

**SCALAR FIELDS AND SPIN-HALF FIELDS ON
MILDLY SINGULAR SPACETIMES**

by

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ABSTRACT OF THE DISSERTATION

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A charged particle-spacetime is a solution to Einstein's equations coupled to a nonlinear electromagnetic theory. It has a mild curvature singularity and a bounded electric potential. Morawetz and Strichartz estimates are proved for spherically symmetric scalar waves on such a spacetime. These spacetimes have conical singularities at their centers. As a first step towards understanding the behavior of scalar waves on such spacetimes, a way to reproduce the known fundamental solution to the scalar wave equation on flat two-dimensional cones is found using Sommerfeld's method. Dirac's equation for a spin-half field is set up on a charged particle-spacetime. The Dirac Hamiltonian is shown to be essentially self-adjoint on smooth functions with compact support away from the center. The essential spectrum and the continuous spectrum of the Hamiltonian are obtained. Under a certain condition, a neighbourhood of zero is shown to be in the resolvent. The existence of infinitely many eigenvalues is shown.

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Dedication

This work is dedicated to my parents Sathy and Balasubramanian.

Table of Contents

Abstract	ii
Acknowledgements	iii
Dedication	iv
List of Figures	vi
1. Introduction to charged particle-spacetimes	1
2. Massless, spinless, real scalar field	10
2.1. Estimates for spherically symmetric waves	12
3. Conical singularities and Sommerfeld's method	18
3.1. Fundamental solution	20
3.2. Solution on a cone over a circle of circumference $2\pi n, n \in \mathbb{N}$	23
3.3. Solution on a cone over a circle of circumference $2\pi\rho, \rho \notin \mathbb{N}$	39
4. Spin-$\frac{1}{2}$ fields on charged particle-spacetimes	42
4.1. Dirac's equation using Cartan's formalism	44
4.2. Separation of the Dirac Hamiltonian	51
4.3. Essential self-adjointness of the Dirac Hamiltonian	55
4.4. Essential spectrum	59
4.5. Spectral gap around 0	63
4.6. Eigenvalues and continous spectrum	65
5. Summary and outlook	68
References	69

List of Figures

3.1. Motivation for Sommerfeld's method with $n = 1$	26
3.2. Motivation for Sommerfeld's method with $n = 2$	27
3.3. The case $\mu > 1$	30
3.4. The case $-1 < \mu < 1$, sub-case 1	32
3.5. The case $-1 < \mu < 1$, sub-case 2	34
3.6. The case $\mu < -1$, sub-case 1	35
3.7. The case $\mu < -1$, sub-case 2	38

Chapter 1

Introduction to charged particle-spacetimes

The main equations of general relativity are Einstein's equations. The unknowns in Einstein's equations are a four-dimensional manifold \mathcal{M} , a Lorentzian metric \mathbf{g} on \mathcal{M} , and in case of non-vacuum spacetimes, a collection of matter fields giving rise to an energy-momentum tensor \mathbf{T} . In Gaussian CGS units, the equations read

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \quad (1.1)$$

Here, G stands for Newton's universal constant of gravitation, c the speed of light in vacuum, and $R_{\mu\nu}, R$ are respectively the Ricci and scalar curvatures of the metric.

In this work, we will consider electromagnetic spacetimes that can represent a point charge. Our principal reference for this is a recent paper of Tahvildar-Zadeh [30]. The study of electromagnetic phenomena involves two 2-forms: the Maxwell tensor \mathbf{M} and the Faraday tensor \mathbf{F} . These are assumed to be source-free except for the point charge itself. That is,

$$d\mathbf{F} = 0, \quad d\mathbf{M} = 0 \quad (1.2)$$

away from the location of the point charge. A key assumption is that the electromagnetic theory comes from a Lagrangian. That is, there is an action

$$S[\mathbf{A}, D] = \int_D \mathbf{L}_{em}(\mathbf{A}, d\mathbf{A}), \quad (1.3)$$

where \mathbf{L}_{em} is a four-form, that we call the Lagrangian, D is an open domain in \mathcal{M} , and \mathbf{A} is a 1-form that is usually called the electromagnetic four potential. Equations of electromagnetism arise as the Euler-Lagrange equations of this action. Suppose \mathbf{A} is a critical point, then the Faraday tensor is obtained by $\mathbf{F} = d\mathbf{A}$, and the Maxwell tensor is obtained by

$$\mathbf{M} = \left. \frac{\partial}{\partial \mathbf{f}} \right|_{\mathbf{a}=\mathbf{A}, \mathbf{f}=\mathbf{F}} \mathbf{L}(\mathbf{a}, \mathbf{f}). \quad (1.4)$$

Lagrangian density $l(\mathbf{a}, \mathbf{f})$ is defined by the relation $\mathbf{L}(\mathbf{a}, \mathbf{f}) = l(\mathbf{a}, \mathbf{f})\mathbf{v}_{\mathbf{g}}$, where $\mathbf{v}_{\mathbf{g}}$ is the volume form of the metric \mathbf{g} . Under assumptions of source-freeness, Lorentz-invariance and Gauge-invariance, it can be shown ([11]) that l depends only on $\mathbf{f} = d\mathbf{a}$, and then only through the invariants $x = \frac{1}{4}\mathbf{f}_{\mu\nu}\mathbf{f}^{\mu\nu}$ and $y = \frac{1}{4}\mathbf{f}_{\mu\nu}(*\mathbf{f})^{\mu\nu}$, where $*$ is the Hodge star operator. Therefore $l = l(x(\mathbf{f}), y(\mathbf{f}))$. The energy(density)-momentum(density)-stress tensor \mathbf{T} is expressed in terms of the Lagrangian density l , as

$$T_{\mu\nu} = 2\frac{\partial l}{\partial g^{\mu\nu}} - g_{\mu\nu}l. \quad (1.5)$$

The tensor \mathbf{T} is assumed to satisfy the *Dominant Energy Condition*, that is,

1. for every future-directed timelike vector field \mathbf{Y} ,

$$T_{\mu\nu}Y^\mu Y^\nu \geq 0, \quad (1.6)$$

2. whenever \mathbf{Z} is a future-directed causal vector, the vector $-T_{\nu}^{\mu}Z^{\nu}$ is future-directed causal.

The four one-forms of electromagnetism, \mathfrak{E} , \mathfrak{D} , \mathfrak{H} and \mathfrak{B} are derived from Maxwell and Faraday tensors by contracting with \mathbf{K} , a time-like, hypersurface-orthogonal Killing vector field. With $i_{\mathbf{K}}F$ denoting the interior product $(i_{\mathbf{K}}F)_{\nu} = K^{\mu}F_{\mu\nu}$,

$$\mathfrak{E} := i_{\mathbf{K}}\mathbf{F} \quad (1.7)$$

$$\mathfrak{B} := i_{\mathbf{K}}*\mathbf{F} \quad (1.8)$$

$$\mathfrak{D} := i_{\mathbf{K}}\mathbf{M} \quad (1.9)$$

$$\mathfrak{H} := -i_{\mathbf{K}}*\mathbf{M}. \quad (1.10)$$

The one-forms \mathfrak{E} , \mathfrak{D} , \mathfrak{H} and \mathfrak{B} are called (flattened) electric field, electric displacement, magnetic field, and magnetic induction, respectively.

In [30], the system of equations (1.1),(1.2) (which is called the Einstein-Maxwell system) is considered, with \mathbf{T} as in (1.5). The main assumptions made are

1. the spacetime is static, that is, there exists a twist-free time-like Killing vector field for the metric;

2. there is an electric field present, but no magnetic field;
3. the spacetime is spherically symmetric;
4. the spacetime is asymptotically Minkowski;
5. the energy corresponding to the ADM mass of the spacetime is equal to the energy carried by the electric field. That is, the mass of the spacetime is entirely of electromagnetic origin.

The resulting solution is a spacetime with an electric potential defined on it. It is specified by three separate entities:

1. a C^2 function called *reduced Hamiltonian* $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$,
2. a parameter $M > 0$ with units of mass,
3. a parameter $Q \neq 0$ with units of charge.

The function ζ specifies the electromagnetic theory. The traditional Maxwell's equations result from taking $\zeta(\mu) = \mu$, and in this case, the resulting system is the called Einstein-Maxwell-Maxwell system of equations. The function ζ is related to the Lagrangian density function l as follows. Let $f(t) := -l(x = -t^2/2, y = 0)$. Then,

$$\zeta(\mu) := f_*(\sqrt{\mu}). \quad (1.11)$$

where f_* stands for the Legendre-Fenchel transform of f :

$$f_*(s) := \sup_t (st - f(t)). \quad (1.12)$$

The function ζ has to satisfy the following conditions:

(a)

$$\zeta(\mu) = \mu + O(\mu^{5/4}) \text{ as } \mu \rightarrow 0. \quad (1.13)$$

This is assumed so that in the weak field limit ζ agrees with that of Maxwell-Maxwell electromagnetics, that is, with $\zeta_0(\mu) := \mu$.

(b)

$$\forall \mu > 0 : \zeta'(\mu) \geq 0, \zeta - \mu\zeta' \geq 0. \quad (1.14)$$

This assumption is made so that the dominant energy condition is satisfied.

(c)

$$\forall \mu > 0 : \zeta'(\mu) + 2\mu\zeta''(\mu) \geq 0. \quad (1.15)$$

This convexity condition is to ensure that ζ can be derived from a Lagrangian.

(d)

$$I_\zeta := 2^{-\frac{11}{4}} \int_0^\infty \mu^{-7/4} \zeta(\mu) d\mu < \infty. \quad (1.16)$$

This assumption ensures that the ADM mass of the spacetime is finite. (Note that this rules out the Maxwell-Maxwell reduced Hamiltonian $\zeta_0(\mu) = \mu$).

(e) There exists positive constants $\mu_0, J_\zeta, K_\zeta, L_\zeta$ such that

$$\forall \mu > \mu_0 : J_\zeta \sqrt{\mu} - K_\zeta \leq \zeta(\mu) \leq J_\zeta \sqrt{\mu}, \quad (1.17)$$

and

$$\forall \mu > \mu_0 : \frac{J_\zeta}{2\mu^{1/2}} - \frac{L_\zeta}{\mu} \leq \zeta'(\mu) \leq \frac{J_\zeta}{2\mu^{1/2}}. \quad (1.18)$$

These conditions are imposed so as to get the mildest possible singularity at the location of the point charge, namely a *conical singularity*.

An important example of a ζ that satisfies all of these conditions is the Born-Infeld Hamiltonian,

$$\zeta_{BI}(\mu) := \sqrt{1 + 2\mu} - 1. \quad (1.19)$$

We now introduce some notation. Using the ζ -dependent constants defined earlier, we define,

$$A(\zeta) := \frac{\sqrt{2}J_\zeta}{I_\zeta^2}, \quad (1.20)$$

and the dimensionless quantity

$$\epsilon := \frac{\sqrt{GM}}{|Q|}, \quad (1.21)$$

Note (Units). We will work in Gaussian CGS units.

Indeed, since the electrostatic force between two charged particles is computed by Coulomb's law in Gaussian CGS units as $F_{em} = \frac{Q_1 Q_2}{r^2}$ while the gravitational force between two particles of masses M_1, M_2 is computed by Newton's law of gravitation as $F_{gr} = G \frac{M_1 M_2}{r^2}$, both $\frac{Q^2}{r^2}$ and $G \frac{M^2}{r^2}$ have the same dimensions. So their ratio $\frac{GM^2}{Q^2}$ is dimensionless.

Given, M, Q, ζ , it is necessary to scale ζ to ensure that the energy of the electric field is equal to that of the ADM mass. We will use the scaled version

$$\zeta_\beta := \frac{1}{\beta^4} \zeta(\beta^4 \mu), \quad (1.22)$$

where

$$\beta := \frac{(|Q|)^{3/2} I_\zeta}{Mc^2}. \quad (1.23)$$

The spacetime \mathcal{M} is diffeomorphic to $\mathbb{R} \times (\mathbb{R}^3 \setminus \mathbf{0})$, and is equipped with the usual coordinates $(t, r, \theta, \phi) \in (-\infty, \infty) \times (0, \infty) \times (0, \pi) \times (0, 2\pi)$. Here, t is a time function, chosen so that the hypersurface orthogonal Killing field is $K = \frac{\partial}{\partial t}$, r is the area radius coordinate, that is, for $p \in \mathcal{M}$, $r(p) = \sqrt{\frac{\text{Ar}(p)}{4\pi}}$, where $\text{Ar}(p)$ is equal to the area of the orbit of the rotation group $SO(3)$ that passes through p , and (θ, ϕ) are the standard spherical coordinates on the orbit sphere. The line element of the metric \mathbf{g} can be expressed in these coordinates as

$$ds_{\mathbf{g}}^2 = -e^{\xi(r)} c^2 dt^2 + e^{-\xi(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (1.24)$$

where

$$e^{\xi(r)} := 1 - 2 \frac{m(r)}{r}, \quad (1.25)$$

and the ‘‘mass function’’ is defined by

$$m(r) := \frac{G}{c^4} \int_0^r \zeta_\beta \left(\frac{Q^2}{2s^4} \right) s^2 ds. \quad (1.26)$$

The spacetime is also endowed with an electric potential

$$\varphi(r) = Q \int_r^\infty \zeta'_\beta \left(\frac{Q^2}{2s^4} \right) \frac{ds}{s^2}. \quad (1.27)$$

This gives the electric field as $\mathfrak{E} = e^{-\xi(r)/2} d\varphi$.

Definition 1.0.1. For a given mass M , charge Q and a reduced Hamiltonian ζ satisfying the aforementioned properties, and the smallness condition

$$\epsilon^2 < \frac{1}{A(\zeta)}, \quad (1.28)$$

the manifold $\mathbb{R} \times (\mathbb{R}^3 \setminus \mathbf{0})$ equipped with the metric (1.24), and the electric potential (1.27) is called a charged *particle-spacetime*. We denote such a spacetime by $\mathcal{M}^{\zeta, M, Q}$ and in short by \mathcal{M} .

The smallness condition is to ensure that the metric coefficient $e^{\xi(r)} > 0 \forall r > 0$. The maximal analytical extension of this metric is indeed \mathbb{R}^4 minus the line $r = 0$, because the metric defined by (1.24) is *always* singular at $r = 0$, for any choice of ζ . The singularity is *conical*, meaning that $\lim_{r \rightarrow 0} \xi(r)$ exists but is not equal to zero.

One should contrast the above charged particle-spacetime with the well-known solution in case $\zeta(\mu) = \mu$ (Maxwell's Hamiltonian), the Reissner-Nordström spacetime.

Definition 1.0.2. The Reissner-Nordström (RN) spacetime is the maximal analytic extension of

$$ds^2 = -f(r)c^2 dt^2 + f(r)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \quad (1.29)$$

where $f(r) = (1 - 2\frac{GM}{c^2}\frac{1}{r} + \frac{G}{c^4}\frac{Q^2}{r^2})$. There is also an electric potential, which is given by

$$\varphi(r) = \frac{Q}{r}. \quad (1.30)$$

If $\epsilon = \frac{\sqrt{GM}}{|Q|} < 1$, then $f(r) > 0 \forall r \in (0, \infty)$, and in this case, the spacetime is called a super-extremal Reissner-Nordström.

The singularity at $r = 0$ in a super-extremal RN is a naked singularity.

We now list the asymptotics near $r = 0$ and $r = \infty$ of the mass function $m(r)$, metric coefficient $e^{\xi(r)}$, and the potential function $\varphi(r)$. Near $r = 0$,

$$\begin{aligned} m(r) &= \frac{A\epsilon^2}{2}r - \frac{B^2\epsilon^6c^4}{2G^2M^2}r^3 + o_2(r^3), \quad B^2 := \frac{2K_\zeta}{3I_\zeta^4}, \\ e^{\xi(r)} &= (1 - A\epsilon^2) + \frac{B^2c^4}{GM^2}\epsilon^6r^2 + o_2(r^2), \\ \text{sgn}(Q)\varphi(r) &= \frac{3\epsilon}{2}\frac{c^2}{\sqrt{G}} - \frac{A\epsilon^3}{2M}\frac{c^4}{G^{3/2}}r + O(r^3), \end{aligned} \quad (1.31)$$

and near $r = \infty$,

$$\begin{aligned} m(r) &= \frac{GM}{c^2} - \frac{GQ^2}{2c^4 r} + O\left(\frac{1}{r^2}\right), \\ e^\xi(r) &= 1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{c^4 r^2} + O\left(\frac{1}{r^2}\right), \\ \varphi(r) &= \frac{Q}{r} + O\left(\frac{1}{r^3}\right). \end{aligned} \tag{1.32}$$

We have

$$(1 - A\epsilon^2) \leq e^{\xi(r)} < 1, \tag{1.33}$$

$$\max\left\{\frac{A\epsilon^2}{2}r - \frac{B^2\epsilon^6 c^4}{2M^2 G^2}r^3, \frac{GM}{c^2} - \frac{GQ^2}{c^4 2r}\right\} \leq m(r) \leq \min\left\{\frac{A\epsilon^2}{2}r, \frac{GM}{c^2}\right\}. \tag{1.34}$$

The metric coefficient $e^{\xi(r)}$ is monotone increasing and potential function $\varphi(r)$ is monotone decreasing in magnitude.

Now, describe some the properties of the spacetime itself. As mentioned earlier, there is a curvature blow-up at $r = 0$, with Kretchman scalar $((R^{abcd}R_{abcd})^{1/2})$, where R^{abcd} is the full Riemann tensor) blowing up like r^{-2} . This should be compared with Schwarzschild spacetime where it blows up like r^{-3} and Reissner-Nordström spacetime, where it blows up like r^{-4} . Thus, the singularity is comparatively mild in our case. In fact, near $r = 0$, the spatial part of the metric takes the form

$$ds^2 = \left((1 - A\epsilon^2)^{-1} + \frac{B^2 c^4 \epsilon^6}{G^2 M^2} r^2 + o_2(r^2) \right) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{1.35}$$

and so, for ϵ small, is close to $a^2 dr^2 + r^2 d\Omega^2$, with $a := (1 - A\epsilon^2)^{-1/2}$ and $d\Omega$ the standard metric on the 2-sphere. Defining $\tilde{r} = ar$, we may write this as $d\tilde{r}^2 + a^{-2}\tilde{r}^2 d\Omega^2$, which brings out the cone-aspect: at a radial distance $r = 1$ from the tip $r = 0$, we have a 2 sphere of radius a^{-1} (instead of the standard sphere).

The singularity at $r = 0$ is naked: radial geodesics fall into $r = 0$ in finite affine parameter. Thus the spacetime, and the constant t -hypersurfaces are geodesically incomplete.

The main objective of this work is to understand physical phenomena on charged particle-spacetimes, and to determine whether they exhibit properties that make charged particle-spacetimes better, in some sense, than super-extremal Reissner-Nordström

spacetimes, as a model for spacetimes around point charges. To begin with, in the case of charged particle-spacetimes, the entirety of the ADM mass is “equal” to the energy carried by the electric field, while in the case of Reissner-Nordström spacetime the latter is not even finite. Charged particle-spacetimes have bounded metric coefficients and the electric potentials, which is an attractive property in itself, considering that it is not true in the case of the super-extremal Reissner-Nordström spacetime.

The first physical phenomenon we investigate is a scalar wave propagating on such a background spacetime. Apart from an intrinsic interest, the knowledge of decay and dispersive properties of scalar waves on these spacetimes is important if one would later choose to study the stability of these charged particle-spacetimes as solutions to the Einstein-Maxwell system, formulated as an initial value problem. In this work, we prove estimates for spherically symmetric scalar waves on charged particle-spacetimes whose conical singularity is mild enough. Our theorem is analogous to a theorem by Stalker and Tahvildar-Zadeh [28] in which similar estimates are shown on a super-extremal Reissner-Nordström background. We use a general theorem stated in that paper, but the computations are much simpler in our case, because we have good bounds on the metric coefficients, like the one coming from the bounds on the mass function in equation (1.34).

In order to remove the spherical symmetry assumption on the scalar wave, one has to understand the effect of the conical singularity on scalar waves. We start with the simplest conical singularity: the one at the vertex of a two-dimensional flat exact cone. In particular, the effect of the conical singularity on the fundamental solution to the scalar wave equation on this cone needs to be investigated. This fundamental solution is known, and is given in a work of Cheeger and Taylor [10], where it is derived using a functional calculus developed for exact cones. One may ask if the fundamental solution could be obtained by simpler means, using a method introduced by Sommerfeld [33]. In our work, we answer this question in the affirmative. So, one could possibly use Sommerfeld’s method to derive fundamental solutions on other flat two-dimensional spaces. This is left for future investigation.

Since these spacetimes represent spacetimes around a charged particle, an interesting

phenomenon is the interaction of this spacetime with other test particles, for instance, a spin-half particle like an electron. The wave function of such particles obeys Dirac's equation. In this case, distinct advantages of charged particle-spacetimes over the naked Reissner-Nordström spacetimes come to the fore. For one, on charged particle-spacetimes, the Dirac Hamiltonian is essentially self-adjoint on the domain consisting of smooth functions with compact support away from the center. This means that there is no confusion about which self-adjoint extension we mean by the Dirac Hamiltonian, because there is only one. This is not true on a naked Reissner-Nordström background, where there are multiple self-adjoint extensions to the Dirac Hamiltonian. Secondly, apart from the smallness condition on ϵ in the definition of charged particle-spacetimes, no restriction on the magnitude of the charge is necessary to ensure essential self-adjointness. This contrasts with Dirac Hamiltonian with a Coulomb potential on Minkowski space, where one needs to assume (in appropriate units) that $|eQ| < \frac{\sqrt{3}}{2}$ where $-e$ with $e > 0$ is the charge of an electron and Q is the charge at the center. Similarly, the existence of discrete spectrum in our case is also independent of the size of the charge. These advantages arise from the fact that the electric potential $\varphi(r)$ on a charged particle-spacetime is finite even near $r = 0$, the location of the charge. Naturally, the spectral properties of the Dirac Hamiltonian on charged particle-spacetimes are also investigated. The essential spectrum and the continuous spectrum is the same as for the Dirac Hamiltonian on Minkowski spacetime with a Coulomb potential. We also show that, under certain conditions, there is an infinity of eigenvalues. However, we have not determined the location of these eigenvalues. We leave this problem for a future investigation.

The remainder of this work is organized as follows. In chapter 2, we obtain estimates for scalar waves. In chapter 3, we use Sommerfeld's method to reproduce the known fundamental solution of the scalar wave equation on flat two-dimensional cone. In chapter 4, Dirac's equation is set up on charged particle-spacetimes, and the spectral properties of the Dirac Hamiltonian are investigated.

Chapter 2

Massless, spinless, real scalar field

An important long-term goal is to show that charged particle-spacetimes are stable solutions to the Einstein-Maxwell system, when it is formulated as an initial value problem. There have been several works that studied the stability of particular solutions to Einstein's equations with or without matter fields. For instance, the linear stability of the Schwarzschild solution was studied by Regge and Wheeler in [26], and by Moncrief in [22]; the linear stability of the Reissner-Nordström solution was studied by Moncrief in [23]. It has become evident that the study of linear stability of Einstein's equations involves a close look at the decay and dispersive properties of the scalar wave equation on the given stationary background. One way to measure dispersion is to show that the scalar wave is in a Lebesgue space on the spacetime (with possibly different powers for space and time). Such estimates, now called Strichartz estimates, began with work of Strichartz [29] on the wave equation on \mathbb{R}^{1+3} and were generalized later by others, for instance in [18]. In recent years, research has been directed at obtaining these estimates on various curved spacetimes, like Schwarzschild and Kerr, for example in [32], [21].

Stalker and Tahvildar-Zadeh in [28] show that on a naked Reissner-Nordström background, spherically symmetric solutions to the wave equation have their norms bounded in two function spaces by an appropriately defined energy of the initial data, provided the mass-to-charge ratio of the spacetime, in geometrized units, is less than one-half. Both function spaces are Lebesgue spaces over the whole spacetime: the first, is a weighted L^2 space and the second, the L^4 space. The corresponding estimate for the former is called a Morawetz estimate and the latter a Strichartz estimate. Our main theorem in this chapter is Theorem 2.1.1, and it is an application of Theorem (3) in [28]. We will not reproduce the proof of that theorem. In a nutshell, the idea behind

the proof is to transform the wave equation on the curved spacetime to wave equation on Minkowski space with a potential and use estimates from an earlier work [7], which holds under certain conditions on the potential. One of those conditions (condition (c) in section 2.1.1) turns out to be hard to verify. In our case, that difficulty is absent because the metric coefficient is comparatively well-behaved. In the remainder of this section we will introduce the main definitions and the set-up.

Note. In this chapter, we will set $c = 1$.

Definition 2.0.3. A *scalar wave* on \mathcal{M} is a real-valued function $u : \mathcal{M} \rightarrow \mathbb{R}$ that is a critical point of the the following action functional.

$$\mathbf{A}[u] = \int_{\mathcal{M}} \mathbf{g}(du, du) d\mathbf{v}_{\mathbf{g}} \quad (2.1)$$

For the purposes of this chapter, we change variables from r to \tilde{r} , designed to get the line element in a certain form. Notice that we may write the metric as $e^{\xi(r)} (-dt^2 + e^{-2\xi(r)} dr^2) + r^2 d\Omega^2$, where $d\Omega^2$ is the volume form on the standard sphere. We introduce \tilde{r} related to r by

$$d\tilde{r} = e^{-\xi(r)} dr, \quad \tilde{r}(r = 0) = 0. \quad (2.2)$$

Properties of this variable change are analyzed in Lemma 2.1.2. Now, if we define

$$\alpha(\tilde{r}) := e^{\frac{\xi(r(\tilde{r}))}{2}}, \quad (2.3)$$

then the metric can be rewritten as

$$ds^2 = \alpha(\tilde{r})^2 (-dt^2 + d\tilde{r}^2) + r(\tilde{r})^2 (d\theta^2 + \sin^2(\theta) d\phi^2). \quad (2.4)$$

With the metric in this form, the scalar wave equation can be shown to satisfy ([28]) the equation,

$$(\partial_t^2 + \mathcal{A})u = 0 \quad (2.5)$$

where the operator \mathcal{A} is

$$\mathcal{A} := -\frac{1}{r^2} \partial_{\tilde{r}} (r^2 \partial_{\tilde{r}}) - \frac{\alpha^2}{r^2} \Delta. \quad (2.6)$$

This operator is initially defined on $C_c^\infty(\mathbb{R}^3 \setminus \{0\})$ due to the singularity at $\tilde{r} = 0$. On $L^2(\mathbb{R}^3, r^2 d\tilde{r} d\Omega)$, this operator is symmetric and positive definite. To get a well-defined

evolution, we consider Friedrichs self-adjoint extension \mathcal{A}_F of \mathcal{A} , ([25] Theorem X.23) which is the unique self-adjoint extension whose domain is contained in the domain of the closure of the quadratic form associated to \mathcal{A} . So, the Cauchy problem considered is

$$(\partial_t^2 + \mathcal{A}_F)u = 0, \quad u(0) = u_0, \quad \partial_t u(0) = u_1. \quad (2.7)$$

We recall relevant spaces and norms from [28]. By Σ_t we denote the spatial slices of \mathcal{M} , with metric in the form (2.4). Let $d\sigma'' = r \sin(\theta) d\theta d\phi d\tilde{r}$. We define Sobolev spaces $\mathbf{H}^1(\Sigma_t)$ as completion $C_c^\infty(\Sigma_t \setminus \{0\})$ with respect to the norm given by $\|f\|_{\mathbf{H}^1}^2 = \int_{\Sigma_t} |f_{\tilde{r}}|^2 + \frac{\alpha^2}{\tilde{r}^2} |\nabla f|^2 d\sigma''$ and $\mathbf{H}^0(\Sigma_t)$ as completion of $C_c^\infty(\Sigma_t \setminus \{0\})$ with respect to the norm given by $\|f\|_{\mathbf{H}^0}^2 = \int_{\Sigma_t} |f|^2 d\sigma''$. For $s \in (0, 1)$, $\mathbf{H}^s(\Sigma_t)$ is defined by interpolation and further by duality to $s \in (-1, 0)$. The energy

$$\mathbf{E}_{1/2}[u] = \frac{1}{2} \left(\|u\|_{\mathbf{H}^{1/2}}^2 + \|u_t\|_{\mathbf{H}^{-1/2}}^2 \right) \quad (2.8)$$

is conserved along the flow of the evolution equation (2.7).

2.1 Estimates for spherically symmetric waves

We now state the main theorem of this chapter. Recall that ϵ is the dimensionless mass-to-charge ratio of the spacetime, defined by (1.21). Also, $\mathbf{E}_{1/2}[u]$ is the energy defined by equation (2.8).

Theorem 2.1.1. *Assume that the reduced Hamiltonian ζ is fixed. Then, there exists positive constants $C(\zeta)$ and $\epsilon_0(\zeta)$ such that for all the charged particle-spacetimes $\mathcal{M} = \mathcal{M}^{\zeta, M, Q}$ for which $\epsilon < \epsilon_0$, any spherically symmetric, scalar wave u propagating on \mathcal{M} (that is a solution to equation (2.7)) satisfies the estimate:*

$$\|r^{-1}u\|_{L^2(\mathcal{M}, \mathbf{v}_{\mathbf{g}})} + \|u\|_{L^4(\mathcal{M}, \mathbf{v}_{\mathbf{g}})} \leq C \mathbf{E}_{1/2}[u]. \quad (2.9)$$

The constant ϵ_0 is prescribed as follows. Let $A(\zeta)$, $I(\zeta)$ be the constants in (1.20) and (1.16), respectively. Let $K(\zeta) = \sup_{\mu > 0} |\frac{\zeta(\mu)}{\sqrt{\mu}}|$, $T(\zeta) = \sup_{\mu > 0} |\zeta'(\mu)\sqrt{\mu}|$, both of which are finite. Let $T_1 = \frac{K}{\sqrt{2}I^2} + A$, $T_2 = \frac{2\sqrt{2}}{I^2} (\frac{K}{2} + T)$. We pick an arbitrary $0 < a < 1$ (optimized later so that the right hand side of the equation below is maximized) and

define $T_3 = 6T_1 + 4T_1^2 \frac{(1-a)}{A} + 2T_2$. Then, one choice of ϵ_0 is by

$$\epsilon_0^2 = \min \left\{ \frac{1-a}{A}, 1, \frac{1}{4} a^3 T_3^{-1}, \frac{1}{8} a T_1^{-1} \right\}. \quad (2.10)$$

2.1.1 Proof of Theorem 2.1.1

We use Theorem 3, reproduced below, from [28], with notation changed to match ours (their ρ is our r , their r is our \tilde{r}). Thus, the framework and conclusion of their theorem is the same as ours.

Theorem (Theorem 3, [28]). *Let \mathcal{M} be a Lorentzian manifold that is homeomorphic to \mathbb{R}^4 , admitting a timelike \mathbb{R} action and a spacelike $SO(3, \mathbb{R})$ action commuting with it, in such a way that exactly one \mathbb{R} orbit, called Γ , is $SO(3, \mathbb{R})$ -fixed. Let (t, \tilde{r}, Ω) be the coordinate system on \mathcal{M} as in section 2 ([28]) with $\Gamma = \{\tilde{r} = 0\}$. Let g be a Lorentzian metric on \mathcal{M} that is of the form (2.4), is C^3 outside Γ and is such that the functions r and α satisfy the following conditions*

- (a) $\sup_{\tilde{r}>0} \tilde{r}^2 V(\tilde{r}) < \infty$,
- (b) $\inf_{\tilde{r}>0} \tilde{r}^2 V(\tilde{r}) > -\frac{1}{4}$,
- (c) $\sup_{\tilde{r}>0} \tilde{r}^2 \frac{d}{d\tilde{r}}(\tilde{r}V(\tilde{r})) < \frac{1}{4}$,
- (d) $\inf_{\tilde{r}>0} \left(\frac{r}{\alpha\tilde{r}}\right) > 0$,

where

$$V(\tilde{r}) := \frac{1}{r} \frac{d^2 r}{d\tilde{r}^2}. \quad (2.11)$$

Then there exists a constant $C > 0$, depending only on the quantities on the left in the conditions above, such that any spherically symmetric solution of

$$\partial_t^2 \psi + \mathcal{A}_F \psi = 0 \quad (2.12)$$

satisfies

$$\|r^{-1}\psi\|_{L^2(\mathcal{M})} + \|\psi\|_{L^4(\mathcal{M})} \leq C \mathbf{E}_{1/2}[\psi]. \quad (2.13)$$

Our metric (2.4) is precisely in the form that is needed in the theorem. From the form of the metric, that we see there is a timelike \mathbb{R} action which commutes with the spacelike action of the rotation group. The metric has to be C^3 outside the fixedpoints of the rotation action, which we do have in (2.4). Indeed, the function $\zeta(\mu)$ is C^2 by definition and thus $e^{\xi(r)}$ is C^3 . From definition of $\alpha(\tilde{r}), r(\tilde{r})$ it is then clear they are also C^3 . The remaining conditions are on the potential $V(\tilde{r}) = \frac{1}{r} \frac{d^2 r}{d\tilde{r}^2}$. These conditions are verified below in section 2.1.2, and so the Theorem 2.1.1 is proved. We simplify the potential as $V(\tilde{r}) = \frac{1}{r(\tilde{r})} \frac{d}{d\tilde{r}}(e^{\xi(r)})$, which is the same as $\frac{1}{r(\tilde{r})} \frac{d}{dr} e^{\xi(r)} \frac{dr}{d\tilde{r}} = \frac{1}{r(\tilde{r})} (-2 \frac{m'(r)}{r} + 2 \frac{m(r)}{r^2}) e^{\xi(r)}$. So we obtain,

$$V(\tilde{r}) = \frac{2}{r^2} \left(-m'(r) + \frac{m(r)}{r} \right) \left(1 - 2 \frac{m(r)}{r} \right). \quad (2.14)$$

Here r on the right hand side depends on \tilde{r} via the coordinate change relation (2.2).

2.1.2 Checking conditions on the potential

We first analyze the relation between \tilde{r} and r defined by the equation (2.2).

Lemma 2.1.2 (Analysing variable change). *Under the differential relation in equation (2.2) between r and \tilde{r} , the following are true:*

- (a) *the function $\tilde{r} : (0, \infty) \rightarrow (0, \infty)$ is a diffeomorphism,*
- (b) $\lim_{r \rightarrow 0} \frac{\tilde{r}}{r} = (1 - A\epsilon^2)^{-1}$,
- (c) $\lim_{r \rightarrow \infty} \frac{\tilde{r}}{r} = 1$,
- (d) $\forall r > 0 : 1 < \frac{\tilde{r}}{r} \leq (1 - A\epsilon^2)^{-1}$,

where $A(\zeta), \epsilon$ are defined in equations (1.20), (1.21) respectively.

Proof. Notice that $\tilde{r}(r) = \int_0^r e^{-\xi(s)} ds$. So \tilde{r} is C^1 , with derivative $e^{-\xi(r)} > 0$, that tends to 1 at ∞ , by (1.32). Given any $\epsilon > 0$, there exist $a > 0$ such that

$$1 - \epsilon < e^{-\xi(r)}, \forall r > a.$$

This implies that $\tilde{r}(r) > \tilde{r}(a) + (1 - \epsilon)(r - a)$, so that $\lim_{r \rightarrow \infty} \tilde{r}(r) = \infty$. Along with the condition $\tilde{r}(0) = 0$, this implies part (a) of the lemma, since the function is monotone.

From the integral expression for \tilde{r} , we see that $\lim_{r \rightarrow 0} \frac{\tilde{r}}{r} = \lim_{r \rightarrow 0} e^{-\xi(r)}$. This limit as seen in equations (1.31) has the value $(1 - A\epsilon^2)^{-1}$. This proves part (b).

Now, $|\frac{\tilde{r}}{r} - 1| = |\frac{\int_0^r (e^{-\xi(s)} - 1) ds}{r}| \leq \frac{\int_0^r |e^{-\xi(s)} - 1| ds}{r}$. For any $\tilde{\epsilon} > 0$, there is an $a > 0$, such that $|e^{-\xi(s)} - 1| < \tilde{\epsilon} \forall s > a$. So, $\forall r > a$, $|\frac{\tilde{r}}{r} - 1| \leq \frac{\int_0^a |e^{-\xi(s)} - 1|}{r} + \frac{\tilde{\epsilon}(r-a)}{r}$, from which part (c) follows because for large r both the terms can be made to be less than $\tilde{\epsilon}$.

By the mean value theorem, $\frac{\tilde{r}}{r} = e^{-\xi(s)}$, for some $s \in (0, r)$. From bounds on $m(r)$ in equation (1.34), we see $m(r) \leq A\epsilon^2/2$. So, $e^\xi = 1 - 2m(r)/r \geq (1 - A\epsilon^2)$, and one side of the required inequality follows. The other inequality is easily seen from $e^\xi < 1$, as stated in equation (1.33). □

First, we list a few facts we will use. Recall that the factor used to scale the reduced Hamiltonian is given by (as $c = 1$)

$$\beta = \frac{|Q|^{3/2} I}{M}. \quad (2.15)$$

Now, $m'(r) = G\zeta_\beta(\frac{Q^2}{2r^4})r^2$. Substituting $\mu = \frac{Q^2}{2r^4}$, and using the expression of β , $m'(r) = \frac{\epsilon^2}{\sqrt{2}I^2} \frac{\zeta(\mu)}{\sqrt{\mu}}$. Let $K = \sup_\mu |\frac{\zeta(\mu)}{\sqrt{\mu}}|$. Because of the asymptotics of ζ we know that $0 < K < \infty$. Therefore we see,

$$|m'(r)| \leq \frac{\epsilon^2}{\sqrt{2}I^2} K. \quad (2.16)$$

Recall that the metric coefficient $e^\xi = 1 - 2\frac{m(r)}{r}$. From chapter 1 we quote,

$$1 - A\epsilon^2 \leq e^\xi < 1 \quad (2.17)$$

which is same as

$$0 < \frac{m(r)}{r} \leq \frac{A\epsilon^2}{2}. \quad (2.18)$$

In this paragraph we show, with T_2 as in the theorem statement,

$$|m''(r)r| \leq T_2\epsilon^2 \quad (2.19)$$

Note that $m''(r) = G\left(\zeta_\beta(\frac{Q^2}{2r^4})2r - \zeta'_\beta(\frac{Q^2}{2r^4})\frac{2Q^2}{r^3}\right)$. So,

$$rm''(r) = G\left(\zeta_\beta(\frac{Q^2}{2r^4})2r^2 - \zeta'_\beta(\frac{Q^2}{2r^4})\frac{2Q^2}{r^2}\right) \quad (2.20)$$

We know from equation (2.16) that $G|\zeta_\beta(\frac{Q^2}{2r^4})2r^2| \leq 2\frac{\epsilon^2 K}{\sqrt{2}I^2}$. Notice,

$$\zeta'_\beta\left(\frac{Q^2}{2r^4}\right)\frac{1}{r^2} = \frac{M^2\sqrt{2}}{I^2|Q|^4}\zeta'(\mu)\sqrt{\mu}, \text{ where } \mu = \frac{Q^2}{2r^4}. \quad (2.21)$$

Since $0 < \zeta'(\mu) \leq \frac{\zeta(\mu)}{\mu}$ and $\zeta(\mu) \sim \mu, \mu \rightarrow 0$ we can see that $\zeta'(s)\sqrt{s} \rightarrow 0, s \rightarrow 0$. Also, $\zeta'(s) \sim \frac{J_\zeta}{2s^{1/2}}$, and this implies that $\zeta'(s)\sqrt{s}$ is bounded near ∞ as well. So the constant T defined in the statement of the theorem is finite.

This proves inequality (2.19).

Condition (a)

Rewrite (a) as $\sup_{\tilde{r}>0}(\frac{\tilde{r}}{r})^2 r^2 V(r)$. Since, $(\frac{\tilde{r}}{r})$ is bounded by Lemma 2.1.2, we just need to show $\sup_{r>0} r^2 V(r) < \infty$. But, $r^2 V(r) = 2(-m'(r) + 2\frac{m(r)}{r})e^\xi$, and so using equations (2.16), (2.18), we see

$$|r^2 V(r)| \leq 2T_1 \epsilon^2. \quad (2.22)$$

Thus condition (a) is verified.

Condition (d)

We need to show that $\inf_{\tilde{r}>0} \frac{r(\tilde{r})}{\tilde{r}e^{\frac{\xi(r(\tilde{r}))}{2}}} > 0$. This is equivalent to showing $\inf_{r>0} \frac{r}{\tilde{r}e^{\frac{\xi(r)}{2}}} > 0$. Using part (d) of Lemma 2.1.2 and inequality 2.17, we see

$$\frac{r}{\tilde{r}e^{\frac{\xi(r)}{2}}} > (1 - A\epsilon^2). \quad (2.23)$$

Since, $\epsilon < \epsilon_0$, the right side is positive and so part (d) of the potential condition is verified.

Condition (b)

As before, $\tilde{r}^2 V(\tilde{r}) = (\frac{\tilde{r}}{r})^2 2\left(-m'(r) + \frac{m(r)}{r}\right)e^{\xi(\tilde{r})}$.

Using equation (2.22) and part (d) of Lemma 2.1.2, $|\tilde{r}^2 V(\tilde{r})| \leq (1 - A\epsilon^2)^{-1}(2)(1)T_1 \epsilon^2$. Therefore since $\epsilon^2 < \epsilon_0^2 < a\frac{1}{8}T_1$, then condition (b) is true.

Condition (c)

Observe that

$$\tilde{r}^3 \frac{d}{d\tilde{r}} V(\tilde{r}) = \left(\frac{\tilde{r}}{r}\right)^3 \left(\frac{dr}{d\tilde{r}}\right) r^3 \frac{d}{dr} V(\tilde{r}). \quad (2.24)$$

Taking into account the expression for $V(\tilde{r})$

$$\begin{aligned}
r^3 \frac{d}{dr} V(\tilde{r}) &= 2r^3 \left[\frac{-2}{r^3} \left(-m'(r) + \frac{m(r)}{r} \right) e^{\xi(r)} \right. \\
&+ \frac{1}{r^2} \left(-m''(r) + \frac{m'(r)}{r} - \frac{m(r)}{r^2} \right) e^{\xi(r)} \\
&\left. + \frac{1}{r^2} \left(-m'(r) + \frac{m(r)}{r} \right) 2 \left(\frac{-m'(r)}{r} + \frac{m(r)}{r^2} \right) \right] \quad (2.25)
\end{aligned}$$

The first term is $-4 \left(-m'(r) + \frac{m(r)}{r} \right) e^{\xi(r)}$. So, $|\text{first term}| \leq 4T_1 \epsilon^2(1)$, as shown while verifying previous conditions. The third term simplifies to $4 \left(-m'(r) + \frac{m(r)}{r} \right)^2$. So, $|\text{third term}| \leq 4T_1^2 \epsilon^4$.

The second term simplifies to $2 \left(-m''(r)r + m'(r) - \frac{m(r)}{r} \right) e^{\xi(r)}$. The absolute value of this is bounded by $2T_1 \epsilon^2 + 2T_2 \epsilon^2$.

Finally, using $\frac{dr}{d\tilde{r}} = e^\xi$ which is less than 1 by equation (2.17), and $\frac{\tilde{r}}{r} \leq (1 - A\epsilon^2)^{-1}$ by part (d) of Lemma 2.1.2, from equation (2.24), we get

$$\left| \tilde{r}^3 \frac{dV(\tilde{r})}{d\tilde{r}} \right| \leq (1 - A\epsilon^2)^{-3} T_3 \epsilon^2 \leq a^{-3} T_3 \epsilon^2 \quad (2.26)$$

Using this, and assumptions that $(1 - A\epsilon^2) \leq a^{-1}$ and $\epsilon^2 \leq \frac{1}{4} a^3 T_3^{-1}$ in the theorem, condition (c) is verified.

Chapter 3

Conical singularities and Sommerfeld's method

The charged particle-spacetime introduced in chapter 1 has a conical singularity at the location of the charge. Deriving good estimates for solutions of the wave equation often involves obtaining the fundamental solution. In this chapter, we will obtain the known fundamental solution (Schwartz kernel) of the wave equation on a two-space dimensional exact circular cone, starting from the standard fundamental solution to the wave equation on \mathbb{R}^2 and performing a method used by Sommerfeld to construct branched solutions to Laplace's and Helmholtz equations in \mathbb{R}^3 . The main theorem of this chapter is Theorem 3.1.1. In the remainder of this section, we will explain the origin of our interest in Sommerfeld's method.

The estimates of chapter 2 (Theorem 2.1.1) are only for spherically symmetric solutions. To weaken that hypothesis, we believe that it is important to understand how a conical singularity affects the fundamental solution of the wave equation. Naturally, one should first study the simplest conical singularity: one that exists at the tip of a two dimensional cone. Things are simpler in that case since this space is flat except at the tip.

In a recent work of Blair, Ford and Marzuola [5], Strichartz estimates are obtained for waves on a flat, exact two dimensional cone. The starting point of their proof is a formula for the Schwartz kernel of the wave propagator $\frac{\sin(t|\xi|)}{|\xi|}$ made from the Friedrichs extension of the Laplacian. (The usage "fundamental solution" and "Schwartz kernel" are be taken to be synonymous in the present work. Intuitively, it is a solution with initial data being a pulse "delta distribution" at a specified point "source" on the cone.) We found the formula for the kernel to be interesting in itself because one could pinpoint the contribution coming from the cone tip: the integral that appears in equation

(3.12). It seems to be integrating over the fundamental solution in standard \mathbb{R}^2 in some way.

The formula for the Schwartz kernel in [5] is quoted from a work of Cheeger and Taylor [10],[9] where they use functional calculus on cones over compact manifolds to study diffraction by conical singularities. Since our objective is to study waves on the charged particle-spacetime (Definition 1.0.1), the spatial part of which is *not* globally an exact cone over a compact manifold like S^2 (because of the nature of the metric coefficients), we look for alternative ways to derive the same formula.

The formula for the cone over a circle of circumference 4π (which is then a two sheeted cover of the punctured plane) is often attributed to Sommerfeld who did, for the first time, a rigorous analysis of diffraction problems in [27], and later in his study of branched potentials [33]. For example, he obtained diffractive solutions of the wave equation in a region that is \mathbb{R}^3 from which a vertical half-plane over the positive x axis is removed. His method was to take a solution from \mathbb{R}^3 , which is 2π periodic in the azimuthal angle in cylindrical coordinates, and obtain a new one that is, say, 4π periodic in the angle: so that it may be thought to be defined on a Riemann space consisting of two copies of \mathbb{R}^3 joined along the aforementioned plane.

Sommerfeld's method has since been used many times, for instance by Carslaw [8] and others, [24], [1], [13], [1], [15]. All of them consider the equations $(-\Delta + k^2)u = 0$, $-\Delta u = 0$ in three space dimensions. None of these works treat the wave equation in two dimensional space directly using Sommerfeld's method. The formula for the fundamental solution to the wave equation on an exact 2-D cone is derived in chapter 5 of Friedlander's book [17], but it is obtained by Fourier expansion of the delta function in the angle.

We perform Sommerfeld's method, starting with the standard fundamental solution $u_1(t, x, x_0) = \frac{1}{2\pi} \frac{H(t-|x-x_0|)}{\sqrt{t^2-|x-x_0|^2}}$, where H is the Heaviside function. We consider u_1 as the real part of a complex valued function \tilde{u}_1 . Then, we perform Sommerfeld's method on \tilde{u}_1 , obtaining complex valued functions \tilde{u}_n which are $2\pi n$ periodic in the angle (i.e., defined on a cone over a circle of circumference $2\pi n$). We show that when one takes the real part of \tilde{u}_n we get the known formula. One may then use an image method to find

solutions that are periodic with period that is a rational multiple of 2π . Afterwards, we may observe that taking formal limits of the formula, one can get the formula of the Schwartz kernel of a cone over a circle of circumference that is an irrational multiple of π .

In this work, we do not address the question of why the procedure does produce a fundamental solution on the cone (apart from the fact that we do obtain the known formula). This is left for a future study. Another avenue for future exploration is to find out why the procedure gives the Schwartz kernel of the wave propagator coming from the Friedrichs Laplacian. Thus, here we confine ourselves to the study of the effectiveness of Sommerfeld's method to treat the wave equation on two-dimensional manifolds with conical singularities.

3.1 Fundamental solution

Let S_ρ^1 denote a circle of radius ρ , with local coordinate function θ , which for a fixed point $p \in S_\rho^1$, maps $S_\rho^1 \setminus \{p\}$ *isometrically* into $(0, 2\pi\rho)$.

Definition 3.1.1. The cone on S_ρ^1 is defined as $C(S_\rho^1) := \mathbb{R}_+ \times S_\rho^1$, with the metric line element given by $ds^2 = dr^2 + r^2 d\theta^2$.

For example, $C(S_1^1) \cong \mathbb{R}^2 \setminus \{0\}$ with the Euclidean metric in polar coordinates. The Laplace-Beltrami operator on $C(S_\rho^1)$ is

$$\Delta := \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2. \quad (3.1)$$

Information about ρ is hidden in the coordinate θ . We say $u : \mathbb{R} \times C(S_\rho^1) \rightarrow \mathbb{R}$ is a solution to the wave equation if

$$\square u := \partial_t^2 u - \Delta u = 0. \quad (3.2)$$

Definition 3.1.2. For every point (r_1, θ_1) which we call a *source* in $C(S_\rho^1)$ we are interested in a distribution u_ρ on $[0, \infty) \times C(S_\rho^1)$ such that

$$\square u_\rho = \frac{1}{r_1} \delta(t) \delta_{r_1}(r_2) \delta_{\theta_1}(\theta_2), \quad (3.3)$$

where the right-hand side is the delta distribution on $[0, \infty) \times C(S_\rho^1)$ with support at $(0, r_1, \theta_1)$ and the left side is \square taken in the variables (t, r_2, θ_2) . Such a distribution u_ρ is called a (*forward*) *fundamental solution* to the wave equation on $C(S_\rho^1)$. (Ref. [17], chapter 5, equation 5.2.7).

For instance, on \mathbb{R}^2 with Euclidean metric, the fundamental solution in rectangular coordinates with source (x_0, y_0) , for $t \geq 0$ is given by

$$u_1^{\text{rect}}(t, (x, y)) = \text{Re} \left[\frac{1}{2\pi \sqrt{t^2 - |(x, y) - (x_0, y_0)|^2}} \right] \quad (3.4)$$

and written in polar coordinates, takes the form in $u_1 = \text{Re}(\tilde{u}_1)$, defined in equation (3.9).

The expression for the fundamental solution u_ρ is what we derive. As mentioned earlier, Cheeger and Taylor studied this problem in the 1980s in [9], [10], where they used the functional calculus (spectral theory) developed for a cone on any compact manifold. We do not use spectral theory. We note that we are really trying to find a solution to the equation (3.3) that is $2\pi\rho$ periodic in the angular variables, because S_ρ^1 is just the Riemannian quotient manifold $\frac{\mathbb{R}}{2\pi\rho\mathbb{Z}}$. This is the perspective we will take- that we just want to change the periodicity in the angular variable of the known fundamental solution from 2π to $2\pi\rho$.

The distance function on $C(S_\rho^1)$ is given by the following [5] expression:

$$d((r_2, \theta_2), (r_1, \theta_1)) = \begin{cases} (r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2))^{1/2} & |\theta_1 - \theta_2| < \pi \\ r_1 + r_2 & |\theta_1 - \theta_2| \geq \pi. \end{cases} \quad (3.5)$$

We point out that by $|\theta_1 - \theta_2|$ we mean the distance on S_ρ^1 . So in other-words,

$$|\theta_1 - \theta_2| := \min \{|\theta_1 - \theta_2 + 2\pi\rho k| \mid k \in \mathbb{Z}\}. \quad (3.6)$$

We define three regions in $\mathbb{R}_+ \times C(S_\rho^1) \times C(S_\rho^1)$:

1. Region-1 = $\{(t, (r_1, \theta_1), (r_2, \theta_2)) \mid 0 < t < d((r_1, \theta_1), (r_2, \theta_2))\}$
2. Region-2 = $\{(t, (r_1, \theta_1), (r_2, \theta_2)) \mid d((r_1, \theta_1), (r_2, \theta_2)) < t < r_1 + r_2\}$
3. Region-3 = $\{(t, (r_1, \theta_1), (r_2, \theta_2)) \mid t > r_1 + r_2\}$

We use the following expression μ throughout this chapter:

$$\mu := \frac{r_1^2 + r_2^2 - t^2}{2r_1r_2}. \quad (3.7)$$

Let us denote the standard 2π -periodic-in-angles fundamental solution in \mathbb{R}^{2+1} with polar spatial coordinates as $u_1((r_1, \theta_1), t, (r_2, \theta_2))$. Then

$$u_1((r_1, \theta_1), t, (r_2, \theta_2)) := \operatorname{Re}(\tilde{u}_1) \quad (3.8)$$

with

$$\tilde{u}_1((r_1, \theta_1), t, (r_2, \theta_2)) = \frac{1}{2\pi} (2r_1r_2)^{-1/2} \frac{1}{\sqrt{(-\mu + \cos(\theta_1 - \theta_2))}}. \quad (3.9)$$

This is just a rewriting of equation (3.4), in polar coordinates using cosine rule. The fundamental solution on $C(S_\rho^1)$ with source (r_1, θ_1) , given in Cheeger and Taylor [10], can now be expressed as follows.

In region 1,

$$K^{(1)}((r_1, \theta_1), t, (r_2, \theta_2)) \equiv 0. \quad (3.10)$$

In region 2,

$$K^{(2)}((r_1, \theta_1), t, (r_2, \theta_2)) = \sum_{j: 0 < |\theta_1 - \theta_2 + 2\pi\rho j| < \cos^{-1}(\mu)} u_1((r_1, \theta_1 + 2\pi\rho j), t, (r_2, \theta_2)) \quad (3.11)$$

In region 3,

$$K^{(3)}((r_1, \theta_1), t, (r_2, \theta_2)) = K_1^{(3)} + K_2^{(3)}, \quad (3.12)$$

where,

$$K_1^{(3)} = \sum_{j: 0 < |\theta_1 - \theta_2 + 2\pi\rho j| < \pi} u_1((r_1, \theta_1 + 2\pi\rho j), t, (r_2, \theta_2)) \quad (3.13)$$

$$K_2^{(3)} = -\frac{1}{4i\pi^2\rho} \int_0^{\cosh^{-1}(-\mu)} (\mu + \cosh(s))^{-1/2} C(s, \theta_1, \theta_2, \rho) ds. \quad (3.14)$$

$$C(s, \theta_1, \theta_2, \rho) = C_1(s, \theta_1, \theta_2, \rho) + C_2(s, \theta_1, \theta_2, \rho) \quad (3.15)$$

$$C_1(s, \theta_1, \theta_2, \rho) = \frac{\sin((\theta_1 - \theta_2 + \pi)\rho^{-1})}{\cosh(s\rho^{-1}) - \cos((\theta_1 - \theta_2 + \pi)\rho^{-1})} \quad (3.16)$$

$$C_2(s, \theta_1, \theta_2, \rho) = \frac{\sin((\theta_1 - \theta_2 - \pi)\rho^{-1})}{\cosh(s\rho^{-1}) - \cos((\theta_1 - \theta_2 - \pi)\rho^{-1})}. \quad (3.17)$$

Now, we are in a position to state the main theorem of this chapter.

Theorem 3.1.1. *Suppose that $\rho \in \mathbb{N}$. Fix a source point $(r_1, \theta_1) \in C(S_\rho^1)$. Let $(r_2, \theta_2) \in C(S_\rho^1)$ be a test point and $t > 0$. Let μ be computed as in equation (3.7). Assume that $|\mu| \neq 1$, $\cos(\theta_2 - \theta_1) \neq \mu$, and $|\theta_1 - \theta_2| \neq \pi \pmod{2\pi\rho}$. Then, Sommerfeld's method starting from the seed solution u_1 , results in a fundamental solution u_ρ on $C(S_\rho^1)$ that agrees with the formulae given by equations (3.10), (3.11), (3.12).*

The next section is the proof of this theorem.

3.2 Solution on a cone over a circle of circumference $2\pi n, n \in \mathbb{N}$

We observe that, $\tilde{u}_1((r_1, \alpha), t, (r_2, \theta_2)), \alpha \in \mathbb{C}$ is well defined and is complex-analytic in \mathbb{C} away from branch cuts, which are necessary because of the square-root.

Notation. Throughout, we will use $\tilde{u}_1(\alpha)$ with the implicit assumption that the other variables are fixed. That is,

$$\tilde{u}_1(\alpha) := \tilde{u}_1((r_1, \alpha), t, (r_2, \theta_2)). \quad (3.18)$$

Later we will fix the branch cuts to be vertical lines. We will use that branch for which $\tilde{u}_1(\alpha = \theta_2)$ is either positive real or purely imaginary with positive imaginary part.

Lemma 3.2.1 (Determining the square-root in $\tilde{u}_1(\alpha)$). *Suppose $a, b \in \mathbb{R}, \theta_2$ and let $w = -\mu + \cos(a + ib - \theta_2)$. If b is non-zero and fixed, then as a increases, w moves clockwise if $b > 0$ and counter clockwise if $b < 0$, with period 2π on the ellipse $\frac{(x+\mu)^2}{\cosh(b)^2} + \frac{y^2}{\sinh(b)^2} = 1$ and if a is fixed such that $a - \theta_2 \neq k\frac{\pi}{2} \forall k \in \mathbb{Z}$ and b is varied, w moves on the hyperbola $\frac{(x+\mu)^2}{\cos(a-\theta_2)^2} - \frac{y^2}{\sin(a-\theta_2)^2} = 1$.*

Proof. This is inferred from writing $-\mu + \cos(a - \theta_2 + ib) = -\mu + \cosh(a - \theta_2) \cosh(b) + i \sin(a - \theta_2) \sinh(b)$. \square

The ellipse has center $(-\mu, 0)$ with semi-major axis of length $\cosh(b)$ along the x -axis and semi-minor axis of length $\sinh(b)$ along the y -axis. So $w = 0$ lies inside the

ellipse iff $\cosh(b) > |\mu|$. Hence, if branch cuts are chosen on vertical lines going off to $\pm i\infty$, the resulting branches will be 2π periodic.

We now introduce an *auxiliary function* with a parameter $n \in \mathbb{R}$, used by Sommerfeld:

$$f_n(\alpha, \theta_1) := \frac{i}{n \left(1 - e^{i\left(\frac{\theta_1 - \alpha}{n}\right)}\right)}. \quad (3.19)$$

Notation. We will often use $f_n(\alpha) := f_n(\alpha, \theta_1)$, taking θ_1 to be understood from the context.

This function is $2\pi n$ periodic in both θ_1 and α . For a fixed θ_1 , it is complex analytic in $\alpha \in \mathbb{C}$ with simple poles at $\alpha = \theta_1 + 2\pi nk$, $\forall k \in \mathbb{Z}$, with residue equal to 1. Let C_0 denote a path going once around θ_1 and not containing any other poles of f_n and not intersecting any branch cuts of \tilde{u}_1 . By the residue theorem,

$$\tilde{u}_1(\alpha = \theta_1) = \frac{1}{2\pi i} \int_{C_0} \tilde{u}_1(\alpha) f_n(\alpha, \theta_1) d\alpha. \quad (3.20)$$

The periodicity of the integrand plays a decisive role in the development of the ideas in this chapter. The following proposition shows that it can only be an integer multiple of 2π . Also, the values $n \in \mathbb{Z}$ are enough to attain all possible periodicities of the integrand. We state this without proof.

Proposition 3.2.2. *Let p be the period of $\tilde{u}_1(\alpha)f_n(\alpha, \theta_1)$ in the variable α .*

1. *Let $k, l \neq 0 \in \mathbb{Z}$, $\gcd(k, l) = 1$ and $n = k/l$. Then, $p = 2\pi k$.*
2. *For n irrational, there is no periodicity in α .*

Now, fix n to be a positive integer. Let us pick an arbitrary $\theta \in \mathbb{R}$. We focus our attention on the closed vertical strip of width $2\pi n$, given by $\mathfrak{S} = \{z | \theta \leq \operatorname{Re}(z) \leq \theta + 2\pi n\}$ and assume $\theta_1 \in \operatorname{interior}(\mathfrak{S}) \cap \mathbb{R}$. Because the poles are precisely those of f_n , there is only one pole in \mathfrak{S} , and it is a simple pole, at θ_1 , with residue $\tilde{u}_1(\theta_1)$. The branch points, however, are 2π periodic since they arise from \tilde{u}_1 . In fact, we know they appear in pairs, located at $\alpha = \pm \cos^{-1}(\mu) + \theta_2 + 2\pi k$, $k \in \mathbb{Z}$. Therefore, generally, in \mathfrak{S} there will be $2n$ branch points. This number could go up by one or two, depending

on whether there are branch points on the boundary of \mathfrak{G} . We may choose the branch cuts to not intersect the vertical boundaries of \mathfrak{G} .

We make observations about the integrand in equation (3.20) in the following lemma.

Lemma 3.2.3 (Limits at $x \pm i\infty$). *Suppose $x, y \in \mathbb{R}, \alpha = x + iy$. Then*

(a)

$$\lim_{y \rightarrow \pm\infty} \tilde{u}_1(\alpha) = 0 \quad (3.21)$$

(b)

$$\lim_{y \rightarrow \infty} f_n(\alpha, \theta_1) = 0 \quad (3.22)$$

(c)

$$\lim_{y \rightarrow -\infty} f_n(\alpha, \theta_1) = \frac{i}{n} \quad (3.23)$$

Proof. We deduce (a) from the following identity,

$$\tilde{u}_1(x + iy) = \frac{(2r_1 r_2)^{\frac{1}{2}}}{2\pi} \frac{1}{\sqrt{-\mu + \frac{1}{2} \{e^{i(x-\theta_2)} e^{-y} + e^{-i(x-\theta_2)} e^y\}}}. \quad (3.24)$$

The parts (b), (c) become evident from

$$f_n(\alpha, \theta_1) = \left(\frac{i}{n}\right) \frac{1}{\left(1 - e^{\frac{y}{n}} e^{i\left(\frac{\theta_1 - x}{n}\right)}\right)}. \quad (3.25)$$

□

The counter-clockwise integral over C_0 is therefore equal to the sum of integrals taken clockwise on contours around the branch cuts and the vertical boundaries of \mathfrak{G} , as shown in Figure 3.1 below. The integrals on the two vertical portions cancel each other because of the $2\pi n$ periodicity of the integrand. We make another crucial observation: *The standard 2π periodic solution u_1 is the real part of an integral on a contour going clockwise around the branch cuts involving the $2n$ branch points.* Consider the vertical strip $\mathfrak{G}_{\theta_2} := \{\alpha | \theta_2 - \pi \leq \text{Re}(\alpha) \leq \theta_2 + \pi\}$. As noted at the beginning, branch cuts will always be taken to be vertical, going from the branch points to $\pm\infty$, so that the function $\tilde{u}_1(\alpha)$ is 2π periodic.

Definition 3.2.1 (Contour C_n). Given θ_2, θ_1 (and t, r_1, r_2), consider all branch cuts in the strip \mathfrak{S}_{θ_2} . Let C_n be contour that stays in the interior of \mathfrak{S}_{θ_2} , and such that for each branch cut in \mathfrak{S}_{θ_2} , C_n either goes around the branch cut (if it lies in the interior of $\mathfrak{S}_{2\pi}$) or along the branch cut (if it is on the boundary of $\mathfrak{S}_{2\pi}$). The direction of the path is downwards when it lies to the right of a branch cut, and upwards if it lies to the left.

We now give the expression for the $2\pi n$ periodic solution. See figure 3.2.

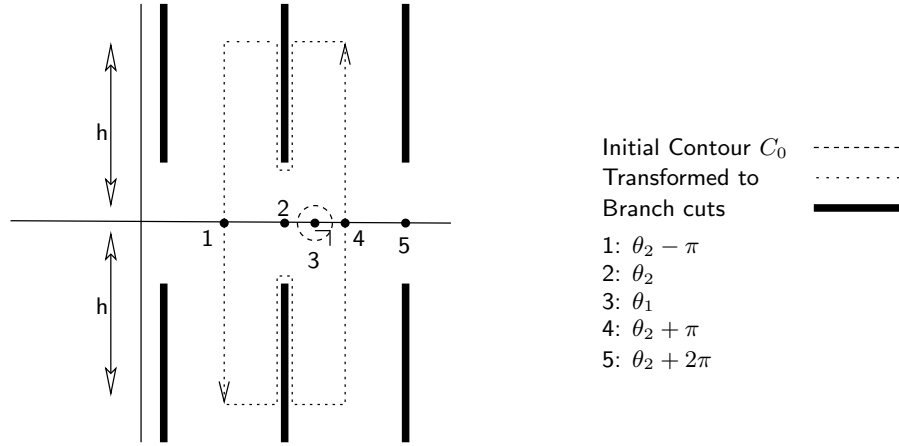


Figure 3.1: Rationale behind Sommerfeld's method illustrated in the case $n = 1, \mu > 1$, and integrand $\tilde{u}_1(\alpha)f_2(\alpha)$. Poles are at $\theta_1 + 4\pi k, k \in \mathbb{Z}$. Branch points at $\theta_2 \pm \cosh^{-1}(\mu) + 2\pi k, k \in \mathbb{Z}$. Integral at extreme left and right vertical edges cancel. Integral along horizontal parts go to zero as $h \rightarrow \infty$. So, $\tilde{u}_1(\alpha = \theta_1)$ is equal to a sum of integrals over 1 pair of branch cuts, over θ_2 .

Definition 3.2.2 (Sommerfeld's solution). Let

$$\tilde{u}_n((r_1, \theta_1), t, (r_2, \theta_2)) = \frac{1}{2\pi i} \int_{C_n} \tilde{u}_1(\alpha) f_n(\alpha, \theta_1) d\alpha.$$

Then the $2\pi n$ periodic solution produced by Sommerfeld's method is given by

$$u_n = \text{Re}(\tilde{u}_n). \quad (3.26)$$

We shall now make a few remarks about this solution.

Definition 3.2.3. For a given fixed $n \in \mathbb{N}$ and source angle θ_1 , let $1 \leq k \leq n$. Then the k -th sheet is defined as $S_k = \{\alpha | \theta_1 + (k-2)\pi < \text{Re}(\alpha) < \theta_1 + k\pi\}$.

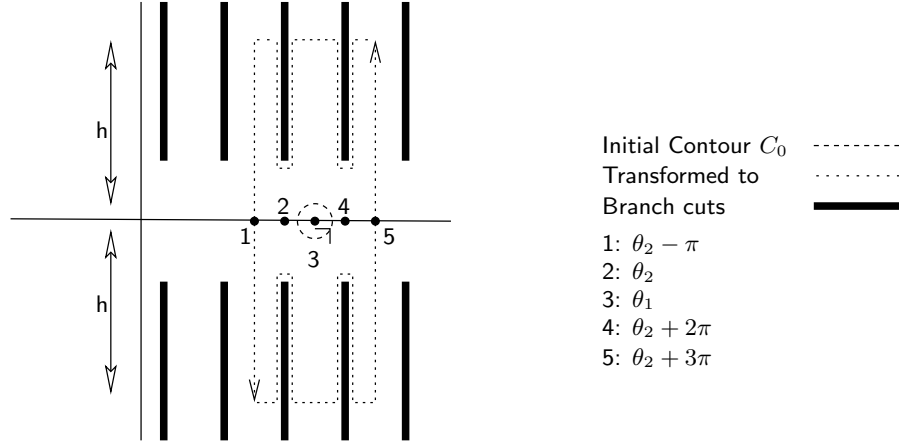


Figure 3.2: Rationale behind Sommerfeld's method illustrated in the case $n = 2, \mu > 1$ and integrand $\tilde{u}_1(\alpha)f_2(\alpha)$. Poles are at $\theta_1 + 4\pi k, k \in \mathbb{Z}$. Branch points at $\theta_2 \pm \cosh^{-1}(\mu) + 2\pi k, k \in \mathbb{Z}$. Integral at extreme left and right vertical edges cancel. Integral along horizontal parts go to zero as $h \rightarrow \infty$. So, $\tilde{u}_1(\alpha = \theta_1)$ is equal to a sum of integrals over 2 pairs of branch cuts, one over θ_2 and another over $\theta_2 + 2\pi$.

Clearly, $\mathbb{C} = \cup_{k \in \mathbb{Z}} (S_k + 2\pi n) \cup \{\alpha \in \mathbb{C} | \text{Re}(\alpha) - \theta_1 = \pi(2k + 1), k \in \mathbb{Z}\}$. Also, we note that $m_k(z) = z + 2\pi(k - 1)$ is a bijection from S_1 to S_k . The next proposition shows the relation between the \tilde{u}_n and \tilde{u}_1 .

Proposition 3.2.4 (Sum over sheets). *Fix $n \in \mathbb{N}, \theta_1 \in \mathbb{R}, \theta_2 \in \mathbb{R}$. Let us denote the restriction to the k -th sheet by $\tilde{u}_n^{(k)} = \tilde{u}_n|_{S_k}$. Then, for $\theta_1 - \pi < \theta_2 < \theta_1 + \pi$,*

$$\tilde{u}_1(\theta_2) = \sum_{1 \leq k \leq n} \tilde{u}_n^{(k)}(m_k(\theta_2)). \quad (3.27)$$

Proof. Evident from the Figure 3.2, and the way we defined the u_n . \square

We now calculate the branch points of \tilde{u}_1 in terms of μ .

Lemma 3.2.5 (Branch points). *The relation between μ and the branch points is:*

1. $\mu < -1$: Branch points are at $\alpha = \theta_2 + \pi + 2\pi k \pm i \cosh^{-1}(-\mu), \forall k \in \mathbb{Z}$, where we take $\cosh^{-1}(-\mu) \geq 0$.
2. $-1 \leq \mu \leq 1$: Branch points are at $\alpha = \theta_2 + \cos^{-1}(\mu) + 2\pi k, \alpha = \theta_2 - \cos^{-1}(\mu) + 2\pi k, \forall k \in \mathbb{Z}$ where we take $\cos^{-1}(\mu)$ to be in $[0, \pi]$.

3. $1 < \mu$: Branch points are at $\alpha = \theta_2 + 2\pi k \pm i \cosh^{-1}(\mu), \forall k \in \mathbb{Z}$, where we take $\cosh^{-1}(\mu) \geq 0$.

Proof. Let us assume $\alpha = a + ib$ with $a, b \in \mathbb{R}$. Then,

$$\cos(\alpha - \theta_2) = \cos(a - \theta_2) \cosh(b) + i \sin(a - \theta_2) \sinh(b).$$

Now, if this is equal to μ , which is real, we obtain either $\sin(a - \theta_2) = 0$ or $\sinh(b) = 0$. In the former case we have $a = \theta_2 + \pi k, k \in \mathbb{Z}$, which results in $\mu = \pm \cosh(b)$, depending on whether k even, odd. If k is even, then, we need $\mu > 1$ giving $b = \cosh^{-1}(\mu)$. If k is odd, we need $\mu < -1$ in which case $b = \cosh^{-1}(-\mu)$. Let us now consider $\sinh(b) = 0$, which immediately results in $b = 0$, leading to $\mu = \cos(a - \theta_2)$ which has a solution iff $|\mu| \leq 1$, in which case we get case 2.

□

Locations of the branch points as calculated from Lemma 3.2.5 makes it natural to consider the three cases $\mu < -1, -1 < \mu < 1, \mu > 1$ separately. We note that μ does not depend on the angular variables. So, dividing up the spacetime based on μ alone is unlikely to give the three regions in section 3.1, which are defined using the distance function on the cone given in equation (3.5)- which in turn depends on the angles. The following lemma reconciles the two ways of dividing up spacetime.

Lemma 3.2.6 (The parameter μ and the three regions). *Let the regions 1, 2, 3 of the spacetime $([0, \infty) \times C(S^1_\rho))$ be as defined in section 3.1. Then, it is also true that region 1 is*

$$\{((r_1, \theta_1), t, (r_2, \theta_2)) | \mu > 1\} \cup \{((r_1, \theta_1), t, (r_2, \theta_2)) | -1 < \mu < 1, |\theta_1 - \theta_2| > \cos^{-1}(\mu)\},$$

region 2 is

$$\{((r_1, \theta_1), t, (r_2, \theta_2)) | -1 < \mu < 1, |\theta_1 - \theta_2| < \cos^{-1}(\mu)\},$$

and region 3 is

$$\{((r_1, \theta_1), t, (r_2, \theta_2)) | \mu < -1\}.$$

Proof. Let us take $p_i = (r_i, \theta_i), i = 1, 2$. First of all, since the definition of region 3 does not involve angles, we can immediately write down the condition in terms of μ . The definition is $t > r_1 + r_2$, which, upon squaring and simplifying, is equivalent to $\mu < -1$. We now move on to analyze the definition of regions 1 and 2 in terms of μ , and we need only bother about $\mu > -1$.

Suppose that $|\theta_1 - \theta_2| > \pi$. Then, from the definition, $d(p_1, p_2) = r_1 + r_2$. Let us now see what the definitions of the regions become in this case, $0 < t < d(p_1, p_2)$ is equivalent to $0 < t < r_1 + r_2$, which is the same as $t > 0, t^2 < r_1^2 + r_2^2 + 2r_1r_2$, which is the same as $t > 0, \mu > -1$, since we assume $r_1, r_2 > 0$. Secondly, region 2, which is same as $d((p_1, p_2)) < t < r_1 + r_2$, is an empty set.

Now, suppose that $|\theta_1 - \theta_2| < \pi$. In that case, $0 < t < d(p_1, p_2)$ becomes, $t > 0, t^2 < r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)$, which is the same as $t > 0, \cos(\theta_1 - \theta_2) < \mu$. On the other hand, the region 2, $d(p_1, p_2) < t < r_1 + r_2$, becomes $t > 0, \cos(\theta_1 - \theta_2) > \mu$ and $\mu > -1$.

□

3.2.1 The case $\mu > 1$

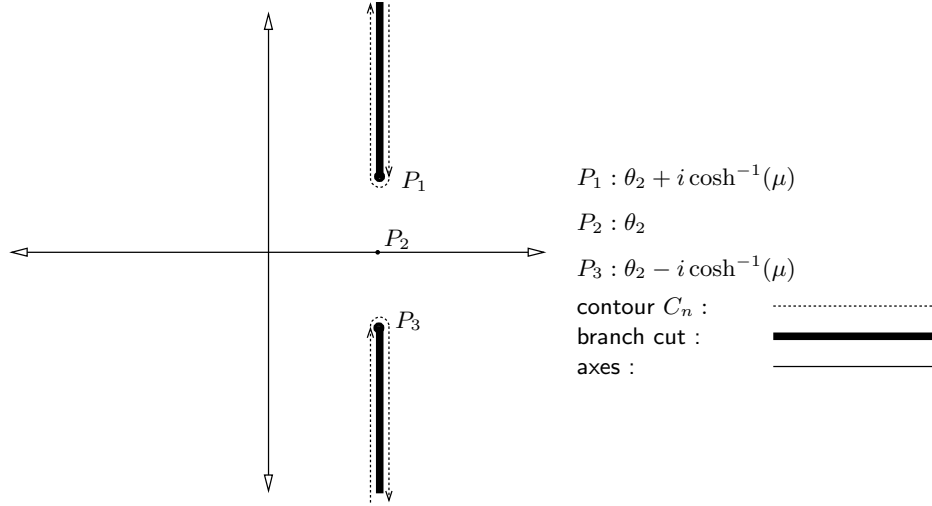
The branch points are located at $\theta_2 \pm i \cosh^{-1}(\mu) + 2\pi k, \forall k \in \mathbb{Z}$. We join these to $\pm i\infty$ respectively, along vertical lines and designate these lines as branch cuts. In our general Definition 3.2.2, we see the contour C_n consist of two disconnected pieces, going clockwise around the branch cuts at $\theta_2 \pm \cosh^{-1}(\mu)$, as shown in figure 3.3 below.

Lemma 3.2.7 (Too far from the source). *Assume that t, r_1, r_2 are such that $\mu > 1$.*

Then $\forall \theta_1, \theta_2$, we have

$$u_n((r_1, \theta_1), t, (r_2, \theta_2)) = 0.$$

Proof. The square-root term in the integral has opposite signs on both sides of the branch cut, and it is traversed in opposite directions on either side. So the integral on both sides of the cut is just double the integral on one side. Consider $w(\alpha) = -\mu + \cos(\alpha - \theta_2)$. Then, $w(\theta_2 - \pi + i\beta) = -\mu - \cosh(\beta) < 0$. By our choice of branch, $\sqrt{w(\theta_2)}$ (which is purely imaginary) is chosen to have positive imaginary part. So,

Figure 3.3: The case $\mu > 1$

$\sqrt{w(\theta)}$, $\theta \in \mathbb{R}$ is also purely imaginary with positive imaginary part. Along the vertical line $\text{Re}(\alpha) = \theta_2 - \pi + i\beta$, therefore, the same is true. Now, consider a $\beta_0 > \cosh^{-1}(\mu) > 0$. Then by Lemma 3.2.1, as x varies in $[0, \pi]$, $w(x + \theta_2 - \pi + i\beta_0)$, moves clockwise around 0 and reaches positive x axis, and encounters a branch cut. So, to the left of the upwards branch cut at $\theta_2 + \cosh^{-1}(\mu)$, $\sqrt{w(\theta_2 + i\beta)}$ is positive real, and hence to its right, negative real. By similar reasoning, we see to the left of downwards branch cut at $\theta_2 - \cosh^{-1}(\mu)$, $\sqrt{w(\theta_2 + i\beta)}$ is negative real, and to its right positive real.

These observations enable us to write:

$$\tilde{u}_n = \frac{2(2r_1r_2)^{-1/2}}{2\pi i} \int_{\cosh^{-1}(\mu)}^{+\infty} \frac{1}{2\pi \sqrt{-\mu + \cos(i\beta)}} \{f_n(\theta_2 + i\beta, \theta_1) - f_n(\theta_2 - i\beta, \theta_1)\} i d\beta. \quad (3.28)$$

Since $\beta \geq \cosh^{-1}(\mu)$, we have $\cosh(\beta) \geq \mu$, and so the quantity under the square-root is non-negative. The quantity between the braces is purely imaginary, from part (e) of Lemma 3.2.8. We conclude that \tilde{u}_n is purely imaginary. Since $u_n = \text{Re}(\tilde{u}_n)$, the lemma is proved. \square

Lemma 3.2.8 (Operations on the auxiliary function). *Suppose that $\theta, \phi, x, a, b \in \mathbb{R}$. Then, the following identities hold true.*

$$(a) \quad \frac{1}{1 - e^{i(\theta+\phi)}} - \frac{1}{1 - e^{i(\theta-\phi)}} = \frac{-i \sin(\phi)}{\cos(\theta) - \cos(\phi)}, \quad (3.29)$$

$$(b) \quad \frac{1}{1 - e^{i\theta}e^x} = \frac{e^{-x} - e^{-i\theta}}{2(\cosh(x) - \cos(\theta))}, \quad (3.30)$$

$$(c) \quad f_n(a + ib, \theta_1) = \left(\frac{i}{n}\right) \frac{e^{-\frac{b}{n}} - e^{-i(\frac{\theta_1-a}{n})}}{2\left(\cosh\left(\frac{b}{n}\right) - \cos\left(\frac{\theta_1-a}{n}\right)\right)}, \quad (3.31)$$

$$(d) \quad \text{Im} [f_n(a + ib, \theta_1) + f_n(a - ib, \theta_1)] = \frac{1}{n}, \quad (3.32)$$

$$(e) \quad f_n(a + ib, \theta_1) - f_n(a - ib, \theta_1) = -\left(\frac{i}{n}\right) \frac{\sinh\left(\frac{b}{n}\right)}{\left(\cosh\left(\frac{b}{n}\right) - \cos\left(\frac{\theta_1-a}{n}\right)\right)}, \quad (3.33)$$

$$(f) \quad \text{Re} [f_n(a + ib, \theta_1) + f_n(a - ib, \theta_1)] = \left(\frac{1}{n}\right) \frac{-\sin\left(\frac{\theta_1-a}{n}\right)}{\left(\cosh\left(\frac{b}{n}\right) - \cos\left(\frac{\theta_1-a}{n}\right)\right)}. \quad (3.34)$$

Proof. The left side of (a) simplifies to $\frac{-e^{i(\theta-\phi)} + e^{i(\theta+\phi)}}{1 - (e^{i(\theta+\phi)} + e^{i(\theta-\phi)}) + e^{2i\theta}}$, which is same as $\frac{e^{i\phi} - e^{-i\phi}}{e^{i\theta} + e^{-i\theta} - (e^{i\phi} + e^{-i\phi})}$, the right side of (a). Now, in (b), the left side is equal to $\frac{(1 - e^{-i\theta}e^x)}{(1 - e^{i\theta}e^x)(1 - e^{-i\theta}e^x)}$, which is same as $\frac{(1 - e^{-i\theta}e^x)}{(1 - e^x(e^{i\theta} + e^{-i\theta}) + e^{2x})}$ and this simplifies to the right side of (b) on dividing numerator and denominator by e^{-x} .

To prove (c), we observe from f_n in equation (3.19) that $f_n(a + ib, \theta_1) \frac{n}{i}$ is of the form of the left side of (b) with $x = \frac{b}{n}, \theta = \frac{\theta_1-a}{n}$. To prove (d), notice that $\text{Im}(f_n(a + ib, \theta_1)) = \left(\frac{1}{n}\right) \frac{e^{-\frac{b}{n}} - \cos\left(\frac{\theta_1-a}{n}\right)}{2\left(\cosh\left(\frac{b}{n}\right) - \cos\left(\frac{\theta_1-a}{n}\right)\right)}$. From this (d) is easily deduced. Part (f) follows easily from (c) like in (d). \square

3.2.2 The case $-1 < \mu < 1$

The branch points are real and located at $\theta_2 \pm \cos^{-1}(\mu) + 2\pi k, \forall k \in \mathbb{Z}$. We take branch cuts to be vertical lines to $\pm\infty$ respectively. So, C_n will be the contour that

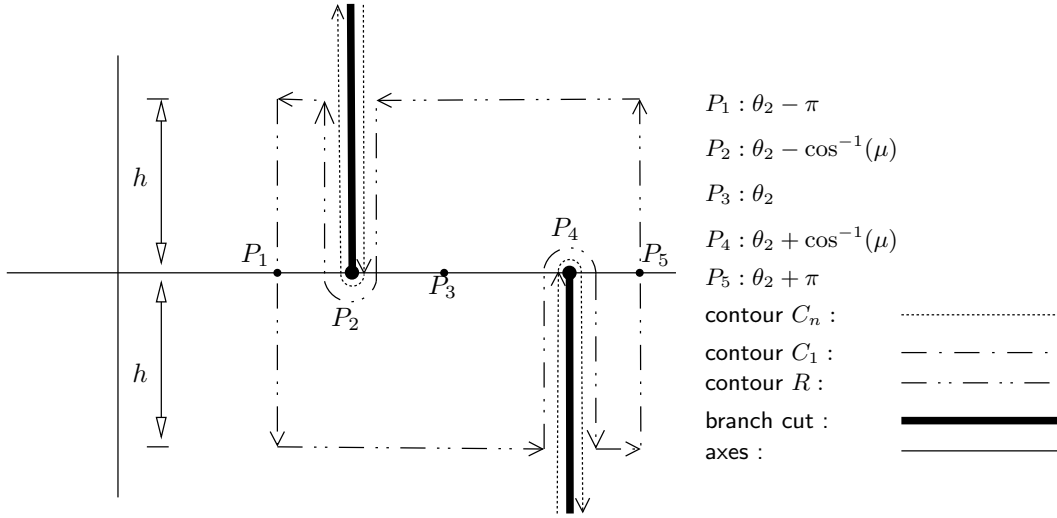


Figure 3.4: The case $-1 < \mu < 1$, sub-case 1 : $|\theta_1 - \theta_2| \bmod 2\pi n > \pi$. We let $h \rightarrow \infty$.

consists of two pieces, going clockwise around the branch cuts at $\theta_2 \pm \cos^{-1}(\mu)$, as shown in the figure.

Lemma 3.2.9 (Signals directly from the source). *Let $\theta_1, \theta_2 \in \mathbb{R}$, and $0 < \cos^{-1}(\mu) < \pi$. If $\forall k \in \mathbb{Z}, |\theta_1 + 2\pi nk - \theta_2| > \cos^{-1}(\mu)$, then $u_n((r_1, \theta_1), t, (r_2, \theta_2)) = 0$. If $\exists k \in \mathbb{Z}, |\theta_1 + 2\pi nk - \theta_2| < \cos^{-1}(\mu)$, then $u_n((r_1, \theta_1), t, (r_2, \theta_2)) = u_1((r_1, \theta_1), t, (r_2, \theta_2))$, the standard 2π periodic solution.*

Proof. We start by computing the integral along a vertical path $\alpha = x + i\beta$ from $\beta = -\infty$ to $\beta = +\infty$. Assume that this path does not hit any branch points or poles.

Let

$$I(x) := \frac{1}{2\pi i} \int_{\beta=-\infty}^{\beta=+\infty} \tilde{u}_1(x + i\beta) f_n(x + i\beta, \theta_1) i d\beta \quad (3.35)$$

Now, $\tilde{u}(\theta_2 \pm \pi + i\beta) = \frac{1}{2\pi} \frac{(2r_1 r_2)^{\frac{1}{2}}}{\sqrt{-\mu - \cosh(\beta)}}$. The precise sign of the square root can be determined. On the line $\alpha = \theta_2 + iy, y \in \mathbb{R}$, $\tilde{u}_1(\alpha)$ is positive real. Pick any $y_0 > 0$. Then, by Lemma 3.2.1 the horizontal path $\alpha = x + iy_0, x \in [\theta_2, \theta_2 + \pi]$ when mapped onto $w = -\mu + \cos(\alpha - \theta_2)$ starts on the positive real axis and moves clockwise on an ellipse and ends on the negative real axis, and so, since the square-root is positive at the starting point and the path α did not encounter a branch cut, we can deduce that on this branch of $\tilde{u}_1(\alpha), \sqrt{w(\theta_2 + \pi + iy_0)}$ is purely imaginary with a negative imaginary

part. So, we see $\tilde{u}(\theta_2 + \pi + i\beta) = \frac{1}{2\pi i} \frac{i(2r_1 r_2)^{\frac{1}{2}}}{\sqrt{+\mu + \cosh(\beta)}}$. By 2π periodicity of \tilde{u}_1 , $\tilde{u}_1(\theta_2 - \pi + i\beta)$ has the same value.

We can rewrite

$$I(x) = \frac{1}{2\pi i} \int_{\beta=0}^{+\infty} \{\tilde{u}_1(x + i\beta) f_n(x + i\beta, \theta_1) + \tilde{u}_1(x - i\beta) f_n(x - i\beta, \theta_1) i d\beta\}. \quad (3.36)$$

Since $\tilde{u}_1(\theta_2 + \pi - i\beta) = \tilde{u}_1(\theta_2 + \pi + i\beta)$, we see

$$I(\theta_2 + \pi) = \frac{1}{2\pi i} \int_0^{\infty} \frac{1}{2\pi} \frac{i(2r_1 r_2)^{\frac{1}{2}}}{\sqrt{+\mu + \cosh(\beta)}} \{f_n(\theta_2 + \pi + i\beta) + f_n(\theta_2 + \pi - i\beta)\} i d\beta. \quad (3.37)$$

The equations (3.37) is unchanged if $\theta_2 + \pi$ is replaced by $\theta_2 - \pi$. So by equation (3.2.8), we see

$$\operatorname{Re}(I(\theta_2 \pm \pi)) = \frac{-(2r_1 r_2)^{\frac{1}{2}}}{4\pi^2 n} \int_0^{\infty} \frac{1}{\sqrt{\mu + \cosh(\beta)}} d\beta. \quad (3.38)$$

Sub-case 1:

Assume that θ_2 doesn't lie in the same sheet as the source angle θ_1 . That is

$$\forall k \in \mathbb{Z}, |\theta_2 - \theta_1 + 2\pi n k| > \pi. \quad (3.39)$$

This case is illustrated in figure 3.4.

Consider a ‘‘rectangular’’ loop of height h above and below the x -axis, such that the vertical sides are along $\operatorname{Re}(\alpha) = \theta_2 \pm \pi$, horizontal sides are along $y = \pm h$, but going around the two branch cuts. Let $h \rightarrow \infty$. From Lemma 3.2.3, we see that the contribution from the horizontal portions of the contour goes to zero. Also, by the hypothesis for this case, there are no poles or branch points of the integrand within this loop. So we get

$$\tilde{u}_n - I(\theta_2 - \pi) + I(\theta_2 + \pi) = 0. \quad (3.40)$$

Taking real part of this equation and using equation (3.39), we see that

$$u_n(\theta_2) = \operatorname{Re}(\tilde{u}_n(\theta_2)) = 0 \quad (3.41)$$

Sub-case 2:

Let us assume that θ_2 is in the same sheet as the source angle θ_1 . That is,

$$\exists k, |\theta_1 + 2\pi n k - \theta_2| < \pi. \quad (3.42)$$

This case is illustrated in figure 3.5. Since the integrand is $2\pi n$ periodic, without loss of generality, we can assume $k = 0$. In particular, this means that there is a pole on the real interval between the two branch points on whose branch cuts we integrate to get the solution u_n . So, the equation (3.38) in case 1 won't hold true in this case.

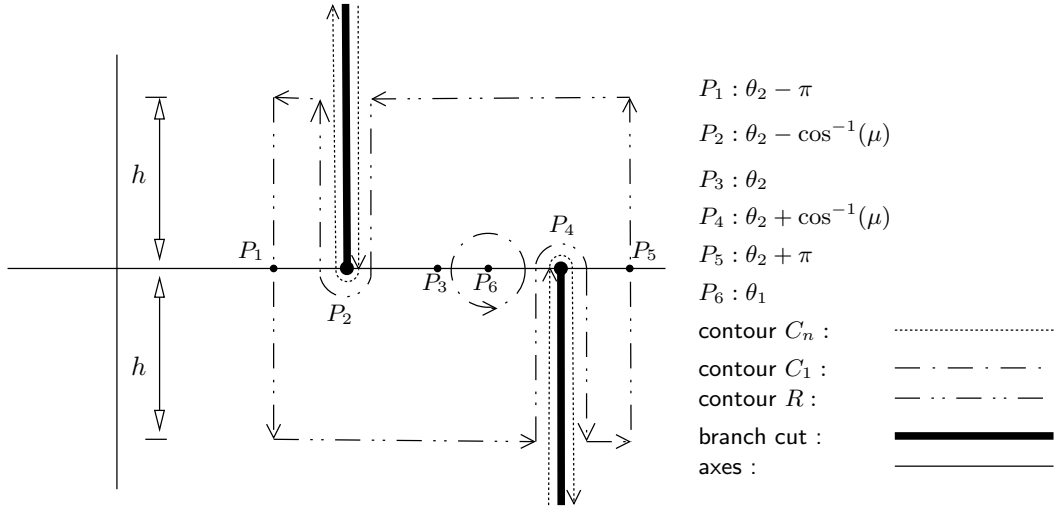


Figure 3.5: The case $-1 < \mu < 1$, sub-case 2 : $|\theta_1 - \theta_2| \bmod 2\pi n < \pi$. We let $h \rightarrow \infty$.

However, we may use the sum-over-sheets Proposition 3.2.4. We note that for $1 < k \leq n$, $|\theta_2 + (k-1)2\pi - \theta_1| > |(k-1)2\pi - |(\theta_1 - \theta_2)|| > 2\pi - \pi = \pi$. To rephrase, the points over θ_2 in other sheets satisfy the hypothesis for case 1. That is, for $1 < k \leq n$,

$$\forall l \in \mathbb{Z} : |m_k(\theta_2) - \theta_1 + 2\pi nl| > \pi. \quad (3.43)$$

So, by case 1,

$$\forall 1 < k \leq n : u_n(\theta_2 + 2(k-1)\pi) = 0. \quad (3.44)$$

The sum-over-sheets proposition 3.27 enables us to deduce the conclusion of the lemma. \square

3.2.3 The case $\mu < -1$

The branch points are at $\theta_2 \pm \pi \pm \cosh^{-1}(-\mu)i + 2\pi k, \forall k \in \mathbb{Z}$. We take branch cuts to be vertical lines to $i\infty$ or $-i\infty$ depending on whether the imaginary part is positive

or negative respectively. By definition, the contour path C_n in this case consists of four disconnected pieces.

The four parts are denoted $C_n^{ul}, C_n^{ur}, C_n^{dl}, C_n^{dr}$ with the superscript used to indicate the position of the path relative to θ_2 , by using u, d, r, l to stand for up, down, right, left respectively. Now, C_n^{ul} goes along $\text{Re}(\alpha) = \theta_2 - \pi$, to its right, starting at $\theta_2 - \pi + i\infty$ and coming down to $\theta_2 - \pi + \cosh^{-1}(-\mu)i$. Similarly, C_n^{dl} starts goes along $\text{Re}(\alpha) = \theta_2 - \pi$, to its right, starting at $\theta_2 - \pi - \cosh^{-1}(-\mu)i$ going down to $\theta_2 - \pi - i\infty$. In the same way, C_n^{dr} starts at $\theta_2 + \pi - i\infty$ goes up along $\text{Re}(\alpha) = \theta_2 + \pi$ to its left, until $\theta_2 + \pi - i \cosh^{-2}(-\mu)$. Finally, C_n^{ur} starts at $\theta_2 + \pi + i \cosh^{-1}(-\mu)$ and goes up along $\text{Re}(\alpha) = \theta_2 + \pi$ to its left, up to $\theta_2 + \pi + i\infty$.

Sub-case 1: Test point not in the same sheet as the source

We know that the test point is not in the same sheet as the source. That is,

$$\forall k \in \mathbb{Z}, |\theta_2 - \theta_1 - 2\pi nk| > \pi. \quad (3.45)$$

This case is illustrated in figure 3.6.

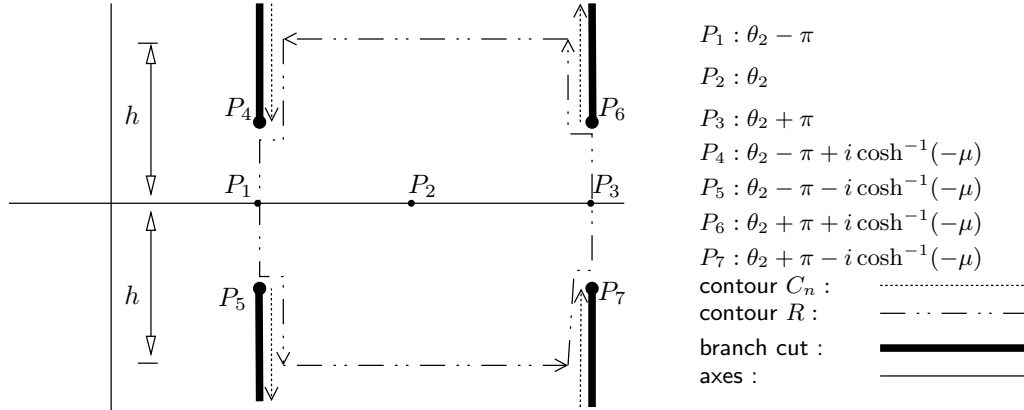


Figure 3.6: The case $\mu < -1$, sub-case 1 : $|\theta_1 - \theta_2| \bmod 2\pi n > \pi$. We let $h \rightarrow \infty$.

By this assumption the open vertical strip $\{\alpha | \theta_2 - \pi < \text{Re}(\alpha) < \theta_2 + \pi\}$ contains no poles (because poles are at $\theta_1 + 2\pi nk, k \in \mathbb{Z}$) and no branch points. On its boundary there are four branch points $\theta_2 \pm \pi \pm \cosh^{-1}(-\mu)$, and the branch cuts go along the boundary to infinity.

Consider the counter-clockwise rectangular loop R inside the strip, such that the horizontal sides go along $Im(\alpha) = \pm h$, and vertical sides along $Re(\alpha) = \theta_2 \pm \pi$. Since the integrand is analytic inside the loop, we have

$$\frac{1}{2\pi i} \int_R \tilde{u}_1(\alpha) f_n(\alpha, \theta_1) d\alpha = 0. \quad (3.46)$$

The loop skips the branch points by going around it, but goes to zero as their radius goes to zero. As the height h is taken to infinity, the contribution from the horizontal portions goes to zero, by Lemma 3.2.3.

Let R_b consist of two disconnected vertical segments R_b^l, R_b^r of the loop R joining the branch points, such that R_b^l goes down from $\theta_2 - \pi + i \cosh^{-1}(-\mu)$ to $\theta_2 - \pi - i \cosh^{-1}(-\mu)$ and R_b^r goes up from $\theta_2 + \pi - i \cosh^{-1}(-\mu)$ to $\theta_2 + \pi + i \cosh^{-1}(-\mu)$.

Rewriting equation (3.46) in the limit as h goes to infinity, we get

$$\tilde{u}_n(\theta_1) + \frac{1}{2\pi i} \int_{R_b} \tilde{u}_1(\alpha) f_n(\alpha, \theta_1) d\alpha = 0. \quad (3.47)$$

Taking the real part, we see that

$$u_n(\theta_1) = -\frac{1}{2\pi} Im \left\{ \int_{R_b} \tilde{u}_1(\alpha) f_n(\alpha, \theta_1) d\alpha \right\}. \quad (3.48)$$

Now, we split the integral on the right side as

$$\int_{R_b} \tilde{u}_1(\alpha) f_n(\alpha, \theta_1) d\alpha = \left(\int_{R_b^l} + \int_{R_b^r} \right) \tilde{u}_1(\alpha) f_n(\alpha, \theta_1) d\alpha. \quad (3.49)$$

We have

$$\int_{R_b^l} \tilde{u}_1(\alpha) f_n(\alpha, \theta_1) d\alpha = - \int_{-\cosh^{-1}(-\mu)}^{\cosh^{-1}(-\mu)} \tilde{u}_1(\theta_2 - \pi + i\beta) f_n(\theta_2 - \pi + i\beta) i d\beta. \quad (3.50)$$

Since $\tilde{u}_1(\theta_2 - \pi \pm i\beta) = \frac{(2r_1 r_2)^{1/2}}{2\pi} \frac{1}{\sqrt{-\mu + \cos(\theta_2 - \pi \pm i\beta - \theta_2)}}$ and we want $\tilde{u}_1(\theta_2)$ to be positive real, we have for $|\beta| < \cosh^{-1}(-\mu)$,

$$\tilde{u}_1(\theta_2 - \pi \pm i\beta) = \frac{(2r_1 r_2)^{1/2}}{2\pi} \frac{1}{\sqrt{-\mu - \cosh(\beta)}}, \quad (3.51)$$

and so,

$$\int_{R_b^l} \tilde{u}_1(\alpha) f_n(\alpha, \theta_1) d\alpha = - \int_0^{\cosh^{-1}(-\mu)} \frac{(2r_1 r_2)^{1/2}}{2\pi} \frac{1}{\sqrt{-\mu - \cosh(\beta)}} S(\beta) i d\beta, \quad (3.52)$$

where we use the temporary variable

$$S(\beta) = \{f_n(\theta_2 - \pi + i\beta) + f_n(\theta_2 - \pi - i\beta)\}. \quad (3.53)$$

Since the square-root term is real and there being an $i = \sqrt{-1}$ already in the integrand, the imaginary part of the integral involves the real part of $S(\beta)$. We compute that, using equation (3.34),

$$\operatorname{Re}(S(\beta)) = \left(\frac{1}{n}\right) \frac{-\sin\left(\frac{\theta_1 - \theta_2 + \pi}{n}\right)}{\left(\cosh\left(\frac{\beta}{n}\right) - \cos\left(\frac{\theta_1 - \theta_2 + \pi}{n}\right)\right)}. \quad (3.54)$$

Finally, we see that

$$\begin{aligned} & \operatorname{Im} \left(\int_{R_b^l} \tilde{u}_1(\alpha) f_n(\alpha) d\alpha \right) = \\ & -\frac{(2r_1 r_2)^{1/2}}{2\pi n} \int_0^{\cosh^{-1}(-\mu)} \frac{1}{\sqrt{-\mu - \cosh(\beta)}} \frac{-\sin\left(\frac{\theta_1 - \theta_2 + \pi}{n}\right)}{\left(\cosh\left(\frac{\beta}{n}\right) - \cos\left(\frac{\theta_1 - \theta_2 + \pi}{n}\right)\right)} d\beta. \end{aligned} \quad (3.55)$$

Moving on the right integral along R_b^r , we note that equations (3.50),(3.51),(3.53), (3.54) hold true, when R_b^l, π are replaced by $R_b^r, -\pi$, respectively, and the minus sign in the front is dropped (which accounts for the change in direction of integration). So,

$$\begin{aligned} & \operatorname{Im} \left(\int_{R_b^r} \tilde{u}_1(\alpha) f_n(\alpha) d\alpha \right) = \\ & \frac{(2r_1 r_2)^{1/2}}{2\pi n} \int_0^{\cosh^{-1}(-\mu)} \frac{1}{\sqrt{-\mu - \cosh(\beta)}} \frac{-\sin\left(\frac{\theta_1 - \theta_2 - \pi}{n}\right)}{\left(\cosh\left(\frac{\beta}{n}\right) - \cos\left(\frac{\theta_1 - \theta_2 - \pi}{n}\right)\right)} d\beta. \end{aligned} \quad (3.56)$$

In order that coefficient of π inside the sine is positive, may take the negative sign inside. In equation (3.55), on the other hand, the two minus signs, one outside the integral and one inside next to sine, cancel.

So using the previous two equations (3.55), (3.56), via equation (3.49) in equation (3.48), we get

$$u_n((r_1, \theta_1), t, (r_2, \theta_2)) = -\frac{(2r_1 r_2)^{1/2}}{4\pi^2 n} \int_0^{\cosh^{-1}(-\mu)} \frac{1}{\sqrt{-\mu - \cosh(\beta)}} S(s, \theta_1, \theta_2, n) d\beta \quad (3.57)$$

where,

$$S(s, \theta_1, \theta_2, n) = \frac{\sin\left(\frac{\theta_1 - \theta_2 + \pi}{n}\right)}{\left(\cosh\left(\frac{\beta}{n}\right) - \cos\left(\frac{\theta_1 - \theta_2 + \pi}{n}\right)\right)} + \frac{\sin\left(\frac{-\theta_1 + \theta_2 + \pi}{n}\right)}{\left(\cosh\left(\frac{\beta}{n}\right) - \cos\left(\frac{-\theta_1 + \theta_2 + \pi}{n}\right)\right)}. \quad (3.58)$$

This completes the proof in the case where the test point doesn't lie in the same sheet as the source point. We remark that this matches equation (3.12), because $K_1^{(3)} = 0$ by the hypothesis of this case, $\rho = n$, as the radius of a circle at a distance 1 from the cone tip is $2\pi n$, and the i in the denominator in (3.12) comes from taking -1 out of the square-root in (3.57).

Sub-case 2: The test point is in the same sheet as the source point

In this case, we assume that there the test point is in the same sheet as the source point. That is,

$$\exists k \in \mathbb{Z} : |\theta_2 - \theta_1 - 2\pi k| < \pi. \quad (3.59)$$

This case is illustrated in figure 3.7.

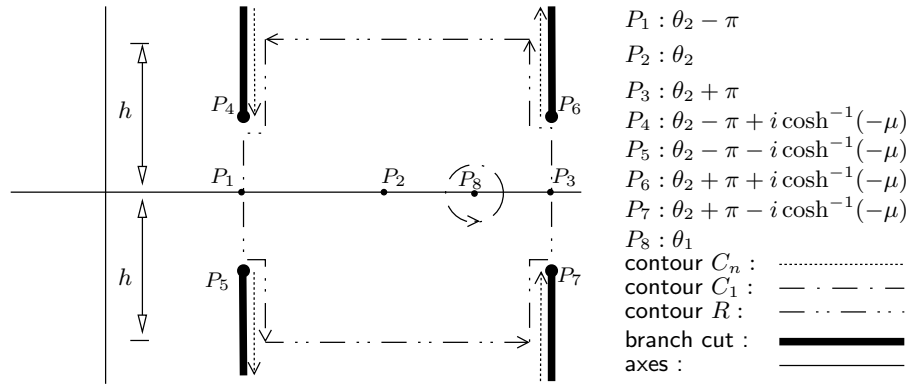


Figure 3.7: The case $\mu < -1$, sub-case 2: $|\theta_1 - \theta_2| \bmod 2\pi n < \pi$. We let $h \rightarrow \infty$.

Without loss of generality we may assume that $k = 0$ above. As in case 1 we consider the rectangular loop R . Now, there is a pole in the open vertical strip $\{\theta_2 - \pi < \text{Re}(\alpha) < \theta_2 + \pi\}$ at θ_1 . Since that is the only pole in the interior of the region specified by the loop R and since the auxiliary function $f_n(\alpha, \theta_1)$ has residue 1 at the pole, by the residue theorem, we see that the analogue of equation (3.46) in the current case is

$$\frac{1}{2\pi i} \int_R \tilde{u}_1(\alpha) f_n(\alpha, \theta_1) d\alpha = \tilde{u}_1(\theta_1). \quad (3.60)$$

From here on, the steps are exactly the same as in case 1. Letting $h \rightarrow \infty$, taking the real part and noting that $\text{Re}(\tilde{u}_1(\theta_1)) = u_1(\theta_1)$, the standard 2π periodic solution, we get

the analogue of equation (3.48)

$$u_n((r_1, \theta_1), t, (r_2, \theta_2)) = u_1((r_1, \theta_1), t, (r_2, \theta_2)) - \operatorname{Re} \left\{ \frac{1}{2\pi i} \int_R \tilde{u}_1(\alpha) f_n(\alpha, \theta_1) d\alpha \right\} \quad (3.61)$$

The second term (including the minus sign) on the right side is precisely the integral we evaluated in case 2, and therefore it simplifies to the right side of equation (3.57)

Noting that $K_1^{(3)} = u_1((r_1, \theta_1), t, (r_2, \theta_2))$, because only $j = 0$ is active in equation (3.12), and that $\rho = n$, we note that equation (3.61) matches that given by (3.12). This completes the proof in case 2.

3.3 Solution on a cone over a circle of circumference $2\pi\rho, \rho \notin \mathbb{N}$

In Theorem 3.1.1, we showed that the fundamental solution to the wave equation on a cone over circle of circumference $2\pi n$ could be obtained using Sommerfeld's method. Here, we will address the case when the circumference is $2\pi\rho, \rho \in \mathbb{Q}$. The main message is that using the fundamental solutions with period $2\pi k$, on application of the image method one obtains the fundamental solution with period $2\pi \frac{k}{l}$. This is stated in Proposition 3.3.1. As for $\rho \notin \mathbb{Q}$, the only inference we make is simply that once we obtain solution for $\rho \in \mathbb{Q}$, that same formula works in the case of $\rho \notin \mathbb{Q}$.

In order to convey what we mean by image method we will use the following example. Suppose we have $u_1((r_1, \theta_1), t, (r_2, \theta_2))$, a fundamental solution that is periodic in the angles with period 2π and has a source at (r_1, θ_1) , and that we want to find a fundamental solution that is $\frac{2\pi}{3}$ periodic in the angles. Then the image method dictates, given a source point (r_1, θ_1) , that such a solution is given by summing three 2π -periodic solutions with sources at the images of the actual source point (r_1, θ_1) . That is,

$$u_{2/3}((r_1, \theta_1), t, (r_2, \theta_2)) = \sum_{j=0}^{j=2} u_1 \left((r_1, \theta_1 + \frac{2\pi}{3}j), t, (r_2, \theta_2) \right). \quad (3.62)$$

Proposition 3.3.1. *Suppose u_ρ indicate the fundamental solution on $C(S_\rho^1)$ given by formulae (3.10), (3.11), (3.12). Suppose also that k, l are two natural numbers with $\gcd(k, l) = 1$. Then, the following equality holds true:*

$$u_{k/l}((r_1, \theta_1), t, (r_2, \theta_2)) = \sum_{j=0}^{l-1} u_k \left((r_1, \theta_1 + 2\pi \frac{k}{l}j), t, (r_2, \theta_2) \right). \quad (3.63)$$

Proof. We consider three cases, one by one. Let us indicate by $p_i, i = 1, 2$ the points (r_i, θ_i) on $C(S_{k/l}^1)$ respectively. Also, let us denote $(r_1, \theta_1 + 2\pi \frac{k}{l} j)$ by p_1^j . First, (p_1, t, p_2) is in region 1 of $[0, \infty) \times C(S_{k/l}^1)$, so that $t < d(p_1, p_2)$. It can be seen by using the definition of the distance function that $d(p_1, p_2) \leq d(p_1^j, p_2)$, where the first one is the distance on $C(S_{k/l}^1)$ and the second is the distance on $C(S_k^1)$. So, all (p_1^j, t, p_2) lie in region 1 on $[0, \infty) \times C(S_k^1)$. Therefore, both sides of equation (3.63) evaluate to zero in this case.

Now, suppose that (p_1, t, p_2) is in region 2 on $[0, \infty) \times C(S_{k/l}^1)$, so that $d(p_1, p_2) < t < r_1 + r_2$. In this case, by definition of the distance function $|\theta_1 - \theta_2| \bmod 2\pi \frac{k}{l} < \pi$. Now, if $d(p_1^j, p_2) = r_1 + r_2$, then (p_1^j, t, p_2) is in region 1 of $[0, \infty) \times C(S_k^1)$ and the solution is zero there. If not, $d(p_1^j, p_2) = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 + 2\pi \frac{k}{l} j - \theta_2)$, and using that (p_1^j, t, p_2) belongs to region 1 of $[0, \infty) \times C(S_k^1)$ if $\cos(\theta_1 + 2\pi \frac{k}{l} j - \theta_2) < \mu$, we see that u_k evaluates to zero, and otherwise to region 2. So, the right hand side of equation (3.63) evaluates to
$$\sum_{j: 0 < |\theta_1 + 2\pi \frac{k}{l} j - \theta_2| < \cos^{-1}(\mu)} u_k(p_1^j, t, p_2).$$
 Since the evaluation on the right is on points that are in region 2, we can be seen from formula for region (3.11) that $u_k(p_1^j, t, p_2)$ is equal to $u_1(p_1^j, t, p_2)$. Thus, even in this case equation (3.63) holds true.

Finally, assume that (p_1, t, p_2) belongs to region 3 on $[0, \infty) \times C(S_{k/l}^1)$. Then, $t > r_1 + r_2$. So, all the points (p_1^j, t, p_2) belong to region 3 on $[0, \infty) \times C(S_k^1)$. Then, it is easily seen that the term $K_1^3(p_1, t, p_2)$ in (3.12) for $\rho = k/l$ is just the sum $\sum_{j=0}^{l-1} K_1^3(p_1^j, t, p_2)$ with $\rho = k$. Now, to deal with the other term, we start with the following identity, where we assume $z \in \mathbb{C}$ is such that both sides of the equation are defined, and $l \in \mathbb{N}$:

$$\sum_{j=0}^{l-1} \frac{1}{1 - e^{i2\pi \frac{j}{l}} z} = \frac{l}{1 - z^l}. \quad (3.64)$$

Let us introduce the notation

$$C_1^j(\rho) = \frac{\sin((\theta_1 + 2\pi \frac{k}{l} j - \theta_2 + \pi)\rho^{-1})}{\cosh(s\rho^{-1}) - \cos((\theta_1 - \theta_2 + \pi)\rho^{-1})}. \quad (3.65)$$

As we have seen earlier, while dealing with the case $\mu < -1$, in equation (3.54),

$$C_1^j(\rho) = -\rho \operatorname{Re}\{f_\rho\left(\theta_2 - \pi + is, \theta_1 + 2\pi \frac{k}{l} j\right) + f_\rho\left(\theta_2 - \pi - is, \theta_1 + 2\pi \frac{k}{l} j\right)\}. \quad (3.66)$$

Here, we have used f_ρ to mean the same expression for f_n with n replaced by ρ . Now,

$$\sum_{j=0}^{l-1} f_k \left(\theta_2 \pm \pi + is, \theta_1 + 2\pi \frac{k}{l} j \right) = \frac{i}{k} \sum_{j=0}^{l-1} \frac{1}{1 - e^{2\pi \frac{i}{l}} e^{\frac{1}{k}(\theta_2 \pm \pi + is - \theta_1)}} \quad (3.67)$$

$$= \frac{i}{k} \frac{l}{1 - e^{\frac{1}{k}(\theta_2 \pm \pi + is - \theta_1)}} \quad (3.68)$$

$$= f_{k/l}(\theta_2 \pm \pi + is, \theta_1) \quad (3.69)$$

From this we see that

$$\sum_{j=0}^{l-1} \frac{1}{k} C_1^j(k) = \frac{l}{k} C_1^0(k/l). \quad (3.70)$$

The previous steps can be repeated with π in $C_1^j(\rho)$ replaced by $-\pi$. Therefore, we see that in equation (3.12),

$$K_2^3 \left(p_1, t, p_2, \rho = \frac{k}{l} \right) = \sum_{j=0}^{l-1} K_2^3(p_1^j, t, p_2, \rho = k) \quad (3.71)$$

This completes the proof. □

Chapter 4

Spin- $\frac{1}{2}$ fields on charged particle-spacetimes

In classical non-relativistic quantum mechanics, the wave function ψ of an electron in the presence of an externally generated potential energy field $V(t, x)$, obeys Schrödinger's equation,

$$i\hbar\partial_t\psi(t, x) = \left(-\frac{1}{2m}\hbar^2\Delta + V(t, x)\right)\psi(t, x), \quad (4.1)$$

where $\psi(t, x) : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}$, \hbar is Planck's constant divided by 2π , and m is the mass of the electron. In special relativistic quantum mechanics, the wave function of an electron obeys Dirac's equation on Minkowski space:

$$i\hbar\partial_t\psi = i\hbar c(\alpha \cdot \nabla)\psi + mc^2\beta\psi + V(t, x)\psi, \quad (4.2)$$

where, $\psi(t, x) : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}^4$, c is the speed of light in vacuum, m is the mass of the electron, $V(t, x)$ is a 4×4 matrix that represents the effects of the external potential if it is present, $\alpha = (\alpha^1, \alpha^2, \alpha^3)$ and β are four 4×4 matrices which satisfy

$$\alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij}, \forall i, j \in \{1, 2, 3\} \quad (4.3)$$

$$\alpha^i\beta + \beta\alpha^i = 0 \quad \forall i \in \{1, 2, 3\} \quad (4.4)$$

$$\beta^2 = 1 \quad (4.5)$$

An example of such matrices are the Dirac matrices given in (4.76). The right side of equation (4.2) is called the Dirac Hamiltonian, indicated with the letter H_0 . It is studied as an unbounded operator on $L^2(\mathbb{R}^3, \mathbb{C}^4)$. The spectral properties of H_0 can be obtained using the Fourier transform. The spectrum is just $(-\infty, -mc^2] \cup [mc^2, \infty)$. One can also write down the unitary map that converts H_0 into a multiplication operator.

An important situation is when an external field is present, for instance, a Coulomb potential. In this case, one is interested in the spectral properties of $H_0 - \frac{\gamma}{|x|}$, $\gamma \in \mathbb{R}$.

Since the potential is spherically symmetric, it is natural to separate the operator into radial and angular parts. The radial Dirac Hamiltonian (or reduced Dirac Hamiltonian) has the form

$$H_{\kappa}^{red} := \begin{pmatrix} mc^2 + \frac{\gamma}{r} & -\frac{d}{dr} + \frac{\kappa}{r} \\ \frac{d}{dr} + \frac{\kappa}{r} & -mc^2 + \frac{\gamma}{r} \end{pmatrix}, \quad (4.6)$$

where κ is the eigenvalue of an angular momentum (“spin-orbit”) operator, and H_{κ}^{red} is an operator on $L^2(0, \infty)^2$. It is initially defined only on $C_c^{\infty}(0, \infty)^2$. This symmetric operator has a self-adjoint extension uniquely if and only if $|\gamma| \leq \frac{\sqrt{3}}{2}$, when \hbar, c, m are taken to be 1 (Theorem 2.1.6 of [2]). The essential spectrum is still that of the free Dirac Hamiltonian, namely, $(-\infty, -mc^2] \cup [mc^2, \infty)$. For such values of γ , in the gap of the essential spectrum there are eigenvalues. These are explicitly computable.

In general-relativistic quantum mechanics, Dirac’s equation is posed on a curved spacetime and there are several formulations on how to define the spinor-connection. Once properly formulated, we can ask questions about essential self-adjointness and the spectrum. For instance Finster, Smoller, and Yau in [16] consider the Dirac equation on a Reissner-Nordström (RN) black hole background and prove that there are no normalizable time periodic solutions. In particular, there are no bound states. In [12], Cohen and Powers show that on the RN black hole background, the Dirac Hamiltonian is essentially self adjoint, the essential spectrum is the whole real line, and there are no eigenvalues. They also find that in the naked case, there are multiple self-adjoint extensions, all of which have the same essential spectrum. They conjecture that there are also eigenvalues in the gap. Belgiorno in [3] produces similar results.

In this chapter, we prove some results about the Dirac Hamiltonian on charged particle-spacetimes. First in section, 4.1, we set up Dirac’s equation using the frame formalism of Cartan. Then in section 4.2, we perform the separation of variables and decompose the Hilbert space into partial wave subspaces on which the Dirac Hamiltonian reduces to a first order differential operator on a wave function with two components. In section 4.3, we use a version of Weyl’s limit-point/limit-circle theory to show that the Dirac Hamiltonian is essentially self-adjoint on functions that are compactly supported away from the origin. We are helped by the fact that unlike in the Reissner-Nordström

case, the metric coefficients do not blow up. In section 4.4, we implement a standard perturbation argument and determine that the essential spectrum is the same as in the free case. In section 4.5, we show that the resolvent set contains a neighbourhood of the origin. Finally, in section 4.6, we use oscillation properties of certain ordinary differential equations to show that the gap in the essential spectrum contains infinitely many eigenvalues.

4.1 Dirac's equation using Cartan's formalism

4.1.1 Definition of Dirac's equation

In this section we derive Dirac's equation on a spherically symmetric spacetime. We use the so called *frame formalism* of Cartan. Our principal reference is a paper by Brill and Cohen [6]. We have seen this method implemented in other references such as, Cohen and Powers [12], and more recently by Kiessling and Tahvildar-Zadeh [20] while studying Dirac's equation on zero gravity ($G \rightarrow 0$) Kerr-Newman spacetimes .

We first choose an orthonormal co-frame consisting of four one forms $\omega^{(\mu)}$, $\mu = 0, 1, 2, 3$.

Applying the exterior differential operator to these one-forms and writing the resulting two-forms as wedge combinations of the frame forms, we arrive at the sixteen connection one-forms, denoted $\Omega_{\kappa}^{(\mu)}$, $0 \leq \mu, \kappa \leq 3$. That is,

$$d\omega^{(\mu)} = -\Omega_{\kappa}^{(\mu)} \wedge \omega^{(\kappa)}, \quad (4.7)$$

where we have used the Einstein summation convention (that indices that appear on the subscripts and superscripts are to be summed over). In [6], it is explained that if one demands skew-symmetry

$$\Omega_{\mu\kappa} = -\Omega_{\kappa\mu} \quad (4.8)$$

then, by the orthonormality of the frame, equation (4.7) results in a unique set of Ω_{κ}^{μ} . Lowering and raising of indices is performed using the Minkowski metric $\eta := \text{diag}(-1, 1, 1, 1)$. That is,

$$\Omega_{\mu\kappa} := \eta_{\mu\gamma} \Omega_{\kappa}^{\gamma}. \quad (4.9)$$

Ricci rotation coefficients, $\omega_{\mu\kappa\lambda}, 0 \leq \mu, \kappa, \lambda, \leq 3$, are then obtained from the connection one-forms by expressing them as a combination of the co-frame forms, as follows.

$$\Omega_{\mu\kappa} = \Omega_{\mu\kappa\lambda}\omega^\lambda. \quad (4.10)$$

Let $\mathbf{e}_\mu, 0 \leq \mu \leq 3$ be the orthonormal frame of vector fields dual to the frame ω^μ . That is,

$$\omega^\kappa(\mathbf{e}_\mu) = \delta_\mu^\kappa, \quad (4.11)$$

where the right side is the Kronecker delta.

Let $\gamma_\mu, \mu = 0, 1, 2, 4$ be a particular representation of gamma matrices of Minkowski spacetime. That is, a set of four matrices that satisfy the relation

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}1_{4 \times 4}. \quad (4.12)$$

The covariant derivative of a spinor field $\psi : \mathcal{M} \rightarrow \mathbb{C}^4$ (where \mathcal{M} is the spacetime) takes the following form

$$\nabla_\mu\psi := \mathbf{e}_\mu\psi - \Gamma_\mu\psi, \quad (4.13)$$

where the four connection matrices Γ_μ are defined by

$$\Gamma_\mu := -\frac{1}{4}\Omega_{\nu\mu}^\alpha\gamma_\alpha\gamma^\nu + \frac{ie}{\hbar c}A_\mu 1_{4 \times 4} \quad (4.14)$$

Here A_μ are components of an electromagnetic one form \mathbf{A} when expressed in the frame, and $-e$, with $e > 0$ is the charge of the electron.

Definition 4.1.1. With the above notations, Dirac's equation, for a spinor field $\psi : \mathcal{M} \rightarrow \mathbb{C}^4$ in the orthonormal frame is

$$i\gamma^\mu\nabla_\mu\psi + \frac{mc}{\hbar}\psi = 0, \quad (4.15)$$

where m is the mass of the electron, c is speed of light in vacuum, and \hbar is $\frac{1}{2\pi}$ times Planck's constant h .

4.1.2 Dirac's equation on a spherically symmetric spacetime

Let us consider a spherically symmetric, static, Lorentzian manifold \mathcal{M} , that is diffeomorphic to $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$, with coordinates t, r, θ, ϕ taking the usual meaning as

in Chapter 1. Also, let

$$f : (0, \infty) \rightarrow (0, \infty). \quad (4.16)$$

Suppose that the metric takes the following form,

$$ds^2 = -f(r)^2 c^2 dt^2 + f(r)^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (4.17)$$

Remark. In case of charged particle-spacetimes, $f(r) = e^{\frac{\xi(r)}{2}}$, where $e^{\xi(r)}$ is given by equation (1.25).

We will now derive the covariant Dirac equation for such a metric, using the steps in section 4.1.1. This computation follows closely the same computation done in the work of Cohen and Powers [12]. Since the metric has the form (4.17), the following one forms make an orthonormal frame:

$$\tilde{\omega}^0 = f(r)c dt, \tilde{\omega}^1 = f^{-1} dr, \tilde{\omega}^2 = r d\theta, \tilde{\omega}^3 = r \sin(\theta) d\phi \quad (4.18)$$

We will not, however, use this frame for our computations. Instead we will use a frame that matches the Cartesian frame if $f \equiv 1$. Henceforth, let

$$x = r \sin(\theta) \cos(\phi), y = r \sin(\theta) \sin(\phi), z = r \cos(\theta). \quad (4.19)$$

We will work with the following frame

$$\omega^0 = \tilde{\omega}^0 \quad (4.20)$$

$$\omega^1 = \sin(\theta) \cos(\phi) \tilde{\omega}^1 + \cos(\theta) \cos(\phi) \tilde{\omega}^2 - \sin(\phi) \tilde{\omega}^3 \quad (4.21)$$

$$\omega^2 = \sin(\theta) \sin(\phi) \tilde{\omega}^1 + \cos(\theta) \sin(\phi) \tilde{\omega}^2 + \cos(\phi) \tilde{\omega}^3 \quad (4.22)$$

$$\omega^3 = \cos(\theta) \tilde{\omega}^1 - \sin(\theta) \tilde{\omega}^2 \quad (4.23)$$

$$(4.24)$$

Orthonormality of this frame can be seen from the fact that $\tilde{\omega}^\mu$ is an orthonormal frame.

Also, we note that if $f \equiv 1$ then, $\omega^0 = dt, \omega^1 = dx, \omega^2 = dy, \omega^3 = dz$. We now write the above in another way, to make computations easier. Let

$$g(r) := r^{-1} f(r)^{-1} - r^{-1} \quad (4.25)$$

Then, the frame in (4.24), can be re-written as

$$\omega^0 = cf(r) dt \quad (4.26)$$

$$\omega^1 = dx + g(r)x dr \quad (4.27)$$

$$\omega^2 = dy + g(r)y dr \quad (4.28)$$

$$\omega^3 = dz + g(r)z dr. \quad (4.29)$$

Since $r^2 = x^2 + y^2 + z^2$, we have $r dr = x dx + y dy + z dz$, and

$$x\omega^1 + y\omega^2 + z\omega^3 = r dr + g(r)r^2 dr = r dr + (f^{-1} - 1)r dr = f^{-1}r dr \quad (4.30)$$

Let us denote

$$\mathbf{s} := x\omega^1 + y\omega^2 + z\omega^3 \quad (4.31)$$

then,

$$dr = \mathbf{s}f(r)r^{-1} \quad (4.32)$$

and so dx, dy, dz are respectively

$$dx = \omega^1 - g(r)f(r)xr^{-1}\mathbf{s} \quad (4.33)$$

$$dy = \omega^2 - g(r)f(r)yr^{-1}\mathbf{s} \quad (4.34)$$

$$dz = \omega^3 - g(r)f(r)zr^{-1}\mathbf{s} \quad (4.35)$$

From the above we compute

$$d\omega^0 = f' dr dt = f' r^{-1} \mathbf{s} \wedge \omega^0 \quad (4.36)$$

$$= -(f' r^{-1} x \omega^0) \wedge \omega^1 - (f' r^{-1} y \omega^0) \wedge \omega^2 - (f' r^{-1} z \omega^0) \wedge \omega^3 \quad (4.37)$$

$$d\omega^1 = g dx dr = g f r^{-1} \omega^1 \wedge \mathbf{s} \quad (4.38)$$

$$= g f r^{-1} y \omega^1 \wedge \omega^2 + g f r^{-1} z \omega^1 \wedge \omega^3 \quad (4.39)$$

$$d\omega^2 = g dy dr = g f r^{-1} \omega^2 \wedge \mathbf{s} \quad (4.40)$$

$$= g f r^{-1} z \omega^2 \wedge \omega^3 + g f r^{-1} x \omega^2 \wedge \omega^1 \quad (4.41)$$

$$d\omega^3 = g dz dr = g f r^{-1} \omega^3 \wedge \mathbf{s} \quad (4.42)$$

$$= g f r^{-1} x \omega^3 \wedge \omega^1 + g f r^{-1} y \omega^3 \wedge \omega^2 \quad (4.43)$$

Now, to satisfy the skew symmetry assumption (equation (4.8)), we rewrite the above as

$$d\omega^0 = -(f'r^{-1}x\omega^0) \wedge \omega^1 - (f'r^{-1}y\omega^0) \wedge \omega^2 - (f'r^{-1}z\omega^0) \wedge \omega^3 \quad (4.44)$$

$$\begin{aligned} d\omega^1 &= -(f'r^{-1}x\omega^0) \wedge \omega^0 + (gfr^{-1}y\omega^1 - (gfr^{-1}x\omega^2)) \wedge \omega^2 \\ &\quad + (gfr^{-1}z\omega^1 - gfr^{-1}x\omega^3) \wedge \omega^3 \end{aligned} \quad (4.45)$$

$$\begin{aligned} d\omega^2 &= -(f'r^{-1}y\omega^0) \wedge \omega^0 + (-(gfr^{-1}y\omega^3) + gfr^{-1}z\omega^2) \wedge \omega^3 \\ &\quad + (gfr^{-1}x\omega^2 - (gfr^{-1}y\omega^1)) \wedge \omega^1 \end{aligned} \quad (4.46)$$

$$\begin{aligned} d\omega^3 &= -(f'r^{-1}z\omega^0) \wedge \omega^0 + (gfr^{-1}x\omega^3 - (gfr^{-1}z\omega^1)) \wedge \omega^1 \\ &\quad + (gfr^{-1}y\omega^3 - (gfr^{-1}z\omega^2)) \wedge \omega^2 \end{aligned} \quad (4.47)$$

From the above table we determine the Ricci connection coefficients, using equation (4.10). The non-zero ones are

$$\Omega_{010} = -f'r^{-1}x = -\Omega_{100} \quad ; \quad \Omega_{020} = -f'r^{-1}y = -\Omega_{200} \quad (4.48)$$

$$\Omega_{030} = -f'r^{-1}z = -\Omega_{300} \quad ; \quad \Omega_{121} = -gfr^{-1}y = -\Omega_{211} \quad (4.49)$$

$$\Omega_{131} = -gfr^{-1}z = -\Omega_{311} \quad ; \quad \Omega_{232} = -gfr^{-1}z = -\Omega_{322} \quad (4.50)$$

$$\Omega_{323} = -gfr^{-1}y = -\Omega_{233} \quad ; \quad \Omega_{212} = -gfr^{-1}x = -\Omega_{122} \quad (4.51)$$

$$\Omega_{313} = -gfr^{-1}x = -\Omega_{133} \quad (4.52)$$

Though inconsequential, we point out the pattern - (a) there are two distinct indices in each subscript, (b) exactly one of x, y, z appear depending on whether the non-repeating index is 1, 2, 3, respectively, (c) also if 0 appears in the subscript, $f'r^{-1}$ appears on the right and if not, gfr^{-1} appears, and (d) if the repeated index occurs together on the second and third positions only, and in this case the sign is positive and otherwise negative.

The spin connection matrices can be computed from the Ricci rotation coefficients.

$$\begin{aligned} \Gamma_0 &= -1\frac{1}{4}\Omega_{ab}\gamma^a\gamma^b + ie\hbar^{-1}c^{-1}A_0 \\ &= -1\frac{1}{4}f'r^{-1}[-x\gamma^0\gamma^1 - y\gamma^0\gamma^2 - z\gamma^0\gamma^3 \\ &\quad + x\gamma^1\gamma^0 + y\gamma^2\gamma^0 + z\gamma^3\gamma^0 +] + ie\hbar^{-1}c^{-1}A_0 \\ &= \frac{1}{2}f'r^{-1}\gamma^0[x\gamma^1 + y\gamma^2 + z\gamma^3] + ie\hbar^{-1}c^{-1}A_0 \end{aligned} \quad (4.53)$$

Here we have used the fact that if $\mu \neq \nu$, then $\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu$. Now,

$$\begin{aligned}\Gamma_1 &= -\frac{1}{4}\Omega_{ab1}\gamma^a\gamma^b + ie\hbar^{-1}c^{-1}A_1 \\ &= \frac{1}{2}gfr^{-1}\gamma^1(y\gamma^2 + z\gamma^3) + ie\hbar^{-1}c^{-1}A_1.\end{aligned}\quad (4.54)$$

Similarly we see

$$\begin{aligned}\Gamma_2 &= \frac{1}{2}gfr^{-1}\gamma^2(z\gamma^3 + x\gamma^1) + ie\hbar^{-1}c^{-1}A_2 \\ \Gamma_3 &= \frac{1}{2}gfr^{-1}\gamma^3(x\gamma^1 + y\gamma^2) + ie\hbar^{-1}c^{-1}A_3.\end{aligned}\quad (4.55)$$

We now make use of the assumption that in the spacetime that we are studying there is only an electric potential present, so that $A_0 = \varphi(r)f(r)^{-1}$, where φ is a function of r alone, and $A_1 = A_2 = A_3 = 0$.

Remark. The electric potential $\varphi(r)$ is specified on charged particle-spacetimes by the expression given in the first chapter in equation (1.27).

Suppose $\tilde{g}(r)$ is a function that satisfies $\tilde{g}(r) + g(r) + \tilde{g}(r)g(r)r = 0$. That is,

$$\tilde{g}(r) := r^{-1}(f - 1).\quad (4.56)$$

Also, note that $\partial_r = \frac{1}{r}(x\partial_x + y\partial_y + z\partial_z)$. Then, the dual frame of orthonormal vector fields consists of

$$\mathbf{e}_0 = f(r)^{-1}c^{-1}\partial_t\quad (4.57)$$

$$\mathbf{e}_1 = \partial_x + \tilde{g}(r)x\partial_r\quad (4.58)$$

$$\mathbf{e}_2 = \partial_y + \tilde{g}(r)y\partial_r\quad (4.59)$$

$$\mathbf{e}_3 = \partial_z + \tilde{g}(r)z\partial_r\quad (4.60)$$

Therefore the covariant Dirac equation (4.15) simplifies to

$$\begin{aligned}0 &= \gamma^0 \left(f(r)^{-1}c^{-1}\partial_t - \frac{1}{2}f'r^{-1}\gamma^0 [x\gamma^1 + y\gamma^2 + z\gamma^3] - ie\hbar^{-1}c^{-1}\varphi(r)f(r)^{-1} \right) \psi \\ &\quad + \gamma^1 \left(\partial_x + \tilde{g}(r)x\partial_r - \frac{1}{2}gfr^{-1}\gamma^1 (y\gamma^2 + z\gamma^3) \right) \psi \\ &\quad + \gamma^2 \left(\partial_y + \tilde{g}(r)y\partial_r - \frac{1}{2}gfr^{-1}\gamma^2 (z\gamma^3 + x\gamma^1) \right) \psi \\ &\quad + \gamma^3 \left(\partial_z + \tilde{g}(r)z\partial_r - \frac{1}{2}gfr^{-1}\gamma^3 (x\gamma^1 + y\gamma^2) \right) \psi + \frac{mc}{\hbar}\psi\end{aligned}\quad (4.61)$$

We introduce the notations

$$T\psi := \gamma^1 \partial_x \psi + \gamma^2 \partial_y \psi + \gamma^3 \partial_z \psi \quad (4.62)$$

$$\gamma^r := \frac{1}{r} (x\gamma^1 + y\gamma^2 + z\gamma^3) \quad (4.63)$$

Using these we may rewrite Dirac's equation as

$$\begin{aligned} 0 = & \gamma^0 f(r)^{-1} c^{-1} \partial_t \psi + \frac{1}{2} f' \gamma^r \psi - \gamma^0 i e \hbar^{-1} c^{-1} \varphi(r) f(r)^{-1} \psi \\ & + T\psi + \tilde{g}(r) r \gamma^r \partial_r \psi - g f \gamma^r \psi + m c \hbar^{-1} \psi. \end{aligned} \quad (4.64)$$

From this, observing that $g(r)f(r) = (1 - f(r))r^{-1}$, the definition of $\tilde{g} = r^{-1}(f - 1)$ from equation (4.56), and $(\gamma^0)^2 = -1$, we get

$$\begin{aligned} i\hbar \partial_t \psi = & i\hbar c \{ [2^{-1} f f' - r^{-1} f(1 - f)] \gamma^0 \gamma^r \psi + f \gamma^0 T\psi + f \gamma^0 m c \hbar^{-1} \psi \\ & + f(f - 1) \gamma^0 \gamma^r \partial_r \psi \} - e \varphi(r) \psi. \end{aligned} \quad (4.65)$$

In the next section, we will separate the angular and the radial parts of the above equation. In order to set the stage for it, we present the matrices $\alpha^k, k = 1, 2, 3$, and β defined as

$$\alpha^k = -\gamma^0 \gamma^k; \beta = i\gamma^0. \quad (4.66)$$

The choice of γ_μ is made in equation (4.75). Furthermore,

$$p_1 := -i\partial_x; p_2 := -i\partial_y; p_3 := -i\partial_z \quad (4.67)$$

$$p_r := -i\partial_r; \alpha^r := \frac{1}{r} (x\alpha^1 + y\alpha^2 + z\alpha^3). \quad (4.68)$$

Then, the right side of equation (4.65) can be written as

$$\begin{aligned} H\psi := & \hbar c [2^{-1} f f' + r^{-1} f(f - 1)] (-i\alpha^r \psi) + \hbar c f (\alpha^k p_k + \beta \hbar^{-1} m c) \psi \\ & + \hbar c f(f - 1) \alpha^r p_r \psi - e \varphi(r) \psi. \end{aligned} \quad (4.69)$$

This is equation (3.1) in [12].

4.1.3 Hilbert space of spinors

In general, the inner product on spinors defined on a hyper-surface Σ is written in terms of the conjugate spinor which is defined as

$$\bar{\psi} := \psi^\dagger \gamma^0, \quad (4.70)$$

where ψ^\dagger denote the conjugate transpose, namely, as

$$(\psi_1, \psi_2)_{gen} := - \int_{\Sigma} \bar{\psi} \tilde{\gamma}^\nu \psi \eta_\nu d\mu_\Sigma; \quad (4.71)$$

here, the measure is the induced measure on the hypersurface and η is a unit normal vector field, and $\tilde{\gamma}$ are the gamma matrices expressed in the coordinate frame computed as $\tilde{\gamma}^\nu = \mathbf{e}_\mu^\nu \gamma^\mu$. On constant- t hypersurfaces $\eta = (1, 0, 0, 0)$, and because of the γ^0 appearing in the definition of the conjugate spinor, the integrand reduces to $\psi^\dagger \gamma^0 \tilde{\gamma}^0$ which in turn simplifies to $-f(r)^{-1} \psi^\dagger$ because the only nonzero \mathbf{e}_μ^0 is when $\mu = 0$. Thus, on constant t hyper-surfaces the inner product reduces to

$$(\psi_1, \psi_2) := \int_0^\infty \int_0^\pi \int_0^{2\pi} \psi_1(r, \theta, \phi)^\dagger \psi_2(r, \theta, \phi) f(r)^{-1} r^2 \sin(\theta) d\theta d\phi dr. \quad (4.72)$$

So, we define the following Hilbert space of four-component spinors.

$$\mathfrak{H} := \{ \psi : \mathbb{R}^3 \rightarrow \mathbb{C}^4 \mid (\psi, \psi) < \infty \} \quad (4.73)$$

Finally, we define

Definition 4.1.2. The operator H of equation (4.69) on the Hilbert space \mathfrak{H} is called the Dirac Hamiltonian.

4.2 Separation of the Dirac Hamiltonian

In this section we separate the angular and radial parts of the Dirac Hamiltonian. The Hilbert space is expressed as direct sum of invariant subspaces, on which the action of the Hamiltonian reduces to that of a first order differential system with two components depending just on the radial coordinate r .

Separation of Dirac's equation on Minkowski space with a radially symmetric electrostatic potential is well known. It is presented, for instance, in Thaller's book [31]. The particular form of the Dirac Hamiltonian in equation (4.69) makes it easy to work out the details of the separation. This is because the Minkowskian Dirac operator appears verbatim in it. Therefore the angular dependence occurs the same way as in the Minkowskian Dirac operator. We remark that this is an advantage arising from the fact that we used an orthonormal Cartan frame that reduces to the Cartesian frame if $f \equiv 1$.

Representation of gamma matrices

The metric of charged particle-spacetime has signature $(-1, 1, 1, 1)$. We take the following representation for the gamma matrices. First, let $\sigma^k, k = 1, 2, 3$ denote the Pauli matrices, given by

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.74)$$

Then, with 1 denoting $I_{2 \times 2}$, the 4×4 gamma matrices are given by

$$\gamma^0 = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^j = -i \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}, \quad j = 1, 2, 3 \quad (4.75)$$

Remark. These are $(-i)$ times the matrices in the Dirac representation used when the metric has signature $(+, -, -, -)$.

Then, the alpha matrices determined by equation (4.66) are

$$\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad \alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}, \quad k = 1, 2, 3. \quad (4.76)$$

These are the Dirac matrices in the standard representation introduced by Dirac. With the notation introduced in equation (4.68), the (free) Dirac Hamiltonian in Minkowski space is

$$H_0 := \hbar c \alpha^k p_k + \beta m c^2, \quad (4.77)$$

where we have once again used Einstein's summation convention in k , which varies over $1, 2, 3$. The radial and angular parts of H_0 become when it is written in polar coordinates (from Thaller [31] section 4.6.3),

$$H_0 = -i \hbar c (\alpha^r) \left(\frac{d}{dr} + \frac{1}{r} - \frac{1}{r} \beta K \right) + \beta m c^2, \quad (4.78)$$

where K , called "spin-orbit operator", is given by

$$K := \beta (2\mathbf{S} \cdot \mathbf{L} + 1), \quad (4.79)$$

where

$$\mathbf{S} := \frac{1}{2} \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \quad (4.80)$$

$$\mathbf{L} := \mathbf{x} \times -i \nabla, \quad (4.81)$$

are the *spin* and the *orbital angular momentum* operators, respectively. The components of the former are obtained from $\sigma := (\sigma^1, \sigma^2, \sigma^3)$.

We will now describe the decomposition of \mathfrak{H} into partial wave subspaces. The treatment follows Thaller [31], sections (4.6.4) and (4.6.5). Firstly, the spin-orbit operator K acts on $L^2(S^2)^4$ (with the standard measure on the sphere), and it has a purely discrete spectrum and a complete system of orthonormal eigenvectors. Suppose j is an index that varies in positive half-integers,

$$j = 1/2, 3/2, 5/2, \dots, \quad (4.82)$$

while for each such j , m_j is allowed to take values

$$m_j = -j, -j+1, \dots, +j, \quad (4.83)$$

and κ_j is allowed to take the two values

$$\kappa = -(j+1/2), +(j+1/2). \quad (4.84)$$

Specifying a triplet (j, m_j, κ_j) gives two orthonormal eigenvectors of Φ_{m_j, κ_j}^\pm . The explicit expression for these are available in terms of spherical harmonics, but we will not need those here. These form a complete orthonormal family of eigenvectors of K in $L^2(S^2)^4$. The indices are actually eigenvalues of operators J^2, J_3, K (where $\mathbf{J} := \mathbf{L} + \mathbf{S}$ is the *total angular momentum* operator), given by

$$J^2 \Phi_{m_j, \kappa_j}^\pm = j(j+1) \Phi_{m_j, \kappa_j}^\pm, \quad (4.85)$$

$$J_3 \Phi_{m_j, \kappa_j}^\pm = m_j \Phi_{m_j, \kappa_j}^\pm, \quad (4.86)$$

$$K \Phi_{m_j, \kappa_j}^\pm = -\kappa_j \Phi_{m_j, \kappa_j}^\pm. \quad (4.87)$$

These eigenfunctions satisfy

$$i\alpha^r \Phi_{m_j, \kappa_j}^\pm = -\pm \Phi_{m_j, \kappa_j}^\pm. \quad (4.88)$$

Another property of these that we use is that the lower two components of Φ_{m_j, κ_j}^+ are zero while the top two components of Φ_{m_j, κ_j}^- are zero. Thus, to summarize,

$$L^2(S^2)^4 = \bigoplus_{j=1/2, 3/2, \dots} \bigoplus_{m_j=-j, -j+1, \dots, j} \bigoplus_{\kappa_j=\pm(j+1/2)} \mathfrak{K}_{m_j, \kappa_j}, \quad (4.89)$$

where

$$\mathfrak{K}_{m_j, \kappa_j} = \{c^+ \Phi_{m_j, \kappa_j}^+ + c^- \Phi_{m_j, \kappa_j}^- | c^\pm \in \mathbb{C}\}. \quad (4.90)$$

Let us define the subspace

$$\mathfrak{H}_{m_j, \kappa_j}^\sim = L^2((0, \infty); f(r)^{-1} dr) \otimes \mathfrak{K}_{m_j, \kappa_j}, \quad (4.91)$$

which we note is isomorphic to $L^2((0, \infty); f(r)^{-1} dr)^2$, and the map $U_{m_j, \kappa_j}^\sim : \mathfrak{H}_{m_j, \kappa_j}^\sim \rightarrow \mathfrak{H}$ by

$$U_{m_j, \kappa_j}^\sim : (g^+(r), g^-(r)) \rightarrow \frac{1}{r} \left(g^+(r) \Phi_{m_j, \kappa_j}^+ + g^-(r) \Phi_{m_j, \kappa_j}^- \right). \quad (4.92)$$

This map preserves the inner product. We present the action of H on $\mathfrak{H}_{m_j, \kappa_j}^\sim$, that is, compute $(U_{m_j, \kappa_j}^\sim)^{-1} H U_{m_j, \kappa_j}^\sim$. In terms of the basis $\Phi_{m_j, \kappa_j}^+, \Phi_{m_j, \kappa_j}^-$, this is just an operator on $L^2((0, r); f(r)^{-1} dr)^2$.

Let us express the Dirac Hamiltonian from equation (4.69), using the free Dirac Hamiltonian on Minkowski, as

$$H\psi = \hbar c \left[\frac{1}{2} f f' + \frac{1}{r} f(f-1) \right] (-i\alpha^r \psi) + f H_0 \psi + \hbar f(f-1) c \alpha^r p_r \psi - e\varphi \psi. \quad (4.93)$$

Now, from Theorem (4.14) in Thaller [31], the $(U_{m_j, \kappa_j}^\sim)^{-1} H U_{m_j, \kappa_j}^\sim$ on $(g^+(r), g^-(r))$ is given by the operator

$$\begin{pmatrix} mc^2 & -cd_r + \hbar c \frac{\kappa_j}{r} \\ \hbar c d_r + \hbar c \frac{\kappa_j}{r} & -mc^2 \end{pmatrix}. \quad (4.94)$$

Using this we compute, with temporary notation $g^\pm = g^\pm(r)$, $\Phi^\pm = \Phi_{m_j, \kappa_j}^\pm(\theta, \phi)$,

$$\begin{aligned} H \left(g^+ \frac{\Phi^+}{r} \right) &= [mc^2 f g^+(r) - e\varphi(r)] \frac{\Phi^+}{r} + \hbar \left[c g^+ \frac{1}{2} f f' + c f^2 d_r g^+ \right] \frac{\Phi^-}{r}, \\ H \left(g^- \frac{\Phi^-}{r} \right) &= [-f mc^2 - e\varphi(r)] \frac{\Phi^-}{r} + \left[-\hbar c f^2 d_r g^- + \kappa_j \hbar \frac{f}{r} g^- - \frac{1}{2} f f' g^- \right] \frac{\Phi^+}{r}. \end{aligned} \quad (4.95)$$

Therefore the operator H on $\mathfrak{H}_{m_j, \kappa_j}^\sim = L^2((0, \infty), f(r)^{-1} dr)^2$ is given by

$$H_{m_j, \kappa_j}^{\sim 1} = \begin{pmatrix} c^2 f m - e\varphi & \hbar c \left(-f^2 d_r + f \frac{\kappa_j}{r} - r \frac{1}{2} f f' \right) \\ \hbar c \left(f^2 d_r + f \frac{\kappa_j}{r} + \frac{1}{2} f f' \right) & -c^2 m f - e\varphi \end{pmatrix}. \quad (4.96)$$

We can remove the $f(r)f'(r)$ term by a variable change. Define

$$U^v : (g^+(r), g^-(r)) \mapsto \left(f(r)^{+1/2} g^+(r), f(r)^{+1/2} g^-(r) \right), \quad (4.97)$$

and it is clear that this is a unitary isomorphism

$$U^v : L^2((0, \infty), f(r)^{-1} dr)^2 \rightarrow L^2((0, \infty); f(r)^{-2} dr)^2.$$

Further,

$$H_{m_j, \kappa_j}^{\sim 2} := U^v H_{m_j, \kappa_j}^{\sim 1} (U^v)^{-1} = \begin{pmatrix} c^2 f(r)m - e\varphi(r) & -c\hbar f(r)^2 d_r + \hbar f(r)c \frac{\kappa_j}{r} \\ c\hbar f(r)^2 d_r + \hbar c f(r) \frac{\kappa_j}{r} & -c^2 m f(r) - e\varphi(r) \end{pmatrix}. \quad (4.98)$$

In the last step, we change r to x according to

$$\frac{dx}{dr} = f(r)^{-2}, \quad x(r=0) = 0. \quad (4.99)$$

Remark. The function $x(r)$ is same as the function $\tilde{r}(r)$ defined in equation (2.2).

Then the map $U^w : L^2((0, \infty), f^{-2}(r)dr)^2 \rightarrow L^2((0, \infty); dx)^2$ given by

$$U^w(g^1(r), g^2(r)) = (g^1(r(x)), g^2(r(x))) \quad (4.100)$$

is a unitary isomorphism, as the integration measure $f^{-2}dr$ is equal to dx . Also, $f(r)^2 \frac{d}{dx}$ turns into $f(r)^{-2} f(r)^2 \frac{d}{dx}$ which is equal to $\frac{d}{dx}$.

Thus we have proved the following theorem.

Theorem 4.2.1. *The Hilbert space \mathfrak{H} is a direct sum of the subspaces $\mathfrak{H}_{m_j, \kappa_j}^{\sim}$, each of which is isomorphic to $\mathfrak{H}_{m_j, \kappa_j}^{red} := L^2((0, \infty); dx)^2$, and each of these subspaces are mapped into itself by the Dirac Hamiltonian. Thus, the Dirac Hamiltonian H on \mathfrak{H} is a direct sum of $H_{m_j, \kappa_j} := U^w H_{m_j, \kappa_j}^{\sim 2} (U^w)^{-1}$, on $\mathfrak{H}_{m_j, \kappa_j}^{red}$, and,*

$$H_{m_j, \kappa_j} = \begin{pmatrix} c^2 f(r)m - e\varphi(r) & -c\hbar \frac{d}{dx} + \hbar f(r)c \frac{\kappa_j}{r} \\ \hbar c \frac{d}{dx} + \hbar c f(r) \frac{\kappa_j}{r} & -c^2 m f(r) - e\varphi(r) \end{pmatrix}, \quad (4.101)$$

where $r(x)$ is obtained from the solution of the differential equation (4.99).

4.3 Essential self-adjointness of the Dirac Hamiltonian

Here we prove that H_{m_j, κ_j} defines a unique self-adjoint operator on $\mathfrak{H}_{m_j, \kappa_j}^{red}$. To do this we first recall the following definition.

Definition 4.3.1. A symmetric operator T with domain D in a Hilbert space is said to be essentially self-adjoint if its closure \bar{T} is self-adjoint.

The usefulness of this definition is that if T is essentially self-adjoint then it has one and only one self-adjoint extension. The main theorem of this section is the following.

Theorem 4.3.1. *Every reduced Dirac Hamiltonian H_{m_j, κ_j} is essentially self-adjoint on the domain $D_0 = C_c^\infty(0, \infty)^2 \subset L^2((0, \infty), dx)^2 = \mathfrak{H}_{m_j, \kappa_j}^{red}$.*

Notation. Henceforth by we will denote the unique self-adjoint extension also by H_{m_j, κ_j} .

Proof. The proof proceeds by a version of Weyl's limit-point/ limit-circle criterion adapted for Dirac type systems. The original argument of Weyl was for Sturm Liouville systems, introduced in his seminal work in 1910. In his book, [34], Weidmann considers general formal differential expressions τ of the form

$$\begin{aligned} \tau u := & r(x)^{-1} \left\{ \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \left(p_j(x) u^{(j)}(x) \right)^{(j)} \right. \\ & \left. + \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^j \left[\left(q_j(x) u^{(j)}(x) \right)^{j+1} - \left(q_j^*(x) u^{(j+1)}(x) \right)^{(j)} \right] \right\}, \quad (4.102) \end{aligned}$$

where the u are \mathbb{C}^m valued functions defined on (a, b) , $-\infty \leq a \leq b \leq \infty$, n is the order of the differential expression, and the coefficients r, p_j, q_j are $m \times m$ matrix-valued functions on (a, b) , $r(x)$ is positive definite, and the $p_j(x)$ are all Hermitian.

In our case, $m = 2$, $n = 1$, so that $j = 0$ is the only index (which we will leave out from now), $r(x) = 1_{2 \times 2}$, and $q(x) = q = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}$, so that τ takes the following form:

Definition 4.3.2. Suppose $P(x)$ is a real symmetric matrix, for each $x \in (a, b)$. Then, by a Dirac type differential expression τ , associated to $P(x)$, we mean, for $u : (a, b) \rightarrow \mathbb{C}^2$,

$$\tau u := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u' + Pu \quad (4.103)$$

The version of Weyl's alternative stated for Dirac type systems is

Theorem 4.3.2 ([34], Theorem 5.6 (Weyl's alternative)). *Suppose τ is a Dirac type differential expression. Then, either:*

- (a) *for every $\lambda \in \mathbb{C}$ all solutions of $(\tau - \lambda)u = 0$ lie in L^2 near b (that is, for every solution u there is a $c \in (a, b)$ such that $u \in L^2((c, b), dx)$), or :*
- (b) *for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exists a unique (up to a multiplicative constant) solution u of $(\tau - \lambda)u = 0$ which is in L^2 near b .*

In the first case, τ is said to be in the limit circle case (l.c.c.) at b , and in the second case, τ is said to be in the limit point case (l.p.c.) The same result holds with L^2 near b replaced with L^2 near a in both cases.

The importance of this theorem is that it helps us to compute the deficiency indices of τ .

Theorem 4.3.3 ([34], Theorem 5.7). *Suppose τ is a Dirac-type differential expression. Then, the deficiency indices are*

- (a) $(2, 2)$ *if τ is l.c.c. at both a, b ,*
- (b) $(1, 1)$ *if τ is l.c.c. at one end point and l.p.c. at the other, and*
- (c) $(0, 0)$ *if τ is l.p.c. at both end points a, b .*

In the last case, if the deficiency indices are $(0, 0)$, then the minimal operator (defined by τ by taking the closure of the operator τ defined on $C_c^\infty(0, \infty)^2$) is the only self-adjoint extension of the minimal operator.

The following theorem, paraphrased from Weidmann for our specific type of τ , says that we need only worry about the endpoint 0.

Theorem 4.3.4 ([34], Theorem 6.8, Corollary). *Suppose τ is a Dirac type differential expression in (a, ∞) . Then, τ is in the limit point case at ∞ .*

Now, let us determine the type at the other end point 0. We take $\lambda = 0$. Write

$$H_{m_j, \kappa_j} = \hbar c \begin{pmatrix} 0 & -d_x + f \frac{\kappa_j}{r} \\ d_x + f \frac{\kappa_j}{r} & 0 \end{pmatrix} + \begin{pmatrix} c^2 f m - e\varphi & 0 \\ 0 & -c^2 m f - e\varphi \end{pmatrix}. \quad (4.104)$$

From chapter 1, since $|\varphi(r)|$ is decreasing and $|\varphi(0)|$ is finite, and $f(r) = e^{\xi(r)/2}$ is increasing with $\lim_{r \rightarrow \infty} f(r) = 1$, the second matrix here is bounded. Therefore in order to determine the type at 0 we may only look at the first matrix. So consider

$$\begin{pmatrix} 0 & -d_x + f(r) \frac{\kappa_j}{r} \\ d_x + f(r) \frac{\kappa_j}{r} & 0 \end{pmatrix} (u_1(x), u_2(x))^T = 0. \quad (4.105)$$

This simplifies to two decoupled equations,

$$-u_2'(x) + f(r) \frac{\kappa_j}{r} u_2(x) = 0, \quad (4.106)$$

$$u_1'(x) + f(r) \frac{\kappa_j}{r} u_1(x) = 0, \quad (4.107)$$

whence

$$u_2(x) = c_2 e^{\int_{x_0}^x f(r(s)) \frac{\kappa_j}{r(s)} ds}, \quad (4.108)$$

$$u_1(x) = c_1 e^{\int_{x_0}^x f(r(s)) \frac{-\kappa_j}{r(s)} ds}, \quad (4.109)$$

where $x_0 \in (0, \infty)$ is fixed below.

From the asymptotics near 0 from equation (1.31) in chapter 1, $f(0) = \sqrt{1 - A\epsilon^2}$, where ϵ is defined by equation (1.21). Further, using the fact that $f(r)$ is increasing, and by continuity, given an $\eta > 0$, there exists an r neighbourhood $(0, \delta)$ such that

$$b_1 := \sqrt{1 - A\epsilon^2} < f(r) < \sqrt{1 - A\epsilon^2} + \eta =: b_2, \forall r \in (0, \delta), \quad (4.110)$$

and so, with $x_0 = x(r = \delta)$, for all $0 < x < x_0$,

$$b_2^{-2} < f(r)^{-2} < b_1^{-2} \quad (4.111)$$

$$\Rightarrow b_2^{-2} r < x < b_1^{-2} r \quad (4.112)$$

$$\Rightarrow b_2^{-2} \frac{1}{x} < \frac{1}{r} < b_1^{-2} \frac{1}{x} \quad (4.113)$$

$$\Rightarrow b_1 b_2^{-2} \frac{1}{x} < f(r) \frac{1}{r} < b_2 b_1^{-2} \frac{1}{x} \quad (4.114)$$

$$\Rightarrow b_1 b_2^{-2} (\ln(x_0) - \ln(x)) < \int_x^{x_0} f(r(s)) \frac{1}{r(s)} ds < b_2 b_1^{-2} (\ln(x_0) - \ln(x)) \quad (4.115)$$

$$\Rightarrow -b_2 b_1^{-2} (\ln(x_0) - \ln(x)) < \int_{x_0}^x f(r(s)) \frac{1}{r(s)} ds < -b_1 b_2^{-2} (\ln(x_0) - \ln(x)) \quad (4.116)$$

Now, suppose κ_j is positive. Then, $\kappa_j \geq 1$, from the fact about eigenvalues of the spin-orbit operator K . Then,

$$u_1(x) > c_1 e^{-b_1 b_2^{-2} \kappa_j \ln(x)} > x^{-b_1 b_2^{-2} \kappa_j} > x^{-b_1 b_2^{-2}}. \quad (4.117)$$

Therefore, if $b_1 b_2^{-2} > 1/2$, u_1 won't be in $L^2((0, x_0), dx)$. Similarly, if κ_j were negative, the same can be said about u_1 provided $b_2 b_1^{-2} > 1/2$. Now, $\sqrt{1 - A\epsilon^2}(\sqrt{1 - A\epsilon^2})^{-2} > 1 > 1/2$ so one may choose an η so that both $b_1 b_2^2$ and $b_2 b_1^2$ are greater than $1/2$.

Thus, we have verified that H_{m_j, κ_j} is the limit point case at the boundary point 0. So H_{m_j, κ_j} is *l.p.c* at both boundary points $0, \infty$, and therefore H_{m_j, κ_j} is essentially self-adjoint on $C_c^\infty(0, \infty)^2$.

This ends the proof of Theorem 4.3.1. □

4.4 Essential spectrum

In this subsection, we prove the theorem stated below.

Theorem 4.4.1. *For every m_j, κ_j , the essential spectrum of H_{m_j, κ_j} is given by*

$$\sigma_{ess}(H_{m_j, \kappa_j}) = (-\infty, -mc^2] \cup [mc^2, \infty). \quad (4.118)$$

Remark. Because the essential spectrum of the Dirac Hamiltonian is the closure of the union of the essential spectrums of the reduced Dirac Hamiltonians, the theorem implies that the essential spectrum of the Dirac Hamiltonian is also $(-\infty, -mc^2] \cup [mc^2, \infty)$.

To begin with, spectrum of the free Dirac operator H_0 on Minkowski space is precisely the same set. This is easily shown using Fourier transform (Lemma 4.4.2 below). Then, we set up the perturbation argument, which follows from the local compactness property of the free Dirac H_0 , from which we show the same for the reduced maps H_{0, m_j, κ_j} (defined by taking $f \equiv 1, e = 0$). The proof of the theorem begins after the lemmas 4.4.2, 4.4.3, 4.4.4 and 4.4.5. We follow the arguments in section 4.3.4 of Thaller's book [31]. We remark that this proof won't work in the case of Reissner-Nordström (RN) spacetime as we use the boundedness of the metric coefficient $f(r)$ and the potential φ . We also use the fact that the metric at infinity is like RN, that is f tends to 1 and the potential asymptotically is Coulombic.

Lemma 4.4.2. *The essential spectrum of the free Dirac operator $H_0 = -i\boldsymbol{\alpha}\cdot\nabla + \beta mc^2$ on $L^2(\mathbb{R}^3)$ is given by $(-\infty, -mc^2] \cup [mc^2, \infty)$.*

Proof. In Fourier (momentum) space the operator H_0 becomes the multiplication operator

$$h(\mathbf{p}) := (\mathcal{F}H_0\mathcal{F}^{-1})(\mathbf{p}) = \boldsymbol{\alpha}\mathbf{p} + \beta mc^2, \quad (4.119)$$

where \mathcal{F} denotes the Fourier transform. The four eigenvalues of the matrix on the right are easily computed as $\lambda(p) := \pm\sqrt{p^2c^2 + m^2c^4}$ with each eigenvalue repeated twice. The unitary transformation $u(\mathbf{p}) := a_+(p)\mathbf{1} + a_-(p)\beta\frac{\boldsymbol{\alpha}\mathbf{p}}{p}$ with $a_{\pm}(p) = \frac{1}{\sqrt{2}}\sqrt{1 \pm \frac{mc^2}{p^2c^2 + m^2c^4}}$ diagonalizes $h(p)$. Therefore the spectrum consists of all possible values of $\lambda(p)$, which is $\mathbb{R} \setminus (-mc^2, mc^2)$. By definition of the essential spectrum, the lemma follows. \square

We reproduced the following perturbation result, from Thaller [31] section 4.3.4.

Lemma 4.4.3. *Suppose that $H_0, H_0 + V$ are two self-adjoint operators such that V is H_0 -bounded, and*

$$\lim_{R \rightarrow \infty} \|V(H_0 - z)^{-1}\chi(|x| > R)\| = 0. \quad (4.120)$$

Suppose also that 0 is not in the spectrum of H_0 , and H_0 possess local compactness with $|H_0|^{-1}\chi(|x| < R)$ compact for all R , then $\sigma_{ess}(H) = \sigma_{ess}(H_0)$.

Proof. We have the famous theorem of H.Weyl that states that if H, H_0 are self-adjoint operators such that for one (and hence all) $z \in \mathbb{C} \setminus \mathbb{R}$, the operator $(H - z)^{-1} - (H_0 - z)^{-1}$ is compact, then the essential spectrums match, that is $\sigma_{ess}(H) = \sigma_{ess}(H_0)$. If $H = H_0 + V$, and V is H_0 -bounded, this resolvent difference can be rewritten using the resolvent formula as $-(H - z)^{-1}V(H_0 - z)^{-1}$. To ensure compactness, it is enough to have $B := V(H_0 - z)^{-1}$ is compact. This can further be split as $B\chi(|x| \leq R) + B\chi(|x| > R)$, where χ is the indicator function. By assumption, the potential we take satisfies $\lim_{R \rightarrow \infty} \|V(H_0 - z)^{-1}\chi(|x| > R)\| = 0$. Because the norm limit of compact operators is compact, to establish the required compactness it is enough to show $B\chi(|x| \leq R)$ is compact. We write $B\chi(|x| \leq R) = V(H_0 - z)^{-1}|H_0|^{-1}|H_0|\chi(|x| \leq R)$. This is compact, by assumption on the last two terms. \square

Next, we state a lemma about reduced the free Dirac Hamiltonian H_{0,m_j,κ_j} obtained by taking $f(r) \equiv 0, e = 0$.

Lemma 4.4.4. *If $H_0 = H_{0,m_j,\kappa_j}$, a reduced free Dirac operator, then requirement (4.120) is fulfilled provided $V(x)$ vanishes at infinity.*

Proof. Suppose $f_R : [0, \infty) \rightarrow [0, 1]$ is a smooth function such that $f_R(x) = 0, x < R/2$ and $f_R(x) = 1, x \geq R$, and also that $|f'_R(x)| \leq R/4\forall x$. Then, $\chi(|x| \geq R) = f_R\chi(|x| \geq R)$. With $R_z := (H_{0,m_j,\kappa_j} - z)^{-1}$, $\chi_R := \chi(|x| > R)$,

$$\begin{aligned} R_z\chi_R &= R_z f_R \chi_R \\ &= f_R R_z \chi_R + [R_z, f_R] \chi_R \\ &= f_R R_z \chi_R + R_z [f_R, H_{0,m_j,\kappa_j} - z] R_z \chi_R. \end{aligned} \quad (4.121)$$

Since

$$H_{0,m_j,\kappa_j} u(x) = -i\hbar c \sigma^2 u'(x) + \hbar c \sigma^1 \frac{\kappa}{r(x)} u(x) + m c^2 \sigma^3 u(x), \quad (4.122)$$

we have

$$[f_R \chi_R, H_{0,m_j,\kappa_j}] = -i\hbar c \sigma^2 f'_R \Rightarrow \|[f_R \chi_R, H_{0,m_j,\kappa_j}]\| \leq \frac{1}{4} R. \quad (4.123)$$

So because R_z is bounded (by the definition of resolvent) and $V R_z$ which is equal to $V H_{0,m_j,\kappa_j} H_{0,m_j,\kappa_j}^{-1} (H_{0,m_j,\kappa_j} - z)^{-1}$, is bounded as V is H_{0,m_j,κ_j} -bounded, we have

$$\|V R_z \chi_R\| \leq \|V f_R\| \|R_z \chi_R\| + \|V R_z\| \|[f_R \chi_R, H_{0,m_j,\kappa_j}]\| \|R_z \chi_R\| \quad (4.124)$$

$$\leq \left(\sup_{x>R/2} |V(x)| \|R_z\| + \frac{1}{4R} \|V R_z\| \|R_z\| \right). \quad (4.125)$$

Therefore, if V is vanishing at infinity, the requirement (4.120) holds true for the reduced free Dirac operators. \square

If H_0 is the free Dirac operator, then the local compactness in Lemma 4.4.3 holds. The operator $|H_0|^{-1} \chi(|x| \leq R)$ is compact by the general result that if we have two functions $g_1, g_2 : [0, \infty) \rightarrow \mathbb{C}$ that vanish at infinity, $\lim_{r \rightarrow \infty} g_1(r) = \lim_{r \rightarrow \infty} g_2(r) = 0$, then $g_1(p^2) g_2(x^2)$ is compact, and by taking $|H_0|^{-1}(\mathbf{p}) = (c^2 p^2 + m^2 c^4)^{-1/2} = g_1(p^2)$ and $\chi(|x| \leq R) = g_2(x^2)$. From this, since H_0 is unitarily equivalent to $\bigoplus_{j,m_j,\kappa_j} H_{0,m_j,\kappa_j}$, we have that $|H_{0,m_j,\kappa_j}|^{-1} \chi(|x| \leq R)$ is also compact for every j, m_j, κ_j .

Now, the essential spectrum of every H_{0,m_j,κ_j} is the same.

Lemma 4.4.5. *Suppose $H_1 = H_{0,m_{j_1}^1, \kappa_{j_1}^1}$, $H_2 = H_{0,m_{j_2}^2, \kappa_{j_2}^2}$, are obtained by reduction of the free Dirac operator (obtained by taking $f \equiv 1, e = 0$ in the Dirac Hamiltonian H). Then, $\sigma_{ess}(H_1) = \sigma_{ess}(H_2)$.*

Proof. Firstly, we see that by adapting the proof of Theorem 4.3.1, we see H_1 and H_2 are essentially self-adjoint on $D = C_0^\infty(0, \infty)^2 \subset L^2((0, \infty), dx)^2$. Now, $V := H_1 - H_2 = \hbar c(\kappa_{j_1}^1 - \kappa_{j_2}^2) \frac{1}{r(x)} \sigma^1$, where σ^1 is the first Pauli matrix defined in (4.74). We note that V is vanishing at ∞ , so by Lemma 4.4.4, we see that the requirements in equation (4.120) are met. Notice that, with $\kappa = \kappa_{j_1}^1$, for $u = (u_1(x), u_2(x))^T$, $H_1 = -i\hbar c \sigma^2 u'(x) + \hbar c \sigma^1 \frac{\kappa}{r(x)} u(x) + mc^2 \sigma^3 u(x)$. From this, with

$$s = \max \left\{ 1, \frac{|\kappa_{j_1}^1 - \kappa_{j_2}^2|}{\kappa} \right\},$$

we see that

$$\begin{aligned} s^2 \|H_1 u\|^2 &= s^2 \hbar^2 c^2 \|u'(x)\|^2 + s^2 \hbar^2 c^2 \kappa^2 \left\| \frac{u(x)}{r(x)} \right\|^2 + m^2 c^4 \|u\|^2 \\ &\geq \|Vu\|^2 = \hbar^2 c^2 \left(\kappa_{j_1}^1 - \kappa_{j_2}^2 \right)^2 \left\| \frac{u(x)}{r(x)} \right\|^2. \end{aligned} \quad (4.126)$$

The last inequality follows from $|\kappa| \geq 1$ and that s can be chosen. Thus V is H_1 bounded. So, by Lemma 4.4.3, the proof is finished. Since H_0 is unitarily equivalent to $\bigoplus_{j, m_j, \kappa_j} H_{0, m_j, \kappa_j}$, we have that $\sigma_{ess}(H_0) = \overline{\cup \sigma_{ess}(H_{0, m_j, \kappa_j})}$, and therefore using Lemma 4.4.5 and Lemma 4.4.5 we arrive at

$$\sigma_{ess}(H_{0, m_j, \kappa_j}) = (-\infty, -mc^2] \cup [mc^2, \infty) \quad \forall j, m_j, \kappa_j^j. \quad (4.127)$$

□

Finally we are in a position to present the proof of the main theorem.

Proof. (Theorem 4.4.1) We note that the reduced Dirac Hamiltonian on the charged particle-spacetime is $H_{m_j, \kappa} = H_{0, m_j, \kappa_j} + V$, where

$$V := \hbar c \sigma^1 \left(f(r) \frac{\kappa}{r(x)} - \frac{\kappa}{x} \right) + (-1 + f(r)) mc^2 \sigma^3 - e\varphi(r). \quad (4.128)$$

So, since σ^1, σ^2 have eigenvalues ± 1 , we have

$$\|Vu\| \leq \hbar c \kappa \|g_1(x)u(x)\| + mc^2 \|g_2(x)u(x)\| + e \|g_3(x)u(x)\|, \quad (4.129)$$

where $g_1(x) = \frac{f(r)}{r(x)} - \frac{1}{x}$, $g_2(x) = (1 - f(r))$, $g_3(x) = \varphi(r(x))$. The $x(r)$ used here satisfies equation (4.99), which is the same as the one satisfied by \tilde{r} in equation (2.2) as $f(r)^2 = e^{\xi(r)}$. So, by Lemma 2.1.2, part (d), $1/r \leq (1 - A\epsilon^2)^{-1}\frac{1}{x}$. Also, $0 < f(r) < 1$. So, $|g_1(x)| \leq ((1 - A\epsilon^2)^{-1} + 1)(\frac{1}{x})$. Since $f(r), \varphi(r)$ are bounded, $g_2(x), g_3(x)$ are in $L^\infty(0, \infty)$ with $|g_2(x)| \leq 2, |g_3(x)| \leq \varphi(0)$ (because we know φ is decreasing). So, we have

$$\|Vu\| \leq \hbar c \kappa ((1 - A\epsilon^2)^{-1} + 1) \left\| \frac{u(\cdot)}{x} \right\| + 2mc^2 \|u\| + \hbar^2 e \varphi(0) \|u\| \quad (4.130)$$

The norm of the reduced free Dirac operator acting on u on the other hand evaluates to

$$\|H_{0,m_j,\kappa_j} u\|^2 = \hbar^2 c^2 \|u'(x)\|^2 + \hbar^2 c^2 \kappa^2 \left\| \frac{u(x)}{x} \right\|^2 + m^2 c^4 \|u\|^2. \quad (4.131)$$

Therefore, if $a := \max \left\{ (1 + A\epsilon^2)^{-1} + 1, \frac{2m^2 c^2 + \hbar e |\varphi(0)|}{mc^2} \right\}$, then $\|Vu\| \leq a \|H_{0,m_j,\kappa_j} u\|$; that is, V is H_{0,m_j,κ_j} bounded. By Lemma 4.4.3 and Lemma 4.4.4, we have that essential spectrum of the reduced Dirac Hamiltonian on our charged particle-spacetime matches that of the reduced free Dirac operator. Since the Dirac Hamiltonian itself unitarily equivalent to a direct sum of reduced Hamiltonians, the theorem is proven. \square

4.5 Spectral gap around 0

Recall $\epsilon = \sqrt{G} \frac{M}{|Q|}$. Let us define

$$\epsilon' := \sqrt{G} \frac{m}{e}. \quad (4.132)$$

Using a result of Hinton, Mingarelli, Read and Shaw from [19], we prove the following theorem.

Theorem 4.5.1. *Suppose that*

$$\epsilon'(1 - A\epsilon^2)^{1/2} - \frac{3}{2}\epsilon > 0 \quad (4.133)$$

and define η by

$$\eta := m(1 - A\epsilon^2)^{1/2} - e \frac{3}{2} \frac{c^2}{\sqrt{G}}. \quad (4.134)$$

Then, the spectral subset $\sigma(H_{m_j,\kappa_j}) \cap (-\eta, \eta)$ is empty.

Proof. From [19], we use Theorem 3.3, which states (with notations changed to match ours) :

Theorem 4.5.2 ([19]Thm3.3). *Let $-i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $P(r)$ be a real symmetric 2×2 matrix for $r \in (0, \infty)$. Let L be the maximal operator obtained from operator*

$$y \rightarrow -i\sigma^2 y' - P(r)y.$$

Suppose U is an orthogonal matrix such that $\operatorname{Re} \int_0^\infty y^ U(-i\sigma^2)y' dx = 0$ and $UP + P^*U^* \geq 2\eta$, or $UP + P^*U^* \leq -2\eta$. Then, for any $y \in C_c(0, \infty)$, $\|Ly\| \geq \eta\|y\|$.*

We will apply this with $L = \hbar^{-1}c^{-1}H_{m_j, \kappa_j}$. We take $U = -\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, which is

unitary. Then, $-iU\sigma^2 = -\sigma_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, and $-y^*U(-i\sigma^2)y' = \bar{y}_1 y'_2 + \bar{y}_2 y'_1$, whose real part is the derivative of $\operatorname{Re}(\bar{y}_1 y_2)$, and so the requirement on U is met. Finally, keeping in mind that the matrix P has real entries, we compute

$$UP + P^*U^* = 2 \begin{pmatrix} -P_{11} & 0 \\ 0 & P_{22} \end{pmatrix} \quad (4.135)$$

We may write the operator $\hbar^{-1}c^{-1}H_{m_j, \kappa_j}$, as

$$\hbar^{-1}c^{-1}H_{m_j, \kappa_j}y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} y' - \hbar^{-1}c^{-1} \begin{pmatrix} e\varphi(r) - mf(r) & -f(r)\frac{\kappa_j}{r} \\ -f(r)\frac{\kappa_j}{r} & e\varphi(r) + mf(r) \end{pmatrix} y. \quad (4.136)$$

For us, $P_{11} = \hbar^{-1}c^{-1}(e\varphi - mf)$, $P_{22} = \hbar^{-1}c^{-1}(e\varphi + mf)$, and so the matrix $UP + P^*U^*$ is diagonal with diagonal entries $2\hbar^{-1}c^{-1}(mf(r) - e\varphi(r))$, $2\hbar^{-1}c^{-1}(mf(r) + e\varphi(r))$. Since $|\varphi|$ is decreasing and f is increasing, we see that both $-2P_{11}$ and $2P_{22}$ are the least when $\eta_1 = \hbar^{-1}c^{-1}(mf(0) - |e\varphi(0)|)$. We now express this in terms of the dimensionless mass-to-charge ratio ϵ . Notice $f(0) = (1 - A\epsilon^2)^{1/2}$, $\operatorname{sgn}(Q)\phi(0) = \frac{3\epsilon}{2} \frac{c^2}{\sqrt{G}}$, so the gap $\eta_1 = \hbar^{-1}c^{-1}\{m(1 - A\epsilon^2)^{1/2} - \frac{3}{2}|e|\epsilon \frac{c^2}{\sqrt{G}}\}$. Now, by the theorem above, the quadratic form associated to the square of operator L , satisfies $(L^2y, y) = (Ly, Ly) \geq \eta^2\|y\|^2$. But in the charged particle spacetime case, the operator defining L is essentially self-adjoint. So, the minimal and the maximal operators are the same. Therefore, the

spectrum of the self-adjoint $L^2 = \hbar^{-2}c^{-2}H_{m_j, \kappa_j}^2$ skips the interval $(-\eta_1, \eta_1)$, and the same is true about $\hbar^{-1}c^{-1}H_{m_j, \kappa_j}$. This proves the theorem. \square

4.6 Eigenvalues and continuous spectrum

We show that there are infinitely many eigenvalues in the gap of the essential spectrum. We came across the method of proof adopted here in a paper by Belgiorno [4].

Theorem 4.6.1. *Suppose that $-eQ \neq 0$. Then, the following statement holds: the spectral subset $\sigma(H_{m_j, \kappa_j}) \cap (-mc^2, mc^2)$ is non-empty and infinite.*

Remark. Since $(-mc^2, mc^2)$ does not belong to the essential spectrum, the theorem shows existence of infinite number of eigenvalues. Also, since the full Hamiltonian H is a direct sum of the reduced Hamiltonians, it also has infinitely many eigenvalues in the gap.

Proof. We use a theorem from [19]. A positive linear functional on real $n \times n$ matrices is one that evaluates to a non-negative value on symmetric, positive semi-definite matrices.

Theorem 4.6.2 ([19], 2.3). *Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $P(x) = \begin{pmatrix} V_2(x) - c_2 & p(x) \\ p(x) & V_1(x) + c_1 \end{pmatrix}$, where c_1, c_2 are positive numbers, and $V_1(x), V_2(x), p(x)$ are real-valued, locally integrable functions defined on $(0, \infty)$. Let L_1 be any self-adjoint operator defined by*

$$L_1 y = Jy' - P(x)y, \quad (4.137)$$

Suppose that $d > 0$ and g is a positive linear functional on the real $n \times n$ matrices. Then following statements about the operator L_1 are equivalent:

(a) $|\sigma(L_1) \cap (-d, d)| = \infty,$

(b) *the differential equation below is oscillatory at zero or infinity:*

$$-g[I]z'' + g \left[P^2 - d^2 I^2 + \frac{P'J - JP'}{2} \right] z = 0. \quad (4.138)$$

We will apply the theorem to $L_1 = (\hbar c)^{-1}H_{m_j, \kappa_j}$. Let us take $g[B] = (Bu, u)$ with the vector $u = (1, 0)^T$. So, $g[I] = 1$, $g[B] = B_{11}$. Secondly, in our case,

$$\Gamma(x) := g \left[P^2 - d^2 I + \frac{P'J - JP'}{2} \right] = (-mc^2 + e\varphi)^2 (\hbar c)^{-2} + \left(\frac{kf}{r} \right)^2 - d^2 + k \left(\frac{f}{r} \right)'. \quad (4.139)$$

Note that as before a prime indicates differentiation in the variable x . From Corollary 37 in [14], every solution of $-z'' + \Gamma(x)z = 0$ has an infinite number of zeros on a neighbourhood $[a, \infty)$ of ∞ if $\lim_{x \rightarrow \infty} x^2 \Gamma(x) < -\frac{1}{4}$. That is, the differential equation is oscillatory. As $x \rightarrow \infty$, we have, $\varphi \sim \frac{Q}{r}$, $f \sim 1$, $r \sim x$, and $d = mc^2 (\hbar c)^{-1}$, so we see that $\lim_{x \rightarrow \infty} x^2 \Gamma(x) = \lim_{x \rightarrow \infty} x^2 (-2me\varphi(r)) = -\infty$, if $-eQ < 0$.

Now, if $-eQ > 0$, we may take $g[B] = (Bu, u)$, $u = (0, 1)^T$. Then, the only changes in the expression for $\Gamma(x)$ is that $-mc^2$ becomes $+mc^2$ and $+k(f/r)'$ turns into $-\kappa(f/r)'$. So, the same argument above helps us conclude that the resulting differential equation is oscillatory.

Therefore, there are infinitely many eigenvalues of $\hbar^{-1}c^{-1}H_{m_j, \kappa_j}$ in $(-mc^2 (\hbar c)^{-1}, mc^2 (\hbar c)^{-1})$, which proves our theorem. \square

In the next theorem, we determine the continuous spectrum.

Theorem 4.6.3. *For each j, m_j, κ_j , the reduced Dirac Hamiltonian H_{m_j, κ_j} has purely absolutely continuous spectrum in $(-\infty, -mc^2) \cup (mc^2, \infty)$.*

Proof. We prove this theorem using a result from Weidmann's book, which we have paraphrased below.

Theorem 4.6.4 (Theorem 16.7, [34]). *Consider a Dirac type expression τ (definition 4.3.2) on (a, ∞) , for which the matrix $P(r)$ can be written as $P_1(r) + P_2(r)$, where for some $c \in (a, \infty)$ the components of $P_1(r)$ are in $L^1((c, \infty))$, and the components of $P_2(r)$ are of bounded variation in $[c, \infty)$. Suppose also that*

$$\lim_{r \rightarrow \infty} P_2(r) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a \geq b. \quad (4.140)$$

Then, every self-adjoint realization of τ has purely absolutely continuous spectrum in $(-\infty, b) \cup (a, \infty)$.

In our case, we have

$$H_{m_j, \kappa_j} = \hbar c \begin{pmatrix} 0 & -d_x \\ d_x & 0 \end{pmatrix} + \begin{pmatrix} mc^2 f(r) - e\varphi(r) & f(r) \frac{\kappa_j}{r} \\ f(r) \frac{\kappa_j}{r} & -mc^2 f(r) - e\varphi(r) \end{pmatrix}. \quad (4.141)$$

Now, the functions $f(r)$, $\varphi(r)$, $f(r)/r$, considered as functions of x , are of bounded variation in $[1, \infty)$. This is because they are all differentiable functions whose derivative is in $L^1(1, \infty)$, for, from the asymptotics near ∞ given in equations (1.32), $f(r)^2 \sim 1 - 2\frac{GM}{c^2 r} + \frac{GQ^2}{c^4 r^2}$ and $\varphi(r) \sim \frac{Q}{r}$ and $\frac{dr}{dx} = f(r)^2$. Therefore, we may take $P_1(x)$ to be zero and $P_2(x)$ to be the second matrix on the right in equation (4.141). Immediately we see that,

$$\lim_{x \rightarrow \infty} P_2(x) = \begin{pmatrix} mc^2 & 0 \\ 0 & -mc^2 \end{pmatrix}, \quad (4.142)$$

and so our theorem is proved by applying Theorem 4.6.4 with $a = mc^2, b = -mc^2$.

□

Chapter 5

Summary and outlook

We have shown a Strichartz estimate and Morawetz estimate for spherically symmetric waves on a charged particle-spacetime \mathcal{M} . Moving forward in this direction one has to be able to relax the requirement of spherical symmetry on the scalar waves; perhaps, replace it with a milder symmetry assumption, for instance axial symmetry. In that case, we conjecture that the study of solutions of the wave equation on a flat two dimensional cone becomes relevant.

We showed Sommerfeld's method computes the known fundamental solution for the wave equation on a flat two dimensional cone. We still have to investigate the question of why the method works, and in particular, why it results in the fundamental solution to the wave equation with the Friedrichs Laplacian. After that, applications to Riemann surfaces would be the next step. For example, two copies of \mathbb{R}^2 , joined along the two edges of the line segment joining $(1, 0)$ and $(-1, 0)$, thus creating a two-sheeted branched surface. Another interesting question is whether Sommerfeld's method can be generalized to dimensions higher than two.

As for Dirac's equation on a charged particle-spacetime, we have determined the essential spectrum and the continuous spectrum, and settled the question of well-posedness (unique self-adjoint extension). We also showed the existence of a gap around zero in the full spectrum of the Dirac Hamiltonian. While we have shown that there are infinitely many eigenvalues, it would be good to compute the eigenvalues exactly or numerically.

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