}_i \cdot \frac{\partial L}{\partial \dot{\boldsymbol{r}}_i} + \varepsilon \; \delta_d \boldsymbol{r}_i \cdot \frac{\partial L}{\partial \boldsymbol{r}_i} - L\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right) \right] \; dt \\ &= \int_{t_1}^{t_2} \left[\; \delta_d \dot{\boldsymbol{r}}_i \cdot \frac{\partial L}{\partial \dot{\boldsymbol{r}}_i} + \; \delta_d \boldsymbol{r}_i \cdot \frac{\partial L}{\partial \boldsymbol{r}_i} \right] \; dt. \end{aligned} \tag{3.15}$$

Since the algebraic relation between $\delta_d \dot{\boldsymbol{r}}_i$ and $\delta_d \boldsymbol{r}_i$ is not known (unless the problem is solved), Equation 3.15 is not accessible for further analysis. This difficulty can be overcome if the displacements are known at t_1 and t_2 [29]. Integrating Equation 3.15 by parts as

$$\int_{t_1}^{t_2} \delta_d L \; dt = \int_{t_1}^{t_2} \; \delta_d \boldsymbol{r}_i \cdot \frac{\partial L}{\partial \boldsymbol{r}_i} \; dt + \left[\delta_d \boldsymbol{r}_i \cdot \frac{\partial L}{\partial \dot{\boldsymbol{r}}_i} \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta_d \boldsymbol{r}_i \cdot \frac{d}{dt} \left(\frac{\partial L}{\partial \boldsymbol{r}_i} \right) \; dt, \tag{3.16}$$

and since the displacements are assumed to be known at t_1 and t_2, the virtual displacements are zero and Equation 3.16 becomes

$$\int_{t_1}^{t_2} \delta_d L \; dt = \int_{t_1}^{t_2} \; \delta_d \boldsymbol{r}_i \cdot \left[\frac{\partial L}{\partial \boldsymbol{r}_i} - \; \frac{d}{dt} \left(\frac{\partial L}{\partial \boldsymbol{r}_i} \right) \right] dt. \tag{3.17}$$

Substituting Equation 3.17 into Equation 3.12, we obtain

$$\int_{t_1}^{t_2} \left[\; \frac{d}{dt} \left(\frac{\partial L}{\partial \boldsymbol{r}_i} \right) - \frac{\partial L}{\partial \boldsymbol{r}_i} \right] \cdot \delta_d \boldsymbol{r}_i \; dt = 0, \tag{3.18}$$

and, since $\delta_d \boldsymbol{r}_i$ are arbitrary non-zero vectors, the *Lagrangian equations of motion* is

obtained by setting the terms inside square brackets equal to zero,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{r}_i}\right) - \frac{\partial L}{\partial \boldsymbol{r}_i} = 0. \tag{3.19}$$

Since Jourdain's principle and the *Lagrangian equations of motion* are connected to each other via d'Alembert's principle, we expect that the energy rate equations can be obtained from Jourdain's principle. We do this first by obtaining the energy equation for a system described in the Lagrangian reference frame. Afterwards, the energy equation is obtained for the same system described in the Eulerian frame of reference.

In Section 2.2, Jourdain's principle was obtained by differentiating d'Alembert's principle with respect to time, and then, by setting the virtual displacement to be equal to zero. Similarly, we start by considering the rate of a Lagrangian function,

$$\frac{d}{dt}L = \dot{L}\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right), \tag{3.20}$$

and define the respective action integral as

$$\int_{t_1}^{t_2} \delta_d \dot{L}\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right)\ dt = 0. \tag{3.21}$$

Substituting $\delta\dot{L}$ instead of δL into Equation 3.13, we obtain

$$\begin{aligned}
\int_{t_1}^{t_2} \delta_d \dot{L}\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right)\ dt &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t_1}^{t_2} \left[\dot{L}\left(\frac{d}{dt}\left(\boldsymbol{r}_i + \varepsilon\ \delta_d \boldsymbol{r}_i\right), \boldsymbol{r}_i + \varepsilon\ \delta_d \boldsymbol{r}_i, t \right) \right. \\
&\quad \left. -\dot{L}\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right) \right]\ dt \\
&= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t_1}^{t_2} \left[\dot{L}\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right) + \varepsilon\ \delta_d \dot{\boldsymbol{r}}_i \cdot \frac{\partial \dot{L}}{\partial \dot{\boldsymbol{r}}_i} + \varepsilon\ \delta_d \boldsymbol{r}_i \cdot \frac{\partial \dot{L}}{\partial \boldsymbol{r}_i} \right. \\
&\quad \left. -\dot{L}\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right) \right]\ dt \\
&= \int_{t_1}^{t_2} \left[\frac{\partial \dot{L}}{\partial \dot{\boldsymbol{r}}_i} \cdot \delta_d \dot{\boldsymbol{r}}_i + \frac{\partial \dot{L}}{\partial \boldsymbol{r}_i} \cdot \delta_d \boldsymbol{r}_i \right]\ dt \\
&= 0. \qquad (3.22)
\end{aligned}$$

Similar to our approach in Section 2.2, Jourdain's variational principle in terms of Lagrangian function L is obtained by imposing the Jourdain's variational constraint $\delta_d \boldsymbol{r}_i = 0$ on Equation 3.22, resulting in

$$\int_{t_1}^{t_2} \delta\dot{L}\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right)\ dt = \int_{t_1}^{t_2} \frac{\partial \dot{L}}{\partial \dot{\boldsymbol{r}}_i} \cdot \delta\dot{\boldsymbol{r}}_i\ dt = 0, \qquad (3.23)$$

or,

$$\delta\dot{L}\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right) = \frac{\partial \dot{L}}{\partial \dot{\boldsymbol{r}}_i} \cdot \delta\dot{\boldsymbol{r}}_i. \qquad (3.24)$$

Therefore, the energy rate equation $\dot{L}\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right)$ can be obtained from Jourdain's principle using Equation 3.24. It is important to note that the variation of acceleration is not considered in the derivation of Equation 3.24 since the acceleration is second order with respect to $\delta_d \boldsymbol{r}_i$. It is emphasized that the only requirement for using Equation 3.24 is that Jourdain's variational operator must commute with the differential operator d/dt.

Next, we consider two simple examples to clarify the procedure.

Example 1 *Consider the single degree of freedom mass-spring system shown in Figure 3.1, where $x\left(t\right)$ is the Lagrangian position of the box of mass m and k is the*

stiffness of the massless spring.We wish to obtain the energy rate equation by integration and by using Equation 3.24.

Using Newton's second law, the equation of motion is

$$m\ddot{x}\left(t\right)+kx\left(t\right)=0. \tag{3.25}$$

The acceleration $\ddot{x}\left(t\right)$ *can be converted as,*

$$\begin{aligned}\ddot{x}\left(t\right) &= \frac{d\dot{x}\left(t\right)}{dt}\\ &= \frac{d\dot{x}\left(t\right)}{dx}\frac{dx\left(t\right)}{dt}\\ &= \frac{d}{dx}\left(\frac{1}{2}\dot{x}^{2}\left(t\right)\right).\end{aligned} \tag{3.26}$$

Substituting Equation 3.26 into Equation 3.25, it becomes

$$m\frac{d}{dx}\left(\frac{1}{2}\dot{x}^{2}\left(t\right)\right)+kx\left(t\right)=0, \tag{3.27}$$

and by integrating this equation with respect to x*, energy equation is obtained as*

$$\frac{1}{2}m\dot{x}^{2}\left(t\right)+\frac{1}{2}kx^{2}\left(t\right)=C, \tag{3.28}$$

where C *is a constant. Finally, the energy rate equation is obtained by differentiating Equation 3.28 with respect to time as*

$$m\ddot{x}\dot{x}+kx\dot{x}=0. \tag{3.29}$$

Now, we wish to obtain the energy rate equation using Jourdain's principle. Multiplying Equation 3.25 by $\delta\dot{x}(t)$, Jourdain's principle is expressed by

$$(m\ddot{x} + kx)\,\delta\dot{x} = 0. \tag{3.30}$$

Expanding Equation 3.30 and using Equation 3.24, we obtain

$$\begin{aligned}
(m\ddot{x} + kx)\,\delta\dot{x} &= m\ddot{x}\delta\dot{x} + kx\delta\dot{x} \\
&= \frac{\partial}{\partial\dot{x}}(m\ddot{x}\dot{x})\,\delta\dot{x} + \frac{\partial}{\partial\dot{x}}(kx\dot{x})\,\delta\dot{x} \\
&= \frac{\partial}{\partial\dot{x}}(m\ddot{x}\dot{x} + kx\dot{x})\,\delta\dot{x} \\
&= \delta(m\ddot{x}\dot{x} + kx\dot{x}) \\
&= \delta\dot{L}(\dot{x}, x, t).
\end{aligned}$$

Therefore, in the absence of nonconservative forces, the energy rate equation is obtained as

$$\dot{L}(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t) = m\ddot{x}\dot{x} + kx\dot{x} = 0, \tag{3.31}$$

which is the same as the energy rate equation obtained by integration.

This example considered a linear system. The next example considers a simple nonlinear problem.

Example 2 *Consider the simple pendulum shown in Figure 3.2, where the mass m is supported by the massless rod of the length L, oscillating with angle θ. The angle of oscillation is large enough that the small angle approximation no longer holds. We wish to obtain the energy rate equations by integrating the EOM as well as by using Jourdain's principle.*

The equation of motion is obtained from Newton's second law (Euler's equation)

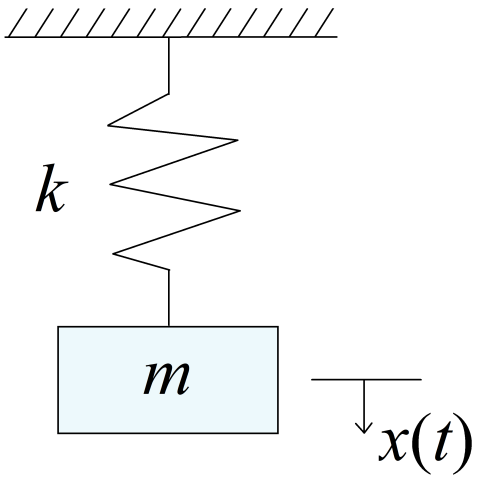

Figure 3.1: The mass-spring system.

to be

$$mL^2\ddot{\theta} + mgL\sin\theta = 0, \tag{3.32}$$

where g is the gravitational acceleration. Converting the acceleration as

$$\begin{aligned} \ddot{\theta} &= \frac{d\dot{\theta}}{dt} \\ &= \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} \\ &= \frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right), \end{aligned} \tag{3.33}$$

and substituting it back into Equation 3.32, we obtain

$$\frac{d}{d\theta}\left(\frac{1}{2}mL^2\dot{\theta}^2\right) + mgL\sin\theta = 0. \tag{3.34}$$

Integration of this equation with respect to θ yields to the energy equation

$$\frac{1}{2}mL^2\dot{\theta}^2 - mgL\cos\theta = C, \tag{3.35}$$

and by differentiating it with respect to time, the energy rate equation is obtained as

$$mL^2\ddot{\theta}\dot{\theta} + mgL\dot{\theta}\sin\theta = 0. \tag{3.36}$$

Jourdain's variational formulation of the simple pendulum problem is obtained by multiplying the EOM by $\delta\dot{\theta}$ as

$$\left(mL^2\ddot{\theta} + mgL\sin\theta\right)\delta\dot{\theta} = 0. \tag{3.37}$$

The energy rate equation is obtained by utilizing Equation 3.24,

$$\begin{aligned}
\left(mL^2\ddot{\theta} + mgL\sin\theta\right)\delta\dot{\theta} &= mL^2\ddot{\theta}\delta\dot{\theta} + mgL\sin\theta\delta\dot{\theta} \\
&= \frac{\partial}{\partial\dot{\theta}}\left(mL^2\ddot{\theta}\dot{\theta}\right)\delta\dot{\theta} + \frac{\partial}{\partial\dot{\theta}}\left(mgL\dot{\theta}\sin\theta\right)\delta\dot{\theta} \\
&= \delta\left(mL^2\ddot{\theta}\dot{\theta} + mgL\dot{\theta}\sin\theta\right) \\
&= \delta\dot{L}\left(\dot{\theta}, \theta, t\right).
\end{aligned}$$

Therefore,

$$\dot{L}\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right) = mL^2\ddot{\theta}\dot{\theta} + mgL\dot{\theta}\sin\theta = 0 \tag{3.38}$$

is the same result as that obtained from integrating the EOM.

As evident from the two examples, the energy rate equation can straightforwardly be obtained from Jourdain's principle by using Equation 3.24. From the variational point of view, the only requirement is that the commutation rule must hold (note that we used the commutation rules in our derivation in Equation 3.22).

While the commutation rule holds for a system described in the Lagrangian reference frame, it does not hold if it is described in the Eulerian reference frame. This is shown in the following section.

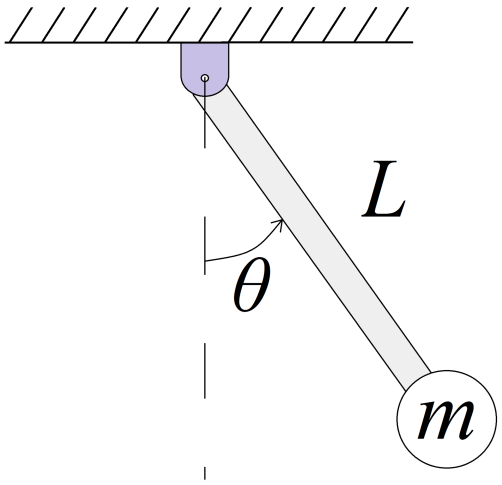

Figure 3.2: The simple pendulum.

3.4 Obtaining the Energy Rate Equation in the Eulerian Reference Frame

As mentioned in the previous section, Lagrange's equation is a monogenic function of generalized displacements, generalized velocities, and time, as is the rate of Lagrange's equation, which contains acceleration terms as shown in the two prior examples. We saw that the energy rate equations can be obtained from Jourdain's principle by using Equation 3.24 if the commutation rule holds.

For the rate of Lagrange's equation described in the Eulerian frame of reference, we show next that Jourdain's variational operator and the material derivative do not commute by first considering the acceleration $D\boldsymbol{u}/Dt$ and obtaining $D\left(\delta\boldsymbol{u}\right)/Dt$, and then by finding $\delta\left(D\boldsymbol{u}/Dt\right)$.

We start with $D\left(\delta\boldsymbol{u}\right)/Dt$. Since the variation is imposed prior to differentiation,

we have

$$\begin{aligned}\frac{D\left(\delta\boldsymbol{u}\right)}{Dt} &= \frac{\partial\left(\delta\boldsymbol{u}\right)}{\partial t}+\left(\boldsymbol{u}+\delta\boldsymbol{u}\right)\cdot\nabla\left(\delta\boldsymbol{u}\right)\\ &= \frac{\partial\left(\delta\boldsymbol{u}\right)}{\partial t}+\boldsymbol{u}\cdot\nabla\left(\delta\boldsymbol{u}\right)+\delta\boldsymbol{u}\cdot\nabla\left(\delta\boldsymbol{u}\right).\end{aligned} \tag{3.39}$$

The last term on the right of Equation 3.39 is second-order with respect to $\delta\boldsymbol{u}$, thus it must be neglected according to Jourdain's principle. Therefore, the material derivative of the virtual velocity is obtained to be

$$\frac{D\left(\delta\boldsymbol{u}\right)}{Dt}=\frac{\partial\left(\delta\boldsymbol{u}\right)}{\partial t}+\boldsymbol{u}\cdot\nabla\left(\delta\boldsymbol{u}\right). \tag{3.40}$$

Regarding the variation of acceleration $\delta\left(D\boldsymbol{u}/Dt\right)$, we consider the definition of Jourdain's variational operator, Equation 3.6, as

$$\begin{aligned}\delta\left(\frac{D\boldsymbol{u}}{Dt}\right) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\frac{D}{Dt}\left(\boldsymbol{u}+\varepsilon\ \delta\boldsymbol{u}\right)-\frac{D}{Dt}\left(\boldsymbol{u}\right)\right]\\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\frac{\partial}{\partial t}\left(\boldsymbol{u}+\varepsilon\ \delta\boldsymbol{u}\right)+\left(\boldsymbol{u}+\varepsilon\ \delta\boldsymbol{u}\right)\cdot\nabla\left(\boldsymbol{u}+\varepsilon\ \delta\boldsymbol{u}\right)-\frac{D}{Dt}\left(\boldsymbol{u}\right)\right]\\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\frac{\partial\boldsymbol{u}}{\partial t}+\varepsilon\frac{\partial\left(\delta\boldsymbol{u}\right)}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\varepsilon\ \delta\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\left(\varepsilon\ \delta\boldsymbol{u}\right)\right.\\ &\quad\left.+\varepsilon^{2}\delta\boldsymbol{u}\cdot\nabla\left(\delta\boldsymbol{u}\right)-\frac{D}{Dt}\left(\boldsymbol{u}\right)\right].\end{aligned} \tag{3.41}$$

Taking the limit, the above expression becomes

$$\delta\left(\frac{D\boldsymbol{u}}{Dt}\right)=\frac{\partial\left(\delta\boldsymbol{u}\right)}{\partial t}+\boldsymbol{u}\cdot\nabla\left(\delta\boldsymbol{u}\right)+\delta\boldsymbol{u}\cdot\nabla\boldsymbol{u}. \tag{3.42}$$

Comparing Equations 3.40 and 3.42 yields

$$\delta\left(\frac{D\boldsymbol{u}}{Dt}\right)=\frac{D\left(\delta\boldsymbol{u}\right)}{Dt}+\delta\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \tag{3.43}$$

where $\delta\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ is the non-commuting part of the acceleration in the Eulerian reference

frame.

Since the commutation rule does not hold in the Eulerian reference frame, Equation 3.24 cannot be used. This difficulty can be overcome if the non-commuting part is extracted from the rate of Lagrange's function $\dot{L}\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right)$ as follows.

In order to demonstrate how the non-commuting part can be separated in the Eulerian reference frame, we consider Newton's second law as described in the Eulerian reference frame,

$$\frac{D\boldsymbol{u}\left(\boldsymbol{x}, t\right)}{Dt} = \boldsymbol{f}\left(\boldsymbol{x}, t\right), \tag{3.44}$$

where $\boldsymbol{f}$ is the resultant of external loads per unit mass acting at the spatial point $\boldsymbol{x}$. Multiplication of Equation 3.44 by the virtual velocity, $\delta\boldsymbol{u}$, results in Jourdain's principle,

$$\frac{D\boldsymbol{u}}{Dt} \cdot \delta\boldsymbol{u} = \boldsymbol{f} \cdot \delta\boldsymbol{u}. \tag{3.45}$$

Regarding the acceleration term, $D\boldsymbol{u}/Dt$, the commutation rule does not hold due to $\delta\boldsymbol{u} \cdot \nabla\boldsymbol{u}$ in Equation 3.43. However, the acceleration term of a system is reversible (conservative) in the Eulerian reference frame if $\delta\boldsymbol{u} \cdot \nabla\boldsymbol{u} = 0$. Therefore, $D\boldsymbol{u}/Dt$ contains both the conservative (reversible) and nonconservative (irreversible, non-commuting) terms.

Expanding the left-hand side of Equation 3.45 as,

$$\frac{D\boldsymbol{u}}{Dt} \cdot \delta\boldsymbol{u} = \frac{D}{Dt}\left(\boldsymbol{u} \cdot \delta\boldsymbol{u}\right) - \frac{D}{Dt}\left(\delta\boldsymbol{u}\right) \cdot \boldsymbol{u}, \tag{3.46}$$

and applying Equation 3.43 to the last term on the right, we obtain

$$\begin{aligned} \frac{D\boldsymbol{u}}{Dt} \cdot \delta\boldsymbol{u} &= \frac{D}{Dt}\left(\boldsymbol{u} \cdot \delta\boldsymbol{u}\right) - \left[\delta\left(\frac{D\boldsymbol{u}}{Dt}\right) - \delta\boldsymbol{u} \cdot \nabla\boldsymbol{u}\right] \cdot \boldsymbol{u} \\ &= \frac{D}{Dt}\left(\boldsymbol{u} \cdot \delta\boldsymbol{u}\right) - \delta\left(\frac{D\boldsymbol{u}}{Dt}\right) \cdot \boldsymbol{u} + \delta\boldsymbol{u} \cdot \nabla\boldsymbol{u} \cdot \boldsymbol{u}, \end{aligned} \tag{3.47}$$

and substituting back into Equation 3.45, we obtain

$$\frac{D}{Dt}\left(\boldsymbol{u}\cdot\delta\boldsymbol{u}\right)-\delta\left(\frac{D\boldsymbol{u}}{Dt}\right)\cdot\boldsymbol{u}+\delta\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}=\boldsymbol{f}\cdot\delta\boldsymbol{u}. \tag{3.48}$$

Considering Equation 3.48, if the commutation rule holds, $\delta\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}$ would not be present ($\delta\boldsymbol{u}\cdot\nabla\boldsymbol{u}=0$). Thus, the remainder of the terms on the left-hand side are those that are reversible with respect to $\delta\boldsymbol{u}$ (the terms for which the commutation rule holds). Therefore, by taking the term $\delta\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}$ to the right-hand side of the equation, the left-hand side becomes purely conservative,

$$\overbrace{\frac{D}{Dt}\left(\boldsymbol{u}\cdot\delta\boldsymbol{u}\right)-\delta\left(\frac{D\boldsymbol{u}}{Dt}\right)\cdot\boldsymbol{u}}^{\text{conservative terms}}=\overbrace{\boldsymbol{f}\cdot\delta\boldsymbol{u}-\delta\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}}^{\text{nonconservative terms}}. \tag{3.49}$$

Using the commutation rule for the conservative terms and rearranging the nonconservative terms, Jourdain's variational principle is obtained alternatively as,

$$\frac{D\boldsymbol{u}}{Dt}\cdot\delta\boldsymbol{u}=\left(\boldsymbol{f}-\nabla\boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot\delta\boldsymbol{u}. \tag{3.50}$$

Therefore, the energy rate can be obtained for the left-hand side of Equation 3.50 by using Equation 3.24, having considered the non-commuting part as a nonconservative force.

The nonconservative terms reduce the generality of variational methods. A commonly used tool to overcome this challenge is Rayleigh's dissipation function (RDF), which is reviewed in the following section. RDF together with the discussion presented in this section will be used in our manipulation of the extended Jourdain's principle obtained in Chapter 2.

3.5 Rayleigh's Dissipation Function

Generally, variational methods are powerful tools for reversible processes (conservative systems). Regarding non-conservative systems, especially frictional dissipative systems (viscous fluids), variational methods lose their generality. Frictional (viscous) forces originate from a transfer of macroscopic into microscopic motions. Therefore, the number of degrees of freedom needed to describe the motion must be increased to include the dissipation process in variational formulations requiring statistical principles [29, p. 359]. However, as mentioned in Section 2.7, Bateman argued that any set of equations of motion, conservative or nonconservative, corresponds to a variational principle. In the presence of dissipative forces, there must exist a set of complimentary equations that act as a secondary process absorbing the energy from the preliminary set [23]. However, Bateman concluded his paper by stating that such secondary functions may not always be found for a nonlinear system.

Alternatively, the effects of nonconservative forces can be added to a variational formulation in the same manner that external forces are considered. This method, which is justified by the law of conservation of energy, reduces the generality of the variational methods to varying extents depending on the dynamics and properties of the system. Having added the terms representing the irreversible processes, dissipation functions may be guessed whose variations represent dissipative forces. If such functions are to be guessed, it must be noted that those potential functions are not necessarily unique (the reason is explained in the next paragraph). Alternatively, dissipation functions may be obtained from experimental data, and then be added to the variational formulation. Consequently, the EOMs can be obtained by taking the variation afterwards. Since dissipation functions have dimensions of energy or energy rate, they might be preferred when it comes to semi-analytical methods.

Variational methods are based on the first order approximation with respect to

a variable. Therefore, corresponding to a nonconservative force, several dissipation functions may be found that differ only in the higher order terms. As long as the higher order terms do not impose additional constraints on the system, no variational rule is broken. In a way, the use of dissipation functions along with variational methods is an approximation method. Thus, a consideration of experimental data may be a requirement when using dissipation functions.

Regarding the dissipation functions, Rayleigh's dissipation function is widely utilized in the literature. It has been applied to important types of forces, for instance, forces due to linear dampers and electromagnetic forces. While our discussion in this chapter is focused on Rayleigh's dissipation function, it worth nothing that there have been many attempts to develop variational methods for nonconservative systems. A good representative work is the paper by Riewe [33], where he proposed the use of fractional derivatives for Lagrangian and Hamiltonian mechanics.

Considering the viscous terms of the Navier-Stokes equations (Equation 2.42), the symmetric part of $\nabla \boldsymbol{u}$ is only included (Equation 2.41) assuming that the rotating flow has no effect on shear stress (assuming rigid body rotation). Since the virtual velocities must be compatible with the system's constraints (or here, assumption), the directions of the virtual velocities are constrained to be the same as those of the actual velocities, otherwise, the rotation caused by $\delta \boldsymbol{u}$ will perform work on the system. This is the assumption that Rayleigh's dissipation function is based on. To make the last statement clear, we consider next a nonconservative force due to a linear damper as an example.

For a linear damper, the force is proportional to the velocity components. Therefore, the component of damping force in x-direction, F_{Cx}, can be expressed by

$$F_{Cx} = -C_x u_x, \tag{3.51}$$

where C_x is the damping coefficient associated with x-direction, and u_x is the component of $\boldsymbol{u}$ in the x-direction. The dissipation function associated with this type of frictional force is called Rayleigh's dissipation function, which is expressed as

$$\phi_C = \frac{1}{2}\left(C_x u_x^2 + C_y u_y^2 + C_z u_z^2\right). \tag{3.52}$$

Differentiation of ϕ_C with respect to u_x yields the force component in x-direction,

$$F_{Cx} = \frac{\partial \phi_C}{\partial u_x} = -C_x u_x. \tag{3.53}$$

Therefore, in vector notation it can be expressed as

$$\boldsymbol{F}_C = \nabla_{\boldsymbol{u}} \phi_C, \tag{3.54}$$

where $\nabla_{\boldsymbol{u}}\ \phi_C$ denotes the differentiation of ϕ_C with respect to $\boldsymbol{u}$ in direction of $\boldsymbol{u}$ [34, pp. 22-24], that is

$$\nabla_{\boldsymbol{u}} \phi_C = \left.\frac{\partial \phi_C}{\partial \boldsymbol{u}}\right|_{\boldsymbol{u}\text{-direction}}. \tag{3.55}$$

The directional differential of any function ϕ can be obtained using Equation 3.7 by setting $\boldsymbol{w} = \boldsymbol{u}$ as

$$\nabla_{\boldsymbol{u}} \phi \cdot \boldsymbol{u} = \left.\frac{\partial \phi\left(\boldsymbol{u} + \varepsilon \boldsymbol{u}\right)}{\partial \varepsilon}\right|_{\varepsilon = 0}. \tag{3.56}$$

Therefore, defining ϕ such that $\nabla_{\boldsymbol{u}}\ \phi = \boldsymbol{F}$, the term $\boldsymbol{F}\cdot\ \delta\boldsymbol{u}$ in the variational formulation can be replaced by $\delta\phi$, since

$$\delta\phi = \nabla_{\boldsymbol{u}} \phi \cdot \delta \boldsymbol{u}, \tag{3.57}$$

where this is the Rayleigh's dissipation function in vector notation for a system.

Comparing Equation 3.57 with Equation 3.9, Rayleigh's dissipation function constrains $\delta\boldsymbol{u}$ to be in the same direction as $\boldsymbol{u}$ for the nonconservative terms at any point

in the domain. However, this constraint was already imposed by the assumptions that were made in obtaining the constitutive relation for Newtonian fluids (*see* Section 2.6 and [28, pp. 100-103]). The reason is that the proportionality of the viscous force and $\nabla \boldsymbol{u}$ was the result of observing laminar flows in the first place.

Regarding Rayleigh's dissipation function, an important note, which is often not mentioned in textbooks, is that the work (power) of the Rayleigh's dissipation function in the variational energy (power) formulation is half of that in the corresponding energy (power) equation [13, Equations 5 and 6]. In order to make this statement more clear, we consider the variational energy formulation of a system with frictional dissipative forces as

$$\delta_d \left(T + \Pi - \Phi - W_{ext} \right) = 0, \tag{3.58}$$

where δ_d is d'Alembert's variational operator, T is the kinetic energy, Π denotes the potential energy, Φ is Rayleigh's dissipation function and W_{ext} is the work due to external loads. Assuming that Equation 3.58 is obtained from the variational manipulation, the energy equation corresponding to it is obtained by multiplying the Rayleigh's dissipation function by a factor of two as,

$$T + \Pi - 2\Phi - W_{ext} = 0. \tag{3.59}$$

The reason is that the frictional dissipation only depends on the velocity and it is unaffected by the displacement. Regarding Jourdain's variational operator, we have

$$\delta \left(\dot{T} + \dot{\Pi} - \dot{\Phi} - \dot{W}_{ext} \right) = 0, \tag{3.60}$$

where the overdot is to denote that the terms have dimensions of power and not work. Regarding Jourdain's variational operator, the frictional dissipation remains unaffected by acceleration and it is similar otherwise, that is, the power equation

becomes

$$\dot{T} + \dot{\Pi} - 2\dot{\Phi} - \dot{W}_{ext} = 0. \tag{3.61}$$

Next, we use the discussions provided so far in this chapter to modify the extended Jourdain's principle derived in the previous chapter. We will refer to both types of equations expressed by Equations 3.60 and 3.61 as the energy rate equation, assuming that the reader would distinguish them from each other by δ.

3.6 Energy Rate Equation from Extended JP

In Section 2.5, the extended Jourdain's principle was expressed in Equation 2.34. If we substitute the forces from the constitutive relations for a Newtonian incompressible viscid fluid (Equations 2.37 to 2.41), the extended JP for a general control volume becomes

$$\int_{CV} \rho \frac{D\boldsymbol{u}}{Dt} \cdot \delta\boldsymbol{u} \; dV = \int_{CV} \left\{ \nabla \cdot \left[-p\bar{\boldsymbol{I}} + \mu \left(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u} \right) \right] \right\} \cdot \; \delta\boldsymbol{u} \; dV, \tag{3.62}$$

where $\nabla^T\boldsymbol{u} = (\nabla\boldsymbol{u})^T$ and body forces are neglected. If the only body force present is due to gravity, its potential function can easily be obtained, since gravitational forces are conservative and independent of the fluid velocity field. Regarding the *model problem* defined in Chapter 1, the gravitational forces can be neglected.

In Section 3.3, it was shown that the energy rate equation can be obtained using Equation 3.24 for systems described in the Lagrangian reference frame. Also, in Section 3.4, it was shown that the commutation rule does not hold in the Eulerian reference frame. However, this difficulty was overcome by extracting the non-commuting term from the acceleration as explained in deriving Equation 3.50. Therefore, using

the same procedure explained there, Equation 3.62 becomes

$$\int_{CV} \rho \frac{D\boldsymbol{u}}{Dt} \cdot \delta\boldsymbol{u}\, dV = \int_{CV} \left\{ \nabla \cdot \left[-p\bar{\boldsymbol{I}} + \mu \left(\nabla\boldsymbol{u} + \nabla^T \boldsymbol{u} \right) \right] \right\} \cdot \ \delta\boldsymbol{u}\, dV - \int_{CV} \rho \left(\nabla\boldsymbol{u} \cdot \boldsymbol{u} \right) \cdot \delta\boldsymbol{u}\, dV, \quad (3.63)$$

where the left-hand side of the equation now represents only the conservative terms and the right-hand side represents the nonconservative terms. Therefore, we continue our derivation by considering each side of Equation 3.63 separately, and use Equation 3.24 for the conservative and Equation 3.57 for the dissipative terms to obtain the energy rate equation as follows.

3.6.1 The left-hand side of Equation 3.63

As mentioned earlier, the term on the left-hand side of Equation 3.63 is reversible and conservative and therefore δ and D/Dt commute. Therefore, the energy rate equation is obtained by using Equation 3.24 and following the same steps as in Examples 1 and 2 as

$$\begin{aligned} \int_{CV} \rho \frac{D\boldsymbol{u}}{Dt} \cdot \delta\boldsymbol{u}\, dV &= \int_{CV} \left[\frac{\partial}{\partial \boldsymbol{u}} \left(\rho \frac{D\boldsymbol{u}}{Dt} \cdot \boldsymbol{u} \right) \right] \cdot \delta\boldsymbol{u}\, dV \\ &= \int_{CV} \delta \left(\rho \frac{D\boldsymbol{u}}{Dt} \cdot \boldsymbol{u} \right) dV \\ &= \delta \int_{CV} \left(\rho \frac{D\boldsymbol{u}}{Dt} \cdot \boldsymbol{u} \right) dV \\ &= \delta \int_{CV} \frac{D}{Dt} \left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) dV, \end{aligned} \quad (3.64)$$

where Jourdain's variational operator commutes with integration over the volume since it assumes zero virtual displacements, that is, the same particles occupy a control

volume before and after imposing the virtual velocities. Note that the same statement is not true for d'Alembert's variational operator since the virtual displacements result in a virtual flux across the control surfaces.

Equation 3.64 can be modified further as follows. The total kinetic energy of a control volume, T, is defined as

$$T = \int_{CV} \frac{1}{2}\rho \ \boldsymbol{u} \cdot \boldsymbol{u} \ dV. \tag{3.65}$$

The rate of kinetic energy of a control volume can be obtained by using the Reynolds transport theorem in the form of Equation 3.5 as

$$\begin{aligned} \frac{DT}{Dt} &= \frac{D}{Dt}\int_{CV} \frac{1}{2}\rho \ \boldsymbol{u} \cdot \boldsymbol{u} \ dV \\ &= \int_{CV} \frac{D}{Dt}\left(\frac{1}{2}\rho \ \boldsymbol{u} \cdot \boldsymbol{u}\right) dV - \int_{CS} \left(\frac{1}{2}\rho \ \boldsymbol{u} \cdot \boldsymbol{u}\right)(\boldsymbol{u} - \boldsymbol{v}_{CS}) \cdot \boldsymbol{n} \ dA. \end{aligned} \tag{3.66}$$

Rearranging Equation 3.66 yields

$$\int_{CV} \frac{D}{Dt}\left(\frac{1}{2}\rho \ \boldsymbol{u} \cdot \boldsymbol{u}\right) dV = \frac{D}{Dt}\int_{CV} \frac{1}{2}\rho \ \boldsymbol{u} \cdot \boldsymbol{u} \ dV + \int_{CS} \left(\frac{1}{2}\rho \ \boldsymbol{u} \cdot \boldsymbol{u}\right)(\boldsymbol{u} - \boldsymbol{v}_{CS}) \cdot \boldsymbol{n} \ dA. \tag{3.67}$$

Finally, by substituting Equation 3.67 into Equation 3.64, the left-hand side of Equation 3.63 is obtained as

$$\int_{CV} \rho \frac{D\boldsymbol{u}}{Dt} \cdot \delta\boldsymbol{u} \ dV = \delta \left[\frac{D}{Dt}\int_{CV} \frac{1}{2}\rho \ \boldsymbol{u} \cdot \boldsymbol{u} \ dV + \int_{CS} \left(\frac{1}{2}\rho \ \boldsymbol{u} \cdot \boldsymbol{u}\right)(\boldsymbol{u} - \boldsymbol{v}_{CS}) \cdot \boldsymbol{n} \ dA\right]. \tag{3.68}$$

Equation 3.68 represents the energy rate of the conservative terms of the extended Jourdain's principle. The energy rate of the nonconservative terms are obtained in the following section.

3.6.2 The right-hand side of Equation 3.63

In this section, we wish to obtain a function ϕ such that

$$\delta\phi = \int_{CV} \left\{ \nabla \cdot \left[-p\bar{\boldsymbol{I}} + \mu \left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u} \right) \right] \right\} \cdot \delta \boldsymbol{u} \; dV - \int_{CV} \rho \left(\nabla \boldsymbol{u} \cdot \boldsymbol{u} \right) \cdot \delta \boldsymbol{u} \; dV, \quad (3.69)$$

where ϕ is a scalar-valued potential function that gives the terms on the right-hand side of the energy rate equation.

For simplicity, since the manipulations are lengthy, we separate the terms of Equation 3.69 according to their physical meanings and manipulate each of them in separate sections as follows.

Power due to pressure

Considering the term containing pressure, p, we have

$$\int_{CV} \left[\nabla \cdot \left(-p\bar{\boldsymbol{I}} \right) \right] \cdot \delta \boldsymbol{u} \; dV = \int_{CV} -\nabla p \cdot \delta \boldsymbol{u} \; dV. \quad (3.70)$$

Since ∇p is assumed to be independent of $\boldsymbol{u}$, it does not vary due to $\delta \boldsymbol{u}$. Thus, since $\delta \left(\nabla p \right) = 0$, we can write

$$\begin{aligned} \int_{CV} -\nabla p \cdot \delta \boldsymbol{u} \; dV &= \int_{CV} \left[-\delta \left(\nabla p \right) \cdot \boldsymbol{u} - \nabla p \cdot \delta \boldsymbol{u} \right] dV \\ &= \delta \int_{CV} -\nabla p \cdot \boldsymbol{u} \; dV. \end{aligned} \quad (3.71)$$

We also know that

$$\nabla \cdot \left(p\boldsymbol{u} \right) = \boldsymbol{u} \cdot \nabla p + p \left(\nabla \cdot \boldsymbol{u} \right) \; = \nabla p \cdot \boldsymbol{u}, \quad (3.72)$$

since $\nabla \cdot \boldsymbol{u} = 0$ for an incompressible fluid. Using Equations 3.70, 3.71 and 3.72, the volume integral of the pressure term of Equation 3.69 becomes

$$\begin{aligned}
\int_{CV} \left[\nabla \cdot \left(-p\bar{\boldsymbol{I}}\right)\right] \cdot \delta\boldsymbol{u}\ dV &= \delta \int_{CV} -\nabla p \cdot \boldsymbol{u}\ dV \\
&= -\delta \int_{CV} -\nabla \cdot (p\boldsymbol{u})\ dV \\
&= -\delta \int_{CS} p\boldsymbol{u} \cdot \boldsymbol{n}\ dA, \qquad (3.73)
\end{aligned}$$

where we used the divergence theorem to obtain the last integral.

Substituting Equation 3.73 back into Equation 3.69, we have

$$\delta\phi = \delta \int_{CS} -p\boldsymbol{u}\cdot\boldsymbol{n}\ dA + \int_{CV} \left[\nabla \cdot \mu \left(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u}\right)\right]\cdot\delta\boldsymbol{u}\ dV - \int_{CV} \rho\left(\nabla\boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot\delta\boldsymbol{u}\ dV. \quad (3.74)$$

Identifying the external and the dissipative forces

Regarding the second integral on the RHS of Equation 3.74, the integrand can be modified as follows,

$$\begin{aligned}
\left[\nabla \cdot \left(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u}\right)\right] \cdot \delta\boldsymbol{u} &= \left[\mathbf{e}_i \frac{\partial}{\partial x_i} \cdot \left(\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k}\right)\mathbf{e}_j\mathbf{e}_k\right] \cdot (\delta u)_m\, \mathbf{e}_m \\
&= \left[\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right)\mathbf{e}_k\right] \cdot (\delta u)_m\, \mathbf{e}_m \\
&= \left[\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right)\right](\delta u)_k \\
&= \frac{\partial}{\partial x_i}\left[\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right)(\delta u)_k\right] - \left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right)\frac{\partial (\delta u)_k}{\partial x_i} \\
&= \overbrace{\nabla \cdot \left[\left(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u}\right)\cdot\delta\boldsymbol{u}\right]}^{\text{due to the external shear stresses}} \\
&\quad \overbrace{-tr\left[\left(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u}\right)\cdot\nabla\left(\delta\boldsymbol{u}\right)\right]}^{\text{viscous dissipation}}, \qquad (3.75)
\end{aligned}$$

where $\mathbf{e}_i$ are the unit vectors on an orthonormal basis and tr denotes the trace of a tensor. The index notation used here follows the rules used by Dill [31]. We will continue our derivation using this notation. Since it is often more convenient to manipulate the equations considered in this section in various notations, we will occasionally switch between component, index and vector notations.

Regarding the first term on the right of Equation 3.75, $\nabla\cdot$ can be eliminated by using Gauss' divergence theorem resulting in $\mu\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\delta\boldsymbol{u}\right]\cdot\boldsymbol{n}$. Therefore, it represents the virtual power due to the external shear stresses applied on the control surface, having compared it with the definition of the stress tensor given by the constitutive relation for Newtonian incompressible fluids (Equation 2.40). Consequently, the second term on the right of Equation 3.75 represents the viscous dissipation of energy inside the control volume.

Power lost due to viscous dissipation

As discussed in the previous section, the function $\left[\nabla\cdot\mu\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\right]\cdot\delta\boldsymbol{u}$ contains both the virtual power due to the viscous forces acting as an external shear load on the control surface and the virtual power due to viscous dissipative forces inside the control volume. As mentioned earlier, we wish to utilize Rayleigh's dissipation function for nonconservative (viscous dissipative) terms. However, applying Equation 3.57 to the last term of Equation 3.75 requires extensive mathematical manipulation due to the term $\nabla\left(\delta\boldsymbol{u}\right)$.

Alternatively, the amount of derivations can be reduced by obtaining the scalar potential function of $\left[\nabla\cdot\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\right]\cdot\delta\boldsymbol{u}$ in direction of $\boldsymbol{u}$ using Equation 3.57 and then subtracting the scalar potential function of the external loads obtained similarly. The remaining terms will be Rayleigh's dissipation function.

We begin with the function $\mu\left[\nabla\cdot\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\right]\cdot\boldsymbol{u}$, and applying Equation 3.56 and obtaining its directional derivative with respect to $\boldsymbol{u}$ in the direction of $\boldsymbol{u}$, we

have

$$
\begin{aligned}
&\nabla_{\boldsymbol{u}} \ \left\{\mu\left[\nabla\cdot\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\right]\cdot\boldsymbol{u}\right\}\cdot\boldsymbol{u}\\
&=\mu\,\frac{\partial}{\partial\varepsilon}\left\{\left\{\nabla\cdot\left[\nabla\left(\boldsymbol{u}+\varepsilon\boldsymbol{u}\right)+\nabla^T\left(\boldsymbol{u}+\varepsilon\boldsymbol{u}\right)\right]\right\}\cdot\left(\boldsymbol{u}+\varepsilon\boldsymbol{u}\right)\right\}\Big|_{\varepsilon=0}\\
&=\mu\,\frac{\partial}{\partial\varepsilon}\left\{\left[\nabla\cdot\left(\nabla\boldsymbol{u}+\varepsilon\ \nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}+\varepsilon\ \nabla^T\boldsymbol{u}\right)\right]\cdot\left(\boldsymbol{u}+\varepsilon\boldsymbol{u}\right)\right\}\Big|_{\varepsilon=0}\\
&=\mu\,\frac{\partial}{\partial\varepsilon}\left\{\left\{\nabla\cdot\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)+\varepsilon\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\right]\right\}\cdot\left(\boldsymbol{u}+\varepsilon\boldsymbol{u}\right)\right\}\Big|_{\varepsilon=0}\\
&=\mu\left\{\left[\nabla\cdot\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\right]\cdot\left(\boldsymbol{u}+\varepsilon\boldsymbol{u}\right)+\left\{\nabla\cdot\left[\left(1+\varepsilon\right)\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\right]\right\}\cdot\boldsymbol{u}\right\}_{\varepsilon=0}\\
&=2\mu\left[\nabla\cdot\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\right]\cdot\boldsymbol{u},
\end{aligned}
\tag{3.76}
$$

or,

$$
\nabla_{\boldsymbol{u}}\ \left\{\frac{1}{2}\mu\left[\nabla\cdot\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\right]\cdot\boldsymbol{u}\right\}=\mu\left[\nabla\cdot\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\right],
\tag{3.77}
$$

where $\frac{1}{2}\mu\left[\nabla\cdot\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\right]\cdot\boldsymbol{u}\equiv\phi$ in Equation 3.57, and therefore,

$$
\delta\left\{\frac{1}{2}\mu\left[\nabla\cdot\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\right]\cdot\ \boldsymbol{u}\right\}=\mu\left[\nabla\cdot\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\right]\cdot\delta\boldsymbol{u}.
\tag{3.78}
$$

By expanding the left-hand side of Equation 3.78, we obtain

$$
\begin{aligned}
\delta\left\{\frac{1}{2}\mu\left[\nabla\cdot\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\right]\cdot\ \boldsymbol{u}\right\} &= \delta\left\{\frac{1}{2}\mu\left[\mathbf{e}_i\frac{\partial}{\partial x_i}\cdot\left(\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}\right)\mathbf{e}_j\mathbf{e}_k\right]\cdot u_m\mathbf{e}_m\right\}\\
&= \frac{1}{2}\mu\ \delta\left\{\frac{\partial}{\partial x_i}\left[\left(\frac{\partial u_k}{\partial x_i}+\frac{\partial u_i}{\partial x_k}\right)u_k\right]\right.\\
&\quad \left.-\left(\frac{\partial u_k}{\partial x_i}+\frac{\partial u_i}{\partial x_k}\right)\frac{\partial u_k}{\partial x_i}\right\}\\
&= \overbrace{\delta\left\{\frac{1}{2}\mu\nabla\cdot\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\boldsymbol{u}\right]\right\}}^{\text{external load}}\\
&\quad -\overbrace{\delta\left\{\frac{1}{2}\mu\ tr\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\nabla\boldsymbol{u}\right]\right\}}^{\text{viscous dissipation}}.
\end{aligned}
\tag{3.79}
$$

Now we consider the term representing the external shear load in Equation 3.75 and, using Equations 3.56 and 3.57, we obtain its potential function in direction of $\boldsymbol{u}$

$$\begin{aligned}
\int_{CV} \mu\nabla\cdot\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\delta\boldsymbol{u}\right]dV &= \int_{CS}\mu\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\delta\boldsymbol{u}\right]\cdot\boldsymbol{n}dA \\
&= \int_{CS}\left\{\nabla_{\boldsymbol{u}}\left[\frac{1}{2}\mu\left(\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\boldsymbol{u}\right)\right]\cdot\delta\boldsymbol{u}\right\}\cdot\boldsymbol{n}dA \\
&= \int_{CS}\delta\left\{\frac{1}{2}\mu\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\boldsymbol{u}\right]\right\}\cdot\boldsymbol{n}dA \\
&= \int_{CV}\delta\left\{\frac{1}{2}\mu\nabla\cdot\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\boldsymbol{u}\right]\right\}dV, \qquad (3.80)
\end{aligned}$$

where we used Gauss' (divergence) theorem to switch between integration over the control surface and control volume.

Substituting Equation 3.79 into the left-hand side of Equation 3.75 and replacing the term representing the external shear load on the right-hand side of Equation 3.75 by Equation 3.80, we obtain

$$\begin{aligned}
&\delta\left\{\frac{1}{2}\mu\nabla\cdot\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\boldsymbol{u}\right]\right\}-\delta\left\{\frac{1}{2}\mu\ tr\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\nabla\boldsymbol{u}\right]\right\} \\
&\qquad = \delta\left\{\frac{1}{2}\mu\nabla\cdot\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\boldsymbol{u}\right]\right\}-tr\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\nabla\left(\delta\boldsymbol{u}\right)\right]. \qquad (3.81)
\end{aligned}$$

Therefore, Rayleigh's viscous dissipation function is obtained by eliminating the term representing the external shear load from both sides, resulting in

$$\mu\ tr\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\nabla\left(\delta\boldsymbol{u}\right)\right]=\delta\left\{\frac{1}{2}\mu\ tr\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\nabla\boldsymbol{u}\right]\right\}. \qquad (3.82)$$

So far we have obtained the scalar potential functions for the pressure and viscous

dissipation terms. Substituting Equation 3.82 back into Equation 3.75, we have

$$\left[\nabla\cdot\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\right]\cdot\,\delta\boldsymbol{u}=\nabla\cdot\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\;\delta\boldsymbol{u}\right]-\delta\left\{\frac{1}{2}\mu tr\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\;\nabla\boldsymbol{u}\right]\right\}. \tag{3.83}$$

And introducing Equation 3.83 into Equation 3.74, we obtain the right-hand side of Equation 3.63,

$$\begin{aligned}\delta\phi=\delta\int_{CS}-p\;\boldsymbol{u}\cdot\boldsymbol{n}\;dA-\delta\int_{CV}\frac{1}{2}\mu\;tr\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\;\nabla\boldsymbol{u}\right]dV\\+\int_{CV}\mu\nabla\cdot\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\;\delta\boldsymbol{u}\right]\;dV-\int_{CV}\rho\left(\nabla\boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot\delta\boldsymbol{u}\;dV.\end{aligned} \tag{3.84}$$

In the following two sections, we obtain the scalar potentials associated with the remaining terms, that is, the external viscous loads and the non-commuting terms.

Power due to shear forces acting on the control surfaces

For the external viscous shear loads in Equation 3.84, we have

$$\begin{aligned}&\int_{CV}\mu\nabla\cdot\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\;\delta\boldsymbol{u}\right]\;dV\\&\quad=\delta\int_{CV}\mu\nabla\cdot\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\;\boldsymbol{u}\right]\;dV-\int_{CV}\mu\nabla\cdot\left[\delta\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\;\boldsymbol{u}\right]\;dV.\end{aligned} \tag{3.85}$$

The external loads must remain unchanged with respect to $\delta\boldsymbol{u}$ since they are the known parameters of the system, that is, the velocity of the flow is known at the control surfaces. Therefore, the second integral on the right of Equation 3.85 equals zero, resulting in

$$\int_{CV}\mu\nabla\cdot\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\;\delta\boldsymbol{u}\right]\;dV=\delta\int_{CV}\mu\nabla\cdot\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\;\boldsymbol{u}\right]\;dV. \tag{3.86}$$

Substituting Equation 3.86 back into Equation 3.84, we obtain

$$\delta\phi = \delta \int_{CS} -p\ \boldsymbol{u}\cdot\boldsymbol{n}\ dA - \delta \int_{CV} \frac{1}{2}\mu\ tr\left[\left(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u}\right)\cdot\nabla\boldsymbol{u}\right] dV + \delta \int_{CV} \mu\nabla\cdot\left[\left(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u}\right)\cdot\boldsymbol{u}\right]\ dV - \int_{CV} \rho\left(\nabla\boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot\delta\boldsymbol{u}\ dV. \quad (3.87)$$

The non-commuting term

The only term left on the right-hand side of Equation 3.63 is the non-commuting part of the rate of the kinetic energy due to variation of the acceleration $(\rho\left(\nabla\boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot\delta\boldsymbol{u})$. In order to obtain the scalar potential function corresponding to the non-commuting part, we start by considering the vector identity,

$$\begin{aligned} \frac{1}{2}\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right) &= \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right) \\ &= \nabla\boldsymbol{u}\cdot\boldsymbol{u}. \end{aligned} \quad (3.88)$$

Therefore, we have

$$\int_{CV} \rho\left(\nabla\boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot\delta\boldsymbol{u}\ dV = \int_{CV} \frac{1}{2}\rho\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot\delta\boldsymbol{u}\ dV. \quad (3.89)$$

The scalar-valued function $\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot\delta\boldsymbol{u}$ in index notation is

$$\begin{aligned} \nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot\delta\boldsymbol{u} &= \mathbf{e}_i\frac{\partial}{\partial x_i}\left(u_j u_j\right)\cdot\left(\delta u\right)_k\mathbf{e}_k \\ &= \left[\frac{\partial}{\partial x_i}\left(u_j u_j\right)\right]\left(\delta u\right)_i. \end{aligned} \quad (3.90)$$

For an incompressible fluid, we have

$$\nabla\cdot\boldsymbol{u} = \frac{\partial u_i}{\partial x_i} = 0. \quad (3.91)$$

Since the virtual velocities must be compatible with the system constraints, for an incompressible fluid the virtual velocities must be divergence free (solenoidal) as well,

$$\nabla \cdot \delta \boldsymbol{u} = \frac{\partial \left(\delta u\right)_i}{\partial x_i} = 0. \tag{3.92}$$

Therefore,

$$\frac{\partial \left(\delta u\right)_i}{\partial x_i} \left(u_j u_j\right) = 0, \tag{3.93}$$

must also equal zero.

Adding Equation 3.93 to Equation 3.90 yields

$$\begin{aligned} \nabla \left(\boldsymbol{u} \cdot \boldsymbol{u}\right) \cdot \delta \boldsymbol{u} &= \left[\frac{\partial}{\partial x_i} \left(u_j u_j\right)\right] \left(\delta u\right)_i + \frac{\partial \left(\delta u\right)_i}{\partial x_i} \left(u_j u_j\right) \\ &= \frac{\partial}{\partial x_i} \left[\left(u_j u_j\right) \left(\delta u\right)_i\right] \\ &= \nabla \cdot \left[\left(\boldsymbol{u} \cdot \boldsymbol{u}\right) \ \delta \boldsymbol{u}\right]. \end{aligned} \tag{3.94}$$

Substituting Equation 3.94 into Equation 3.89 yields

$$\begin{aligned} \int_{CV} \rho \left(\nabla \boldsymbol{u} \cdot \boldsymbol{u}\right) \cdot \delta \boldsymbol{u} \ dV &= \int_{CV} \frac{1}{2} \rho \nabla \cdot \left[\left(\boldsymbol{u} \cdot \boldsymbol{u}\right) \ \delta \boldsymbol{u}\right] \ dV \\ &= \int_{CS} \frac{1}{2} \rho \left(\boldsymbol{u} \cdot \boldsymbol{u}\right) \left(\delta \boldsymbol{u} \cdot \boldsymbol{n}\right) \ dA, \end{aligned} \tag{3.95}$$

where we used Gauss' theorem.

Since the velocity of the flow at the control surfaces are assumed to be known, the forcing function at the control surfaces was considered to be invariant with respect to virtual velocity. Similarly, the non-commuting part of the rate of the kinetic energy, as obtained in Equation 3.95, only represents itself on the control surfaces. Thus, the term $\frac{1}{2}\rho \left(\boldsymbol{u} \cdot \boldsymbol{u}\right)$ in the integrand of Equation 3.95 is invariant with respect to $\delta \boldsymbol{u}$.

Therefore,

$$\begin{aligned}\int_{CV} \rho\left(\nabla \boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot\delta\boldsymbol{u}\; dV &= \int_{CS}\frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\left(\delta\boldsymbol{u}\cdot\boldsymbol{n}\right)\; dA \\ &= \delta\int_{CS}\frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)\; dA. \end{aligned} \tag{3.96}$$

When using Jourdain's variational principle, it is important to recall that Jourdain's virtual displacement does not correspond to any displacement field. Therefore, when the velocity is known, the virtual velocity becomes the actual velocity and not zero.

Finally, by substituting Equation 3.96 into Equation 3.87, the right-hand side of Equation 3.63 is obtained as

$$\begin{aligned}\delta\phi = \delta\int_{CS} -p\;\boldsymbol{u}\cdot\boldsymbol{n}\; dA - \delta\int_{CV}\frac{1}{2}\mu\; tr\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\nabla\boldsymbol{u}\right] dV \\ + \delta\int_{CV}\mu\nabla\cdot\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\boldsymbol{u}\right]\; dV - \delta\int_{CS}\frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)\; dA. \end{aligned} \tag{3.97}$$

Having obtained the scalar potential functions associated with the both right-hand and left-hand sides of Equation 3.63, we combine them and obtain the energy rate equation in the following section.

3.6.3 Extended JP in terms of energy

Substituting Equations 3.64 and 3.97 into Equation 3.63, the energy rate equation for a general control volume of Newtonian incompressible viscous fluids is obtained

as

$$\delta \int_{CV} \frac{D}{Dt}\left(\frac{1}{2}\rho \boldsymbol{u}\cdot\boldsymbol{u}\right) dV = \delta \int_{CS} -p\ \boldsymbol{u}\cdot\boldsymbol{n} dA - \delta \int_{CV} \frac{1}{2}\mu\ tr\left[\left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}\right)\cdot\nabla \boldsymbol{u}\right] dV + \delta \int_{CV} \mu\nabla\cdot\left[\left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}\right)\cdot\boldsymbol{u}\right] dV - \delta \int_{CS} \frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\left(\boldsymbol{u}\cdot\boldsymbol{n}\right) dA. \quad (3.98)$$

Alternatively, substituting Equation 3.67 instead of Equation 3.64 yields

$$\delta\left[\frac{D}{Dt}\int_{CV}\frac{1}{2}\rho\ \boldsymbol{u}\cdot\boldsymbol{u} dV + \int_{CS}\left(\frac{1}{2}\rho\ \boldsymbol{u}\cdot\boldsymbol{u}\ \right)\left(\boldsymbol{u}-\boldsymbol{v}_{CS}\right)\cdot\boldsymbol{n} dA\right] = \overbrace{\delta \int_{CS} -p\ \boldsymbol{u}\cdot\boldsymbol{n} dA + \delta \int_{CV} \mu\nabla\cdot\left[\left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}\right)\cdot\boldsymbol{u}\right] dV}^{\text{power due to external loads}} - \overbrace{\delta \int_{CV} \frac{1}{2}\mu\ tr\left[\left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}\right)\cdot\nabla \boldsymbol{u}\right] dV}^{\text{power loss due to viscous dissipation}} - \overbrace{\delta \int_{CS} \frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\left(\boldsymbol{u}\cdot\boldsymbol{n}\right) dA.}^{\text{non-commuting term}} \quad (3.99)$$

Equations 3.98 and 3.99 can both be used in modeling fluid dynamic systems and FSI, as they are essentially similar equations. Their advantages over one another mainly rely on the choice of the control volume and how the boundary conditions are implemented.

As mentioned earlier, in order to obtain the energy rate equation corresponding to a variational energy rate equation, Rayleigh's dissipation function must be multiplied by a factor of two. Therefore, the energy rate equation corresponding to Equation

3.99 is

$$\frac{D}{Dt}\int_{CV}\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}dV+\int_{CS}\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}_{CS})\cdot\boldsymbol{n}dA$$
$$=\overbrace{\int_{CS}-p\boldsymbol{u}\cdot\boldsymbol{n}dA+\int_{CV}\mu\nabla\cdot\left[\left(\nabla\boldsymbol{u}+\nabla^{T}\boldsymbol{u}\right)\cdot\boldsymbol{u}\right]dV}^{\text{power due to external loads}}$$
$$-\overbrace{\int_{CV}\mu\ tr\left[\left(\nabla\boldsymbol{u}+\nabla^{T}\boldsymbol{u}\right)\cdot\nabla\boldsymbol{u}\right]dV}^{\text{power loss due to viscous dissipation}}-\overbrace{\int_{CS}\frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)dA.}^{\text{non-commuting term}}\quad(3.100)$$

Equations 3.98 and 3.99 are the rate of energy equations. However, for FSI systems, the boundary conditions on the solid surfaces cannot be easily applied to these equations. The following section will show that the viscous terms can be modified further to become more applicable in the FSI and in the VIV cases.

3.7 An Expanded Form of the Energy Rate Equation

Our ultimate goal is the reduced-order modeling of FSI systems, but the energy rate equation obtained in the previous section (Equation 3.99) is not easily accessible, mainly, due to two reasons. First, it is not clear how the boundary conditions at the solid surface can be included. Second, the tensor term $\nabla^{T}\boldsymbol{u}$ provides some challenges, for example, when changing the variables of the EOM. Therefore, we continue with our derivation by modifying the viscous dissipation and external load terms in order to have them in a more meaningful form. Similar to the previous section, the viscous dissipative forces and external forces are considered separately for convenience, as follows.

3.7.1 Viscous dissipation function

In the following, we proceed through a series of manipulations in order to obtain an alternate expression for the viscous dissipation function.

Regarding the viscous dissipation function, $\frac{1}{2}\mu\, tr\left[\left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}\right) \cdot \nabla \boldsymbol{u}\right]$, from Equation 3.75 we have

$$\int_{CV} \frac{1}{2}\mu\, tr\left[\left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}\right) \cdot \nabla \boldsymbol{u}\right] dV = \int_{CV} \frac{1}{2}\mu \left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right) \frac{\partial u_k}{\partial x_i} dV. \tag{3.101}$$

Considering the integrand of the right-hand integral and setting $i = 1, 2, 3$ and $k = 1, 2, 3$, we obtain the expansion

$$\begin{aligned}
\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right) \frac{\partial u_k}{\partial x_i} &= \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_i} \\
&= \frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial x} + \frac{\partial u_z}{\partial x}\frac{\partial u_z}{\partial x} \\
&\quad + \frac{\partial u_x}{\partial y}\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial y}\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial y}\frac{\partial u_z}{\partial y} \\
&\quad + \frac{\partial u_x}{\partial z}\frac{\partial u_x}{\partial z} + \frac{\partial u_y}{\partial z}\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial z}\frac{\partial u_z}{\partial z} \\
&\quad + \frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial y}\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial z}\frac{\partial u_z}{\partial x} \\
&\quad + \frac{\partial u_y}{\partial x}\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial y}\frac{\partial u_y}{\partial y} + \frac{\partial u_y}{\partial z}\frac{\partial u_z}{\partial y} \\
&\quad + \frac{\partial u_z}{\partial x}\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial y}\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial z}\frac{\partial u_z}{\partial z},
\end{aligned} \tag{3.102}$$

and by rearranging, it becomes

$$\begin{aligned}\left(\frac{\partial u_k}{\partial x_i}+\frac{\partial u_i}{\partial x_k}\right)\frac{\partial u_k}{\partial x_i} &= 2\left(\frac{\partial u_x}{\partial x}\right)^2+2\left(\frac{\partial u_y}{\partial y}\right)^2+2\left(\frac{\partial u_z}{\partial z}\right)^2 \\ &+\left(\frac{\partial u_y}{\partial x}\right)^2+\left(\frac{\partial u_z}{\partial x}\right)^2+\left(\frac{\partial u_x}{\partial y}\right)^2 \\ &+\left(\frac{\partial u_z}{\partial y}\right)^2+\left(\frac{\partial u_x}{\partial z}\right)^2+\left(\frac{\partial u_y}{\partial z}\right)^2 \\ &+2\frac{\partial u_x}{\partial y}\frac{\partial u_y}{\partial x}+2\frac{\partial u_x}{\partial z}\frac{\partial u_z}{\partial x}+2\frac{\partial u_z}{\partial y}\frac{\partial u_y}{\partial z}. \end{aligned} \tag{3.103}$$

Following the procedure by Lamb for an incompressible viscid fluid [35, Article 229], the incompressibility condition implies that

$$\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}=0, \tag{3.104}$$

therefore,

$$2\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)^2=0, \tag{3.105}$$

as well. By expanding Equation 3.105 as

$$2\left(\frac{\partial u_x}{\partial x}\right)^2+2\left(\frac{\partial u_y}{\partial y}\right)^2+2\left(\frac{\partial u_z}{\partial z}\right)^2+4\frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial y}+4\frac{\partial u_y}{\partial y}\frac{\partial u_z}{\partial z}+4\frac{\partial u_x}{\partial x}\frac{\partial u_z}{\partial z}=0, \tag{3.106}$$

and subtracting it from Equation 3.103, we obtain

$$\begin{aligned}\left(\frac{\partial u_k}{\partial x_i}+\frac{\partial u_i}{\partial x_k}\right)\frac{\partial u_k}{\partial x_i} &= \left(\frac{\partial u_y}{\partial x}\right)^2+\left(\frac{\partial u_z}{\partial x}\right)^2+\left(\frac{\partial u_x}{\partial y}\right)^2 \\ &+\left(\frac{\partial u_z}{\partial y}\right)^2+\left(\frac{\partial u_x}{\partial z}\right)^2+\left(\frac{\partial u_y}{\partial z}\right)^2 \\ &+2\frac{\partial u_x}{\partial y}\frac{\partial u_y}{\partial x}+2\frac{\partial u_x}{\partial z}\frac{\partial u_z}{\partial x}+2\frac{\partial u_z}{\partial y}\frac{\partial u_y}{\partial z} \\ &-4\frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial y}-4\frac{\partial u_y}{\partial y}\frac{\partial u_z}{\partial z}-4\frac{\partial u_x}{\partial x}\frac{\partial u_z}{\partial z}. \end{aligned} \tag{3.107}$$

Next, we consider the function $(\nabla \times \boldsymbol{u}) \cdot (\nabla \times \boldsymbol{u})$ and obtain it in index notation as

$$\begin{aligned}
(\nabla \times \boldsymbol{u}) \cdot (\nabla \times \boldsymbol{u}) &= \left(\mathbf{e}_i \frac{\partial}{\partial x_i} \times u_j \mathbf{e}_j\right) \cdot \left(\mathbf{e}_k \frac{\partial}{\partial x_k} \times u_m \mathbf{e}_m\right) \\
&= \frac{\partial u_j}{\partial x_i} \frac{\partial u_m}{\partial x_k} e^{ijp} e_{kms} \delta^p_s \\
&= \frac{\partial u_j}{\partial x_i} \frac{\partial u_m}{\partial x_k} e^{ijp} e_{kmp} \\
&= \frac{\partial u_j}{\partial x_i} \frac{\partial u_m}{\partial x_k} \left(\delta^i_k \delta^j_m - \delta^i_m \delta^j_k\right) \\
&= \left(\frac{\partial u_j}{\partial x_i}\right)^2 - \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j}, \qquad (3.108)
\end{aligned}$$

where e is the permutation symbol, and δ^i_j is the Kronecker delta. Also, in component format, we have

$$\begin{aligned}
(\nabla \times \boldsymbol{u}) \cdot (\nabla \times \boldsymbol{u}) &= \left(\frac{\partial u_y}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial x}\right)^2 + \left(\frac{\partial u_x}{\partial y}\right)^2 \\
&\quad + \left(\frac{\partial u_z}{\partial y}\right)^2 + \left(\frac{\partial u_x}{\partial z}\right)^2 + \left(\frac{\partial u_y}{\partial z}\right)^2 \\
&\quad - 2\frac{\partial u_x}{\partial y}\frac{\partial u_y}{\partial x} - 2\frac{\partial u_x}{\partial z}\frac{\partial u_z}{\partial x} - 2\frac{\partial u_z}{\partial y}\frac{\partial u_y}{\partial z}. \qquad (3.109)
\end{aligned}$$

Substitution of Equation 3.109 into Equation 3.107 yields

$$\begin{aligned}
\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right)\frac{\partial u_k}{\partial x_i} &= (\nabla \times \boldsymbol{u}) \cdot (\nabla \times \boldsymbol{u}) + 4\frac{\partial u_x}{\partial y}\frac{\partial u_y}{\partial x} \\
&\quad + 4\frac{\partial u_x}{\partial z}\frac{\partial u_z}{\partial x} + 4\frac{\partial u_z}{\partial y}\frac{\partial u_y}{\partial z} - 4\frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial y} \\
&\quad - 4\frac{\partial u_y}{\partial y}\frac{\partial u_z}{\partial z} - 4\frac{\partial u_x}{\partial x}\frac{\partial u_z}{\partial z}. \qquad (3.110)
\end{aligned}$$

Also, the function $\nabla \cdot [\boldsymbol{u} \times (\nabla \times \boldsymbol{u})]$ in index notation is

$$\begin{aligned}
\nabla \cdot [\boldsymbol{u} \times (\nabla \times \boldsymbol{u})] &= \mathbf{e}_i \frac{\partial}{\partial x_i} \cdot \left[u_j \mathbf{e}_j \times \left(\mathbf{e}_k \frac{\partial}{\partial x_k} \times u_m \mathbf{e}_m \right) \right] \\
&= \mathbf{e}_i \frac{\partial}{\partial x_i} \cdot \left[u_j \mathbf{e}_j \times \left(\mathbf{e}_k \frac{\partial u_m}{\partial x_k} e_{kms} \mathbf{e}_s \right) \right] \\
&= \mathbf{e}_i \frac{\partial}{\partial x_i} \cdot \left(u_j \frac{\partial u_m}{\partial x_k} e^{kms} e_{jsp} \mathbf{e}_p \right) \\
&= \frac{\partial}{\partial x_i} \left(u_j \frac{\partial u_m}{\partial x_k} e^{kms} e_{jsi} \right) \\
&= \frac{\partial}{\partial x_i} \left(u_j \frac{\partial u_m}{\partial x_k} \delta_{ij}^{km} \right) \\
&= \frac{\partial}{\partial x_i} \left(u_j \frac{\partial u_m}{\partial x_k} \right) \delta_i^k \delta_j^m - \frac{\partial}{\partial x_i} \left(u_j \frac{\partial u_m}{\partial x_k} \right) \delta_j^k \delta_i^m \\
&= \frac{\partial}{\partial x_i} \left(u_j \frac{\partial u_j}{\partial x_i} \right) - \frac{\partial}{\partial x_i} \left(u_j \frac{\partial u_i}{\partial x_j} \right) \\
&= \frac{\partial u_j}{\partial x_i} \frac{\partial u_j}{\partial x_i} + u_j \frac{\partial^2 u_j}{\partial x_i \partial x_i} - \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial^2 u_j}{\partial x_i \partial x_j} \\
&\quad \left(\frac{\partial u_j}{\partial x_i} \right)^2 + u_j \frac{\partial^2 u_j}{\partial x_i \partial x_i} - \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j}. \qquad (3.111)
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\nabla^2 (\boldsymbol{u} \cdot \boldsymbol{u}) &= (\nabla \cdot \nabla) (\boldsymbol{u} \cdot \boldsymbol{u}) \\
&= \left(\mathbf{e}_i \frac{\partial}{\partial x_i} \cdot \mathbf{e}_k \frac{\partial}{\partial x_k} \right) (u_j \mathbf{e}_j \cdot u_m \mathbf{e}_m) \\
&= \left(\frac{\partial^2}{\partial x_i \partial x_k} \delta_k^i \right) \left(u_j u_m \delta_m^j \right) \\
&= \frac{\partial^2}{\partial x_i \partial x_i} (u_j u_j) \\
&= 2 u_j \frac{\partial^2 u_j}{\partial x_i \partial x_i}. \qquad (3.112)
\end{aligned}$$

Multiplying Equation 3.111 by two and subtracting it from Equation 3.112, we find

$$\nabla^2 (\boldsymbol{u} \cdot \boldsymbol{u}) - 2 \, \nabla \cdot [\boldsymbol{u} \times (\nabla \times \boldsymbol{u})] = 2 \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} - 2 \frac{\partial u_j}{\partial x_i} \frac{\partial u_j}{\partial x_i}, \qquad (3.113)$$

and then substituting the result (Equation 3.113) into Equation 3.110, we obtain

$$\left(\frac{\partial u_k}{\partial x_i}+\frac{\partial u_i}{\partial x_k}\right)\frac{\partial u_k}{\partial x_i}=(\nabla\times\boldsymbol{u})\cdot(\nabla\times\boldsymbol{u})+\nabla^2(\boldsymbol{u}\cdot\boldsymbol{u})-2\,\nabla\cdot[\boldsymbol{u}\times(\nabla\times\boldsymbol{u})]\,. \quad (3.114)$$

Therefore, by substituting Equation 3.114 into Equation 3.99, the viscous dissipation function can be alternatively expressed as

$$\begin{aligned}&\int_{CV}\frac{1}{2}\mu\ tr\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\nabla\boldsymbol{u}\right]dV\\&\quad=\delta\int_{CV}\frac{1}{2}\mu\left\{(\nabla\times\boldsymbol{u})\cdot(\nabla\times\boldsymbol{u})+\nabla^2(\boldsymbol{u}\cdot\boldsymbol{u})-2\,\nabla\cdot[\boldsymbol{u}\times(\nabla\times\boldsymbol{u})]\right\}dV. \quad (3.115)\end{aligned}$$

Next, we consider the viscous forces acting on the control surfaces as external forces.

3.7.2 Power due to external viscous forces

Regarding the viscous forces external to the control volume, we found its scalar potential function (Equation 3.86) to be

$$\delta\int_{CV}\mu\nabla\cdot\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\ \boldsymbol{u}\right]\ dV. \quad (3.116)$$

Since $\nabla\cdot$ can be eliminated by using the divergence theorem, we consider the function $\left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}\right) \cdot \boldsymbol{u}$ and write it in component notation,

$$\begin{aligned}
\left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}\right) \cdot \boldsymbol{u} &= \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) u_j \mathbf{e}_i \\
&= \left[\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial x}\right) u_x + \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) u_y + \left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) u_z\right] \mathbf{i} \\
&\quad + \left[\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) u_x + \left(\frac{\partial u_y}{\partial y} + \frac{\partial u_y}{\partial y}\right) u_y \right. \\
&\quad \left. + \left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) u_z\right] \mathbf{j} + \left[\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) u_x \right. \\
&\quad \left. + \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) u_y + \left(\frac{\partial u_z}{\partial z} + \frac{\partial u_z}{\partial z}\right) u_z\right] \mathbf{k}. \qquad (3.117)
\end{aligned}$$

In order to modify Equation 3.117, we consider the two functions $\nabla\left(\boldsymbol{u} \cdot \boldsymbol{u}\right)$ and $\boldsymbol{u} \times \left(\nabla \times \boldsymbol{u}\right)$, and write them in component notation as well. For $\nabla\left(\boldsymbol{u} \cdot \boldsymbol{u}\right)$, we have

$$\begin{aligned}
\nabla\left(\boldsymbol{u} \cdot \boldsymbol{u}\right) &= \mathbf{e}_i \frac{\partial}{\partial x_i}\left(u_j \; u_j\right) = 2\frac{\partial u_j}{\partial x_i} u_j \\
&= 2\left(\frac{\partial u_x}{\partial x} u_x + \frac{\partial u_y}{\partial x} u_y + \frac{\partial u_z}{\partial x} u_z\right) \mathbf{i} \\
&\quad + 2\left(\frac{\partial u_x}{\partial y} u_x + \frac{\partial u_y}{\partial y} u_y + \frac{\partial u_z}{\partial y} u_z\right) \mathbf{j} \\
&\quad + 2\left(\frac{\partial u_x}{\partial z} u_x + \frac{\partial u_y}{\partial z} u_y + \frac{\partial u_z}{\partial z} u_z\right) \mathbf{k}, \qquad (3.118)
\end{aligned}$$

and for $\boldsymbol{u} \times \left(\nabla \times \boldsymbol{u}\right)$ (for index derivation, *see* Equation 3.111, without considering "$\nabla\cdot$"),

$$\begin{aligned}
\boldsymbol{u} \times \left(\nabla \times \boldsymbol{u}\right) &= u_i \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \mathbf{e}_j \\
&= \left[u_y \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) + u_z \left(\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z}\right)\right] \mathbf{i} \\
&\quad + \left[u_x \left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right) + u_z \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\right] \mathbf{j} \\
&\quad + \left[u_x \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) + u_y \left(\frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y}\right)\right] \mathbf{k}. \qquad (3.119)
\end{aligned}$$

Therefore, from Equations 3.117, 3.118 and 3.119, it can be seen that

$$\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)=\left(\nabla\boldsymbol{u}+\nabla^{T}\boldsymbol{u}\right)\cdot\boldsymbol{u}. \tag{3.120}$$

From Equations 3.116 and 3.120, the power due to external viscous forces can be alternatively expressed as

$$\delta\int_{CV}\mu\nabla\cdot\left[\left(\nabla\boldsymbol{u}+\nabla^{T}\boldsymbol{u}\right)\cdot\boldsymbol{u}\right]dV=\delta\int_{CV}\mu\nabla\cdot\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]dV. \tag{3.121}$$

Thus far, we have obtained extended expressions for the power lost due to viscous dissipation of energy and for the power of the external viscous forces. Therefore, an alternative form of the rate of energy equation can be obtained by using those terms instead, as follows.

3.7.3 An Extended Form of the Rate of Energy Equation

Substituting Equations 3.115 and 3.121 into Equation 3.98, we obtain the energy rate equation as derived from the extended Jourdain's principle,

$$\begin{aligned}&\delta\int_{CV}\frac{D}{Dt}\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)dV\\&\qquad=\overbrace{\delta\int_{CS}-p\ \boldsymbol{u}\cdot\boldsymbol{n}dA+\ \delta\int_{CV}\left\{\mu\nabla\cdot\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\right.}^{\text{external loading}}\\&\qquad\overbrace{\left.-\frac{1}{2}\mu\left\{\left(\nabla\times\boldsymbol{u}\right)\cdot\left(\nabla\times\boldsymbol{u}\right)+\nabla^{2}\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-2\ \nabla\cdot\left[\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\right\}\right\}dV}^{\text{viscous dissipation}}\\&\qquad\qquad-\overbrace{\delta\int_{CS}\frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)dA}^{\text{non-commuting term}},\end{aligned} \tag{3.122}$$

where the terms of the form $\nabla \cdot [\boldsymbol{u} \times (\nabla \times \boldsymbol{u})]$ will cancel each other out, and the terms $\nabla \cdot [\nabla (\boldsymbol{u} \cdot \boldsymbol{u})]$ and $\nabla^2 (\boldsymbol{u} \cdot \boldsymbol{u})$, which are identical, can be combined. However, since these terms represent different concepts, they are kept separate. The left-hand side of Equation 3.122 is the summation of the rate of change of kinetic energy ($\sum \frac{DT}{Dt}$) for all points in the domain. For reduced-order modeling, the rate of change of kinetic energy of the control volume ($\frac{d}{dt} \sum T$) might be of more interest. Also, for a control volume, in the absence of body forces, the external loads can only be applied at the control surfaces. Therefore, by replacing the left-hand side integral in Equation 3.122 by its equivalent Equation 3.67 and applying Gauss' divergence theorem (Equation 1.27) to the control volume integral on the right-hand side, Equation 3.122 becomes

$$\delta \left\{ \frac{D}{Dt} \overbrace{\int_{CV} \frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} dV}^{\text{kinetic energy}} + \overbrace{\int_{CS} \left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) (\boldsymbol{u} - \boldsymbol{v}_{CS}) \cdot \boldsymbol{n} dA}^{\text{flux of kinetic energy}} \right\}$$

$$= \delta \left\{ \int_{CS} \overbrace{\{ - p \ \boldsymbol{u} \cdot \boldsymbol{n} \ + \mu [\nabla (\boldsymbol{u} \cdot \boldsymbol{u}) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u})] \cdot \boldsymbol{n}}^{\text{external loads}} \right.$$

$$\left. \overbrace{- \frac{1}{2} \mu [\nabla (\boldsymbol{u} \cdot \boldsymbol{u}) - 2 \ \boldsymbol{u} \times (\nabla \times \boldsymbol{u})] \cdot \boldsymbol{n}}^{\text{viscous dissipation}} \right\} dA - \frac{1}{2} \int_{CV} \mu (\nabla \times \boldsymbol{u}) \cdot (\nabla \times \boldsymbol{u}) \, dV$$

$$\left. - \overbrace{\int_{CS} \frac{1}{2} \rho (\boldsymbol{u} \cdot \boldsymbol{u}) (\boldsymbol{u} \cdot \boldsymbol{n}) \, dA}^{\text{non-commuting term}} \right\}. \quad (3.123)$$

Similar to the previous section, it is important to have in mind that the energy rate equation corresponding to Equation 3.123 is obtained by multiplying Rayleigh's

dissipation function by a factor of two, that is,

$$\overbrace{\frac{D}{Dt}\int_{CV}\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}dV}^{\text{kinetic energy}}+\overbrace{\int_{CS}\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}_{CS})\cdot\boldsymbol{n}dA}^{\text{flux of kinetic energy}}$$
$$=\int_{CS}\overbrace{\{-\,p\;\boldsymbol{u}\cdot\boldsymbol{n}\;+\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}}^{\text{external loads}}$$
$$\overbrace{-\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-2\;\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}\}\,dA-\int_{CV}\mu\left(\nabla\times\boldsymbol{u}\right)\cdot\left(\nabla\times\boldsymbol{u}\right)dV}^{\text{viscous dissipation}}$$
$$-\overbrace{\int_{CS}\frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)dA.}^{\text{non-commuting term}}\quad(3.124)$$

In Chapters 4 and 5 we will use Equations 3.122, 3.123 and 3.124 for FSI problems.

3.8 Conclusions

In the absence of known particle displacements at two instances of time, Hamilton's principle, and consequently, Lagrange's equations do not result in conservation of energy directly. As suggested by Lanczos [29] and shown by Benaroya and Wei [1] for a control volume containing a Newtonian incompressible viscous fluid, an energy rate equation can be approximated by assuming that the virtual displacements and actual displacements coincide for all time. This is reviewed in detail in the following chapter.

In this chapter, we considered that Hamilton's principle, Lagrange's equation and Jourdain's principle are all derivable from d'Alembert's principle. We used this common feature and showed that the energy rate equation can be obtained from Jourdain's principle when the system is described in the Lagrangian frame of reference. On the contrary, for a system described in the Eulerian reference frame, it was shown

that the commutation rule does not hold. However, this difficulty was overcome by extracting the non-commuting terms from the acceleration.

Moreover, we used the idea of the Rayleigh dissipation function to obtain the energy terms. The energy dissipation potential due to viscosity was obtained as

$$\phi_\mu = \frac{1}{2}\int_{CV} -\mu\left(\nabla\times\boldsymbol{u}\right)\cdot\left(\nabla\times\boldsymbol{u}\right)dV + \frac{1}{2}\int_{CS} -\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right) - 2\ \boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}\ dA, \tag{3.125}$$

which is similar to the dissipation function found by Lamb [35, Article 229]. However, it is presented here in vector form. The signs in our dissipation function differs from Lamb's equation because he considered the vector $\boldsymbol{n}$ to be the inward normal vector, whereas we take it to be the outward normal.

As evident from our derivations, the classical assumption that the rotating fluid does not effect the shear stress is identical to the Rayleigh's assumption for obtaining the dissipation function. Thus, the energy equation obtained is compatible with the Navier-Stokes equations.

During our derivations, we kept the viscous dissipative terms separate from the viscous forces external to the control volume. However, an interesting result is obtained if we combined those terms in the variational energy rate equation and the energy rate equation as follows.

We consider Equation 3.123. Summation of the viscous terms results in

$$\begin{aligned}\delta\left\{\frac{D}{Dt}\int_{CV}\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}dV + \int_{CS}\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)\left(\boldsymbol{u}-\boldsymbol{v}_{CS}\right)\cdot\boldsymbol{n}dA\right\}&\\ = \delta\left\{\int_{CS}\left[-\ p\ \boldsymbol{u}\cdot\boldsymbol{n}\ + \frac{1}{2}\mu\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot\boldsymbol{n}\right]dA\right.&\\ \left. - \frac{1}{2}\int_{CV}\mu\left(\nabla\times\boldsymbol{u}\right)\cdot\left(\nabla\times\boldsymbol{u}\right)dV - \int_{CS}\frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)dA\right\}.&\end{aligned} \tag{3.126}$$

Similarly, summation of viscous terms in Equation 3.124 yields

$$\frac{D}{Dt}\int_{CV}\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}dV+\int_{CS}\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}_{CS})\cdot\boldsymbol{n}dA$$
$$=\int_{CS}\left[-\,p\;\boldsymbol{u}\cdot\boldsymbol{n}\;+\mu\boldsymbol{u}\times(\nabla\times\boldsymbol{u})\cdot\boldsymbol{n}\;\right]dA$$
$$-\int_{CV}\mu\,(\nabla\times\boldsymbol{u})\cdot(\nabla\times\boldsymbol{u})\,dV-\int_{CS}\frac{1}{2}\rho\,(\boldsymbol{u}\cdot\boldsymbol{u})\,(\boldsymbol{u}\cdot\boldsymbol{n})\,dA. \quad (3.127)$$

Comparing Equations 3.126 and 3.127, we notice a difference regarding the viscous terms on their second lines. For many FSI problems where the fluid viscosity is negligible, the *boundary layer approximation* method has been widely used in modeling the flow [28]. In the boundary layer method, the viscosity is neglected everywhere in the fluid domain except in the vicinity of the structure. Similarly, if we neglect the dissipation inside the control volume, the only terms remaining are $\frac{1}{2}\mu\nabla\,(\boldsymbol{u}\cdot\boldsymbol{u})\cdot\boldsymbol{n}$ in Equation 3.126 and $\mu\boldsymbol{u}\times(\nabla\times\boldsymbol{u})\cdot\boldsymbol{n}$ in Equation 3.127. This difference makes the variational energy rate equation preferable over the energy rate equation when it comes to reduced-order modeling, since the kinetic energy term can readily be obtained as $\frac{1}{2}\mu\nabla\,(\boldsymbol{u}\cdot\boldsymbol{u})\cdot\boldsymbol{n}=\frac{\mu}{\rho}\nabla\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot\boldsymbol{n}$; whereas the term $\mu\boldsymbol{u}\times(\nabla\times\boldsymbol{u})\cdot\boldsymbol{n}$ in Equation 3.127 makes the manipulations challenging. This fact and observation will be utilized in our reduced-order modeling of the model problem in Chapter 5.

Detailed and comparative discussions on the energy equation are put off until we obtain it for fluid-structure interaction systems in the following chapter, where the energy equation is derived for FSI with the addition of an internal structure/solid.

Chapter 4

Modeling Fluid-Structure Interaction: Single Governing Equation of Motion

Modeling the dynamics of a system of fluid particles, alone, is a very challenging problem due to the nonlinear and chaotic behavior of fluids. Considering fluid-structure interaction (FSI) problems, the difficulties increase since the nature of the FSI is not fully understood. Consequently, there has been no compelling theoretical model that explains the nature of fluid-structure interaction. Thus, the choice of the boundary conditions (BCs) on the solid surfaces has remained to some extent just assumptions. Generally, having obtained a theoretical solution by using a set of assumed BCs, the applicability of the BCs are examined by a comparison with the experimental results. Therefore, we begin this chapter with a discussion of boundary conditions.

Regarding fluid-structure interaction and specifically vortex induced vibration (VIV), there exist many empirical models that only include single forcing functions. In Chapter 1, we referred to these models as *experimental force coefficient models*. While these *ad hoc* models have had relative success in capturing the dynamics of solid structures, they fail to provide insights into the dynamic behavior of the fluid systems. Moreover, each structural configuration requires its own model because these empirical models use assumed forcing functions and curve fittings to a set of data points obtained experimentally.

Unlike empirical models, there have been very few efforts to model the VIV problem using a first principles approach. One of the earliest attempts is the work of

Benaroya and Wei [1] where they extended Hamilton's principle for the VIV problem. In this chapter, we use their methodology to obtain a single governing equation for the VIV problem from the extended Jourdain's principle and the energy equation obtained in previous chapters. Consequently, we compare our model with the model of McIver for FSI, and the model of Benaroya and Wei for VIV.

Benaroya and Wei neglected the viscous dissipation of energy in their extension of Hamilton's principle since their experiments found the viscous forces to be of three orders of magnitude smaller than the kinetic energy flux. They showed that their result is a statement of the first law of thermodynamics where heat transfer and dissipation has been omitted. In Section 4.8, we first use their derivation and include the viscous dissipation function (for generality) to obtain the classical energy equation in integral form for a Newtonian incompressible viscous fluid, and then use it to explain the limitation of the classical equation. Subsequently, we compare our equation with the classical one.

In Section 4.10, we explain how the experimental data can be introduced into our energy rate equation and discuss the difficulties we anticipate. In Section 4.11, an alternative energy equation is proposed for two situations where the use of the energy equation might face serious challenges. The chapter is summarized in Section 4.12.

4.1 Boundary Conditions at the Surface of Solids

From the time of Newton up to the early 20th century, one topic of intensive discussions has been the boundary conditions on solid surfaces that are interacting with viscous fluid particles. Many great scientists and engineers, including Navier, Stokes and Prandtl, have considered the topic. For a long time, there was no general agreement on the type of the required BCs with an exception in the case of a very slow motion of viscous fluids where the no-slip condition was accepted [36].

During the past century, many experiments were conducted and compared to the theoretical and numerical results using the no-slip condition, resulting in an almost unified agreement on the no-slip condition [37]. Yet, the no-slip condition remains an assumption since it cannot be proven using a first principles approach. The general understanding is that some intermolecular interactions result in zero relative velocity, $\boldsymbol{u}_r$, of fluid particles at a solid surface [28, pp. 295-296], that is,

$$\boldsymbol{u}_r = \boldsymbol{u} - \boldsymbol{v} = \boldsymbol{0}, \tag{4.1}$$

where $\boldsymbol{u}$ is the velocity of the fluid at the surface of a solid moving with the velocity $\boldsymbol{v}$.

Alternatively, another definition of the no-slip condition is that the tangential components of the relative velocity must vanish [38], that is,

$$\boldsymbol{u}_r \cdot \boldsymbol{t} = (\boldsymbol{u} - \boldsymbol{v}) \cdot \boldsymbol{t} = 0, \tag{4.2}$$

or,

$$\boldsymbol{u}_r - (\boldsymbol{u}_r \cdot \boldsymbol{n})\, \boldsymbol{n} = 0, \tag{4.3}$$

where $\boldsymbol{t}$ is the unit tangent vector to the surface of the solid. This condition must be considered together with the no penetration condition for the normal direction, that is, no fluid particles can penetrate into the solid.

While the no-slip condition seems to be compatible with many experimental observations on macroscopic scales, some experiments have shown that it is violated at microscopic levels for Newtonian fluids. The wetting property of the surface, the velocity of the flow, surface roughness and gas bubbles are among the factors shown to affect the no-slip boundary condition [39], [40], [41], and [42]. More discussion can be found in [37, p. 1222].

We continue our derivation by considering the no-slip boundary condition, as it is widely accepted in the literature. However, before starting our derivations for the *model problem*, we describe the control volume of interest in the following section.

4.2 Control Volume Definition

Generally, care must be exercised in the selection of the control volume, as it greatly affects the applicability of the analytical formulations. Moreover, if the analytical formulations are to be coupled with experimental results, choices of the control volume may be restricted due to experimental limitations, for example, the measurement equipment used and wind/water tunnel dimensions.

In Chapters 2 and 3, results were obtained for a general control volume, that is, a control volume that can deform and move. Therefore, the derived variational formulations did not face any restrictions in the selection of the control volume. Regarding the FSI and VIV problems, the methodology used in this chapter imposes a limitation in selecting the CV in that some portions of the control surface must be comprised of solid surfaces. Since no fluid can cross the solid surface, these control surfaces are referred to as closed control surfaces, CS_C. The other surfaces, where fluid can cross, are called open control surfaces, CS_O.

Since we are interested in reduced-order modeling of the *model problem*, we start by defining a control volume for the two types of *model problem* shown in Figures 1.2 and 1.1. While our efforts in this chapter are to derive the EOM for a single DOF rigid body, the same method can be used for modeling a general deformable solid as well.

Consider the control volume shown in Figure 4.1, where the control surface is comprised of three section, CS_O, CS_C, and the surfaces set apart by the distance α. If $\alpha \rightarrow 0$ and if there exist no singularities between the gap, then the surfaces set

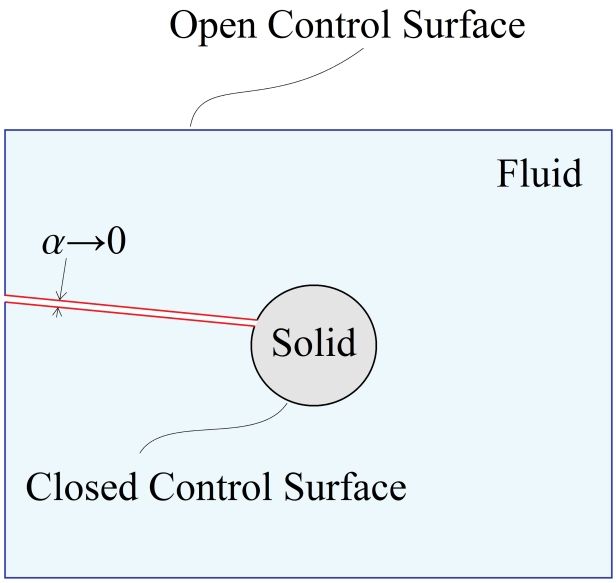

Figure 4.1: Control volume of interest.

apart by the distance α can be neglected. Also, we assume that both the rectangular and circular control surfaces move rigidly and independently of each other as shown in Figure 4.2.

The shape of the open portion of the control volume is selected to be rectangular for simplicity. Moreover, the rectangular control surface can represent solid surfaces, for instance, they can be chosen to coincide with a water tunnel's walls.

So far, we have not considered any solid body in our derivations. In order to include a solid structure, we start by considering the original statement of Jourdain's principle for a system volume.

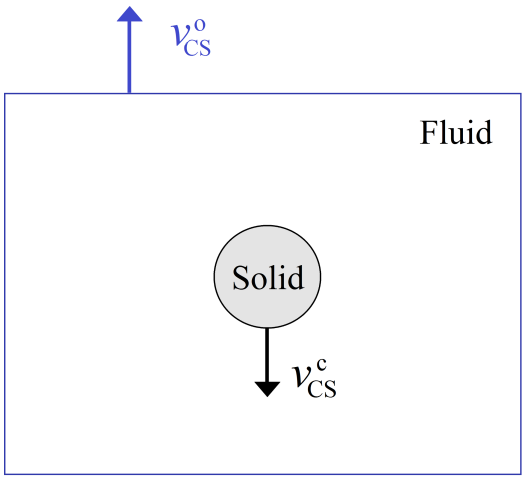

Figure 4.2: Selected control volume; rectangular and circular control surfaces can move independently.

4.3 Extended Jourdain's Principle for FSI Systems

Jourdain's variational principle for a set of N fluid and solid particles is (Equation 2.2)

$$\sum_{i=1}^{N} (m_i \ \ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i) \cdot \delta\dot{\boldsymbol{r}}_i = 0, \qquad \text{where} \qquad \delta\boldsymbol{r}_i = 0 \qquad \text{and} \qquad \delta t = 0.$$

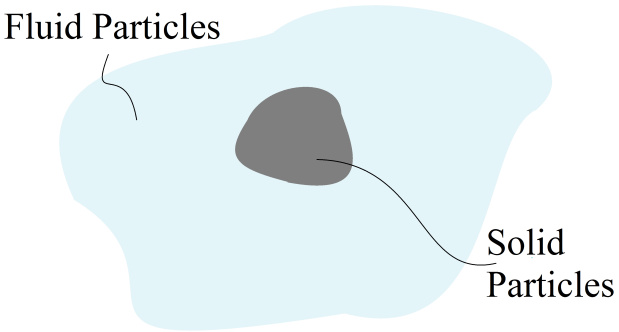

Figure 4.3: A continuous set of fluid and solid particles.

Consider a continuous system of fluid and solid particles as shown in Figure 4.3,

and assume that the system consists of M fluid particles where $M < N$. Jourdain's principle then becomes

$$\sum_{i=1}^{M} (m_i \ \ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i) \cdot \delta\dot{\boldsymbol{r}}_i + \sum_{j=M+1}^{N} (m_j \ \ddot{\boldsymbol{r}}_j - \boldsymbol{F}_j) \cdot \delta\dot{\boldsymbol{r}}_j = 0,$$

$$\text{where} \qquad \delta\boldsymbol{r}_i = 0, \qquad \delta\boldsymbol{r}_j = 0 \qquad \text{and} \qquad \delta t = 0. \tag{4.4}$$

In Section 2.4, it was shown that Jourdain's principle can be alternatively expressed as Equation 2.16. Similarly, Equation 4.4 can be written as

$$\begin{cases} \dfrac{d}{dt}\left[\displaystyle\sum_{i=1}^{M} (m_i \ \ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i) \cdot \delta\boldsymbol{r}_i + \sum_{j=M+1}^{N} (m_j \ \ddot{\boldsymbol{r}}_j - \boldsymbol{F}_j) \cdot \delta\boldsymbol{r}_j\right] = 0 \\ \delta t = 0 \\ \delta\boldsymbol{r}_i = 0 \\ \delta\boldsymbol{r}_j = 0 \\ \dfrac{d}{dt}(\delta\boldsymbol{r}_i) \neq 0 \\ \dfrac{d}{dt}(\delta\boldsymbol{r}_j) \neq 0. \end{cases} \tag{4.5}$$

If the set of particles shown in Figure 4.3 is continuous during some time interval,

then Equation 4.5 can be expressed in integral form as

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_{V_f}\left[\rho\frac{d\boldsymbol{u}\left(\boldsymbol{r},t\right)}{dt}-\boldsymbol{f}\left(\boldsymbol{r},t\right)\right]\cdot\delta\boldsymbol{\Lambda}\left(\boldsymbol{r},t\right)dV_f \\ +\dfrac{d}{dt}\displaystyle\int_{V_s}\left[\rho_s\frac{d\dot{\boldsymbol{r}}_s}{dt}-\boldsymbol{f}_s\left(\boldsymbol{r}_s\right)\right]\cdot\delta\boldsymbol{r}_s dV_s-\frac{d}{dt}\int_{\substack{\text{Solid}\\\text{Surface}}}\boldsymbol{f}_s^*\left(\boldsymbol{r}_s\right)\cdot\delta\boldsymbol{r}_s dA_s=0 \\ \delta t=0 \\ \delta\boldsymbol{\Lambda}\left(\boldsymbol{r},t\right)=0 \\ \delta\boldsymbol{r}_s=0 \\ \dfrac{d}{dt}\left[\delta\boldsymbol{\Lambda}\left(\boldsymbol{r},t\right)\right]\neq 0 \\ \dfrac{d}{dt}\left(\delta\boldsymbol{r}_s\right)\neq 0, \end{cases} \tag{4.6}$$

where V_f is the material volume of the fluid particles, V_s is the material volume of solid particles, ρ_s denotes the density of solid, $\boldsymbol{r}_s$ is the Lagrangian position of the solid particles, $\boldsymbol{f}_s\left(\boldsymbol{r}_s\right)$ denotes the force density (force per unit volume) in the solid domain at $\boldsymbol{r}_s$, and $\boldsymbol{f}_s^*\left(\boldsymbol{r}_s\right)$ is the force density of the fluid dynamic forces acting on the solid surface. Note that $\boldsymbol{f}_s\left(\boldsymbol{r}_s\right)$ does not include any force due to the fluid dynamics. Also, in deriving Equation 4.6, no constraints are included except those imposed by Jourdain's principle.

Regarding the integral over the fluid system in Equation 4.6, the manipulations

shown in Chapter 2 remain valid for a Newtonian incompressible viscous fluid. Therefore, the integral over the fluid system can be replaced using Equation 2.42 as

$$\begin{cases} \int_{CV(t)} \left(\rho \dfrac{D\boldsymbol{u}(\boldsymbol{x},t)}{Dt} - \boldsymbol{f}_b(\boldsymbol{x},t) + \nabla p(\boldsymbol{x},t) - \mu \nabla^2 \boldsymbol{u}(\boldsymbol{x},t) \right) \cdot \delta \boldsymbol{u}(\boldsymbol{x},t)\, dV(t) \\ \qquad + \dfrac{d}{dt} \int\limits_{V_s} \left[\rho_s \dfrac{d\dot{\boldsymbol{r}}_s}{dt} - \boldsymbol{f}_s(\boldsymbol{r}_s) \right] \cdot \delta \boldsymbol{r}_s dV_s - \dfrac{d}{dt} \int\limits_{\substack{\text{Solid} \\ \text{Surface}}} \boldsymbol{f}_s^*(\boldsymbol{r}_s) \cdot \delta \boldsymbol{r}_s dA_s = 0 \\ \delta t = 0 \\ \delta \boldsymbol{r}_s = 0 \\ \dfrac{d}{dt}(\delta \boldsymbol{r}_s) \neq 0. \end{cases} \tag{4.7}$$

In Chapter 3, the energy equation for a general control volume of fluid particles was obtained in Equations 3.122 and 3.123. Introducing Equation 3.123 into Equation 4.7 results in

$$\begin{cases} \delta \left\{ \dfrac{D}{Dt} \int\limits_{CV} \dfrac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} dV + \int\limits_{CS} \left(\dfrac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) (\boldsymbol{u} - \boldsymbol{v}_{CS}) \cdot \boldsymbol{n} dA \right. \\ \qquad \overbrace{- \int\limits_{CS} \{ - p\boldsymbol{u} \cdot \boldsymbol{n} + \mu [\nabla (\boldsymbol{u} \cdot \boldsymbol{u}) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u})] \cdot \boldsymbol{n}}^{\text{loading external to the fluid}} \\ \overbrace{- \dfrac{1}{2} \mu [\nabla (\boldsymbol{u} \cdot \boldsymbol{u}) - 2\, \boldsymbol{u} \times (\nabla \times \boldsymbol{u})] \cdot \boldsymbol{n} \Big\} dA + \dfrac{1}{2} \int\limits_{CV} \mu (\nabla \times \boldsymbol{u}) \cdot (\nabla \times \boldsymbol{u}) dV}^{\text{viscous dissipation}} \\ \qquad \left. \overbrace{+ \int\limits_{CS} \dfrac{1}{2} \rho (\boldsymbol{u} \cdot \boldsymbol{u}) (\boldsymbol{u} \cdot \boldsymbol{n}) dA}^{\text{non-commuting term}} \right\} \\ \qquad + \dfrac{d}{dt} \int\limits_{V_s} \left[\rho_s \dfrac{d\dot{\boldsymbol{r}}_s}{dt} - \boldsymbol{f}_s(\boldsymbol{r}_s) \right] \cdot \delta \boldsymbol{r}_s\, dV_s - \dfrac{d}{dt} \int\limits_{\substack{\text{Solid} \\ \text{Surface}}} \boldsymbol{f}_s^*(\boldsymbol{r}_s) \cdot \delta \boldsymbol{r}_s dA_s = 0 \\ \delta t = 0 \\ \delta \boldsymbol{r}_s = 0 \\ \dfrac{d}{dt}(\delta \boldsymbol{r}_s) \neq 0, \end{cases} \tag{4.8}$$

where body forces are neglected for fluid particles.

In deriving Equation 4.8 we have not considered the boundary conditions on the solid surface. Therefore, we continue our derivation by considering the no-slip condition.

In order to include the no-slip condition, we consider the second line of Equation 4.8 and divide it to integrals over the closed and the open portions of control surface,

$$\begin{aligned}\delta \int_{CS} \{-p\boldsymbol{u}\cdot\boldsymbol{n} + \mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}\}\, dA \\ = \delta \int_{CS_C} \{-p\boldsymbol{u}\cdot\boldsymbol{n} + \mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}\}\, dA \\ + \delta \int_{CS_O} \{-p\boldsymbol{u}\cdot\boldsymbol{n} + \mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}\}\, dA. \quad (4.9)\end{aligned}$$

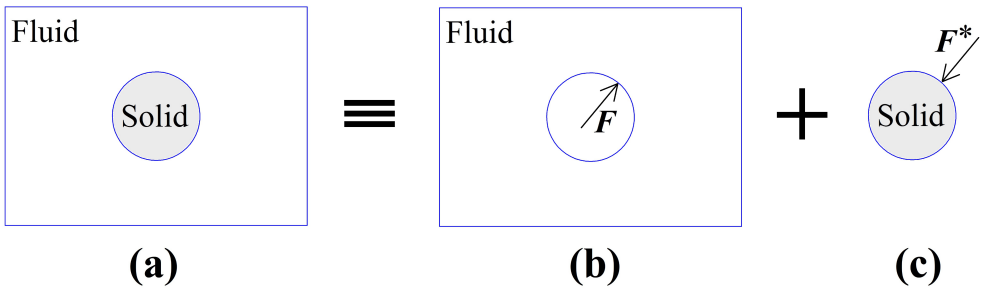

Figure 4.4: Schematics of FSI between a rigid solid and a fluid, where $\boldsymbol{F}$ denotes the resultant of the forces applied to the fluid by the solid structure and $\boldsymbol{F}^*$ is the resultant of forces applied to the solid structure by the flow.

Considering Figure 4.4, we denote the virtual power corresponding to the resultant of the forces acting on the closed control surface, $\boldsymbol{F}$, with δP, and define it as

$$\delta P = \delta \int_{CS_C} \{-p\boldsymbol{u}\cdot\boldsymbol{n} + \mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}\}\, dA. \quad (4.10)$$

Also, denoting the virtual power of the resultant force of fluid dynamic forces applied to the solid structure, $\boldsymbol{F}^*$ by δP^*, it is defined as

$$\begin{aligned}\delta P^* &= \frac{d}{dt}\int\limits_{\substack{\text{Solid}\\\text{Surface}}} \boldsymbol{f}_s^*(\boldsymbol{r}_s)\cdot\delta\boldsymbol{r}_s dA_s \\ &= \int\limits_{\substack{\text{Solid}\\\text{Surface}}} \frac{d}{dt}\left[\boldsymbol{f}_s^*(\boldsymbol{r}_s)\right]\cdot\delta\boldsymbol{r}_s dA_s + \int\limits_{\substack{\text{Solid}\\\text{Surface}}} \boldsymbol{f}_s^*(\boldsymbol{r}_s)\cdot\frac{d}{dt}\left(\delta\boldsymbol{r}_s\right)dA_s, \end{aligned} \tag{4.11}$$

and by considering the Jourdain's constraint, $\delta\boldsymbol{r}_s = 0$, it becomes

$$\delta P^* = \int\limits_{\substack{\text{Solid}\\\text{Surface}}} \boldsymbol{f}_s^*(\boldsymbol{r}_s)\cdot\frac{d}{dt}\left(\delta\boldsymbol{r}_s\right)dA_s. \tag{4.12}$$

Assuming the no-slip condition, and considering that Jourdain's principle does not permit any displacement, the particles at the solid surface remain in the same relative position. Moreover, the shear terms of Equation 4.10 are representative of external loads and are not dissipative. Therefore, in the absence of dissipative terms, the power lost by the structure must be absorbed by the fluid and vice versa. Therefore, $\delta P = -\,\delta P^*$.

Moreover, the virtual velocities must be compatible with system's constraints, thus

$$\delta\dot{\boldsymbol{r}}_s = \delta\boldsymbol{u}, \tag{4.13}$$

at the closed control surface. Therefore, the fluid external forces and $\boldsymbol{f}_s^*(\boldsymbol{r}_s)$ are action and reaction forces at any points on the solid surface. Thus, using the terms for external load from Equation 3.75, $\boldsymbol{f}_s^*$ is obtained to be

$$\boldsymbol{f}_s^* = -p\boldsymbol{n}^* + \mu\left(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u}\right)\cdot\boldsymbol{n}^*, \tag{4.14}$$

or,

$$-\delta \int_{CS_C} \{-\, p\boldsymbol{u}\cdot\boldsymbol{n} \;+\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}\}\, dA$$

$$= \frac{d}{dt} \int_{\substack{\text{Solid}\\ \text{Surface}}} \boldsymbol{f}_s^*\left(\boldsymbol{r}_s\right)\cdot\delta\boldsymbol{r}_s dA_s, \quad (4.15)$$

where $\boldsymbol{n}^*$ is the unit vector outward with respect to the solid. Consequently, for the rigid structure shown in Figure 4.4, $\boldsymbol{F}^* = -\boldsymbol{F}$ are action and reaction forces.

Substituting Equation 4.15 into Equation 4.9, we have

$$\delta \int_{CS} \{-\, p\boldsymbol{u}\cdot\boldsymbol{n} \;+\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}\}\, dA$$

$$= -\frac{d}{dt} \int_{\substack{\text{Solid}\\ \text{Surface}}} \boldsymbol{f}_s^*\left(\boldsymbol{r}_s\right)\cdot\delta\boldsymbol{r}_s dA_s$$

$$+\,\delta \int_{CS_O} \{-\, p\;\boldsymbol{u}\cdot\boldsymbol{n} \;+\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}\}\, dA,$$

and introducing it into Equation 4.8, we obtain

$$\begin{cases} \delta\left\{\dfrac{D}{Dt}\displaystyle\int_{CV}\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}dV+\int_{CS}\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}_{CS})\cdot\boldsymbol{n}dA\right. \\ \qquad \overbrace{-\displaystyle\int_{CS_O}\{-p\boldsymbol{u}\cdot\boldsymbol{n}+\mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u})-\boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}\}dA}^{\text{loading external to the fluid}} \\ \overbrace{+\displaystyle\int_{CS}\frac{1}{2}\mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u})-2\,\boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}dA+\frac{1}{2}\int_{CV}\mu(\nabla\times\boldsymbol{u})\cdot(\nabla\times\boldsymbol{u})\,dV}^{\text{viscous dissipation}} \\ \qquad \left.\overbrace{+\displaystyle\int_{CS}\frac{1}{2}\rho(\boldsymbol{u}\cdot\boldsymbol{u})(\boldsymbol{u}\cdot\boldsymbol{n})\,dA}^{\text{non-commuting term}}\right\} \\ \qquad +\dfrac{d}{dt}\displaystyle\int_{V_s}\left[\rho_s\frac{d\dot{\boldsymbol{r}}_s}{dt}-\boldsymbol{f}_s(\boldsymbol{r}_s)\right]\cdot\delta\boldsymbol{r}_s dV_s=0 \\ \delta t=0 \\ \delta\boldsymbol{r}_s=0 \\ \dfrac{d}{dt}(\delta\boldsymbol{r}_s)\neq 0. \end{cases} \tag{4.16}$$

Equation 4.16 is valid for any control volume of a Newtonian incompressible viscous fluid and any solid body (rigid or deformable). We next continue our derivation for the *model problem* which assumes the solid body to be rigid. For a deformable solid body, similar manipulations need to be performed. However, the stress-strain relations must be known in order to obtain $\boldsymbol{f}_s(\boldsymbol{r}_s,t)$.

4.4 Single Governing EOM for the Translating Cylinder

Consider the model problem in the form of the translating cylinder shown in Figure 1.2. One generalized coordinate is required to fully describe the motion of the cylinder. Let the generalized coordinate be the Lagrangian coordinate $\boldsymbol{x}_s(t)$ measured from the center of the cylinder when it is at rest, perpendicular to the axis of symmetry of the

cylinder and transverse to the flow direction.

Setting $\boldsymbol{r}_s(t) = \boldsymbol{x}_s(t)$ and introducing the force density function, $\boldsymbol{f}_s$,

$$\boldsymbol{f}_s(\boldsymbol{x}_s) = \frac{1}{V_s}\left(-C\dot{\boldsymbol{x}}_s - k\boldsymbol{x}_s\right) \tag{4.17}$$

into Equation 4.16, we obtain

$$\begin{cases} \delta\left\{\dfrac{D}{Dt}\displaystyle\int_{CV}\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}dV + \int_{CS}\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}_{CS})\cdot\boldsymbol{n}dA \right. \\ \qquad\qquad - \displaystyle\int_{CS_O} -p\ \boldsymbol{u}\cdot\boldsymbol{n}\ + \mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}dA \\ + \displaystyle\int_{CS}\frac{1}{2}\mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - 2\ \boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}dA + \frac{1}{2}\int_{CV}\mu(\nabla\times\boldsymbol{u})\cdot(\nabla\times\boldsymbol{u})\,dV \\ \qquad\qquad\qquad + \left.\displaystyle\int_{CS}\frac{1}{2}\rho(\boldsymbol{u}\cdot\boldsymbol{u})(\boldsymbol{u}\cdot\boldsymbol{n})\,dA\right\} \\ \qquad\qquad\qquad\qquad + \dfrac{d}{dt}\left[(m_s\ddot{\boldsymbol{x}}_s + C\dot{\boldsymbol{x}}_s + k\boldsymbol{x}_s)\cdot\delta\boldsymbol{x}_s\right] = 0 \\ \delta t = 0 \\ \delta\boldsymbol{x}_s = 0 \\ \dfrac{d}{dt}(\delta\boldsymbol{x}_s) \neq 0, \end{cases} \tag{4.18}$$

where C is the net support damping constant and k denotes the net support spring constant. The rigidity of the cylinder is considered in the integration over the solid volume.

Differentiating the structural terms, we obtain

$$\begin{aligned} &\frac{d}{dt}\left[(m_s\ddot{\boldsymbol{x}}_s + C\dot{\boldsymbol{x}}_s + k\boldsymbol{x}_s)\cdot\delta\boldsymbol{x}_s\right] \\ &\qquad = \frac{d}{dt}(m_s\ddot{\boldsymbol{x}}_s + C\dot{\boldsymbol{x}}_s + k\boldsymbol{x}_s)\cdot\delta\boldsymbol{x}_s + (m_s\ddot{\boldsymbol{x}}_s + C\dot{\boldsymbol{x}}_s + k\boldsymbol{x}_s)\cdot\delta\dot{\boldsymbol{x}}_s, \end{aligned} \tag{4.19}$$

and imposing the variational constraint ($\delta \boldsymbol{x}_s = 0$), we obtain

$$\frac{d}{dt}\left[(m_s \ddot{\boldsymbol{x}}_s + C\dot{\boldsymbol{x}}_s + k\boldsymbol{x}_s)\cdot \delta \boldsymbol{x}_s\right] = (m_s \ddot{\boldsymbol{x}}_s + C\dot{\boldsymbol{x}}_s + k\boldsymbol{x}_s)\cdot \delta \dot{\boldsymbol{x}}_s. \tag{4.20}$$

Substituting Equation 4.20 into Equation 4.18 yields

$$\begin{aligned}\delta \Bigg\{ \frac{D}{Dt}\int_{CV} \frac{1}{2}\rho \boldsymbol{u}\cdot\boldsymbol{u} dV + \int_{CS}\left(\frac{1}{2}\rho \boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}_{CS})\cdot \boldsymbol{n} dA \\ - \int_{CS_O}\left\{-\,p\;\boldsymbol{u}\cdot\boldsymbol{n}\; + \mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}\right\} dA \\ + \int_{CS}\left\{\frac{1}{2}\mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - 2\;\boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}\right\} dA \\ + \frac{1}{2}\int_{CV}\mu(\nabla\times\boldsymbol{u})\cdot(\nabla\times\boldsymbol{u})\, dV + \int_{CS}\frac{1}{2}\rho(\boldsymbol{u}\cdot\boldsymbol{u})(\boldsymbol{u}\cdot\boldsymbol{n})\, dA\Bigg\} \\ + (m_s \ddot{\boldsymbol{x}}_s + C\dot{\boldsymbol{x}}_s + k\boldsymbol{x}_s)\cdot\delta\dot{\boldsymbol{x}}_s = 0.\end{aligned} \tag{4.21}$$

Similar to the energy rate terms obtained for the control volume of fluid particles, we seek the energy rate terms corresponding to the structural terms.

Regarding the structural terms, $\boldsymbol{x}_s$ is a Lagrangian coordinate, therefore Equation 3.24 can be used to obtain the energy rate equation for the conservative structural terms,

$$\begin{aligned}(m_s\;\ddot{\boldsymbol{x}}_s + k\;\boldsymbol{x}_s)\cdot\delta\dot{\boldsymbol{x}}_s &= m_s\;\ddot{\boldsymbol{x}}_s\cdot\delta\dot{\boldsymbol{x}}_s + k\;\boldsymbol{x}_s\cdot\delta\dot{\boldsymbol{x}}_s \\ &= \frac{\partial(m_s\;\ddot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s)}{\partial\dot{\boldsymbol{x}}_s}\cdot\delta\dot{\boldsymbol{x}}_s + \frac{\partial(k\;\boldsymbol{x}_s\cdot\dot{\boldsymbol{x}}_s)}{\partial\dot{\boldsymbol{x}}_s}\cdot\delta\dot{\boldsymbol{x}}_s \\ &= \delta\left(m_s\;\ddot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s + k\;\boldsymbol{x}_s\cdot\dot{\boldsymbol{x}}_s\right).\end{aligned} \tag{4.22}$$

Regarding the nonconservative term due to the linear damper, we use Rayleigh's dissipation function. Therefore, we consider the function $\frac{1}{2}C\;\dot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s$ and obtain its

directional derivative with respect to $\dot{\boldsymbol{x}}_s$ and in the direction of $\dot{\boldsymbol{x}}_s$ as

$$\begin{aligned}\nabla_{\dot{\boldsymbol{x}}_s}\left(\frac{1}{2}C\dot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s\right)\cdot\dot{\boldsymbol{x}}_s &= \frac{1}{2}C\,\frac{\partial}{\partial\varepsilon}\left[(\dot{\boldsymbol{x}}_s+\varepsilon\;\dot{\boldsymbol{x}}_s)\cdot(\dot{\boldsymbol{x}}_s+\varepsilon\;\dot{\boldsymbol{x}}_s)\right]\Big|_{\varepsilon=0}\\ &= \frac{1}{2}C\,\frac{\partial}{\partial\varepsilon}\left[\dot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s+2\varepsilon\;\dot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s+\varepsilon^2\;\dot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s\right]\Big|_{\varepsilon=0}\\ &= C\dot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s, \end{aligned} \tag{4.23}$$

or,

$$\nabla_{\dot{\boldsymbol{x}}_s}\left(\frac{1}{2}C\dot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s\right)=C\dot{\boldsymbol{x}}_s. \tag{4.24}$$

Therefore, the variation of the function above can be obtained from the definition (Equation 3.57)

$$\delta\phi=\nabla_{\dot{\boldsymbol{x}}_s}\;\phi\cdot\delta\dot{\boldsymbol{x}}_s, \tag{4.25}$$

as,

$$\delta\left(\frac{1}{2}C\dot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s\right)=C\dot{\boldsymbol{x}}_s\cdot\delta\dot{\boldsymbol{x}}_s. \tag{4.26}$$

Substituting Equations 4.22 and 4.26 into Equation 4.21, the energy rate equation for the translating cylinder problem is obtained to be

$$\begin{aligned}\delta\Bigg\{&\frac{D}{Dt}\int_{CV}\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\;dV+\int_{CS}\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}_{CS})\cdot\boldsymbol{n}dA\\ &-\int_{CS_O}\left\{-\;p\boldsymbol{u}\cdot\boldsymbol{n}\;+\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}\right\}dA\\ &+\int_{CS}\left\{\frac{1}{2}\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-2\;\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}\right\}dA\\ &+\frac{1}{2}\int_{CV}\mu\left(\nabla\times\boldsymbol{u}\right)\cdot\left(\nabla\times\boldsymbol{u}\right)dV+\int_{CS}\frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)dA\Bigg\}\\ &+\delta\left(m_s\ddot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s+\frac{1}{2}C\dot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s+k\boldsymbol{x}_s\cdot\dot{\boldsymbol{x}}_s\right)=0. \end{aligned} \tag{4.27}$$

In deriving Equation 4.27, we did not add any constraints with regard to the nature

of solid-fluid interaction other than action and reaction forces. In order to include the boundary conditions, we separate the integrals over the control surface into integrals over its open and closed portions. Regarding the flux of kinetic energy, we have

$$\int_{CS} \left(\frac{1}{2}\rho \boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}_{CS})\cdot\boldsymbol{n}dA = \int_{CS_C} \left(\frac{1}{2}\rho \boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}^c_{CS})\cdot\boldsymbol{n}dA + \int_{CS_O} \left(\frac{1}{2}\rho \boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}^o_{CS})\cdot\boldsymbol{n}dA, \quad (4.28)$$

where $\boldsymbol{v}^o_{CS}$ and $\boldsymbol{v}^c_{CS}$ are the velocities of the open and closed portions of the control volume, respectively. The first integral on the right-hand side of Equation 4.28 is equal to zero due to the *no-penetration* condition, therefore,

$$\int_{CS} \left(\frac{1}{2}\rho \boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}_{CS})\cdot\boldsymbol{n}dA = \int_{CS_O} \left(\frac{1}{2}\rho \boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}^o_{CS})\cdot\boldsymbol{n}dA. \quad (4.29)$$

Regarding the fluid dissipative forces at the control surfaces, we expand the fourth integral in Equation 4.27 as follows

$$\int_{CS} \frac{1}{2}\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right) - 2\ \boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}dA = \int_{CS_O} \frac{1}{2}\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right) - 2\ \boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}dA + \int_{CS_C} \frac{1}{2}\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right) - 2\ \boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}dA. \quad (4.30)$$

Regarding the last integral on the right-hand side, it is not clear whether this term must be considered at the closed control surface or not. On one hand, we expect the energy to be dissipated at higher rates near the structure due to the high velocity gradient. Moreover, the velocity measurements of the flow are not generally feasible at the surface of the solid structure. On the other hand, the no-slip boundary condition

implies that no energy is lost due to friction because the relative motion between the fluid and solid is zero. Yet again, the viscous dissipation can be envisioned to be between the particles at the solid surface and those adjacent to them. Therefore, we choose to keep the dissipative terms at the closed control surfaces and continue our derivations.

Regarding the viscous terms, a couple of notes are worth mentioning. First, for a solid structure that does not move, the dissipative terms at the closed boundary disappear [35, article 329]. Therefore, one choice seems to be that $\boldsymbol{u}$ in Equation 4.30 can be replaced with $\boldsymbol{u}_r$, especially when experimental data is considered. Second, the same discussion as that made for dissipative terms applies to viscous forces acting external to the fluid and one may decide to keep these in the energy equation. However, the pressure terms are conservative forces for incompressible fluids and they can be neglected.

Finally, regarding the non-commuting terms, we also consider the no-slip condition that states that at the closed control surface $\boldsymbol{u} = \dot{\boldsymbol{x}}_s$ for all time. Replacing $\boldsymbol{u}$ by $\dot{\boldsymbol{x}}_s$ at closed surfaces and considering that the commutation rule holds for $\dot{\boldsymbol{x}}_s$, the non-commuting terms disappear. Therefore, in Equation 4.27 we have

$$\begin{aligned}\int_{CS} \frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)dA &= \int_{CS_O} \frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)dA + \int_{CS_C} \frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)dA \\ &= \int_{CS_O} \frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)dA. \qquad (4.31)\end{aligned}$$

Having specified the boundary condition, the energy equation for the translating cylinder problem is obtained by substituting Equations 4.29, 4.30 and 4.31 into Equation

4.27, resulting in

$$\delta\left\{\frac{D}{Dt}\int\limits_{CV}\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\;dV+\int\limits_{CS_O}\left(\frac{1}{2}\rho\;\boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}_{CS}^{o})\cdot\boldsymbol{n}dA\right.$$
$$-\int\limits_{CS_O}\overbrace{\{-\,p\boldsymbol{u}\cdot\boldsymbol{n}\;+\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}}^{\text{external load at open CS}}$$
$$\left.\overbrace{-\frac{1}{2}\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-2\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}}^{\text{viscous dissipation at open CS}}\right\}dA$$
$$+\overbrace{\int\limits_{CS_C}\frac{1}{2}\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-2\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}\;dA}^{\text{viscous dissipation at solid surface}}$$
$$+\overbrace{\int\limits_{CV}\frac{1}{2}\mu\left(\nabla\times\boldsymbol{u}\right)\cdot\left(\nabla\times\boldsymbol{u}\right)dV}^{\text{viscous dissipation inside the CV}}+\int\limits_{CS_O}\frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)\;dA$$
$$\left.+\;m_s\;\ddot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s+\frac{1}{2}C\;\dot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s+k\;\boldsymbol{x}_s\cdot\dot{\boldsymbol{x}}_s\right\}=0. \quad (4.32)$$

As mentioned in Section 3.5, Rayleigh's dissipation function must be multiplied by a factor of two in order to obtain the energy rate equation. Therefore, the energy rate equation for the translating cylinder problem is

$$\frac{D}{Dt}\int\limits_{CV}\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\;dV+\int\limits_{CS_O}\left(\frac{1}{2}\rho\;\boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}_{CS}^{o})\cdot\boldsymbol{n}\;dA$$
$$-\int\limits_{CS_O}\left[-\;p\;\boldsymbol{u}\cdot\boldsymbol{n}\;+\mu\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\cdot\boldsymbol{n}\right]dA$$
$$+\int\limits_{CS_C}\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-2\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}\;dA$$
$$+\int\limits_{CV}\mu\left(\nabla\times\boldsymbol{u}\right)\cdot\left(\nabla\times\boldsymbol{u}\right)dV+\int\limits_{CS_O}\frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)\;dA$$
$$+\;m_s\;\ddot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s+C\;\dot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s+k\;\boldsymbol{x}_s\cdot\dot{\boldsymbol{x}}_s=0, \quad (4.33)$$

where we summed the viscous terms at open control surface.

The *model problem* in the form of inverted pendulum is considered in the following section.

4.5 Single Governing EOM for the Inverted Pendulum

Similar to the translating cylinder problem, the inverted pendulum problem (Figure 1.1) is a single DOF structure. Thus, the single generalized coordinate is selected to be the angle of rotation of the rigid cylinder θ (radian) about its support. Also, the same assumptions as those made by Benaroya and Wei [1] are used, which are: three-dimensional effects can be ignored (vortices remain two-dimensional), weight and buoyancy forces must be included for the structure, and the resultant of all the forces act at the geometric center of the circular cylinder. Moreover, the horizontal plane passing through the center of geometry contains the two dimensional vortices.

Therefore, by replacing the last term in Equation 4.16 by $\left[I_0\ddot{\theta} + C_T\dot{\theta} + k_T\theta - (m_s g - B)\,\frac{L}{2}\sin\theta \mathbf{e}_\theta\right] \cdot \delta\theta \mathbf{e}_\theta$, it becomes

$$
\begin{cases}
\delta \left\{ \dfrac{D}{Dt}\displaystyle\int_{CV} \frac{1}{2}\rho \boldsymbol{u}\cdot\boldsymbol{u} dV + \int_{CS}\left(\frac{1}{2}\rho \boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}_{CS})\cdot\boldsymbol{n} dV \right. \\
\quad - \displaystyle\int_{CS_O} \{-p\boldsymbol{u}\cdot\boldsymbol{n} + \mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}\} dA \\
\qquad + \displaystyle\int_{CS} \frac{1}{2}\mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - 2\,\boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n} dA \\
\quad + \dfrac{1}{2}\displaystyle\int_{CV} \mu(\nabla\times\boldsymbol{u})\cdot(\nabla\times\boldsymbol{u}) dV + \int_{CS}\frac{1}{2}\rho(\boldsymbol{u}\cdot\boldsymbol{u})(\boldsymbol{u}\cdot\boldsymbol{n}) dA \\
+ \dfrac{d}{dt}\left\{\left[I_0\ddot{\theta} + C_T\dot{\theta} + k_T\theta - (m_s g - B)\frac{L}{2}\sin\theta\right]\mathbf{e}_\theta\cdot\delta\theta\mathbf{e}_\theta\right\} = 0 \\
\delta t = 0 \\
\delta\theta = 0 \\
\dfrac{d}{dt}(\delta\theta) \neq 0,
\end{cases}
\tag{4.34}
$$

where I_0 denotes the mass moment of inertia for the circular cylinder about its support, L is the length of the cylinder, $\mathbf{e}_\theta$ denotes the unit vector perpendicular to the plane of movement of the cylinder, g is the gravitational acceleration, B is the total buoyancy force (equal to the weight of the displaced fluid), C_T denotes the torsional damping constant, and k_T is the torsional stiffness constant of the spring.

Similar to our derivation in the previous section, by differentiating the structural terms with respect to time and imposing the variational constraints, we obtain

$$\frac{d}{dt}\left\{\left[I_0\ddot{\theta} + C_T\dot{\theta} + k_T\theta - (m_s g - B)\,\frac{L}{2}\sin\theta\right]\mathbf{e}_\theta \cdot \delta\theta\mathbf{e}_\theta\right\} = \left[I_0\ddot{\theta} + C_T\dot{\theta} + k_T\theta - (m_s g - B)\,\frac{L}{2}\sin\theta\right]\delta\dot{\theta}. \quad (4.35)$$

Regarding the forces due to acceleration, damping and spring, the derivations are similar to those shown in the previous section. The results are

$$\delta\left(I_0\ddot{\theta}\;\dot{\theta} + k_T\theta\;\dot{\theta}\right) = I_0\ddot{\theta}\;\delta\dot{\theta}, \quad (4.36)$$

and

$$\delta\left(\frac{1}{2}C_T\;\dot{\theta}^2\right) = C_T\;\dot{\theta}\;\delta\dot{\theta}. \quad (4.37)$$

Also, regarding the weight and buoyancy forces, we use Equation 3.6 and obtain

$$\begin{aligned}\left[(m_s g - B)\,\frac{L}{2}\sin\theta\right]\delta\dot{\theta} &= \left\{\frac{\partial}{\partial\dot{\theta}}\left[(m_s g - B)\,\frac{L}{2}\dot{\theta}\sin\theta\right]\right\}\delta\dot{\theta} \\ &= \delta\left\{\left[(m_s g - B)\,\frac{L}{2}\sin\theta\right]\dot{\theta}\right\}. \end{aligned} \quad (4.38)$$

Introducing Equations 4.35 to 4.38 into Equation 4.34 and imposing Jourdain's constraints, it becomes

$$\delta\left\{\frac{D}{Dt}\int_{CV}\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}dV+\int_{CS_O}\left(\frac{1}{2}\rho\ \boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}^o_{CS})\cdot\boldsymbol{n}dA\right.$$
$$-\int_{CS_O}\{-p\boldsymbol{u}\cdot\boldsymbol{n}\ +\mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u})-\boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}$$
$$\left.-\frac{1}{2}\mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u})-2\boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}\right\}dA$$
$$+\int_{CS_C}\frac{1}{2}\mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u})-2\boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}dA$$
$$+\int_{CV}\frac{1}{2}\mu(\nabla\times\boldsymbol{u})\cdot(\nabla\times\boldsymbol{u})\,dV+\int_{CS_O}\frac{1}{2}\rho(\boldsymbol{u}\cdot\boldsymbol{u})(\boldsymbol{u}\cdot\boldsymbol{n})\,dA$$
$$\left.+I_0\ddot{\theta}\dot{\theta}+\frac{1}{2}C_T\dot{\theta}^2+k_T\theta\dot{\theta}-\left[(m_sg-B)\frac{L}{2}\sin\theta\right]\dot{\theta}\right\}=0,\quad(4.39)$$

where the *no-penetration* and *no-slip boundary* conditions are considered in a similar way as in Section 4.4.

Therefore, multiplying Rayleigh's dissipation function by two, the energy rate equation is found to be

$$\frac{D}{Dt}\int_{CV}\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}dV+\int_{CS_O}\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}^o_{CS})\cdot\boldsymbol{n}dA$$
$$-\int_{CS_O}[-p\boldsymbol{u}\cdot\boldsymbol{n}\ +\mu\boldsymbol{u}\times(\nabla\times\boldsymbol{u})\cdot\boldsymbol{n}]\,dA$$
$$+\int_{CS_C}\mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u})-2\boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}dA$$
$$+\int_{CV}\mu(\nabla\times\boldsymbol{u})\cdot(\nabla\times\boldsymbol{u})\,dV+\int_{CS_O}\frac{1}{2}\rho(\boldsymbol{u}\cdot\boldsymbol{u})(\boldsymbol{u}\cdot\boldsymbol{n})\,dA$$
$$\left.+I_0\ddot{\theta}\dot{\theta}+C_T\dot{\theta}^2+k_T\theta\dot{\theta}-\left[(m_sg-B)\frac{L}{2}\sin\theta\right]\dot{\theta}\right\}=0.\quad(4.40)$$

So far, we have obtained our variational formulations of the energy rate equations for a general control volume of fluid particles (Equation 3.123), a general control volume enclosing a solid body (Equation 4.16), and the model problems (Equations 4.32 and 4.39). In order to compare our formulations, we next review the works of McIver [12] and Benaroya and Wei [1] in more detail than done in Chapter 1.

4.6 Comparison with McIver's Extension of Hamilton's Principle

As mentioned in Section 1.1, McIver's attempt was to extend Hamilton's principle for systems involving fluid-structure interaction [12]. He specifically considered the FSI system where the fluid is internal to the structure. In this section, we compare our derivations to those of McIver.

The first difference is due to McIver's assumption that the only virtual work applied to a control volume is due to the surface traction over the control surfaces [12, p. 251]. Therefore, McIver started his derivations by assuming that there is no dissipation of energy due to viscosity. If we had made the same assumption, the terms marked *viscous dissipation* would not be present in Equation 3.122.

In his paper, McIver also presented his energy equation in the absence of any interacting system (similar to systems considered in Chapter 3) as [12, Equation 23]

$$\frac{d}{dt}\left(K+E\right)_O = \iint\limits_{B_O(t)} \left[\boldsymbol{u}\cdot\bar{\boldsymbol{\sigma}} + \rho\left(\frac{1}{2}u^2+e\right)\left(\boldsymbol{V}-\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}ds, \tag{4.41}$$

where K is the kinetic energy, E and e denote the potential energy and its density, respectively; $\boldsymbol{V}$ is the velocity of the control surface, $B_O\left(t\right)$, and surface element ds. Neglecting the potential energy terms (as we did) and substituting the terms K and

$\bar{\boldsymbol{\sigma}}$, Equation 4.41 becomes

$$\frac{d}{dt}\int\limits_{CV}\frac{1}{2}\rho u^2 dV = \int\limits_{CS}\left\{\boldsymbol{u}\cdot\left[-p\bar{\boldsymbol{I}}+\mu\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\right]+\left(\frac{1}{2}\rho u^2\right)\left(\boldsymbol{v}_{CS}-\boldsymbol{u}\right)\right\}\cdot\boldsymbol{n}dA, \quad (4.42)$$

where we have changed McIver's notations to match ours to avoid confusion.

For comparison, if we neglect the terms corresponding to viscous dissipation in Equation 3.99, the energy rate equation becomes

$$\frac{D}{Dt}\int\limits_{CV}\frac{1}{2}\rho u^2 dV = \int\limits_{CS}\left\{\boldsymbol{u}\cdot\left[-p\bar{\boldsymbol{I}}+\mu\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\right]+\left(\frac{1}{2}\rho u^2\right)\left(\boldsymbol{v}_{CS}-\boldsymbol{u}\right)+\left(\frac{1}{2}\rho u^2\right)\boldsymbol{u}\right\}\cdot\boldsymbol{n}dA. \quad (4.43)$$

Comparing Equation 4.43 with Equation 4.42, another important difference becomes clear, that is, the additional term: power due to the non-commuting part of the rate of kinetic energy, $\left(\frac{1}{2}\rho u^2\right)\boldsymbol{u}$. We obtained this term after examining the commutation rule in the Eulerian reference frame. However, McIver assumed the commutation rule holds. Moreover, he did not consider the relations between the Lagrangian and the Eulerian reference frames.

Having neglected the viscous dissipation, McIver extended Hamilton's principle by utilizing the Reynolds' transport theorem, and obtained it to be (Equation 1.3)

$$\delta\int_{t_1}^{t_2}(K-E)_{CV}dt+\int_{t_1}^{t_2}\int_{CS_O}\left[\delta\boldsymbol{r}\cdot\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{n}+\rho\left(\boldsymbol{u}\cdot\delta\boldsymbol{r}\right)\left(\boldsymbol{u}_r\cdot\boldsymbol{n}\right)\right]dAdt + \int_{t_1}^{t_2}\int_{CS_C}\delta\boldsymbol{r}\cdot\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{n}dAdt=0,$$

where K and E are the kinetic and potential energy terms, and $\bar{\boldsymbol{\sigma}}$ is the stress dyadic tensor.

For a system of fluid particles, the particle trajectories cannot be obtained from the Eulerian observations of a control volume as we discussed in Chapter 1. In order to overcome this difficulty, McIver followed Lanczos [29, pp. 119-124] and replaced the virtual displacement $\delta \boldsymbol{r}$ by

$$\delta \boldsymbol{r} = \boldsymbol{u} dt. \tag{4.44}$$

For an incompressible fluid, the thermodynamic pressure can be considered to be a conservative force. As mentioned earlier, he considered the energy equation for a control volume of fluid particles in the absence of a solid system to be that expressed by Equation 4.41, and discussed that the energy of the control volume would be conserved if the control volume is selected such that the terms on the right-hand side of Equation 4.41 disappear for all time. Therefore, the control surface must satisfy the relation,

$$\boldsymbol{v}_{CS} \cdot \boldsymbol{n} = -\frac{\boldsymbol{u} \cdot \bar{\boldsymbol{\sigma}} \cdot \boldsymbol{n}}{\rho \left(\frac{1}{2} u^2 + e \right)} + \boldsymbol{u} \cdot \boldsymbol{n}. \tag{4.45}$$

If a control volume with this requirement could be found, then the corresponding extended Hamilton's principle is expressed by

$$\delta \int_{t_1}^{t_2} (K - E)_{CV} dt + \int_{t_1}^{t_2} \int_{CS_O} \left\{ \delta \, \boldsymbol{r} \cdot \left[\bar{\boldsymbol{I}} - \frac{\boldsymbol{u}\boldsymbol{u}}{\rho \left(\frac{1}{2} u^2 + e \right)} \right] \cdot \bar{\boldsymbol{\sigma}} \cdot \boldsymbol{n} \right\} dA dt = 0, \tag{4.46}$$

given that the fluid-structure interacting forces are conservative (Equation 4.46 does not contain any integrals over the closed control system.) Similarly, in Section 4.3, we discussed that the no-slip condition may justify that the fluid-structure interacting forces are conservative.

McIver's extension of Hamilton's principle is an interesting and noteworthy work.

However, a few remarks, as follow, merit attention when applying McIver's formulation to a FSI system, and especially for problems involving VIV.

As mentioned earlier, McIver neglected the viscous dissipation of the energy inside the control volume. This simplification reduces the applicability of the assumed conservation of energy. Moreover, he did not consider the relations between the Lagrangian and the Eulerian frames of reference. While Equation 4.44 seems appropriate since the virtual velocities are arbitrary, McIver did not consider that $\boldsymbol{r}$ and $\boldsymbol{u}$ are not described in the same reference frame. Setting $\delta(\ ) = dt\ d(\ )/dt$ implies that the variation of a system variable at each instant coincides with that actual variable [29, p.120]. Thus, Equation 4.44 suggests that the particle trajectories could be found by integrating the Eulerian velocities. We have discussed the difficulties faced in relating the Lagrangian and the Eulerian velocities in detail in Chapter 1. Moreover, the relation $\delta(\ ) = dt\ d(\ )/dt$ imposes additional constraints on the velocity field functions, which are discussed in Section 4.8.

Another difference becomes apparent if we start the derivation by neglecting the viscous dissipation terms with the result expressed by Equation 4.43. Alternatively, we can neglect the viscous terms left inside the control volume in Equation 3.123, and obtain the corresponding energy equation to be

$$\begin{aligned}
&\delta\left\{\frac{D}{Dt}\int_{CV}\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\ dV+\int_{CS}\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}_{CS})\cdot\boldsymbol{n}\ dA\right\}\\
&\qquad=\delta\left\{\int_{CS}\{-\ p\ \boldsymbol{u}\cdot\boldsymbol{n}\ +\mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u})-\boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}\right.\\
&\left.-\frac{1}{2}\mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u})-2\ \boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}\right\}\ dA-\int_{CS}\frac{1}{2}\rho(\boldsymbol{u}\cdot\boldsymbol{u})(\boldsymbol{u}\cdot\boldsymbol{n})\ dA\Bigg\},
\end{aligned}\tag{4.47}$$

which is a different result than that expressed by Equation 4.43.

For many fluid-structure interaction systems, the dissipation of energy due to viscosity can be neglected. Equations 4.43 and 4.47 both assume that no energy is lost inside the control volume. Equation 4.43 totally neglects the dissipation while Equation 4.47 expresses the power added or subtracted from the system at the boundaries due to external loads corrected with a consideration of viscous dissipation. At the solid surfaces, there is no difference between the two equations if the no-slip condition is assumed (if we neglect the dissipation at the solid surfaces, as discussed earlier as an option).

Therefore, if we neglect the dissipative terms after expanding the energy rate equation, one more difference with McIver's model is the presence in our model of the viscous dissipative terms at both the open and the closed control surfaces.

Modeling fluid-structure interaction is a challenging task. Therefore, the assumptions made by McIver may be appropriate in modeling some systems. However, the selection of control volumes such as specified by McIver is very challenging. It might be easily recognizable for systems where an approximately laminar flow is internal to the structure, but it is ambiguous otherwise. Thus, it is very challenging to compare McIver's method with ours any further since McIver's appropriate control volume cannot be distinguished without knowing the velocity field of the flow.

Another interesting extension of Hamilton's principle was by Benaroya and Wei [1] which is considered in the following section.

4.7 Comparison with Benaroya and Wei's Extension of Hamilton's Principle

In the previous section, we discussed that the applicability of McIver's approach is greatly limited by the constraint it imposes on selecting the appropriate control volume. Such control volumes might be easily distinguished when the flow is internal

to the structure and it is ambiguous for external flows. Having considered McIver's method, Benaroya and Wei [1] took on a more challenging task to extend Hamilton's principle for VIV problems where the flow is external to the structure. As mentioned earlier, we used their methodology in obtaining the single governing EOM for the model problem. Thus, a short summary of their work suffices.

Benaroya and Wei started with Hamilton's principle and applied the Reynolds transport theorem in order to modify the formulation for systems of changing mass. They discussed in detail that in the absence of known particle trajectories for the fluid flow, stationarity cannot be ensured. Thus, using Equation 4.44 would lead to the conservation of energy equation. Similar to McIver's approach, they neglected the viscous dissipation of energy inside the control volume and obtained the energy equation

$$\frac{d}{dt}\left(T_{structure}+\Pi_{structure}\right)=\int_{CS}\frac{1}{2}\rho u^2\left(\boldsymbol{u}_r\cdot\boldsymbol{n}\right)dA + \int_{CS}\left(-p\boldsymbol{n}+\boldsymbol{\tau}\right)\cdot\boldsymbol{u}dA-\left(m_{fluid}u\dot{u}\right)_{CV}, \quad (4.48)$$

where $\boldsymbol{\tau}$ is the shear stress, T is the kinetic energy, and the potential energy is denoted by Π.

Regarding the translating cylinder problem (Figure 1.2), they derived the governing EOM to be,

$$\dot{\boldsymbol{x}}_s\left(m_s\ \ddot{\boldsymbol{x}}_s+k\boldsymbol{x}_s\right)+\left(m_{fluid}u\dot{u}\right)_{CV}= \int_{CS}\frac{1}{2}\rho u^2\left(\boldsymbol{u}_r\cdot\boldsymbol{n}\right)dA+\int_{CS_O}\left(-p\boldsymbol{n}+\boldsymbol{\tau}\right)\cdot\boldsymbol{u}\ dA+\int_{CS_C}\left(-p\boldsymbol{n}+\boldsymbol{\tau}\right)\cdot\boldsymbol{u}\ dA, \quad (4.49)$$

and for the inverted pendulum problem, they derived the governing EOM

$$\dot{\theta}\left[I_0\,\ddot{\theta} + k_T\theta\ - (m_s g - B)\,\frac{L}{2}\sin\theta\ \right] + (m_{fluid}u\dot{u})_{CV} = \int_{CS}\frac{1}{2}\rho u^2\,(\boldsymbol{u}_r\cdot\boldsymbol{n})\,dA + \int_{CS_O}(-p\boldsymbol{n}+\boldsymbol{\tau})\cdot\boldsymbol{u}\ dA + \int_{CS_C}(-p\boldsymbol{n}+\boldsymbol{\tau})\cdot\boldsymbol{u}\ dA, \tag{4.50}$$

where they have ignored the structural damping.

Comparing Benaroya and Wei's results (Equations 4.49 and 4.50) to ours (Equations 4.33 and 4.40), we notice that the sign of the flux of kinetic energy in our equation is different than theirs. The reason is that Benaroya and Wei used the RTT to relate the integration over the control system to that over the control volume while we used the RTT twice; once for the same purpose and once to relate the change of control volume to itself (*See* Benaroya and Wei's Equation 29 and Equation 3.1 of ours).

Moreover, Benaroya and Wei consider the pressure and shear forces at the closed surface which we neglect by considering them to be action and reaction forces that do not appear in the energy equation. Regarding these terms, their equation also differs from that obtained by McIver.

Additionally, we notice similar differences to those observed in Section 4.6 where we compared our derivations to those by McIver. Recall that the differences observed were the dissipation terms and the non-commuting term of the rate of kinetic energy.

Discussed in the following section is an important conclusion that can be made from the work of McIver and of Benaroya and Wei, after which the differences between our method and McIver's and Benaroya and Wei's methods becomes more clear.

4.8 Limitations of the Classic Energy Equation in Integral Form

Having reviewed the Hamilton's extension by McIver, and Benaroya and Wei, we next consider the energy equation of a control volume containing only fluid again.

In classic fluid mechanics, conservation laws in integral form are generally obtained by assuming that they are related to the local differential laws via Reynolds' transport theorem. The energy equation is derived in the similar manner, by setting $b = \rho\left(e + \frac{1}{2}u^2 + gZ\right)$ in the RTT (Equations 3.1, 3.4 and 3.5), where e is the internal energy, g is the gravitational acceleration, and Z is the potential for the gravity force $F_i = -\partial\left(gZ\right)/\partial x_i$ [43, pp. 100-101]. If the relaxation time of the fluid molecules is small compared to the time scale of the flow (which is justifiable for the majority of FSI systems), thermodynamic pressure becomes independent of velocity [28, p. 100] and e becomes a function of pressure and shear stresses. Therefore, the energy equation in integral form for a control volume of a Newtonian incompressible viscous flow becomes

$$\frac{d}{dt}\int_{CV}\left(\frac{1}{2}\rho u^2\right)dV = -\int_{CS}\left(\frac{1}{2}\rho u^2\right)\left(\boldsymbol{u}_r\cdot\boldsymbol{n}\right)dA + \int_{CS_O}\left(-p\boldsymbol{n}+\boldsymbol{\tau}\right)\cdot\boldsymbol{u}\;dA + \int_{CS_C}\left(-p\boldsymbol{n}+\boldsymbol{\tau}\right)\cdot\boldsymbol{u}\;dA - \int_{CV}\mu\frac{\partial}{\partial x_i}\left[\left(\frac{\partial u_k}{\partial x_i}+\frac{\partial u_i}{\partial x_k}\right)u_k\right]dV, \quad (4.51)$$

where we neglected the gravitational body forces for simplicity, as we have done throughout this work. If such body forces are required they can be easily considered by adding $\int_{CV}\rho\left(-\partial\left(gZ\right)/\partial x_i\right)u_i dV$ to the right-hand side of Equation 4.51.

Next we obtain the same result by utilizing McIver's and Benaroya and Wei's methods.

As mentioned earlier, Benaroya and Wei [1] extended Hamilton's principle for systems of fluid-structure interaction. In the absence of known particle trajectories,

they followed Lanczos [29] and assumed that the virtual displacements coincide with the actual displacements of the fluid particles. By assuming that the control volume contains only fluid particles, we exclude the structural and potential terms from the Benaroya and Wei formulation and obtain the energy rate equation to be

$$\begin{aligned}&\int_{CV} \frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) dV \\ &= \int_{CS} \left(\frac{1}{2}\rho u^2\right)(\boldsymbol{u}_r \cdot \boldsymbol{n})\, dA + \int_{CS_O} (-p\boldsymbol{n} + \boldsymbol{\tau}) \cdot \boldsymbol{u}\; dA + \int_{CS_C} (-p\boldsymbol{n} + \boldsymbol{\tau}) \cdot \boldsymbol{u}\; dA.\end{aligned} \tag{4.52}$$

In Section 1.1, we mentioned that one of the differences between McIver's and Benaroya and Wei's extensions of Hamilton's principle was their interpretation of the RTT. While Benaroya and Wei utilized the RTT to relate a material volume to a control volume, McIver's used the RTT to relate a control volume to a system volume. Both are essentially the same. However, McIver's RTT result is the form we are interested in here ($\frac{d}{dt}\int (\;)\, dV$).

Therefore, if we use McIver's method in applying the RTT,

$$\frac{d}{dt}\int_{\substack{\text{Control}\\\text{Volume}}} (\;)\, dV = \frac{D}{Dt}\int_{\substack{\text{System}\\\text{Volume}}} (\;)\, dV + \int_{\substack{\text{Control}\\\text{Surface}}} (\;)\, (\boldsymbol{v}_{cs} - \boldsymbol{u}) \cdot \boldsymbol{n} dA, \tag{4.53}$$

to Hamilton's principle, and then follow Benaroya and Wei's approach, we obtain Equation 4.52 alternatively as

$$\begin{aligned}&\frac{d}{dt}\int_{CV} \left(\frac{1}{2}\rho u^2\right) dV \\ &= -\int_{CS} \left(\frac{1}{2}\rho u^2\right)(\boldsymbol{u}_r \cdot \boldsymbol{n})\, dA + \int_{CS_O} (-p\boldsymbol{n} + \boldsymbol{\tau}) \cdot \boldsymbol{u}\; dA + \int_{CS_C} (-p\boldsymbol{n} + \boldsymbol{\tau}) \cdot \boldsymbol{u}\; dA.\end{aligned} \tag{4.54}$$

Benaroya and Wei conducted a series of experiments on the VIV of the model problem

in the form of the inverted pendulum in parallel to their analytical work. From their experimentally acquired data, they reported that the rate of work done by viscous forces was found to be three orders of magnitude smaller than the rate of kinetic energy flux. Thus, their disregard of the viscous dissipation was justified by their experimental findings. However, an interesting result can be obtained by including the viscous dissipation, that is, the limitations of the classical energy equation becomes clear, as follows.

Including the viscous dissipation from Equation 3.75 in Benaroya and Wei's terms for work done by non-conservative forces [1, *see* Section 4.1] as,

$$\delta W = \int_{CS_C} (-p\boldsymbol{n} + \boldsymbol{\tau}) \cdot \delta \boldsymbol{r}\, dA + \int_{CS_O} (-p\boldsymbol{n} + \boldsymbol{\tau}) \cdot \delta \boldsymbol{r}\, dA - \int_{CV} \mu \frac{\partial}{\partial x_i} \left[\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k} \right) (\delta r)_k \right] dV, \quad (4.55)$$

and,

$$dt \frac{dW}{dt} = dt \int_{CS_C} (-p\boldsymbol{n} + \boldsymbol{\tau}) \cdot \boldsymbol{u}\, dA + dt \int_{CS_O} (-p\boldsymbol{n} + \boldsymbol{\tau}) \cdot \boldsymbol{u}\, dA - dt \int_{CV} \mu \frac{\partial}{\partial x_i} \left[\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k} \right) (u_i)_k \right] dV, \quad (4.56)$$

Equation 4.54 becomes,

$$\frac{d}{dt} \int_{CV} \left(\frac{1}{2} \rho u^2 \right) dV = - \int_{CS} \left(\frac{1}{2} \rho u^2 \right) (\boldsymbol{u}_r \cdot \boldsymbol{n})\, dA + \int_{CS_O} (-p\boldsymbol{n} + \boldsymbol{\tau}) \cdot \boldsymbol{u}\, dA + \int_{CS_C} (-p\boldsymbol{n} + \boldsymbol{\tau}) \cdot \boldsymbol{u}\, dA - \int_{CV} \mu \frac{\partial}{\partial x_i} \left[\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k} \right) u_k \right] dV, \quad (4.57)$$

which is the classical *energy equation in integral form* [43, pp. 100-101]. Note that Benaroya and Wei mentioned that the result is the first law of thermodynamics. Here, we just added the viscous dissipation term.

There are no doubts about the generality of Hamilton's principle. However, we mentioned that Benaroya and Wei obtained their energy equation by using the relation $\delta(\,) = dt\ d(\,)/dt$ and Equation 4.44. Therefore, the applicability of the *energy equation in integral form* is limited to the systems where the assumption of Equation 4.44 holds.

As mentioned earlier, for a general system, Hamilton's principle results in Lagrange's equations which fully describe the dynamics of a system. Therefore, we explain the limitation of the mentioned assumption by considering the derivation of Lagrange's equation, as shown in Section 3.3. Assuming that the commutation rule holds for Lagrange's equation in the Eulerian reference frame, the variation of $L(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t)$ is

$$\delta_d L = \frac{\partial L}{\partial \dot{\boldsymbol{r}}_i} \cdot \delta_d \dot{\boldsymbol{r}}_i + \frac{\partial L}{\partial \boldsymbol{r}_i} \cdot \delta_d \boldsymbol{r}_i, \tag{4.58}$$

as was obtained from Equation 3.15. Since the algebraic relation between $\delta_d \dot{\boldsymbol{r}}_i$ and $\delta_d \boldsymbol{r}_i$ is not known, Equation 4.58 is not accessible for further analysis. When the configuration is not known at t_1 and t_2, a conservation of L equation can be obtained by setting $dt \equiv \varepsilon$ and $\delta(\,) = dt\ d(\,)/dt$, as mentioned by Lanczos and used by Benaroya and Wei. Therefore, Equation 3.15 becomes

$$\frac{d}{dt}L = \frac{\partial L}{\partial \boldsymbol{r}_i} \cdot \dot{\boldsymbol{r}}_i + \frac{\partial L}{\partial \dot{\boldsymbol{r}}_i} \cdot \ddot{\boldsymbol{r}}_i. \tag{4.59}$$

The Lagrangian function, L, is a function of velocities, displacements and time. Velocities and displacements can be chosen to be the generalized velocities, $\dot{q}_i$, and generalized displacements, q_i, as well. Therefore, the time derivative of L is obtained by using the *chain rule* as

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t}, \tag{4.60}$$

given that q_i and $\dot{q}_i$ are independent of each other.

Comparing Equations 4.59 and 4.60, the limitations the conservation equation obtained by assuming $dt \equiv \varepsilon$ becomes clear. The first difference is the absence of $\partial L/\partial t$ in Equation 3.18, meaning that the conservation equation is only valid for scleronomic systems where $\partial L/\partial t = 0$. The second important note is that for the chain rule expressed by Equation 4.60 to hold, q_i and $\dot{q}_i$ must be independent of each other. Since this is not generally true, therefore, the terms of Lagrangian function L must each be just a function of either q_i or $\dot{q}_i$ and not both. Considering the definition of the Lagrangian function (Equation 3.11), the last statement implies that the kinetic energy must be a quadratic function of generalized velocity, the potential energy shall not contain any velocity terms, and $\dot{q}_i$ cannot have any explicit terms of q_i. These conditions were also mentioned by Lanczos [29, p. 122]; however, we tried here to make these restrictions more clear. Lanczos adds to these limitations that this method is not suitable for mechanical systems with *gyroscopic terms* since the velocity terms are linear and not quadratic.

It is important to note that neither Hamilton's principle nor Lagrange's variational equations require that the kinetic energy to be a quadratic form. The only requirement set by these principles for the kinetic energy term is that it must be a positive-valued function of $\dot{q}_i$, q_i, t and homogeneous of second order with respect to $\dot{q}_i$ [44, p. 23], so that,

$$T\left(\lambda\dot{q}_i, q_i, t\right) = \lambda^2 T\left(\dot{q}_i, q_i, t\right), \tag{4.61}$$

where λ is any arbitrary constant.

Another important point is that the classical *energy equation in integral form* contains acceleration terms that are unaffected by the virtual displacements as they are of second order with respect to $\delta_d \boldsymbol{r}_i$. Thus, it is essentially impossible to investigate the irreversibility of the acceleration term in the Eulerian reference frame using d'Alembert's virtual displacements. Inherently, this method obtains the conservation

of the energy rate of a control volume by retrieving the acceleration from the zero order approximation of the velocities of the actual particles. The difficulties associated with the last statement, especially when using experimentally acquired data, is discussed in Section 4.10.

Having discussed the derivation and limitations of the classical energy equation in integral form, it is compared with the energy equation, obtained in Chapter 3, in the following section.

4.9 Comparison with the Classical Energy Equation in Integral Form

In the previous section, it was shown that the classical energy equation in integral form (Equation 4.51), for a control volume of Newtonian incompressible viscous flow and in the absence of body forces, is expressed by

$$\begin{aligned}\frac{d}{dt}\int_{CV}\left(\frac{1}{2}\rho u^2\right)dV = &-\int_{CS}\left(\frac{1}{2}\rho u^2\right)(\boldsymbol{u}_r\cdot\boldsymbol{n})\,dA\\ &+\int_{CS}-p\boldsymbol{u}\cdot\boldsymbol{n}\;dA+\int_{CS}\mu\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\boldsymbol{u}\right]\cdot\boldsymbol{n}dV\\ &-\overbrace{\int_{CV}\mu\;tr\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\nabla\boldsymbol{u}\right]dV}^{\text{viscous dissipation}}.\end{aligned}\tag{4.62}$$

Comparing Equation 4.62 with our energy rate equation, Equation 3.100,

$$\begin{aligned}\frac{D}{Dt}\int_{CV}\frac{1}{2}\rho\ u^2 dV + \int_{CS}\left(\frac{1}{2}\rho\ u^2\right)(\boldsymbol{u}-\boldsymbol{v}_{CS})\cdot\boldsymbol{n}dA \\ = \int_{CS} -p\boldsymbol{u}\cdot\boldsymbol{n}dA + \int_{CS}\mu\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\boldsymbol{u}\right]\cdot\boldsymbol{n}\ dV \\ \overbrace{-\int_{CV}\mu\ tr\left[\left(\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u}\right)\cdot\nabla\boldsymbol{u}\right]dV}^{\text{viscous dissipation}} \overbrace{-\int_{CS}\frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)\ dA}^{\text{non-commuting term}},\end{aligned} \quad (4.63)$$

one difference is noticed to be the existence of the non-commuting acceleration term. Examining the irreversibility of the Eulerian acceleration with respect to Jourdain's variational operator in Section 3.4, the non-commuting term was a result of obtaining the acceleration from the first order velocities, as per the definition of Jourdain's variational operator. Instead, if we choose the zeroth order velocities, the non-commuting term disappears and the conservative terms of our energy rate equation would match the classical ones.

Therefore, our energy rate equation essentially assumes that the particle accelerations can be obtained from first order approximation of the velocities. On the other hand, the classical energy equation considers the Eulerian acceleration obtained by a zeroth order approximation of the velocity field as the actual acceleration of the fluid particles.

Both energy rate equations, ours and the classical one, first integrate the kinetic energy of the control volume and then differentiate it to obtain the rate of kinetic energy. Therefore, both methods obtain the rate of average kinetic energy in the Eulerian reference frame. Conservation laws, however, are defined in the Lagrangian reference frame. As mentioned earlier, Stokes [30] calculated the first order velocities and showed that the time average of the Eulerian velocities lags the average of the Lagrangian velocities for oscillatory waves. Similarly, our energy rate equation

calculates the rate of change of kinetic energy of a control volume via first order velocities.

Our discussion here with regard to the non-commuting term will become more clear after the following section where the use of experimental data in the energy rate equation is considered.

4.10 Coupling the Energy Rate Equation with Experimental Data

As mentioned earlier, we have followed the methodology proposed by Benaroya and Wei in this chapter. They suggested that the terms representing the fluid dynamics in the energy rate equation can be evaluated experimentally, and then the structural response can be obtained by integrating the energy equation twice. Following their approach, we assume that the fluid dynamics terms are known. However, the only known parameter is usually the approximated velocity field obtained by *particle image velocimetry* (PIV) or *digital particle image velocimetry* (DPIV) methods. Thus, $\nabla \boldsymbol{u}$ and $D\boldsymbol{u}/Dt$ are not known, however, they can be approximated at the cost of accuracy.

In PIV or DPIV, generally, a group of physical particles are added in order to track the flow. A laser beam is converted to a planar sheet of light with some thickness that illuminates the particles. Then, a series of images are recorded that are apart from each other by a sufficiently small time. Comparing two consecutive images, there are various methods that can be used to estimate the displacements of the particles. A commonly used method is the *cross-correlation method*, where the images are each divided into a large number of windows, each containing a certain number of pixels. Then all the pixels in the first frame are correlated with those of the second frame.

Finally, the cross-correlation estimates the most probable displacement in each window, and the velocity is obtained by dividing the displacement by the time interval between the two images [45, pp.277-281]. Therefore, the velocity field measured via PIV or DPIV is partial and incomplete. Consequently, the applicability of the conservation equations, which are Lagrangian concepts, depends on how accurately the Lagrangian parameters can be estimated from the Eulerian incomplete observation.

One of the most sensible way to obtain a conservation law is by using Hamilton's principle which requires the configuration (particle position) to be known at least at two points in time. Therefore one measure of accuracy of the conservative laws in the Eulerian reference frame can be chosen to be how accurately they can estimate the actual (Lagrangian) particle trajectories. An interesting discussion about the effects of order of approximation on estimating the particle paths using the partially known Eulerian velocity field is provided by Price [5], which is summarized next.

Price considered the velocity field of a steady, irrotational vortex in a two dimensional plane by

$$\boldsymbol{u} = (u_r, u_a) = \left(0, \frac{c}{2\pi r}\right), \tag{4.64}$$

where u_r denotes the radial component of the velocity, u_a is the azimuthal component of the velocity, c is a constant, and r is the distance from the vortex center.

An irrotational vortex is an idealized flow with circular streamlines produced by the convergent flow into a drain. For an incompressible fluid, the net viscous force per unit volume is zero. Thus the EOMs of the flow are the inviscid Euler equations [28, pp.70, 142-143][5, pp.61-68].

Price chose a representative point (at $(x, y) = (0, 1)$ of Figure 4.5) and assumed that the Eulerian velocity is known at this point. Then, he estimated the Lagrangian path of a particle occupying that point at some time, say t_0, by using the zeroth order and first-order velocities of the Taylor series. For zeroth order the particle trajectory,

$\boldsymbol{x}_0(t)$, is obtained by

$$\boldsymbol{x}_0(t) = \int \boldsymbol{u}_0 dt, \tag{4.65}$$

where $\boldsymbol{u}_0$ is the zero-order velocity. Similarly, for the first order approximation, he used the velocity

$$\boldsymbol{u}_1 = \boldsymbol{u}_0 + \nabla \boldsymbol{u} \cdot \boldsymbol{x}(t), \tag{4.66}$$

and estimated the trajectory of the same particle as

$$\boldsymbol{x}_1(t) = \int \boldsymbol{u}_1 dt. \tag{4.67}$$

Then, Price computed the exact trajectory of that particle in cylindrical coordinates and compared them. His results are shown in Figure 4.5.

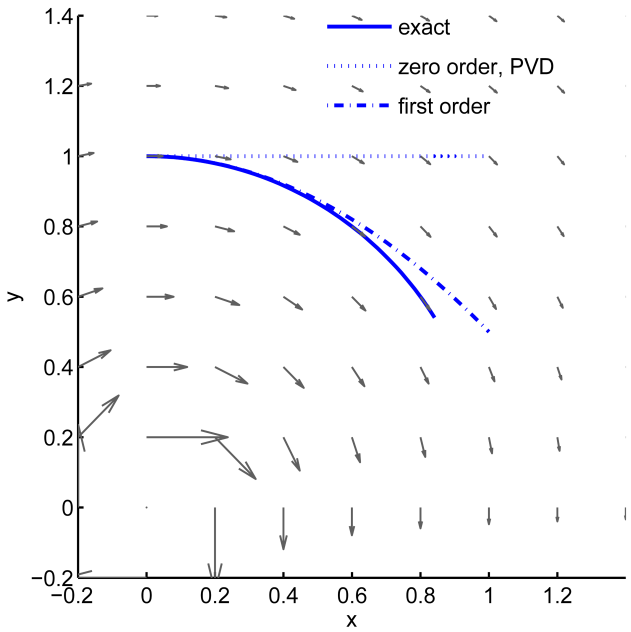

Figure 4.5: Comparison of the exact particle trajectory with those obtained by Taylor series up to zeroth and first order approximations of the Eulerian velocity [5] (with permission from Dr. James F. Price of Woods Hole Oceanographic Institution).

Based on the above discussion, let us assume that all the terms in the energy rate

equations are known except for the rate of kinetic energy term for the irrotational flow considered by Price. The particle path can be obtained by integrating the rate of kinetic energy twice (integrating the acceleration twice). Therefore, the results obtained from our energy rate equation would match the first order approximation of Figure 4.5, while the classical energy equation would result in the zeroth order approximation of that figure. Since the actual path represents the conservation law in the Lagrangian frame of reference, then we expect that our energy equation would increase the accuracy of the energy method. Note that we obtain the first order term purely theoretically. The discussion presented here with regard to experimental data is solely for clarity purposes and does not affect the theoretical developments in this work.

In deriving the energy equation, we utilized the incompressibility condition, $\nabla \cdot \boldsymbol{u} = 0$, along with Gauss' theorem to bring the non-commuting term and the viscous dissipative terms from inside the control volume to the control surface. When using PIV and DPIV, the velocity field is obtained partially. There exist different techniques to estimate the velocity field from those partial velocity fields. However, these techniques do not necessarily satisfy the incompressibility condition. For many VIV problems, the shear forces are order(s) of magnitude smaller than the kinetic energy terms. Thus using Gauss' theorem most likely does not cost much accuracy. However, regarding the non-commuting terms, care must be exercised and the order at which the incompressibility is violated must be compared with the kinetic energy terms. If $\nabla \cdot \boldsymbol{u}$ is significant, then perhaps the non-commuting terms must be brought back inside the control volume.

In Chapter 1, it was mentioned that the experiments of Benaroya and Wei [1] and their later experiments with Dong [4] proved that the accuracy of the structural response obtained using the energy equation depends on the choice of the control volume, specifically, the location of the control surface downstream to the structure.

The non-commuting term obtained also presents itself on the control surfaces. For the control surface upstream to the structure, the non-commuting term would disappear since the flow is nearly laminar. Therefore, the effect of the non-commuting term essentially would be on the control surface downstream to the flow.

The energy equation obtained and utilized by Benaroya, Wei and Dong differs from ours in the kinetic flux terms, mainly due to the non-commuting term, and in the viscous dissipation, as discussed. While we do not have access to their experimental data, a few remarks are worth mentioning, as follows.

Benaroya, Wei and Dong's preferred control surface (downstream to structure) was located at about the position where the vortices are well organized. Thus, perhaps the non-commuting term and viscous dissipation have cancelled each other out at that location. Viscosity exhausts the energy of the vortices and makes the streamlines expand, which at some point causes the zeroth order and the first order approximations almost coincide with the actual path. However, a bit further downstream, the dissipation would increase as the vortices structure break down to many small eddies of different sizes. At this point, since the dissipation scale may be smaller than the scale of the measurements, the velocities obtained experimentally no longer can be considered reversible by using the first order approximation.

Theoretically, there might exist a way to anticipate the correct energy equation when the downstream control surface is either close or far from the structure. This is the focus of our discussion in the following section.

4.11 Energy Equation for CVs with Extreme CSs

In Chapter 3, we obtained the conservative terms of the energy rate equation by applying Equation 3.24, which assumes the virtual velocity $\delta \boldsymbol{u}$ to be an arbitrary vector field if no constraints are present. Also, the viscous dissipative terms were obtained

by utilizing the Rayleigh dissipation function idea that constrained the direction of $\delta\boldsymbol{u}$ to that of $\boldsymbol{u}$. This dual use of $\delta\boldsymbol{u}$ is possible if the number of degrees of freedom affected by viscosity is substantially less than the total number of DOFs. For example, the relative velocities of some particles are small enough that the viscous work can be neglected for them. This assumption is a good approximation for the VIV problem where the majority of the control volume is occupied by fully-structured vortices. Also, we discussed that the non-commuting term of acceleration is of most interest for the control surface downstream to the structure. Therefore, regarding the choice of this control surface, two cases require specific consideration: one where it is very close to the solid structure, and the other where it is very far from the structure. We refer to these control surfaces as *extreme control surfaces*.

In VIV experiments, the existence of a formation region (cavity) is observed in vicinity of the structure and downstream to it [6]. Usually, this formation region contains a stagnation region where the fluid particles become trapped. In the formation region, the velocity of the flow is relatively slow. Thus we expect the viscous terms to be more dominant. Moreover, $\nabla\boldsymbol{u}$ is quite large in vicinity of structural surfaces (side faces which are nearly parallel to the flow). Thus the particles experience high accelerations. Downstream to the formation regions, the vortices become structured and the scale of dissipation is quite large. Therefore, the kinetic energy terms are more dominant. Depending on the dynamics of the system, at some point the large vortices brake down into smaller eddies of different sizes. Thus the turbulence scale of dissipation decreases and dissipation becomes more important. However, for a such control volume the kinetic energy terms are expected to be more dominant since the control volume still contains the region with structured vortices. If the scale of dissipation is smaller than what can be measured (due to the resolution of the measurements), the flow seems dissipative to the experimental observation, that is, some information is lost.

In the two cases mentioned, especially for the case where the control surface is too close to the structure, the conserved terms of the energy equation can no longer be considered reversible. If we choose not to use a statistical method, an estimate of energy equation can be obtained from the variational formulation as explained next.

As mentioned earlier, the Rayleigh dissipation function (RDF) constrains the virtual velocities to be in the same direction as the actual velocities, at the points where dissipation exists. If the dissipation is dominant in a majority of the control volume, the constraints imposed by viscosity on $\delta \boldsymbol{u}$ can be envisioned to be distributed throughout the domain covered by the control volume. Therefore, the virtual velocities corresponding to all of the terms in the extended Jourdain's principle are constrained similarly to the viscous terms. Consequently, we can obtain an energy rate equation by utilizing Equation 3.57 for all the terms (conservative or not) as follows.

The extended Jourdain's principle for a control volume of only fluid particles was obtained (Equation 3.62) as

$$\int_{CV} \rho \frac{D\boldsymbol{u}}{Dt} \cdot \delta \boldsymbol{u} \; dV = \int_{CV} \left\{ \nabla \cdot \left[-p\bar{\boldsymbol{I}} + \mu \left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u} \right) \right] \right\} \cdot \; \delta \boldsymbol{u} \; dV.$$

Regarding the left-hand side term of Equation 3.62, we wish to find a function ϕ_{left}, such that

$$\begin{aligned} \delta \phi_{\text{left}} &= \nabla_{\boldsymbol{u}} \phi_{\text{left}} \cdot \delta \boldsymbol{u} \\ &= \int_{CV} \rho \frac{D\boldsymbol{u}}{Dt} \cdot \delta \boldsymbol{u} \; dV. \end{aligned} \tag{4.68}$$

Considering the function

$$\phi_{\text{left}} = \frac{1}{2} \frac{D}{Dt} \left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right), \tag{4.69}$$

and using the definition of directional derivative in direction of $\boldsymbol{u}$ (Equation 3.56), we

obtain

$$\begin{aligned}
\nabla_{\boldsymbol{u}}\left[\frac{1}{2}\frac{D}{Dt}\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)\right]\cdot\boldsymbol{u} &= \frac{\partial}{\partial\varepsilon}\left\{\frac{1}{2}\frac{D}{Dt}\left[\frac{1}{2}\rho\left(\boldsymbol{u}+\varepsilon\boldsymbol{u}\right)\cdot\left(\boldsymbol{u}+\varepsilon\boldsymbol{u}\right)\right]\right\}\Bigg|_{\varepsilon=0} \\
&= \frac{\partial}{\partial\varepsilon}\left\{\frac{1}{2}\frac{D}{Dt}\left[\frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}+2\varepsilon\boldsymbol{u}\cdot\boldsymbol{u}+\varepsilon^2\boldsymbol{u}\cdot\boldsymbol{u}\right)\right]\right\}\Bigg|_{\varepsilon=0} \\
&= \frac{1}{2}\frac{D}{Dt}\left[\frac{1}{2}\rho\left(2\boldsymbol{u}\cdot\boldsymbol{u}+2\varepsilon\boldsymbol{u}\cdot\boldsymbol{u}\right)\right]\Bigg|_{\varepsilon=0} \\
&= \frac{1}{2}\frac{D}{Dt}\left(\rho\boldsymbol{u}\cdot\boldsymbol{u}\right) \\
&= \rho\frac{D\boldsymbol{u}}{Dt}\cdot\boldsymbol{u}. \qquad (4.70)
\end{aligned}$$

Therefore, by canceling the velocities from both sides,

$$\frac{1}{2}\nabla_{\boldsymbol{u}}\left[\frac{D}{Dt}\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)\right]=\rho\frac{D\boldsymbol{u}}{Dt}. \qquad (4.71)$$

Using Equation 4.71, the LHS of Equation 3.62 can be modified as

$$\begin{aligned}
\int_{CV}\rho\frac{D\boldsymbol{u}}{Dt}\cdot\delta\boldsymbol{u}\,dV &= \int_{CV}\frac{1}{2}\nabla_{\boldsymbol{u}}\left[\frac{D}{Dt}\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)\right]\cdot\delta\boldsymbol{u}\,dV \\
&= \frac{1}{2}\delta\int_{CV}\frac{D}{Dt}\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)\,dV. \qquad (4.72)
\end{aligned}$$

By applying the Reynolds' transport theorem in the form of Equation 3.5 to Equation 4.72, the LHS of Equation 3.62 can be expressed as

$$\int_{CV}\rho\frac{D\boldsymbol{u}}{Dt}\cdot\delta\boldsymbol{u}\,dV=\frac{1}{2}\delta\left\{\frac{D}{Dt}\int_{CV}\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\,dV+\int_{CS}\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)\left(\boldsymbol{u}-\boldsymbol{v}_{CS}\right)\cdot\boldsymbol{n}\,dV\right\}. \qquad (4.73)$$

We continue our derivation by considering the RHS of Equation 3.62. The pressure term is independent of velocity and the results remain the same as that obtained by

Equation 3.73, that is,

$$\int_{CV} \left[\nabla \cdot \left(-p\bar{\boldsymbol{I}}\right)\right] \cdot \delta \boldsymbol{u}\, dV = -\delta \int_{CS} p\boldsymbol{u} \cdot \boldsymbol{n}\, dA. \tag{4.74}$$

From Equation 3.77 for the terms containing the viscosity, it was shown that

$$\nabla_{\boldsymbol{u}} \left\{\frac{1}{2}\mu \left[\nabla \cdot \left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}\right)\right] \cdot \boldsymbol{u}\right\} = \mu \left[\nabla \cdot \left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}\right)\right], \tag{4.75}$$

where $\frac{1}{2}\mu \left[\nabla \cdot \left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}\right)\right] \cdot \boldsymbol{u} \equiv \phi$ in Equation 3.57. Therefore,

$$\delta \left\{\frac{1}{2}\mu \left[\nabla \cdot \left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}\right)\right] \cdot \ \boldsymbol{u}\right\} = \mu \left[\nabla \cdot \left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}\right)\right] \cdot \delta \boldsymbol{u}. \tag{4.76}$$

Substituting Equations 4.73, 4.74 and 4.76 into 3.62 yields

$$\begin{aligned} \frac{1}{2}\delta \left\{\frac{D}{Dt}\int_{CV} \frac{1}{2}\rho \boldsymbol{u} \cdot \boldsymbol{u} dV + \int_{CS} \left(\frac{1}{2}\rho \boldsymbol{u} \cdot \boldsymbol{u}\right)(\boldsymbol{u} - \boldsymbol{v}_{CS}) \cdot \boldsymbol{n} dV\right\} \\ = \delta \int_{CS} -p\ \boldsymbol{u} \cdot \boldsymbol{n} dA + \delta \int_{CV} \frac{1}{2}\mu \left[\nabla \cdot \left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}\right)\right] \cdot \boldsymbol{u} dV. \end{aligned} \tag{4.77}$$

In order to compare the new energy rate equation with the classical one, we replace $\mu \left[\nabla \cdot \left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}\right)\right] \cdot \boldsymbol{u}$ by its equivalent from Equation 3.75 and multiply both sides by two (as mentioned before, Rayleigh's dissipation function must be multiplied by two in order to obtain the energy rate equation), and obtain

$$\begin{aligned} \frac{D}{Dt}\int_{CV} \frac{1}{2}\rho \boldsymbol{u} \cdot \boldsymbol{u} dV + \int_{CS} \left(\frac{1}{2}\rho \boldsymbol{u} \cdot \boldsymbol{u}\right)(\boldsymbol{u} - \boldsymbol{v}_{CS}) \cdot \boldsymbol{n} dV \\ = \int_{CS} \left\{-2p\ \boldsymbol{u} \cdot \boldsymbol{n} + \mu \left[\left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}\right) \cdot \boldsymbol{u}\right] \cdot \boldsymbol{n}\right\} dA \\ - \int_{CV} \mu\ tr\left[\left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}\right) \cdot \nabla \boldsymbol{u}\right] dV. \end{aligned} \tag{4.78}$$

Comparing Equation 4.51 and Equation 4.78, we observe that all the terms are identical but the power due to pressure. The VIV systems are generally pressure driven fluid-structure interaction [4]. Therefore, Equation 4.78 predicts the structural oscillations with amplitudes as high as almost twice that estimated using the classical energy equation.

It is interesting to note that Benaroya and Wei [1] used their energy rate equation for the inverted pendulum problem. Their experimental control volume started at about the structure and mainly contained the formation region. The amplitude of the structural oscillation predicted from their energy equation was about half of that observed in their experiments.

Equation 4.78 implies that the viscosity might play a third role in addition to acting as external loads and to dissipate energy, that is, it can act as a nonholonomic constraint. Thus, for regions where the viscosity is dominant, the viscous forces may determine the direction of the motion of the fluid particles. While, as mentioned, this can explain the results of Benaroya and Wei's experiments, additional experiments must be conducted to investigate if this interpretation holds.

Equation 4.77 can be expanded similarly as in Section 3.7 in order to obtain terms that are more useful when modeling FSI systems. The result is

$$\frac{1}{2}\delta\left\{\frac{D}{Dt}\int\limits_{CV}\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\;dV+\int\limits_{CS}\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}_{CS})\cdot\boldsymbol{n}\;dA\right\}$$
$$=\delta\left\{\overbrace{\int\limits_{CS}-p\;\boldsymbol{u}\cdot\boldsymbol{n}\;+\frac{1}{2}\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}}^{\text{external loading}}\right.$$
$$\left.\overbrace{-\frac{1}{2}\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-2\;\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}\;dA-\frac{1}{2}\int\limits_{CV}\mu\left(\nabla\times\boldsymbol{u}\right)\cdot\left(\nabla\times\boldsymbol{u}\right)dV}^{\text{viscous dissipation}}\right\}. \quad (4.79)$$

In the following section, the chapter is concluded with a summary and few additional

remarks.

4.12 Summary

In this chapter, we obtained an energy rate equation for FSI systems. As examples we considered the model problem in the form of a translating cylinder and inverted pendulum. Then, the energy rate equation was compared as best as possible to McIver's extension of Hamilton's principle. Our discussions were limited to the theoretical aspects, since the required control volume for McIver's method cannot be easily distinguished for flows external to the structure. Also, we compared our formulation of the energy equation to that of Benaroya and Wei for both model problems.

Moreover, we used Benaroya and Wei's extension of Hamilton's principle to show the limitations of the classical energy equation in integral form. Afterwards, a comparison was made between our energy rate equation and the classical one.

Additionally, we considered the difficulties that might be faced when using the experimental data. Two specific extreme control volumes were considered and an energy rate equation was proposed which might be helpful in finding an estimated structural response. We anticipate that this equation would be useful for control volumes very close to the structure. Also, it may be of use when the control surface is far downstream to the solid structure.

We used Benaroya and Wei's methodology to extend our energy rate equation for systems of fluid-structure interaction. This method requires the pressure to be known at the surface of the structure. PIV and DPIV methods only estimate the velocity field. Thus, the pressure term is problematic. To obtain the pressure, Benaroya and Wei integrated the Navier-Stokes equation numerically. In our derivations we neglect the pressure terms by considering them to be action and reaction forces. However, the pressure remains a requirement for the open sections of the control surface.

The energy equation obtained here is a reduced-order model for FSI systems. It obtains the hydrodynamic forces on the structure by approximating the conservation of energy. For many applications, reduced-order models where two equations are coupled might be of interest. These models, which often referred to as wake-oscillator models, are considered in the next chapter.

Chapter 5

Modeling Fluid-Structure Interaction: Nonlinear Coupled Equations of Motion

In this chapter, we wish to obtain reduced-order models that are similar to wake-oscillator models. The main challenge is how to reduce the number of required equations without neglecting the main characteristics of the system. Reduced-order mathematical models are the backbone of our understanding of complex phenomena. The number of required variables, and consequently the number of required differential equations, are reduced by considering only certain physical variables and assuming boundary conditions. Then, the less important variables are neglected by the analytical formulation. Physical variables are selected based on experimental observations, having considered their participation in the solution of certain variables of interest. For example, the structural response is the focus of this research work, making the variables of interest to be the amplitudes of structural oscillation and the frequency at which it oscillates. However, this is a two-way street in that are approximate analytical model is required in advance of even a crude experiment.

Regarding vortex-induced vibration, as mentioned briefly in Chapter 1, there exist many parameters that have been found to affect the structural response. Consequently, many different interpretations of VIV have been made in the literature thus making it nearly impossible to derive physically-based mathematical models. As mentioned earlier, two well-written reviews of VIV experiments are the works by Gabbai and Benaroya [6] and by Sarpkaya [8]. Readers new to the subject may find themselves overwhelmed with number of different findings and occasionally contradictory

observations. The reduced-order models in the literature are mainly based on a set of assumed equations and curve fitting to a set of experimental data. As a result, it is nearly impossible to use those models to interpret and understand the vast amount of experimental findings. Therefore, in Section 5.1, we start by proposing a method that can be applied to the energy rate equation obtained earlier so that wake-oscillator type models are derived from first-principles.

In the previous chapter, we obtained a single degree-of-freedom (SDOF) model for fluid-structure interaction (FSI) systems. For the model of the previous chapter, the boundary conditions are important but are not crucial to the modeling process, since the structural response is estimated via averaging the energy that must be transferred to the structure. On the other hand, when obtaining a reduced-order model in the form of the wake-oscillator model, the boundary condition is extremely important as it essentially determines the nature of the coupling between the governing equations of motion.

As mentioned in Section 4.1, the no-slip condition is widely accepted in the literature. However, applicability of the no-slip condition or any other condition must be justified by a comparison with experimental observations. The no-slip condition, in its most complete form as expressed by Equation 4.1, satisfies the no-penetration conditions as well. Therefore, for reduced-order models, care must be exercised so that the reduced number of equations can capture the main characteristics of the FSI. For example, if the no-slip condition is strongly imposed, then the resulting reduced-order model most likely cannot model the separation of the vortices from solid structure, unless additional assumptions are made. Consequently, it might be necessary to relax the no-slip condition to some extent in analytical or numerical modeling. In fact, the no-slip condition has not been applied in its most complete form in many computational simulations or analytical models of FSI. It is implicitly implemented, for example, via other constraints in the system [46] or through an assumed force field

[47]. Moreover, the no-slip condition is often explicitly implemented and then is relaxed or corrected in some later steps in modeling process [48]. Alternatively, some authors decided to weakly impose the no-slip condition [38].

Regarding reduced-order models, the difficulties increase since they are less flexible with respect to the choice of boundary conditions. Consequently, a boundary condition other than the no-slip condition may be required for some problems. Therefore, we derive the variational energy rate equation without any consideration of boundary condition in Section 5.2. Then, we continue our derivations by considering the no-slip condition and implementing it implicitly in Section 5.3 and explicitly in Section 5.4. This terminology will become clear as we progress.

For reduced-order modeling, dimensional analysis, similarity methods and perturbation techniques have been acceptable methods. They have been used extensively in modeling fluid systems. However, such analysis may not be accurately performed unless experimental data is available for a problem in hand. A good reference regarding these methods is [49].

Therefore, we use the experimental findings of Dong *et al.* [4] to show the steps required for obtaining a model similar to the lift-oscillator model of Hartlen and Currie [9] in Section 5.6. We choose their model since it is one of the earliest and most successful wake-oscillator models. To accomplish this, we will not use similarity methods in a traditional way. However, we will make a few assumptions based on the experimental observations of Benaroya and Wei [1] and Dong et al. [4]. Then, we conclude the chapter in Section 5.7.

5.1 Coupled Equations of Motion: Conceptual Approach

Having obtained the variation of the rate of the Lagrangian function L in Chapter 4, we wish to obtain two nonlinear coupled equations that can fully describe the main

characteristics of the dynamics of a FSI system.

In analytical mechanics, the partial derivatives of the Lagrangian function with respect to velocities result in a fundamental concept called *generalized momentum* [29], p_i, defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \tag{5.1}$$

where $\dot{q}_i$ is the generalized velocity that can be selected to be either the Lagrangian or the Eulerian velocities.

The following derivations in many ways are similar to that concept as a result of the definition of Jourdain's variational operator (Equation 3.9). However, we are dealing with the rate of the Lagrangian function, $\dot{L}$ (or in the Eulerian reference frame, DL/Dt), instead. Therefore, applying Equation 3.9 would result in the rate of *generalized momenta.*

A fundamental concept of variational calculus states that, if $\psi\left(\boldsymbol{u}, \dot{\boldsymbol{r}}_s\right)$ is a set of monogenic potential scalar functions satisfying $\delta\psi\left(\boldsymbol{u}, \dot{\boldsymbol{r}}_s\right) = 0$, the relation $\psi\left(\boldsymbol{u}, \dot{\boldsymbol{r}}_s\right) = 0$ is a conservation equation. Consequently, the corresponding differential equations can be obtained by using the definition of Jourdain's principle and by choosing $\boldsymbol{u}$ and $\dot{\boldsymbol{r}}_s$ to be the independent generalized velocities. Thus,

$$\delta\psi = \frac{\partial \psi}{\partial \boldsymbol{u}} \cdot \delta \boldsymbol{u} + \frac{\partial \psi}{\partial \dot{\boldsymbol{r}}_s} \cdot \delta \dot{\boldsymbol{r}}_s = 0, \tag{5.2}$$

where $\partial\psi/\partial\boldsymbol{u}$ and $\partial\psi/\partial\dot{\boldsymbol{r}}_s$ are defined from Equation 3.8, as

$$\begin{aligned} \frac{\partial \psi}{\partial \boldsymbol{u}} \cdot \delta \boldsymbol{u} &= \left. \frac{\partial \psi\left(\boldsymbol{u} + \varepsilon \delta \boldsymbol{u}\right)}{\partial \varepsilon} \right|_{\varepsilon = 0} \\ &\equiv \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\psi\left(\boldsymbol{u} + \varepsilon\ \delta \boldsymbol{u}\right) - \psi\left(\boldsymbol{u}\right)\right], \end{aligned} \tag{5.3}$$

and

$$\begin{aligned}\frac{\partial \psi}{\partial \dot{\boldsymbol{r}}_s} \cdot \delta \dot{\boldsymbol{r}}_s &= \left.\frac{\partial \psi \left(\dot{\boldsymbol{r}}_s + \varepsilon \delta \dot{\boldsymbol{r}}_s\right)}{\partial \varepsilon}\right|_{\varepsilon=0} \\ &\equiv \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\psi \left(\dot{\boldsymbol{r}}_s + \varepsilon \delta \dot{\boldsymbol{r}}_s\right) - \psi \left(\dot{\boldsymbol{r}}_s\right)\right]. \end{aligned} \tag{5.4}$$

Using Equation 5.2, where $\psi = \dot{L}$,

$$\delta \dot{L} = \frac{\partial \dot{L}}{\partial \boldsymbol{u}} \cdot \delta \boldsymbol{u} + \frac{\partial \dot{L}}{\partial \dot{\boldsymbol{r}}_s} \cdot \delta \dot{\boldsymbol{r}}_s, \tag{5.5}$$

and considering that $\delta \dot{L} = 0$, results in

$$\frac{\partial \dot{L}}{\partial \boldsymbol{u}} \cdot \delta \boldsymbol{u} + \frac{\partial \dot{L}}{\partial \dot{\boldsymbol{r}}_s} \cdot \delta \dot{\boldsymbol{r}}_s = 0. \tag{5.6}$$

Since $\delta \boldsymbol{u}$ and $\delta \dot{\boldsymbol{r}}_s$ are arbitrary, non-zero vectors, Equation 5.6 results in

$$\frac{\partial \dot{L}}{\partial \boldsymbol{u}} = 0, \tag{5.7}$$

$$\frac{\partial \dot{L}}{\partial \dot{\boldsymbol{r}}_s} = 0, \tag{5.8}$$

that is, six coupled differential equations for a general fluid-solid system.

Regarding the model problem, the results are four coupled equations, Equation 5.8 is a single differential equation (since the structure has a single DOF) representing the solid structure and Equation 5.7 are three differential equations describing the flow.

Consequently, if we use $\partial \dot{L} / \partial \dot{\boldsymbol{r}}_s = 0$ to obtain the structural governing equation and substitute it in Equation 5.6, instead of Equation 5.7, we have

$$\frac{\partial \dot{L}}{\partial \boldsymbol{u}} \cdot \delta \boldsymbol{u} = 0. \tag{5.9}$$

Defining the partial scalar potential function $\delta\dot{L}_{\boldsymbol{u}}$ as

$$\delta\dot{L}_{\boldsymbol{u}} = \frac{\partial\dot{L}}{\partial\boldsymbol{u}} \cdot \delta\boldsymbol{u}, \tag{5.10}$$

the coupled set of equations represented by Equations 5.7 and 5.8 becomes

$$\begin{cases} \delta\dot{L}_{\boldsymbol{u}} = 0 \\ \dfrac{\partial\dot{L}}{\partial\dot{\boldsymbol{r}}_s} = 0. \end{cases} \tag{5.11}$$

Since $\dot{L}_{\boldsymbol{u}}$ is a set of potential functions satisfying $\delta\dot{L}_{\boldsymbol{u}} = 0$, then $\dot{L}_{\boldsymbol{u}} = 0$ is the corresponding conservation equation. Therefore, Equations 5.11 become

$$\begin{cases} \dot{L}_{\boldsymbol{u}} = 0 \\ \dfrac{\partial\dot{L}}{\partial\dot{\boldsymbol{r}}_s} = 0, \end{cases} \tag{5.12}$$

that is, two coupled equations; $\partial\dot{L}/\partial\dot{\boldsymbol{r}}_s = 0$ is an ordinary differential equation representing the participation of the structural terms, and $\dot{L}_{\boldsymbol{u}} = 0$ is an integral equation representing the fluid dynamics.

Next, we derive the variational energy rate equation without implementation of any boundary conditions.

5.2 General Variational Energy Rate Equation for FSI

We start the modeling process from the variational formulation of the FSI without considering the boundary conditions, as follows. Considering Equation 4.8, we neglect the fluid-structure interaction force density, $\boldsymbol{f}_s^*(\boldsymbol{r}_s)$, and obtain the variational

formulation in its general form to be

$$
\begin{cases}
\delta \Bigg\{ \dfrac{D}{Dt} \displaystyle\int_{CV} \frac{1}{2}\rho \boldsymbol{u}\cdot\boldsymbol{u} dV + \int_{CS} \left(\frac{1}{2}\rho \boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}_{CS})\cdot\boldsymbol{n} dA \\
\qquad \overbrace{- \displaystyle\int_{CS} \{- p\boldsymbol{u}\cdot\boldsymbol{n} + \mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}}^{\text{loading external to the fluid}} \\
\overbrace{- \frac{1}{2}\mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - 2\,\boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}}^{\text{viscous dissipation}} \Bigg\} dA + \frac{1}{2}\displaystyle\int_{CV} \mu(\nabla\times\boldsymbol{u})\cdot(\nabla\times\boldsymbol{u}) dV \\
\qquad \overbrace{+ \displaystyle\int_{CS} \frac{1}{2}\rho(\boldsymbol{u}\cdot\boldsymbol{u})(\boldsymbol{u}\cdot\boldsymbol{n}) dA}^{\text{non-commuting term}} \Bigg\} \\
\qquad + \dfrac{d}{dt}\displaystyle\int_{V_s} \left[\rho_s \frac{d\dot{\boldsymbol{r}}_s}{dt} - \boldsymbol{f}_s(\boldsymbol{r}_s)\right]\cdot\delta\boldsymbol{r}_s dV = 0 \\
\delta t = 0 \\
\delta \boldsymbol{r}_s = 0 \\
\dfrac{d}{dt}(\delta\boldsymbol{r}_s) \neq 0,
\end{cases}
\tag{5.13}
$$

where the boundary conditions are not included. Equation 5.13 can be applied to any FSI system where a Newtonian incompressible viscous fluid is considered. The boundary conditions can be included via a variety of methods, explicitly or implicitly.

For the model problem in the form of the translating cylinder, Equation 5.13

becomes

$$\delta\left\{\frac{D}{Dt}\int\limits_{CV}\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}dV+\int\limits_{CS}\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\boldsymbol{v}_{CS})\cdot\boldsymbol{n}dA\right.$$
$$\overbrace{-\int\limits_{CS}\{-\,p\boldsymbol{u}\cdot\boldsymbol{n}\;+\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}}^{\text{loading external to the fluid}}$$
$$\overbrace{-\frac{1}{2}\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-2\;\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}\Big\}dA+\frac{1}{2}\int\limits_{CV}\mu\left(\nabla\times\boldsymbol{u}\right)\cdot\left(\nabla\times\boldsymbol{u}\right)dV}^{\text{viscous dissipation}}$$
$$\left.+\overbrace{\int\limits_{CV}\nabla\cdot\left[\frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\boldsymbol{u}\right]dV}^{\text{non-commuting term}}\right\}$$
$$+\delta\left(m_s\ddot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s+\frac{1}{2}C\dot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s+k\boldsymbol{x}_s\cdot\dot{\boldsymbol{x}}_s\right)=0,\quad(5.14)$$

where we used the Gauss' divergence theorem to take the non-commuting term inside the control volume for the reasons explained in Section 4.10.

Equation 5.14 can also be expressed as

$$\delta\left\{\int\limits_{CV}\frac{D}{Dt}\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)dV\overbrace{-\int\limits_{CS}\{-\,p\boldsymbol{u}\cdot\boldsymbol{n}\;+\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}}^{\text{loading external to the fluid}}\right.$$
$$\overbrace{-\frac{1}{2}\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-2\;\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}\Big\}dA+\frac{1}{2}\int\limits_{CV}\mu\left(\nabla\times\boldsymbol{u}\right)\cdot\left(\nabla\times\boldsymbol{u}\right)dV}^{\text{viscous dissipation}}$$
$$\left.+\overbrace{\int\limits_{CV}\nabla\cdot\left[\frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\boldsymbol{u}\right]dV}^{\text{non-commuting term}}\right\}$$
$$+\delta\left(m_s\ddot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s+\frac{1}{2}C\dot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s+k\boldsymbol{x}_s\cdot\dot{\boldsymbol{x}}_s\right)=0,\quad(5.15)$$

where we used Equation 3.122 instead of Equation 3.123 for the fluid terms. Equations

5.14 and 5.15 are similar equations, yet they have some advantages over one another depending on the selection of the control volume.

Regarding the inverted pendulum problem, the equations are similar. The only difference is that $m_s\ddot{\boldsymbol{x}}_s \cdot \dot{\boldsymbol{x}}_s + \frac{1}{2}C\dot{\boldsymbol{x}}_s \cdot \dot{\boldsymbol{x}}_s + k\boldsymbol{x}_s \cdot \dot{\boldsymbol{x}}_s$ must be replaced with $I_0\ddot{\theta}\dot{\theta} + \frac{1}{2}C_T\dot{\theta}^2 + k_T\theta\dot{\theta} - \left[(m_s g - B)\,\frac{L}{2}\sin\theta\right]\dot{\theta}$.

Having obtained the variational energy rate equation for the model problem (Equations 5.14 and 5.15), we continue by implementing the no-slip condition implicitly and explicitly in the following two sections.

5.3 Reduced-Order Model with an Implicitly Implemented BC

The applicability of an analytical formulation is greatly affected by the selection of the control volume. For reasons that will become apparent, we consider a more restricted form of the control volume shown in Figure 4.2, where the control volume moves rigidly with the structure. Therefore,

$$\boldsymbol{v}_{CS} = \boldsymbol{v}_{\text{structure}}. \tag{5.16}$$

No other requirements are necessary other than some portions of the control surface must be the solid surfaces. A schematic drawing of this control volume is shown in Figure 5.1, where the open portion of the control volume is selected to be rectangular for simplicity.

We denote the velocity of the structure with $\dot{\boldsymbol{x}}_s$, therefore,

$$\dot{\boldsymbol{x}}_s = \boldsymbol{v}_{CS}. \tag{5.17}$$

In Section 4.3, we discussed that by excluding the dissipative terms the remaining

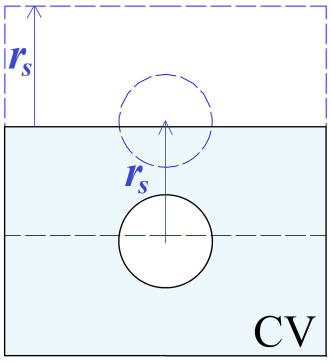

Figure 5.1: The selected control volume where the control volume moves rigidly with the structure. $\boldsymbol{r}_s$ denotes the displacement of structure.

viscous terms together with the pressure represent the external loads on the closed surface and they can be viewed as conservative action and reaction forces where the no-slip condition is considered. Therefore, going through the same steps shown there, Equation 5.14 becomes

$$\delta\left\{\frac{D}{Dt}\int_{CV}\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\,dV+\int_{CV}\nabla\cdot\left[\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\dot{\boldsymbol{x}}_s)\right]dV\right.$$
$$-\int_{CS_O}\left[-p\boldsymbol{u}\cdot\boldsymbol{n}+\frac{1}{2}\mu\nabla(\boldsymbol{u}\cdot\boldsymbol{u})\cdot\boldsymbol{n}\right]dA$$
$$+\int_{CS_C}\frac{1}{2}\mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u})-2\,\boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}\,dA$$
$$\left.+\frac{1}{2}\int_{CV}\mu(\nabla\times\boldsymbol{u})\cdot(\nabla\times\boldsymbol{u})\,dV+\int_{CV}\nabla\cdot\left[\frac{1}{2}\rho(\boldsymbol{u}\cdot\boldsymbol{u})\,\boldsymbol{u}\right]\,dV\right\}$$
$$+\delta\left(m_s\ddot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s+\frac{1}{2}C\dot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s+k\boldsymbol{x}_s\cdot\dot{\boldsymbol{x}}_s\right)=0,\quad(5.18)$$

where the term $\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\dot{\boldsymbol{x}}_s)$ is taken inside the control volume, assuming that the fluid particles can shed from the structure. Thus, the no-slip condition is implemented by neglecting the viscous and pressure force at the solid structure in Equation

5.18. The no-penetration condition will be satisfied by integration over the appropriate control volume.

Having obtained the variational energy rate equation, we wish to obtain two coupled governing equations by using Equation 5.12, via the methodology explained in Section 5.1. We start by applying Equation 5.3 to Equation 5.18 in order to obtain the rate of generalized momentum associated with $\boldsymbol{u}$. However, we are just interested in the partial scalar potential $\dot{L}_{\boldsymbol{u}}$. Clearly, the potential terms not containing the flow velocity $\boldsymbol{u}$ will drop out and the terms purely expressed by $\boldsymbol{u}$ remain the same. Moreover, the potential functions of both velocities $\boldsymbol{u}$ and $\dot{\boldsymbol{x}}_s$ remain the same. However, the last statement is shown mathematically for clarity, as follows.

The only integral containing both velocities $\boldsymbol{u}$ and $\dot{\boldsymbol{x}}_s$ in Equation 5.18 is

$$\delta \int_{CV} \nabla \cdot \left[\left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) (\boldsymbol{u} - \dot{\boldsymbol{x}}_s) \right] dV \\ = \delta \int_{CV} \nabla \cdot \left[\left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) \boldsymbol{u} \right] dV - \delta \int_{CV} \nabla \cdot \left[\left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) \dot{\boldsymbol{x}}_s \right] dV, \quad (5.19)$$

where we expanded the equation. The first integral on the right is a function of just velocity $\boldsymbol{u}$, thus it remains unchanged.

We apply the Gauss' divergence theorem to the second integral on the right of Equation 5.19,

$$\delta \int_{CV} \nabla \cdot \left[\left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) \dot{\boldsymbol{x}}_s \right] dV = \delta \int_{CS} \left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) (\dot{\boldsymbol{x}}_s \cdot \boldsymbol{n}) \, dA. \quad (5.20)$$

We need to ensure that the variation of the integral on the right-hand side is non-zero.

Considering the integral on the right-hand side of Equation 5.20, we have

$$\delta \int_{CS} \left[\left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) (\dot{\boldsymbol{x}}_s \cdot \boldsymbol{n}) \right] dA$$
$$= \int_{CS} \left[\delta \left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) (\dot{\boldsymbol{x}}_s \cdot \boldsymbol{n}) \right] dA + \int_{CS} \left[\left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) \delta (\dot{\boldsymbol{x}}_s \cdot \boldsymbol{n}) \right] dA. \quad (5.21)$$

Since we are considering the variational operator with respect to $\boldsymbol{u}$ only, we have

$$\delta (\dot{\boldsymbol{x}}_s \cdot \boldsymbol{n}) = 0. \quad (5.22)$$

The integrand of the first integral on the right-hand side of Equation 5.21 becomes

$$\int_{CS} \left[\delta \left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) (\dot{\boldsymbol{x}}_s \cdot \boldsymbol{n}) \right] dA = \int_{CS} (\rho \boldsymbol{u} \cdot \delta \boldsymbol{u}) (\dot{\boldsymbol{x}}_s \cdot \boldsymbol{n}) \, dA. \quad (5.23)$$

Substituting Equations 5.22 and 5.23 into Equation 5.21, we obtain

$$\begin{aligned} \delta \int_{CS} \left[\left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) (\dot{\boldsymbol{x}}_s \cdot \boldsymbol{n}) \right] dA &= \int_{CS} (\rho \boldsymbol{u} \cdot \delta \boldsymbol{u}) (\dot{\boldsymbol{x}}_s \cdot \boldsymbol{n}) \, dA \\ &\neq 0. \end{aligned} \quad (5.24)$$

Therefore, the term $\left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u}\right) (\dot{\boldsymbol{x}}_s \cdot \boldsymbol{n})$ remains in the formulation, and $\delta \dot{L}_{\boldsymbol{u}}$ is found

to be

$$\delta\dot{L}_{\boldsymbol{u}} = \delta\left\{\frac{D}{Dt}\int\limits_{CV}\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\;dV + \int\limits_{CV}\nabla\cdot\left[\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\dot{\boldsymbol{x}}_s)\right]dV\right.$$
$$-\int\limits_{CS_O}\left[-p\boldsymbol{u}\cdot\boldsymbol{n} + \frac{1}{2}\mu\nabla(\boldsymbol{u}\cdot\boldsymbol{u})\cdot\boldsymbol{n}\right]dA$$
$$+\int\limits_{CS_C}\frac{1}{2}\mu\left[\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - 2\;\boldsymbol{u}\times(\nabla\times\boldsymbol{u})\right]\cdot\boldsymbol{n}\;dA$$
$$\left.+\frac{1}{2}\int\limits_{CV}\mu(\nabla\times\boldsymbol{u})\cdot(\nabla\times\boldsymbol{u})\,dV + \int\limits_{CV}\nabla\cdot\left[\frac{1}{2}\rho(\boldsymbol{u}\cdot\boldsymbol{u})\,\boldsymbol{u}\right]\;dV\right\}. \quad (5.25)$$

The equation representing the participation of the fluid dynamic parameters is obtained from Equation 5.12 by setting $\dot{L}_{\boldsymbol{u}} = 0$.

For the structural terms, we apply Equation 5.4 to the remaining terms of energy equation (Equation 5.18), which are

$$\delta\int\limits_{CV}\nabla\cdot\left[\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)(\boldsymbol{u}-\dot{\boldsymbol{x}}_s)\right]dV +$$
$$\delta\left(m_s\ddot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s + \frac{1}{2}C\dot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s + k\boldsymbol{x}_s\cdot\dot{\boldsymbol{x}}_s\right) = 0. \quad (5.26)$$

Considering the structural terms, we have

$$\begin{aligned}\delta\left(m_s\ddot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s + \frac{1}{2}C\dot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s + k\boldsymbol{x}_s\cdot\dot{\boldsymbol{x}}_s\right) &= (m_s\ddot{\boldsymbol{x}}_s + C\dot{\boldsymbol{x}}_s + k\boldsymbol{x}_s)\cdot\delta\dot{\boldsymbol{x}}_s \\ &= (m_s\ddot{x}_s + C\dot{x}_s + kx_s)\,\delta\dot{x}_s, \quad (5.27)\end{aligned}$$

and for the flux of the kinetic energy term, we have

$$\delta \int_{CV} \nabla \cdot \left[\left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) (\boldsymbol{u} - \dot{\boldsymbol{x}}_s) \right] dV$$

$$= \overbrace{\delta \int_{CV} \nabla \cdot \left[\left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) \boldsymbol{u} \right] dV}^{=0} - \delta \int_{CV} \nabla \cdot \left[\left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) \dot{\boldsymbol{x}}_s \right] dV. \quad (5.28)$$

Manipulating it by using the Gauss' theorem to obtain

$$\begin{aligned} \delta \int_{CV} \nabla \cdot \left[\left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) \dot{\boldsymbol{x}}_s \right] dV &= \delta \int_{CS} \left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) (\dot{\boldsymbol{x}}_s \cdot \boldsymbol{n}) \, dA \\ &= \int_{CS} \left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) (\delta \dot{\boldsymbol{x}}_s \cdot \boldsymbol{n}) \, dA \\ &= \delta \dot{x}_s \left\{ \int_{CV} \nabla \left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) \cdot \mathbf{e}_x dV \right\}, \quad (5.29) \end{aligned}$$

where $\mathbf{e}_x$ is the unit vector in direction of $\dot{\boldsymbol{x}}_s$. Substituting Equation 5.29 into Equation 5.28 yields

$$\delta \int_{CV} \nabla \cdot \left[\left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) (\boldsymbol{u} - \dot{\boldsymbol{x}}_s) \right] dV = \delta \dot{x}_s \left\{ \int_{CV} \nabla \left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) \cdot \mathbf{e}_x dV \right\}. \quad (5.30)$$

Introducing Equations 5.27 and 5.30 into Equation 5.26 as

$$-\delta \dot{x}_s \left\{ \int_{CV} \nabla \left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) \cdot \mathbf{e}_x dV \right\} + \left(m_s \ddot{x}_s + C \dot{x}_s + k x_s \right) \delta \dot{x}_s = 0, \quad (5.31)$$

and considering the $\delta \dot{x}_s$ is arbitrary, the structural term is obtained to be

$$m_s \, \ddot{x}_s + C \, \dot{x}_s + k \, x_s = \int_{CV} \nabla \left(\frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u} \right) \cdot \mathbf{e}_x dV. \quad (5.32)$$

Finally, the coupled equations are obtained from Equations 5.25 and 5.32 to be

$$\begin{cases} m_s\ddot{x}_s + C\dot{x}_s + kx_s = \int\limits_{CV} \nabla\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot\mathbf{e}_x dV, \\ \\ \frac{D}{Dt}\int\limits_{CV}\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\;dV + \int\limits_{CV}\nabla\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot\boldsymbol{u}dV \\ \qquad\qquad - \int\limits_{CS_O}\left[-\,p\boldsymbol{u}\cdot\boldsymbol{n}\; + \frac{1}{2}\mu\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot\boldsymbol{n}\right]dA \\ \qquad\quad + \int\limits_{CS_C}\frac{1}{2}\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right) - 2\;\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}\;dA \\ + \int\limits_{CV}\frac{1}{2}\mu\left(\nabla\times\boldsymbol{u}\right)\cdot\left(\nabla\times\boldsymbol{u}\right)dV + \int\limits_{CV}\nabla\left[\frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\right]\cdot\boldsymbol{u}\;dV \\ \qquad\qquad = \int\limits_{CV}\nabla\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot\dot{\boldsymbol{x}}_s dV. \end{cases} \tag{5.33}$$

Having obtained the coupled equations for the model problem, next we show that a different equation can be obtained by a different approach towards the boundary conditions.

5.4 Reduced-Order Model with Explicit Implementation of the No-Slip Condition

In the previous section, we obtained a reduced-order model by implementing the no-slip condition implicitly, that is, by assuming the fluid forces at the solid surface have an equal and opposite reaction. Therefore, they do not affect the total energy of a system (based on d'Alembert's principle). Alternatively, in this section, we show that a different approach can be used if we wish to keep the interacting forces at the solid surface. Similar to the previous section, we assume that the no-slip condition holds (note that Dong *et al.* [4] did not consider any boundary conditions and yet they obtained a control volume where the predictions of their analytical model matched

the experimental observation with very good accuracy). Moreover, we choose the control volume shown in Figure 4.2. The open portion of control volume can contain closed surfaces and it may be selected to have other shapes other than the rectangle shown in that figure.

The no-slip condition implies that

$$\boldsymbol{u} - \dot{\boldsymbol{x}}_s = 0, \qquad \text{or,} \qquad \dot{\boldsymbol{x}}_s - \boldsymbol{u} = 0, \qquad \text{at the solid surface.} \tag{5.34}$$

Therefore, we wish to introduce Equation 5.34 to Equation 5.15. Since the virtual velocities must be compatible with system constraints and the no-slip condition is simply a nonholonomic constraint, we have

$$\delta\boldsymbol{u} - \delta\dot{\boldsymbol{x}}_s = 0, \qquad \text{or,} \qquad \delta\dot{\boldsymbol{x}}_s - \delta\boldsymbol{u} = 0, \qquad \text{at the solid surface.} \tag{5.35}$$

From the derivations of the previous chapter and by dimensional consideration, we can add the zero term

$$\delta \int\limits_{CS_C} \left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)(\dot{\boldsymbol{x}}_s - \boldsymbol{u})\cdot\boldsymbol{n}dA = 0. \tag{5.36}$$

Since Equation 5.36 equals zero for all time, we can add it to Equation 5.15 to obtain

$$\delta\left\{\int_{CV}\frac{D}{Dt}\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)dV+\overbrace{\int_{CS_C}\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)(\dot{\boldsymbol{x}}_s-\boldsymbol{u})\cdot\boldsymbol{n}dA}^{\text{no-slip condition}}\right.$$
$$\overbrace{-\int_{CS}\{-p\boldsymbol{u}\cdot\boldsymbol{n}+\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}}^{\text{loading external to the fluid}}$$
$$\overbrace{-\frac{1}{2}\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)-2\,\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}}^{\text{viscous dissipation}}\Big\}dA+\int_{CV}\frac{1}{2}\mu\left(\nabla\times\boldsymbol{u}\right)\cdot\left(\nabla\times\boldsymbol{u}\right)dV$$
$$\left.+\overbrace{\int_{CV}\nabla\cdot\left[\frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\boldsymbol{u}\right]\,dV}^{\text{non-commuting term}}\right\}$$
$$+\delta\left(m_s\ddot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s+\frac{1}{2}C\dot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s+k\boldsymbol{x}_s\cdot\dot{\boldsymbol{x}}_s\right)=0. \quad (5.37)$$

Summation of the viscous terms and expansion of the second and last integrals of Equation 5.37 yields for the problem at hand

$$\delta\left\{\int_{CV}\rho\frac{D}{Dt}\left(\frac{1}{2}u^2\right)dV+\int_{CS_C}\rho\left(\frac{1}{2}u^2\right)(\dot{\boldsymbol{x}}_s\cdot\boldsymbol{n})\,dA-\int_{CS_C}\rho\left(\frac{1}{2}u^2\right)(\boldsymbol{u}\cdot\boldsymbol{n})\,dA\right.$$
$$-\int_{CS}\left[-p\boldsymbol{u}\cdot\boldsymbol{n}+\mu\nabla\left(\frac{1}{2}u^2\right)\cdot\boldsymbol{n}\right]dA+\int_{CV}\frac{1}{2}\mu\left(\nabla\times\boldsymbol{u}\right)\cdot\left(\nabla\times\boldsymbol{u}\right)dV$$
$$\left.+\int_{CS_C}\rho\left(\frac{1}{2}u^2\right)(\boldsymbol{u}\cdot\boldsymbol{n})\;dA+\int_{CS_O}\rho\left(\frac{1}{2}u^2\right)(\boldsymbol{u}\cdot\boldsymbol{n})\;dA\right\}$$
$$+\delta\left(m_s\ddot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s+\frac{1}{2}C\dot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s+k\boldsymbol{x}_s\cdot\dot{\boldsymbol{x}}_s\right)=0. \quad (5.38)$$

Summing terms of the form $\int_{CS_C}\rho\left(\frac{1}{2}u^2\right)(\boldsymbol{u}\cdot\boldsymbol{n})\,dA$ and applying Gauss' theorem to

the viscous terms in the integrand of the forth integral as

$$\begin{aligned}\int_{CS} \frac{1}{2}\mu\nabla\left(u^2\right)\cdot\boldsymbol{n}dA &= \int_{CV} \mu\nabla\cdot\left[\nabla\left(\frac{1}{2}u^2\right)\right]dV \\ &= \int_{CV} \mu\nabla^2\left(\frac{1}{2}u^2\right)dV, \end{aligned} \tag{5.39}$$

we obtain,

$$\begin{aligned}\delta\Bigg\{\int_{CV}\rho\frac{D}{Dt}\left(\frac{1}{2}u^2\right)dV + \int_{CS_C}\rho\left(\frac{1}{2}u^2\right)\left(\dot{\boldsymbol{x}}_s\cdot\boldsymbol{n}\right)dA + \int_{CS_O}\rho\left(\frac{1}{2}u^2\right)\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)\,dA \\ + \int_{CV}\left[\nabla p\cdot\boldsymbol{u} - \mu\nabla^2\left(\frac{1}{2}u^2\right)\cdot\boldsymbol{n} + \frac{1}{2}\mu\left(\nabla\times\boldsymbol{u}\right)\cdot\left(\nabla\times\boldsymbol{u}\right)\right]dV \\ + m_s\ddot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s + \frac{1}{2}C\dot{\boldsymbol{x}}_s\cdot\dot{\boldsymbol{x}}_s + k\boldsymbol{x}_s\cdot\dot{\boldsymbol{x}}_s\Bigg\} = 0. \end{aligned} \tag{5.40}$$

Applying Equation 5.12 to Equation 5.40 results in

$$\begin{cases} m_s\ddot{x}_s + C\dot{x}_s + kx_s = -\displaystyle\int_{CS_C}\rho\left(\frac{1}{2}u^2\right)\left(\mathbf{e}_x\cdot\boldsymbol{n}\right)dA, \\ \\ \displaystyle\int_{CV}\rho\frac{D}{Dt}\left(\frac{1}{2}u^2\right)dV + \int_{CS_O}\rho\left(\frac{1}{2}u^2\right)\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)\,dA \\ \displaystyle + \int_{CV}\left[\nabla p\cdot\boldsymbol{u} - \mu\nabla^2\left(\frac{1}{2}u^2\right) + \frac{1}{2}\mu\left(\nabla\times\boldsymbol{u}\right)\cdot\left(\nabla\times\boldsymbol{u}\right)\right]dV \\ \displaystyle = -\dot{x}_s\int_{CS_C}\rho\left(\frac{1}{2}u^2\right)\left(\mathbf{e}_x\cdot\boldsymbol{n}\right)dA, \end{cases} \tag{5.41}$$

where the required steps are similar to those of the previous section.

Thus far, we have obtained two possible reduced-order models of the model problem in the form of translating cylinder. Similar reduced-order models for the inverted pendulum problem are shown next.

5.5 Reduced-Order Models: Inverted Pendulum Problem

The required steps for obtaining reduced-order models for the inverted pendulum is very similar to those of the translating cylinder problem. The following are the results for the two methods of considering the no-slip condition shown in Sections 5.3 and 5.4.

Implementing the no-slip condition implicitly as was done in Section 5.3, we found the reduced-order model of the inverted pendulum to be

$$
\begin{cases}
I_0\ddot{\theta} + C_T\dot{\theta} + k_T\theta - (m_s g - B)\,\dfrac{L}{2}\sin\theta = \displaystyle\int_{CV} \nabla\left(\frac{1}{2}\rho \boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot \mathbf{e}_\theta dV, \\[2ex]
\dfrac{D}{Dt}\displaystyle\int_{CV}\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\;dV + \int_{CV}\nabla\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot\boldsymbol{u}dV \\
\qquad - \displaystyle\int_{CS_O}\left[-\,p\boldsymbol{u}\cdot\boldsymbol{n} \;+\frac{1}{2}\mu\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot\boldsymbol{n}\right]dA \\
\qquad + \displaystyle\int_{CS_C}\frac{1}{2}\mu\left[\nabla\left(\boldsymbol{u}\cdot\boldsymbol{u}\right) - 2\;\boldsymbol{u}\times\left(\nabla\times\boldsymbol{u}\right)\right]\cdot\boldsymbol{n}\;dA \\
+\displaystyle\int_{CV}\frac{1}{2}\mu\left(\nabla\times\boldsymbol{u}\right)\cdot\left(\nabla\times\boldsymbol{u}\right)dV + \int_{CV}\nabla\left[\frac{1}{2}\rho\left(\boldsymbol{u}\cdot\boldsymbol{u}\right)\right]\cdot\boldsymbol{u}\;dV \\
\qquad = \dot{\theta}\displaystyle\int_{CV}\nabla\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot\mathbf{e}_\theta dV.
\end{cases}
\tag{5.42}
$$

Also, implementation of the no-slip condition explicitly as shown in Section 5.4, yields

$$\begin{cases} I_0\ddot{\theta} + C_T\dot{\theta} + k_T\theta - (m_s g - B)\,\dfrac{L}{2}\sin\theta = -\displaystyle\int_{CS_C} \rho\left(\frac{1}{2}u^2\right)(\mathbf{e}_x\cdot\boldsymbol{n})\,dA, \\ \displaystyle\int_{CV}\rho\frac{D}{Dt}\left(\frac{1}{2}u^2\right)dV + \int_{CS_O}\rho\left(\frac{1}{2}u^2\right)(\boldsymbol{u}\cdot\boldsymbol{n})\;dA \\ \qquad + \displaystyle\int_{CV}\left[\nabla p\cdot\boldsymbol{u} - \mu\nabla^2\left(\frac{1}{2}u^2\right) + \frac{1}{2}\mu\,(\nabla\times\boldsymbol{u})\cdot(\nabla\times\boldsymbol{u})\right]dV \\ \qquad\qquad = -\dot{\theta}\displaystyle\int_{CV}\nabla\left(\frac{1}{2}\rho\boldsymbol{u}\cdot\boldsymbol{u}\right)\cdot\mathbf{e}_\theta dV. \end{cases} \tag{5.43}$$

As evident from Equations 5.33, 5.41, 5.42 and 5.43 depending on our interpretation of the boundary conditions and on how we choose to implement it, the resulting reduced order model would differ to some extent. Both Equations 5.33 and 5.41 or any other models obtained by implementing the boundary conditions differently than those shown can be simplified further by dimensional analysis, similarity methods and perturbation techniques. However, experimental data and observations are required for further simplifications. In order to show the steps required, we consider the *lift-oscillator model* proposed by Hartlen and Currie and show how a similar equation can be obtained.

5.6 Comparison with Hartlen and Currie's Method

Our main goal in this research work has been to obtain wake-oscillator type equations, which we accomplished in the previous sections with Equations 5.33, 5.41, 5.42 and 5.43. As shown, the results are two coupled governing equations, one a differential equation representing the structure and the other an integral equation representing the fluid dynamics. However, the wake-oscillator models in the literature are generally two non-linear coupled differential equations. In order to show that these type of

equations can be obtained from, as an example, Equation 5.41, we consider the *lift-oscillator* model of Hartlen and Currie [9].

In Chapter 1, we mentioned that one of the earliest and the most noteworthy of the wake-oscillator models is the lift-oscillator model proposed by Hartlen and Currie. Next, we review their work briefly, and then, we show how a similar equation can be obtained from our variational method.

5.6.1 Hartlen and Currie's *lift-oscillator* model

Hartlen and Currie considered the model problem in the form of translating cylinder. Then, they consider an instantaneous lift coefficient, c_L, to be a representative variable for the oscillatory lift force. Therefore, they expressed the governing equation of motion for the structure as

$$m\ddot{x}_s + C\dot{x}_s + kx_s = \frac{1}{2}\rho u^2 DLc_L, \tag{5.44}$$

where D is the cylinder's diameter and L is the cylinder's length. Then, the governing equation was nondimensionalized as

$$\ddot{x}_r + 2\zeta\dot{x}_r + x_r = a\omega_0^2 c_L, \tag{5.45}$$

where x_r is the dimensionless structural displacement, ζ is a reduced damping coefficient, ω_0 is the dimensionless wind speed, and a is a known dimensionless constant. Since the structure exhibits oscillatory behavior, they assumed that the lift coefficient must also behave as an oscillator. Therefore, they sought oscillatory equations in the form of

$$\ddot{c}_L + (\text{damping term}) + \omega_0^2 c_L = (\text{forcing term})\,. \tag{5.46}$$

Based on the nature of VIV systems, Hartlen and Currie discussed that the oscillator must be a self-exited and self-limited oscillator. Thus, they selected damping terms such that the resulting equation becomes a van der Pol type oscillator so that Equation 5.46 was modified to

$$\ddot{c}_L - \alpha\omega_0\dot{c}_L + \frac{\gamma}{\omega_0}\left(\dot{c}_L\right)^3 + \omega_0^2 c_L = (\text{forcing term})\,, \tag{5.47}$$

where α and γ are found experimentally. The remaining term is the forcing term, which was selected arbitrarily to be $b\dot{x}_r$, where b is a constant to be estimated experimentally. Therefore, Hartlen and Currie expressed their coupled differential equations as,

$$\begin{cases} \ddot{x}_r + 2\zeta\dot{x}_r + x_r = a\omega_0^2 c_L \\ \ddot{c}_L - \alpha\omega_0\dot{c}_L + \dfrac{\gamma}{\omega_0}\left(\dot{c}_L\right)^3 + \omega_0^2 c_L = b\dot{x}_r, \end{cases} \tag{5.48}$$

where only two of the fluid unknown parameters, α, b, and γ, must be selected in order to obtain the best fit to an experimental data set.

As it is obvious from the summary above, the model proposed by Hartlen and Currie does not provide any insights into the fluid dynamics parameters and their effects on the structural response, but it still is the most noteworthy of all such approaches. Next, we use our proposed variational method to obtain similar types of equations.

5.6.2 Reduction of Equation 5.41 to Hartlen-Currie

We start our manipulations with some general assumptions that can be made for many VIV problems and apply these to the reduced-order model Equation 5.41.

For fluids with very low viscosity, the *boundary layer approximation method,* proposed by Prandtl, assumes that the viscosity only exists in the vicinity of the solid structure, and is neglected elsewhere. Considering Equation 5.41, the term $\mu\nabla^2\left(\frac{1}{2}u^2\right)$

is the sum of both the viscous forces acting as external loads and those dissipating energy on the surface of structure and on the open control surfaces. Therefore, following Prandtl's hypothesis, we neglect the remaining dissipative terms inside the control volume,

$$\int_{CV} \frac{1}{2}\mu \left(\nabla \times \boldsymbol{u}\right) \cdot \left(\nabla \times \boldsymbol{u}\right) dV \approx 0. \tag{5.49}$$

Also, we define the kinetic energy density function as

$$\hat{T} = \frac{\frac{1}{2}\rho u^2}{\rho} = \frac{1}{2}u^2. \tag{5.50}$$

Substituting of Equations 5.49 and 5.50 into Equation 5.41 results in

$$\begin{cases} m_s\ddot{x}_s + C\dot{x}_s + kx_s = -\displaystyle\int_{CS_C} \rho\hat{T}\left(\mathbf{e}_x \cdot \boldsymbol{n}\right) dA, \\ \\ \displaystyle\int_{CV} \rho\frac{D\hat{T}}{Dt} dV + \int_{CS_O} \rho\hat{T}\left(\boldsymbol{u} \cdot \boldsymbol{n}\right)\ dA + \int_{CV} \left(\nabla p \cdot \boldsymbol{u} - \mu\nabla^2\hat{T}\right) dV \\ \qquad\qquad = -\dot{x}_s \displaystyle\int_{CS_C} \rho\hat{T}\left(\mathbf{e}_x \cdot \boldsymbol{n}\right) dA. \end{cases} \tag{5.51}$$

In order to simplify Equations 5.51 further, we consider some experimental observations of Dong *et al.* [4].

As mentioned in Chapter 1, Dong *et al.* performed a series of experiments on the VIV of the model problem in the form of an inverted pendulum. They found an optimum control volume for which the analytical model proposed by Benaroya and Wei [1] predicted the structural response up to an excellent accuracy. The phase-averaged terms of Benaroya and Wei's energy rate equation (Equation 4.50) obtained by their experiments are shown in Figure 5.2, where the figure depicts the results for the optimum control volume. Also, the corresponding spectra of the fluid energy transport terms (of Figure 5.2) are shown in Figure 5.3.

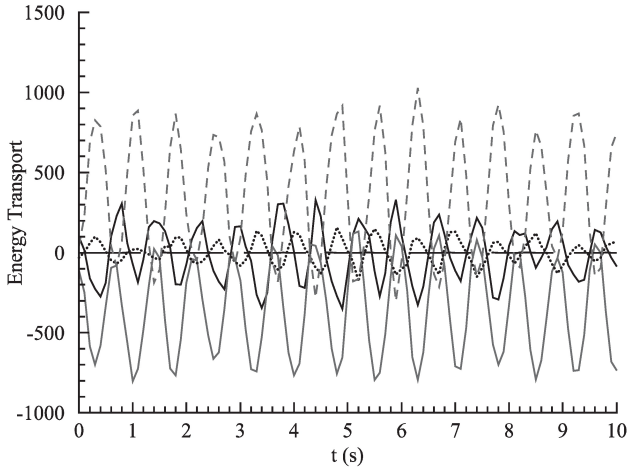

Figure 5.2: Phase-averaged terms of Equation 4.50 for the inverted pendulum model where the dark solid line denotes the time rate of change of the fluid kinetic energy in CV, the light solid line is the flux of kinetic energy, the dashed lined is the rate of work done on the cylinder by pressure forces, and the dotted line is the time rate of change of mechanical energy of the cylinder. The work done by the viscous forces are included in the figure, however, they are too small to be visible ([4], with permission from Elsevier).

Considering Figure 5.3, Dong *et al.* reported that their careful examination indicates that the flux of the fluid kinetic energy across the open control surface and the work done by the pressure correlated with the vortex shedding while the flux of the kinetic energy around the cylinder correlated with the cylinder oscillation. We expected the last statement is a result of the no-slip condition.

These experiments were conducted on an inverted pendulum, while we have considered the model problem in the form of translating cylinder. However, due to similarities between the two problem, we assume similar results would be found for the problem at hand. Moreover, the control volume of their experiments was selected such that the downstream control surface views the structured vortices, that is, it was close enough to the structure so that the shed vortices had not yet broken down to smaller eddies.

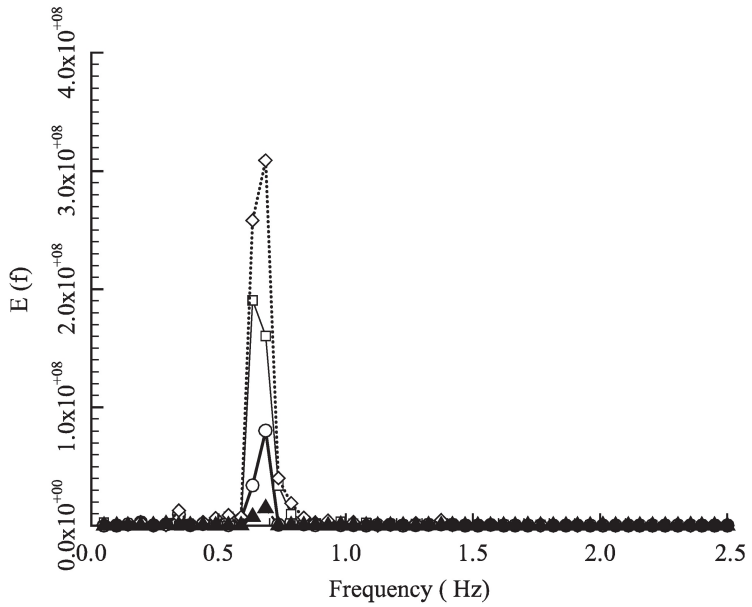

Figure 5.3: Spectra of the energy terms shown in Figure 5.2, where —◯— corresponds to the time rate of change of the fluid kinetic energy inside the CV, $\cdots\diamond\cdots$ is for the flux of kinetic energy, —□— is for the rate of work done by pressure on the structure, - -△- - represents the rate of mechanical energy of the cylinder ([4], with permission from Elsevier).

Since Dong *et al.* considered the rate of kinetic energy as $\int_{CV} \frac{\partial T}{\partial t} dV$, we continue by expanding the first integral in the second equation of Equation 5.51 and obtain it to be

$$\begin{cases} m_s \ddot{x}_s + C\dot{x}_s + kx_s = -\int\limits_{CS_C} \rho\hat{T}\left(\mathbf{e}_x \cdot \boldsymbol{n}\right) dA, \\ \int\limits_{CV} \rho \dfrac{\partial \hat{T}}{\partial t} dV + \int\limits_{CS_C} \rho\hat{T}\left(\boldsymbol{u} \cdot \boldsymbol{n}\right) dA + 2\int\limits_{CS_O} \rho\hat{T}\left(\boldsymbol{u} \cdot \boldsymbol{n}\right) dA \\ + \int\limits_{CV} \left(\nabla p \cdot \boldsymbol{u} - \mu\nabla^2\hat{T}\right) dV = -\dot{x}_s \int\limits_{CS_C} \rho\hat{T}\left(\mathbf{e}_x \cdot \boldsymbol{n}\right) dA. \end{cases} \tag{5.52}$$

Now we try to formulate the experimental observations of Dong *et al.* so that they become applicable to Equation 5.52.

Based on the observation by Dong *et al.* that the flux term at the closed surface

corresponds to the structural oscillation, we assume the relation

$$\int_{CS_C} \rho\hat{T}\left(\boldsymbol{x},t\right)\left[\boldsymbol{u}\left(\boldsymbol{x},t\right)\cdot\boldsymbol{n}\right]\ dA = \int_{CS_C} \rho\hat{T}\left(\boldsymbol{x},t+\frac{1}{f_s}\right)\left[\boldsymbol{u}\left(\boldsymbol{x},t+\frac{1}{f_s}\right)\cdot\boldsymbol{n}\right]\ dA, \quad (5.53)$$

where f_s (Hz) is the frequency of structural oscillation. Equation 5.53 represents a boundary condition and since we have two coupled equations it must only be applied to one of them since the similar term in the other equation would be relaxed due to this boundary condition. Since the no-slip condition is a boundary condition for the fluid and not the structure, we make the substitution in the governing equation of the fluid (second equation) in Equation 5.53. Since the terms $\int_{CS_C} \rho\hat{T}\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)dA$ and $\dot{x}_s\int_{CS_C} \rho\hat{T}\left(\mathbf{e}_x\cdot\boldsymbol{n}\right)dA$ vary with the same frequency, their superposition also varies with the same frequency. Therefore,

$$\Theta\dot{x}_s \cong \dot{x}_s\int_{CS_C} \rho\hat{T}\left(\mathbf{e}_x\cdot\boldsymbol{n}\right)dA + \int_{CS_C} \rho\hat{T}\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)dA, \quad (5.54)$$

where Θ is a constant with dimension of force and the integrals are evaluated from the experimental measurements in the vicinity of the structure. This relaxes the condition on the corresponding term in the structural differential equation. In this equation,

$$\int_{CS_C} \rho\hat{T}\left(\boldsymbol{x},t\right)\left(\mathbf{e}_x\cdot\boldsymbol{n}\right)dA \equiv \int_{CS_C} \rho\hat{T}\left(\boldsymbol{x},t+\frac{1}{f_v}\right)\left(\mathbf{e}_x\cdot\boldsymbol{n}\right)dA, \quad (5.55)$$

where f_v (Hz) is the frequency of vortex shedding.

Considering the second equation in Equation 5.52 and introducing Equation 5.54, we have

$$\int_{CV} \rho\frac{\partial\hat{T}}{\partial t}dV + 2\int_{CS_O} \rho\hat{T}\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)\ dA + \int_{CV}\left(\nabla p\cdot\boldsymbol{u} - \mu\nabla^2\hat{T}\right)dV = -\Theta\dot{x}_s. \quad (5.56)$$

Applying Gauss' divergence theorem to the pressure and viscous terms, we have

$$\int_{CV} \rho \frac{\partial \hat{T}}{\partial t} dV + 2 \int_{CS_O} \rho \hat{T} \left(\boldsymbol{u} \cdot \boldsymbol{n} \right) \, dA + \int_{CS} \left(p\boldsymbol{u} - \mu \nabla \hat{T} \right) \cdot \boldsymbol{n} dA = -\Theta \dot{x}_s, \tag{5.57}$$

so that no pressure or viscous terms are present inside the control volume. Note that the viscous terms were neglected by the hypothesis of Equation 5.49. Also, Dong *et al.* found the rate of work done by the viscous forces to be so small that its corresponding trace is not visible in Figure 5.2. They also found that the rate of kinetic energy of the system correlated with the vortex shedding frequency. Therefore,

$$\int_{CV} \rho \frac{\partial}{\partial t} \hat{T} \left(\boldsymbol{x}, t \right) dV = \int_{CV} \rho \frac{\partial}{\partial t} \hat{T} \left(\boldsymbol{x}, t + \frac{1}{f_v} \right) dV, \tag{5.58}$$

or,

$$\frac{\partial}{\partial t} \int_{CV} \rho \hat{T} \left(\boldsymbol{x}, t \right) dV = \frac{\partial}{\partial t} \int_{CV} \rho \hat{T} \left(\boldsymbol{x}, t + \frac{1}{f_v} \right) dV. \tag{5.59}$$

Before continuing our derivations, we choose the direction of free-stream flow to be the z axis of the Cartesian coordinate system defined in Figure 5.4. Therefore, $\hat{T} \left(\boldsymbol{x}, t \right) = \hat{T} \left(x, z, t \right)$.

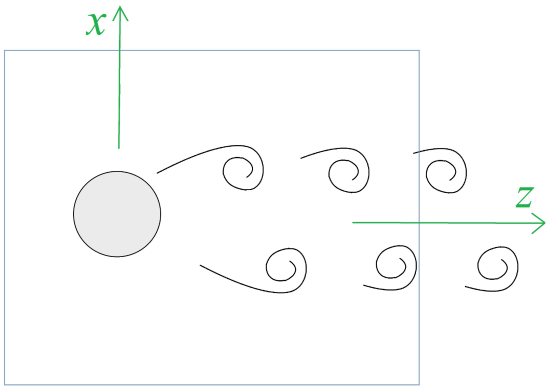

Figure 5.4: A schematic diagram of the experiments of *Dong et al.* where the shedded vortices are $P+S$ type vortices. For definition of vortices names one may refer to [6].

From Equation 5.57, since no external or dissipative forces are present inside the control volume, a shed vortex must repeat its pattern periodically as it travels in the direction of free-stream flow. Therefore, we have

$$\hat{T}\left(x,z,t\right)d=\hat{T}\left(x,z-c_{v}t,t\right), \tag{5.60}$$

where c_v is the velocity of shed vortices.

Considering Equations 5.59 and 5.60 together with Figure 5.2, $\hat{T}\left(\boldsymbol{x},t\right)$ has the characteristics of a propagating wave with velocity c_v and frequency f_v. Therefore, from the wave equation we have

$$\frac{\partial^{2}\hat{T}\left(\boldsymbol{x},t\right)}{\partial t^{2}}\equiv c_{v}^{2}\nabla^{2}\hat{T}\left(\boldsymbol{x},t\right), \tag{5.61}$$

or

$$\nabla^{2}\hat{T}\left(\boldsymbol{x},t\right)=\frac{1}{c_{v}^{2}}\frac{\partial^{2}\hat{T}\left(\boldsymbol{x},t\right)}{\partial t^{2}}. \tag{5.62}$$

Introducing Equation 5.62 to Equation 5.56, yields

$$\int\limits_{CV}\rho\frac{\partial\hat{T}}{\partial t}dV+2\int\limits_{CS_{O}}\rho\hat{T}\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)\,dA+\int\limits_{CV}\nabla p\cdot\boldsymbol{u}dV-\int\limits_{CV}\frac{\mu}{c_{v}^{2}}\frac{\partial^{2}\hat{T}\left(\boldsymbol{x},t\right)}{\partial t^{2}}dV=-\Theta\dot{x}_{s}. \tag{5.63}$$

Based on the experiments of *Dong et al.*, the flux of the kinetic energy across the open control volume was found to vary with the frequency of the vortex shedding, that is,

$$\int\limits_{CS_{O}}\rho\hat{T}\left(\boldsymbol{x},t\right)\left[\boldsymbol{u}\left(\boldsymbol{x},t\right)\cdot\boldsymbol{n}\right]\,dA=\int\limits_{CS_{O}}\rho\hat{T}\left(\boldsymbol{x},t+\frac{1}{f_{v}}\right)\left[\boldsymbol{u}\left(\boldsymbol{x},t+\frac{1}{f_{v}}\right)\cdot\boldsymbol{n}\right]\,dA. \tag{5.64}$$

However, the kinetic energy inside the control volume must also vary with the same frequency when Equation 5.59 is considered. Moreover, the flux of the kinetic energy adds and subtracts energy from the system very similarly to a spring. Since, $\hat{T}\left(\boldsymbol{x},t\right)$,

$\boldsymbol{u}(\boldsymbol{x},t)$ and $\hat{T}(\boldsymbol{x},t)\,\boldsymbol{u}(\boldsymbol{x},t)$ have the same frequency, and since the solution of the velocity field inside the control volume is a function of the boundary conditions, we can write

$$\int_{CS_O} \rho\hat{T}(\boldsymbol{x},t)\left[\boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{n}\right]\, dA \underset{\sim}{\propto} -\int_{CV} \rho f_v \hat{T}(\boldsymbol{x},t)\, dV, \tag{5.65}$$

where $\underset{\sim}{\propto}$ means they are approximately proportional and the negative sign is introduced to show that the right-hand side is positive when the left-hand side is negative. Assuming proportionality, we can write

$$\int_{CS_O} \rho\hat{T}(\boldsymbol{x},t)\left[\boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{n}\right]\, dA = -\int_{CV} \beta\rho f_v \hat{T}(\boldsymbol{x},t)\, dV, \tag{5.66}$$

where β is a constant to be determined. Substituting Equation 5.66 into Equation 5.63, we obtain

$$-\frac{\mu}{c_v^2}\int_{CV}\frac{\partial^2\hat{T}(\boldsymbol{x},t)}{\partial t^2}dV + \int_{CV}\rho\frac{\partial\hat{T}}{\partial t}dV - \int_{CV} 2\beta f_v\rho\hat{T}dV + \int_{CV}\nabla p\cdot\boldsymbol{u}dV = -\Theta\dot{x}_s. \tag{5.67}$$

Also, from Gauss' divergence theorem we expect that the term $\int_{CS_C}\rho\hat{T}(\mathbf{e}_x\cdot\boldsymbol{n})\,dA$ in Equation 5.52 can be related in the following way,

$$\int_{CS_C}\rho\hat{T}(\mathbf{e}_x\cdot\boldsymbol{n})\,dA \underset{\sim}{\propto} \int_{CV}\nabla\left(\rho\hat{T}\right)\cdot\mathbf{e}_x dV, \tag{5.68}$$

and, since $\hat{T}(\boldsymbol{x},t)$ has the characteristics of a propagating wave,

$$\frac{\partial\hat{T}(\boldsymbol{x},t)}{\partial t} \equiv -c_v\nabla\hat{T}(\boldsymbol{x},t). \tag{5.69}$$

Equation 5.68 becomes

$$\int_{CS_C} \rho\hat{T}\left(\mathbf{e}_x \cdot \boldsymbol{n}\right) dA \underset{\sim}{\propto} - \int_{CV} \frac{1}{c_v}\rho\frac{\partial\hat{T}\left(\boldsymbol{x},t\right)}{\partial t}dV. \tag{5.70}$$

Therefore, we assume proportionality and obtain

$$\int_{CS_C} \rho\hat{T}\left(\mathbf{e}_x \cdot \boldsymbol{n}\right) dA = - \int_{CV} \gamma\frac{1}{c_v}\rho\frac{\partial\hat{T}\left(\boldsymbol{x},t\right)}{\partial t}dV, \tag{5.71}$$

where γ is a constant to be determined. It is important to note that in Equations 5.66 and 5.71, constants β and γ can more generally be replaced with functions of f_v and f_v/c_v, respectively.

Substituting Equations 5.67 and 5.71 into Equation 5.52 yields

$$\begin{cases} m_s\ddot{x}_s + C\dot{x}_s + kx_s = \int_{CV} \gamma\frac{1}{c_v}\rho\frac{\partial\hat{T}\left(\boldsymbol{x},t\right)}{\partial t}dV, \\ \\ -\frac{\mu}{c_v^2}\int_{CV} \frac{\partial^2\hat{T}\left(\boldsymbol{x},t\right)}{\partial t^2}dV + \int_{CV} \rho\frac{\partial\hat{T}}{\partial t}dV - \int_{CV} 2\beta f_v\rho\hat{T}dV + \int_{CV} \nabla p\cdot\boldsymbol{u}dV = -\Theta\dot{x}_s. \end{cases} \tag{5.72}$$

Thus far, we have not considered the pressure terms, since they are perhaps the most complex and challenging terms. As mentioned in Section 4.8, the pressure is the thermodynamic pressure and it is assumed to be independent of velocity in the derivation of the constitutive relation for Newtonian incompressible fluids. Yet, its distribution is often found to be a function of velocity. For example, for the problems with steady low speed flow, the pressure distribution is found via Bernoulli's equation to be

$$p + \frac{1}{2}\rho u^2 = p_\infty + \frac{1}{2}\rho u_\infty^2, \tag{5.73}$$

which leads to defining a nondimensional excess pressure, called the *pressure coefficient* C_p,

$$C_p \equiv \frac{p - p_\infty}{\frac{1}{2}\rho u_\infty^2}, \tag{5.74}$$

where p_∞ and u_∞ are the pressure and velocity at infinity.

While the pressure coefficient was originally defined for steady flow, it has been used extensively in similarity analysis of complicated non-linear problems. For example, it is used to relate the local pressure at a point, $p(\boldsymbol{x})$, to other parameters of the nondimensional Navier-Stokes equations via the relation

$$\frac{p(\boldsymbol{x}) - p_\infty}{\frac{1}{2}\rho u_\infty^2} = f\left(\mathrm{Fr},\, \mathrm{Re};\, \frac{\boldsymbol{x}}{l}\right), \tag{5.75}$$

where Fr denotes the Froude number, Re is the Reynolds number, l is a length scale, and $f(\ )$ is a function [28].

As evident from Equation 5.72, pressure is an external load to the control volume. Also, since we are seeking a reduced-order model that can be coupled with the experiments, a set of experimental velocities is already the result of the pressure field. In Chapter 4, we argued that the classical energy equation is a zeroth order approximation. Moreover, when deriving our variational energy equation we assumed that the pressure remains constant with respect to first order velocities, and there seems no reason why we cannot assume it to vary with first order velocities. Given the difficulties faced in acquiring experimental data (as mentioned in Section 4.10), we are encouraged to consider an approximation of pressure at least up to first order. However, the zeroth order terms must be neglected since their effects must be observable from the measured velocities.

The experiments of Dong *et al.* found that the work done by the pressure correlated with the rate of kinetic energy, that is,

$$p\left(\boldsymbol{x},t\right)=p\left(\boldsymbol{x},t+\frac{1}{f_v}\right). \tag{5.76}$$

Also, from Equation 5.74, we expect a relation between pressure and kinetic energy,

$$p\left(\boldsymbol{x},t\right)=f\left[\rho\hat{T}\left(\boldsymbol{x},t\right)\right], \tag{5.77}$$

where f is a function to be determined. Moreover, since $\hat{T}\left(\boldsymbol{x},t\right)$ behaves similarly to a wave, that is,

$$\begin{aligned} p\left(\boldsymbol{x},t\right) &= f\left[\rho\hat{T}\left(\boldsymbol{x},t\right)\right] \\ &= f\left[\rho\hat{T}\left(\boldsymbol{x},t+\frac{1}{f_v}\right)\right] \\ &= f\left[\rho\hat{T}\left(\boldsymbol{x}-c_v t,t\right)\right], \end{aligned} \tag{5.78}$$

p itself must be in the form of a wave. Therefore,

$$\frac{\partial p\left(\boldsymbol{x},t\right)}{\partial t}\equiv -c_v\nabla p\left(\boldsymbol{x},t\right). \tag{5.79}$$

Consequently, we can say $\nabla p\left(\boldsymbol{x},t\right)$ must be a function of $\frac{\partial}{\partial t}\hat{T}\left(\boldsymbol{x},t\right)$, that is,

$$\nabla p\left(\boldsymbol{x},t\right)=-\frac{1}{c_v}\boldsymbol{f}\left[\frac{\partial}{\partial t}\hat{T}\left(\boldsymbol{x},t\right)\right], \tag{5.80}$$

where $\boldsymbol{f}$ is a vector-valued function to be determined. Substituting Equation 5.80

into Equation 5.72, we obtain

$$\begin{cases} m_s\ddot{x}_s + C\dot{x}_s + kx_s = \dfrac{\gamma}{c_v}\displaystyle\int_{CV}\rho\frac{\partial\hat{T}\left(\boldsymbol{x},t\right)}{\partial t}dV, \\ \\ \dfrac{\mu}{c_v^2}\displaystyle\int_{CV}\frac{\partial^2\hat{T}\left(\boldsymbol{x},t\right)}{\partial t^2}dV - \int_{CV}\rho\frac{\partial\hat{T}}{\partial t}dV + \int_{CV}\frac{1}{c_v}\boldsymbol{f}\left[\frac{\partial\hat{T}\left(\boldsymbol{x},t\right)}{\partial t}\right]\cdot\boldsymbol{u}dV \\ \qquad\qquad + \displaystyle\int_{CV}2\beta f_v\rho\hat{T}dV = \Theta\dot{x}_s. \end{cases} \tag{5.81}$$

Defining an instantaneous kinetic energy function for the control volume of an incompressible fluid as

$$\begin{aligned} T\left(t\right) &= \int_{CV}\frac{1}{2}\rho u^2 dV \\ &= \int_{CV}\rho\hat{T}dV, \end{aligned} \tag{5.82}$$

and introducing it to Equation 5.81, we obtain

$$\begin{cases} m_s\ddot{x}_s + C\dot{x}_s + kx_s = \dfrac{\gamma}{c_v}\dot{T} \\ \\ \dfrac{\mu}{\rho c_v^2}\ddot{T} - \dot{T} + f\left(\dot{T}\right) + 2\beta f_v T = \Theta\dot{x}_s, \end{cases} \tag{5.83}$$

where Θ, β and γ are defined by Equations 5.54, 5.66 and 5.71, respectively; $f\left(\dot{T}\right)$ is a function with an order higher than one with respect to $\dot{T}$ and an appropriate form of $f\left(\dot{T}\right)$ must be selected based on experimental considerations. Since the sign of $f\left(\dot{T}\right)$ must be positive, Equation 5.83 is a van der Pol type oscillator.

For comparison with Hartlen and Currie, we may assume that $f\left(\dot{T}\right)$ is the first

order term of a sine function. Therefore, we select

$$f\left(\dot{T}\right) = \frac{\eta}{\rho f_v c_v^4}\dot{T}^3, \tag{5.84}$$

where negative sign is eliminated, since $f\left(\dot{T}\right)$ must be a positive-valued function, and η is a constant found experimentally. Therefore, the reduced-order model becomes

$$\begin{cases} m_s\ddot{x}_s + C\dot{x}_s + kx_s = \dfrac{\gamma}{c_v}\dot{T} \\ \\ \dfrac{\mu}{\rho c_v^2}\ddot{T} - \dot{T} + \dfrac{\eta}{\rho f_v c_v^4}\dot{T}^3 + 2\beta f_v T = \Theta\dot{x}_s. \end{cases} \tag{5.85}$$

Having obtained a reduced-order model similar to that of Hartlen and Currie, we conclude the chapter next.

5.7 Conclusions

In this chapter, we proposed a method, based on the important concept of generalized momentum in analytical mechanics, that the rate of generalized momenta can be used to obtain reduced-order coupled governing equations from the variational energy rate equation derived in Chapters 3 and 4. Depending on the type of equations required, two possible methods are those expressed by Equations 5.7, 5.8, and by Equation 5.12.

Regarding the model problem, we chose Equation 5.12 and obtained two coupled equations, one a differential equation and the other an integral equation. Moreover, we showed the importance of boundary conditions and their implementation by obtaining two different possible reduced-order models for each of the model problem (Equations 5.33, 5.41, 5.42 and 5.43).

As mentioned earlier, the wake-oscillator models available in the literature are *ad*

hoc models, obtained by guessing a function that can capture some characteristics of a system. Therefore, they do not provide any insights into the fluid dynamic parameters that can effect the outcome. As an example, we considered one the most noteworthy of those models proposed by Hartlen and Currie. Their *lift-oscillator* model has proven to capture many features of the structural response. However, it neglects the fluid dynamics.

As evident from Equation 5.83, the method developed in this chapter has the advantage that the resulting reduced-order model is expressed in terms of the kinetic energy of the control volume. Each of the terms in Equation 5.83 has a specific meaning and can be traced back to the energy equation, and even further back to the Navier-Stokes equation. However, we expect the energy framework presented to be more beneficial to the analysis of such systems given that the energy terms are more meaningful.

As mentioned in Chapter 1, there exist no reduced-order models in the literature that are obtained without an assumed lift or drag coefficient function. However, the reduced-order modeling method proposed here does not require any such *ad hoc* assumptions. The coupled equations are obtained directly from the energy equation by using a variational method and it is a first principles approach, which was the ultimate goal of this research work.

In the following chapter, we summarize and conclude our research work.

Chapter 6
Conclusions and Future Work

Our review of the literature in Chapter 1 revealed that the classical variational principles have had very limited success in modeling fluid dynamic systems. Their applicability has been mainly limited to the cases where an ideal fluid was considered.

Regarding fluid-structure interaction, and specifically vortex-induced vibration, there have been a very limited number of efforts to model these phenomena using a first principles approach. The majority of the efforts have been devoted to guessing a hydrodynamic forcing function which can be curve fit to a set of experimental data. Consequently, no general conclusions could be made from such models, except those obtained from experimental observations.

Therefore, in Chapter 1 the main challenges were identified to be:

1. The nature of the Lagrangian-Eulerian transformation has not been fully understood.

2. The concept of virtual displacements, a Lagrangian concept, does not have an Eulerian counterpart. Thus, using variational principles based on d'Alembert's virtual displacement face serious challenges that have not been overcome. Consequently, there have been no compelling variational methods that can be used in modeling the behavior of fluid systems or FSI.

3. While the classical energy equation for Newtonian incompressible fluids is expected to be independent of the choice of control volume, there is a dependency that requires attention.

4. There has been no analytical method proposed that can be used to obtain wake-oscillator type coupled equations without any *ad hoc* assumptions regarding the lift or drag forcing functions.

Therefore, we continued our research work with a thorough review of the relations between the Lagrangian and the Eulerian reference frames so as to clarify some of the inconsistencies in the literature. Then, we proposed the use of Jourdain's variational principle for modeling fluid and fluid-structure interaction systems in Chapter 2, and extended the principle for it to become applicable for modeling fluid dynamics. This extension was verified by using it to derive the Navier-Stokes equations. We did not add any assumptions to those usually made in deriving the constitutive relations for Newtonian incompressible viscous fluids except to stipulate the existence of the mapping function, $\Lambda(\boldsymbol{x}, t)$, defined by

$$\begin{cases} \boldsymbol{r} = \boldsymbol{\Lambda}(\boldsymbol{r}, t) \\ \dfrac{d}{dt}\boldsymbol{r} = \dfrac{D}{Dt}\boldsymbol{\Lambda}(\boldsymbol{x}, t)\Big|_{\boldsymbol{x}=\boldsymbol{r}} \end{cases}. \tag{6.1}$$

The mapping function $\Lambda(\boldsymbol{x}, t)$ was only required for mathematical consistency so that Jourdain's principle could be expressed purely in the Eulerian frame of reference. As evident from our subsequent manipulations, a specific mapping function $\Lambda(\boldsymbol{x}, t)$ is not required.

In Chapter 3, we proposed a method to obtain the energy rate equations from Jourdain's principle. Having investigated the commutation rule for the conserved terms of the Navier-Stokes equations, it was shown that the commutation rule does not hold, that is, the variation of the Eulerian acceleration does not behave similarly to the acceleration of the Eulerian virtual velocity. This suggested that the Eulerian acceleration is not necessarily a representation of the Lagrangian acceleration of particles. Therefore, we extracted the non-commuting part which resulted in obtaining

an additional term in the classical energy equation in integral form for Newtonian incompressible viscous fluids. We interpreted this to mean that the Eulerian energy rate of the system does not match the Lagrangian energy rate. This result is extremely important when it comes to modeling fluid-structure interactions, as the structural terms are defined in a Lagrangian reference frame.

Also, we obtained an expanded form of the energy rate equation. It was shown that the energy rate equation becomes simpler when it is expressed in a variational form, since the term $\boldsymbol{u} \times (\nabla \times \boldsymbol{u})$ drops out of the equation. We took advantage of this fact in our reduced-order modeling in Chapter 5.

Having obtained the variational equation in differential and integral form for a control volume of fluid particles, fluid-structure interaction was considered in Chapter 4. A structure was introduced internal to the fluid and the energy rate equation was obtained. As examples, we considered the model problem in the form of a translating cylinder and in the form of an inverted pendulum. In was shown that the result is an energy rate equation where the fluid dynamic terms can be evaluated experimentally, and then the structural terms can be found. For example, for the translating cylinder problem, the relevant equation is (Equation 4.33)

$$
\begin{aligned}
m_s\ \ddot{\boldsymbol{x}}_s \cdot \dot{\boldsymbol{x}}_s + C\ \dot{\boldsymbol{x}}_s \cdot \dot{\boldsymbol{x}}_s + k\ \boldsymbol{x}_s \cdot \dot{\boldsymbol{x}}_s = \\
&- \frac{D}{Dt} \int\limits_{CV} \frac{1}{2} \rho \boldsymbol{u} \cdot \boldsymbol{u}\ dV - \int\limits_{CS_O} \left(\frac{1}{2} \rho\ \boldsymbol{u} \cdot \boldsymbol{u} \right) (\boldsymbol{u} - \boldsymbol{v}_{CS}^o) \cdot \boldsymbol{n}\ dA \\
&\quad + \int\limits_{CS_O} \left[-\ p\ \boldsymbol{u} \cdot \boldsymbol{n}\ + \mu \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) \cdot \boldsymbol{n} \right] dA \\
&\quad - \int\limits_{CS_C} \mu \left[\nabla (\boldsymbol{u} \cdot \boldsymbol{u}) - 2\boldsymbol{u} \times (\nabla \times \boldsymbol{u}) \right] \cdot \boldsymbol{n}\ dA \\
&\quad - \int\limits_{CV} \mu (\nabla \times \boldsymbol{u}) \cdot (\nabla \times \boldsymbol{u})\, dV - \int\limits_{CS_O} \frac{1}{2} \rho (\boldsymbol{u} \cdot \boldsymbol{u}) (\boldsymbol{u} \cdot \boldsymbol{n})\ dA, \quad (6.2)
\end{aligned}
$$

where the right-hand side includes terms obtained experimentally. The importance

of Equation 6.2 is that it provides an equation where each term has a physical meaning and we know all the assumptions used for its derivation. For Equation 6.2, the assumptions are the existence of the mapping function defined by Equation 6.1, the assumptions made in the classical derivation of the constitutive relations for Newtonian incompressible viscous fluids, and the no-slip condition that was implemented implicitly.

Also, in Chapter 4, we compared our results with the models proposed by Benaroya and Wei and by McIver. Additionally, we used the methodology of Benaroya and Wei in extending Hamilton's principle and added the dissipation effect to obtain the classical energy rate equation. Then, based on the fact that Hamilton's principle is a conservative principle, we specified the limitation of the classical formulation of energy.

Moreover, in Section 4.11 we considered the cases where the control volume is selected very close to or very far from the structure. In particular, for the case where the CV is selected too close to the structure, we argued that the viscous terms become more important as the velocities are lower. Therefore, we assumed all the fluid particles to be constrained by Rayleigh's constraints. The energy equation thus obtained showed that the power due to pressure is twice as important when compared with the classical energy rate equation, meaning that while the viscous forces are not performing work themselves they are acting as nonholonomic constraints. This can explain why the results of Benaroya and Wei's energy equation were half of that observed experimentally. However, this remains to be verified by a series of experiments where the control volume is selected very close to the structure, containing the formation region. Then, the validity of this idea can be examined.

In Chapter 5, we proposed our ideas on how the reduced-order coupled nonlinear equations can be obtained using a method similar to the classical utilization of the generalized momenta. We discussed the importance of boundary conditions to the

proposed modeling method. As examples, we chose the no-slip condition and implemented it both implicitly and explicitly. It is important to note that no additional assumptions were made to those mentioned so far in this chapter. Therefore, the reduced-order modeling could be done without any *ad hoc* assumptions regarding the fluid forcing function, and the method remained a purely variational method.

Additionally, in order to show the generality of the method, we chose one of the most used wake-oscillator models, proposed by Hartlen and Currie, and showed that their model is a specific case of our model. As evident from our results, the reduced-order model was expressed in terms of the kinetic energy of the control volume where the contributions of the fluid dynamic parameters were specified. This was the very goal of this research work.

6.1 Future Work

First of all, the variational approach can be extended for compressible fluids. We introduced the incompressibility condition via Equation 2.25,

$$\det\left[\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}}\right] = 1, \tag{6.3}$$

when mapping the system from the Lagrangian frame of reference to the Eulerian representation. The derivations prior to Equation 2.25 were kept general. For a compressible fluid, the determinant of the Jacobian $\partial \boldsymbol{r}/\partial \boldsymbol{x}$ can be used as a means to introduce compressibility into a variational approach. For example, following Eckart [16], we may utilize

$$\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{r}} = \frac{1}{\rho}. \tag{6.4}$$

Many of the manipulations and derivations shown can still be used. However, additional efforts are required for extra terms due to Equation 6.4. Regarding Equation

6.4, a suggestion can be made that the density at some initial time, say t_0, be selected as a datum, say ρ_0. It can be assumed that the Eulerian and the Lagrangian reference frames coincide at t_0. Then, the reference frames must be normalized accordingly to ensure one-to-one mapping and derivations can be continued.

Generally, variational methods in integral form are based on a conservative equation in terms of positive-valued scalar potential functions. In the absence of known particle trajectories, the conservation of energy may not be fully ensured and the resulting potential functions are approximations of the conservative process. As mentioned in Section 4.8, a method can be used to obtain an approximate conservative equation from Hamilton's principle and Lagrange's equations by assuming that the virtual displacements match with the actual displacement, that is, by using the relation $\delta(\ ) = dt\ d(\ )/dt$. Similar ideas can also be applied to the approach proposed in Section 5.1, that is, assuming $dt \equiv \varepsilon$ and $\delta(\ ) \equiv \frac{d}{dt}(\ )$ for functions described in the Lagrangian reference frame and $\delta(\ ) \equiv \frac{D}{Dt}(\ )$ for those described in the Eulerian reference frame, coupled equations can be obtained via the relations

$$\begin{cases} \dfrac{D}{Dt}\left(\dot{L}_{\boldsymbol{u}}\right) = 0 \\ \\ \dfrac{\partial \dot{L}}{\partial \dot{\boldsymbol{r}}_s} = 0. \end{cases} \tag{6.5}$$

These equations will lead to equations with higher order differentials with respect to time.

Regarding the fluid-structure interaction and vortex-induced vibration, the most challenging modeling aspects seem to be the pressure terms. While the independence of the pressure from the velocity field is an assumption made in deriving the constitutive relations, the pressure distribution is generally found to be dependent on the velocity field. It seems that the linear assumption made with regard to the pressure term may not be sufficient for modeling complicated phenomena such as VIV.

From the experimental point of view, obtaining the pressure terms for FSI has been focused mainly on manipulations of the Navier-Stokes equations and by integrating these numerically. Regarding incompressible fluids, the thermodynamic pressure and mechanical pressure must match. Therefore, VIV problems are perhaps a good way to measure the applicability of an assumed pressure field, since its effects are observable from the structural response. Moreover, Equation 5.81 suggests higher order terms might be necessary.

The only way to obtain an exact conservative energy equation is by fully understanding the nonconservative processes in a system and by knowing the position of all of the particles, for Hamilton's principle, at least at two points in time. However, we discussed in detail the difficulties with regard to the Lagrangian-Eulerian transformation in Chapter 1. One important note there was that to obtain the Eulerian velocities from the Lagrangian velocity, the inverse of a composite function was required in order to obtain the initial position $\boldsymbol{A}(\boldsymbol{r}, t)$ from the Lagrangian particle paths $\boldsymbol{r}(\boldsymbol{A}, t)$. Generally, this is impossible in the absence of any information on $\boldsymbol{A}(\boldsymbol{r}, t)$ or $\boldsymbol{r}(\boldsymbol{A}, t)$. Consequently, the efforts in the literature to extend Hamilton's principle to fluids could not pass this roadblock, unless the flow is a steady-state simple laminar flow where some assumptions could be made. An interesting way to obtain the inverse function of an analytic function is by the Lagrange inversion theorem, thus possibly extending Hamilton's principle for fluid systems. However, this theorem is often neglected even in text books of mathematics and we could not find a good reference to recommend.

Also, in Section 3.5, we mentioned an interesting work by Riewe [33] were he proposed the use of fractional derivatives for Lagrangian and Hamiltonian mechanics. It might be interesting to perform an analysis on the applicability of that method in extending Hamilton's principle for fluid mechanics problems.

We believe that there are elements of this work that have broad applicability to

many engineering problems.

Appendix A
Variational Principles and How They are Related

The main important feature of variational principles is that they deal with *functionals* instead of functions. Functionals are scalar-valued functions of functions. A variational principle is an assertion stating that some functional defined for all possible states of a dynamic system reaches its minimum, maximum, or stationary value for the actual physical state. Functionals are obtained by performing the dot product of a virtual variable with a set of differential equations, usually Newton's second law. Unlike Newton's second law, the resulting functional is a scalar-valued function allowing scalar variables to be selected to describe the dynamics of a system. These scalar variables (usually called generalized coordinates, generalized velocities, and generalized forces) can be defined instead of the actual vector variables in modeling the dynamics of a system [29], [44].

The above statement is true for all the classical variational principles. However, it is important to note that these characteristics result from different variational principles and extend to other principles due to the relations between all of them. In this appendix, our aim is to show the importance of some of the variational principles and how they are related. As will become clear, each of the principles considered is an independent principle. Thus, assumptions must be made when obtaining the relations.

The following are some key features and the relations between d'Alembert's principle, Jourdain's principle, Hamilton's principle and Lagrange's equations.

A.1 Newton's Second Law and d'Alembert's Principle

Newton's second law for a set of N particles is

$$m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i - \boldsymbol{R}_i, \quad \text{for} \quad i = 1, \ldots, N, \tag{A.1}$$

where $\boldsymbol{r}_i$ is the Lagrangian coordinate of particle i, $\boldsymbol{F}_i$ is the external force impressed, and $\boldsymbol{R}_i$ is the resultant of the reaction forces applied on the particle. If the nature of the interaction between the particles is not known, Equation A.1 is indeterminate. This is because the total of $2N$ unknowns cannot be found with N equations.

To overcome this challenge, d'Alembert's principle multiplies Equation A.1 by an arbitrary but reversible virtual displacement $\delta_d \boldsymbol{r}_i$ that is compatible with system constraints,

$$\sum_{i=1}^{N} (m_i \ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i - \boldsymbol{R}_i) \cdot \delta_d \boldsymbol{r}_i = 0. \tag{A.2}$$

Further, d'Alembert's principle considers the case of ideal constraints for which the virtual work is zero, that is,

$$\sum_{i=1}^{N} \boldsymbol{R}_i \cdot \delta_d \boldsymbol{r}_i = 0, \tag{A.3}$$

thus, reducing Equation A.2 to

$$\sum_{i=1}^{N} (m_i \ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i) \cdot \delta_d \boldsymbol{r}_i = 0. \tag{A.4}$$

This equation is d'Alembert's principle. Equation A.4 implies that the acceleration of each particle is uniquely determined by the external force applied to it for systems with holonomic constraints [50]. Moreover, d'Alembert's principle introduces a new force, the force of inertia ($m_i \ddot{\boldsymbol{r}}_i$). Therefore, it extends the principle of virtual work from static systems to dynamic system [29]. D'Alembert's principle is also important in that it is the starting principle from which other variational formulations can be

derived.

The external forces are monogenic differential, meaning that they are derivable from a single functional. However, the inertial forces must be obtained separately.

A.2 Relation Between d'Alembert's and Jourdain's Principles

In Section 2.2, it was shown that JP can be obtained by differentiating d'Alembert's principle with respect to time and then imposing Jourdain's constraints ($\delta_d \boldsymbol{r}_i = 0$ and $\delta_d t = 0$) as

$$\begin{aligned}
& \frac{d}{dt}\left[\sum_{i=1}^{N}\left(m_i\ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i\right)\cdot\delta_d\boldsymbol{r}_i\right] \\
= & \sum_{i=1}^{N}\left[\frac{d}{dt}\left(m_i\ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i\right)\cdot\delta_d\boldsymbol{r}_i + \left(m_i\ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i\right)\cdot\frac{d}{dt}\left(\delta_d\boldsymbol{r}_i\right)\right] \\
= & \sum_{i=1}^{N}\left[\frac{d}{dt}\left(m_i\ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i\right)\cdot\delta_d\boldsymbol{r}_i + \left(m_i\ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i\right)\cdot\delta_d\dot{\boldsymbol{r}}_i\right] \\
= & \sum_{i=1}^{N}\left[\frac{d}{dt}\left(m_i\ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i\right)\cdot\delta_d\boldsymbol{r}_i + \left(m_i\ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i\right)\cdot\delta_d\dot{\boldsymbol{r}}_i\right] \\
= & \sum_{i=1}^{N}\left(m_i\ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i\right)\cdot\delta\dot{\boldsymbol{r}}_i. \qquad \text{(A.5)}
\end{aligned}$$

Therefore, substituting Equation A.5 into the time derivative of Equation A.4 results in Jourdain's principle,

$$\sum_{i=1}^{N}\left(m_i\ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i\right)\cdot\delta\dot{\boldsymbol{r}}_i = 0, \qquad \text{where} \qquad \delta\boldsymbol{r} = 0,\ \delta t = 0. \qquad \text{(A.6)}$$

A.3 Relation Between Hamilton's Principle and d'Alembert's Principle

We begin with an expansion of d'Alembert's principle,

$$\begin{aligned}
\sum_{i=1}^{N} \left(m_i \ddot{\boldsymbol{r}}_i - \boldsymbol{F}_i\right) \cdot \delta_d \boldsymbol{r}_i &= \sum_{i=1}^{N} m_i \ddot{\boldsymbol{r}}_i \cdot \delta_d \boldsymbol{r}_i - \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta_d \boldsymbol{r}_i \\
&= \sum_{i=1}^{N} \left[m_i \frac{d}{dt} \left(\dot{\boldsymbol{r}}_i \cdot \delta_d \boldsymbol{r}_i \right) - m_i \dot{\boldsymbol{r}}_i \cdot \delta_d \dot{\boldsymbol{r}}_i \right] - \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta_d \boldsymbol{r}_i \\
&= \sum_{i=1}^{N} \left[m_i \frac{d}{dt} \left(\dot{\boldsymbol{r}}_i \cdot \delta_d \boldsymbol{r}_i \right) - \frac{1}{2} m_i \delta \left(\dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i \right) \right] - \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta_d \boldsymbol{r}_i \\
&= -\sum_{i=1}^{N} \frac{1}{2} m_i \delta_d \left(\dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i \right) - \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta_d \boldsymbol{r}_i + \sum_{i=1}^{N} m_i \frac{d}{dt} \left(\dot{\boldsymbol{r}}_i \cdot \delta_d \boldsymbol{r}_i \right) \\
&= 0. \qquad \text{(A.7)}
\end{aligned}$$

Introducing the variation of the total kinetic energy,

$$\begin{aligned}
\delta_d T &= \delta_d \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i \\
&= \sum_{i=1}^{N} \frac{1}{2} m_i \delta_d \left(\dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i \right), \qquad \text{(A.8)}
\end{aligned}$$

and the total virtual work,

$$\delta_d W = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i, \qquad \text{(A.9)}$$

into Equation A.7 yields

$$\delta_d T + \delta_d W = \sum_{i=1}^{N} m_i \frac{d}{dt} \left(\dot{\boldsymbol{r}}_i \cdot \delta_d \boldsymbol{r}_i \right). \qquad \text{(A.10)}$$

Integrating both sides of this equation with respect to time from t_1 to t_2,

$$\begin{aligned}\int_{t_1}^{t_2} (\delta_d T + \delta_d W)\, dt &= \int_{t_1}^{t_2} \sum_{i=1}^{N} m_i \frac{d}{dt} (\dot{\boldsymbol{r}}_i \cdot \delta_d \boldsymbol{r}_i)\, dt \\ &= \int_{t_1}^{t_2} \sum_{i=1}^{N} m_i d (\dot{\boldsymbol{r}}_i \cdot \delta_d \boldsymbol{r}_i) \\ &= \sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_i \cdot \delta_d \boldsymbol{r}_i \Big|_{t_1}^{t_2}, \end{aligned} \tag{A.11}$$

results in Hamilton's principle of varying action,

$$\int_{t_1}^{t_2} (\delta_d T + \delta_d W)\, dt - \sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_i \cdot \delta_d \boldsymbol{r}_i \Big|_{t_1}^{t_2} = 0. \tag{A.12}$$

If the configuration is known at t_1 and t_2, no variation is allowed at $t = t_1$ and $t = t_2$, resulting in

$$\int_{t_1}^{t_2} (\delta_d T + \delta_d W)\, dt = 0. \tag{A.13}$$

For a conservative system, $\delta_d W = -\delta_d \Pi$, where Π is the potential energy. Therefore, Hamilton's principle for a conservative system becomes,

$$\int_{t_1}^{t_2} \delta_d L dt = 0, \tag{A.14}$$

where L is called the Lagrangian function and is defined as $L = T - \Pi$. Therefore, Hamilton's principle shows that an integration with respect to time brings the work done by inertial forces into a monogenic form.

It is important to note that d'Alembert's principle considers $\delta_d \boldsymbol{r}_i$ to be a local virtual displacement which is not necessarily integrable. However, Hamilton's principle deals with $\delta_d \boldsymbol{r}_i$ that is smooth and integrable between end times t_1 and t_2 when the

configuration is known. Therefore, these principles essentially deal with alternative spaces with differences.

More detailed analyses of the problem is given in [26], [29] and [22].

A.4 Obtaining Lagrange's Equation from Hamilton's Principle

As shown in Section 3.3, Lagrange's equation is obtained from Hamilton's principle for conservative systems. The derivations are summarized as follows,

$$\begin{aligned}
& \int_{t_1}^{t_2} \delta_d L\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right) \\
= & \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t_1}^{t_2} \left[L\left(\frac{d}{dt}\left(\boldsymbol{r}_i + \varepsilon\ \delta_d \boldsymbol{r}_i\right), \boldsymbol{r}_i + \varepsilon\ \delta_d \boldsymbol{r}_i, t \right) - L\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right) \right]\ dt \\
= & \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t_1}^{t_2} \left[L\left(\dot{\boldsymbol{r}}_i + \varepsilon\ \delta_d \dot{\boldsymbol{r}}_i, \boldsymbol{r}_i + \varepsilon\ \delta_d \boldsymbol{r}_i, t\right) - L\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right) \right]\ dt \\
= & \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t_1}^{t_2} \left[L\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right) + \varepsilon\ \delta_d \dot{\boldsymbol{r}}_i \cdot \frac{\partial L}{\partial \dot{\boldsymbol{r}}_i} + \varepsilon\ \delta_d \boldsymbol{r}_i \cdot \frac{\partial L}{\partial \boldsymbol{r}_i} - L\left(\dot{\boldsymbol{r}}_i, \boldsymbol{r}_i, t\right) \right]\ dt \\
= & \int_{t_1}^{t_2} \left[\delta_d \dot{\boldsymbol{r}}_i \cdot \frac{\partial L}{\partial \dot{\boldsymbol{r}}_i} + \ \delta_d \boldsymbol{r}_i \cdot \frac{\partial L}{\partial \boldsymbol{r}_i} \right]\ dt \\
= & \int_{t_1}^{t_2} \delta_d \boldsymbol{r}_i \cdot \frac{\partial L}{\partial \boldsymbol{r}_i}\ dt + \left[\delta_d \boldsymbol{r}_i \cdot \frac{\partial L}{\partial \dot{\boldsymbol{r}}_i} \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta_d \boldsymbol{r}_i \cdot \frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{r}_i} \right)\ dt \\
= & \int_{t_1}^{t_2} \delta_d \boldsymbol{r}_i \cdot \left[\frac{\partial L}{\partial \boldsymbol{r}_i} - \ \frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{r}_i} \right) \right] dt \\
= & \ 0.
\end{aligned} \tag{A.15}$$

Since $\delta_d \boldsymbol{r}_i$ are arbitrary vectors, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{r}_i} \right) - \frac{\partial L}{\partial \boldsymbol{r}_i} = 0. \tag{A.16}$$

Equation A.16 is called Lagrange's equation. Lagrange's equation has two important consequences. First, Lagrange's equation shows that a set of generalized coordinates

can replace the rectangular coordinates. Second, conservation of energy is a consequence of the Lagrange's equation. More detailed analyses of the problem can be found in [26], [29] and [22].

A.5 Obtaining Lagrange's Equation from d'Alembert's Principle

D'Alembert's principle is expressed by

$$\sum_{i=1}^{N} m_i \ddot{\boldsymbol{r}}_i \cdot \delta_d \boldsymbol{r}_i = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta_d \boldsymbol{r}_i, \tag{A.17}$$

for a system of N particles. If the system has M degrees of freedom, M generalized coordinates are required to describe the system. For such systems, the equation of condition in differential form (not necessarily integrable) is

$$d\boldsymbol{r}_i = \sum_{j=1}^{M} \frac{\partial \boldsymbol{p}_i}{\partial q_j} dq_j + \boldsymbol{p}_i dt, \tag{A.18}$$

where $\boldsymbol{p}_i$ are any functions of q and t. Therefore, $\delta_d \boldsymbol{r}_i$ is obtained as

$$\delta_d \boldsymbol{r}_i = \sum_{j=1}^{M} \frac{\partial \boldsymbol{p}_i}{\partial q_j} \delta_d q_j, \tag{A.19}$$

since $\delta_d t = 0$. Therefor, the left-hand side of Equation A.17 can be expanded as

$$\begin{aligned} & \sum_{i=1}^{N} m_i \ddot{\boldsymbol{r}}_i \cdot \delta_d \boldsymbol{r}_i \\ = & \sum_{i=1}^{N} \frac{d}{dt} \left(m_i \dot{\boldsymbol{r}}_i \cdot \delta_d \boldsymbol{r}_i \right) - \sum_{i=1}^{N} m_i \ddot{\boldsymbol{r}}_i \cdot \frac{d}{dt} \left(\delta_d \boldsymbol{r}_i \right) \\ = & \sum_{j=1}^{M} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \delta_d q_j \right) - \sum_{j=1}^{M} \frac{\partial T}{\partial \dot{q}_j} \frac{d}{dt} \left(\delta_d q_j \right) - \sum_{j=1}^{M} \sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_i \cdot \delta_d q_j \frac{d}{dt} \frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_j}, \end{aligned} \tag{A.20}$$

where T is the kinetic energy, defined as

$$T = \sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i. \tag{A.21}$$

For a holonomic system, it can be shown that

$$\frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_j} = \frac{\partial \boldsymbol{r}_i}{\partial q_j}, \tag{A.22}$$

and

$$\frac{d}{dt}\frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_j} = \frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_j}. \tag{A.23}$$

Substituting Equations A.22 and A.23 into Equation A.20 yields

$$\begin{aligned}
&\sum_{i=1}^{N} m_i \ddot{\boldsymbol{r}}_i \cdot \delta_d \boldsymbol{r}_i \\
= &\sum_{j=1}^{M} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\delta_d q_j\right) - \sum_{j=1}^{M} \frac{\partial T}{\partial \dot{q}_j}\frac{d}{dt}\left(\delta_d q_j\right) - \sum_{j=1}^{M}\sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_i \cdot \frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_j}\delta_d q_j \\
= &\sum_{j=1}^{M} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\delta_d q_j\right) - \sum_{j=1}^{M} \frac{\partial T}{\partial \dot{q}_j}\frac{d}{dt}\left(\delta_d q_j\right) - \sum_{j=1}^{M} \frac{\partial T}{\partial q_j}\delta_d q_j \\
= &\sum_{j=1}^{M} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\delta_d q_j - \frac{\partial T}{\partial q_j}\delta_d q_j.
\end{aligned} \tag{A.24}$$

Introducing Equation A.24 into Equation A.17 results in

$$\sum_{j=1}^{M} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\delta_d q_j - \frac{\partial T}{\partial q_j}\delta_d q_j = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta_d \boldsymbol{r}_i. \tag{A.25}$$

Regarding the right-hand side, since $\boldsymbol{F}_i$ is monogenic, we have

$$\sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta_d \boldsymbol{r}_i = \sum_{j=1}^{M}\left(-\frac{\partial \Pi}{\partial q_j}\delta_d q_j + Q_j \delta_d q_j\right), \tag{A.26}$$

where Π is the potential energy and Q_j is the generalized force. Substituting Equation A.26 into Equation A.25 results in

$$\sum_{j=1}^{M}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}+\frac{\partial \Pi}{\partial q_j}\right]\delta_d q_j=\sum_{j=1}^{M}Q_j\delta_d q_j, \tag{A.27}$$

or,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}+\frac{\partial \Pi}{\partial q_j}=Q_j \qquad \text{for} \qquad j=1,2,\ldots,M, \tag{A.28}$$

which is Lagrange's equation.

A more detailed discussion can be found in [24].

A.6 Obtaining Lagrange's Equation from Jourdain's Principle

Jourdain's principle for a system of N particles is

$$\sum_{i=1}^{N} m_i\ddot{\boldsymbol{r}}_i\cdot\delta\dot{\boldsymbol{r}}_i \;=\sum_{i=1}^{N}\boldsymbol{F}_i\cdot\delta\dot{\boldsymbol{r}}_i\;, \qquad \text{where} \qquad \delta\boldsymbol{r}_i=0,\;\delta t=0. \tag{A.29}$$

Similar to the previous section, we assume the system has M degrees of freedom, therefore,

$$\delta\dot{\boldsymbol{r}}_i=\sum_{j=1}^{M}\frac{\partial \boldsymbol{p}_i}{\partial \dot{q}_j}\delta\dot{q}_j, \tag{A.30}$$

since $\delta q_j=0$. The left-hand side of Equation A.29 can be expanded as

$$\begin{aligned}
&\sum_{i=1}^{N} m_i\ddot{\boldsymbol{r}}_i\cdot\delta\dot{\boldsymbol{r}}_i \\
=\;& \frac{d}{dt}\left[\sum_{i=1}^{N} m_i\dot{\boldsymbol{r}}_i\cdot\delta\dot{\boldsymbol{r}}_i\right]-\sum_{i=1}^{N} m_i\dot{\boldsymbol{r}}_i\cdot\frac{d}{dt}\left(\delta\dot{\boldsymbol{r}}_i\right) \\
=\;& \frac{d}{dt}\left(\delta T\right)-\sum_{j=1}^{M}\frac{\partial T}{\partial \dot{q}_j}\frac{d}{dt}\left(\delta\dot{q}_j\right)-\sum_{j=1}^{M}\sum_{i=1}^{N} m_i\dot{\boldsymbol{r}}_i\cdot\frac{d}{dt}\left(\frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_j}\right)\delta\dot{q}_j,
\end{aligned} \tag{A.31}$$

where T was defined in Equation A.21. Introducing Equation A.23 into Equation A.31 yields

$$\int_{t_1}^{t_2} \sum_{i=1}^{N} m_i \ddot{\boldsymbol{r}}_i \cdot \delta \dot{\boldsymbol{r}}_i dt$$
$$= \int_{t_1}^{t_2} \frac{d}{dt} (\delta T)\, dt - \int_{t_1}^{t_2} \sum_{j=1}^{M} \frac{\partial T}{\partial \dot{q}_j} \frac{d}{dt} (\delta \dot{q}_j)\, dt - \int_{t_1}^{t_2} \sum_{j=1}^{M} \left(\frac{\partial T}{\partial q_j} \right) \delta \dot{q}_j dt, \quad \text{(A.32)}$$

where we also integrate both sides of the equation with respect to time from t_1 to t_2. Integrating Equation A.32 by parts,

$$\begin{aligned}
&\int_{t_1}^{t_2} \sum_{i=1}^{N} m_i \ddot{\boldsymbol{r}}_i \cdot \delta \dot{\boldsymbol{r}}_i dt \\
= \;& [\delta T]_{t_1}^{t_2} - \left[\sum_{j=1}^{M} \frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \sum_{j=1}^{M} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) \delta \dot{q}_j dt - \int_{t_1}^{t_2} \sum_{j=1}^{M} \left(\frac{\partial T}{\partial q_j} \right) \delta \dot{q}_j dt \\
= \;& \left[\delta T - \sum_{j=1}^{M} \frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \sum_{j=1}^{M} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) \delta \dot{q}_j dt \\
& - \int_{t_1}^{t_2} \sum_{j=1}^{M} \left(\frac{\partial T}{\partial q_j} \right) \delta \dot{q}_j dt, \qquad \text{(A.33)}
\end{aligned}$$

and noting that from the definition of Jourdain's variational operator (Equation 3.9)

$$\delta T = \sum_{j=1}^{M} \frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j, \qquad \text{(A.34)}$$

we obtain,

$$\int_{t_1}^{t_2} \sum_{i=1}^{N} m_i \ddot{\boldsymbol{r}}_i \cdot \delta \dot{\boldsymbol{r}}_i dt = \int_{t_1}^{t_2} \sum_{j=1}^{M} \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \left(\frac{\partial T}{\partial q_j} \right) \right] \delta \dot{q}_j dt. \qquad \text{(A.35)}$$

Considering the right-hand side of Equation A.29, we define

$$\int_{t_1}^{t_2} \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \dot{\boldsymbol{r}}_i dt = \int_{t_1}^{t_2} \sum_{j=1}^{M} \left(-\frac{\partial \Pi}{\partial q_j} + Q_j \right) \delta \dot{q}_j dt. \tag{A.36}$$

Substituting Equations A.35 and A.36 into Equation A.29, we obtain Lagrange's equation,

$$\int_{t_1}^{t_2} \sum_{j=1}^{M} \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \left(\frac{\partial T}{\partial q_j} \right) \right] \delta \dot{q}_j dt = \int_{t_1}^{t_2} \sum_{j=1}^{M} \left(-\frac{\partial \Pi}{\partial q_j} + Q_j \right) \delta \dot{q}_j dt, \tag{A.37}$$

or,

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \left(\frac{\partial T}{\partial q_j} \right) = -\frac{\partial \Pi}{\partial q_j} + Q_j \qquad \text{for} \qquad j = 1, 2, \ldots, M. \tag{A.38}$$

An important conclusion is that

$$\left[\delta T - \sum_{j=1}^{M} \frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j \right]_{t_1}^{t_2}$$

is satisfied identically regardless of knowledge of the end time configuration. We believe that a similar connection is possible with Hamilton's principle.

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