ON SYNCHRONOUS BEHAVIOR IN COMPLEX NONLINEAR DYNAMICAL SYSTEMS

BY ZAHRA AMINZARE

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ABSTRACT OF THE DISSERTATION

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The purpose of this dissertation is to study synchronous behavior of certain nonlinear dynamical systems by the method of *contraction* theory.

Contraction theory provides an elegant way to analyze the behavior of certain nonlinear systems. Under sometimes easy to check hypotheses, systems can be shown to have the incremental stability property that trajectories converge to each other. This work provides a self contained introduction to some of the basic concepts and results in contraction theory. As we will discuss later, contractivity is not a topological, but is instead a metric property: it depends on the norm being used in contraction theory and in fact an appropriate choice of norms is critical. One of the main contributions of this dissertation is to generalize some of the existing results in the literature which are based on L^2 norms to results based on non L^2 norms using some modern techniques from nonlinear functional analysis.

The focus of the first main part of this dissertation is on the application of contraction theory and graph theory to synchronization in complex interacting systems that can be modeled as an interconnected network of identical systems. We base our approach on contraction theory, using norms that are not induced by inner products. Such norms are the most appropriate in many applications, but proofs cannot rely upon

Lyapunov-like linear matrix inequalities, and different techniques, such as the use of the Perron-Frobenious Theorem in the cases of L^1 or L^∞ norms, must be introduced.

On the second main part of this work, using the method of contraction theory based on non L^2 norms, spatial uniformity for the asymptotic behavior of the solutions of a reaction diffusion PDE with Neumann boundary conditions will be studied.

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Dedication

TO MY DEAR PARENTS.

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Chapter 1

Introduction

1.1 Background and previous work

Synchrony can be divided into two categories: *external synchronization* which refers to synchronization by an external force, [1, 2, 3], and *mutual synchronization* of two or more coupled nonlinear systems. Discovered by the Dutch scientist Christiaan Huygens, inventor of the pendulum clock in 1657, mutual synchronization of two periodic oscillators was first analytically studied by Maier [4].

The analysis of mutual synchronization in networks of identical components involves a variety of research fields in science and engineering as well as in mathematics. In biology, the synchronization phenomenon is exhibited at the physiological level, for example in neuronal interactions, in the generation of circadian rhythms, or in the emergence of organized bursting in pancreatic beta-cells, [5, 6, 7, 8, 9, 10]. It is also exhibited at the population level, for example in the simultaneous flashing of fireflies, [11, 12]. In engineering, one finds applications of synchronization ideas in areas as varied as robotics or autonomous vehicles, [13, 14]. For more references, see also [15, 16, 17, 18].

In modeling such networks, a direct communication between nodes is often assumed. However, there are many natural examples that nodes rather to communicate through the environment than direct communication. *Quorum sensing* is an example of this kind of communication [19].

We will restrict attention to interconnections given by diffusion, where each pair of “adjacent” components exchange information and adjust in the direction of the difference with each other.

Many different theoretical methods, based on Lyapunov exponents [20], Master stability function [21] (see Section 3.3.4 for a discussion), graph theory [22], and Lyapunov functions [23], have been employed to approach the problem of synchronization. Another useful method to show synchronization, which we are interested in, is *contraction theory*.

A proper tool for characterizing contractivity for nonlinear systems is provided by the matrix measures, or logarithmic norms, [24, 25], of the Jacobian of the vector field, evaluated at all possible states. This idea is a classical one, and can be traced back at least to work of D.C. Lewis in the 1940s, [26, 27]. Dahlquist's 1958 thesis under Hörmander ([28] for a journal paper) used matrix measures to show contractivity of differential equations, and more generally of differential inequalities, the latter applied to the analysis of convergence of numerical schemes for solving differential equations. Several authors have independently rediscovered the basic ideas. For example, in the 1960s, Demidovič [29, 30] established basic convergence results with respect to Euclidean norms, as did Yoshizawa [31, 32]. In control theory, the field attracted much attention after the work of Lohmiller and Slotine [33], and follow-up papers by Slotine and collaborators, see for example [34, 35, 36, 37]. These papers showed the power of contraction techniques for the study of not only stability, but also observer problems, nonlinear regulation, and consensus problems in complex networks. (See also the work of Nijmejer and coworkers [38].) We refer the reader to the historical analysis given in [39, 40] and the survey [41].

Synchronization results based on contraction-based techniques, typically employing measures derived from L^2 or weighted L^2 norms, [33, 35, 42, 43, 44, 45, 46] have been already well studied. The proofs rely upon Lyapunov functions.

Our interest here is in using matrix measures derived from norms that are not induced by inner products, such as L^1 and L^∞ norms, because these are the most appropriate in many applications, such as the biochemical examples discussed as illustrations in this dissertation (see Section 3.3.3). For such more general norms, proofs cannot rely upon Lyapunov-like linear matrix inequalities. We remark that other authors have also previously studied matrix measures based on non L^2 norms, see for instance [33];

however, rigorous proofs of the types of results proved here have not been given in [33]. In [47], the author studies synchronization using matrix measures for L^1, L^2, and L^∞ norms; we will compare our results to this and other papers later in this dissertation (see Section 3.3.4). Also, in [48, 49], a sufficient condition for synchronization based on matrix measure induced by an arbitrary norm is given for *linear* systems, but in this work we focus on *nonlinear* systems.

1.2 The main contribution of the current work

We are interested in approaching the problem of synchronization by the methods of contraction theory for non L^2 norms and graph theory. Unlike the usual methods appropriate for Hilbert spaces, based on Lyapunov functions, our method is based on techniques from modern functional analysis. The reason that we are interested in L^p norms rather than just L^2 norms is that we were motivated by a desire to understand an important biological system, for which contractivity holds for diagonally weighted L^1 norms, but not with respect to diagonally weighted L^p norms, for any $1 < p \leq \infty$.

System of ODEs

In Chapter 2 and 3, we study the global convergence of the solution of a network with N compartments (nodes) $x_1, \dots, x_N$, $x_i \in \mathbb{R}^n$, with identical dynamics $\dot{x}_i = F(x_i, t)$, which are connected through an undirected, connected graph $\mathcal{G}$ with N vertices, m edges and (graph) Laplacian $\mathcal{L}$. The following system of ODEs describes the evolution of the x_i's in the network:

$$\dot{x} = \tilde{F}(x, t) - \mathcal{L} \otimes D(t)x, \tag{1.1}$$

where x is the column of the x_i's and $\tilde{F}(x, t)$ is the column of $F(x_i, t)$'s and $D(t)$ is the diffusion matrix.

Let $(X, \|\cdot\|_X)$ be a finite dimensional normed vector space over $\mathbb{R}$ or $\mathbb{C}$. The space $\mathcal{L}(X, X)$ of linear transformations $A\colon X \to X$ is also a normed vector space with the induced operator norm $\|A\|_{X\to X} = \sup_{\|x\|_X=1} \|Ax\|_X$. The logarithmic norm (also called matrix measure) $\mu_X(\cdot)$ induced by $\|\cdot\|_X$ is defined as the directional derivative of the matrix norm, that is,

$$\mu_X[A] \;=\; \lim_{h\to 0^+} \frac{1}{h}\left(\|I+hA\|_{X\to X}-1\right),$$

where I is the identity operator on X.

We say that the system (1.1) is *contractive*, if any two solutions of the network converge to each other exponentially and with no overshoot (see Remark 1). In Chapter 2, we provide conditions on (only) F that guarantee contractivity of the system (1.1). In this work, J_f denotes the Jacobian of f.

Theorem 1. *Consider the system (1.1) and let* $c := \sup_{(x,t)} \mu_{p,Q}[J_F(x,t)]$, *where* $\mu_{p,Q}$ *is the logarithmic norm induced by the* Q*-weighted* L^p *norm,* $\|\cdot\|_{p,Q}$, *on* $\mathbb{R}^n$ *defined by* $\|x\|_{p,Q} := \|Qx\|_p$, *for some* $1 \le p \le \infty$ *and a positive diagonal matrix* Q. *Then for any two solutions* x *and* y *of (1.1), we have*

$$\|x(t)-y(t)\|_{p,Q} \le e^{ct}\,\|x(0)-y(0)\|_{p,Q}\,.$$

In particular, when $c<0$, *the system (1.1) is contractive.*

We say that the system (1.1) *synchronizes*, if any solution $x = \left(x_1^T,\dots,x_N^T\right)^T$ of the system converges to a uniform solution exponentially; in other words, $x_i(t)-x_j(t)\to 0$, exponentially as $t\to\infty$. See Figure 1.1 for graphical illustrations.

Although the "contractivity" condition provided in Theorem 1, namely $c<0$, guarantees synchrony of the system (1.1), we are interested in a weaker condition. In Chapter 3, we provide such a condition.

Theorem 2. *Consider the system (1.1). For an arbitrary orientation of* $\mathcal{G}$, *let* E *be the directed incidence matrix of* $\mathcal{G}$, *and pick any* $m\times m$ *matrix* K *satisfying* $E^T\mathcal{L} = KE^T$. *Denote*

$$c \;:=\; \sup_{(w,t)} \mu\left[J(w,t) - K\otimes D(t)\right],$$

where μ *is the logarithmic norm induced by an arbitrary norm on* $\mathbb{R}^{mn}$, $\|\cdot\|$, *and for* $w = \left(w_1^T,\dots,w_m^T\right)^T$, $J(w,t)$ *is defined as follows:*

$$J(w,t) = diag\left(J_F(w_1,t),\dots,J_F(w_m,t)\right),$$

and $J_F(\cdot,t)$ *denotes the Jacobian of* F *with respect to the first variable. Then*

$$\left\|\left(E^T \otimes I\right) x(t)\right\| \;\leq\; e^{ct} \left\|\left(E^T \otimes I\right) x(0)\right\|.$$

Note that $\left(E^T \otimes I\right) x$ is a column vector whose entries are the differences $x_i(t) - x_j(t)$, for each edge $e = \{i,j\}$ in $\mathcal{G}$. Therefore, if $c < 0$, the system synchronizes.

As a direct application of Theorem 2, we conclude the following results for *complete* and *path* graphs, in particular.

Proposition 1. *Let* $(x_1^T, \ldots, x_N^T)^T$ *be a solution of*

$$\dot{x}_i = F(x_i,t) + D(t)(x_{i-1} - x_i + x_{i+1} - x_i), \quad i = 1, \ldots, N,$$

assuming $x_0 = x_1$ *and* $x_N = x_{N+1}$. *For* $1 \leq p \leq \infty$ *and a positive diagonal matrix* Q, *let*

$$c = \sup_{(x,t)} \mu_{p,Q} \left[J_F(x,t) - 4\sin^2\left(\pi/2N\right) D(t)\right].$$

Then

$$\|e(t)\|_{p,Q_p\otimes Q} \;\leq\; e^{ct}\|e(0)\|_{p,Q_p\otimes Q},$$

where $e = \left(e_1^T, \ldots, e_{N-1}^T\right)^T$ *with* $e_i = x_i - x_{i+1}$ *denotes the vector of all edges of the path graph, and* $\|\cdot\|_{p,Q_p\otimes Q}$ *denotes the weighted* L^p *norm with the weight* $Q_p \otimes Q$, *where*

$$\begin{aligned} Q_p &= diag\left(p_1^{\frac{2-p}{p}}, \ldots, p_{N-1}^{\frac{2-p}{p}}\right), \quad \textit{for} \quad 1 \leq p < \infty \\ Q_\infty &= diag\left(1/p_1, \ldots, 1/p_{N-1}\right), \end{aligned}$$

and for $1 \leq k \leq N-1$, $p_k = \sin(k\pi/N)$. *In addition,* $4\sin^2\left(\pi/2N\right)$ *is the smallest nonzero eigenvalue of the Laplacian matrix of* $\mathcal{G}$.

Proposition 2. *Let* $\|\cdot\|$ *be an arbitrary norm on* $\mathbb{R}^n$ *with corresponding logarithmic norm* μ, $x = (x_1^T, \ldots, x_N^T)^T$ *be a solution of*

$$\dot{x}_i = F(x_i,t) + D(t)\sum_{j=1}^{N}(x_j - x_i),$$

and

$$c := \sup_{(x,t)} \mu[J_F(x,t) - ND(t)].$$

Then

$$\sum_{k=1}^{m} \|e_k(t)\| \;\leq\; e^{ct} \sum_{k=1}^{m} \|e_k(0)\| \,,$$

where e_k*'s, for* $k = 1, \ldots, m$ *are the edges of* $\mathcal{G}$*, meaning the differences* $x_{i_k}(t) - x_{j_k}(t)$ *for* $i_k < j_k \in \{1, \ldots, N\}$*.*

Note that N is the smallest nonzero eigenvalue of the Laplacian matrix of a complete graph with N vertices.

In Propositions 1, and Propositions 2, when $c < 0$, the systems synchronize.

In addition to path and complete graphs, we provide conditions for synchronization in systems that are connected through a *star* graph: if $\sup_{(x,t)} \mu[J_F(x,t) - D(t)] < 0$, then the system synchronizes. Also, a *Cartesian product* of the above graphs would synchronize under an appropriate condition.

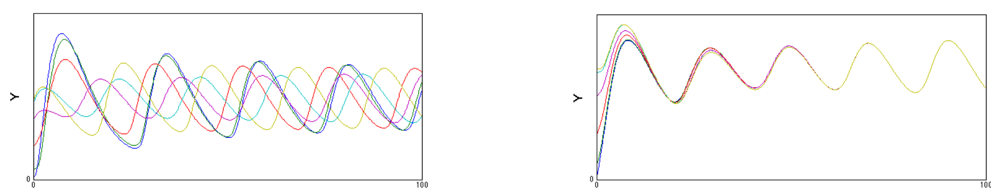


Figure 1.1: (left) 6 isolated compartments (right) complete interconnection

System of PDEs

As in the discrete case, we are also interested in finding conditions that guarantee the *synchronous* behavior of the solutions of the following reaction diffusion PDEs defined on $\Omega \times [0, \infty)$ for a smooth domain $\Omega \subset \mathbb{R}^m$, and subject to Neumann boundary conditions:

$$\begin{aligned}
&\frac{\partial u_i}{\partial t}(\omega, t) = F_i(u(\omega, t), t) + d_i(t)\Delta u_i(\omega, t), && i = 1, \ldots, n, \\
&\frac{\partial u_i}{\partial \mathbf{n}}(\xi, t) = 0 && \forall \xi \in \partial\Omega, \;\; \forall t \in [0, \infty).
\end{aligned} \tag{1.2}$$

An analogous result to Theorem 1 for a reaction diffusion PDE (1.2), is the following theorem proved in Section 4.2.

Theorem 3. *Consider the PDE (1.2) and let*

$$c = \sup_{(x,t)} \mu_{p,Q}[J_F(x,t)] \ ,$$

for some $1 \leq p \leq \infty$ *and some positive diagonal matrix* Q. *Then for any two solutions* u *and* v *of the reaction diffusion PDE defined on* $[0,T)$ *for some* $T \in (0,\infty]$,

$$\|u(\cdot,t) - v(\cdot,t)\|_{p,Q} \leq e^{ct} \|u(\cdot,0) - v(\cdot,0)\|_{p,Q} \, .$$

We say that the system (1.2) *synchronizes*, if for any solution $u(\omega,t)$ of the system, there exists a uniform solution $\bar{u}(\omega,t) = \bar{u}(t)$, such that $u(\omega,t) - \bar{u}(t) \to 0$ as $t \to \infty$ for all $\omega \in \Omega$ (understood in an appropriate topology). See Figure 1.2a and Figure 1.2b for graphical illustrations.

Similar to Theorem 1, Theorem 3 guarantees synchrony of reaction diffusion PDE system (1.2), namely when $c = \sup_{(x,t)} \mu_{p,Q}[J_F(x,t)] < 0$, any solution u, converges to a uniform solution $\bar{u}(\omega,t) = \bar{u}(t)$. But more interestingly, we are looking for a weaker condition. In Section 4.4, we will provide such a condition for 1-dimensional space and for L^1 norms.

Theorem 4. *Let* $u(\omega,t)$ *be a solution of*

$$\begin{aligned} \frac{\partial u}{\partial t}(\omega,t) &= F(u(\omega,t),t) + D(t)\,\frac{\partial^2 u}{\partial \omega^2}(\omega,t) \quad \text{on } (0,L) \\ \frac{\partial u}{\partial \omega}(0,t) \;&=\; \frac{\partial u}{\partial \omega}(L,t) \;=\; 0, \end{aligned}$$

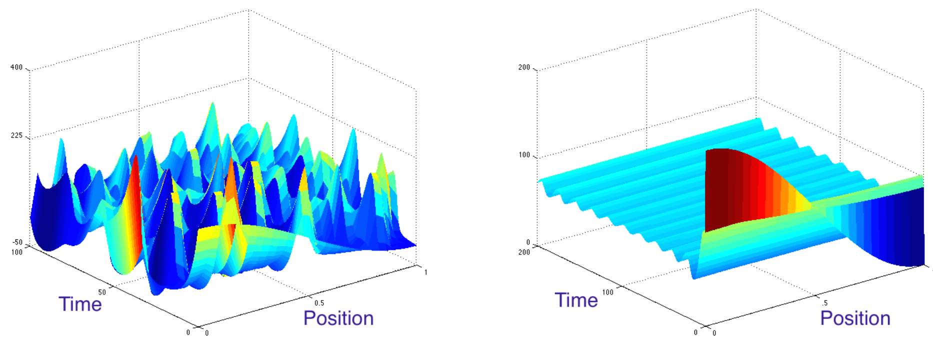
defined for all $t \in [0,T)$ *for some* $0 < T \leq \infty$. *In addition, assume that* $u(\cdot,t) \in C^3(\Omega)$, *for all* $t \in [0,T)$. *Let*

$$c \;=\; \sup_{t\in[0,T)} \sup_{x\in V} \mu_{1,Q}\left[J_F(x,t) - \frac{\pi^2}{L^2} D(t)\right],$$

where $\mu_{1,Q}$ *is the logarithmic norm induced by* $\|\cdot\|_{1,Q}$ *for a positive diagonal matrix* Q. *Then for all* $t \in [0,T)$,

$$\left\|\frac{\partial u}{\partial \omega}(\cdot,t)\right\|_{1,\phi,Q} \leq e^{ct} \left\|\frac{\partial u}{\partial \omega}(\cdot,0)\right\|_{1,\phi,Q},$$

where $\|\cdot\|_{1,\phi,Q} := \|\sin(\pi\omega/L)(\cdot)\|_{1,Q}$.



(a) (b)

Figure 1.2: (a) shows the oscillatory behavior of a solution of a PDE system when there is no diffusion, $D = 0$. (b) shows the spatially uniformity of the solution of the same PDE when diffusion occurs, $D \neq 0$

1.3 Publications associated to this work

Many portions of this dissertation have been published. These publications are all self contained, and the reader is invited to look at them for discussions that are independent of the rest of the material.

- A part of Chapter 2 was published in the proceedings for the 2014 IEEE conference on Decision and Control [50], (jointly authored with Eduardo Sontag).

- A version of Section 3.2 and Section 4.2 were published in the Journal of Nonlinear Analysis: Theory, Methods and Applications [51], (jointly authored with Eduardo Sontag).

- A version of Section 3.4 and Section 4.5 were published in a joint book chapter in Springer-Verlag [46] and in proceedings for the 2013 American Control Conference [52] (coauthored with Yusef Shafi, Murat Arcak, and Eduardo Sontag).

- Section 3.3 was published in the proceedings for the 2014 IEEE conference on Decision and Control [53] and in the Journal of IEEE Transactions on Network Science and Engineering [54], (jointly authored with Eduardo Sontag).

Chapter 2

Contraction theory for nonlinear ODE systems

Acknowledgement of conference publication:

Parts of the material in this chapter have been published in the conference paper [50].

2.1 Introduction

Global stability is a central research topic in dynamical systems theory. Stability properties are typically defined in terms of attraction to an invariant set, for example to an equilibrium or a periodic orbit, often coupled with a Lyapunov stability requirement that trajectories that start near the attractor must stay close to the attractor for all future times.

A far stronger requirement than attraction to a pre-specified target set is to ask that any two trajectories should (exponentially, and with no overshoot (see Remark 1 below)), converge to each other, or, in more abstract mathematical terms, that the flow be a contraction in the state space. While this requirement will be less likely to be satisfied for a given system, it is sometimes comparatively easier to check. Indeed, checking stability properties often involves constructing an appropriate Lyapunov function, which, in turn, requires a priori knowledge of the attractor location. In contrast, contraction-based methods, discussed here, do not require the prior knowledge of attractors. Instead, one checks an infinitesimal property, that is to say, a property of the vector field defining the system, which guarantees exponential contractivity of the induced flow.

It is useful to first discuss the relatively trivial case of linear time-invariant systems of differential equations $\dot{x} = Ax$, with Euclidean norm. Since differences of solutions are also solutions, contractivity amounts simply to the requirement that there exists a positive number c such that, for all solutions, $|x(t)| \leq e^{-ct}|x(0)|$, where $|\cdot|$ refers to the Euclidean norm. This is clearly equivalent to the requirement that $A + A^T$ be a negative definite matrix. In Lyapunov-function terms, $x^T Px$ is a Lyapunov function for the system, when $P = I$.

This property is of course stronger than merely asymptotic stability of the zero equilibrium of $\dot{x} = Ax$, that is, that A be a Hurwitz matrix (all eigenvalues with negative real part). Of course, asymptotic stability is equivalent to the existence of some positive definite matrix P (but not necessarily the identity) so that $x^T Px$ is a Lyapunov function, and this can be interpreted, as remarked later, as a contractivity property with respect to a weighted Euclidean norm associated to P. This simple example with linear systems already illustrates why an appropriate choice of norms when defining "contractivity" is critical; even for linear systems, contractivity is not a topological, but is instead a metric property: it depends on the norm being used, in close analogy to the choice of an appropriate Lyapunov function.

Remark 1. *Our results provide a far stronger property than asymptotic stability of solutions. Consider for example the system*

$$\begin{aligned} \dot{x} &= -x \\ \dot{y} &= (x-1)y, \end{aligned} \tag{2.1}$$

which has the origin as a globally asymptotically stable state. This system cannot be contractive under any possible norm, since solutions starting with large x initially diverge from each other. Figure 2.1 shows the x components (left) and y components (right) of two solutions of Equation (2.1). It is clear from the right figure that the solutions diverge after a while and so the solutions are not contractive.

Figure 2.2 shows that a nonlinear system could be contractive in one norm (here L^1 norm) while it cannot be contractive in other norms (e.g., L^2 norm). For more details about the system see the biochemical example explained in Section 3.3.3.

In this section, we first discuss the most basic results regarding contraction for ODE systems. We frame our discussion in the language of modern nonlinear functional analysis in the style of [55]. This language provides the natural concepts needed to

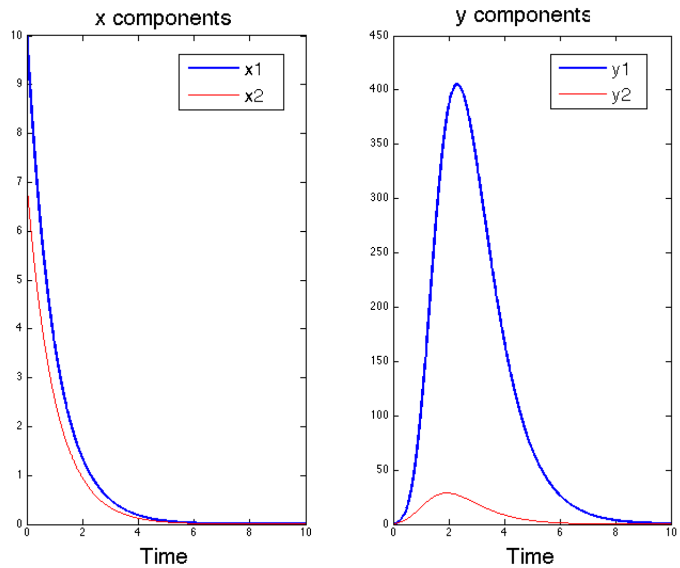


Figure 2.1: Two solutions of Equation (2.1)

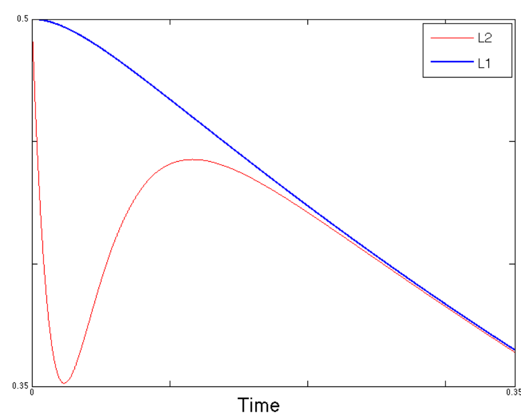


Figure 2.2: In this figure the difference between two solutions of a nonlinear system (biochemical example) is given in L^2 norm (red and thin graph) and L^1 norm (blue and thick graph). As it is clear, in L^2 norm, there is an overshoot between the solutions (the graph of their difference increases after a while)

understand abstract norms as well as extensions to infinite-dimensional spaces, including partial differential equations. We then turn to certain new developments regarding diffusive synchronization of ODE systems. The emphasis of this section would be on contractions with respect to non-Euclidean norms, and for which many problems remain open (see Section 5). We only consider deterministic systems; see [36] for applications of contraction theory to the analysis of certain stochastic systems.

In this section we study systems described by nonlinear deterministic systems of differential equations

$$\dot{x} \;=\; f(x,t)\,, \tag{2.2}$$

where $x(t) \in V \subseteq \mathbb{R}^n$ is an n dimensional vector corresponding to the state of the system, $t \in [0,\infty)$ is the time, and f is a nonlinear vector field which is differentiable on x with Jacobian matrix denoted by J_f and is continuous on (x,t). The goal is to find a condition that guarantees that any two trajectories of (2.2) converge to each other exponentially.

As mentioned in the introduction, Section 1, we focus here on conditions based on matrix measures. We recall (see for instance [24] or [25]) that, given a vector norm on Euclidean space ($|\cdot|$), with its induced matrix norm $\|A\|$, the associated *matrix measure* μ is defined as the directional derivative of the matrix norm in the direction of A and evaluated at the identity matrix, that is:

$$\mu[A] \;:=\; \lim_{h\to 0^+} \frac{1}{h}\left(\|I+hA\|-1\right).$$

The limit is known to exist (see Remark 4), and the convergence is monotonic, see [28, 56].

Matrix measure, also known as "*logarithmic norm*" of a square matrix A, was independently introduced by Germund Dahlquist [28], and Sergei Lozinskii [57], in 1959. In 1965, W. A. Coppel [58, page 58] showed that μ can be used to bound solutions of linear differential equation $\dot{x} = A(t)x$, (see Theorem 5). In 1970, R. H. Martin [59] extended the definition of μ to functions which satisfy a Lipschitz condition on bounded subsets of a Banach space (see Definition 2 in Section 2.2) and used this extension to bound solutions of the corresponding differential equations (see Theorem 6).

2.2 Preliminaries: logarithmic Lipschitz constants

We now define and state elementary properties of logarithmic Lipschitz constants based on the definitions in [41, 55].

Definition 1. *Let $(X, \|\cdot\|_X)$ be a normed space and $f\colon Y \to X$ be a function, where $Y \subseteq X$. The least upper bound (lub) Lipschitz constant of f induced by the norm $\|\cdot\|_X$, on Y, is defined by*

$$L_{Y,X}[f] = \sup_{u \neq v \in Y} \frac{\|f(u) - f(v)\|_X}{\|u - v\|_X}.$$

Note that $L_{Y,X}[f] < \infty$ if and only if f is Lipschitz on Y.

When identifying a linear operator $f : \mathbb{R}^n \to \mathbb{R}^n$ with its matrix representation A with respect to the canonical basis, $L_{Y,X}[A] = \|A\|_{X \to X}$, where $\|\cdot\|_{X\to X}$ is the operator norm induced by $\|\cdot\|_X$.

Definition 2. *Let $(X, \|\cdot\|_X)$ be a normed space and $f\colon Y \to X$ be a Lipschitz function. The least upper bound (lub) logarithmic Lipschitz constant of f induced by the norm $\|\cdot\|_X$, on $Y \subseteq X$, is defined by*

$$\mu_{Y,X}[f] = \lim_{h \to 0+} \frac{1}{h}\left(L_{Y,X}[I + hf] - 1\right),$$

or equivalently,

$$\mu_{Y,X}[f] = \lim_{h\to 0^+} \sup_{u \neq v \in Y} \frac{1}{h}\left(\frac{\|u - v + h(f(u) - f(v))\|_X}{\|u - v\|_X} - 1\right). \tag{2.3}$$

If $X = Y$, we write μ_X instead of $\mu_{X,X}$. Whenever it is clear from the context, we drop the subscript and simply write μ instead of $\mu_{Y,X}$.

When identifying a linear operator $f : \mathbb{R}^n \to \mathbb{R}^n$ with its matrix representation A with respect to the canonical basis, we call μ a "matrix measure" or a "logarithmic norm".

Notation 1. *Let $(X, \|\cdot\|_X)$ be a normed space and $f\colon Y \to X$ be a function. Denote $\mu^+_{Y,X}$ as follows*

$$\mu^+_{Y,X}[f] := \sup_{u \neq v \in Y} \lim_{h\to 0^+} \frac{1}{h}\left(\frac{\|u - v + h(f(u) - f(v))\|_X}{\|u - v\|_X} - 1\right). \tag{2.4}$$

If $X = Y$, we write μ^+_X instead of $\mu^+_{X,X}$. Whenever it is clear from the context, we drop the subscript and simply write μ^+ instead of $\mu^+_{Y,X}$.

Similarly, denote $\mu^{-}_{Y,X}$ *as follows*

$$\mu^{-}_{Y,X}[f] := \sup_{u\neq v\in Y} \lim_{h\to 0^{-}} \frac{1}{h}\left(\frac{\|u-v+h(f(u)-f(v))\|_X}{\|u-v\|_X}-1\right). \tag{2.5}$$

Remark 2. In general $\mu^{+}_{Y,X}[f]$ and $\mu^{-}_{Y,X}[f]$ are not equal. Consider $X = Y = \mathbb{R}^2$ with L^1 norm, i.e., $\|x\|_1 = |x_1| + |x_2|$, where $x = (x_1, x_2)^T \in \mathbb{R}^2$. For any $x \in \mathbb{R}^2$, let $f(x) := Ax$, where $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. We will show that $\mu^{+}_{X}[f] = 3$, and $\mu^{-}_{X}[f] = -1$.

Let $\|\cdot\|_{1\to 1}$ be the operator norm induced by L^1 norm. By the definitions of μ^+, we have,

$$\begin{aligned}
\mu^{+}_{X}[A] &= \lim_{h\to 0^+} \frac{\|I+hA\|_{1\to 1}-1}{h} \\
&= \lim_{h\to 0^+} \frac{\max\{|1+h|, |1+h|+|2h|\}-1}{h} \\
&= \lim_{h\to 0^+} \frac{1+3h-1}{h} \\
&= 3.
\end{aligned}$$

Also by the definitions of μ^-, we have,

$$\begin{aligned}
\mu^{-}_{X}[A] &= \lim_{h\to 0^+} \frac{\|I-hA\|_{1\to 1}-1}{-h} \\
&= \lim_{h\to 0^+} \frac{\max\{|1-h|, |1-h|+|-2h|\}-1}{-h} \\
&= \lim_{h\to 0^+} \frac{1+h-1}{-h} \\
&= -1.
\end{aligned}$$

Therefore, in general $\mu^{+}_{Y,X}[f] \neq \mu^{-}_{Y,X}[f]$.

Remark 3. Another way to define μ^+ (and μ^-) is by the concept of semi inner product which is in fact a generalization of inner product to non Hilbert spaces. Let $(X, \|\cdot\|_X)$ be a normed space. For $x_1, x_2 \in X$, the right semi inner product is defined by

$$(x_1, x_2)_+ = \|x_1\|_X \lim_{h\to 0^+} \frac{1}{h}\left(\|x_1 + hx_2\|_X - \|x_1\|_X\right). \tag{2.6}$$

Using this definition,

$$\mu^{+}_{Y,X}[f] = \sup_{u\neq v\in Y} \frac{(u-v, f(u)-f(v))_+}{\|u-v\|_X^2}. \tag{2.7}$$

Note that one can also define the left semi inner product as follows

$$(x_1, x_2)_- = \|x_1\|_X \lim_{h\to 0^-} \frac{1}{h} \left(\|x_1 + hx_2\|_X - \|x_1\|_X\right), \tag{2.8}$$

and

$$\mu^-_{Y,X}[f] = \sup_{u\neq v\in Y} \frac{(u-v, f(u)-f(v))_-}{\|u-v\|^2_X} \ . \tag{2.9}$$

Both the right and left semi inner products $(\cdot,\cdot)_\pm$, induce the norm $\|\cdot\|_X$ in the usual way: $(x,x)_\pm = \|x\|^2_X$. Conversely, if the norm arises from an inner product $(\cdot,\cdot)$, as when X is a Hilbert space, $(x_1,x_2)_\pm = (x_1,x_2)$. Moreover the semi inner products satisfy the Cauchy-Schwarz inequalities:

$$-\|x\|\cdot\|y\| \leq (x,y)_\pm \leq \|x\|\cdot\|y\|.$$

Remark 4. *As every norm possesses right (left) Gâteaux-differentials, the limit in (2.6) (in (2.8)) exists and is finite. For more details, see [60].*

The following elementary properties of semi inner products are consequences of the properties of norms. See [41, 55] for proofs.

Proposition 3. *For $x, y, z \in X$ and $\alpha \geq 0$,*

1. $(x, -y)_\pm = -(x,y)_\mp$;
2. $(x, \alpha y)_\pm = \alpha(x,y)_\pm$;
3. $(x,y)_- + (x,z)_\pm \leq (x, y+z)_+ \leq (x,y)_+ + (x,z)_\pm$.

Remark 5. *In general, the semi inner products are not symmetric:*

$$(x,y)_\pm \neq (y,x)_\pm.$$

In this work, we mostly use the right semi inner product and μ^+. See Definition 6 for the only application of the left semi inner product and μ^- in this work.

Remark 6. For any Lipschitz operator $f\colon Y\subset X \to X$:

$$\mu^+_{Y,X}[f] \leq \mu_{Y,X}[f].$$

However, $\mu^+[f] = \mu[f]$ if the norm is induced by an inner product.

Proof. For any fixed $u \neq v \in Y$,

$$\frac{\|u - v + h(f(u) - f(v))\|_X}{\|u - v\|_X} \leq \sup_{u \neq v \in Y} \frac{\|u - v + h(f(u) - f(v))\|_X}{\|u - v\|_X}.$$

Now using this inequality, we have:

$$\begin{aligned}
&\lim_{h \to 0^+} \frac{1}{h} \left(\frac{\|u - v + h(f(u) - f(v))\|_X}{\|u - v\|_X} - 1 \right) \\
&\leq \lim_{h \to 0^+} \sup_{u \neq v \in Y} \frac{1}{h} \left(\frac{\|u - v + h(f(u) - f(v))\|_X}{\|u - v\|_X} - 1 \right).
\end{aligned}$$

Taking sup over all $u \neq v \in Y$, we have:

$$\begin{aligned}
&\sup_{u \neq v \in Y} \lim_{h \to 0^+} \frac{1}{h} \left(\frac{\|u - v + h(f(u) - f(v))\|_X}{\|u - v\|_X} - 1 \right) \\
&\leq \lim_{h \to 0^+} \sup_{u \neq v \in Y} \frac{1}{h} \left(\frac{\|u - v + h(f(u) - f(v))\|_X}{\|u - v\|_X} - 1 \right),
\end{aligned}$$

from which the conclusion follows by Definition 2 and Equation (2.4). $\square$

In what follows we show that for linear f, one has the reverse of the inequality in Remark 6 as well.

Proposition 4. *Let $(X, \|\cdot\|_X)$ be a finite dimensional normed space. For any linear operator $A\colon X \to X$,*

$$\mu_X[A] \;=\; \mu_X^+[A].$$

See the Appendix, Section 2.6, for a proof.

Remark 7. For a linear operator f, μ and μ^+ can be written as follows:

$$\mu_{Y,X}[f] = \lim_{h \to 0^+} \sup_{u \neq 0 \in Y} \frac{1}{h} \left(\frac{\|u + hf(u)\|_X}{\|u\|_X} - 1 \right), \tag{2.10}$$

and

$$\mu_{Y,X}^+[f] = \sup_{u \neq 0 \in Y} \lim_{h \to 0^+} \frac{1}{h} \left(\frac{\|u + hf(u)\|_X}{\|u\|_X} - 1 \right). \tag{2.11}$$

Notation 2. *In this work, for $(X, \|\cdot\|_p)$, where $\|\cdot\|_p$ is the L^p norm on X, for some $1 \leq p \leq \infty$, we sometimes use the notation "μ_p" instead of μ_X for the least upper bound logarithmic Lipschitz constant or logarithmic norm, and by "$\mu_{p,Q}$" we denote the least upper bound logarithmic Lipschitz constant or logarithmic norm induced by the weighted L^p norm, $\|u\|_{p,Q} := \|Qu\|_p$, where Q is a fixed nonsingular matrix.*

The following elementary properties of logarithmic norms are well-known. For more properties of logarithmic norms, see e.g., [61].

Lemma 1. *For any square matrix A,*

1. *Let $\lambda_{\max}(A)$ be the largest real part of an eigenvalue of A. Then, for an arbitrary norm, namely $\|\cdot\|$, and logarithmic norm μ induced by $\|\cdot\|$,*

$$\lambda_{\max}(A) \leq \mu[A] \leq \|A\|.$$

2. *$\mu_{p,Q}[A] = \mu_p[QAQ^{-1}]$ for $1 \leq p \leq \infty$, and nonsingular matrix Q.*

Remark 8. In Table 2.1, the algebraic expression of the least upper bound logarithmic Lipschitz constant induced by the L^p norm for $p = 1, 2$, and ∞ for matrices (i.e., logarithmic norm) are shown.

Remark 9. 1. Note that unlike norms, logarithmic norms could be negative. For example, consider the following matrix

$$A = \begin{pmatrix} -3 & -1 \\ 2 & -4 \end{pmatrix}.$$

Using Table 2.1, observe that

$$\lambda_{\max}(A) = -3.5 < \mu_1[A] = \max\{-3 + |2|, -4 + |-1|\} = -1 < 0 < \|A\|_1 = 5,$$

and

$$\lambda_{\max}(A) = -3.5 < \mu_2[A] \simeq -2.8 < 0 < \|A\|_2 \simeq 4.5.$$

2. Unlike norms, one can not compare the logarithmic norms. For example, as it is shown above, $\mu_2[A] < \mu_1[A] < 0$, while for

$$B = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix},$$

$\mu_2[B] \simeq 2.2 > \mu_1[B] = 2 > 0$.

The following subadditivity property, from [41], is key to diffusive interconnection analysis.

vector norm, $\|\cdot\|$	induced matrix measure, $\mu[A]$
$\|x\|_1 = \sum_{i=1}^{n} \|x_i\|$	$\mu_1[A] = \max_j \left(a_{jj} + \sum_{i \neq j} \|a_{ij}\| \right)$
$\|x\|_2 = \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}}$	$\mu_2[A] = \max_{\lambda \in \text{spec} \frac{1}{2}(A+A^T)} \lambda$
$\|x\|_\infty = \max_{1 \leq i \leq n} \|x_i\|$	$\mu_\infty[A] = \max_i \left(a_{ii} + \sum_{i \neq j} \|a_{ij}\| \right)$

Table 2.1: Standard matrix measures for a real $n \times n$ matrix, $A = [a_{ij}]$

Proposition 5. *Let $(X, \|\cdot\|_X)$ be a normed space. For any f, $g\colon Y \to X$ and any $Y \subseteq X$:*

1. $\mu^+_{Y,X}[f+g] \leq \mu^+_{Y,X}[f] + \mu^+_{Y,X}[g]$.

2. $\mu^+_{Y,X}[\alpha f] = \alpha \mu^+_{Y,X}[f] \quad$ *for* $\alpha \geq 0$.

In addition, for Lipschitz maps f and g,

1. $\mu_{Y,X}[f+g] \leq \mu_{Y,X}[f] + \mu_{Y,X}[g]$.

2. $\mu_{Y,X}[\alpha f] = \alpha \mu_{Y,X}[f] \quad$ *for* $\alpha \geq 0$.

Proof. Since the proofs for μ and μ^+ are exactly the same, we only give the proof for $\mu+$.

1. By the definition of $\mu^+_{Y,X}$, and the triangle inequality for norms, we have:

$$
\begin{aligned}
\mu^+_{Y,X}[f+g] &= \sup_{u \neq v \in Y} \lim_{h \to 0^+} \frac{1}{h} \left(\frac{\|u - v + h((f+g)(u) - (f+g)(v))\|_X}{\|u-v\|_X} - 1 \right) \\
&= \sup_{u \neq v \in Y} \lim_{h \to 0^+} \frac{1}{2h} \left(\frac{\|2(u - v) + 2h((f+g)(u) - (f+g)(v))\|_X}{\|u-v\|_X} - 2 \right) \\
&\leq \sup_{u \neq v \in Y} \lim_{h \to 0^+} \frac{1}{2h} \left(\frac{\|u - v + 2h(f(u) - f(v))\|_X}{\|u-v\|_X} - 1 \right) \\
&\quad + \sup_{u \neq v \in Y} \lim_{h \to 0^+} \frac{1}{2h} \left(\frac{\|u - v + 2h(g(u) - g(v))\|_X}{\|u-v\|_X} - 1 \right) \\
&= \mu^+_{Y,X}[f] + \mu^+_{Y,X}[g].
\end{aligned}
$$

2. For $\alpha = 0$, the equality is trivial, because both sides are equal to zero. For $\alpha > 0$:

$$\begin{aligned}
\mu^{+}_{Y,X}[\alpha f] &= \sup_{u \neq v \in Y} \lim_{h \to 0^+} \frac{1}{h} \left(\frac{\|u - v + h(\alpha f(u) - \alpha f(v))\|_X}{\|u - v\|_X} - 1 \right) \\
&= \sup_{u \neq v \in Y} \lim_{h \to 0^+} \frac{\alpha}{\alpha h} \left(\frac{\|u - v + (\alpha h)(f(u) - f(v))\|_X}{\|u - v\|_X} - 1 \right) \\
&= \alpha \mu^{+}_{Y,X}[f].
\end{aligned}$$

□

The (lub) logarithmic Lipschitz constant makes sense even if f is not differentiable. However, the constant can be tightly estimated, for differentiable mappings on convex subsets of finite-dimensional spaces, by means of Jacobians, [62].

Lemma 2. *For any given norm on $X = \mathbb{R}^n$, let μ be the (lub) logarithmic Lipschitz constant induced by this norm. Let Y be a connected subset of $X = \mathbb{R}^n$. Then for any Lipschitz and continuously differentiable function $f \colon Y \to \mathbb{R}^n$,*

$$\sup_{x \in Y} \mu_X[J_f(x)] \leq \mu_{Y,X}[f] \, .$$

Moreover, if Y is convex, then

$$\sup_{x \in Y} \mu_X[J_f(x)] = \mu_{Y,X}[f] \, .$$

Note that for any $x \in Y$, $J_f(x) \colon X \to X$. Therefore, we use μ_X instead of $\mu_{X,X}$, as we said in Definition 2.

We also recall a notion of generalized derivative, that can be used when taking derivatives of norms (which are not differentiable).

Definition 3. *For any continuous function, $\Psi \colon [0, \infty) \to \mathbb{R}$, the upper right Dini derivative is defined by*

$$\left(D^{+}\Psi\right)(t) = \limsup_{h \to 0^+} \frac{1}{h} \left(\Psi(t + h) - \Psi(t)\right).$$

Note that $D^{+}\Psi$ might be infinite.

One can also define the upper *left* Dini derivative as follows

$$\left(D^{-}\Psi\right)(t) = \limsup_{h \to 0^-} \frac{1}{h} \left(\Psi(t + h) - \Psi(t)\right).$$

Notation	Definition	Equivalent Definition	Equivalent Definition
$\mu_X[A]$	$\lim\limits_{h\to 0^+} \frac{1}{h}\left(\|I+hA\|_{X\to X}-1\right)$	$\lim\limits_{h\to 0^+} \sup\limits_{\|x\|_X=1} \frac{1}{h}\left(\|x+hAx\|_X-1\right)$	$\sup\limits_{\|x\|_X=1} \lim\limits_{h\to 0^+} \frac{1}{h}\left(\|x+hAx\|_X-1\right)$
$L_{Y,X}[f]$	$\sup\limits_{u\neq v\in Y} \frac{\|f(u)-f(v)\|_X}{\|u-v\|_X}$		
$\mu_{Y,X}[f]$	$\lim\limits_{h\to 0+} \frac{1}{h}\left(L_{Y,X}[I+hf]-1\right)$	$\lim\limits_{h\to 0^+} \sup\limits_{u\neq v\in Y} \frac{1}{h}\left(\frac{\|u-v+h(f(u)-f(v))\|_X}{\|u-v\|_X}-1\right)$	
$(x,y)_\pm$	$\|x\|_X \lim\limits_{h\to 0^\pm} \frac{1}{h}\left(\|x+hy\|_X-\|x\|_X\right)$		
$\mu^\pm_{Y,X}[f]$	$\sup\limits_{u\neq v\in Y} \frac{(u-v,f(u)-f(v))_\pm}{\|u-v\|^2_X}$		$\sup\limits_{u\neq v\in Y} \lim\limits_{h\to 0^\pm} \frac{1}{h}\left(\frac{\|u-v+h(f(u)-f(v))\|_X}{\|u-v\|_X}-1\right)$

Table 2.2: Basic concepts

In this work, we only use the upper right Dini derivative.

For ease of reference, we summarize the main notations and definitions in Table 2.2.

In what follows, we state and prove a few technical lemmas that we will use in the following chapters.

Lemma 3. *Let $(X, \|\cdot\|_X)$ be a normed space and $G\colon Y\times[0,\infty)\to X$ be a Lipschitz function on its first argument, where $Y\subseteq X$. Let $u,v\colon[0,\infty)\to Y$ be two solutions of*

$$\frac{du(t)}{dt} = G_t(u(t)),$$

where $G_t(u)=G(u,t)$. Then for all $t\in[0,\infty)$,

$$D^+\|(u-v)(t)\|_X = \frac{((u-v)(t), G_t(u(t))-G_t(v(t)))_+}{\|(u-v)(t)\|^2_X}\|(u-v)(t)\|_X. \tag{2.12}$$

(When $u(t)=v(t)$, we understand the right hand side through the limit in (2.14).)

In addition,

$$D^+\|(u-v)(t)\|_X \le \mu^+[G_t]\|(u-v)(t)\|_X\ . \tag{2.13}$$

Proof. By definition of the right semi inner product, the right hand side of (2.12) is:

$$\lim_{h\to 0^+}\frac{1}{h}\left(\|(u-v)(t)+h(G_t(u(t))-G_t(v(t)))\|_X - \|(u-v)(t)\|_X\right), \tag{2.14}$$

so it suffices to show that

$$D^+\|(u-v)(t)\|_X = \lim_{h\to 0^+}\frac{1}{h}\left(\|(u-v)(t)+h(G_t(u(t))-G_t(v(t)))\|_X - \|(u-v)(t)\|_X\right).$$

Now using the definition of Dini derivative, we have:

$$\begin{aligned} D^+\|(u-v)(t)\|_X &= \limsup_{h\to 0^+}\frac{1}{h}\left(\|(u-v)(t+h)\|_X - \|(u-v)(t)\|_X\right) \\ &= \limsup_{h\to 0^+}\frac{1}{h}\left(\|(u-v)(t)+h(\dot u-\dot v)(t)+o(h)\|_X - \|(u-v)(t)\|_X\right) \end{aligned}$$

$$
\begin{aligned}
&= \limsup_{h\to 0^+} \frac{1}{h}\left(\|(u-v)(t)+h(\dot u-\dot v)(t)\|_X - \|(u-v)(t)\|_X\right)\\
&= \lim_{h\to 0^+} \frac{1}{h}\left(\|(u-v)(t)+h(\dot u-\dot v)(t)\|_X - \|(u-v)(t)\|_X\right)\\
&= \lim_{h\to 0^+} \frac{1}{h}\left(\|(u-v)(t)+h(G_t(u)-G_t(v))\|_X - \|(u-v)(t)\|_X\right).
\end{aligned}
$$

(Note that the fourth equality holds because of Remark 4.)

Inequality (2.13) holds by the definition of μ^+ and Equation (2.12). □

Lemma 4. *Let A be an $mn\times mn$ block diagonal matrix with $n\times n$ matrices $A_1,\dots,A_m$ on its diagonal. Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$ and define $\|\cdot\|_*$ on $\mathbb{R}^{mn}$ as follows. For any $e=\left(e_1^T,\dots,e_m^T\right)^T$ with $e_i\in\mathbb{R}^n$, and any $1\le p\le\infty$,*

$$
\|e\|_* := \left\|(\|e_1\|,\dots,\|e_m\|)^T\right\|_p .
$$

Then

$$
\mu_*[A] \le \max\left\{\mu[A_1],\dots,\mu[A_m]\right\},
$$

where μ and μ_* are the logarithmic norms induced by $\|\cdot\|$ and $\|\cdot\|_*$ respectively.

Proof. By the definition, for $p\neq\infty$, $\mu_*[A]$ can be written as follows.

$$
\mu_*[A] = \sup_{e\neq 0}\lim_{h\to 0^+}\frac{1}{h}\left\{\left(\frac{\sum_{i=1}^m \|(I+hA_i)e_i\|^p}{\sum_{i=1}^m \|e_i\|^p}\right)^{\frac{1}{p}} - 1\right\}.
$$

For a fixed $e=(e_1^T,\dots,e_m^T)^T\neq 0$, there exists some $k\in\{1,\dots,m\}$, depends on e, such that for all $i\in\{1,\dots,m\}$

$$
\|(I+hA_i)e_i\| \le \frac{\|(I+hA_k)e_k\|}{\|e_k\|}\|e_i\| ,
$$

after raising to the power p and taking $\sum$ over all i's, we get

$$
\sum_{i=1}^m \|(I+hA_i)e_i\|^p \le \frac{\|(I+hA_k)e_k\|^p}{\|e_k\|^p}\ \sum_{i=1}^m \|e_i\|^p .
$$

Therefore

$$
\begin{aligned}
\lim_{h\to 0^+}\frac{1}{h}\left\{\left(\frac{\sum_{i=1}^n \|(I+hA_i)e_i\|^p}{\sum_{i=1}^n \|e_i\|^p}\right)^{\frac{1}{p}} - 1\right\} &\le \lim_{h\to 0^+}\frac{1}{h}\left\{\frac{\|(I+hA_k)e_k\|}{\|e_k\|} - 1\right\}\\
&\le \mu[A_k] \le \max\left\{\mu[A_1],\dots,\mu[A_m]\right\}.
\end{aligned}
$$

Now by taking sup over all $e\neq 0$, we get the desired result.

For $p = \infty$,

$$\mu_*[A] = \sup_{e \neq 0} \lim_{h \to 0^+} \frac{1}{h} \left\{ \frac{\max\limits_i \|(I + hA_i)e_i\|}{\max\limits_i \|e_i\|} - 1 \right\}.$$

Note that

$$\frac{\max\limits_i \|(I + hA_i)e_i\|}{\max\limits_i \|e_i\|} = \max_i \frac{\|(I + hA_i)e_i\|}{\max\limits_i \|e_i\|} \leq \max_i \frac{\|(I + hA_i)e_i\|}{\|e_i\|} .$$

Therefore,

$$\frac{\max\limits_i \|(I + hA_i)e_i\|}{\max\limits_i \|e_i\|} - 1 \leq \max_i \frac{\|(I + hA_i)e_i\|}{\|e_i\|} - 1 = \max_i \left\{ \frac{\|(I + hA_i)e_i\|}{\|e_i\|} - 1 \right\},$$

dividing both sides by $h > 0$, taking $\lim_{h \to 0^+}$, and taking sup over all $e \neq 0$, we get

$$\begin{aligned} \mu_*[A] &\leq \sup_{e \neq 0} \max_i \lim_{h \to 0^+} \frac{1}{h} \left\{ \frac{\|(I + hA_i)e_i\|}{\|e_i\|} - 1 \right\} \\ &= \max_i \sup_{e \neq 0} \lim_{h \to 0^+} \frac{1}{h} \left\{ \frac{\|(I + hA_i)e_i\|}{\|e_i\|} - 1 \right\} \\ &= \max \{\mu[A_1], \ldots, \mu[A_m]\} . \end{aligned}$$

□

Also see [63] for another proof of Lemma 4.

2.3 Single system of ODEs

In 1965, Coppel showed that the logarithmic norm can be used to bound solutions of a linear differential equation.

Theorem 5. *[58, page 58] If $A(t)$ is a continuous matrix function defined for $t \geq t_0$, then for any solution of $\dot{x} = A(t)x$, and any $t \geq t_0$,*

$$|x(t_0)| \exp\left(-\int_{t_0}^{t} \mu[-A(s)]\, ds \right) \;\leq\; |x(t)| \;\leq\; |x(t_0)| \exp\left(\int_{t_0}^{t} \mu[A(s)]\, ds \right) .$$

In 1970, Martin introduced a generalization of the logarithmic norm and proved Theorem 5 for nonlinear differential equations.

Theorem 6. *[59] In Equation (2.2), suppose that f is a Lipschitz function of x and a continuous function of t for $t \geq t_0$. Then for any solution x of (2.2), and any $t \geq t_0$,*

$$|x(t_0)| \exp\left(-\int_{t_0}^{t} \mu[-f_t]\, ds\right) \;\leq\; |x(t)| \;\leq\; |x(t_0)| \exp\left(\int_{t_0}^{t} \mu[f_t]\, ds\right),$$

where $f_t(\cdot) = f(\cdot, t)$.

The following theorem gives an upper bound for the difference between the solutions of (2.2) using the logarithmic norm of the Jacobian of f, see [26, 27, 33, 40, 64].

Theorem 7. *Suppose that V is a convex subset of $\mathbb{R}^n$ and let*

$$c = \sup_{(x,t)\in V\times[0,\infty)} \mu\left[J_f\left(x,t\right)\right].$$

Then, for any two solutions $x(t)$ and $y(t)$ of (2.2), that remain in V, it holds that:

$$\|x(t) - y(t)\| \;\leq\; e^{ct}\|x(0) - y(0)\|, \qquad \forall\, t \geq 0\,. \tag{2.15}$$

To prove Theorem 7, we will use the following general result which estimates rates of contraction, if $c < 0$ (or expansion, if $c > 0$) among any two functions, even functions that are not solutions of the same system of ODEs (see comment on observers to follow, Remark 11):

Lemma 5. *Let $(X, \|\cdot\|_X)$ be a normed space and $G\colon Y \times [0,\infty) \to X$ be a Lipschitz function on its first argument, where $Y \subseteq X$. Suppose $u, v\colon [0,\infty) \to Y$ satisfy*

$$(\dot{u} - \dot{v})(t) \;=\; G_t(u(t)) - G_t(v(t)),$$

where $G_t(u) = G(u,t)$. Let $c := \sup_{t\in[0,\infty)} \mu_{Y,X}\left[G_t\right]$. Then for all $t \in [0,\infty)$,

$$\|u(t) - v(t)\|_X \;\leq\; e^{ct}\|u(0) - v(0)\|_X. \tag{2.16}$$

Proof. By the definition of Dini derivative, we have (dropping some t's for simplicity)

$$\begin{aligned}
D^+\|(u-v)(t)\| &= \limsup_{h\to 0^+} \frac{1}{h}\left(\|(u-v)(t+h)\|_X - \|(u-v)(t)\|_X\right)\\
&= \limsup_{h\to 0^+} \frac{1}{h}\left(\|u - v + h(\dot{u}-\dot{v})\|_X - \|u-v\|_X\right)\\
&= \lim_{h\to 0^+} \frac{1}{h}\left(\|u - v + h(G_t(u) - G_t(v))\|_X - \|u-v\|_X\right)\\
&\leq \mu^+_{Y,X}[G_t]\;\|(u-v)(t)\|_X \quad \text{(by definition of } \mu^+\text{)}\\
&\leq \mu_{Y,X}[G_t]\;\|(u-v)(t)\|_X \quad \text{(by Remark 6)}\\
&= c\;\|(u-v)(t)\|_X\,.
\end{aligned}$$

The third equality holds because since every norm possesses right (and left) Gâteaux-differentials, the limit exists. Using Gronwall's Lemma for Dini derivatives, we obtain (2.16). A version of Gronwall's Lemma is given in [65, Appendix A]. For ease of reference, we will give a proof in the Appendix. □

Remark 10. In the finite-dimensional case, and when G is continuously differentiable with respect to its first argument, Lemma 5 can be verified in terms of Jacobians. Indeed, suppose that $X = \mathbb{R}^n$, and Y is a convex subset of $\mathbb{R}^n$. Then, by Lemma 2,

$$c \;=\; \tilde{c} := \sup_{(w,t)\in Y\times[0,\infty)} \mu_X\left[J_{G_t}(w)\right].$$

Therefore,

$$\|u(t)-v(t)\|_X \;\leq\; e^{\tilde{c}t}\|u(0)-v(0)\|_X\ .$$

In fact, in the finite-dimensional case, a more direct proof of Lemma 5 can instead be given. We sketch it next. Let $z(t) = u(t) - v(t)$. We have that

$$\dot{z}(t) = A(t)z(t),$$

where $A(t) = \int_0^1 \frac{\partial}{\partial x} G_t\left(su(t) + (1-s)v(t)\right)\, ds$. Now, by subadditivity of matrix measures (Proposition 5), which, by continuity, extends to integrals, we have:

$$\mu[A(t)] \leq \sup_{(w,t)\in Y\times[0,\infty)} \mu\left[\frac{\partial}{\partial x}G_t(w)\right].$$

Applying Theorem 5 gives the result.

Proof of Theorem 7. Since $c = \sup_{(x,t)} \mu\left[J_f\left(x,t\right)\right]$, and $\dot{x} - \dot{y} \;=\; f(x,t) - f(y,t)$, by Remark 10, (2.15) can be obtained. □

Definition 4. *[66] Given a norm $\|\cdot\|$, the system (2.2), or the time-dependent vector field f, is said to be* infinitesimally contracting with respect to this norm *on a set $V \subseteq \mathbb{R}^n$ if there exists some norm in V, with associated matrix measure μ, such that, for some constant $c > 0$ (the* contraction rate*), it holds that:*

$$\mu\left[J_f\left(x,t\right)\right] \leq -c, \quad \forall x \in V, \quad \forall t \geq 0\ . \tag{2.17}$$

The key result is that by Theorem 7 infinitesimal contractivity implies global contractivity.

Note that we use the convexity of V to apply Remark 10 (or Lemma 2). One can prove Theorem 7 for any arbitrary V but for

$$c = \sup_{(x,t)\in V\times[0,\infty)} \mu\left[f\left(x,t\right)\right],$$

instead of

$$c = \sup_{(x,t)\in V\times[0,\infty)} \mu\left[J_f\left(x,t\right)\right].$$

In addition, one can prove the converse of Theorem 7 for any arbitrary V, but for

$$c = \sup_{(x,t)\in V\times[0,\infty)} \mu^+\left[f\left(x,t\right)\right],$$

see Proposition 6 below for more details.

Remark 11. The statement of Lemma 5 allows for considerably more generality than Theorem 7. Suppose for example that we consider a standard observer configuration:

$$\begin{aligned} \dot{x} &= f(x,u) \\ \dot{z} &= f(z,u) + K(h(z) - h(x)), \end{aligned}$$

where h is an output function and K is an observer gain matrix. Let

$$G_t(y) := f(y,u(t)) + Kh(y),$$

evaluated along any given solution with an input u. Then, $\dot{z} - \dot{x} = G_t(z) - G_t(x)$, and thus, if G_t has a contractivity property, it follows that $z - x$ converges exponentially to zero, by Lemma 5. (Theorem 7 does not apply, since x and z solve different equations.) This recovers the standard Luenberger observer construction for linear time-invariant systems.

Corollary 1. *Under the assumptions of Theorem 7, and for $c < 0$, the following statements hold.*

- *If $\mathcal{A}$ is a non-empty forward-invariant set for the dynamics, then every solution must approach $\mathcal{A}$. Indeed, take any trajectory $x(t)$ and a trajectory $y(t)$ with $y(0) \in \mathcal{A}$. Then, as $t \to \infty$,*

$$dist\,(x(t), \mathcal{A}) \leq \|(x-y)(t)\| \;\leq\; e^{ct}\|(x-y)(0)\| \to 0.$$

- *If an equilibrium exists, then it must be unique and globally asymptotically stable.*

When contractive systems are forced by periodic signals, they are "entrained", in the sense that solutions converge to unique limit cycles. This property is very important in applications, see for example [64, 67].

Definition 5. *Given a number $T > 0$, we will say that system (2.2) is T-*periodic *if it holds that $f(x, t+T) = f(x, t), \quad \forall\, t \geq 0,\, x \in V$. Notice that a system $\dot{x} = f\left(x, u(t)\right)$ with input $u(t)$ is T-periodic if $u(t)$ is itself a periodic function of period T.*

The basic theoretical result about periodic orbits is as follows. For more details see [33, 34, 66].

Theorem 8. *Suppose that*

1. V is a closed convex subset of $\mathbb{R}^n$;

2. f is infinitesimally contracting with contraction rate c, i.e., $\mu[J_f] \leq -c$, for some $c > 0$;

3. f is T-periodic.

Then, there is a unique periodic solution $\hat{x}(t) : [0, \infty) \to V$ of (2.2) of period T and, for every solution $x(t)$, it holds that $\|x(t) - \hat{x}(t)\| \to 0$ as $t \to \infty$.

Proof. We denote by $\varphi(t, s, \xi)$ the value of the solution $x(t)$ at time t of the differential equation (2.2) with initial value $x(s) = \xi$. Define now $P(\xi) = \varphi(T, 0, \xi)$, where $\xi = x\left(0\right) \in V$.

Claim. $P^k(\xi) = \varphi(kT, 0, \xi)$ for all positive integers k and $\xi \in V$.

We will prove the claim by recursion. In particular, the statement is true by definition when $k = 1$. Inductively, assuming it is true for k, we have:

$$\begin{aligned} P^{k+1}(\xi) &= P(P^k(\xi)) = \varphi(T, 0, P^k(\xi)) \\ &= \varphi(T, 0, \varphi(kT, 0, \xi)) = \varphi(kT + T, 0, \xi). \end{aligned}$$

This proves the claim.

Observe that P is a contraction with factor $e^{-cT} < 1$: $\|P(\xi) - P(\zeta)\| \leq e^{-cT}\|\xi - \zeta\|$ for all $\xi, \zeta \in V$, as a consequence of Theorem 7. The set V is a closed subset of $\mathbb{R}^n$ and hence is complete as a metric space with respect to the distance induced by the norm being considered. Thus, by the contraction mapping theorem, there is a (unique) fixed point $\bar{\xi}$ of P. Let $\hat{x}(t) := \varphi(t, 0, \bar{\xi})$. Since $\hat{x}(T) = P(\bar{\xi}) = \bar{\xi} = \hat{x}(0)$, $\hat{x}(t)$ is a periodic orbit of period T. Moreover, again by Theorem 7, we have that $\|x(t) - \hat{x}(t)\| \leq e^{-ct}\|\xi - \bar{\xi}\| \to 0$. Uniqueness is clear, since two different periodic orbits would be disjoint compact subsets, and hence at positive distance from each other, contradicting convergence. This completes the proof. $\square$

The next result is for the special case of Euclidean norms.

Lemma 6. *Suppose that P is a positive definite matrix and A is an arbitrary matrix.*

1. *If $\mu_{2,P}[A] = \mu$, then $QA + A^TQ \leq 2\mu Q$, where $Q = P^2$.*
2. *If for some positive definite matrix Q, $QA + A^TQ \leq 2\mu Q$, then there exists a positive definite matrix P such that $P^2 = Q$ and $\mu_{2,P}[A] \leq \mu$.*

Proof. First suppose $\mu_{2,P}[A] = \mu$. By definition of μ:

$$\frac{1}{2}\left(PAP^{-1} + \left(PAP^{-1}\right)^T\right) \leq \mu I.$$

Since P is symmetric, so is P^{-1},

$$PAP^{-1} + P^{-1}A^TP \leq 2\mu I.$$

Now multiplying the last inequality by P on the right and the left, we get:

$$P^2A + A^TP^2 \leq 2\mu P^2.$$

This proves 1. Now assume that for some positive definite matrix Q, $QA+A^TQ \leq 2\mu Q$. Since $Q > 0$ (positive definite), there exists $P > 0$ such that $P^2 = Q$; moreover, because Q is symmetric, so is P. Hence we have:

$$P^2A + A^TP^2 \leq 2\mu P^2.$$

Multiplying the last inequality by P^{-1} on the right and the left, we conclude 2. $\square$

Remark 12. Lemma 6 implies that, *for linear time-invariant systems* $\dot{x} = Ax$*, contractivity with respect to some weighted* L^2 *norm* (with a not necessarily diagonal weighting matrix) *is equivalent to* A *being a Hurwitz matrix.* One direction is clear, as contractivity obviously implies stability. Conversely, suppose that A is Hurwitz. Then, one may pick a quadratic Lyapunov function $V(x) = x^T Qx$, where Q is a positive definite matrix. By definition of Lyapunov function, $QA + A^T Q \leq -\beta I$, for some $\beta > 0$. Letting $\gamma := \beta/\lambda_{\max}$, where $\lambda_{\max}$ is the largest eigenvalue of Q, we have that also $QA + A^T Q \leq -\gamma Q$. (Conversely, an inequality of the type $QA + A^T Q \leq -\mu Q$ implies that $QA + A^T Q \leq -\beta I$, if we define $\beta := \mu\lambda_{\min}$ and $\lambda_{\min}$ is the smallest eigenvalue of Q.) Thus, there is a positive definite P so that $\mu_{2,P}[A] \leq -\gamma/2 < 0$, showing contractivity with respect to the P-weighted L^2 norm. Of course, contractivity with respect to a *diagonally* weighted norm, a property which is required in the interconnection and PDE results mentioned later in Chapter 4, imposes additional requirements even in the linear time invariant case, amounting to asking that a quadratic Lyapunov function's principal axes align with the natural coordinates in $\mathbb{R}^n$. Systems admitting such Lyapunov functions are often called "diagonally stable" [68], and the study of diagonal stability is closely related to passivity [69] (see Section 2.5).

The significance of Theorem 7 is that it is true for any norm. Different norms are appropriate to different problems, just as different Lyapunov functions have to be carefully chosen when analyzing a nonlinear system. The choice of norms is a key step in the application of contraction techniques. Non-Euclidean (i.e., not weighted L^2) norms have been found to be useful in the study of many problems. To illustrate this fact, we next provide an example of a biochemical model which can be shown to be contractive by applying Theorem 7 when using a weighted L^1 norm, but which is not contractive in any weighted L^p norm, for any $p > 1$. The proof that L^1 norms suffice for this example is from [64], and the proof of non-contractivity in L^p, $p > 1$, is shown below.

2.4 Example: biochemical model

A typical biochemical reaction is one in which a molecule X (whose concentration is quantified by the non-negative variable $x = x(t)$) binds to a second molecule S (whose concentration is quantified by $s = s(t) \geq 0$), to produce a dimer Y (whose concentration is quantified by $y = y(t) \geq 0$), and the molecule X is subject to degradation and dilution (at rate δx, where $\delta > 0$) and production according to an external signal $z = z(t) \geq 0$. Examples of such reactions might be an enzyme binding to a substrate to produce a complex, or a transcription factor binding to an unoccupied promoter to make an active promoter, and the enzyme or the transcription factor is itself being continuously created and destroyed. The diagram for such a reaction is as follows:

$$0 \xrightarrow{z} X \xrightarrow{\delta} 0\,, \quad X + S \underset{k_1}{\overset{k_2}{\rightleftharpoons}} Y.$$

Using mass-action kinetics, and assuming a well-mixed reaction in a large volume, the system of chemical reactions is given by:

$$\begin{aligned}
\dot{x} &= z(t) - \delta x + k_1 y - k_2 s x \\
\dot{y} &= -k_1 y + k_2 s x \\
\dot{s} &= k_1 y - k_2 s x.
\end{aligned}$$

We observe that $y(t) + s(t) = S_Y$ remains constant along solutions. Thus we can study the following reduced system:

$$\begin{aligned}
\dot{x} &= z(t) - \delta x + k_1 y - k_2 (S_Y - y) x \\
\dot{y} &= -k_1 y + k_2 (S_Y - y) x.
\end{aligned}$$

Note that

$$(x(t), y(t)) \in V = [0, \infty) \times [0, S_Y],$$

for all $t \geq 0$ (V is convex and forward-invariant), and S_Y, k_1, k_2, δ, d_1, and d_2 are arbitrary positive constants.

For any t, the Jacobian of $F_t = (z(t) - \delta x + k_1 y - k_2(S_Y - y)x, -k_1 y + k_2(S_Y - y)x)^T$, is as follows

$$J_{F_t}(x, y) \ := \left(\begin{array}{cc} -\delta - k_2(S_Y - y) & k_1 + k_2 x \\ k_2(S_Y - y) & -(k_1 + k_2 x) \end{array} \right).$$

Following [64], we show

$$\sup_t \sup_{(x,y)\in V} \mu_{1,Q}\left[J_{F_t}(x,y)\right] < 0,$$

where

$$Q = \operatorname{diag}\left(1, 1 + \delta/(k_2 S_Y) - \zeta\right),$$

and we will pick a suitable $0 < \zeta < \frac{\delta}{k_2 S_Y}$. Equivalently, for fixed t and (x,y), we show that $\mu_{1,Q}\left[J_{F_t}(x,y)\right] < 0$ and for simplicity we drop the argument (x,y).

We will find a $q > 1$ such that $\mu_{1,Q}\left[J_{F_t}(x,y)\right] < 0$ holds with $Q = \operatorname{diag}(1,q)$. For any such q, we can always find ζ such that $q := 1 + \frac{\delta}{k_2 S_Y} - \zeta > 1$. With this form for Q,

$$QJ_{F_t}Q^{-1} \;=\; \begin{pmatrix} -\delta - a & \dfrac{b}{q} \\ aq & -b \end{pmatrix},$$

where $a = k_2(S_Y - y) \in [0, k_2 S_Y]$ and $b = k_1 + k_2 x \in [k_1, \infty)$. Since $a \geq 0$, $b > 0$, and $q > 1$, by Table 2.1, we have:

$$\begin{aligned} \mu_{1,Q}[J_{F_t}] &= \mu_1\left[QJ_{F_t}Q^{-1}\right] \\ &= \max\{-\delta - a + |aq|\,, -b + |b/q|\} \\ &= \max\{-\delta + a(q-1), b\,(1/q - 1)\}. \end{aligned}$$

So to show that $\mu_{1,Q}[J_{F_t}] < 0$, since we assume $q > 1$, we need to find an upper bound for the values of q such that:

$$-\delta + a(q-1) < 0. \tag{2.18}$$

Observe that

$$-\delta + a(q-1) \;<\; 0 \quad \text{iff} \quad q \;<\; 1 + \frac{\delta}{a} \;=\; 1 + \frac{\delta}{k_2(S_Y - y)} \;<\; 1 + \frac{\delta}{k_2 S_Y}.$$

Hence for $Q = \operatorname{diag}(1,q)$, with $1 < q < 1 + \frac{\delta}{k_2 S_Y}$, $\mu_{1,Q}[J_{F_t}] < 0$. Therefore, by Theorem 7, the system is contracting. Note that a *weighted* L^1 norm is necessary, since with $Q = I$ we obtain $\mu_1 = 0$.

We next show that for a fixed t, any $p > 1$, and any positive diagonal Q, it is not true that $\mu_{p,Q}[J_{F_t}(x,y)] < 0$ for all $(x,y) \in V$.

We first consider the case $p \neq \infty$ and show that there exists $(x_0, y_0) \in V$ such that for any small $h > 0$, $\|I + hQJ_{F_t}(x_0,y_0)Q^{-1}\|_p > 1$. This will imply $\mu_{p,Q}\left[J_{F_t}(x_0,y_0)\right] \geq 0$.

Computing explicitly, we have the following expression:

$$\|I + hQJ_{F_t}Q^{-1}\|_p = \sup_{(\xi_1,\xi_2)\neq(0,0)} \frac{\left(|\xi_1 - h(\delta+a)\xi_1 + hb\xi_2/q|^p + |haq\xi_1 + \xi_2 - hb\xi_2|^p\right)^{1/p}}{(|\xi_1|^p + |\xi_2|^p)^{1/p}}$$
$$\geq \frac{\left(|1 - h(\delta+a) + hb\lambda/q|^p + |haq + \lambda - hb\lambda|^p\right)^{1/p}}{(1+|\lambda|^p)^{1/p}},$$

where we take a point of the form $(\xi_1, \xi_2) = (1, \lambda)$, for a $\lambda > 0$ which will be determined later. To show

$$\frac{\left(\left|1 - h(\delta+a) + h\frac{b\lambda}{q}\right|^p + |haq + \lambda - hb\lambda|^p\right)^{\frac{1}{p}}}{(1+|\lambda|^p)^{\frac{1}{p}}} > 1,$$

we equivalently show that for any small enough $h > 0$:

$$\frac{1}{h}\left(\left|1 - h(\delta+a) + h\frac{b\lambda}{q}\right|^p + |haq + \lambda - hb\lambda|^p - 1 - |\lambda|^p\right) > 0\,. \tag{2.19}$$

Note that the $\lim_{h\to 0^+}$ of the left hand side of the above inequality is $f'(0)$ where

$$f(h) \;=\; |1 + h\,(b\lambda/q - (\delta+a))|^p + |\lambda + h(aq - b\lambda)|^p\,.$$

Therefore, it suffices to show that $f'(0) > 0$ for some value $(x_0, y_0) \in V$ (because $f'(0) > 0$ implies that there exists $h_0 > 0$ such that for $0 < h < h_0$, the expression in (2.19) is positive). Since $p > 1$, by assumption, f is differentiable and

$$f'(h) = p\,(b\lambda/q - (\delta+a))\,|1 + h\,(b\lambda/q - (\delta+a))|^{p-2}\,(1 + h\,(b\lambda/q - (\delta+a)))$$
$$+\, p(aq - b\lambda)\,|\lambda + h(aq - b\lambda)|^{p-2}\,(\lambda + h(aq - b\lambda))\,.$$

(Note that $\frac{d}{dx}|u(x)|^p = |u(x)|^{p-2}u(x)\,\frac{du}{dx}(x)$.)

Hence, since $\lambda > 0$

$$f'(0) = p\,(b\lambda/q - (\delta+a)) + p(aq - b\lambda)\lambda^{p-1} = p\,(b\lambda/q - a)\,(1 - \lambda^{p-1}q) - p\delta.$$

Choosing λ small enough such that $1 - \lambda^{p-1}q > 0$ and choosing x, or equivalently b, large enough, we can make $f'(0) > 0$.

For $p = \infty$, using Table 2.1,

$$\mu_\infty\left[QJ_{F_t}Q^{-1}\right] = \max\left\{-\delta - a + |b/q|\,, -b + |aq|\right\}.$$

For large enough x, $-\delta - a + |b/q| > 0$ (and $-b + aq < 0$) and hence $\mu_\infty\left[QJ_{F_t}Q^{-1}\right] > 0$.

2.5 Some relations to accretive and dissipative operators

In this section, we show that the converse of Theorem 7 is true as well, and in fact that contractivity is equivalent to a number of other inequalities. After that, we review the definitions of accretive and dissipative operators on Banach spaces, and see how these are related to contractive operators.

The following result summarizes the basic equivalences.

Proposition 6. *Consider (2.2) where $x(t) \in Y \subset X$, X is a Banach space with norm $\|\cdot\|$, and $t \in [0, \infty)$. We assume that*

$$f : Y \times [0, \infty) \to X,$$

is a Lipschitz vector field on x and continuous on (x, t).

Then the following statements are equivalent.

1. *For any two solutions x, y of (2.2), and all $t, s \geq 0$,*

$$\|x(t+s) - y(t+s)\| \leq e^{ct} \|x(s) - y(s)\|.$$

2. *For any $t \geq 0$, let $f_t(x) = f(x, t)$. Then*

$$\mu^+[f_t] \leq c\ .$$

3. *For any two points x, y, and any $t \geq 0$*

$$(x - y, f_t(x) - f_t(y))_+ \leq c\|x - y\|^2.$$

4. *For any two solutions x, y of (2.2), and all $t \geq 0$,*

$$D^+\|(x - y)(t)\| \leq c\|(x - y)(t)\|.$$

Proof. $1 \Rightarrow 2$. Fix $s \geq 0$ and let $a \neq b \in Y$ be arbitrary. For $t \geq s$, let $x(t), y(t)$ be the solutions of (2.2) with $x(s) = a$ and $y(s) = b$ respectively.

$$\begin{aligned}
\|x(s+h) - y(s+h)\| &= \|x(s) - y(s) + h(f_s(x(s)) - f_s(y(s))) + o(h)\| \\
&\leq e^{ch}\|x(s) - y(s)\|.
\end{aligned}$$

Therefore, by subtracting $\|x(s)-y(s)\|$ from both sides of the above inequality, dividing by $h>0$, and taking the $\lim_{h\to 0^+}$, we get:

$$\lim_{h\to 0^+} \frac{\|a-b+h(f_s(a)-f_s(b))+o(h)\|-\|a-b\|}{h} \quad\leq\quad \lim_{h\to 0^+} \frac{e^{ch}-1}{h}\,\|a-b\|.$$

Dividing by $\|a-b\|$, we get:

$$\lim_{h\to 0^+} \frac{\|a-b+h(f_s(a)-f_s(b))\|-\|a-b\|}{h\|a-b\|} \quad\leq\quad c,$$

and now taking sup over all $a\neq b\in Y$, we get:

$$\mu^+[f_s]\leq c.$$

$2\Rightarrow 3$. For any fixed t, and any $x\neq y\in Y$

$$\begin{aligned}
(x-y, f_t(x)-f_t(y))_+ &= \|x-y\| \lim_{h\to 0^+} \frac{\|x-y+h(f_t(x)-f_t(y))\|-\|x-y\|}{h} \\
&= \|x-y\|^2 \lim_{h\to 0^+} \frac{\|x-y+h(f_t(x)-f_t(y))\|-\|x-y\|}{h\|x-y\|} \\
&\leq \mu^+[f_t]\|x-y\|^2 \\
&\leq c\|x-y\|^2.
\end{aligned}$$

$3\Rightarrow 4$. Using the definition of upper Dini derivative, we have: (we drop the argument t for simplicity)

$$\begin{aligned}
D^+\|(x-y)(t)\| &= \limsup_{h\to 0^+} \frac{1}{h}\,\left(\|(x-y)(t+h)\|-\|(x-y)(t)\|\right) \\
&= \limsup_{h\to 0^+} \frac{1}{h}\left(\|x-y+h(\dot{x}-\dot{y})\|-\|x-y\|\right) \\
&= \lim_{h\to 0^+} \frac{1}{h}\left(\|x-y+h(f_t(x)-f_t(y))\|-\|x-y\|\right).
\end{aligned}$$

Note that if $(x-y)(t)=0$, then using the above inequality $D^+\|(x-y)(t)\|=0$, and therefore, 4 holds. Assume that $(x-y)(t)\neq 0$. Multiplying both sides of the above equality by $\|(x-y)(t)\|$, we get:

$$\begin{aligned}
&\|(x-y)(t)\|D^+\|(x-y)(t)\| \\
&= \|x-y\| \lim_{h\to 0^+} \frac{1}{h}\left(\|x-y+h(f_t(x)-f_t(y))\|-\|x-y\|\right) \\
&= ((x-y)(t), f_t(x(t))-f_t(y(t)))_+ \\
&\leq c\|(x-y)(t)\|^2 \quad \text{using 3.}
\end{aligned}$$

Dividing by $\|(x-y)(t)\| \neq 0$, we get 4.

$4 \Rightarrow 1$. Let $\phi(t) := \|(x-y)(t)\|$. A simple calculation shows that (see the proof of Lemma 9 for the calculations.)

$$D^{+}\left(\phi(t)e^{-ct}\right) \leq 0.$$

Applying Gronwall's inequality for Dini derivatives (see Lemma 9, in the Appendix), we have that

$$\phi(t+s) \leq e^{ct}\phi(s),$$

for all $t, s \geq 0$, as desired.

$\square$

Note that even if Y is a convex subset of X, $1 \iff 2$, in Proposition 6 is not a generalization of Theorem 7, because $\mu^{+}[f] \leq c$ doesn't imply $\mu[f] \leq c$, in general.

Definition 6. *[55] An $F \colon Y \subset X \to X$ satisfying*

$$(x-y, F(x)-F(y))_{+} \geq 0, \quad \textit{for any } x, y \in Y$$

is said to be accretive (monotone when $(\cdot,\cdot)_{+}$ is a true inner product), while F is dissipative if $-F$ is accretive. Equivalently, by the definition of $\mu^{\pm}$, F is said to be accretive if $\mu^{+}[F] \geq 0$ and F is dissipative if $\mu^{+}[-F] \geq 0$, i.e., $\mu^{-}[F] \leq 0$, (by the definition of $\mu^{\pm}$ and Proposition 3, part 1).

Note that in Hilbert spaces, $\mu^{+}[F] = \mu^{-}[F] = \mu[F]$. Therefore, $F - cI$ is dissipative, if $\mu^{-}[F] = \mu^{+}[F] \leq c$. In particular, when $c < 0$, F is dissipative if and only if F is infinitesimally contractive.

2.6 Appendix

Proof of Proposition 4

To prove this result, we first review some basic minimax optimization facts.

Proposition 7. *[70] Let X and Y be arbitrary sets and $\varphi : X \times Y \to \mathbb{R}$ be an arbitrary function. For any $y \in Y$ and $c \in \mathbb{R}$, denote $H_{y,c} = \{x \in X : \varphi(x, y) \geq c\}$ and $\mathcal{C}$ the set of all real numbers c such that for all $y \in Y$, $H_{y,c} \neq \emptyset$, and let $c^* = \sup \mathcal{C}$. Then*

$$(B =) \sup_{x \in X} \inf_{y \in Y} \varphi(x, y) = \inf_{y \in Y} \sup_{x \in X} \varphi(x, y)(= J),$$

if and only if for every $c < c^*$, $\bigcap_{y \in Y} H_{y,c} \neq \emptyset$. In this case $B = J = c^*$.

A proof is outlined in [70]. For ease of reference, we provide a self-contained proof next. First we need the following

Lemma 7. *For any fixed $y \in Y$, if $c_1 < c_2$ then $H_{y,c_2} \subset H_{y,c_1}$.*

Proof. Pick any $x \in H_{y,c_2}$. By the definition of H_{y,c_1}, $\varphi(x, y) \geq c_2 > c_1$ and hence $x \in H_{y,c_1}$. □

Corollary 2. *If $c_1 < c_2$ then $\bigcap_{y \in Y} H_{y,c_2} \subset \bigcap_{y \in Y} H_{y,c_1}$.*

Proof of Proposition 7. *First we assume that for every $c < c^*$, $\bigcap_{y \in Y} H_{y,c} \neq \emptyset$. Using this assumption, we will show that $B = J = c^*$. To this end, we will show the following three inequalities:*

1. *$B \leq J$.*

$$\begin{aligned} \varphi(x, y) \leq \sup_{x \in X} \varphi(x, y) \qquad \forall x, y \quad &\Rightarrow \quad \inf_{y \in Y} \varphi(x, y) \leq \inf_{y \in Y} \sup_{x \in X} \varphi(x, y) \qquad \forall x \\ &\Rightarrow \quad \sup_{x \in X} \inf_{y \in Y} \varphi(x, y) \leq \inf_{y \in Y} \sup_{x \in X} f(x, y). \end{aligned}$$

   Hence, $B \leq J$.

2. *$J \leq c^*$. For an arbitrary $c > c^*$ we show that $J \leq c$. Since $c > c^*$, there exists $y_0 \in Y$ such that $H_{y_0,c} = \emptyset$ (otherwise $c \in \mathcal{C}$ and so $c \leq c^*$). This means that for all $x \in X$, $\varphi(x, y_0) < c$ which implies $\sup_{x \in X} \varphi(x, y_0) \leq c$ and hence $J = \inf_{y \in Y} \sup_{x \in X} \varphi(x, y_0) \leq c$.*

3. *$c^* \leq B$. For an arbitrary $c < c^*$ we show that $B \geq c$. Since $c < c^*$, there exists $c_0 \in \mathcal{C}$ such that $c < c_0$ (otherwise $c = \sup \mathcal{C}$). Since we assumed that for every $c < c^*$, $\bigcap_{y \in Y} H_{y,c} \neq \emptyset$, and since $c_0 \leq c^*$, $c_0 \in \mathcal{C}$, we have $\bigcap_{y \in Y} H_{y,c_0} \neq \emptyset$. By*

Corollary 2, because $c < c_0$, then $\bigcap_{y \in Y} H_{y,c} \neq \emptyset$, i.e., there exists $x_0 \in X$ such that for all $y \in Y$, $\varphi(x_0, y) \geq c$, which implies $\inf_{y \in Y} \varphi(x_0, y) \geq c$ and hence $B = \sup_{x \in X} \inf_{y \in Y} \varphi(x, y) \geq c$.

Now we suppose $B = J$ and for fixed $c < c^*$ show that $\bigcap_{y \in Y} H_{y,c} \neq \emptyset$. Since $c < c^*$, there exists $c_0 \in \mathcal{C}$ such that $c < c_0$. This means that there exists $x_0 \in X$ such that for all $y \in Y$, $\varphi(x_0, y) \geq c_0$, i.e., $J \geq c_0$. Since $J = B$, there exists $x_1 \in X$ such that for all $y \in Y$, $\varphi(x_1, y) \geq c_0 > c$, i.e., $\bigcap_{y \in Y} H_{y,c} \neq \emptyset$. □

For a fixed arbitrary norm $\|\cdot\|$ on $\mathbb{R}^{nN}$ and a fixed arbitrary matrix $A \in \mathbb{R}^{nN \times nN}$, define $\varphi : S^{nN-1} \times (0,1) \to \mathbb{R}$ by

$$\varphi(v, h) = \frac{1}{h} \left(\|v + hAv\| - 1 \right),$$

where $S^{nN-1} = \{v \in \mathbb{R}^{nN} : \|v\| = 1\}$. For any $h \in (0,1)$ and $c \in \mathbb{R}$, let

$$H_{h,c} = \left\{ v \in S^{nN-1} : \varphi(v, h) \geq c \right\},$$

and let $\mathcal{C}$ be the set of all real numbers c such that $H_{h,c} \neq \emptyset$ whenever $h \in (0,1)$. Let $c^* = \sup \mathcal{C}$.

Lemma 8. *$\varphi(v, h) = \frac{1}{h}(\|v + hAv\| - 1)$ is non-increasing as $h \to 0^+$.*

Proof. Let $\alpha < 1$. Since $\|v\| = 1$,

$$\begin{aligned}
\varphi(v, \alpha h) &= \frac{1}{\alpha h} \left(\|v + \alpha hAv\| - 1 \right) = \frac{1}{h} \left(\left\| \frac{v}{\alpha} + hAv \right\| - \frac{1}{\alpha} \right) \\
&= \frac{1}{h} \left(\left\| \left(\frac{v}{\alpha} - v \right) + (v + hAv) \right\| - \left(\frac{1}{\alpha} - 1 \right) - 1 \right) \\
&\leq \frac{1}{h} \left(\left\| \frac{v}{\alpha} - v \right\| + \|v + hAv\| - \left(\frac{1}{\alpha} - 1 \right) - 1 \right) \\
&= \frac{1}{h} (\|v + hAv\| - 1) = \varphi(v, h).
\end{aligned}$$

□

Corollary 3. *For any matrix A, $\frac{1}{h}(\|I + hA\|_{op} - 1)$ is non-increasing as $h \to 0^+$, where $\|\cdot\|_{op}$ is the operator norm induced by $\|\cdot\|$.*

Proof. Let $\alpha < 1$. By Lemma 8, we have

$$\begin{aligned}
\frac{1}{\alpha h}(\|I + \alpha h A\|_{op} - 1) &= \sup_{\|v\|=1} \frac{1}{\alpha h}(\|v + \alpha h A v\| - 1) \\
&= \sup_{\|v\|=1} \varphi(v, \alpha h) \\
&\leq \sup_{\|v\|=1} \varphi(v, h) \\
&= \frac{1}{h}(\|I + hA\|_{op} - 1).
\end{aligned}$$

□

Proof of Proposition 4. ***Claim 1.***

$$\sup_{v \in S^{nN-1}} \inf_{h \in (0,1)} \varphi(v, h) = \inf_{h \in (0,1)} \sup_{v \in S^{nN-1}} \varphi(v, h). \tag{2.20}$$

Proof of Claim 1. *First we show that for $c < c^*$, $\bigcap_{h \in (0,1)} H_{h,c} \neq \emptyset$, where $H_{h,c} = \{v \in S^{nN-1} : \varphi(v, h) \geq c\}$ and c^* is defined as above. By Lemma 8, $\varphi(v, h)$ is decreasing in h which implies $H_{h_1,c} \subset H_{h_2,c}$ when $h_1 < h_2$. Also by the definition of c^*, $c < c^*$ implies that $H_{h,c} \neq \emptyset$ for any $h \in (0, 1)$. On the other hand, each $H_{h,c}$ is a closed subset of S^{nN-1}, so they are all compact. Hence their intersection is non-empty. By applying Proposition 7, we obtain Equation (2.20).*

Claim 2.

$$\sup_{v \in S^{nN-1}} \lim_{h \to 0^+} \varphi(v, h) = \sup_{v \in S^{nN-1}} \inf_{h \in (0,1)} \varphi(v, h), \tag{2.21}$$

and

$$\lim_{h \to 0^+} \sup_{v \in S^{nN-1}} \varphi(v, h) = \inf_{h \in (0,1)} \sup_{v \in S^{nN-1}} \varphi(v, h). \tag{2.22}$$

Proof of claim 2. *By Lemma 8, since $f(v, h)$ is non-increasing as $h \to 0^+$, (2.21) holds. By Corollary 3, since $\frac{1}{h}(\|I + hA\|_{op} - 1)$ is non-increasing as $h \to 0^+$, (2.22) holds. By claim 1, the right hand sides of (2.21) and (2.22) are equal, and therefore so are their left hand sides:*

$$\sup_{v \in S^{nN-1}} \lim_{h \to 0^+} \varphi(v, h) = \lim_{h \to 0^+} \sup_{v \in S^{nN-1}} \varphi(v, h),$$

which implies $\mu[A] = \sup_{\|v\|=1} \lim_{h \to 0^+} \frac{1}{h}(\|v + hAv\| - 1) = \mu^+[A]$. □

Gronwall's Lemma for Dini derivatives

Lemma 9. *Suppose $D^+\phi(t) \leq L(t)\phi(t)$, where $\phi \geq 0$, and ϕ and L are continuous functions of t for $t \geq t_0$. Then,*

$$\phi(t) \leq \phi(t_0) \exp\left(\int_{t_0}^{t} L(s)\, ds\right).$$

Proof. Let $\psi(t) = \phi(t) \exp\left(-\int_{t_0}^{t} L(s)\, ds\right)$. Then by the definition of Dini derivative and $D^+\phi(t) \leq L(t)\phi(t)$, we have

$$\begin{aligned}
& D^+\psi(t) \\
= & \limsup_{h\to 0^+} \frac{1}{h}\left\{\phi(t+h)\exp\left(-\int_{t_0}^{t+h} L(s)\, ds\right) - \phi(t)\exp\left(-\int_{t_0}^{t} L(s)\, ds\right)\right\} \\
= & \limsup_{h\to 0^+} \frac{1}{h}\left\{\phi(t+h)\exp\left(-\int_{t_0}^{t} L(s)\, ds\right) - \phi(t)\exp\left(-\int_{t_0}^{t} L(s)\, ds\right)\right\} \\
+ & \limsup_{h\to 0^+} \frac{1}{h}\left\{\phi(t+h)\exp\left(-\int_{t_0}^{t+h} L(s)\, ds\right) - \phi(t+h)\exp\left(-\int_{t_0}^{t} L(s)\, ds\right)\right\} \\
\leq & \limsup_{h\to 0^+} \frac{\phi(t+h)-\phi(t)}{h}\exp\left(-\int_{t_0}^{t} L(s)\, ds\right) \\
+ & \limsup_{h\to 0^+} \phi(t+h)\frac{1}{h}\left\{\exp\left(-\int_{t_0}^{t+h} L(s)\, ds\right) - \exp\left(-\int_{t_0}^{t} L(s)\, ds\right)\right\} \\
= & D^+\phi(t)\exp\left(-\int_{t_0}^{t} L(s)\, ds\right) + \phi(t)\left(-L(t)\exp\left(-\int_{t_0}^{t} L(s)\, ds\right)\right) \\
= & \left(D^+\phi(t) - L(t)\phi(t)\right)\exp\left(-\int_{t_0}^{t} L(s)\, ds\right) \\
\leq & 0.
\end{aligned}$$

Therefore, for $t \geq t_0$,

$$\psi(t) \leq \psi(t_0),$$

which means

$$\phi(t) \leq \phi(t_0) \exp\left(\int_{t_0}^{t} L(s)\, ds\right).$$

□

Note that a generalized form of Lemma 9, but restricted to non-negative L, is given in [65, Appendix A, Proposition 2] as follows.

Lemma 10. *Let* $\psi\colon [a,b] \to \mathbb{R}$ *and* $f, g \in C^0([a,b],\mathbb{R})$ *satisfy* $f, g \geq 0$*, and*

$$\begin{aligned} 0 \leq \psi(t) &\leq \limsup_{h\to 0^+} \psi(t-h) \\ \psi(t) &\geq \limsup_{h\to 0^+} \psi(t+h) \\ D^+\psi(t) &\leq f(t) \limsup_{h\to 0^+} \psi(t-h) + g(t). \end{aligned}$$

Then, for every $t \in [a,b]$*, the function* ϕ *fulfills the upper estimate*

$$\psi(t) \leq \psi(a)\exp\left(\xi(t)\right) + \int_a^t \exp(\xi(t)-\xi(s))g(s)\, ds,$$

where $\xi(t) = \int_a^t f(s)\, ds$*.*

Chapter 3

Diffusive interconnection of identical nonlinear ODE systems

In this Chapter, we study networks consisting of identical systems, described by ordinary differential equations, which are diffusively interconnected. The state of the system will be described by a vector x which one may interpret as a vector collecting the states x_i's (each of them itself possibly a vector) of identical "agents" which tend to follow each other according to a diffusion rule, with interconnections specified by an undirected graph. Another interpretation, useful in the context of biological modeling, is a set of chemical reactions among species that evolve in separate compartments (e.g., nucleus, cytoplasm, membrane in a cell); then the x_i's represent the vectors of concentrations of the species in each separate compartment.

The techniques of this chapter are based on Lipschitz norms (as defined in Section 2.2) and graph theory. For ease of reference, in Section 3.1, we recall some definitions and ideas in graph theory and introduce some notations that we will use through this work (for more details see e.g. [71]).

3.1 Preliminaries: graph theory

In this work, we let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote a finite, undirected, connected, simple (graph that has no loops, i.e., edges are not connected at both ends to the same vertex) graph (a "graph" for short) where $\mathcal{V} = \{x_1, \dots, x_N\}$ is the set of vertices (nodes) and $\mathcal{E} = \{e_1, \dots, e_m\}$ is the set of edges, where $e_k = x_{i_k} x_{j_k}$ is the kth edge that interconnects x_{i_k} and x_{j_k} for some $i_k, j_k \in \{1, \dots, N\}$. (Sometimes we indicate e_k by $x_{i_k} - x_{j_k}$ for some $i_k < j_k$.)

Two vertices x_i and x_j are *adjacent* if there exists an edge between them, i.e., $x_i x_j \in \mathcal{E}$.

The *neighborhood* $\mathcal{N}_i \subset \mathcal{V}$ of vertex x_i is the set of all vertices that are adjacent to x_i.

In what follows we define some standard classes of graphs that we use in this work.

Complete graph. In a complete graph every vertex is adjacent to every other vertex. Figure 3.1(left) indicates a complete graph with 5 vertices.

Path graph. In a path graph with $\mathcal{V} = \{x_1, \ldots, x_N\}$, the set of edges is as follows:

$$\mathcal{E} = \{x_1x_2, x_2x_3, \ldots, x_{N-1}x_N\}.$$

Figure 3.1(middle) indicates a path graph with 5 vertices.

Star graph. In a star graph with $\mathcal{V} = \{x_1, \ldots, x_N\}$, the set of edges is as follows:

$$\mathcal{E} = \{x_1x_2, x_1x_3, \ldots, x_1x_N\}.$$

Figure 3.1(right) indicates a star graph with 5 vertices.

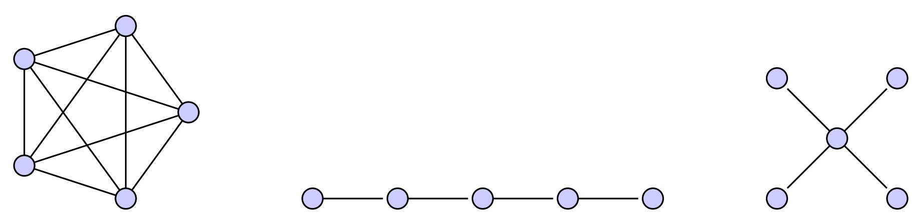


Figure 3.1: (left) complete graph, (middle) path graph, (right) star graph

Tree. A tree is a simple, undirected graph with no *cycles*, i.e., any two vertices are connected by exactly one "path" (a sequence of edges that connects two vertices). Path graphs and star graphs are two simple examples of trees.

Next we recall the definition of *Cartesian product* of graphs and some of its properties. The Cartesian product of two graphs, namely $\mathcal{G}_1$ and $\mathcal{G}_2$, indicated by $\mathcal{G}_1 \times \mathcal{G}_2$ is a graph such that

- the vertex set of $\mathcal{G}_1 \times \mathcal{G}_2$ is the Cartesian product $\mathcal{V}_1 \times \mathcal{V}_2$; and
- any two vertices (x_i, y_j) and (x_k, y_l) are adjacent in $\mathcal{G}_1 \times \mathcal{G}_2$ if and only if either $x_i = x_k$ and y_j is adjacent to y_l in $\mathcal{G}_2$, or $y_j = y_l$ and x_i is adjacent to x_k in $\mathcal{G}_1$.

Figure 3.2 indicates the Cartesian product of two complete graphs with 3 nodes (called "Rook") and the Cartesian product of three path graphs with 2 nodes (hypercube or "lattice").

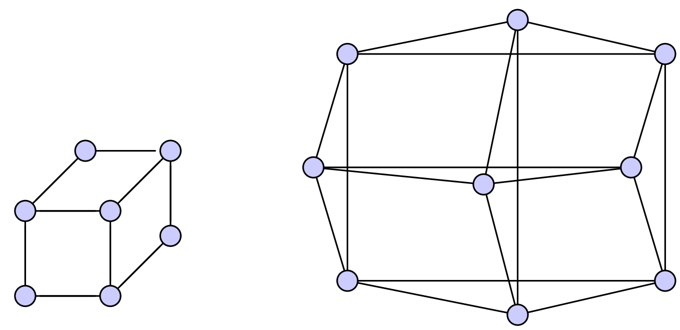


Figure 3.2: (left) hypercube, (right) Rook

Weighted graph. More generally, a weighted graph is a graph that a number (weight) is assigned to each edge. A weight matrix $W = (w_{ij})$, associated to a weighted graph $\mathcal{G}_w$ (the subscript w refers to weight), is defined as follows.

$$(W)_{ij} = \begin{cases} w_{ij} \neq 0 & x_i \text{ and } x_j \text{ are adjacent with weight } w_{ij}, \\ 0 & x_i \text{ and } x_j \text{ are not adjacent.} \end{cases}$$

Since there are exactly m (the number of edges) non-zero w_{ij}'s, they can be ordered as $\omega_1, \ldots, \omega_m$. Let $\mathcal{W}$ be an $m \times m$ diagonal matrix with ω_i's on its diagonal.

$$\mathcal{W} = \operatorname{diag}(\omega_1, \ldots, \omega_m). \tag{3.1}$$

Graph and matrices

The incidence matrix

A *directed* graph is a set of nodes that are connected by directed edges, see e.g. Figure 3.3(left).

Consider a graph $\mathcal{G}$ whose edges have been arbitrary oriented. The $N \times m$ *incidence*

Figure 3.3: (left) a directed path graph, (right) the associated undirected graph

matrix E is defined as follows.

$$(E)_{ij} = \begin{cases} 1 & \text{if edge } e_j \text{ originates from vertex } x_i, \\ -1 & \text{if edge } e_j \text{ terminates at vertex } x_i, \\ 0 & \text{if edge } e_j \text{ and vertex } x_i \text{ are not incident.} \end{cases}$$

The following matrix indicates the incidence matrix associated to the (directed) graph in Figure 3.3.

$$E_{4\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

An incidence matrix for a weighted graph is defined as follows (see [72] for more details).

$$E_w := E\sqrt{\mathcal{W}}, \tag{3.2}$$

where E is an incidence matrix of the associated unweighted graph, and

$$\sqrt{\mathcal{W}} = \operatorname{diag}(\sqrt{\omega_1}, \ldots, \sqrt{\omega_m}).$$

The graph Laplacian matrix

Another matrix representation of an (undirected, unweighted) graph $\mathcal{G}$ is the graph Laplacian (or simply Laplacian) $\mathcal{L}$ and is defined as follows.

$$(\mathcal{L})_{ij} = \begin{cases} -|\mathcal{N}_i| & \text{when } i = j, \\ 1 & \text{when } i \neq j \text{ and } x_i \text{ and } x_j \text{ are adjacent,} \\ 0 & \text{when } i \neq j \text{ and } x_i \text{ and } x_j \text{ are not adjacent,} \end{cases}$$

where $|\mathcal{N}_i|$ indicates the number of the neighbors of vertex x_i. The following matrix indicates the (graph) Laplacian matrix associated to the undirected graph in Figure

3.3.

$$\mathcal{L}_{4\times 4} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

In what follows we recall a few properties of $\mathcal{L}$ that we need in this work.

1. $\mathcal{L}$ is a positive semidefinite matrix, hence its real eigenvalues can be ordered as

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N.$$

   - Since $\mathcal{L}\mathbf{1} = 0$, $\lambda_1 = 0$ with corresponding eigenvector $\mathbf{1} = (1, \ldots, 1)^T$.
   - $\mathcal{G}$ is connected if and only if $\lambda_2 > 0$. The second smallest eigenvalue, λ_2, is called the *algebraic connectivity* of the graph. This number helps to quantify "how connected" the graph is; for example, a complete graph is "more connected" than a path graph with the same number of nodes, and this is reflected in the fact that the second eigenvalue of the Laplacian matrix of a complete graph ($\lambda_2 = N$) is larger that the second eigenvalue of the Laplacian matrix of a path graph ($\lambda_2 = 4\sin^2(\pi/2N)$).

2. For any incidence matrix E of a graph, $\mathcal{L} = EE^T$.

Examples

1. The following $N \times N$ matrix indicates the Laplacian matrix of a complete graph with N nodes,

$$\mathcal{L} = \begin{pmatrix} N-1 & -1 & \ldots & -1 \\ -1 & N-1 & \ldots & -1 \\ & \ddots & & \\ -1 & \ldots & -1 & N-1 \end{pmatrix},$$

   with $\lambda_1 = 0$ and $\lambda_2 = N$.

2. The following tridiagonal $N \times N$ matrix indicates the Laplacian matrix of a path graph with N nodes:

$$\mathcal{L} = \begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}. \tag{3.3}$$

with $\lambda_1 = 0$ and $\lambda_2 = 4\sin^2(\pi/2N)$.

3. The following $(N+1) \times (N+1)$ matrix indicates the Laplacian matrix of a star graph of $N+1$ nodes:

$$\mathcal{L} = \begin{pmatrix} 1 & & & & -1 \\ & 1 & & & -1 \\ & & \ddots & & \\ & & & 1 & -1 \\ -1 & & & -1 & N \end{pmatrix}.$$

with $\lambda_1 = 0$ and $\lambda_2 = 1$ (which is independent of the number of vertices).

4. Laplacian spectrum of the Cartesian product $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_K$ is

$$\{\lambda_{i_1}(\mathcal{G}_1) + \cdots + \lambda_{i_K}(\mathcal{G}_K) \mid i_j = 1, \ldots, N_j\},$$

where N_j is the number of vertices of $\mathcal{G}_j$. Therefore, since for any j, $\lambda_1(\mathcal{G}_j) = 0$,

$$\lambda_2(\mathcal{G}) = \min\{\lambda_2(\mathcal{G}_1), \ldots, \lambda_2(\mathcal{G}_K)\}.$$

Similar to the unweighted graphs, the graph Laplacian can be defined for a weighted graph with a weight matrix $W = (w_{ij})$ as follows (see [72] for more details).

$$(\mathcal{L}_w)_{ij} = \begin{cases} -\sum_{k \in \mathcal{N}_i} w_{ik} & \text{when } i = j, \\ w_{ij} & \text{when } i \neq j \text{ and } x_i \text{ and } x_j \text{ are adjacent,} \\ 0 & \text{when } i \neq j \text{ and } x_i \text{ and } x_j \text{ are not adjacent.} \end{cases}$$

Similar properties to those we listed above also hold for unweighted graphs as follows.

1. $\mathcal{L}_w$ is a positive semidefinite matrix, hence its real eigenvalues can be ordered as $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$.

2. For any incidence matrix E_w of a weighted graph, $\mathcal{L}_w = E_w E_w^T$.

The edge Laplacian matrix

For any graph $\mathcal{G}$, the edge Laplacian is an $m \times m$ matrix which is defined as follows.

$$\mathcal{K} = E^T E. \tag{3.4}$$

The following lemma describes the eigenvalues of the edge Laplacian of a graph and their relations to the eigenvalues of the graph Laplacian.

Lemma 11. *[73] Let $\mathcal{G}$ be a connected graph with incidence matrix E, edge Laplacian $\mathcal{K} = E^T E$, and (graph) Laplacian $\mathcal{L} = EE^T$. Then*

1. *The nonzero eigenvalues of $\mathcal{K}$ are equal to the nonzero eigenvalues of $\mathcal{L}$.*

2. *The null space of the edge Laplacian depends on the number of cycles in the graph. In particular, the null space of a tree is equal to 0, i.e., all the eigenvalues are nonzero.*

For any weighted graph $\mathcal{G}_w$, the edge Laplacian is an $m \times m$ matrix which is defined as follows (see [72] for more details).

$$\mathcal{K}_w = E_w^T E_w. \tag{3.5}$$

The following lemma describes the eigenvalues of the edge Laplacian of a graph and their relations to the eigenvalues of the graph Laplacian.

Lemma 12. *[72, 73] Let $\mathcal{G}_w$ be a connected undirected weighted graph with weighted incidence matrix E_w, edge Laplacian matrix $\mathcal{K}_w = {E_w}^T E_w$, and (graph) Laplacian matrix $\mathcal{L}_w = E_w {E_w}^T$. Then*

1. *The nonzero eigenvalues of $\mathcal{K}_w$ are equal to the nonzero eigenvalues of $\mathcal{L}_w$.*

2. *The null space of the edge Laplacian depends on the number of cycles in the graph. In particular, the null space of a tree is equal to 0, i.e., all the eigenvalues are nonzero.*

3.2 Contractivity of diffusively-connected ODEs: weighted L^p norm approaches

Acknowledgement of journal publication:

Parts of the material in this section have been published in the journal paper [51].

In this section we first describe the networks consisting of identical systems, which are diffusively interconnected. In order to formally describe the interconnections, we introduce the following concepts.

- For a fixed convex subset of $\mathbb{R}^n$, say V, $\tilde{F}\colon V^N \times [0,\infty) \to \mathbb{R}^{nN}$ is a function of the form:
$$\tilde{F}(x,t) = \left(F(x_1,t)^T, \ldots, F(x_N,t)^T\right)^T,$$
where $x = \left(x_1^T, \ldots, x_N^T\right)^T$, with $x_i \in V$ for each i, and $F(z,t)$ is a Lipschitz function on z and a continuous function on (z,t).

- For any $x \in V^N$ we define $\|x\|_{p,I\otimes Q}$ as follows:
$$\|x\|_{p,I\otimes Q} = \left\|(\|Qx_1\|_p, \ldots, \|Qx_N\|_p)^T\right\|_p,$$
where $Q = \mathrm{diag}\,(q_1, \ldots, q_n)$ is a positive diagonal matrix and $1 \le p \le \infty$.

  When $N = 1$, we simply have a norm in $\mathbb{R}^n$:
$$\|x\|_{p,Q} := \|Qx\|_p.$$
  We also let $\mu_{p,Q}$ denote the (lub) logarithmic Lipschitz constant (or logarithmic norm) induced by $\|\cdot\|_{p,Q}$ defined on an appropriate space.

- $D(t) = \mathrm{diag}\,(d_1(t), \ldots, d_n(t))$, where $d_i(t) \ge 0$ is a continuous functions of t. The matrix $D(t)$ is called the diffusion matrix.

- $\mathcal{L} \in \mathbb{R}^{N\times N}$ is a symmetric matrix and $\mathcal{L}\mathbf{1} = 0$, where $\mathbf{1} = (1, \ldots, 1)^T$. We think of $\mathcal{L}$ as the Laplacian of a graph that describes the interconnections among component subsystems.

- $\otimes$ denotes the Kronecker product of two matrices.

We recall that if $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is a $p \times q$ matrix, then the Kronecker product, denoted by $A \otimes B$, is the $mp \times nq$ block matrix defined as follows:

$$A \otimes B \ := \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix},$$

where $a_{ij}B$ denote the following $p \times q$ matrix:

$$a_{ij}B \ := \begin{bmatrix} a_{ij}b_{11} & \dots & a_{ij}b_{1q} \\ \vdots & \ddots & \vdots \\ a_{ij}b_{p1} & \dots & a_{ij}b_{pq} \end{bmatrix}.$$

The following are some properties of Kronecker product (for more properties see e.g. [74]):

1. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$;

2. If A and B are invertible, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

3. Suppose that A and B are square matrices of size n and m respectively. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A and $\mu_1, \dots, \mu_m$ be those of B (listed according to multiplicity). Then the eigenvalues of $A \otimes B$ are $\lambda_i \mu_j$ for $i = 1, \dots, n$, and $j = 1, \dots, m$.

Definition 7. *For any arbitrary graph $\mathcal{G}$ with the associated (graph) Laplacian matrix $\mathcal{L}$, any diagonal matrix $D(t)$, and any $F \colon V \times [0, \infty) \to \mathbb{R}^n$, the associated $\mathcal{G}-$compartment system, denoted by $(F, \mathcal{G}, D)$, is defined by*

$$\dot{x}(t) = \tilde{F}(x(t), t) - (\mathcal{L} \otimes D(t))x(t), \tag{3.6}$$

where x and $\tilde{F}$ are as defined above.

Definition 8. *We say that the $\mathcal{G}-$compartment system (3.6) is contractive, if for any two solutions $x = \left(x_1^T, \dots, x_N^T\right)^T$ and $y = \left(y_1^T, \dots, y_N^T\right)^T$ of (3.6), $x(t) - y(t) \to 0$ as $t \to \infty$.*

A discrete analog of the Turing instability phenomenon (discussed again in Section 4.2) is that when a dynamic equilibrium $\bar{x}$ of the non-diffusing ODE system $\dot{x} = F(x,t)$ is stable, but, at least for some diagonal positive matrices D, the corresponding interconnected system (3.6) is unstable.

In what follows, we show that, for infinitesimally contractive reaction F (which implies, in particular, that any two trajectories of F converge to each other, see Theorem 7), no diffusion instability will occur, no matter what is the size of the diffusion matrix D.

Theorem 9. *Consider the system (3.6) and let $c = \mu_{p,Q}[F]$. Then for any two solutions x, y of (3.6), we have*

$$\|x(t) - y(t)\|_{p,I\otimes Q} \;\leq\; e^{ct}\|x(0) - y(0)\|_{p,I\otimes Q} \,. \tag{3.7}$$

In addition, if F is C^1 on x, then

$$\|x(t) - y(t)\|_{p,I\otimes Q} \;\leq\; e^{\tilde{c}t}\|x(0) - y(0)\|_{p,I\otimes Q} \,,$$

where $\tilde{c} = \sup_{(x,t)} \mu_{p,Q}[J_F(x,t)]$.

In particular, when $c < 0$ (or $\tilde{c} < 0$), the system (3.6) is contractive.

To prove this theorem, we first prove the following technical lemma.

Lemma 13. *For any $1 \leq p \leq \infty$, and positive diagonal matrix Q,*

1. $\mu_{p,I\otimes Q}[-\mathcal{L} \otimes D(t)] = 0$.
2. $\mu^{+}_{p,I\otimes Q}[\tilde{F}] \;\leq\; \mu_{p,Q}[F]$.

Proof. To prove 1 we need the following two steps.

Step 1. $\mu_{p,I\otimes Q}[-\mathcal{L} \otimes D(t)] = \mu_p[-\mathcal{L} \otimes D(t)]$.

By the properties of Kronecker product mentioned above, we have:

$$\begin{aligned}
\mu_{p,I\otimes Q}[-\mathcal{L} \otimes D(t)] &= \mu_p\left[(I \otimes Q)(-\mathcal{L} \otimes D(t))(I \otimes Q^{-1})\right] \\
&= \mu_p[-\mathcal{L} \otimes QD(t)Q^{-1}] \\
&= \mu_p[-\mathcal{L} \otimes D(t)].
\end{aligned}$$

The last equality holds because both Q and $D(t)$ are diagonal, and thus they commute: $QD(t)Q^{-1} = D(t)QQ^{-1} = D(t)$.

Step 2. $\mu_p[-\mathcal{L} \otimes D(t)] = 0$.

For any fixed $t \geq 0$, let $\tilde{\mathcal{L}}(t) = -\mathcal{L} \otimes D(t) = (\tilde{\mathcal{L}}(t)_{ij})$. Note that since $\mathcal{L}\mathbf{1} = 0$, by the definition of Kronecker product, $\tilde{\mathcal{L}}(t)\mathbf{1} = 0$. In addition because $\mathcal{L}$ is symmetric and $D(t)$ is diagonal, $\tilde{\mathcal{L}}(t)$ is also symmetric and therefore $\tilde{\mathcal{L}}(t)\mathbf{1} = \mathbf{1}\tilde{\mathcal{L}}(t) = 0$. Also the off diagonal entries of $\tilde{\mathcal{L}}(t)$, like those of $-\mathcal{L}$, are positive. We first show that $\mu_p\left[\tilde{\mathcal{L}}(t)\right] = 0$ for $p = 1, \infty$. For $p = 1$,

$$\mu_1\left[\tilde{\mathcal{L}}(t)\right] \;=\; \max_j \sum_{i \neq j, i=1,\ldots,nN} \left(\tilde{\mathcal{L}}_{ii}(t) + \left|\tilde{\mathcal{L}}_{ij}(t)\right|\right) = \max_j 0 = 0.$$

Similarly, for $p = \infty$,

$$\mu_\infty\left[\tilde{\mathcal{L}}(t)\right] \;=\; \max_i \sum_{i \neq j, j=1,\ldots,nN} \left(\tilde{\mathcal{L}}_{ii}(t) + \left|\tilde{\mathcal{L}}_{ij}(t)\right|\right) = \max_j 0 = 0.$$

Now suppose $p \neq 1, \infty$. By Lemma 1, $\mu_p\left[\tilde{\mathcal{L}}(t)\right] \geq \lambda_{\max}$, where $\lambda_{\max}$ is the largest real part of an eigenvalue of $\tilde{\mathcal{L}}(t)$. Because $\tilde{\mathcal{L}}(t)\mathbf{1} = 0$, $\lambda = 0$ is an eigenvalue of $\tilde{\mathcal{L}}(t)$; therefore $\mu_p\left[\tilde{\mathcal{L}}(t)\right] \geq 0$. To show that $\mu_p\left[\tilde{\mathcal{L}}(t)\right] \leq 0$, by Lemma 3, it suffices to show that $D^+\|u\|_p \leq 0$ where u is a solution of $\dot{u} = \tilde{\mathcal{L}}(t)u$. By the definition of Dini derivative, it suffices to show that $\|u(t)\|_p$ is a non-increasing function of t. Let $\Phi(u(t)) := \|u(t)\|_p^p$, where $u = \left(u_1^T, \ldots, u_N^T\right)^T$ with $u_i = \left(u_i^1, \ldots, u_i^n\right)^T \in V^n$. Here we abuse the notation and assume that $u = (u_1, \ldots, u_{nN})^T$. We will show that $\frac{d\Phi}{dt}(u(t)) \leq 0$.

We will use the following simple fact (which is proved in the Appendix):

For any real α, β, and $1 \leq p$,

$$\left(|\alpha|^{p-2} + |\beta|^{p-2}\right)\alpha\beta \;\leq\; |\alpha|^p + |\beta|^p.$$

As we explained above, $\tilde{\mathcal{L}}(t)$ is symmetric and $\tilde{\mathcal{L}}(t)\mathbf{1} = 0$. Using this information and the above inequality:

$$\begin{aligned}
\frac{d\Phi}{dt}(u(t)) &= \sum_{i=1}^{nN} \frac{d\Phi}{du_i}\frac{du_i}{dt} \\
&= \nabla\Phi \cdot \dot{u} \\
&= \nabla\Phi \cdot \tilde{\mathcal{L}}(t)u \\
&= p\left(|u_1|^{p-2}u_1, \ldots, |u_{nN}|^{p-2}u_{nN}\right)\tilde{\mathcal{L}}(t)\left(u_1, \ldots, u_{nN}\right)^T
\end{aligned}$$

$$
\begin{aligned}
&= p\sum_{i,j}|u_i|^{p-2}u_i\tilde{\mathcal{L}}_{ij}(t)u_j \\
&= p\sum_{i}|u_i|^{p}\tilde{\mathcal{L}}_{ii}(t)+p\sum_{i<j}\tilde{\mathcal{L}}_{ij}(t)\left(|u_i|^{p-2}+|u_j|^{p-2}\right)u_iu_j \\
&\leq p\sum_{i}|u_i|^{p}\tilde{\mathcal{L}}_{ii}(t)+p\sum_{i<j}\tilde{\mathcal{L}}_{ij}(t)\left(|u_i|^{p}+|u_j|^{p}\right) \\
&= p\sum_{i}|u_i|^{p}\tilde{\mathcal{L}}_{ii}(t)+p\sum_{i\neq j}\left(\tilde{\mathcal{L}}_{ij}(t)|u_i|^{p}+\mathcal{L}_{ji}(t)|u_j|^{p}\right) \\
&= p\sum_{i}|u_i|^{p}\left(\tilde{\mathcal{L}}_{ii}(t)+\sum_{i\neq j}\tilde{\mathcal{L}}_{ij}(t)\right) \\
&= 0,
\end{aligned}
$$

since $\frac{d\Phi}{du_i}=\frac{d}{du_i}|u_i|^p=p|u_i|^{p-2}u_i$ (recall that $|x|^p$ is differentiable for $p>1$).

Next we prove part 2 of the lemma. In this part, for any t, we let $F_t(\cdot):=F(\cdot,t)$ and $\tilde{F}_t(\cdot):=\tilde{F}(\cdot,t)$. By the definition of $\tilde{c}_t:=\mu_{p,Q}[F_t]$, we have

$$
\lim_{h\to 0^+}\frac{1}{h}\sup_{x_1\neq y_1\in V}\left(\frac{\|x_1-y_1+h(F_t(x_1)-F_t(y_1))\|_{p,Q}}{\|x_1-y_1\|_{p,Q}}-1\right)=\tilde{c}_t.
$$

Fix an arbitrary $\epsilon>0$. Then there exists $h_0>0$ such that for all $0<h<h_0$,

$$
\frac{1}{h}\sup_{x_1\neq y_1\in V}\left(\frac{\|x_1-y_1+h(F_t(x_1)-F_t(y_1))\|_{p,Q}}{\|x_1-y_1\|_{p,Q}}-1\right)<\tilde{c}_t+\epsilon.
$$

Therefore, for any $x_1\neq y_1$, and $0<h<h_0$

$$
\|x_1-y_1+h(F_t(x_1)-F_t(y_1))\|_{p,Q}<((\tilde{c}_t+\epsilon)h+1)\|x_1-y_1\|_{p,Q}. \tag{3.8}
$$

Consider (3.8) for N pairs (x_i,y_i)'s with $x_i\neq y_i$. Raising Inequality (3.8) to the power p, summing over all N pairs (x_i,y_i), dividing by $\sum\|x_i-y_i\|^p_{p,Q}$, raising to the power $\frac{1}{p}$, subtracting by 1, and dividing by h, we get:

$$
\frac{1}{h}\left(\frac{\left(\sum_i\|x_i-y_i+h(F_t(x_i)-F_t(y_i))\|^p_{p,Q}\right)^{\frac{1}{p}}}{\left(\sum_i\|x_i-y_i\|^p_{p,Q}\right)^{\frac{1}{p}}}-1\right)<\tilde{c}_t+\epsilon.
$$

Now by letting $\epsilon\to 0$ and taking sup over all pairs $(x_1^T,\ldots,x_N^T)^T\neq(y_1^T,\ldots,y_N^T)^T$, and by definition of $\tilde{F}_t$, and $\mu^+_{p,I\otimes Q}$, for ant t, we get, $\mu^+_{p,I\otimes Q}[\tilde{F}_t]\leq\tilde{c}_t=\mu_{p,Q}[F_t]$. Therefore, $\mu^+_{p,I\otimes Q}[\tilde{F}]\leq\mu_{p,Q}[F]$. □

Proof of Theorem 9. As in the proof of Lemma 13, for any t, we let $F_t(\cdot) := F(\cdot, t)$ and $\tilde{F}_t(\cdot) := \tilde{F}(\cdot, t)$. By subadditivity of $\mu^+_{p,I\otimes Q}$, Proposition 5, and Lemma 13, for any $t > 0$:

$$\begin{aligned}\mu^+_{p,I\otimes Q}[\tilde{F}_t - \mathcal{L}\otimes D(t)] &\leq \mu^+_{p,I\otimes Q}[\tilde{F}_t] + \mu^+_{p,I\otimes Q}[-\mathcal{L}\otimes D(t)] \\ &\leq \mu_{p,Q}[F_t] + \mu_{p,I\otimes Q}[-\mathcal{L}\otimes D(t)] \\ &\leq \mu_{p,Q}[F].\end{aligned}$$

Therefore, $\sup_{t\in[0,\infty)} \mu^+_{p,I\otimes Q}[\tilde{F}_t - \mathcal{L}\otimes D(t)] \leq c$. Now using Theorem 7, we have

$$\|x(t) - y(t)\|_{p,I\otimes Q} \leq e^{ct}\|x(0) - y(0)\|_{p,I\otimes Q}.$$

In addition, when F is C^1, by Proposition 2, $c \leq \tilde{c}$ and the following inequality holds:

$$\|x(t) - y(t)\|_{p,I\otimes Q} \leq e^{\tilde{c}t}\|x(0) - y(0)\|_{p,I\otimes Q}.$$

□

Lemma 14. *Assume F is a linear operator. Then*

$$\mu_{p,I\otimes Q}\left[\tilde{F} - \mathcal{L}\otimes D(t)\right] \leq \mu_{q,Q}[F] \quad \text{if} \quad p = q. \tag{3.9}$$

Proof. Note that for a linear operator, $\mu^+ = \mu$ (Proposition 4). Therefore, by subadditivity of μ and Lemma 13 we have:

$$\mu_{p,I\otimes Q}\left[\tilde{F} - \mathcal{L}\otimes D(t)\right] \leq \mu_{p,I\otimes Q}[\tilde{F}] = \mu^+_{p,I\otimes Q}[\tilde{F}] \leq \mu_{p,I\otimes Q}[F].$$

□

Remark 13. *Note that (3.9) does not need to hold if $p \neq q$. Indeed, consider the following system:*

$$\begin{aligned}\dot{x}_1 &= Ax_1 + D(x_2 - x_1) \\ \dot{x}_2 &= Ax_2 + D(x_1 - x_2),\end{aligned}$$

where $x_i \in \mathbb{R}^2$, $A = \begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}$ and $D = diag(d_1, d_2)$. In this example $\mathcal{L} = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$, $F(u) = Au$, and $\tilde{F}(u) = diag(Au, Au)$. We show that for $Q = diag(3,1)$, $\mu_{2,Q}[A] < 0$ while $\mu_{1,I\otimes Q}[\tilde{F} - \mathcal{L}\otimes D] > 0$. By Table 2.1,

$$\mu_{2,Q}[A] = \mu_2[QAQ^{-1}] = \mu_2\begin{bmatrix}-2 & 3\\ \frac{1}{3} & -2\end{bmatrix} < 0,$$

and

$$\mu_{1,I\otimes Q}\left[\tilde{F}-\mathcal{L}\otimes D(t)\right] = \mu_{1,I\otimes Q}\begin{bmatrix} -2-d_1 & 1 & d_1 & 0 \\ 1 & -2-d_2 & 0 & d_2 \\ d_1 & 0 & -2-d_1 & 1 \\ 0 & d_2 & 1 & -2-d_2 \end{bmatrix}$$

$$= \mu_1\begin{bmatrix} -2-d_1 & 3 & d_1 & 0 \\ \frac{1}{3} & -2-d_2 & 0 & d_2 \\ d_1 & 0 & -2-d_1 & 3 \\ 0 & d_2 & \frac{1}{3} & -2-d_2 \end{bmatrix} = 1 > 0.$$

3.3 Synchronization of diffusively-connected ODEs: non L^2 norm approaches

Acknowledgement of journal and conference publications:

Parts of the material in this section have been published in the conference paper [53] and the journal paper [54].

In this section we use the contraction theory to show synchronization (or "consensus") in diffusively connected identical ODE systems. Synchronization results based on contraction-based techniques, typically employing measures derived from L^2 or weighted L^2 norms, have been already well studied [33, 35, 42, 43, 44, 45].

Our interest here is in using matrix measures derived from norms that are not induced by inner products, such as L^1 and L^∞ norms, because these are the most appropriate in many applications, such as the biochemical examples discussed as illustrations in Section 3.3.3. For such more general norms, proofs cannot rely upon Lyapunov-like linear matrix inequalities. We remark that other authors have also previously studied matrix measures based on non L^2 norms, see for instance [33]; however, rigorous proofs of the types of results proved here have not been given in [33]. In [47], the author studies synchronization using matrix measures for L^1, L^2, and L^∞ norms; we compare our results to this and other papers in Section 3.3.4. Also, in [48, 49] a sufficient condition for synchronization based on matrix measure induced by an arbitrary norm is given for *linear* systems, see Remark 14 in Section 3.3.1 below with slightly different proof. In this work, we are interested in *nonlinear* systems.

Definition 9. *We say that the $\mathcal{G}-$compartment system (3.6) synchronizes, if for any solution $x = \left(x_1^T, \ldots, x_N^T\right)^T$ of (3.6), and $\forall i, j \in \{1, \ldots, N\}$, $(x_i - x_j)(t) \to 0$ as $t \to \infty$.*

An easy first result is as follows.

Proposition 8. *Suppose that x is a solution of (3.6), $F(x,t)$ is C^1 on x, and let $c = \sup_{(x,t)} \mu_{p,Q}[J_F(x,t)] < 0$. Then the $\mathcal{G}-$compartment system (3.6) synchronizes.*

Proof. Note that $z(t) := \left(z_1^T(t), \ldots, z_1^T(t)\right)^T$ is a solution of (3.6), where $z_1(t)$ is a solution of $\dot{x} = F(x,t)$. Then by Equation (3.7),

$$\|x(t) - z(t)\|_{p,I\otimes Q} \leq e^{ct}\|x(0) - z(0)\|_{p,I\otimes Q}.$$

When $c < 0$, $\forall i$, $(x_i - z_1)(t) \to 0$, hence for any pair (i,j), $(x_i - x_j)(t) \to 0$ as $t \to \infty$. $\square$

In Proposition 8, we imposed a strong condition on F, which in turn leads to the very strong conclusion that all solutions should converge exponentially to a particular solution, no matter what is the strength of the interconnection (choice of diffusion matrix). A more interesting and challenging problem is to provide a condition that links the vector field, the graph structure, and the diffusion matrix, so that interesting dynamical behaviors (such as oscillations in autonomous systems, which are impossible in contractive systems) can be exhibited by the individual systems, and yet the components synchronize. The example in Section 3.3.3 illustrates this question.

3.3.1 Synchronization based on contractions

In this section, we discuss several matrix measure based conditions that guarantee synchronization of ODE systems.

Consider a $\mathcal{G}-$compartment system, $(F, \mathcal{G}, D)$, where $\mathcal{G}$ is any arbitrary graph and F is C^1 on its first argument. The following re-phrasing of a theorem from [42] provides sufficient conditions on F and D (D is time invariant in [42]), based upon contractions with respect to L^2 norms, that guarantee synchrony of the associated $\mathcal{G}-$compartment system. We have translated the result to the language of contractions. (Actually, the result in [42] is stronger, in that it allows for certain non-diagonal diffusion and also certain non-diagonal weighting matrices Q, by substituting these assumptions by a commutativity type of condition.)

Theorem 10. *[42] Consider a $\mathcal{G}-$compartment as defined in Equation (3.6) and suppose that $V \subseteq \mathbb{R}^n$ is convex. For a given positive diagonal matrix Q, let*

$$c \;:=\; \sup_{(x,t)} \mu_{2,Q}[J_F(x,t) - \lambda_2 D]. \tag{3.10}$$

Then for every forward-complete solution $x = (x_1^T, \ldots, x_N^T)^T$ that remains in V, the following inequality holds:

$$\|\tilde{x}(t)\|_{2,I\otimes Q} \;\leq\; e^{ct}\|\tilde{x}(0)\|_{2,I\otimes Q},$$

where $\tilde{x} = \left((x_1 - \bar{x})^T, \ldots, (x_N - \bar{x})^T\right)^T$ and $\bar{x} = (x_1 + \cdots + x_N)/N$. In particular, if $c < 0$, then for any pair $i, j \in \{1, \ldots, N\}$, $(x_i - x_j)(t) \to 0$ exponentially as $t \to \infty$.

We next turn to general norms. The following theorem provides a sufficient condition on F, D, and $\mathcal{G}$ that guarantees synchrony of the associated $\mathcal{G}-$compartment system in any norm.

Theorem 11. *Consider a $\mathcal{G}-$compartment system, $(F, \mathcal{G}, D)$, where $\mathcal{G}$ is an arbitrary graph with N nodes and m edges. For an arbitrary orientation of $\mathcal{G}$, let E be the directed incidence matrix of $\mathcal{G}$, and pick any $m \times m$ matrix K satisfying*

$$E^T\mathcal{L} = KE^T\,. \tag{3.11}$$

Denote:

$$c \;:=\; \sup_{(w,t)} \mu\left[J(w,t) - K \otimes D(t)\right], \tag{3.12}$$

where μ is the logarithmic norm induced by an arbitrary norm on $\mathbb{R}^{mn}$, say $\|\cdot\|$, and for $w = \left(w_1^T, \ldots, w_m^T\right)^T$, $J(w,t)$ is defined as follows:

$$J(w,t) = diag\left(J_F(w_1,t), \ldots, J_F(w_m,t)\right).$$

Then

$$\left\|\left(E^T \otimes I\right) x(t)\right\| \;\leq\; e^{ct} \left\|\left(E^T \otimes I\right) x(0)\right\|.$$

Proof. Assume that x is a solution of $\dot{x} = \tilde{F}(x,t) - (\mathcal{L} \otimes D(t))\, x$. For any t, define y as follows.

$$y(t) := \left(E^T \otimes I\right) x(t).$$

Notice that for $k = 1, \ldots, m$, the kth entry of $\left(E^T \otimes I\right) x(t)$ is $x_{i_k} - x_{j_k}$ which indicates the kth edge of $\mathcal{G}$, i.e., the difference between states associated to the two nodes that constitute the edge, and I is the $n \times n$ identity matrix. Then, using the Kronecker product identity $(A \otimes B)(C \otimes D) = AC \otimes BD$, for matrices A, B, C, and D of appropriate dimensions, we have:

$$\begin{aligned}
\dot{y} &= \left(E^T \otimes I\right) \dot{x} \\
&= \left(E^T \otimes I\right) \left(\tilde{F}(x,t) - (\mathcal{L} \otimes D(t))\, x\right) \\
&= \left(E^T \otimes I\right) \tilde{F}(x,t) - \left(E^T \mathcal{L} \otimes D(t)\right) x \\
&= \left(E^T \otimes I\right) \tilde{F}(x,t) - \left(K E^T \otimes D(t)\right) x \\
&= \left(E^T \otimes I\right) \tilde{F}(x,t) - (K \otimes D(t)) \left(E^T \otimes I\right) x \\
&= \left(E^T \otimes I\right) \tilde{F}(x,t) - (K \otimes D(t))\, y,
\end{aligned}$$

where for $i = 1, \ldots, m$, $\left(E^T \otimes I\right) \tilde{F}(x,t)$ can be written as follows:

$$\left(E^T \otimes I\right) \tilde{F}(x,t) = \operatorname{diag}\left(F(x_{i_1},t) - F(x_{j_1},t), \ldots, F(x_{i_m},t) - F(x_{j_m},t)\right).$$

Now let $u = \left(x_{i_1}^T, \ldots, x_{i_m}^T\right)^T$, $v = \left(x_{j_1}^T, \ldots, x_{j_m}^T\right)^T$, and for any t, G_t be as follows:

$$G_t(u) := \begin{pmatrix} F(x_{i_1},t) \\ \vdots \\ F(x_{i_m},t) \end{pmatrix} - (K \otimes D(t)) \begin{pmatrix} x_{i_1} \\ \vdots \\ x_{i_m} \end{pmatrix},$$

then $\dot{u} - \dot{v} = G_t(u) - G_t(v)$, and by Remark 10,

$$\|u(t) - v(t)\| \;\leq\; e^{ct} \|u(0) - v(0)\|,$$

where $u - v = \left(E^T \otimes I\right) x$ and $c = \sup_{(w,t)} \mu\left[J_{G_t}(w)\right] = \sup_{(w,t)} \mu\left[J(w,t) - K \otimes D(t)\right].$

□

Since $\left(E^T \otimes I\right) x$ is a column vector whose entries are the differences $x_{i_k} - x_{j_k}$, when $c < 0$, the system synchronizes.

In Section 3.3.2, we will see the application of Theorem 11 to path graphs (Proposition 10) and complete graphs (Proposition 11) . We remark that, at least for certain graphs, one can recover the L^2 result from [42] as a corollary of Theorem 11 (see Remark 17 in Section 3.5).

We next specialize to the linear case, when $F(x,t) = A(t)x$.

Remark 14. *Our interest in this work is in nonlinear systems. For the special case of linear dynamics, a general result is easy, and well-known. Consider a $\mathcal{G}-$compartment system, $(F, \mathcal{G}, D)$, and suppose that $F(x,t) = A(t)x$, i.e.,*

$$\dot{x}(t) \;=\; \left(I \otimes A(t) - \mathcal{L} \otimes D(t)\right) x(t). \tag{3.13}$$

For a given arbitrary norm in $\mathbb{R}^n$, say $\|\cdot\|$, suppose that $\sup_t \mu[A(t) - \lambda_2 D(t)] < 0$. Then, for any $i, j \in \{1, \ldots, N\}$, $(x_i - x_j)(t) \to 0$, exponentially as $t \to \infty$.

Proof. Note that any solution x of Equation (3.13) can be written as follows:

$$x(t) \;=\; \sum_{i=1,\ldots,N} \; \sum_{j=1,\ldots,n} c_{ij}(t)\,(v_i \otimes e_j),$$

where the v_i's, $v_i \in \mathbb{R}^N$, are a set of orthonormal eigenvectors of $\mathcal{L}$ (that make up a basis for $\mathbb{R}^N$), corresponding to the eigenvalues λ_i's of $\mathcal{L}$, and the e_j's are the standard basis of $\mathbb{R}^n$. In addition, the c_{ij}'s are the coefficients that satisfy

$$\dot{C}(t) = \begin{pmatrix} A(t) - \lambda_1 D(t) & & \\ & \ddots & \\ & & A(t) - \lambda_N D(t) \end{pmatrix} C(t),$$

with appropriate initial conditions, where $C = (c_{11}, \ldots, c_{1n}, \ldots, c_{N1}, \ldots, c_{Nn})^T$. For an incidence matrix E, let $y = \left(E^T \otimes I\right) x$, then

$$y(t) = \sum_{i=1,\ldots,N} \; \sum_{j=1,\ldots,n} c_{ij}(t)\;\left(E^T v_i \otimes e_j\right) = \sum_{i=2,\ldots,N} \; \sum_{j=1,\ldots,n} c_{ij}(t)\;\left(E^T v_i \otimes e_j\right),$$

because $E^T v_1 = 0$ (where $v_1 = (1/\sqrt{n})\mathbf{1}$). Therefore, if $\sup_t \mu[A(t) - \lambda_2 D(t)] < 0$, then $\sup_t \mu[A(t) - \lambda_i D(t)] < 0, \forall i = 2, \ldots, N$, and by Lemma 5, the $c_{ij}(t)$'s, for $j \geq 2$, and hence also $y(t)$, converge to 0 exponentially as $t \to \infty$. $\square$

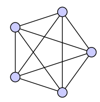
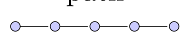
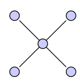
graph	second eigenvalue, λ_2	synchronization condition
complete 	N	$\mu[J_F - \lambda_2 D] < 0$
path 	$4\sin^2(\pi/2N)$	$\mu_{p,Q}\left[J_F - \lambda_2 D\right] < 0$
star 	1	$\mu\left[J_F - \lambda_2 D\right] < 0$

Table 3.1: Sufficient conditions for synchronization in complete, line and star graphs with N nodes. If no subscript is used in μ, the result has been proved for arbitrary norms

For a different proof of Remark 14, see [48, 49].

3.3.2 Synchronization based on graph structure

While the results for measures based on Euclidean norm are quite general, in the nonlinear case and for L^p norms, $p \neq 2$, we separately establish results for special cases, depending on the graph structure. We present sufficient conditions for synchronization for some general families of graphs (path, complete, star), and compositions of them (Cartesian product graphs).

See Table 3.1 and Table 3.2 for a summary of the results that will be stated in this section.

Note that the results presented in Propositions 10 and 11 below are derived from Theorem 11 directly. But to prove Proposition 12 (star graph), we use different techniques.

Two compartments

We first study the relatively trivial case of a system with two compartments, $N = 2$. Since it makes no difference in the proof, we allow in this case a “nonlinear diffusion”

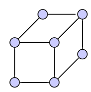
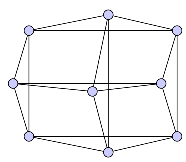
graph	second eigenvalue, λ_2	synchronization condition
hypercube 	$4 \min_{1\le i\le K} \{\sin^2(\pi/2N_i)\}$	$\mu_{p,Q}[J_F - \lambda_2 D] < 0$
Rook 	$\min\{N_1, \ldots, N_K\}$	$\mu[J_F - \lambda_2 D] < 0$

Table 3.2: Sufficient conditions for synchronization in cartesian products of K line and complete graphs, (if no subscript is used in μ, the result has been proved for arbitrary norms)

term represented by a function h which need not be linear:

$$\begin{aligned} \dot{x}_1 &= F(x_1, t) + h_1(x_2, t) - h_1(x_1, t) \\ \dot{x}_2 &= F(x_2, t) + h_2(x_1, t) - h_2(x_2, t) \end{aligned} \tag{3.14}$$

Proposition 9. *Let* $c = \sup_{(x,t)} \mu\left[J_F(x,t) - (J_{h_1} + J_{h_2})(x,t)\right]$, *and* $\left(x_1^T, x_2^T\right)^T$ *be a solution of (3.14). Then*

$$\|x_1(t) - x_2(t)\| \;\le\; e^{ct}\|x_1(0) - x_2(0)\|,$$

where $\|\cdot\|$ *is an arbitrary norm in* $\mathbb{R}^n$ *and* μ *is the logarithmic norm induced by* $\|\cdot\|$*.*

Proof. Note that $\dot{x}_1 - \dot{x}_2 = G_t(x_1) - G_t(x_2)$ where $G_t(x) = F(x,t) - (h_1 + h_2)(x,t)$. By Remark 10,

$$\|x_1(t) - x_2(t)\| \;\le\; e^{ct}\|x_1(0) - x_2(0)\|,$$

where $c = \sup_{(x,t)} \mu\left[J_{G_t}(x)\right] = \sup_{(x,t)} \mu\left[J_F(x,t) - (J_{h_1} + J_{h_2})(x,t)\right]$. □

Our interest here is in an arbitrary number of compartments, and we turn to that general problem next.

Path graphs

Consider a system of N compartments, $x_1, \ldots, x_N$, that are connected to each other by a path graph $\mathcal{G}$. Assuming $x_0 = x_1,\ x_{N+1} = x_N$, the following system of ODEs describes the evolution of the individual agent x_i, for $i = 1, \ldots, N$:

$$\dot{x}_i \;=\; F(x_i, t) + D(t)(x_{i-1} - x_i + x_{i+1} - x_i). \tag{3.15}$$

The following result is an application of Theorem 11 to path graphs.

Proposition 10. *Let $(x_1^T, \ldots, x_N^T)^T$ be a solution of (3.15), and for $1 \le p \le \infty$ and a positive diagonal matrix Q, let*

$$c = \sup_{(x,t)} \mu_{p,Q} \left[J_F(x,t) - 4\sin^2(\pi/2N)\, D(t) \right]. \tag{3.16}$$

Then

$$\|e(t)\|_{p,Q_p \otimes Q} \;\le\; e^{ct} \|e(0)\|_{p,Q_p \otimes Q}, \tag{3.17}$$

where $e = \left(e_1^T, \ldots, e_{N-1}^T\right)^T$ with $e_i = x_i - x_{i+1}$ denotes the vector of all edges of the path graph, and $\|\cdot\|_{p,Q_p \otimes Q}$ denotes the weighted L^p norm with the weight $Q_p \otimes Q$, where for any $1 \le p < \infty$,

$$Q_p := diag\left(p_1^{\frac{2-p}{p}}, \ldots, p_{N-1}^{\frac{2-p}{p}} \right),$$

and for $1 \le k \le N-1$, $p_k = \sin(k\pi/N)$. In addition, $4\sin^2(\pi/2N)$ is the smallest nonzero eigenvalue of the Laplacian matrix of $\mathcal{G}$. Note that

$$Q_\infty = diag\left(1/p_1, \ldots, 1/p_{N-1}\right).$$

The significance of Proposition 10 is as follows: since the numbers $p_k = \sin(k\pi/N)$ are nonzero, we have, when $c < 0$, exponential convergence to uniform solutions in a weighted L^p norm, the weights being specified in each compartment by the matrix Q and the relative weights among compartments being weighted by the numbers p_k's.

Before we prove Proposition 10, we will explain where the p_i's and $4\sin^2(\pi/2N)$ come from. For a path graph with N nodes, consider the following $N \times N-1$ directed incidence matrix E and the $N-1 \times N-1$ edge Laplacian $\mathcal{K} := E^T E$:

$$E = \begin{pmatrix} -1 & & & \\ 1 & \ddots & & \\ & \ddots & -1 \\ & & 1 \end{pmatrix}, \quad \mathcal{K} = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \\ & & -1 & 2 \end{pmatrix}. \tag{3.18}$$

Note that since $-\mathcal{K}$ is a Metzler (a matrix with non-negative off diagonal entries) and irreducible matrix, it follows by the Perron-Frobenius Theorem (see the Appendix for the statement of the theorem) that it has a positive eigenvector $(v_1, \ldots, v_{N-1})$ corresponding to $-\lambda$, the largest eigenvalue of $-\mathcal{K}$, (λ is the smallest eigenvalue of $\mathcal{K}$), i.e.,

$$(p_1, \ldots, p_{N-1})(-\mathcal{K}) = -\lambda (p_1, \ldots, p_{N-1}). \tag{3.19}$$

Note that by Lemma 11, the smallest eigenvalue of $\mathcal{K}$ is equal to the smallest non-zero eigenvalue of the corresponding Laplacian matrix of $\mathcal{G}$. A simple calculation shows that $p_k = \sin(k\pi/N)$ and $\lambda = 4\sin^2(\pi/2N)$ (see the Appendix for more details).

To prove Proposition 10, we first prove the following Lemma.

Lemma 15. *Let $\mathcal{K}$ be the edge Laplacian of a path graph with $N \geq 3$ nodes as shown in (3.18). Then for any $1 \leq p \leq \infty$,*

$$\mu_{p,Q_p \otimes Q}\left[4\sin^2(\pi/2N)\, I \otimes D(t) - \mathcal{K} \otimes D(t)\right] \leq 0, \tag{3.20}$$

where Q and Q_p are as in Proposition 10.

Proof. To prove (3.20), we will show that $\mu_p[\mathcal{A}] \leq 0$, where $\mathcal{A}$ is defined as follows:

$$\mathcal{A} = (Q_p \otimes Q)\left(4\sin^2(\pi/2N)\, I \otimes D(t) - \mathcal{K} \otimes D(t)\right)\left(Q_p^{-1} \otimes Q^{-1}\right).$$

(Recall that $\mu_{p,Q}[A] = \mu_p\left[QAQ^{-1}\right]$, and $A^{-1} \otimes B^{-1} = (A \otimes B)^{-1}$.)

We first show that for $p = 1$, $\mu_p[\mathcal{A}] = 0$. A simple calculation shows that, for $p = 1$, $\mathcal{A}$

can be written as follows:

$$\begin{pmatrix} (\lambda-2)D(t) & \frac{p_1}{p_2}D(t) & & & \\ \frac{p_2}{p_1}D(t) & (\lambda-2)D(t) & \frac{p_2}{p_3}D(t) & & \\ & \ddots & \ddots & & \\ & & & & \\ & & & \frac{p_{N-1}}{p_{N-2}}D(t) & (\lambda-2)D \end{pmatrix},$$

where as mentioned before, $\lambda = 4\sin^2(\pi/2N)$. For $\mathbf{1} = (1,\ldots,1)^T$, and $p = 1$, since $\mathbf{1}^T Q_p = (p_1,\ldots,p_{N-1})$, it follows by Equation (3.19) that $\mathbf{1}^T Q_p(-\mathcal{K})Q_p^{-1} = -\lambda\mathbf{1}^T$, therefore,

$$-2 + \frac{p_2}{p_1} = -2 + \frac{p_1}{p_2} + \frac{p_3}{p_2} = \cdots = -2 + \frac{p_{N-2}}{p_{N-1}} = -\lambda. \tag{3.21}$$

Hence, by the definition of μ_1, $\mu_1[A] = \max_j \left(a_{jj} + \sum_{i\neq j}|a_{ij}|\right)$, and because $D(t)$ is diagonal, $\mu_1[\mathcal{A}] = 0$.

Now, we show that $\mu_\infty[\mathcal{A}] = 0$. A simple calculation shows that, for $p = \infty$, since $Q_\infty = \text{diag}\,(1/p_1,\ldots,1/p_{N-1})$, $\mathcal{A}$ can be written as follows:

$$\begin{pmatrix} (\lambda-2)D(t) & \frac{p_2}{p_1}D(t) & & & \\ \frac{p_1}{p_2}D & (\lambda-2)D(t) & \frac{p_3}{p_2}D(t) & & \\ & \ddots & \ddots & & \\ & & & & \\ & & & \frac{p_{N-2}}{p_{N-1}}D(t) & (\lambda-2)D(t) \end{pmatrix}.$$

Therefore, by the definition of μ_∞, $\mu_\infty[A] = \max_i \left(a_{ii} + \sum_{i\neq j}|a_{ij}|\right)$, and because $D(t)$ is diagonal, $\mu_\infty[\mathcal{A}] = \max\left\{\lambda - 2 + \frac{p_2}{p_1},\ldots,\lambda - 2 + \frac{p_{N-2}}{p_{N-1}}\right\} = 0$.

Next we show for $1 < p < \infty$, $\mu_p[\mathcal{A}] \leq 0$. A simple calculation shows that $\mathcal{A}$ can be written as follows:

$$\begin{pmatrix} (\lambda_2-2)D(t) & \alpha_1^{-1}D(t) & & & \\ \alpha_1 D(t) & (\lambda_2-2)D(t) & \alpha_2^{-1}D(t) & & \\ & \ddots & \ddots & & \\ & & & & \\ & & & \alpha_{N-2}D(t) & (\lambda_2-2)D(t) \end{pmatrix},$$

where $\alpha_i = \left(\frac{p_{i+1}}{p_i}\right)^{\frac{2-p}{p}}$. To show $\mu_p[\mathcal{A}] \leq 0$, using Lemma 3 and the definition of μ, it suffices to show that $D^+\|u\|_p \leq 0$, where $u = (u_{11}, \ldots, u_{1n}, \ldots, u_{N-11}, \ldots, u_{N-1n})^T$ is the solution of $\dot{u} = \mathcal{A}u$, or equivalently, $\frac{d\Phi}{dt}(u(t)) \leq 0$, where $\Phi(t) = \|u(t)\|_p^p$. In the calculations below, we use the following simple fact (which is proved in the Appendix). For any real α and β and $1 \leq p$:

$$\left(|\alpha|^{p-2} + |\beta|^{p-2}\right)\alpha\beta \leq |\alpha|^p + |\beta|^p.$$

In the calculations below, we let $\beta_i = \alpha_i^{\frac{2}{2-p}}$. We also use the fact that $|x|^p$ is differentiable for $p > 1$ and

$$\frac{d\Phi}{du_i} = \frac{d}{du_i}|u_i|^p = p|u_i|^{p-1}\frac{u_i}{|u_i|} = p|u_i|^{p-2}u_i.$$

Observe that

$$\begin{aligned}
\frac{d\Phi}{dt}(u(t)) &= \sum_{i,k}\frac{d\Phi}{du_{ik}}\frac{du_{ik}}{dt} = \nabla\Phi\cdot\dot{u} = \nabla\Phi\cdot\mathcal{A}u \\
&= p\left(|u_{11}|^{p-2}u_{11}, \ldots, |u_{nN-1}|^{p-2}u_{N-1n}\right)\mathcal{A}(u_{11}, \ldots, u_{N-1n})^T \\
&= p\sum_{k=1}^{n} d_k\mathcal{Q}_k,
\end{aligned}$$

where $\mathcal{Q}_k$ is the following expression:

$$\begin{aligned}
&\sum_{i=1}^{N-1}(\lambda-2)\,|u_{ik}|^p + \sum_{i=1}^{N-2}\left(\alpha_i\,|u_{i+1k}|^{p-2}\,u_{i+1k}u_{ik} + \alpha_i^{-1}\,|u_{ik}|^{p-2}\,u_{i+1k}u_{ik}\right) \\
&= \sum_{i=1}^{N-1}(\lambda-2)\,|u_{ik}|^p + \sum_{i=1}^{N-2}\frac{\alpha_i}{\beta_i}\left(|u_{i+1k}|^{p-2}\,u_{i+1k}(\beta_i u_{ik}) + |\beta_i u_{ik}|^{p-2}\,u_{i+1k}(\beta_i u_{ik})\right) \\
&\leq \sum_{i=1}^{N-1}(\lambda-2)\,|u_{ik}|^p + \sum_{i=1}^{N-2}\frac{\alpha_i}{\beta_i}\left(|u_{i+1k}|^p + |\beta_i u_{ik}|^p\right) \\
&= \sum_{i=1}^{N-1}(\lambda-2)\,|u_{ik}|^p + \sum_{i=1}^{N-2}\frac{\alpha_i}{\beta_i}\,|u_{i+1k}|^p + \alpha_i\beta_i^{p-1}\,|u_{ik}|^p \\
&= \sum_{i=1}^{N-1}(\lambda-2)\,|u_{ik}|^p + \sum_{i=1}^{N-2}\frac{p_i}{p_{i+1}}\,|u_{i+1k}|^p + \frac{p_{i+1}}{p_i}\,|u_{ik}|^p \\
&= |u_{1k}|^p\left(\lambda - 2 + \frac{p_2}{p_1}\right) + \cdots + |u_{N-1k}|^p\left(\lambda - 2 + \frac{p_{N-2}}{p_{N-1}}\right),
\end{aligned}$$

and this last term vanishes by Equation (3.21). $\square$

Proof of Proposition 10. *Let $\mathcal{K}$ be as defined in (3.18) and for $w = (w_1^T, \ldots, w_{N-1}^T)^T$, let*

$$J(w,t) = diag\left(J_F(w_1,t), \ldots, J_F(w_{N-1},t)\right).$$

By subadditivity of μ, and Lemma 15, for any $1 \leq p \leq \infty$,

$$\begin{aligned}
&\mu_{p,Q_p\otimes Q}\left[J(w,t) - \mathcal{K}\otimes D(t)\right]\\
&\leq \mu_{p,Q_p\otimes Q}\left[J(w,t) - \lambda_2 I\otimes D(t)\right] + \mu_{p,Q_p\otimes Q}\left[\lambda_2 I\otimes D(t) - \mathcal{K}\otimes D(t)\right]\\
&\leq \mu_{p,Q_p\otimes Q}\left[J(w,t) - \lambda_2 I\otimes D(t)\right]\\
&\leq \max_i \left\{\mu_{p,Q}\left[J_F(w,t) - \lambda_2 D(t)\right]\right\}.
\end{aligned}$$

The last inequality holds by Lemma 4. Note that Q_p does not appear in the last equation. Now by taking sup *over all $w = (w_1^T, \ldots, w_{N-1}^T)^T$ and all $t \geq 0$, we get*

$$\sup_{(w,t)} \mu_{p,Q_p\otimes Q}\left[J(w,t) - \mathcal{K}\otimes D(t)\right] \;\leq\; \sup_t \sup_{x\in\mathbb{R}^n} \mu_{p,Q}\left[J_F(x,t) - \lambda_2 D(t)\right]. \tag{3.22}$$

Now by applying Theorem 11, we obtain the desired inequality, Equation (3.17). □

Remark 15. Under the conditions of Proposition 10, the following inequality holds:

$$\sum_{i=1}^{N-1} \|e_i(t)\|_{p,Q} \;\leq\; \alpha e^{ct} \sum_{i=1}^{N-1} \|e_i(0)\|_{p,Q},$$

where $\alpha = \frac{\max_k\{(Q_p)_k\}}{\min_k\{(Q_p)_k\}}(N-1)^{1-1/p} > 0$, and $(Q_p)_k$ is the kth diagonal entry of Q_p.

Proof. Using Equation (3.17) and the following inequality for L^p norms, $p \geq 1$, on $\mathbb{R}^{N-1}$:

$$\|\cdot\|_p \leq \|\cdot\|_1 \leq (N-1)^{1-1/p}\|\cdot\|_p\,, \tag{3.23}$$

we conclude the desired result. □

Complete graphs

Consider a $\mathcal{G}-$compartment system with an undirected complete graph $\mathcal{G}$. The following system of ODEs describes the evolution of the interconnected agents x_i's:

$$\dot{x}_i \;=\; F(x_i,t) + D(t)\sum_{j=1}^{N}(x_j - x_i)\,. \tag{3.24}$$

Proposition 11. *Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$ with corresponding logarithmic norm μ, $x = (x_1^T, \ldots, x_N^T)^T$ be a solution of Equation (3.24), and*

$$c := \sup_{(x,t)} \mu[J_F(x,t) - ND(t)] \ .$$

Then

$$\sum_{k=1}^{m} \|e_k(t)\| \ \leq \ e^{ct} \sum_{k=1}^{m} \|e_k(0)\| \ , \tag{3.25}$$

where the e_k's, for $k = 1, \ldots, m$ are the edges of $\mathcal{G}$, meaning the differences $x_{i_k}(t) - x_{j_k}(t)$ for $i_k < j_k \in \{1, \ldots, N\}$.

Proof. Let E be an incidence matrix of $\mathcal{G}$. We first show that $E^T E E^T = N E^T$. For any orientation of $\mathcal{G}$, E^T is an $\binom{N}{2} \times N$ matrix such that its $i-$th row looks like $(\epsilon_{i1}, \ldots, \epsilon_{iN})$, where for exactly one j, $\epsilon_{ij} = 1$, for exactly one j, $\epsilon_{ij} = -1$, and for the rest of j's, $\epsilon_{ij} = 0$. Observe that for any row i, $\sum_j \epsilon_{ij} = 0$, and

$$\left(E^T \mathcal{L}\right)_{(ij)} = \left(E^T\right)_{r_i} (\mathcal{L})_{c_j} \ ,$$

where $(A)_{(ij)}$ denotes the $(i,j)-$th entry of matrix A, and $(A)_{r_i}$ and $(A)_{c_i}$ denote the ith row and ith column of A, respectively. Hence,

$$\begin{aligned} \left(E^T \mathcal{L}\right)_{(ij)} &= (\epsilon_{i1}, \ldots, \epsilon_{iN}) \begin{pmatrix} -1 \\ \vdots \\ N-1 \\ \vdots \\ -1 \end{pmatrix} \begin{matrix} \\ \\ \leftarrow j^{th} \\ \\ \\ \end{matrix} \\ &= -\epsilon_{i1} - \cdots + (N-1)\epsilon_{ij} - \cdots - \epsilon_{iN} \\ &= N\epsilon_{ij} - \sum_k \epsilon_{ik} = N\epsilon_{ij}. \end{aligned}$$

This proves $E^T \mathcal{L} = E^T E E^T = N E^T$.

Thus we may apply Theorem 11 with $K = NI$. Then $\mathcal{J} = J(w,t) - K \otimes D(t)$ can be written as follows:

$$\mathcal{J} = \operatorname{diag}\left(J_F(w_1,t) - ND(t), \ldots, J_F(w_m,t) - ND(t)\right).$$

For $u = (u_1^T, \ldots, u_m^T)^T$, with $u_i \in \mathbb{R}^n$, let $\|u\|_* := \left\| (\|u_1\|, \ldots, \|u_m\|\|)^T \right\|_1$, where $\|\cdot\|_1$ is L^1 norm on $\mathbb{R}^m$, and let μ_* be the logarithmic norm induced by $\|\cdot\|_*$. Then by the definition of μ_* and Lemma 4,

$$\mu_*[J(w,t) - K \otimes D(t)] \leq \max_i \{\mu[J_F(w_i,t) - ND(t)]\}.$$

By taking sup over all possible w's in both sides of the above inequality, we get:

$$\sup_w \mu_*[J(w,t) - K \otimes D(t)] \leq \sup_{(x,t)} \mu[J_F(x,t) - ND(t)] = c.$$

Applying Theorem 11, we conclude (3.25). □

Star graphs

Consider a $\mathcal{G}-$compartment system, where $\mathcal{G}$ is a star graph with $N+1$ nodes. The following system of ODEs describes the evolution of the whole system:

$$\begin{aligned} \dot{x}_i &= F(x_i,t) + D(t)\,(x_0 - x_i)\,,\ i = 1, \ldots, N \\ \dot{x}_0 &= F(x_0,t) + D(t) \sum_{i \neq 0} (x_i - x_0)\,. \end{aligned} \tag{3.26}$$

Proposition 12. *Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$ with corresponding logarithmic norm μ, $x = (x_0^T, x_1^T, \ldots, x_N^T)^T$ be a solution of Equation (3.26), and*

$$c := \sup_{(x,t)} \mu[J_F(x,t) - D(t)].$$

Then for any $i \in \{1, \ldots, N\}$,

$$\|(x_i - x_0)(t)\| \ \leq \ (1 + \alpha_i t) e^{ct} \ \|(x_i - x_0)(0)\|\,, \tag{3.27}$$

where $\alpha_i = \sum_{j \neq i,0} \|(x_j - x_i)(0)\|$.

Proof. Using (3.26), for all $i, j = 1, \ldots, N$,

$$\dot{x}_i - \dot{x}_j = (F(x_i,t) - D(t)x_i) - (F(x_j,t) - D(t)x_j)\,.$$

Applying Lemma 5, we get

$$\|(x_i - x_j)(t)\| \ \leq \ e^{ct} \ \|(x_i - x_j)(0)\|. \tag{3.28}$$

For any $i = 1, \ldots, N$, we have:

$$\begin{aligned} \dot{x}_i - \dot{x}_0 &= F(x_i,t) - F(x_0,t) - D(t)\left((x_i - x_0) - \sum_{j=1}^{N}(x_j - x_0)\right) \\ &= F(x_i,t) - F(x_0,t) - D(t)(N+1)(x_i - x_0) - D(t)\sum_{j=1}^{N}(x_j - x_i)\,. \end{aligned}$$

(In line 2, we added and subtracted $ND(t)x_i$.)

Now using the Dini derivative of $\|x_i - x_0\|$ and using the upper bound for $\|x_i - x_j\|$ derived in (3.28), we get:

$$D^+\|(x_i - x_0)(t)\| \;\leq\; \tilde{c}\|(x_i - x_0)(t)\| + \alpha_i e^{ct},$$

where, $\alpha_i = \sum_{j\neq i,0}\|(x_j - x_i)(0)\|$ and by subadditivity of μ,

$$\begin{aligned} \tilde{c} &:= \sup_x \mu[J_F(x,t) - (N+1)D(t)] \\ &\leq \sup_x \mu[J_F(x,t) - D(t)] + \sup_x \mu[-ND(t)] \\ &\leq \sup_{(x,t)} \mu[J_F(x,t) - D(t)] = c \quad \text{since } \mu[-ND(t)] < 0. \end{aligned}$$

Applying Gronwall's Lemma (Lemma 9) to the above inequality, we get Equation (3.27).

□

Observe that, as a consequence, when $c < 0$, we have synchronization, i.e., for any $i \in \{1, \ldots, N\}$, $(x_i - x_0)(t) \to 0$, as $t \to \infty$.

Corollary 4. *Under the conditions of Proposition 12, the following inequality holds:*

$$\sum_{i\neq 0}\|(x_i - x_0)(t)\| \;\leq\; Pe^{ct}\sum_{i\neq 0}\|(x_i - x_0)(0)\|\,, \tag{3.29}$$

where $P = 1 + 2(N-1)\;t\;\sum_{i\neq 0}\|(x_i - x_0)(0)\|$.

Proof. For any $i \neq 0$, using the triangle inequality, we have

$$\begin{aligned} \alpha_i = \sum_{j\neq i,0}\|(x_j - x_i)(0)\| &\leq \sum_{j\neq i,0}\|(x_j - x_0)(0)\| + \sum_{j\neq i,0}\|(x_i - x_0)(0)\| \\ &= \sum_{j\neq i,0}\|(x_j - x_0)(0)\| + (N-1)\|(x_i - x_0)(0)\|, \end{aligned}$$

taking sum over all $i \neq 0$, we get

$$\sum_{i\neq 0} \alpha_i \ \leq \ (N-1)\sum_{j\neq 0} \|(x_j - x_0)(0)\| + (N-1)\sum_{i\neq 0} \|(x_i - x_0)(0)\|.$$

Therefore, since the α_i's are nonnegative, for any i,

$$1 + \alpha_i t \ \leq \ 1 + t\sum_{i\neq 0} \alpha_i \leq 1 + 2(N-1)\ t\ \sum_{i\neq 0} \|(x_i - x_0)(0)\| := P\ ,$$

and hence, using Equation (3.27),

$$\|(x_i - x_0)(t)\| \leq Pe^{ct}\|(x_i - x_0)(t)\|.$$

Now taking sum over all $i \neq 0$, we obtain Equation (3.29) as we wanted. □

Cartesian products

Consider a network with $N_1 \times N_2$ compartments that are connected to each other by a 2-D, $N_1 \times N_2$ lattice (grid) graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where

$$\mathcal{V} = \{x_{ij},\ i = 1, \ldots, N_1,\ j = 1, \ldots, N_2\}\,,$$

is the set of all vertices and $\mathcal{E}$ is the set of all edges of $\mathcal{G}$.

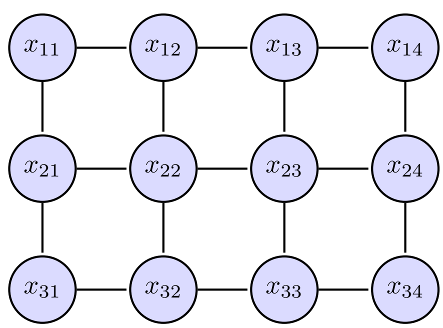


Figure 3.4: An example of a lattice graph: 3×4 nodes

The following system of ODEs describes the evolution of the x_{ij}'s: for any $i = 1, \ldots, N_1$, and $j = 1, \ldots, N_2$

$$\dot{x}_{ij} \ = \ F(x_{ij}, t) + D(t)\,(x_{i-1j} - 2x_{ij} + x_{i+1j}) + D(t)\,(x_{ij-1} - 2x_{ij} + x_{ij+1})\,, \qquad (3.30)$$

assuming Neumann boundary conditions, i.e., $x_{i0} = x_{i1}$, $x_{iN_2} = x_{iN_2+1}$, etc.

Proposition 13. *Let $x = \{x_{ij}\}$ be a solution of Equation (3.30) and $c = \max\{c_1, c_2\}$, where for $i = 1, 2$,*

$$c_i := \sup_{(x,t)} \mu_{p,Q} \left[J_F(x,t) - 4\sin^2(\pi/2N_i)\, D(t) \right],$$

and $1 \le p \le \infty$. Then, there exist a positive constant $\alpha \ge 1$, and a positive function of time, $\beta(t)$, such that

$$\sum_{e\in\mathcal{E}} \|e(t)\|_{p,Q} \;\le\; (\alpha + \beta(t)t)\, e^{ct} \sum_{e\in\mathcal{E}} \|e(0)\|_{p,Q}. \tag{3.31}$$

In particular, when $c < 0$, the system (3.30) synchronizes, i.e., $\forall i, j, k, l$, as $t \to \infty$, $(x_{ij} - x_{kl})(t) \to 0$, exponentially.

Proof. For $i = 1, \ldots, N_1$, let $x_i = \left(x_{i1}^T, \ldots, x_{iN_2}^T\right)^T$, and assume that x_i's are diffusively interconnected by a path graph of N_1 nodes.

For ease of notation, we assume that for $i = 1, \ldots, N_1$, $\mathcal{E}_{(i)}$ is the set of all edges in the compartment i, i.e., all the edges in each row in Figure 3.4. We let $\mathcal{E}_h = \bigcup_{i=1}^{N_1} \mathcal{E}_{(i)}$ denote all the horizontal edges in $\mathcal{G}$. Also we assume that for $i = 1, \ldots, N_1$, $\mathcal{E}^{(i)}$ is the set of all edges that connect the compartment i to the other compartments. In addition, we let $\mathcal{E}_v = \bigcup_{i=1}^{N_1} \mathcal{E}^{(i)}$ denote all the vertical edges in $\mathcal{G}$.

For each $i = 1, \ldots, N_1$, and fixed t, let

$$G(x_i, t) := \tilde{F}(x_i, t) - \mathcal{L}_2 \otimes D(t) x_i,$$

where $\mathcal{L}_2$ is the Laplacian matrix of the path graph of N_2 nodes; and $\tilde{F}(x_i, t) = \left(F(x_{i1}, t)^T, \ldots, F(x_{iN_2}, t)^T\right)^T$. We can think of G as the reaction operator that acts in each compartment x_i. Then the system (3.30) can be written as:

$$\begin{aligned}
\dot{x}_1 &= G(x_1, t) + (I_{N_2} \otimes D(t))\,(x_2 - x_1)\\
\dot{x}_2 &= G(x_2, t) + (I_{N_2} \otimes D(t))\,(x_1 - 2x_2 + x_3)\\
&\;\;\vdots\\
\dot{x}_{N_1} &= G(x_{N_1}, t) + (I_{N_2} \otimes D(t))\,(x_{N_1-1} - x_{N_1}).
\end{aligned}$$

By Remark 15, if for $1 \le p \le \infty$, c_1 is defined as follows

$$c_1 = \sup_{(x,t)} \mu_{p, I_{N_2}\otimes Q} \left[J_G(x,t) - 4\sin^2(\pi/2N_1)\,(I_{N_2} \otimes D(t)) \right],$$

then,

$$\sum_{e\in\mathcal{E}_v} \|e(t)\|_{p,Q} \;\le\; \alpha_1\, e^{c_1 t} \sum_{e\in\mathcal{E}_v} \|e(0)\|_{p,Q}\,, \tag{3.32}$$

where

$$\alpha_1 = \max_k \{\sin(k\pi/N_1)\} \,/\, \min_k \{\sin(k\pi/N_1)\}\,(N_1-1)^{1-1/p}.$$

By Lemma 4, for any p,

$$\begin{aligned} c_1 &= \sup_{(x,t)} \mu_{p,I_{N_2}\otimes Q}\left[J_G(x,t) - 4\sin^2(\pi/2N_1)\,(I_{N_2}\otimes D(t))\right] \\ &\le \sup_{(x,t)} \mu_{p,Q}\left[J_F(x,t) - 4\sin^2(\pi/2N_1)\,D(t)\right] \le c. \end{aligned} \tag{3.33}$$

Therefore, using Equations (3.32) and (3.33), we have

$$\sum_{e\in\mathcal{E}_v} \|e(t)\|_{p,Q} \;\le\; \alpha_1\, e^{ct} \sum_{e\in\mathcal{E}_v} \|e(0)\|_{p,Q}\,. \tag{3.34}$$

Now let's look at each compartment x_i which contains N_2 sub-compartment that are connected by a path graph. For example, for $i=1$:

$$\begin{aligned} \dot{x}_{11} &= F(x_{11},t) + D(t)(x_{12} - x_{11} + x_{21} - x_{11}) \\ \dot{x}_{12} &= F(x_{12},t) + D(t)(x_{11} - 2x_{12} + x_{13} + x_{22} - x_{12}) \\ &\;\;\vdots \\ \dot{x}_{1N_2} &= F(x_{1N_2},t) + D(t)(x_{1N_2-1} - x_{1N_2} + x_{2N_2} - x_{1N_2}). \end{aligned}$$

Let $u := \left(x_{11}^T,\ldots,x_{1N_2-1}^T\right)^T$, $v := \left(x_{12}^T,\ldots,x_{1N_2}^T\right)^T$, and for any fixed t, define $\tilde{G}$ as follows:

$$\tilde{G}(u,t) := \begin{pmatrix} F(x_{11},t) \\ F(x_{12},t) \\ \vdots \\ F(x_{1N_2-1},t) \end{pmatrix} - \mathcal{K}\otimes D(t) \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1N_2-1} \end{pmatrix},$$

where $\mathcal{K}$ is as defined in (3.18). Then

$$\dot{u} - \dot{v} \;=\; \tilde{G}(u,t) - \tilde{G}(v,t) + \begin{pmatrix} (x_{21} - x_{11}) - (x_{22} - x_{12}) \\ \vdots \\ (x_{2N_2-1} - x_{1N_2-1}) - (x_{2N_2} - x_{1N_2}) \end{pmatrix} \otimes D(t).$$

Using the Dini derivative, for any p, and Q_p as defined in Proposition 10, we have: (for ease of the notation let $\|\cdot\| := \|\cdot\|_{p,Q_p\otimes Q}$.)

$$\begin{aligned}
D^+\|(u-v)(t)\| &= \limsup_{h\to 0^+}\frac{1}{h}\ \left(\|(u-v)(t+h)\| - \|(u-v)(t)\|\right) \\
&= \limsup_{h\to 0^+}\frac{1}{h}\left(\|(u-v+h(\dot{u}-\dot{v}))(t)\| - \|(u-v)(t)\|\right) \\
&\leq \lim_{h\to 0^+}\frac{1}{h}\left(\|(u-v)(t)+h(\tilde{G}(u,t)-\tilde{G}(v,t))\| - \|(u-v)(t)\|\right) \\
&\quad + \left\|\begin{pmatrix}(x_{21}-x_{11})-(x_{22}-x_{12})\\ \vdots \\ (x_{2N_2-1}-x_{1N_2-1})-(x_{2N_2}-x_{1N_2})\end{pmatrix}\otimes D(t)\right\| \\
&\leq \sup_{(w,t)}\mu_{p,P\otimes Q}\left[J_{\tilde{G}}(w,t)\right]\|(u-v)(t)\| \\
&\quad + \left\|\begin{pmatrix}(x_{21}-x_{11})-(x_{22}-x_{12})\\ \vdots \\ (x_{2N_2-1}-x_{1N_2-1})-(x_{2N_2}-x_{1N_2})\end{pmatrix}\otimes D(t)\right\|.
\end{aligned}$$

Note that the last term is the difference between some of the vertical edges of $\mathcal{G}$. Therefore by Equation (3.34), and the triangle inequality, we can approximate the last term as follows:

$$\left\|\begin{pmatrix}(x_{21}-x_{11})-(x_{22}-x_{12})\\ \vdots \\ (x_{2N_2-1}-x_{1N_2-1})-(x_{2N_2}-x_{1N_2})\end{pmatrix}\otimes D(t)\right\|_{p,Q_p\otimes Q} \leq 2d(t)a\alpha_1\sum_{e\in\mathcal{E}^{(1)}}\|e(t)\|_{p,Q},$$

where $a = \max_i\{(Q_p)_i\}$, $d(t) = \max\{d_1(t),\ldots,d_n(t)\}$, and $\mathcal{E}^{(1)}$ is the set of edges of $\mathcal{G}$ which connect the compartment x_1 to the compartment x_2.

By Equation (3.22), for any $1\leq p\leq\infty$,

$$\sup_{(u,t)}\mu_{p,Q_p\otimes Q}\left[J_{\tilde{G}}(u,t)\right] \leq \sup_{(x,t)}\mu_{p,Q}\left[J_F(x,t) - 4\sin^2\left(\pi/2N_2\right)D(t)\right] \leq c.$$

Therefore for x_1, we have:

$$D^+ \sum_{e\in\mathcal{E}_{(1)}} \|\phi_e e(t)\|_{p,Q} \leq c \sum_{e\in\mathcal{E}_{(1)}} \|\phi_e e(t)\|_{p,Q} + 2d(t)\ a\ \alpha_1 \sum_{e\in\mathcal{E}^{(1)}} \|e(t)\|_{p,Q}\,,$$

where $\phi_e = (Q_p)_k$, when $e = e_k$ is the k-th edge of the N_2-path graph.

Repeating the same process for other compartments, $x_2, \ldots, x_{N_1}$, and adding them up, we get the following inequality

$$\begin{aligned} D^+ \sum_{e\in\mathcal{E}_h} \|\phi_e e(t)\|_{p,Q} &\leq c \sum_{e\in\mathcal{E}_h} \|\phi_e e(t)\|_{p,Q} + 2\times 2d(t)\ a\ \alpha_1 \sum_{e\in\mathcal{E}_v} \|e(t)\|_{p,Q} \\ &\leq c \sum_{e\in\mathcal{E}_h} \|\phi_e e(t)\|_{p,Q} + 4d(t)\ a\ \alpha_1 e^{ct} \sum_{e\in\mathcal{E}^{(1)}} \|e(0)\|_{p,Q}\,. \end{aligned}$$

Note that in the first inequality, the coefficient 2 appears because each edge e that connects the ith compartment to the jth compartment is counted twice: once when we do the process for x_i and once when we do it for x_j.

Applying Gronwall's inequality allows us to conclude:

$$\begin{aligned} \sum_{e\in\mathcal{E}_h} \|\phi_e e(t)\|_{p,Q} &\leq e^{ct} \sum_{e\in\mathcal{E}_h} \|\phi_e e(0)\|_{p,Q} + 4d(t)\ a\ \alpha_1 t e^{ct} \sum_{e\in\mathcal{E}_v} \|e(0)\|_{p,Q} \\ &\leq e^{ct} \sum_{e\in\mathcal{E}_h} \|\phi_e e(0)\|_{p,Q} + 4d(t)\ a\ \alpha_1 t e^{ct} \sum_{e\in\mathcal{E}} \|e(0)\|_{p,Q}\,. \end{aligned}$$

Now using Equation (3.23) and the following inequalities:

$$\min_k \{(Q_p)_k\} \|e(t)\|_{p,Q} \leq \|\phi_e e(t)\|_{p,Q}\,, \quad \|\phi_e e(0)\|_{p,Q} \leq \max_k \{(Q_p)_k\} \|e(0)\|_{p,Q}\,,$$

we get

$$\sum_{e\in\mathcal{E}_h} \|e(t)\|_{p,Q} \leq \alpha_2 e^{ct} \sum_{e\in\mathcal{E}_h} \|e(0)\|_{p,Q} + \beta(t) t e^{ct} \sum_{e\in\mathcal{E}} \|e(0)\|_{p,Q}\,. \tag{3.35}$$

where

$$\alpha_2 = \frac{\max_k \{(Q_p)_k\}}{\min_k \{(Q_p)_k\}} (N_2 - 1)^{1-1/p}, \quad \beta(t) = \frac{4d(t)\ a\ \alpha_1}{\alpha_2}.$$

Let $\alpha = \max\{\alpha_1, \alpha_2\}$, then Equations (3.34) and (3.35), imply (3.31). □

Remark 16. We proved Proposition 13 for two path graphs. One can generalize the result of Proposition 13 for K ($K \geq 2$) arbitrary graphs. A sketch of proof is as follows. For $k = 1, \ldots, K$, let $\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k)$ be an arbitrary graph, with $|\mathcal{V}_k| = N_k$ and Laplacian matrix $\mathcal{L}_{\mathcal{G}_k}$. Consider a system of $N = \Pi_{k=1}^K N_k$ compartments $x_{i_1,\ldots,i_K} \in \mathbb{R}^n$, for

$i_j = 1, \ldots, N_j$, which are interconnected by $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_K$, where $\times$ denotes the Cartesian product. The following system of ODEs describes the evolution of the $x_{i_1,\ldots,i_K}$'s:

$$\dot{x} \; = \; \tilde{F}(x,t) - (\mathcal{L} \otimes D(t))\, x, \tag{3.36}$$

where $x = (x_{i_1,\ldots,i_K})$ is the vector of all N compartments, $\tilde{F}(x,t) = (F(x_{i_1,\ldots,i_K}, t))$, and $\mathcal{L} = \sum_i I_{N_1} \otimes \cdots \otimes \mathcal{L}_{\mathcal{G}_i} \otimes \cdots \otimes I_{N_K}$. Recall that $\lambda_2(\mathcal{G}) = \min\{\lambda_2(\mathcal{G}_1), \ldots, \lambda_2(\mathcal{G}_K)\}$. (See Section 3.1.)

Given graphs $\mathcal{G}_k$, $k = 1, \ldots, K$ as above, suppose that for each k, there are a norm $\|\cdot\|_{(k)}$ on $\mathbb{R}^n$, a real nonnegative number $\lambda_{(k)}$, and a polynomial $P_{(k)}(z,t)$ on $\mathbb{R}^2_{\geq 0}$, with the property that for each z, $P_{(k)}(z,0) \geq 1$, such that for any solution x of (3.36),

$$\sum_{e \in \mathcal{E}_k} \|e(t)\|_{(k)} \; \leq \; P_{(k)}\left(\sum_{e \in \mathcal{E}_k} \|e(0)\|_{(k)}, t\right) \, e^{c_k t} \sum_{e \in \mathcal{E}_k} \|e(0)\|_{(k)}, \tag{3.37}$$

holds, where $c_k := \sup_{(x,t)} \mu_{(k)} \left[J_F(x,t) - \lambda_{(k)} D(t)\right]$, and $\mu_{(k)}$ is the logarithmic norm induced by $\|\cdot\|_{(k)}$. Then for any norm $\|\cdot\|$ on $\mathbb{R}^n$, there exists a polynomial $P(z,t)$ on $\mathbb{R}^2_{\geq 0}$, with the property that for each z, $P(z,0) \geq 1$, such that

$$\sum_{e \in \mathcal{E}} \|e(t)\| \;\; \leq \;\; P\left(\sum_{e \in \mathcal{E}} \|e(0)\|, t\right) \, e^{ct} \sum_{e \in \mathcal{E}} \|e(0)\|,$$

where $c := \max\{c_1, \ldots, c_K\}$, and $\mathcal{E}$ is the set of the edges of $\mathcal{G}$. Observe that if all $c_i < 0$, then also $c < 0$, and this guarantees synchronization, i.e., $e(t) \to 0$ as $t \to \infty$. The proof of this result is by induction on the number of graphs K and is similar to the proof of Proposition 13 but the notations are very involved.

Note that for $K = 1$, Remark 15, Proposition 11, and Corollary 4 show that (3.37) holds when $\mathcal{G}_k$ is a path, complete or star graph, for $P_{(k)}(z,t) = \alpha, 1, 1 + 2(N-1)tz$, respectively. Therefore, for a hypercube (cartesian product of K path graphs) with $N_1 \times \cdots \times N_K$ nodes, if for some p, $1 \leq p \leq \infty$, and some positive diagonal matrix Q, and $\lambda_2 = 4 \min_i \left\{\sin^2(\pi/2N_i)\right\}$, $\sup_{(x,t)} \mu_{p,Q} \left[J_F(x,t) - \lambda_2 D(t)\right] < 0$, then the system synchronizes. For a Rook (cartesian product of K complete graphs) with $N_1 \times \cdots \times N_K$ nodes, if for any given norm, and $\lambda_2 = \min_i \{N_i\}$, $\sup_{(x,t)} \mu \left[J_F(x,t) - \lambda_2 D(t)\right] < 0$, then the system synchronizes.

Trees

The following remark for the L^2 case is already known for constant diffusion D, see [42], but we show here how it follows from Theorem 11 as a special case and for time varying diffusion $D(t)$.

Remark 17. *Consider a $\mathcal{G}-$compartment system, $(F, \mathcal{G}, D)$, where $\mathcal{G}$ is a tree and denote*

$$c \;:=\; \sup_{(w,t)} \mu_{2,Q}\left[J_F(w,t) - \lambda_2 D(t)\right],$$

for a positive diagonal matrix Q. Then

$$\left\|\left(E^T \otimes I\right) x(t)\right\|_{2,I\otimes Q} \;\;\leq\;\; e^{ct} \left\|\left(E^T \otimes I\right) x(0)\right\|_{2,I\otimes Q},$$

where I is the identity matrix of appropriate size and E is a directed incidence matrix of $\mathcal{G}$.

Proof. Let $\mathcal{K} = E^T E$ and $J = \operatorname{diag}\left(J_F(w_1,t), \ldots, J_F(w_m,t)\right)$, where m is the number of edges of $\mathcal{G}$. By subadditivity of μ, for fixed $w = (w_1^T, \ldots, w_m^T)^T$ and t, we have:

$$\mu_{2,I\otimes Q}\left[J - \mathcal{K}\otimes D(t)\right] \leq \mu_{2,I\otimes Q}\left[J - \lambda_2 I \otimes D(t)\right] + \mu_{2,I\otimes Q}\left[\lambda_2 I \otimes D(t) - \mathcal{K}\otimes D(t)\right]. \tag{3.38}$$

We first show that the second term of the right hand side of the above inequality is zero. By Lemma 11, λ_2 is the smallest eigenvalue of the edge Laplacian, $E^T E$, so the largest eigenvalue of $\lambda_2 I - \mathcal{K}$ and hence $(\lambda_2 I - \mathcal{K}) \otimes D(t)$ is zero. Therefore,

$$\begin{aligned}
\mu_{2,I\otimes Q}[(\lambda_2 I - \mathcal{K}) \otimes D(t)] \;&=\; \mu_2\left[(I\otimes Q)\left((\lambda_2 I - \mathcal{K}) \otimes D(t)\right)\left(I \otimes Q^{-1}\right)\right] \\
&=\; \mu_2\left[(\lambda_2 I - \mathcal{K}) \otimes D(t)\right] \\
&=\; \text{largest eigenvalue of } (\lambda_2 I - \mathcal{K}) \otimes D(t) = 0,
\end{aligned}$$

(since $(\lambda_2 I - \mathcal{K}) \otimes D(t)$ is symmetric, $\mu_2\left[(\lambda_2 I - \mathcal{K}) \otimes D(t)\right]$ is equal to the largest eigenvalue of $(\lambda_2 I - \mathcal{K}) \otimes D(t)$). Next, we will show that the first term of the right hand side of Equation (3.38) is $\leq c$. By Lemma 4,

$$\mu_{2,I\otimes Q}\left[J - \lambda_2 I \otimes D(t)\right] \leq \max_i \left\{\mu_{2,Q}\left[J_F(w_i,t) - \lambda_2 D(t)\right]\right\},$$

where $J = J(w,t)$. By taking sup over all $t \geq 0$ and w, we get

$$\sup_{(w,t)} \mu_{2,I\otimes Q}\left[J(w,t) - \mathcal{K}\otimes D(t)\right] \quad \leq \quad \sup_{t}\sup_{x\in\mathbb{R}^n} \mu_{2,Q}\left[J_F(x,t) - \lambda_2 D(t)\right] = c.$$

Now by applying Theorem 11, we obtain the desired inequality. □

3.3.3 Examples

We discuss here two examples that illustrate the power of our estimates.

A biomolecular reaction

We revisit the biochemical example described in Section 2.4. As we showed there, the following set of ODEs describes the system.

$$\begin{aligned} \dot{x} &= z(t) - \delta x + k_1 y - k_2(S_Y - y)x \\ \dot{y} &= -k_1 y + k_2(S_Y - y)x, \end{aligned} \tag{3.39}$$

where $(x(t), y(t)) \in V = [0,\infty)\times[0, S_Y]$ for all $t \geq 0$ (V is convex and forward-invariant), and S_Y, k_1, k_2, and δ are arbitrary positive constants.

It was shown in [64] that this system entrains to the external signal $z(t)$, and therefore, even for isolated systems, we will see synchronization behavior. We show next how to obtain estimates on how the speed of synchronization improves under diffusion.

Figure 3.5 shows the solutions of the system (3.39) for 6 different initial conditions (6 identical compartments with dynamics described by the system (3.39)) for periodic function $z(t) = 20(1 + \sin(10t))$, and the following set of parameters:

$$\delta = 20,\ k_1 = 0.5,\ k_2 = 5,\ S_Y = 0.1.$$

As it is clear from the figure, all the solutions converge to a periodic solution; in other words, the system (3.39) synchronizes.

In what follows, we first show that one cannot apply Theorem 17 to show synchronous behavior of (3.39). Then, we show how to justify the synchronous behavior of the solutions of the system (3.39) by applying Proposition 8.

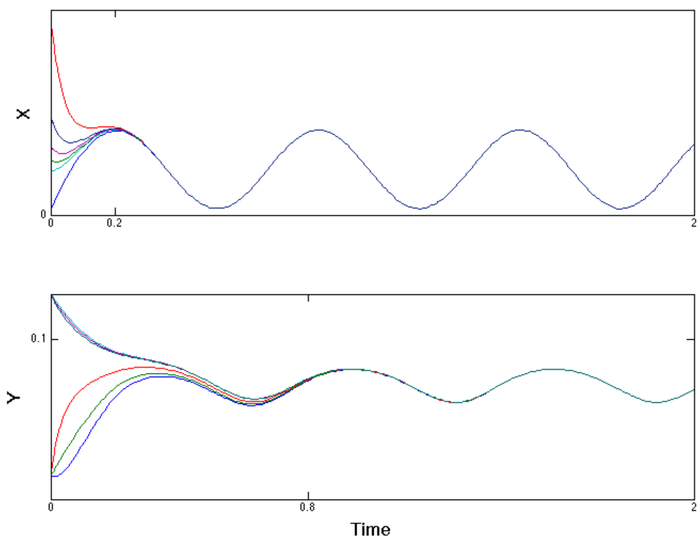


Figure 3.5: Biochemical example: 6 isolated compartments

Let J_{F_t} be the Jacobian of $F_t(x,y) := (z(t)-\delta x+k_1 y-k_2(S_Y-y)x, -k_1 y+k_2(S_Y-y)x)^T$:

$$J_{F_t}(x,y) \;=\; \begin{pmatrix} -\delta - k_2(S_Y-y) & k_1+k_2x \\ k_2(S_Y-y) & -(k_1+k_2x) \end{pmatrix}.$$

In Section 2.4, it has been shown that for any $p>1$, and any positive diagonal Q,

$$c = \sup_t \sup_{(x,y)\in V} \mu_{p,Q}[J_{F_t}(x,y)] \geq 0\,.$$

Here, we will show that not only $c \geq 0$, but

$$\sup_t \sup_{(x,y)\in V} \mu_{2,Q}[J_{F_t}(x,y) - \lambda D] \geq 0, \tag{3.40}$$

for any positive diagonal matrix Q, any $\lambda > 0$, and any constant diffusion matrix $D = \mathrm{diag}\,(d_1, d_2)$:

Without loss of generality we assume $Q = \mathrm{diag}\,(1, q)$. Then

$$QJ_{F_t}(x,y)Q^{-1} \;=\; \begin{pmatrix} -\delta - a & \dfrac{b}{q} \\ aq & -b \end{pmatrix},$$

where $a = k_2(S_Y - y) \in [0, k_2S_Y]$ and $b = k_1 + k_2x \in [k_1, \infty)$. By definition of $\mu_{2,Q}$, we know that, $\mu_{2,Q}\left[J_{F_t}(x,y) - \lambda D\right] \;=\; \lambda_{\max}(R)$, where $\lambda_{\max}(R)$ denotes the largest

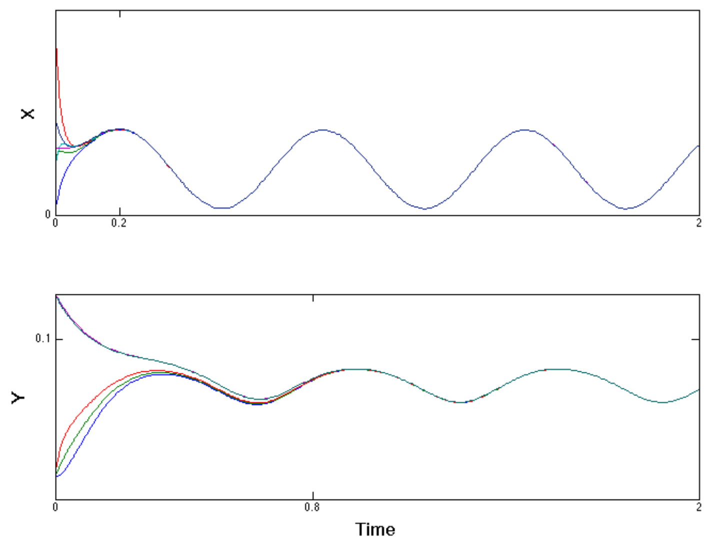


Figure 3.6: Biochemical example: 6 compartments interconnected with a path graph with strength constant $d_1 \neq 0$, and $d_2 = 0$ (note the faster synchronization when there is diffusion)

eigenvalue of

$$R := \frac{1}{2}\left(Q(J_{F_t}(x,y) - \lambda D)Q^{-1} + \left(Q(J_{F_t}(x,y) - \lambda D)Q^{-1}\right)^T\right).$$

A simple calculation shows that the eigenvalues of R are as follows:

$$\lambda_{\pm} \;=\; -(\delta + a + b + (d_1 + d_2)\lambda) \pm \Delta,$$

where

$$\Delta = \sqrt{((d + a + d_1\lambda) - (b + d_2\lambda))^2 + (aq + b/q)^2}\,.$$

We can pick $x = x^*$ large enough (i.e., b large enough) and $y = y^* = S_Y$ (i.e., $a = 0$), such that $\lambda_+ > 0$ and hence $\mu_{2,Q}\left[J_{F_t}(x^*, y^*) - \lambda D\right] > 0$. Therefore, (3.10) doesn't hold and one cannot apply the existing result in L^2 norms, [42], to justify the synchronous behavior of the solutions of the system (3.39). But on the other hand, In [64], it has been shown that $\sup_t \sup_{(x,y)\in V} \mu_{1,Q}[J_{F_t}(x,y)] < 0$, for some non-identity, positive diagonal matrix Q. Therefore, by Proposition 8, the system (3.39) synchronizes.

Figure 3.6 shows the solutions of the system (3.39) for the same initial conditions and parameters as when the x's of the 6 compartments are connected to each other by a path graph with strength constant $d_1 \neq 0$. Observe that in this case the system synchronizes faster than when the compartments are isolated.

Synchronous autonomous oscillators

We consider the following three-dimensional system (all variables are non-negative and all coefficients are positive):

$$\begin{aligned} \dot{x} &= \frac{a}{k+z} - bx \\ \dot{y} &= \alpha x - \beta y \\ \dot{z} &= \gamma y - \frac{\delta z}{k_M + z}\ , \end{aligned} \tag{3.41}$$

where x, y, and z are functions of t.

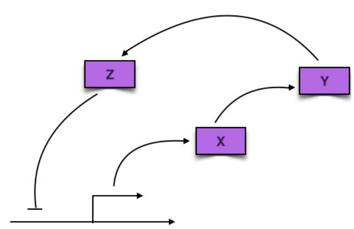
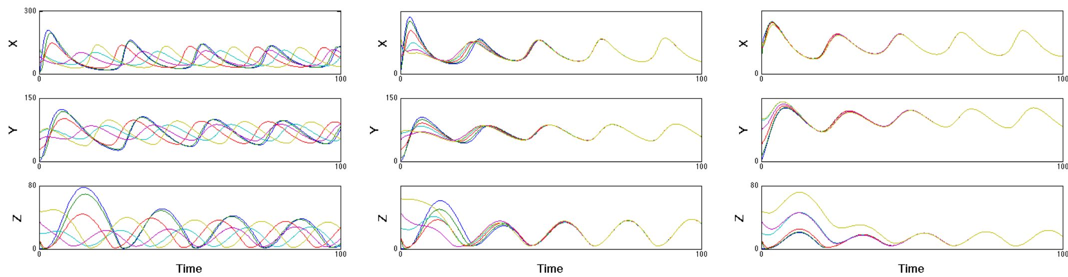


Figure 3.7: The Goodwin autoregulation model

This system is a variation of a model ([75]), often called in mathematical biology the "Goodwin model", that was proposed in order to describe a generic model of an oscillating autoregulated gene, and its oscillatory behavior has been well-studied [76]. In Goodwin's original formulation, as is sketched in Figure 3.7, X is the mRNA transcribed from a given gene, Y an enzyme translated from this mRNA, and Z a metabolite whose production is catalyzed by Y. It is assumed that Z, in turn, can inhibits the expression of the original gene. However, many other interpretations are possible. Figure 3.8a shows non-synchronized oscillatory solutions of (3.41) for 6 different initial conditions, using the following parameter values from the textbook [77]:

$$a = 150,\ k = 1,\ b = \alpha = \beta = \gamma = 0.2,\ \delta = 15,\ K_M = 1.$$

Figure 3.8b shows the solutions of the same system (6 compartments, with the same initial conditions as in Figure 3.8a) that are now interconnected diffusively by a path



(a) 6 isolated compartments (b) Linear interconnection (c) Complete interconnection

graph in which only X diffuses, that is, $D = \operatorname{diag}(d, 0, 0)$. The following system of ODEs describes the evolution of the full system: (in all equations, $i = 1, \ldots, N$):

$$
\begin{aligned}
\dot{x}_i &= \frac{a}{k + z_i} - b\, x_i \; + d\,(x_{i-1} - 2x_i + x_{i+1}) \\
\dot{y}_i &= \alpha\, x_i - \beta\, y_i \\
\dot{z}_i &= \gamma\, y_i - \frac{\delta z_i}{k_M + z_i}\ ,
\end{aligned}
$$

where for convenience we are writing $x_0 = x_1$ and $x_N = x_{N+1}$.

Figure 3.8c shows the solutions of the same system (6 compartments with the same initial conditions as in Figure 3.8a) that are now interconnected, with the same D as in Figure 3.8b, by a complete graph. Observe that the second and "more connected" graph structure (reflected, as discussed in the magnitude of its second Laplacian eigenvalue), leads to much faster synchronization.

Let us now compute, using our theory, for what values of d, the system synchronizes. For this end, we need to compute $\sup_{(x,t)} \mu_{1,Q}[J_F(x,t) - \lambda_2 D]$ for $Q = \operatorname{diag}(1, 12, 11)$. It is easy to see that $Q(J_F - \lambda_2 D)Q^{-1}$ is equal to:

$$
\begin{pmatrix}
-0.2 - \lambda_2 d & 0 & \frac{-150/11}{(1+z)^2} \\
(0.2)(12) & -0.2 & 0 \\
0 & (0.2)(11/12) & \frac{-15}{(1+z)^2}
\end{pmatrix}.
$$

A calculation shows that $\sup_z \mu_1\left[Q(J_F(z) - \lambda_2 D)Q^{-1}\right] < 0$, when $2.2 < \lambda_2 d$. For instance, in a complete graph with 6 nodes (Figure 3.8c), $d > \frac{2.2}{6}$ guarantees synchronization.

3.3.4 Comparison with other synchronization conditions

Master stability function (MSF)

In order to study the synchronous behavior of $\dot{x} = F(x) + \sigma\mathcal{L} \otimes H(x)$, where σ is the coupling strength, $\mathcal{L}$ is the Laplacian of the interconnected graph and H is used for coupling (in our case, $\sigma H(x) = D(t)x$), one can transform the stability of the synchronization manifold $x_1 = \cdots = x_N$, into the following master stability equation

$$\dot{\xi} = (DF + (\alpha + \beta i)DH)\xi \ , \tag{3.42}$$

where $\alpha + \beta i$ is an eigenvalue of $\sigma\mathcal{L}$, [21, 78, 79]. One can write the maximum Floquet or Lyapunov exponents $\lambda_{\max}$ of Equation (3.42) as a function of α and β. The signs of the various numbers $\lambda_{\max}$ at the points $\alpha + \beta i$ reveal the stability of Equation (3.42). If for all the eigenvalues of $\mathcal{L}$, $\lambda_{\max}$ is negative, then the system synchronizes.

- The MSF approach provides local conditions for synchronization, while contraction theory provides global conditions.
- The condition in MSF depends on all the eigenvalues of the interconnected graph, while our condition depends only on one eigenvalue, λ_2.
- Our approach is effective for autonomous and non-autonomous systems.
- In the MSF approach, the conditions need to be checked numerically, while we prove our results analytically.

See also [45] for more details about the two approaches (contraction and MSF) to study synchronization.

A matrix measure approach using L^1 and L^∞ norms

In [47], the author studies the system (3.6) for a weighted and time varying matrix $\mathcal{L}$ but restricted to a time invariant reaction operator $F = F(x)$ (it seems that the result can be generalized to time varying reaction operator $F = F(x, t)$). In order to compare with the result of the current work, in the following theorem, we only mention the result of [47] for unweighted and time invariant Laplacian and $D(t) = dI$, and matrix measure induced by L^1 and L^∞ norms.

Theorem 12. *Let $X_{1j} = x_j - x_1$ and $A = diag(a_1, \ldots, a_n)$ with $a_i \geq 0$. For $\mathcal{L} = (l_{ij})$, let $S = dS_1$, where S_1 is defined as follows:*

$$S_1 = \begin{pmatrix} -\sum_{j=1}^{N} l_{2j} - l_{12} & l_{23} - l_{13} & \cdots & l_{2N} - l_{1N} \\ l_{32} - l_{12} & -\sum_{j=1}^{N} l_{3j} - l_{13} & \cdots & l_{3N} - l_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ l_{N2} - l_{12} & l_{N3} - l_{13} & \cdots & -\sum_{j=1}^{N} l_{Nj} - l_{1N} \end{pmatrix}.$$

Assume that

1. *for $j = 2, \ldots, N$,*

$$\dot{X}_{1j} = \left[\int_0^1 J_F(sx_j + (1-s)x_j)ds - A\right] X_{1j} \ ,$$

is globally stabilized in the sense of a Lyapunov function $V_{1j} = \frac{1}{2} X_{1j}^T X_{1j}$.

2. *for $p = 1, \infty$, and $a = \max\{a_1, \ldots, a_n\} \geq 0$*

$$2a + \mu_p \left[S + S^T\right] < 0 \ .$$

Then $\lim_{t\to\infty}(x_j - x_1)(t) = 0$, i.e., the system (3.6) synchronizes.

Now let $\mathcal{G}$ be a path graph with $N = 4$ nodes and $F(x) = x$. Then for $D = dI$, S would be as follows:

$$S = \begin{pmatrix} -3d & d & 0 \\ 0 & -2d & d \\ d & d & -d \end{pmatrix}.$$

A simple calculation shows that $\mu_1[S + S^T] = \mu_\infty[S + S^T] = d$. Therefore, the second condition of Theorem 12 is not satisfied for any $d > 0$ and one cannot apply the result of [47]. By Proposition 10, if $\mu_1[J_F - 4\sin^2(\pi/8)dI] < 0$, where $4\sin^2(\pi/8)$ is the second eigenvalue of the Laplacian of a path graph with 4 nodes, then the system synchronizes. Note that for this example,

$$J_F - 4\sin^2(\pi/8)dI = \left(1 - 4\sin^2(\pi/8)d\right) I.$$

Therefore, $\mu_1[J_F - 4\sin^2(\pi/8)dI] = 1 - 4\sin^2(\pi/8)d < 0$ when $d > \frac{1}{4\sin^2(\pi/8)} \approx 1.7$.

A matrix measure approach using an arbitrary norm

The paper [63] presents a contraction-based network small-gain theorem which has some relation to the results given here. In that result, a given "global" partitioned matrix $A_\mathrm{G} \in \mathbb{R}^{N\times N}$ is given, where $N = n_1 + n_2 + \cdots + n_k$:

$$A_\mathrm{G} \; = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ A_{21} & A_{22} & \dots & A_{2k} \\ \vdots & \vdots & \dots & \vdots \\ A_{k1} & A_{k2} & \dots & A_{kk} \end{pmatrix},$$

as well as a set of "local" norms

$$|\xi_i|_{\mathrm{L},i} \text{ on } \mathbb{R}^{n_i},\; i = 1,\dots,k,$$

and one introduces the induced norms of interconnections, as well as the measures of each subsystem, as follows:

$$\rho_{ij} := \sup_{|x|_{\mathrm{L},j}=1} |A_{ij}x|_{\mathrm{L},i}\;, \quad \mu_i := \mu_i(A_{ii}),$$

as well as a "structure matrix" that encodes all these numbers:

$$A_\mathrm{S} \; := \begin{pmatrix} \mu_1 & \rho_{12} & \cdots & \rho_{1k} \\ \rho_{21} & \mu_2 & \cdots & \rho_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k1} & \rho_{k2} & \cdots & \mu_k \end{pmatrix}.$$

Figure 3.9 shows a schematic of the interconnection and the quantities in question. The main theorem in [63] states that, given any monotone ("interconnection" or "structure") norm $|x|_\mathrm{S}$ on $\mathbb{R}^k$, and defining a "global" norm by:

$$|\xi|_\mathrm{G} \; := \left|\left(|\xi_1|_{\mathrm{L},1}\,, \dots, |\xi_k|_{\mathrm{L},k}\right)^\mathrm{T}\right|_\mathrm{S} \text{ on } \mathbb{R}^{n_1+n_2+\cdots+n_k},$$

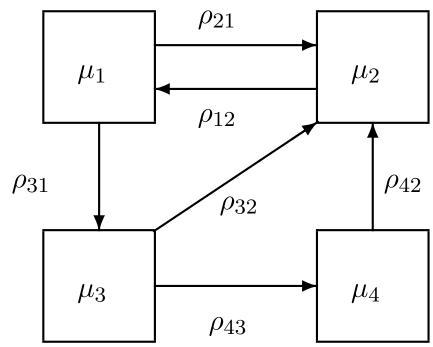


Figure 3.9: An interconnection of four subsystems

then

$$\mu_{\mathrm{G}}[A_{\mathrm{G}}] \;\leq\; \mu_{\mathrm{S}}[A_{\mathrm{S}}].$$

(The theorem is applied to nonlinear systems by considering all possible Jacobians.)

The main objective of [63] was to apply this result to networks of dynamical systems, allowing one to show global stability, and even contraction, of interconnected systems, based only estimates on upper bounds on norms of interconnections as well as on "certificates" given by upper bounds on matrix measures of the Jacobians of each component. In principle, this result applies, in particular, to diffusive interconnections: just take local systems equal to each other (and with the same local norms), let the off-diagonal terms in the global matrix be obtained from the diffusion terms (i.e., $A_{ij} = D$ for all $i \neq j$), and adjust the diagonal terms by subtracting D. However, this theorem is in essence a small-gain theorem, and as such is too conservative compared to our results in this work, even for linear systems. To see this, let us consider a diffusive interconnection of two identical linear systems with dynamics $F(x) = -Dx$, where $D = \mathrm{diag}\,(1, 3)$, (observe that $\mu_1[J_F] = -1$ and hence the system is contractive)

$$\begin{aligned} \dot{x}_1 &= F(x_1) + D(x_2 - x_1) \\ \dot{x}_2 &= F(x_2) + D(x_1 - x_2), \end{aligned}$$

which gives

$$A_{\mathrm{G}} \;=\; \begin{pmatrix} -2D & D \\ D & -2D \end{pmatrix}.$$

Thus, for any given local norm, we have

$$A_{\mathrm{S}} := \begin{pmatrix} \mu[-2D] & \|D\| \\ \|D\| & \mu[-2D] \end{pmatrix}.$$

Note that $\mu_1[A_{\mathrm{G}}] = -1 < 0$. In what follows, we show that for any structure norm $\|\cdot\|_S$, $\mu_S[A_S] > 0$, which implies that one cannot apply the result of [63] to conclude $\mu_1[A_{\mathrm{G}}] < 0$. Since $\mu[-2D] \geq \lambda_{\max}(-2D)$ (where $\lambda_{\max}(A)$ indicates the largest eigenvalue of A), and $\lambda_{\max}(-2D) = -2$, we have that $\alpha := \mu[-2D] \geq -2$. Also, $\|D\| = \max\{d_1, d_2\} = 3$. Therefore, for any norm $\|\cdot\|_S$, we have that $\mu_S[A_S] \geq \lambda_{\max}(A_S) = \alpha + 3 \geq 1$.

By Proposition 10, if $\sup_x \mu[J_F(x) - 2D] < 0$ (where 2 is the second eigenvalue of a path graph of two nodes), then the interconnected system synchronizes. A simple calculation shows that $\sup_x \mu_1[J_F(x) - 2D] = \mu_1[-3D] = -3 < 0$.

3.3.5 Synchronization of diffusively-connected ODEs: weighted graphs

In this section, we study the generalizations of Theorem 11, Proposition 10, and Proposition 11 for identical systems which are connected through an undirected weighted graph. Let $\mathcal{G}_w$ (the subscript w refers to weighted graph) be a weighted graph, with time varying weight matrix $W(t) = (w_{ij}(t))$ where for each $i, j \in \{1, \ldots, N\}$, $w_{ij} \colon [0, \infty) \to \mathbb{R}_{\geq 0}$ is a continuous function of t. For $i = 1, \ldots, N$, the following system describes the evolution of the x_i's.

$$\dot{x}_i(t) \;=\; F(x_i(t), t) \;+ \sum_{j \in \mathcal{N}(i)} w_{ij}(t) D(t)(x_j - x_i)(t),$$

where the ith subsystem (or "agent") has state $x_i(t)$, the weight matrix $W(t) = (w_{ij}(t))$, provides the adjacency structure, the indices in $\mathcal{N}(i)$ represent the "neighbors" of the ith subsystem in this graph, and F and D are as defined in (3.6). We also assume that for each pair (i, j), $i \in \mathcal{N}_j \iff j \in \mathcal{N}_i$ and $w_{ij} = w_{ji}$. Recall that $w_{ij} = w_{ji} = 0$ when x_i and x_j are not connected or $i = j$, and $w_{ij}(t) > 0$ when x_i and x_j are connected by an edge.

Definition 10. *For any arbitrary undirected weighted graph $\mathcal{G}_w$ with weight matrix $W = (w_{ij})$ and the associated (graph) Laplacian matrix $\mathcal{L}_w$, any diagonal matrix $D(t)$,* *and any* $F: V \times [0, \infty) \to \mathbb{R}^n$*, the associated* $\mathcal{G}_w-$*compartment system, denoted by* $(F, \mathcal{G}_w, D)$*, is defined by*

$$\dot{x}(t) = \tilde{F}(x(t), t) - (\mathcal{L}_w(t) \otimes D(t))x(t), \tag{3.43}$$

where $x, \tilde{F}$*, and* D *are as defined in (3.6) and* $\mathcal{L}_w$ *is as defined in Section 3.1.*

Theorem 13. *Consider a* $\mathcal{G}_w-$*compartment system,* $(F, \mathcal{G}_w, D)$*, where* $\mathcal{G}_w$ *is an arbitrary undirected, weighted graph with* N *nodes and* m *edges. For an arbitrary orientation of* $\mathcal{G}_w$*, let* E_w *be a directed incidence matrix of* $\mathcal{G}_w$*, and pick any* $m \times m$ *matrix* $K_w(t)$ *satisfying*

$$E_w(t)^T \mathcal{L}_w(t) = K_w(t) E_w(t)^T . \tag{3.44}$$

Denote:

$$c \ := \ \sup_{(z,t)} \mu \left[J_w(z,t) - K_w(t) \otimes D(t) \right], \tag{3.45}$$

where μ *is the logarithmic norm induced by an arbitrary norm on* $\mathbb{R}^{mn}$*,* $\|\cdot\|$*, and for* $z = \left(z_1^T, \ldots, z_m^T\right)^T$*,* $J_w(z,t)$ *is defined as follows.*

$$J_w(z,t) = \sqrt{\mathcal{W}(t)}\,diag\left(J_F(z_1,t), \ldots, J_F(z_m,t)\right),$$

where $\mathcal{W}$ *is defined as (3.1) and* $\sqrt{\mathcal{W}} = diag(\sqrt{\omega_1}, \ldots, \sqrt{\omega_m})$*. Then*

$$\left\| \left(E_w(t)^T \otimes I\right) x(t) \right\| \ \leq \ e^{ct} \left\| \left(E_w(0)^T \otimes I\right) x(0) \right\| .$$

Proof. Assume that x is a solution of $\dot{x} = \tilde{F}(x,t) - (\mathcal{L}_w(t) \otimes D(t))\,x$. Let's define y as follows. For any t,

$$y(t) := \left(E_w(t)^T \otimes I\right) x(t).$$

Note that for $k = 1, \ldots, m$, the kth entry of $\left(E_w(t)^T \otimes I\right) x(t)$ is $\sqrt{\omega_k(t)}(x_{i_k} - x_{j_k})$ which indicates the kth edge of $\mathcal{G}_w$, i.e., the difference between states associated to the two nodes that constitute the edge, and I is the $n \times n$ identity matrix. Then, using the Kronecker product identity $(A \otimes B)(C \otimes D) = AC \otimes BD$, for matrices A, B, C, and D of appropriate dimensions, we have:

$$\begin{aligned}
\dot{y} \ &= \ \left(E_w(t)^T \otimes I\right) \dot{x} \\
&= \ \left(E_w(t)^T \otimes I\right) \left(\tilde{F}(x,t) - (\mathcal{L}_w(t) \otimes D(t))\,x\right) \\
&= \ \left(E_w(t)^T \otimes I\right) \tilde{F}(x,t) - \left(E_w(t)^T \mathcal{L}_w(t) \otimes D(t)\right) x
\end{aligned}$$

$$
\begin{aligned}
&= \left(E_w(t)^T \otimes I\right) \tilde{F}(x,t) - \left(K_w(t)E_w(t)^T \otimes D(t)\right) x \\
&= \left(E_w(t)^T \otimes I\right) \tilde{F}(x,t) - \left(K_w(t) \otimes D(t)\right) \left(E_w(t)^T \otimes I\right) x \\
&= \left(E_w(t)^T \otimes I\right) \tilde{F}(x,t) - \left(K_w(t) \otimes D(t)\right) y,
\end{aligned}
$$

where for $i = 1, \ldots, m$, $\left(E_w(t)^T \otimes I\right) \tilde{F}(x,t)$ can be written as follows:

$$\sqrt{\mathcal{W}(t)}\,\mathrm{diag}\left(F(x_{i_1},t) - F(x_{j_1},t), \ldots, F(x_{i_m},t) - F(x_{j_m},t)\right).$$

Now let $u(t) = \sqrt{\mathcal{W}(t)}(x_{i_1}, \ldots, x_{i_m})^T$, $v(t) = \sqrt{\mathcal{W}(t)}(x_{j_1}, \ldots, x_{j_m})^T$, and for any t, G_t be as follows:

$$G_t(u) := \sqrt{\mathcal{W}(t)} \begin{pmatrix} F(x_{i_1},t) \\ \vdots \\ F(x_{i_m},t) \end{pmatrix} - \left(K_w(t) \otimes D(t)\right) \sqrt{\mathcal{W}(t)} \begin{pmatrix} x_{i_1} \\ \vdots \\ x_{i_m} \end{pmatrix},$$

then $\dot{u} - \dot{v} = G_t(u) - G_t(v)$. By Lemma 5 and Remark 10,

$$\|u(t) - v(t)\| \;\le\; e^{ct}\|u(0) - v(0)\|,$$

where $c = \sup_{(x,t)} \mu\left[J_{G_t}(x)\right] = \sup_{(x,t)} \mu\left[J_w(x,t) - K_w(t) \otimes D(t)\right]$ and for any fixed t, $u(t) - v(t) = \left(E_w(t)^T \otimes I\right) x(t)$. $\square$

Remark 18. For each t, $\left(E_w(t)^T \otimes I\right) x$ is an m column vector whose entries are the differences $\sqrt{\omega_k(t)}(x_{i_k} - x_{j_k})(t)$, for each weighted edge $e_k = \{i_k, j_k\}$ in $\mathcal{G}_w$. Therefore, if there exists an $a > 0$, such that $\omega_k(t) > a$ for all $t \ge 0$ and every k, and if $c < 0$, then the system synchronizes.

In Proposition 14 below, we will see the application of Theorem 13 to weighted path graphs.

Consider a system of N compartments, $x_1, \ldots, x_N$, that are connected to each other by a weighted path graph $\mathcal{G}_w$. Assuming $x_0 = x_1,\ x_{N+1} = x_N$, the following system of ODEs describes the evolution of the individual agent x_i, for $i = 1, \ldots, N$:

$$\dot{x}_i \;=\; F(x_i,t) + D(t)\left(\omega_{i-1}(x_{i-1} - x_i) + \omega_i(x_{i+1} - x_i)\right). \tag{3.46}$$

Note that the ω_i's are positive constants (and independent of t). The following result is a generalization of Proposition 10 to weighted path graphs.

Proposition 14. *Let $(x_1^T, \ldots, x_N^T)^T$ be a solution of (3.46), and for $1 \leq p \leq \infty$ and a positive diagonal matrix Q, let*

$$c = \sup_{(x,t)} \mu_{p,Q} \left[\sqrt{\mathcal{W}} J_F(x,t) - \lambda D(t) \right], \tag{3.47}$$

where $\sqrt{\mathcal{W}} = diag\left(\sqrt{\omega_1}, \ldots, \sqrt{\omega_{N-1}}\right)$, and λ is the smallest nonzero eigenvalue of the associated graph Laplacian $\mathcal{L}_w$. Then

$$\|e(t)\|_{p,Q_p \otimes Q} \ \leq \ e^{ct} \|e(0)\|_{p,Q_p \otimes Q}, \tag{3.48}$$

where for any t, $e(t) = \left(\sqrt{\omega_1}(x_1 - x_2)^T(t), \ldots, \sqrt{\omega_{N-1}}(x_{N-1} - x_N)^T(t)\right)^T$ denotes the vector of all edges of the path graph, and Q_p is defined as follows:

$$Q_p(t) = diag\left(p_1^{\frac{2-p}{p}}, \ldots, p_{N-1}^{\frac{2-p}{p}}\right),$$

and $(p_1, \ldots, p_{N-1})$ is a positive eigenvector associated to $-\lambda$ for $-\mathcal{K}_w = -E_w^T E_w$, the weighted edge Laplacian, i.e.,

$$(p_1, \ldots, p_{N-1})\,(-\mathcal{K}_w) = -\lambda\,(p_1, \ldots, p_{N-1})\,.$$

Note that $Q_\infty = diag\,(1/p_1, \ldots, 1/p_{N-1})$.

Note that for $\mathcal{W} = I$ (unweighted path graph), we saw in the previous section that $\lambda = 4\sin^2(\pi/2N)$, and for $1 \leq k \leq N-1$, $p_k = \sin(k\pi/N)$.

The proof of Proposition 14, which is similar to the proof of Proposition 10, is as follows. For a path graph with N nodes, consider the following $N \times N-1$ directed incidence matrix E_w and the $N-1 \times N-1$ weighted edge Laplacian $\mathcal{K}_w := {E_w}^T E_w$:

$$E_w = \begin{pmatrix} -\sqrt{\omega_1} & & \\ \sqrt{\omega_2} & \ddots & \\ & \ddots & -\sqrt{\omega_{N-2}} \\ & & \sqrt{\omega_{N-1}} \end{pmatrix},$$

$$\mathcal{K}_w = \begin{pmatrix} 2\omega_1 & -\sqrt{\omega_1\omega_2} & & \\ -\sqrt{\omega_1\omega_2} & 2\omega_2 & -\sqrt{\omega_2\omega_3} & \\ & \ddots & \ddots & \\ & & -\sqrt{\omega_{N-2}\omega_{N-1}} & 2\omega_{N-1} \end{pmatrix}. \tag{3.49}$$

Since $-\mathcal{K}_w$ is a Metzler and irreducible matrix, it follows by the Perron-Frobenius Theorem (see the Appendix) that it has a positive eigenvector $(p_1, \ldots, p_{N-1})$ corresponding to $-\lambda$, the largest eigenvalue of $-\mathcal{K}_w$, (λ is the smallest eigenvalue of $\mathcal{K}_w$), i.e.,

$$(p_1, \ldots, p_{N-1})\,(-\mathcal{K}_w) = -\lambda\,(p_1, \ldots, p_{N-1})\,. \tag{3.50}$$

To prove Proposition 14, we first prove the following Lemma which is a generalization of Lemma 15.

Lemma 16. *Let $\mathcal{K}_w$ be the weighted edge Laplacian of a path graph with $N \geq 3$ nodes as shown in (3.49). Then for any $1 \leq p \leq \infty$,*

$$\mu_{p,Q_p\otimes Q}\left[\lambda I \otimes D(t) - \mathcal{K}_w \otimes D(t)\right] \leq 0, \tag{3.51}$$

where Q and Q_p are as in Proposition 14.

Proof. To prove (3.51), we will show that for any t, $\mu_p[\mathcal{A}_w(t)] \leq 0$, where $\mathcal{A}_w(t)$ is defined as follows:

$$\mathcal{A}_w(t) = (Q_p \otimes Q)\,(\lambda I \otimes D(t) - \mathcal{K}_w \otimes D(t))\left(Q_p^{-1} \otimes Q^{-1}\right).$$

We fix $t \geq 0$ and first show that for $p = 1$, $\mu_p[\mathcal{A}_w(t)] = 0$. A simple calculation shows that, for $p = 1$, $\mathcal{A}_w$ is as follows:

$$\begin{pmatrix} (\lambda - 2\omega_1) & \frac{p_1}{p_2}\sqrt{\omega_1\omega_2} & & \\ \frac{p_2}{p_1}\sqrt{\omega_1\omega_2} & (\lambda - 2\omega_2) & \frac{p_2}{p_3}\sqrt{\omega_2\omega_3} & \\ & \ddots & \ddots & \\ & & & \\ & & \frac{p_{N-1}}{p_{N-2}}\sqrt{\omega_{N-2}\omega_{N-1}} & (\lambda - 2\omega_{N-1}) \end{pmatrix} \otimes D(t),$$

For $\mathbf{1} = (1, \ldots, 1)^T$, and $p = 1$, since $\mathbf{1}^T Q_p = (p_1, \ldots, p_{N-1})$, it follows by Equation (3.50) that $\mathbf{1}^T Q_p(-\mathcal{K}_w)Q_p^{-1} = -\lambda \mathbf{1}^T$, therefore,

$$-2\omega_1 + \frac{p_2}{p_1}\sqrt{\omega_1\omega_2} = -2\omega_2 + \frac{p_1}{p_2}\sqrt{\omega_1\omega_2} + \frac{p_3}{p_2}\sqrt{\omega_2\omega_3} = \cdots = -\lambda. \tag{3.52}$$

Hence, by the definition of μ_1, $\mu_1[A] = \max_j \left(a_{jj} + \sum_{i\neq j} |a_{ij}|\right)$, and because $D(t)$ is diagonal, for any fixed t, $\mu_1[\mathcal{A}_w(t)] = 0$.

Now we show that for any fixed t, $\mu_\infty[\mathcal{A}_w(t)] = 0$. A simple calculation shows that, for $p = \infty$, since $Q_\infty = \text{diag}\ (1/p_1, \dots, 1/p_{N-1})$,

$$A_w(t) = \begin{pmatrix} (\lambda - 2\omega_1) & \frac{p_2}{p_1}\sqrt{\omega_1\omega_2} & & \\ \frac{p_1}{p_2}\sqrt{\omega_1\omega_2} & (\lambda - 2\omega_2) & \frac{p_3}{p_2}\sqrt{\omega_2\omega_3} & \\ & \ddots & \ddots & \\ & & & \\ & & \frac{p_{N-2}}{p_{N-1}}\sqrt{\omega_{N-2}\omega_{N-1}} & (\lambda - 2\omega_{N-1}) \end{pmatrix} \otimes D(t),$$

Therefore, by the definition of μ_∞, $\mu_\infty[A] = \max_i \left(a_{ii} + \sum_{i\neq j} |a_{ij}|\right)$, and because $D(t)$ is diagonal, $\mu_\infty[\mathcal{A}_w(t)] = 0$.

Next we show for $1 < p < \infty$, $\mu_p[\mathcal{A}_w(t)] \leq 0$. Note that for a fixed t, $\mathcal{A}_w(t)$ can be written as follows:

$$\begin{pmatrix} \lambda - 2\omega_1 & \alpha_1^{-1}\sqrt{\omega_1\omega_2} & & \\ \alpha_1\sqrt{\omega_1\omega_2} & \lambda - 2\omega_2 & \alpha_2^{-1}\sqrt{\omega_2\omega_3} & \\ & \ddots & \ddots & \\ & & & \\ & & \alpha_{N-2}\sqrt{\omega_{N-2}\omega_{N-1}} & \lambda - 2\omega_{N-1} \end{pmatrix} \otimes D(t),$$

where $\alpha_i = \left(\frac{p_{i+1}}{p_i}\right)^{\frac{2-p}{p}}$. To show $\mu_p[\mathcal{A}_w(t)] \leq 0$, using Lemma 3 and the definition of μ, it suffices to show that $D^+\|u\|_p \leq 0$, where $u = (u_{11}, \dots, u_{1n}, \dots, u_{N-11}, \dots, u_{N-1n})^T$ is the solution of $\dot{u} = \mathcal{A}_w u$, or equivalently, $\frac{d\Phi}{dt}(u(t)) \leq 0$, where $\Phi(t) = \|u(t)\|_p^p$. In the calculations below, we use the following simple fact (which is proved in the Appendix). For any real α and β and $1 \leq p$:

$$\left(|\alpha|^{p-2} + |\beta|^{p-2}\right)\alpha\beta \leq |\alpha|^p + |\beta|^p.$$

In the calculations below, we let $\beta_i = \alpha_i^{\frac{2}{2-p}}$. We also use the fact that $|x|^p$ is differentiable for $p > 1$ and

$$\frac{d\Phi}{du_i} = \frac{d}{du_i}|u_i|^p = p|u_i|^{p-1}\frac{u_i}{|u_i|} = p|u_i|^{p-2}u_i.$$

Observe that

$$\frac{d\Phi}{dt}(u(t)) = \sum_{i,k} \frac{d\Phi}{du_{ik}}\frac{du_{ik}}{dt} = \nabla\Phi \cdot \dot{u} = \nabla\Phi \cdot \mathcal{A}_w u$$

$$
\begin{aligned}
&= p\left(|u_{11}|^{p-2}u_{11},\ldots,|u_{nN-1}|^{p-2}u_{N-1n}\right)\mathcal{A}_w(u_{11},\ldots,u_{N-1n})^T \\
&= p\sum_{k=1}^{n} d_k\mathcal{Q}_k,
\end{aligned}
$$

where $\mathcal{Q}_k$ is the following expression:

$$
\begin{aligned}
&\sum_{i=1}^{N-1}(\lambda-2\omega_i)|u_{ik}|^p\sum_{i=1}^{N-2}\sqrt{\omega_i\omega_{i+1}}\left(\alpha_i\,|u_{i+1k}|^{p-2}\,u_{i+1k}u_{ik}+\alpha_i^{-1}\,|u_{ik}|^{p-2}\,u_{i+1k}u_{ik}\right) \\
&= \sum_{i=1}^{N-1}(\lambda-2\omega_i)|u_{ik}|^p\sum_{i=1}^{N-2}\sqrt{\omega_i\omega_{i+1}}\frac{\alpha_i}{\beta_i}\left(|u_{i+1k}|^{p-2}\,u_{i+1k}(\beta_i u_{ik})+|\beta_i u_{ik}|^{p-2}\,u_{i+1k}(\beta_i u_{ik})\right) \\
&\leq \sum_{i=1}^{N-1}(\lambda-2\omega_i)|u_{ik}|^p+\sum_{i=1}^{N-2}\sqrt{\omega_i\omega_{i+1}}\frac{\alpha_i}{\beta_i}\left(|u_{i+1k}|^p+|\beta_i u_{ik}|^p\right) \\
&= \sum_{i=1}^{N-1}(\lambda-2\omega_i)|u_{ik}|^p+\sum_{i=1}^{N-2}\sqrt{\omega_i\omega_{i+1}}\left(\frac{\alpha_i}{\beta_i}|u_{i+1k}|^p+\alpha_i\beta_i^{p-1}|u_{ik}|^p\right) \\
&= \sum_{i=1}^{N-1}(\lambda-2\omega_i)|u_{ik}|^p+\sum_{i=1}^{N-2}\sqrt{\omega_i\omega_{i+1}}\left(\frac{p_i}{p_{i+1}}|u_{i+1k}|^p+\frac{p_{i+1}}{p_i}|u_{ik}|^p\right) \\
&= |u_{1k}|^p\left(\lambda-2\omega_1+\frac{p_2}{p_1}\sqrt{\omega_1\omega_2}\right)+\cdots+|u_{N-1k}|^p\left(\lambda-2\omega_{N-1}+\frac{p_{N-2}}{p_{N-1}}\sqrt{\omega_{N-2}\omega_{N-1}}\right),
\end{aligned}
$$

and this last term vanishes by Equation (3.52). $\square$

Proof of Proposition 14. *Let $\mathcal{K}_w$ be as defined in (3.49) and for $u=\left(u_1^T,\ldots,u_{N-1}^T\right)^T$, let*

$$J_w(u,t)=\sqrt{\mathcal{W}}diag\left(J_F(u_1,t),\ldots,J_F(u_{N-1},t)\right).$$

By subadditivity of μ, and Lemma 16, for any $1\leq p\leq\infty$,

$$
\begin{aligned}
&\mu_{p,Q_p\otimes Q}\left[J_w(u,t)-\mathcal{K}_w\otimes D(t)\right] \\
&\leq \mu_{p,Q_p\otimes Q}\left[J_w(u,t)-\lambda I\otimes D(t)\right]+\mu_{p,Q_p\otimes Q}\left[\lambda I\otimes D(t)-\mathcal{K}_w\otimes D(t)\right] \\
&\leq \mu_{p,Q_p\otimes Q}\left[J_w(u,t)-\lambda I\otimes D(t)\right] \\
&\leq \max_i\left\{\mu_{p,Q}\left[\sqrt{\mathcal{W}}J_F(u_i,t)-\lambda D(t)\right]\right\}.
\end{aligned}
$$

The last inequality holds by Lemma 4. Note that Q_p does not appear in the last equation. Now by taking sup *over all $u=\left(u_1^T,\ldots,u_{N-1}^T\right)^T$ and all $t\geq 0$, we get*

$$\sup_{(u,t)}\mu_{p,Q_p\otimes Q}\left[J_w(u,t)-\mathcal{K}_w\otimes D(t)\right]\leq\sup_t\sup_{x\in\mathbb{R}^n}\mu_{p,Q}\left[\sqrt{\mathcal{W}}J_F(x,t)-\lambda D(t)\right].$$

Now by applying Theorem 13, we obtain the desired inequality, Equation (3.48). $\square$

We remark that, at least for certain graphs, one can recover the L^2 result from [42] as a corollary of Theorem 13.

Remark 19. *Consider a $\mathcal{G}_w$−compartment system, $(F, \mathcal{G}_w, D)$, where $\mathcal{G}_w$ is an undirected weighted tree with weight $\mathcal{W} = diag\,(\omega_1, \ldots, \omega_{N-1})$ and denote*

$$c \;:=\; \sup_{(x,t)} \mu_{2,Q}\left[\sqrt{\mathcal{W}} J_F(x,t) - \lambda D(t)\right],$$

where λ is the smallest nonzero eigenvalue of the Laplacian of $\mathcal{G}_w$ and Q is a positive diagonal matrix. Then

$$\left\|\left({E_w}^T \otimes I\right) x(t)\right\|_{2,I\otimes Q} \;\;\leq\;\; e^{ct}\left\|\left({E_w}^T \otimes I\right) x(0)\right\|_{2,I\otimes Q}.$$

where I is the identity matrix of appropriate size and E_w is a weighted incidence matrix of $\mathcal{G}_w$.

Proof. Let $\mathcal{K}_w = E_w^T E_w$ and J_w be as defined in Theorem 13. By subadditivity of μ,

$$\begin{aligned}\mu_{2,I\otimes Q}\left[J_w(x,t) - \mathcal{K}_w \otimes D(t)\right] \leq\; & \mu_{2,I\otimes Q}\left[J_w(x,t) - \lambda I \otimes D(t)\right] \\ & + \mu_{2,I\otimes Q}\left[\lambda I \otimes D(t) - \mathcal{K}_w \otimes D(t)\right].\end{aligned} \tag{3.53}$$

We first show that the second term of the right hand side of the above inequality is zero. By Lemma 12, λ is the smallest eigenvalue of the edge Laplacian, $\mathcal{K}_w$, so the largest eigenvalue of $\lambda I - \mathcal{K}_w$ and hence $(\lambda I - \mathcal{K}_w) \otimes D(t)$ is 0. Therefore,

$$\begin{aligned}\mu_{2,I\otimes Q}[(\lambda I - \mathcal{K}_w) \otimes D(t)] \;&=\; \mu_2\left[(I \otimes Q)\left((\lambda I - \mathcal{K}_w) \otimes D(t)\right)\left(I \otimes Q^{-1}\right)\right] \\ &=\; \mu_2\left[(\lambda I - \mathcal{K}_w) \otimes D(t)\right] \\ &=\; \text{largest eigenvalue of } (\lambda I - \mathcal{K}_w) \otimes D(t) = 0.\end{aligned}$$

Since $(\lambda I - \mathcal{K}_w) \otimes D(t)$ is symmetric, $\mu_2\left[(\lambda I - \mathcal{K}_w) \otimes D(t)\right]$ is equal to the largest eigenvalue of $(\lambda I - \mathcal{K}_w) \otimes D(t)$. Next, we will show that the first term of the right hand side of Equation (3.53) is $\leq c$. By Lemma 4,

$$\mu_{2,I\otimes Q}\left[J_w(x,t) - \lambda I \otimes D(t)\right] \leq \max_i \left\{\mu_{2,Q}\left[\sqrt{\mathcal{W}} J_F(x_i,t) - \lambda D(t)\right]\right\}.$$

By taking sup over all $t \geq 0$ and $x = \left(x_1^T, \ldots, x_m^T\right)^T$, we get

$$\sup_{(x,t)} \mu_{2,I\otimes Q}\left[J_w(x,t) - \mathcal{K}_w \otimes D(t)\right] \leq \sup_t \sup_{x\in\mathbb{R}^n} \mu_{2,Q}\left[\sqrt{\mathcal{W}} J_F(x,t) - \lambda D(t)\right] = c.$$

Now by applying Theorem 13, we obtain the desired result. $\square$

3.4 Synchronization of diffusively-connected ODEs: weighted L^2 norm approaches

Acknowledgement of book chapter and conference publications:

Parts of the material in this section have been published in the conference paper [52] and the book chapter [46].

In this section, we study the asymptotic behavior of the solutions of the following system which is a generalized form of system (3.6),

$$\dot{x}(t) = \tilde{F}(x(t), t) - \left(\sum_{i=1}^{r} \mathcal{L}_i \otimes D_i(t) \right) x(t), \tag{3.54}$$

where $x, \tilde{F}$, and $\mathcal{L}_i$'s are as defined in (3.6). For each $i = 1, \ldots, r$, $D_i(t)$ is an $n \times n$ diagonal matrix with entries $[D_i]_{jj}(t) = d_{ij}(t)$, for $j = i, \ldots, n_i$ (where $n_1 + \cdots + n_r = n$) and 0 elsewhere. Note that (3.6) is a version of (3.54) for $r = 1$.

Before we state the main result of this section, we introduce some notations.

For a fixed $i \in \{1, \ldots, r\}$, let λ_k^i be the k-th eigenvalue of the matrix $\mathcal{L}_i$ and e_k^i be the corresponding normalized eigenvector. Note that by the definition of $\mathcal{L}_i$, $\lambda_1^i = 0$ and $e_1^i = \frac{1}{\sqrt{N}} \mathbf{1}_N$. For any fixed t, let

$$\Lambda_k(t) := \sum_{i=1}^{r} \lambda_k^i D_i(t). \tag{3.55}$$

For each $k \in \{1, \ldots, N\}$, let E_k^i be the subspace spanned by the first k eigenvectors:

$$E_k^i := \langle e_1^i, \ldots, e_k^i \rangle.$$

For each $k \in \{2, \ldots, N\}$, let $\pi_{k,i}$ be the orthogonal projection map from $\mathbb{R}^N$ onto E_{k-1}^i. Namely for any vector $v \in \mathbb{R}^N$ with $v = \sum_{j=1}^{N} (v \cdot e_j^i) e_j^i$,

$$\pi_{k,i}(v) \ := \ \sum_{j=1}^{k-1} (v \cdot e_j^i) e_j^i,$$

and for $k = 1$, let $\pi_{1,i}(v) = 0$.

For each $k \in \{2, \ldots, N\}$, and any $u = \left({u^1}^T, \ldots, {u^N}^T \right)^T$ with $u^j \in \mathbb{R}^n$, we define $\pi_k \colon \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ as follows.

$$\pi_k(u) \ := \ \sum_{j=1}^{n} \pi_{k,j}(u_j) \otimes e_j, \tag{3.56}$$

where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$, and $u_j := \left(u^1 \cdot e_j, \ldots, u^N \cdot e_j\right)^T \in \mathbb{R}^N$; and for $k = 1$, let $\pi_1(u) = 0$.

Note that for each k and any $u, v \in \mathbb{R}^{nN}$, by the definition of orthogonal maps, we have

$$(u - \pi_k(u))^T \pi_k(v) \;=\; \sum_{j=1}^{n} (u_j - \pi_{k,j}(u_j))^T \pi_{k,j}(v_j) \;=\; 0. \tag{3.57}$$

We also can define $\pi_k(u)$ as follows. For $i = 1, \ldots, n$, let $e^i := \sum_{j=1}^{N} e^i_j \otimes e_j$. It is straightforward to show that $e^1, \ldots, e^n$ are linearly independent and for any $i, j \in \{1, \ldots, n\}$, ${e^i}^T e^j = 0$. Hence, one can extend $\left\{e^i\right\}_{1\le i\le n}$ to an orthogonal basis for $\mathbb{R}^{nN}$, $\left\{e^i\right\}_{1\le i\le nN}$. Then for each $k = 2, \ldots, nN$, and any $u \in \mathbb{R}^{nN}$,

$$\pi_k(u) \;=\; \sum_{j=1}^{k-1} \left(u \cdot e^j\right) e^j,$$

and $\pi_1(u) = 0$. Note that for $k = 1, \ldots, N$, this definition is compatible with (3.56).

The goal of this section is to prove the following theorem which its first part is an analogous of Theorem 9 for system (3.54), and for non diagonal Q but restricted to $p = 2$; and its second part is an analogous of [42, Theorem 4] for system (3.54).

Theorem 14. *Consider the system (3.54) and for $k = 1, 2$, let*

$$\mu_k := \sup_{(x,t)\in V\times[0,\infty)} \mu_{2,P}[J_F(x, t) - \Lambda_k],$$

where P is a positive definite matrix such that for every $i = 1, \ldots, r$, and any t

$$P^2 D_i(t) + D_i(t) P^2 > 0.$$

Then for any two solutions, namely x and y, of (3.54), we have:

1.

$$\|(x - y)(t)\|_{2,I_N\otimes P} \;\le\; e^{\mu_1 t} \|(x - y)(0)\|_{2,I_N\otimes P}. \tag{3.58}$$

2.

$$\|(x - \pi_2(x))(t)\|_{2,I_N\otimes P} \;\le\; e^{\mu_2 t} \|(x - \pi_2(x))(0)\|_{2,I_N\otimes P}, \tag{3.59}$$

where I_N is the identity matrix of order N.

To prove Theorem 14, we need the following lemmas. We first state Courant-Fischer Minimax Theorem, from [80].

Lemma 17. *Let $\mathcal{L}$ be a positive semidefinite matrix in $\mathbb{R}^{N\times N}$ and let $\lambda_1 \leq \cdots \leq \lambda_N$ be its eigenvalues with corresponding normalized orthogonal eigenvectors $v_1, \ldots, v_N$. For any $v \in \mathbb{R}^N$, if $v^T v_j = 0$, for $1 \leq j \leq k-1$, then*

$$v^T \mathcal{L} v \geq \lambda_k v^T v.$$

Lemma 18. *Let $w := x - z$, where x is a solution of (3.54) and $z = y$ is either another solution of (3.54) or $z = \pi_2(x)$, i.e., $z = \mathbf{1}_N \otimes \left(\frac{1}{N}\sum_{j=1}^N x^j\right)$. For a positive semidefinite matrix Q, let*

$$\Phi(w) \;:=\; \frac{1}{2} w^T \left(I_N \otimes Q\right) w. \tag{3.60}$$

Then

$$\frac{d\Phi}{dt}(w) = w^T \left(I_N \otimes Q\right) \left(\tilde{F}(x,t) - \tilde{F}(z,t)\right) - w^T \left(I_N \otimes Q\right) \mathfrak{L}(t) w. \tag{3.61}$$

where $\mathfrak{L}(t) = \sum_{i=1}^r \mathcal{L}_i \otimes D_i(t)$.

Proof. When $z = y$, the claim is trivial because both x and y satisfy (3.54). When $z = \pi_2(x)$, then, by Equation (3.57), and the definition of π_2, we have:

$$\begin{aligned}
\frac{d\Phi}{dt}(w) &= (x - \pi_2(x))^T \left(I_N \otimes Q\right) \left(\tilde{F}(x,t) - \pi_2(\tilde{F}(x,t))\right) + w^T \left(I_N \otimes Q\right) \mathfrak{L}(t) w \\
&= (x - \pi_2(x))^T \left(I_N \otimes Q\right) \tilde{F}(x,t) + w^T \left(I_N \otimes Q\right) \mathfrak{L}(t) w \\
&= (x - \pi_2(x))^T \left(I_N \otimes Q\right) \left(\tilde{F}(x,t) - \tilde{F}(\pi_2(x),t)\right) + w^T \left(I_N \otimes Q\right) \mathfrak{L}(t) w.
\end{aligned}$$

The last equality holds because

$$\begin{aligned}
(x - \pi_2(x))^T \left(I_N \otimes Q\right) \tilde{F}(\pi_2(x),t)) &= \sum_{j=1}^N (x^j - \bar{x}) Q F(\bar{x},t) \\
&= \left(\sum_{j=1}^N x^j - N\bar{x}\right) Q F(\bar{x},t) = 0,
\end{aligned}$$

where $\bar{x} = \frac{1}{N}\sum_{j=1}^N x^j$. □

Proof of Theorem 14. *By Lemma 6,*

$$Q(J_F - \Lambda_k) + (J_F - \Lambda_k)^T Q \leq 2\mu_k Q, \tag{3.62}$$

where $Q = P^2$. *Define* w *and* $\Phi(w)$ *as in Lemma 18 for* $Q = P^2$. *Since* $\Phi(w) = \frac{1}{2}\|Pw\|_2^2$, *to prove (3.58) and (3.59), it suffices to show that for* $k = 1, 2$

$$\frac{d}{dt}\Phi(w) \leq 2\mu_k \Phi(w).$$

We rewrite the second term of the right hand side of (3.61) as follows. Since $Q = P^2$ *and for any* t, $P^2 D_i(t) + D_i(t)P^2 > 0$, *there exists a positive definite matrix* $M_i(t)$ *such that* $QD_i(t) + D_i(t)Q = 2M_i(t)^T M_i(t)$. *Therefore,*

$$\begin{aligned}
w^T(I_N \otimes Q)\mathfrak{L}(t)w &= w^T(I_N \otimes Q)\left(\sum_{i=1}^{r} \mathcal{L}_i \otimes D_i(t)\right) w \\
&= w^T\left(\sum_{i=1}^{r} I_N\mathcal{L}_i \otimes QD_i(t)\right) w \\
&= \frac{1}{2}\sum_{i=1}^{r} w^T\left(\mathcal{L}_i \otimes (QD_i(t) + D_i(t)Q)\right) w \\
&= \sum_{i=1}^{r} w^T\left(\mathcal{L}_i \otimes M_i(t)^T M_i(t)\right) w \\
&= \sum_{i=1}^{r} w^T(I_N \otimes M_i(t)^T)\left(\mathcal{L}_i \otimes I_n\right)\left(I_N \otimes M_i(t)\right) w \\
&\geq \sum_{i=1}^{r} \lambda_k^i \left((I_N \otimes M_i(t))w\right)^T (I_N \otimes M_i(t))w \\
&= \sum_{i=1}^{r} \lambda_k^i w^T(I_N \otimes M_i(t)^T M_i(t))w \\
&= \sum_{i=1}^{r} \lambda_k^i w^T(I_N \otimes QD_i(t))w \\
&= w^T(I_N \otimes Q\Lambda_k(t))w \qquad \text{(by Equation (3.55))},
\end{aligned}$$

where the inequality holds for $k = 1, 2$, *by Lemma 17. Therefore,*

$$-w^T(I_N \otimes Q)\mathfrak{L}(t)w \leq -w^T(I_N \otimes Q\Lambda_k(t))w. \tag{3.63}$$

Note that the smallest eigenvalue of $\mathcal{L}_i \otimes I_n$, *similar to* $\mathcal{L}_i$, *is* 0 *with corresponding eigenvector* $\mathbf{1}_{nN}$. *Now by applying Lemma 17 to* $\mathcal{L}_i \otimes I_n$, *since for* $z = \pi_2(x)$, *by definition,* $w^T\mathbf{1}_{nN} = 0$, $(I_N \otimes M_i(t))\, w\mathbf{1}_{nN} = 0$, *and the first inequality holds for* $k = 2$. *It also holds for* $k = 1$, *since* $\mathcal{L}_i$ *and hence* $\mathcal{L}_i \otimes I_n$ *are positive definite, and* $\lambda_1^i = 0$.

Now, by the Mean Value Theorem for integrals and Lemma 6, we rewrite the first term *of the right hand side of (3.61) as follows:*

$$w^T\left(I_N \otimes Q\right)\left(\tilde{F}(x,t) - \tilde{F}(z,t)\right) = \sum_{i=1}^{N} {w^i}^T Q(F(x^i,t) - F(z^i,t))w^i \, ds$$
$$= \sum_{i=1}^{N} \int_0^1 {w^i}^T Q J_F(z^i + sw^i, t) w^i \, ds.$$

This last equality together with (3.63) imply:

$$\begin{aligned}
& w^T\left(I_N \otimes Q\right)\left(\tilde{F}(x,t) - \tilde{F}(z,t)\right) - w^T(I_N \otimes Q)\mathfrak{L}(t)w \\
&= \sum_{i=1}^{N} \int_0^1 {w^i}^T Q\left(J_F(z^i + sw^i, t) - \Lambda_k(t)\right) w^i \, ds \\
&\leq \sum_{i=1}^{N} \frac{2\mu_k}{2} \int_0^1 ds \; {w^i}^T Q w^i \\
&= \frac{2\mu_k}{2} w^T(I_N \otimes Q)w \\
&= 2\mu_k \Phi(w).
\end{aligned}$$

Therefore

$$\frac{d\Phi}{dt}(w) \;\leq\; 2\mu_k \Phi(w).$$

This last inequality implies (3.58) and (3.59) for $k = 1$ and $k = 2$, respectively. □

Corollary 5. *In Theorem 14, if $\mu_1 < 0$, then (3.54) is contracting, meaning that solutions converge (exponentially) to each other, as $t \to +\infty$ in the P-weighted L_2 norm.*

Corollary 6. *In Theorem 14, if $\mu_2 < 0$, then solutions converge (exponentially) to uniform solutions, as $t \to +\infty$ in the P-weighted L_2 norm.*

3.5 Appendix

We state the following lemma about the eigenvalues of tridiagonal matrices. For more details see [81].

Lemma 19. *Denote by $M = M(v, a, b, s, t)$ the $n \times n$ tridiagonal matrix*

$$M = \begin{pmatrix} a+v & t & & & \\ s & v & t & & \\ & \ddots & \ddots & \ddots & \\ & & s & v & t \\ & & & s & b+v \end{pmatrix},$$

where $v, a, b, s, t \in \mathbb{R}$. Let $\sigma = \sqrt{st}$, and assume that $\lambda_1 \leq \cdots \leq \lambda_n$ are the eigenvalues of M. Then,

1. *for $a = b = 0$, and $k = 1, \ldots, n$, $\lambda_k = v - 2\sigma \cos\left(\frac{(n+1-k)\pi}{n+1}\right)$;*

2. *for $a = b = \sigma$, and $k = 1, \ldots, n$, $\lambda_k = v - 2\sigma \cos\left(\frac{(n+1-k)\pi}{n}\right)$.*

Note that in $-\mathcal{L}$, as defined in (3.3), $v = -2$, and $a = b = s = t = \sigma = 1$. Therefore, by Lemma 19,

$$\lambda_2(\mathcal{L}) = -\lambda_2(-\mathcal{L}) = 2 + 2\cos\left((N-1)\pi/N\right) = 4\sin^2(\pi/2N).$$

Next, we state the Perron-Frobenius Theorem that we used in the proofs of Proposition 10 and Proposition 14. For more details see [80].

Perron-Frobenius Theorem *Let A be an $n \times n$ non-negative (means all the entries are non-negative) and irreducible matrix. Then the following statements hold.*

1. *There is a real number λ^*, called the Perron-Frobenius eigenvalue, such that λ^* is an eigenvalue of A and $|\lambda^*| > \lambda$ for any other eigenvalue λ of A.*

2. *The Perron-Frobenius eigenvalue is simple. Consequently, the left and right eigenspaces associated to λ^* are one-dimensional.*

3. *There exist a left and a right eigenvector $v = (v_1, \ldots, v_n)$ of A corresponding to eigenvalue λ^* such that all components of v are positive.*

4. *There are no other positive left and right eigenvectors except positive multiples of v.*

Note that the Perron-Frobenius Theorem can be generalized to any Metzler, irreducible matrix, since for any Metzler matrix A, there exists a real positive number α such that $A - \alpha I$ is a non-negative matrix.

We used the following lemma in the proof of Proposition 13 and Lemma 16.

Lemma 20. *For any real α and β and $1 \leq p$:*

$$(|\alpha|^{p-2} + |\beta|^{p-2})\alpha\beta \leq |\alpha|^p + |\beta|^p.$$

Proof. For $\alpha\beta \leq 0$, the inequality is trivial. Suppose $\alpha\beta > 0$, and w.l.o.g $|\beta| \geq |\alpha|$ and let $\lambda = \frac{\beta}{\alpha}$. Then it suffices to prove that for $\lambda \geq 1$,

$$(1 + \lambda^{p-2})\lambda \leq \lambda^p + 1.$$

Let $f(\lambda) = \lambda^{p-1} + \lambda - \lambda^p - 1$. We want to show that $f(\lambda) \leq 0$ for $\lambda \geq 1$. Since $f(1) = f'(1) = 0$ and $f''(\lambda) \leq 0$ for $\lambda \geq 1$, indeed $f(\lambda) \leq 0$. $\square$

Chapter 4

Reaction diffusion PDEs with Neumann and Dirichlet boundary conditions

4.1 Introduction

In this chapter we study reaction diffusion PDE systems of the general form:

$$\begin{aligned}\frac{\partial u_1}{\partial t}(\omega,t) &= F_1(u(\omega,t),t) + d_1(t)\Delta u_1(\omega,t)\\ &\vdots\\ \frac{\partial u_n}{\partial t}(\omega,t) &= F_n(u(\omega,t),t) + d_n(t)\Delta u_n(\omega,t),\end{aligned} \tag{4.1}$$

subject to the Neumann boundary condition:

$$\frac{\partial u_i}{\partial \mathbf{n}}(\xi,t) = 0, \quad \forall \xi \in \partial\Omega, \ \ \forall t \in [0,\infty), \ \ \forall i = 1,\dots,n, \tag{4.2}$$

or subject to the Dirichlet boundary condition:

$$u_i(\xi,t) = 0, \quad \forall \xi \in \partial\Omega, \ \ \forall t \in [0,\infty), \ \ \forall i = 1,\dots,n, \tag{4.3}$$

which can be written as the following closed form:

$$\frac{\partial u}{\partial t}(\omega,t) = F(u(\omega,t),t) + D(t)\boldsymbol{\Delta} u(\omega,t),$$

where we assume

- Ω is a bounded domain of $\mathbb{R}^m$ with smooth boundary $\partial\Omega$ and outward normal $\mathbf{n}$.
- $u(\omega,t) = (u_1(\omega,t),\dots,u_n(\omega,t))^T$, where for any i, $u_i\colon \bar{\Omega}\times[0,\infty)\to\mathbb{R}$ is twice continuously differentiable on the first argument and continuously differentiable function on the second argument.

- $F\colon V \times [0,\infty) \to \mathbb{R}^n$ is a vector field with components $F_i\colon V \times [0,\infty) \to \mathbb{R}$, where V is a convex subset of $\mathbb{R}^n$: $F(x,t) = (F_1(x,t),\ldots,F_n(x,t))^T$. Also, we assume that for each i, $F_i(x,t)$ is Lipschitz on x and continuous on (x,t).

- $D(t) = \operatorname{diag}(d_1(t),\ldots,d_n(t))$, where for each i, $d_i(t) \geq 0$ is a continuous function of t. The matrix $D(t)$ is called the diffusion matrix.

- $\boldsymbol{\Delta} u = (\Delta u_1,\ldots,\Delta u_n)^T$, where $\Delta = \nabla \cdot \nabla$ is the Laplacian operator defined by $\Delta v = \sum_{i=1}^{m} \frac{\partial^2 v}{\partial \omega_i^2}$ for $v = v(\omega_1,\ldots,\omega_m)$.

In biology, a PDE system of this form describes individuals (particles, chemical species, etc.) of n different types, with respective abundances $u_i(\omega,t)$ at time t and location $\omega \in \Omega$, that can react instantaneously, guided by the interaction rules encoded into the vector field F, and can diffuse due to random motion. reaction diffusion PDEs play a key role in modeling intracellular dynamics and protein localization in cell processes such as cell division and eukaryotic chemotaxis (e.g. [82, 83, 84, 85]) as well as in the modeling of differentiation in multi-cellular organisms, through the diffusion of morphogens which control heterogeneity in gene expression in different cells (e.g. [86, 87]). From a bioengineering perspective, reaction diffusion models can be used to model artificial mechanisms for achieving cellular heterogeneity in tissue homeostasis (e.g. [88, 89]).

Definition 11. *By a solution of the PDE*

$$\begin{aligned}
\frac{\partial u}{\partial t}(\omega,t) &= F(u(\omega,t),t) + D(t)\boldsymbol{\Delta} u(\omega,t)\\
\frac{\partial u}{\partial \mathbf{n}}(\xi,t) &= 0, \quad \forall \xi \in \partial\Omega, \ \ \forall t \in [0,\infty),
\end{aligned}$$

on an interval $[0,T)$*, where* $0 < T \leq \infty$*, we mean a function* $u = (u_1,\ldots,u_n)^T$*, with* $u\colon \bar{\Omega} \times [0,T) \to V$*, such that:*

1. *for each* $\omega \in \bar{\Omega}$*,* $u(\omega,\cdot)$ *is continuously differentiable;*

2. *for each* $t \in [0,T)$*,* $u(\cdot,t)$ *is in* $\mathbf{Y}_V^{(n)}$ *(the superscript* (n) *is for Neumann), where*

$$\mathbf{Y}_V^{(n)} = \left\{ v\colon \bar{\Omega} \to V, v = (v_1,\ldots,v_n)^T, \ v_i \in C^2_{\mathbb{R}}\left(\bar{\Omega}\right), \ \frac{\partial v_i}{\partial \mathbf{n}}(\xi) = 0, \ \forall \xi \in \partial\Omega, \ \forall i \right\}, \tag{4.4}$$

and $C^2_{\mathbb{R}}\left(\bar{\Omega}\right)$ is the set of twice continuously differentiable functions $\bar{\Omega} \to \mathbb{R}$; and

3. *for each $\omega \in \bar{\Omega}$, and each $t \in [0, T)$, u satisfies the above PDE.*

Definition 12. *By a solution of the PDE*

$$\frac{\partial u}{\partial t}(\omega, t) = F(u(\omega, t), t) + D(t)\mathbf{\Delta} u(\omega, t)$$

$$u(\xi, t) = 0, \quad \forall \xi \in \partial\Omega, \ \ \forall t \in [0, \infty),$$

on an interval $[0, T)$, where $0 < T \leq \infty$, we mean a function $u = (u_1, \ldots, u_n)^T$, with $u \colon \bar{\Omega} \times [0, T) \to V$, such that:

1. *for each $\omega \in \bar{\Omega}$, $u(\omega, \cdot)$ is continuously differentiable;*

2. *for each $t \in [0, T)$, $u(\cdot, t)$ is in $\mathbf{Y}^{(d)}_V$, (the superscript (d) is for Dirichlet), where*

$$\mathbf{Y}^{(d)}_V \ = \ \left\{ \ v \colon \bar{\Omega} \to V, \ v = (v_1, \ldots, v_n)^T, \ v_i \in C^2_{\mathbb{R}}\left(\bar{\Omega}\right), \ v_i(\xi) = 0, \ \forall \xi \in \partial\Omega, \ \forall i \ \right\}, \tag{4.5}$$

and $C^2_{\mathbb{R}}\left(\bar{\Omega}\right)$ is the set of twice continuously differentiable functions $\bar{\Omega} \to \mathbb{R}$; and

3. *for each $\omega \in \bar{\Omega}$, and each $t \in [0, T)$, u satisfies the above PDE.*

Under the additional assumptions that $F(x, t)$ is twice continuously differentiable on x and continuous on (x, t), theorems on existence and uniqueness for PDEs such as (4.1) can be found in standard references, e.g., [90, 91, 92]. One must impose appropriate conditions on the vector field, on the boundary of V, to insure invariance of V (i.e., the solutions with initial conditions $u \colon \bar{\Omega} \to V$ remain in V.). Convexity of V insures that the Laplacian also preserves V. Since we are interested here in estimates relating pairs of solutions, we do not need to deal with well-posedness of the solutions. Our results will refer to solutions already assumed to exist.

Therefore, in the current work, we assume that (4.1) has a unique solution on $[0, T)$ for some $0 < T \leq \infty$. In addition, when we discuss contractivity and synchronous behavior of the solutions of (4.1), we assume that the solutions are defined globally (otherwise $t \to \infty$ and the concepts of contraction and synchronization don't make sense). We will discuss (global) existence and uniqueness of the solutions in Section 4.6 below.

For any $1 \leq p \leq \infty$, and any nonsingular, diagonal matrix $Q = \operatorname{diag}(q_1, \ldots, q_n)$, we introduce a Q-weighted norm on $\mathbf{X} = C_{\mathbb{R}^n}\left(\bar{\Omega}\right)$ as follows:

$$\|v\|_{p,Q} := \left\| Q \left(\|v_1\|_p, \ldots, \|v_n\|_p\right)^T \right\|_p . \tag{4.6}$$

Since

$$\|v\|_{p,Q} = \begin{cases} \left(\sum_i |q_i|^p \|v_i\|_p^p\right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \sup_i |q_i| \|v_i\|_p & p = \infty, \end{cases}$$

without loss of generality we will assume $q_i > 0$ for each i. Note that $\|v\|_{p,Q}$ is finite, for any p, Q, because each v_i is a continuous function on $\bar{\Omega}$ and $\bar{\Omega}$ is a compact subset of $\mathbb{R}^m$.

With a slight abuse of notation, we use the same symbol for a norm in $\mathbb{R}^n$:

$$\|x\|_{p,Q} := \|Qx\|_p.$$

Lemma 21. *For any $v \in \mathbf{X} = C_{\mathbb{R}^n}\left(\bar{\Omega}\right)$, $\|v\|_{p,Q} = \|v\|^*{}_{p,Q}$, where*

$$\|v\|^*{}_{p,Q} = \begin{cases} \left(\int_\Omega \|Qv(\omega)\|_p^p \, d\omega\right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \sup_\omega \|Qv(\omega)\|_\infty & p = \infty. \end{cases} \tag{4.7}$$

Note that $\|Qv(\omega)\|_p^p = \sum_{i=1}^n |q_i v_i(\omega)|^p$ and $\|Qv(\omega)\|_\infty = \max_i |q_i v_i(\omega)|$.

Proof. Let $Q = \operatorname{diag}(q_1, \ldots, q_n)$, $q_i > 0$. For $1 \leq p < \infty$ (the proof is analogous when $p = \infty$), by the definitions of $\|\cdot\|_{p,Q}$ and $\|\cdot\|^*{}_{p,Q}$

$$\begin{aligned} \|v\|^*{}_{p,Q} &= \left(\int_\Omega \|Qv(\omega)\|_p^p \, d\omega\right)^{\frac{1}{p}} \\ &= \left(\int_\Omega \left\|(q_1 v_1(\omega), \ldots, q_n v_n(\omega))^T\right\|_p^p \, d\omega\right)^{\frac{1}{p}} \\ &= \left(\int_\Omega |q_1 v_1(\omega)|^p + \cdots + |q_n v_n(\omega)|^p \, d\omega\right)^{\frac{1}{p}} \\ &= \left(\|q_1 v_1\|_p^p + \cdots + \|q_n v_n\|_p^p\right)^{\frac{1}{p}} \\ &= \left\|(q_1\|v_1\|_p, \ldots, q_n\|v_n\|_p)^T\right\|_p \\ &= \left\|Q\left(\|v_1\|_p, \ldots, \|v_n\|_p\right)^T\right\|_p \\ &= \|v\|_{p,Q} . \end{aligned}$$

□

Note that this equality between weighted L^p norms of functions and of vectors depends on our having taken the matrix Q to be diagonal. This is the key place where the assumption that Q is diagonal is being used.

4.2 Contractivity of reaction diffusion PDEs: weighted L^p norm approaches

Acknowledgement of journal publication:

Parts of the material in this section have been published in the journal paper [51].

The "symmetry breaking" phenomenon of diffusion-induced, or Turing instability refers to the case where a dynamic equilibrium $\bar{u}$ of the non-diffusing ODE $\frac{du}{dt} = F(u,t)$ is stable, but, at least for some diagonal positive matrices D, the corresponding uniform state $u(\omega,t) = \bar{u}$ is unstable for the PDE system $\frac{\partial u}{\partial t} = F(u,t) + D\boldsymbol{\Delta} u$. This phenomenon has been studied at least since Turing's seminal work on pattern formation in morphogenesis [93], where he argued that chemicals might react and diffuse so as result in heterogeneous spatial patterns.

Subsequent work by Gierer and Meinhardt [94, 95] produced a molecularly plausible minimal model, using two substances that combine local autocatalysis and long-ranging inhibition. Since that early work, a variety of processes in physics, chemistry, biology, and many other areas have been studied from the point of view of diffusive instabilities, and the mathematics of the process has been extensively studied [86, 87, 96, 97, 98, 99, 100, 101, 102, 103]. Most past work has focused on local stability analysis, through the analysis of the instability of nonuniform spatial modes of the linearized PDE. Nonlinear, global results are usually proved under strong constraints on diffusion constants as they compare to the growth of the reaction part.

In this work, we are interested in conditions on the reaction part F that guarantee that no diffusion instability will occur, no matter what is the size of the diffusion matrix D. We show that if the reaction system is "contractive" in the sense that trajectories globally and exponentially converge to each other with respect to a diagonally weighted L^p norm, then the same property is inherited by the PDE. In particular, if there exists a homogeneous steady state $\bar{u}$, it will follow that this steady state is globally exponentially stable for the PDE system. We were motivated by the desire to understand the important biological systems described in [64, 104] for which, as we will show, contractivity holds for diagonally weighted L^1 norms, but not with respect to diagonally weighted L^p norms, for any $1 < p \leq \infty$.

In what follows, we first state and prove the main result of this section and then we provide some examples to support the result. Note that the following theorem is an analogous version of Theorem 9 for reaction diffusion PDEs.

Theorem 15. *Consider the reaction diffusion PDE (4.1) defined on* $[0,T)$*, subject to Neumann boundary conditions (4.2). Let* $c := \mu_{p,Q}[F]$ *for some* $1 \leq p \leq \infty$*, and some positive diagonal matrix* Q *. Then for any two solutions* u, v *of the PDE and all* $t \in [0,T)$*:*

$$\|u(\cdot,t) - v(\cdot,t)\|_{p,Q} \leq e^{ct} \|u(\cdot,0) - v(\cdot,0)\|_{p,Q}.$$

To prove the theorem, we need the following two lemmas. We first show how to write the reaction diffusion PDE (4.1) as an ODE.

Lemma 22. *Pick any* $0 < T \leq \infty$ *and suppose that* u *is a solution of (4.1) and (4.2) defined on* $\bar{\Omega} \times [0,T)$*. In addition, we denote* $\mathbf{X} = C_{\mathbb{R}^n}\left(\bar{\Omega}\right)$*, where* $C_{\mathbb{R}^n}\left(\bar{\Omega}\right)$ *is the set of all continuous functions* $\bar{\Omega} \to \mathbb{R}^n$*. For each* $t \in [0,T)$*,* $\omega \in \bar{\Omega}$*, and* $u \in \mathbf{Y}_V^{(n)}$ *(as defined in (4.4)) define the following functions:*

- $\hat{u}\colon [0,T) \to \mathbf{Y}_V^{(n)}$*,* $\hat{u}(t)(\omega) \;:=\; u(\omega,t)$*.*
- $\tilde{F}_t\colon \mathbf{Y}_V^{(n)} \to \mathbf{X}$*,* $\tilde{F}_t(u)(\omega) \;:=\; F(u(\omega),t)$*.*
- $A_{p,Q}(t)\colon \mathbf{Y}_V^{(n)} \to \mathbf{X}$*,* $A_{p,Q}(t)(u) = \mathit{diag}\,(d_1(t)\Delta u_1, \ldots, d_n(t)\Delta u_n)$*.*
- $\hat{v}\colon [0,T) \to \mathbf{X}$*,* $\hat{v}(t)(\omega) = v(\omega,t) = \frac{\partial u}{\partial t}(\omega,t)$*.*

Then, $\hat{v}(t)$ *is the derivative of* $\hat{u}(t)$ *in the space* $(\mathbf{X}, \|\cdot\|_{p,Q})$*, that is:*

$$\lim_{h\to 0} \left\| \frac{1}{h}\left[\hat{u}(t+h) - \hat{u}(t)\right] - \hat{v}(t) \right\|_{p,Q} \;=\; 0,$$

for all $t \in [0,T)$*. Moreover,*

$$\frac{d\hat{u}}{dt}(t) = \tilde{F}_t(\hat{u}(t)) + A_{p,Q}(t)(\hat{u}(t)). \tag{4.8}$$

Proof. Fix $t \in [0,T)$ and $i \in \{1,\ldots,n\}$. Using the definition of v, we have:

$$\lim_{h\to 0}\left|\frac{1}{h}\left[u_i(\omega,t+h) - u_i(\omega,t)\right] - v_i(\omega,t)\right| = 0,$$

for any $\omega \in \bar{\Omega}$. Hence for any $\epsilon > 0$, there exists $h_\omega > 0$ such that for any $0 < h < h_\omega$,

$$\left|\frac{1}{h}\left[u_i(\omega,t+h) - u_i(\omega,t)\right] - v_i(\omega,t)\right| < \frac{\epsilon}{2}\,.$$

Now since u_i is a continuous function of ω, there exists a ball B_ω centered at ω such that for any $0 < h < h_\omega$,

$$\left|\frac{1}{h}\left[u_i(\tilde{\omega},t+h) - u_i(\tilde{\omega},t)\right] - v_i(\tilde{\omega},t)\right| < \epsilon,$$

for all $\tilde{\omega} \in B_\omega$. Since $\{B_\omega : \omega \in \bar{\Omega}\}$ is an open cover of $\bar{\Omega}$ and $\bar{\Omega}$ is a compact subset of $\mathbb{R}^m$, finitely many of these balls, namely $B_{\omega_1},\ldots,B_{\omega_k}$, cover $\bar{\Omega}$. Now let $h_0 = \min\{h_{\omega_1},\ldots,h_{\omega_k}\}$. Then, for any $0 < h < h_0$ and any $\omega \in \bar{\Omega}$, we have

$$\left|\frac{1}{h}\left[u_i(\omega,t+h) - u_i(\omega,t)\right] - v_i(\omega,t)\right| < \epsilon.$$

Raising to the p-th power and taking the integral over Ω of the above inequality, we get

$$\int_\Omega \left|\frac{1}{h}\left[u_i(\omega,t+h) - u_i(\omega,t)\right] - v_i(\omega,t)\right|^p \, d\omega < |\Omega|\,\epsilon^p,$$

which by the definition of $\|\cdot\|_{p,Q}$, it implies that for any $0 < h < h_0$,

$$\left\|\frac{1}{h}\left[u(\cdot,t+h) - u(\cdot,t)\right] - v(\cdot,t)\right\|_{p,Q} < c\epsilon,$$

where $c = (|\Omega| \sum_{i=1}^n q_i^p)^{\frac{1}{p}}$. Since $\epsilon > 0$ is arbitrary, we have proved that

$$\lim_{h\to 0}\left\|\frac{1}{h}\left[\hat{u}(t+h) - \hat{u}(t)\right] - \hat{v}(t)\right\|_{p,Q} = 0.$$

For a fixed $t \in [0,T)$ and any $\omega \in \bar{\Omega}$:

$$\begin{aligned}
\hat{v}(t)(\omega) = v(t,\omega) &= \frac{\partial u}{\partial t}(\omega,t) \\
&= F(u(\omega,t),t) + D(t)\boldsymbol{\Delta} u(\omega,t) \\
&= \tilde{F}_t(\hat{u}(t))(\omega) + A_{p,Q}(t)(\hat{u}(t))(\omega),
\end{aligned}$$

and therefore Equation (4.8) holds. $\square$

Lemma 23. *For any t,*

1. $\mu^{+}_{\mathbf{Y}_V^{(n)},\mathbf{X}}[A_{p,Q}(t)] \ \leq\ 0.$

2. $\mu^{+}_{\mathbf{Y}_V^{(n)},\mathbf{X}}[\tilde{F}_t] \ \leq\ \mu_{p,Q}[F_t].$

Proof. To prove the first part of the lemma, we consider the following three cases. Fix $t \geq 0$. We drop the arguments ω and t for simplicity.

Case 1. $1 < p < \infty$. By the definition of $\mu^{+}_{\mathbf{Y}_V^{(n)},\mathbf{X}}[A_{p,Q}(t)]$, it is enough to show that for any $u \in \mathbf{Y}_V^{(n)}$ with $\|u\|_{p,Q} \neq 0$, and any $\epsilon > 0$, there exists $h_\epsilon > 0$, depending on ϵ, such that for $0 < h < h_\epsilon$,

$$\frac{1}{h}\left(\frac{\|u + hD\mathbf{\Delta}u\|_{p,Q}}{\|u\|_{p,Q}} - 1\right) = \frac{1}{h}\left(\frac{(\sum_i q_i^p\|u_i + hd_i\Delta u_i\|_p^p)^{\frac{1}{p}}}{(\sum_i q_i^p\|u_i\|_p^p)^{\frac{1}{p}}} - 1\right) < \epsilon.$$

(As $A_{p,Q}(t)u = D(t)\mathbf{\Delta}u$, we write $D(t)\mathbf{\Delta}u$ instead of $A_{p,Q}(t)u$.)

Therefore, we will show that for h small enough

$$\sum_i q_i^p\|u_i + hd_i\Delta u_i\|_p^p < (1+\epsilon h)^p \sum_i q_i^p\|u_i\|_p^p\ . \tag{4.9}$$

Let $k\colon [0,1] \to \mathbb{R}$ be as follows:

$$k(h) = \sum_i q_i^p\|u_i + hd_i\Delta u_i\|_p^p - (1+\epsilon h)^p \sum_i q_i^p\|u_i\|_p^p\ .$$

Observe that k is continuously differentiable, and

$$\begin{aligned} k'(h) &= \frac{d}{dh}\sum_i q_i^p \int_\Omega |u_i + hd_i\Delta u_i|^p \ d\omega - p\epsilon(1+\epsilon h)^{p-1}\sum_i q_i^p\|u_i\|_p^p \\ &= \sum_i q_i^p \int_\Omega p\,|u_i + hd_i\Delta u_i|^{p-2}\,(u_i + hd_i\Delta u_i)\,d_i\Delta u_i \ d\omega \\ &\quad -p\epsilon(1+\epsilon h)^{p-1}\sum_i q_i^p\|u_i\|_p^p\ . \end{aligned}$$

Note that in general $|g|^p$ is differentiable for $p > 1$ and its derivative is $p\,|g|^{p-2}\,gg'$. Now by Green's Identity, the Neumann boundary condition, and by the assumption that $\sum_i q_i^p\|u_i\|_p^p \neq 0$, it follows integrating by parts that:

$$\begin{aligned} k'(0) &= p\sum_i q_i^p \int_\Omega |u_i|^{p-2}\,u_i d_i\Delta u_i \ d\omega - p\epsilon\sum_i q_i^p\|u_i\|_p^p \\ &= -p(p-1)\sum_i q_i^p d_i \int_\Omega |u_i|^{p-2}\,|\nabla u_i|^2 \ d\omega - p\epsilon\sum_i q_i^p\|u_i\|_p^p \\ &< 0\ . \end{aligned}$$

(Note that by the definition, any $u \in \mathbf{Y}_V^{(n)}$ satisfies the Neumann boundary condition.) Since $k'(0) < 0$, k' is continuous, and $k(0) = 0$, $k(h) < 0$ for h small enough and therefore Inequality (4.9) holds.

Case 2. $p = 1$. Let

$$g(p) := \lim_{h \to 0^+} \frac{1}{h} \left(\frac{(\sum_i q_i^p \|u_i + h d_i \Delta u_i\|_p^p)^{\frac{1}{p}}}{(\sum_i q_i^p \|u_i\|_p^p)^{\frac{1}{p}}} - 1 \right).$$

Since $g(p)$ is a continuous function at $p = 1$, and since in Case 1, we showed that $g(p) \leq 0$ for any $p > 1$, we conclude that $g(1) \leq 0$.

Case 3. $p = \infty$. Before proving this case we need the following lemma, which is an easy exercise in real analysis. (For completeness, we include a proof in the Appendix.)

Lemma 24. *Let $\Omega \subset \mathbb{R}^m$ be a Lebesgue measurable set with finite measure $|\Omega|$ and let f be a bounded, continuous function on $\mathbb{R}$. Then $F(p) := \left(\frac{1}{|\Omega|}\int_\Omega |f|^p\right)^{\frac{1}{p}}$ is an increasing function of p and its limit as $p \to \infty$ is $\|f\|_\infty$.*

For a fixed $p_0 > 1$, pick $u \in \mathbf{Y}_V^{(n)}$ with $\|u\|_{p_0,Q} \neq 0$. By the definition of the norm, $\|u\|_{p_0,Q} \neq 0$ implies that for some $i_0 \in \{1, \ldots, n\}$, $\|u_{i_0}\|_{p_0} \neq 0$. Let

$$\varphi(p) := \frac{1}{|\Omega|^{\frac{1}{p}}} \|u_{i_0}\|_p .$$

Since by Lemma 24, φ is an increasing function of p, for any $p > p_0$,

$$\|u_{i_0}\|_p \geq \|u_{i_0}\|_{p_0} > 0 .$$

Now for fixed $i \in \{1, \ldots, n\}$, $p > p_0$, and $\epsilon > 0$, let $k(h)$ be as follows:

$$k(h) = \begin{cases} \|u_i + h d_i \Delta u_i\|_p^p - (1 + \epsilon h)^p \|u_{i_0}\|_p^p & \text{if } \|u_{i_0}\|_p \geq \|u_i\|_p \\ \\ \|u_i + h d_i \Delta u_i\|_p^p - (1 + \epsilon h)^p \|u_i\|_p^p & \text{if } \|u_{i_0}\|_p \leq \|u_i\|_p . \end{cases}$$

In both cases $k(0) \leq 0$ and $k'(0) < 0$ (the proof is similar to the proof of $k'(0) < 0$ in Case 1, since $\|u_{i_0}\|_p > 0$ and $\|u_i\|_p > 0$). Therefore, for a small enough h, $k(h) \leq 0$, which implies that:

$$\lim_{h \to 0^+} \frac{1}{h} \left(\frac{\|u_i + h d_i \Delta u_i\|_p}{\|u_i\|_p} - 1 \right) \leq 0 .$$

Now by Lemma 24, since

$$\frac{1}{|\Omega|^{\frac{1}{p}}}\|u_i + hd_i\Delta u_i\|_p \to \|u_i + hd_i\Delta u_i\|_\infty \quad \text{and} \quad \frac{1}{|\Omega|^{\frac{1}{p}}}\|u_i\|_p \to \|u_i\|_\infty, \quad \text{as} \quad p \to \infty,$$

we can conclude that

$$\lim_{h\to 0^+} \frac{1}{h}\left(\frac{\|u_i + hd_i\Delta u_i\|_\infty}{\|u_i\|_\infty} - 1\right) \le 0.$$

In other words, for a fixed $\epsilon > 0$, there exists $h_i > 0$ such that for any $0 < h < h_i$,

$$\|u_i + hd_i\Delta u_i\|_\infty \le (1 + \epsilon h)\|u_i\|_\infty \quad \text{for any} \quad i \in \{1, \dots, n\}.$$

Let $h_0 = \min_i h_i$. Then for any $0 < h < h_0$,

$$\max_i q_i\|u_i + hd_i\Delta u_i\|_\infty \ =: \ q_j\|u_j + hd_j\Delta u_j\|_\infty \ \le \ q_j(1+\epsilon h)\|u_j\|_\infty \ \le \ (1+\epsilon h)\max_i q_i\|u_i\|_\infty,$$

which implies

$$\lim_{h\to 0^+} \frac{1}{h}\left(\frac{\max_i q_i\|u_i + hd_i\Delta u_i\|_\infty}{\max_i q_i\|u_i\|_\infty} - 1\right) \le 0\ .$$

This prove the first part of the lemma.

We next prove the second part of the lemma. By the definition of $c := \mu_{p,Q}[F_t]$, for any fixed t, we have

$$\lim_{h\to 0^+} \frac{1}{h} \sup_{x\neq y\in V}\left(\frac{\|x - y + h(F(x,t) - F(y,t))\|_{p,Q}}{\|x-y\|_{p,Q}} - 1\right) = c.$$

Fix an arbitrary $\epsilon > 0$. Then there exists $h_0 > 0$ such that for all $0 < h < h_0$,

$$\frac{1}{h} \sup_{x\neq y\in V}\left(\frac{\|x - y + h(F(x,t) - F(y,t))\|_{p,Q}}{\|x-y\|_{p,Q}} - 1\right) < c + \epsilon.$$

Therefore, for any $x \neq y$, and $0 < h < h_0$

$$\frac{\|x - y + h(F(x,t) - F(y,t))\|_{p,Q}}{\|x-y\|_{p,Q}} < (c+\epsilon)h + 1. \tag{4.10}$$

For fixed $u \neq v \in \mathbf{Y}_V^{(n)}$, let $\Omega_1 = \{\omega \in \bar{\Omega} : u(\omega) \neq v(\omega)\}$. Fix $\omega \in \Omega_1$, and let $x = u(\omega)$ and $y = v(\omega)$. We give a proof for the case $p < \infty$; the case $p = \infty$ is analogous. Using Equation (4.10), we have:

$$\frac{\left(\sum_i q_i^p \,|u_i - v_i + h(F_i(u,t) - F_i(v,t))|^p\right)^{\frac{1}{p}}}{\left(\sum_i q_i^p \,|u_i - v_i|^p\right)^{\frac{1}{p}}} < (c+\epsilon)h + 1. \tag{4.11}$$

Multiplying both sides by the denominator and raising to the power p, we have:

$$\sum_i q_i^p \left|u_i - v_i + h\left(F_i(u,t) - F_i(v,t)\right)\right|^p < ((c+\epsilon)h+1)^p \sum_i q_i^p \left|u_i - v_i\right|^p . \tag{4.12}$$

Since $\tilde{F}_t(u)(\omega) = F(u(\omega),t)$, Equation (4.12) can be written as:

$$\sum_i q_i^p \left|u_i - v_i + h\left(\tilde{F}_{t,i}(u) - \tilde{F}_{t,i}(v)\right)\right|^p < ((c+\epsilon)h+1)^p \sum_i q_i^p \left|u_i - v_i\right|^p . \tag{4.13}$$

Now by taking the integral over $\bar{\Omega}$, using Lemma 21, we get:

$$\left\|u - v + h\left(\tilde{F}_t(u) - \tilde{F}_t(v)\right)\right\|_{p,Q} < ((c+\epsilon)h+1)\left\|u-v\right\|_{p,Q} .$$

(Note that for $\omega \notin \Omega_1$, $((c+\epsilon)h+1)^p \sum_i q_i^p |u_i - v_i|^p = 0$ which we can add to the right hand side of (4.13), and $\sum_i q_i^p |u_i - v_i + h(F_i(u,t) - F_i(v,t))|^p = 0$ which we can add to the left hand side of (4.13), and hence we can indeed take the integral over all $\bar{\Omega}$.)

Hence,

$$\lim_{h\to 0^+} \frac{1}{h}\left(\frac{\left\|u - v + h\left(\tilde{F}_t(u) - \tilde{F}_t(v)\right)\right\|_{p,Q}}{\left\|u-v\right\|_{p,Q}} - 1\right) \le c+\epsilon.$$

Now by letting $\epsilon \to 0$ and taking sup over $u \neq v \in \mathbf{Y}_V^{(n)}$, we get $\mu^+_{\mathbf{Y}_V^{(n)},\mathbf{X}}[\tilde{F}_t] \le c$. □

Proof of Theorem 15. *For any $1 \le p \le \infty$ and any fixed t, by subadditivity of μ^+, Equation (4.8), and Lemma 23, we have*

$$\mu^+_{\mathbf{Y}_V^{(n)},\mathbf{X}}\left[\tilde{F}_t + A_{p,Q}(t)\right] \;\le\; \mu^+_{\mathbf{Y}_V^{(n)},\mathbf{X}}\left[\tilde{F}_t\right] \;\le\; \mu_{p,Q}[F_t].$$

Now using Theorem 7, for any t,

$$\left\|\hat{u}(t) - \hat{v}(t)\right\|_{p,Q} \;\le\; e^{ct}\left\|\hat{u}(0) - \hat{v}(0)\right\|_{p,Q},$$

which is equivalent to the following inequality

$$\left\|u(\cdot,t) - v(\cdot,t)\right\|_{p,Q} \;\le\; e^{ct}\left\|u(\cdot,0) - v(\cdot,0)\right\|_{p,Q}.$$

□

Corollary 7. *In addition to the conditions of Theorem 15, assume that F is C^1 on x and let $c = \sup_{(x,t)} \mu[J_F(x,t)]$. Then*

$$\left\|u(\cdot,t) - v(\cdot,t)\right\|_{p,Q} \;\le\; e^{ct}\left\|u(\cdot,0) - v(\cdot,0)\right\|_{p,Q}.$$

Proof. The proof is immediate from Theorem 15 and Proposition 2. □

Corollary 8. *Under the conditions of Theorem 15 and Corollary 7, if $c < 0$ and the solutions of (4.1) are defined globally (for more discussion see Section 4.6), then the reaction diffusion PDE is contracting, meaning that solutions converge (exponentially) to each other, as $t \to +\infty$.*

Examples

Example 1. In Section 2.4, we provided an example of a biochemical model which is modeled by a system of ODEs and is contractive when using a weighted L^1 norm [64], but it is not contractive in any weighted L^p norm, $p > 1$. In what follows, we consider the spatial dependence version of the same example which is modeled by a reaction diffusion PDE subject to Neumann boundary conditions. We use the result of this section, namely Theorem 15 or Corollary 8, to show that the system is contractive.

As discussed in Section 2.4, a typical biochemical reaction is one in which an enzyme X (whose concentration is quantified by the non-negative variable $x = x(\omega, t)$) binds to a substrate S (whose concentration is quantified by $s = s(\omega, t) \geq 0$), to produce a complex Y (whose concentration is quantified by $y = y(\omega, t) \geq 0$), and the enzyme is subject to degradation and dilution (at rate δx, where $\delta > 0$) and production according to an external signal $z = z(t)$. We let the domain Ω (here $\Omega = (0, 1)$) represents the part of the cytoplasm where these chemicals are free to diffuse. Taking equal diffusion constants for S and Y (which is reasonable since typically S and Y have approximately the same size), a natural model is given by a reaction diffusion system

$$
\begin{aligned}
\frac{\partial x}{\partial t} &= z(t) - \delta x + k_1 y - k_2 s x + d_1 \Delta x \\
\frac{\partial y}{\partial t} &= -k_1 y + k_2 s x + d_2 \Delta y \\
\frac{\partial s}{\partial t} &= k_1 y - k_2 s x + d_2 \Delta s,
\end{aligned}
$$

subject to the Neumann boundary condition, $\frac{\partial x}{\partial \omega}(0, t) = \frac{\partial x}{\partial \omega}(1, t) = 0$, etc. Note that $\frac{\partial}{\partial t}(y + s)(\omega, t) = d_2 \, \frac{\partial^2}{\partial \omega^2}(y + s)(\omega, t)$. Therefore, if we assume that initially S and Y are uniformly distributed, i.e., $(y + s)(\omega, 0) = S_Y$, for a positive constant S_Y, it follows that, by the uniqueness of the solutions of heat equation, for any t,

$$(y+s)(\omega,t) = (y+s)(\omega,0) = S_Y.$$

Thus, we can study the following reduced system:

$$\begin{aligned}\frac{\partial x}{\partial t} &= z(t) - \delta x + k_1 y - k_2(S_Y - y)x + d_1 \Delta x\\ \frac{\partial y}{\partial t} &= -k_1 y + k_2(S_Y - y)x + d_2 \Delta y.\end{aligned} \tag{4.14}$$

Note that $(x(\omega,t), y(\omega,t)) \in V = [0,\infty) \times [0, S_Y]$ for all $\omega \in (0,1)$, and $t \geq 0$ (V is convex and forward-invariant), and k_1, k_2, δ, d_1, and d_2 are arbitrary positive constants. Let J_{F_t} be the Jacobian of $F_t(x,y) := (z(t) - \delta x + k_1 y - k_2(S_Y - y)x, -k_1 y + k_2(S_Y - y)x)^T$:

$$J_{F_t}(x,y) \;=\; \begin{pmatrix} -\delta - k_2(S_Y - y) & k_1 + k_2 x \\ k_2(S_Y - y) & -(k_1 + k_2 x) \end{pmatrix}.$$

In Section 2.4, following [64], we showed

$$\sup_t \sup_{(x,y)\in V} \mu_{1,Q}\left[J_{F_t}(x,y)\right] < 0,$$

for some positive diagonal matrix Q. Therefore, by Corollary 8, the reaction diffusion PDE system (4.14) is contractive.

The following example, from the literature on pattern formation, also illustrates the need to choose norms judiciously.

Example 2. [86] In this example, we study the Thomas mechanism, which is based on a specific reaction, involving the substrates oxygen, v, and uric acid, u. The dimensionless form of the reaction diffusion equations for the oxygen and the uric acid concentrations are as follows:

$$\begin{aligned}\frac{\partial u}{\partial t} &= a - u - \rho R(u,v) + d_1 \Delta u\\ \frac{\partial v}{\partial t} &= \alpha(b - v) - \rho R(u,v) + d_2 \Delta v,\end{aligned} \tag{4.15}$$

where $R(u,v) = \dfrac{uv}{1+u+Ku^2}$. We assume :

1. a, b, ρ, α, K, d_1, and d_2 are all positive constants,

2. for all $t \geq 0$, $(u(t), v(t)) \in V = [0, 2a] \times [0, \infty)$,

3. $a < \dfrac{1}{2\sqrt{K}}$.

Note that V is convex and forward-invariant.

In this model, u and v are subject to production at constant rates a and αb, and are subject to degradation at rates $-u$ and $-\alpha v$ respectively; and both are used up in the reaction at a rate $\rho R(u,v)$. The form $R(u,v)$ exhibits substrate inhibition: For small u, namely $u < 2a$, R is increasing and for u large enough, $u > 2a$, R is decreasing.

Let J_F be the Jacobian of $F = (a - u - \rho R(u,v), \alpha(b-v) - \rho R(u,v))^T$:

$$J_F(u,v) \;=\; \begin{pmatrix} -1-\rho R_u(u,v) & -\rho R_v(u,v) \\ -\rho R_u(u,v) & -\alpha - \rho R_v(u,v) \end{pmatrix},$$

where

$$R_u(u,v) = \frac{v\left(1 - Ku^2\right)}{\left(1+u+Ku^2\right)^2}\,, \quad R_v(u,v) = \frac{u}{1+u+Ku^2}\,,$$

are the partial derivatives of R with respect to u and v, respectively. Note that, by assumptions 2 and 3, both R_u and R_v are non-negative on V. Hence, for any $(u,v) \in V$,

$$\begin{aligned} \mu_1[J_F(u,v)] &= \max\left\{-1-\rho R_u(u,v) + \left|-\rho R_u(u,v)\right|, -\alpha - \rho R_v(u,v) + \left|-\rho R_v(u,v)\right|\right\} \\ &= \max\left\{-1-\rho R_u(u,v) + \rho R_u(u,v), -\alpha - \rho R_v(u,v) + \rho R_v(u,v)\right\} \\ &= \max\{-1, -\alpha\} < 0. \end{aligned}$$

Therefore, by Corollary 8, the system is contractive.

We next show that for any positive diagonal matrix Q, and $p > 1$,

$$\sup_{(u,v)\in V} \mu_{p,Q}[J_F(u,v)] \;\geq\; 0.$$

Let $Q = \operatorname{diag}(1,q)$ and $u = 0$. Then for any $v \in [0,\infty)$:

$$J_0(v) \;:=\; I + hQJ_F(0,v)Q^{-1} \;=\; \begin{pmatrix} 1 - h(1+\rho v) & 0 \\ -q\rho v h & 1 - h\alpha \end{pmatrix}.$$

We first consider $p \neq \infty$ and will show that $\mu_{p,Q}[J_F(0,v)] \geq 0$ for some $v \in [0,\infty)$. To this end, by the definition of the logarithmic norm, we show that there exists $v \in [0,\infty)$ such that for all small enough $h > 0$, $\|J_0(v)\|_p > 1$. Computing explicitly, we have:

$$\begin{aligned} \|J_0(v)\|_p &= \sup_{(\xi_1,\xi_2)\neq(0,0)} \frac{\left(|\xi_1 - h(1+\rho v)\xi_1|^p + |-q\rho v h\xi_1 + \xi_2 - \alpha h \xi_2|^p\right)^{\frac{1}{p}}}{\left(|\xi_1|^p + |\xi_2|^p\right)^{\frac{1}{p}}} \\ &\geq \frac{\left(|1 - h(1+\rho v)|^p + |-q\rho v h + \lambda - \alpha h\lambda|^p\right)^{\frac{1}{p}}}{\left(1 + |\lambda|^p\right)^{\frac{1}{p}}}, \end{aligned}$$

where we take a point of the form $(\xi_1, \xi_2) = (1, \lambda)$, for a $\lambda < 0$ which will be determined later. To show

$$\frac{(|1-h(1+\rho v)|^p + |-qh\rho v + \lambda - \alpha h\lambda|^p)^{\frac{1}{p}}}{(1+|\lambda|^p)^{\frac{1}{p}}} \;>\; 1,$$

we equivalently show that for any small enough $h > 0$:

$$\frac{1}{h}\left(|1-h(1+\rho v)|^p + |-q\rho v h + \lambda - \alpha h\lambda|^p - 1 - |\lambda|^p\right) > 0. \tag{4.16}$$

Note that the $\lim_{h\to 0^+}$ of the left hand side of the above inequality is $f'(0)$ where

$$f(h) \;=\; |1-h(1+\rho v)|^p + |-q\rho v h + \lambda - \alpha h\lambda|^p\,.$$

Therefore, it suffices to show that $f'(0) > 0$ for some value $v \in [0, \infty)$ (because $f'(0) > 0$ implies that there exists $h_0 > 0$ such that for $0 < h < h_0$, (4.16) holds). Since $p > 1$, by assumption, f is differentiable and

$$\begin{aligned} f'(h) \;&=\; -p(1+\rho v)\,|1-h(1+\rho v)|^{p-2}\,(1-h(1+\rho v)) \\ &+\; p(-q\rho v - \alpha\lambda)\,|-q\rho v h + \lambda - \alpha h\lambda|^{p-2}\,(-q\rho v h + \lambda - \alpha h\lambda)\,. \end{aligned}$$

Hence, since $\lambda < 0$

$$\begin{aligned} f'(0) \;&=\; -p(1+\rho v) + p(-q\rho v - \alpha\lambda)\,|\lambda|^{p-2}\,\lambda \\ &=\; -p(1+\rho v) + p(q\rho v + \alpha\lambda)(-\lambda)^{p-1} \\ &=\; p\rho\left(-1 + q(-\lambda)^{p-1}\right)v - p\left(1 + \alpha(-\lambda)^{p-1}\right). \end{aligned}$$

Choosing $\lambda < 0$ small enough such that $-1 + q(-\lambda)^{p-1} > 0$ and choosing v large enough, we can make $f'(0) > 0$.

Now we show that for large v, $\mu_\infty[J_0(v)] > 0$. Using Table 2.1,

$$\mu_\infty[J_0(v)] \;=\; \max\{-\alpha + q\rho v, -1 - \rho v\}\,,$$

which is positive for $v > \dfrac{\alpha}{q\rho}$.

Remark 20. *For a system*

$$\begin{aligned} \frac{\partial x}{\partial t} \;&=\; f(x,y) + d_1\Delta x \\ \frac{\partial y}{\partial t} \;&=\; g(x,y) + d_2\Delta y, \end{aligned}$$

with a steady state (x^*, y^*), a set of necessary and sufficient conditions for diffusive instability are as follows (for a proof see e.g., [86, 87]):

1. $f_x + g_y < 0$,

2. $f_x g_y - f_y g_x > 0$,

3. $d_2 f_x + d_1 g_y > 0$,

4. $(d_2 f_x + d_1 g_y)^2 - 4 d_2 d_1 (f_x g_y - f_y g_x) > 0$;

where f_x denote the partial derivative of f, with respect to x, at the steady state (x^*, y^*), etc. The first two conditions say that (u^*, v^*) is (locally) stable before diffusion. Note that the derivatives f_x and g_y must be of opposite sign.

In Example 2, the first two conditions hold for all $(u, v) \in V$, so if there exists a steady state in V, it must be asymptotically stable (without diffusion terms). But since R_u and R_v are both non-negative on V (because of the choice of V and the parameters), the third condition is violated. Hence, if there exists a steady state in V, it remains locally asymptotically stable after diffusion; and we showed that it is in fact globally *stable on V. One may get diffusive instability with choosing parameters appropriately.*

4.3 Contractivity of reaction diffusion PDEs with space dependent diffusion: weighted L^p norm approaches

In this section, we generalize the main result of Section 4.2, namely Theorem 15, to the following space dependent reaction diffusion system which is a generalization of reaction diffusion system (4.1).

$$\begin{aligned}
\frac{\partial u_1}{\partial t}(\omega, t) &= F_1(u(\omega, t), t) + d_1(t)\nabla \cdot (A_1(\omega)\nabla u_1(\omega, t)) \\
&\vdots \\
\frac{\partial u_n}{\partial t}(\omega, t) &= F_n(u(\omega, t), t) + d_n(t)\nabla \cdot (A_n(\omega)\nabla u_n(\omega, t)),
\end{aligned} \tag{4.17}$$

subject to the Neumann boundary condition:

$$\frac{\partial u_i}{\partial \mathbf{n}}(\xi, t) = 0, \quad \forall \xi \in \partial\Omega, \;\; \forall t \in [0, \infty), \;\; \forall i = 1, \ldots, n, \tag{4.18}$$

where u_i's, F_i's, d_i's, Ω, $\partial\Omega$, and $\mathbf{n}$ are as defined in Equation (4.1). For each i, $A_i \colon \Omega \to \mathbb{R}^{m\times m}$ is a symmetric matrix and there exist $\alpha_i, \beta_i > 0$ such that for all $\omega \in \Omega$, and $\zeta = (\zeta_1, \ldots, \zeta_m)^T \in \mathbb{R}^m$,

$$\alpha_i \left|\zeta\right|^2 \;\leq\; \zeta^T A_i(\omega)\zeta \;\leq\; \beta_i \left|\zeta\right|^2 . \tag{4.19}$$

In addition, $A_i(\omega)$ is a C^1 function of ω. Note that in Equation (4.1), $A_i(\omega) = I$.

Theorem 16. *Consider the reaction diffusion PDE (4.17) defined on $[0, T)$ for some $T \in (0, \infty]$, subject to Neumann boundary conditions (4.18). Let $c := \mu_{p,Q}[F]$ for some $1 \leq p \leq \infty$, and some positive diagonal matrix Q. Then for any two solutions u, v of the PDE and all $t \in [0, T)$:*

$$\left\|u(\cdot, t) - v(\cdot, t)\right\|_{p,Q} \leq e^{ct} \left\|u(\cdot, 0) - v(\cdot, 0)\right\|_{p,Q} .$$

Proof. The proof of Theorem 16 is exactly similar to the proof of Theorem 15. We only need to generalize Lemma 22 and the first part of Lemma 23 for

$$\mathcal{A}_{p,Q}(t)(u) = \operatorname{diag}\left(d_1(t)\nabla \cdot (A_1(\omega)\nabla u_1), \ldots, \nabla \cdot (A_n(\omega)\nabla u_n)\right).$$

By the definition, it is straightforward that one can generalize Lemma 22 for $\mathcal{A}_{p,Q}$. Next we show that $\mu^{+}_{\mathbf{Y}_V^{(n)},\mathbf{X}}[\mathcal{A}_{p,Q}(t)] \;\leq\; 0$. Similar to the proof of part 1 of Lemma 23, we consider the following three cases. Note that for each i, $A_i = A_i(\omega)$, $u_i = u_i(\omega)$, and $d_i = d_i(t)$. Fix $t \geq 0$.

Case 1. $1 < p < \infty$. By the definition of $\mu^{+}_{\mathbf{Y}_V^{(n)},\mathbf{X}}[\mathcal{A}_{p,Q}(t)]$, it is enough to show that for any $u \in \mathbf{Y}_V^{(n)}$ with $\left\|u\right\|_{p,Q} \neq 0$, and any $\epsilon > 0$, there exists $h_\epsilon > 0$, depending on ϵ, such that for $0 < h < h_\epsilon$,

$$\frac{1}{h}\left(\frac{(\sum_i q_i^p \|u_i + hd_i\nabla \cdot (A_i\nabla u_i)\|_p^p)^{\frac{1}{p}}}{(\sum_i q_i^p \|u_i\|_p^p)^{\frac{1}{p}}} - 1\right) < \epsilon.$$

Therefore, we will show that for h small enough

$$\sum_i q_i^p \|u_i + hd_i\nabla \cdot (A_i\nabla u_i)\|_p^p < (1 + \epsilon h)^p \sum_i q_i^p \|u_i\|_p^p . \tag{4.20}$$

Let $k \colon [0, 1] \to \mathbb{R}$ be as follows:

$$k(h) = \sum_i q_i^p \|u_i + hd_i\nabla \cdot (A_i\nabla u_i)\|_p^p - (1 + \epsilon h)^p \sum_i q_i^p \|u_i\|_p^p .$$

Observe that k is continuously differentiable and $k'(h)$ is as follows

$$\begin{aligned}
&\frac{d}{dh}\sum_i q_i^p \int_\Omega |u_i + hd_i\nabla\cdot(A_i\nabla u_i)|^p \; d\omega - p\epsilon(1+\epsilon h)^{p-1}\sum_i q_i^p\|u_i\|_p^p \\
&= \sum_i q_i^p \int_\Omega p\,|u_i + hd_i\nabla\cdot(A_i\nabla u_i)|^{p-2}\,(u_i + hd_i\nabla\cdot(A_i\nabla u_i))\,d_i\nabla\cdot(A_i\nabla u_i)\;d\omega \\
&\quad -p\epsilon(1+\epsilon h)^{p-1}\sum_i q_i^p\|u_i\|_p^p \,.
\end{aligned}$$

Now by Green's Identity, the Neumann boundary condition, and by the assumption that $\sum_i q_i^p\|u_i\|_p^p \neq 0$, it follows integrating by parts that:

$$\begin{aligned}
k'(0) &= p\sum_i q_i^p \int_\Omega |u_i|^{p-2}\,u_i d_i\nabla\cdot(A_i\nabla u_i)\;d\omega - p\epsilon\sum_i q_i^p\|u_i\|_p^p \\
&= -p(p-1)\sum_i q_i^p d_i \int_\Omega |u_i|^{p-2}\,\nabla {u_i}^T A_i\nabla u_i\;d\omega - p\epsilon\sum_i q_i^p\|u_i\|_p^p \\
&< -p(p-1)\sum_i q_i^p d_i \int_\Omega \alpha_i\,|u_i|^{p-2}\,|\nabla u_i|^2\;d\omega - p\epsilon\sum_i q_i^p\|u_i\|_p^p \\
&< 0\,.
\end{aligned}$$

The first inequality holds by Equation (4.19). Note that by the definition of $\mathbf{Y}_V^{(n)}$, any $u \in \mathbf{Y}_V^{(n)}$ satisfies the Neumann boundary condition. Since $k'(0) < 0$ and k' is continuous and $k(0) = 0$, $k(h) < 0$ for h small enough and therefore Inequality (4.20) holds.

Case 2. $p = 1$. Let

$$g(p) := \lim_{h\to 0^+}\frac{1}{h}\left(\frac{(\sum_i q_i^p\|u_i + hd_i\nabla\cdot(A_i\nabla u_i)\|_p^p)^{\frac{1}{p}}}{(\sum_i q_i^p\|u_i\|_p^p)^{\frac{1}{p}}} - 1\right).$$

Since $g(p)$ is a continuous function at $p = 1$, and since in Case 1, we showed that $g(p) \leq 0$ for any $p > 1$, we conclude that $g(1) \leq 0$.

Case 3. $p = \infty$. For a fixed $p_0 > 1$, pick $u \in \mathbf{Y}_V^{(n)}$ with $\|u\|_{p_0,Q} \neq 0$. By the definition of the norm, $\|u\|_{p_0,Q} \neq 0$ implies that for some $i_0 \in \{1,\ldots,n\}$, $\|u_{i_0}\|_{p_0} \neq 0$. Let

$$\varphi(p) := \frac{1}{|\Omega|^{\frac{1}{p}}}\|u_{i_0}\|_p\,.$$

Since by Lemma 24, φ is an increasing function of p, for any $p > p_0$,

$$\|u_{i_0}\|_p \;\geq\; \|u_{i_0}\|_{p_0} \;>\; 0\,.$$

Now for fixed $i \in \{1, \ldots, n\}$, $p > p_0$, and $\epsilon > 0$, we define k as follows:

$$k(h) = \begin{cases} \|u_i + hd_i \nabla \cdot (A_i \nabla u_i)\|_p^p - (1+\epsilon h)^p \|u_{i_0}\|_p^p & \text{if } \ \|u_{i_0}\|_p \geq \|u_i\|_p \\ \\ \|u_i + hd_i \nabla \cdot (A_i \nabla u_i)\|_p^p - (1+\epsilon h)^p \|u_i\|_p^p & \text{if } \ \|u_{i_0}\|_p \leq \|u_i\|_p \ . \end{cases}$$

In both cases $k(0) \leq 0$ and $k'(0) < 0$ (the proof is similar to the proof of $k'(0) < 0$ in Case 1, since $\|u_{i_0}\|_p > 0$ and $\|u_i\|_p > 0$). Therefore, for some small h, $k(h) \leq 0$, which implies that:

$$\lim_{h \to 0^+} \frac{1}{h} \left(\frac{\|u_i + hd_i \Delta u_i\|_p}{\|u_i\|_p} - 1 \right) \leq 0 \ .$$

Now by Lemma 24, since

$$\frac{1}{|\Omega|^{\frac{1}{p}}} \|u_i + hd_i \nabla \cdot (A_i \nabla u_i)\|_p \ \to \ \|u_i + hd_i \nabla \cdot (A_i \nabla u_i)\|_\infty, \quad \text{as} \quad p \to \infty,$$

and

$$\frac{1}{|\Omega|^{\frac{1}{p}}} \|u_i\|_p \ \to \ \|u_i\|_\infty, \quad \text{as} \quad p \to \infty,$$

we can conclude that

$$\lim_{h \to 0^+} \frac{1}{h} \left(\frac{\|u_i + hd_i \nabla \cdot (A_i \nabla u_i)\|_\infty}{\|u_i\|_\infty} - 1 \right) \leq 0 \ .$$

In other words, for a fixed $\epsilon > 0$, there exists $h_i > 0$ such that for any $0 < h < h_i$,

$$\|u_i + hd_i \nabla \cdot (A_i \nabla u_i)\|_\infty \leq (1+\epsilon h) \|u_i\|_\infty \quad \text{for any} \quad i \in \{1, \ldots, n\}.$$

Let $h_0 = \min_i h_i$. Then for any $0 < h < h_0$,

$$\begin{aligned} \max_i q_i \|u_i + hd_i \nabla \cdot (A_i \nabla u_i)\|_\infty \quad &=: \quad q_j \|u_j + hd_j \nabla \cdot (A_j \nabla u_j)\|_\infty \\ &\leq \quad q_j (1+\epsilon h) \|u_j\|_\infty \\ &\leq \quad (1+\epsilon h) \max_i q_i \|u_i\|_\infty, \end{aligned}$$

which implies

$$\lim_{h \to 0^+} \frac{1}{h} \left(\frac{\max_i q_i \|u_i + hd_i \nabla \cdot (A_i \nabla u_i)\|_\infty}{\max_i q_i \|u_i\|_\infty} - 1 \right) \leq 0.$$

□

Remark 21. Using the result of this section, namely Theorem 16, one can study Example 1 and Example 2 from Section 4.2 for space dependent reaction diffusion equations.

Generalization of Example 1 from Section 4.2.

$$\begin{aligned}\frac{\partial x}{\partial t} &= z(t) - \delta x + k_1 y - k_2(S_Y - y)x + d_1 \nabla \cdot (A_1 \nabla x)\\ \frac{\partial y}{\partial t} &= -k_1 y + k_2(S_Y - y)x + d_2 \nabla \cdot (A_2 \nabla y).\end{aligned} \tag{4.21}$$

where A_i's are as defined in Equation (4.17). As we explained in Example 1, there exists a positive diagonal matrix Q, such that $\sup_{(x,y)} \mu_{1,Q}[J_F(x,y)] < 0$. Therefore, by Theorem 16, Equation (4.21) is also contractive.

Generalization of Example 2, Section 4.2.

$$\begin{aligned}\frac{\partial u}{\partial t} &= a - u - \rho R(u,v) + d_1 \nabla \cdot (A_1 \nabla u)\\ \frac{\partial v}{\partial t} &= \alpha(b - v) - \rho R(u,v) + d_2 \nabla \cdot (A_2 \nabla v),\end{aligned} \tag{4.22}$$

As we explained in Example 2, $\sup_{(u,v)} \mu_1[J_F(u,v)] < 0$. Therefore, by Theorem 16, Equation (4.22) is also contractive.

4.4 Spatial uniformity of solutions of reaction diffusion PDEs: non L^2 norm approaches

The convergence to uniform solutions in reaction diffusion partial differential equations $\partial u/\partial t = F(u,t) + D(t)\mathbf{\Delta} u$ where $u = u(\omega, t)$, is a formal analogue of the synchronization of ODE systems. In the analogy, we think of $u(\omega, \cdot)$ as representing an individual system or agent (the index "i" in the synchronization problem) whose state is described at time t by $u = u(\omega, t)$. (So $u = u(\omega, t)$ plays the role of $x_i(t)$. We use "u" to denote the state, instead of x, so as to be consistent with standard PDE notations.) Questions of convergence to uniform solutions in reaction diffusion PDEs are also a classical topic of research. We think of convergence to spatially uniform solutions as a sort of "synchronization" of independent "agents", one at each spatial location, and each evolving according to a dynamics specified by an ODE. In that interpretation, our work is related to a large literature on synchronization of discrete groups of agents connected by diffusion, whose interconnections are specified by an undirected graph.

Definition 13. *We say that the reaction diffusion PDE (4.1) synchronizes, if for any global solution u of (4.1), subject to Neumann or Dirichlet boundary conditions, there exists $\bar{u}(t)$ such that $\|u(\cdot,t) - \bar{u}(t)\| \to 0$ as $t \to \infty$, or equivalently $\|\nabla u(\cdot,t)\| \to 0$ as $t \to \infty$.*

Remark 22. Under the conditions of Theorem 15, if $c = \sup_{(x,t)} \mu_{p,Q}[J_F(x,t)] < 0$, any global solution u of the PDE (4.1) with $u(\omega, 0) = u_0(\omega)$ exponentially converges to the spatially uniform solution $\bar{u}(t)$ which is itself the solution of the following ODE system:

$$\dot{x} = F(x,t), \quad x(0) = \frac{1}{|\Omega|} \int_\Omega u_0(\omega)\, d\omega. \tag{4.23}$$

But, note that the condition $c < 0$ rules out any interesting non-equilibrium behavior. For instance in Goodwin's oscillatory system, Section 3.3.3, $c < 0$ kills out the oscillation. So we look for a weaker condition than $c < 0$, that guarantees spatial uniform convergence result (which is a weaker property than contraction) while keeps interesting non-equilibrium behavior, like oscillatory in Goodwin example.

Recall, [105], that for any bounded, open subset $\Omega \subset \mathbb{R}^m$, there exists a sequence of non-negative eigenvalues $0 \le \lambda_1^{(n)} \le \lambda_2^{(n)} \le \cdots$ going to ∞, (superscript (n) for Neumann) and a sequence of corresponding orthonormal eigenfunctions $\phi_1^{(n)}, \phi_2^{(n)}, \ldots$ (defining a Hilbert basis of $L^2(\Omega)$) satisfying the following Neumann eigenvalue problem:

$$\begin{aligned} -\Delta \phi_i^{(n)} &= \lambda_i^{(n)} \phi_i^{(n)} \quad &&\text{in } \Omega \\ \nabla \phi_i^{(n)} \cdot \mathbf{n} &= 0 &&\text{on } \partial\Omega. \end{aligned} \tag{4.24}$$

Note that the first eigenvalue is always zero, $\lambda_1^{(n)} = 0$, and the corresponding eigenfunction is a nonzero constant ($\phi_1^{(n)}(\omega) = 1/\sqrt{|\Omega|}$).

The following re-phasing of a theorem from [42], provides a sufficient condition on F and D (time invariant diffusion matrix) using the Jacobian matrix of the reaction term and the second Neumann eigenvalue of the Laplacian operator on the given spatial domain to insure the convergence of trajectories, in this case to their space averages in weighted L^2 norms. The proof in [42] is based on the use of a quadratic Lyapunov function, which is appropriate for Hilbert spaces. We have translated the result to the language of contractions. (Actually, the result in [42] is stronger, in that it allows for non-diagonal diffusion and also non-diagonal weighting matrices Q, by substituting these assumptions by a commutativity type of condition, see Section 4.5 for more details and a generalization to spatially-varying diffusion.)

Theorem 17. *Consider the reaction diffusion system (4.1) subject to the Neumann boundary condition and assume that $F(x,t)$ is C^1 on x. Let*

$$c := \sup_{(x,t)\in V\times[0,\infty)} \mu_{2,Q}\left[J_F(x,t) - \lambda_2^{(n)} D\right],$$

where Q is a positive diagonal matrix. Then

$$\|u(\cdot,t) - \tilde{u}(t)\|_{2,Q} \leq e^{ct}\|u(\cdot,0) - \tilde{u}(0)\|_{2,Q},$$

where $\tilde{u}(t) = \frac{1}{|\Omega|}\int_\Omega u(\omega,t)\,d\omega$.

Note that when $c < 0$, the reaction diffusion system (4.1) synchronizes. As we discussed in the biochemical example, in Section 4.2,

$$\sup_{(x,t)\in V\times[0,\infty)} \mu_{2,Q}\left[J_F(x,t) - \lambda_2^{(n)} D\right] \geq 0,$$

therefore, conditions given in [42] do not hold for the biochemical example.

We next prove an analogous result to Theorem 17 for any norm but restricted to the linear operators F, $F(u,t) = A(t)u$, where for any t, $A(t) \in \mathbb{R}^{n\times n}$.

Theorem 18. *Consider the reaction diffusion system (4.1), for a linear operator F. For a given norm $\|\cdot\|$ in $\mathbb{R}^n$, let*

$$c := \sup_{(x,t)\in V\times[0,\infty)} \mu\left[J_F(x,t) - \lambda_2^{(n)} D(t)\right],$$

where μ is the logarithmic norm induced by $\|\cdot\|$. Then for any $\omega \in \Omega$ and any $t \geq 0$,

$$\|u(\omega,t) - \bar{u}(t)\| \;\leq\; \sum_{i\geq 2}\left\|\alpha_i(t)\phi_i^{(n)}(\omega)\right\| \;\leq\; e^{ct}\sum_{i\geq 2}\left\|\alpha_i(0)\phi_i^{(n)}(\omega)\right\|,$$

where $\bar{u}(t)$ is the solution of the system (4.23) with initial condition $u_0(\omega) = u(\omega,0)$, and $\alpha_i(t) = \int_\Omega u(\omega,t)\phi_i^{(n)}(\omega)\,d\omega$. In particular, when $c < 0$,

$$\|u(\omega,t) - \bar{u}(t)\| \to 0 \quad \textit{exponentially, as } t \to \infty.$$

Proof. We first show that the solution of Equation (4.23), namely $\bar{u}$, is equal to $\tilde{u}(t) = \frac{1}{|\Omega|}\int_\Omega u(\omega,t)\ d\omega$. Note that both $\bar{u}$ and $\tilde{u}$ satisfy $\dot{x} = A(t)x$. In addition, by the definition, $\bar{u}(0) = \tilde{u}(0) = \frac{1}{|\Omega|}\int_\Omega u(\omega,0)\ d\omega$. Therefore, by uniqueness of the solutions of ODEs, $\bar{u}(t) = \tilde{u}(t)$. The solution $u(\omega,t)$ can be written as follows:

$$u(\omega,t) \ = \ \sum_{i\geq 1}\phi_i^{(n)}(\omega)\alpha_i(t), \tag{4.25}$$

where for any t, $\alpha_i(t) = \int_\Omega u(\omega,t)\phi_i^{(n)}(\omega)\ d\omega \in \mathbb{R}^n$ and $\phi_i^{(n)}$'s are the eigenfunctions of (4.24).

Claim 1.

$$u(\omega,t) - \bar{u}(t) \ = \ \sum_{i\geq 2}\alpha_i(t)\phi_i^{(n)}(\omega). \tag{4.26}$$

Using the expansion of u as in (4.25), we have

$$u(\omega,t) - \bar{u}(t) \quad = \quad \alpha_1(t)\phi_1^{(n)}(\omega) - \bar{u}(t) + \sum_{i\geq 2}\phi_i^{(n)}(\omega)\alpha_i(t).$$

Multiplying both sides of the above equality by the constant eigenfunction $\phi_1^{(n)}$ and taking integral over Ω, by orthonormality of the $\phi_i^{(n)}$'s, we get:

$$\int_\Omega (u(\omega,t) - \bar{u}(t))\ d\omega = \alpha_1(t).$$

We showed that $\bar{u}(t) = \frac{1}{|\Omega|}\int_\Omega u(\omega,t)\ d\omega$, hence $\alpha_1(t) = 0$. This proves Claim 1.

Claim 2. Fix $\omega \in \Omega$. Then for any $i \geq 1$,

$$\dot{\alpha}_i(t) \quad = \quad \left(A(t) - \lambda_i^{(n)}D(t)\right)\alpha_i(t).$$

Using the expansion of u as in (4.25) and after omitting the arguments ω and t for simplicity, we have the following expression for $\dot{u}$:

$$\sum_{i\geq 1}\dot{\alpha}_i\phi_i^{(n)} = Au + D\boldsymbol{\Delta}u = A\sum_{i\geq 1}\alpha_i\phi_i^{(n)} + D\boldsymbol{\Delta}\sum_{i\geq 1}\alpha_i\phi_i^{(n)} = \sum_{i\geq 1}\left(A - \lambda_i^{(n)}D\right)\alpha_i\phi_i^{(n)}.$$

Multiplying both sides of the above equality by $\phi_i^{(n)}$ and taking integral over Ω, by orthonormality of $\phi_i^{(n)}$'s we get:

$$\dot{\alpha}_i(t) \ = \ \left(A - \lambda_i^{(n)}D\right)\alpha_i(t).$$

This proves Claim 2.

By subadditivity of μ and because $\lambda_2^{(n)} \leq \lambda_3^{(n)} \leq \cdots$, if $\forall t$, $\mu\left[A(t) - \lambda_2^{(n)} D(t)\right] \leq c$, then $\forall t$ and $\forall i > 2$, $\mu\left[A(t) - \lambda_i^{(n)} D(t)\right] \leq c$. Therefore, by Claim 2 and Lemma 5:

$$\|\alpha_i(t)\| \leq e^{ct} \|\alpha_i(0)\|.$$

Using the above inequality and triangle inequality in Equation (4.26), for any $\omega \in \Omega$ and any t, we get the following inequality:

$$\|u(\omega, t) - \bar{u}(t)\| \;\leq\; \sum_{i \geq 2} \left\|\alpha_i(t) \phi_i^{(n)}(\omega)\right\| \;\leq\; e^{ct} \sum_{i \geq 2} \left\|\alpha_i(0) \phi_i^{(n)}(\omega)\right\|.$$

Specifically, when $c < 0$, $\|u(\omega, t) - \bar{u}(t)\| \to 0$, exponentially as $t \to \infty$. $\square$

In what follows, we first present some conditions, analogous to the conditions in Theorem 15, which guarantee contractivity of the solutions of a reaction diffusion PDE with *Dirichlet* boundary conditions. Then, we show that how contractivity of the reaction diffusion PDE with Dirichlet boundary conditions implies spatial uniformity for the asymptotic behavior of the solutions of a reaction diffusion PDE with *Neumann* boundary conditions. As with synchronization, for non-Euclidean norms we only provide results in special cases, and the general problem being open (see Section 5).

Recall, [105], that for any bounded, open subset $\Omega \subset \mathbb{R}^m$, there exist a sequence of positive eigenvalues $0 < \lambda_1^{(d)} \leq \lambda_2^{(d)} \leq \cdots$ going to ∞ (superscript (d) for Dirichlet), and a sequence of corresponding orthonormal eigenfunctions $\phi_1^{(d)}, \phi_2^{(d)}, \ldots$ (defining a Hilbert basis of $L^2(\Omega)$) satisfying the following Dirichlet eigenvalue problem:

$$\begin{aligned} -\Delta \phi_i^{(d)} &= \lambda_i^{(d)} \phi_i^{(d)} \quad && \text{in } \Omega \\ \phi_i^{(d)} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{4.27}$$

Let us assume that Ω is a connected open set. Then the first eigenvalue $\lambda_1^{(d)}$ is simple and the first eigenfunction $\phi_1^{(d)}$ has a constant sign on Ω. Without loss of generality, $\phi_1^{(d)}$ can be assumed to be everywhere positive on Ω.

We next prove an analogues result to Theorem 15 (restricted to $p = 1$), for reaction diffusion PDE (4.1) subject to the Dirichlet boundary condition (4.3).

Theorem 19. *Consider the reaction diffusion PDE (4.1) subject to the Dirichlet boundary condition (4.3) and assume that $F(x,t)$ is C^1 on x. Let*

$$c = \sup_{(x,t)} \mu_{1,Q}\left[J_F(x,t) - \lambda_1^{(d)} D(t)\right],$$

and let $u(\omega,t)$ and $v(\omega,t)$ be two solutions of (4.1) and (4.3). Then

$$\|(u-v)(\cdot,t)\|_{1,\phi,Q} \leq e^{ct}\|(u-v)(\cdot,0)\|_{1,\phi,Q}, \tag{4.28}$$

where $\|u\|_{1,\phi,Q} = \|\phi u\|_{1,Q}$ and $\phi = \phi_1^{(d)} \geq 0$ is an eigenfunction corresponding to $\lambda_1^{(d)}$.

To prove Theorem 19, we need the following lemmas:

Lemma 25. *Let Ω be an open subset of $\mathbb{R}^m$. For any fixed t, let $\mathcal{A}(t)$ denote an $n \times n$ diagonal matrix of operators on $\mathbf{Y}_V^{(d)}$ with operators $d_i(t)\Delta$ on the diagonal. Let $\Lambda^{(d)}(t)$ denote an $n \times n$ diagonal matrix of operators on $\mathbf{Y}_V^{(d)}$ with operators $\Lambda_i^{(d)}(t)$ on the diagonal which are defined as follows:*

$$\Lambda_i^{(d)}(t)(\psi)(\omega) := \lambda_1^{(d)} d_i(t)\psi_i(\omega).$$

Then

$$\mu_{1,\phi,Q}^{+}\left[\mathcal{A} + \Lambda^{(d)}\right] = 0, \tag{4.29}$$

where $\mu_{1,\phi,Q}^{+}$ is induced by $\|\cdot\|_{1,\phi,Q}$.

See the Appendix for a proof.

Lemma 26. *For a Lipschitz function $G\colon \mathbb{R}^n \to \mathbb{R}^n$, define $\hat{G}\colon \mathbf{Y}_V^{(d)} \to \mathbb{R}^n$ as follows:*

$$\hat{G}(u)(\omega) := G(u(\omega)).$$

Then,

$$\mu_{p,\phi,Q}^{+}[\hat{G}] \leq \mu_{p,Q}[G]. \tag{4.30}$$

In addition, if G is continuously differentiable then

$$\mu_{p,\phi,Q}^{+}[\hat{G}] \leq \mu_{p,Q}[G] = \sup_x \mu_{p,Q}[J_G(x)].$$

Note that although we will need this lemma only for $p = 1$, the result of this lemma is true for any $1 \le p \le \infty$. See the Appendix for a proof.

Proof of Theorem 19. Suppose that u is a solution of Equation (4.1) defined on $\Omega \times [0, T)$. Define $\hat{u}$, $\mathcal{H}_t$, and $\mathcal{A}$ as follows:

- $\hat{u}\colon [0,T) \to \mathbf{Y}_V^{(d)}, \quad \hat{u}(t)(\omega) := u(\omega, t).$
- $\mathcal{H}_t\colon \mathbf{Y}_V^{(d)} \to \mathbb{R}^n, \quad \mathcal{H}_t(\psi)(\omega) := F(\psi(\omega), t), \quad \forall \psi \in \mathbf{Y}_V^{(d)}, \quad \forall \omega \in \Omega.$
- $\mathcal{A}$ is as defined in Lemma 25.

Then by the definition,

$$\frac{d\hat{u}}{dt}(t) \;=\; (\mathcal{H}_t + \mathcal{A})\,(\hat{u}(t))\,. \tag{4.31}$$

Suppose u and v are two solutions of Equation (4.1). By Lemma 3 and Equation (4.31) we have:

$$D^+\|(\hat{u} - \hat{v})(t)\|_{1,\phi,Q} \le \mu^+_{1,\phi,Q}[\mathcal{H}_t + \mathcal{A}]\|(\hat{u} - \hat{v})(t)\|_{1,\phi,Q}. \tag{4.32}$$

Let $\Lambda^{(d)}$ be as in Lemma 25. By subadditivity of μ^+ (Proposition 5), Lemma 25 and Lemma 26, we have:

$$\begin{aligned}
\mu^+_{1,\phi,Q}[\mathcal{H}_t + \mathcal{A}] &\le \mu^+_{1,\phi,Q}[\mathcal{H}_t - \Lambda^{(d)}] + \mu^+_{1,\phi,Q}[\mathcal{A} + \Lambda^{(d)}] \\
&\le \mu^+_{1,\phi,Q}\left[\mathcal{H}_t - \Lambda^{(d)}\right] \\
&\le \sup_{x \in V} \mu_{1,Q}\left[J_F(x,t) - \lambda_1^{(d)} D(t)\right] \\
&\le \sup_{t \in [0,T)} \sup_{x \in V} \mu_{1,Q}\left[J_F(x,t) - \lambda_1^{(d)} D(t)\right] \\
&= c.
\end{aligned} \tag{4.33}$$

By (4.32), (4.33), and Lemma 5, we get:

$$\|(\hat{u} - \hat{v})(t)\|_{1,\phi,Q} \le e^{ct}\|(\hat{u} - \hat{v})(0)\|_{1,\phi,Q}.$$

In terms of the PDE (4.1), this last estimate can be equivalently written as:

$$\|(u - v)(\cdot, t)\|_{1,\phi,Q} \le e^{ct}\|(u - v)(\cdot, 0)\|_{1,\phi,Q}.$$

□

Note that unlike in Neumann boundary problems, one cannot conclude synchronization from contraction in the Dirichlet boundary problems unless for any t, $F(0,t) = 0$:

Corollary 9. *Under the conditions of Theorem 19, if* $F(0,t) = 0$*, then* $v = 0$ *is a uniformly spatial solution of Equations (4.1) and (4.3), and therefore, for any solution* u *of Equations (4.1) and (4.3),*

$$\|u(\cdot,t)\|_{1,\phi,Q} \le e^{ct}\|u(\cdot,0)\|_{1,\phi,Q},$$

Hence, when $c < 0$ *and* u *is defined globally, the PDE system synchronizes.*

The following theorem provides a sufficient condition for synchronization of reaction diffusion systems subject to the Neumann boundary condition restricted to one dimensional space and $p = 1$. The proof is based on the results of Theorem 19.

Theorem 20. *Let* $u(\omega,t)$ *be a solution of*

$$\begin{aligned} \frac{\partial u}{\partial t}(\omega,t) &= F(u(\omega,t),t) + D(t)\,\frac{\partial^2 u}{\partial \omega^2}(\omega,t) \quad \text{on } (0,L) \\ \frac{\partial u}{\partial \omega}(0,t) \;&=\; \frac{\partial u}{\partial \omega}(L,t) \;=\; 0, \end{aligned} \tag{4.34}$$

defined for all $t \in [0,T)$ *for some* $0 < T \le \infty$*. In addition, assume that* $u(\cdot,t) \in C^3(\Omega)$*, for all* $t \in [0,T)$*. Let*

$$c \;=\; \sup_{t\in[0,T)}\sup_{x\in V}\mu_{1,Q}\left[J_F(x,t) - \frac{\pi^2}{L^2}D(t)\right].$$

Then for all $t \in [0,T)$*:*

$$\left\|\frac{\partial u}{\partial \omega}(\cdot,t)\right\|_{1,\phi,Q} \le e^{ct}\left\|\frac{\partial u}{\partial \omega}(\cdot,0)\right\|_{1,\phi,Q}, \tag{4.35}$$

where

$$\|\cdot\|_{1,\phi,Q} := \|\sin(\pi\omega/L)(\cdot)\|_{1,Q}\,.$$

The significance of Theorem 20 lies in the fact that $\sin(\pi\omega/L)$ is nonzero everywhere in the domain (except at the boundary). In that sense, we have exponential convergence to uniform solutions in a weighted L^1 norm, the weights being specified in V by the matrix Q and in space by the function $\sin(\pi\omega/L)$.

Proof. Suppose that u is a solution of Equation (4.34) defined on $[0,L]\times[0,T)$, and let $v = \frac{\partial u}{\partial \omega}$. Then by taking $\frac{\partial}{\partial \omega}$ from the both sides of Equation (4.34), we get the following PDE:

$$\frac{\partial v}{\partial t} = J_F(u,t)v + D(t)\,\frac{\partial^2 v}{\partial \omega^2}, \tag{4.36}$$

subject to Dirichlet boundary condition: $v(0) = v(L) = 0$.

For $\Omega = (0, L)$, the first Dirichlet eigenvalue is π^2/L^2 and a corresponding eigenfunction is $\sin(\pi\omega/L)$. Therefore, by Equation (4.36) and Corollary 9,

$$\|v(\cdot,t)\|_{1,\phi,Q} \leq e^{ct}\|v(\cdot,0)\|_{1,\phi,Q},$$

where $c = \sup_{t\in[0,T)} \sup_{x\in V} \mu_{1,Q}\left[J_F(x,t) - \frac{\pi^2}{L^2}D(t)\right]$. □

Another proof of Theorem 20 using the method of discretization is given in the Appendix.

Remark 23. In the case of $\Omega = (0, L)$, $\lambda_1^{(d)} = \lambda_2^{(n)}$. Therefore, one can state the conditions of Theorem 20 in terms of the second Neumann eigenvalue instead of the first Dirichlet eigenvalue.

Examples

Example 1. In the biochemical model, we showed that there exists a positive diagonal matrix Q such that

$$c := \sup_{(x,y)\in V} \mu_{1,Q}[J_F(x,y)] < 0.$$

This condition implies that any solution of (4.14) converges to a uniform solution with at least rate c (Remark 22). Next, we show that by Theorem 20, any solution of (4.14) converges to a uniform solution at a better rate than c:

By subadditivity of μ, we have:

$$\sup_{(x,y)\in V} \mu_{1,Q}[J_F(x,y) - \pi^2 D] \leq \sup_{(x,y)\in V} \mu_{1,Q}[J_F(x,y)] - \pi^2 d, \quad \text{where } d = \min\{d_1, d_2\}.$$

Therefore, $c_0 := \sup_{(x,y)\in V} \mu_{1,Q}[J_F(x,y) - \pi^2 D] < c < 0$. Hence, by Theorem 20, for any solution $u = (x,y)^T$ of (4.14):

$$\left\|\frac{\partial u}{\partial\omega}(\cdot,t)\right\|_{1,\phi,Q} \leq e^{c_0 t} \left\|\frac{\partial u}{\partial\omega}(\cdot,0)\right\|_{1,\phi,Q},$$

where in this example $\phi(\omega) = \sin(\pi\omega)$, since $\Omega = (0, 1)$.

Figure 4.1 indicates two different solutions of the biochemical model, Equation (4.14), namely $(x_1, y_1)^T$ and $(x_2, y_2)^T$ on $\Omega = (0, 2)$ for specific initial conditions, and for a periodic input z, namely $z(t) = 20(1+\sin(10t))$, and for the following set of parameters:

$$\delta = 20,\ k_1 = 0.5,\ k_2 = 5,\ S_Y = 0.1,\ d_1 = 0.001,\ d_2 = 0.1.$$

Also, in Figure 4.1, the difference between the two solutions has been shown that goes to zero as expected.

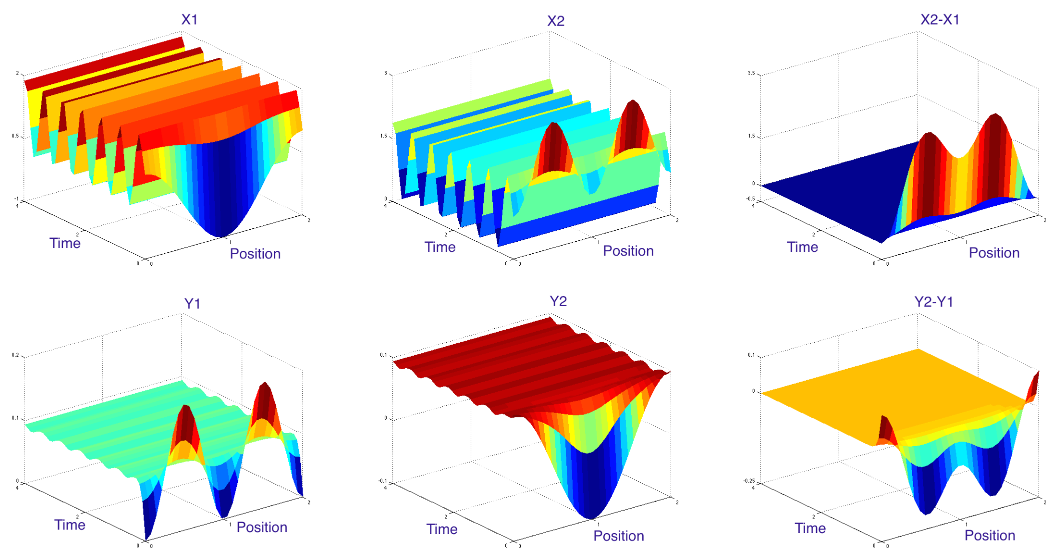


Figure 4.1: Two solutions of Equation (4.14) and their differences

Example 2. In 1965, Brian Goodwin proposed a differential equation model, that describes the generic model of an oscillating autoregulatory gene, and studied its oscillatory behavior [76]. In Section 3.3.3, we studied the following systems of ODEs which is a variant of Goodwin's model [75]:

$$\begin{aligned}
\dot{x} &= \frac{a}{k + z(t)} - bx \\
\dot{y} &= \alpha x - \beta y \\
\dot{z} &= \gamma y - \frac{\delta z}{k_M + z}.
\end{aligned}$$

In this section, we assume a continuous model where species diffuse in space. This example has been studied in [42]. The following system of PDEs, subject to Neumann boundary conditions, describe the evolution of X, Y, and Z on $(0,1)\times[0,\infty)$:

$$
\begin{aligned}
\frac{\partial x}{\partial t} &= \frac{a}{k+z} - b\,x + d_1\Delta x \\
\frac{\partial y}{\partial t} &= \alpha\,x - \beta\,y + d_2\Delta y \\
\frac{\partial z}{\partial t} &= \gamma\,y - \frac{\delta z}{k_M + z} + d_3\Delta z
\end{aligned}
\tag{4.37}
$$

Figure 4.2 provides plots of solutions x, y, and z of (4.37), using the following parameter values from the textbook [77]:

$$a = 150,\ k = 1,\ b = \alpha = \beta = \gamma = 0.2,\ \delta = 15,\ K_M = 1, \tag{4.38}$$

which oscillate when there is no diffusion ($d_1 = d_2 = d_3 = 0$).

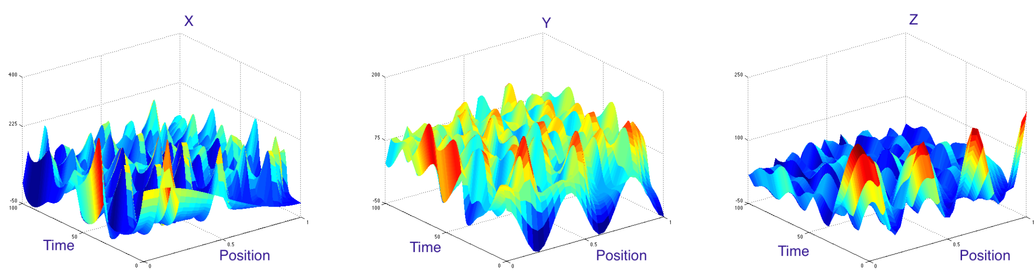


Figure 4.2: Goodwin oscillator, no diffusion (parameters as in Equation (4.38))

A simple calculation shows that for the weighted matrix $Q = \operatorname{diag}(1, 12, 11)$, and for $2.2/\pi^2 < d_1$, and $d_2 = d_3 = 0$,

$$\sup_{w=(x,y,z)^T} \mu_{1,Q}[J_F(w) - \pi^2 D] < 0.$$

Applying Theorem 20, we conclude that for $2.2/\pi^2 < d_1$ and $d_2 = d_3 = 0$, (4.37) synchronizes, meaning that solutions tend to uniform solutions. Note that to have synchronization, $2.2/\pi^2$ is not a sharp lower bound for d_1, i.e., the system would synchronize even for smaller values of d_1.

Figure 4.3 shows the spatially uniformity of the solutions of (4.37), for the same parameter values and initial conditions as in Figure 4.2, when $2.2/\pi^2 < d_1$, here $d_1 = 0.3$, and $d_2 = d_3 = 0$.

In what follows, we compare our result with the results in [42] and [106] using the Goodwin example.

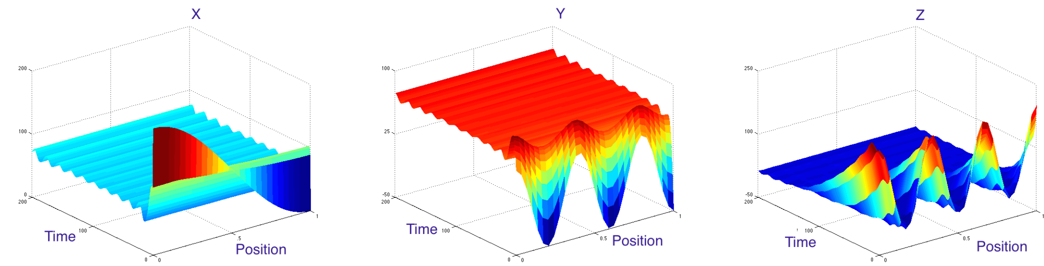


Figure 4.3: Goodwin oscillator, X diffuses (parameters as in Equation (4.38))

Considering Equation (4.37), the following sufficient condition in ([42], Equation 55) is given for synchronization:

$$\frac{\alpha\gamma a}{k(b+\lambda d_1)(\beta+\lambda d_2)\lambda d_3} \; < \; 4, \tag{4.39}$$

where $\lambda = \pi^2$. Note that when $d_3 = 0$, one cannot apply (4.39) directly to get synchronization.

In ([106], Equation 3), Othmer provides a sufficient condition for uniform behavior of the solutions of the reaction diffusion (4.1) on $(0, L)$, subject to Neumann boundary conditions:

$$\sup_{w} \|J_F(w)\| < \pi^2/L^2 \min_{i} d_i. \tag{4.40}$$

In Goodwin's example (4.37), $\sup_w \|J_F(w)\|$ is positive and finite (the sup is taken at $z = 0$), and $\min_i d_i = 0$, hence (4.40) doesn't hold and this condition is not applicable for this example.

4.5 Spatial uniformity of solutions of reaction diffusion PDEs: weighted L^2 norm approaches

Acknowledgement of book chapter publication:

Parts of the material in this section have been published in the book chapter [46].

In this section, we study the asymptotic behavior of the solutions of the following reaction diffusion system which is a generalized form of system (4.1) and another generalized form of system (4.17). For any $t > 0$ and $\omega \in \Omega$:

$$\frac{\partial u}{\partial t}(\omega, t) = F(u(\omega,t),t) + \sum_{i=1}^{r} D_i(t)\mathcal{L}_i u(\omega,t), \tag{4.41}$$

subject to the Neumann boundary condition:

$$\frac{\partial u_i}{\partial \mathbf{n}}(\xi,t) = 0, \quad \forall \xi \in \partial\Omega, \;\; \forall t \in [0,\infty), \;\; \forall i = 1,\ldots,n, \tag{4.42}$$

where $u = (u_1,\ldots,u_n)^T$, F, Ω, $\partial\Omega$, and $\mathbf{n}$ are as defined in Equation (4.1) and (4.17). In addition, we assume that $F(x,t)$ is C^1 on x. For each $i = 1,\ldots,r$, $D_i(t)$ is an $n\times n$ diagonal matrix with entries $[D_i(t)]_{jj} = d_{ij}(t)$, for $j = i,\ldots,n_i$ and 0 elsewhere, where d_{ij}'s are non-negative continuous functions of t, and $n_1 + \cdots + n_r = n$. For example, $D_1(t) = \operatorname{diag}(d_{11}(t),\ldots,d_{1n_1}(t),0,\ldots,0)$ and $D_r(t) = \operatorname{diag}(0,\ldots,0,d_{r1}(t),\ldots,d_{rn_r}(t))$. For each $i = 1,\ldots,r$,

$$\mathcal{L}_i u = \left(\nabla\cdot(A_i(w)\nabla u_1),\ldots,\nabla\cdot(A_i(w)\nabla u_n)\right)^T,$$

where A_i's are as defined in Section 4.3.

Note that in Equation (4.1), $r = 1$ ($n_1 = n$), and $A_1(w) = I$. Therefore,

$$\sum_{i=1}^{r} D_i(t)\mathcal{L}_i u = D_1(t)\left(\nabla\cdot\nabla u_1,\ldots,\nabla\cdot\nabla u_n\right)^T = \left(d_1(t)\Delta u_1,\ldots,d_n(t)\Delta u_n\right)^T,$$

where $d_j(t) = d_{1j}(t)$.

In Equation (4.17), $r = n$ ($n_1 = \cdots = n_r = 1$), and A_i's are not necessarily identity. In this case $D_1(t) = \operatorname{diag}(d_1(t),0,\ldots,0)$, $D_2(t) = \operatorname{diag}(0,d_2(t),0,\ldots,0)$, etc. where $d_i(t) = d_{i1}(t)$. Therefore,

$$\sum_{i=1}^{r} D_i(t)\mathcal{L}_i u = \left(d_1(t)\nabla\cdot(A_1(w)\nabla u_1),\ldots,d_n(t)\nabla\cdot(A_n(w)\nabla u_n)\right)^T.$$

For a fixed $i \in \{1,\ldots,r\}$, let λ_k^i be the k-th Neumann eigenvalue of the operator $-\nabla\cdot(A_i\nabla)$ and e_k^i be the corresponding normalized eigenfunction:

$$\begin{aligned} -\nabla\cdot\left(A_i(\omega)\nabla e_k^i(\omega)\right) &= \lambda_k^i e_k^i(\omega), \qquad \omega \in \Omega \\ \nabla e_k^i(\xi)\cdot\mathbf{n} &= 0, \qquad \xi \in \partial\Omega. \end{aligned} \tag{4.43}$$

Note that for any i, $0 = \lambda_1^i < \lambda_2^i < \cdots \to \infty$ and $e_1^i(\omega) = \text{constant}$. For any fixed t, let

$$\Lambda_k(t) := \sum_{i=1}^{r} \lambda_k^i D_i(t). \tag{4.44}$$

For each $k \in \{1, 2, \ldots\}$, let E_k^i be the subspace spanned by the first k eigenfunctions:

$$E_k^i \;=\; \langle e_1^i, \ldots, e_k^i \rangle.$$

Now define the map $\Pi_{k,i}$ on $L^2(\Omega)$ as follows:

$$\Pi_{k,i}(v) \;=\; v - \pi_{k,i}(v),$$

where $\pi_{k,i}$ is the orthogonal projection map onto E_{k-1}^i, and we let $E_0^i = 0$. Namely for any $v = \sum_{j=1}^{\infty}(v, e_j^i)e_j^i$,

$$\begin{aligned}&\pi_{k,i}(v) = \sum_{j=1}^{k-1}(v, e_j^i)e_j^i, \quad \text{and} \quad \Pi_{k,i}(v) \;=\; \sum_{j=k}^{\infty}(v, e_j^i)e_j^i, \quad \text{for } k > 1,\\ &\pi_{1,i}(v) = 0, \quad \text{and} \quad \Pi_{1,i}(v) = v;\end{aligned} \tag{4.45}$$

where $(x, y) := \int x^T y$. Note that for any $i = 1, \ldots, r$,

$$\Pi_{2,i}(v) \;=\; v - \frac{1}{|\Omega|}\int_{\Omega} v, \tag{4.46}$$

and therefore,

$$\int_{\Omega} \Pi_{2,i}(v) \;=\; 0. \tag{4.47}$$

For any $v = (v_1, \ldots, v_n)^T$, define $\Pi_k(v) = v - \pi_k(v)$, where $\pi_k(v)$ is defined as follows:

$$\pi_k(v) = (\pi_{k,1}(v_1), \ldots, \pi_{k,1}(v_{n_1}), \ldots, \pi_{k,r}(v_{n-n_r+1}), \ldots, \pi_{k,r}(v_n))^T.$$

Observe that $\pi_k(v)$ is the orthogonal projection map onto

$$\underbrace{E_{k-1}^1 \times \cdots \times E_{k-1}^1}_{n_1 \text{ times}} \times \cdots \times \underbrace{E_{k-1}^r \times \cdots \times E_{k-1}^r}_{n_r \text{ times}}.$$

The goal of this section is to prove the following theorem which its first part is an analogous of Theorem 15 for system (4.41) for non diagonal Q but restricted to $p = 2$; and its second part is an analogous of [42, Theorem 1] to spatially-varying diffusion. In addition, the following theorem is an analogous of Theorem 14 for reaction diffusion equations.

Theorem 21. *Consider the reaction diffusion system (4.41) subject to the Neumann boundary condition. For $k = 1, 2$, let*

$$\mu_k := \sup_{(x,t) \in V \times [0,\infty)} \mu_{2,P}[J_F(x, t) - \Lambda_k],$$

for a positive definite matrix P such that for any $i = 1, \ldots, r$:

$$P^2 D_i + D_i P^2 \; > \; 0. \tag{4.48}$$

Then for any two solutions, namely u and v, of (4.41), we have:

1.
$$\|u(\cdot,t) - v(\cdot,t)\|_{2,P} \leq e^{\mu_1 t} \|u(\cdot,0) - v(\cdot,0)\|_{2,P}. \tag{4.49}$$

2.
$$\|\Pi_2(u(\cdot,t))\|_{2,P} \; \leq \; e^{\mu_2 t} \|\Pi_2(u(\cdot,0))\|_{2,P}. \tag{4.50}$$

To prove Theorem 21, we need the following two lemmas. Lemma 27 below is an analogous of Lemma 18.

Lemma 27. *Let $w = u - x$, where u is a solution of Equation (4.41) subject to the Neumann boundary condition and either $x = \pi_2(u)$ or $x = v$ is another solution of Equation (4.41). For a positive definite matrix Q, let*

$$\Phi(w) \; := \; \frac{1}{2}(w, Qw). \tag{4.51}$$

Then

$$\frac{d\Phi}{dt}(w) = (w, Q(F(u,t) - F(x,t))) + (w, Q\mathfrak{L}w)\,, \tag{4.52}$$

where $\mathfrak{L} := \sum_{i=1}^{r} D_i(t)\mathcal{L}_i$.

See the Appendix for a proof.

Lemma 28. *Suppose $u \in L^2(\Omega)$ satisfies the Neumann boundary conditions. For any $k \in \{1, 2, \ldots\}$,*

$$(\Pi_k(u), \mathfrak{L}\Pi_k(u)) \; \leq \; -(\Pi_k(u), \Lambda_k \Pi_k(u))\,. \tag{4.53}$$

where $\mathfrak{L} := \sum_{i=1}^{r} D_i(t)\mathcal{L}_i$. In addition for $k = 1, 2$ and any $n \times n$ symmetric matrix Q with the following property:

$$QD_i + D_i Q \; > \; 0 \qquad i = 1, \ldots, r, \tag{4.54}$$

we have:

$$(\Pi_k(u), Q\mathfrak{L}\Pi_k(u)) \; \leq \; -(\Pi_k(u), Q\Lambda_k \Pi_k(u))\,. \tag{4.55}$$

See the Appendix for a proof.

Proof of Theorem 21

By Lemma 6,

$$Q(J_F - \Lambda_k) + (J_F - \Lambda_k)^T Q \leq 2\mu_k Q, \tag{4.56}$$

where $Q = P^2$. Define w and $\Phi(w)$ as in Lemma 27 for $Q = P^2$. Since $\Phi(w) = \frac{1}{2}\|Pw\|_2^2$, to prove (4.49) and (4.50), it suffices to show that for $k = 1, 2$,

$$\frac{d}{dt}\Phi(w) \leq 2\mu_k \Phi(w).$$

Note that by Lemma 28, and the fact that $w = \Pi_1(u - v)$ or $w = \Pi_2(u)$, the second term of the right hand side of (4.52) satisfies:

$$(w, Q\mathfrak{L}w) \leq -(w, Q\Lambda_k w). \tag{4.57}$$

Next, by the Mean Value Theorem for integrals, and using (4.56), we rewrite the first term of the right hand side of (4.52) as follows:

$$\begin{aligned}
(w, Q(F(u,t) - F(x,t))) &= \int_\Omega w^T(\omega,t) Q(F(u(\omega,t),t) - F(x,t))\, d\omega \\
&= \int_\Omega w^T(\omega,t) Q \int_0^1 J_F(x + sw(\omega,t),t) \cdot w(\omega,t)\, ds\, d\omega \\
&= \int_0^1 \int_\Omega w^T(\omega,t) Q J_F(x + sw(\omega,t),t) \cdot w(\omega,t)\, d\omega\, ds.
\end{aligned}$$

This last equality together with (4.57) imply:

$$\begin{aligned}
&(w, Q(F(u,t) - F(x,t))) + (w, Q\mathfrak{L}w) \\
&\leq \int_0^1 \int_\Omega w^T(\omega,t) Q\big(J_F(x + sw(\omega,t),t) - \Lambda_k\big) \cdot w(\omega,t)\, d\omega\, ds \\
&\leq \frac{2\mu_k}{2} \int_0^1 ds \int_\Omega w^T Q w\, d\omega = \frac{2\mu_k}{2} \int_\Omega w^T Q w\, d\omega = 2\mu_k \Phi(w).
\end{aligned}$$

Therefore $\frac{d\Phi}{dt}(w) \leq 2\mu_k \Phi(w)$. □

Corollary 10. *In Theorem 21, if $\mu_1 < 0$ and the solutions are defined globally, then the reaction diffusion system (4.41) is contracting, meaning that solutions converge (exponentially) to each other, in P weighted L_2 norm, i.e., $\|u(\cdot,t) - v(\cdot,t)\|_{2,P} \to 0$, as $t \to \infty$.*

Corollary 11. *In Theorem 21, if* $\mu_2 < 0$ *and the solutions are defined globally, then any solution of the reaction diffusion system (4.41) converges (exponentially) to a uniform solution, in* P *weighted* L_2 *norm, i.e.,* $\|\Pi_2(u(\cdot,t))\|_{2,P} \to 0$, *as* $t \to \infty$.

Note that (4.55) does not necessarily hold for any $k > 2$, since for $k > 2$, the $\Pi_{k,i}$'s could be different for different i's. In the following lemma we provide a condition for which (4.55) holds for any k.

Lemma 29. *Assume* $P\mathfrak{L} = \mathfrak{L}P$, *where* P *is a positive definite* $n \times n$ *matrix, and* $\mathfrak{L} := \sum_{i=1}^{r} D_i(t)\mathcal{L}_i$. *Then for any* $k = 1, 2, \ldots$,

$$(\Pi_k(u), Q\mathfrak{L}\Pi_k(u)) \;\leq\; -\left(\Pi_k(u), Q\Lambda_k\Pi_k(u)\right),$$

where $Q = P^2$,

Proof. The proof is analogous to the proof of (4.55), using the fact that $P\mathfrak{L} = \mathfrak{L}P$ implies that P is diagonal (if all $\mathcal{L}_i$'s are different) or block diagonal (for equal Laplacian operators). $\square$

Remark 24. *Note that Theorem 21 is valid for* $Q = P^2$, *if* $P\mathfrak{L} = \mathfrak{L}P$ *is assumed instead of (4.54), because (4.55) holds by Lemma 29 and this is all that is needed in the proof. In the following theorem we use this condition to generalize the result of Theorem 21 for any arbitrary* k *but restricted to linear systems. We omit the proof, which is analogous.*

Theorem 22. *Consider the reaction diffusion system (4.41) and assume that* F *is a linear function. For* $k \in \{1, 2, \ldots\}$, *let*

$$\mu_k := \sup_{(x,t)\in V\times[0,\infty)} \mu_{2,P}[J_F(x,t) - \Lambda_k],$$

where P *is a positive definite matrix and* $P\mathfrak{L} = \mathfrak{L}P$. *Then for any two solutions of (4.41), namely* u *and* v, *we have:*

$$\|\Pi_k(u-v)(\cdot,t)\|_{2,P} \leq e^{\mu_k t}\|\Pi_k(u-v)(\cdot,0)\|_{2,P}.$$

Example

In Section 4.4 we studied the following system:

$$\begin{aligned}\dot{x} &= z(t) - \delta x + k_1 y - k_2(S_Y - y)x + d_1 \Delta x \\ \dot{y} &= -k_1 y + k_2(S_Y - y)x + d_2 \Delta y,\end{aligned} \tag{4.58}$$

and showed that there exists a positive diagonal matrix Q such that for Q-weighted L^1 norm, $\sup_{(x,y)} \mu_{1,Q}[J_F(x,y)] < 0$ and concluded that the reaction diffusion PDE is contractive. In addition, we had shown before that for any $p > 1$ and any positive diagonal matrix Q , $\sup_{(x,y)} \mu_{p,Q}[J_F(x,y)] \geq 0$.

Now we show that there exists some positive definite (but non-diagonal) matrix P such that for all $(x,y) \in V$, $\mu_{2,P}[J_F(x,y)] < 0$ and $P^2D + DP^2 > 0$, where $D = \mathrm{diag}\,(d_1, d_2)$. Then by Theorem 21 (for $r = 1$ and $\mathcal{L}_i u_i = \Delta u_i$), and Corollary 10, one can conclude that the system is contractive.

Claim. Let $Q = \begin{pmatrix} 1 & 1 \\ 1 & q \end{pmatrix}$, where $q > \max\left\{1 + \frac{\delta}{4k_1}, \left(\frac{1}{2\sqrt{d}} + \frac{\sqrt{d}}{2}\right)^2\right\}$, and $d = \frac{d_1}{d_2}$. Then $QJ_F + (QJ_F)^T < 0$ and $QD + DQ > 0$.

Proof of Claim. First note that Q is positive definite (because $q > 1$). We next compute QJ_F:

$$QJ_F = \begin{pmatrix} 1 & 1 \\ 1 & q \end{pmatrix} \begin{pmatrix} -\delta - a & b \\ a & -b \end{pmatrix} = \begin{pmatrix} -\delta & 0 \\ -\delta + (q-1)a & -b(q-1) \end{pmatrix}.$$

So

$$QJ_F + (J_F Q)^T = \begin{pmatrix} -2\delta & -\delta + (q-1)a \\ -\delta + (q-1)a & -2b(q-1) \end{pmatrix}.$$

To show $QJ_F + J_F^T Q < 0$, since the first order leading principal minor, i.e., -2δ, is negative, it suffices to show that the second order leading principal minor, i.e., $\det\left(QJ_F + J_F^T Q\right)$, is positive:

$$\det\left(QJ_F + J_F^T Q\right) = 4\delta b(q-1) - \left(-\delta + (q-1)a\right)^2.$$

Note that for any $q > 1$,

$$f(a) := \left(-\delta + (q-1)a\right)^2 \leq \delta^2 \quad \text{on} \quad [0, k_2 S_Y],$$

and

$$g(b) := 4\delta b(q-1) \geq 4\delta k_1(q-1) \quad \text{on} \quad [k_1, \infty].$$

Therefore, to have det > 0, it is enough to have $4\delta k_1(q-1) - \delta^2 > 0$, i.e., $q - 1 > \frac{\delta^2}{4\delta k_1}$, i.e., $q > 1 + \frac{\delta}{4k_1}$.

Now we compute $QD + DQ$:

$$QD + DQ = \begin{pmatrix} 2d_1 & d_1 + d_2 \\ d_1 + d_2 & 2qd_2 \end{pmatrix}.$$

since the first order leading principal minor of $QD + DQ$, i.e., $2d_1$, is positive, to show $QD + DQ > 0$, it suffices to show that the second order leading principal minor, i.e., $\det(QD + DQ) > 0$.

$$\det(QD + DQ) \quad = \quad 4d_1d_2q - (d_1 + d_2)^2 > 0 \quad \text{iff} \quad q > \left(\frac{1}{2\sqrt{d}} + \frac{\sqrt{d}}{2}\right)^2,$$

where $d = \frac{d_1}{d_2}$. This completes the proof of the Claim.

Now by Lemma 6 and Remark 12, $\mu_{2,P}[J_F(x,y)] < 0$, for all $(x,y) \in V$, where $P^2 = Q$. Hence by Theorem 21, the system (4.58) is contractive using P- weighted L^2 norm where P is not diagonal.

In what follows, we consider the biochemical example with space dependent diffusions (see Remark 21) and show that the result of this section cannot be applied to this system.

$$\begin{aligned} \frac{dx}{dt} &= z(t) - \delta x - k_2\,(S_Y - y)x + k_1\,y + d_1\,\nabla\cdot(A_1(\omega)\nabla x(\omega,t)) \\ \frac{dy}{dt} &= k_2\,(S_Y - y)x - k_1\,y + d_2\,\nabla\cdot(A_2(\omega)\nabla y(\omega,t))\,, \end{aligned} \tag{4.59}$$

where we assume $A_1 \neq A_2$ and define D_1 and D_2 as follows:

$$D_1 = \begin{pmatrix} d_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad D_2 = \begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}.$$

Claim. There exists no positive definite matrix P such that for $i = 1, 2$,

$$PD_i + D_iP > 0.$$

Proof of Claim. Let $P = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ be an arbitrary positive definite matrix. Then

$$PD_1 + D_1P = \begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} d_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} d_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 2ad_1 & bd_1 \\ bd_1 & 0 \end{pmatrix}.$$

Since the determinant of PD_1+D_1P is negative, it has a negative eigenvalue and cannot be positive definite.

Recall that in Remark 21 we showed that the system (4.59) is contractive using a weighted L^1 norm.

4.6 Global existence and uniqueness of the solutions of reaction diffusion PDEs

In this section, we review the conditions that guarantee the existence and uniqueness of the solutions of reaction diffusion equation (4.1) and provide some conditions for global existence of the solutions.

Theorem 23. *[92, Proposition 2.2] Consider the reaction diffusion equation (4.1) with either Neumann (4.2) or Dirichlet (4.3) boundary condition and initial condition $u(\cdot, 0) = u_0(\cdot) \in [L^\infty(\Omega)]^n$. Then there exist $T_{\max} > 0$ and $N_i \in C([0, T_{\max}))$ such that*

1. *System (4.1) has a unique, classical, noncontinuable solution $u(x, t)$ on $\bar{\Omega} \times [0, T_{\max})$;*

2. *$\|u_i(\cdot, t)\|_\infty \le N_i(t)$ for all $1 \le i \le n$ and $t \in [0, T_{\max})$.*

Moreover, if $T_{\max} < \infty$ then $\|u_i(\cdot, t)\|_\infty \to \infty$ as $t \to T_{\max}^-$, for some $1 \le i \le n$.

See also [90, Theorem 3.1] and [91, Theorem 1.11, Theorem 1.12].

Note that under the conditions of Theorem 23, if the system admits a compact invariant region $\Sigma \subset \mathbb{R}^n$, i.e., if $u(\omega, 0) \in \mathrm{int}(\Sigma)$, then $u(\omega, t) \in \Sigma$ for all $\omega \in \Omega$ and $0 < t < T_{\max}$ (as in the case of many applications, including the examples mentioned in the current work), then $T_{\max} = \infty$.

In [92], Morgan made the assumption that there exists a Lyapunov structure function $H \in C^2(M, \mathbb{R}_{\geq 0})$, where M is a unbounded region of $\mathbb{R}^n$ for which (4.1) is invariant. Under some assumptions on H, he obtained boundedness and stability results for (4.1). In this work, instead of finding a Lyapunov function, we provide a condition based on contraction theory for global existence of the solutions.

Theorem 24. *Consider the reaction diffusion equation (4.1) with Neumann boundary condition (4.2). Let* $c = \mu_\infty[F]$ *(or* $c = \sup_{(x,t)} \mu_\infty[J_F(x,t)]$*, if* F *is differentiable) be finite, and* u *be a solution of (4.1) defined on* $[0, T_{\max})$ *with initial condition* $u(\cdot, 0) = u_0(\cdot) \in [L^\infty(\Omega)]^n$*. Then* $T_{\max} = \infty$*.*

Proof. By Theorem 23, it suffices to show that $\limsup_{t \to T_{\max}^-} \|u(\cdot, t)\|_\infty$ is finite. Let v be a solution of $\dot{x} = F(x,t)$ with constant initial condition $v(0) = v_0$. Note that v is defined globally since F is globally Lipschitz (because F is assumed to be Lipschitz and $c < \infty$). Therefore, v is a global solution of (4.1) and by Theorem 15, for any $t < T_{\max}$,

$$\|u(\cdot,t) - v(t)\|_\infty \leq e^{ct} \|u_0(\cdot) - v_0\|_\infty \,.$$

Therefore, for any $t < T_{\max}$,

$$\|u(\cdot,t)\|_\infty \leq N(T_{\max}) + e^{c\,T_{\max}} \|u_0(\cdot) - v_0\|_\infty =: B(T_{\max}) < \infty \,,$$

where $N(T_{\max})$ is the upper bound of v on $[0, T_{\max})$, and hence

$$\limsup_{t \to T_{\max}^-} \|u(\cdot,t)\|_\infty \leq B(T_{\max}) < \infty.$$

□

4.7 Appendix

Proof of Lemma 24

Fix $p < q$. Then there exists $r > 0$ such that $\frac{1}{q} + \frac{1}{r} = \frac{1}{p}$. Indeed $r = \frac{1}{1/p - 1/q}$. Using Hölder Inequality,

$$\left(\int_\Omega |f|^p\right)^{\frac{1}{p}} \;\leq\; \left(\int_\Omega |f|^q\right)^{\frac{1}{q}} \left(\int_\Omega 1^r\right)^{\frac{1}{r}} \leq \left(\int_\Omega |f|^q\right)^{\frac{1}{q}} |\Omega|^{\frac{1}{p} - \frac{1}{q}} \,,$$

therefore,

$$\left(\frac{1}{|\Omega|}\int_\Omega |f|^p\right)^{\frac{1}{p}} \leq \left(\frac{1}{|\Omega|}\int_\Omega |f|^q\right)^{\frac{1}{q}}.$$

The above inequality shows that F is an increasing function of p. Now we show that as $p \to \infty$, $F(p) \to \|f\|_\infty$. Since for any p, $F(p) \leq \|f\|_\infty$, $\lim_{p\to\infty} F(p) \leq \|f\|_\infty$. To prove the converse inequality, for any $\epsilon > 0$, we define $E_\epsilon := \{\omega \in \Omega : |f(\omega)| > \|f\|_\infty - \epsilon\}$ which by the definition of $\|f\|_\infty$ has positive measure, i.e., $|E_\epsilon| > 0$. Note that

$$\left(\int_\Omega |f|^p\right)^{\frac{1}{p}} \geq \left(\int_{E_\epsilon} |f|^p\right)^{\frac{1}{p}} \geq (\|f\|_\infty - \epsilon)\left(|E_\epsilon|/|\Omega|\right)^{\frac{1}{p}}.$$

Since $|E_\epsilon|/|\Omega| > 0$, $\left(|E_\epsilon|/|\Omega|\right)^{\frac{1}{p}} \to 1$ as $p \to \infty$. Therefore, for any arbitrary $\epsilon > 0$,

$$\lim_{p\to\infty} F(p) \geq \|f\|_\infty - \epsilon,$$

which implies

$$\lim_{p\to\infty} F(p) \geq \|f\|_\infty.$$

Proof of Lemma 25

To prove Lemma 25, we need the following lemma:

Lemma 30. *For any $u\colon \Omega \subset \mathbb{R}^m \to \mathbb{R}$, assume that Δu is defined on Ω. Then, there exists a set $I \subset \Omega$ such that:*

- *$\mu(I) = 0$, where μ denote the measure; and*
- *$\Delta |u|$ is defined on $\Omega \setminus I$.*

In fact, $I = \{\omega \in \Omega \,:\, u(\omega) = 0,\ \nabla u(\omega) \neq 0\}$.

Proof. We only prove the special case $\Omega = (a, b)$. The proof for a general domain Ω is analogue. We show that I is countable, and hence of measure zero:

Fix $\omega^* \in I$ such that $\frac{\partial u}{\partial \omega}(\omega^*) \neq 0$. Since u is continuous and $\frac{\partial u}{\partial \omega}(\omega^*) \neq 0$, there exists an open subinterval I^* around ω^* such that $u(\omega) \neq 0$ for all $\omega \neq \omega^* \in I^*$. Pick a rational number in I^*. Since the intersection of two such subintervals is empty (if not, there exists e sequence $\{\omega_n\}$, $u(\omega_n) = 0$ and $\omega_n \to \omega^*$. By Mean Value Theorem, there exists a sequence $\{\nu_n\}$, $\omega_n < \nu_n < \omega_{n+1}$, such that $\frac{\partial u}{\partial \omega}(\nu_n) = 0$. Since $\nu_n \to \omega^*$, and $\frac{\partial u}{\partial \omega}(\nu_n) = 0$, by continuity, $\frac{\partial u}{\partial \omega}(\omega^*) = 0$, that contradicts the choice of ω^*), every member of I is in one of these subinterval. Hence, I is countable.

If $u > 0$ or < 0, then it is trivial that $\Delta |u| = |\Delta u|$. Suppose that $u(\omega^*) = 0$ and $\dfrac{\partial u}{\partial \omega}(\omega^*) = 0$. Then $u(\omega) = (\omega - \omega^*)^2 v(\omega)$ for some function v. Then

$$\Delta u(\omega) = 2v(\omega) + (\omega - \omega^*)^2 \Delta v(\omega) + 4(\omega - \omega^*)\frac{\partial v}{\partial \omega}(\omega). \tag{4.60}$$

On the other hand,

$$\frac{d}{d\omega}|u|(\omega) = \begin{cases} \left|2(\omega - \omega^*)v(\omega) + (\omega - \omega^*)^2 \dfrac{\partial v}{\partial \omega}(\omega)\right| & v(\omega) \neq 0 \\ 0 & v(\omega) = 0. \end{cases}$$

Therefore,

$$\Delta |u|(\omega) = \begin{cases} \left|2v(\omega) + (\omega - \omega^*)^2 \Delta v(\omega) + 4(\omega - \omega^*)\dfrac{\partial v}{\partial \omega}(\omega)\right| & v(\omega) \neq 0 \\ \lim\limits_{\nu \to \omega} \dfrac{1}{\nu - \omega}\left|2(\omega - \omega^*)v(\omega) + (\omega - \omega^*)^2 \dfrac{\partial v}{\partial \omega}(\omega)\right| & v(\omega) = 0. \end{cases} \tag{4.61}$$

Hence, by computing (4.60) and (4.61) at $\omega = \omega^*$, we get:

$$\Delta |u|(\omega^*) = |2v(\omega^*)| = |\Delta u(\omega^*)|.$$

$\square$

Proof of Lemma 25. By definition of $\mu^+_{1,\phi,Q}$ we have:

$$\mu^+_{1,\phi,Q}[\mathcal{A} + \Lambda^{(d)}] = \sup_{u \in \mathbf{Y}^{(d)}_V} \lim_{h \to 0^+} \frac{1}{h}\left\{ \frac{\sum_i q_i \int_\Omega \phi(\omega)\left|u_i + h d_i(t)(\Delta + \lambda^{(d)}_1) u_i(\omega)\right| \, d\omega}{\sum_i q_i \int_\Omega \phi(\omega)\, |u_i(\omega)| \, d\omega} - 1 \right\},$$

it is enough to show that for a fixed $u \neq 0 \in \mathbf{Y}^{(d)}_V$ and a fixed $i = 1, \ldots, n$, and fixed t:

$$\lim_{h \to 0^+} \frac{1}{h}\left\{ \int_\Omega \phi(\omega)\left|u_i(\omega) + h d_i(t)(\Delta + \lambda^{(d)}_1)u_i(\omega)\right| \, d\omega - \int_\Omega \phi(\omega)\, |u_i| \, d\omega \right\} = 0. \tag{4.62}$$

Or equivalently, after dividing by $d_i(t)\int_\Omega \phi(\omega)\,|u_i| \, d\omega$, (note that if $d_i(t) = 0$, then the left hand side of (4.62) is zero, so we assume that $d_i(t) \neq 0$) and renaming $d_i(t)h$ as h, and dropping i, we need to show that:

$$\lim_{h \to 0^+} \frac{1}{h}\left\{ \frac{\int_\Omega \phi(\omega)\left|u(\omega) + h(\Delta + \lambda^{(d)}_1)u(\omega)\right| \, d\omega}{\int_\Omega \phi(\omega)\, |u| \, d\omega} - 1 \right\} = 0. \tag{4.63}$$

Let I be as in Lemma 30: the set of points of Ω such that for any $\omega \in I$, $u(\omega)=0$ and $\nabla u(\omega) \neq 0$.

To show (4.63), we add and subtract $\phi(\omega)\left(|u| + h\Delta|u| + \lambda_1^{(d)}|u|\right)$ in the integral of the numerator of the left hand side of (4.63), and get:

$$\begin{aligned}
&\lim_{h\to 0^+}\frac{1}{h}\left\{\frac{\int_\Omega \phi(\omega)\left|u + h(\Delta+\lambda_1^{(d)})u\right|\,d\omega}{\int_\Omega \phi(\omega)\,|u|\,d\omega} - 1\right\}\\
&= \lim_{h\to 0^+}\frac{1}{h}\left\{\frac{\int_\Omega \phi(\omega)\left(|u| + h(\Delta+\lambda_1^{(d)})\,|u|\right)\,d\omega}{\int_\Omega \phi(\omega)\,|u|\,d\omega} - 1\right\} \qquad (4.64)\\
&\quad + \lim_{h\to 0^+}\frac{1}{h}\left\{\frac{\int_\Omega \phi(\omega)\left(\left|u + h(\Delta+\lambda_1^{(d)})u\right| - |u| - h(\Delta+\lambda_1^{(d)})\,|u|\right)\,d\omega}{\int_\Omega \phi(\omega)\,|u|\,d\omega}\right\}.
\end{aligned}$$

First, we show that the first term of the right hand side of (4.64) is 0. By Divergence Theorem and Dirichlet boundary conditions, we have (recall that $\phi = \phi_1^{(d)}$):

$$\begin{aligned}
\int_\Omega \phi_1^{(d)}\Delta|u| &= \int_{\partial\Omega}\phi_1^{(d)}\nabla|u|\cdot\mathbf{n} - \int_\Omega \nabla|u|\cdot\nabla\phi_1^{(d)} \quad (\phi_1 = 0 \text{ on } \partial\Omega)\\
&= -\int_{\partial\Omega}\nabla\phi_1^{(d)}\,|u|\cdot\mathbf{n} + \int_\Omega |u|\,\Delta\phi_1^{(d)} \quad (u = 0 \text{ on } \partial\Omega)\\
&= \int_\Omega |u|\,\Delta\phi_1^{(d)}\\
&= -\int_\Omega |u|\,\lambda_1^{(d)}\phi_1^{(d)}.
\end{aligned}$$

Therefore,

$$\int_\Omega \phi(\omega)\left(\lambda_1^{(d)} + \Delta\right)|u|(\omega)\,d\omega = 0,$$

and so:

$$\lim_{h\to 0^+}\frac{1}{h}\left\{\frac{\displaystyle\int_\Omega \phi(\omega)\left(|u| + h(\Delta+\lambda_1^{(d)})\,|u|\right)\,d\omega}{\displaystyle\int_\Omega \phi(\omega)\,|u|\,d\omega} - 1\right\} = 0.$$

Next, we show that the second term of the right hand side of (4.64) is 0:

$$\lim_{h\to 0^+}\frac{1}{h}\left\{\frac{\displaystyle\int_\Omega \phi(\omega)\left(\left|u + h\left(\Delta+\lambda_1^{(d)}\right)u\right| - |u| - h\left(\Delta+\lambda_1^{(d)}\right)|u|\right)\,d\omega}{\displaystyle\int_\Omega \phi(\omega)\,|u|\,d\omega}\right\} = 0. \qquad (4.65)$$

In this part, we drop the superscript (d) for the ease of notation: $\lambda_1 = \lambda_1^{(d)}$. For a fixed $u \in \mathbf{Y}_V^{(d)}$, we define F_h, for any $0 < h$, as follows:

$$F_h(\omega) := \frac{1}{h}\left\{\phi(\omega)\left(|u + h(\Delta+\lambda_1)u| - |u| - h(\Delta+\lambda_1)\,|u|\right)(\omega)\right\}.$$

1. First, we will show that there exists $M > 0$ such that for all h positive, $|F_h| < M$ almost everywhere:

   We study F_h, for any $0 < h$, on the following possible subsets of $W := \Omega \setminus I$:

   - $W_1 := \{\omega : u(\omega) > 0,\ (\Delta + \lambda_1)u(\omega) \geq 0\}$. By definition,
   $$F_h(\omega) = \frac{\phi(\omega)}{h}\left(u + h(\Delta + \lambda_1)u - u - h(\Delta + \lambda_1)u\right)(\omega) = 0.$$
   - $W_2 := \{\omega : u(\omega) > 0,\ (\Delta + \lambda_1)u(\omega) < 0,\ u > |(\Delta + \lambda_1)u|\, h\}$. By definition,
   $$F_h(\omega) = \frac{\phi(\omega)}{h}\left(u + h(\Delta + \lambda_1)u - u - h(\Delta + \lambda_1)u\right)(\omega) = 0.$$
   - $W_3 = \{\omega : u(\omega) > 0,\ (\Delta + \lambda_1)u(\omega) < 0,\ u < |(\Delta + \lambda_1)u|\, h\}$. By definition,
   $$F_h(\omega) = \frac{\phi(\omega)}{h}\left(-u - h(\Delta + \lambda_1)u - u - h(\Delta + \lambda_1)u\right)(\omega).$$
     Using the triangle inequality and the assumption $u < |(\Delta + \lambda_1)u|\, h$, we get:
   $$\begin{aligned} |F_h| &< \frac{2}{h}\max_{\Omega}|\phi|\left(|u| + h\,|(\Delta + \lambda_1)u|\right) \\ &< 4\max_{\Omega}|\phi|\,|(\Delta + \lambda_1)u| \\ &\leq 4\max_{\Omega}|\phi|\left(\max_{\Omega}|\Delta u| + \lambda_1\max_{\Omega}|u|\right) =: M. \end{aligned} \tag{4.66}$$
     (Note that, without loss of generality, we assume that $M \neq 0$; otherwise, $u = 0$. Therefore $F_h = 0$ on Ω.)
   - $W_4 := \{\omega : u(\omega) < 0,\ (\Delta + \lambda_1)u(\omega) \leq 0\}$. By definition,
   $$F_h(\omega) = \frac{\phi(\omega)}{h}\left(-u - h(\Delta + \lambda_1)u + u + h(\Delta + \lambda_1)u\right)(\omega) = 0.$$
   - $W_5 := \{\omega : u(\omega) < 0,\ (\Delta + \lambda_1)u(\omega) > 0,\ |u| < h(\Delta + \lambda_1)u\}$. Similar to the case of W_3, $|F_h| < M$.
   - $W_6 := \{\omega : u(\omega) < 0,\ \Delta u(\omega) > 0,\ |u| > (\Delta + \lambda_1)uh\}$. By definition,
   $$F_h(\omega) = \frac{\phi(\omega)}{h}\left(-u - h(\Delta + \lambda_1)u + u + h(\Delta + \lambda_1)u\right)(\omega) = 0.$$
   - $W_7 := \{\omega : u(\omega) = 0, u_\omega(\omega) = 0\}$. In this case, by definition of $\Delta|u|$, we have $\Delta|u|(\omega) = |\Delta u(\omega)|$. Therefore, $F_h(\omega) = 0$.

2. Next, we will show that as $h \to 0$, $F_h \to 0$ almost everywhere. Fix $\omega \in \Omega \setminus I$ and consider the following cases:

   - $u(\omega) > 0$. We can choose h small enough, such that
     $$|u(\omega) + h(\Delta + \lambda_1)u(\omega)| = u(\omega) + h(\Delta + \lambda_1)u(\omega).$$
     Therefore,
     $$F_h(\omega) = \frac{1}{h}\phi(\omega)(u(\omega) + h(\Delta + \lambda_1)u(\omega) - u(\omega) - h(\Delta + \lambda_1)u(\omega)) = 0.$$
   - $u(\omega) < 0$. We can choose h small enough, such that
     $$|u(\omega) + h(\Delta + \lambda_1)u(\omega)| = -u(\omega) - h(\Delta + \lambda_1)u(\omega).$$
     Therefore,
     $$F_h(\omega) = \frac{1}{h}\phi(\omega)(-u(\omega) - h(\Delta + \lambda_1)u(\omega) + u(\omega) + h(\Delta + \lambda_1)u(\omega)) = 0.$$
   - $u(\omega) = 0$. Then as we discussed before, on W_7, $F_h(\omega) = 0$.

Using 1 and 2, and the Dominated Convergence Theorem, we can conclude (4.65). □

Proof of Lemma 26

By the definition of $c := \mu_{p,Q}[G]$, we have
$$\lim_{h\to 0^+} \frac{1}{h} \sup_{x\neq y\in V} \left(\frac{\|x - y + h(G(x) - G(y))\|_{p,Q}}{\|x - y\|_{p,Q}} - 1 \right) = c.$$
Fix an arbitrary $\epsilon > 0$. Then there exists $h_0 > 0$ such that for all $0 < h < h_0$,
$$\frac{1}{h} \sup_{x\neq y\in V} \left(\frac{\|x - y + h(G(x) - G(y))\|_{p,Q}}{\|x - y\|_{p,Q}} - 1 \right) < c + \epsilon.$$
Therefore, for any $x \neq y$, and $0 < h < h_0$
$$\frac{\|x - y + h(G(x) - G(y))\|_{p,Q}}{\|x - y\|_{p,Q}} < (c + \epsilon)h + 1. \tag{4.67}$$

For fixed $u \neq v \in \mathbf{Y}_V^{(d)}$, let $\Omega_1 = \{\omega \in \bar{\Omega} : u(\omega) \neq v(\omega)\}$. Fix $\omega \in \Omega_1$, and let $x = u(\omega)$ and $y = v(\omega)$. We give a proof for the case $p < \infty$; the case $p = \infty$ is analogous. Using equation (4.67), we have:
$$\frac{\left(\sum_i q_i^p \left|u_i(\omega) - v_i(\omega) + h(G_i(u(\omega)) - G_i(v(\omega)))\right|^p\right)^{\frac{1}{p}}}{\left(\sum_i q_i^p \left|u_i(\omega) - v_i(\omega)\right|^p\right)^{\frac{1}{p}}} < (c + \epsilon)h + 1. \tag{4.68}$$

Multiplying both sides by the denominator and raising to the power p, we have:

$$\sum_i q_i^p \left|(u_i - v_i)(\omega) + h\left(G_i(u) - G_i(v)\right)(\omega)\right|^p < ((c+\epsilon)h+1)^p \sum_i q_i^p \left|(u_i - v_i)(\omega)\right|^p . \tag{4.69}$$

Since $\hat{G}(u)(\omega) = G(u(\omega))$, Equation (4.68) can be written as:

$$\sum_i q_i^p \left|(u_i - v_i)(\omega) + h\left(\hat{G}_i(u) - \hat{G}_i(v)\right)(\omega)\right|^p < ((c+\epsilon)h+1)^p \sum_i q_i^p \left|(u_i - v_i)(\omega)\right|^p .$$

Now by multiplying both sides of the above inequality by $\phi(\omega)$ which is nonnegative, and taking the integral over $\bar{\Omega}$, we get:

$$\|u - v + h\left(\hat{G}(u) - \hat{G}(v)\right)\|_{p,\phi,Q} < ((c+\epsilon)h+1)\|u-v\|_{p,\phi,Q}.$$

(Note that for $\omega \notin \Omega_1$,

$$((c+\epsilon)h+1)^p \sum_i q_i^p \left|u_i(\omega) - v_i(\omega)\right|^p = 0,$$

which we can add to the right hand side of (4.69), and also

$$\sum_i q_i^p \left|u_i(\omega) - v_i(\omega) + h(G_i(u(\omega)) - G_i(v(\omega)))\right|^p = 0,$$

which we can add to the left hand side of (4.69), and hence we can indeed take the integral over all $\bar{\Omega}$.)

Hence,

$$\lim_{h\to 0^+} \frac{1}{h}\left(\frac{\left\|u - v + h\left(\hat{G}(u) - \hat{G}(v)\right)\right\|_{p,\phi,Q}}{\|u-v\|_{p,\phi,Q}} - 1\right) \le c + \epsilon.$$

Now by letting $\epsilon \to 0$ and taking sup over $u \neq v \in \mathbf{Y}_V^{(d)}$, we get $\mu_{p,Q}^+[\hat{G}] \le c$.

Another proof of Theorem 20

Proof by discretization:

Let $0 = \omega_0 < \omega_1 < \cdots < \omega_{N+1} = L$ be the mesh points of the closed interval $[0, L]$ with equal mesh size $\Delta\omega = \frac{L}{N+1}$. For $i = 0, \ldots, N+1$, define

$$x_i(t) := u(\omega_i, t),$$

By the Neumann boundary condition, we have:

$$0 = u_\omega(0,t) \simeq \frac{u(\omega_1,t) - u(\omega_0,t)}{\Delta\omega} \quad \Rightarrow \quad u(\omega_1,t) = u(\omega_0,t),$$

where $u_\omega = \frac{\partial u}{\partial \omega}$. Therefore for any t, $x_0(t) = x_1(t)$, and similarly, $x_N(t) = x_{N+1}(t)$. Now using the definition of $u_{\omega\omega}$, we have the following expressions for $u_{\omega\omega}$ at mesh points:

$$\begin{aligned} u_{\omega\omega}(\omega_i,t) &= \lim_{\Delta\omega\to 0} \frac{u(\omega_{i-1},t) - 2u(\omega_i,t) + u(\omega_{i+1},t)}{\Delta\omega^2} \\ &= \lim_{N\to\infty} \frac{(N+1)^2}{L^2} \left(x_{i-1} - 2x_i + x_{i+1}\right)(t), \end{aligned} \tag{4.70}$$

we can write Equation (4.1) for the mesh points as follows:

$$\begin{aligned} \dot{x}_1 &= F(x_1,t) + \frac{(N+1)^2}{L^2}\, D(t)\, (x_2 - x_1) \\ \dot{x}_2 &= F(x_2,t) + \frac{(N+1)^2}{L^2}\, D(t)\, (x_1 - 2x_2 + x_3) \\ &\vdots \\ \dot{x}_N &= F(x_N,t) + \frac{(N+1)^2}{L^2}\, D(t)\, (x_{N-1} - x_N). \end{aligned} \tag{4.71}$$

Note that the ODE system (4.71) describes the dynamics of N identical compartments that are connected through a path graph with diffusion matrix $\frac{(N+1)^2}{L^2} D(t)$. Therefore, by Proposition 10, if

$$c_N := \sup_{(x,t)} \mu_{1,Q} \left[J_F(x,t) - 4\sin^2(\pi/2N) \frac{(N+1)^2}{L^2} D(t) \right],$$

where $-4\sin^2(\pi/2N)$ is the second eigenvalue of (graph) Laplacian of path graph, then

$$\sum_{k=1}^{N-1} \sin(k\pi/N) \left\|(x_k - x_{k+1})(t)\right\|_{1,Q} \;\leq\; e^{c_N t} \sum_{k=1}^{N-1} \sin(k\pi/N) \left\|(x_k - x_{k+1})(t)\right\|_{1,Q}. \tag{4.72}$$

Now dividing both sides of (4.72) by $\Delta\omega = \frac{L}{N+1}$ and letting $N \to \infty$, we get:

$$\int_0^L \sin(\pi\omega) \left\| \frac{\partial u}{\partial \omega}(t) \right\|_{1,Q} d\omega \leq e^{\lim_{N\to\infty} c_N t} \int_0^L \sin(\pi\omega) \left\| \frac{\partial u}{\partial \omega}(t) \right\|_{1,Q} d\omega, \tag{4.73}$$

where

$$\begin{aligned} \lim_{N\to\infty} c_N &= \lim_{N\to\infty} \sup_{(x,t)} \mu_{1,Q} \left[J_F(x,t) - 4\sin^2(\pi/2N) \frac{(N+1)^2}{L^2} D(t) \right] \\ &= \sup_{(x,t)} \mu_{1,Q} \left[J_F(x,t) - 4 \lim_{N\to\infty} \sin^2(\pi/2N) \frac{(N+1)^2}{L^2} D(t) \right] \\ &= \sup_{(x,t)} \mu_{1,Q} \left[J_F(x,t) - \frac{\pi^2}{L^2} D(t) \right] = c. \end{aligned}$$

Proof of Lemma 27

To prove lemma 27, we need the following lemma.

Lemma 31. *Suppose $u \in L^2(\Omega)$ satisfies Neumann boundary conditions. Then, for any $k \in \{1, 2, \ldots\}$, $\Pi_k(\mathfrak{L}u) = \mathfrak{L}\Pi_k(u)$, where $\mathfrak{L}$ is as defined in Lemma 27.*

Proof. First note that $\mathfrak{L}$ can be written as $\mathfrak{L} := \operatorname{diag}(\mathfrak{L}_1, \ldots, \mathfrak{L}_n)$, where for example, $\mathfrak{L}_1 = d_{11}\mathcal{L}_1$, $\mathfrak{L}_2 = d_{12}\mathcal{L}_1$, etc.

By the definition of Π_k and $\mathfrak{L} = \operatorname{diag}(\mathfrak{L}_1, \ldots, \mathfrak{L}_n)$, it suffices to show that for any fixed $i \in \{1, \ldots, n\}$,

$$\Pi_{k,i}(\mathfrak{L}_i u_i) = \mathfrak{L}_i \Pi_{k,i}(u_i). \tag{4.74}$$

Using the fact that $\mathfrak{L}_i = d_{pq}(t)\mathcal{L}_p$, for some p, q, and $\mathcal{L}_p e_j^p = -\lambda_j^p e_j^p$, the right hand side of (4.74) becomes:

$$\mathfrak{L}_i \Pi_{k,i}(u_i) = d_{pq}(t)\mathcal{L}_p \sum_{j=k}^{\infty}(u_i, e_j^p)e_j^p = d_{pq}(t)\sum_{j=k}^{\infty}(u_i, e_j^p)\mathcal{L}_p e_j^p = -d_{pq}(t)\sum_{j=k}^{\infty}(u_i, e_j^p)\lambda_j^p e_j^p;$$

and using the orthogonality of the e_j^p's, the left hand side of (4.74) becomes:

$$\begin{aligned}
\Pi_{k,i}(\mathfrak{L}_i u_i) &= \sum_{j=k}^{\infty}\left(d_{pq}(t)\mathcal{L}_p u_i, e_j^p\right)e_j^p \\
&= \sum_{j=k}^{\infty}\left(d_{pq}(t)\mathcal{L}_p \sum_{l=1}^{\infty}(u_i, e_l^p)e_l^p \ , \ e_j^p\right)e_j^p \\
&= \sum_{j=k}^{\infty}\left(\sum_{l=1}^{\infty}(u_i, e_l^p)d_{pq}(t)\mathcal{L}_p e_l^i \ , \ e_j^p\right)e_j^p \\
&= -\sum_{j=k}^{\infty}\left(\sum_{l=1}^{\infty}(u_i, e_l^p)d_{pq}(t)\lambda_l^p e_l^p \ , \ e_j^p\right)e_j^p \\
&= -d_{pq}(t)\sum_{j=k}^{\infty}(u_i, e_j^p)\lambda_j^p e_j^p.
\end{aligned}$$

Hence, (4.74) holds. □

Proof of Lemma 27. For $x = v$,

$$\begin{aligned}
\frac{d\Phi}{dt}(w) &= \left(u - v, Q\,\frac{\partial}{\partial t}(u - v)\right) \\
&= (w, Q(F(u,t) - F(v,t))) \ + \ (w, Q\mathfrak{L}(u - v)) \\
&= (w, Q(F(u,t) - F(x,t))) + (w, Q\mathfrak{L}w)\,.
\end{aligned}$$

For $x = \pi_2(u)$, i.e., $w = \Pi_2(u)$,

$$
\begin{aligned}
\frac{d\Phi}{dt}(w) &= \left(\Pi_2(u), Q\,\frac{\partial}{\partial t}(\Pi_2(u))\right) \\
&= (\Pi_2(u), Q\Pi_2(F(u,t))) \;+\; (w, Q\Pi_2(\mathfrak{L}u)) \\
&= (\Pi_2(u), Q\Pi_2(F(u,t))) \;+\; (w, Q\mathfrak{L}\Pi_2(u)) \quad \text{by Lemma 31} \\
&= (\Pi_2(u), Q(F(u,t) - \pi_2(F(u,t)))) \;+\; (w, Q\mathfrak{L}w) \\
&= (\Pi_2(u), Q(F(u,t) - F(\pi_2(u),t))) \;+\; (w, Q\mathfrak{L}w) \\
&\quad\; + \;(\Pi_2(u), Q(\pi_2(F(u,t)) - F(\pi_2(u),t))) \\
&= (w, Q(F(u,t) - F(x,t))) + (w, Q\mathfrak{L}w)\,.
\end{aligned}
$$

Note that the last equality holds because $Q(\pi_2(F(u,t)) - F(\pi_2(u),t))$ is independent of ω and $\int_\Omega \Pi_{2,i}(u) = 0$ (by Equation (4.47)). $\square$

Proof of Lemma 28

To prove Lemma 28, we first recall a result following from the Poincaré principle as in [105], which gives a variational characterization of the eigenvalues of an elliptic operator. The following lemma is an analogous of Lemma 17 for PDEs.

Lemma 32. *Let A be a matrix as defined in (4.19), and consider the elliptic operator $-\nabla \cdot (A\nabla(\cdot))$ with eigenvalues λ_j's and corresponding orthonormal eigenfunctions e_i's. Let $v = v(\omega)$ be a function not identically zero in $L^2(\Omega)$ with derivatives $\frac{\partial v}{\partial \omega_j} \in L^2(\Omega)$ that satisfies the Neumann boundary condition, $\nabla v \cdot \mathbf{n} = 0$, and $\forall j \in \{1, \ldots, k-1\}$, $\int_\Omega v e_j = 0$. Then the following inequality holds, for any $k \geq 1$:*

$$\int_\Omega \nabla v \cdot (A(\omega)\nabla v)\, d\omega \geq \lambda_k \int_\Omega v^2\, d\omega.$$

Proof of Lemma 28.

$$
\begin{aligned}
&(\Pi_k(u), \mathfrak{L}\Pi_k(u)) \\
= \;& \sum_{i=1}^{n_1} \int_\Omega \Pi_{k,i}(u_i)^T \mathfrak{L}_i \Pi_{k,i}(u_i)\; d\omega + \cdots + \sum_{i=n-n_r+1}^{n} \int_\Omega \Pi_{k,i}(u_i)^T \mathfrak{L}_i \Pi_{k,i}(u_i)\; d\omega \\
= \;& \sum_{i=1}^{n_1} d_{1i}(t) \int_\Omega \Pi_{k,i}(u_i) \nabla \cdot (A_1(\omega)\nabla\Pi_{k,i}(u_i))\; d\omega \\
& + \cdots + \sum_{i=n-n_r+1}^{n} d_{ri}(t) \int_\Omega \Pi_{k,i}(u_i) \nabla \cdot (A_r(\omega)\nabla\Pi_{k,i}(u_i))\; d\omega
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n_1} d_{1i}(t)\left\{\int_{\partial\Omega}\Pi_{k,i}(u_i)A_1\nabla\Pi_{k,i}(u_i)\cdot\mathbf{n}\,dS-\int_{\Omega}\nabla\Pi_{k,i}(u_i)^T A_1\nabla\Pi_{k,i}(u_i)\,d\omega\right\}+\cdots\\
&\quad+\sum_{i=n-n_r+1}^{n} d_{1i}(t)\left\{\int_{\partial\Omega}\Pi_{k,i}(u_i)A_r\nabla\Pi_{k,i}(u_i)\cdot\mathbf{n}\,dS-\int_{\Omega}\nabla\Pi_{k,i}(u_i)^T A_r\nabla\Pi_{k,i}(u_i)\,d\omega\right\}\\
&\le -\sum_{i=1}^{n_1} d_{1i}(t)\lambda_k^1\int_{\Omega}\Pi_{k,i}^2(u_i)\,d\omega-\cdots-\sum_{i=n-n_r+1}^{n} d_{1i}(t)\lambda_k^r\int_{\Omega}\Pi_{k,i}^2(u_i)\,d\omega\\
&= -\left(\Pi_k(u),\Lambda_k\Pi_k(u)\right).
\end{aligned}
$$

The first and second equalities follow by the definition of $\mathfrak{L}$ (recall that we can write $\mathfrak{L}=\mathrm{diag}\left(\mathfrak{L}_1,\ldots,\mathfrak{L}_n\right)$). The third equality holds by Green's Identity.

For any $n_1+\cdots+n_{s-1}+1\le i\le n_1+\cdots+n_s$,

- $\nabla\Pi_{k,i}(v(\xi))\cdot\mathbf{n}=\sum_{j=k}^{\infty}(v,e_j^s)\nabla e_j^s(\xi)\cdot\mathbf{n}=0,$
- and for any $j=1,\ldots,k-1$, $\int_{\Omega}\Pi_{k,i}(v)e_j^s\,d\omega \;=\; 0,$

using the above equalities and applying Lemma 32 to A_i's, the fourth inequality holds. The last equality holds by the definition of Λ_k.

Next we prove the second part of the lemma. Since for each $i=1,\ldots,r$, $QD_i+D_iQ>0$, there exists positive definite matrix M_i, such that $QD_i+D_iQ=2M_i^TM_i$. Note that

$$
\begin{aligned}
2\left(\Pi_k(u),QD_i\mathcal{L}_i\Pi_k(u)\right)&=\left(\Pi_k(u),(QD_i+D_iQ)\mathcal{L}_i\Pi_k(u)\right)\\
&\quad+\;\left(\Pi_k(u),(QD_i-D_iQ)\mathcal{L}_i\Pi_k(u)\right).
\end{aligned}
\tag{4.75}
$$

A simple calculation shows that $\left(\Pi_k(u),(QD_i-D_iQ)\mathcal{L}_i\Pi_k(u)\right)=0$:

$$
\begin{aligned}
\left(\Pi_k(u),D_iQ\mathcal{L}_i\Pi_k(u)\right)&=\left(QD_i\Pi_k(u),\mathcal{L}_i\Pi_k(u)\right)\\
&=\left(QD_i\Pi_k(u),\nabla\cdot(A_i\nabla\Pi_k(u))\right)\\
&=-\left(\nabla(QD_i\Pi_k(u)),A_i\nabla\Pi_k(u)\right)\\
&=-\left(QD_i\nabla\Pi_k(u),A_i\nabla\Pi_k(u)\right)\\
&=-\left(A_iQD_i\nabla\Pi_k(u),\nabla\Pi_k(u)\right)\\
&=-\left(\nabla\Pi_k(u),A_iQD_i\nabla\Pi_k(u)\right).
\end{aligned}
$$

The first equality holds because both Q and D_i are symmetric and hence QD_i, i.e., $(D_iQ)^T=QD_i$. The second equality holds by the definition of $\mathcal{L}_i$. The third equality holds by Green's Identity and the Neumann boundary condition. The fourth equality holds because both Q and D_i are independent of space and ∇ does not affect them. The fifth equality holds because A_i is symmetric. Also,

$$\begin{aligned}
(\Pi_k(u), QD_i\mathcal{L}_i\Pi_k(u)) &= (\Pi_k(u), \mathcal{L}_iQD_i\Pi_k(u)) \\
&= (\Pi_k(u), \nabla\cdot(A_i\nabla(QD_i\Pi_k(u)))) \\
&= -(\nabla\Pi_k(u), A_i\nabla(QD_i\Pi_k(u))) \\
&= -(\nabla\Pi_k(u), A_iQD_i\nabla\Pi_k(u)).
\end{aligned}$$

The first equality holds because by the definition of $\mathcal{L}_i$, for any vector u and matrix A, $A\mathcal{L}_iu = \mathcal{L}_iAu$. Therefore, for $k = 1, 2$, we get

$$\begin{aligned}
(\Pi_k(u), Q\mathfrak{L}\Pi_k(u)) &= \sum_{i=1}^{r}(\Pi_k(u), QD_i\mathcal{L}_i\Pi_k(u)) \\
&= \frac{1}{2}\sum_{i=1}^{r}(\Pi_k(u), (QD_i + D_iQ)\mathcal{L}_i\Pi_k(u)) \\
&= \sum_{i=1}^{r}\left(\Pi_k(u), M_i^TM_i\mathcal{L}_i\Pi_k(u)\right) \\
&= \sum_{i=1}^{r}(M_i\Pi_k(u), M_i\mathcal{L}_i\Pi_k(u)) \\
&= \sum_{i=1}^{r}(M_i\Pi_k(u), \mathcal{L}_iM_i\Pi_k(u)) \\
&= \sum_{i=1}^{r}(\Pi_k(M_iu), \mathcal{L}_i\Pi_k(M_iu)) \\
&= -\sum_{i=1}^{r}(\nabla\Pi_k(M_iu), A_i\nabla\Pi_k(M_iu)) \\
&\leq -\sum_{i=1}^{r}\lambda_k^i\,(\Pi_k(M_iu), \Pi_k(M_iu)) \\
&= -\sum_{i=1}^{r}\lambda_k^i\,(\Pi_k(u), QD_i\Pi_k(u)).
\end{aligned}$$

The first equality holds by the definition of $\mathfrak{L}$. The second equality holds by Equation (4.75), and the third equality holds by the choice of M_i. For any v, $M_i\mathcal{L}_iv = \mathcal{L}_iM_iv$ and this justify the fifth equality. To justify the sixth equality we show that for $k = 1, 2$, $M_i\Pi_k(u) = \Pi_k(M_iu)$: For $k = 1$, $\Pi_k(u) = u$, hence $M_i\Pi_1(u) = M_iu = \Pi_1(M_iu)$. Also, for $k = 2$,

$$\Pi_2(u) = (\Pi_{2,1}(u_1), \ldots, \Pi_{2,n}(u_n))^T = \left(u_1 - \frac{1}{|\Omega|} \int_\Omega u_1, \ldots, u_n - \frac{1}{|\Omega|} \int_\Omega u_n \right)^T.$$

It is easy to see that $M_i \Pi_2(u) = \Pi_2(M_i u)$. The seventh equality holds by Green's Identity and the Neumann boundary condition. The inequality holds for $k = 1$, because $\lambda_1^i = 0$ and A_i is positive definite and it holds for $k = 2$ by Lemma 32 and the fact that for any l, $\int \Pi_{2,l} = 0$. $\square$

Chapter 5

Discussion

In this work we studied the contractivity behavior of the solutions of diffusively interconnected ODEs (3.6) and reaction diffusion PDEs (4.1) by the tool of matrix measures induced by non L^2 norms.

Although the problem is well studied in weighted L^2 norms, in Section 2.4, we saw that the existing results in L^2 norms do not justify the contractivity behavior of simple biochemical examples which one sees in simulations; while the results presented in this work, mainly Theorem 9 and Theorem 15, can justify it.

We also studied synchronous behavior of the solutions of diffusively interconnected ODEs (3.6) and reaction diffusion PDEs (4.1) by the method of contraction theory.

The synchronization problem is also a well-understood problem in weighted L^2 norms. However, in this work, we saw that the synchronous behavior of the biochemical examples cannot be justified by the existing results in weighted L^2 norms.

In Theorem 11 we provided a general sufficient condition based on the edge Laplacian, for arbitrary norms, which guarantees the synchronous behavior of diffusively interconnected ODEs (3.6).

We then simplified the conditions in Theorem 11 for path and complete graphs and provided conditions in terms of second eigenvalue of the graph Laplacian. Using different techniques from those used to prove the results for path and complete graphs, we showed an analogous result in non L^2 norms for star graphs and the Cartesian products of path, complete and star graphs and saw how these results explain the synchronization in biochemical examples. However, obtaining generalizations to arbitrary graphs remains the subject of future research.

As in the ODE case, the synchronization behavior of the solutions of PDE system (4.1) is a well-understood problem in weighted L^2 norms, but the results are not applicable to the biochemical examples; while our result in weighted L^1 norms but restricted to one dimensional spaces (Theorem 20) is applicable to more examples (see the discussion in Section 4.4, Example 2).

The problem of synchronization for PDE system (4.1) is still open for general norms and higher dimension spaces.

Another important topic for further research is to generalize the current results to non-constant norms, i.e., when the weighted matrix Q depends on x, $Q = Q(x)$.

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