SOME PARABOLIC AND ELLIPTIC PROBLEMS IN COMPLEX RIEMANNIAN GEOMETRY

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ABSTRACT OF THE DISSERTATION

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Dissertation Director: Professor Jian Song

This dissertation consists of three parts, the first one is on the blow-up behavior of Kähler Ricci flow on $\mathbb{CP}^n$ blown-up at one point, and the second one on the convergence of Kähler Ricci flow on minimal projective manifolds of general type, and the last one is on the existence of canonical conical Kähler metrics on toric manifolds.

In the first part, we consider the Ricci flow on $\mathbb{CP}^n$ blown-up at one point starting with any rotationally symmetric Kähler metric. We show that if the total volume does not go to zero at the singular time, then any parabolic blow-up limit of the Ricci flow along the exceptional divisor is a non-compact complete shrinking Kähler Ricci soliton with rotational symmetry on $\mathbb{C}^n$ blown-up at one point, hence the FIK soliton constructed in [36].

In the second part, we consider the Kähler Ricci flow on a smooth minimal model of general type, following the ideas of Song ([76, 77]), we show that if the Ricci curvature is uniformly bounded below along the Kähler-Ricci flow, then the diameter is uniformly bounded. As a corollary we show that under the Ricci curvature lower bound assumption, the Gromov-Hausdorff limit of the flow is homeomorphic to the canonical model of the manifold. Moreover, we will give a purely analytic proof of a recent result of Tosatti-Zhang ([102]) that if the canonical line bundle K_X is big and nef, but not

ample, then the Ricci flow is of Type IIb.

In the last part, we give criterion for the existence of toric conical Kähler-Einstein and Kähler-Ricci soliton metrics on any toric manifold in relation to the greatest Ricci lower bound and Bakry-Emery-Ricci lower bound. It is shown that any two toric manifolds with the same dimension can be joined by a continuous path of toric manifolds with conical Kähler-Einstein metrics in the Gromov-Hausdorff topology.

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Dedication

This thesis is dedicated to my family

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Chapter 1

Introduction

1.1 Background

1.1.1 Kähler Ricci flow

On a compact Riemannian manifold (M, g_0), the Ricci flow equation

$$\frac{\partial}{\partial t} g = -2\mathrm{Ric}(g), \quad g(0) = g_0, \tag{1.1}$$

introduced by Hamilton ([46]), has been a useful tool in geometric analysis. It gives a canonical way to deform a metric along its Ricci curvatures. It is used by Hamilton ([46, 47]) to classify closed three manifolds with positive Ricci curvature and four manifolds with positive curvature operator. And he introduced the notion *Ricci flow with surgery* and laid out a program to prove the Poincaré conjecture and geometrization conjecture. Perelman ([64, 65, 66]) developed new techniques and completed Hamilton's program, hence proving the conjectures mentioned above.

In the complex setting, if the Ricci flow (1.1) starts at a Kähler metric, the evolving metrics remain to be Kähler ([13]), and the Ricci flow is called *Kähler Ricci flow*. Using this flow together with the parabolic version of Yau's estimates ([107, 3]), Cao ([13]) proved the long-time existence of (1.1) and gave an alternative proof of the existence of Kähler Einstein metrics in the cases when the first Chern class is negative or zero. In the Fano case, the convergence of Kähler Ricci flow is related to some stability conditions ([67, 69, 70, 97, 25] etc). Perelman ([73]) proves that in a Fano manifold X, the normalized Kähler Ricci flow

$$\frac{\partial}{\partial t} \omega = -\mathrm{Ric}(\omega) + \omega, \quad \omega(0) = \omega_0 \tag{1.2}$$

starting at a metric $\omega_0 \in C_1(X)$, has uniformly bounded diameter and scalar curvature.

He also proves that if X admits a Kähler Einstein metric, then the flow (1.2) converges to a Kähler Einstein metric (see [100] for the more general case when X admits a Kähler Ricci soliton). Recently, Chen and Wang ([25]) prove that on any Fano manifold, the Kähler Ricci flow (1.2) converges to a Kähler Ricci soliton with mild singularities on a $\mathbb{Q}$-Fano normal variety (the three dimensional case is also proved by Tian and Zhang ([97])), hence proving the Hamilton-Tian conjecture.

However, most projective manifolds do not have definite or trivial first Chern class. Tsuji ([104]) applies the Kähler Ricci flow and proves the existence of a singular Kähler Einstein metric on a minimal projective manifold of general type. It is the first attempt to relate the Kähler Ricci flow and canonical metrics to the minimal model program. Since then much progress has been made in this direction. It is proved by Tian and Zhang in [98] (when K_X is nef, this is obtained by Tsuji [104]) that the maximal existence time for the unnormalized flow

$$\frac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega), \quad \omega(0) = \omega_0 \tag{1.3}$$

is given by

$$T = \sup\{t > 0 | [\omega_0] + tK_X \text{ is Kähler}\}. \tag{1.4}$$

The regularity of the canonical singular Kähler Einstein metrics on minimal projective manifolds of general type was studied in [35, 110, 79]. The analytic minimal model program, introduced in [79, 80], aims to find the minimal model of an algebraic variety, by running the Kähler-Ricci flow. If the minimal projective manifold has positive Kodaira dimension and is not of general type, it admits an Iitaka fiberation over its canonical model. Song and Tian ([78, 79]) define on the canonical model a new family of generalized Kähler Einstein metrics twisted by a canonical form of Weil-Petersson type from the fiberation structure, and they prove that the normalized Kähler Ricci flow

$$\frac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega) - \omega \tag{1.5}$$

converges to such a canonical metric in some weak sense if the canonical bundle is semi-ample. If K_X is not nef, the unnormalized Kähler Ricci flow (1.3) must be singular at some finite time $T < \infty$ given in (1.4). At the singular time T, the flow either develops singularities on a subvariety of X or X admits a Fano fiberation and the flow is expected to collapse along the fibers. The singularities can only appear where K_X is negative, and the flow might perform an analytic or geometric surgery equivalent to an algebraic surgery such as divisorial contraction or a flip, and replace X by a new projective variety X'. In [80], Song and Tian show the existence of weak solutions to the Kähler Ricci flow on singular projective varieties, and it is expected that the flow can flow through singularity and continue to exist on the possibly singular variety X' ([80]). In a series paper of Song and Weinkove ([83, 84, 85]), for a divisorial contraction map $\pi : X \to Y$, under some cohomology class assumptions, they show the flow (1.3) starting at X can flow through the singularity and uniquely extend to the manifold Y. When the flow collapses onto a new projective variety, it is expected ([80]) that the flow can also be continued, and we can repeat the above procedures until either the flow exists for all time or it collapses to a point. If the flow exits for all time, it should converge to a generalized Kähler Einstein metric on its canonical model or a Ricci flat metric after normalization, if we assume the abundance conjecture.

1.1.2 Conical metrics on toric manifolds

The existence of Kähler-Einstein metrics has been a central problem in Kähler geometry since Yau's celebrated solution [107] to the Calabi conjecture. Constant scalar curvature metrics with conical singularities have been extensively studied in [63, 103, 60] for Riemann surfaces. In general, one considers a pair (X, D) for an n-dimensional compact Kähler manifold and a smooth complex hypersurface D of X. A conical Kähler metric g on X with cone angle $2\pi\beta$ along D is locally equivalent to the following model edge metric

$$g = |z_1|^{-2(1-\beta)} dz_1 \otimes d\bar{z}_1 + \sum_{j=2}^{n} dz_j \otimes d\bar{z}_j$$

if D is locally defined by $z_1 = 0$. Applications of conical Kähler metrics are proposed and applied to obtain various Chern number inequalities [90, 82]. Donaldson has developed the linear theory to study the existence of canonical conical Kähler metrics in [33]. It plays an essential role in the recent breakthrough of the Yau-Tian-Donalson conjecture

[88, 20, 92, 21, 22, 23]. Brendle [9] solves Yau's Monge-Ampère equations for conical Kähler metrics with cone angle $2\pi\beta$ for $\beta \in (0, 1/2)$ along a smooth divisor D. The general case is settled by Jeffres, Mazzeo and Rubinstein [49] for all $\beta \in (0, 1)$. As an immediate consequence, there always exist conical Kähler-Einstein metrics with negative or zero constant scalar curvature with cone angle $2\pi\beta$ along a smooth divisor D for $\beta \in (0, 1)$. When X is a Fano manifold, Donaldson [33] proposes to study the conical Kähler-Einstein equation

$$Ric(\omega) = \beta\omega + (1 - \beta)[D], \tag{1.6}$$

where D is smooth simple divisor in the anticanonical class $[-K_X]$ and $\beta \in (0, 1)$.

The solvability of equation (1.6) is closely related to the following holomorphic invariant for Fano manifolds which is known as the greatest Ricci lower bound first introduced by Tian in [89].

Definition 1.1.1. *Let X be a Fano manifold. The greatest Ricci lower bound $R(X)$ is defined by*

$$R(X) = \sup\{\beta \mid Ric(\omega) \geq \beta\omega, \ \textit{for some } \omega \in c_1(X) \cap \mathcal{K}(X)\}, \tag{1.7}$$

where $\mathcal{K}(X)$ is the space of all Kähler metrics on X.

It is proved by Szekelyhidi in [86] that $[0, R(X))$ is the maximal interval for the continuity method to solve the Kähler-Einstein equation on a Fano manifold X. In particular, it is independent of the choice for the initial Kähler metric when applying the continuity method. The invariant $R(X)$ is explicitly calculated for $\mathbb{P}^2$ blown up at one point by Szekelyhidi [86], and for all toric Fano manifolds by Li [56]. Recent results [58] show that $R(X) = 1$ if and only if X is K semi-stable, and such a Fano manifold satisfies the Chern-Miyaoka inequality [82]. It is shown in [82, 59] that (1.6) cannot be solved for $\beta > R(X)$, answering a question of Donaldson [33] while it can always be solved for $\beta \in (0, R(X))$ if one replace D by a smooth divisor in the pluri-anticanonical system of X.

1.2 Main results

1.2.1 Blow up behavior of Kähler Ricci flow

We will study the unnormalized Kähler Ricci flow

$$\frac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega), \quad \omega(0) = \omega_0. \tag{1.8}$$

on $X = \mathbb{CP}^n \# \overline{\mathbb{CP}^n}$, $\mathbb{CP}^n$ blown-un at one point, with the initial metric ω_0 $U(n)$-invariant. It is proved ([83]) that the flow (1.8) must develop finite time singularity and it either shrinks to a point, collapses to $\mathbb{CP}^{n-1}$ or contracts an exceptional divisor, in the Gromov-Hausdorff topology.

When the flow shrinks to a point, the initial Kähler class is proportional to the first Chern class. Zhu ([111]) shows that the flow must develop Type I singularities and the rescaled Ricci flow converges in the Cheeger-Gromov-Hamilton sense to the unique compact shrinking Kähler Ricci soliton on X constructed in [14, 51, 105].

When the flow collapses to $\mathbb{CP}^{n-1}$, it is shown by Fong ([39]) that the flow must develop Type I singularities and the rescaled flow converges in Cheeger-Gromov-Hamilton sense to the ancient solution that splits isometrically as $\mathbb{C}^{n-1} \times \mathbb{CP}^1$.

When the flow does not collapse, Song ([75]) shows that the flow (1.8) must develop Type I singularities and the parabolic blow up of the Type I Ricci flow along the exceptional divisor converges to a complete non-flat shrinking Kähler Ricci soliton on a complete manifold *diffeomorphic* to $\mathbb{C}^n$ blown-up at one point.

Our main goal is to show that in the non-collapsed case as done by Song ([75]), the blow-up limit of the Kähler Ricci flow is *biholomorphic* to $\mathbb{C}^n$ blown-up at one point and the limit Kähler Ricci soliton is the FIK soliton constructed in [36] on $\mathbb{C}^n$ blown-up at one point. The main theorem is

Theorem 1.2.1 ([45])**.** *Suppose ω_0 is a Kähler metric on $X = \mathbb{CP}^n \# \overline{\mathbb{CP}^n}$ with Calabi symmetry and $\omega_0 \in b_0[D_\infty] - a_0[D_0]$ with $0 < a_0(n-1) < b_0(n+1)$. Choose $p \in D_0$, let $(X_\infty, p_\infty, g_\infty)$ be the Cheeger-Gromov-Hamilton limit of $(X_\infty, p, g_j(t))$, then $(X_\infty, p_\infty, g_\infty)$ is biholomorphic to $\mathbb{C}^n$ blown up at one point and g_∞ is a complete, $U(n)$*

symmetric Kähler Ricci soliton metric, hence is one of the FIK solitons constructed in [36].

For the notions of D_0, D_∞ and $g_j(t)$ and $U(n)$-invariant Kähler metrics, see Section 3.1. The proof of Theorem 1.2.1 will be given in Chapter 3.

1.2.2 Convergence of Kähler Ricci flow

Let X be a projective n-dimensional manifold with the canonical bundle K_X big and nef. We consider the Kähler-Ricci flow

$$\frac{\partial \omega}{\partial t} = -\mathrm{Ric}(\omega) - \omega, \quad \omega(0) = \omega_0, \tag{1.9}$$

where ω_0 is a Kähler metric on X. It's well-known that the equation (1.9) is equivalent to the following complex Monge-Ampere equation

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} = \log \dfrac{(\chi + e^{-t}(\omega_0 - \chi) + i\partial\bar{\partial}\varphi)^n}{\Omega} - \varphi \\ \varphi(0) = 0, \end{cases} \tag{1.10}$$

where Ω is a smooth volume form, $\chi = i\partial\bar{\partial}\log\Omega \in c_1(K_X) = -c_1(X)$, and $\omega(t) = \chi + e^{-t}(\omega_0 - \chi) + i\partial\bar{\partial}\varphi$. It's also well-known ([104, 98]) that the equation (1.10) has long time existence, if K_X is nef. Our first result is

Theorem 1.2.2 ([44])**.** *Let X be a projective manifold with K_X big and nef. If along the Kähler Ricci flow* (1.9)*, the Ricci curvature is uniformly bounded below for any $t \geq 0$, i.e.,*

$$\mathrm{Ric}(\omega(t)) \geq -K\omega(t),$$

for some $K > 0$, then there is a constant $C > 0$ such that the diameter of $(X, \omega(t))$ remain bounded, i.e.,

$$\mathrm{diam}(X, \omega(t)) \leq C.$$

Remark 1.2.1. *If we use Kawamata's theorem ([50]) that the nef and big canonical line bundle K_X is semi-ample, by [110, 81] the scalar curvature along the Kähler Ricci flow* (1.9) *is uniformly bounded, hence Ricci curvature lower bound implies that Ricci curvature is uniformly bounded on both sides. Then in the proof of Theorem 1.2.2, we* *can use Cheeger-Colding-Tian ([16] or see Theorem 2.2.8) theory to identify the regular sets. Moreover, if K_X is semi-ample and big, then the L^∞ bound of φ in* (1.10) *will simplify the proof. However, following Song's ([77]) recent analytic proof of base point freeness for nef and big K_X, our proof of Theorem 1.2.2 does not rely on Kawamata's theorem.*

It is conjectured by Song-Tian in [80] that the Kähler Ricci flow (1.9) will converge to the the canonical model of X coupled with the unique Kähler Einstein current with bounded potential, in the Gromov-Hausdorff sense. Under the assumption that the Ricci curvature is uniformly bounded below, we can confirm this conjecture.

Corollary 1.2.1 ([44])**.** *Under the same assumptions as Theorem 1.2.2, then as $t \to \infty$,*

$$(X, \omega(t)) \xrightarrow{d_{GH}} (X_\infty, d_\infty),$$

the limit space X_∞ is homeomorphic to the canonical model X_{can} of X. Moreover, (X_∞, d_∞) is isometric to the metric completion of (X°_{can}, g_{KE}), where g_{KE} is the unique Kähler-Einstein current with bounded local potentials and X°_{can} is the regular part of X_{can}.

Consider the unnormalized Kähler Ricci flow

$$\frac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega), \quad \omega(0) = \omega_0, \tag{1.11}$$

with long time existence. The flow (1.11) is said to be of Type III, if

$$\sup_{X\times[0,\infty)} t|Rm|(x,t) < \infty,$$

otherwise it is of Type IIb, here $|Rm|(\omega(t))$ denotes the Riemann curvature of $\omega(t)$. It's well-known that Type III condition is equivalent to the curvature is uniformly bounded along the normalized Kähler Ricci flow (1.9). As a by-product of our proof of Theorem 1.2.2, we obtain a purely analytic proof of a recent result of Tosatti-Zhang ([102]), namely,

Theorem 1.2.3 ([102])**.** *Let X be a projective manifold with K_X big and nef, if the Kähler Ricci flow* (1.9) *is of type III, then the canonical line bundle K_X is ample.*

1.2.3 Existence of canonical conical Kähler metrics on toric manifolds

In the joint paper with Datar, Song and Wang ([29]), we give various generalizations of the greatest Ricci lower bound (1.7).

The Bakry-Emery-Ricci curvature on a Riemannian manifold (M, g) is defined by

$$Ric_f(g) = Ric(g) + Hess f$$

for a smooth real valued function f on M [4]. If (M, g, f) satisfies the equaiton $Ric_f(g) = \lambda g$ for some $\lambda \in \mathbb{R}$, it is called a gradient Ricci soliton with the gradient vector field $V = \nabla f$. We can define the greatest Bakry-Emery-Ricci lower bound on Fano manifolds as an analogue of the greatest Ricci lower bound.

Definition 1.2.1. *Let X be a Fano manifold. The greatest Bakry-Emery-Ricci lower bound $R_{BE}(X)$ is defined by*

$$R_{BE}(X) = \sup\{\beta \mid Ric(\omega) \geq \beta\omega + \mathcal{L}_{Re\xi}\omega, \textit{for some } \omega \in c_1(X) \cap \mathcal{K}(X) \textit{ and } \xi \in H^0(X, TX)\}.$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ.

Since X is Fano, it is simply connected and $\mathcal{L}_{Re\xi}\omega = -i\partial\bar{\partial}f_\xi$ for some real-valued smooth function f_ξ with $\nabla_{z_i}\nabla_{z_j}f_\xi = 0$ in holomorphic coordinates. This implies that $Ric(\omega) \geq \beta\omega + \mathcal{L}_{Re\xi}\omega$ is equivalent to

$$R_{ij} + \nabla_i\nabla_j f_\xi \geq \beta g_{ij}$$

in real coordinates. Hence

$$R_{BE}(X) = \sup\{\beta \mid Ric(\omega) + i\partial\bar{\partial}f \geq \beta\omega,\ \omega \in c_1(X) \cap \mathcal{K}(X),$$
$$f \in C^\infty(X), \uparrow \bar{\partial}f \text{ is holomorphic }\}.$$

One can relate $R_{BE}(X)$ to the continuity method for solving the Kähler-Ricci soliton equation on X as introduced in [99] as analogue of $R(X)$ and explicitly calculate the value of $R_{BE}(X)$ for toric Fano manifolds. In fact, we conjecture that $R_{BE}(X) = 1$ for any Fano manifold X.

We start with a few definitions. Let X be an n-dimensional toric manifold and L a Kähler class (or equivalently, an ample $\mathbb{R}$ divisor) on X. In [32, 82], smooth toric conical Kähler metrics are defined and studied in detail. We let $\mathcal{K}_c(X)$ be the set of all smooth toric conical Kähler metrics with each cone angle in $(0, 2\pi]$.

Definition 1.2.2. *Let X be a toric manifold. Let $\omega \in \mathcal{K}_c(X)$ be a smooth toric conical Kähler metric on X. We say*

$$Ric(\omega) > \alpha\omega$$

if there exists $\eta \in \mathcal{K}_c(X)$ with the same cone angle along an effective toric divisor D such that

$$Ric(\omega) = \alpha\omega + \eta + [D].$$

Definition 1.2.3. *A smooth toric conical Kähler metric $\omega \in \mathcal{K}_c(X)$ is called a conical Kähler-Ricci soliton metric if it satsifies*

$$Ric(\omega) = \alpha\omega + \mathcal{L}_\xi\omega + [D]$$

for some holomorphic vector field ξ and effective toric divisor D. If $\xi = 0$, the metric is a smooth toric conical Kähler-Einstein metric.

Associated to any toric Kähler class, we define the following geometric invariants $\mathcal{R}(X, L)$, $\mathcal{R}_{BE}(X, L)$ and $\mathcal{S}(X, L)$.

Definition 1.2.4. *Let X be a toric manifold and L be a Kähler class on X. Let $\{D_j\}_{j=1}^N$ be the set of all prime toric divisors on X. Then we define*

1. $\mathcal{R}(X, L) = \sup\{\alpha \mid Ric(\omega) > \alpha\omega$ *for some* $\omega \in c_1(L) \cap \mathcal{K}_c(X)\}$,
2. $\mathcal{R}_{BE}(X, L) = \sup\{\alpha \mid Ric(\omega) + \mathcal{L}_\xi\omega > \alpha\omega$ *for a* $\omega \in c_1(L) \cap \mathcal{K}_c(X)$ *and a toric* $\xi \in H^0(X, TX)\}$,
3. $\mathcal{S}(X, L) = \sup\left\{\alpha \mid \text{there exists } D = \sum_{j=1}^N a_j D_j \sim -K_X - \alpha L \text{ with } a_j \in [0, 1)\right\}$.

$\mathcal{R}(X, L)$ and $\mathcal{R}_{BE}(X, L)$ are natural generalizations of $R(X)$ and $R_{BE}(X)$ for log Fano manifolds with polarization L. $\mathcal{S}(X, L)$ characterizes when (X, D) is log Fano as by definition $K_X + D$ is klt and negative. In the special case that X is toric Fano and $L = -K_X$, $\mathcal{R}(X, -K_X)$ is the usual greatest Ricci lower bound studied in [86] and $\mathcal{S}(X, -K_X) = 1$. In fact, for any toric pair (X, L), $\mathcal{R}(X, L)$ and $\mathcal{S}(X, L)$ are both positive.

Any toric manifold X is induced by an integral Delzant polytope P and P determines a Kähler class on X. Without loss of generality, we let

$$P = \{x \in \mathbb{R}^n \mid l_j(x) > 0,\ j = 1, ..., N\}, \tag{1.12}$$

where $l_j(x) = v_j \cdot x + \lambda_j$, v_j is a prime integral integral vector in $\mathbb{Z}^n$ and $\lambda_j \in \mathbb{R}$ for all $j = 1, ..., N$. As a special case, when X is Fano, one can choose $\lambda_j = 1$ for all j and the polytope gives the anti-canonical polarization of X. The existence of smooth toric Kähler-Einstein and Kähler-Ricci soliton metrics on toric Fano manifolds is completely settled by Wang-Zhu [105]. We generalize their results to toric conical Kähler-Einstein and Kähler-Ricci soliton metrics on any toric manifold.

Theorem 1.2.4 ([29]). *Let X be an n-dimensional toric Kähler manifold and L be the Kähler class on X induced by the Delzant polytope P. Then*

1. $\mathcal{R}_{BE}(X, L) = \mathcal{S}(X, L) > 0$ *and*

$$\mathcal{R}_{BE}(X, L) = \sup\{\alpha \mid \text{there exists } \tau \in P \text{ with } 1 - \alpha l_j(\tau) > 0, j = 1, ..., N\}, \tag{1.13}$$

2. *For any $\alpha \in (0, \mathcal{S}(X, L))$ and $\tau \in P$ satisfying $1 - \alpha l_j(\tau) \geq 0$ for all j, there exists a unique $\omega \in L \cap \mathcal{K}_c(X)$ solving the Kähler-Ricci soliton equation*

$$Ric(\omega) = \alpha\omega + \mathcal{L}_\xi \omega + [D]. \tag{1.14}$$

Moreover the divisor D and the vector field ξ are given by

$$D = \sum_{j=1}^{N} (1 - \alpha l_j(\tau)) D_j, \quad \xi = \sum_{i=1}^{n} c_i z_i \frac{\partial}{\partial z_i}, \tag{1.15}$$

where z_i's are the standard coordinates on $(\mathbb{C}^*)^n$ and $c \in \mathbb{R}^n$ is uniquely given by

$$\tau = \frac{\int_P x e^{c\cdot x}\, dx}{\int_P e^{c\cdot x}\, dx}. \tag{1.16}$$

3. *There does not exist a toric conical Kähler-Ricci soliton metric $\omega \in L \cap \mathcal{K}_c(X)$ solving the soliton equation (1.14) for any $\alpha > \mathcal{R}_{BE}(X, L)$.*

The existence of toric conical Kähler-Ricci soliton metrics on log Fano toric varieties is derived in [6] and for general toric manifolds by allowing the cone angle in $(0,\infty)$ [55]. Our result gives a complete classification for the existence of toric conical Kähler-Einstein and Kähler-Ricci soliton metrics using the invariants $\mathcal{R}(X,L)$ and $\mathcal{R}_{BE}(X,L)$ for any Kähler class. We are only interested in the toric conical Kähler metrics with cone angle in $(0,2\pi)$ since the smooth part is geodesic convex and various Riemannian geometric properties can be applied. In particular, it gives optimal regularity and a complete classification of smooth toric conical Kähler-Ricci soliton metrics for any toric pair (X,L).

As a special case, we obtain an existence result for conical Kähler-Einstein metrics on toric manifolds and apply it to characterize the invariant $\mathcal{R}(X,L)$ in terms of the polytope data.

Theorem 1.2.5 ([29]). *Let X be an n-dimensional toric Kähler manifold and L be the Kähler class on X induced by the Delzant polytope P. Let P_C be the barycenter of P. Then*

1. *$\mathcal{R}(X,L)>0$ and*

$$\mathcal{R}(X,L)=\sup\{\alpha \mid 1-\alpha l_j(P_C)>0,\ j=1,...,N\}. \tag{1.17}$$

2. *For all $\alpha\in(0,\mathcal{R}(X,L)]$, there exists a unique toric conical Kähler-Einstein metric $\omega\in L\cap\mathcal{K}_c(X)$ solving*

$$Ric(\omega)=\alpha\omega+[D]. \tag{1.18}$$

Moreover the divisor D is given by

$$D=\sum_{j=1}^{N}(1-\alpha l_j(P_c))D_j. \tag{1.19}$$

3. *There does not exist a toric conical Kähler-Einstein metric $\omega\in L\cap\mathcal{K}_c(X)$ solving the equation (1.18) for any $\alpha>\mathcal{R}(X,L)$.*

In the special case when X is Fano and $L=-K_X$, $l_j(0)=1$ and so $1-\alpha l_j(P_C)=(1-\alpha)l_j(\frac{-\alpha P_C}{1-\alpha})$. By the theorem, $\mathcal{R}(X,L)$ is the maximum of all α such that $\frac{-\alpha P_C}{1-\alpha}$ remains inside the polytope, generalizing the results in the smooth case in [86] and [56].

Smooth Kähler-Einstein and Kähler-Ricci soliton metrics on Fano manifolds are unique, while the space of conical Kähler-Einstein and Kähler-Ricci soliton metrics is much bigger. One would ask under what assumptions the space of conical Kähler-Einstein metrics is connected. In other words, given two log Fano manifolds X_0 and X_1, we ask when and how one can connect X_0 and X_1 by a family of conical Kähler-Einstein spaces in Gromov-Hausdorff topology. To answer this question in the toric case, we have

Theorem 1.2.6 ([29])**.** *Let X_0 and X_1 be two n-dimensional toric manifolds. Suppose $\omega_0 \in \mathcal{K}_c(X_0)$ and $\omega_1 \in \mathcal{K}_c(X_1)$ are two smooth toric conical Kähler-Einstein metrics on X_0 and X_1 respectively. Then, there exist a family $\{(X_t, \omega_t)\}_{t\in[0,1]}$ of n-dimensional toric manifolds X_t with smooth toric conical Kähler-Einstein metrics $\omega_t \in \mathcal{K}_c(X_t)$ for $t \in [0, 1]$, such that*

1. *(X_t, ω_t) is a continuous path in Gromov-Hausdorff topology for $t \in [0, 1]$,*

2. *ω_t is piecewise smooth in t on the complex torus $(\mathbb{C}^*)^n$.*

Theorem 1.2.6 can be considered to be an analytic analogue of the weak factorization theorem for toric varieties in algebraic geometry (see Theorem 2.3.1). Combined with Theorem 1.2.5, it implies that any two toric manifolds of same dimension can be joined by a continuous path of conical Kähler-Einstein spaces in Gromov-Hausdorff topology. It is a natural question to ask if for any two birationally equivalent Fano manifolds, there exists a continuous path connecting them by Fano varieties coupled with conical Kähler-Einstein metrics, in Gromov-Hausdorff topology. This is related to the connectedness of moduli space of log Fano varieties coupled with conical Kähler-Einstein metrics.

Chapter 2

Preliminaries

In this chapter, we will collect some basic facts and definitions from Kähler geometry, Riemannian geometry and metric geometry. They are basically well-known and will be stated without proofs.

2.1 Kähler geometry

Let (X, ω, J) be a compact complex manifold. The metric form ω is called Kähler if it is closed, i.e. $d\omega = 0$, or in local coordinates $(z_1, \ldots, z_n)$,

$$\frac{\partial g_{i\bar{j}}}{\partial z_k} = \frac{\partial g_{k\bar{j}}}{\partial z_i}, \quad \frac{\partial g_{i\bar{j}}}{\partial \bar{z}_k} = \frac{\partial g_{i\bar{k}}}{\partial \bar{z}_j}, \quad \forall i, j, k,$$

where $g_{i\bar{j}}$ is the components of ω in these coordinates, i.e.,

$$\omega = \sqrt{-1} g_{i\bar{j}} dz_i \wedge d\bar{z}_j.$$

The Kähler metric ω lies in a cohomology class $[\omega] \in H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$. By the $i\partial\bar{\partial}$-lemma ([40]) for any other $(1,1)$-form ω' in the same cohomology class as ω, there exists a smooth real function φ such that

$$\omega' = \omega + i\partial\bar{\partial}\varphi.$$

Hence all the Kähler metrics in the Kähler class $[\omega]$ can be written as the form $\omega + i\partial\bar{\partial}\varphi$ for some $\varphi \in PSH(X, \omega)$ where

$$PSH(X, \omega) = \{\varphi \in C^\infty(X, \mathbb{R}) | \omega + i\partial\bar{\partial}\varphi > 0\}.$$

The Riemannian curvature of ω is equal to (in locally coordinates $(z_1, \ldots, z_n)$)

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l},$$

and the Ricci curvature is

$$R_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \det g_{k\bar{l}}, \tag{2.1}$$

and the scalar curvature $R = g^{i\bar{j}} R_{i\bar{j}}$. Here and in the rest of the thesis, we denote $g^{i\bar{j}}$ the inverse of $g_{i\bar{j}}$, i.e., $g^{i\bar{j}} g_{k\bar{j}} = \delta^i_k$. The first and second Bianchi identities say that the indices with or without bar are all symmetric in the local components: $R_{i\bar{j}k\bar{l}}$, $R_{i\bar{j}k\bar{l},p}$ and $R_{i\bar{j}k\bar{l},\bar{p}}$.

The Ricci form

$$\mathrm{Ric}(\omega) = \frac{\sqrt{-1}}{2\pi} R_{i\bar{j}} dz_i \wedge d\bar{z}_j = \frac{1}{2\pi} i\partial\bar{\partial} \log \det g_{i\bar{j}}$$

is a closed $(1,1)$-form, and its cohomology class (denoted by $C_1(X)$) is called the first Chern class of X.

A holomorphic line bundle L over the Kähler manifold X is a vector bundle over X with fiber $\mathbb{C}$, and the transition functions h_{ij} over $U_i \cap U_j$ are never zero holomorphic functions, where $L|_{U_i} \cong U_i \times \mathbb{C}$ is a local trivialization of L, and $X = \cup_i U_i$. The transition functions satisfy

$$h_{ij} h_{ji} = 1, \quad \text{on } U_i \cap U_j \neq \emptyset,$$

and

$$h_{ij} h_{jk} h_{ki} = 1, \quad \text{on } U_i \cap U_j \cap U_k \neq \emptyset.$$

These equations implies that $\{h_{ij}\}$ defines a 1 co-cycle hence a cohomology class the Cech group $H^1(X, \mathcal{O}^*)$.

A holomorphic section s of L is defined locally by $s = s_i e_i$ on each U_i, where e_i is a local frame of L over U_i and s_i is a holomorphic function on U_i. s is globally defined iff $s_i = h_{ij} s_j$ over $U_i \cap U_j$, since $e_i h_{ij} = e_j$. A Hermitian metric h on L is given by positive local functions $\{h_i\}$ over U_i such that $h_j = |h_{ij}|^2 h_i$ on $U_i \cap U_j$. Hence the $(1,1)$-form $-(2\pi)^{-1} i\partial\bar{\partial} \log h_i$ is globally defined, noting that $i\partial\bar{\partial} \log |h_{ij}|^2 = 0$. Actually, this $(1,1)$-form is called the curvature of the Hermitian metric h, and we will denote it by $\mathrm{Ric}(h)$. It represents a cohomology class in the Dolbeault cohomology group $H^{1,1}(X, \mathcal{O}) \cap H^2(X, \mathbb{Z})$. It is not hard to see for all Hermtian metrics on L, their curvatures lie in the same cohomology class, and we will denote this class by $C_1(L)$.

The set of holomorphic sections of L is denoted by $H^0(X, L)$, and the norm of $s \in H^0(X, L)$ with respect to a Hermtian metric h on L is defined by

$$|s|_h^2 := s_i \bar{s}_i h_i \quad \text{on } U_i,$$

and it is each to check this norm is globally defined by the transition laws of s_i and h_i.

The canonical line bundle K_X of X is defined to the determinant line bundle of $T^*_{(1,0)}M$, and $dz_1 \wedge \cdots \wedge dz_n$ is a local section of K_X on any local coordinates chart $(z_1, \ldots, z_n)$. The holomorphic sections of K_X are holomorphic n-forms. For any Kähler metric ω, $\frac{1}{\det g_{i\bar{j}}}$ defines a Hermtian metric on K_X. And its associated curvature is given

$$-(2\pi)^{-1} i\partial\bar{\partial} \log \frac{1}{\det g_{i\bar{j}}} = (2\pi)^{-1} i\partial\bar{\partial} \log \det g_{i\bar{j}} = -\mathrm{Ric}(\omega).$$

Hence we see that $-C_1(X) = C_1(K_X)$, or $C_1(X) = C_1(-K_X)$, where $-K_X$ is dual line bundle of K_X.

We recall a few notions about the line bundles.

Definition 2.1.1. *Given a holomorphic line bundle L over a compact Kähler manifold X, then*

(1) L is called ample, *if the linear system $|kL|$ for some $k \in \mathbb{N}$ gives an embedding of X to some projective space $\mathbb{CP}^N$, i.e., $kL = \mathcal{O}_{\mathbb{CP}^N}(1)$. In this case, X is necessarily projective by definition. The Kodaira embedding theorem implies that this is equivalent to the existence of a Hermtian metric h on L with curvature* $\mathrm{Ric}(h) > 0$.

(2) L is called numerical effective *(or* nef*) if for any irreducible curve $C \subset X$,* $\int_C C_1(L) \geq 0$.

(3) L is big, *if the Kodaira dimension $\kappa(L) = n$, or (in the* nef *case) equivalently*

$$\int_X C_1(L)^n > 0.$$

(4) L is called base point free, *if for any point $x \in X$, there exist a section $s \in H^0(X, L)$ such that $s(x) \neq 0$.*

When the canonical line bundle K_X of a Kähler manifold X is big and nef, then X is said to be a minimal manifold of general type. Kawamata's base point free theorem implies that K_X is semiample, in the sense that the linear system $|kK_X|$ for some $k \in \mathbb{N}$ gives a morphism of X to an ambient projective space.

We recall Hörmander's L^2 existence theorem on pseudo convex domains in $\mathbb{C}^n$.

Theorem 2.1.1 ([48])**.** *Let $\Omega \subset \mathbb{C}^n$ be an open set with C^2 pseudo convex boundary. Let $\varphi \in C^2(\bar{\Omega})$ be a strictly plurisubharmonic function in ω satisfying $i\partial\bar{\partial}\varphi \geq c\omega_E$ for some $c > 0$. Then for any $\sigma \in L^2_{(0,1)}(\Omega, e^{-\varphi})$ such that $\bar{\partial}\sigma = 0$ and*

$$\int_\Omega |\tau|^2 e^{-\varphi} < \infty,$$

then we can find a function $u \in L^2(\Omega, e^{-\varphi})$ solving the equation $\bar{\partial}u = \tau$ and

$$\int_\Omega |u|^2 e^{-\varphi}\omega_E^n \leq \int_\Omega \frac{|\tau|^2}{c} e^{-\varphi}\omega_E^n.$$

We can also reduce the C^2 regularity of φ by smooth approximation. In particular, the following global version of L^2 estimates due to Demailly ([30]) is very useful in our applications.

Theorem 2.1.2. *Suppose X is an n-dimensional projective manifold equipped with a smooth Kähler metric ω. Let L be a holomorphic line bundle over X equipped with a possibly singular hermitian metric h such that $\mathrm{Ric}(h) + \mathrm{Ric}(\omega) \geq \delta\omega$ in the current sense for some $\delta > 0$. Then for any L-valued $(0,1)$-form τ satisfying*

$$\bar{\partial}\tau = 0, \qquad \int_X |\tau|^2_{h,\omega}\omega^n < \infty,$$

there exists a smooth section u of L such that $\bar{\partial}u = \tau$ and

$$\int_X |u|^2_{h,\omega}\omega^n \leq \frac{1}{2\pi\delta}\int_X |\tau|^2_{h,\omega}\omega^n.$$

2.2 Riemannian geometry and metric geometry

2.2.1 Riemannian geometry

Let (M, g) be a Riemannian manifold and $p \in M$ be a point. The cut-locus of p is defined to be the points $q \in M$ either q is a conjugate point of p or there exists at least two distinct minimal geodesics from p to q. It is known that the cut-locus has measure zero by an application of Sard's theorem. The exponential map $\exp_p : T_pM \to M$ is local diffeomorphism in the interior of cut-locus. Denote $\Omega = M \backslash \{$the cut-locus of $p\}$, then $\exp_p^{-1}(\Omega)$ is a star-shaped domain in $T_pM \cong \mathbb{R}^n$. It is also well-known that the distance function $d(x) = d(p, x)$ is smooth in $\Omega \backslash \{p\}$. The injectivity radius of p is defined to be

$$i_p = inj_g(p) := \sup\{r > 0 | \ B(p, r) \subset \Omega\}$$

where $B(p, r)$ is the geodesic ball centered at p. And it is clear that $\exp_p : B(0, i_p) \subset T_pM \to B(p, i_p) \subset M$ is a diffeomorphism.

The space forms are simply connected manifolds with constant sectional curvature, which by the uniformization theorem are S^n, $\mathbb{R}^n$ and $\mathbb{H}^n$, with curvatures normalized being $1, 0, -1$, respectively. The metric with constant sectional curvature K is given by (see [17])

$$dr^2 + \mathrm{sn}_K(r)^2 g_{S^{n-1}},$$

where $g_{S^{n-1}}$ is the standard metric on S^{n-1} with curvature 1, and

$$\mathrm{sn}_K(r) = \begin{cases} \frac{1}{\sqrt{K}} \sin(\sqrt{K} r), & \text{if } K > 0 \\ r, & \text{if } K = 0 \\ \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|} r), & \text{if } K < 0. \end{cases}$$

Theorem 2.2.1 (Hessian comparison ([17])). *Let (M, g) be a complete Riemannian manifold with dimension n and $p \in M$ be a fixed point. Suppose the sectional curvature of g satisfies*

$$\kappa \leq \mathrm{sect}_g \leq K$$

for some $\kappa, K \in \mathbb{R}$, then the Hessian of $r(x) = d(p, x)$ satisfies

$$\mathrm{Hess}_{r_K}(r(x)) \leq \mathrm{Hess}_r(x) \leq \mathrm{Hess}_{r_\kappa}(r(x)),$$

at x where $r(\cdot)$ is smooth, and r_κ and r_K are the distance functions on the space forms with constant curvature κ, K, respectively.

Theorem 2.2.2 (comparison for Jacobi fields)**.** *Let (M,g) be a complete Riemannian manifold with $\kappa \leq \mathrm{sect}_g \leq K$. For a point $p \in M$, $\gamma(t)$ is a normal geodesic with initial point p, and we suppose along γ there is no conjugate point, then for any Jacobi field $J(t)$ along γ with $J(0) = 0$, we have*

$$|J_K(t)| \leq |J(t)| \leq |J_\kappa(t)|,$$

where $J_\kappa(t)$ is a Jacobi field along a normal geodesic $\gamma_\kappa(t) \subset S^n_\kappa$, the space form with constant sectional curvature κ, such that

$$J_\kappa(0) = 0, \quad |J'(0)| = |J'_\kappa(0)|, \quad \langle J'(0), \gamma'(0)\rangle = \langle J'_\kappa(0), \gamma'_\kappa(0)\rangle.$$

And similar choice of the Jacobi field $J_K(t)$ in S^n_K.

The Jacobi field $J_K(t)$ on the space form S^n_K along a normal geodesic $\gamma_K(t)$ with $J_K(0) = 0$ has norm

$$|J_K(t)| = \mathrm{sn}_K(t)|J'_K(0)|.$$

And the Jacobi vector field $J(t)$ along $\gamma(t)$ with vanishing initial has the form

$$J(t) = (d\exp_p)_{t\gamma'(0)}\Big(tJ'(0)\Big) = t(d\exp_p)_{t\gamma'(0)}\Big(J'(0)\Big).$$

Take $J'(0) = X_0 \in T_pM$ and by the comparison Theorem 2.2.2 we have for $t > 0$ small,

$$\frac{\mathrm{sn}_K(t)}{t}|X_0| \leq |(d\exp_p)_{t\gamma'(0)}(X_0)| \leq \frac{\mathrm{sn}_\kappa(t)}{t}|X_0|. \tag{2.2}$$

In particular, when $K = 0$, (2.2) implies the exponential map $\exp_p : T_pM \to M$ is distance non-increasing.

When we have only Ricci curvature lower bound, we have the Bonnet-Myers' theorem, Laplacian comparison theorem and Bishop-Gromov volume comparison theorem:

Theorem 2.2.3 (Bonnet-Myers' theorem)**.** *Suppose (M,g) is a complete Riemannian manifold with $\mathrm{Ric}(g) \geq (n-1)K > 0$, then the diameter of (M,g) is bounded above by $\frac{\pi}{\sqrt{K}}$.*

Theorem 2.2.4 (Laplacian comparison)**.** *Let (M,g) be a complete Riemannian manifold with $\mathrm{Ric} \geq (n-1)K$ for some $K \in \mathbb{R}$, $r(x) = d(x,p)$ for some $p \in M$, then*

$$\Delta r(x) \leq \Delta_K r_K(r(x)),$$

smoothly when $r(\cdot)$ is smooth at x and globally in the sense of distributions, where r_K is the distance function in the space form S^n_K.

Theorem 2.2.5 (Volume comparison)**.** *Let (M, g) be a complete Riemannian manifold with* $\mathrm{Ric} \geq (n-1)K$ *for some $K \in \mathbb{R}$, $r(x) = d(x, p)$ for some $p \in M$, then the function*

$$r \mapsto \frac{Vol_g(B(p, r))}{Vol_K(B_K(r))}$$

is non-increasing, where $Vol_K(B_K(r))$ is the volume of geodesic ball of radius r in the space form S^n_K.

2.2.2 Metric geometry

Definition 2.2.1. *Given any two compact metrics spaces (X, d_X) and (Y, d_Y), the Gromov-Hausdorff (GH) distance $d_{GH}(X, Y)$ of X, Y is defined to be the infimum of all $\epsilon > 0$ such that there is a map (continuous or not) $f : X \to Y$ which is called ϵ-Gromov-Hausdorff approximation (ϵ-GHA) such that*

(1) f is ϵ-onto, i.e., the image $f(X)$ is ϵ-dense in (Y, d_Y),

(2) f is ϵ-isometry, i.e., for any $x_1, x_2 \in X$,

$$|d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)| \leq \epsilon.$$

There are also other equivalent definitions of GH distance, for example, $d_{GH}(X, Y)$ can also be defined as the infimum of $\epsilon > 0$ over all compatible metrics on $X \sqcup Y$ such that both components are ϵ-dense. These two definitions may not be the same, but they are equivalent and hence do not affect our applications.

We say sequence of compact metric spaces (X_i, d_i) converges to (X_∞, d_∞) in GH topology, if $d_{GH}(X_i, X_\infty) \to 0$ as $i \to \infty$.

One of the fundamental results in metric geometry is the Gromov pre-compactness theorem:

Theorem 2.2.6 (Gromov pre-compactness)**.** *The set $\mathcal{M}(n, \Lambda, D)$ of n dimensional compact Riemannian manifolds (M, g) such that*

$$\mathrm{Ric}(g) \geq \Lambda, \quad \mathrm{diam}(M, g) \leq D$$

is pre-compact in the GH topology.

In the case manifolds not having finite diameter, we consider the *pointed* -GH convergence. We say

$$(X_i, d_i, p_i) \xrightarrow{p-GH} (X_\infty, d_\infty, p_\infty),$$

if for any $R > 0$, the metric balls $B_i(p_i, R) \xrightarrow{GH} B_\infty(p_\infty, R)$. Hence by Gromov precompactness theorem, for any sequence of complete Riemannian manifolds (M_i^n, g_i, p_i) with $\mathrm{Ric}(g_i) \geq \Lambda$, there exists a subsequence which converges in pointed GH sense.

In general the GH limit space of a sequence of metric spaces does not have good regularities. Under some geometric assumptions, Cheeger-Colding prove that the limit space does have some regularities:

Theorem 2.2.7 ([15, 26])**.** *Let (M_i^n, g_i, p_i) be a sequence of smooth Riemannian manifolds with*

$$\mathrm{Ric}(g_i) \geq -(n-1), \quad Vol(B(p_i, 1)) \geq v_0 > 0,$$

then any GH limit of (M_i, g_i, p_i), $(M_\infty, d_\infty, p_\infty)$ satisfies

(1) Volume converges, $\lim_{i\to\infty} Vol_{g_i}(B(p_i, R)) = \mathcal{H}^n(B_\infty(p_\infty, R))$ for any $R > 0$, where $\mathcal{H}^n$ is a suitable n-dimensional Hausdorff measure on (M_∞, d_∞).

(2) M_∞ has a regular-singular decomposition, $M_\infty = \mathcal{R} \cup \mathcal{S}$, where $\mathcal{R}$ is defined to be the points whose tangent cones are $\mathbb{R}^n$, and $\mathcal{S} = M_\infty \backslash \mathcal{R}$.

(3) The Hausdorff dimension of $\mathcal{S} \leq n-2$.

Recall a tangent cone at $q \in M_\infty$ is the GH limit of the spaces $(M_\infty, r_i^{-2} d_\infty, q)$ for a sequence $r_i \to 0$. The tangent cone at a point $q \in M_\infty$ may not be unique, and it depends on the choice of sequence $r_i \to 0$. We remark that by definition no tangent cone at $q \in \mathcal{S}$ can be $\mathbb{R}^n$. And if a tangent cone at some point splits off a Euclidean factor $\mathbb{R}^{n-1}$, then it must be $\mathbb{R}^n$, hence the point is regular.

If we assume Ricci curvature uniformly bounded, instead of lower bound, then Cheeger-Colding-Tian theory says more about the regularity of the limit space.

Theorem 2.2.8 ([16])**.** *Suppose a sequence of Riemannian manifolds (M_i, g_i, p_i) converges in GH sense to $(M_\infty, d_\infty, p_\infty)$. Suppose*

$$|\mathrm{Ric}(g_i)| \le n-1, \quad Vol_{g_i}(B(p_i,1)) \ge v_0 > 0,$$

then we have

(1) In the regular-singular decomposition $M_\infty = \mathcal{R} \cup \mathcal{S}$, $\mathcal{R}$ is an open $C^{2,\alpha}$ manifold with a $C^{1,\alpha}$ Riemannian metric compatible with the distance d_∞ on M_∞. $\mathcal{S}$ is closed and of Hausdorff codimension ≥ 2.

(2) If (M_i, g_i) are Kähler, then $\mathcal{S}$ is of Hausdorff codimension ≥ 4.

Recently the solution to the codimension 4 conjecture of Cheeger-Naber ([18]) implies that the Kähler condition in (2) above can be removed. And Colding-Naber ([27]) prove that the regular set $\mathcal{R}$ is geodesically convex, in the sense that any minimal geodesic starting at a point in $\mathcal{R}$ will remain inside $\mathcal{R}$.

2.3 Toric manifolds

In this section, let us collect some well-known facts of projective toric varieties.

Definition 2.3.1. *A Kähler manifold (X, ω) is called a toric manifold if it admits an effective Hamiltonian action of the compact torus $(S^1)^n$, which can be extended to a holomorphic action of the complex torus $(\mathbb{C}^*)^n$ with an open dense orbit $\cong (\mathbb{C}^*)^n$.*

Since the $(S^1)^n$ action is Hamiltonian, the moment map $\mu : X \to \mathbb{R}^n$ is defined to satisfy the equation

$$\langle d\mu, V\rangle = i_{\tilde{V}}\omega,$$

where $V \in \mathbb{R}^n = Lie((S^1)^n)$ and $\tilde{V}$ denotes the infinitesimal action of V on X. The Atiyah-Guillemin-Sternberg theorem implies that the image $\mu(X)$ of this map is the closure of a Delzant polytope $P \subset \mathbb{R}^n$, whose definition is given below.

Definition 2.3.2. *A convex polytope $P \subset \mathbb{R}^n$ is called a* Delzant polytope *if a neighborhood of any vertex $p \in P$ is $SL(n, \mathbb{Z})$ equivalent to $\{x_j \ge 0, j = 1, ..., n\} \subset \mathbb{R}^n$. P is called* an integral Delzant *polytope if each vertex $p \in P$ is a lattice point in $\mathbb{Z}^n \subset \mathbb{R}^n$.*

Let P be an integral Delzant polytope in $\mathbb{R}^n$ defined by

$$P = \{x \in \mathbb{R}^n \mid l_j(x) > 0, j = 1, ..., N\}, \tag{2.3}$$

where

$$l_j(x) = v_j \cdot x + \lambda_j$$

and v_i is a primitive integral vector in $\mathbb{Z}^n$ and $\lambda_j \in \mathbb{Z}_+$ for all $j = 1, ..., N$. Let Σ_P be the fan consisting of the cones over the faces of the *polar polytope*

$$\check{P} = \{y \in \mathbb{R}^n \mid \langle y, x\rangle_{\mathbb{R}} \geq -1 \text{ for all } x \in P\}.$$

Then Σ_P defines an n-dimensional smooth projective toric variety X_P. Its Picard group $\mathrm{Pic}(X)$ is generated by D_i's, the toric divisors corresponding to the generators of edges e_i's of the fan Σ_P. For any toric divisor $D = \sum a_i D_i$, it determines a rational convex polyhedron

$$P_D \quad = \quad \{\alpha \in \mathbb{R}^n \mid \langle \alpha, v_i\rangle \geq -a_i \text{ for all } i\} \subset \mathbb{R}^n \ ,$$

and the space of global sections of the line bundle $\mathcal{O}_X(D)$ is given by

$$H^0(X, \mathcal{O}_X(D)) = \bigoplus_{\alpha \in P_D \cap \mathbb{Z}^n} \mathbb{C} \cdot \chi^\alpha, \tag{2.4}$$

where χ's are the characters $\mathrm{Hom}(T, \mathbb{C}^*)$. In particular, we have

$$\dim H^0(X, \mathcal{O}(kD)) = k^n \mathrm{Vol}(P_D) + O(k^{n-1}), \tag{2.5}$$

where $\mathrm{Vol}(P_D)$ denote the Euclidean volume of $P_D \in \mathbb{R}^n$. In particular, for the anti-canonical divisor $-K_{X_P} = \sum_i D_i$, we have $P_{-K_X} = \{\alpha \in \mathbb{R}^n \mid \langle \alpha, v_i\rangle \geq -1 \text{ for all } i\}$, hence $\mathrm{Vol}(P_{-K_X}) > 0$. By (2.5), we conclude with the following well-known lemma.

Lemma 2.3.1. *Let X be a smooth projective toric variety, then $-K_X$ is big.*

2.3.1 Smooth toric Kähler metrics

A smooth toric Kähler metric ω is invariant under the $(S^1)^n$ action on the toric manifold. In the open dense $(\mathbb{C}^*)^n$, the metric $\omega = i\partial\bar{\partial}\varphi$ for some smooth function

φ by the Poincare lemma. We may suppose the function φ is also invariant under $(S^1)^n$ action, so $\varphi(z)$ depends only on $|z_1|, \ldots, |z_n|$, where $z_1, \ldots, z_n$ are the standard coordinates on $(\mathbb{C}^*)^n$. Introducing the logarithmic coordinates $\rho_j = \log|z_j|^2$ and $\theta_j = \arg(z_j)$, φ can be viewed as a function of $(\rho_1, \ldots, \rho_n) \in \mathbb{R}^n$, and under those coordinates,

$$\omega = \frac{\partial^2 \varphi}{\partial \rho_i \partial \rho_j} d\rho_i \wedge d\theta_j. \tag{2.6}$$

Since ω is positive, a necessary condition for the choice of φ is φ is a strictly convex function on $\mathbb{R}^n$. It can be shown that the moment map when restricted to $(\mathbb{C}^*)^n$ of the $(S^1)^n$ action with Kähler metric $\omega = i\partial\bar{\partial}\varphi$ is given by $\nabla\varphi : \mathbb{R}^n \to \mathbb{R}^n$, the gradient map of $\varphi : \mathbb{R}^n \to \mathbb{R}$. And the Atiyah-Guillemin-Sternberg theorem implies that the image of $\nabla\varphi$ is an open convex polytope P, given by the form (2.3). Moreover, Guillemin shows that a convex function φ on $\mathbb{R}^n$ defines a toric Kähler metric on X, if and only if it satisfies the *Guillemin boundary condition*:

$$u(x) = \sum_j l_j(x) \log l_j(x) + f(x), \quad x \in P \tag{2.7}$$

where $f \in C^\infty(\bar{P})$ is chosen such that u is strictly convex, and

$$u(x) = \sup_{\rho \in \mathbb{R}^n} x \cdot \rho - \varphi(\rho)$$

is the Legendre transform of the convex function φ, which will be called *symplectic potential* of the metric $i\partial\bar{\partial}\varphi$ in this thesis.

Finally we recall the following weak factorization theorem which was first proved in [106] (cf. also [1]) and reduces the proof of our main result to the case of a simple blow-up or blow-down of a smooth toric center.

Theorem 2.3.1. *Let $f : X \dashrightarrow Y$ be a toric birational map between two complete nonsingular toric varieties X and Y over $\mathbb{C}$, and let $U \subset X$ be an open set where f is an isomorphism. Then f can be factored into a sequence of blow-ups and blow-downs with nonsingular irreducible toric centers disjoint from U, namely, there is a sequence of birational maps between complete nonsingular toric varities*

$$X = X_0 \overset{f_1}{\dashrightarrow} X_1 \overset{f_2}{\dashrightarrow} \cdots \overset{f_i}{\dashrightarrow} X_i \overset{f_{i+1}}{\dashrightarrow} \cdots \overset{f_n}{\dashrightarrow} X_n = Y,$$

where

1. $f = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$*,*

2. f_i *is an isomorphism on* U*, and*

3. either $f_i : X_{i-1} \dashrightarrow X_i$ *or* $f_i^{-1} : X_i \dashrightarrow X_{i-1}$ *is a morphism obtained by blowing up a nonsingular irreducible toric center disjoint from* U*.*

2.4 Existence of uniform holomorphic coordinates

In this section, we will show the following result, which will be used in proving Proposition 4.1.3:

Proposition 2.4.1 ([74, 96]). *Given a complete Kähler manifold* (M, g)*, suppose the sectional curvature and its derivatives are uniformly bounded, i.e.*

$$\sup_M |\nabla^k Rm| \leq C(k)$$

for $k = 0, 1$ *and the injectivity radius at a point* p *is bounded below by* $r_0 > 0$*, then there is a uniform* $\theta \in (0, 1)$ *such that on* $B(p, \theta r_0)$*, there exists a local holomorphic coordinates* $z_1, \ldots, z_n$ *such that* $z(p) = 0$ *and for the metric* $g_{i\bar{j}} = g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})$

$$\frac{1}{10}\delta_{ij} \leq g_{i\bar{j}} \leq 10\delta_{ij}, \quad \|g_{i\bar{j}}\|_{C^{1,\alpha}} \leq 10,$$

for some $\alpha \in (0, 1)$*.*

This result is basically well-known, by combining the arguments from [74] and [96]. But we cannot find a complete proof of it in the literature, so we will give a full proof below. The proof is still due to [74] and [96].

2.4.1 An inequality of Siu-Yau ([74])

In their paper [74] Siu and Yau show an inequality, which says that if sectional curvature is nonpositive and bounded below, i.e. $-A_1 \leq \mathrm{sect}_g \leq 0$, then for a system normal coordinates x_i at $p \in M$, let $r = r_p(\cdot)$ be the distance function, and $\gamma(t)$ is a normal geodesic starting from p, then we have

$$|\nabla_t \nabla_t (t\frac{\partial}{\partial x_i})| \leq 4A_1 t \exp(A_1 t^2). \tag{2.8}$$

By slightly modifying their arguments, we will first show that under more general assumption on sectional curvature, say, $-A_1 \leq \mathrm{sect}_g \leq A_2$, then (2.8) still holds with the RHS replaced by $4(A_1 + A_2)t\exp(A_1 t^2)$.

It's well-known that $J(t) = t\frac{\partial}{\partial x_i}$ is a Jacobi field with vanishing initial along $\gamma(t)$, so

$$\nabla_t \nabla_t (t\frac{\partial}{\partial x_i}) = J(t)'' = -R(J, \gamma')\gamma'$$

by the Jacobi equation. We only need to estimate $|R(J, \gamma')\gamma'|$, which is equal to

$$|R(J, \gamma')\gamma'| = \sup_{Z \in T_p M, |Z|=1} \langle R(J, \gamma')\gamma', Z\rangle = \sup_{Z} R(J, \gamma', Z, \gamma').$$

On the other hand, the bi-linear form (in the following we will write $\gamma' = \mathbf{e}$)

$$Z \in T_p M \to L(Z) := -R(Z, \mathbf{e}, Z, \mathbf{e}) + A_2 \langle Z, Z\rangle,$$

is nonnegative definite, hence by Cauchy-Schwarz inequality $L(Z, J) = -R(Z, \mathbf{e}, J, \mathbf{e}) + A_2\langle Z, J\rangle$

$$|L(Z, J)|^2 \leq L(Z)L(J),$$

which implies

$$|-R(Z, \mathbf{e}, J, \mathbf{e}) + A_2\langle Z, J\rangle|^2 \leq (-R(Z, \mathbf{e}, Z, \mathbf{e}) + A_2\langle Z, Z\rangle)(-R(J, \mathbf{e}, J, \mathbf{e}) + A_2\langle J, J\rangle)$$

hence for $|Z| = 1$,

$$|-R(Z, \mathbf{e}, J, \mathbf{e}) + A_2\langle Z, J\rangle|^2 \leq (A_1 + A_2)^2 |J|^2.$$

Therefore

$$\sup_{Z, |Z|=1} |R(J, \mathbf{e}, Z, \mathbf{e})| \leq 2A_2|J| + (A_1 + A_2)|J| \leq 3(A_1 + A_2)|J|. \tag{2.9}$$

We are going to estimate the norm of J. We write the decomposition of J as

$$J(t) = (\alpha + \beta t)\mathbf{e} + V.$$

Since $J(0) = 0$, we see that $\alpha = 0$ and $V(0) = 0$, $V \perp \mathbf{e}$.

$$|J(t)|^2 = \beta^2 t^2 + |V|^2,$$

and V is also a Jacobi field. Taking derivative on both sides

$$\langle J(t), J'(t)\rangle = t\beta^2 + \langle V, V'\rangle.$$

For any fixed $t_0 > 0$ suitably small, since $V \perp \mathbf{e}$ is a Jacobi field, we have by the second variational formula

$$\langle V, V'\rangle(t_0) = \text{Hess } r_p(V,V)(t_0) = \int_0^{t_0} (|V'|^2 - R(V, \mathbf{e}, V, \mathbf{e}))dt = I_0^{t_0}(V),$$

where $I_0^{t_0}$ is the index along the normal geodesic. Now choose $E(t) = \frac{t}{t_0}\bar{V}$, where $\bar{V}$ is the parallel transport of $V(t_0)$ along the geodesic $\gamma(t)$, by index theorem ([17]) we have $I_0^{t_0}(V) \leq I_0^{t_0}(E)$, hence

$$I_0^{t_0}(V) \leq \int_0^{t_0} \frac{|\bar{V}|^2}{t_0^2} - \frac{t^2|\bar{V}|^2}{t_0^2} R\left(\frac{\bar{V}}{|\bar{V}|}, \mathbf{e}, \frac{\bar{V}}{|\bar{V}|}, \mathbf{e}\right) \leq \frac{|\bar{V}|^2}{t_0} + \frac{A_1}{3} t_0 |\bar{V}|^2,$$

Hence

$$\frac{1}{2}(|J|^2)'(t_0) = \langle J, J'\rangle(t_0) \leq \beta^2 t_0 + |V(t_0)|^2\Big(\frac{1}{t_0} + A_1 t_0\Big) \leq |J(t_0)|^2\Big(\frac{1}{t_0} + A_1 t_0\Big)$$

since t_0 is arbitrary

$$\frac{d}{dt}\log|J(t)|^2 \leq \frac{2}{t} + A_1 t.$$

Integrate from s to t, we get

$$\log\frac{|J(t)|^2}{|J(s)|^2} \leq \log\frac{t^2}{s^2} + \frac{A_1}{2}(t^2 - s^2),$$

that is,

$$|J(t)|^2 \leq \frac{|J(s)|^2}{s^2} t^2 \exp(A_1(t^2 - s^2)/2),$$

since

$$\frac{|J(s)|^2}{s^2} \to |\frac{\partial}{\partial x_i}|^2_{\gamma(0)} = 1, \text{ as } s \to 0,$$

so

$$|J(t)|^2 \leq t^2 \exp(A_1 t^2).$$

Thus combined with (2.9), we get

$$|R(J, \mathbf{e})\mathbf{e}|^2 \leq 3(A_1 + A_2)^2 t^2 \exp(A_1 t^2/2).$$

2.4.2 Good closed $(0,1)$-forms

From section 2.4.1, we know that if $|\text{sect}| \le C_0$ on a complete Kähler manifold, then at least for small radius we have at any point p, along a normal geodesic $\gamma(t)$ starting from p, the local normal coordinates $\{x_i\}$ satisfy

$$|\nabla_t \nabla_t (t \frac{\partial}{\partial x_i})| \le Ct \exp(Ct^2), \quad \text{for } t \text{ small.} \tag{2.10}$$

We will use C to denote a uniform constant depending only on C_0, n. We aim to construct local $\bar\partial$-closed $(0,1)$-forms near any point p by L^2 method. Without loss of generality, by rotating x_i by an $SO(2n)$ action, we may assume the functions

$$z_i := x_i + \sqrt{-1} x_{n+i}, \quad \forall i = 1, \dots, n$$

are holomorphic at p, i.e. $\bar\partial z_i(p) = 0$, or $\{\frac{\partial}{\partial z_i}\}_{i=1}^n|_p \in T_p^{1,0}M$ form a holomorphic basis for that bundle, or $dz_i(p) \in T^*_{1,0}M$ form a basis. We formally define

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\Big(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial x_{n+i}}\Big), \quad \frac{\partial}{\partial \bar z_i} = \frac{1}{2}\Big(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial x_{n+i}}\Big)$$

to be a first order differential operators, which coincide with $\partial_i, \bar\partial_i$ at the point p. We also define two projection operators

$$P_{1,0} : T_{\mathbb{C}}M \to T^{1,0}M, \quad P_{0,1} : T_{\mathbb{C}}M \to T^{0,1}M.$$

And for any $X \in T_{\mathbb{C}}M = TM \otimes \mathbb{C}$, since g is Kähler we have by a simple calculation

$$|X|_g^2 = |X^{1,0}|_g^2 + |X^{0,1}|_g^2, \tag{2.11}$$

where $X^{1,0} = P_{1,0}(X)$ is the $(1,0)$ component of the (complex-valued) vector X. Since the metric is Kähler, the associated Levi-Civita covariant derivative ∇_g preserves the type of vector fields, hence

$$\nabla P_{1,0} = P_{1,0} \nabla.$$

Our first goal is to estimate $|P_{0,1}\frac{\partial}{\partial z_i}|$ along a given normal geodesic $\gamma(t)$ with $\gamma(0) = p$. By (2.10) we have

$$|\nabla_t \nabla_t (t \frac{\partial}{\partial z_i})| \le Ct \exp(Ct^2),$$

and the $(0,1)$-component of $\nabla_t\nabla_t(t\frac{\partial}{\partial z_i})$ is $\nabla_t\nabla_t\Big(P_{0,1}(t\frac{\partial}{\partial z_i})\Big)$ since ∇_t preserves the type of vector fields. By (2.11) we have

$$|\nabla_t\nabla_t\Big(P_{0,1}(t\frac{\partial}{\partial z_i})\Big)| \leq Ct\exp(Ct^2). \tag{2.12}$$

Choose an orthonormal basis of $T_p^{0,1}M$ $\{X_i\}$ and parallel transport them along $\gamma(t)$ to get an orthonormal basis of $T_{\gamma(t)}^{0,1}M$, which we still denote by $\{X_i\}$. Since

$$P_{0,1}(t\frac{\partial}{\partial z_i}) = \sum_{j=1}^{n}\langle P_{0,1}(t\frac{\partial}{\partial z_i}), X_j\rangle X_j.$$

And

$$|P_{0,1}(t\frac{\partial}{\partial z_i})|^2 = \sum_{j=1}^{n}|\langle P_{0,1}(t\frac{\partial}{\partial z_i}), X_j\rangle|^2 := \sum_{j=1}^{n} f_j^2(t).$$

(2.12) implies that

$$|f_j''(t)| \leq Ct\exp(Ct^2). \tag{2.13}$$

And it's obvious that $f_j(0) = 0$ and $f_j'(0) = \langle 0\cdot\nabla_0 P_{1,0}(\frac{\partial}{\partial z_i}) + P_{0,1}(\frac{\partial}{\partial z_i})(0), X_j\rangle = 0$ since $\frac{\partial}{\partial z_i}|_p \in T_p^{1,0}M$. Therefore, we get

$$|f_j(t)| \leq Ct^3\exp(Ct^2)$$

by integrating (2.13) on both sides, therefore we get

$$|P_{0,1}(t\frac{\partial}{\partial z_i})| \leq Ct^3\exp(Ct^2),$$

for some different but uniform constant $C > 0$, or

$$|P_{0,1}(\frac{\partial}{\partial z_i})| \leq Ct^2\exp(Ct^2). \tag{2.14}$$

By taking conjugation we also have

$$|P_{0,1}(\frac{\partial}{\partial \bar{z}_i})| \leq Ct^2\exp(Ct^2). \tag{2.15}$$

Next we **claim** that $|\bar{\partial} z_i| \leq Ct^2\exp(Ct^2)$. To see this, for any

$$X = \sum_{j=1}^{n}\lambda_j\frac{\partial}{\partial z_j} + \mu_j\frac{\partial}{\partial \bar{z}_j} \in T_p^{\mathbb{C}}M,$$

since the sectional curvature is bounded above we can assume for t_0 small enough but definite, we have from (2.2) that

$$|(d\exp_p)_{t\gamma'(0)}(X)|_{\gamma(t)} = |X|_g(\gamma(t)) \geq \frac{1}{2}|X| = \frac{1}{2}\sqrt{\sum_j |\lambda_j|^2 + |\mu_j|^2}, \quad \text{for } t \leq t_0.$$

Then (we will use $dz_i(\frac{\partial}{\partial \bar{z}_j}) = 0$ by the definition of those objects.)

$$\begin{aligned}(\bar{\partial} z_i)(X) &= dz_i(P_{0,1}(X)) \\ &= dz_i(\lambda_j P_{0,1}\frac{\partial}{\partial z_j} + \mu_j P_{0,1}\frac{\partial}{\partial \bar{z}_j}) \\ &= dz_i\Big(\lambda_j P_{0,1}\frac{\partial}{\partial z_j} + \mu_j \frac{\partial}{\partial \bar{z}_j} - \mu_j P_{0,1}\frac{\partial}{\partial \bar{z}_j}\Big) \\ &= dz_i\Big(\lambda_j P_{0,1}\frac{\partial}{\partial z_j} - \mu_j P_{0,1}\frac{\partial}{\partial \bar{z}_j}\Big)\end{aligned}$$

Thus for $t \leq t_0$, by (2.14) and (2.15) we have

$$|\bar{\partial} z_i(X)| \leq \sum_j (|\lambda_j| + |\mu_j|) C t^2 \exp(Ct^2)|dz_i| \leq Ct^2 \exp(Ct^2)|X|,$$

where we use

$$|dz_i(X)| = |\lambda_i| \leq 2|X|, \text{ when } t \leq t_0,$$

so we have

$$|\bar{\partial} z_i| \leq Ct^2 \exp(Ct^2). \tag{2.16}$$

2.4.3 Proof of Proposition 2.4.1

Suppose (M, ω) is a complete Kähler manifold with sectional curvature and its derivatives uniformly bounded. Without loss of generality we assume

$$|\nabla^k \mathrm{Rm}| \leq 1, \quad \forall k = 0, 1$$

and the injectivity radius at p is uniformly bounded below, by rescaling we may assume $inj_g(p) \geq 1$, hence $\exp_p : B_1(0) \subset T_pM \to B_1(p)$ is a diffeomorphism map.

By the Hessian comparison (Theorem 2.2.1) we know that $\mathrm{Hess} r_p \geq \mathrm{Hess} r_{\bar{p}}$ where $r_{\bar{p}}$ is the distance function from a fixed point $\bar{p} \in S_1^n$, the simply connected space form with constant curvature $+1$. Then it's not hard to see (see [**?**]) that for the function $\log r^2$ (here $r = r_p$ the distance function from p) is a quasi-plurisubharmonic on $B_1(p)$, and r^2 is strictly plurisubharmonic (and this implies $B_1(p)$ is pseudo convex), i.e. there exists a constant $c > 0$ such that

$$i\partial\bar{\partial}r^2 \geq c\,\omega, \text{ in } B_1(p).$$

Here ω is the Kähler form associated to the metric g. Now take a cut-off function $\beta(r)$ which is 1 on $[0, 1/2]$ and vanishes on $[1, \infty)$, consider $\sigma_i := \bar{\partial}(\beta \cdot z_i)$, a $\bar{\partial}$-closed $(0, 1)$-form on $B_1(p)$. Take

$$\varphi = (n + 1)\log r^2 + \chi(r^2)$$

where χ is a smooth rapidly increasing and strictly convex function on $[0, 1]$. We are going to solve the $\bar{\partial}$-equation

$$\bar{\partial}u_i = \sigma_i = \bar{\partial}(\beta z_i), \text{ in } B_1(p) \tag{2.17}$$

by Hormander's L^2 estimates. Since

$$i\partial\bar{\partial}\varphi + \mathrm{Ric}(\omega) \geq i\partial\bar{\partial}(\log r^{2(n+1)} + \chi(r^2)) - (n-1)\omega \geq c_0\omega, \text{ in the current sense}$$

on $B_1(p)$ for a positive constant $c_0 > 0$ if we choose χ increasing very rapidly. We can directly check by (2.16) that the integral

$$A_0 := \int_{B_1(p)} \frac{|\sigma_i|^2}{c_0} e^{-\varphi}\omega^n \leq \int_{B_1(p)} \frac{2|\bar{\partial}\beta|^2|z_i|^2 + 2|\beta|^2|\bar{\partial}z_i|^2}{c_0} \frac{C}{r^{2(n+1)}}\omega^n < \infty$$

since the pole order at p is $2n - 2$, hence integrable. Thus Hormander's L^2 estimate (Theorem 2.1.1) implies that equation (2.17) can be solved with

$$\int_{B_1(p)} |u_i|^2 e^{-\varphi}\omega^n \leq \frac{1}{c_0}\int_{B_1(p)} |\sigma_i|^2 e^{-\varphi}\omega^n = A_0 < \infty.$$

Since the pole order of $e^{-\varphi}$ at p is $2n+2$, we see that $u_i(p) = 0$ and $du_i(p) = 0$. Hence $\beta_i := \beta z_i - u_i$ is a nontrivial holomorphic function on $B_1(p)$ since $d\beta_i(p) = dz_i(p) \neq 0$. And the estimate above implies that

$$\int_{B_1(p)} |\beta_i|^2\omega^n \leq B_0 < \infty.$$

We claim that for small ball $B_{r_0}(p)$, the functions $\{\beta_i\}_{i=1}^n$ form a holomorphic coordinates system. Since $d\beta_i(p) = dz_i(p) \in (T_p^*M)^{1,0}$ we know that β_i do form coordinates near p, but we need a uniform small ball where this holds. To see this, define

$$\alpha = d\beta_1 \wedge \cdots \wedge d\beta_n$$

to be a holomorphic n-form on $B_1(p)$, and clearly $\alpha(p) \neq 0$ and $|\alpha(p)|_\omega = 1$ by the choice of normal coordinates. We are going to see that $|\alpha| \neq 0$ on a definite small ball around p and this will prove that β_i's form local holomorphic coordinates.

Observing that the Sobolev constant on the ball $(B_1(p), \omega)$ is uniformly bounded, and by direct calculations, for the holomorphic function β_i we have

$$\Delta|\nabla\beta_i|^2 = \mathrm{Ric}(\nabla\beta_i, \bar\nabla\beta_i) + |\nabla\nabla\beta_i|^2 \geq -(n-1)|\nabla\beta_i|^2 + |\nabla\nabla\beta_i|^2.$$

Combined with the Kato's inequality that $|\nabla|\nabla\beta_i|| \leq \frac{|\nabla\nabla\beta_i| + |\nabla\bar\nabla\beta_i|}{2} = \frac{|\nabla\nabla\beta_i|}{2}$, we have

$$\Delta|\nabla\beta_i| \geq -\frac{n-1}{2}|\nabla\beta_i|$$

Hence the Moser iteration implies that

$$\sup_{B_{1/2}(p)} |\nabla\beta_i| \leq C\Big(\int_{B_{3/4}(p)} |\nabla\beta_i|^2\omega^n\Big)^{1/2}.$$

By the inverse Poincare estimate for harmonic functions, i.e. take a cut-off function ϕ supported on $B_{4/5}$ and 1 on $B_{3/4}$, then apply IBP to

$$\int_{B_{3/4}} \phi^2|\nabla\beta_i|^2$$

and Holder inequality gives the desired upper bound

$$\int_{B_{3/4}} |\nabla\beta_i|^2\omega^n \leq C\int_{B_{4/5}} |\beta_i|^2\omega^n \leq C.$$

Hence we get gradient bound for holomorphic functions β_i.

$$\sup_{B_{1/2}} |\nabla\beta_i| \leq C. \tag{2.18}$$

On the other hand, for a holomorphic function $f := \beta_i$, we have the following equation

$$\begin{aligned}\Delta|\nabla\nabla f|^2 = {} & f_{mk}R_{i\bar m j\bar k}\overline{f_{ij}} + f_{ijk}\overline{f_{ijk}} + f_{ij\bar k}\overline{f_{ij\bar k}} + f_m R_{i\bar m,j}\overline{f_{ij}} \\ & + 2f_{mj}R_{i\bar m}\overline{f_{ij}} + f_{ij}\overline{f_{mk}R_{i\bar m j\bar k}} + f_{ij}\overline{f_m R_{i\bar m,j}},\end{aligned}$$

by the Kato's inequality

$$2|\nabla|\nabla\nabla f||^2 \leq |\nabla\nabla\nabla f|^2 + |\bar\nabla\nabla\nabla f|^2,$$

and the curvature and derivatives bound, we have

$$\Delta|\nabla\nabla f| \geq -C|\nabla\nabla f| - |\nabla f|.$$

Hence on $B_{1/2}$ we have $\Delta|\nabla\nabla f| \geq -|\nabla\nabla f| - 1$ (modulo the constants multiples) and by Moser iteration, we have

$$\sup_{B_{1/8}} |\nabla\nabla f| \leq C\Big(\int_{B_{1/4}} |\nabla\nabla f|^2\Big)^{1/2}.$$

Similar to what we did before, let ϕ be a cut-off function supported at $B_{1/2}$ and equals to 1 on $B_{1/4}$, we have

$$\int_{B_{1/4}} |\nabla\nabla f|^2 \leq \int \phi^2|\nabla\nabla f|^2 = -\int 2\phi\phi_j f_i \overline{f_{ij}} - \phi^2 \mathrm{Ric}(\nabla f, \nabla f)$$

which is less than

$$2\Big(\int \phi^2|\nabla\nabla f|^2\Big)^{1/2}\Big(\int |\nabla\phi|^2|\nabla f|^2\Big)^{1/2} + C\int_{B_{1/2}} |\nabla f|^2$$

by Holder inequality and then Schwartz inequality implies the bound

$$\int_{B_{1/4}} |\nabla\nabla f|^2 \leq C\int_{B_{1/2}} |\nabla f|^2 \leq C,$$

hence we get the desired bound

$$\sup_{B_{1/8}} |\nabla\nabla\beta_i| = \sup_{B_{1/8}} |\nabla\nabla f| \leq C. \tag{2.19}$$

Combined with the estimates (2.18) and (2.19) we get that

$$\sup_{B_{1/8}} |\nabla|\alpha|_\omega| \leq C,$$

and since $|\alpha|_\omega(p) = 1$ thus on the ball $B_{1/2C}(p)$,

$$|\alpha|_\omega \geq 1/2,$$

and this gives a positive lower bound of $|\alpha|_\omega$ and hence $\{\beta_i\}$ form a holomorphic coordinates system on the ball $B_{1/2C}(p)$, moreover, it's clear that all such functions β_i are uniformly bounded on the ball $B_{1/2}$ since $|\beta_i|^2$ is a subharmonic function and the mean value inequality would give the upper bound. So under these coordinates $\{\beta_i\}$

on $B_{1/2C}(p)$, we have for $g_{i\bar{j}} = g(\nabla\beta_i, \bar{\nabla}\beta_j)$, (noting that $g_{i\bar{j}}(p) = \delta_{ij}$ by the choice of normal coordinates)

$$g_{i\bar{j}} \sim \delta_{ij}, \quad \|g_{i\bar{j}}\|_{C^1(B_{1/2C})} \leq C.$$

Moreover, by the expression of Ricci curvature (2.1), we get

$$R_{i\bar{j}} = -\Delta_g g_{i\bar{j}} + g^{p\bar{q}} g^{k\bar{l}} \frac{\partial g_{i\bar{l}}}{\partial \beta_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{\beta}_l}$$

which is an elliptic equation and elliptic regularity implies that $C^{1,\alpha}$ of $g_{i\bar{j}}$ in the small ball $B_{1/4C}(p)$(in fact, we can get $C^{2,\alpha}$ bound of the metric $g_{i\bar{j}}$).

In particular this implies that the Euclidean metric under those coordinates

$$\omega_E = \sum_i \sqrt{-1} d\beta_i \wedge d\bar{\beta}_i$$

is uniformly equivalent to ω in the ball $B_{1/2C}$. We can also get higher order estimates of $g_{i\bar{j}}$ under those coordinates if the higher order derivative of the sectional curvature is also bounded. But the above is already enough for our applications. □

Chapter 3

Kähler Ricci flow with symmetry on $\mathbb{CP}^n \# \overline{\mathbb{CP}^n}$

This chapter is from the joint work [45] with Jian Song.

3.1 Known results

In this section, we collect some backgrounds and known results about the flow (1.8).

3.1.1 Calabi symmetry

Let X be $\mathbb{CP}^n$ blown-up at one point and it is a $\mathbb{CP}^1$ bundle over $\mathbb{CP}^{n-1}$ given by

$$X = \mathbb{P}(\mathcal{O}_{\mathbb{CP}^{n-1}} \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}(-1)). \tag{3.1}$$

Let D_0 be the exceptional divisor of X defined by the image of the section $(1, 0)$ of $\mathcal{O}_{\mathbb{CP}^{n-1}} \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$ and D_∞ be the divisor defined by the image of the section $(0, 1)$ of $\mathcal{O}_{\mathbb{CP}^{n-1}} \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$. Both divisors D_0 and D_∞ are complex hypersurfaces isomorphic to $\mathbb{CP}^{n-1}$. The Kähler cone on X is given by

$$\mathcal{K} = \{-a[D_0] + b[D_\infty] \mid 0 < a < b\}.$$

Let $z = (z_1, \ldots, z_n)$ be the standard complex coordinates on $\mathbb{C}^n$. Let $\rho = \log |z|^2 = \log(|z_1|^2 + \cdots + |z_n|^2)$.

Definition 3.1.1. *A smooth convex function $u = u(\rho)$ for $\rho \in (-\infty, \infty)$ is said to satisfy the* Calabi symmetry conditions*, if*

(1) $u''(\rho) > 0, u'(\rho) > 0$ for $\rho \in (-\infty, \infty)$,

(2) There exist $0 < a < b$ and smooth functions $U_0, U_\infty : [0, \infty) \to \mathbb{R}$ such that

$$U_0'(0) > 0, \quad U_\infty'(0) > 0,$$

$$u(\rho) = a\rho + U_0(e^{\rho}) \quad near\ \rho = -\infty,$$

$$u(\rho) = b\rho + U_\infty(e^{-\rho}) \quad near\ \rho = +\infty.$$

It is known ([11]) that a metric $\omega = i\partial\bar{\partial}u$ which defines a smooth Kähler metric on $\mathbb{C}^n\backslash\{0\}$ extends to a Kähler metric on $X = \mathbb{CP}^n \# \overline{\mathbb{CP}^n}$ if and only if u satisfies the Calabi symmetry condition, and it defines a Kähler metric in the class $-a[D_0] + b[D_\infty]$.

On $\mathbb{C}^n\backslash\{0\}$, the Kähler metric $\omega = i\partial\bar{\partial}u$ is given by

$$\omega = \sqrt{-1}g_{i\bar{j}}dz_i \wedge d\bar{z}_j = \Big(e^{-\rho}u'\delta_{ij} + e^{-2\rho}\bar{z}_i z_j(u'' - u')\Big)\sqrt{-1}dz_i \wedge d\bar{z}_j. \tag{3.2}$$

This metric ω is invariant under the standard unitary $U(n)$-actions on $\mathbb{C}^n$, hence also invariant under the induced $U(n)$-actions on X, i.e. $U(n) \subset \mathrm{Isom}(X, \omega)$, the isometry group of ω.

On $\mathbb{C}^n\backslash\{0\}$, $\det(g_{i\bar{j}}) = e^{-n\rho}(u')^{n-1}u''$ and the Ricci potential of $\omega = i\partial\bar{\partial}u$ is

$$v = -\log\det g_{i\bar{j}} = n\rho - (n-1)\log u' - \log u'',$$

and Ricci curvature tensor of ω is given by

$$R_{i\bar{j}} = e^{-\rho}v'\delta_{ij} + e^{-2\rho}\bar{z}_i z_j(v'' - v').$$

It is known ([83]) that the Calabi symmetry is preserved by the Kähler Ricci flow (1.8), in other words, the evolving Kähler metrics $\omega(t)$ of (1.8) is invariant under $U(n)$-action if the initial metric ω_0 is $U(n)$-invariant. In [83] it is shown that (1.8) can be reduced to the following parabolic equation for $u = u(\rho, t)$

$$\frac{\partial}{\partial t}u(\rho, t) = \log u''(\rho, t) + (n-1)\log u'(\rho, t) - n\rho, \tag{3.3}$$

where the evolving metrics $\omega(t)$ are given by $\omega(t) = i\partial\bar{\partial}u(\rho, t)$. If the initial Kähler metric $\omega(0) \in -a_0[D_0] + b_0[D_\infty]$, then the evolving Kähler class is given by

$$\omega(t) \in -a_t[D_0] + b_t[D_\infty], \quad \text{with } a_t = a_0 - (n-1)t,\ b_t = b_0 - (n+1)t.$$

We will identify the zero divisor $D_0 \subset X$ as the exceptional divisor $E \cong \mathbb{CP}^{n-1}$ in $\mathbb{C}^n$ blown-up at the origin, and $\mathbb{C}^n \subset \mathbb{CP}^n$. Under the rotationally symmetric metric $g = \omega = i\partial\bar{\partial}u$, the distance from a point $z \in \mathbb{C}^n\backslash\{0\}$ to E is given by

$$d_g(z, E) = \frac{1}{2}\int_{-\infty}^{\log|z|^2} \sqrt{u''(\rho)}d\rho.$$

The Calabi symmetry condition (2) above implies this distance is finite.

We define the tubular neighborhood $B_g(E,R)$ of E (in the following we also call $B_g(E,R)$ as metric balls centered at E) as

$$B_g(E,R) := \{q \in X | d_g(q,E) < R\},$$

which (for R small) can be identified as $\pi^{-1}(B)$ for some Euclidean ball $B \subset \mathbb{C}^n$ centered at 0 and $\pi : \widetilde{\mathbb{C}^n} \to \mathbb{C}^n$ is the blown-up map of $\mathbb{C}^n$ at 0. The volume of $B_g(E,R)$ with respect to the metric $\omega = i\partial\bar{\partial}u$ is given by

$$\int_{B(E,R)} \omega^n = C(n) \int_{-\infty}^{\rho_R} (u'(\rho))^{n-1} u''(\rho) d\rho, \tag{3.4}$$

for some constant $C(n)$ depending only on the dimension and ρ_R is the unique constant determined by the equation

$$R = \frac{1}{2} \int_{-\infty}^{\rho_R} \sqrt{u''(\rho)} d\rho, \tag{3.5}$$

i.e., a point $z \in \mathbb{C}^n \backslash \{0\}$ with $\log |z|^2$ satisfies $z \in \partial B_g(E,R)$.

We recall the following formulas of gradient and Laplacian of a rotationally symmetric function, which follow from direct calculations so we omit the proof.

Lemma 3.1.1. *Suppose f is a $U(n)$-invariant function on X, then with respect to the metric $\omega = i\partial\bar{\partial}u$, we have*

$$|\nabla f|^2_\omega = \frac{(f')^2}{u''}, \quad \Delta_\omega f = (n-1)\frac{f'}{u'} + \frac{f''}{u''},$$

where as usual for the function f, $f' = \frac{\partial}{\partial \rho} f$, $f'' = \frac{\partial^2 f}{\partial \rho^2}$.

3.1.2 Type I solutions

Recall the Ricci flow (1.8) is said to have *Type I singularity* if

$$\sup_{(x,t)\in X\times[0,T)} (T-t)|Rm|(x,t) < \infty,$$

where T is the singular time.

Theorem 3.1.1 ([75, 39, 111]). *Let X be $\mathbb{CP}^n$ blown-up at one point. Then the Kähler Ricci flow* (1.8) *on X must develop Type I singularities for any $U(n)$ invariant initial Kähler metric.*

Let $g(t)$ be the solution on $[0,T)$. For any $t_j \to T$, we consider the rescaled flows $(X, g_j(t))$ defined on $[-\frac{t_j}{T-t_j}, 1)$ by

$$g_j(t) = \frac{1}{T-t_j} g(t_j + t(T-t_j)).$$

Then one and only one of the following must occur.

(1) ([75]) If $\liminf_{t\to T}(T-t)^{-1}Vol(X, g(t)) = \infty$, then $(X, g_j(t), p)$ sub-converges in C^∞ Cheeger-Gromov-Hamilton (CGH) sense to a complete shrinking non-flat gradient Kähler Ricci soliton on a complete Kähler manifold diffeomorphic *to $\widetilde{\mathbb{C}^n}$, for any fixed point $p \in E$, the exceptional divisor.*

(2) ([39]) If $\liminf_{t\to T}(T-t)^{-1}Vol(X, g(t)) \in (0,\infty)$, then $(X, g_j(t), p_j)$ sub-converges in C^∞-CGH sense to $(\mathbb{C}^{n-1}\times\mathbb{CP}^1, g_{\mathbb{C}^{n-1}} \oplus (-t)g_{FS})$, where $g_{\mathbb{C}^{n-1}}$ is the standard flat metric on $\mathbb{C}^{n-1}$ and g_{FS} is the Fubini-Study metric on $\mathbb{CP}^1$ for any sequence of points p_j.

(3) ([111]) If $\liminf_{t\to T}(T-t)^{-1}Vol(X, g(t)) = 0$, then $(X, g_j(t))$ converges in C^∞-CGH sense to the unique compact shrinking Kähler Ricci soliton on $\mathbb{CP}^n$ blown-up at one point.

Our main result in this paper is to show the limit Kähler Ricci soliton in case (1) is in fact one of the FIK solitons constructed in [36], and the limit space is biholomorphic to $\widetilde{\mathbb{C}^n}$.

Suppose the initial $U(n)$-invariant Kähler metric lies in the class $-a_0[D_0] + b_0[D_\infty]$. It is proved ([83]) that the condition in case (1) above that $\liminf_{t\to T}(T-t)^{-1}Vol(X, g(t)) = \infty$ is equivalent to the inequality

$$0 < a_0(n+1) < b_0(n-1).$$

And the Kähler Ricci flow (1.8) will contract the exceptional divisor D_0 at the singular time

$$T = \frac{a_0}{n-1}. \tag{3.6}$$

Throughout this paper we will assume $0 < a_0(n+1) < b_0(n-1)$.

3.1.3 Cheeger-Gromov convergence

Let $g_j := g_j(0) = \frac{1}{T-t_j} g(t_j)$ and $X_j = X, p_j = p \in D_0 = E$ be a fixed point, then from case (1) in Theorem 3.1.1, we know the pointed manifolds (X_j, p_j, g_j) converge in C^∞ Cheeger-Gromov (CG) sense to a complete Kähler manifolds $(X_\infty, p_\infty, g_\infty)$ and g_∞ is a nontrivial complete Kähler Ricci soliton. Recall the CG convergence means that there exists a sequence of increasing relatively compact exhaustion $\{U_j\}$ of X_∞, and diffeomorphisms (onto its image) $\phi_j : U_j \to X_j$ satisfying $\phi_j(p_\infty) = p_j$ and

$$\phi_j^* g_j \xrightarrow{C^\infty_{loc}} g_\infty, \quad \phi_j^* J_j \xrightarrow{C^\infty_{loc}} J_\infty, \tag{3.7}$$

where J_j, J_∞ are the complex structures on X_j, X_∞, respectively, compatible with the Kähler metrics g_j, g_∞.

Since the restriction of the metrics g_j to E are $(n-1)g_{FS}$ where g_{FS} is the Fubini-Study metric on $\mathbb{CP}^{n-1}$, we have

Lemma 3.1.2 (see also [75])**.** *The diameter of $(E, g_j|_E)$ is $D_n = \alpha_n(n-1)^{1/2}$, hence uniformly bounded. Here $\alpha_n =$ the diameter of $(\mathbb{CP}^{n-1}, g_{FS})$.*

3.2 A priori estimates

As we mentioned before, we will assume the initial Kähler metric lies in $-a_0[D_0] + b_0[D_\infty]$ with $0 < a_0(n+1) < b_0(n-1)$. The evolution equations for the evolving metrics $\omega(t) = i\partial\bar\partial u(\rho, t)$ for $\rho \in (-\infty, \infty)$ and $t \in [0, T)$, where T is given in (3.6), are given by ([83, 75])

$$\frac{\partial}{\partial t} u' = \frac{u'''}{u''} + (n-1)\frac{u''}{u'} - n, \tag{3.8}$$

$$\frac{\partial}{\partial t} u'' = \frac{u^{(4)}}{u''} - \frac{(u''')^2}{(u'')^2} + (n-1)\frac{u'''}{u'} - (n-1)\frac{(u'')^2}{(u')^2}, \tag{3.9}$$

$$\begin{aligned} \frac{\partial}{\partial t} u''' =& \frac{u^{(5)}}{u''} - \frac{3u'''u^{(4)}}{(u'')^2} + \frac{2(u''')^3}{(u'')^3} + (n-1)\frac{u^{(4)}}{u'} \\ & - 3(n-1)\frac{u''u'''}{(u')^2} + 2(n-1)\frac{(u'')^3}{(u')^3}. \end{aligned} \tag{3.10}$$

Along the flow (1.8) or (3.3), we have (see ([83, 75]))

Lemma 3.2.1. *There exists $C > 0$ such that for all $t \in [0, T)$ and $\rho \in (-\infty, \infty)$ such that*

$$(n-1)(T-t) = a_t \leq u' \leq C, \tag{3.11}$$

and

$$0 \leq \frac{u''}{u'} \leq C, \quad -C \leq \frac{u'''}{u''} \leq C. \tag{3.12}$$

Lemma 3.2.2. *There exists a constant $C > 0$ such that*

$$C^{-1}(u' - a_t)(b_t - u') \leq u'' \leq C(u' - a_t)(b_t - u'). \tag{3.13}$$

Proof. The proof of the second inequality is given in Lemma 4.5 of [83], and the first inequality can be proved following the same argument as in [83]. For readers' convenience we include the proof below.

Consider the quantity $H = \log u'' - \log(u' - a_t) - \log(b_t - u')$, using the evolution equations (3.8) and (3.9)

$$\begin{aligned}\frac{\partial H}{\partial t} =& \frac{1}{u''}\Big(\frac{u^{(4)}}{u''} - \frac{(u''')^2}{(u'')^2} + (n-1)\frac{u'''}{u'} - (n-1)\frac{(u'')^2}{(u')^2}\Big) \\ &- \frac{1}{u' - a_t}\Big(\frac{u'''}{u''} + (n-1)\frac{u''}{u'} - 1\Big) - \frac{1}{b_t - u'}\Big(-\frac{u'''}{u''} - (n-1)\frac{u''}{u'} - 1\Big).\end{aligned} \tag{3.14}$$

It can be checked by Calabi symmetry condition that for each fixed $t \in [0, T)$

$$\lim_{\rho \to \pm\infty} \frac{u''(\rho, t)}{(b_t - u'(\rho, t))(u'(\rho, t) - a_t)} = \frac{1}{b_t - a_t},$$

which is uniformly bounded above and below in our case.

For any $T' \in (0, T)$, suppose the minimum of H on $X \times [0, T']$ is obtained at some (ρ_0, t_0), then at this point we have $\frac{\partial}{\partial t} H \leq 0$, $H' = 0$, and $H'' \geq 0$, i.e.

$$\frac{u'''}{u''} - \frac{u''}{u' - a_t} + \frac{u''}{b_t - u'} = 0, \tag{3.15}$$

$$\frac{u^{(4)}}{u''} - \frac{(u''')^2}{(u'')^2} - \frac{u'''}{u' - a_t} + \frac{(u'')^2}{(u' - a_t)^2} + \frac{u'''}{b_t - u'} + \frac{(u'')^2}{(b_t - u')^2} \geq 0, \tag{3.16}$$

combining with (3.14), (3.15) and (3.16) we have at (ρ_0, t_0),

$$-u''\Big(\frac{1}{(u' - a_t)^2} + \frac{1}{(b_t - u')^2}\Big) - (n-1)\frac{u''}{(u')^2} + \frac{1}{u' - a_t} + \frac{1}{b_t - u'} \leq 0. \tag{3.17}$$

Hence

$$\frac{u''}{(b_t-u')(u'-a_t)}+\frac{(n-1)u''(b_t-u')(u'-a_t)}{(u')^2\big((u'-a_t)^2+(b_t-u')^2\big)}\geq\frac{b_t-a_t}{(b_t-u')^2+(u'-a_t)^2}.$$

We observe that

$$\frac{(b_t-u')(u'-a_t)}{(u')^2((u'-a_t)^2+(b_t-u')^2)}\leq\frac{1}{(b_t-u')(u'-a_t)},$$

and

$$(u'-a_t)^2+(b_t-u')^2\leq 2(b_t-a_t)^2,$$

hence at (ρ_0,t_0),

$$\frac{u''}{(u'-a_t)(b_t-u')}\geq\frac{1}{2n(b_t-a_t)}\geq C^{-1},$$

as b_t-a_t is uniformly bounded above. The maximum principle implies the minimum of H on $X\times[0,T']$ is uniformly bounded below independent of the choice of T', hence we conclude that $\inf_{X\times[0,T)}H\geq -C$. And we finish the proof the first inequality in (3.13).

□

3.2.1 Estimates for the sequence g_j

Recall that the Cheeger-Gromov limit $(X_\infty,g_\infty,p_\infty)$ of (X_j,g_j,p) is a complete Kähler Ricci soliton. Hence by a theorem of Cao-Zhou (Theorem 1.2 in [**?**]), there exists a constant $C_0>0$ such that the volume of geodesic balls $B_{g_\infty}(p_\infty,R)$ satisfies

$$Vol_{g_\infty}(B_{g_\infty}(p_\infty,R))\leq C_0R^{2n}. \tag{3.18}$$

And Perelman's non-collapsing implies there exists a $\kappa>0$ for any $q\in X_\infty$, the volume $Vol_{g_\infty}(B_{g_\infty}(q,r_0))\geq\kappa r_0^{2n}$, for any $r_0^2\leq\frac{1}{C}$, where C is the constant in Type I condition. In particular, we have $Vol_{g_\infty}(B_{g_\infty}(p_\infty,R))\to\infty$ as $R\to\infty$.

Lemma 3.2.3. *For any $R>0$, there exist constants $c(n,R)>0$ and $C(n,R)=O(R^2)$ such that for $j\geq 1$ large enough, then in the metric balls $B_{g_j}(E_j,R)$, we have*

$$(n-1)(T-t_j)=a_{t_j}\leq u'(\rho,t_j)\leq C(n,R)(T-t_j),\quad u''(\rho,t_j)\leq C(n,R)(T-t_j). \tag{3.19}$$

Moreover, on $\partial B_{g_j}(E_j, R)$, for j large enough, we have

$$\begin{aligned} c(n,R)(T-t_j)+(n-1)(T-t_j) \leq u'(\rho_{j,R},t_j) \leq C(n,R)(T-t_j), \\ c(n,R)(T-t_j) \leq u''(\rho_{j,R},t_j) \leq C(n,R)(T-t_j) \end{aligned} \tag{3.20}$$

and $c(n,R) \to +\infty$ as $R \to \infty$, $\rho_{j,R}$ is defined in (3.5), corresponding to points in $\partial B_{g_j}(E_j, R)$.

Proof. For any fixed $R > 0$, by the C^∞-CG convergence (3.7) we have

$$Vol_{g_j}(B_{g_j}(p_j, R)) \to Vol_{g_\infty}(B_{g_\infty}(p_\infty, R)), \quad \text{as } j \to \infty,$$

in particular, we have both $Vol_{g_j}(B_{g_j}(p_j, R))$ and $Vol_{g_j}(B_{g_j}(p_j, R+D_n))$ are uniformly bounded above and below, for j large enough, where D_n is the diameter of E_j given by Lemma 3.1.2. Noting that

$$B_{g_j}(p_j, R) \subset B_{g_j}(E_j, R) \subset B_{g_j}(p_j, R+D_n),$$

hence there are two constants $c_1 = c_1(n,R)$ and $C_1 = C_1(n,R)$

$$c_1(n,R) \leq Vol_{g_j}(B_{g_j}(E_j, R)) \leq C_1(n,R), \tag{3.21}$$

By (3.18), it is easy to see that when j is large enough, we can choose $C_1(n,R) = O(R^{2n})$. Moreover, by the volume formula (3.4)

$$\begin{aligned} Vol_{g_j}(B_{g_j}(E_j,R)) &= \frac{C(n)}{(T-t_j)^n}\int_{-\infty}^{\rho_{j,R}} (u'(\rho,t_j),t_j)^{n-1}u''(\rho,t_j)d\rho \\ &= \frac{C(n)}{n(T-t_j)^n}\Big((u'(\rho_{j,R},t_j))^n - a_{t_j}^n\Big), \end{aligned} \tag{3.22}$$

where $a_{t_j} = (n-1)(T-t_j)$, and $\rho_{j,R}$ is a constant determined by the equation (3.5) with u'' replaced by $\frac{u''(\rho,t_j)}{T-t_j}$ i.e. a point $z \in \mathbb{C}^n \backslash \{0\}$ with $\log|z|^2 = \rho_{j,R}$ lies in $\partial B_{g_j}(E_j, R)$. Combining (3.21) and (3.22), there are constants $c_2(n,R)$ and $C_2(n,R)$ such that

$$c_2(n,R) + (n-1) \leq \frac{u'(\rho_{j,R},t_j)}{T-t_j} \leq C_2(n,R). \tag{3.23}$$

Combining with the fact that $u'(\rho,t_j)$ is increasing in ρ and Lemmas 3.2.1 and 3.2.2, if j is large enough, (3.19) and (3.20) hold. □

3.2.2 X_j as $\mathbb{CP}^1$-bundle over $\mathbb{CP}^{n-1}$

Recall the manifold X_j can be viewed as a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^{n-1}$ (see (3.1)). Let

$$F_j : X_j \to \mathbb{CP}^{n-1}$$

be the holomorphic bundle map.

Lemma 3.2.4. *The holomorphic maps $F_j : (X_j, \omega_j) \to (\mathbb{CP}^{n-1}, \omega_{FS})$ have uniformly bounded derivatives. And the derivatives $dF_j : TX_j \to T\mathbb{CP}^{n-1}$ have full rank $n-1$ everywhere. Here we refer to the "operator norm" of the derivatives.*

Proof. Note that $F_j^*\omega_{FS} = i\partial\bar{\partial}\rho = i\partial\bar{\partial}\log|z|^2$. As a map $F_j : (X_j, \omega_j) \to (\mathbb{CP}^{n-1}, \omega_{FS})$, its energy density $e(F_j) := tr_{\omega_j} f_j^*\omega_{FS} = \Delta_{\omega_j}\rho$ is equal to

$$\frac{(T-t_j)(n-1)}{u'(\rho, t_j)} \leq 1,$$

hence the differential of maps F_j, $dF_j : TX_j \to T\mathbb{CP}^{n-1}$ is uniformly bounded.

Moreover, by (3.19), in the balls $B_{g_j}(E_j, R)$,

$$e(F_j) \geq \frac{(n-1)}{C(n,R)}. \tag{3.24}$$

By the symmetry of F_j and ω_j, it is not hard to see the $(n-1)$-many nonzero eigenvalues of $\omega_j^{-1} \cdot F_j^*\omega_{FS}$ are bounded below by $\frac{1}{C(n,R)}$ in $B_{g_j}(E_j, R)$. And this implies that the rank of the differential map $dF_j : TX_j \to T\mathbb{CP}^{n-1}$ is $n-1$. □

Lemma 3.2.5. *There exists a constant $C = C(n) > 0$, such that for any $j \geq 1$,*

$$|\nabla\nabla F_j|^2_{g_j} \leq C.$$

Hence we have uniform C^2 bound of the maps F_j.

Proof. Since F_j is holomorphic and ω_j and ω_{FS} are Kähler metrics, $F_j : X_j \to \mathbb{CP}^{n-1}$ is also a harmonic map. Hence by the Bochner formula

$$\Delta e(F_j) = |\nabla\nabla F_j|^2 + \mathrm{Ric}_{\omega_j}(\nabla F_j, \overline{\nabla F_j}) - (R_{\omega_{FS}})^{\alpha}_{\beta\bar{\gamma}\delta}\overline{(F_j)^{\alpha}_{i}}(F_j)^{\delta}_{i}(F_j)^{\beta}_{k}\overline{(F_j)^{\gamma}_{k}}. \tag{3.25}$$

On the other hand, by direct calculations we have

$$\begin{aligned} \Delta_{\omega_j} e(F_j) &= -\frac{(n-1)^2(T-t_j)^2 u''}{(u')^3} - \frac{(n-1)(T-t_j)^2 u'''}{u''(u')^2} \\ &\leq -\frac{(n-1)(T-t_j)^2 u'''}{u''(u')^2} \leq C(n), \end{aligned} \tag{3.26}$$

where in the last inequality we use Lemma 3.2.1. Combining (3.25), (3.26) and the Type I condition $|\mathrm{Ric}_{\omega_j}| \leq C$, we have

$$|\nabla\nabla F_j|^2 \leq C(n) + Ce(F_j) + Ce(F_j)^2 \leq C(n).$$

Therefore, the maps $F_j : X_j \to \mathbb{CP}^{n-1}$ have uniform second order estimates. □

Remark 3.2.1. *Actually, the harmonicity of the maps F_j implies F_j satisfy uniform C^k estimates, for any $k \in \mathbb{Z}$. But the second order estimate is enough for our applications.*

The target manifold of F_j is the compact $(\mathbb{CP}^{n-1}, \omega_{FS})$, and by Lemmas 3.2.4 and 3.2.5, the maps $F_j : X_j \to \mathbb{CP}^{n-1}$ have uniformly bounded C^1, C^2 bounds, hence F_j converge in $C^{1,\alpha}$ topology to a limit map $F_\infty : X_\infty \to \mathbb{CP}^{n-1}$, where by definition $X_j \to X_\infty$ in the C^∞-CG sense with the Riemannian metrics and complex structures converging smoothly. Since F_j are holomorphic with the given complex structures, the limit map $F_\infty : X_\infty \to \mathbb{CP}^{n-1}$ is also holomorphic with respect to the limit complex structure J_∞ on X_∞. And the $C^{1,\alpha}$ convergence and Lemma 3.2.4 imply that the differential map $dF_\infty : TX_\infty \to T\mathbb{CP}^{n-1}$ has full rank $n-1$ at any point, hence implicit function theorem implies that the fibers of F_∞ are smooth complete Riemann surfaces.

We remark that the convergence of $F_j \to F_\infty$ is in the Cheeger-Gromov sense, that is, the maps $\phi_j^* F_j$ converge to F_∞ in uniform $C^{1,\alpha}_{loc}$ topology on any compact subset of X_∞, where $\phi_j : U_j \to X_j$ is the diffeomorphism we chose in Section 3.1.3 realizing the C^∞-Cheeger-Gromov convergence.

3.2.3 Holomorphic vector fields

Let $V = \sum_i z_i \frac{\partial}{\partial z_i}$ be a holomorphic vector field on $\mathbb{C}^n \backslash \{0\}$, which extends to a holomorphic vector field on X, and vanishes on the exceptional divisor E. Clearly V is tangential to the fibers of $F_j : X_j \to \mathbb{CP}^{n-1}$.

Lemma 3.2.6. *With respect to a Kähler metric $\omega = i\partial\bar{\partial}u$ with Calabi symmetry, the imaginary part Im(V) of V is a Killing vector field. Moreover, Im(V) is also Killing with respect to the restriction of the metric on each fiber of $F_j : X_j \to \mathbb{CP}^{n-1}$.*

Proof. This is by direct calculation. Observe that $Vu = u'$ and

$$L_V\omega = d(\iota_V i\partial\bar{\partial}u) = d\big(-i\bar{\partial}(u')\big) = -i\partial\bar{\partial}u',$$

taking conjugate on both sides we have $L_{\bar{V}}\omega = -i\partial\bar{\partial}u'$, hence we have $L_{V-\bar{V}}\omega = 0$ and this implies the imaginary part of V, Im(V), is a Killing vector field with respect to the metric ω, i.e.

$$L_{\mathrm{Im}(V)}\omega = 0. \tag{3.27}$$

On the other hand, for any fiber F_p of $F_j : X_j \to \mathbb{CP}^{n-1}$, we denote the restriction of the metric ω on this fiber by $\omega_|$ and $i : F_p \to X$ the inclusion map of the fiber in X. Using the fact that Im(V) is tangential to F_p and $L_{\mathrm{Im}(V)}\omega = 0$ when pulled back by the map i, we have

$$L_{\mathrm{Im}(V)}(\omega_|) = 0,$$

hence Im(V) is also a Killing vector field on $(F_p, \omega_|)$.

□

Remark 3.2.2. *We remark that the equation* (3.27) *only involves the first order derivatives of V.*

Lemma 3.2.7. *For any $R > 0$, there exist $c(n, R) > 0$ which goes to ∞ as $R \to \infty$ and $C(n, R) > 0$, such that if j is large enough, then on the boundary $\partial B_{g_j}(E_j, R)$,*

$$|V|^2_{g_j} \geq c(n, R),$$

and

$$\sup_{B_{g_j}(E_j,R)} |V|^2_{g_j} \leq C(n, R).$$

Proof. Applying the expansion formula (3.2) of the metric g_j we have

$$|V|^2_{g_j} = \frac{u''(\rho, t_j)}{T - t_j}, \tag{3.28}$$

so by Lemma 3.2.3, we have

$$|V|^2_{g_j} \le C(n,R) = O(R^2), \quad \text{in } B_{g_j}(E_j,R) \tag{3.29}$$

and

$$\text{on } \partial B_{g_j}(E_j,R), \quad |V|^2_{g_j} \ge c(n,R) \to \infty, \ \text{ as } R \to \infty. \tag{3.30}$$

In particular, in $B_{g_j}(E_j,R)$ V is a nontrivial holomorphic vector field which vanishes exactly at E_j. □

We will calculate the derivatives of V with respect to g_j.

Lemma 3.2.8. *There exists a constant $C = C(n) > 0$ such that $|\nabla_j V|^2_{g_j} \le C$ for any j, where $\nabla_j V$ denote the covariant derivative of V with respect to the metric g_j.*

Proof. We will write $V = V^i \frac{\partial}{\partial z_i}$ for $V^i = z_i$, then $|\nabla V|^2 = V^i_{,k}\overline{V^i_{,k}}$, where

$$V^i_{,k} = \frac{\partial}{\partial z_k} V^i + \Gamma^i_{kl} V^l$$

is the covariant derivative of V and $\Gamma^i_{kl} = g^{i\bar{p}} \frac{\partial}{\partial z_l} g_{k\bar{p}}$ is the Levi-Civita connection of a Kähler metric g. Use the expansion formula (3.2) (multiplied by $(T-t_j)^{-1}$) of the metric g_j, we have

$$V^i_{,k} = \frac{u''}{u'}\delta_{ik} + \Big(\frac{u'''}{u''} - \frac{u''}{u'}\Big) z_i \bar{z}_k e^{-\rho},$$

observe that when restricted to the exceptional divisor $E = (\rho = -\infty)$ the matrix $(V^i_{,k})$ is of the form (hence has rank 1)

$$\nabla V|_E = \mathrm{diag}(1, 0, \ldots, 0). \tag{3.31}$$

We calculate the norm of ∇V:

$$\begin{aligned} |\nabla V|^2_{g_j} &= n\Big(\frac{u''}{u'}\Big)^2 + 2\frac{u''}{u'}\Big(\frac{u'''}{u''} - \frac{u''}{u'}\Big) + \Big(\frac{u'''}{u''} - \frac{u''}{u'}\Big)^2 \\ &= (n-1)\Big(\frac{u''}{u'}\Big)^2 + \Big(\frac{u'''}{u''}\Big)^2. \end{aligned} \tag{3.32}$$

Hence Lemma 3.2.8 follows from the estimates in Lemma 3.2.1. □

So we have uniform C^1 bounds of V with respect to g_j. Next we would derive the C^2 bounds of V with respect to the metrics g_j on any metric balls $B_{g_j}(E_j,R)$.

Lemma 3.2.9. *For any $R > 0$, there is a constant $C(n,R) > 0$ such that for j large enough we have*

$$\sup_{B_{g_j}(E_j,R)} \Big(|\nabla\nabla V|^2_{\omega_j} + |\bar\nabla\nabla V|^2_{\omega_j} \Big) \le C(n,R),$$

i.e., the C^2 bounds of V with respect to the Kähler metrics ω_j hold uniformly on any metric ball $B_{g_j}(E_j,R)$.

To prove Lemma 3.2.9, we need the following Bochner type identity.

Lemma 3.2.10. *We have the Bochner type identity: for a Kähler metric ω,*

$$\begin{aligned}\Delta_\omega |\nabla V|^2 =& |\nabla\nabla V|^2_\omega + |\bar\nabla\nabla V|^2_\omega + R_{l\bar m} V^i_{,m}\overline{V^i_{,l}} - R_{m\bar i} V^m_{,l}\overline{V^i_{,l}} \\ & - 2Re\Big(R_{\bar i m\bar k l} V^m_{,k}\overline{V^i_{,l}} + R_{m\bar i,l} V^m \overline{V^i_{,l}}\Big).\end{aligned} \tag{3.33}$$

Proof. This is a direct calculation.

$$\begin{aligned}\Delta|\nabla V|^2 &= (V^i_{,l}\overline{V^i_{,l}})_{k\bar k} \\ &= V^i_{,lk\bar k}\overline{V^i_{,l}} + V^i_{,lk}\overline{V^i_{,lk}} + V^i_{,l\bar k}\overline{V^i_{,l\bar k}} + V^i_{,l}\overline{V^i_{,l\bar k k}}.\end{aligned} \tag{3.34}$$

By changing the indices, we have

$$\begin{aligned}V^i_{,lk\bar k} &= V^i_{,l\bar k k} + V^i_{,m} R_{l\bar m k\bar k} - V^m_{,l} R^i{}_{m\bar k k} \\ &= V^i_{,l\bar k k} + V^i_{,m} R_{l\bar m} - V^m_{,l} R^i{}_m,\end{aligned} \tag{3.35}$$

and

$$V^i_{,l\bar k k} = \Big(V^i_{,\bar k l} - V^m R^i{}_{m\bar k l}\Big)_k = -V^m_{,k} R^i{}_{m\bar k l} - V^m R^i{}_{m,l}, \tag{3.36}$$

where we use the fact that V is a holomorphic holomorphic vector field and the second Bianchi identity. Combining the formulas (3.34), (3.35) and (3.36), we can see (3.33). □

Lemma 3.2.11. *On the balls $B_{g_j}(E_j,R) \subset X_j$, there exists a constant $C(n,R) > 0$ such that*

$$\Delta_{\omega_j} |\nabla V|^2_{\omega_j} \le C(n,R), \quad \forall j >> 1.$$

Proof. From (3.1.1) and (3.32) we have

$$\Delta_{\omega_j} |\nabla V|^2 = (n-1)(T-t_j)\frac{(|\nabla V|^2)'}{u'} + \frac{T-t_j}{u''}\Big(|\nabla V|^2\Big)'', \tag{3.37}$$

where as before $u' = \frac{\partial}{\partial\rho}u(\rho, t_j)$, etc. Our goal is to show that both terms on RHS of (3.37) are uniformly bounded on the balls $B_{g_j}(E_j, R)$. To begin with, we need to estimate $u^{(4)}$.

Claim: There is a uniform constant $C = C(n) > 0$ such that

$$|u^{(4)}| \leq C\frac{(u'')^2}{T-t} + \frac{(u''')^2}{u''}.$$

Proof of the **Claim**. By the formula of scalar curvature (see [75]), we have

$$R(\omega(t)) = -\frac{u^{(4)}}{(u'')^2} + \frac{(u''')^2}{(u'')^3} - 2(n-1)\frac{u'''}{u'u''} - (n-1)(n-2)\frac{u''}{(u')^2} + \frac{n(n-1)}{u'}. \quad (3.38)$$

And by Type I condition we have $|R| \leq \frac{C}{T-t}$. Combining with Lemma 3.2.1, it is easy to see the bound on $|u^{(4)}|$. $\square$

The first term on RHS of (3.37) is equal to

$$\begin{aligned}&\frac{(n-1)(T-t_j)}{u'}\Big(2(n-1)\frac{u''}{u'}\frac{u'''u'-(u'')^2}{(u')^2} + 2\frac{u'''}{u''}\frac{u^{(4)}u''-(u''')^2}{(u'')^2}\Big)\\ =&\frac{2(n-1)^2(T-t_j)}{u'}\frac{u''}{u'}\cdot\frac{u'''u'-(u'')^2}{(u')^2} + \frac{2(n-1)(T-t_j)}{u'}\frac{u'''}{u''}\cdot\frac{u^{(4)}u''-(u''')^2}{(u'')^2},\end{aligned}$$

by examining the terms above using Lemma 3.2.1 and **Claim** we see that the first term on RHS of (3.37) is uniformly bounded above by $C = C(n) > 0$.

The second term in RHS of (3.37) is a little complicated, after some calculations and replacing the $u^{(4)}$ by the scalar curvature (3.38), we have the second term in RHS of (3.37) is equal to

$$\begin{aligned}&4(n-1)(n-3)\frac{T-t_j}{u''}\Big(\frac{\big((u''')^2+u''u^{(4)}\big)u' - u'''(u'')^2}{(u')^3}\Big) - 6(n-1)\frac{T-t_j}{u''}\Big(\frac{u''}{u'}\Big)^2\frac{u'''u'-(u'')^2}{(u')^2}\\ &-\frac{T-t_j}{u''}\Big(2R'u''' + 2Ru^{(4)}\Big) - 2(n-1)\frac{T-t_j}{u''}\Big(\frac{2u'u''u'''u^{(4)} - (u''')^2(u'u'''+(u'')^2)}{(u'u'')^2}\Big)\\ &+2n(n-1)\frac{T-t_j}{u''}\frac{u^{(4)}u'-u'''u''}{(u')^2}.\end{aligned} \quad (3.39)$$

We look the third term in (3.39). By the Type I condition and Shi's derivative estimate along Ricci flow, we know $|\nabla R(\omega(t_j))| \leq \frac{C}{(T-t_j)^{3/2}}$, and also we know $|\nabla R|^2 = \frac{(R')^2}{u''}$, hence

$$|R'| \leq C\frac{\sqrt{u''}}{(T-t_j)^{3/2}},$$

so we have

$$\Big| -\frac{T-t_j}{u''}\Big(2R'u''' + 2Ru^{(4)}\Big)\Big| \leq C\frac{T-t_j}{u''}\Big(\frac{\sqrt{u''}}{(T-t_j)^{3/2}}|u'''| + \frac{(u'')^2}{(T-t_j)^2} + \frac{1}{T-t_j}\frac{(u''')^2}{u''}\Big)$$
$$\leq C(n,R),$$

by the Lemmas 3.2.1, 3.2.3 and the **Claim**.

The other terms in (3.39) can be estimated similarly using the lemmas above, and we can see they are all uniformly bounded. Hence we finish the proof of Lemma 3.2.11. □

Proof of Lemma 3.2.9. Combining with the Bochner identity (3.33), Type I condition and Shi's derivative estimates, i.e., $|Rm(g_j)|_{g_j}, |\nabla Rm(g_j)|_{g_j} \leq C$, and Lemma 3.2.11, we can get the bound on $|\nabla\nabla V|^2_{\omega_j} + |\bar{\nabla}\nabla V|^2_{\omega_j}$.

□

Proposition 3.2.1. *There exists a nontrivial holomorphic vector field V_∞ as the subsequential limit of V along the Cheeger-Gromov convergence, such that V_∞ is tangential to the fibers of $F_\infty : X_\infty \to \mathbb{CP}^{n-1}$ and $Im(V_\infty)$ is a Killing vector field on the fibers.*

Proof. Along the Cheeger-Gromov convergence (3.7), by the (locally) uniform C^0, C^1, C^2 bound of the holomorphic vector fields $V_j = V$ with respect to the metrics ω_j, up to a subsequence, V_j converge in $C^{1,\alpha}$ norm (in the Cheeger-Gromov sense) to a vector field V_∞ on X_∞, which is holomorphic with respect to the complex structure J_∞. The holomorphic vector field V_∞ satisfies similar C^0, C^1, C^2 bounds as V_j, when restricted on the balls $B_{g_\infty}(p_\infty, R)$.

To see V_∞ is nontrivial, there exists a sequence of points $x_j \in \partial B_{g_j}(E_j, R)$ converging to an $x_\infty \in X_\infty$, by (3.30), we see that $|V_\infty|(x_\infty) \geq c(n,R) > 0$, hence V_∞ is nontrivial.

On the other hand, the vector fields V_j vanish identically on the exceptional divisors E_j in $B_{g_j}(E_j, R)$, and by taking limits, V_∞ also has zero points, e.g. $V_\infty(p_\infty) = 0$. Hence the zero set of V_∞ is a nonempty analytic set, since V_∞ is a holomorphic vector field, and we denote this zero set by $\tilde{E}_\infty$. It's clear that if a sequence of points $x_j \in E_j$ converges to $x_\infty \in X_\infty$, then $x_\infty \in \tilde{E}_\infty$.

Since V_j is tangential to the fibers of $F_j : X_j \to \mathbb{CP}^{n-1}$, $dF_j(V_j) = 0$, from the $C^{1,\alpha}$ convergence of F_j, V_j, the limit vector field V_∞ satisfies $dF_\infty(V_\infty) = 0$, i.e., V_∞ is tangential to the fibers of $F_\infty : X_\infty \to \mathbb{CP}^{n-1}$.

Choose a fiber $F_\infty^{-1}(y)$ of F_∞ (here $y \in \mathbb{CP}^{n-1}$). There exists a sequence of points $x_j \in F_j^{-1}(y) \cap E_j$ which converge up to a subsequence to $x_\infty \in X_\infty \cap F_\infty^{-1}(y)$, such that $V_\infty(x_\infty) = 0$. On the other hand, for any other point $x'_\infty \in F_\infty^{-1}(y)$, we may assume $d_\infty(x_\infty, x'_\infty) = R > 0$ and there exists a subsequence of $x'_j \in X_j \cap F_j^{-1}(y)$ with $d_{g_j}(x_j, x'_j) > R/2 > 0$ which converges to x'_∞, then by (3.30), we see $|V_\infty|_{g_\infty}(x'_\infty) \geq c(n, R) > 0$. We remark that (3.32) implies at this zero point $|\nabla V_\infty|^2_\infty = 1$.

Thus on each fiber $F_\infty^{-1}(y)$ ($y \in \mathbb{CP}^{n-1}$), V_∞ is a holomorphic vector field with simple single zero point. (3.27) implies that for V_∞, $\mathrm{Im}(V_\infty)$ is a Killing vector field of g_∞. Since $\mathrm{Im}(V_\infty)$ is tangential to the fiber, it follows that on the fiber $F_\infty^{-1}(y)$, with respect to the restriction metric $g_\infty|$ of g_∞ on $F_\infty^{-1}(y)$, the vector field $\mathrm{Im}(V_\infty)$ is also Killing. □

Corollary 3.2.1. *The fibers of $F_\infty : X_\infty \to \mathbb{CP}^{n-1}$ are either biholomorphic to $\mathbb{C}$ or the disk $D \subset \mathbb{C}$.*

Proof. Fix any fiber $F_\infty^{-1}(y)$ of F_∞, which is a complete noncompact Riemann surface. From the proof of Proposition 3.2.1, we know the vector field $\mathrm{Im}(V_\infty)$ is Killing in $F_\infty^{-1}(y)$ and has a single zero point in $F_\infty^{-1}(y)$, from Lemma 1 in [24], we conclude that topologically $F_\infty^{-1}(y)$ is $\mathbb{R}^2$, which in particular is simply connected. By Riemann uniformization theorem for Riemann surfaces, $F_\infty^{-1}(y)$ is either $\mathbb{C}$ or the holomorphic disk $D \subset \mathbb{C}$. □

3.3 $U(n)$-actions on Limiting space X_∞

3.3.1 Define $U(n)$-actions

We first define a metric on the compact Lie group $U(n)$ by

$$d_0(\sigma_1, \sigma_2) := \max\{d_E(\sigma_1(x), \sigma_2(x))| \text{ for all } x \in S^{2n-1} \subset \mathbb{C}^n\} \tag{3.40}$$

where d_E is the Euclidean distance on $\mathbb{C}^n$ and $\sigma_1, \sigma_2 \in U(n)$ act in the standard way on $S^{2n-1} \subset \mathbb{C}^n$. We remark that the metrics on the compact group $U(n)$ are all equivalent, so any other metrics on $U(n)$ will do.

Lemma 3.3.1. *d_0 defines a metric on the compact group $U(n)$.*

Proof. We only need to prove that d_0 satisfies the triangle inequality, since the $U(n)$-action on S^{2n-1} is effective. For any $\sigma_1, \sigma_2, \sigma_3 \in U(n)$, any $\epsilon > 0$, there exists an $x_\epsilon \in S^{2n-1}$ such that $d_0(\sigma_1, \sigma_2) \leq d_E(\sigma_1(x_\epsilon), \sigma_2(x_\epsilon)) + \epsilon$, then

$$d_0(\sigma_1, \sigma_2) \leq d_E(\sigma_1(x_\epsilon), \sigma_3(x_\epsilon)) + d_E(\sigma_2(x_\epsilon), \sigma_3(x_\epsilon)) + \epsilon \leq d_0(\sigma_1, \sigma_3) + d_0(\sigma_2, \sigma_3) + \epsilon,$$

then letting $\epsilon \to 0$ we can get the triangle inequality. $\square$

For each $\sigma \in U(n)$, we consider the map $\chi_{j,\sigma}$ defined by

$$\chi_{j,\sigma} : (X_j, g_j, J_j) \to (X_j, g_j, J_j), \tag{3.41}$$

by $\chi_{j,\sigma}(x) = \sigma(x)$. Recall σ acts isometrically and holomorphically on (X_j, g_j, J_j), $\chi_{j,\sigma}$ is also a holomorphic isometry. Hence the energy density of $\chi_{j,\sigma}$, $|\nabla_j \chi_{j,\sigma}|^2_{g_j} = n$, where ∇_j is the connection induced from g_j and $\chi^*_{j,\sigma} g_j$. Since $\chi_{j,\sigma}$ is holomorphic, hence also a harmonic map. For notation convenience we denote $F = \chi_{j,\sigma}$, then by Bochner formula,

$$0 = \Delta_j |\nabla_j F|^2 = |\nabla \nabla_j F|^2 + \mathrm{Ric}_{g_j}(\nabla_j F, \overline{\nabla_j F}) - R(F^* g_j)^\alpha_{\beta\bar{\gamma}\delta} \overline{F^\alpha_{,i}} F^\delta_{,i} F^\beta_{,k} \overline{F^\gamma_{,k}}, \tag{3.42}$$

where $\Delta_j = \Delta_{g_j}$ and $R(F^* g_j)^\alpha_{\beta\bar{\gamma}\delta}$ denotes the sectional curvature of the pulled-back metric $F^* g_j$, which is uniformly bounded by the Type I condition, so is the Ricci curvature of g_j. Hence by (3.42) and the bound $|\nabla_j F|^2 = n$, we see that $|\nabla \nabla_j F|^2 \leq C$ by a uniform constant $C = C(n)$. Therefore, we get the uniform C^2 bound of the maps $\chi_{j,\sigma}$, independent of j, σ.

Since $\chi_{j,\sigma}$ is an isometry and maps E_j to itself, which has fixed diameter D_n under the metrics g_j, we have for any $R > 0$, the image of $B_{g_j}(p_j, R)$ under $\chi_{j,\sigma}$ is contained in $B_{g_j}(p_j, R + D_n)$. Therefore, the maps $\chi_{j,\sigma}$ are locally uniformly bounded, and satisfy uniform C^1, C^2 bounds, so along the Cheeger-Gromov convergence (3.7), up to a subsequence of j, $\chi_{j,\sigma}$ converge to a limit map

$$\chi_{\infty,\sigma} : (X_\infty, g_\infty, J_\infty) \to (X_\infty, g_\infty, J_\infty),$$

which preserves the metric g_∞ and complex structure J_∞, hence an isometry and holomorphic map. The map $\chi_{\infty,\sigma}$ is defined through a subsequence of $\chi_{j,\sigma}$, for different σ, the subsequence might be different. Our next lemma will show that there exists a subsequence of j, such that for *all* $\sigma \in U(n)$, $\chi_{j,\sigma}$ converge to limit maps $\chi_{\infty,\sigma}$.

Lemma 3.3.2. *For any $R > 0$, there exists a $C(n,R) > 0$ such that for j large enough, we have*

$$d_{g_j}(\sigma_1(x), \sigma_2(x)) \le C(n,R) d_0(\sigma_1, \sigma_2), \quad \forall \sigma_1, \sigma_2 \in U(n)$$

and $x \in B_{g_j}(E_j, R) \subset (X_j, g_j, J_j)$, where d_0 is the metric on $U(n)$ defined in (3.40)

Proof. By the expansion formula of $g_j = \frac{1}{T-t_j} g(t_j)$ in (3.2), and Lemma 3.2.3 we have on $B_{g_j}(E_j, R) \backslash E_j \subset \mathbb{C}^n \backslash \{0\}$, (we identify $B_{g_j}(E_j, R) \backslash E_j$ as a punctured ball in $\mathbb{C}^n \backslash \{0\}$)

$$\begin{aligned} g_j &= \frac{u'}{T-t_j}\Big(\frac{\delta_{ik}}{|z|^2} - \frac{\bar{z}_i z_k}{|z|^4}\Big) dz_i \wedge d\bar{z}_k + \frac{u''}{T-t_j}\frac{\bar{z}_i z_k}{|z|^4} dz_i \wedge d\bar{z}_k \\ &\le \frac{C(n,R)}{|z|^2}\omega_E, \end{aligned} \tag{3.43}$$

and ω_E is the Euclidean metric on $\mathbb{C}^n$, so for any $x \in B_{g_j}(E_j, R) \backslash E_j \subset \mathbb{C}^n \backslash \{0\}$, and $\sigma_1, \sigma_2 \in U(n)$, $\sigma_1(x), \sigma_2(x)$ remain in $B_{g_j}(E_j, R) \backslash E_j \subset \mathbb{C}^n \backslash \{0\}$ and the Euclidean norm $|\sigma_1(x)| = |\sigma_2(x)| = |x|$. Choose a curve $\gamma \subset S^{2n-1}_{|x|}$, the Euclidean sphere in $\mathbb{C}^n \backslash \{0\}$ with radius $|x|$, connecting $\sigma_1(x)$ and $\sigma_2(x)$ and the Euclidean length $L_E(\gamma) \le 2d_E(\sigma_1(x), \sigma_2(x))$. Hence by the estimate (3.43), we have

$$\begin{aligned} d_{g_j}(\sigma_1(x), \sigma_2(x)) &\le d_{g_j}(\gamma) \\ &\le \frac{C(n,R)}{|x|} L_E(\gamma) \\ &\le \frac{2C(n,R)}{|x|} d_E(\sigma_1(x), \sigma_2(x)) \\ &= 2C(n,R) d_E\Big(\sigma_1\Big(\frac{x}{|x|}\Big), \sigma_2\Big(\frac{x}{|x|}\Big)\Big) \\ &\le 2C(n,R) d_0(\sigma_1, \sigma_2). \end{aligned} \tag{3.44}$$

By continuity, (3.44) also holds for $x \in E_j$.

□

If we define maps

$$\chi_j : (X_j, g_j, J_j) \times (U(n), d_0) \to (X_j, g_j, J_j) \tag{3.45}$$

by $\chi_j(x, \sigma) = \chi_{j,\sigma}(x)$, which are holomorphic in x and satisfy

$$d_{g_j}(\chi_j(x, \sigma), \chi_j(y, \sigma)) = d_{g_j}(x, y), \quad \forall x, y \in X_j, \sigma \in U(n),$$

and by Lemma 4.2.2 we also have

$$d_{g_j}(\chi_j(x, \sigma_1), \chi_j(x, \sigma_2)) \leq C(n, R) d_0(\sigma_1, \sigma_2), \quad \forall x \in B_{g_j}(E_j, R), \sigma_1, \sigma_2 \in U(n).$$

Hence for any $x, y \in B_{g_j}(E_j, R)$ and $\sigma_1, \sigma_2 \in U(n)$

$$\begin{aligned} d_{g_j}(\chi_j(x, \sigma_1), \chi_j(y, \sigma_2)) &\leq d_{g_j}(\chi_j(x, \sigma_1), \chi_j(y, \sigma_1)) + d_{g_j}(\chi_j(y, \sigma_1), \chi_j(y, \sigma_2)) \\ &\leq d_{g_j}(x, y) + C(n, R) d_0(\sigma_1, \sigma_2) \end{aligned} \tag{3.46}$$

which implies the maps χ_j defined in (3.45) are locally uniformly bounded and locally equi-continuous with respect to the given product metrics. Moreover the maps $\chi_j(\cdot, \sigma)$ satisfy uniform C^1, C^2 bounds for any $\sigma \in U(n)$, hence by Arzela-Ascoli theorem, up to a subsequence of j, χ_j converge to a map

$$\chi_\infty : (X_\infty, g_\infty, J_\infty) \times (U(n), d_0) \to (X_\infty, g_\infty, J_\infty), \tag{3.47}$$

and for each $\sigma \in U(n)$, the map

$$\chi_\infty(\cdot, \sigma) : (X_\infty, g_\infty, J_\infty) \to (X_\infty, g_\infty, J_\infty)$$

is an isometry and J_∞-holomorphic.

Lemma 3.3.3. *The map χ_∞ defined in* (3.47) *satisfies*

$$\chi_\infty(x, \sigma_1\sigma_2) = \chi_\infty(\chi_\infty(x, \sigma_2), \sigma_1), \quad \forall x \in X_\infty, \sigma_1, \sigma_2 \in U(n). \tag{3.48}$$

Proof. For any $x \in X_\infty$ and $\sigma_1, \sigma_2 \in U(n)$, choose a sequence of $x_j \in X_j$ converging to x_∞. For each j from the definition we have

$$\begin{aligned} \chi_j(x_j, \sigma_1\sigma_2) &= \chi_{j,\sigma_1\sigma_2}(x_j) = \sigma_1\sigma_2(x_j) = \sigma_1\big(\sigma_2(x)\big) \\ &= \chi_j(\sigma_2(x_j), \sigma_1) = \chi_j\big(\chi_j(x_j, \sigma_2), \sigma_1\big), \end{aligned}$$

taking $j \to \infty$ and by the definition of χ_∞ we have

$$\chi_\infty(x, \sigma_1\sigma_2) = \chi_\infty\big(\chi_\infty(x, \sigma_2), \sigma_1\big).$$

□

Remark 3.3.1. *If we define the "action" of $\sigma \in U(n)$ on X_∞, $\sigma : X_\infty \to X_\infty$ by $\sigma \cdot x = \chi_\infty(x, \sigma)$, then Lemma 3.3.3 means that for any $\sigma_1, \sigma_2 \in U(n)$, $(\sigma_1\sigma_2) \cdot x = \sigma_1 \cdot (\sigma_2 \cdot x)$, for any $x \in X_\infty$.*

It is clear that the identity element $e \in U(n)$ satisfies $\chi_\infty(x, e) = x$, i.e., $e \cdot x = x$ for any $x \in X_\infty$. Hence the $U(n)$-action on X_∞ defined above is a group action.

3.3.2 $U(n)$-action and fiber map F_∞

Recall in Section 3.2.2, we define a holomorphic map $F_\infty : X_\infty \to \mathbb{CP}^{n-1}$, as the limit map of $F_j : X_j \to \mathbb{CP}^{n-1}$. It is clear that F_j is $U(n)$-equivariant with respect to the $U(n)$-action on $X_j = \mathbb{CP}^n \# \overline{\mathbb{CP}^n}$ and the standard action on $\mathbb{CP}^{n-1}$, i.e.

$$F_j(\sigma \cdot x_j) = \sigma \cdot F_j(x_j), \quad \forall x_j \in X_j \ \ \forall \sigma \in U(n).$$

Now for any $x \in X_\infty$, there is a sequence $x_j \in X_j$ converging to x, taking $j \to \infty$ and by the smooth convergence of F_j to F_∞, we have

$$F_\infty(\sigma \cdot x) = \sigma \cdot F_\infty(x),$$

i.e. F_∞ is $U(n)$-equivariant. Hence for any $y \in \mathbb{CP}^{n-1}$, $\sigma \in U(n)$ maps the fiber $F_\infty^{-1}(y)$ to $F_\infty^{-1}(\sigma \cdot y)$.

Lemma 3.3.4. *The restriction of $\sigma : X_\infty \to X_\infty$ to the fiber $F_\infty^{-1}(y)$*

$$\sigma|_{F_\infty^{-1}(y)} : F_\infty^{-1}(y) \to F_\infty^{-1}(\sigma \cdot y)$$

is a biholomorphic map.

Proof. This follows from the fact that

$$\sigma\sigma^{-1} = e = id : F_\infty^{-1}(\sigma \cdot y) \to F_\infty^{-1}(\sigma \cdot y),$$

and

$$\sigma^{-1}\sigma = e = id : F_\infty^{-1}(y) \to F_\infty^{-1}(y).$$

And both σ and σ^{-1} are holomorphic maps. $\square$

Corollary 3.3.1. *The fibers of $f_\infty : X_\infty \to \mathbb{CP}^{n-1}$ are all biholomorphic to each other.*

This follows from the previous lemma and the fact that $U(n)$ action on $\mathbb{CP}^{n-1}$ is transitive.

Fix $p = [1 : 0 : \cdots : 0] \in \mathbb{CP}^{n-1}$, and denote the fiber $F_\infty^{-1}(p) = F_p$. We know from the Corollary 3.3.1 all fibers of F_∞ are isomorphic. It is hoped that F_∞ is in fact a fiber bundle over $\mathbb{CP}^{n-1}$ with fiber F_p.

Proposition 3.3.1. *The map $F_\infty : X_\infty \to \mathbb{CP}^{n-1}$ is a fiber bundle with fibers isomorphic to F_p.*

Proof. The compact group $SU(n)$-action on X_∞ induces an action of the complexified group $SL(n, \mathbb{C})$ of $SU(n)$, which is defined through the infinitesimal action: for any $\xi + \sqrt{-1}\eta \in \mathfrak{su}(n) \oplus \sqrt{-1}\mathfrak{su}(n) = \mathfrak{sl}(n, \mathbb{C})$, we define $\exp(\xi + \sqrt{-1}\eta) \cdot x = \exp(\xi) \exp(J_\infty \eta) \cdot x$, where J_∞ is the complex structure on X_∞.

Define a map

$$\pi : SL(n, \mathbb{C}) \times F_p \to X_\infty, \quad (\sigma, x) \mapsto \sigma \cdot x.$$

This is indeed a *surjective* map by the property of group actions. If $\pi(\sigma_1, x_1) = \pi(\sigma_2, x_2)$ for some $\sigma_1, \sigma_2 \in SL(n, \mathbb{C})$ and $x_1, x_2 \in F_p$. Then $\sigma_1 \cdot x_1 = \sigma_2 \cdot x_2$, and $\sigma_1 \cdot p = \sigma_1 \cdot f_\infty(x_1) = f_\infty(\sigma_1 \cdot x_1) = f_\infty(\sigma_2 \cdot x_2) = \sigma_2 \cdot p$, therefore, $\sigma_1^{-1} \circ \sigma_2 \in$ isotropic subgroup B of $SL(n, \mathbb{C})$ acting on $\mathbb{CP}^{n-1}$, which is given by the matrices of the form

$$\sigma_1^{-1} \circ \sigma_2 = \begin{pmatrix} a & * \\ \mathbf{0} & A \end{pmatrix}$$

where $a \in \mathbb{C}^*$ and $A \in GL(n-1, \mathbb{C})$ such that $a \det A = 1$ and $*$ denotes a vector in $\mathbb{C}^{n-1}$. Hence we have $x_1 = \begin{pmatrix} a & * \\ \mathbf{0} & A \end{pmatrix} \cdot x_2$.

We define an equivalence relation on $SL(n, \mathbb{C}) \times F_p$ as

$$(\sigma_1, x_1) \sim (\sigma_2, x_2)$$

if there exists a matrix $\begin{pmatrix} a & * \\ \mathbf{0} & A \end{pmatrix} \in B$ such that $\sigma_2 = \sigma_1 \circ \begin{pmatrix} a & * \\ \mathbf{0} & A \end{pmatrix}$ and $x_2 = \begin{pmatrix} a & * \\ \mathbf{0} & A \end{pmatrix}^{-1} \cdot x_1$. Then we can see that if $(\sigma_1, x_1) \sim (\sigma_2, x_2)$, then $\pi(\sigma_1, x_1) = \pi(\sigma_2, x_2)$. Hence the quotient map

$$\bar{\pi} : SL(n, \mathbb{C}) \times F_p/_{\sim} \to X_\infty$$

is bijective and also a biholomorphic map, since each action $\sigma \in SL(n, \mathbb{C})$ on X_∞ is holomorphic and $SL(n, \mathbb{C})$ is a complex manifold.

Claim: $SL(n, \mathbb{C}) \times F_p/_{\sim}$ is a fiber bundle over $\mathbb{CP}^{n-1}$ with fibers isomorphic to F_p.

Proof of the claim: Define the projection map $pr : SL(n, \mathbb{C}) \times F_p/_{\sim} \to SL(n, \mathbb{C})/B \cong \mathbb{CP}^{n-1}$, by $pr(\sigma, x) = Q(\sigma)$, where $Q : SL(n, \mathbb{C}) \to SL(n, \mathbb{C})/B$ is the quotient map. pr is clearly well-defined and we want to show pr is locally trivial. The principal B-bundle Q is locally trivial, so around any point in $\mathbb{CP}^{n-1} \cong SL(n, \mathbb{C})/B$, there is an open set U such that $Q^{-1}(U) \cong U \times B$, i.e. there is a local trivialization $\varphi : Q^{-1}(U) \to U \times B$, and we will denote $\varphi = (\varphi_1, \varphi_2)$. By the definition of quotient map Q, it is clear that $\varphi_1(\sigma\mathbf{b}) = \varphi_1(\sigma)$ for any $\sigma \in Q^{-1}(U)$ and $\mathbf{b} \in B$. Thus we can define a local section $s : U \to Q^{-1}(U)$ of Q by $s(y) = \varphi^{-1}(y, \mathbf{e})$ with $\mathbf{e} \in B$ being the identity matrix.

Define a map $\tilde{\varphi} : pr^{-1}(U) = Q^{-1}(U) \times F_p/_{\sim} \to U \times F_p$ by

$$\tilde{\varphi}(\sigma, x) = (\varphi_1(\sigma), s(\varphi_1(\sigma))^{-1} \cdot \sigma \cdot x)$$

which by the property of φ_1 is clearly well-defined. We want to show $\tilde{\varphi}$ is bijective. $\tilde{\varphi}$ is clearly surjective. To see that it is also injective, suppose $\tilde{\varphi}(\sigma_1, x_1) = \tilde{\varphi}(\sigma_2, x_2)$, then $\varphi_1(\sigma_1) = \varphi_1(\sigma_2)$, so there exists a matrix $\mathbf{b} \in B$ such that $\sigma_2 = \sigma_1\mathbf{b}$. Since $s(\varphi_1(\sigma_1))^{-1} : F_{\sigma_1 \cdot p} \to F_p$ is an isomorphism, we must have $\sigma_1 \cdot x_1 = \sigma_2 \cdot x_2$, and this implies $x_2 = \mathbf{b}^{-1} \cdot x_1$, and hence $(\sigma_1, x_1) \sim (\sigma_2, x_2)$, and the map $\tilde{\varphi}$ is injective. In the definition of $\tilde{\varphi}$, all maps are holomorphic hence $\tilde{\varphi}$ is also holomorphic, and $\tilde{\varphi}$ provides the local trivialization of $SL(n, \mathbb{C}) \times F_p/_{\sim}$ over $\mathbb{CP}^{n-1}$. □

Fix the point $p = [1 : 0 : \cdots : 0] \in \mathbb{CP}^{n-1}$, it is well-known that the isotropic subgroup U_p at p of the $U(n)$-action on $\mathbb{CP}^{n-1}$ is isomorphic to $U(1) \times U(n-1)$ and given by the the matrices of the form

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & A \end{pmatrix}, \quad \text{for some } A \in U(n-1), \quad e^{i\theta} \in U(1).$$

Each $\sigma \in U_p$ induces an isomorphism of the fiber F_p, which is either $\mathbb{C}$ or the unit disk $D \subset \mathbb{C}$.

Lemma 3.3.5. *There exists an $x_0 \in F_p$ such that for any $\sigma \in U_p$, $\sigma \cdot x_0 = x_0$. Moreover, if $\sigma \in U_p$ fixes all $x \in F_p$, then $\sigma \in \{1\} \times U(n-1)$, i.e., σ is of the form*

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \quad \textit{for some } A \in U(n-1).$$

Proof. It is clear that $\sigma \in U_p$ also induces an isomorphism of the fibers $F_j^{-1}(p)$. For each j, there exists an $x_j \in F_j^{-1}(p) \cap E_j$ which is fixed by all $\sigma \in U_p$. Let x_0 be a limit point of x_j, then $x_0 \in F_\infty^{-1}(p) = F_p$ is the fixed point of all $\sigma \in U_p$.

Suppose there exists a $\sigma \in U_p$ such that $\sigma \cdot x = x$ for all $x \in F_p$. Fix a large $R > 0$ and then for any $x_j \in F_j^{-1}(p) \cap B_{g_j}(E_j, R)$, $d_{g_j}(\sigma \cdot x_j, x_j) \leq \epsilon_j \to 0$ as $j \to \infty$, since $d_{g_j}(\sigma \cdot x_j, x_j) \to d_{g_\infty}(\sigma \cdot x_\infty, x_\infty) = 0$, for some $x_\infty \in F_p$. On $\partial B_{g_j}(E_j, R)$, by Lemma 3.2.3 and the expansion formula of g_j in (3.43) there exists a constant $c(n, R) > 0$ such that

$$g_j \geq c(n, R) \frac{\omega_E}{|z|^2}. \tag{3.49}$$

For any $x_j \in F_j^{-1}(p) \cap \partial B_{g_j}(E_j, R)$, the minimal geodesic (with respect to g_j) γ_j connecting x_j and $\sigma \cdot x_j$ must be contained in the annulus $B_{g_j}(E_j, R+\epsilon_j) \backslash B_{g_j}(E_j, R - \epsilon_j) \subset \mathbb{C}^n \backslash \{0\}$, where the estimate (3.49) still holds with some different $c(n, R) > 0$, therefore we have

$$\begin{aligned} \epsilon_j \geq d_{g_j}(\sigma \cdot x_j, x_j) &= L_{g_j}(\gamma_j) \\ &\geq c(n, R) d_{\frac{\omega_E}{|z|^2}}(\gamma_j) \\ &\geq c(n, R) d_{g_{S^{2n-1}}} \Big(\sigma(\frac{x_j}{|x_j|}), \frac{x_j}{|x_j|} \Big), \end{aligned} \tag{3.50}$$

where $g_{S^{2n-1}}$ is the standard metric on the unit sphere $S^{2n-1} \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ and we use the fact that the metric

$$\frac{\omega_E}{|z|^2} = (d \log |z|)^2 + g_{S^{2n-1}},$$

is a product metric, so the distance of x_j and $\sigma \cdot x_j \in \mathbb{C}^n \backslash \{0\}$ with respect to $\frac{\omega_E}{|z|^2}$ is equal to $d_{g_{S^{2n-1}}}\Big(\sigma(\frac{x_j}{|x_j|}), \frac{x_j}{|x_j|}\Big)$, since the Euclidean norms $|x_j| = |\sigma \cdot x_j|$. Suppose $\sigma \in U_p \subset U(n)$ is given by $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & A \end{pmatrix}$ for some $A \in U(n-1)$, and it acts on the big circle $F_j^{-1}(p) \cap S^{2n-1}$ by rotation by angle θ. Then (3.50) means that for any $x \in F_j^{-1}(p) \cap S^{2n-1}$, $d_{g_{S^{2n-1}}}(\sigma \cdot x, x)$ is arbitrarily small, hence equals to zero, so the rotation angle $\theta = 0$, and $\sigma \in U_p$ is of the form $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$ for some $A \in U(n-1)$. $\square$

Remark 3.3.2. *Noting that the automorphism groups of D and $\mathbb{C}$ are given by*

$$\mathrm{Aut}(D) = \Big\{ f_{a,\theta} | f_{a,\theta}(\zeta) = e^{i\theta}\frac{\zeta - a}{1 - \bar{a}\zeta}, \quad \theta \in S^1, \ a \in D \Big\},$$

$$\mathrm{Aut}(\mathbb{C}) = \Big\{ a\zeta + b | a, b \in \mathbb{C}, \ a \neq 0 \Big\},$$

respectively. The action of each nonidentity $\sigma \in U_p$ on F_p is of one of the above, hence has one and only one fixed point in F_p.

We know the line bundles of $\mathbb{CP}^{n-1}$ are given by $\mathcal{O}_{\mathbb{CP}^{n-1}}(k)$ for some $k \in \mathbb{Z}$. And each F_p can be embedded in a complex line $\mathbb{C}$ with the fixed point x_0 identified as $0 \in \mathbb{C}$ hence the fiber bundle $F_\infty : X_\infty \to \mathbb{CP}^{n-1}$ can be embedded into some line bundle $\mathcal{O}_{\mathbb{CP}^{n-1}}(k)$, so that X_∞ is either the line bundle $\mathcal{O}_{\mathbb{CP}^{n-1}}(k)$ or the disk subbundle of $\mathcal{O}_{\mathbb{CP}^{n-1}}(k)$.

Lemma 3.3.6. *We have $k = -1$.*

Proof. We have known from Theorem 3.1.1 (1) (see also [75]) that X_∞ is *diffeomorphic* to $\widetilde{\mathbb{C}^n}$, so k must be negative and odd. On the other hand, if $k \neq -1$, then the U_p actions on the fiber of $\mathcal{O}_{\mathbb{CP}^{n-1}}(k)$ over $p \in \mathbb{CP}^{n-1}$ are not "effective" in the sense that a matrix of the form $\begin{pmatrix} e^{2\pi i/k} & 0 \\ 0 & A \end{pmatrix}$ inducing the identity action on the fiber of $\mathcal{O}_{\mathbb{CP}^{n-1}}(k)$ over $p \in \mathbb{CP}^{n-1}$, and inducing the identity action on F_p. This contradicts Lemma 3.3.5. $\square$

Corollary 3.3.2. *X_∞ is either the holomorphic line bundle $\mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$ or the holomorphic disk bundle of $\mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$.*

3.4 Proof of Theorem 1.2.1

We first show that the limit metric g_∞ on $X_\infty \subset \mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$ is rotationally symmetry with respect to the natural coordinates of $\mathbb{C}^n\backslash\{0\} = \mathcal{O}_{\mathbb{CP}^{n-1}}(-1)\backslash E_\infty$.

Lemma 3.4.1. *There exists a smooth function U_∞ on X_∞, such that*

$$g_\infty = (n-1)F_\infty^*\omega_{FS} + i\partial\bar{\partial}U_\infty,$$

where $F_\infty : X_\infty \to \mathbb{CP}^{n-1}$ is the map constructed in Section 3.2, and ω_{FS} is the Fubini-Study metric on $\mathbb{CP}^{n-1}$.

Proof. Let $R > 0$ be large number, on $B_{g_j}(E_j, R)$, the metrics

$$g_j = i\partial\bar{\partial}\Big(\frac{u(t_j,\rho)}{T-t_j}\Big) = (n-1)F_j^*\omega_{FS} + i\partial\bar{\partial}\Big(\frac{u(t_j,\rho)}{T-t_j} - (n-1)\rho\Big). \tag{3.51}$$

By the Calabi symmetry condition, the Kähler potentials $u(t,\rho) = (n-1)(T-t)\rho + U_0(t, e^\rho)$ near $\rho = -\infty$, and we can normalize for each $t \in [0,T)$, $U_0(t,0) = 0$, hence the smooth functions $\Big(\frac{u(t_j,\rho)}{T-t_j} - (n-1)\rho\Big)|_{E_j} = \Big(\frac{u(t_j,\rho)}{T-t_j} - (n-1)\rho\Big)|_{\rho=-\infty} = 0$ for any $j \geq 1$. Set $U_j = \frac{u(t_j,\rho)}{T-t_j} - (n-1)\rho$. The gradient of U_j with respect to g_j is

$$\begin{aligned}
|\nabla U_j|^2_{g_j} &= (T-t_j)\frac{(U_j')^2}{u''(t_j,\rho)} \\
&= (T-t_j)\frac{\Big(\frac{u'(\rho,t_j)}{T-t_j} - (n-1)\Big)^2}{u''(t_j,\rho)} \\
&\leq \frac{(u'(t_j,\rho) - (n-1)(T-t_j))}{T-t_j}\frac{u'(\rho,t_j) - a_{t_j}}{u''(t_j,\rho)} \\
&\leq C(n,R) \quad \text{ on } B_{g_j}(E_j,R)
\end{aligned}$$

for j large enough, where in the last inequality we use Lemmas 3.2.1 and 3.2.3. Hence $\|U_j\|_{C^0(B_{g_j}(E_j,R))} \leq C(n,R)$ for some $C(n,R) > 0$. Moreover, the Laplacian of U_j

$$\Delta_{g_j}U_j = n - (n-1)tr_{\omega_j}f_j^*\omega_{FS} = n - (n-1)\frac{(n-1)(T-t_j)}{u'(t_j,\rho)}$$

satisfies $\Delta_{g_j}U_j|_{E_j} = 1$ and

$$|\nabla\Delta_j U_j|^2_{g_j} = C(n)\frac{(T-t_j)^3u''(t_j)}{(u'(t_j))^4} \leq C(n),$$

so

$$\|\Delta_j U_j\|_{C^1(g_j, B_{g_j}(E_j,R))} \leq C(n,R).$$

Hence by elliptic estimate

$$\|U_j\|_{C^{2,\alpha}(g_j, B_{g_j}(E_j,R/2))} \leq C(n,R).$$

Therefore the functions U_j are locally uniformly bounded in $C^{2,\alpha}$ norm on any compact subset $B_{g_j}(E_j,R)$ of X_j. Taking a subsequence and using a diagonal argument, U_j converge (in the Cheeger-Gromov sense) locally uniformly in $C^{2,\alpha}$ topology to some $C^{2,\alpha}$ function U_∞ on X_∞, therefore from (3.51), $C^{1,\alpha}_{loc}$ convergence of the holomorphic maps F_j to F_∞ and smooth convergence of complex structures, the metrics g_j converge in C^α norm to

$$g_\infty = (n-1)F_\infty^*\omega_{FS} + i\partial\bar{\partial}U_\infty. \tag{3.52}$$

Since g_∞ and $F_\infty^*\omega_{FS}$ are both smooth, U_∞ is also a smooth function on X_∞. □

Take the natural coordinates of $\mathcal{O}_{\mathbb{CP}^{n-1}}(-1) \cong \widetilde{\mathbb{C}^n}$, $\mathbb{C}^n$ blown up at the origin, $\zeta = z_1(\neq 0)$, $w_2 = z_2/z_1$, …, $w_n = z_n/z_1$, where $z_1, \ldots, z_n$ are the natural coordinates on $\mathbb{C}^n$, and ζ is the coordinate of fibers and $w_2, \ldots, w_n$ are coordinates of $\mathbb{CP}^{n-1}$. Set $\rho = \log|z|^2 = \log\Big(|\zeta|^2(1+|w|^2)\Big)$, our goal in this subsection is to show

Lemma 3.4.2. *The function U_∞ constructed in* (3.52) *can be modified to depend only on ρ. That is, $U_\infty(\zeta,w) = \tilde{U}_\infty(|\zeta|^2(1+|w|^2))$ for some single-variable function $\tilde{U}_\infty(\cdot)$: $\mathbb{R}\to\mathbb{R}$. Hence U_∞ is rotationally symmetric on $\mathbb{C}^n\backslash\{0\}$.*

Proof. By construction the limit metric g_∞ is invariant under the $U(n)$-action defined in section 3.3, so we have

$$\sigma^* g_\infty = g_\infty, \quad \forall \sigma \in U(n).$$

By (3.52) we have

$$\sigma^*(i\partial\bar{\partial}U_\infty) = i\partial\bar{\partial}\sigma^* U_\infty = i\partial\bar{\partial}U_\infty, \quad \forall\sigma\in U(n).$$

By averaging the function U_∞ over the compact group $U(n)$ using the Harr measure, we may assume $\sigma^* U_\infty = U_\infty$ for all $\sigma \in U(n)$. Since we identify the unique fixed point of the U_p action in the fiber F_p with the origin in $\mathbb{C}$, the zero section (denoted by E_∞ which is locally given by $\zeta = 0$) of the line bundle $\mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$ coincide with the fixed point of the $U(n)$-actions in each fiber of $F_\infty : X_\infty \to \mathbb{CP}^{n-1}$. And $U_\infty|_{E_\infty} = 0$. Since $U(n)$-action is transitive on $\mathbb{CP}^{n-1}$, for any $w = (\zeta, w_2, \dots w_n) \in X_\infty$, there is some $\sigma_w \in U(n)$ mapping w to $(\zeta', 0, \dots, 0) \in X_\infty$ for some $\zeta' \in \mathbb{C}$ satisfying $|\zeta'|^2 = |\zeta|^2(1 + |w|^2)$. So (writing $Z = (w_2, \dots, w_n)$)

$$U_\infty(\zeta, \bar{\zeta}, Z, \bar{Z}) = U_\infty(\sigma_w \cdot (\zeta, Z)) = U_\infty(\zeta', \bar{\zeta}', 0, 0). \tag{3.53}$$

On the other hand, for $p = (0, \dots, 0) \in \mathbb{CP}^{n-1}$, the isotopy group $U_p \subset U(n)$ at p preserves the fiber $F_\infty^{-1}(p)$, which is either $D \subset \mathbb{C}$ or $\mathbb{C}$. The subgroup U_p fixes the point $(\zeta = 0, 0, \dots, 0) \in E_\infty$, which can be viewed as the origin in the fiber $F_\infty^{-1}(p)$. Noting that the automorphism groups of D and $\mathbb{C}$ are given by

$$\mathrm{Aut}(D) = \Big\{ f_{a,\theta} | f_{a,\theta}(\zeta) = e^{i\theta} \frac{\zeta - a}{1 - \bar{a}\zeta}, \quad \theta \in S^1, \ a \in D \Big\},$$

$$\mathrm{Aut}(\mathbb{C}) = \Big\{ a\zeta + b | a, b \in \mathbb{C}, \ a \neq 0 \Big\},$$

respectively. We see from both cases that the U_p action on the fiber $F_\infty^{-1}(p)$ is given by $\sigma_\theta(\zeta) = e^{i\theta}\zeta$ for $\theta \in S^1$, which means that the U_p action on the fiber is the rotation action of S^1 on $\mathbb{C}$. The property that U_∞ is invariant under the U_p action implies that

$$U_\infty(\zeta', \bar{\zeta}', 0, 0) = U_\infty(|\zeta'|, |\zeta'|, 0, 0), \quad \forall \zeta' \in \mathbb{C},$$

combining with (3.53), we see for any $(\zeta, w_2, \dots, w_n) \in X_\infty$

$$U_\infty(\zeta, w_2, \dots, w_n) = \tilde{U}_\infty\big(|\zeta|^2(1 + |w|^2)\big)$$

for some single variable function $\tilde{U}_\infty$. $\square$

3.4.1 Proof of Theorem 1.2.1

So far we have shown that $X_\infty \subset \mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$ is a fiber bundle with fibers either disk D or the line $\mathbb{C}$ and the metric g_∞ is rotationally symmetric on $\mathbb{C}^n \backslash \{0\} \cap (X_\infty \backslash E_\infty)$. We know (Theorem 3.1.1, or [75]) that the metric g_∞ is a *complete* gradient Kähler

Ricci soliton, i.e., for some $f_\infty \in C^\infty(X_\infty)$ with $\nabla_{g_\infty} f_\infty$ a holomorphic vector field, such that

$$\mathrm{Ric}(g_\infty) + i\partial\bar{\partial} f_\infty = g_\infty, \qquad \nabla\nabla f_\infty = 0 \tag{3.54}$$

Without loss of generality, we can choose f_∞ such that it's invariant under the $U(n)$-action and rotationally symmetric on $\mathbb{C}^n\backslash\{0\}$, since both g_∞ and $\mathrm{Ric}(g_\infty)$ are invariant under $U(n)$-action. $X_\infty\backslash E_\infty$ can be identified with either a punctured ball $B^* \subset \mathbb{C}^n\backslash\{0\}$, or $\mathbb{C}^n\backslash\{0\}$, on which the metric g_∞ can be written as $g_\infty = i\partial\bar{\partial} u_\infty$ satisfying the Calabi symmetry condition near $z = 0 \in \mathbb{C}^n$, i.e.,

$$u_\infty = u_\infty(\rho) = (n-1)\rho + U_0(e^\rho), \quad \text{near } \rho = -\infty$$

for some smooth $U_0 : (-\epsilon, \epsilon) \to \mathbb{R}$ such that $U_0'(0) > 0$, $U_0''(0) > 0$, where $\rho = \log |z|^2$, and $u_\infty', u_\infty'' > 0$. Then the equation (3.54) is equivalent to the following equation on $X_\infty\backslash E_\infty$,

$$u_\infty^{(4)} - 2\frac{(u_\infty''')^2}{u_\infty''} + n u_\infty''' - (n-1)\frac{(u_\infty'')^3}{(u_\infty')^2} - \left(u_\infty''' u_\infty' - (u_\infty'')^2\right) = 0,$$

where $u_\infty' = \frac{d}{d\rho} u_\infty$. Denote $\phi = u_\infty'$, then by some calculation reduction we see that the above equation is equivalent to

$$(\log\phi')' + (n-1)(\log\phi)' - \mu\phi' + \phi - n = 0, \quad \text{for some } \mu \in \mathbb{R}. \tag{3.55}$$

Lemma 3.4.3. *$\mu \neq 0$.*

Proof. If $\mu = 0$, then for $Q := \log\det g_\infty + u_\infty = -n\rho + (n-1)\log\phi + \log(\phi') + u_\infty$, we have $Q' = 0$, and this implies the metric g_∞ is KE with $\mathrm{Ric}(g_\infty) = g_\infty$. Myers' theorem from Riemannian geometry implies the diameter of (X_∞, g_∞) is bounded, however, from previous arguments we know the diameter of (X_∞, g_∞) is infinity, hence a contradiction. Thus $\mu \neq 0$. □

As in [36], since $\phi' = u_\infty'' > 0$, we may write $\phi' = F(\phi)$ for some smooth function F on $\mathbb{R}^+$, in terms of which (3.55) can be written as

$$F' + \Big(\frac{n-1}{\phi} - \mu\Big)F - (n - \phi) = 0, \tag{3.56}$$

and one can solve this first order ODE

$$\phi' = F(\phi) = \frac{\nu e^{\mu\phi}}{\phi^{n-1}} + \frac{\phi}{\mu} - \frac{\mu - 1}{\mu^{n+1}} \sum_{j=0}^{n-1} \frac{n!}{j!} \mu^j \phi^{j+1-n}, \tag{3.57}$$

for some constant $\nu \in \mathbb{R}$.

At any zero point ϕ_0 of $F(\phi)$, by (3.56), we know $F(\phi_0)' = n - \phi_0$, and by the intermediate value theorem this implies that $F(\phi)$ has at most two positive zeros $0 < a \leq b$ satisfying $0 < a \leq n \leq b$. By the Calabi symmetry, we have

$$\lim_{\rho \to -\infty} \phi(\rho) = n - 1, \quad \lim_{\rho \to -\infty} \phi'(\rho) = 0,$$

hence $0 = \lim_{\rho \to -\infty} \phi' = \lim_{\rho \to -\infty} F(\phi) = F(n-1)$, and $a = n - 1$ is a zero of F. Plugging $a = n - 1$ into (3.57) we get

$$\frac{\nu e^{\mu a}}{a^{n-1}} + \frac{a}{\mu} - \frac{\mu - 1}{\mu^{n+1}} \sum_{j=0}^{n-1} \frac{n!}{j!} \mu^j a^{j+1-n} = 0 \tag{3.58}$$

Proposition 3.4.1. *We must have $\mu > 0$ and $\nu = 0$.*

Proof. Suppose $\mu < 0$, then for large $\phi > 0$, the leading term on the RHS of (3.57) is ϕ/μ, hence the solution to (3.57) exists for all $\rho \in (-\infty, \infty)$, and $\phi(\rho)$ is uniformly bounded for $\rho \in \mathbb{R}$, we have

$$\lim_{\rho \to \infty} \phi(\rho) = b, \quad \lim_{\rho \to \infty} \phi'(\rho) = 0, \text{ for some } b > 0.$$

So we have $a \leq \phi \leq b$. However, the volume of (X_∞, g_∞) is given by

$$Vol(X_\infty, g_\infty) = C(n) \int_{-\infty}^{\infty} (\phi)^{n-1} \phi' d\rho = C(n) \Big((\lim_{\rho \to \infty} \phi(\rho))^n - a^n \Big),$$

and we know $Vol(X_\infty, g_\infty)$ is unbounded, hence $\lim_{\rho \to \infty} \phi(\rho)$ is not bounded, and we get a contradiction.

Suppose $\nu < 0$, then for large ϕ, $F(\phi)$ is dominated by $\nu \phi^{1-n} e^{\mu\phi} < 0$, and this implies $F(\phi)$ has another zero $b > a$, which contradicts the unboundedness of the volume of (X_∞, g_∞) as before. If $\nu > 0$, F is controlled by the term $\nu \phi^{1-n} e^{\mu\phi} > 0$ when ϕ is large enough, so there is no second zero b of F, and $F > 0$ on $\phi \in (a, \infty)$, $\phi(\rho) \to \infty$ as ρ converges to a maximal value $\rho_0 < \infty$.

For ϕ large enough, we have $\phi' \geq ce^{\mu\phi}$ for some small constant $c = c(\nu) > 0$, integrating over $[\rho, \rho_0)$, we have

$$e^{\mu\phi(\rho)} \leq \frac{1}{c\mu(\rho_0 - \rho)},$$

and hence for ϕ large

$$u''_\infty = \phi' \leq \frac{2\nu}{a^{n-1}} \frac{1}{c\mu(\rho_0 - \rho)},$$

then the integral

$$\int_0^{\rho_0} \sqrt{u''_\infty} d\rho \leq C \int_0^{\rho_0} \frac{1}{\sqrt{\rho_0 - \rho}} d\rho < \infty$$

contradicting the completeness of the metric g_∞ on X_∞. □

Hence from (3.57) we know that the solution ϕ exists for all $\rho \in (-\infty, \infty)$ since the leading term on RHS is the linear ϕ/μ when ϕ is large and this implies X_∞ is the line bundle $\mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$, and from (3.58) we have

$$\frac{a}{\mu} - \frac{\mu - 1}{\mu^{n+1}} \sum_{j=0}^{n-1} \frac{n!}{j!} \mu^j a^{j+1-n} = 0,$$

which must have a positive root $\mu = \mu(n)$ for the given $a = n - 1$ by the intermediate value theorem, and for this root μ, the solution ϕ to (3.57) defines a complete Kähler Ricci soliton, which must be one of the FIK solutions constructed in [36].

Chapter 4

Kähler Ricci flow on projective manifolds

In this chapter we will prove Theorems 1.2.2 and 1.2.3.

As in section 1.2.2, let X be a projective n-dimensional manifold, with the canonical bundle K_X big and nef. We consider the Kähler-Ricci flow

$$\frac{\partial \omega}{\partial t} = -\mathrm{Ric}(\omega) - \omega, \quad \omega(0) = \omega_0, \tag{4.1}$$

where ω_0 is a Kähler metric on X. It's well-known that the equation (4.1) is equivalent to the following complex Monge-Ampere equation

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} = \log \dfrac{(\chi + e^{-t}(\omega_0 - \chi) + i\partial\bar{\partial}\varphi)^n}{\Omega} - \varphi \\ \varphi(0) = 0, \end{cases} \tag{4.2}$$

where Ω is a smooth volume form, $\chi = i\partial\bar{\partial} \log \Omega \in c_1(K_X) = -c_1(X)$, and $\omega(t) = \chi + e^{-t}(\omega_0 - \chi) + i\partial\bar{\partial}\varphi$.

Throughout this chapter, the constants C may be different from lines to lines, but they are all uniform. We also use g as the associated Riemannian metric of a Kähler form ω, for example the metric space $(X, \omega(t))$ means the space $(X, g(t))$.

4.1 Identify the regular sets

Since K_X is big and nef, by Kodaira lemma, there exists an effective divisor $D \subset X$, such that $K_X - \varepsilon D$ is ample, hence there exists a hermitian metric on $[D]$ such that

$$\chi - \varepsilon \mathrm{Ric}(h_D) > 0.$$

Let's recall a few known estimates of the flow, (see [68, 104] or [76, 77] without assuming that K_X is semi-ample)

Lemma 4.1.1. *(i) There is a constant $C > 0$ such that for any $t \geq 0$,*

$$\sup_X \varphi(t) \leq C, \quad \sup_X \dot{\varphi}(t) = \sup_X \frac{\partial}{\partial t}\varphi \leq C.$$

(ii) For any $\delta \in (0,1)$, there is a constant $C_\delta > 0$ such that

$$\varphi \geq \delta \log |\sigma_D|^2_{h_D} - C_\delta,$$

where σ_D is a holomorphic section of the line bundle $[D]$ associated to the divisor D.

(iii) Along the Kähler Ricci flow, there exist constants $C > 0, \lambda > 0$ such that

$$\mathrm{tr}_{\omega_0}\omega(t) \leq C|\sigma_D|^{-2\lambda}_{h_D}$$

(iv) For any compact subset $K \subset X \backslash D$, any $\ell \in \mathbb{Z}^+$, there exists a constant $C_{\ell,K} > 0$ such that

$$\|\varphi\|_{C^\ell(K)} \leq C_{\ell,K}.$$

Hence we can conclude that

$$\omega(t) \xrightarrow{C^\infty_{loc}(X\backslash D)} \omega_\infty,$$

for some smooth Kähler metric ω_∞ on $X\backslash D$. On the other hand, it can be shown that $\dot{\varphi}(t) \to 0$ on any $K \subset X\backslash D$ as $t \to \infty$, hence ω_∞ satisfies the equation

$$\omega^n_\infty = (\chi + i\partial\bar{\partial}\varphi_\infty)^n = e^{\varphi_\infty}\Omega, \text{ on } X\backslash D,$$

and ω_∞ is a Kähler-Einstein metric on $X\backslash D$, i.e.,

$$\mathrm{Ric}(\omega_\infty) = -\omega_\infty.$$

Proposition 4.1.1. *[77] For any holomorphic section $\sigma \in H^0(X, mK_X)$, there is a constant $C = C(\sigma)$ such that for any $t \geq 0$, we have*

$$\sup_X |\sigma|^2_{h_t^m} \leq C, \qquad \sup_X |\nabla_t \sigma|^2_{h_t^m} \leq C$$

where $h_t = \frac{1}{\omega(t)^n}$ is the hermtian metric on K_X induced by the Kähler metric $\omega(t)$ and ∇_t is the covariant derivative with respect to h_t^m.

Letting $t \to \infty$, we have $h_t \to h_\infty = h_{KE} = h_\chi e^{-\varphi_\infty}$ on $X \backslash D$ (here $h_\chi = \frac{1}{\Omega}$), and

$$\sup_{X \backslash D} |\sigma|^2_{h^m_\infty} \leq C, \quad \sup_{X \backslash D} |\nabla_\infty \sigma|^2_{h^m_\infty} \leq C. \tag{4.3}$$

Definition 4.1.1. *We define a set $\mathcal{R}_X \subset X$ to be the points $p \in X$ such that the μ-jets at p are generated by global sections of mK_X for some $m \in \mathbb{Z}^+$, for any $\mu \in \mathbb{N}^n$ with $|\mu| \leq 2$.*

Proposition 4.1.2 ([77]). *$\mathcal{R}_X$ is an open dense set of X and on $\mathcal{R}_X$ we have locally smooth convergence of $\omega(t)$ to ω_∞.*

By the smooth convergence of $\omega(t)$ on $X \backslash D$, we can choose a point $p \in X \backslash D$ and a small $r_0 > 0$ such that (we write the associated Riemannian metric of $\omega(t)$ as $g(t)$)

$$B_{g(t)}(p, r_0) \subset\subset X \backslash D, \qquad Vol_{g(t)}(B_{g(t)}(p, r_0)) \geq v_0, \quad \forall t \geq 0$$

for some $v_0 > 0$. For any sequence $t_i \to \infty$, $(X, g(t_i), p)$ is a sequence of almost Kähler-Einstein manifolds (see the Appendix), in the sense of Tian-Wang ([95]). By the structure theorem in Tian-Wang ([95]), we have

$$(X, g(t_i), p) \xrightarrow{d_{GH}} (X_\infty, d_\infty, p_\infty). \tag{4.4}$$

Moreover, X_∞ has a regular-singular decomposition, $X_\infty = \mathcal{R} \cup \mathcal{S}$; the singular $\mathcal{S}$ is closed and of Hausdroff dimension $\leq 2n-4$; the regular set $\mathcal{R}$ is an open smooth Kähler manifold, and $d_\infty|_\mathcal{R}$ is induced by some smooth Kähler-Einstein metric g'_∞, i.e. on $\mathcal{R}$, $\mathrm{Ric}(g'_\infty) = -g'_\infty$.

We define a subset $\mathcal{S}_X \subset X_\infty$ to be a set consisting of the points $q \in X_\infty$ such that there exist a sequence of points $q_k \in X \backslash \mathcal{R}_X$ such that $q_k \to q$ along the Gromov-Hausdroff convergence.

By a theorem of Rong-Zhang (see Theorem 4.1 in [72]), there exists a surjective map

$$\overline{(\mathcal{R}_X, g_\infty)} \to (X_\infty, d_\infty),$$

where $\overline{(\mathcal{R}_X, g_\infty)}$ denotes the metric completion of the metric space $(\mathcal{R}_X, g_\infty)$, and a homeomorphism $(\mathcal{R}_X, g_\infty) \to (X_\infty \backslash \mathcal{S}_X, d_\infty)$ which is a local isometry.

It's not hard to see that $\mathcal{S}_X$ is closed in X_∞ and any tangent cone at $q \notin \mathcal{S}_X$ is $\mathbb{R}^{2n}$, hence $X_\infty \backslash \mathcal{S}_X \subset \mathcal{R}$, i.e., $\mathcal{S} \subset \mathcal{S}_X$.

Proposition 4.1.3. *We have*

$$\mathcal{S}_X \subset \mathcal{S}$$

hence $\mathcal{S} = \mathcal{S}_X$.

Proof. Suppose not, there exists $q \in \mathcal{S}_X \cap \mathcal{R}$, then there exist $q_k \in (X\backslash \mathcal{R}_X, g(t_k))$ converging to q along the Gromov-Hausdroff convergence (4.4). Since $\mathcal{R}$ is open and tangent cones at points in $\mathcal{R}$ is the Euclidean space $\mathbb{R}^{2n}$, for any small $\delta > 0$, there exists a sufficiently small $r_0 > 0$ such that

$$B_{d_\infty}(q, 3r_0) \subset\subset \mathcal{R}, \qquad Vol_{g'_\infty}(B_{d_\infty}(q, 3r_0)) > (1 - \delta/2)Vol_{g_E}(B(0, 3r_0)),$$

where g_E is the standard Euclidean metric on $\mathbb{R}^{2n}$ and $B(0, 3r_0)$ is the Euclidean ball. Since Ricci curvatures are bounded below, by volume continuity for the Gromov-Hausdorff convergence ([26]) we have for k large enough,

$$Vol_{g(t_k)}(B_{g(t_k)}(q_k, 3r_0)) > (1 - \delta)Vol_{g_E}(B(0, 3r_0)).$$

By assumption that the Ricci curvature is uniformly bounded below along the Kähler Ricci flow, hence Perelman's pseudo-locality ([64, 95]) implies that if δ is small enough, there exists a small but uniform constant $\varepsilon_0 > 0$ such that

$$\sup_{B_{g(t_k)}(q_k, 2r_0)} |Rm(g(t_k + \varepsilon_0))| \le \frac{2}{\varepsilon_0}.$$

Moreover, by Theorem 4.2 in [95],

$$(B_{g(t_k)}(q_k, 2r_0), g(t_k + \varepsilon_0), q_k) \xrightarrow{d_{GH}} (B_{d_\infty}(q, 2r_0), d_\infty, q). \tag{4.5}$$

By Shi's derivative estimate, we have

$$\sup_{B_{g(t_k)}(q_k, 3r_0/2)} |\nabla^l Rm(t_k + \varepsilon_0)| \le C(\varepsilon_0, l),$$

for any $l \in \mathbb{N}$ and some constant $C(\varepsilon_0, l)$. Thus we have smooth convergence of $g(t_k+\varepsilon_0)$ to a Kähler metric $\tilde{g}_\infty$ on $(B_{d_\infty}(q, r_0), J_\infty)$ along the Gromov-Hausdorff convergence (4.5), where J_∞ is the limit complex structure.

Without loss of generality we can assume the injectivity radii of $g(t_k+\varepsilon_0)$ at q_k are bounded below by r_0 ([19]), since the Riemann curvatures and volumes of $B_{g(t_k)}(q_k, r_0)$ are uniformly bounded. For k large enough, there exists (see Proposition 2.4.1) a local holomorphic coordinates system $\{z_\alpha^{(k)}\}_{\alpha=1}^n$ on the ball $(B_{g(t_k)}(q_k, r_0), g(t_k+\varepsilon_0))$ such that $|z^{(k)}|^2 = \sum_{\alpha=1}^n |z_\alpha^{(k)}|^2 \le r_0^2$, $|z^{(k)}|^2(q_k) = 0$ and under these coordinates $g_{\alpha\bar\beta} = g_{t_k+\varepsilon_0}(\nabla z_\alpha^{(k)}, \bar\nabla z_\beta^{(k)})$ satisfies

$$\frac{1}{C}\delta_{\alpha\beta} \le g_{\alpha\bar\beta} \le C\delta_{\alpha\beta}, \quad \|g_{\alpha\bar\beta}\|_{C^{1,\gamma}} \le C, \quad \text{for some } \gamma \in (0,1).$$

This implies that the Euclidean metric under these coordinates

$$\sum_{\alpha=1}^n \sqrt{-1} dz_\alpha^{(k)} \wedge d\bar z_\alpha^{(k)} \tag{4.6}$$

is uniformly equivalent to $g(t_k+\varepsilon_0)$ on the ball $B_{g(t_k)}(q_k, r_0)$.

Recall that along Kähler-Ricci flow

$$\mathrm{Ric}(\omega(t)) = -\chi - i\partial\bar\partial(\varphi + \dot\varphi).$$

Take a cut-off function η on $\mathbb{R}$ such that $\eta(x) = 1$ for $x \in (-\infty, 1/2)$ and vanishes for $x \in [1, \infty)$. Choose a function

$$\Phi_k = (|\mu| + 1 + n)\eta\Big(\frac{|z^{(k)}|^2}{r_0^2/2}\Big)\log|z^{(k)}|^2 + \varphi(t_k+\varepsilon_0) + \dot\varphi(t_k+\varepsilon_0).$$

Note that Φ_k is a globally defined function on X (with a log-pole at q_k) when k is large enough.

Since the metrics (4.6) and $g(t_k+\varepsilon_0)$ are uniformly equivalent for k large enough on the support of $i\partial\bar\partial\Big((n+1+|\mu|)\eta\Big(\frac{|z^{(k)}|^2}{r_0^2/2}\Big)\log|z^{(k)}|^2\Big)$, we see that there is a uniform constant Λ independent of k such that

$$i\partial\bar\partial\Big((n+1+|\mu|)\eta\Big(\frac{|z^{(k)}|^2}{r_0^2/2}\Big)\log|z^{(k)}|^2\Big) \ge -\Lambda\omega(t_k+\varepsilon_0).$$

We will fix an integer $m \ge 10\Lambda$.

Define a (singular) hermitian metric on K_X by

$$h_k = h_\chi e^{-\frac{\varphi(t_k+\varepsilon_0)}{2} - \frac{\epsilon}{m}\log|\sigma_D|^2_{h_D}},$$

for some small $\epsilon > 0$. Then we have for k large enough (we denote below $\omega_k = \omega(t_k+\varepsilon_0)$, and $[D]$ the current of integration over the divisor D.)

$$\begin{aligned}\mathrm{Ric}(h_k^m) + \mathrm{Ric}(\omega_k) + i\partial\bar\partial\Phi_k =& m\chi + \frac{1}{2} i\partial\bar\partial\varphi - \epsilon\mathrm{Ric}(h_D) + \epsilon[D] - \chi \\ &+ i\partial\bar\partial\Big((n+1+|\mu|)\eta\Big(\frac{|z^{(k)}|^2}{r_0^2/2}\Big)\log|z^{(k)}|^2\Big) \\ =& \frac{m}{2}\omega_k + \frac{m\chi}{2} - \epsilon\mathrm{Ric}(h_D) - \frac{m}{2}e^{-t_k-\varepsilon_0}(\omega_0 - \chi) \\ &+ \epsilon[D] + i\partial\bar\partial\Big((n+1+|\mu|)\eta\Big(\frac{|z^{(k)}|^2}{r_0^2/2}\Big)\log|z^{(k)}|^2\Big) \\ \geq& \frac{m}{4}\omega_k,\end{aligned}$$

in the current sense, for k large enough. The above inequality follows since $\frac{m}{2}\chi - \epsilon\mathrm{Ric}(h_D)$ is a fixed Kähler metric, which is greater than $\frac{m}{2}e^{-t_k-\varepsilon_0}(\omega_0 - \chi)$ for k large enough.

Define an mK_X-valued $(0,1)$ form

$$\eta_{k,\mu} = \bar\partial\Big(\eta\Big(\frac{|z^{(k)}|^2}{r_0^2/2}\Big)(z^{(k)})^\mu\Big),$$

where

$$(z^{(k)})^\mu = \prod_{\alpha=1}^n (z_\alpha^{(k)})^{\mu_\alpha}, \quad \mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n.$$

It's not hard to see (noting that the pole order along D is $\leq \epsilon$)

$$\int_X |\eta_{k,\mu}|^2_{h_k^m} e^{-\Phi_k}\omega_k^n < \infty.$$

Then we can apply the Hormander's L^2 estimate (see Theorem 2.1.2 with $L = mK_X$) to solve the following $\bar\partial$-equation

$$\bar\partial u_{k,\mu} = \eta_{k,\mu},$$

with $u_{k,\mu}$ a smooth section of mK_X satisfying

$$\int_X |u_{k,\mu}|_{h_k^m} e^{-\Phi_k}\omega_k^n \leq \frac{4}{m}\int_X |\eta_{k,\mu}|^2_{h_k^m} e^{-\Phi_k}\omega_k^n < \infty.$$

By checking the pole order of $e^{-\Phi_k}$ at q_k we can see that $u_{k,\mu}$ vanishes at q_k up to order $|\mu|$, and hence

$$\sigma_{k,\mu} := u_{k,\mu} - \eta\Big(\frac{|z^{(k)}|^2}{r_0^2/2}\Big)(z^{(k)})^\mu$$

is a nontrivial global holomorphic section of mK_X. Hence we see the global sections of mK_X generates the μ-jets at q_k for k large enough. This gives the contradiction. Hence $\mathcal{S}_X \subset \mathcal{S}$. □

Thus we have a local isometry homeomorphism

$$(\mathcal{R}_X, g_\infty) \to (X_\infty \backslash \mathcal{S}_X, d_\infty) = (\mathcal{R}, d_\infty).$$

Hence we can identify $\mathcal{R}_X$ and $\mathcal{R}$, and $d_\infty|_{\mathcal{R}}$ is induced by the Kähler-Einstein metric $g_\infty|_{\mathcal{R}_X}$.

4.2 Estimates near the singular set

Throughout this section, we fix an effective divisor $D \subset X$ such that

$$K_X - \varepsilon[D] > 0$$

for sufficiently small $\varepsilon > 0$. By the previous section, we see $X \backslash D \subset \mathcal{R}_X$. Choose a log-resolution of (X, D),

$$\pi_1 : Z \to X$$

such that $\pi_1^{-1}(D)$ is a smooth divisor with simple normal crossings. Fix a point O in a smooth component of $\pi_1^{-1}(D)$ and blow up Z at the point O, we get a map

$$\pi_2 : \tilde{X} \to Z,$$

for some smooth projective manifold $\tilde{X}$. Denote $\pi = \pi_1 \circ \pi_2 : \tilde{X} \to X$.

By Adjunction formula, we have

$$K_{\tilde{X}} = \pi^* K_X + (n-1)E + F, \qquad F = \sum_k a_k F_k,$$

where E is the exceptional locus of the blow up π_2, and F_k is a prime divisor in the exceptional locus of π. We also note that $a_k > 0$ for any k.

Since $\tilde{\chi} = \pi^* \chi \in \pi^* K_X$ is big and nef, Kodaira's lemma implies there exists an effective divisor $\tilde{D}$ whose support coincide with the exceptional locus E, F and

$$\tilde{\chi} - \varepsilon[\tilde{D}] \text{ is Kähler},$$

hence there exists a hermitian metric $h_{\tilde{D}}$ on the line bundle associated to D such that

$$\tilde{\chi} - \varepsilon \mathrm{Ric}(h_{\tilde{D}}) > 0.$$

We write $\tilde{D} = \tilde{D}' + \tilde{D}''$, where supp$D'' = E$, and $E \not\subset \tilde{D}'$. Let σ_E, σ_F, $\sigma_{\tilde{D}}$ be the defining section of E, F and $\tilde{D}$, respectively. Here these sections are multi-valued holomorphic sections which become global after taking some power. There also exist hermtian metrics h_E, h_F, and $h_{\tilde{D}}$ such that

$$\pi^*\Omega = |\sigma_E|_{h_E}^{2(n-1)} |\sigma_F|_{h_F}^2 \tilde{\Omega},$$

for some smooth volume form $\tilde{\Omega}$ on $\tilde{X}$.

We fix a Kähler metric $\tilde{\omega}$ on $\tilde{X}$. The Kähler Ricci flow on X is pulled back to $\tilde{X}$ by the map π, and it safeties the equation

$$\frac{\partial}{\partial t}\pi^*\varphi = \log \frac{(\tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + i\partial\bar{\partial}\pi^*\varphi)^n}{|\sigma_E|_{h_E}^{2(n-1)} |\sigma_F|_{h_F}^2 \tilde{\Omega}} - \pi^*\varphi, \tag{4.7}$$

with the initial $\pi^*\varphi(0) = 0$. By the previous estimates, we see that $\pi^*\varphi$ satisfies the estimates

$$\delta \log |\sigma_{\tilde{D}}|_{h_{\tilde{D}}}^2 - C_\delta \le \pi^*\varphi(t) \le C, \forall t \ge 0 \text{ and } \forall \delta \in (0,1).$$

We will consider a family of perturbed parabolic Monge-Ampere equations for $\epsilon \in (0,1)$

$$\begin{cases} \dfrac{\partial}{\partial t}\tilde{\varphi}_\epsilon = \log \dfrac{(\tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} + i\partial\bar{\partial}\pi^*\varphi)^n}{(|\sigma_E|_{h_E}^{2(n-1)} + \epsilon)(|\sigma_F|_{h_F}^2 + \epsilon)\tilde{\Omega}} - \tilde{\varphi}_\epsilon, \\ \tilde{\varphi}_\epsilon(0) = 0 \end{cases} \tag{4.8}$$

where $\tilde{\varphi}_\epsilon(t) \in PSH(\tilde{X}, \tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega})$. The equation (4.8) has long time existence [98], and we will show that solutions to (4.8) converge to that of (4.7) in some sense.

It's easy to check that the Kähler metrics $\tilde{\omega}_\epsilon = \tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} + i\partial\bar{\partial}\tilde{\varphi}_\epsilon$ satisfies the following evolution equation

$$\frac{\partial}{\partial t}\tilde{\omega}_\epsilon = -\mathrm{Ric}(\tilde{\omega}_\epsilon) - \tilde{\omega}_\epsilon + \tilde{\chi} + \epsilon\tilde{\omega} - i\partial\bar{\partial}\log\Big((|\sigma_E|_{h_E}^{2(n-1)} + \epsilon)(|\sigma_F|_{h_F}^2 + \epsilon)\tilde{\Omega}\Big). \tag{4.9}$$

By direct calculations, for any smooth nonnegative function f, we have

$$i\partial\bar{\partial}\log(\epsilon + f) = \frac{i\partial\bar{\partial}f}{\epsilon + f} - \frac{\partial f \wedge \bar{\partial} f}{(f+\epsilon)^2}$$

$$= \frac{f}{f+\epsilon} i\partial\bar{\partial}\log f + \frac{\epsilon}{f(f+\epsilon)^2}\partial f \wedge \bar{\partial} f$$
$$\geq \frac{f}{f+\epsilon} i\partial\bar{\partial}\log f,$$

in the smooth sense on $\tilde{X}\backslash\{f=0\}$ and globally as currents. So

$$i\partial\bar{\partial}\log\Big((|\sigma_E|_{h_E}^{2(n-1)}+\epsilon)(|\sigma_F|_{h_F}^2+\epsilon)\tilde{\Omega}\Big)$$
$$\geq -\frac{(n-1)|\sigma_E|_{h_E}^{2(n-1)}}{|\sigma_E|_{h_E}^{2(n-1)}+\epsilon}\mathrm{Ric}(h_E) - \frac{|\sigma_F|_{h_F}^2}{|\sigma_F|_{h_F}^2+\epsilon}\mathrm{Ric}(h_F) - \mathrm{Ric}(\tilde{\Omega})$$
$$\geq -C\tilde{\omega},$$

for some uniform constant C independent of ϵ. Thus away from $\mathrm{supp}\tilde{D} = \mathrm{supp}E \cup \mathrm{supp}F$, we have

$$\frac{\partial}{\partial t}\tilde{\omega}_\epsilon \leq -\mathrm{Ric}(\tilde{\omega}_\epsilon) - \tilde{\omega}_\epsilon + C\tilde{\omega}. \tag{4.10}$$

Lemma 4.2.1. *Let $\tilde{\varphi}_\epsilon$ be the solution to (4.8), then there exists a constant $C>0$ such that for any $t\geq 0$, $\epsilon\in(0,1)$, we have*

$$\sup_{\tilde{X}}\tilde{\varphi}_\epsilon(t,\cdot)\leq C, \qquad \sup_{\tilde{X}}\frac{\partial\tilde{\varphi}_\epsilon}{\partial t}(t,\cdot)\leq C.$$

Proof. Let

$$V_\epsilon = \int_{\tilde{X}}(|\sigma_E|_{h_E}^{2(n-1)}+\epsilon)(|\sigma_F|_{h_F}^2+\epsilon)\tilde{\Omega}$$

be the volume with respect to the volume form $(|\sigma_E|_{h_E}^{2(n-1)}+\epsilon)(|\sigma_F|_{h_F}^2+\epsilon)\tilde{\Omega}$. We see that

$$V_1 \geq V_\epsilon \geq V_0 = \int_X \Omega,$$

hence V_ϵ is uniformly bounded. We consider (for simplicity we denote $\tilde{\Omega}_\epsilon = (|\sigma_E|_{h_E}^{2(n-1)}+\epsilon)(|\sigma_F|_{h_F}^2+\epsilon)\tilde{\Omega}$)

$$\frac{\partial}{\partial t}\Big(\frac{1}{V_\epsilon}\int_{\tilde{X}}\tilde{\varphi}_\epsilon\tilde{\Omega}_\epsilon\Big)$$
$$=\frac{1}{V_\epsilon}\int_{\tilde{X}}\log\frac{(\tilde{\chi}+e^{-t}(\pi^*\omega_0-\tilde{\chi})+\epsilon\tilde{\omega}+i\partial\bar{\partial}\tilde{\varphi}_\epsilon)^n}{(|\sigma_E|_{h_E}^{2(n-1)}+\epsilon)(|\sigma_F|_{h_F}^2+\epsilon)\tilde{\Omega}}\tilde{\Omega}_\epsilon - \frac{1}{V_\epsilon}\int_{\tilde{X}}\tilde{\varphi}_\epsilon\tilde{\Omega}_\epsilon$$
$$\leq\log\Big(\int_{\tilde{X}}(\tilde{\chi}+e^{-t}(\pi^*\omega_0-\tilde{\chi})+\epsilon\tilde{\omega}+i\partial\bar{\partial}\tilde{\varphi}_\epsilon)^n\Big) - \frac{1}{V_\epsilon}\int_{\tilde{X}}\tilde{\varphi}_\epsilon\tilde{\Omega}_\epsilon$$
$$\leq C - \frac{1}{V_\epsilon}\int_{\tilde{X}}\tilde{\varphi}_\epsilon\tilde{\Omega}_\epsilon,$$

where for the first inequality we use Jensen's inequality. From the above we see that

$$\frac{1}{V_\epsilon}\int_{\tilde{X}}\tilde{\varphi}_\epsilon\tilde{\Omega}_\epsilon \le C.$$

Since $\tilde{\varphi}_\epsilon \in PSH(\tilde{X}, \tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega})$, the mean value inequality implies the uniform upper bound of $\tilde{\varphi}_\epsilon$

$$\sup_{\tilde{X}} \tilde{\varphi}_\epsilon(t) \le C.$$

Direct calculations show that (we will denote $\dot{\tilde{\varphi}}_\epsilon = \frac{\partial}{\partial t}\tilde{\varphi}_\epsilon$)

$$\begin{aligned}\frac{\partial}{\partial t}\dot{\tilde{\varphi}}_\epsilon &= \Delta_{\tilde{\omega}_\epsilon}\dot{\tilde{\varphi}}_\epsilon - tr_{\tilde{\omega}_\epsilon}e^{-t}(\pi^*\omega_0 - \tilde{\chi}) - \dot{\tilde{\varphi}}_\epsilon \\ &= \Delta_{\tilde{\omega}_\epsilon}\dot{\tilde{\varphi}}_\epsilon - n + tr_{\tilde{\omega}_\epsilon}\tilde{\chi} + \epsilon tr_{\tilde{\omega}_\epsilon}\tilde{\omega} + \Delta_{\tilde{\omega}_\epsilon}\tilde{\varphi}_\epsilon - \dot{\tilde{\varphi}}_\epsilon.\end{aligned}$$

Hence

$$(\frac{\partial}{\partial t} - \Delta_{\tilde{\omega}_\epsilon})\Big((e^t - 1)\dot{\tilde{\varphi}}_\epsilon - \tilde{\varphi}_\epsilon\Big) = -tr_{\tilde{\omega}_\epsilon}\pi^*\omega_0 + n - \epsilon tr_{\tilde{\omega}_\epsilon}\tilde{\omega} \le n, \tag{4.11}$$

then maximum principle implies

$$\dot{\tilde{\varphi}}_\epsilon(t) \le \frac{\tilde{\varphi}_\epsilon + nt}{e^t - 1} \le C, \quad \forall t > 0. \tag{4.12}$$

□

Lemma 4.2.2. *For any $\delta \in (0, 1)$, there is a constant $C = C_\delta$ such that for any $t \ge 0$, we have*

$$\tilde{\varphi}_\epsilon(t) \ge \delta \log|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - C_\delta.$$

Proof. We will apply the maximum principle. For any small $\delta > 0$ such that

$$\tilde{\chi} - \delta\mathrm{Ric}(h_{\tilde{D}}) > 0,$$

where we may also assume $|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} \le 1$ by rescaling the metric $h_{\tilde{D}}$. We consider the function $H := \tilde{\varphi}_\epsilon - \delta\log|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}}$. On $\tilde{X}\backslash\tilde{D}$, H satisfies the equation

$$\frac{\partial H}{\partial t} = \log\frac{(\tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} - \delta\mathrm{Ric}(h_{\tilde{D}}) + i\partial\bar{\partial}H)^n}{(|\sigma_E|^{2(n-1)}{}_{h_E} + \epsilon)(|\sigma_F|^2_{h_F} + \epsilon)\tilde{\Omega}} - H - \delta\log|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}}. \tag{4.13}$$

By choosing δ even smaller, we may obtain $\omega'_\epsilon := \tilde{\chi} - \delta\mathrm{Ric}(h_{\tilde{D}}) + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega}$ is a Kähler metric on $\tilde{X}$ for all $t \ge 0$, and these metrics are uniformly equivalent to $\tilde{\omega}$, i.e., there exists $C_0 > 0$ such that

$$C_0^{-1}\tilde{\omega}_\epsilon \le \omega'_\epsilon \le C_0\tilde{\omega}.$$

Consider the Monge-Ampere equations

$$(\tilde{\chi}-\delta\mathrm{Ric}(h_{\tilde{D}})+e^{-t}(\pi^*\omega_0-\tilde{\chi})+\epsilon\tilde{\omega}+i\partial\bar{\partial}\psi_\epsilon)^n = e^{\psi_\epsilon}(|\sigma_E|^{2(n-1)}_{h_E}+\epsilon)(|\sigma_F|^2_{h_F}+\epsilon)\tilde{\Omega}, \quad (4.14)$$

where $\psi_\epsilon \in PSH(\tilde{X},\omega'_\epsilon)$. By the Aubin-Yau theorem, (4.14) admits a unique smooth solution ψ_ϵ for any $\epsilon \in (0,1)$ and $t \geq 0$. It can be seen that

$$\frac{1}{V_\epsilon}\int_{\tilde{X}} e^{\psi_\epsilon}\tilde{\Omega}_\epsilon \leq \frac{1}{V_\epsilon}\int_{\tilde{X}}(\omega'_\epsilon)^n \leq C.$$

Hence mean value inequality implies $\sup_{\tilde{X}}\psi_\epsilon \leq C$. Then by [35], we have $\inf_{\tilde{X}}\psi_\epsilon \geq -C$, hence

$$\|\psi_\epsilon\|_{L^\infty} \leq C,$$

for some C independent of $\epsilon \in (0,1)$ and $t \geq 0$.

Denote $\omega'_\epsilon(t) = \omega'_\epsilon + i\partial\bar{\partial}\psi_\epsilon$. Taking derivative with respective to t on both sides of (4.14), we get

$$\Delta_{\omega'_\epsilon(t)}\dot{\psi}_\epsilon - tr_{\omega'_\epsilon(t)}e^{-t}(\pi^*\omega_0-\tilde{\chi}) = \dot{\psi}_\epsilon. \quad (4.15)$$

$$\Delta_{\omega'_\epsilon(t)}\psi_\epsilon = n - tr_{\omega'_\epsilon(t)}\Big(\tilde{\chi}-\delta\mathrm{Ric}(h_{\tilde{D}})+e^{-t}(\pi^*\omega_0-\tilde{\chi})+\epsilon\tilde{\omega}\Big),$$

Hence

$$\Delta_{\omega'_\epsilon(t)}(\dot{\psi}_\epsilon-\psi_\epsilon) = \dot{\psi}_\epsilon - n + tr_{\omega'_\epsilon(t)}\Big(\tilde{\chi}-\delta\mathrm{Ric}(h_{\tilde{D}})+2e^{-t}(\pi^*\omega_0-\tilde{\chi})+\epsilon\tilde{\omega}\Big)$$

Noting that $\tilde{\chi}-\delta\mathrm{Ric}(h_{\tilde{D}})+2e^{-t}(\pi^*\omega_0-\tilde{\chi})+\epsilon\tilde{\omega} > 0$ is δ is chosen appropriately, hence at the maximum point of $\dot{\psi}_\epsilon-\psi_\epsilon$, we have $\dot{\psi}_\epsilon \leq n$, thus

$$\dot{\psi}_\epsilon \leq C+n \leq C.$$

Let $G = H-\psi_\epsilon = \tilde{\varphi}_\epsilon - \delta\log|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - \psi_\epsilon$. On $\tilde{X}\backslash\tilde{D}$ it satisfies the equation

$$\frac{\partial G}{\partial t} = \log\frac{(\tilde{\chi}-\delta\mathrm{Ric}(h_{\tilde{D}})+e^{-t}(\pi^*\omega_0-\tilde{\chi})+\epsilon\tilde{\omega}+i\partial\bar{\partial}\psi_\epsilon+i\partial\bar{\partial}G)^n}{(\tilde{\chi}-\delta\mathrm{Ric}(h_{\tilde{D}})+e^{-t}(\pi^*\omega_0-\tilde{\chi})+\epsilon\tilde{\omega}+i\partial\bar{\partial}\psi_\epsilon)^n} - G - \delta\log|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - \dot{\psi}_\epsilon. \quad (4.16)$$

The minimum of G cannot be at $\tilde{D}$ for any $t \geq 0$, since it tends to $+\infty$ when approaching $\tilde{D}$. For any $T > 0$, suppose $(p_0,t_0) \in \tilde{X}\backslash\tilde{D}\times[0,T]$ is the minimum point of G, then we have at this point $\frac{\partial G}{\partial t} \leq 0$ and $i\partial\bar{\partial}G \geq 0$, hence maximum principle implies at this point

$$G \geq -\delta \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - \dot{\psi}_\epsilon \geq -C_\delta,$$

combining with L^∞ bound of ψ_ϵ, we have

$$\tilde{\varphi}_\epsilon \geq \delta \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - C_\delta.$$

□

Lemma 4.2.3. *There exist two constants $C > 0, \lambda > 0$ such that for all $t \geq 0$,*

$$\tilde{\omega}_\epsilon \leq C|\sigma_{\tilde{D}}|^{-2\lambda}_{h_{\tilde{D}}}\tilde{\omega}. \tag{4.17}$$

Proof. By the classical C^2 estimate for Monge-Ampere equations, there is a constant C_1 depending on the lower bound of bisectional curvature of $\tilde{\omega}$, such that

$$\Delta_{\tilde{\omega}_\epsilon} \log tr_{\tilde{\omega}}\tilde{\omega}_\epsilon \geq -C_1 tr_{\tilde{\omega}_\epsilon}\tilde{\omega} - C_1 - \frac{tr_{\tilde{\omega}}\mathrm{Ric}(\tilde{\omega}_\epsilon)}{tr_{\tilde{\omega}}\tilde{\omega}_\epsilon}. \tag{4.18}$$

By the inequality (4.10), we have on $\tilde{X}\backslash\tilde{D}$

$$\frac{\partial}{\partial t}\log tr_{\tilde{\omega}}\tilde{\omega}_\epsilon = \frac{tr_{\tilde{\omega}}\frac{\partial}{\partial t}\tilde{\omega}_\epsilon}{tr_{\tilde{\omega}}\tilde{\omega}_\epsilon} \leq \frac{-tr_{\tilde{\omega}}\mathrm{Ric}(\tilde{\omega}_\epsilon)}{tr_{\tilde{\omega}}\tilde{\omega}_\epsilon} - 1 + \frac{C}{tr_{\tilde{\omega}}\tilde{\omega}_\epsilon}. \tag{4.19}$$

So we have on $\tilde{X}\backslash\tilde{D}$,

$$\begin{aligned}
&(\frac{\partial}{\partial t} - \Delta_{\tilde{\omega}_\epsilon})\Big(\log tr_{\tilde{\omega}}\tilde{\omega}_\epsilon - A\tilde{\varphi}_\epsilon + A\delta\log|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}}\Big)\\
\leq&\frac{C}{tr_{\tilde{\omega}}\tilde{\omega}_\epsilon} + Ctr_{\tilde{\omega}_\epsilon}\tilde{\omega} - A\log\frac{\tilde{\omega}^n_\epsilon}{\tilde{\Omega}_\epsilon} + A\tilde{\varphi}_\epsilon + C\\
&\quad - Atr_{\tilde{\omega}_\epsilon}(\tilde{\chi} - \delta\mathrm{Ric}(h_{\tilde{D}}) + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega})\\
\leq&\frac{C}{tr_{\tilde{\omega}}\tilde{\omega}_\epsilon} - 2tr_{\tilde{\omega}_\epsilon}\tilde{\omega} - A\log\frac{\tilde{\omega}^n_\epsilon}{\tilde{\omega}^n} - A\log\frac{\tilde{\omega}^n}{\tilde{\Omega}_\epsilon} + C\\
\leq&\frac{C}{tr_{\tilde{\omega}}\tilde{\omega}_\epsilon} - tr_{\tilde{\omega}_\epsilon}\tilde{\omega} + C,
\end{aligned}$$

if we choose A sufficiently large and δ suitably small, and in the last inequality we use the facts that

$$\log\frac{\tilde{\omega}^n}{\tilde{\Omega}_\epsilon} \leq C,$$

and

$$-tr_{\tilde{\omega}_\epsilon}\tilde{\omega} - A\log\frac{\tilde{\omega}^n_\epsilon}{\tilde{\omega}^n} \leq -tr_{\tilde{\omega}_\epsilon}\tilde{\omega} + An\log tr_{\tilde{\omega}_\epsilon}\tilde{\omega} \leq C,$$

since the function $x(\in \mathbb{R}^+) \mapsto -x + An \log x$ is bounded above.

Using the inequality

$$tr_{\tilde{\omega}}\tilde{\omega}_\epsilon \leq (tr_{\tilde{\omega}_\epsilon}\tilde{\omega})^{n-1}\frac{\tilde{\omega}^n}{\tilde{\omega}_\epsilon^n} \leq (tr_{\tilde{\omega}_\epsilon}\tilde{\omega})^{n-1}e^{\tilde{\varphi}_\epsilon+\dot{\tilde{\varphi}}_\epsilon}\frac{\tilde{\Omega}_\epsilon}{\tilde{\omega}^n} \leq C(tr_{\tilde{\omega}_\epsilon}\tilde{\omega})^{n-1}.$$

The maximum of $\log tr_{\tilde{\omega}}\tilde{\omega}_\epsilon - A\tilde{\varphi}_\epsilon + A\delta \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}}$ cannot be at $\tilde{D}$, then maximum principle implies that at the maximum of $\log tr_{\tilde{\omega}}\tilde{\omega}_\epsilon - A\tilde{\varphi}_\epsilon + A\delta \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}}$, we have

$$0 \leq \frac{C}{tr_{\tilde{\omega}}\tilde{\omega}_\epsilon} - C(tr_{\tilde{\omega}}\tilde{\omega}_\epsilon)^{\frac{1}{n-1}} + C,$$

that is, at the maximum point,

$$tr_{\tilde{\omega}}\tilde{\omega}_\epsilon \leq C.$$

Thus we get the desired estimate. □

By standard Schauder estimate ([68]) for parabolic equations, we have

Lemma 4.2.4. *For any integer $\ell \in \mathbb{Z}_+$, on any compact subset $K \subset\subset \tilde{X}\backslash\tilde{D}$, there is a constant $C_{\ell,K}$ such that for any $t \geq 0$*

$$\|\tilde{\varphi}_\epsilon\|_{C^\ell(K)} \leq C_{\ell,K}.$$

From Lemma 4.2.4, we see that $\tilde{\varphi}_\epsilon(t)$ converge to a smooth function φ_∞ on $\tilde{X}\backslash\tilde{D}$ as $t \to \infty$ and $\epsilon \to 0$, which satisfies the estimates

$$\delta \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - C_\delta \leq \varphi_\infty \leq C, \text{ on } \tilde{X}\backslash\tilde{D}, \text{ for any } \delta \in (0,1)$$

$$\tilde{\omega}_\infty := \tilde{\chi} + i\partial\bar{\partial}\varphi_\infty \leq C|\sigma_{\tilde{D}}|^{-2\lambda}_{h_{\tilde{D}}}.$$

Moreover, from (4.12) we see that

$$\frac{\partial}{\partial t}(\tilde{\varphi}_\epsilon + Ce^{-t/2}) \leq 0.$$

Hence on any compact subset $K \subset \tilde{X}\backslash\tilde{D}$, the function $\tilde{\varphi}_\epsilon(t) + Ce^{-t/2}$ decreases to a function $\varphi_{\infty,\epsilon}$ as $t \to \infty$. Hence $\dot{\tilde{\varphi}}_\epsilon(t)|_K$ approaches zero as $t \to \infty$ and $\epsilon \to 0$. Thus the metric $\tilde{\omega}_\infty$ satisfies the equation

$$\tilde{\omega}_\infty^n = (\tilde{\chi} + i\partial\bar{\partial}\varphi_\infty)^n = e^{\varphi_\infty}|\sigma_E|^{2(n-1)}_{h_E}|\sigma_F|^2_{h_F}\tilde{\Omega}, \text{ on } \tilde{X}\backslash\tilde{D}.$$

Let $\epsilon \to 0$, $\tilde{\varphi}_\epsilon(t)$ tends to a function $\tilde{\varphi}_0(t) \in PSH(\tilde{X}, \tilde{\chi}+e^{-t}(\pi^*\omega_0-\tilde{\chi}))\cap C^\infty(\tilde{X}\backslash\tilde{D})$ in $C^\infty_{loc}(\tilde{X}\backslash\tilde{D}\times[0,\infty))$-topology, which satisfies the degenerate parabolic Monge-Ampere equation

$$\begin{cases} \dfrac{\partial\tilde{\varphi}_0}{\partial t} = \log \dfrac{(\tilde{\chi}+e^{-t}(\pi^*\omega_0-\tilde{\chi})+i\partial\bar{\partial}\tilde{\varphi}_0)^n}{|\sigma_E|^{2(n-1)}_{h_E}|\sigma_F|^2_{h_F}\tilde{\Omega}} - \tilde{\varphi}_0, \\ \tilde{\varphi}_0(0) = 0 \end{cases} \tag{4.20}$$

with the estimates

$$\delta\log|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - C_\delta \le \tilde{\varphi}_0 \le C, \text{ on } \tilde{X}\backslash\tilde{D}, \forall\delta\in(0,1), \tag{4.21}$$

$$\tilde{\omega}_0(t) := \tilde{\chi}+e^{-t}(\pi^*\omega_0-\tilde{\chi})+i\partial\bar{\partial}\tilde{\varphi}_0 \le C|\sigma_{\tilde{D}}|^{-2\lambda}_{h_{\tilde{D}}}, \text{ on } \tilde{X}\backslash\tilde{D}. \tag{4.22}$$

When the solutions $\tilde{\varphi}_0$ to (4.20) are in $PSH(\tilde{X}, \tilde{\chi}+e^{-t}(\pi^*\omega_0-\tilde{\chi}))\cap L^\infty(\tilde{X})$, the uniqueness of such solutions has been proved in [80]. In the following, we will adapt their method to prove the uniqueness when solutions satisfy (4.21), instead of global L^∞-bound.

Proposition 4.2.1. *Let $\varphi' \in PSH(\tilde{X}, \tilde{\chi}+e^{-t}(\pi^*\omega_0-\tilde{\chi}))\cap C^\infty(\tilde{X}\backslash\tilde{D}\times[0,\infty))$ be a solution to the equation* (4.20) *with the estimate* (4.21)*, then*

$$\varphi' = \tilde{\varphi}_0.$$

Proof. We consider the following perturbed equation

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi_{\epsilon,\gamma} = \log \dfrac{(\tilde{\chi}+e^{-t}(\pi^*\omega_0-\tilde{\chi})+\epsilon\tilde{\omega}+i\partial\bar{\partial}\varphi_{\epsilon,\gamma})^n}{(|\sigma_E|^{2(n-1)}_{h_E}+\gamma)(|\sigma_F|^2_{h_F}+\gamma)\tilde{\Omega}} - \varphi_{\epsilon,\gamma} \\ \varphi_{\epsilon,\gamma}(0) = 0, \end{cases}$$

for any $\epsilon\in(0,1)$, $\gamma\in(0,1)$. By similar arguments as in Lemmas 4.2.1, 4.2.2, 4.2.3, we can get the following estimates for $\varphi_{\epsilon,\gamma}$,

$$\sup_{\tilde{X}}\varphi_{\epsilon,\gamma} \le C, \quad \sup_{\tilde{X}}\dot{\varphi}_{\epsilon,\gamma} \le C. \tag{4.23}$$

For any $\delta\in(0,1)$, there is a constant C_δ such that

$$\varphi_{\epsilon,\gamma} \ge \delta\log|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - C_\delta. \tag{4.24}$$

$$\omega_{\epsilon,\gamma}(t) := \tilde{\chi}+e^{-t}(\pi^*\omega_0-\tilde{\chi})+\epsilon\tilde{\omega}+i\partial\bar{\partial}\varphi_{\epsilon,\gamma} \le C|\sigma_{\tilde{D}}|^{-2\lambda}_{h_{\tilde{D}}}\tilde{\omega}. \tag{4.25}$$

$$\|\varphi_{\epsilon,\gamma}\|_{C^\ell(K)} \leq C_{\ell,K}, \text{ for any compact } K \subset\subset \tilde{X}\backslash\tilde{D}. \tag{4.26}$$

Moreover, by maximum principle, we have the following monotonicity properties

$$\varphi_{\epsilon,\gamma_1} \geq \varphi_{\epsilon,\gamma_2}, \text{ for any } \gamma_1 \leq \gamma_2, \forall \epsilon \in (0,1);$$

$$\varphi_{\epsilon_1,\gamma} \leq \varphi_{\epsilon_2,\gamma}, \text{ for any } \epsilon_1 \leq \epsilon_2, \forall \gamma \in (0,1).$$

We can define a function

$$\varphi_\epsilon := (\lim_{\gamma\to 0} \varphi_{\epsilon,\gamma})^*,$$

where $f^*(z) = \lim_{r\to 0} \sup_{w\in B(z,r)\backslash\{z\}} f(w)$ is the upper regularization of a function. Then φ_ϵ satisfies the equation

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi_\epsilon = \log \dfrac{(\tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} + i\partial\bar{\partial}\varphi_\epsilon)^n}{|\sigma_E|^{2(n-1)}_{h_E}|\sigma_F|^2_{h_F}\tilde{\Omega}} - \varphi_\epsilon, \text{ on } \tilde{X}\backslash\tilde{D} \\ \varphi_\epsilon(0) = 0, \end{cases}$$

And we have the monotonicity

$$\varphi_{\epsilon_1} \leq \varphi_{\epsilon_2}, \text{ for any } \epsilon_1 \leq \epsilon_2, \text{ on } \tilde{X}\backslash\tilde{D}.$$

So we can define $\varphi_0 := \lim_{\epsilon\to 0}\varphi_\epsilon$, which satisfies the equation

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi_0 = \log \dfrac{(\tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + i\partial\bar{\partial}\varphi_0)^n}{|\sigma_E|^{2(n-1)}_{h_E}|\sigma_F|^2_{h_F}\tilde{\Omega}} - \varphi_0, \text{ on } \tilde{X}\backslash\tilde{D} \\ \varphi_0(0) = 0, \end{cases} \tag{4.27}$$

The estimates (4.23), (4.24), (4.25) and (4.26) implies that

$$\varphi_\epsilon \xrightarrow{C^\infty(K)} \varphi_0, \text{ as } \epsilon \to 0,$$

for any compact $K \subset\subset \tilde{X}\backslash\tilde{D}$, and φ_0 satisfies similar estimates as in (4.23), (4.24), (4.25) and (4.26).

For any φ' as in Proposition 4.2.1, define a function

$$\psi := \varphi_\epsilon - \varphi' - \varepsilon_0\epsilon\log|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}}$$

on $\tilde{X}\backslash\tilde{D}$, where ε_0 is a small number such that $\tilde{\omega} - \varepsilon_0\mathrm{Ric}(h_{\tilde{D}}) > 0$. For any ϵ,

$$\psi \geq \delta\log|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - C_\delta - C - \varepsilon_0\epsilon\log|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} \to +\infty,$$

as the point approaching $\tilde{D}$, if δ is small enough, say, $\delta \leq \varepsilon_0\epsilon/2$. Hence the minimum of $\psi(\cdot, t)$ can only be at $\tilde{X}\backslash\tilde{D}$. And on $\tilde{X}\backslash\tilde{D}$, ψ satisfies the equation

$$\frac{\partial \psi}{\partial t} = \log \frac{(\tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + i\partial\bar{\partial}\varphi' + \epsilon(\tilde{\omega} - \omega_0 \mathrm{Ric}(h_{\tilde{D}})) + i\partial\bar{\partial}\psi)^n}{(\tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + i\partial\bar{\partial}\varphi')^n} - \psi - \varepsilon_0\epsilon \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}}.$$

Maximum principle argument implies that $\psi_{\min} = \inf_{\tilde{X}\backslash\tilde{D}} \psi(\cdot, t) \geq 0$. (Recall we assume $|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} \leq 1$.) Hence

$$\varphi_\epsilon \geq \varphi' + \varepsilon_0\epsilon \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}}, \text{ on } \tilde{X}\backslash\tilde{D}.$$

On any compact $K \subset\subset \tilde{X}\backslash\tilde{D}$, letting $\epsilon \to 0$, we get

$$\varphi_0 \geq \varphi', \text{ on } K,$$

then let $K \to \tilde{X}\backslash\tilde{D}$, we see that

$$\varphi_0 \geq \varphi'. \tag{4.28}$$

To show the uniqueness, we only need to show $\varphi_0 \leq \varphi'$, and this will be done by another perturbed equation, as Song-Tian do in [80].

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi^{(r)}_{\epsilon,\gamma} = \log \dfrac{(\tilde{\chi} + e^{-t}((1-r)\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} + i\partial\bar{\partial}\varphi_{\epsilon,\gamma})^n}{(|\sigma_E|^{2(n-1)}_{h_E} + \gamma)(|\sigma_F|^2_{h_F} + \gamma)\tilde{\Omega}} - \varphi^{(r)}_{\epsilon,\gamma} \\ \varphi^{(r)}_{\epsilon,\gamma}(0) = 0, \end{cases}$$

It's not hard to see that $\varphi^{(r)}_{\epsilon,\gamma} \to \varphi_{\epsilon,\gamma}$ as $r \to 0$. Denote $\hat{\omega} = \tilde{\chi} + e^{-t}((1-r)\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} + i\partial\bar{\partial}\varphi_{\epsilon,\gamma}$.

Lemma 4.2.5. *For some constant $C > 0$, we have*

$$\sup_{\tilde{X}} \varphi^{(r)}_{\epsilon,\gamma} \leq C, \quad C \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - C \leq \frac{\partial}{\partial r}\varphi^{(r)}_{\epsilon,\gamma} \leq 0$$

Proof. The upper bound of $\varphi^{(r)}_{\epsilon,\gamma}$ follows similarly as the proof in Lemma 4.2.1.

$$\frac{\partial}{\partial t}\Big(\frac{\partial \varphi^{(r)}_{\epsilon,\gamma}}{\partial r}\Big) = \Delta_{\hat{\omega}}\frac{\partial \varphi^{(r)}_{\epsilon,\gamma}}{\partial r} - e^{-t} tr_{\hat{\omega}}\pi^*\omega_0 - \frac{\partial \varphi^{(r)}_{\epsilon,\gamma}}{\partial r} \leq \Delta_{\hat{\omega}}\frac{\partial \varphi^{(r)}_{\epsilon,\gamma}}{\partial r} - \frac{\partial \varphi^{(r)}_{\epsilon,\gamma}}{\partial r}.$$

Maximum principle argument implies $\frac{\partial \varphi^{(r)}_{\epsilon,\gamma}}{\partial r} \leq 0$.

Let $H := \frac{\partial \varphi^{(r)}_{\epsilon,\gamma}}{\partial r} + A\varphi^{(r)}_{\epsilon,\gamma} - A\varepsilon_0 \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}}$, where $\varepsilon_0 > 0$ is a small number such that

$$\tilde{\chi} + e^{-t}((1-r)\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} - \varepsilon_0 \mathrm{Ric}(h_{\tilde{D}}) \geq c_0\tilde{\omega},$$

for all $t \geq 0$ and $c_0 > 0$ is a uniform constant.

On $\tilde{X} \backslash \tilde{D}$, if we choose A sufficiently large, we have

$$\begin{aligned}
(\frac{\partial}{\partial t} - \Delta_{\hat{\omega}})H = & - e^{-t} tr_{\hat{\omega}} \pi^* \omega_0 - \frac{\partial \varphi_{\epsilon,\gamma}^{(r)}}{\partial r} + A \log \frac{\hat{\omega}^n}{\tilde{\Omega}_\gamma} - A\varphi_{\epsilon,\gamma}^{(r)} - An \\
& + Atr_{\hat{\omega}}(\tilde{\chi} + e^{-t}((1-r)\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} - \varepsilon_0 \mathrm{Ric}(h_{\tilde{D}})) \\
\geq & - H - A\varepsilon_0 \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - C \\
\geq & - H - C.
\end{aligned}$$

Since the minimum of H cannot occur at $\tilde{D}$, maximum principle argument implies that $H \geq -C$, combing with the uniform upper bound of $\varphi_{\epsilon,\gamma}^{(r)}$, we conclude that

$$\frac{\partial}{\partial r} \varphi_{\epsilon,\gamma}^{(r)} \geq C \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - C.$$

□

Let

$$\varphi_\epsilon^{(r)} := (\lim_{\gamma \to 0} \varphi_{\epsilon,\gamma}^{(r)})^*,$$

then it satisfies the equation

$$\begin{cases}
\frac{\partial}{\partial t} \varphi_\epsilon^{(r)} = \log \frac{(\tilde{\chi} + e^{-t}((1-r)\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} + i\partial\bar{\partial}\varphi_\epsilon^{(r)}}{|\sigma_E|^{2(n-1)}_{h_E} |\sigma_F|^2_{h_F} \tilde{\Omega}} - \varphi_\epsilon^{(r)}, \text{ on } \tilde{X} \backslash \tilde{D} \\
\varphi_\epsilon^{(r)}(0) = 0.
\end{cases}$$

We have the monotonicity $\varphi_{\epsilon_1}^{(r)} \leq \varphi_{\epsilon_2}^{(r)}$ for any $\epsilon_1 \leq \epsilon_2$. Define

$$\varphi^{(r)} = \lim_{\epsilon \to 0} \varphi_\epsilon^{(r)}.$$

From Lemma 4.2.5 it's not hard to see that

$$|\varphi^{(r_1)} - \varphi^{(r_2)}| \leq C(1 - \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}})|r_1 - r_2|, \text{ on } \tilde{X} \backslash \tilde{D},$$

hence on any compact subset $K \subset\subset \tilde{X} \backslash \tilde{D}$, $\varphi^{(r)} \to \varphi_0$ in the C^∞ sense as $r \to 0$, where φ_0 is the solution constructed in (4.27).

Now we are ready to finish the proof of Proposition 4.2.1. Define $G := \varphi' - \varphi^{(r)} - e^{-t} r \varepsilon_0 \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}}$. By the assumption on φ', for any fixed $t \geq 0$, $r \in (0, 1)$

$$G \geq \delta \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - C_\delta - C - e^{-t} r \varepsilon_0 \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} \to +\infty,$$

as approaching $\tilde{D}$, if δ is smaller than $e^{-t}r\varepsilon_0$, hence the minimum of G cannot be at $\tilde{D}$. On the other hand, on $\tilde{X}\backslash\tilde{D}$, we have

$$\begin{aligned}\frac{\partial}{\partial t}G &= \log\frac{(\tilde{\chi}+e^{-t}((1-r)\pi^*\omega_0-\tilde{\chi})+i\partial\bar{\partial}\varphi^{(r)}+re^{-t}(\pi^*\omega_0-\varepsilon_0\mathrm{Ric}(h_{\tilde{D}}))+i\partial\bar{\partial}G)^n}{(\tilde{\chi}+e^{-t}((1-r)\pi^*\omega_0-\tilde{\chi})+i\partial\bar{\partial}\varphi^{(r)})^n}-G\\ &\geq \log\frac{(\tilde{\chi}+e^{-t}((1-r)\pi^*\omega_0-\tilde{\chi})+i\partial\bar{\partial}\varphi^{(r)}+i\partial\bar{\partial}G)^n}{(\tilde{\chi}+e^{-t}((1-r)\pi^*\omega_0-\tilde{\chi})+i\partial\bar{\partial}\varphi^{(r)})^n}-G,\end{aligned}$$

by maximum principle, we have $G\geq 0$, i.e.,

$$\varphi' \geq \varphi^{(r)}+e^{-t}r\varepsilon_0\log|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}}.$$

On any compact subset $K\subset\subset\tilde{X}\backslash\tilde{D}$, letting $r\to 0$, we get

$$\varphi'\geq\varphi_0,\qquad \text{on } K.$$

Then let $K\to\tilde{X}\backslash\tilde{D}$, we see that $\varphi'\geq\varphi_0$ on $\tilde{X}\backslash\tilde{D}$, combing with (4.28), we show that $\varphi'=\varphi_0$. Hence we finish the proof of uniqueness of solutions.

□

From the uniqueness of solutions to (4.20) and estimates of $\pi^*\varphi$, we see that

$$\tilde{\omega}_\epsilon(t)\xrightarrow{C^\infty_{loc}(\tilde{X}\backslash\tilde{D})}\pi^*\omega(t),\ \text{as }\epsilon\to 0,\tag{4.29}$$

where $\omega(t)$ is the solution to the Kähler Ricci flow (4.1) on X.

We will come back to equation (4.8).

Let $O\in B_O\subset Z$ be a small Euclidean ball, $\tilde{B}_O=\pi_2^{-1}(B_O)\subset\tilde{X}$. The divisors $\tilde{D}'$ and $\pi^{-1}(D)-E$ (the proper transform of D) lie in the zero set of of a local holomorphic function w in $\tilde{B}_O$. By Lemma 4.2.3, we have

Lemma 4.2.6.

$$\tilde{\omega}_\epsilon(t)\leq\frac{C}{|w|^{2\lambda}}\tilde{\omega},\qquad \textit{on } \partial\tilde{B}_O.$$

Let $\hat{\omega}:=\pi_2^*\omega_{Eucl}$, where ω_{Eucl} is the Euclidean metric on B_O, then local calculation shows that (see [84, 76])

$$C_0^{-1}\hat{\omega}\leq\tilde{\omega}\leq\frac{C_0}{|\sigma_E|^2_{h_E}}\hat{\omega},\quad \text{in } \tilde{B}_O,\tag{4.30}$$

and

$$\tilde{\chi}-\varepsilon_0\mathrm{Ric}(h_E)>0,\qquad \text{in } \tilde{B}_O.$$

Proposition 4.2.2. *There exist a small* $\delta \in (0,1)$ *and* $\lambda > 0$ *such that for any* $t \geq 0, \epsilon > 0$*, we have*

$$\tilde{\omega}_\epsilon(t) \leq \frac{C}{|\sigma_E|_{h_E}^{2(1-\delta)}|w|^{2\lambda}}\tilde{\omega}, \quad \textit{in } \tilde{B}_O. \tag{4.31}$$

Proof. We will do the calculation in $\tilde{B}_O \backslash E \cup \{w = 0\}$. Since $\hat{\omega}$ has flat curvature in $\tilde{B}_O \backslash E \cup \{w = 0\}$, we have

$$\Delta_{\tilde{\omega}_\epsilon(t)} \log tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t) \geq -\frac{tr_{\hat{\omega}}\mathrm{Ric}(\tilde{\omega}_\epsilon(t))}{tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)},$$

and by (4.10)

$$\frac{\partial}{\partial t}\log tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t) \leq \frac{tr_{\hat{\omega}}(-\mathrm{Ric}(\tilde{\omega}_\epsilon(t)) - \tilde{\omega}_\epsilon(t) + C\tilde{\omega})}{tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)}.$$

So

$$(\frac{\partial}{\partial t} - \Delta_{\tilde{\omega}_\epsilon(t)})\log tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t) \leq -1 + C\frac{tr_{\hat{\omega}}\tilde{\omega}}{tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)} \leq \frac{C}{|\sigma_E|_{h_E}^2 tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)},$$

where we have used (4.30).

So we have (r is a sufficiently small number)

$$(\frac{\partial}{\partial t} - \Delta_{\tilde{\omega}_\epsilon(t)})\log(|\sigma_E|_{h_E}^{2(1+r)}|w|^{2\lambda}tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)) \leq \frac{C}{|\sigma_E|_{h_E}^2 tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)} + (1+r)tr_{\tilde{\omega}_\epsilon(t)}\mathrm{Ric}(h_E).$$

$$\begin{aligned}
&(\frac{\partial}{\partial t} - \Delta_{\tilde{\omega}_\epsilon(t)})\Big(\log|\sigma_E|_{h_E}^{2(1+r)}|w|^{2\lambda}tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t) - A\tilde{\varphi}_\epsilon\Big)\\
\leq& C - A\log\frac{\tilde{\omega}_\epsilon(t)^n}{\tilde{\omega}^n} + \frac{C}{|\sigma_E|_{h_E}^2 tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)} + (1+r)\mathrm{Ric}(h_E)\\
&- Atr_{\tilde{\omega}_\epsilon(t)}\tilde{\chi} - Ae^{-t}tr_{\tilde{\omega}_\epsilon(t)}(\pi^*\omega_0 - \tilde{\chi}) - A\epsilon tr_{\tilde{\omega}_\epsilon(t)}\tilde{\omega}\\
\leq& C - tr_{\tilde{\omega}_\epsilon(t)}\tilde{\omega} + \frac{C}{|\sigma_E|_{h_E}^2 tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)},
\end{aligned}$$

if A is sufficiently large, and in the last inequality we have used the fact that $\tilde{\chi} - \varepsilon\mathrm{Ric}(h_E)$ is a Kähler metric on $\tilde{B}_O$ when ε is small.

On the other hand, similar calculation shows that

$$(\frac{\partial}{\partial t} - \Delta_{\tilde{\omega}_\epsilon(t)})\log tr_{\tilde{\omega}}\tilde{\omega}_\epsilon(t) \leq C_1 tr_{\tilde{\omega}_\epsilon(t)}\tilde{\omega} + C_1 + \frac{C_1}{tr_{\tilde{\omega}}\tilde{\omega}_\epsilon(t)},$$

where C_1 depends on the lower bound of the bisectional curvature of $\tilde{\omega}$.

Define

$$G = \log |\sigma_E|^{2(1+r)}_{h_E} |w|^{2\lambda} tr_{\hat{\omega}} \tilde{\omega}_\epsilon(t) - A\tilde{\varphi}_\epsilon + \frac{1}{2C_1} \log |w|^{2\lambda+2} tr_{\tilde{\omega}} \tilde{\omega}_\epsilon(t),$$

by the calculations above, we have

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta_{\tilde{\omega}_\epsilon(t)})G \leq & C - \frac{1}{2} tr_{\tilde{\omega}_\epsilon(t)} \tilde{\omega} + \frac{C}{|\sigma_E|^2_{h_E} tr_{\hat{\omega}} \tilde{\omega}_\epsilon(t)} + \frac{1}{tr_{\tilde{\omega}} \tilde{\omega}_\epsilon(t)} \\ \leq & C_2 - \frac{1}{2} tr_{\tilde{\omega}_\epsilon(t)} \tilde{\omega} + \frac{C}{|\sigma_E|^2_{h_E} tr_{\hat{\omega}} \tilde{\omega}_\epsilon(t)},\end{aligned}$$

where in the last inequality we use (4.30).

For any small positive r, $|\sigma_E|^{2(1+r)}_{h_E} |w|^{2\lambda} tr_{\hat{\omega}} \tilde{\omega}_\epsilon(t)$ tends to 0 as approaching E and $\{w = 0\}$, so for any $t \geq 0$, G cannot obtain its maximum at $\tilde{B}_O \cap E \cap \{w = 0\}$. Moreover, we know

$$\tilde{\varphi}_\epsilon \geq \delta \log |w| - C_\delta, \text{ on } \partial \tilde{B}_O,$$

for any small $\delta > 0$. Hence by Lemma 4.2.6, we have

$$\sup_{\partial \tilde{B}_O} G \leq C.$$

For any $T > 0$, assume $(p_0, t_0) \in \overline{\tilde{B}_O} \backslash E \cup \{w = 0\} \times [0, T]$ is the maximum point of G. If $p_0 \in \partial \tilde{B}_O$, then we are done. Otherwise, we have at this maximum point

$$|\sigma_E|^2_{h_E} tr_{\hat{\omega}} \tilde{\omega}_\epsilon(t) \big(tr_{\tilde{\omega}_\epsilon(t)} \tilde{\omega} - 2C_2 \big) \leq C.$$

By the inequality

$$tr_{\tilde{\omega}} \tilde{\omega}_\epsilon(t) \leq \frac{\tilde{\omega}_\epsilon(t)^n}{\tilde{\omega}^n} (tr_{\tilde{\omega}_\epsilon(t)} \tilde{\omega})^{n-1} = (tr_{\tilde{\omega}_\epsilon(t)} \tilde{\omega})^{n-1} e^{\tilde{\varphi}_\epsilon + \dot{\tilde{\varphi}}_\epsilon} \frac{\tilde{\Omega}_\epsilon}{\tilde{\omega}^n} \leq C_3 (tr_{\tilde{\omega}_\epsilon(t)} \tilde{\omega})^{n-1}.$$

So

$$|\sigma_E|^2_{h_E} tr_{\hat{\omega}} \tilde{\omega}_\epsilon(t) \big((tr_{\tilde{\omega}} \tilde{\omega}_\epsilon(t))^{1/(n-1)} - C_4 \big) \leq C, \quad \text{at } (p_0, t_0).$$

If $tr_{\tilde{\omega}} \tilde{\omega}_\epsilon(t)(p_0, t_0) \leq 2^{n-1} C_4^{n-1}$, then $|\sigma_E|^2_{h_E} tr_{\hat{\omega}} \tilde{\omega}_\epsilon(t)(p_0, t_0) \leq 2^{n-1} C C_4^{n-1}$. Noting that in $\tilde{B}_O$,

$$\tilde{\varphi}_\epsilon \geq \delta \log |\sigma_E|^2_{h_E} + \delta \log |w|^2 - C_\delta,$$

hence G is bounded above by a uniform constant, if we choose δ small enough in the above inequality.

If $tr_{\tilde{\omega}}\tilde{\omega}_\epsilon(t)(p_0,t_0) \geq 2^{n-1}C_4^{n-1}$, then we have

$$|\sigma_E|^2_{h_E} tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)(p_0,t_0) \leq C.$$

Then for δ small enough,

$$G(p_0,t_0) \leq r\log|\sigma_E|^2_{h_E} - \delta\log|\sigma_E|^2_{h_E} - \delta\log|w|^2 + (\lambda+1)\log|w|^2 + C \leq C.$$

In sum, in all cases, we have $\sup_{\tilde{B}_O\times[0,T]} G \leq C$. Then

$$\log\Big(|\sigma_E|^{2(1+r)}_{h_E} tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)|w|^{2\lambda+(2+2\lambda)(2C_1)^{-1}}(tr_{\tilde{\omega}}\tilde{\omega}_\epsilon(t))^{(2C_1)^{-1}}\Big) \leq \tilde{\varphi}_\epsilon + C \leq C,$$

noting that $tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t) \geq C_0^{-1} tr_{\tilde{\omega}}\tilde{\omega}_\epsilon(t)$, we have

$$\Big(tr_{\tilde{\omega}}\tilde{\omega}_\epsilon(t)\Big)^{1+(2C_1)^{-1}} \leq \frac{C}{|\sigma_E|^{2(1+r)}_{h_E}|w|^{2\Lambda}}.$$

If we choose r sufficiently small, say, $r \leq (10C_1)^{-1}$, then $\frac{1+r}{1+(2C_1)^{-1}} = 1-\delta$ for some $\delta \in (0,1)$, and hence

$$tr_{\tilde{\omega}}\tilde{\omega}_\epsilon(t) \leq \frac{C}{|\sigma_E|^{2(1-\delta)}_{h_E}|w|^{2\Lambda}}, \quad \text{in } \tilde{B}_O\backslash E \cup \{w=0\}.$$

□

Corollary 4.2.1. *By letting $\epsilon \to 0$ in (4.31) and the convergence (4.29), we have for any $t \geq 0$*

$$\pi^*\omega(t) \leq \frac{C}{|\sigma_E|^{2(1-\delta)}_{h_E}|w|^{2\lambda}}\tilde{\omega}, \text{ in } \tilde{B}_O\backslash(E\cup\{w=0\}). \tag{4.32}$$

Letting $t \to \infty$, we have

$$\pi^*\omega_\infty \leq \frac{C}{|\sigma_E|^{2(1-\delta)}_{h_E}|w|^{2\lambda}}\tilde{\omega}, \text{ in } \tilde{B}_O\backslash(E\cup\{w=0\}).$$

Lemma 4.2.7. *For any $q \in D \subset X$, there exists a smooth curve $\gamma(s) :\in [0,1] \to X$ such that*

(1) $\gamma([0,1)) \subset X\backslash D$, and $\gamma(1) = q$;

(2) γ is transversal to D;

(3) for any $\varepsilon > 0$, there exists an $s_0 > 0$, such that for all $s \in [s_0,1)$

$$d_{g(t)}(q,\gamma(s)) \leq \varepsilon, \quad \forall t \geq 0.$$

Proof. We take the resolution $\pi_1 : Z \to X$ and choose a point O in a smooth component of $\pi_1^{-1}(D)$ with $\pi_1(O) = q$, and blow up O, $\pi_2 : \tilde{X} \to Z$, and $\pi = \pi_1 \circ \pi_2 : \tilde{X} \to X$. We choose an appropriate smooth path $\tilde{\gamma}([0,1)) \subset \tilde{B}_O \backslash E \cup \{w = 0\}$ which keeps away from $\{w = 0\}$ and $\tilde{\gamma}(1) \subset E$. Then $\gamma = \pi(\tilde{\gamma})$ is the desired path, and last item follows from the uniform estimate (4.32). □

Corollary 4.2.2. *For a fixed $p \in X \backslash D$, any $q \in D$, there exists a constant $C = C_q$ such that for any $t \geq 0$*

$$d_{g(t)}(p, q) \leq C_q.$$

Hence along the convergent sequence $(X, g(t_i), p) \xrightarrow{d_{GH}} (X_\infty, d_\infty, p_\infty)$, $q \in (X, g(t_i))$ converges (up to a subsequence) to some $q_\infty \in X_\infty$ in the Gromov-Hausdorff sense.

Since we aim to give a purely analytic proof of our main results, without using of Kawamata's base point free theorem, we need the local freeness of some power of the canonical line bundle K_X as proved in [77], for completeness we give a sketched proof.

Proposition 4.2.3. *[77] For any $q \in D$, there exists $\sigma \in H^0(X, mK_X)$ for some $m \in \mathbb{Z}_+$ such that*

$$\sigma(q) \neq 0.$$

Proof. By Corollary 4.2.2, we can take q_∞ as the limit point of q. By [77], there exists a $\sigma \in H^0(X, mK_X)$ such that

$$|\sigma|^2_{h^m_{KE}}(q_\infty) > 1,$$

where $|\sigma|^2_{h^m_{KE}}$ is a Lipschitz continuous function on X_∞.

By Lemma 4.2.7, there exists a sequence of points $\{q_k\} \subset \mathcal{R}_X$ which transversely tend to q and

$$d_{g(t_i)}(q, q_k) \leq k^{-1}, \quad \forall i.$$

We may assume $q_k \in \mathcal{R}_X$ converge to the same point $q_k \in \mathcal{R} = \mathcal{R}_X$. Hence

$$d_\infty(q_\infty, q_k) = \lim_{i \to \infty} d_{g(t_i)}(q, q_k) \leq k^{-1}.$$

By the continuity of $|\sigma|^2_{h^m_{KE}}$, for k large enough,

$$e^{-m\varphi_{KE}} |\sigma|^2_{h^m_\chi}(q_k) = |\sigma|^2_{h^m_{KE}}(q_k) \geq \frac{1}{2},$$

i.e.

$$\varphi_{KE}(q_k) \le C + C\log|\sigma|^2_{h_X^m}(q_k).$$

On the other hand, for any $\delta \in (0,1)$, we have

$$\delta \log|\sigma_D|^2_{h_D}(q_k) - C_\delta \le \varphi_{KE}(q_k),$$

hence

$$|\sigma_D|^{2\delta}_{h_D}(q_k) \le C|\sigma|^2_{h_X^m}(q_k).$$

Since q_k approaches D transversely, and δ is any arbitrarily small number, we see that σ cannot vanish at q. Thus complete the proof. □

By a compactness argument and the previous proposition, we have:

Proposition 4.2.4. *There exists an integer $m \in \mathbb{Z}_+$ such that for any $q \in X$, there exists a holomorphic section $\sigma \in H^0(X, mK_X)$ such that $\sigma(q) \neq 0$, i.e., mK_X is base point free. Thus a basis $\{\sigma_0, \ldots, \sigma_{N_m}\}$ of $H^0(X, mK_X)$ gives a morphism*

$$\Phi_m : X \to X_{can} \subset \mathbb{CP}^{N_m},$$

where X_{can} is the image of X under Φ_m.

Remark 4.2.1. *Proposition 4.2.4 is well-known from Kawamata's base point free theorem. It follows from algebraic geometry [54] that when mK_X is base point free the maps Φ_m stabilize when m is sufficiently large, i.e., Φ_m is independent of m when m is large enough and we will denote this map by Φ.*

For the given basis $\{\sigma_0, \ldots, \sigma_N\}$ of $H^0(X, mK_X)$, we have

$$\sum_{i=0}^N |\sigma_i|^2_{h_t^m} = \sum_{i=0}^N |\sigma_i|^2_{h_X^m} e^{-\varphi-\dot\varphi} \ge c_0 \sum_{i=0}^N |\sigma_i|_{h_X^m} \ge c_1 > 0.$$

Moreover, by Proposition 4.1.1, we know

$$\sup_X \sum_{i=1}^N |\sigma_i|^2_{h_t^m} \le C, \quad \sup_X |\nabla_{\omega(t)}\sigma_i|^2_{h_t^m} \le C, \forall i, \quad \forall t \ge 0,$$

thus the map

$$\Phi_i : (X, \omega(t_i)) \to (X_{can}, \omega_{FS}), \quad x \mapsto [\sigma_0(x) : \ldots : \sigma_N(x)] \in \mathbb{CP}^N$$

has uniformly bounded derivatives (see e.g. [34]). Since the target space (X_{can}, ω_{FS}) is compact, by Arzela-Ascoli theorem, the map extends to the Gromov-Hausdorff limit space

$$\Phi_\infty : (X_\infty, d_\infty) \to (X_{can}, \omega_{FS}),$$

which is Lipschitz continuous.

Under our assumption that the Ricci curvature is uniformly bounded below, Tian-Wang's theory ([95]) on the structure of limit of almost Kähler Einstein manifolds implies that the singular set is closed and of Hausdorff codimension at least 4, which also implies that any tangent cone in the limit space is good (see [34]), in the sense that there exists a tangent cone $C(Y)$, such that for any $\eta > 0$ there exists a cut-off function β which is 1 on a small neighborhood of the singular set $S_Y \subset Y$, and vanishes outside the η-neighborhood of S_Y, and $\|\nabla\beta\|_{L^2(Y)} \leq \eta$, then following Donaldson-Sun's idea ([34]) on partial C^0 estimates (see also [94]), by similar arguments as in [76], for any two distinct points $p, q \in X_\infty$, one can construct two holomorphic sections $\sigma_1, \sigma_2 \in H^0(X_\infty, mK_{X_\infty})$ which separate p, q, hence we have

Proposition 4.2.5 ([76])**.** Φ_∞ *is injective.*

4.3 Proof of Theorems 1.2.2 and 1.2.3

Proof of Theorem 1.2.2. To prove Theorem 1.2.2, we will argue by contradiction. Following the ideas in [77], we need the following lemma:

Lemma 4.3.1. *Suppose* $\mathrm{diam}(X, g(t_i)) \to \infty$*, then we have* $\mathrm{diam}(X_\infty, d_\infty) = \infty$*,*

(1) $\Phi_\infty : (X_\infty, d_\infty) \to (X_{can}, \omega_{FS})$ *is not surjective;*

(2) For $p \in X_{can} \backslash \Phi_\infty(X_\infty)$ *and any sequence of points* $q_j \in X_\infty$ *with* $d_{\omega_{FS}}(\Phi_\infty(q_j), p) \to 0$ *as* $j \to \infty$*, we have*

$$d_\infty(p_\infty, q_j) \to \infty.$$

Proof. (1) Suppose Φ_∞ is surjective. Since $\mathrm{diam}(X_\infty, d_\infty) = \infty$, there exists a sequence of points $q_j \in \mathcal{R} \subset X_\infty$ with $d_\infty(p_\infty, q_j) \to \infty$. (X_{can}, ω_{FS}) is a compact metric space, hence there exists a convergent subsequence of $\{\Phi_\infty(q_j)\}$ which converge to some $q'_\infty \in X_{can}$ with respect to the metric ω_{FS}. Then there is a point $q_\infty \in X_\infty$ such that $\Phi_\infty(q_\infty) = q'_\infty$. We claim that the ball $B_{d_\infty}(q_\infty, 1)$ contains all but finitely many q_j's. Assuming this claim, we get that the distance of q_j and p_∞ is bounded by $d_\infty(p_\infty, q_\infty)+1$ contradicting the choice of q_j which converge to ∞ under d_∞ as $j \to \infty$. To see the claim, suppose not, there exists a subsequence $q_{j_l} \subset \{q_j\}$ such that $d_\infty(x_{j_l}, q_{j_l}) \geq 1$, where x_j is a sequence of points contained in $\mathcal{R}$ which converge to q_∞ under the metric d_∞. Since $\Phi_\infty(q_{j_l})$ and $\Phi_\infty(x_{j_l})$ are both in X_{can}^{reg} which is connected and these points both converge to q'_∞, so we can choose a curve $\gamma_{j_l} \subset X_{can}^{reg}$ whose length under ω_{FS} tend to 0 as $j_l \to \infty$. Then $\Phi_\infty^{-1}(\gamma_{j_l}) \subset \mathcal{R} \subset X_\infty$ is a connected curve connecting x_{j_l} and q_{j_l} which has d_∞-length greater than 1, so we can take a point $y_{j_l} \in \Phi_\infty^{-1}(\gamma_{j_l})$ such that $1/2 \leq d_\infty(x_{j_l}, y_{j_l}) \leq 1$. Then by compactness we may assume that up to a subsequence y_{j_l} converge to a point $y_\infty \in X_\infty$ which satisfies $1/2 \leq d_\infty(q_\infty, y_\infty) \leq 1$. It's not hard to see by triangle inequality that $d_{\omega_{FS}}(\Phi_\infty(y_{j_l}), q'_\infty) \to 0$, hence $d_{\omega_{FS}}(\Phi_\infty(y_\infty), q'_\infty) = 0$ and $\Phi_\infty(y_\infty) = q'_\infty$, and this contradicts the property that Φ_∞ is injective. Hence we prove the claim.

(2) Suppose $d_\infty(p_\infty, q_j) \leq A$ for some constant $A > 0$. By compactness we can assume a subsequence of q_j converges to a point q_∞, with $d_\infty(p_\infty, q_\infty) \leq A$.

Then we have $d_{\omega_{FS}}(p, \Phi_\infty(q_j)) \to d_{\omega_{FS}}(p, \Phi_\infty(q_\infty)) = 0$ as $j \to \infty$, thus $p = \Phi_\infty(q_\infty)$ and this contradicts the choice of p. □

Since Φ_∞ is not surjective, there exists a $q' \in X_{can} \backslash \Phi_\infty(X_\infty)$. Consider a point $q \in D \subset X$ with $\Phi(q) = q'$, for a sequence of points $\{q_j\} \subset \mathcal{R}_X = \mathcal{R}$ in the path constructed in Lemma 4.2.7 with $q_j \to q$, i.e. $d_{\omega_{FS}}(q', \Phi_\infty(q_j)) \to 0$, we have

$$\sup_j d_\infty(p_\infty, q_j) < \infty.$$

This contradicts item (2) in Lemma 4.3.1. Hence the diameter of $(X, g(t_i))$ is uniformly bounded. And we finish the proof of Theorem 1.2.2 □

Proof of Corollary 1.2.1. We will show $\Phi_\infty : X_\infty \to X_{can}$ is surjective. Suppose not, there is $p \notin \Phi_\infty(X_\infty)$. Since $\Phi_\infty(\mathcal{R})$ is dense in X_{can}, there exists a sequence of points

$q_j \in \mathcal{R}$ such that $d_{\omega_{FS}}(p, \Phi_\infty(q_j)) \to 0$ as $j \to \infty$. We have shown $\mathrm{diam}(X_\infty, d_\infty)$ is bounded, hence q_j would converge to some point $q_\infty \in X_\infty$ under d_∞. Hence

$$d_{\omega_{FS}}(p, \Phi_\infty(q_\infty)) \ = \lim_{j\to\infty} d_{\omega_{FS}}(p, \Phi_\infty(q_j)) = 0,$$

and we conclude that $p = \Phi_\infty(q_\infty)$, and thus a contradiction. Hence Φ_∞ is surjective. Combining with Song's result that Φ_∞ is also injective, we see that Φ_∞ is a Lipschitz continuous homeomorphism of (X_∞, d_∞) and X_{can}, since (X_∞, d_∞) is a compact space. Moreover, $\Phi_\infty|_{\mathcal{R}} : (\mathcal{R}, d_\infty) \to (\mathcal{R}_X, g_\infty)$ is an isometry so Φ_∞ induces an isometry between (X_∞, d_∞) and $\overline{(\mathcal{R}_X, g_\infty)} = (X_{can}, g_\infty)$. Hence the Gromov-Hausdroff limit of the Kähler Ricci flow (4.1) is the canonical model of X, with the limit metric of the flow, under the assumption of Ricci curvature lower bound along the flow. □

Proof of Theorem 1.2.3. Suppose the flow (4.1) is of Type III, i.e. $|Rm|(g(t))$ is uniformly bounded, by Shi's derivative estimates all derivatives of Rm are bounded. Fix a point $p \in X\backslash D$, for any sequence $t_i \to \infty$, by the smooth convergence of $\omega(t_i)$ on $X\backslash D$ (Lemma 4.1.1), the volumes of unit balls $B_{g(t_i)}(p, 1) \subset (X, g(t_i), p)$ are bounded below by a uniform positive constant, the limit space $(X_\infty, d_\infty, p_\infty)$ is smooth. Hence $X_\infty = \mathcal{R}$ and $\mathcal{S} = \mathcal{S}_X = \emptyset$. Then $\mathcal{R}_X = X$, otherwise, if there exists $q \in X\backslash\mathcal{R}_X$, then by Corollary 4.2.2 we have $d_{g(t_i)}(p, q) \leq C_q$ for any t_i and a uniform constant C_q depending only on q, hence q must converge to some point $q_\infty \in X_\infty$ along the Gromov-Hausdorff convergence, by the definition of $\mathcal{S}_X$, $q \in \mathcal{S}_X \neq \emptyset$, thus a contradiction. So we have X is a compact Kähler manifold admitting a smooth Kähler Einstein metric ω_{KE} with $\mathrm{Ric}(\omega_{KE}) = -\omega_{KE}$, hence K_X is ample. □

Appendix

In this appendix, we will show that along the Kähler-Ricci flow (4.1), assume Ricci curvature is uniformly bounded below for all $t \geq 0$, then for any sequence $t_i \to \infty$, $(X, \omega(t_i), p)$ is a sequence of almost Kähler-Einstein manifolds in the sense of Tian-Wang ([95]), where $p \in X\backslash D$ is a fixed point. Recall a sequence of Kähler manifolds (X_i, ω_i, p_i) is called almost Kähler Einstein if the following conditions are satisfied.

(1) $\mathrm{Ric}(\omega_i) \geq -\omega_i$

(2) $Vol_{\omega_i}(B(p_i, r_0)) \geq v_0 > 0$, for two fixed constants $r_0 > 0$ and v_0.

(3) The flow $\frac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega) + \lambda_i \omega$ has a solution $\omega(t)$ with $\omega(0) = \omega_i$ on $X_i \times [0,1]$, where $\lambda_i \in [-1,1]$ is a constant. Moreover, $\int_0^1 \int_{X_i} |R(\omega(t)) - n\lambda_i|\omega(t)^n dt \to 0$ as $i \to \infty$.

We may assume $\mathrm{Ric}(\omega(t)) \geq -K$ for a constant $K > 0$ (we may assume $K \geq 1$) and any $t \geq 0$. Let $\tilde{\omega}_i = K\omega(t_i)$, then $\mathrm{Ric}(\tilde{\omega}_i) \geq -1$. Since $(X, \omega(t_i))$ is non-collapsed at the point $p \in X \backslash D$ due to the smooth convergence, we have $(X, \tilde{\omega}_i)$ is also non-collapsed at p, i.e., there exists $v_0 > 0$ such that $Vol_{\tilde{\omega}_i}(B_{\tilde{\omega}_i}(p, r_0)) \geq v_0$ for some small $r_0 > 0$.

$\tilde{\omega}_i(t) := K\omega(t_i + K^{-1}t)$ with $t \in [0,1]$ satisfies the (normalized) Kähler Ricci flow equation

$$\frac{\partial}{\partial t}\tilde{\omega}_i(t) = -\mathrm{Ric}(\tilde{\omega}_i(t)) - K^{-1}\tilde{\omega}_i(t),$$

with the initial $\tilde{\omega}_i(0) = \tilde{\omega}_i$. From the evolution equation for the scalar curvature $R(\omega(t))$ ($\omega(t)$ is the solution to (4.1))

$$\frac{\partial}{\partial t}R = \Delta_{\omega(t)}R + |\mathrm{Ric}|^2 + R,$$

by maximum principle, at the minimum point of $R(\omega(t))$ for each t, $R_{\min} = \min_X R(\omega(t))$, we have

$$\frac{d}{dt}R_{\min}(t) \geq |\mathrm{Ric}|^2 + R_{\min}(t) \geq \frac{R_{\min}(t)^2}{n} + R_{\min}(t).$$

Standard comparison theorem of ODE implies that

$$R_{\min}(t) \geq -n - \frac{R_{\min}(0)n + n^2}{R_{\min}(0)e^t - R_{\min}(0) - n} \geq -n - O(e^{-t}).$$

Hence for $t \in [0,1]$, we have

$$R(\tilde{\omega}_i(t)) = K^{-1}R(\omega(t_i + K^{-1}t)) \geq -K^{-1}n - O(e^{-t_i}).$$

Then

$$\begin{aligned} &\int_0^1 \int_X |R(\tilde{\omega}_i(t)) + K^{-1}n|\tilde{\omega}_i(t)^n dt \\ \leq &\int_0^1 \int_X \big((R(\tilde{\omega}_i(t)) + K^{-1}n) + O(e^{-t_i})\big)\tilde{\omega}_i(t)^n dt \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \int_X n(\mathrm{Ric}(\tilde{\omega}_i(t)) + K^{-1}\tilde{\omega}_i(t)) \wedge \tilde{\omega}_i(t)^{n-1} dt + O(e^{-t_i}) \\
&= \int_0^1 \int_X n(e^{-t_i - K^{-1}t}(\omega_0 - \chi) - i\partial\bar{\partial}\dot{\varphi}) \wedge \tilde{\omega}_i(t)^{n-1} dt + O(e^{-t_i}) \\
&= \int_0^1 \int_X n e^{-t_i - K^{-1}t}(\omega_0 - \chi) \wedge \tilde{\omega}_i(t)^{n-1} dt + O(e^{-t_i}) \\
&\leq O(e^{-t_i}) \to 0, \quad \text{as } t_i \to \infty.
\end{aligned}
$$

Chapter 5

Canonical conical Kähler metrics on toric manifolds

In this chapter, we will give proof of Theorems 1.2.4, 1.2.5 and 1.2.6 (see also the joint work [29] with Ved Datar, Jian Song and Xiaowei Wang).

5.1 Toric conical Kähler metrics

In this section, we introduce the conical Kähler metrics on a smooth projective toric variety X following [82]. Recall in section 2.3.1, the metric (on the open dense $(\mathbb{C}^*)^n$) $\omega = i\partial\bar{\partial}\varphi$ defines a smooth toric Kähler metric on X if and only if φ satisfies the Guillemin boundary condition (2.7).

Now we extend the Guillemin boundary condition to conical toric Kähler metrics on X_P, as a generalization of orbifold Kähler metrics. On each coordinate chart determined by the pair $(p, \{v_{p,i}\}_{i=1}^n)$ associated to a vertex of P, we let $z = (z_1, ..., z_n)$ be the coordinates on $\mathbb{C}^n$. The closure of $\{z_i = 0\} \subset X$ gives rise to a smooth toric divisor of X_P. Let D be a toric divisor of X_P and suppose D restricted to this coordinate chart is given by

$$\sum_{i=1}^n a_i \{z_i = 0\}.$$

For any function $f(z)$ invariant under the $(S^1)^n$-action, we can lift it to a function

$$\tilde{f}(w) = f(z)$$

by letting

$$|w_i| = |z_i|^{\beta_i},\ w = (w_1, ..., w_n) \in \mathbb{C}^n,$$

and clearly $\tilde{f}(w)$ is also $(S^1)^n$-invariant. $w \in \mathbb{C}^n$ can be regarded a β-covering of $z \in \mathbb{C}^n$.

Now we introduce consider the $(S^1)^n$-invariant function space for $k \in \mathbb{Z}_+$ and $\gamma \in [0,1]$

$$C^{k,\gamma}_{\beta,p} = \{f(z) = f(|z_1|, ..., |z_n|) \mid \tilde{f}(w) \in C^{k,\gamma}(\mathbb{C}^n)\}.$$

This in turn defines the weighted function space

$$C^{k,\gamma}_{\beta}(X_P), \beta = (\beta_1, ..., \beta_N) \in (\mathbb{R}_+)^N$$

whose restriction on each chart belongs to $C^{k,\gamma}_{\beta}$ with respect to the weight β and β_j corresponding to the divisor D_j induced by $l_j(x) = 0$.

Definition 5.1.1. *A Kähler current $\omega \in c_1(L)$ is said to be a smooth β-conical metric if for each vertex p of the polytope P determined by L,*

$$\omega|_{U_p} = i\partial\bar{\partial}\varphi_p$$

for some $\varphi_p \in C^{\infty}_{\beta,p}$. Such a metric naturally has a cone angle of $2\pi\beta_j$ along the divisor D_j.

The local lifting $\tilde{\varphi}(w)$ is a smooth plurisubharmonic function on the lifting space $w \in \mathbb{C}^n$. We have the following conical extension of the Guillemin condition for toric Kähler metrics.

Proposition 5.1.1. *Let φ be a toric potential on $(\mathbb{C}^*)^n$, u it's Legendre transform and P the image of $\mathbb{R}^n$ under $\nabla\varphi$. Then $\omega = i\partial\bar{\partial}\varphi$ extends to a global smooth β-conical metric on X_P if and only if*

$$u(x) = \sum_{j=1}^{N} \beta_j^{-1} l_j(x) \log l_j(x) + f(x) \tag{5.1}$$

for some $f \in C^{\infty}(\overline{P})$ and $0 < \beta_j \leq 1$. Moreover, the angle along the divisor corresponding to l_j is precisely $2\pi\beta_j$.

The main advantage of dealing with conical metrics on toric manifolds is that one has all the curvature bounds.

Lemma 5.1.1 ([82]). *Let g be a smooth toric conical metric on a toric manifold X. Let D be a toric divisor consist of all toric prime divisors and let* Rm *denote the full* *curvature tensor of g. Then for any $k \geq 0$ there exists a constant C_k such that for all $p \in X \backslash D$,*

$$|\nabla_g^k \mathrm{Rm}|_g^2(p) \leq C_k. \tag{5.2}$$

5.1.1 Comparison theorems for conical Kähler-Einstein metrics

Next, consider a conical Kähler-Einstein metric (not necessarily toric) ω on X such that

$$Ric(\omega) = \alpha\omega + [D],$$

where D like before, is a simple normal crossing k.l.t divisor. Then it is shown in [28] that the regular set $X \setminus D$ is convex. As a consequence some classical comparison theorems such as Bishop-Gromov volume comparison and Myers' diameter bound extend to this singular setting. Before stating the theorems, we remark that since D simple normal crossing and k.l.t, $X \setminus D$ is of full measure in X, and hence volumes of sets can be calculated simply by integrating $\omega^n/n!$ over the intersection with $X \setminus D$.

Theorem 5.1.1 (Relative Volume Comparison)**.** *Let (X, D, ω) as above, and d be the induced distance by ω. For any point $p \in X \setminus D$, let $V(p, r) = \mathrm{Vol}_\omega(B(p, r))$. Also, let M_α^m denote the simply-connected m-dimensional space form of constant curvature α, and $V_\alpha(r)$ the volume of geodesic ball in M_α^m with radius r. Then if A is any star-convex set centered at p,*

$$\frac{V(B(p, r_2 \cap A)) - V(B(p, r_2) \cap A)}{V_k(r_2) - V_k(r_1)} \leq \frac{V(\partial B(p, r_1))}{V_k(\partial B_{r_1}^\alpha)},$$

for any $r_1 \leq r_2$.

Theorem 5.1.2 (Myers Theorem)**.** *With (X, ω, D) as above, if $\alpha > 0$, then*

$$diam(X, \omega) \leq \pi\sqrt{\frac{2n-1}{\alpha}}.$$

Next we prove a lemma due to Gromov [41] which will be the main technical tool in proving Gromov-Hausdorff convergence in the final section.

Lemma 5.1.2. *Let (X, D, ω) as above, and $\alpha > 0$. Let E be a small tubular neighborhood of D. Let $\varepsilon > 0$ and p_1, p_2 be two points in $X \backslash E$ such that $B(p_i, \varepsilon) \cap E = \emptyset$ $(i = 1, 2)$. If for any point $p \in B(p_2, \varepsilon)$, there is a minimal geodesic connecting p_1 and p and intersecting with ∂E, then there exists a constant $c = c(n, \varepsilon, \alpha)$ such that*

$$\mathrm{Vol}(\partial E) \geq c\mathrm{Vol}(B(p_2, \varepsilon)).$$

Proof. We proceed the proof following Gromov [41]. For any $p \in B(p_2, \varepsilon)$, let γ_p be a minimal geodesic connecting p_1 and p, by assumption, $\gamma_p \cap \partial E \neq \emptyset$, denoted by $\tilde{p} \in \gamma \cap \partial E$ (if there are more than one intersection points, we take the one closest to p). Set $d_1(\gamma_p) := d(p_1, \tilde{p})$ and $d_2(\gamma_p) := d(p, \tilde{p})$. Then clearly, $\varepsilon < d_i(\gamma_p) \leq \Lambda$, $i = 1, 2$, and here Λ is the diameter of (X, ω), which is bounded above by $\pi\sqrt{2n - 1/\alpha}$. Applying the relative volume comparison theorem, we have

$$\frac{\mathrm{Vol}(B(p_2, \varepsilon))}{\mathrm{Vol}(\partial E)} \leq \sup_{\gamma} \frac{V_k(d_1(\gamma) + d_2(\gamma)) - V_k(d_1(\gamma))}{V_k(\partial B_k(d_1(\gamma)))} \leq \frac{V_k(D) - V_k(\varepsilon)}{V_k(\partial B_k(\varepsilon))} =: c(n, \varepsilon, \alpha).$$

□

5.2 Toric Kähler-Ricci solitons with conical singularities

In this section, we will prove Theorem 1.2.4 and Theorem 1.2.5. The key ingredient is to prove a family version of C^0 estimates in [105] which will help us prove Theorem 1.2.6 for a possibly degenerate family of toric manifolds.

5.2.1 Setting up the continuity method

For the rest of this section, X will be a toric manifold with a fixed polarization by an ample line bundle L. We borrow notation from the last section and section 2.3. In particular, recall that the polytope given by L is fixed to be

$$P = \{x \in \mathbb{R}^n \mid l_j(x) = v_j \cdot x + \lambda_j > 0\}.$$

Also, we once and for all fix a reference metric in $c_1(L)$ by setting $\hat{\omega} = i\partial\bar{\partial}\hat{\varphi}$ on $(\mathbb{C}^*)^n$ where,

$$\hat{u}(x) = \sum_{j=1}^{N} \frac{l_j(x) \log(l_j(x))}{\beta_j} \tag{5.3}$$

and $\hat{\varphi}$ is the Legendre transform of $\hat{u}$. Then, by the discussion in the last section, $\hat{\omega}$ is a global toric smooth conical metric with angles $2\pi\beta_j$ along the divisor D_j.

Our aim in this section is to solve the following conical soliton equation

$$Ric(\omega) = \alpha\omega + \mathcal{L}_\xi \omega + [D], \tag{*}$$

where $\alpha > 0$, ω is a smooth toric conical Kähler metric, ξ is a holomorphic toric vector field on X and D is an effective toric $\mathbb{R}$-divisor. This is a generalization of Wang-Zhu [105] in the case of smooth Fano manifolds. We will prove our estimates in the framework of [105] combined with some techniques of [32]. On the open part $(\mathbb{C}^*)^n$, we can write $\omega = i\partial\bar{\partial}\varphi$. ξ being holomorphic then implies that $\mathcal{L}_\xi\omega = \partial\bar{\partial}\xi(\varphi)$. Since ξ is also toric, it is generated by the standard vector fields $\{z_i\partial/\partial z_i\}$. Consequently, there exists a vector $\vec{c} \in \mathbb{R}^n$ such that

$$\xi(\varphi) = \sum_{i=1}^{n} c_i \frac{\partial\varphi}{\partial\rho_i}.$$

Since on the open part one does not see the divisor, the soliton equation can be re-written as a real Monge-Ampere equation

$$\det(\nabla^2\varphi) = e^{-\alpha\varphi - c\cdot\nabla\varphi + \alpha\tau\cdot\rho} \tag{5.4}$$

for some $\tau \in \mathbb{R}^n$. Here the linear part shows up when one gets rid of the $\partial\bar{\partial}$ and in some sense corresponds to the divisor and controls the blow up of the metric as is seen below.

Lemma 5.2.1. *If there exists a solution of* (5.4) *then τ and the vector $\vec{c}$ must satisfy*

$$\tau = \frac{\int_P x e^{c\cdot x}\, dx}{\int_P e^{c\cdot x}\, dx}. \tag{5.5}$$

Moreover, the divisor D in (*) *is given by*

$$D = \sum_j (1 - \alpha l_j(\tau)) D_j$$

and so the cone angle along each D_j is $2\pi\beta_j$ where $\beta_j = \alpha l_j(\tau)$.

Proof. Since $\det(\nabla^2\varphi) \to 0$ as $|\rho| \to \infty$, for $\alpha > 0$,

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \nabla(e^{-\alpha\varphi+\alpha\tau\cdot\rho})\, d\rho \\ &= \alpha\tau \int_P e^{c\cdot x}\, dx - \alpha \int_P x e^{c\cdot x}\, dx. \end{aligned}$$

To compute the cone angles we consider the asymptotics at infinity. First, the equation for the symplectic potential u, the Legendre transform of φ, is given by

$$\det(\nabla^2 u) = e^{-\alpha u+\alpha(x-\tau)\cdot\nabla u - c\cdot x}. \tag{5.6}$$

By the conic Gullemein boundary conditions (see Proposition 5.1.1),

$$u = \hat{u} + f(x) = \sum_{j=1}^{N} \frac{l_j(x)\log(l_j(x))}{\beta_j} + f(x)$$

for some $f \in C^\infty(\bar{P})$. By direct computation it can be seen that

$$\det(\nabla^2 u) = \frac{G(x)}{l_1(x)\dots l_N(x)}$$

for some non vanishing $G \in C^\infty(\bar{P})$. On the other hand, once again using the formula for u, the order of l_j on the right hand side of equation (5.6) can be seen to be $\alpha l_j(\tau)/\beta_j$. Comparing the orders of $l_j(x)$ on both sides of the equation we conclude that $\beta_j = \alpha l_j(\tau)$. □

Hence D is effective if and only if $1 - \alpha l_j(\tau) > 0$ for all j and in this case Lemma 5.2.1 implies that $\tau \in P$. Conversely, we have the following lemma due to Wang-Zhu ([105]) and Donaldson ([32]):

Lemma 5.2.2. *For each $\tau \in P$, there exists a unique vector $\vec{c} \in \mathbb{R}^n$ satisfying*

$$\tau = \frac{\int_P x e^{c\cdot x}\, dx}{\int_P e^{c\cdot x}\, dx}. \tag{5.7}$$

Proof. By translating the polytope by τ, we can assume without loss of generality that $\tau = 0$. Consider the function

$$F(\vec{c}) = \int_P e^{c\cdot x}\, dx.$$

Clearly this function is strictly convex as can be seen by differentiating it twice. It is also proper. This follows from 0 being an interior point. Hence the function has a unique minimum $\vec{c}$. But then $\nabla F(\vec{c}) = 0$ which is precisely what we need. □

By the Cartan formula, for any Kähler metric ω, $\mathcal{L}_\xi\omega = di_\xi\omega$. Since ξ is holomorphic, clearly $\bar{\partial}i_\xi\omega = 0$. Now, all toric manifolds are simply connected i.e $H^{0,1}(X,\mathbb{C}) = 0$. So there exists a potential function θ_ξ such that $\xi = \nabla\theta_\xi$. Of course the function also depends on the metric. The Lie derivative is now given by

$$\mathcal{L}_\xi\omega = i\partial\bar{\partial}\theta_\xi. \tag{5.8}$$

From now on, we fix $\tau \in P$ with $1-\alpha l_j(\tau) > 0$ for all j and ξ is the unique holomorphic vector field determined by τ as in Lemma 5.2.1.

For the continuity method, we need to set up the Monge-Ampere equation. For that we need an analogue of the $\partial\bar{\partial}$-lemma in this conical setting. We also set $\beta(\alpha) = (\alpha l_1(\tau), \ldots, \alpha l_N(\tau))$. By lifting the smooth conical Kähler metric $\hat{\omega}$ to each uniformization covering, we can obtain the following lemma.

Lemma 5.2.3. *There exists a unique (up to constants) function $h \in C^\infty_{\beta(\alpha)}(X)$ satisfying*

$$Ric(\hat{\omega}) - \alpha\hat{\omega} - [D] - \mathcal{L}_\xi\hat{\omega} = i\partial\bar{\partial}h. \tag{5.9}$$

We now write the Monge-Ampere equation for the conical soliton. Set $\omega = \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi$. Then the equation for the conical soliton is

$$\begin{cases} (\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{-\alpha\psi - \xi(\psi) + h}\hat{\omega}^n \\ \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi > 0 \\ \psi \in C^\infty_{\beta(\alpha)}(X), \end{cases} \tag{$**$}$$

where h is from the above lemma.

To solve this equation, like usual, we introduce a parameter $s \in [0,\alpha]$ and look at the following family of equations.

$$\begin{cases} (\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi_s)^n = e^{-s\psi_s - \xi(\psi_s) + h}\hat{\omega}^n \\ \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi_s > 0 \\ \psi_s \in C^\infty_{\beta(\alpha)}(X) \end{cases} \tag{$**$}_s$$

or equivalently

$$Ric(\omega_s) = s\omega_s + (\alpha - s)\hat{\omega} + \mathcal{L}_\xi\omega_s + D. \tag{$**$}_s$$

The corresponding linearized operator is given by

$$L_s(\psi) = L(\psi) + s\psi = \Delta\psi + \xi(\psi) + s\psi.$$

Recall that we are only looking at the space of functions invariant under the toric action. One can define an inner product by

$$(\psi_1, \psi_2) = \int_X \psi_1 \bar{\psi}_2 e^{\theta_\xi} \omega^n$$

and denote the corresponding Hilbert space of square integrable functions by $L^2(e^{\theta_\xi})$. Then L restricted to $C^\infty_\beta(X)$ is self adjoint and hence can be thought of as an operator from $L^2(e^{\theta_\xi})$ to itself. Also, by virtue of being self adjoint, L only has real eigenvalues. The linear theory for the spaces $C^{k,\gamma}_\beta(X)$ is summarized below.

Lemma 5.2.4. *Let ω be a β-conical metric, Δ be the corresponding Laplacian and the linear operator L be defined as above. Then*

(1) For $k \geq 2$, $\Delta : C^{k,\gamma}_\beta(X) \to C^{k-2,\gamma}_\beta(X)$ is an invertible operator, modulo constants

(2) The Fredholm alternative holds for L.

(3) All nonzero eigenvalues of $-L$ are positive. Moreover, if $Ric(\omega) > t\omega + \mathcal{L}_\xi\omega$ and $-L\psi = \lambda\psi$, then $\lambda > t$.

The lemma that follows is essentially an observation of Zhu [112], adapted to the conical setting and is required in all the subsequent estimates. The proof in the toric case is in fact much easier.

Lemma 5.2.5. *There exists a uniform constant C depending only on $\hat{\omega}$ and ξ such that, for any function $\psi \in C^\infty_\beta(X) \cap PSH(X, \hat{\omega})$*

$$|\xi(\psi)| \leq C.$$

Proof. Locally on $(\mathbb{C}^*)^n$, $\hat{\omega} = \partial\bar{\partial}\hat{\varphi}$ and for any $\varphi = \hat{\varphi} + \psi$, since ψ is globally bounded and plurisubharmonic, it is easy to see that $\nabla\varphi(\mathbb{R}^n) = \nabla\hat{\varphi}(\mathbb{R}^n) = P$ and $\partial\bar{\partial}\varphi$ extends to a global Kähler metric. So there exists a uniform constant C such that

$$|\nabla\psi| \leq C.$$

But then, since ξ is given by

$$\xi = \sum_{i=1}^{n} c_i z_i \frac{\partial}{\partial z_i},$$

we have that

$$\xi(\psi) = c \cdot \nabla \psi.$$

This gives us the required bound.

□

The following proposition shows that there exists a solution to equation $(**)_s$ at $s = 0$

Proposition 5.2.1. *For any $h \in C^{\infty}_{\beta(\alpha)}(X)$ there exists a unique function $\psi \in C^{\infty}_{\beta(\alpha)}(X)$ satisfying*

$$\begin{cases} (\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{h-\xi(\psi)}\hat{\omega}^n \\ \sup \psi = 0 \\ \omega = \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi > 0. \end{cases}$$

Proof. We proceed by the continuity method. Consider the family of equations

$$\begin{cases} (\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi_s)^n = e^{sh-\xi(\psi_s)}\hat{\omega}^n \\ \sup \psi_s = 0 \\ \omega = \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi_s > 0 \end{cases} \tag{5.10}$$

and set $S = \{s \in [0,1] |$ equation (5.10) has a solution $\psi_s \in C^{3,\gamma}_{\beta(\alpha)}(X)$ at $s\}$. The set S is clearly nonempty, since $0 \in S$. In what follows we suppress the index s for convinience.

Openness. This follows straight from part(a) of Lemma 5.2.4 and the implicit function theorem on the space $C^{\infty}_{\beta(\alpha)}(X)$, since the linearized operator is just L.

C^0 *estimates.* By Lemma 5.2.5, the right hand side of the equation is uniformly bounded in s and hence in particular there exists a uniform L^p bound for any $p > 1$. Now, Kolodziej's results and their generalizations [53, 35, 109] give a uniform C^0 bound.

Second order estimates. Consider the quantity

$$H_s = \log tr_{\hat{\omega}}\omega_s - A\psi,$$

where A is some large number to be chosen later. Since both $\hat{\omega}$ and ω_s have poles of same order, the quantity is bounded. Let $\sup_X H_s = H_s(q)$. We lift all the local calculations to the $(S^1)^n$ invariant β- covering space. The second order estimates easily follow from [107] and [99].

C^3 *and higher order estimates.* Calabi's method for third order estimates can again be carried out by lifting the calculations to the β-cover. The reader should refer to [68] for the simplified computations. Higher order derivatives can be obtained by a standard bootstrapping argument. Closedness now follows from Ascoli-Arzela. Hence $1 \in S$.

□

5.2.2 Estimates and proofs of Theorems 1.2.4 and 1.2.5

For later applications, we need to get the precise dependence of the C^0 estimate on the polytope. So we introduce some notation. Recall that

$$P = \{x \in \mathbb{R}^n \mid l_j(x) = v_j \cdot x + \lambda_j > 0\}.$$

We let ν and σ be two constants such that

$$\nu^{-1} < Vol(P) < \nu,$$

$$(\sigma)^{-1} < diam(P) < \sigma.$$

On $(\mathbb{C}^*)^n$ we write $\hat{\omega} = i\partial\bar{\partial}\hat{\varphi}$ and $\omega_s = i\partial\bar{\partial}\varphi_s$. Using the standard logarithmic coordinates like before one can rewrite equation $(**)_s$ as a real Monge-Ampere on $\mathbb{R}^n$

$$\begin{cases} \det(\nabla^2\varphi_s) = e^{-s(\varphi_s - \tau\cdot\rho) - (\alpha - s)(\hat{\varphi} - \tau\cdot\rho) - c\cdot\nabla\varphi_s}, \\ u_s = \mathcal{L}\varphi_s = \sum_{j=1}^N (\beta_j)^{-1} l_j(x)\log l_j(x) + f(x), \qquad f \in C^\infty(\overline{P}) \\ \nabla^2\varphi_s > 0. \end{cases} \tag{5.11}$$

Proposition 5.2.2. *For any $s_0 \in (0,\alpha)$ there exists a constant $C = C(n, s_0, \nu, \sigma, \Lambda, \sup_{j,P}|l_j|)$ such that*

$$|\varphi_s - \hat{\varphi}| \leq C$$

for all $s \in [s_0, \alpha)$. Here Λ is the constant from Lemma 5.2.7 below.

We use the arguments in [105] with inputs and simplifications from [32], most notably the last step which helps us avoid the Harnack inequality. We first need two technical lemmas.

Lemma 5.2.6. *Suppose $v \geq 0$ is a strictly convex function on $\mathbb{R}^n$ such that $v(0) = 0$ and $\det(\nabla^2 v) \geq \lambda$ on $v \leq 1$. Then there exists $C > 0$ such that*

$$Vol(v \leq 1) \leq C\lambda^{-1/2}. \tag{5.12}$$

The proof is a standard barrier function argument and so we skip it.

Lemma 5.2.7. *If $\hat{\varphi}$ is the Legendre transform of $\hat{u}$ and we define the function*

$$g_j(\rho) = \log(l_j(\nabla\hat{\varphi}(\rho))). \tag{5.13}$$

Then, there exists a constant Λ such that

$$\sup_{\mathbb{R}^n} |\nabla g_j| \leq \Lambda. \tag{5.14}$$

Here, Λ depends only on β_j, N, n and the normal vectors v_j.

The proof of this lemma is by a long but direct computation, so we omit it and refer to our paper [29].

Proof of Proposition 5.2.2 There are several steps following [105] and [32] combined with Lemma 5.2.7. Let $\phi_s = \varphi_s - \tau \cdot \rho$, $\hat{\phi} = \hat{\varphi} - \tau \cdot \rho$ and define

$$w_s = s\phi_s + (\alpha - s)\hat{\phi}. \tag{5.15}$$

Set

$$m_s := \inf_{\mathbb{R}^n} w_s = w_s(\rho_s)$$

Step 1. We claim that there exist $C, \zeta > 0$ independent of s such that for all $s \in [s_0, \alpha]$,

$$(a)\ |m_s| \leq C \tag{5.16}$$

$$(b)\ w_s \geq \zeta|\rho - \rho_s| - C. \tag{5.17}$$

It follows from the definition of w_s that $\det(\nabla^2 w_s) \geq s^n \det(\nabla^2 \phi_s) \geq s_0^n \det(\nabla^2 \phi_s)$. Set $K = \{m_s \leq w_s \leq m_s + 1\}$, $K_\mu = \{m_s \leq w_s \leq m_s + \mu\}$ and $V_\mu = Vol(K_\mu)$. From the equation, $\det(\nabla^2 \phi_s) = e^{-w_s - c\cdot\nabla\varphi_s}$ and so on K,

$$\begin{aligned}\det(\nabla^2 w_s) &\geq s_0^n \det(\nabla^2 \phi_s)\\ &= s_0^n e^{-w_s - c\cdot\nabla\varphi_s}\\ &\geq Ce^{-m_s},\end{aligned}$$

where C only depends on s_0 and σ which is an upper bound for $|\nabla\varphi_s|$. So, Lemma 5.2.6 applied to $v = w_s - m_s$ implies that $Vol(K) \leq Ce^{m_s/2}$. But $K_\mu \subseteq \mu K$, where by μK, we mean dilation with center ρ_s. So we have the volume estimate

$$V_\mu \leq C\mu^n e^{m_s/2}.$$

Now,

$$\begin{aligned}\nu^{-1} \leq Vol(X) &= \int_{\mathbb{R}^n} \det(\nabla^2 \phi_s)\, d\rho\\ &= \int_{\mathbb{R}^n} e^{-w_s - c\cdot\nabla\varphi_s}\, d\rho\\ &\leq Ce^{-m_s} \int_0^\infty e^{-\mu} V_\mu\, d\mu\\ &\leq Ce^{-m_s/2}\end{aligned}$$

and so $m_s \leq C(n, s_0, \nu, \sigma)$. For the lower bound, notice that $\nabla w_s(\mathbb{R}^n) = P - \tau$ and so $|\nabla w_s| \leq 2\sigma$. This implies that K contains a ball of radius $1/2\sigma$. But the volume of K is bounded above by $Ce^{m_s/2}$ and so we immediately have a lower bound for m_s. Hence (a) is proved with $C = C(n, s_0, \nu, \sigma)$.

Suppose now there exists a point $\rho \in K$ such that $|\rho - \rho_s| = R$. Because $B = B(\rho_s, 1/(2\sigma)) \subseteq K$, by convexity, the entire cone κ with vertex at ρ and base as B lies inside K. So, $Vol(K) \geq CR$, where C depends only on dimension and σ. But

$Vol(K) \leq Ce^{m_s/2}$ and so is less than some fixed constant C by part (a). Hence R is uniformly bounded. That is, there exists a uniform R such that $K \subseteq B(\rho_s, R)$. But then, convexity implies that $K_\mu \subseteq B(\rho_s, \mu R)$. From this and the lower bound on m_s, it easily follows that

$$w_s \geq \frac{1}{R}|\rho - \rho_s| - C.$$

This proves (b) with $\zeta = 1/R$.

Step 2. We now claim that, there exists uniform constant C such that

$$|\rho_s| \leq C. \tag{5.18}$$

We first observe,

$$0 = \int_{\mathbb{R}^n} \nabla(e^{-w_s}) = -\int_{\mathbb{R}^n} [s(\nabla\varphi_s - \tau) + (\alpha - s)(\nabla\hat{\varphi} - \tau)]e^{-w_s}\, d\rho$$
$$= -\int_{\mathbb{R}^n} (\alpha - s)(\nabla\hat{\varphi} - \tau)]e^{-w_s},$$

where we use the change of coordinates $x = \nabla\varphi_s(\rho)$ along with the equation $e^{-w_s} = \det(\nabla^2\varphi_s)e^{c\cdot\nabla\varphi_s}$ and the fact that $\vec{c}$ and τ are compatible to conclude that the first term is zero. This computation gives us the crucial identity

$$\int_{\mathbb{R}^n} (\nabla\hat{\varphi} - \tau)e^{-w_s}\, d\rho = 0,$$

or equivalently,

$$\frac{1}{\tilde{V}_s}\int_{\mathbb{R}^n} (\nabla\hat{\varphi})e^{-w_s}\, d\rho = \tau, \tag{5.19}$$

where $\tilde{V}_s$ is the weighted volume given by

$$\tilde{V}_s = \int_{\mathbb{R}^n} e^{-w_s}\, d\rho.$$

Note that when the Futaki invariant vanishes, this is precisely the identity in the paper of Wang and Zhu since in that case τ is the barycenter which is zero.

Suppose the claim is false i.e for all $M > 0$ there exists a pair (s, ρ_s) with $|\rho_s| > M$. Applying l_j to both sides of the identity (5.19),

$$\frac{1}{\tilde{V}_s}\int_{\mathbb{R}^n} l_j(\nabla\hat{\varphi})e^{-w_s}\, d\rho = l_j(\tau) > \delta \tag{5.20}$$

for some j and some $\delta > 0$. Fix an $\epsilon > 0$. From the estimates in the previous step there exists an $R_\epsilon >> 1$ such that

$$\int_{\mathbb{R}^n \setminus B(\rho_s, R_\epsilon)} e^{-w_s}\, d\rho \leq \epsilon. \tag{5.21}$$

Recall that as ρ goes to infinity, the image under $\nabla\hat{\varphi}$ goes to the boundary of P. So, by hypothesis, on can choose a big $M >> 1$ such that $|\rho_s| > M$ and $\log\left(l_j(\nabla\hat{\varphi}(\rho_s))\right) < -M$ for some s and some face l_j. By the gradient estimate in Lemma 5.2.7 there exists a constant Λ (which does not depend on s) such that on $B = B(\rho_s, R_\epsilon)$

$$\log\left(l_j(\nabla\hat{\varphi}(\rho))\right) < -M + \Lambda R_\epsilon < \frac{-M}{2} < \log\epsilon \tag{5.22}$$

for M sufficiently big. So combining 5.21 and 5.22 we estimate the integral in 5.20,

$$\begin{aligned} \frac{1}{\tilde{V}}\int_{\mathbb{R}^n} l_j(\nabla\hat{\varphi})e^{-w_s} &= \frac{1}{\tilde{V}}\int_B l_j(\nabla\hat{\varphi})e^{-w_s} + \frac{1}{\tilde{V}}\int_{\mathbb{R}^n\setminus B} l_j(\nabla\hat{\varphi})e^{-w_s} \\ &\leq \epsilon + C\epsilon, \end{aligned}$$

where C only depends on an upper bound for the image of P under l_j and a lower bound for the total volume of X. Now choose ϵ small enough so that $\epsilon + C\epsilon < \delta/2$. But then, this contradicts (5.20), completing Step 2.

Step 3. We first observe the elementary identity from convex analysis

$$\sup_{\mathbb{R}^n} |\varphi_s - \hat{\varphi}| = \sup_P |u_s - \hat{u}|. \tag{5.23}$$

So, to complete the proof, one only needs to control the C^0 norm of u_s since from the definition it is easy to see that the bound for $\hat{u}$ only depends on β_j and an upper bound on l_j. From (5.17) and (5.18),

$$w_s(\rho) \geq \zeta|\rho| - C. \tag{5.24}$$

Let u_s be the Legendre transform of φ_s, then for any $p > n$,

$$\begin{aligned} \int_P |\nabla u_s|^p\, dx &= \int_{\mathbb{R}^n} |\rho|^p e^{-w_s - c\cdot\nabla\varphi_s}\, d\rho \\ &\leq C\int_{\mathbb{R}^n} |\rho|^p e^{-\zeta|\rho|}\, d\rho \\ &\leq C(p). \end{aligned}$$

By Morrey's inequality $osc_{\bar{P}} u_s < C$ for some C independent of s. Now, if we set $x_s = \nabla\varphi_s(\rho_s)$, then,

$$u_s(x_s) = \rho_s \cdot x_s - \varphi_s(\rho_s).$$

The first term is clearly bounded from Step-2. Moreover by Step-1, $w_s(\rho_s)$ is bounded. Since ρ_s stays bounded, there is a uniform bound on $\hat{\varphi}(\rho_s)$, which in turn gives a uniform bound on $\varphi_s(\rho_s)$. This shows that $|u_s(x_s)|$ is uniformly bounded. Hence the oscillation bound implies

$$|u_s|_{C^0(P)} \leq C.$$

This completes the proof of the proposition.

We are now in a position to prove Theorems 1.2.4 and 1.2.5.

Proof of Theorem 1.2.4. We divide our proof into three steps:

Step 1. We first characterize the invariant $\mathcal{S}(X, L)$ in terms of the polytope data as follows - The polytope for the linear system $|-K_X - \alpha L|$ can be taken to be $P^\alpha = \{x \in \mathbb{R}^n \mid l_j^\alpha = v_j \cdot x + 1 - \alpha\lambda_j\}$. For any $\alpha > 0$ and any j,

$$\begin{aligned} \tau \in P \text{ with } 1 - \alpha l_j(\tau) &\geq 0 \\ \Leftrightarrow 0 \leq 1 - \alpha l_j(\tau) &\leq 1 \\ \Leftrightarrow 0 \leq v_j \cdot (-\alpha\tau) + 1 - \alpha\lambda_j &\leq 1 \\ \Leftrightarrow 0 \leq l_j^\alpha(-\alpha\tau) &\leq 1. \end{aligned}$$

But then divisor $D = \sum l_j^\alpha(-\alpha\tau) D_j$ is an effective divisor in $|-K_X - \alpha L|$ with coefficients less than 1.

Step 2. We next outline a proof of the existence of solutions to the soliton equation. Let $S = \{s \in [0, \alpha] | \exists$ a solution $\psi \in C^{3,\gamma}_{\beta(\alpha)}$ to eqn. $(**)_s$ $\}$. By proposition 5.2.1, $0 \in S$ and hence S is nonempty. We now need to show that S is both open and closed.

Openness- The linearized operator for equation $(**)_s$ is $L_s = \Delta_s + \xi + sI$. Since $[D] \geq 0$, $Ric(\omega_s) > s\omega_s + \mathcal{L}_\xi \omega_s$. By lemma 5.2.4 all eigenvalues of $-L_s$ are strictly positive and hence the Fredholm alternative implies that L_s is invertible. Implicit function theorem then implies that S is open.

C^0 *estimates*- Since there is a solution at $s = 0$ by openness there exists an s_0 such that there is a solution on $[0, s_0]$. With this choice of s_0, by proposition 5.2.2 there exists a constant C independent of s such that

$$|\psi_s| = |\varphi_s - \hat{\varphi}| \leq C.$$

C^2 *and higher order estimates* - Once the uniform bound is obtained, the argument for the second and higher order estimates is the same as that in the proof of proposition 5.2.1. Hence the upshot is that S is nonempty, open and closed. Hence $S = [0, \alpha]$ and in particular $\alpha \in S$. This completes the proof of the second part of the theorem.

Step 3. Finally, to complete the proof of Theorem 1.2.4, we now prove that $\mathcal{R}_{BE}(X, L) = \mathcal{S}(X, L)$. From the existence part of the theorem, it is easy to see that $\mathcal{R}_{BE}(X, L) \geq \mathcal{S}(X, L)$. In order to prove the reverse inequality, let $\alpha \in (0, \mathcal{R}_{BE})$. Then by definition, there exist smooth toric β-conical metrics $\omega = i\partial\bar{\partial}\varphi$ and $\eta = i\partial\bar{\partial}\psi$, and a holomorphic vector field ξ vector $\tau \in \mathbb{R}^n$, such that

$$Ric(\omega) = \alpha\omega + \mathcal{L}_\xi\omega + \eta + [D]$$

for some smooth conical Kähler metric η and some effective divisor D. Note that the volume form can be expressed as

$$\omega^n = \frac{\Omega}{\prod_{j=1}^N |s_j|_{h_j}^{2(1-\beta_j)}}$$

for some global volume form Ω with $\log \Omega$ bounded. From this, it is clear that the divisor is given by

$$D = \sum_{j=1}^N (1 - \beta_j) D_j$$

and consequently one can take the polytope for η to be

$$P^\eta = \{x \in \mathbb{R}^n | l_j^\eta = v_j \cdot x + \beta_j - \alpha\lambda_j > 0 \ j = 1, \dots, N\}.$$

Locally on $(\mathbb{C}^*)^n$, $\omega = i\partial\bar{\partial}\varphi$ and $\eta = i\partial\bar{\partial}\psi$, and the corresponding real Monge-Ampere equation reads

$$\det \nabla^2 \varphi = e^{-\alpha\varphi - \psi - c\cdot\nabla\varphi - \tau\cdot\rho}$$

for some $\tau \in \mathbb{R}^n$. As before, we take φ so that $\nabla\varphi(\mathbb{R}^n) = P$. Furthermore we normalize ψ so that $\nabla\psi(\mathbb{R}^n) = P^\eta$. With this normalization, we claim that $\tau = 0$.

Since $\nabla\varphi$ is bounded, it suffices to prove that

$$|\log\det\nabla^2\varphi + \alpha\varphi + \psi| \tag{5.25}$$

is bounded. Let $\overline{\varphi} = \alpha\varphi + \psi$ be the potential for the smooth β-conical metric $\overline{\omega} = \alpha\omega + \eta$. Then $\overline{\omega}^n/\omega^n$ is a global bounded function. This is because both the metrics have the same poles at the divisors. Consequently it is enough to show that

$$|\log\det\nabla^2\overline{\varphi} + \overline{\varphi}|$$

is bounded. But, as in the proof of Lemma 5.2.1,

$$\begin{aligned} |\log\det\nabla^2\overline{\varphi} + \overline{\varphi}| &\leq |\sum_{j=1}^N \Big(1 + \frac{x\cdot v_j}{\beta_j}\Big)\log\bar{l}_j| + C \\ &\leq |\sum_{j=1}^N \Big(1 - \frac{\bar{l}_j(0)}{\beta_j}\Big)\log\bar{l}_j| + C \\ &\leq C, \end{aligned}$$

where $\bar{l}_j(x) = v_j\cdot x + \beta_j$ and we used the fact that $\bar{l}_j\log\bar{l}_j$ is a bounded function in the second line. Note that the polytope for $\overline{\varphi}$ is given precisely by the intersection of $\bar{l}_j > 0$ for $j = 1, \ldots, N$. This completes the proof of (5.25) and hence proves the claim that $\tau = 0$. But then using the integration by parts trick from the proof of Lemma 5.2.1

$$0 = \int_{\mathbb{R}^n}\nabla(e^{-\alpha\varphi-\psi})d\rho = -\alpha\int_P xe^{c\cdot x}dx - \int_P \nabla\psi e^{c\cdot x}dx.$$

So, if we set

$$\bar{\tau} = \frac{\int_P xe^{c\cdot x}dx}{\int_P e^{c\cdot x}dx} = \frac{-\int_P\nabla\psi e^{c\cdot x}dx}{\alpha\int_P e^{c\cdot x}dx}.$$

Obviously, $\bar{\tau}\in P$ and applying l_j , we have

$$1 - \alpha l_j(\bar{\tau}) = \frac{\int_P(1 + v_j\cdot\nabla\psi - \alpha\lambda_j)e^{c\cdot x}dx}{\int_P e^{c\cdot x}dx} \geq 1 - \beta_j \geq 0.$$

where we used the definition of P^η for the first inequality and $D \geq 0$ for the second inequality. Hence $\alpha < \mathcal{S}(X.L)$ and this completes the proof of the theorem. □

Proof of Theorem 1.2.5. This theorem follows directly from Theorem 1.2.4 by taking $\tau = P_C$. For uniqueness we refer to results of Berndtsson [8]. The only slightly subtle point is the existence of conical Kähler-Einstein metrics for $\alpha = \mathcal{R}(X.L)$. This follows from the fact that barycenter always stays in the interior of the polytope and hence Proposition 5.2.2 also holds for this choice of α (Contrast this, for instance, with the case when $\alpha = \mathcal{S}(X, L)$ as discussed in Example **??** below). All the higher order estimates then follow from the C^0 bound exactly as in the proof above. □

5.2.3 Relation to the K-stability

Let (X, D) be a smooth log Fano manifold, that is X is smooth and $D = \sum(1-\beta_i)D_i$ is a *simple normal crossing* (snc) divisor with D_i's being smooth prime divisors. Let $\hat{\omega}$ be a conical Kähler form on X with cone angle β_i along D_i. Let h_i be smooth Hermitian metric for the line bundle $\mathcal{O}_X(D_i)$ and Ω be a smooth volume form on X, then locally near a point $x \in X$ we have

$$\hat{\omega}^n|_x = \frac{\Omega}{\prod_{i=1}^{r_x} |s_{D_i}|_{h_i}^{2(1-\beta_i)}} \text{ and } \mathrm{Ric}(\hat{\omega}) - [D] = \mathrm{Ric}(\Omega) - c_1([D], h) \tag{5.26}$$

where $c_1([D], h)$ is the *smooth* Chern form for the $\mathbb{R}$-line bundle $\mathcal{O}_X(D)$ with respect to the smooth metric $h = \prod_i h_i^{1-\beta_i}$. In [33] and [57], log Donaldson-Futaki invariant was defined as follows

Definition 5.2.1.

$$\mathrm{DF}(X, D; L)(\xi) = -\int_{X\setminus \mathrm{Supp}(D)} \hat{\theta}_\xi \{\mathrm{Ric}(\hat{\omega}) - [D] - \mu\hat{\omega}\} \frac{\hat{\omega}^{n-1}}{(n-1)!} \tag{5.27}$$

where $\mu := \frac{(-K_X - D)\cdot L^{n-1}}{L^n}$, $\mathrm{Supp}(D)$ *is the support of the divisor* D*,* ξ *is a holomorphic vector field on* X *tangent to* D *and the function* $\hat{\theta}_\xi$ *is determined via* $d\hat{\theta}_\xi := i_\xi\hat{\omega}$ *which is unique up to a constant.*

In particular, when X is toric then for any $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$, we may define the corresponding vector field $\xi = \sum_i \lambda_i z_i \frac{\partial}{\partial z_i}$ then we may regard $\mathrm{DF}(X, D; L)$ as a linear functional on $\mathbb{R}^n$, which can be identified as a vector via the Euclidean inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$. With these understood, we have

Lemma 5.2.8. *Let $\tau \in P_{-K_X-\alpha L} \subset \mathbb{R}^n$ and D_τ be the corresponding divisor in the linear system $|-K_X-\alpha L|$. Suppose $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ and $\xi = \sum_{i=1}^n \lambda_i z_i \frac{\partial}{\partial z_i}$ on a toric pair (X, D_τ)*

$$\mathrm{DF}(X, D_\tau; L) = -V(P)(\alpha P_C + \tau) \in \mathbb{R}^n. \tag{5.28}$$

In particular, $\mathrm{DF}(X, D_\tau; L) = 0$ *if and only if* $\tau = -\alpha P_C$.

Proof. Since $\hat{\omega}$ is a conical Kähler form, there is a $\hat{h} \in L^\infty(X)$ satisfying $\mathrm{Ric}(\hat{\omega}) - [D] - \mu_1\hat{\omega} = i\partial\bar{\partial}\hat{h}$. By definition 5.2.1, we have

$$\begin{aligned} \mathrm{DF}(X, D; L)(\xi) &= -\int_{X\setminus \mathrm{Supp}(D)} \hat{\theta}_\xi \{\mathrm{Ric}(\hat{\omega}) - [D] - \mu_1\hat{\omega}\} \frac{\hat{\omega}^{n-1}}{(n-1)!} = \int_X \xi\hat{h}\frac{\hat{\omega}^n}{n!} \\ &= \int_{\mathbb{R}^n} \sum_{i=1}^n \lambda_i \frac{\partial}{\partial\rho_i}(-\log\det\nabla^2\varphi - \alpha\varphi - \tau\cdot\rho)\det\nabla^2\varphi d\rho \\ &= \left\langle -\alpha\int_P x dx - \tau\int_P dx, \lambda\right\rangle_{\mathbb{R}^n} \\ &= \langle -V(P)(\alpha P_C + \tau), \lambda\rangle_{\mathbb{R}^n} \end{aligned}$$

Notice that for the second identity, we need to subtract the linear growth term $\tau \cdot \rho$ in order to keep $\hat{h}$ bounded. □

This combined with the work of Berman [7], we may reformulate Theorem 1.2.5 as follows

Theorem 5.2.1 (Theorem 1.2.5)**.** *Let X be an n-dimensional projective toric manifold polarized by an ample line bundle L. Then there exists a conical Kähler-Einstein metric with cone angle along a effective toric divisor $D \in |-K_X-\alpha L|$ if and only if the log pair (X, D) is log K-polystable.*

Remark 5.2.1. *This can be viewed as toric version of generalizing the work of [21, 22, 23] to the log Fano pair (X, D) with simple normal crossing divisors.*

5.3 Connectedness of the space of toric conical Kähler-Einstein metrics

We will prove Theorem 1.2.6 in this section.

5.3.1 Reducing the proof of Theorem 1.2.6 to the case of one blow-up

Let us fix a toric manifold Y with an ample toric line bundle $\mathcal{L}$ and corresponding polytope P. Let X be the blow-up of Y along a k-dimensional smooth toric variety V with $\pi : X \to Y$ as the blow-down map. Set $L_t = \pi^*\mathcal{L} + tA$ for some ample toric line bundle A on X and for $t \in [0, 1]$. Recalling the definition of the invariant $\mathcal{R}$, we make the following simple observation

Lemma 5.3.1. *Let (X_t, L_t) be a family of toric manifolds with ample $\mathbb{R}$-line bundles L_t for $t \in (0, 1]$. Then as long as the corresponding polytopes P_t stay bounded, we have*

$$\inf_t \mathcal{R}(X_t, L_t) > 0. \tag{5.29}$$

Proof. By Theorem 1.2.4

$$\begin{aligned}\mathcal{R}(X, L) &= \sup\{\alpha | 1 - \alpha l_j(P_C) > 0,\ j = 1, \ldots, N\}\\ &= \inf_j \{\frac{1}{l_j(P_C)}\},\end{aligned}$$

which stays bounded away from zero if the polytopes stay bounded. □

For $t \in [0, t_0]$ we can now choose a continuous path α_t such that

$$0 < \alpha_t < \min(R(X, L_t), R(Y, \mathcal{L}))$$

and let ω_t be the unique toric conical Kähler-Einstein metrics in $c_1(L_t) \cap \mathcal{K}_c(X)$ with Einstein constant α_t. We also let $\omega_Y \in c_1(\mathcal{L}) \cap \mathcal{K}_c(Y)$ be the the toric conical Kähler-Einstein metric on Y with Einstein constant α_0. Denoting the corresponding Riemannian metrics by g_t and g_Y, we have the following proposition.

Proposition 5.3.1. *(X, g_t) is a continuous path in the Gromov-Hausdorff topology and*

$$(X, g_t) \xrightarrow{t \to 0} (Y, g_Y).$$

By restricting α_t to be less than $R(X, L_t)$ we ensure that g_t are geodesically convex, thus facilitating the application of tools from comparison geometry. In particular, we will make use of lemma 5.1.2. Taking the above proposition for granted, we now prove Theorem 1.2.6

Proof of Theorem 1.2.6. We fix (X_j, ω_j) for $j = 0, 1$ as in the statement of the theorem and we let L_j be the Kähler class of ω_j and $\alpha_j < R(X, L_j)$, and we call $(L_j, \omega_j, \alpha_j)$ compatible triples on X_j. By the factorization theorem 2.3.1, there exist a sequence $0 = t_0 < t_1 < \ldots < t_k = 1$ and pairs (X_{t_i}, f_{t_i}) such that

$$X = X_0 \overset{f_{t_1}}{\dashrightarrow} X_{t_1} \overset{f_{t_2}}{\dashrightarrow} \cdots \overset{f_{t_i}}{\dashrightarrow} X_{t_i} \overset{f_{t_{i+1}}}{\dashrightarrow} \cdots \overset{f_{t_k}}{\dashrightarrow} X_{t_k} = X_1.$$

We start from the left and construct the family of metrics inductively. Suppose we are at stage t_i i.e we have already constructed $(X_{t_i}, L_{t_i}, \alpha_{t_i}, \omega_{t_i})$. Then there are two cases

Case-1 - $f_{t_{i+1}}$ is a blow-down map.

On $X_{t_{i+1}}$, we take an arbitrary choice of compatible triples $(L_{t_{i+1}}, \alpha_{t_{i+1}}, \omega_{t_{i+1}})$. Then we connect this to $(X_{t_i}, L_{t_i}, \alpha_{t_i}, \omega_{t_i})$ in two steps. Fix a $\mu \in (t_i, t_{i+1})$ and ample line bundle A on X_{t_i}.

Step-1 For $t \in [\mu, t_{i+1}]$, set

$$\begin{aligned} X_t &= X_{t_i} \\ L_t &= f^*_{t_{i+1}} L_{t_{i+1}} + (t_{i+1} - t)A \\ \alpha_t &< \min(R(X_t, L_t), R(X_{t_{i+1}}, L_{t_{i+1}})) \end{aligned}$$

where we choose α_t to be continuous. We now let ω_t be the α_t - conical Kähler-Eisntein metric in L_t. Then by Proposition 5.3.1, (X_t, ω_t) is continuous for $t \in [\mu, t_{i+1})$ and converges to $(X_{t_{i+1}}, \omega_{t_{i+1}})$ in the Gromov-Hausdorff topology.

Step-2 - For $t \in [t_i, \mu]$ set

$$\begin{aligned} X_t &= X_{t_i} \\ L_t &= \frac{\mu - t}{\mu - t_i} L_{t_i} + \frac{t - t_i}{\mu - t_i} L_\mu \\ \alpha_t &< R(X_t, L_t) \end{aligned}$$

Again, let ω_t be the corresponding α_t - conical Kähler-Einstein metrics. Since in this case, L_t is uniformly Kähler all the estimates of Proposition 5.3.1 go through and we in fact get that (X_{t_i}, ω_t) is continuous in t in the smooth topology on X_{t_i}.

Case 2. $f_{t_{i+1}}$ ***is a blow-up map.*** This can be treated by the same argument as in Case 1 by moving t backward from t_{i+2} to t_{i+1}.

The smoothness of g_t on the complex torus $(\mathbb{C}^*)^n$ follows if we take α_t to be a smooth path in t.

□

5.3.2 Uniform estimates and proof of proposition 5.3.1

In this section we prove Proposition 5.3.1, thereby completing the proof of Theorem 1.2.6. Throughout the section we fix an $\alpha > 0$, such that $\alpha \in (0, \min(R(X, L_t), R(Y, \mathcal{L})))$.

Without loss of generality, we can assume that the polytope P that induces the toric manifold Y is given by $(N-1)$ defining functions $l_j(x) = v_j \cdot x + \lambda_j \geq 0$, $j = 1, \ldots, N-1$. Let P^A be the poytope corresponding to the ample line bundle A on X with N defining functions $l_j^A(x) = v_j \cdot x + \lambda_j^A$ for $j = 1, \ldots, N$. The blow-up process corresponds to the $(n-1)-$dimensional face given by l_N contracting to a k-dimensional face given by the intersection of $(n-k)$ co-dimension one faces, say $l_1, \ldots, l_{n-k}$. We denote the divisor corresponding to l_j on X by D_j with defining section s_j, while on Y we denote the divisor corresponding to l_j by $\tilde{D}_j$ and the corresponding section by $\tilde{s}_j$. Then it follows from the definition of blow-ups that

$$\begin{cases} \pi^* \tilde{D}_j = D_j + D_N, & j = 1, \ldots, n-k, \\ \pi^* \tilde{D}_j = D_j, & j = n-k+1, \ldots, N-1, \end{cases} \tag{5.30}$$

where as before D_j denotes the divisor corresponding to l_j and π^* is the total transform. Using this fact, one can explicitly write down the polytope P^t for L_t by defining

$$\begin{cases} l_j^t(x) = v_j \cdot x + \lambda_j + t\lambda_j^A, & j = 1, \ldots, N-1, \\ l_N^t(x) = v_N \cdot x + (\sum_{j=1}^{n-k} \lambda_j) + t\lambda_N^A, \end{cases}$$

where $v_N = \sum_{j=1}^{n-k} v_j$.

We denote the barycenters of the evolving polytopes by P_C^t and the corresponding angles by $\beta_j^t = \alpha l_j^t(P_C^t)$. We then set l_j^0, P_C^0 and β_j^0 to be the limit of the respective quantities as t goes to zero. We first prove an important identity that will be very useful, among other things, in proving that the limiting Monge-Ampere equation descends to the Einstein equation on Y.

Lemma 5.3.2.

$$(1-\beta_N^0) - \sum_{j=1}^{n-k}(1-\beta_j^0) = -(n-k-1). \tag{5.31}$$

Proof. Since D_N is obtained by blowing up the intersection of $D_1, \ldots, D_{n-k}$, it is well known that

$$v_N = \sum_{j=1}^{n-k} v_j.$$

Now at $t=0$, there are $(n-k+1)$ affine linear functions $l_1, \ldots, l_{n-k}$ and l_N vanishing on a k- dimensional face (see the figure given below). So they must be linearly related i.e there exist real numbers a_j so that

$$l_N^0 = \sum_{j=1}^{n-k} a_j l_j^0.$$

But then, since v_j's are linearly independent, the two equations together force the a_j's to be one i.e

$$l_N^0 = \sum_{j=1}^{n-k} l_j^0.$$

The lemma now follows.

□

Example. Let $X = \mathbb{P}^2 \# \overline{\mathbb{P}^2}$ and $Y = \mathbb{P}^2$. On Y we take $\mathcal{L}$ to be the anti-canonical bundle and the corresponding $P \subset \mathbb{R}^2$ to be defined by $\{x+1 \geq 0, y+1 \geq 0, 1-x-y \geq 0\}$. One can view X as a projective bundle over $\mathbb{P}^1$ with a zero section D_0 and a section D_∞ at infinity. We take A to be $2[D_\infty] - [D_0]$, with the polytope P^A given by $\{x \geq 0, y \geq 0, 2-x-y \geq 0, -1+x+y \geq 0\}$. It follows from the Nakai criteria that A is ample. Then the polytope for L_t is given by the inequalities $\{x+1 \geq 0, y+1 \geq 0, 1+2t-x-y \geq 0, 2-t+x+y \geq 0\}$. Computing the β_j^0 for this example we see that $\beta_1^0 = 1, \beta_2^0 = 1, \beta_3^0 = 1, \beta_4^0 = 2$. One can now easily verify Lemma 5.3.2 for this simple example.

Now let ω_t and ω_Y be the unique toric conical Kähler-Einstein metrics with Einstein constant α on X and Y in the class $c_1(L_t)$ and $c_1(\mathcal{L})$ respectively. In section 5.2, we worked with conical reference metrics coming from the symplectic potential. However, for dealing with convergence issues as the Kähler class degenerates, it is more convenient to work with smooth reference forms. So we pick a Kähler form $\tilde{\omega}_Y \in c_1(\mathcal{L})$ and a Kähler form $\chi \in c_1(A)$. More explicitly, by taking the embedding of Y into a big projective space via the sections coming from the lattice points of P, we can set $\tilde{\omega}_Y$ to be the pull-back of the Fubini-Study metric. One can make a similar choice for χ. We then set $\tilde{\omega}_t = \pi^*\tilde{\omega}_Y + t\chi$. Clearly, there exist locally bounded functions ψ_t such that $\omega_t = \tilde{\omega}_t + \sqrt{-1}\partial\bar{\partial}\psi_t$. We similarly have a potential ψ_Y for ω_Y.

Lemma 5.3.3. *There exists a uniform constant C independent of t such that*

$$\sup_X \psi_t - \inf_X \psi_t \leq C. \tag{5.32}$$

Proof. We fix a volume form on X say $\Omega = \chi^n$. Recall from section 3, that ω_t is given locally on $(\mathbb{C}^*)^n$ by $\omega_t = i\partial\bar{\partial}\varphi_t$, where φ_t is a function only of $\rho \in \mathbb{R}^n$ and satisfies the real Monge-Ampere

$$\det(\nabla^2\varphi_t) = e^{-\alpha(\varphi_t - P_C^t \cdot \rho)} = e^{-w_t}$$

The volume for ω_t^n is given by

$$\omega_t^n = (\det \nabla^2\varphi_t) d\rho_1 \wedge \ldots \wedge \rho_n \wedge d\theta_1 \ldots \wedge d\theta_n$$

All the estimates in the proof of Proposition 5.2.2 remain uniform under small perturbations of the polytope. In particular, the estimate (5.24) holds with constants ζ and C independent of t. That is

$$\det(\nabla^2\varphi_t) < Ce^{-\zeta|\rho|}.$$

Similarly on $(\mathbb{C}^*)^n$, $\chi = i\partial\bar{\partial}\phi$. Since χ is the pull back of the Fubini-Study metric, one can take

$$\phi = \log\Big(\sum_{\nu \in P^A \cap \mathbb{Z}^n} e^{\nu \cdot \rho}\Big).$$

By an elementary calculation, there exist constant B_1, B_2, B_3, B_4 depending only on P^A such that

$$B_3 e^{-B_1|\rho|} < \det(\nabla^2\phi) < B_4 e^{-B_2|\rho|}.$$

Now, consider the trivial identity

$$(\tilde{\omega}_t + i\partial\bar{\partial}\psi_t)^n = \omega_t^n = \frac{\omega_t^n}{\Omega}\Omega. \tag{5.33}$$

We claim that ω_t^n/Ω is in $L^{1+\epsilon}(X,\Omega)$ for some $\epsilon > 0$. This is because

$$\begin{aligned}
\int_X \left(\frac{\omega_t^n}{\Omega}\right)^{1+\epsilon} \Omega &= \int_{(S^1)^n}\int_{\mathbb{R}^n} \left(\frac{\det(\nabla^2\varphi_t)}{\det(\nabla^2\phi)}\right)^{1+\epsilon} \det(\nabla^2\phi)\, d\rho d\theta \\
&\leq C\int_{\mathbb{R}^n} \left(\frac{e^{-\zeta|\rho|}}{e^{-B_1|\rho|}}\right)^{\epsilon} e^{-B_2|\rho|}\, d\rho \\
&\leq C\int_{\mathbb{R}^n} e^{-(B_2+\epsilon(B_1-\zeta))|\rho|}\, d\rho \\
&\leq C
\end{aligned}$$

if ϵ is small enough. Then in view of (5.33), since we have a uniform $L^{1+\epsilon}(X,\Omega)$ bound on ω_t^n/Ω, applying [53, 35, 109] we directly obtain a uniform bound on the oscillation of ψ_t.

□

We now spend some time in deriving a complex Monge-Ampere equation satisfied by ψ_t. Let Ω and Ω_Y be two fixed volume forms on X and Y respectively and let ξ_Y, ξ_A be metrics on $\mathcal{L}$ and A such that $\omega_Y = -i\partial\bar{\partial}\log\xi_Y$ and $\chi = -i\partial\bar{\partial}\log\xi_A$. One can also view Ω and Ω_Y as metrics on $-K_X$ and $-K_Y$. We recall the adjunction formula

$$K_X = \pi^* K_Y + (n-k-1)[D_N].$$

By the $\partial\bar{\partial}$-lemma there exists a metric h_N on $[D_N]$ such that

$$\Omega = \frac{\pi^*\Omega_Y}{|s_N|_{h_N}^{2(n-k-1)}}. \tag{5.34}$$

Next, since $-K_Y = \alpha\mathcal{L} + \tilde{D}$, one can choose smooth hermitian metrics $\tilde{h}_1, \ldots, \tilde{h}_{N-1}$ on $\tilde{D}_1, \ldots, \tilde{D}_{N-1}$ such that

$$\prod_{j=1}^{N-1} \tilde{h}_j^{(1-\beta_j^Y)} = \pi^*\left(\frac{\Omega_Y}{(\xi_Y)^\alpha}\right). \tag{5.35}$$

Using 5.30, we then define smooth metrics on D_j for $j < N$ by setting

$$\begin{cases} h_j = \pi^*\tilde{h}_j/h_N & j = 1, \ldots, n-k \\ h_j = \pi^*\tilde{h}_j & j = n-k+1, \ldots, N-1. \end{cases} \tag{5.36}$$

Finally we define a family of metrics on $[D_N]$ by

$$h_N(t) = \left(\frac{\Omega \xi_A^{-t\alpha} \pi^*(\xi_Y^{-\alpha})}{\prod_{j=1}^{N-1} h_j^{(1-\beta_j^t)}} \right)^{\frac{1}{1-\beta_N^t}}. \tag{5.37}$$

We claim

Lemma 5.3.4. *At all points of X,*

$$\lim_{t \to 0} \frac{h_N(t)}{h_N} = 1. \tag{5.38}$$

Proof. If we consider Ω as a metric on $-K_X$ and $\pi^*\Omega_Y$ as a pull back metric on $-\pi^*K_Y$, then by equation (5.34), $\Omega = \pi^*\Omega_Y h_N^{-(n-k-1)}$.

$$\begin{aligned}
\lim_{t \to 0} \frac{h_N(t)}{h_N} &= \left(\frac{\Omega \pi^*(\xi_Y)^{-\alpha}}{h_N^{(1-\beta_N^0)} \prod_{j=1}^{N-1} h_j^{(1-\beta_j^0)}} \right)^{\frac{1}{1-\beta_N^0}} \\
&= \left(\frac{\pi^*(\Omega_Y \xi_Y^{-\alpha}) h_N^{-(n-k-1)}}{h_N^{(1-\beta_N^0)} \prod_{j=1}^{N-1} h_j^{(1-\beta_j^0)}} \right)^{\frac{1}{1-\beta_N^0}} \\
&= \left(\frac{\pi^*(\Omega_Y \xi_Y^{-\alpha})}{\prod_{j=1}^{N-1} \pi^* \tilde{h}_j^{(1-\beta_j^Y)}} \right)^{\frac{1}{1-\beta_N^0}} \\
&= 1,
\end{aligned}$$

where we used lemma 5.3.2, equation (5.36) in line three and equation (5.35) in line four.

□

By applying logarithm and taking $\partial\bar{\partial}$ we see that $h_1, \ldots, h_{N-1}$ and $h_N(t)$ satisfy

$$-\partial\bar{\partial} \log \Omega = \alpha \tilde{\omega}_t - \sum_{j=1}^{N-1} (1 - \beta_j^t) \partial\bar{\partial} \log h_j - (1 - \beta_N^t) \partial\bar{\partial} \log h_N(t).$$

The purpose of the above constructions is that ψ_t and ψ_Y now satisfy, possibly after modification by some constant, the following Monge-Ampere equations

$$(\tilde{\omega}_t + i\partial\bar{\partial}\psi_t)^n = \frac{e^{-\alpha\psi_t}\Omega}{|s_N|_{h_N(t)}^{2(1-\beta_N^t)} \prod_{j=1}^{N-1} |s_j|_{h_j}^{2(1-\beta_j^t)}}, \tag{$(*)_t$}$$

$$(\tilde{\omega}_Y + i\partial\bar{\partial}\psi_Y)^n = \frac{e^{-\alpha\psi_Y}\Omega_Y}{\prod_{j=1}^{N-1}|\tilde{s}_j|_{\tilde{h}_j}^{2(1-\beta_j^Y)}}. \qquad (*)_Y$$

We remark that since modification by a constant doesn't change the oscillation, the estimate of lemma 5.32 holds for this modified ψ_t. Immediately, we have the following corollary from Lemma 5.3.3 because the total volume of (X, ω_t) is $[L_t]^n$ and is uniformly bounded .

Corollary 5.3.1. *There exists a unique constant $C > 0$ such that for all $t \in (0, 1]$,*

$$||\psi_t||_{L^\infty(X)} \leq C. \qquad (5.39)$$

We next prove uniform estimates away from D on all derivatives of ψ_t

Proposition 5.3.2. *For all $l \geq 0$ and $K \subset\subset X\backslash D$, there exist constants $C_{K,l}$ independent of t such that*

$$||\psi_t||_{C^l(K)} < C_{K,l}. \qquad (5.40)$$

Here the norm is with respect to some fixed reference metric.

Proof. Since the usual arguments for $C^{2,\gamma}$ estimates, using the theory of Evans, Kryllov and Safanov, are local in nature they can be used in the present context. Hence, to prove the proposition, we only need uniform C^2 estimates.

We follow the argument in [79] using Tsuji's trick [104]. By Kodaira's lemma, $L_\epsilon = \pi^*\tilde{\omega}_y - \epsilon[D_N] > 0$ for small $\epsilon > 0$. So, we pick a new smooth hermitian metric ξ_N on $[D_N]$ such that $\eta = \pi^*\tilde{\omega}_Y + \epsilon\partial\bar{\partial}\log\xi_N > 0$ and consider

$$Q_t = \log\left(|s_N|_{\xi_N}^{2(1+A\epsilon)}\prod_{j=1}^{N-1}|s_j|_{h_j}^2 tr_\eta\omega_t\right) - A\psi_t$$

for some big constant A to be chosen later. We note that Q goes to negative infinity near D. This is because the order of poles of ω_t near each D_j is $2(1-\beta_j^t)$ which is strictly less than two. So for each t, the maximum is attained in $X\backslash D$. Following Yau, we compute $\Delta_t Q_t$ where Δ_t is the Laplacian with respect to ω_t. Since on $X\backslash D$, $Ric(\omega_t) = \alpha\omega_t$, standard calculations show that there exists a constant C depending only on the dimension and curvature of η such that

$$\Delta_t Q_t \geq -Ctr_{\omega_t}\eta + (1 + A\epsilon)\Delta_t\log\xi_N + \sum_{j=1}^{N-1}\Delta_t\log h_j + Atr_{\omega_t}\tilde{\omega}_t - C.$$

Also, there exists a constant C' independent of t such that

$$\Delta_t \log \xi_N > -C' tr_{\omega_t}\eta,$$
$$\Delta_t \log h_j > -C' tr_{\omega_t}\eta.$$

Combining this with the above estimate

$$\begin{aligned}\Delta_t Q_t &\geq -C tr_{\omega_t}\eta + A tr_{\omega_t}(\tilde{\omega}_t + \epsilon\partial\bar{\partial}\log \xi_N) - C \\ &= -C tr_{\omega_t}\eta + A tr_{\omega_t}(\eta + t\chi) - C \\ &> tr_{\omega_t}\eta - C,\end{aligned}$$

where we choose $A = C + 1$. So, at the maximum point of Q_t, $tr_{\omega_t}\eta(p_t) < C$. Now, standard arguments show

$$\left(|s_N|_{\xi_N}^{2(1+A\epsilon)} \prod_{j=1}^{N-1} |s_j|_{h_j}^2\right) tr_\eta \omega_t < Ce^{\sup\psi_t - \inf\psi_t}\left(|s_N|_{\xi_N}^{2(1+A\epsilon)} \prod_{j=1}^{N-1} |s_j|_{h_j}^2 \frac{\omega_t^n}{\eta^n}\right)(p_t).$$

Using the equation and the oscillation bound on ψ_t and the fact that $\beta_j^t < 1$,

$$tr_\eta\omega_t < \frac{C}{\left(|s_N|_{\xi_N}^{2(1+A\epsilon)} \prod_{j=1}^{N-1} |s_j|_{h_j}^2\right)}.$$

Hence, we have uniform second order estimates away from D and this completes the proof of the proposition.

$\square$

With the above uniform local estimates and the uniqueness of ω_Y, we have the following local uniform convergence away from the divisors.

Proposition 5.3.3. *For any compact subset $K \subset\subset X \setminus D$, we have the following uniform convergence*

$$\omega_t \xrightarrow{C^\infty(K)} \omega_Y.$$

Using Moser iteration, one can in fact show that ψ_t converges to $\pi^*\psi_Y$ globally in L^∞. We now have to prove the global convergence, in Gromov-Hausdorff topology, of (X, ω_t) to (Y, ω_Y).

Proof of Proposition 5.3.1. We let $t \to 0$. Fix an $\epsilon > 0$. Let E be a tubular neighborhood of $D \subset Y$ such that $A = Y \backslash E$ is ϵ-dense in Y with respect to the metric g_Y. Note, that since X and Y are bi-holomorphic away from D, A can be identified as a subset of X. We also pick E close enough to D so that $Vol_{g_Y}(E) < \epsilon^{4n}$ and we set $\tilde{E} = \pi^*(E)$. Finally, we denote the distances with respect to g_t and g_Y, by d_t and d_Y respectively.

Claim 1. For t small enough, $A = Y \backslash E = X \backslash \tilde{E}$ is ϵ - dense in (X, g_t).

Proof. If not, then there exists a sequence $t_k \to 0$ and points $x_k \in \tilde{E}$ such that $B_{g_{t_k}}(x_k, \epsilon) \subset \tilde{E}$. Using volume comparison, uniform diameter bounds and the fact that the volumes converge, for small t_k

$$\kappa\epsilon^{2n} < Vol_{g_k}(B_{g_k}(x_k, \epsilon)) < Vol_{g_k}(\tilde{E}) < 2Vol_{g_Y}(E) < 2\epsilon^{4n}$$

for some constant κ if k is sufficiently large. But if ϵ is small enough, this is a contradiction.

Claim 2. There exists a $t(\epsilon)$ such that for all $0 < t < t(\epsilon)$ and for all $p, q \in A$,

$$d_t(p, q) < d_Y(p, q) + \epsilon.$$

Proof. By the geodesic convexity of $Y \backslash D$, one can choose a small tubular neighborhood, $T \subset E$ of D in Y such that any two points in A can be connected by a g_Y - minimal geodesic in $Y \backslash T$. Set $\tilde{T} = \pi^{-1}(T)$. Let γ be a g_Y - minimal geodesic connecting p and q lying in $Y \backslash T$. Since the metrics converge uniformly on compact sets of $X \setminus \tilde{E}$, for t sufficiently small,

$$d_t(p, q) < \mathcal{L}_t(\gamma) < \mathcal{L}_Y(\gamma) + \epsilon = d_Y(p, q) + \epsilon,$$

where $\mathcal{L}$ denotes the length functional.

Claim 3. There exists a $t(\epsilon)$ such that for all $0 < t < t(\epsilon)$ and for all $p, q \in A$,

$$d_t(p, q) > d_Y(p, q) - \epsilon.$$

Proof. The proof of this claim relies on the generalization of Gromov's lemma to the conical setting (Lemma 5.1.2). We can a tubular neighborhood T of D contained in E

with smooth boundary, $\epsilon_1 = \epsilon_1(A) \leq \epsilon$, and a sufficiently small $\delta > 0$ to be fixed later, such that for all $q \in A$,

$$B_{g_Y}(q, \epsilon_1/2) \subset Y \backslash T$$
$$Vol_{g_Y}(\partial T) < \delta/2$$

and set $\tilde{T} = \pi^*(T)$. D being of real co-dimension two, one can choose δ to be as small as needed. Since away from D the metric converges uniformly, we can assume that $Vol_{g_t}(\partial T) < \delta$ by choosing $t < t(\delta)$ sufficiently small. Furthermore, since $d_{g_Y}(q, \partial T) > \epsilon_1/2$, once again by the uniform convergence of the metric on $X\backslash\tilde{T}$, for small t, $d_{g_t}(q, \partial\tilde{T}) > \epsilon_1/4$, i.e., $B_{g_t}(q, \epsilon_1/4) \subset X\backslash\tilde{T}$.

We claim that there exists at least one minimal g_t-geodesic from p to a point in $B_{g_t}(q, \epsilon_1/4)$ that lies entirely in $X\backslash\tilde{T}$. If not, then by Gromov's lemma there exists a constant c uniform in t (but depending on ϵ_1) such that

$$\kappa\epsilon_1^{2n} < Vol_{g_t}(B_{g_t}(q, \epsilon_1/4)) < cVol_{g_t}(\partial\tilde{T}) < c\delta.$$

By choosing δ such that $c\delta < \kappa\epsilon_1/2$ to start off with, we get a contradiction.

So there exists at least one g_t - minimal geodesic γ_t connecting p to a point $\tilde{q}_t \in B_{g_t}(q, \epsilon_1/4)$. By compactness, there exists a $\tilde{q} \in B_{g_Y}(q, \epsilon_1/2)$ such that $\tilde{q}_t \to \tilde{q}$ and moreover the geodesics γ_t converge to a curve, denoted by γ, joining p to $\tilde{q}$.

$$\begin{aligned} d_{g_t}(p, q) &> \mathcal{L}_{g_t}(\gamma_t) - \epsilon_1/4 \\ &> \mathcal{L}_{g_Y}(\gamma) - \epsilon_1/2 \\ &> d_{g_Y}(p, \tilde{q}) - \epsilon_1/2 \\ &> d_{g_Y}(p, q) - \epsilon_1 \\ &> d_{g_Y}(p, q) - \epsilon \end{aligned}$$

and this proves Claim 3.

Now we complete the proof of the proposition. For sufficiently small t,

$$\begin{aligned} &d_{GH}((X, d_t), (Y, d_Y)) \\ \leq &d_{GH}((X, d_t), (A, d_t)) + d_{GH}((A, d_t), (A, d_Y)) + d_{GH}((A, d_Y), (Y, d_Y)) \end{aligned}$$

$$< 3\epsilon,$$

where we use Claim 1 to bound the first term, Claim 2 and Claim 3 to bound the second term, while the last term is bound by ϵ from the choice of A. Now, letting ϵ go to zero, we see that (X, g_t) converges in Gromov-Hausdorff distance to (Y, g_Y). □

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