STABILITY RESULTS IN ADDITIVE COMBINATORICS AND GRAPH THEORY

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ABSTRACT OF THE DISSERTATION

Stability Results in Additive Combinatorics and Graph Theory

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A general problem in Extremal Combinatorics asks about the maximum size of a collection of finite objects satisfying certain restrictions, and an ideal solution to it presents to you the objects which attain the maximum size.

In several problems, it is the case that any large set satisfying the given property must be similar to one of the few extremal examples.

Such *stability results* give us a complete understanding of the problem, and also make the result more flexible to be applied as a tool in other mathematical problems.

Stability results in additive combinatorics and graph theory constitute the main topic of this thesis, in which we solve a question of Erdös and Sárközy on sums of integers, and reprove a conjecture of Posa and Seymour on powers of hamiltonian cycles.

Along the way we prove stronger structural statements that have as a corollary the optimal solution to these problems. We also introduce a counting technique and two graph theory tools which we believe to be of great interest in their own right. Namely the *shifting method*, the *connecting lemma*, and a robust version of the classic Erdos-Stone Simonovits theorem.

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This thesis is a result of joint work with Asif Jamshed, Aiman Khalfalah and Endre Szemerédi. Chapter 2 springs from a collaboration with Aiman Khalfalah and Endre Szemerédi. Section 2.3 is a presentation of the work of Lagarias, Odlyzko and Shearer [27], and Section 2.5 of A.Khalfalah, S. Lodha, E. Szemerédi [19]. No originality is claimed over the material in these two sections, which are included in the thesis for the sake of completeness, and for a correct presentation of our own results. Chapter 3 is the product of a collaboration with Asif Jamshed and Endre Szemerédi.

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Dedication

To my grandparents, parents, brothers, sister, and all my family:

for giving me love and stability.

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Chapter 1

Introduction

A typical type of problem in additive combinatorics is to determine the maximum size of a set of integers satisfying some (additive) property.

For instance, Roth's theorem can be formulated as determining the maximum size of a set $A \subseteq \{1, \cdots, N\}$, with no three-term arithmetic progressions (i.e. with no solutions to $x + y = 2z$). The best known upper and lower bounds to this problem are still very far apart, and it seems hard to understand what large sets might look like.

In many other additive problems, however, we know exactly the extremal solutions.

An example is that of determining the maximum size of a sum-free set $A \subseteq \{1, \cdots, N\}$ (i.e. with no solutions to $x + y = z$). It is easy to see that such a set can have at most $\frac{N}{2}$ elements, and that there are only two extremal examples. Namely, the set of odd integers, and the set $\left\{\frac{N}{2} + 1, \ldots, N\right\}$.

The next interesting question to ask in such a problem is whether any large set satisfying the given property has to be similar to one of the extremal examples.

Such theorems are called *stability results* and constitute the main topic of this thesis. They are central to the solution of the two problems we study:

- Erdös and Sárközy's question on the maximum size of a set $A \subseteq \{1, \ldots, N\}$ with no two elements adding up to a perfect square (i.e. no solutions to $x + y = t^2$, with $t \in \mathbf{N}$)

- The Posa-Seymour conjecture that any graph G of order n, and minimum degree at least $(\frac{k-1}{k})n$, contains the $(k-1)^{th}$ power of a Hamiltonian cycle (defined to be a Hamiltonian cycle where each vertex is connected to the $k - 1$ following ones)

Whereas the first question is very much in the spirit of the examples we mentioned, the second problem is more of a classical graph theory one. This should come as no surprise, as stability results arise naturally in any problem where one tries to find how large a collection of finite objects (numbers, vectors, sets) can be, satisfying certain restrictions. They have, therefore, a natural relevance in Extremal Combinatorics, and in Extremal Graph Theory in particular.

Our goal is to show the utility of stability results through the solution of these two very different combinatorial problems.[1]

On the Erdös and Sárközy problem, a good first guess to a large set with the given property is that of all integers congruent to 1 (mod 3) (since 2 is not a square (mod 3)). Massias, in [30], found another set with slightly higher density $\frac{11}{32}$. Namely, that of all integers $x \in \{1, \ldots, N\}$, with $x \equiv 1 \pmod 4$, or with $x \equiv 14, 26, 30 \pmod{32}$.

Over a finite cyclic group Lagarias et al. [27] showed that one cannot find more than a $\frac{11}{32}$ fraction of the residue classes, with the sum of any two a quadratic nonresidue.

In chapter 2, we start by characterizing all the sets that achieve maximum density in the modular setting. Considering the corresponding sets of integers we prove a stability result, which states that if $A \subseteq \{1, \cdots, N\}$ has density slightly smaller than $\frac{11}{32}$ and no two elements that add up to a perfect square, then it has to be close to one of those extremal sets.

For a set A close to the extremal examples, we then use their structure together with a so-called shifting method and analytic techniques to count the number of solutions of $a_1 + a_2 = t^2$, with $a_1, a_2 \in A$, and $t \in \{1, \cdots, N\}$. We prove that this is greater than 0, so there exists two elements in A that add up to a perfect square, assuming $|A| \geq \frac{11}{32}$.

In conclusion, we determine not only the solution to the Erdös and Sárközy question $\left(\text{namely } \frac{11}{32}, \text{ for sufficiently large } N\right)$ but more interestingly, that any subset of the first N integers, with size close to $\frac{11}{32}N$ and no solutions to the given equation, has to

[1] It is nevertheless interesting to recall the historical interplay between the fields, as in Szemeredi's theorem, Plunnecke-Ruzsa inequality, or Balog-Szemeredi-Gowers theorem, only to mention a few. In Tao and Vu's [34] words *Additive combinatorics is a subfield of combinatorics so it is no surprise that graph theory plays an important role in this theory.*

be close to the extremal example given by Massias, or to a similar one we shall present.

The Posa-Seymour Conjecture was first proved for large graphs, via the Regularity Lemma, in [24]. In Chapter 3 we reprove this conjecture without using Regularity, thus showing that it also holds for graphs with a much smaller number of vertices.

A key ingredient is a stability version of the classic Erdos-Stone-Simonovits theorem. Say a graph G is α-*extremal* if it has a subset $A \subseteq V(G)$ with $(\frac{1}{k}-\alpha)n \leq |A| \leq (\frac{1}{k}+\alpha)n$, and edge density $d(A) < \alpha$. We show that a graph with minimum degree at least $\left(\frac{k-1}{k} - \epsilon\right)n$, which is not α-extremal, contains a complete balanced $(k+1)$-partite subgraph with color classes of size $\log n$.

Using this stability result and a new kind of connecting lemma, we prove not only Posa-Seymour conjecture, but a characterization of graphs with large minimum degree and no $(k-1)$-Hamiltonian cycle. That is the content of the main result in the chapter, theorem 21, which says that any graph G with $|V(G)| = n > n_0(\alpha)$, minimum degree at least $(\frac{k-1}{k} - \epsilon(\alpha))n$, and not containing a $(k-1)^{th}$ power of a Hamiltonian cycle, has to be α-extremal.

It is our belief that both the robust version of the Erdos-Stone-Simonovits theorem and the Connecting Lemma are very interesting in their own right, and will probably have many applications in Graph Theory in the near future.

Hopefully the reader will be inspired by the power of stability results demonstrated by the solution of these two problems, and will want to keep this work in the back of his mind, or at hand, or at least out of the shelf!

Chapter 2

Solution to the Erdös-Sárközy prolem

2.1 Introduction of the problem

In their seminal paper in 1981, on the sum and difference of a set of integers, Erdös and Sárközy [6] posed the following question:

What is the maximum cardinality of a subset of the first N integers, with the property (NS) that no two elements add up to a perfect square?

The set of all integers congruent to 1 (mod 3) is easily seen to satisfy this (since 2 is not a square (mod 3)), and it has density $\frac{1}{3}$. Massias [30] found another set with the slightly higher density $\frac{11}{32}$. Namely, the set of all integers $x \in \{1, \ldots, N\}$, with $x \equiv 1 \pmod 4$, or with $x \equiv 14, 26, 30 \pmod{32}$.

Over a cyclic group $\mathbb{Z}_m$ Lagarias et al. [27] showed that one cannot find more than $\frac{11}{32}m$ residue classes, with the sum of any two a square nonresidue. Using the result over cyclic groups, their authors gave in a subsequent paper [28] the first non-trivial upper bound of 0.475, for the density of an arbitrary set of integers with the property (NS)

An almost complete answer to the original problem was given by A.Khalfalah, S.Lodha, and E.Szemeredi [19]. Building as well on the modular counterpart result of Lagarias et al, they showed that for any given $\delta > 0$, and sufficiently large N, every subset of $\{1, \ldots, N\}$ with density at least $\frac{11}{32} + \delta$ contains two elements that add up to a perfect square.

It is relevant to mention here that the upper bound of $\frac{11}{32}N$ does not always hold though. Some abnormally behaving counter-examples can be found with a computer for small values of N, as $N = 79$.

The goal of the work in this chapter is to establish the exact value of $\frac{11}{32}N$ for the maximum possible density, for every N bigger than a well defined constant N_0 (and conclude that those small counterexamples are indeed an exception).

2.2 Main Lemmas

We manage to achieve the optimal bound here, building upon the work of [19] and [27]. For the sake of completeness, and for an understandable exposition of the material, we present the material in those two papers. This is done in Sections 2.3 and 2.5, in which we claim no originality, and only include the proofs necessary for a self-contained reading of the new material.

The solution for the problem over $\mathbb{Z}_n$ is explained in Section 2.3. In Section 2.4, we refine their study, characterizing the sets that achieve maximal density over $\mathbb{Z}_n$. It turns out that the example presented by Massias, and an identical one, are the only two possible extremal sets. We show a stability result: any set with property (NS), and density close to the maximum possible, is similar to one of these two extremal examples.

In Section 2.5, we explain the proof of the bound $(\frac{11}{32} + \delta)$ for subsets of the first N integers, studying their distribution when reduced modulo q_i, for distinct integers q_i. In Section 2.6, using the stability lemma of Section 2.4, we show that such argument can be extended to any set of integers of size at least $\frac{11}{32}N$, which is far from both the Massias examples.

Lastly, we prove in Section 2.7 that any subset of the first N integers of size at least $\frac{11}{32}N$ which is sufficiently close to one of the massias examples, has to have two elements that add to a perfect square. Therefore proving our main result

Theorem 1. *There is a constant N_0, such that for every $N > N_0$, any subset of the first N integers with property (NS) has density at most $\frac{11}{32}N$.*

A suggested road map to a first read of this chapter is to take theorem 2 for granted, and directly start with Section 2.5. We invite the reader to then move on to Sections 2.6, and 2.7, only borrowing the stable characterization of the extremal sets of theorem 13, which will make him fully equipped to understand our main theorem 1.

2.3 Exact solution over a finite cyclic group

The work of Lagarias, Odlyzko and Shearer

Let S be a subset of $\mathbb{Z}_n$, with the property

(NS) the sum of any two of its elements is not a square residue

In a remarkable work, Lagarias, Odlyzko and Shearer prove that, such a set must have density at most $\frac{11}{32}$. In this Section we reproduce their exact same proof, for the sake of completeness and to be able to build upon it. No originality is claimed in any of the arguments till Section 2.4. Only the proofs necessary for a clear phrasing of the exposition are included. For the remaining material we direct the reader to the original work [27].

We will explain their following main result

Theorem 2. *Let S be a subset of $\mathbb{Z}_n$, such that the sum of any two of its elements is not a square residue. Then, denoting $d(S)$ the density of S, we have that $d(S) \leq \frac{11}{32}$.*

To prove this theorem, the mentioned authors first transfer the problem to the following graph and corresponding linear program.

Let Q_n denote the graph with vertex set $\mathbb{Z}_n$, and $(x, y) \in \mathbb{Z}_n \times \mathbb{Z}_n$ an edge if and only if $x + y \equiv t^2 \pmod{n}$, for some $t \in \mathbb{Z}_n$.

Finding the maximum size of a subset of $\mathbb{Z}_n$ with property (NS), is the same as finding the maximum size of an independence set in the graph Q_n. Let $\alpha(n)$ denote the independence number of the graph Q_n, and $i(n) = \frac{\alpha(n)}{n}$ its independence ratio. We aim to show that $i(n) \leq \frac{11}{32}$, for any integer n.

Since the independence number of a graph is by definition the size of its maximal independent set, we can determine it by solving the linear program $L(Q_n)$, with objective function

$$z = \sum_{i \in Z_n} x_i$$

and constraints

$$x_i + x_j \leq 1, \text{ for any } i, j \in Q_n,$$

$$x_i = 0, 1, \text{ for any } i$$

If we replace a subset of these constraints by a different one satisfied by all the solutions of $L(Q_n)$, we get a different linear program whose solution is an upper bound for the original one. This is called *loosening* the problem. For example one can replace some of the conditions by

$$\sum_{i \in H} x_i \leq \alpha(H)$$

for any subgraph H of Q_n. In particular:

-if three vertices x_i, x_j, x_k of Q_n form a triangle, we can add the constraint

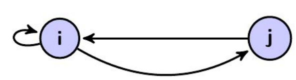
$$x_i + x_j + x_k \leq 1$$

-if Q_n has a loop on a vertex i, all solutions to $L(Q_n)$ must have $x_i = 0$. So we can add the constraint

$$3x_i \leq 1$$

-if Q_n has as a subgraph a collapsed triangle, $\{i, k\}$, with a loop on i, we can add the constraint

$$2x_i + x_j \leq 1$$



(the choice of these coefficients might seem mysterious and artificial at the moment, but it will reveal its utility further on in the argument)

Also, if our objective function $z = \sum z_j$, for z_j objective functions of linear programs L_j, whose constraints are a subset of $L(Q_n)$, then the sum of their optimal solutions, is an upper bound for the optimal solution of $L(Q_n)$. This is called *decomposing* the problem.

We will use both the techniques in the following manner.

Definition 1. *Define multiplicity of a vertex of a triangle, collapsed triangle or loop, to be its coefficient in the correspondent equation above.*

Definition 2. *A graph G has a d-uniform covering by some of its subgraphs H_i, which are triangles, collapsed triangles, or loops, if any vertex of G occurs d times in the subgraphs H_i, when counted with multiplicity.*

The following number theoretical result is a cornerstone of the argument. We refer the interested reader to the proof in the original paper [27].

Lemma 3. *If m is an odd number, then Q_m admits a d-uniform covering by triangles, loops, and collapsed triangles; for some integer d.*

By the definition of uniform covering one can also prove

Lemma 4. *If a graph G has a d-uniform covering by subgraphs $H_i, i = 1, \ldots, k$ which are triangles, collapsed triangles, or loops, then*

$$\alpha(G) \leq \frac{1}{d} \sum_{i=1}^{k} \alpha(H_i)$$

From these two lemmas it is easy to conclude

Corollary 2.3.1. *If m is an odd number, then $i(Q_m) \leq \frac{1}{3}$.*

Therefore, we may restrict our attention to $Q_{2^n m}$, with $n \geq 1$, m odd.

Define the *product of two graphs* G and H, to be the graph $G \times H$, with vertex set $V(G) \times V(H)$, and $\Big((x, y), (w, z)\Big)$ an edge of $G \times H$, if and only if (x, w) is an edge of G and (y, z) is an edge of H.

Lemma 5. *If $n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$, then*

$$Q_n \cong Q_{p_1^{a_1}} \times Q_{p_2^{a_2}} \times Q_{p_k^{a_k}}$$

Proof. The proof follows from the chinese remainder theorem.

□

In particular, $Q_{2^n m} \cong Q_{2^n} \times Q_m$, which reduces our problem to showing that $i(Q_{2^n} \times Q_m) \leq \frac{11}{32}$. The following lemma reveals that in fact, it is enough to show that $i(Q_{2^n} \times T) \leq \frac{11}{32}$.

Lemma 6. *If a graph H has a d-uniform covering by triangles, collapsed triangles, and loops, for some integer d, then $i(G \times H) \leq i(G \times T)$.*

Proof. Let $H_i, i = 1, \dots, k$ be a d-uniform covering of H.

For each H_i, let L_i be the 0-1 integer program with objective function

$$z_i = \sum_{j \in G \times H_i} a_j x_j \tag{$\star$}$$

and constraints

$$x_{j_1} + x_{j_2} \leq 1, \text{ for any } j_1, j_2 \text{ which form an edge in } G \times H_i$$

where for $j = (g, h) \in G \times H_i,\ a_j$ denotes the multiplicity of h in H_i.

Let $\beta(G \times H_i)$ denote the optimal solution of this linear program.

Letting $z = \sum_{j \in G \times H} x_j$ be the objective function of the linear program for the independence number of $G \times H$, from the definition of d-uniform covering

$$dz = d \sum_{j \in G \times H} x_j = \sum_{i=0}^{k} \sum_{j \in G \times H_i} a_j x_j = \sum_{i=0}^{k} z_i$$

By decomposition of a linear program, we get

$$d \cdot \alpha(G \times H) \leq \sum_{i=0}^{k} \beta(G \times H_i)$$

If H_i is a triangle T, then all coefficients $a_j = 1$ in the objective function (3.1) of L_i. Thus L_i is just the linear program that computes the independence number of $G \times T$. That is $\beta(G \times T) = \alpha(G \times T)$.

For H_i a collapsed triangle, the linear program L_i can be obtained from the program for the independence number of $G \times T$, by adding extra constraints $x_{(g,1)} = x_{(g,3)}$, for every $g \in G$. Hence $\beta(G \times H_i) \leq \alpha(G \times T)$.

For H_i a loop, the linear program L_i can be obtained from the program for the independence number of $G \times T$, by adding constraints $x_{(g,1)} = x_{(g,2)}$, and $x_{(g,2)} = x_{(g,3)}$, for every $g \in G$. Hence $\beta(G \times H_i) \leq \alpha(G \times T)$.

Therefore

$$\alpha(G \times H) \leq \frac{1}{d} \sum_{i=0}^{k} \alpha(G \times T)$$

By d-uniformity, we have $3k = dm$, and so this yelds

$$\frac{\alpha(G \times H)}{m} \leq \frac{\alpha(G \times T)}{3}$$

that is $i(G \times H) \leq i(G \times T)$.

□

Let us show that $i(Q_n \times T) \leq \frac{11}{32}$.

Partition the set of vertices $V(Q_n \times T)$ into eight classes $V_0, V_1, \ldots, V_7$, according to the residue class (mod 8) of their first coordinate. Namely

$$V_i = \{(x,t)|\ x \equiv i \pmod 8, t = a, b, c\}$$

Let I be an independent set of $Q_n \times T$. Let α_i denote the proportion of elements of V_i that are in I. That is

$$\alpha_i = \frac{|I \cap V_i|}{|V_i|}$$

Our goal is to show that

$$\sum_{i=1}^{8} \alpha_i \leq 2 + \frac{3}{4} \quad \Big(= 8 \cdot \frac{11}{32}\Big)$$

We'll derive it from the following observations:

1. (a) $\alpha_0 = 0$, or $\alpha_1 = 0$, or $(\alpha_0 =\leq 1/3$ and $\alpha_1 =\leq 1/3$)
   (b) $\alpha_2 = 0$, or $\alpha_7 = 0$, or $(\alpha_2 =\leq 1/3$ and $\alpha_7 =\leq 1/3$)
   (c) $\alpha_3 = 0$, or $\alpha_6 = 0$, or $(\alpha_3 =\leq 1/3$ and $\alpha_6 =\leq 1/3$)
   (d) $\alpha_4 = 0$, or $\alpha_5 = 0$, or $(\alpha_4 =\leq 1/3$ and $\alpha_5 =\leq 1/3$)
2. $\alpha_0 + \alpha_4 \leq \frac{11}{16}$
3. (a) $\alpha_1 + \alpha_7 \leq 1$
   (b) $\alpha_3 + \alpha_5 \leq 1$
4. (a) $\alpha_1 + \alpha_3 \leq 1$
   (b) $\alpha_5 + \alpha_7 \leq 1$
5. $2\alpha_6 \leq 1$
6. $2\alpha_2 + \alpha_6 \leq 1$

Proof.

We will prove the above inequality for independent sets of $Q_{2^n m}$, by induction on n . The cases $n = 1, 2$ can be easily verified, so assume $n \geq 3$. Recall that an element $x \in \mathbb{Z}_{2^n}$ is a quadratic residue if and only if $x = 4^k y$, with $y \equiv 1 \pmod 8$ [15].

1. In particular, any $x \in \mathbb{Z}_{2^n}$ with $x \equiv 1 \pmod 8$ is a quadratic residue.

    (a) Suppose that $\alpha_0 \neq 0$.

    Consider an element $(x, t) \in I$, with $x \equiv 0 \pmod 8$, $t = a, b, c$. Without loss of generality, $(x, a) \in I$.

    Then (x, a) is adjacent to (y, b), and (y, c), for any $y \equiv 1 \pmod 8$.

    Therefore $\alpha_1 \leq \frac{1}{3}$.

    (b) (c), and (d) have analogous proofs.

2. Notice that $Q_{2^n m} \times T \restriction_{V_0 \cup V_4} \cong Q_{2^{n-3} m} \times T$, as witnessed by the isomorphism

$$\phi \colon Q_{2^{n-3} m} \times T \to Q_{2^n m} \times T \restriction_{V_0 \cup V_4}$$

$$(x, t) \mapsto (4x, t)$$

By the induction hypothesis we get that $\alpha_0 + \alpha_4 \leq \frac{11}{16}$.

3. We'll exhibit matchings between V_1 and V_7, and between V_3 and V_5.

    Consider the set of edges $\{(x, a), (-x, b)\}, \{(x, b), (-x, c)\}, \{(x, b), (-x, c)\}$

    (a) Letting $x \equiv 1 \pmod 8$, we get a matching between V_1, and V_7.

    Hence $\alpha_1 + \alpha_7 \leq 1$

    (b) Letting $x \equiv 3 \pmod 8$, we get a matching between V_3, and V_5. Hence $\alpha_3 + \alpha_5 \leq 1$.

4. Consider edges $\{(x, a), (4 - x, b)\}, \{(x, b), (4 - x, c)\}, \{(x, c), (4 - x, a)\}$

    (a) Letting $x \equiv 1 \pmod 8$, we get a matching between V_1, and V_3.

    Hence $\alpha_1 + \alpha_3 \leq 1$

    (b) Letting $x \equiv 5 \pmod 8$, we get a matching between V_5, and V_7.

    Hence $\alpha_5 + \alpha_7 \leq 1$

5. The same set of edges of (4), gives us a matching between the vertices of V_6 with $x \equiv 6 \pmod{16}$, and those with $y \equiv 4 - x \equiv 14 \pmod{16}$.

    Hence $2\alpha_6 \leq 1$.

6. For any $x \equiv 1 \pmod 8$, the following are triangles in $Q_{2^n m} \times T$:

$\{(-2x, a), (2x, b), (2x, c))\}, \{(-2x, b), (2x, c), (2x, a))\}, \{(-2x, c), (2x, a), (2x, b))\}.$

   Therefore $2\alpha_2 + \alpha_6 \leq 1$.

□

From these inequalities we can deduce the following ones

7. (a) $\alpha_0 + \alpha_1 \leq 1$
   (b) $\alpha_2 + \alpha_7 \leq 1$
   (c) $\alpha_3 + \alpha_6 \leq 1$
   (d) $\alpha_4 + \alpha_5 \leq 1$
8. $\alpha_2 + \alpha_6 \leq \frac{3}{4}$
9. (a) $\alpha_2 + \alpha_6 + \alpha_1 + \alpha_3 + \alpha_7 \leq 2$
   (b) $\alpha_2 + \alpha_6 + \alpha_3 + \alpha_5 + \alpha_7 \leq 2$
10. (a) $\alpha_0 + \alpha_4 + \alpha_1 + \alpha_3 + \alpha_5 \leq 2$
    (b) $\alpha_0 + \alpha_4 + \alpha_1 + \alpha_5 + \alpha_7 \leq 2$

Proof.

7. (a)-(d) follow easily from 1(a)-(d).
8. Adding inequality (5) with twice inequality (6), we get $\alpha_2 + \alpha_6 \leq \frac{3}{4}$.
9. Follows by a case by case analysis. We refer interested reader to [27].
10. Follows by a case by case analysis. We refer interested reader to [27].

□

With observations (1)-(10) in hand, we may now prove that

$$S = \sum_{i=1}^{8} \alpha_i \leq 2 + \frac{3}{4}$$

Proof. If all the $\alpha_i's$ are non-zero, by (1) they are all at most $\frac{1}{3}$. So $S \leq 2 + \frac{2}{3}$.

We may assume that at least one of α_i equals 0.

Case $\alpha_1 = 0$ Then $S = (\alpha_0 + \alpha_4) + \alpha_2 + \alpha_6 + \alpha_3 + \alpha_5 + \alpha_7$. By (2) and (9)(b), we get that $S \leq 2 + \frac{11}{16} < 2 + \frac{3}{4}$.

Case $\alpha_5 = 0$ Then $S = (\alpha_0 + \alpha_4) + \alpha_2 + \alpha_6 + \alpha_1 + \alpha_3 + \alpha_7$. By (2) and (9)(a), we get that $S \leq 2 + \frac{11}{16} < 2 + \frac{3}{4}$.

Case $\alpha_3 = 0$ Then $S = (\alpha_2 + \alpha_6) + \alpha_0 + \alpha_4 + \alpha_1 + \alpha_5 + \alpha_7$. By (8) and (10)(b), we get that $S \leq 2 + (\alpha_2 + \alpha_6) \leq 2 + \frac{3}{4}$

Case $\alpha_7 = 0$ Then $S = (\alpha_2 + \alpha_6) + \alpha_0 + \alpha_4 + \alpha_1 + \alpha_3 + \alpha_5$. By (8) and (10)(a), we get that $S \leq 2 + (\alpha_2 + \alpha_6) \leq 2 + \frac{3}{4}$

Assume $\alpha_1, \alpha_3, \alpha_5, \alpha_7$ all non-zero. So their sum is at most 2, by (3). And also $\alpha_0, \alpha_2, \alpha_4, \alpha_6 \leq \frac{1}{3}$ according to (1).

So if two of $\alpha_0, \alpha_2, \alpha_4, \alpha_6$ are zero, then $S \leq 2 + 2/3$

Hence exactly one of these equals 0.

If for example, $\alpha_0 = 0$ and $\alpha_2, \alpha_4, \alpha_6 \neq 0$, then $\alpha_7, \alpha_5, \alpha_3 \leq \frac{1}{3}$ (by (1)). Therefore, $S = \alpha_2 + \alpha_4 + \alpha_6 + (\alpha_1 + \alpha_3) + \alpha_5 + \alpha_7 \leq 1 + \frac{5}{3} < 2 + \frac{3}{4}$ (according to (4)(a)).

An analogous argument shows that in the remaining cases where only one of $\alpha_0, \alpha_2, \alpha_4, \alpha_6$ is zero, we also get $S \leq 2 + \frac{2}{3}$.

We conclude that $S \leq 2 + \frac{3}{4}$, and so theorem 2 holds.

□

2.4 Stability over a finite cyclic group

Characterization of the extremal examples

Let

$$M_{10} = \{x \in \mathbb{Z}_{2^n} |\ x \equiv 1 \pmod 4, \text{ or } x \equiv 10, 14, 30 \pmod{32}\}$$

$$M_{26} = \{x \in \mathbb{Z}_{2^n} |\ x \equiv 1 \pmod 4, \text{ or } x \equiv 14, 26, 30 \pmod{32}\}$$

We will show that there are only two extremal independent sets of $Q_{2^n} \times T$.

Theorem 7. *Let I be an independent set of $Q_{2^n} \times T$, with $|I| = \frac{11}{32} \cdot 2^n 3$. Then either $I = M_{10} \times T$ or $I = M_{26} \times T$*

Proof. Going through the cases of the proof of theorem 2, notice that for an independent set to demonstrate exact equality $\sum_{i=1}^{8} \alpha_i = 2 + \frac{3}{4}$, one must have:

- $\alpha_1 \neq 0$ (otherwise $S \leq 2 + \frac{11}{16}$)
- $\alpha_5 \neq 0$ (otherwise $S \leq 2 + \frac{11}{16}$)
- $\alpha_2 + \alpha_6 = 3/4$. This implies (by (5) and (6)):
  - $\alpha_2 = \frac{1}{4}$
  - $\alpha_6 = \frac{1}{2}$
- $\alpha_3 = 0$ $\big($since $\alpha_6 = \frac{1}{2}$, and because of (1)$\big)$.
- Hence $S = \alpha_0 + \alpha_1 + \alpha_4 + \alpha_5 + \alpha_7 + \frac{3}{4}$. By the previous points and (1) we know that $\alpha_0 \leq \frac{1}{3}$, $\alpha_1 \neq 0$, $\alpha_4 \leq \frac{1}{3}$, $\alpha_5 \neq 0$, $\alpha_7 \leq \frac{1}{3}$.

  If all of these are at most $\frac{1}{3}$, then $S \leq \frac{5}{3} + \frac{3}{4} < 2 + \frac{3}{4}$.

  Thus either $\alpha_1 > \frac{1}{3}$, or $\alpha_5 > \frac{1}{3}$.

  If $\alpha_1 > \frac{1}{3}$, then $\alpha_0 = 0$ by (1), and $S = \alpha_1 + \alpha_4 + \alpha_5 + \alpha_7 + \frac{3}{4}$, This is $\leq 2 + \frac{3}{4}$, because of (3)(a) and (7)(d), and equality holds if and only if $\alpha_5 = 1$, and $\alpha_4 = 0$, according to (1)(d).

  If $\alpha_5 > \frac{1}{3}$, then $\alpha_4 = 0$, and again equality can hold iff $\alpha_0 = 0$ (analogous argument). We conclude that:

  - $\alpha_0 = 0$
  - $\alpha_4 = 0$

  So $S = \alpha_1 + \alpha_5 + \alpha_7 + \frac{3}{4}$.

  According to (3)(a), 5(a), and the fact that all of the α_i's are at most 1, we see that equality can hold iff

- $\alpha_1 = 1$, and $\alpha_5 = 1$.

  Finally the inequality $S \leq 2 + 3/4$ guarantees that

- $\alpha_7 = 0$.

In conclusion, we have proved that equality $S = 2 + \frac{3}{4}$ occurs only if:

- $\alpha_0 = 0$
- $\alpha_1 = 1$
- $\alpha_2 = 1/4$
- $\alpha_3 = 0$
- $\alpha_4 = 0$
- $\alpha_5 = 1$
- $\alpha_6 = 1/2$
- $\alpha_7 = 0$

Before completing the exact characterization of the extremal sets, let us look at what we have just obtained, and observe how much smaller than $2 + \frac{3}{4}$ does the sum S get, if any of the α_i's differ from the value in the list above.

Proposition 2.4.1. *If* $S \geq 2 + \frac{3}{4} - \delta$*, for some* $0 \leq \delta < \dfrac{1}{16}$*, then*

- $\alpha_0 = 0$
- $\alpha_1 \geq 1 - \delta$
- $\alpha_2 \leq \frac{1}{4} + \delta$ $\quad\big($ *and* $\alpha_2 \geq \frac{1}{4} - \delta$*, by (5) and (8)*$\big)$
- $\alpha_3 = 0$
- $\alpha_4 = 0$
- $\alpha_5 \geq 1 - \delta$
- $\alpha_6 \geq \frac{1}{2} - 2\delta$ $\quad\big($*and* $\alpha_6 \leq \frac{1}{2}$*, by (6)*$\big)$
- $\alpha_7 \leq 2\delta$

Proof. Following the previous proof we see that if $\alpha_1 = 0$, or $\alpha_5 = 0$, then $S = 2 + \frac{11}{16} < 2 + \frac{3}{4} - \delta$. Thus $\alpha_1 \neq 0$, and $\alpha_5 \neq 0$.

Secondly, if $S \geq 2 + \frac{3}{4} - \delta$, then necessarily $\alpha_2 + \alpha_6 \geq \frac{3}{4} - \delta$. This implies by (5) and (6) that

- $\alpha_6 \geq \frac{1}{2} - 2\delta$
- $\alpha_2 \leq \frac{1}{4} + \delta$

The remaining analysis follows each of the cases in the previous proof in an entirely analogous way. Details are left to the reader as a good exercise. □

We have seen so far that a maximal independent set of $Q_{2^n} \times T$, restricted to a class V_i, other than V_2 or V_6 is either full, or it is the empty set. The remaining question is how the elements of the classes V_2, and V_6 are distributed. To finish the characterization, we refine our partition of the vertex set of $Q_{2^n} \times T$ into 32 classes, $U_0, U_1, U_2, \ldots, U_{31}$, according to the residue class (mod 32) of their first coordinate. Namely

$$U_i = \{(x,t) |\ x \equiv i \pmod{32}, t = a, b, c\}$$

Let α_i^* denote the proportion of elements of U_i that are in the independent set I. That is

$$\alpha_i^* = \frac{|I \cap U_i|}{|U_i|}$$

Partitioning each of the classes V_2, and V_6, into the respective four classes (mod 32), we get $V_2 = U_2 \cup U_{10} \cup U_{18} \cup U_{26}$, and $V_6 = U_6 \cup U_{14} \cup U_{22} \cup U_{30}$.

We showed that if $S = 2 + \frac{3}{4}$, then $\alpha_2 = \frac{1}{4}$, and $\alpha_6 = \frac{1}{2}$. That is

$$\alpha_2^* + \alpha_{10}^* + \alpha_{18}^* + \alpha_{26}^* = 1, \text{ and } \alpha_6^* + \alpha_{14}^* + \alpha_{22}^* + \alpha_{30}^* = 2$$

Again we have some constraints among the α_i^*'s.

Recall that an element $x \in \mathbb{Z}_{2^n}$ is a quadratic residue if and only if $x = 4^k y$, with $y \equiv 1 \pmod 8$ [15]. In particular, any $x \equiv 4 \pmod{32}$ is a quadratic residue, and so,

(i) (a) $\alpha_6^* = 0$, or $\alpha_{30}^* = 0$, or $(\alpha_6^* \leq \frac{1}{3}$ and $\alpha_{30}^* \leq \frac{1}{3})$

(b) $\alpha_{10}^* = 0$, or $\alpha_{26}^* = 0$, or $(\alpha_{10}^* \leq \frac{1}{3}$ and $\alpha_{26}^* \leq \frac{1}{3})$

(c) $\alpha_{14}^* = 0$, or $\alpha_{22}^* = 0$, or $(\alpha_{14}^* \leq \frac{1}{3}$ and $\alpha_{22}^* \leq \frac{1}{3})$

(d) $\alpha_{18}^* \leq \frac{1}{3}$

(e) $\alpha_2^* \leq \frac{1}{3}$

Proof. (1)(a):

Assume that $\alpha_6^* \neq 0$.

Consider an element $(x, t) \in I$, with $x \equiv 6 \pmod{32}$, $t = a, b, c$. Without loss of generality, $(x, a) \in I$.

By the above mentioned fact, (x, a) is adjacent to (y, b), and (y, c), for any $y \equiv 30 \pmod 8$.

Therefore $\alpha_{30}^* \leq \frac{1}{3}$.

(b), (c), (d), and (e) have analogous proofs. □

(ii) (a) $\alpha_6^* + \alpha_{26}^* \leq 1$,

(b) $\alpha_{14}^* + \alpha_{18}^* \leq 1$

(c) $\alpha_{22}^* + \alpha_{10}^* \leq 1$

(d) $\alpha_{30}^* + \alpha_2^* \leq 1$

Proof. The proof is entirely analogous to that of (3)(a), (b) (noticing that 0 is the square of a residue class) □

(iii) (a) $\alpha_2^* + \alpha_{14}^* \leq 1$,

(b) $\alpha_{10}^* + \alpha_6^* \leq 1$

(c) $\alpha_{18}^* + \alpha_{30}^* \leq 1$

(d) $\alpha_{26}^* + \alpha_{22}^* \leq 1$

Proof. The proof is entirely analogous to that of (4)(a), (b) (replacing 4 by 16, and noticing that 16 is a quadratic residue in Z_{2^n}) □

Consider $\alpha_6^*, \alpha_{14}^*, \alpha_{22}^*$, and α_{30}^*. We assumed that their sum is 2. So according to (1)(a)-(d), they cannot al be different from zero.

If three of them are zero, since each is at most 1, their sum is at most 1, a contradiction.

If only one is zero, we also get a contradiction. For example if only $\alpha_6^* = 0$, then $\alpha_{14}^* \le \frac{1}{3}$ and $\alpha_{22}^* \le \frac{1}{3}$ (i(c)). And so $\alpha_6^* + \alpha_{14}^* + \alpha_{22}^* + \alpha_{30}^* \le \alpha_{30}^* + \frac{2}{3} \le 1 + \frac{2}{3}$ a contradiction (Analogous proof when only another one of the four $\alpha_i's$ is zero).

So exactly two of $\alpha_6^*, \alpha_{14}^*, \alpha_{22}^*, \alpha_{30}^*$ equal 0, and since their sum equals 2, the remaining two equal 1. By (1)(a), (d), this can only happen in the following four cases:

<u>Case 1: $\alpha_6^* = 1, \alpha_{30}^* = 0, \alpha_{14}^* = 1, \alpha_{22}^* = 0$:</u>

Then by (2), and (3), $\alpha_2^*, \alpha_{10}^*, \alpha_{18}^*, \alpha_{26}^* = 0$, whereas their sum should be 1. Contradiction.

<u>Case 2: $\alpha_6^* = 1, \alpha_{30}^* = 0, \alpha_{14}^* = 0, \alpha_{22}^* = 1$:</u>

By (2)(a), (3)(b), and (1)(d)(e), we get $\alpha_2^* \le \frac{1}{3}, \alpha_{10}^* = 0, \alpha_{18}^* \le \frac{1}{3}, \alpha_{26}^* = 0$, whereas their sum should be 1. Contradiction.

<u>Case 3: $\alpha_6^* = 0, \alpha_{30}^* = 1, \alpha_{14}^* = 0, \alpha_{22}^* = 1$:</u>

By (2) and (3), we get $\alpha_2^*, \alpha_{10}^*, \alpha_{18}^*, \alpha_{26}^* = 0$, whereas their sum should be 1. Contradiction.

<u>Case 4: $\alpha_6^* = 0, \alpha_{30}^* = 1, \alpha_{14}^* = 1, \alpha_{22}^* = 0$:</u>

By (2) and (3), we get $\alpha_2^* = 0$, $\alpha_{18}^* = 0$. By (1)(b) α_{10}^*, and α_{26}^* cannot be both nonzero, since the sum of these four equals 1. One concludes that either ($\alpha_{10}^* = 1$, $\alpha_{26}^* = 0$), or ($\alpha_{10}^* = 0$, $\alpha_{26}^* = 1$)

That is, our set is either $M_{10} \times T$, or $M_{26} \times T$, proving theorem 7 $\square$

As noticed in proposition 2.4.1, if $S \ge 2 + \frac{3}{4} - \delta$, for some $0 \le \delta < \dfrac{1}{16}$, then

- $\alpha_2^* + \alpha_{10}^* + \alpha_{18}^* + \alpha_{26}^* = 4\alpha_2 \le 1 + 4\delta \quad \big(\text{and} \ge 1 - 4\delta\big)$
- $\alpha_6^* + \alpha_{14}^* + \alpha_{22}^* + \alpha_{30}^* = 4\alpha_6 \ge 2 - 8\delta \quad \big(\text{and} \le 2\big)$

Following the argument and cases above, this can only happen if

$$\alpha_6^* = 0,\ \alpha_{30}^* \ge 1 - 8\delta,\ \alpha_{14}^* \ge 1 - 8\delta,\ \alpha_{22}^* = 0$$

and either

$$\alpha_2^* = 0,\ \alpha_{10}^* = 0,\ \alpha_{18}^* = 0,\ (1 - 4\delta) \le \alpha_{26}^* \le 1,$$

or

$$\alpha_2^* = 0,\ (1 - 4\delta) \le \alpha_{10}^* \le 1,\ \alpha_{18}^* = 0,\ \alpha_{26}^* = 0$$

In particular we conclude the following

Theorem 8. *Let $A \subseteq V(Q_{2^n} \times T)$ be an independent set, with symmetric difference $|A \triangle (M_{10} \times T)| \geq \delta 2^n 3$, and $|A \triangle (M_{26} \times T)| \geq \delta 2^n 3$, $\left(0 \leq \delta < \dfrac{1}{2}\right)$. Then $|A| \leq \left(\frac{11}{32} - \frac{\delta}{64}\right) 2^n 3$.*

Proof. Let $A \subseteq Q_{2^n} \times T$ be an independent set.

Suppose the symmetric difference $|A \triangle M_{10}| \geq \delta 2^n$, and $|A \triangle M_{26}| \geq \delta 2^n$.

Then, there is a subclass $U_i, i = 0, \ldots, 31$ for which the proportion of elements of A in it differs from the listed value of α_i^* in an optimal set by at least δ. From the conclusion above, this implies that the sum of the proportion of elements of A in the classes V_i, is at most $2 + \frac{3}{4} - \frac{\delta}{8}$ or equivalently, that the total density of A is at most $\frac{11}{32} - \frac{\delta}{64}$. □

An analogous result holds for $Q_{2^n} \times H$, if H is a collapsed triangle or a loop.

Given $A \subseteq Q_{2^n} \times H$, we may see it as a multiset with the multiplicity of an element $j = (g, h) \in Q_{2^n} \times H_i$, that of h in the equation for the graph H (as in definition 1). Denote by $||A||$ the size of A as a multiset.

Theorem 9. *Let H be a collapsed triangle or a loop. Let $A \subseteq V(Q_{2^n} \times H)$ be an independent set, with symmetric difference $||A \triangle (M_{10} \times H)|| \geq \delta 2^n 3$, and $||A \triangle (M_{26} \times H)|| \geq \delta 2^n 3$, $\left(0 \leq \delta < \dfrac{1}{2}\right)$. Then $||A|| \leq \left(\frac{11}{32} - \frac{\delta}{64}\right) 2^n 3$.*

Proof. For a collapsed triangle or loop subgraph H_i, let L_i be the 0-1 integer program with objective function

$$z_i = \sum_{j \in G \times H_i} a_j x_j \qquad (\star)$$

and constraints

$$x_{j_1} + x_{j_2} \leq 1, \text{ for any } j_1, j_2 \text{ which form an edge in } G \times H_i$$

where given $j = (g, h) \in Q_{2^n} \times H_i$, a_j denotes the multiplicity of h in H_i.

Assume H_i is a collapsed triangle. The maximum size of an independent multiset of $Q_{2^n} \times T$ whose symmetric difference to $M_{10} \times H_i$, and $M_{26} \times H_i$ is at least $\delta 2^n 3$, can be obtained from solving the linear program L_i with extra linear constraints (each imposing that the set is not one of those for which this difference is small). On the other hand the linear program L_i with these extra constraints can be obtained from that for the maximum size of an independent subset of $Q_{2^n} \times T$ whose symmetric difference to $M_{10} \times T$, and $M_{26} \times T$ is at least $\delta 2^n 3$, by adding extra constraints $x_{(g,1)} = x_{(g,3)}$, for every $g \in Q_{2^n}$.

Thus the theorem holds (an analogous argument works when H_i is a loop). $\square$

From $Q_{2^n} \times T$ to $Q_{2^n m}$

So far we only proved that an independent set of $Q_{2^n} \times T$ of maximum cardinality has to be either $M_{10} \times T$ or $M_{26} \times T$. We want to show the following

Theorem 10. *Let S be a set of $Q_{2^n} \times Q_m$ with no two elements adding to a square, and $|S| = \frac{11}{32} 2^n m$. Then either $S = M_{10} \times Q_m$ or $S = M_{26} \times Q_m$.*

Proof. By theorem 6 we know that an independent set $S \subseteq Q_{2^n} \times Q_m$ can attain maximum density only if its restriction to each of the subgraphs of the uniform covering $Q_{2^n} \times H_i$, is a maximum independent set.

Hence, by theorem 7, we know that if $H_i = T$ is a triangle, the restriction of S to $Q_{2^n} \times T$ has to be equal to $M_{10} \times T$, or $M_{26} \times T$.

Analogously, if H_i is a collapsed triangle, or a loop, the restriction of S to $Q_{2^n} \times H_i$ has to be equal to $M_{10} \times H_i$, or $M_{26} \times H_i$ (analogous argument to the proof of theorem 9 from theorem 8).

Therefore, S is a union of sets of the form $M_{10} \times H_i$, or $M_{26} \times H_j$, for subgraphs H_i, H_j. Let us show it is the union of only one such type of set.

Partition the vertex set of Q_m into two sets

$$A = \{x \in Q_m | M_{10} \times \{x\}, \subseteq S\}$$

$$\bar{A} = \{y \in Q_m | M_{26} \times \{y\} \subseteq S\}$$

Observe that no two elements $a \in A$, and $b \in \bar{A}$, can be adjacent in Q_m.

(Otherwise $\{(i,a),(j,b)\}$ would be an edge in $Q_{2^n} \times Q_m$, between two elements of S, for every $i \equiv 10 \pmod{32}$, $j \equiv 26 \pmod{32}$)

But this is a contradiction, as one cannot partition the vertices of Q_m into two sets, with no edges between them. In fact, let us count the number ν of solutions of

$$a + b \equiv t^2 \pmod{m}, \text{ with } a \in A, b \in \bar{A}, t \in Z_m$$

There is at least one edge between A and $\bar{A}$, provided that $\nu > 0$.

Denote $e(x) := e^{2\pi i x}$, and let

$$f_A(\alpha) = \sum_{a \in A} e(\alpha\, a), \quad f_{\bar{A}}(\alpha) = \sum_{b \in \bar{A}} e(\alpha\, b), \quad f_{SQ}(\alpha) = \sum_{z=0}^{m-1} e(\alpha\, z^2)$$

We have that

$$\nu = \frac{1}{m} \sum_{t=0}^{m} f_A\Big(\frac{t}{m}\Big)\, f_{\bar{A}}\Big(\frac{t}{m}\Big) f_{SQ}\Big(-\frac{t}{m}\Big).$$

That is

$$\nu = \frac{1}{m} f_A(0)\, f_{\bar{A}}(0) f_{SQ}(0) + \frac{1}{m} \sum_{t \neq 0} f_A\Big(\frac{t}{m}\Big)\, f_{\bar{A}}\Big(\frac{t}{m}\Big) f_{SQ}\Big(-\frac{t}{m}\Big).$$

The main term is

$$\frac{1}{m} f_A(0)\, f_{\bar{A}}(0) f_{SQ}(0) = \frac{1}{m} \sum_{a \in A} e^0 \sum_{b \in \bar{A}} e^0 \sum_{t \in Z_m} e^0 = |A|\,|\bar{A}|.$$

Let us bound the absolute value of the error term

$$\left| \frac{1}{m} \sum_{t \neq 0} f_A\Big(\frac{t}{m}\Big)\, f_{\bar{A}}\Big(\frac{t}{m}\Big) f_{SQ}\Big(\frac{-t}{m}\Big) \right|.$$

Let $\frac{t}{m} = \frac{a}{q}$, with a and q coprime integers, $f_{SQ}\Big(\frac{-t}{m}\Big) = \sum_{x=0}^{m} e\Big(-\frac{a}{q}\, x^2\Big)$. Write x in base q: $\quad x = kq + l$, with $k = 0, \ldots, \lfloor \frac{m}{q} \rfloor$, $l = 0, \ldots, q-1$.

Then $x^2 = k^2 q^2 + 2lkq + l^2$, and so

$$\begin{aligned} f_{SQ}\Big(\frac{-t}{m}\Big) &= \sum_{k=0}^{\lfloor \frac{m}{q} \rfloor} \sum_{l=0}^{q-1} e\Big(k^2 aq + 2lka + l^2 \frac{a}{q}\Big) \\ &= \sum_{k=0}^{\lfloor \frac{m}{q} \rfloor} \sum_{l=0}^{q-1} e\Big(l^2 \frac{a}{q}\Big) \end{aligned}$$

Thus

$$\left|f_{SQ}\Big(\frac{-t}{m}\Big)\right| \le \left\lfloor\frac{m}{q}\right\rfloor \left|\sum_{l=0}^{q-1} e\Big(l^2\frac{a}{q}\Big)\right| \le \frac{m}{\sqrt{q}} \le \frac{m}{\sqrt{2}}$$

(by the formula for the gauss sum, Lemma 4.3 of [35], $\left|\sum_{l=0}^{q-1} e\Big(l^2\frac{a}{q}\Big)\right| = \sqrt{q}\Big)$.

We also have that

$$f_A\Big(\frac{t}{m}\Big) = \sum_{a\in A} e\Big(a\frac{t}{m}\Big) = \Bigg(\sum_{x\in Z_m} e\Big(x\frac{t}{m}\Big) - \sum_{b\in \bar{A}} e\Big(b\,\frac{t}{m}\Big)\Bigg) = -f_{\bar{A}}\Big(\frac{t}{m}\Big)$$

so that

$$\left|\frac{1}{m}\sum_{t\neq 0} f_A\Big(\frac{t}{m}\Big)\, f_{\bar{A}}\Big(\frac{t}{m}\Big) f_{SQ}\Big(\frac{t}{m}\Big)\right| \le \frac{1}{\sqrt{2}}\sum_{t\neq 0}\left|f_A\Big(\frac{t}{m}\Big)\right|^2.$$

By Parseval's identity the right hand side equals $\frac{1}{\sqrt{2}}\Big(|A|m - |A|^2\Big)$.

We conclude that there is at least one solution to the given equation, if

$$|A|\,|\bar{A}| > \frac{1}{\sqrt{2}}\Big(|A|m - |A|^2\Big) = \frac{1}{\sqrt{2}}\Big(|A|\cdot(m-|A|)\Big) = \frac{1}{\sqrt{2}}\Big(|A|\,|\bar{A}|\Big)$$

which trivially holds for any nonempty set A.

$\square$

We finish the Section with a stability version of theorem 10

Theorem 11. *For every $\delta > 0$, there is a $\epsilon > 0$ such that, if $S \subseteq Q_{2^n} \times Q_m$ is an independent set, $|S \triangle (M_{10} \times Q_m)| \ge \epsilon 2^n m$, and $|S \triangle (M_{26} \times Q_m)| \ge \epsilon 2^n m$, then $|S| < (\frac{11}{32} - \delta)2^n m$ (denoting by $\triangle$ the symmetric difference between sets). In particular we can take $\epsilon \le 200\sqrt{\delta}$.*

Proof. Suppose that $|S| > (\frac{11}{32} - \delta^2)2^n m$.

Let $H_i, \ldots H_k$ be a d-uniform covering of Q_m, as in lemma 4, for some constant d.

By the hypothesis and the definition of $d-$uniform covering, we have that

$$\sum_{H_i} ||S\restriction_{\mathbf{Q}_{2^n}\times H_i}|| = d\sum_{x\in Q_m} |S\restriction_{\mathbf{Q}_{2^n}\times\{x\}}| > d\Big(\frac{11}{32} - \delta^2\Big)2^n m.$$

Noticing that $3k = dm$ we get that, in average, $||S\restriction_{\mathbf{Q}_{2^n}\times H_i}|| > \Big(\frac{11}{32} - \delta^2\Big)2^n 3$.

Hence, for at least $(1-\delta)k$ of the subgraphs H_i, $||S\restriction_{\mathbf{Q}_{2^n}\times H_i}|| > \Big(\frac{11}{32} - \delta\Big)2^n 3$.

Hence at least $(1-\delta)m$ of the vertices of Q_m are in a subgraph H_i for which $||S\restriction_{\mathbf{Q}_{2^n}\times H_i}|| > \Big(\frac{11}{32} - \delta\Big)2^n 3$ (recalling that $3k = dm$).

By theorem 9 all these vertices are in a subgraph H_i such that either

$$\left|\left|S \restriction_{\mathbf{Q}_{2^n} \times H_i} \triangle (M_{10} \times H_i)\right|\right| < (64\delta)\, 2^n 3, \text{ or } \left|\left|S \restriction_{\mathbf{Q}_{2^n} \times H_i} \triangle (M_{26} \times H_i)\right|\right| < (64\delta)\, 2^n 3.$$

Let

$$A := \{a \in Q_m : a \text{ is in some } H_i, \text{ for which } \left|\left|S \restriction_{\mathbf{Q}_{2^n} \times H_i} \triangle (M_{10} \times H_i)\right|\right| < (64\delta)\, 2^n 3\}$$

$$B := \{b \in Q_m : b \text{ is in some } H_i, \text{ for which } \left|\left|S \restriction_{\mathbf{Q}_{2^n} \times H_i} \triangle (M_{26} \times H_i)\right|\right| < (64\delta)\, 2^n 3\}$$

Then $|A| + |B| > (1 - \delta)m$.

For any $a \in A$, there exists an $x \equiv 10 \pmod{32}$, such that $(x, a) \in S$ (by definition of A).

For any $b \in B$, there exists an $y \equiv 26 \pmod{32}$ such that $(y, b) \in S$ (by definition of B).

So, if there is an edge in Q_m, between any two vertices a of A, and b of B, we can consider elements $x, y \in Q_{2^n}$ such that (x, a), and $(y, b) \in S$ are adjacent in $Q_{2^n} \times Q_m$, a contradiction with S being an independent set.

Without loss of generality, assume $|B| \geq |A|$

Let us show that if $|A| \geq 6\delta m$, there is one such edge. Fourier analysis shows that there are at least $(\left(1 - \frac{1}{\sqrt{2}}\right)|A|\,|\bar{A}| \geq \frac{1}{4}|A|\,|\bar{A}| >)\ \delta m^2$ edges from A to $\bar{A}$, and by counting we see that these cannot all go to the small set of δm elements that are not in B.

This leads us to a contradiction. Hence $|A| \leq 6\delta m$, so for at least $(1-7\delta)m$ elements $x \in Q_m$, we have

$$\left|S \restriction_{\mathbf{Q}_{2^n} \times \{x\}} \triangle (M_{26} \times \{x\})\right| < (64\delta)\, 2^n 3.$$

Hence

$$\left|S \triangle (M_{10} \times Q_m)\right| < (64\delta)\, 2^n 3 * (1 - 7\delta)m + \ 7\delta\, 2^n m$$

and so

$$\left|S \triangle (M_{10} \times Q_m)\right| < 200\delta\, 2^n m$$

(Analogously we also get $\left|S \triangle (M_{26} \times Q_m)\right| < 200\delta\, 2^n m$ in the case $|A| > |B|$)

We conclude that the theorem holds with $\epsilon = 200\sqrt{\delta}$.

$\square$

Remark 12. *By using lemma 5 isomorphism from $Q_{2^n m}$ to $Q_{2^n} \times Q_m$*

$$\begin{aligned}\phi\colon \mathbb{Z}_{2^n m} &\to \mathbb{Z}_{2^n} \times \mathbb{Z}_m \\ x &\mapsto \big(x\,(mod\ 2^n),\, x\,(mod\ m)\big)\end{aligned}$$

we can rewrite our last theorem for $Q_{2^n m}$.

Letting

$$M_{10}^* = \{x \in \mathbb{Z}_{2^n m} |\ x \equiv 1 \pmod 4,\ \text{or } x \equiv 10, 14, 30 \pmod{32}\}$$

$$M_{26}^* = \{x \in \mathbb{Z}_{2^n m} |\ x \equiv 1 \pmod 4,\ \text{or } x \equiv 14, 26, 30 \pmod{32}\}$$

we see that $\phi(M_{10}^*) = M_{10} \times Z_m$ and $\phi(M_{26}^*) = M_{26} \times Z_m$.

From theorem 11 we conclude

Theorem 13. *For every $\delta > 0$, there is an $\epsilon > 0$ such that the following holds. Let S be a set of $\mathbb{Z}_{2^n m}$ with no two elements adding to a square.*

$$\textit{If } |S| > (\frac{11}{32} - \delta)2^n m, \textit{ then either } |S \triangle (M_{10}^*)| \le \epsilon 2^n m \textit{ or } |S \triangle (M_{26}^*)| \le \epsilon 2^n m$$

(Furthermore, ϵ can be taken smaller than $200\sqrt{\delta}$).

By abuse of notation, we'll also refer to the sets M_{10}^*, M_{26}^* , just as M_{10}, M_{26}.

2.5 From finite cyclic groups to integers

We will now present the work of Khalfalah, Lodha, and Szemeredi [19], which achieves the bound of $(\frac{11}{32}+\delta)N$.

As in Section 2.3, no originality is claimed in the arguments of this Section, and we will only include the details and proofs necessary for a right understanding of our work (For more details consult the original paper [19]).

The following is their main theorem

Theorem 14. *For any $\delta > 0$, there is a positive integer N_0 such that the following holds. Given $N \geq N_0$, for every $S \subseteq [N]$ with size at least $(\frac{11}{32}+\delta)N$, there are two elements in S that add up to a perfect square.*

Let us start by giving an outline of their proof.

Assume, to get a contradiction, that for some sufficiently large N, we may consider a subset S of $[N]$, with density $d(S) \geq \frac{11}{32}+\delta$, such that $x+y=z^2$ has no solutions, with $x, y \in S$, and $z \in \mathbb{Z}$.

Successively reducing the set (mod q_i), for integers q_i (each one a multiple of the former), by theorem 2 for cyclic groups we see that at some point there must be two classes on which the set S has positive density, that add up to a square residue class. It can still be the case that no pair of integers of S lying in these two classes adds up to the square of an integer.

However, the number of solutions to $x+y=t^2$, with $x \in S$, and $y \in (S+jM)$ for consecutive shifts of the set S by a constant M, has to be large. Precisely letting $j = 1, \ldots, N^{(1-2\epsilon)}$ we will find $10\,\frac{N\sqrt{N}}{\sqrt{P}}\,N^{(1-2\epsilon)}$ many solutions, for P the largest prime divisor of M.

On the other hand, shifting the set S to $S+jM$, for some $j = 1, \ldots, N^{(1-2\epsilon)}$, the number of solutions of $x+y=z^2$, with $x \in S$, $y \in (S+jM)$, and $z \in \mathbb{Z}$, cannot change too much from the original number of pairs of S adding to a perfect square. Since by hypothesis there is no such pair, we will see that the former number of solutions can be at most $10\,\frac{N\sqrt{N}}{\sqrt{P}}$, for every such j. This follows via standard harmonic analysis techniques. In particular we will need our constant M to be a highly composite integer.

The two inequalities give us a contradiction, and therefore the theorem holds.

Many squares in the sumsets of S with translates of it

Consider a sequence of positive integers $q_1, q_2, q_3, \ldots, q_l$, such that :

- q_{i+1} is divisible by all integers smaller than or equal to $\left(\frac{5120}{\delta^2}\, q_i^2\right)^2$

In particular q_i divides q_{i+1}.

Let's study the distribution of S, among residue classes, modulo each q_i.

Definition 3 (density of a set in a residue class)**.**

Let $\epsilon_{i,j}$ denote the density of the set S in the residue class j, modulo q_i.

$$\epsilon_{i,j} = \frac{|\{s \in S: \; s \equiv j \; (mod\; q_i)\}|}{N/q_i}$$

Definition 4 (index of unevenness of distribution modulo q_i)**.**

Let $\alpha_i := \frac{1}{q_i} \sum_{j=0}^{q_i-1} \epsilon_{i,j}^2$

By the Cauchy-Schwartz inequality, and the hypothesis $|S| \geq (\frac{11}{32} + \delta)N$, one can show ([19]) that the unevenness of distribution modulo q_i is increasing and bounded.

Lemma 15. $\alpha_i \leq \alpha_{i+1}, \; \forall i.$

Lemma 16. $(\frac{11}{32} + \delta)^2 \leq \alpha_i \leq (\frac{11}{32} + \delta), \; \forall i.$

For each residue class x modulo q_i, we shall now look at the distribution of the densities of S, on the subclasses $x + kq_i$ modulo q_{i+1}.

Definition 5 (good residue class)**.**

Say the residue class j modulo q_i is bad if

$$\frac{|\, \{0 \leq k \leq q_{i+1}/q_i : |\epsilon_{i,j} - \epsilon_{i+1,j+k\,q_i}| \geq \delta/4\}\, |}{q_{i+1}/q_i} \geq \frac{1}{8}$$

Otherwise, call the residue class j modulo q_i good.

Fixing $\sigma := \dfrac{\delta^3}{480}$, and the length of our initial sequence $l = \left(\frac{\left(\frac{11}{32}+\delta\right) - \left(\frac{11}{32}+\delta\right)^2}{\sigma} \right)$ by lemmas 15 and 16, for some $i \leq l$ we must have that $\alpha_{i+1} - \alpha_i \leq \sigma$.

Recalling the *defect form of Cauchy-Schwartz Inequality*([2], IV- Lemma 27), one can prove that

Lemma 17.

If $\alpha_{i+1} - \alpha_i < \sigma$, the number of bad residue classes modulo q_i is at most $\frac{\delta q_i}{2}$.

With the control on the number of bad classes, we can now apply theorem 2. The extra δ factor in the density of our set give us enough room to find two residue classes (mod q_s) that add up to a square residue (mod q_s), on which S has positive density, that hardly changes for most of the subclasses, $a + kq_s$, and $b + kq_s \pmod{q_{s+1}}$.

This is the key lemma, which we will be able to improve in Section 2.6 (making use of theorem 13's characterization of large sets of residue classes with no two adding to a square residue).

Lemma 18. *If $\alpha_{i+1} - \alpha_i < \sigma$, then there exists two good residue classes, modulo q_i, say a and b, such that,*

- $\epsilon_{i,a} \geq \frac{\delta}{2}$
- $\epsilon_{i,b} \geq \frac{\delta}{2}$
- *$a + b$ is a quadratic residue modulo q_i*

Proof. Let $T = \dfrac{|\{j : j \text{ is a good class and } \epsilon_{i,j} \geq \frac{\delta}{2}\}|}{q_i}$

Let's show that $|T| > \frac{11}{32}$. The proposition then follows from lemma 1.

By the previous lemma, since $\alpha_{i+1} - \alpha_i < \sigma$, the number of bad residue classes modulo q_i is at most $\frac{\delta}{2}\, q_i$. Hence, and since $\epsilon_{i,j} \leq 1,\ \forall j$,

$$\begin{aligned}
\frac{1}{q_i}\sum_{j \text{ good}} \epsilon_{i,j} &= \frac{1}{q_i}\sum_{j=0}^{q_i-1} \epsilon_{i,j} - \frac{1}{q_i}\sum_{j \text{ bad}} \epsilon_{i,j} \\
&= d(S) - \frac{1}{q_i}\sum_{j \text{ bad}} \epsilon_{i,j} \\
&\geq \tfrac{11}{32} + \delta - \delta/2 \\
&= \tfrac{11}{32} + \tfrac{\delta}{2}.
\end{aligned}$$

On the other hand, if $T \leq \frac{11}{32}$ we get

$$\frac{1}{q_i}\sum_{j \text{ good}} \epsilon_{i,j} < T\,1 + (1-T)\,\frac{\delta}{2} \leq \frac{11}{32} + \frac{21}{64}\delta$$

a contradiction.

Thus $T > \frac{11}{32}$, and the conclusion follows from the Lemma 2. ∎

To find elements in a quadratic residue classes which are perfect squares, we will notice that if a long arithmetic progression is contained in a small interval, and if it has base point in a quadratic residue class, then it must contain many perfect squares. Whence the idea of shifting the set S (and counting the number of solutions of $x+y=z^2$, over $S \times (S + jq_{i+1})$ as $j = 0, \ldots, h$) to capture all these squares.

Lemma 19.

Let c be a square residue modulo q_i.

That is $c = r^2 + sq_i$, for some integer $0 \leq r \leq q_i - 1$, $s \in \mathbb{Z}$.

Then any arithmetic progression $\{c + tq_i | t = 0, \ldots, h\}$ of lenght $h \geq 10\sqrt{N}$, which is entirely contained in $[2N] = \{1, 2, 3, \ldots, 2N\}$, contains at least $\frac{h}{8\sqrt{N}}$ perfect squares.

Proof. Let
$$\begin{aligned} Q &= \{c + tq_i |\, t = 0, \ldots, h\} \\ &= \{r^2 + t' q_i |\, t' = s, \ldots, s+h\} \end{aligned}$$

Let's look for perfect squares w^2 in Q, with w integer of the form $w = r + uq_i$.

We want $w^2 = r^2 + (2ru + u^2 q_i)q_i$ to be in Q.

Equivalently, we want integers u such that $s \leq (2ru + u^2 q_i) \leq s + h$.

Since $r \leq q_i$, it is enough that both $s \leq u^2 q_i$, $(u+1)^2 q_i \leq s + h$.

Thus, for each pair of consecutive squares in the interval $[\frac{s}{q_i}, \frac{s+h}{q_i}]$, we will have a perfect square in Q. Since this is an interval of length $\frac{h}{q_i}$, of integers smaller than or equal to $\frac{s+h}{q_i} \leq \frac{2N}{q_i^2}$, the number of perfect squares in it is at least $\left\lfloor \frac{h/q_i}{2\sqrt{2N/q_i^2}} \right\rfloor = \left\lfloor \frac{h}{2\sqrt{2}\sqrt{N}} \right\rfloor \geq \frac{h}{4\sqrt{N}}$.

Whence the total number of such consecutive pairs, is at least $\frac{h}{8\sqrt{N}}$.

□

Let $h := N^{1-2\epsilon}$.

We are now ready to conclude

Proposition 2.5.1 (Lower bound for the number of solutions). *There is an $i \in \{1, \ldots, l\}$, such that the number of solutions (j, x, y) of the equation $x + y = z^2$, with $j \in \{0, 1, \ldots, h\}$, $x \in S$, $y \in (S + jq_{i+1})$, and $z \in \mathbb{Z}$, is at least*

$$\frac{\delta^2 N \sqrt{N}\, h}{512 q_i^2}$$

Proof. By lemma 5, there exist a pair of good residue classes modulo some q_i, denote them by a, b, such that $\epsilon_{i,a} \geq \frac{\delta}{2}$, $\epsilon_{i,b} \geq \frac{\delta}{2}$, and $c := a+b$ is a quadratic residue modulo q_i.

Let

- $D_a = \{0 \leq w_a \leq \frac{q_{i+1}}{q_i} \,|\, \epsilon_{i+1,a+w_a\,q_i} \geq \frac{\delta}{4}\}$
- $D_b = \{0 \leq w_b \leq \frac{q_{i+1}}{q_i} \,|\, \epsilon_{i+1,b+w_b\,q_i} \geq \frac{\delta}{4}\}$
- $Q_c = \{0 \leq w_c \leq \frac{q_{i+1}}{q_i} \,|\, c + w_c\,q_i$ is a quadratic residue $\pmod{q_{i+1}}\}$

By definition of good class, we have that

$$|D_a| \geq \frac{7}{8}\,\frac{q_{i+1}}{q_i}, \text{ and } |D_b| \geq \frac{7}{8}\,\frac{q_{i+1}}{q_i}.$$

Given any $w_c \in Q_c$, let

- $S_{w_c} = \{(w_a, w_b) |\ w_a \in D_a, w_b \in D_b,\ c + w_c q_i = (a + w_a q_i) + (b + w_b q_i)\}$

By the pigeonhole principle,

$$|S_{w_c}| \geq \frac{6}{8}\,\frac{q_{i+1}}{q_i} \geq \frac{1}{2}\,\frac{q_{i+1}}{q_i}.$$

Let $L = |Q_c|$.

By lemma 5, if we shift an element x of a square residue class (mod q_i) along a homogeneous arithmetic progression of step q_i, and length $h\frac{q_{i+1}}{q_i} \gg 10\sqrt{N}$, the number of perfect squares in it is at least $\frac{hq_{i+1}}{8q_i\sqrt{N}}$. Notice that this is the same as if each element of the form $x + k\,q_i$ with $k = 0, \ldots, \frac{q_{i+1}}{q_i} - 1$, was shifted along an arithmetic progression of step q_{i+1}, and length h. Moreover, all perfect squares must lie on shifts of the elements $x + kq_i$, where $k \in Q_c$.

If we repeat the proof of lemma 5 for each of the $x + kq_i$, with $k \in Q_c$, we see that all perfect squares of the form $(r + uq_i)^2$, that were counted as shifts of x, modulo q_i, will also be counted as perfect squares of form $(r' + u'q_{i+1})^2$ for shifts of some $x + kq_i$, with $k \in Q_c$, modulo q_{i+1}. Therefore, and since the number of perfect squares of that form is essentially the same for every $x + kq_i$ (given by the number of pairs $\{u^2, (u+1)^2\}$ on an interval which is essentially the same for all k, by their expression, and because q_{i+1} is constant, $h \gg \sqrt{N}$), this is at least $\frac{1}{2}\frac{1}{L}\frac{hq_{i+1}}{8q_i\sqrt{N}}$.

Therefore, for a fixed w_c in Q_c, and for a given pair $(w_a, w_b) \in S_{w_c}$, the number of solutions (x, y, j), of $x + y = z^2$, with

$$\begin{aligned}
&j = 0, \dots, h-1\\
&x \in S, \text{ and } x \equiv a + w_a q_i \bmod q_{i+1}\\
&y \in S + jq_i, \; y \equiv b + w_b q_i \bmod q_{i+1}
\end{aligned}$$

is at least

$$\frac{\delta N}{4q_{i+1}} \frac{\delta N}{4q_{i+1}} \frac{h\, q_{i+1}}{16 q_i L \sqrt{N}}.$$

If we count the solutions for all possible w_c in Q_c, and for every pair $(w_a, w_b) \in S_{w_c}$, we conclude that the total number of solutions is at least

$$\sum_{w_c \in Q_c} \frac{1}{2} \frac{q_{i+1}}{q_i} \frac{\delta N}{4q_{i+1}} \frac{\delta N}{4q_{i+1}} \frac{h\, q_{i+1}}{16 q_i L \sqrt{N}} = \frac{\delta^2 N \sqrt{N}\, h}{512 q_i^2}.$$

That is the number of solutions of $x+y = z^2$, over $S \times (S + jq_{i+1})$, as $j = 0, \dots, N^{1-2\epsilon}$, is at least

$$\frac{\delta^2 N \sqrt{N}\, h}{512 q_i^2}.$$

□

Analytic upper bound for the number of solutions

Proposition 2.5.2. *Let $S \subseteq [N]$, such that $x + y = z^2$ has no solution, with $x, y \in S$, $z \in \mathbb{Z}$. Let P be a prime, and M some positive integer, divisible by all positive integers smaller than or equal to P.*

Then the number of solutions to $x + y = z^2$, with $x \in S$, and $y \in S + j\,M$, $z \in \mathbb{Z}$, is at most $10\left(\frac{N\sqrt{N}}{\sqrt{P}}\right)$, for any $j = 1, 2, \cdots, N^{1-2\epsilon}$.

Proof. (Sketch. For full detailed computations, consult [19])

Let $S \subseteq [n]$. Denote $e(x) := e^{2\pi i x}$, and let

$$f_S(\alpha) = \sum_{x \in S} e(\alpha\, x), \quad f_{S+jM}(\alpha) = \sum_{y \in S+jM} e(\alpha\, y), \quad f_{SQ}(\alpha) = \sum_{z=0}^{\sqrt{3N}} e(\alpha\, z^2)$$

The following expression counts the number of solutions of $x+y = z^2$, with $x, y \in S$

$$\frac{1}{3N} \sum_{t=0}^{3N-1} f_S\Big(\frac{t}{3N}\Big)\, f_S\Big(\frac{t}{3N}\Big) f_{SQ}\Big(\frac{t}{3N}\Big)$$

Analogously, the number of solutions of $x+y=z^2$, with $x \in S$, and $y \in S+j\,M$, is given by

$$\frac{1}{3N}\sum_{t=0}^{3N-1} f_S\Big(\frac{t}{3N}\Big)\, f_{S+j\,M}\Big(\frac{t}{3N}\Big) f_{SQ}\Big(\frac{t}{3N}\Big).$$

One can estimate this expression, taking advantage of the fact that by assumption, the value of the previous (quite similar) one is 0 (since by hypothesis $x+y=z^2$ has no solutions over $S \times S$). So

$$\frac{1}{3N}\Big|\sum_{t=0}^{3N-1} f_S\Big(\frac{t}{3N}\Big)\, f_{S+j\,M}\Big(\frac{t}{3N}\Big) f_{SQ}\Big(\frac{t}{3N}\Big)\Big| \le 10\frac{N\sqrt{N}}{\sqrt{P}}.$$

for any $j=1,2,\cdots,N^{1-2\epsilon}$. (the interested reader is invited to consult [19] for further details, or to check the analogous estimates in our Section 2.7). Hence the total number of solutions is less than $10\dfrac{N\sqrt{N}}{\sqrt{P}}$.

$\square$

Conclusion

Taking $M=q_{i+1}$, P the largest prime dividing M (which is bigger than $\big(\frac{\delta^2}{5120}\,q_i^2\big)^2$), and $h=N^{1-2\epsilon}$, we get from propositions 2.5.1 and 2.5.2 that the number of solutions of the equation $x+y=z^2$, with $j=1,2,\ldots,h$, $x \in S,\ y \in S+j\,M$, and $z \in \mathbb{Z}$, is

$$\text{at least } \frac{\delta^2 N\sqrt{N}\,h}{512q_i^2}, \text{ and smaller than } \frac{\delta^2 N\sqrt{N}\,h}{512q_i^2}.$$

This is a contradiction. Thus the theorem holds.

2.6 Sets of integers far from the extremal examples

Let S be a subset of $[N]$ with property (NS), and density bigger than $\frac{11}{32}N$.

Let us show that the argument of Section 2.5 still follows for a set S, with $|S| \geq \frac{11}{32}N$ (without the extra room of δ), if the set is not close to one of the two Massias examples.

In fact, recall Lemma 18, and the proof therein, key for proving the existence of many solutions to $x+y=z^2$, with $j=1,2,\ldots,h$, $x \in S,\ y \in S + j\,M$, and $z \in \mathbb{Z}$ (in contradiction with the maximum possible number, determined by analytic techniques).

Lemma. *If $\alpha_{i+1} - \alpha_i < \sigma$, then there exists two good residue classes, modulo q_i, say a and b, such that,*

- $\epsilon_{i,a} \geq \frac{\delta}{2}$
- $\epsilon_{i,b} \geq \frac{\delta}{2}$
- *$a+b$ is a quadratic residue modulo q_i*

Proof. Let $T = \{j : j \text{ is a good class and } \epsilon_{i,j} \geq \frac{\delta}{2}\}$.

By Lemma 13, there is an $\epsilon > 0$ with $\epsilon < 200\sqrt{\delta}$, such that the following holds. If $|T \triangle (M_{10}^*)| \geq \epsilon q_i,\ |T \triangle (M_{26}^*)| \geq \epsilon q_i$, and $|T| > (\frac{11}{32} - \delta)q_i$, then there are two residue classes in T that add up to a square residue. Thus the conclusion of the lemma follows.

In fact since we do have that $|T| > (\frac{11}{32} - \delta)q_i$ (analogous proof as that of Lemma 18), we can then assume that $|T \triangle (M_{10}^*)| \leq \epsilon q_i$, or $|T \triangle (M_{26}^*)| \leq \epsilon q_i$.

Without loss of generality, suppose

$$|T \triangle (M_{10}^*)| \leq \epsilon q_i.$$

Any element of $S \setminus T$ is either in a bad class, or in a class j for which $\epsilon_{i,j} < \frac{\delta}{2}$. Since by Lemma 17 the total number of bad classes is at most $\frac{\delta}{2}N$ we conclude from the definition of T that then

$$|S \setminus M_{10}| \leq (\epsilon + \frac{\delta}{2} + \frac{21}{32} \cdot \frac{\delta}{2})N \leq 2\epsilon N.$$

$\square$

In the next Section we finish this argument, by showing that any set with at least $\frac{11}{32}N$ elements, which is very close to, but different than one of the Massias examples (in this case M_{10}), must have two elements that add to a perfect square, assuming N is large enough.

2.7 Sets of integers close to one of the extremal examples

Let S be a subset of the first N integers, with $|S| = \frac{11}{32}N$. Denote by

$$\begin{aligned} M_{10}^* = &\quad \{x \in \mathbb{Z}_{32} |\ x \equiv 1 \pmod 4),\ \text{ or } x = 10, 14, 30\} \\ M_{10} = &\quad \{x \in [N] \mid x \equiv 1 \pmod 4),\ \text{ or } x \equiv 10, 14, 30 \pmod{32}\} \end{aligned}$$

and assume that $|S \setminus M_{10}| \leq 2\epsilon N$.

Let a be the residue class in $\mathbb{Z}_{32} \setminus M_{10}^*$, where S has maximum density, and

$$A_1 := \{x \in S :\ x \equiv a \pmod{32}\}$$

Let b be a residue class in M_{10}^*, such that $a + b \equiv z^2 \pmod{32}$, for some $z \in \mathbb{Z}_{32}$ (we may consider this since M_{10} is a maximal set of residue classes in $\mathbb{Z}_{32}$ with no two adding to a square residue), and let

$$B = \{x \in S :\ x \equiv b \pmod{32}\}$$

$$\bar{B} = \{x \in [N] \setminus S :\ x \equiv b \pmod{32}\}$$
$$A := A_1 \cap \left[\left\lfloor \frac{N}{2} \right\rfloor \right].$$

Assume, without loss of generality, that $|A| \geq \frac{|A_1|}{2}$ (otherwise consider A the set $A_1 \cap \left\{\frac{N}{2} + 1, \cdots, N\right\}$, and the argument follows mutatis mutandis).

Let

$$SQ_z = \{w^2 \in \left\{ \left\lfloor \frac{N}{2} \right\rfloor + 1, \cdots, N \right\} |\ \text{ with } w \in \mathbb{N},\ \text{ and } w^2 \equiv z^2 \pmod{32}\}.$$

According to Lemma 19, $|SQ_z| \geq \frac{\sqrt{N}}{512}$.

Observe as well that $|A|$ and $|\bar{B}|$ are comparable. In fact, since $|S| = \frac{11}{32}N$, the definition of A_1 implies that $|\bar{B}| \leq 32|A_1| \leq 64|A|$. In the other direction, since no two elements of S add to a perfect square, given a fixed $w^2 \in SQ$, $(w^2 - a)$ has to be an element of $\bar{B}$, for every $a \in A$. Thus $|\bar{B}| \geq |A|$.

Let ν_0 be the number of solutions of

$$a + \bar{b} = w^2, \text{ with } a \in A, \bar{b} \in \bar{B}, \ w^2 \in SQ_z.$$

Since S has no two elements that add to a perfect square, for every $a \in A$, and $w^2 \in SQ_z$, $(w^2 - a)$ has to be an element of $\bar{B}$. Hence

$$\nu_0 = |A||SQ_z|.$$

On the other hand,

$$\nu_0 = \frac{1}{2N} \sum_{t=0}^{2N-1} f_A\Big(-\frac{t}{2N}\Big) f_{\bar{B}}\Big(-\frac{t}{2N}\Big) f_{SQ}\Big(\frac{t}{2N}\Big)$$

where $f_S(\alpha) = \sum_{a \in A} e(\alpha\, a)$, $f_{\bar{B}}(\alpha) = \sum_{\bar{b} \in \bar{B}} e(\alpha\, \bar{b})$, $f_{SQ}(\alpha) = \sum_{w^2 \in SQ} e(\alpha\, w^2)$,
and $SQ = \Big\{ w^2 \in \{\lfloor \frac{N}{2} \rfloor + 1, \cdots, N\} \,|\, w \in \mathbb{N} \Big\}$.

(We can consider in the expression the full set of squares SQ instead of SQ_z, because for any $a \in A$, and $\bar{b} \in \bar{B}$, $a + \bar{b} \equiv z^2 \pmod{32}$. So the two expressions count the same, whereas the chosen one is easier to work with).

By Dirichlet's approximation principle, for every $t \in [2N]$ there exist coprime integers $a(t)$ and $b(t)$, with $b(t) \leq N^{1-\epsilon}$, such that

$$\Big| \frac{t}{2N} - \frac{a(t)}{b(t)} \Big| \leq \frac{1}{b(t) N^{1-\epsilon}}.$$

Let P be a large prime, and denote

$$\sum{}^1 = \frac{1}{2N} \sum_{t: b(t) \leq P} f_A\Big(-\frac{t}{2N}\Big) f_{\bar{B}}\Big(-\frac{t}{2N}\Big) f_{SQ}\Big(\frac{t}{2N}\Big)$$

$$\sum{}^2 = \frac{1}{2N} \sum_{t: b(t) > P} f_A\Big(-\frac{t}{2N}\Big) f_{\bar{B}}\Big(-\frac{t}{2N}\Big) f_{SQ}\Big(\frac{t}{2N}\Big)$$

Let us show that the main contribution to ν_0 comes from $\sum^1$.

Letting $r := \dfrac{t}{2N} - \dfrac{a(t)}{b(t)}$,

$$\begin{aligned}
f_{SQ}\Big(\tfrac{t}{2N}\Big) &= f_{SQ}\Big(\frac{a(t)}{b(t)} + r\Big) \\
&= \sum_{x=\sqrt{\frac{N}{2}}}^{\sqrt{N}} e\Big(\frac{a(t)}{b(t)}\, x^2 + r\, x^2\Big) \\
&= \sum_{x=0}^{\sqrt{N}} e\Big(\frac{a(t)}{b(t)}\, x^2 + r\, x^2\Big) - \sum_{x=0}^{\sqrt{\frac{N}{2}}} e\Big(\frac{a(t)}{b(t)}\, x^2 + r\, x^2\Big).
\end{aligned}$$

Denote by

$$f_{SQ_{\sqrt{N}}}\Big(\frac{t}{2N}\Big) = \sum_{x=0}^{\sqrt{N}} e\Big(\frac{a(t)}{b(t)}\, x^2 + r\, x^2\Big), \quad f_{SQ_{\sqrt{N/2}}}\Big(\frac{t}{2N}\Big) = \sum_{x=0}^{\sqrt{N/2}} e\Big(\frac{a(t)}{b(t)}\, x^2 + r\, x^2\Big).$$

<u>Case $b(t) \leq \sqrt{\frac{N}{2}}$:</u>

Write x in base $b(t)$: $x = kb(t) + l$, with $k = 0, \dots, \lfloor \frac{\sqrt{N}}{b(t)} \rfloor$, $l = 0, \dots, b(t) - 1$.

Then $x^2 = k^2 b(t)^2 + 2lkb(t) + l^2$, and

$$\begin{aligned}
f_{SQ_{\sqrt{N}}}\Big(\tfrac{t}{2N}\Big) &= \sum_{k=0}^{\lfloor \frac{\sqrt{N}}{b(t)} \rfloor} \sum_{l=0}^{b(t)-1} e\Big(k^2 a(t) b(t) + 2lka(t) + l^2 \frac{a(t)}{b(t)}\Big) e\Big(rk^2 b(t)^2 + 2rlkb(t) + rl^2\Big) \\
&= \sum_{k=0}^{\lfloor \frac{\sqrt{N}}{b(t)} \rfloor} e\Big(rk^2 b(t)^2\Big) \sum_{l=0}^{b(t)-1} e\Big(l^2 \frac{a(t)}{b(t)}\Big) e\Big(2rlkb(t) + rl^2\Big).
\end{aligned}$$

Thus

$$\Big| f_{SQ_{\sqrt{N}}}\Big(\frac{t}{2N}\Big)\Big| \leq \Big\lfloor \frac{\sqrt{N}}{b(t)} \Big\rfloor \Big| \sum_{l=0}^{b(t)-1} e\Big(l^2 \frac{a(t)}{b(t)}\Big) e\Big(2rlkb(t) + rl^2\Big)\Big|$$

We want to estimate the absolute value of the sum in this expression.

By the well known formula for the gauss sum (Lemma 4.3 of [35]),

$$\Big| \sum_{l=0}^{b(t)-1} e\Big(l^2 \frac{a(t)}{b(t)}\Big)\Big| \leq \sqrt{b(t)}.$$

By the power series expansion of the exponential function, we have that

$$e\Big(2rlkb(t) + rl^2\Big) = 1 + \sum_{n=1}^{\infty} \frac{\big(2rlkb(t) + rl^2\big)^n}{n!}.$$

Hence

$$\begin{aligned}
&\Big| \sum_{j=0}^{b(t)-1} e\Big(l^2 \frac{a(t)}{b(t)}\Big) e\Big(2rlkb(t) + rl^2\Big)\Big| \leq \\
&\leq \Big| \sum_{j=0}^{b(t)-1} e\Big(l^2 \frac{a(t)}{b(t)}\Big)\Big| + \Big| \sum_{l=0}^{b(t)-1} e\Big(j^2 \frac{a(t)}{b(t)}\Big) \sum_{n=1}^{\infty} \frac{\big(2rlkb(t) + rl^2\big)^n}{n!}\Big|.
\end{aligned}$$

When $b(t) \leq \sqrt{\frac{N}{2}}$, we have that $2rlkb(t) + rl^2 \leq \frac{2}{N^{1/2-\epsilon}}$ and so the sum above is bounded by

$$\sqrt{b(t)} + 2\,b(t)\,\frac{2}{N^{1/2-\epsilon}}.$$

Therefore,

$$\left|f_{SQ_{\sqrt{N}}}\Big(\frac{t}{2N}\Big)\right| \leq \frac{\sqrt{N}}{b(t)}\left(\sqrt{b(t)} + 2\,b(t)\,\frac{2}{N^{1/2-\epsilon}}\right) \leq 2\,\frac{\sqrt{N}}{\sqrt{b(t)}}$$

for sufficiently large N.

Case $b(t) \geq \sqrt{N/2}$ Recall Weyl's inequality, [35]-Lemma 2.4:

Lemma 20. *Suppose that* $(a, q) = 1$. *Let* $\alpha \in \mathbb{R}$, *with* $\left|\alpha - \frac{a}{q}\right| \leq \frac{1}{q^2}$. *Then*

$$|\sum_{x=1}^{Q} e(\alpha\, x^2)| \leq Q^{1+\epsilon}(q^{-1} + Q^{-1} + qQ^{-2})^{\frac{1}{2}}.$$

In our case $Q = \sqrt{N}$, and $\sqrt{\frac{N}{2}} \leq q \leq N^{1-\epsilon}$, and so

$$\left|f_{SQ_{\sqrt{N}}}\Big(\frac{t}{2N}\Big)\right| = o(\sqrt{N}).$$

Hence, for any t with $b(t) \geq P$, and N sufficiently large

$$\left|f_{SQ_{\sqrt{N}}}\Big(\frac{t}{2N}\Big)\right| \leq 2\,\frac{\sqrt{N}}{\sqrt{P}}.$$

An entirely analogous argument (replacing N by $\frac{N}{2}$ whenever appropriate) shows us that, for any t with $b(t) \geq P$, and N sufficiently large

$$\left|f_{SQ_{\sqrt{N/2}}}\Big(\frac{t}{2N}\Big)\right| \leq \frac{2}{\sqrt{2}}\,\frac{\sqrt{N}}{\sqrt{P}}.$$

Thus

$$\left|f_{SQ}\Big(\frac{t}{2N}\Big)\right| \leq \left|f_{SQ_{\sqrt{N}}}\Big(\frac{t}{2N}\Big)\right| + |f_{SQ_{\sqrt{N/2}}}\Big(\frac{t}{2N}\Big)\Big| \leq 4\,\frac{\sqrt{N}}{\sqrt{P}}$$

for any t with $b(t) \geq P$, and N sufficiently large.

We conclude that

$$\begin{aligned}
\left|\sum\nolimits^2\right| = & \ \frac{1}{2N}\left|\sum_{t:b(t)>P} f_A\Big(\frac{t}{2N}\Big)\, f_{\bar{B}}\Big(\frac{t}{2N}\Big) f_{SQ}\Big(\frac{t}{2N}\Big)\right| \\
\leq & \ \frac{1}{2N}\sum_{t:b(t)>P} \left|f_A\Big(\frac{t}{2N}\Big)\right|\left|f_{\bar{B}}\Big(\frac{t}{2N}\Big)\right|\left|f_{SQ}\Big(\frac{t}{2N}\Big)\right| \\
\leq & \ \frac{1}{2N}\Big(\frac{4\sqrt{N}}{\sqrt{P}}\Big)\sum_{t:b(t)>P} \left|f_A\Big(\frac{t}{2N}\Big)\right|\left|f_{\bar{B}}\Big(\frac{t}{2N}\Big)\right| \\
\leq & \ \frac{2}{\sqrt{P}\sqrt{N}}\sqrt{\sum_{t=0}^{2N}\left|f_A\Big(\frac{t}{2N}\Big)\right|}\sqrt{\sum_{t=0}^{2N}\left|f_{\bar{B}}\Big(\frac{t}{2N}\Big)\right|} \\
\leq & \ \frac{2}{\sqrt{P}\sqrt{N}}\sqrt{|A|2N}\,\sqrt{|B|2N} \\
\leq & \ \frac{32}{\sqrt{P}}|A|\sqrt{N} \quad \text{(recall that } |\bar{B}| \leq 64|A|).
\end{aligned}$$

Letting M, and P large, we see that the main contribution to the value of $\nu_0 = |A||SQ_z|$ comes from the terms for which $b(t) \leq P$, that is from $\sum^1$. In particular

$$\left|\sum\nolimits^1\right| \geq |A||SQ_z| - \frac{32}{\sqrt{P}}|A|\sqrt{N} \geq \Big(\frac{1}{512} - \frac{32}{\sqrt{P}}\Big)|A|\sqrt{N}.$$

We will now use the so called *shifting technique*, which also played a crucial role in Section 2.5, in the construction of many solutions to $x + y = z^2$.

Namely, let P be a large prime number (any prime bigger than $512^2 \times 10^4$ will work in subsequent computations) and M some positive integer, divisible by all positive integers smaller than or equal to P. Let us count the total number ν of solutions of

$$a + \bar{b} = w^2 + jM$$

with $a \in A$, $\bar{b} \in \bar{B}$, $w^2 \in SQ$, $j \in 1, \cdots, N^{1-\epsilon}$.

$$\nu = \sum_j \frac{1}{2N}\sum_{t=0}^{2N-1} f_A\Big(-\frac{t}{2N}\Big)\, f_{\bar{B}}\Big(-\frac{t}{2N}\Big) f_{SQ+jM}\Big(\frac{t}{2N}\Big)$$

(once again it is equivalent to consider the sets SQ_Z or SQ in this expression) Taking advantage of the previous estimates, we are now able to upper bound the respective sums for the set of squares shifted by a multiple of M.

Define

$$\sum\nolimits_j^1 = \frac{1}{2N} \sum_{t:b(t)\le P} f_A\Big(-\frac{t}{2N}\Big)\, f_{\bar{B}}\Big(-\frac{t}{2N}\Big) f_{SQ+jM}\Big(\frac{t}{2N}\Big)$$

$$\sum\nolimits_j^2 = \frac{1}{2N} \sum_{t:b(t)> P} f_A\Big(-\frac{t}{2N}\Big)\, f_{\bar{B}}\Big(-\frac{t}{2N}\Big) f_{SQ+jM}\Big(\frac{t}{2N}\Big).$$

Again, we can estimate $\sum_j^2$ easily.

In fact, notice that

$$\begin{aligned} f_{SQ+jM}\Big(\tfrac{t}{2N}\Big) &= \sum_{x^2\in SQ} e\Big((x^2+jM)\frac{t}{2N}\Big) \\ &= e\Big(jM\frac{t}{2N}\Big) \sum_{x^2\in SQ} e\Big(x^2\frac{t}{2N}\Big) \\ &= e\big(jM\Big(\tfrac{a(t)}{b(t)}+r\Big)\big)\, f_{SQ}\Big(\frac{t}{2N}\Big) \\ &= e\big(jM\tfrac{a(t)}{b(t)}\big)\ e(jMr)\, f_{SQ}\Big(\frac{t}{2N}\Big). \end{aligned}$$

And so

$$\begin{aligned} \Big|\sum\nolimits_j^2\Big| &= \frac{1}{2N}\Big|\sum_{t:b(t)>P} f_A\Big(\frac{t}{2N}\Big)\, f_{\bar{B}}\Big(\frac{t}{2N}\Big) f_{SQ+jM}\Big(\frac{t}{2N}\Big)\Big| \\ \Big|\sum\nolimits_j^2\Big| &= \frac{1}{2N}\sum_{t:b(t)>P}\Big| f_A\Big(\frac{t}{2N}\Big)\, f_{\bar{B}}\Big(\frac{t}{2N}\Big)\Big|\Big|f_{SQ}\Big(\frac{t}{2N}\Big)\Big| \\ &\le \frac{4\sqrt{N}}{2N\sqrt{P}}\sum_{t:b(t)>P}\Big|f_A\Big(\frac{t}{2N}\Big)\Big|\,\Big|f_{\bar{B}}\Big(\frac{t}{2N}\Big)\Big| \\ &\le \frac{2}{\sqrt{P}}\,|A|\sqrt{N}. \end{aligned}$$

For $b(t)\le P$, by definition of M as a highly composite number, we have that $b(t)$ divides M. We use this to bound $\sum_j^1$ by comparing its expression with that of $\sum^1$.

When $b(t)\le P$, we get that

$$\begin{aligned} f_{SQ+jM}\Big(\tfrac{t}{2N}\Big) &= e\big(jM\tfrac{a(t)}{b(t)}\big)\ e(jMr)\, f_{SQ}\Big(\frac{t}{2N}\Big) \\ &= e(jMr)\, f_{SQ}\Big(\frac{t}{2N}\Big). \end{aligned}$$

By the power series definition of the exponential function,

$$e(rjM) = 1 + (rjM) + \sum_{n=2}^{\infty}\frac{(rjM)^n}{n!}$$

Therefore,

$$\begin{aligned}
\Big|\sum\nolimits_j^1\Big| &= \frac{1}{2N}\Big|\sum_{t:b(t)\le P} f_A\Big(\frac{t}{2N}\Big)\, f_{\bar B}\Big(\frac{t}{2N}\Big) f_{SQ+jM}\Big(\frac{t}{2N}\Big)\Big| \\
&= \frac{1}{2N}\Big(\Big|\sum_{t:b(t)\le P} f_A\Big(\frac{t}{2N}\Big)\, f_{\bar B}\Big(\frac{t}{2N}\Big) f_{SQ}\Big(\frac{t}{2N}\Big)+ \\
&\qquad + \sum_{t:b(t)\le P} f_A\Big(\frac{t}{2N}\Big)\, f_{\bar B}\Big(\frac{t}{2N}\Big) f_{SQ}\Big(\frac{t}{2N}\Big)\sum_{n=1}^{\infty}\frac{(rjM)^n}{n!}\Big|\Big) \\
&\ge \Big|\sum\nolimits^1\Big| - \frac{1}{2N}\sum_{t:b(t)\le P}\Big| f_A\Big(\frac{t}{2N}\Big)\, f_{\bar B}\Big(\frac{t}{2N}\Big) f_{SQ}\Big(\frac{t}{2N}\Big)\Big|\Big|\sum_{n=1}^{\infty}\frac{(rjM)^n}{n!}\Big| \\
&\ge \Big|\sum\nolimits^1\Big| - \frac{1}{2N}\sum_{t:b(t)\le P}\Big| f_A\Big(\frac{t}{2N}\Big)\, f_{\bar B}\Big(\frac{t}{2N}\Big) f_{SQ}\Big(\frac{t}{2N}\Big)\Big| 2N^{-\epsilon} \\
&\ge \big(\tfrac{1}{512} - \tfrac{32}{\sqrt{P}}\big)|A|\sqrt{N} - 2|A|N^{1/2-\epsilon} \\
&\ge \big(\tfrac{1}{512} - \tfrac{40}{\sqrt{P}}\big)|A|\sqrt{N} \quad \text{(for sufficiently large } N).
\end{aligned}$$

Therefore

$$\Big|\sum\nolimits_j\Big| = \Big|\sum\nolimits_j^1+\sum\nolimits_j^2\Big| \ge \Big|\sum\nolimits_j^1\Big| - \Big|\sum\nolimits_j^2\Big| \ge \Big(\frac{1}{512}-\frac{42}{\sqrt{P}}\Big)|A|\sqrt{N} \ge (\frac{1}{1000})|A|\sqrt{N}$$

for sufficiently large N (recall that $P > 512^2 \times 10^4$).

And so the total number of solutions is at least

$$\nu \ge N^{1-2\epsilon}(\frac{1}{1000})|A|\sqrt{N}.$$

On the other hand, given $a \in A$, $\bar b \in \bar B$,

$$a + \bar b = w^2 + jM$$

has a solution $w^2 \in SQ$, for at most $\frac{MN^{1-2\epsilon}}{\sqrt{N}}$ of the integers $j \in 1, \cdots, N^{1-2\epsilon}$ (since the number of squares in an interval of lenght $MN^{1-2\epsilon}$ contained in $\Big\{\lfloor\frac{N}{2}\rfloor + 1, \cdots, N\Big\}$ is at most $\frac{MN^{1-2\epsilon}}{\sqrt{N}}$).

Hence we also have the upper bound for the total number of solutions

$$\nu \le |A||\bar B|N^{\frac{1}{2}-2\epsilon}.$$

The two inequalities show that $|\bar B| \ge (\frac{1}{1000})N$, which is a contradiction for sufficiently small δ (since $|\bar B| \le 64|A|$, and $|A| < 2\epsilon N \le 2\cdot 200\sqrt{\delta}N$, by the assumption that S is close to M_{10}).

Chapter 3

Proof of the Pósa-Seymour Conjecture

3.1 Introduction of the problem

3.1.1 Notation and Definitions

$V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G. (A, B, E) denotes a bipartite graph $G = (V, E)$, where $V = A \cup B$, A and B are disjoint and $E \subset A \times B$. For a graph G and a subset U of its vertices, $G|_U$ is the restriction of G to U. $N(v)$ is the set of neighbors of v in V, and $N_S(v)$ is the set of neighbors of v in S. $|N_S(v)|$ is the degree of v into S, denoted by $\deg_S(v)$. $\delta(G)$ stands for the minimum and $\Delta(G)$ for the maximum degree of a vertex in G. $K_r(t)$ is the balanced complete r-partite graph with color classes of size t. We write $N(p_1, p_2, \ldots, p_\ell) = \bigcap_{i=1}^{\ell} N(p_i)$ for the set of common neighbors of $p_1, p_2, \ldots, p_\ell$, and, more generally, $N(X) = \bigcap_{x \in X} N(x)$. When A and B are subsets of $V(G)$, we denote by $e(A, B)$ the number of edges of G with one endpoint in A and the other in B. For non-empty A and B,

$$d(A, B) = \frac{e(A, B)}{|A||B|}$$

is the *density* of the graph between A and B. In particular, we write $d(A) = d(A, A)$. A graph G on n vertices is γ-*dense* if it has at least $\gamma\binom{n}{2}$ edges. A bipartite graph $G(A, B)$ is γ-*dense* if it contains at least $\gamma|A||B|$ edges. Throughout the chapter log denotes the base 2 logarithm.

A graph G is α-*extremal*, if there exists an $A \subseteq V(G)$ for which

1. $(\frac{1}{k} - \alpha)n \le |A| \le (\frac{1}{k} + \alpha)n$

2. $d(A) < \alpha$

(We say that G is α-non extremal if no set $A \subseteq V(G)$ satisfies (a) and (b))

$K_{k+1}(t)$ is a complete $k+1$-partite graph where each color class has size t

We call a graph a small multipartite graph if it is either $K_{k+1}(t)$, or $K_k(t)$.

A path P_m means a path of m vertices. Let C be a cycle. Then the k^{th} power of C, denoted by C^k, is defined as follows: $V(C^k) = V(C)$ and uv is an edge in C^k if the distance between u and v in C is at most k. The k^{th} power of a path P is defined in an analogous manner. For notational convenience we call the k^{th} power of a path a k-path.

3.1.2 History

A classical result of Dirac [5] asserts that if $\delta(G) \geq n/2$, then G contains a Hamiltonian cycle. A natural question analog to Dirac's theorem was asked by Pósa (see Erdős [7]) in 1962:

Conjecture 1 (Pósa)**.** *Let G be a graph on n vertices. If $\delta(G) \geq \frac{2}{3}n$, then G contains the square of a Hamiltonian cycle.*

This conjecture was generalized by Seymour in 1974 [32]:

Conjecture 2 (Seymour)**.** *Let G be a graph on n vertices. If $\delta(G) \geq (\frac{k-1}{k})n$, then G contains the $(k-1)^{th}$ power of a Hamiltonian cycle.*

Substantial amount of work has been done on these problems. Jacobson (unpublished) first established that the square of a Hamiltonian cycle can be found in any graph G given that $\delta(G) \geq 5n/6$. Later Faudree, Gould, Jacobson and Schelp [14] improved the result, showing that the square of a Hamiltonian cycle can be found if $\delta(G) \geq (3/4+\varepsilon)n$. The same authors further relaxed the degree condition to $\delta(G) \geq 3n/4$. Fan and Häggkvist lowered the bound first in [8] to $\delta(G) \geq 5n/7$ and then in [9] to $\delta(G) \geq (17n+9)/24$. Faudree, Gould and Jacobson [13] further lowered the minimum degree condition to $\delta(G) \geq 7n/10$. Then Fan and Kierstead [10] achieved the almost optimal bound: they proved that if $\delta(G) \geq \left(\frac{2}{3}+\varepsilon\right)n$, then G contains the square of a Hamiltonian cycle. They also proved in [11] that already $\delta(G) \geq (2n-1)/3$ is sufficient for the existence of the square of a Hamiltonian path. Finally, they proved in [12] that if $\delta(G) \geq 2n/3$ and G contains the square of a cycle with length greater than $2n/3$, then G contains square of a Hamiltonian cycle.

For Conjecture 2, in the above mentioned paper of Faudree, Gould, Jacobson, and Schelp, they proved that for any $\varepsilon > 0$ and positive integer k, if the graph G, on n vertices, satisfies $\delta(G) \geq \left(\frac{2k-1}{2k} + \varepsilon\right) n$, then G contains the k^{th} power of a Hamiltonian cycle.

Using the Regularity Lemma – Blow-up Lemma method first Komlós, Sárközy and Szemerédi [23] proved Conjecture 2 in asymptotic form, then in [21] and [24] they proved both conjectures for $n \geq n_0$. The proofs used the Regularity Lemma [33], the Blow-up Lemma [22, 25] and the Hajnal-Szemerédi Theorem [17]. Since the proofs used the Regularity Lemma, the resulting n_0 is very large (it involves a tower function). The use of the Regularity Lemma was removed by Levitt, Sárközy and Szemerédi in a new proof of Pósa's conjecture in [29].

The purpose of our work is to present a new proof of the Pósa-Seymour conjecture that avoids the use of the Regularity Lemma, thus resulting in a simpler proof and a much smaller n_0 ; and to prove a stability result, namely that if our graph does not contain an almost independent set of size $\frac{n}{k}$, then Seymour conjecture is true even if the minimum degree of our graph G is at least $(\frac{k-1}{k} - \epsilon)n$.

We would like to mention the main ingredient in our proof, a new kind of connecting lemma, which we believe will have a lot of applications.

While proving the Pósa-Seymour Conjecture, we do not try to determine the optimal constants.

3.1.3 Main Results

Theorem 21. *There exists an integer $n_0(\alpha)$, and $\epsilon(\alpha)$ such that any α-non extremal graph G, with $|V(G)| = n > n_0(\alpha)$, and $\delta(G) \geq (\frac{k-1}{k} - \epsilon(\alpha))n$, contains a $(k-1)^{th}$ power of a hamiltonian cycle.*

Theorem 22. *There exists an integer n_o such that any graph G, with $|V(G)| = n > n_0$ and $\delta(G) \geq (\frac{k-1}{k})n$, contains a $(k-1)^{th}$ power of a hamiltonian cycle.*

3.2 Sketch of the proof of theorem 21

In Section 3.4.1, using the tools developed in Section 3.3, we first cover a constant fraction of the vertices in G by $K_{k+1}(t)$'s, and then we cover as many vertices as we can with $K_k(t)$'s, where $t = c\log n$, for a constant $0 < c < 1$. We refer to the sets $K_{k+1}(t)$'s and $K_k(t)$'s by $\mathcal{C}$ and $\mathcal{K}$ respectively.

We would inevitably be left with a set $\mathcal{I}$ that cannot be covered in such a manner. However we show that the number of such vertices is small.

Denote the complete graphs $K_{k+1}(t)$ in our collection $\mathcal{C}$ by $C_1, C_2, \ldots,$ and by $K_1, K_2, \ldots,$ the graphs $K_k(t)$ in our collection $\mathcal{K}$.

For a graph $C_j \in \mathcal{C}$, denote its color classes by $V_j^1, V_j^2, \ldots V_j^{k+1}$. In any color class V_j^i, consider an ordering of its vertices $v_{0,i}, v_{1,i}, v_{2,i}, \ldots, v_{t-1,i}$. Finally let us call the l^{th} column of C_j to the sequence $v_{l,1}, v_{l,2}, v_{l,3}, \ldots, v_{l,k+1}$.

For a graph $K_j \in \mathcal{K}$, denote color classes by W_j^i, and vertices by $w_{l,i}$.

To build a $(k-1)$-path in each of the small multipartite graphs C_j, K_j, sequentially connect the vertices within a column, and then connect its last vertex, to the first vertex of the following column (see figure 3.1).

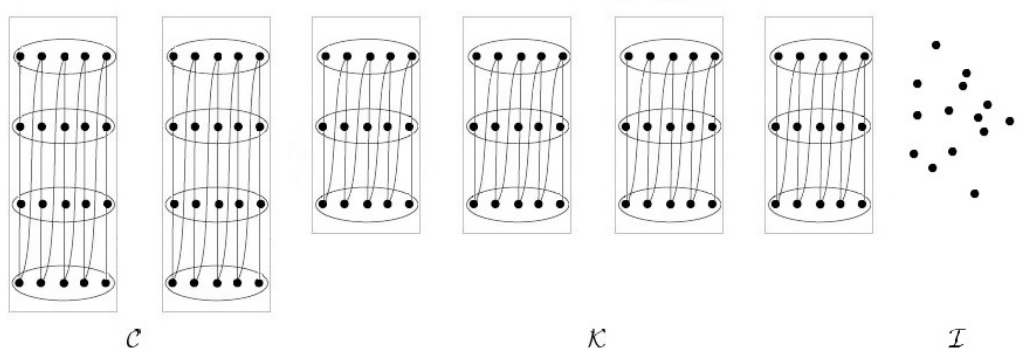


Figure 3.1: $(k-1)$-path inside each small multipartite graph

In Section 3.3.2, we prove that given two cliques $\{a_1, a_2, \ldots, a_{k-1}\}$ and $\{b_{k-1}, b_{k-2}, \ldots, b_1\}$, we can connect a_{k-1} to b_{k-1} with a $(k-1)$-path of length at most $9(k+1)!$, even if we cannot use $o(n)$ vertices of the graph, given in advance. We use this lemma to connect with $(k-1)$-paths (of length at most $9(k+1)!$) the last vertex of the last collumn of a *small multipartite graph* with the first vertex of the first column of the succeeding one.

We will also require that such paths do not use vertices from the first or last columns of any of the small multipartite graphs (which can be done, since $t = \Omega(\log n)$, and therefore the number of vertices in those columns is at most $O\big(\frac{n}{\log n}\big) = o(n)$)

After connecting the graphs in $\mathcal{C}$ and $\mathcal{K}$ we get a $(k-1)^{th}$ power of a cycle covering the vertices in $V(\mathcal{C}) \cup V(\mathcal{K})$(see figure 3.2)

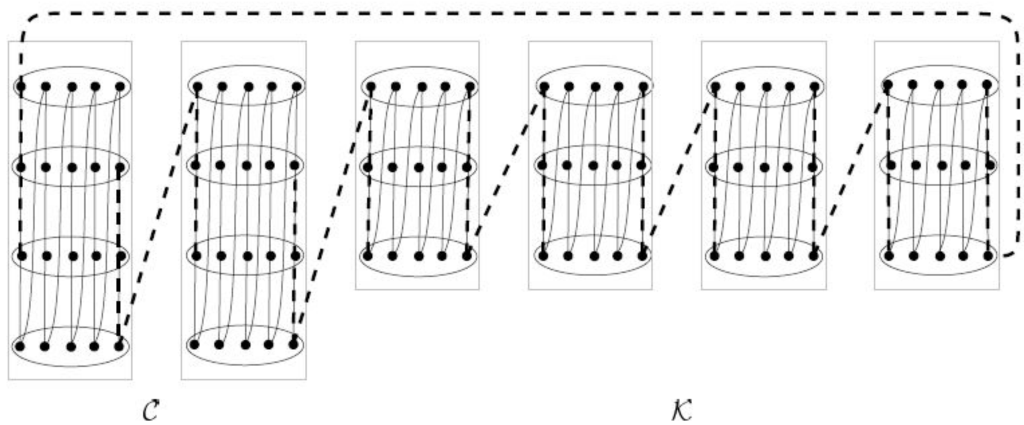


Figure 3.2: Dashed lines represent the $(k-1)$-paths constructed via the Connecting Lemma

Unfortunately, since the $(k-1)$-path connecting the small complete multipartite graphs might use a small number of vertices from some columns $\big($at most $9(k+1)!(k+1)\eta^{-\frac{1}{\eta}}\frac{n}{\log n}$, for some constant $0 \leq \eta \leq 1\big)$, we have to remove the vertices of those columns, and put them in I. This will not increase significantly the size of I.

Each time we remove a column from a small multipartite graph, we have to reconstruct the path inside the graph. We do this by connecting the column preceeding the deleted one, directly to the one following it.

Let $\mathcal{C}^*, \mathcal{K}^*$ denote the collection of multipartite graphs obtained from $\mathcal{C}$ and $\mathcal{K}$ respectively, after the removal of the columns. It will be important that $|\mathcal{C}^*| \geq |\mathcal{C}| - 9(k+1)!(k+1)\eta^{-\frac{1}{\eta}}\frac{n}{\log n}$, which is still much bigger than $|\mathcal{I}|$.

To obtain a $(k-1)$-hamiltonian cycle, we have to insert the vertices of $\mathcal{I}$ in the cycle we constructed in such a way that it remains a $(k-1)$-cycle. Given a vertex $a \in \mathcal{I}$, by the minimum degree condition, it sends at least $(\frac{k-1}{k} - \epsilon)n$ edges to $\mathcal{C}^* \cup \mathcal{K}^*$. We will first try to insert the vertex a in $\mathcal{C}^*$.

If for some graph $C_j \in \mathcal{C}^*$ the degree of a in C_j is at least $(\frac{k-1}{k+1} + \delta)|C_j|$ we can insert it easily in the path inside C_j. Indeed, without loss of generality, there is a vertex $v_{x,y}$ in the path in C_j such that a is connected to all vertices in the path at distance at most k from $v_{x,y}$ (Otherwise, we just reorder the color classes of C_j other than the first and last one, and reconstruct the path inside $|C_j|$ as in the initial procedure. Notice that this does not change the connecting paths, nor the vertices from the multipartite graphs they intersect, nor $\mathcal{I}, \mathcal{C}^*, \mathcal{K}^*$).

We proceed by replacing vertex $v_{x,y}$ by a in the path inside C_j. Notice that $v_{x,y}, v_{x+1,y-1}, v_{x+2,y-2}, \ldots, v_{x+k,y-k}$ (where indices are taken $\pmod{k+1}$) form a clique because they belong to different color classes in C_j. We can remove each one of them, and reconnect the one preceding it directly to the one succeeding it in the original path, still obtaining a $(k-1)$-path.

Finally we insert the path formed by the removed vertices between any two columns of the graph other than the one from which a was removed.

When inserting other vertices in C_j we will be careful not to place them in any of the k columns neighboring each of the affected ones $(x+1, \ldots, x+k)$ to guarantee that the resulting path remains a $(k-1)$-path.

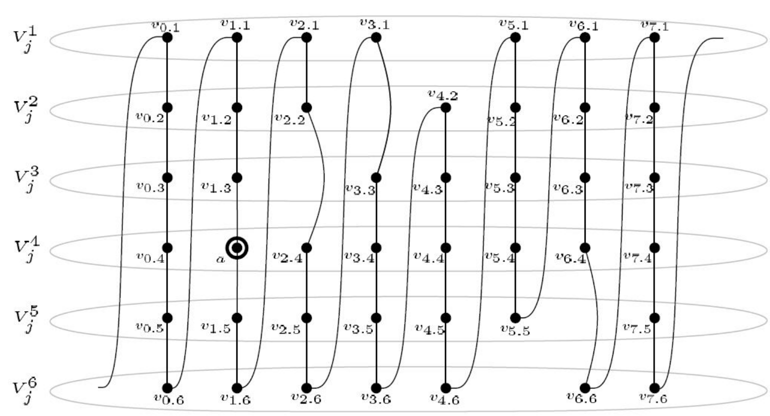


Figure 3.3: Inserting a into the $(k-1)$-path being constructed in the complete balanced $(k+1)$-partite graph C_j^*, where $k = 5$, and $v_{x,y} = v_{1,4}$.

If a vertex $a \in \mathcal{I}$ cannot be inserted in any $(k+1)$-partite graph, the minimum degree condition implies that the vertex has at least $\left(\frac{k-1}{k} + \delta\right)|K|$ neighbors for a big fraction of the graphs $K \in \mathcal{K}^*$. In this case we can assume that a is connected to all vertices in three consecutive columns of K, and just insert it between any two consecutive vertices of the middle column. The resulting path remains a remains a $(k-1)$-path. Registering the three columns used as not available to place new vertices in K, we can insert in the path inside this graph a new element from I.

Repeating this until all elements are used (which is feasible because $|\mathcal{K}^*| \gg |\mathcal{I}|$), we obtain a $(k-1)$-hamiltonian cycle.

3.3 Main Tools

We shall assume that n is sufficiently large and use the following main parameters:

$$0 < \eta \ll \alpha \ll 1,$$

where $a \ll b$ means that a is sufficiently small compared to b. In order to present the results transparently we do not compute the actual dependencies, although it could be done.

3.3.1 Complete k-Partite Subgraphs

In [24] the Regularity Lemma [33] was used to prove the Pósa-Seymour conjecture, however, here we use more elementary methods using only the Bollobás-Erdős-Stone-Simonovits bound [26].

Lemma 23 (Theorem 3.1 on page 328 in [1])**.** *There is an absolute constant* $\beta_1 > 0$ *such that if* $0 < \varepsilon < 1/s$ *and we have an* n*-graph* G *with*

$$|E(G)| \geq \left(1 - \frac{1}{s} + \varepsilon\right) \frac{n^2}{2}$$

then G *contains a* $K_{s+1}(t_1)$*, where*

$$t_1 = \left\lfloor \frac{\beta_1 \log n}{s \log 1/\varepsilon} \right\rfloor.$$

The following two observations will be useful later on.

Lemma 24. *If* $G(A, B)$ *is an* η*-dense bipartite graph, then there must be at least* $\eta|B|/2$ *vertices in* B *for which the degree in* A *is at least* $\eta|A|/2$*.*

Indeed, otherwise the total number of edges would be less than

$$\frac{\eta}{2}|A||B| + \frac{\eta}{2}|A||B| = \eta|A||B|,$$

a contradiction to the fact that $G(A,B)$ is η-dense.

Lemma 25. *Let $G(A,B)$ be a bipartite graph such that $|A| = c_1 \log n$, $|B| = c_2 n^{c_3}$ where $0 < c_1, c_2, c_3 < 1$ are constants and $c_1 \ll c_3$. If for all $b \in B$ we have $deg_A(b) \geq \eta|A|/2$, then we can find a complete bipartite subgraph $K(A', B')$ of G such that $A' \subset A, B' \subset B, |A'| \geq \eta|A|/2$ and $|B'| \geq c_2 n^{(c_3 - c_1)}$.*

To see this consider the neighborhoods in A of the vertices in B. Since there can be at most $2^{|A|} = n^{c_1}$ such neighborhoods, by averaging there must be a neighborhood that appears for at least $\dfrac{c_2 n^{c_3}}{n^{c_1}} = c_2 n^{(c_3 - c_1)}$ vertices of B. This means that we can find the desired complete bipartite graph.

Lemma 26. *Let G be a graph with $V(G)$ partitioned into $A_1, A_2, \ldots, A_k$ and B such that the subsets $A_1, A_2, \ldots, A_k$ form a complete k-partite graph, and for $1 \leq i \leq k$, $|A_i| = c_1 \log n$, $|B| = c_2 n$ for constants $0 < c_1, c_2 < 1$. If for every $b \in B$, $deg_{A_i}(b) \geq \eta|A_i|/2$, then we can find a complete $(k+1)$-partite graph $G(A_1', A_2', \ldots, A_k', B')$ such that $A_i' \subset A_i, B' \subset B, |A_i'| \geq \eta|A_i|/2$ and $|B'| \geq c_2 n^{(1 - kc_1)}$.*

Proof. First consider the bipartite graph $G_1(A_1, B)$. By Lemma 25 we have $A_1' \subset A_1, B_1 \subset B, |A_1'| \geq \eta|A_1|/2$ and $|B_1| \geq c_2 n^{(1-c_1)}$, such that $G_1(A_1', B_1)$ is a complete bipartite subgraph. Now consider the bipartite graph $G_2(A_2, B_1)$. Applying again Lemma 25, we find $A_2' \subset A_2, B_2 \subset B_1, |A_2'| \geq \eta|A_2|/2$ and $|B_2| \geq c_2 n^{(1-2c_1)}$, such that $G_2(A_2', B_2)$ is a complete bipartite subgraph. Note that this gives us a complete tripartite graph with color classes A_1', A_2' and B_2. Proceeding similarly, we can find a complete $(k+1)$-partite graph $G_k(A_1', A_2', \ldots, A_k', B)$ such that for $1 \leq j \leq k$, $A_j' \subset A_j, B_i \subset B, |A_j'| \geq \eta|A_j|/2$ and $|B_k| \geq c_2 n^{(1-kc_1)}$. □

Lemma 27. *There exist two constants n_0 and $\beta_2 > 0$ such that if G is a α-non extremal graph on $n \geq n_0$ vertices with $\delta(G) \geq (\frac{k-1}{k} - \sqrt{\eta})n$, then G contains a $K_{k+1}(t)$, where $t = \lfloor \beta_2 \log n \rfloor$. Here β_2 and n_0 depend on α and η.*

Proof. We apply Lemma 23 to G to get k disjoint sets $A_1, A_2, \ldots, A_k$, each of size $t_1 = \left\lfloor \frac{\beta_1 \log n}{k \log 1/\alpha} \right\rfloor$ such that they form a complete balanced k-partite graph. Define $A := \bigcup_{i=1}^k A_i$ and let $B \subset V(G) \setminus A$ be the set of vertices that have more than $\eta|A_i|$ neighbors in each A_i. Our first observation is that we can assume that $|B| \leq \eta^2 n$. If not, by Lemma 26 we get our desired $K_{k+1}(t)$.

Let $C = V(G) \setminus (A \cup B)$ and for $1 \leq i \leq k$ let $C_i = \{c \in C : deg_{A_i}(c) < \eta|A_i|\}$. By definition of B it follows that $C = \bigcup_{i=1}^k C_i$. From the minimum degree condition and the definition of C_i we have that for every i:

$$\left(\left(\frac{k-1}{k} - \sqrt{\eta}\right) n - |B| - |A|\right) |A_i| \leq e(A_i, C) \leq \eta|A_i||C_i| + |A_i|(|C| - |C_i|)$$

which shows us that $|C_i| \leq (1 + 3\sqrt{\eta})n/k$.

We will now show that from $|C_j| \leq (1 + 3\sqrt{\eta})\frac{n}{k}$, $(1 \leq j \leq k)$, it follows that for every j $(1 \leq j \leq k)$

$$|C_j| \geq (1 - 4k\sqrt{\eta})\frac{n}{k} \tag{3.1}$$

Assume 3.1 does not hold.

Then $n - |B| - kt_1 = \displaystyle\sum_{i=1}^{k} |C_i| \leq (k-1)(1 + 3\sqrt{\eta})n/k + (1 - 4k\sqrt{\eta})\frac{n}{k}$ which is a contradiction because $|B| \leq \eta^2 n$, and $t_1 = O(\log n)$.

From the minimum degree condition and from

$$|B| \leq \eta^2 n \ , \quad e(A_j, C_j) \leq \eta|A_j||C_j| \qquad (1 \leq j \leq k) \tag{3.2}$$

it follows after a little calculation that

$$e(A_j, C_1) \geq (1 - 100k\sqrt{\eta})|A_j||C_1|, \tag{3.3}$$

for $2 \leq j \leq k$.

Let us denote by C_1^j the following set:

$$C_1^j = \left\{ x \in C_1, \deg_{A_j}(x) \leq \frac{2}{3}|A_j| \right\}.$$

Then, by double counting, from (3.3), we get

$$|C_1^j| \leq 300k\sqrt{\eta}|C_1|.$$

Define $C_1^* = \bigcup_{j=2}^{k} C_1^j$. Then $|C_1^*| \leq 300k^2\sqrt{\eta}|C_1|$. We omit C_1^* from C_1. Denote the remaining set by C_1^{**}. Obviously, $|C_1^{**}| \geq (1 - 300k^2\sqrt{\eta})|C_1|$. We group the vertices in C_1^{**} according to their neighborhoods in $A \setminus A_1$. The number of groups is at most 2^{kt_1}.

We again omit the groups containing at most $\sqrt{\eta}n/k2^{kt_1}$ elements. The union of the remaining groups has size greater than $(1 - 400\sqrt{\eta}k^2) \cdot \frac{n}{k}$.

Let's denote these groups by $D_1, D_2, \ldots, D_m$, and $D = \bigcup_{i=1}^{m} D_i$. We know that $|D| > (1 - 400\sqrt{\eta}k^2) \cdot \frac{n}{k} > (1 - \alpha)\frac{n}{k}$, and that $|D| \leq \left(1 + 3\sqrt{n}\right)\frac{n}{k}$.

Since G is not α-extremal, and $(1 - \alpha)\frac{n}{k} < |D| < (1 + \alpha)\frac{n}{k}$, we get that

$$e(G|_D) \geq \alpha \left(\frac{n}{k}\right)^2.$$

There are two cases.

Case 1.1. There is a group, say D_1 such that $e(G|_{D_1}) \geq \eta^2|D_1|^2$. Then, by Lemma 23, with $s = 1$ in $G|_{D_1}$ we have a complete bipartite graph, spanned by two sets, say Q and R, of size greater than $\beta_2 \log n$. Since $N_{A_i}(D_1) > \frac{2}{3}|A_i|$ (for $i \neq 1$) we can find $K_{k-1}(t) \subset N_{A \setminus A_1}(D_1)$ which together with Q and R gives us the required $K_{k+1}(t)$.

Case 1.2. There are two sets, say, D_1 and D_2, such that $e(D_1, D_2) \geq \eta^2|D_1||D_2|$. Then by Lemma 2, we have a set $Q \subset D_1$ and $R \subset D_2$ of size greater than $\beta_2 \log n$ and Q and R form a complete bipartite graph. Since $N_{A_i}(D_1) \cap N_{A_i}(D_2) > \frac{1}{3}|A_i|$ for $i \neq 1$ we can find $K_{k-1}(t) \subset N_{A \setminus A_1}(D_1) \cap N_{A \setminus A_1}(D_2)$ which together with Q and R gives us the required $K_{k+1}(t)$. □

We will use in Section 3.5 the following simple fact on the size of a maximum set of vertex disjoint paths in G (see [1]).

Lemma 28. *In a graph G on n vertices, we have*

$$\nu_1(G) \geq \max\left\{\delta(G), \delta(G)\frac{n}{4\Delta(G)}\right\} \text{ and } \nu_2(G) \geq (\delta(G) - 1)\frac{n}{6\Delta(G)}$$

where $\nu_i(G)$ denotes the size of maximum set of vertex disjoint paths of length i in G.

3.3.2 The Connecting Lemma

Definition 6 (Eligible vertices)**.** *We call a vertex $v \in V(G)$,* eligible*, if for each $\ell \in [1, 9(k+1)!]$, the number of paths of length $\ell+1$ (edges) between any $v_1, v_2 \in N(v)$, and which lie completely in $N(v)$, is at least $\eta^{4\ell} \cdot n^{\ell}$.*

(The paths contain $\ell + 2$ vertices, v_1, v_2 included. The endpoint are fixed, so we may have n^{ℓ} connecting paths and we require a "positive" percentage, $c_k n^{\ell}$ of them. Here $c_k = \eta^{-k^k}$).

It is easy to see that in G there are at least n/k^2 eligible vertices.

Definition 7. *A path of length ℓ is* good *if it is in $N(v)$ for at least $\eta^{k^{k^k}} n$ vertices v. Otherwise it is* bad*. We call a vertex v* good *if $N(v)$ contains at most $100k^2\eta^{k^{k^k}} n^{\ell+1}$ bad paths P_ℓ, for each $\ell \in [1, 9k!]$.*

An easy calculation shows that the number of good vertices is at least

$$n\left(1 - \frac{1}{100k^2}\right)$$

and the number of vertices which are both eligible and good is at least

$$\frac{n}{4k}\left(1 - \frac{1}{100k^2}\right)$$

After these definitions we formulate the Connecting Lemma.

Lemma 29 (Connecting Lemma)**.**
Given a clique $\{a_1, a_2, \ldots, a_{k-1}\}$ and $\{b_{k-1}, b_{k-2}, \ldots, b_1\}$, there is an $\ell \leq 9(k+1)!$ such that we can connect these two cliques with at least $\eta^{k^{k^{k^{k^k}}}} n^{\ell}$ $(k-1)$-paths of lenght $l+1$, even if we forbid using $o(n)$ vertices, given in advance.

Lemma 30. *If G is not an α-extremal graph, then there are $\sqrt{\eta}n$ vertices $v \in V(G)$ such that v is good and eligible and (for each v) there is a $T_v \subseteq N(v)$ for which $|T_v| = \sqrt{\eta}n$, $T_v = \{t_{1,v}, t_{2,v}, \ldots, t_{\sqrt{\eta}n,v}\}$ and $|N(t_{i,v}) \cap \overline{N(v)}| \leq \frac{n}{k} - \sqrt{\eta}n$ for $t_{i,v} \in T_v$ $(1 \leq i \leq \sqrt{\eta}n)$.*

Proof. Notice that the lemma trivially holds for each vertex $v \in V(G)$ for which $|\overline{N(v)}| \leq \frac{n}{k} - \sqrt{\eta}n$.

Assume that there is a $v_0 \in V(G)$ so that for at least $|N(v_0)| - \sqrt{\eta}n$ vertices $w_i \in N(v_0)$,

$$|N(w_i) \cap \overline{N(v_0)}| > \frac{n}{k} - \sqrt{\eta}n.$$

Denote the set of these vertices by $W = \{w_1, w_2, \dots\}$.

Obviously the vertices in W are eligible, because $(W, \overline{N(v_0)})$ is an almost complete bipartite graph. Omit from $N(v_0)$ the vertices $z \in N(v_0)$ for which $|N(z) \cap \overline{N(v_0)}| \leq \frac{n}{k} - \sqrt{\eta}n$. We omitted at most $\sqrt{\eta}n$ vertices. Denote the set of the omitted vertices by T_1. Now we choose a vertex $v_1 \in N(v) \setminus T_1$. If there are at least $\sqrt{\eta}n$ vertices in $N(v_1) \setminus T_1$, such that their neighbors intersect $\overline{N(v_1)}$ in at most $\frac{n}{k} - \sqrt{\eta}n$ points, then we are done. (We can assume that v_1 is good, too. The number of vertices that are not good in $N(v_1)$ is at most $\sqrt{\eta}n$.) If not, then we omit the vertices $z \in N(v_1) \setminus T_1$ for which $|N(z) \cap \overline{N(v_1)}| \leq \frac{n}{k} - \sqrt{\eta}n$. We again omitted at most $\sqrt{\eta}n$ vertices. Denote this set by T_2. Notice that $|\overline{N(v_1)} \cap N(v) \setminus T_1)| \geq \frac{n}{k} - 2\sqrt{\eta}n$ and $|\overline{N(v)} \cap \overline{N(v_1)}| \leq 2\sqrt{\eta}n$. Now choose a vertex $v_2 \in N(v_1) \setminus (T_1 \cup T_2)$. We argue as previously and continue this argument until we find a v_i such that there are at least $\sqrt{\eta}n$ vertices $z \in N(v_i)$ for which $|N(z) \cap \overline{N(v_i)}| \leq \frac{n}{k} - \sqrt{\eta}n$. Then we stop and v_i is our vertex which we mentioned in the Lemma. If we can not find such a v_i $(i < k)$, renaming $\overline{N(v_{i-1})}$ to A_i, then our graph is the union of $A_1, A_2, \dots, A_k, X$ where X is a small vertex set ($|X| \leq 10k\sqrt{\eta}n$); and $|A_i| \geq \frac{n}{k} - \sqrt{\eta}n$ and the bipartite graphs (A_i, A_j), $(i \neq j)$ are almost complete (namely every vertex has degree at least $\frac{n}{k} - 5k\sqrt{\eta}n$). Adding vertices from X, we balance the sets A_i so that all of them will have size $\frac{n}{k}$. Denoting the new classes by $A_1^*, A_2^*, \dots, A_k^*$, then (A_i^*, A_j^*), $(i \neq j)$, is still an almost complete bipartite graph.

In particular, every vertex in $A_1^* \setminus X$, has at least $(\frac{k-1}{k} - 10k\sqrt{\eta})n$ neighbors in the union of $A_2^*, A_3^*, \dots, A_k^*$. Since we assumed that the graph G is not α-extremal, and X is small, there is a vertex $v \in A_1^*$ which has at least $\frac{\alpha}{2}\frac{n}{k}$ neighbors in A_1^*. Hence $|N(v)| \geq (\frac{k-1}{k} + \frac{\alpha}{4})n$, and the lemma follows from the first observation in this proof. $\square$

Iterating the above procedure, we get our set M of size $\sqrt{\eta}n$, such that for every $v \in M$, v is good and eligible and there is a $T_v \subseteq N(v)$ for which $|T_v| = \sqrt{\eta}n$, $T_v = \{t_{1,v}, t_{2,v}, \dots, t_{\sqrt{\eta}n,v}\}$ and $|N(t_{i,v}) \cap \overline{N(v)}| \leq \frac{n}{k} - \sqrt{\eta}n$.

The Extending Lemma

We fix a vertex $v_1 \in M$. For simplicity, we denote $N(v_1)$ by F.

Lemma 31 (Extending Lemma)**.**

Given a clique $A = \{a_1, \dots, a_{k-1}\}$ one can extend it with vertices $w_1 w_2 \dots w_s x_1 x_2 \dots x_{k-1}$ ($0 \le s \le k-1$) so that $a_1 a_2 \dots a_{k-1} w_1 w_2 \dots w_s x_1 x_2 \dots x_{k-1}$ is a $(k-1)$-path and $x_1, x_2, \dots, x_{k-1} \in F$. Moreover, the number of choices of $w_1, \dots, w_s, x_1, x_2 \dots, x_{k-1}$ is at least $\left(\frac{\eta n}{10}\right)^{k-1+s}$.

Proof. Let $W_0 = \{a_1, a_2, \dots, a_{k-1}\}$. The size of $N(W_0)$ is at least $\frac{n}{k}$. We may assume that it is $\frac{n}{k}$, after possibly dropping some vertices from $N(W_0)$. If $|N(W_0) \cap F| \ge \frac{\eta}{10} n$, then we choose w_1 from $N(W_0) \cap F$ and we get a set $W_1 = \{a_2, \dots, a_{k-1}, w_1\}$. Note that we have at least $\frac{\eta n}{10}$ choices for w_1. If $|N(W_1) \cap F| \ge \frac{\eta}{10} n$ then we choose w_2 from $N(W_1) \cap F$. We get a set $W_2 = \{a_3, a_4, \dots, w_1, w_2\}$. Define $W_i = \{a_{i+1}, \dots, a_{k-1}, w_1, \dots, w_i\}$. We proceed the same way as long as

$$|N(W_i) \cap F| > \frac{\eta}{10} n \tag{3.4}$$

holds. If (3.4) holds for all $i \le k-1$, then we have at least $\left(\frac{\eta}{10}\right)^{k-1} n^{k-1}$ $(k-1)$-paths $a_1, a_2, \dots, a_{k-1}, w_1, w_2, \dots, w_{k-1}$. We rename $w_1, w_2, \dots, w_{k-1}$ to $x_1, x_2, \dots, x_{k-1}$. So, in this case we are done with the proof. If $i < k-1$ is the smallest integer for which (3.4) does not hold, then we are going to work with W_i (meaning that instead of $\{a_1, a_2, \dots, a_{k-1}\}$ we start with W_i). For the sake of simplicity, we rename W_i to W. We define W_{low} as follows.

$$W_{low} = \left\{ w \in W : \deg_{N(W)}(w) \le |N(W)| - \eta n \right\}.$$

Since $|N(W) \cap F| \le \frac{\eta}{10} n$, we have that $|N(W)| \le \frac{n}{k} + \frac{\eta}{10} n$, and so $deg_{N(W)}(w) \le \frac{n}{k} + \frac{\eta}{10} n - \eta n$, for any $w \in W_{low}$.

That implies that for any $w \in W_{low}$,

$$|N_F(w)| \ge \frac{k-2}{k} n + \frac{8}{10} \eta n$$

Therefore, if $S \subset F$, $|S| = \frac{n}{k}$, then $|N_S(w)| \ge \frac{8}{10} \eta n$. There are two cases.

Case 2.1. $|W_{low}| \ge \frac{\eta}{10} n$. Choose a vertex $l \in W_{low}$. Since $|N_F(l)| \ge \frac{k-2}{k} n + \frac{8}{10} \eta n$, we get that $|N_F(l, z_1, \dots, z_{k-2})| \ge \frac{8}{10} \eta n$, for any $k-2$ vertices $z_1, \dots, z_{k-2}$.

From this remark, it is easy to see that we can construct $(k-1)$-paths of the form $a_1, \dots, a_{k-1}, w_1, \dots, w_i, l, x_1, x_2, \dots, x_{k-2}$, such that each x_j can be chosen in at least $\frac{8}{10}\eta n$ ways. So we are done with Case 1.

Case 2.2. $|W_{low}| \le \frac{\eta}{10}n$. Choose a clique $C_1 \subseteq N(\{a_1, a_2, \dots, a_{k-1}\})$ with $C_1 = \{c_1, \dots, c_{2k}\}$. It is obvious that there are two distinct vertices, c_i and c_j for which $|N_F(c_i) \cap N_F(c_j)| > \frac{k-3}{k}n + \frac{n}{k^2}$.

We can assume that $i = 1$ and $j = 2$. We will consider only the first $k-1$ elements and denote the set $C_2 = \{c_1, c_2, \dots, c_{k-1}\}$. Then using an argument similar to the previous one, we can find $x_1, x_2, \dots, x_{k-2}$, such that

$$a_1, a_2, \dots, a_{k-1}, c_{k-1}, c_{k-2}, \dots, c_2, c_1 x_1, x_2, \dots, x_{k-2}$$

is a $(k-1)$-path and for each x_i, as before, we have at least $\frac{n}{k^2}$ choices (notice that $c_1, c_2, \dots, c_k$, can also be chosen in at least $\frac{\eta}{10}n$ ways).

If $|N_F(x_1, x_2, \dots, x_{k-3}, x_{k-2}, c_1)| \ge \eta n$, we can choose x_{k-1} from $N_F(x_1, \dots, x_{k-2}, c_1)$ in $\frac{\eta}{10}n$ different ways, such that $a_1, a_2, \dots, a_{k-1}, c_{k-1}, c_{k-2} \dots c_2, c_1, x_1, \dots, x_{k-2}, x_{k-1}$ is a $(k-1)$-path.

If $|N_F(x_1, x_2, \dots, x_{k-2}, c_2)| \ge \eta n$, then we can choose x_{k-1} in at least $\frac{\eta}{10}n$ different ways, so that $a_1, a_2, \dots a_{k-1}, c_{k-1} c_{k-2}, \dots, c_1, c_2, x_1, \dots, x_{k-2}, x_{k-1}$ is a $(k-1)$-path.

If both $|N_F(x_1, \dots, x_{k-2}, c_2)| < \frac{\eta}{10}n$, and $|N_F(x_1, \dots, x_{k-2}, c_1)| < \frac{\eta}{10}n$, then since $|N_F(x_1, x_2, \dots, x_{k-3}, x_{k-2})| \ge \frac{n}{k}$, we have

$$N_F(c_1, c_2) \ge \frac{k-2}{k}n - \frac{2\eta}{10}n,$$

and therefore $|N_F(c_1, c_2, x_1, x_2 \dots x_{k-3})| > \frac{n}{k} - \frac{2\eta}{10}n$. Since our graph G is α-non-extremal, we have at least $\alpha\left(\frac{n}{k}\right)^2$ edges in $N_F(c_1, c_2, x_1, x_2 \dots x_{k-3})$. For x_{k-2} we consider the vertices which have degrees into $N_F(c_1, c_2, x_1, x_2 \dots x_{k-3})$ that are larger than $\frac{\eta}{10}n$. We have at least $\frac{\eta}{10}n$ such large degree vertices. We choose for x_{k-2} the large degree vertices and for x_{k-1} the endpoint of the edges incident to x_{k-2}. It is then obvious that $a_1, \dots, a_{k-1}, c_{k-1}, \dots, c_2, c_1, x_1, x_2, \dots, x_{k-3}, x_{k-2}, x_{k-1}$ is a $(k-1)$-path, and for both x_{k-2} and x_{k-1} we have $(\frac{\eta}{10}n)^2$ choices. So we have proved the extending Lemma. We apply this to the b_i's as well, to get a $(k-1)$-path □

$$b_1, b_2, \dots, b_{k-1} w'_1 w'_2 w_3 \dots w'_t y_1 y_2 y_{k-1}.$$

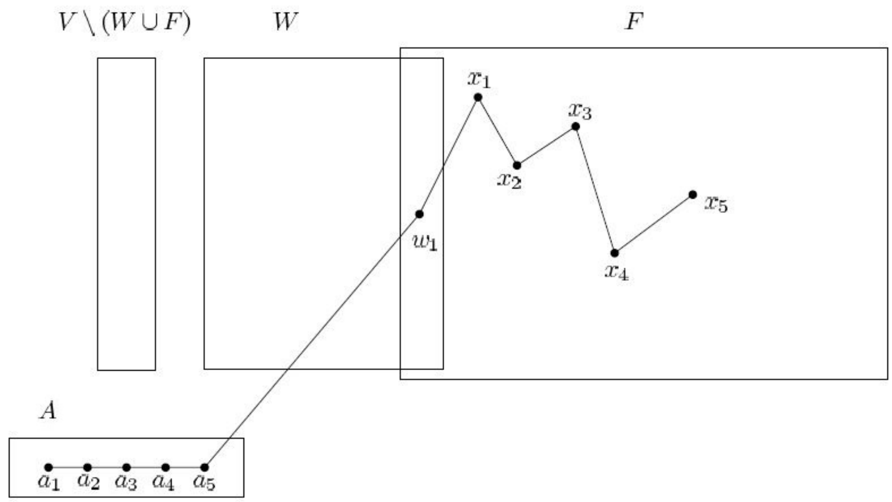


Figure 3.4: The $(k-1)$-path may be extended if $W \cap F$ is large.

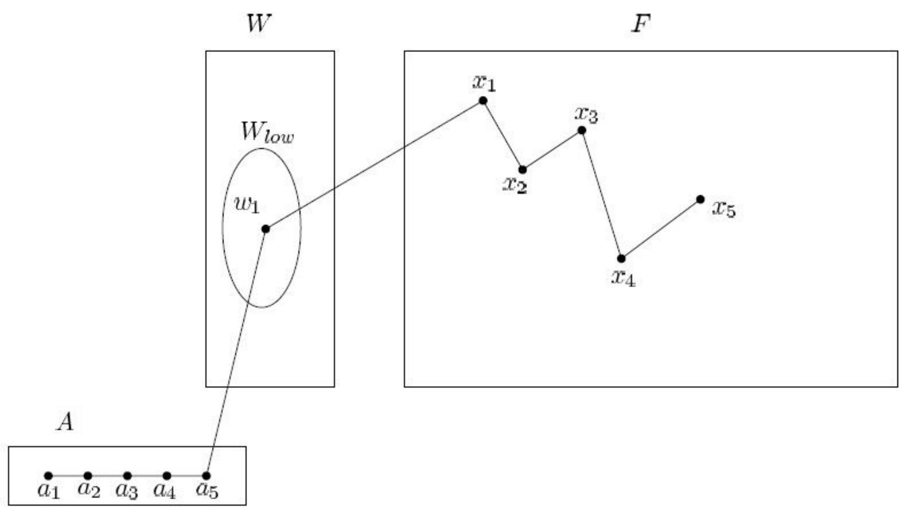


Figure 3.5: The $(k-1)$-path may be extended if W_{low} is large.

Connecting $a_1, a_2, \ldots, a_{k-1}$ and $b_{k-1}, \ldots, b_2, b_1$ inside $N(v)$, where v is a good vertex

First we choose $z_1, z_2, \ldots, z_k \in M$. For every z_i we choose a $(k-1)$-path

$$u_{k-1}^{(i)}, u_{k-2}^{(i)}, \ldots, u_1^{(i)}, z_i, u_{k+1}^{(i)}, \ldots, u_{2k-1}^{(i)}.$$

Because of the properties of z_i, this can be done easily. These paths are vertex-disjoint. The Extending Lemma will be applied to $a_1, \ldots, a_{k-1}$ and also to $b_1, b_2, \ldots, b_{k-1}$: We consider the $(k-1)$-paths

$$a_1a_2\dots a_{k-1}w_1w_2\dots w_sx_1x_2\dots x_{k-1} \quad b_1b_2\dots b_{k-1}w_1'w_2'w_3\dots w_t'y_1y_2y\dots y_{k-1}$$

Now, we connect $x_2x_3\dots x_{k-1}$ to $u_{k-1}^{(1)}u_{k-2}^{(1)}\dots u_2^{(1)}$, using the induction hypothesis. The number of connecting paths is at least $\eta^{(k-1)^{(k-1)^4}}n^{l_1}$, where ℓ_1 is the length of the $(k-2)$-connecting paths (We can assume that all the paths have the same length $\ell_1 \le 10(k-1)(k-1)!$). Now we connect $u_{k+2}^{(1)},\dots,u_{2k-1}^{(1)}$ to $u_{k-1}^{(2)},u_{k-2}^{(2)},\dots,u_2^{(2)}$.

We continue connecting the i^{th} ending segment to the $(i+1)^{th}$ initial segment. Finally we connect $u_{k+1}^{(k)}\dots u_{2k-1}^{(k)}$ to $y_{k-1}y_{k-2}\dots y_2$. They are $(k-2)$-paths and they remain $(k-2)$-paths even after removing any number of z's. Let the number of vertices on the path from x_{k-1} to y_{k-1} be $(k-1)t+r$, where $r \le k-1$. We omit r vertices $z_1,\dots,z_r$ from our path, to get the divisibility with $k-1$ and get $\gamma=(k-1)t$ vertices. If the corresponding lengths are $\ell_1,\dots,\ell_{k+1}$, then we have altogether at least $\eta^{(k+1)(k-1)^{(k-1)^4}}\eta^{2k(k-1)}n^{\gamma}$ paths.

If we consider all the paths for all possible $x_1,x_2,\dots,x_{k-1}$ and $y_{k-1},y_{k-2},\dots,y_1$, then obviously, many of them will be good paths, because the number of paths we constructed is at least Q, where

$$Q=\eta^{(k+1)(k-1)^{(k-1)}\cdot(k-1)^{(k-1)^{(k-1)}}}\left(\frac{\eta}{10}\right)^{2(k+1)(k-1)}n^{\gamma+2(k-1)} \ge 100k^2\eta^{k^{k^k}}n^{\gamma+2(k-1)}.$$

Fix a good path P joining x_1 to y_1. For this P we have a set T_P of size $\eta^{k^{k^k}}n$. We shall change this P into a $(k-1)$-path by inserting t vertices from T_P. We insert them in the following way. We move along P, starting with y_{k-1} inserting the next vertex from T_P and then moving along P for $k-1$ vertices and then again insert a vertex from T_P, again move on along the paths P, ... and continue this until we have inserted a vertex of T_P next to x_{k-1}. This way our path will be a $(k-1)$-path, and we created from a given P at least $|T_P|^t$ $(k-1)$-paths. If we consider all the possible paths and do the same thing, we get at least $Q\eta^{tk^{k^k}}n^t$ $(k-1)$-paths, which is more than the required number of connecting paths in the Lemma.

3.4 The α-non Extremal Case

3.4.1 The Covering Lemma

Lemma 32 (Covering). *We can partition $V(G)$ into $\mathcal{C}, \mathcal{K}, \mathcal{I}$ such that $\mathcal{C}$ is the union of complete $(k+1)$-partite graphs with color classes of size $\eta^{(1/\eta)} \log n$, and $|\mathcal{C}| = \sqrt{\eta} n$; $\mathcal{K}$ is the union of complete k-partite graphs with color classes of size $\eta^{(1/\eta)} \log n$; and $|\mathcal{I}| < \eta n$.*

Proof. Using Lemma 27, we get C. Notice that if a graph G^* on m vertices has density $> \frac{k-2}{k-1} + \frac{1}{k^3}$, then by Lemma 26 we know that G^* contains a complete $K_k(\eta \log m)$. So using Lemma 26, we can construct a set $\mathcal{K}^{(1)}$ of complete k-partite graphs of size $\eta \log n$, whose union has at least ηn vertices. We are left with a set of vertices we denote by $\mathcal{I}^{(1)}$.

We will successively remove vertices from $\mathcal{I}^{(1)}$, and use them to construct more complete k-partite graphs (even if with slightly smaller color classes), until the set of remaining vertices has less than ηn elements.

We proceed by rounds, as explained below. At the i^{th} round we get a set $\mathcal{K}^{(i)}$ of complete k-partite graphs with color classes of size $t^{(i)} := \eta^i \log n$, and whose union has $i\eta n$ vertices. Let $\mathcal{I}^{(i)} = V(G) \setminus (\mathcal{C} \cup \mathcal{K}^{(i)})$. If $|\mathcal{I}^{(i)}| \geq \eta n$ we carry out the $(i+1)^{th}$ round; otherwise we stop and $\mathcal{I}^{(i)}$ is the set $\mathcal{I}$ as in the lemma. Notice that such a procedure terminates after at most $\frac{1}{\eta}$ rounds.

When the set of remaining vertices $\mathcal{I}^{(i)}$ has many more elements than $\mathcal{C}$, say $|\mathcal{I}^{(i)}| \geq \eta^{\frac{1}{4}} n$, we can easily complete a round in the following way.

Case $d(\mathcal{I}^{(i)}) \geq \frac{k-2}{k-1} + \frac{1}{k^3}$ **:** then, since $|\mathcal{I}^{(i)}| \geq \eta^{\frac{1}{4}} n$, by lemma 23 we can get complete k-partite graphs of size $\eta^{i+1} \log n$.

Case $d(\mathcal{I}^{(i)}) < \frac{k-2}{k-1} + \frac{1}{k^3}$ **:** then, because of the minimum degree condition, we get $deg_{\mathcal{K}^{(i)}}(x) \geq \frac{k-1}{k} n + \frac{\eta^{\frac{1}{4}}}{2k^2} n$, for at least $\frac{\eta^{\frac{1}{4}}}{2k^2}$ of the vertices $x \in \mathcal{I}^{(i)}$. Therefore, by lemma 26 we can find a complete k-partite graph $K \in \mathcal{K}^{(i)}$ with color classes $V_1, V_2, \ldots, V_k$, and a set $B \subseteq \mathcal{I}^{(i)}$ with $|B| = k\eta^{i+1} \log n$, such that $N_{V_1 \cup V_2 \cup \cdots \cup V_k}(v)$ is the same for every $v \in B$, and it has size at least $k\eta^{i+1} \log n$. But then some subsets $A_j \subseteq N(B) \cap V_j$, $(1 \leq j \leq k)$, and B will give us a complete $(k+1)$-partite graph with color classes of size $k\eta^{i+1}\log n$.

We break this complete $k+1$-partite graph into complete k-partite graphs of size $\eta^{i+1}\log n$. And we also break the unbalanced complete k-partite graph obtained from K by removing the sets $A_1, A_2, \ldots, A_k$, into k-partite graphs all with color classes of size $\eta^{i+1}\log n$.

We can repeat the above procedure while $|\mathcal{I}^{(i)}| \geq \eta^{\frac{1}{4}}n$, since $|\mathcal{I}^{(i)}| \gg |\mathcal{C}|$ in this situation, and so almost all edges coming out of $\mathcal{I}^{(i)}$ go into $\mathcal{K}^{(i)}$.

Assume now that $\eta n \leq |\mathcal{I}^{(i)}| \leq \eta^{\frac{1}{4}}n$.

Let us show that **for most of the complete k-partite graphs in $\mathcal{K}^{(i)}$, a big fraction of the vertices in $I^{(i)}$ have a large neighborhood in it**.

We can assume that for every $K \in \mathcal{K}^{(i)}$, $\{x \in \mathcal{I}^{(i)} : N_K(x) \geq \left(\frac{k-1}{k}+\eta\right)|K|\}$ has size at most $n^{\frac{1}{2}}$ (Otherwise we find $k\eta^{i+1}\log n$ elements which we can remove from $\mathcal{I}^{(i)}$ and use to construct new complete k-partite graphs, just as in the previous case).

It follows by an elementary averaging that for at least a $(1-\eta^{\frac{1}{8}})$ fraction of the graphs K in $\mathcal{K}^{(i)}$, $|E(K,\mathcal{I}^{(i)})| \geq \left(\frac{k-1}{k} - 4\eta^{\frac{1}{8}}\right)|K||\mathcal{I}^{(i)}|$. Let $\mathcal{K}^*$ be the set of such k-partite graphs in $\mathcal{K}^{(i)}$ (for which a large fraction of the vertices in $\mathcal{I}^{(i)}$ have large neighborhood in it).

Given $K \in \mathcal{K}^*$, by definition of $\mathcal{K}^*$, and after averaging it follows that at least $(1-\eta^{\frac{1}{16}})|\mathcal{I}^{(i)}|$ vertices in $\mathcal{I}^{(i)}$ have more than $\left(\frac{k-1}{k} - 5\eta^{\frac{1}{16}}\right)|K|$ neighbors in K. Call $\mathcal{I}^{*(i)}$ the set formed by those vertices.For simplicity, we denote it by $\mathcal{I}^*$.

By the definition of $\mathcal{I}^*$ (and the assumption that few elements of $\mathcal{I}^{(i)}$ have degree larger than $(\frac{k-1}{k}+\eta)|K_1|$ in K) we see that every vertex in $\mathcal{I}^*$ has almost full degree to $k-1$ of the color classes of K, and very few neighbors in the remaining one. Partition $\mathcal{I}^*$ into sets $I_1, I_2, \ldots, I_k$, where $I_j = \{x \in \mathcal{I}^* : deg_{V^j}(x) \leq 6\eta^{\frac{1}{16}}\}$.

If $|I_1| \geq n^{\frac{1}{2}}$, by lemma 26 we can find a set $I_1' \subseteq I_1$ of ηt elements, and $A_2 \subseteq V^2, \ldots, A_k \subseteq V^k$, which form a complete k-partite graph. Therefore if we remove a set X_1 of η_1 elements from the first color class V^1, and replace it by I_1', we can break the graph into smaller k-partite graphs, with color classes of size ηt. The reader might feel discouraged to notice that we got as many new vertices excluded from $\mathcal{K}^*$ as the ones we were able to remove from $\mathcal{I}^*$. However, if we repeat this procedure for $I_2, I_3, \ldots, I_k$, we obtain excluded sets $X_1, X_2, \ldots, X_k$ from the different color classes in K. And these form a $K_k(\eta t^{(i)})$, which we can add to $\mathcal{K}^{(i)}$.

Therefore, without loss of generality, we can assume that $|I_k| \leq n^{\frac{1}{2}}$. This means that at least $(1-2\eta^{\frac{1}{16}})|\mathcal{I}^{(i)}|$ of the vertices in $\mathcal{I}^{(i)}$ have more than $\left(1-7\eta^{\frac{1}{16}}\right)|V^k|$ neighbors in V^k.

Proceeding the same way with the remaining k-partite graphs in $\mathcal{K}^*$, we can find for each one of them, a big subset of vertices in $I^{(i)}$ which are connected to almost all elements of their last color class.

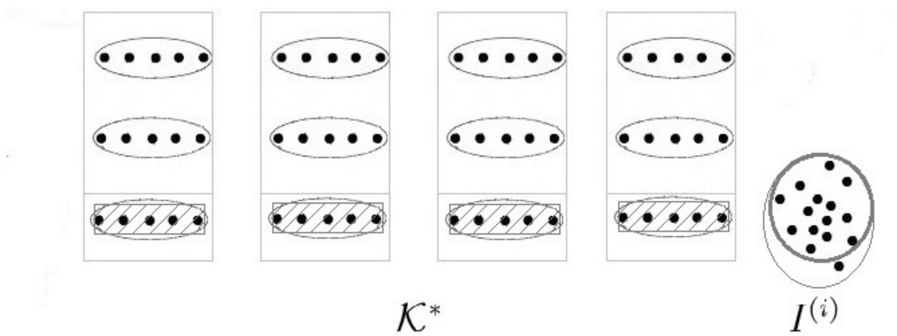


Figure 3.6: The bold circle represents elements in $\mathcal{I}^{(i)}$, which neighbor almost all the vertices of the last class of the first graph.

Given a graph in $\mathcal{K}^*$, we can repeat the above analysis, studying the neighborhood in its first $k-1$ color classes, of those vertices in $I^{(i)}$ that are connected to almost all in the last class.

Iterating the previous argument, we can assume that for each of the k-partite graphs in $\mathcal{K}^*$, there is a big subset of (at least $(1-2\eta^{\frac{1}{16}})^{k-1}|\mathcal{I}^{(i)}|$) vertices in $I^{(i)}$ which are connected to almost all elements of the last $k-1$ color classes (precisely, with at least $\left(1-7\eta^{\frac{1}{16}}\right)t^{(i)}$ neighbors in each class).

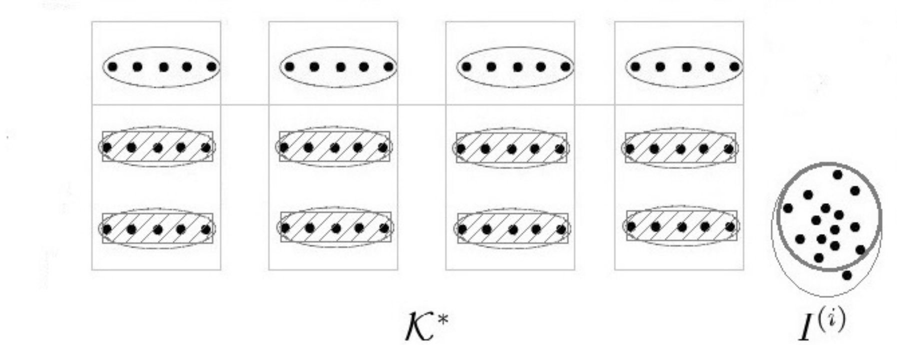


Figure 3.7: The bold circle represents elements in $\mathcal{I}^{(i)}$, which neighbor almost all the vertices of the last $k-1$ classes of the first graph.

At this point of our procedure, given a graph K in $\mathcal{K}^*$, we can replace any $\eta t^{(i)}$ of the elements of its first color class, by a subset of elements from $\mathcal{I}^{(i)}$ which are connected to almost all vertices in the last $k-1$ classes of K. We use lemma 26 as usual.

The reader might feel discouraged to notice that we exclude as many elements from $\mathcal{K}^{(i)}$ as the ones we insert from $\mathcal{I}^*$. The interest in such a step is that we may choose any $\eta t^{(i)}$ elements we like to remove from the first color class of the graph K. Proceeding analogously with the remaining graphs in $\mathcal{K}^*$, we remove a random looking set of their first color classes and replace $\mathcal{I}^{(i)}$ by it. Since G is α-non extremal, we can chose $Y_1, Y_2, \ldots$ such that the set obtained from $\mathcal{I}^{(i)}$ after replacing most of its elements by $Y_1 \cup Y_2 \cup \ldots$ still has the same size as $\mathcal{I}^{(i)}$, but density at least $\frac{\alpha}{2}$.

Restarting the whole procedure with this new set of remaining vertices, and assuming that the algorithm does not the terminate, we reach the last step (as in figure 3.4.1) with a set of vertices with density at least $\frac{\alpha}{4}$. By abuse of notation, we still denote it by $\mathcal{I}^{(i)}$ (though many rounds might be completed).

At this point we can reduce the size of $\mathcal{I}^{(i)}$ in the following way. Given a graph K in $\mathcal{K}^*$, with color classes $V^1, V^2, \ldots, V^k$, partition $\mathcal{I}^{(i)}$ into sets of vertices $D_1, D_2, \ldots, D_l$ that have the same neighborhood in K. We distinguish two cases.

Case 1.1. There is a group, say D_1 such that $e(G|_{D_1}) \geq \eta^2 |D_1|^2$. Then, by Lemma 23, with $s = 1$ in $G|_{D_1}$ we have a complete bipartite graph, spanned by two sets Q and R, of size greater than $\beta_2 \log n$. Since $N_{V_j}(D_1) > \frac{2}{3}|V_j|$ (for $j \neq 1$) we can find $K_{k-1}(t) \subset N_{V(K)\setminus V_1}(D_1)$ which together with Q and R gives us a $(k+1)$-partite graph with color classes of size $k\eta t^{(i)}$.

Case 1.2. There are two sets, say, D_1 and D_2, such that $e(D_1, D_2) \geq \eta^2 |D_1||D_2|$. Then by Lemma 2, we have a set $Q \subset D_1$ and $R \subset D_2$ of size greater than $\beta_2 \log n$ and Q and R form a complete bipartite graph. Since $N_{V_j}(D_1) \cap N_{V_j}(D_2) > \frac{1}{3}|V_j|$ for $j \neq 1$ we can find $K_{k-1}(t) \subset N_{V(K)\setminus V_1}(D_1) \cap N_{V(K)\setminus V_1}(D_2)$ which together with Q and R gives us a $K_{k+1}(k\eta t^i)$.

In either case, we can consider sets $A_2 \subseteq V_2, \ldots, A_k \subseteq V_k$ such that $Q, R, A_2, \ldots, A_K$ form a complete (k+1)-partite graph. Breaking this complete $(k+1)$-partite graph, and also the graph obtained from K by removing $A_1, \ldots, A_k$, into complete k-partite graphs of size $\eta^{i+1} \log n$, we get k-partite graphs with color classes of size $\eta t^{(i)}$.

The key remark is that we inserted in $\mathcal{K}^{(i)}$ new $2\eta tk$ elements from $\mathcal{I}^{(i)}$ (namely the whole sets Q, R), and only removed $|A_1| = \eta tk$ of its vertices. Therefore the size of $\mathcal{I}^{(i)}$ decreases by ηtk.

We continue this procedure until the size of $\mathcal{I}^{(i)}$ is smaller than ηn, in which case we are done. This completes the proof of the Covering Lemma.

□

3.4.2 Constructing the cycle in the non extremal case

Let $t = \eta^{(1/\eta)} \log n$. First we shall find a $(k-1)$-cycle covering $\mathcal{C} \cup \mathcal{K}$, and then insert all vertices from $\mathcal{I}$, thus getting a $(k-1)^{th}$ power of a Hamiltonian cycle.

Connecting the vertices in $\mathcal{C} \cup \mathcal{K}$

We have covered the vertices of $\mathcal{C}$ by vertex-disjoint copies of $K_{k+1}(t)$, and the vertices of $\mathcal{K}$ by copies of $K_k(t)$. We call these blocks small multipartite graphs: some of them have $k+1$ classes; the others have k classes. Denote the graphs in $\mathcal{C}$ by $C_1, C_2, \ldots,$ and by $K_1, K_2, \ldots,$ the graphs in $\mathcal{K}$.

For a graph $C_j \in \mathcal{C}$, denote its color classes by $V_j^1, V_j^2, \ldots V_j^{k+1}$. In every color class V_j^i, consider an ordering of the vertices $v_{0,i}, v_{1,i}, v_{2,i}, \ldots, v_{t-1,i}$. Finally let us denote by the l^{th} column of C_j the sequence $v_{l,1}, v_{l,2}, v_{l,3}, \ldots, v_{l,k+1}$.

For a graph $K_j \in \mathcal{K}$, denote color classes by W_j^i, and vertices by $w_{l,i}$.

To build a $(k-1)$-path in each of the small multipartite graphs C_j, K_j, sequentially connect the vertices within a column, and then connect its last vertex, to the first vertex of the following column (see figure 3.1).

Using lemma 29 we connect with $(k-1)$-paths (of length at most $9(k+1)!$) the last vertex of the last collumn of a *small multipartite graph* with the first vertex of the first column of the succeeding one (see figure 3.2). We impose futher that such paths do not use vertices from the first or last columns of any of the small multipartite graphs (which can be done since $t = \Omega(\log n)$, and therefore the number of vertices in those columns is at most $O\big(\frac{n}{\log n}\big) = o(n)$).

Since the connecting paths might use a small number of vertices of the columns of the small multipartite graphs $\big($at most $9(k+1)!(k+1)\eta^{-\frac{1}{\eta}}\frac{n}{\log n}\big)$, we remove the vertices of those columns, and put them in I. This will not increase significantly the size of I.

Each time we remove a column from a small multipartite graph, we have to reconstruct the path inside the graph. We do this by connecting the column preceeding the deleted one, directly to the one following it.

Let $\mathcal{C}^*, \mathcal{K}^*$ denote the collection of multipartite graphs obtained from $\mathcal{C}$ and $\mathcal{K}$ respectively, after the removal of columns. Notice that $|\mathcal{C}^*| \geq |\mathcal{C}| - 9(k+1)!(k+1)\eta^{-\frac{1}{\eta}}\frac{n}{\log n}$, which is still much bigger than $\mathcal{I}$. Let us now insert the vertices of $\mathcal{I}$ in the cycle in such a way that it remains a $(k-1)$-path.

Adding the vertices of $\mathcal{I}$ to the cycle on $\mathcal{C} \cup \mathcal{K}$

Definition 8 (Rich points)**.** *A vertex $x \notin K_k(t)$ is* rich *for this $K_k(t)$, if it is joined to each class of $K_k(t)$ by at least $k\eta t$ edges. A vertex y is* rich *for $K_{k+1}(t)$ if it is joined to at least k of the classes by at least $k\eta t$ edges.*

We will first prove an easy consequence of the degree condition in G:

Lemma 33. *Every vertex $a \in \mathcal{I}$ is "rich" for at least η fraction of the cliques in $\mathcal{C} \cup \mathcal{K}$.*

Proof. For contradiction, assume that we are given a vertex $a \in \mathcal{I}$ that is not rich to at least an η fraction of the graphs in $\mathcal{C} \cup \mathcal{K}$. Then

$$\begin{aligned} deg_G(a) &< |\mathcal{I}| + \eta n + (|V(\mathcal{C})| - \eta n)\left(\frac{k-1}{k+1} + 2\eta\right) + |\mathcal{K}|\left(\frac{k-1}{k} + \eta\right) \\ &< \frac{k-1}{k} n \qquad (\text{since} |\mathcal{C}| \gg \eta n). \end{aligned}$$

A contradiction to the minimum degree condition. $\square$

We will have two cases.

Case 1: When a is rich to a $(k+1)$-partite graph $C_j = (V_j^1, \ldots, V_j^{k+1}) \in \mathcal{C}^*$, assume that a has at least $k\eta t$ neighbors in all the color classes of C_j except V_j^y. Without loss of generality, there is a vertex $v_{x,y}$ in the path in C_j, such that a is connected to all vertices in the path at distance at most $(k-1)$ from $v_{x,y}$ (Otherwise, we can reorder the vertices inside each color class of C_j other than those in the first or last column, and reconstruct the path inside $|C_j|$ as in the first step. This does not change the connecting paths, nor the vertices of the small multipartite graphs they use, nor the sets $\mathcal{I}, \mathcal{C}^*, \mathcal{K}^*$).

Let us now replace the vertex $v_{x,y}$ by a in the path inside C_j. Notice that the vertices $v_{x,y}, v_{x+1,y-1}, v_{x+2,y-2}, \ldots, v_{x+k,y-k}$ (where indices are taken $\pmod{k+1}$) form a clique, because they belong to different color classes in C_j. We can remove each one of them from the path, connecting the vertex preceding it directly to the one succeeding it in the original path. Finally, we insert the path formed by these $k+1$ vertices between any two the columns of the graph, other than the one from which a was removed.

Notice that the path obtained is still a $k-1$ path, since each vertex was connected to the closest k-neighbors in the original one.

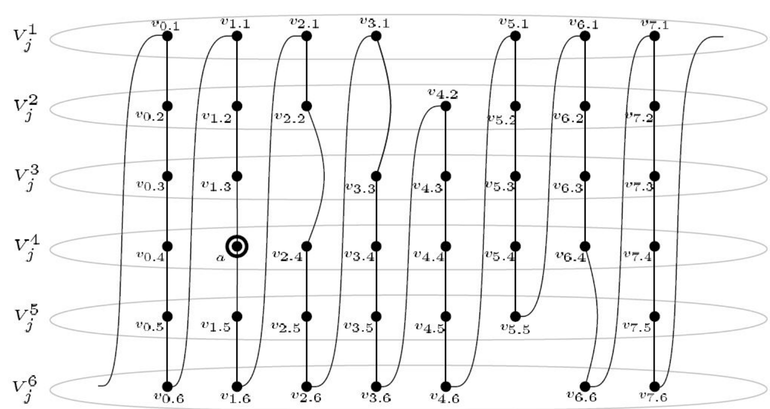


Figure 3.8: Inserting a into the $(k-1)$-path being constructed in the complete balanced $(k+1)$-partite graph C_j, where $k = 5$ and $b = 4$.

We will however call the columns $x+1, \ldots, x+k$ *contaminated*, as well as the $2k$ ones closer to each one of these after repeating the process and possibly reordering the vertices in C_j, because one no longer can use them to insert a new element, and still guarantee a $(k-1)$-path. The process of inserting a into our $(k-1)$-path is depicted in Figure 3.8, for $k = 5$ and $b = 4$.

When inserting a new vertex rich to the graph C_j, at most $(k+1)$ columns of the block got *contaminated* by the previous vertex inserted. So we shall have enough space to insert all the vertices from $\mathcal{I}$ that are "rich" to graphs in $\mathcal{C}^*$.

Case 2: When $a \in \mathcal{I}$ is not "rich" for any $K_{k+1}(t) \in \mathcal{C}^*$ $\Big($and so $\deg_{\mathcal{K}^*}(a) \geq \Big(\frac{k-1}{k} + \frac{\eta^{\frac{1}{2}}}{4k}\Big)|\mathcal{K}^*|$ since $|\mathcal{C}^*| \geq n^{\frac{1}{2}}$ $\Big)$ it follows from an easy computation that a is rich for at least a $\frac{\eta^{\frac{1}{2}}}{8k}$ fraction of the graphs $K \in \mathcal{K}^*$.

As a remark to the reader, this case is the very reason for considering $(k+1)$-partite graphs in the original covering, and for requiring their union to have size much bigger than $|\mathcal{I}|$.

Considering now a k-partite graph $K = (W^1, \ldots, W^{k+1}) \in \mathcal{K}^*$ for which a is rich, we can can assume without loss of generality that a is connected to all vertices in three consecutive columns of K (otherwise we just reorder the vertices inside each color class of K other than those in the first or last column, and reconstruct the path inside $|K|$ as we did initially).

We insert the vertex a in the middle column of those three, between any two consecutive vertices. Again we refer to the three columns as being *contaminated*, for the fact that we do not use them again when inserting in K a new vertex rich to the graph. It is clear that the paths thus obtained in the k-partite graphs after insertion of vertices in non contaminated columns, remain $(k-1)$-paths.

We can repeat this until all the vertices from $\mathcal{I}$ are used up (since $|\mathcal{K}^*| \gg |\mathcal{I}|$). This completes the case of the α-non-extremal graph.

3.5 The α-Extremal Case

Take the maximum number of disjoint α-extremal sets $A_1, A_2, \ldots, A_\ell$, with $|A_i| = \frac{n}{k}$ $(1 \leq i \leq \ell)$.

We let $B = V(G) \setminus (A_1 \cup \cdots \cup A_\ell)$ for $\ell \leq k$.

Furthermore, we say that $v \in A_i$ is *bad* if we have

$$\deg_{A_i}(v) \geq \alpha^{1/3}|A_i|. \tag{3.5}$$

By the fact that $d(A_i) < \alpha$, there are at most $\alpha^{2/3}|A_i|$ *bad* vertices in any A_i. A vertex $v \in A_i$ (or B) is *exceptional for* A_j (for $j \neq i$) if $deg_{A_j}(v) < \alpha^{1/3}|A_j|$. For each $v \in A_i$ (or B), there can be at most one $j \neq i$, such that v is *exceptional* for A_j. We denote the set of vertices in A_i (or B) that are *exceptional for* A_j by $E_i(j)$ (or $E_B(j)$). The following remarks are easy to deduce.

Remark 34. *A vertex can be in $E_i(j)$ for at most one $j \neq i$.*

Remark 35. *If a vertex v is in $E_i(j)$ for some j then it is* bad *- indeed $deg_{A_i}(v) > (1 - \frac{\alpha^{1/3}}{2})|A_i|$.*

Remark 36. *Switching a* bad *vertex in A_i with a vertex in $E_j(i)$ reduces the number of* exceptional *vertices. Hence we may assume that either there are no* bad *vertices in A_i or $E_j(i)$ is empty for every $j \neq i$.*

3.5.1 Finding the cycle in the extremal case

To convey the basic idea of the proof we deal separately with the cases when $\ell = k$ and when $\ell < k$.

G has k extremal sets

In this case the vertex set V can be partitioned into $A_1, A_2, \ldots, A_k$ such that $|A_i| = \lfloor \frac{n}{k} \rfloor$ and $d(A_i) < \alpha$ for $1 \leq i \leq k$, that is, $\ell = k$ (and hence $B = \phi$). We will further subdivide this case into two subcases.

The Clean Case: There are no *bad* or *exceptional* vertices in any A_i, (hence $E_i(j)$ is empty for all i, j by Remark 35). We will cover $A_1 \cup \cdots \cup A_k$ with k-cliques such that every clique uses a vertex from each A_i. Since in this case there are no *bad* vertices, by the minimum degree condition for each $v \in A_i$, we have $deg_{A_j}(v) \geq (1-\alpha^{1/3})|A_j|$ for all $j \neq i$. Furthermore, it is relatively straightforward to find k-cliques by a simple greedy procedure that uses the König-Hall theorem as follows. We first find a perfect matching M_1 between A_1 and A_2. Then we find a perfect matching between M_1 and A_3, such that $e = \{x, y\} \in M_1$ is matched with a vertex $z \in N(x, y) \cap A_3$. We can continue this process to find the desired k-cliques. Indeed, let M_{k-2} be the $(k-1)$-cliques made so far, from $A_1, A_2, \ldots, A_{k-1}$. For any clique $(x_1, x_2, \ldots, x_{k-1})$, $x_i \in A_i$ we have that $|N(x_1, x_2, \ldots x_{k-1}) \cap A_k| \geq (1-\alpha^{1/4})|A_k|$, and also for $y \in A_k$, y is connected to at least $(1-\alpha^{1/10})\frac{n}{k}$ $(k-1)$-cliques of M_{k-2}. Therefore, by König-Hall theorem there exists a perfect matching between the $(k-1)$-cliques and vertices in A_k, so we can extend these $(k-1)$-cliques to k-cliques. Call this clique cover $C_k = \{c_1, c_2, \ldots, c_{\lfloor \frac{n}{k} \rfloor}\}$.

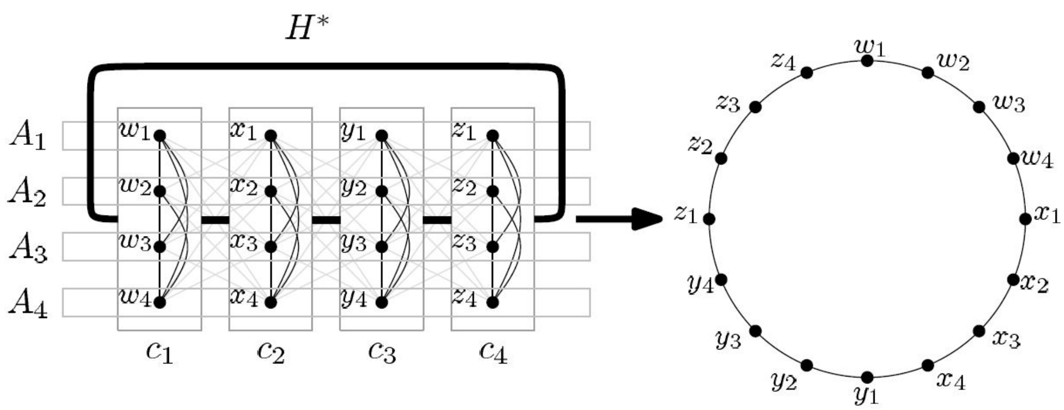


Figure 3.9: Unfolding the cliques in the order defined by H^* gives us the required power of a Hamiltonian cycle.

Let $c_1 = (x_1, x_2, \ldots, x_k)$ and $c_2 = (y_1, y_2, \ldots, y_k)$ be any two such k-cliques in C_k (note that $x_i, y_i \in A_i$). We say that c_1 *precedes* c_2 if x_i is connected to $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_k$, for $1 \leq i \leq k$. c_1 *precedes* c_2 basically means that $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$ is a $(k-1)$-path. We say that $\{c_1, c_2\}$ is a *good* pair, if c_1 *precedes* c_2 and c_2 *precedes* c_1. By the degree conditions above, any $c_i \in C_k$ makes a *good* pair with at least $(1-\alpha^{1/5})|C_k|$ other cliques in C_k.

We define an auxiliary graph G^* in the following way: the vertex set of the graph G^* is $C_k = \{c_1, c_2, \ldots, c_{\lfloor \frac{n}{k} \rfloor}\}$ and $\{c_i, c_j\}$ is an edge in G^* if and only if $\{c_i, c_j\}$ is a *good* pair. By the above observation $\delta(G^*) > |C_k|/2$, so there exists a Hamiltonian cycle H^* in G^*. If we take the cliques in the order of H^* and unfold individual cliques in the natural order defined by $A_1, A_2, \ldots, A_k$, it is easy to see that this gives us the $(k-1)^{th}$ power of a Hamiltonian cycle in G.

Handling the *Exceptional* Vertices: In this case we have some $E_j(i)$'s that are non-empty. The main idea is to reduce this case to the clean case where there are no *exceptional* vertices. Handling of the bad vertices can be reduced to the handling of the exceptional vertices. So we shall discuss only the handling of the exceptional vertices.

Define X_i to be the set of all the vertices that are *exceptional* for A_i, that is, $X_i = \bigcup_{\substack{j=1 \\ j\neq i}}^{k} E_j(i)$.

Case 1: If $|X_i| > 1$, we would want to find paths of length 2 with endpoints in A_i and centers at *exceptional* vertices in $E_j(i)$ for some j. For this purpose we note that $\delta(G|_{A_i \cup X_i}) \geq |X_i|$ by the minimum degree condition. Furthermore, since we assume there are no bad vertices, it follows that $\Delta(G|_{A_i \cup X_i}) \leq \alpha^{1/3}|A_i| + |X_i|$. Thus by Lemma 28 we can find more than $|X_i|$ vertex disjoint paths of length two. However, not all such paths may have their endpoints in A_i or their centers in X_i. This can easily be handled by noting that any vertex in X_i may be switched with any of the vertices in A_i and the exchanged vertices become *exceptional* or not *bad* in their respective new sets. Therefore we may assume that there is a set, P_i, of $|X_i|$ disjoint paths of length 2, such that the two endpoints of each path are vertices in A_i and the center is an *exceptional* vertex in some $E_j(i)$.

We embed each of these paths in a distinct unit of three k-cliques as follows: let $(u_i, c_j, \bar{u}_i) \in P_i$ be one of the paths such that $u_i, \bar{u}_i \in A_i$, and $c_j \in A_j$. Select a clique in the natural order $S = (s_1, s_2, \ldots, s_k)$ such that $s_i = u_i$ and $s_j = c_j$, (so we use the $\{u_i, c_j\}$ edge). Now we select another clique $T = (t_1, t_2, \ldots, t_k)$ such that $t_i = \bar{u}_i$ and S *precedes* T. Then we select a clique $R = (r_1, r_2, \ldots, r_k)$ which *precedes* S.

It is easy to see that there are many cliques with the given restrictions such that only s_i is the *bad* vertex among all the three cliques. The cliques, unfolded in the order R, S, T, make a $(k-1)$-path. We think of this set of three k-cliques as a single k-clique (which we call an *exceptional clique*) with one vertex each from $A_1, \ldots, A_k$. The new

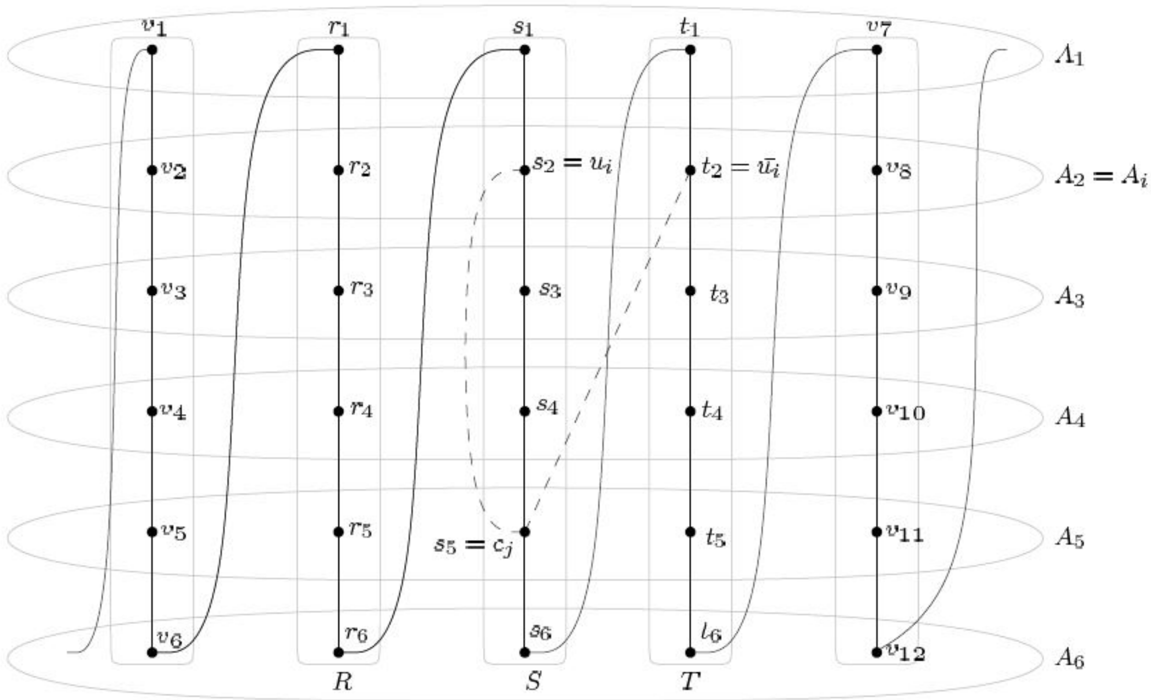


Figure 3.10: Finding the *exceptional clique* when $|X_i| > 1$.

vertex of the *exceptional clique* in A_m is connected to all the common neighbors of r_m and t_m. Since r_m and t_m are not *bad* vertices for $1 \leq m \leq k$, these new vertices have high degree in all the sets A_i where $i \neq m$. We deal with all the *exceptional* vertices in this manner and get *exceptional cliques* for each of them. In the remaining graph, we use the procedure described in the previous Section to find a cover consisting of k-cliques and add the *exceptional cliques* to the cover. Then, as before, we find a Hamiltonian cycle of the cliques in the cover and unfold the vertices in the cliques in the order defined by the cycle to get $(k-1)^{th}$ power of a Hamiltonian cycle. In Figure 3.10 the relevant portion in the final $(k-1)^{th}$ power of a Hamiltonian cycle looks as follows: $(\ldots, v_5, v_6, r_1, r_2, r_3, r_4, r_5, r_6, s_1, s_2, s_3, s_4, s_5, s_6, t_1, t_2, t_3, t_4, t_5, t_6, v_7, v_8, \ldots)$

Case 2:

When $|X_i| = 1$, we may not be able to find the length 2 path as above. The only ways this can happen is when the *exceptional* vertex $c_j \in E_j(i)$ for some j has exactly one neighbor $y \in A_i$ (it has to have at least one neighbor), and all the vertices in A_i (except y) have exactly one neighbor inside A_i. Therefore we find a path $p_i = (u_i, c_j, u_j)$ of length 2, where $u_i \in A_i$ and $c_j, u_j \in A_j$ such that c_j is an *exceptional* vertex for A_i.

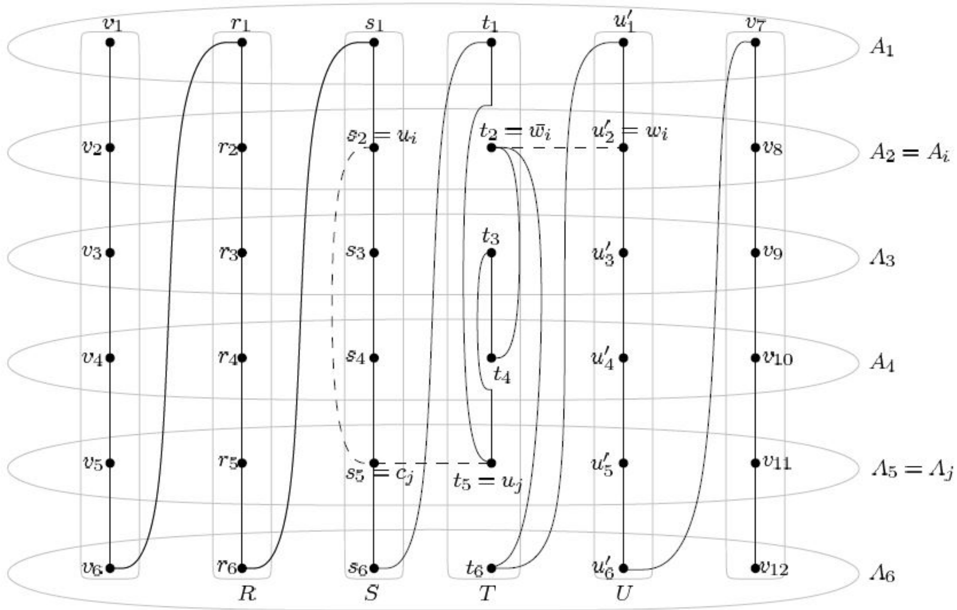


Figure 3.11: Finding the *exceptional clique* when $|X_i| = 1$.

In addition, we select an edge $\{w_i, \bar{w}_i\}$ inside A_i disjoint from all the paths of length 2 that we may have already chosen.

Select a clique in the natural order $S = (s_1, s_2, \dots, s_k)$ such that $s_i = u_i$ and $s_j = c_j$ so that we use the $\{u_i, c_j\}$ edge. Now select another clique $T = (t_1, t_2, \dots, t_k)$ such that $t_i = w_i$ and $t_j = u_j$. However, we are going to consider T in the following order:

$$T' = (t_1, t_2, \dots, t_{i-1}, t_j = u_j, t_{i+1}, \dots, t_{j-1}, t_i = w_i, t_{j+1}, \dots t_k)$$

(i.e. this order switches the positions of t_i and t_j). Note that S *precedes* T', since $\{c_j, u_j\}$ is an edge in our graph.

Next we find a clique U such that T' *precedes* U. Such a clique exists, because $\{w_i, \bar{w}_i\}$ is an edge in our graph. There are many cliques $U = (u'_1, \dots, u'_k)$, and $u'_i = \bar{w}_i$, such that T' *precedes* U. Then we find another clique R which *precedes* S. We think of this set of four k-cliques as a single k-clique (the *exceptional clique*) with one vertex for each of $A_1, \dots, A_k$. As previously, the new vertex of the *exceptional clique* in A_m is connected to all the common neighbors of r_m and u'_m. Since r_m and u'_m are not *bad* vertices for $1 \leq m \leq k$, these new vertices have high degree in all the sets A_i where $i \neq m$. We deal with all the *exceptional* vertices in this manner and get *exceptional cliques* for each of them. We get a $(k-1)^{th}$ power of a Hamiltonian cycle using the same method as that was used in the previous cases. In Figure 3.11 the relevant portion in the final $(k-1)^{th}$ power of a Hamiltonian cycle looks as follows: $(\ldots, v_6, r_1, r_2, r_3, r_4, r_5, r_6, s_1, s_2, s_3, s_4, s_5, s_6, t_1, t_5, t_3, t_4, t_2, t_6, u_1', u_2', u_3', u_4', u_5', u_6', v_7, \ldots)$

G has less than k extremal sets

We first assume that $A_1, A_2, \ldots, A_\ell$ are the extremal sets, where $\ell < k$ and we let $A = \bigcup_{i=1}^{\ell} A_i$ and $B = V(G) \setminus A$. (We remark that $\ell \leq k-2$, otherwise if we omit connecting in the non extremal set, two vertices of degree bigger than $\frac{n}{k}+1$, the resulting graph is an extremal graph and satisfies the minimum degree condition).

We say that $v \in B$ is *exceptional* if $deg_A(v) \leq (\ell - 1 + \alpha^{1/3})|A|$. The *bad* vertices in A_i's are defined exactly as before.

Then

$$\delta(G|_B) \geq \left(\frac{k-1}{k}\right) n - \left(\frac{\ell}{k}\right) n \geq \left(\frac{k-\ell-1}{k}\right) n \geq \left(\frac{k-\ell-1}{k-\ell}\right) |B|$$

Also, since there is no extremal set $A' \subset B$ with $|A'| = \lfloor \frac{n}{k} \rfloor = \lfloor (\frac{1}{k-\ell})|B| \rfloor$, we have that $G|_B$ does not satisfy the α-extremal condition.

By the non-extremality of $G|_B$ and its minimum degree $\delta(G|_B)$, using the procedure given in previous Section on the non-extremal case, we can find $(k-\ell-1)^{th}$ power of a Hamiltonian cycle $H = (p_1, p_2, \ldots, p_{|B|})$ in B. We will insert ℓ vertices after every $k-\ell$ vertices in H such that we get $(k-1)^{th}$ power of a Hamiltonian cycle.

For this purpose we divide H into $\lfloor \frac{n}{k} \rfloor$ consecutive intervals of $k-\ell$ vertices each. We define $B' = \{b_1, b_2, \ldots, b_{\lfloor \frac{n}{k} \rfloor}\}$ in the following way: b_1 corresponds to $\{p_1, p_2, \ldots, p_{(k-\ell)}\}$; b_2 corresponds to $\{p_{(k-\ell)+1}, p_{(k-\ell)+2}, \ldots, p_{2(k-\ell)}\}$, etc., and $b_{\lfloor \frac{n}{k} \rfloor}$ corresponds to the path $\{p_{(\lfloor \frac{n}{k} \rfloor - 1)(k-\ell)+1}, \ldots, p_{|B|}\}$. We also have that $|A_1| = |A_2| = \cdots = |A_\ell| = |B'| = \lfloor \frac{n}{k} \rfloor$.

The Clean Case: Assume there are no *bad* or *exceptional* vertices

As before, we construct cliques of size ℓ. Let the set of these cliques be $\mathcal{C} = \{c_1, c_2, \ldots, c_{n/k}\}$, where $c_i = \{y_1, y_2, \ldots, y_\ell : y_i \in A_i, \text{for } 1 \leq i \leq \ell\}$.

Remarks.

(R1) $N_A(p_{(i-1)(k-\ell)+1}, \ldots, p_{i(k-\ell)}) \geq (1 - k\eta^{1/4})|A|$, where $|A| = \ell\frac{n}{k}$

(R2) For every $y \in A$, $N_B(y) \geq (1 - k\eta^{1/4})|B|)$, where $|B| = (k-\ell)\frac{n}{k}$.

(R3) For every $c_s \in \mathcal{C}$, the number of good pairs is at least $(1-k\eta^{1/4})|\mathcal{C}|$, where $|\mathcal{C}| = \frac{n}{k}$.

After these remarks, we start to build our $(k-1)^{th}$-Hamiltonian cycle. Consider $b_1, b_2, b_3, \ldots, b_{n/k}$, which forms in this order a $(k-1)^{th}$-Hamiltonian path. By (R3), we can easily find $\frac{1}{2}\frac{n}{k}$ c_s so that all the points of c_s are connected to all the points of $b_{2s-1} \cup b_{2s}$. So we must have the following subpath:

$$p_{(2s-1)(k-\ell)+1} \ldots, p_{2s(k-\ell)}, y_1^{(s)} \ldots, y_\ell^{(s)} p_{2s(k-\ell)+1} \ldots, p_{(2s+1)(k-\ell)}.$$

Because of Remarks (R1-R3), using the Kőnig-Hall Theorem, we can order the remaining c_s so that in this ordering say $c_{(n/2k)+1}, c_{(n/2k)+2}, \ldots, c_{(n/k)}$ have the property that $c_{(n/2k)+s}$ is good to c_s and c_{s+1}, and every point of $c_{(n/2k)+s}$ is connected to every point of b_{2s} and b_{2s+1}.

The subpath looks like this.

$$p_{(2s-1)(k-\ell)+1}, \ldots, p_{2s(k-\ell)} y_1^{(s)} \ldots y_\ell^{(s)}, p_{2s(k-\ell)+1}, \ldots, p_{(2s+1)(k-\ell)} \cdots,$$
$$y_1^{(\frac{n}{2k}+s)} \ldots y_\ell^{(\frac{n}{2k}+s)} p_{(2s+1)(k-\ell)+1} \ldots, p_{(2s+2)(k-\ell)}.$$

We got our desired $(k-1)^{th}$-Hamiltonian cycle.

Handling the *Exceptional* Vertices: Let us denote by B^+ the exceptional vertices in B. Notice that B^+ is connected to at least $(k - \ell - \alpha^{\frac{1}{3}}k)\frac{n}{k}$ vertices of $B \setminus B^+$. Therefore we can insert them in the $(k-\ell-1)^{th}$-Hamiltonian cycle [1] covering $B \setminus B^+$, which we have already constructed.

We will insert each vertex $v \in B^+$ into a carefully chosen cycle b_s, so that v is also connected to every point of $b_{s-10}, \ldots, b_s, b_{s+1}, \ldots, b_{s+10}$. Since the degree $d_B(v)$ of every $v \in B^+$ is extremely large, we can do this easily. Notice that our new cycle is still a $(k-\ell-1)^{th}$ Hamiltonian cycle.

[1] When we say Hamiltonian path in a smaller set, we mean it completely covers the set.

Again, we break our Hamiltonian cycle into paths of length $(k - \ell)$. As in the clean case, we denote these paths by $b_1, b_2, \ldots, b_{n/k}$. Since $N_B(v)$ is extremely large for $v \in B^+$, we can insert these vertices so that they are always the initial points of some b_i's.

Now we build above $v \in B^+$ an exceptional clique as we did for the case where $\ell = k$, and we had exceptional vertices. Of course, the points of b_i are also included in these exceptional cliques. So the length of this clique is k. From here the procedure is the same as when B^+ was empty. $\square$

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