SOME RESULTS ON THE REPRESENTATION THEORY OF VERTEX OPERATOR ALGEBRAS AND INTEGER PARTITION IDENTITIES

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ABSTRACT OF THE DISSERTATION

Some results on the representation theory of vertex operator algebras and integer partition identities

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Integer partition identities such as the Rogers-Ramanujan identities have deep relations with the representation theory of vertex operator algebras, among many other fields of mathematics and physics. Such identities, when written in generating function form typically take the shape "product side" = "sum side." In some vertex-operator-algebraic settings, the product sides arise naturally, and the problem is to explain, interpret and prove the sum sides, while some other settings pose an opposite problem. In this thesis, we provide some results on both types of problems. In Part I of this thesis, we interpret the sum sides of the Göllnitz-Gordon identities using Lepowsky-Wilson's Z-algebraic constructions applied to certain principally twisted level 2 standard modules for $A_5^{(2)}$. In Part II, we give, following Dong-Lepowsky, explicit constructions for certain higher level twisted intertwining operators for $\widehat{\mathfrak{sl}_2}$; these constructions are inspired by a desire to interpret Andrews-Baxter's q-series theoretic "motivated proof" of the Rogers-Ramanujan identities and more generally, motivated proofs of the Gordon-Andrews and the Andrews-Bressoud identities given by Lepowsky-Zhu and Kanade-Lepowsky-Russell-Sills, respectively. These motived proofs are about explaining the "sum sides" starting with the "product sides." In Part III, following an idea of J. Lepowsky, we introduce and analyze a Koszul complex related to the principal subspace of the level 1

vacuum module of $\widehat{\mathfrak{sl}_2}$; this construction is expected to yield a "character formula" for the principal subspaces, thereby explaining the emergence of "product sides."

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Dedication

To Aai, Baba, Aajee, Aabaa, Aditya and Anagha.

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Chapter 1

Introduction

Given a positive integer n, by a *partition* π of n we mean a non-increasing sequence of positive integers $\pi = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $n = \lambda_1 + \cdots + \lambda_r$. Each λ_i is called a *part* of π. Identities involving integer partitions have a long-standing history and arise in various branches of mathematics and physics; see for example [A3]. The pair of Rogers-Ramanujan identities is a remarkable example. These identities state that:

1. The partitions of a positive integer n into parts congruent to $1, 4 \pmod 5$ are equinumerous with the partitions in which adjacent parts differ by at least 2.

2. The partitions of a positive integer n into parts congruent to $2, 3 \pmod 5$ are equinumerous with the partitions in which adjacent parts differ by at least 2 and such that smallest part is at least 2.

With q being a purely formal variable, in generating function form, the identities read as:

$$\prod_{j\geq 0} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})} = \sum_{n\geq 0} d_1(n)q^n \tag{1.0.1}$$

$$\prod_{j\geq 0} \frac{1}{(1-q^{5j+2})(1-q^{5j+3})} = \sum_{n\geq 0} d_2(n)q^n, \tag{1.0.2}$$

where $d_i(n)$ for $i = 1, 2$, is the number of partitions of n such that the adjacent parts differ by at least 2 and such that the smallest part is at least i. As a convention, $n = 0$ has exactly one partition — the null partition. The left-hand sides of these identities are called the "product sides," while right-hand sides are referred to as the "sum sides."

These identities and their analogues and generalizations arise in various fields of mathematics and physics, such as representation theory, theory of vertex operator algebras, number theory, knot theory, algebraic geometry, statistical mechanics, conformal field theory, etc. This ubiquity is one of the prime reasons for their importance. The depth of such identities could be gauged by the fact that the bijective proof of the Rogers-Ramanujan identities given by A. Garsia and S. Milne in [GM] runs for about 50 pages!

In the vertex algebraic settings, such identities appear in many contexts. Sometimes, the product sides arise very naturally and the task is to explain, interpret and/or prove the sum sides using vertex operator theoretic mechanisms, and sometimes, the sum sides or the natural recursions governing the sums arise very naturally, and the task is to explain, interpret and/or prove the product sides.

In this thesis, we will present ideas and results based on both of these directions, with the "product to sum" direction explored in Chapters 2 and 3 and the "sum to product" direction explored in Chapter 4. With the introduction that follows, the chapters could be read independently of one another.

1.1 Products to sums: Lepowsky-Wilson's Z-algebras

J. Lepowsky and S. Milne observed in [LM] that the product sides of a certain class of integer partition identities (including the Rogers-Ramanujan identities) arise naturally, up to factors, known as "fudge factors," as the principally specialized characters of standard modules for the affine Lie algebras $A_1^{(1)}$ and $A_2^{(2)}$. Now, the philosophical problem was to "explain" the sum sides using the representation theory of affine Lie algebras. Lepowsky and R. L. Wilson, in a series of papers [LW1]–[LW4] achieved this by inventing "principal Heisenberg subalgebras" (generalized in [KKLW]) and "Z-algebras". They proved that the vacuum spaces, with respect to the principal Heisenberg subalgebra, of the level 3 standard modules for the affine Lie algebra $A_1^{(1)}$ have bases formed by certain monomials in the Z-operators applied to a highest weight vector. They showed that these monomials are enumerated precisely by the partitions satisfying the difference 2 conditions, thereby giving a completely representation theoretic proof of the Rogers-Ramanujan identities. They went on to interpret (by providing "small enough" spanning sets for the vacuum spaces) the Andrews-Gordon and the Andrews-Bressoud identities using the higher level standard modules for $A_1^{(1)}$. Building on these ideas, A. Meurman and M. Primc in [MP1] *proved* all of these identities using the higher level standard modules for $A_1^{(1)}$. The structure of certain standard modules for several affine Lie algebras was analyzed by K. C. Misra in [Mi1]–[Mi4] and by M. Mandia in [Ma]. For a review of these and related developments see for example [L2].

Using this program, S. Capparelli in [C1]–[C2] found remarkable partition identities by investigating the level 3 standard modules for $A_2^{(2)}$. Recently, spectacular new identities have been conjectured by D. Nandi in [N] corresponding to the level 4 standard modules for $A_2^{(2)}$. In [KR], using "experimental mathematics," we have conjectured six new partition identities, three of which are related to the level 3 standard modules for $D_4^{(3)}$, but we mention here that our identities are yet to be interpreted by vertex algebraic methods.

Lepowsky-Wilson's Z-algebras are universal, in the sense that they "work" for any affine Lie algebra at any level; however, their implementation for interpreting (and proving) the sum sides depends on the algebra and the level and can be quite subtle, even in those cases where explicit sum sides have been constructed.

It is worth noting that the invention of Z-algebras was a very important milestone in representation theory. This was the first time vertex operators were invented on the mathematical side. Ideas stemming from Lepowsky-Wilson's work led, along with many other developments, to other fascinating discoveries, for instance, the Frenkel-Lepowsky-Meurman's construction [FLM] of the famous $V^\natural$ — the natural infinite dimensional space on which the Monster (the largest sporadic group) acts, etc. There is a vast literature on the theory of vertex operator algebras; see for instance, [Bor], [FLM], [FHL], [DL], [LL], [HL1]–[HL3], [H4] — works that we will be using in the present work.

In Chapter 2, we carry forward the program of vertex algebraic interpretation of the combinatorial identities and explicit constructions of modules for affine Lie algebras. We use the Z-algebra approach to give, for the first time, a vertex-operator-theoretic interpretation of the pair of Göllnitz-Gordon identities; cf. Chapter 7 of [A3]. We achieve this by analyzing the structure of certain principally twisted level 2 standard modules for the affine Lie algebra $A_5^{(2)}$. We focus on those modules that are contained in the tensor product of the two inequivalent level 1 modules for $A_5^{(2)}$.

We recall here that the pair of Göllnitz-Gordon identities states:

1. The partitions of a positive integer n into parts congruent to $1, 4, 7 \pmod 8$ are equinumerous with the partitions in which adjacent parts differ by at least 2, with adjacent even parts differing by at least 4.

2. The partitions of a positive integer n into parts congruent to $3, 4, 5 \pmod 8$ are equinumerous with the partitions in which adjacent parts differ by at least 2 with adjacent even parts differing by at least 4 such that all of the parts are greater than 2.

Our main contribution in this direction is the following theorem, which interprets the Göllnitz-Gordon identities Z-algebraically:

Theorem 1.1.1. *Enumerate the nodes of affine Dynkin diagram of $A_5^{(2)}$ in the usual way (cf. [K]). With $L(\Lambda_0+\Lambda_1)$ and $L(\Lambda_3)$ being the indicated level 2 standard modules for $A_5^{(2)}$ and with $\Omega(\cdot)$ denoting the vacuum space with respect to the principal Heisenberg subalgebra, we have the following spanning sets constructed from Z-operators applied to highest weight vector:*

$$\Omega(L(\Lambda_0+\Lambda_1)) = Span\{Z_{i_1}\cdots Z_{i_r}\cdot v_{L(\Lambda_0+\Lambda_1)} \,|\, r\in\mathbb{N},\, i_1 < i_2 < \cdots < i_r \leq -1,$$
$$|i_j - i_{j+1}| \geq 2 \text{ with } |i_j - i_{j+1}| \geq 4 \text{ if } i_j, i_{j+1} \text{ are even}\},$$
$$\Omega(L(\Lambda_3)) = Span\{Z_{i_1}\cdots Z_{i_r}\cdot v_{L(\Lambda_3)} \,|\, r\in\mathbb{N},\, i_1 < i_2 < \cdots < i_r \leq -3,$$
$$|i_j - i_{j+1}| \geq 2 \text{ with } |i_j - i_{j+1}| \geq 4 \text{ if } i_j, i_{j+1} \text{ are even}\}.$$

Here, for odd i, Z_i is the coefficient of ζ^i in $Z(\alpha_1,\zeta)$ and for even i, Z_i is the coefficient of ζ^i in $Z(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5,\zeta)$ where α_j for $1\leq j\leq 5$ are simple roots of A_5.

For further q-series theoretic and number theoretic discussions of the Göllnitz-Gordon-Andrews identities, and in particular the Göllnitz-Gordon identities, see [A3], [Göl], [G1]–[G2], [CKLMQRS]. For a short history of these identities, see for instance [SW].

1.2 Products to sums: Motivated proofs and intertwining operators

On the purely combinatorial and/or q-series-theoretic sides, there are many known proofs of the Rogers-Ramanujan identities. However, from the viewpoint of vertex operator algebras, there is one that stands out, namely, the "motivated proof" given by G. Andrews and R. Baxter in [AB], even though Andrews and Baxter were working entirely q-series-theoretically.

Using the right-hand sides of (1.0.1) and (1.0.2), i.e., the sum sides, it is easy to see that the number of partitions of n enumerated by the first Rogers-Ramanujan identity is at least as great as the number of partitions enumerated by the second identity. An explanation of this phenomenon using *only the product sides* was asked for by L. Ehrenpreis. Motivated by this question, Andrews and Baxter were in fact led to a proof of the Rogers-Ramanujan identities in [AB], and they remarked that this proof was essentially the same as an earlier proof of Rogers-Ramanujan and of Baxter.

Let us write the products, i.e., the generating functions of the left-hand sides in the Rogers-Ramanujan identities, as $G_1(q)$ and $G_2(q)$, respectively. Andrews and Baxter consider the following recursively defined sequence of power series:

$$G_i = (G_{i-1} - G_{i-2})/q^{i-2} \quad \text{for } i = 3, 4, \ldots. \tag{1.2.1}$$

They observe empirically that

$$G_i = 1 + q^i + \cdots .$$

Proving this observation (what they called the "Empirical Hypothesis") not only led to an answer of Ehrenpreis's question, but also led to a proof of the Rogers-Ramanujan identities themselves.

Gordon's identities generalize the Rogers-Ramanujan identities to all odd moduli (cf. Chapter 7, [A3]). Recently, Lepowsky and M. Zhu gave a motivated proof of Gordon's identities in [LZ]. This proof is a generalization of the Andrews-Baxter proof, with some new structure; in particular, a certain "shelf picture," which is now known to be fundamental for such motivated proofs in our works in general, and which was implicit in [AB], was made transparent in [LZ]. For a fixed modulus $2k+1$, the given products in

Gordon's identities constitute the zeroth shelf, and then successively higher shelves are created recursively from each previous shelf by the use of appropriate subtractions and divisions by *pure powers of* q (as in equation (1.2.1) for the case $k = 2$). This division by pure powers of q is what will be important to us from vertex-algebraic considerations. We remark here that an analogue of this motivated proof for the Göllnitz-Gordon-Andrews identities is given in [CKLMQRS] and for the Andrews-Bressoud identities in [KLRS]. Note that for a fixed odd modulus $2k + 1$, the Gordon-Andrews identities correspond to the level $2k - 1$ standard modules for $A_1^{(1)}$ and that for a fixed even modulus $2k$, the Andrews-Bressoud identities correspond to the level $2k - 2$ standard modules for $A_1^{(1)}$.

There are important philosophical similarities and differences between Lepowsky-Wilson's Z-algebraic approach and the Andrews-Baxter's motivated proof. For both of these, the starting point is the pair of product sides, and the idea is to both motivate and prove the corresponding sum sides. However, these approaches differ, in that, in some sense, Lepowsky-Wilson's approach treats one module at a time, in other words, proves one identity at a time, while the motivated proof approach moves back and forth between the identities (as is evident from (1.2.1)) and thus alternates between the modules.

It was an idea of Lepowsky and A. Milas that the recursive definition of the G_i's in the motivated proof could be "explained" by means of exact sequences among the vacuum spaces, with respect to the principal Heisenberg subalgebra, of level 3 standard modules for $A_1^{(1)}$, where the maps in these exact sequences should arise from what are known as the relativized and twisted intertwining operators naturally arising in the theory of vertex operator algebras, as developed in [DL] and other works. This program of "categorification" of the motivated proof is ongoing. It is expected that this program, once completed, will provide crucial insight into the representation theory of vertex algebras and will also aid in the discovery and proof of new partition identities.

As a first step, in Chapter 3, we explicitly construct twisted intertwining operators among certain mixed triples of untwisted and twisted modules for the vertex operator algebra $V = L_{\widehat{\mathfrak{sl}_2}}(\ell, 0)$ based on the "vacuum" standard $\widehat{\mathfrak{sl}_2}$-module of level ℓ, a positive integer. We refer the reader to [LL] for notation. Note that twisted intertwining operators among twisted modules for colored vertex superalgebras have been previously considered by X. Xu in [Xu]. We focus our attention to the twisted modules obtained by a certain involution θ of $L_{\widehat{\mathfrak{sl}_2}}(\ell, 0)$. For $\ell = 1$, θ is obtained from the -1 isometry of the root lattice of $\mathfrak{sl}_2$. We start with the twisted intertwining operators for basic modules as in [Ab1] and[ADL], and we give exact analogues of constructions in [DL] for our setting. Specifically, we prove that:

Theorem 1.2.1. *There exist explicitly constructed twisted intertwining operators among certain mixed triples of untwisted and θ-twisted modules for the vertex operator algebra $V = L_{\widehat{\mathfrak{sl}_2}}(\ell, 0)$, for a positive integer ℓ.*

We are currently investigating the properties of these intertwining operators.

In Chapter 3 we also give an abelian intertwining algebra structure that incorporates the untwisted and twisted intertwining operators mentioned above. We give two methods — a direct approach, following a suggestion of C. Sadowski and an approach as carried out in [H1] using Huang-Lepowsky's tensor category theory [HL1]–[HL3], [H4]. We prove that:

Theorem 1.2.2. *Let $V = V_{\mathbb{Z}\alpha}$ be the rank 1 lattice vertex algebra such that $\langle \alpha, \alpha \rangle = 2$. Let θ be a lift to V of the -1 automorphism of the lattice $L = \mathbb{Z}\alpha$, Let V^{T_1}, V^{T_2} be the inequivalent irreducible θ-twisted modules for V. There exists a natural abelian intertwining algebra structure on the space $V \oplus V_{(\mathbb{Z}+1/2)\alpha} \oplus V^{T_1} \oplus V^{T_2}$, such that the grading group is $G \cong \mathbb{Z}/8$ and such that the Y map for the abelian intertwining algebra (see [DL]) is comprised of twisted intertwining operators.*

An abelian intertwining algebra is one of the simplest structures that naturally generalizes the notion of a vertex operator algebra and is essentially comprised of the intertwining operators. Roughly speaking, an abelian intertwining algebra is formed when the fusion is relatively simple, i.e., when the fusion algebra is the group algebra of a finite abelian group. Each of the constituent intertwining operators could be scaled independently of each other and thus normalized 3-cocycles for the fusion group G enter the picture. An even more general structure is that of an intertwining algebra, as defined by Y.-Z. Huang in [H2], which heavily rests on the tensor category structure of the modules for the vertex operator algebra in question.

Whenever one wants to move between the modules, as is the case for the motivated proofs, such intertwining algebras provide the natural setting to look for the maps involved. In a similar setup, untwisted rather than twisted (recalled below), for the principal subspaces, abelian intertwining algebras based on level 1 modules for the affine Lie algebras $A_N^{(1)}, D_N^{(1)}$ and $E_N^{(1)}$ have been successfully employed by C. Calinescu, S. Capparelli, J. Lepowsky, A. Milas and C. Sadowski; see [CLM1]– [CLM2], [CalLM1]–[CalLM4], [Sa1]–[Sa3]. See also [MilP], which generalizes the previous untwisted level 1 constructions.

1.3 Sums to products: Principal subspaces

Motivated by an earlier work of Lepowsky and Primc, [LP], B. Feigin and A. Stoyanovsky in [FS1]–[FS2] introduced and studied "principal subspaces" of standard modules for affine Lie algebras. Assuming a certain generators-and-relations result for the principal subspaces, they demonstrated how principal subspaces of standard modules for $A_1^{(1)}$ at appropriate levels exhibit the difference-2 conditions in the Rogers-Ramanujan identities and more generally, the Andrews-Gordon identities. Employing the geometry of infinite dimensional flag manifolds, they showed how the product sides of these identities arise. In [FFJMM], Feigin et al. found a different way to calculate the "bosonic formulas" (which in our case are infinite products) for the principal subspaces of standard modules for $A_2^{(1)}$. For other relevant works, we refer the reader to [FL] and [FJLMM].

There are natural recursions that govern the sum sides in the Andrews-Gordon identities, called the Rogers-Selberg recursions. For example, in the special case of the Rogers-Ramanujan identities, these recursions specialize to the Rogers-Ramanujan recursion:

$$F(x,q) = F(xq,q) + xqF(xq^2,q), \tag{1.3.1}$$

where

$$F(x,q) = \sum_{m,n\geq 0} d_{m,n} x^m q^n,$$

with $d_{m,n}$ being the number of partitions of n into exactly m parts, such that the parts satisfy the difference-2 condition. Note here that $F(1,q)$ is equal to the right-hand side of (1.0.1) and $F(q,q)$ is equal to the right-hand side of (1.0.2). Capparelli, Lepowsky and Milas in [CLM1] gave an elegant way to interpret the recursion (1.3.1) using exact sequences among the principal subspaces for the basic $A_1^{(1)}$ modules, where the maps came from the intertwining operators among triples of these basic modules. They generalized this interpretation to the Rogers-Selberg recursions using higher level modules for $A_1^{(1)}$ in [CLM2]. However, these works also assumed the presentation result that was assumed by [FS1]–[FS2].

Later, Calinescu, Lepowsky and Milas in [CalLM1]–[CalLM4] developed a systematic vertex-operator-theoretic mechanism to provide "a priori" proofs of these presentation results. "A priori" means that the proofs did not rely on knowledge of bases of the principal subspaces themselves.

Various authors have made remarkable progress in analyzing the structure of principal subspaces, providing recursions for their characters and calculating corresponding "sum side" representations. For a detailed history of the recent progress, we refer the reader to the Introductions of [Sa1]–[Sa2].

One is now naturally led to the question of exhibiting the "product sides" for the characters of principal subspaces using purely vertex-operator-theoretic methods and without invoking the underlying geometric structure. As yet, there is no known general character formula for principal subspaces analogous to the Weyl-Kac character formula for the standard modules; see however [FS1]–[FS2].

It was an idea of Lepowsky that such an "abstract" character formula could be obtained by using a Koszul resolution for the principal subspaces. We note that a precise description for the defining ideal of the principal subspaces is a crucial ingredient for such a construction. Thereafter, one could perhaps use the Garland-Lepowsky resolution of the ambient standard module in terms of the generalized Verma modules to gain information about the homology of this Koszul complex.

Let us work with the algebra $A_1^{(1)}$, i.e., $\widehat{\mathfrak{sl}_2}$, and let

$$\mathfrak{n} = \mathbb{C}x_\alpha,$$

where x_α is a root vector corresponding to the root α, and let $\hat{\mathfrak{n}}$ be its affinization. In general, $\mathfrak{n}$ will be the sum of the positive root spaces. Then, the *principal subspace* associated to a standard $\widehat{\mathfrak{sl}_2}$-module $L(\Lambda)$ is defined as:

$$W_\Lambda = \mathcal{U}(\hat{\mathfrak{n}}) \cdot v_\Lambda,$$

where v_Λ is a highest weight vector and $\mathcal{U}(\cdot)$ denotes the universal enveloping algebra. One of the main theorems of [CalLM1] states that the kernel (called $\mathcal{I}_{\Lambda_0}$) of the natural map

$$f_{\Lambda_0} : \mathcal{U}(\hat{\mathfrak{n}}_-) \longrightarrow W_{\Lambda_0}, \quad a \longmapsto a \cdot v_{\Lambda_0} \tag{1.3.2}$$

is generated (in a natural vertex-algebraic sense) by the singular vector $x_\alpha(-1)^2 \cdot \mathbf{1}$. This theorem has since been generalized, by various authors, to higher levels and ranks.

The case of $\widehat{\mathfrak{sl}_2}$ is peculiar, in that $\mathcal{U}(\hat{\mathfrak{n}})$ is a commutative algebra. Let

$$\mathcal{A} = \mathcal{U}(\hat{\mathfrak{n}}_-) \cong \mathbb{C}[x_{-1}, x_{-2}, x_{-3}, \ldots].$$

For $n \geq 2$, let

$$r_{-n} = \sum_{i=1}^{n-1} x_{-i}x_{-n+i} = x_{-1}x_{-n+1} + x_{-2}x_{-n+2} + \cdots + x_{-n+1}x_{-1}.$$

From [CalLM1], we have the presentation

$$W_{\Lambda_0} \cong \mathcal{A}/\mathcal{A}\langle r_{-n} \,|\, n \geq 2\rangle.$$

Now consider the following complex consisting of free $\mathcal{A}$-modules:

$$\cdots \xrightarrow{\partial_4} C_3 = \bigoplus_{i_1, i_2 \geq 2} \mathcal{A}\xi_{-i_1,-i_2} \xrightarrow{\partial_3} C_2 = \bigoplus_{i_1 \geq 2} \mathcal{A}\xi_{-i_1} \xrightarrow{\partial_2} C_1 = \mathcal{A} \xrightarrow{\partial_1} C_0 = W_{\Lambda_0} \twoheadrightarrow 0$$

where $\xi_{\ldots}$ are formal symbols with

$$\xi_{\ldots,i,\ldots,j,\ldots} = -\xi_{\ldots,j,\ldots,i,\ldots}$$

and such that for $k \geq 1$,

$$\partial_{k+1}(\xi_{-i_1,-i_2,\cdots,-i_k}) = \sum_{n=1}^{k}(-1)^{n-1} \cdot r_{-i_n} \cdot \xi_{-i_1,-i_2,\cdots,\widehat{-i_n},\cdots,-i_k}.$$

The problem now is to find a presentation and the graded dimension of the homology (viewed as a differential-graded algebra) of this complex so that the graded dimension of the bottom level — which is the principal subspace in question — could be obtained by the use of the Euler-Poincaré principle. It should be noted that the sequence of elements r_{-n} for $n = 2, 3, \ldots$ is a non-regular sequence and hence the problem of determining the homology is quite non-trivial.

Interestingly, a certain "finite version" of this very complex is also conjectured to arise in connection with the stable Khovanov homology of the torus knots $T(m, n)$ in the work [GOR] of Gorsky, Oblomkov and Rasmussen. Gorsky et al. also conjecture a generators-and-relations type description of the homology (viewed as a differential-graded algebra) of this "finite version" of the Koszul complex. Our results give evidence for their description and provide hope that vertex-operator-algebraic techniques could provide crucial insights for studying this homology.

The first kernel, $\mathrm{Ker}(\partial_1)$, is precisely the kernel $\mathcal{I}_{\Lambda_0}$ (see above) from [CalLM1]. In Chapter 4, using analogous techniques, we prove that the second homology is generated by the "next" singular vector in the Garland-Lepowsky resolution of the ambient standard module. The precise statement of our theorem is:

Theorem 1.3.1. *The Virasoro operator L_{-1} acting on $\mathcal{A}$ and W_{Λ_0} can be extended naturally to each of the C_j's in such a way that*

$$L_{-1}(r \cdot c) = L_{-1}(r) \cdot c + r \cdot L_{-1}(c),$$

for all $r \in \mathcal{A}$ and $c \in C_j$. Moreover, L_{-1} commutes with $\partial_\bullet$. With this,

$$Ker(\partial_2) = \langle L_{-1}^s \cdot (2\xi_{-2}x_{-2} - \xi_{-3}x_{-1}) \,|\, s \in \mathbb{N}\rangle + Im(\partial_3).$$

The vector $2\xi_{-2}x_{-2} - \xi_{-3}x_{-1}$ is precisely the "next" singular vector in the Garland-Lepowsky resolution of the standard module $L(\Lambda_0)$.

Among other things, this result and its "finite" analogue were conjectured in [GOR]. A similar complex and its homology have been analysed by Feigin in [Fe] in the context of Bernstein-Gelfand-Gelfand-type resolutions of certain minimal models for the Virasoro algebra.

The structure of higher kernels is under investigation. It is interesting to note that in the works [MP2]–[MP3] and [P2], vertex operators parametrized by the natural analogues, to higher ranks and levels, of the singular vector $2\xi_{-2}x_{-2} - \xi_{-3}x_{-1}$ play an important role in the determination of generators of relations for the annihilating fields of standard modules. We are currently investigating how these works could help us generalize our result to higher kernels and to higher ranks and levels.

Chapter 2

From products to sums: The Göllnitz-Gordon identities

As recalled in the Introduction, the pair of Göllnitz-Gordon identities says that:

1. The partitions of a positive integer n into parts congruent to $1, 4, 7 \pmod 8$ are equinumerous with the partitions in which adjacent parts differ by at least 2, with adjacent even parts differing by at least 4.

2. The partitions of a positive integer n into parts congruent to $3, 4, 5 \pmod 8$ are equinumerous with the partitions in which adjacent parts differ by at least 2 with adjacent even parts differing by at least 4 such that all of the parts are greater than 2.

In this chapter, we give a vertex-operator-algebraic interpretation, using the techniques of Z-algebras, of these identities using those level 2 modules for $A_5^{(2)}$ that are contained in the tensor products of two inequivalent level 1 modules for $A_5^{(2)}$. See [K] for affine Lie algebras, but we shall need the vertex-operator-calculus constructions in [L1].

Our vertex-algebraic interpretation clearly exhibits the "asymmetry" between the even parts and the odd parts. For the vacuum spaces of the two level 2 modules in question, we exhibit spanning sets enumerated by partitions counted in the sum sides of the corresponding Göllnitz-Gordon identity. In our spanning sets, the even and the odd parts arise from two distinct families of Z operators, corresponding to two distinct nodes of the Dynkin diagram of $A_5^{(2)}$. In other words, each of the Göllnitz-Gordon identities is interpreted as a two-color identity that in fact "degenerates" to an honest (single-color) partition identity since one of the colors exclusively appears as odd parts and the other as even. Our methods are very similar to the ones used in [T1], [T2] for analyzing the structure of level 2 standard modules for $D_{l+1}^{(2)}$ and $D_4^{(3)}$, respectively.

There exists a natural generalization of the Göllnitz-Gordon identities to higher moduli, namely, the Göllnitz-Gordon-Andrews identities. However, our interpretation of the Göllnitz-Gordon identities points toward a yet another natural and genuinely multi-color generalization of the Göllnitz-Gordon identities arising from the higher level modules for $A_5^{(2)}$.

We remark that a number of times we will need to find power series expansions of certain rational functions. These computations can be done quickly using a computer algebra system. Nonetheless, we shall provide all the details of these calculations for the sake of completeness. For convenience of such calculations, we will need to fix a primitive 10^{th} root of unity, which we take to be the one that has $\pi/5$ as its argument.

2.1 The affine Lie algebra $A_5^{(2)}$ in the principal picture

Closely following [LW3], [L1] and [F], we first construct the affine Lie algebra $A_5^{(2)}$ in the principal picture.

Let Φ be the root system of type A_5, with the system of positive simple roots $\Delta = \{\alpha_1, \ldots, \alpha_5\}$. Let L be the root lattice. Let $\langle \cdot, \cdot \rangle$ be the natural symmetric positive-definite bilinear form on L such that $\langle \alpha_i, \alpha_i \rangle = 2$, $\langle \alpha_i, \alpha_j \rangle = -1$ if $|i-j| = 1$ and $\langle \alpha_i, \alpha_j \rangle = 0$ if $|i-j| > 1$. Let σ be the automorphism of $(L, \langle \cdot, \cdot \rangle)$ induced by the diagram automorphism of Φ:

$$\sigma : \alpha_1 \longleftrightarrow \alpha_5 \tag{2.1.1}$$

$$\alpha_2 \longleftrightarrow \alpha_4 \tag{2.1.2}$$

$$\alpha_3 \longleftrightarrow \alpha_3. \tag{2.1.3}$$

and let σ_i be the reflection about the positive simple root α_i, i.e.,

$$\sigma_i(\alpha) = \alpha - 2\frac{\langle \alpha, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i \tag{2.1.4}$$

As in [F], the twisted Coxeter automorphism of Φ is $\nu = \sigma_1\sigma_2\sigma_3\sigma$.

Let $m = 10$ be the order of ν and let ω be a primitive root m^{th} of unity. For convenience, we may and do choose $\omega = e^{2\pi \mathbf{i}/10}$.

Under ν, the elements of Φ fall into the following orbits:

	[1]	[2]	[3]
$\mathbf{1}$	α_3	$\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5$	α_1
ν^1	$-\alpha_1-\alpha_2-\alpha_3$	$\alpha_2+\alpha_3+\alpha_4+\alpha_5$	α_5
ν^2	$-\alpha_4-\alpha_5$	$\alpha_2+\alpha_3+\alpha_4$	α_2
ν^3	$-\alpha_2-\alpha_3$	$\alpha_3+\alpha_4$	$\alpha_1+\alpha_2+\alpha_3+\alpha_4$
ν^4	$-\alpha_4$	$-\alpha_1-\alpha_2$	$\alpha_3+\alpha_4+\alpha_5$
ν^5	$-\alpha_3$	$-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5$	$-\alpha_1$
ν^6	$\alpha_1+\alpha_2+\alpha_3$	$-\alpha_2-\alpha_3-\alpha_4-\alpha_5$	$-\alpha_5$
ν^7	$\alpha_4+\alpha_5$	$-\alpha_2-\alpha_3-\alpha_4$	$-\alpha_2$
ν^8	$\alpha_2+\alpha_3$	$-\alpha_3-\alpha_4$	$-\alpha_1-\alpha_2-\alpha_3-\alpha_4$
ν^9	α_4	$\alpha_1+\alpha_2$	$-\alpha_3-\alpha_4-\alpha_5$
ν^{10}	α_3	$\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5$	α_1

Following [F], for $\alpha, \beta \in L$ define

$$\varepsilon(\alpha,\beta) = \prod_{p=1}^{m-1} (1-\omega^{-p})^{\langle \nu^p \alpha, \beta \rangle}. \tag{2.1.5}$$

Then, we have that for any $\alpha, \beta, \gamma \in L$,

$$\varepsilon(\alpha+\beta,\gamma) = \varepsilon(\alpha,\gamma)\varepsilon(\beta,\gamma), \tag{2.1.6}$$

$$\varepsilon(\alpha,\beta+\gamma) = \varepsilon(\alpha,\beta)\varepsilon(\alpha,\gamma), \tag{2.1.7}$$

$$\frac{\varepsilon(\alpha,\beta)}{\varepsilon(\beta,\alpha)} = (-1)^{\langle \alpha,\beta \rangle}, \tag{2.1.8}$$

$$\varepsilon(\nu\alpha,\nu\beta) = \varepsilon(\alpha,\beta). \tag{2.1.9}$$

As in [F], [FLM], using $L, \langle \cdot, \cdot \rangle, \nu, \varepsilon(\cdot,\cdot)$, one can construct a finite-dimensional simple Lie algebra $\mathfrak{g}$ of type A_5, with an invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$, and an automorphism ν that preserves this form as follows:

Let $\mathfrak{g}$ be the vector space over $\mathbb{C}$ spanned by the symbols $\Delta \cup \{x_\alpha \,|\, \alpha \in \Phi\}$. Let $\mathfrak{a}$ be the span of Δ. Define the bracket $[\cdot,\cdot]$ on $\mathfrak{g}$ by

$$[\alpha_i, x_\alpha] = \langle \alpha_i, \alpha \rangle x_\alpha = -[x_\alpha, \alpha_i] \tag{2.1.10}$$

$$[x_\alpha, x_\beta] = \begin{cases} \varepsilon(-\alpha,\alpha)\alpha & \text{if } \langle\alpha,\beta\rangle = -2 \\ \varepsilon(\alpha,\beta)x_{\alpha+\beta} & \text{if } \langle\alpha,\beta\rangle = -1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.11}$$

The symmetric invariant bilinear form on $\mathfrak{a}$ can be extended to $\mathfrak{g}$ as:

$$\langle\alpha, x_\beta\rangle = 0 \tag{2.1.12}$$

$$\langle x_\alpha, x_\beta\rangle = \varepsilon(\alpha,\beta)\delta_{\alpha+\beta,0}. \tag{2.1.13}$$

Extend the ν acting on $\mathfrak{a}$ to $\nu : \mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$\nu x_\alpha = x_{\nu\alpha}. \tag{2.1.14}$$

Proceeding as in Capter 6 of [F], we now construct the twisted affine Lie algebras $\widehat{\mathfrak{g}}(\nu)$, $\tilde{\mathfrak{g}}(\nu)$ of type $A_5^{(2)}$. For $p \in \mathbb{Z}_m$, let

$$\mathfrak{g}_{(p)} = \{x \in \mathfrak{g} \,|\, \nu x = \omega^p x\},$$

similarly define $\mathfrak{a}_{(p)}$. Let π_p be the projection map $\mathfrak{g} \longrightarrow \mathfrak{g}_{(p)}$. We denote the map $^-: \mathbb{Z} \longrightarrow \mathbb{Z}_m$. For any $x \in \mathfrak{g}$, let $x_{(j)}$ denote $\pi_{\bar{j}}(x)$.

Let

$$\tilde{\mathfrak{g}}(\nu) = \left(\bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{(\bar{i})} \otimes t^i\right) \oplus \mathbb{C}c \oplus \mathbb{C}d, \hat{\mathfrak{g}}(\nu) = \left(\bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{(\bar{i})} \otimes t^i\right) \oplus \mathbb{C}c. \tag{2.1.15}$$

such that

$$[x \otimes t^i, y \otimes t^j] = [x,y] \otimes t^{i+j} + \frac{1}{m} i\delta_{i+j,0}\langle x,y\rangle c, \tag{2.1.16}$$

$$[c, \tilde{\mathfrak{g}}(\nu)] = 0, \tag{2.1.17}$$

$$[d, x \otimes t^i] = ix \otimes t^i, \tag{2.1.18}$$

for all $x \in \mathfrak{g}_{(\bar{i})}$, $y \in \mathfrak{g}_{(\bar{j})}$ with $i, j \in \mathbb{Z}$. We define $\tilde{\mathfrak{a}}(\nu)$ and $\hat{\mathfrak{a}}(\nu)$ similarly. The Lie algebra $\hat{\mathfrak{a}}(\nu)$ is a Heisenberg subalgebra of $\widehat{\mathfrak{g}}(\nu)$.

From [F] we know that $\widehat{\mathfrak{g}}(\nu)$ is isomorphic to the principal realization of the affine Lie algebra $A_5^{(2)}$. Let $\{h_i, e_i, f_i \,|\, i = 0, \ldots 3\}$ be the canonical generators of $A_5^{(2)}$. We know that

$$c = h_0 + h_1 + 2h_2 + 2h_3. \tag{2.1.19}$$

Let

$$\mathfrak{t} = \mathrm{span}\{h_0, h_1, h_2, h_3\}. \tag{2.1.20}$$

For a dominant integral $\lambda \in \mathfrak{t}^*$, i.e., $\lambda(h_i) \in \mathbb{N}$ for $i = 0, \ldots, 3$, let $L(\lambda)$ be the corresponding standard module of $\tilde{\mathfrak{g}}(\nu)$. For such a $L(\lambda)$, $\lambda(c)$ is called the level. Let Λ_i ($i = 0, \ldots, 3$) be the fundamental weight such that $\Lambda_i(h_j) = \delta_{i,j}$ for all $j = 0, \ldots, 3$. Invoking Theorem 4.6 of [BM2], we get that:

Theorem 2.1.1 ([BM2])**.** *Up to a shift by the imaginary root, the level 2 standard modules $L(\Lambda_0 + \Lambda_1)$ and $L(\Lambda_3)$ for $A_5^{(2)}$ appear as direct summands of $L(\Lambda_0) \otimes L(\Lambda_1)$.*

Let the highest weight vector of L be v_L. It is clear that $\tilde{\mathfrak{g}}(\nu)$ acts on L by specifying an arbitrary scalar action of d on v_L. We let $d \cdot v_L = 0$. With this,

$$L = \bigoplus_{n \geq 0} L_{-n},$$

where L_{-n} is the (finite dimensional) eigenspace for d with eigenvalue $-n$. We call

$$\chi(L) = \sum_{n \geq 0} (\dim L_{-n})\, q^n \tag{2.1.21}$$

the principally specialized character of L.

Given a standard module L for $\widehat{\mathfrak{g}}(\nu)$, let $\Omega(L)$, called the vacuum space, be the space of highest-weight vectors for the Heisenberg algebra $\widehat{\mathfrak{a}}(\nu)$. It is clear that $\Omega(L)$ also breaks up as a direct sum of finite dimensional eigen-spaces for d, with non-positive integral eigenvalues, and hence, we define $\chi(\Omega(L))$ analogously to (2.1.21).

Using the Weyl-Kac character formula and the Lepowsky-Milne numerator formula, it can be easily deduced that:

Theorem 2.1.2. *The principally specialized characters of $\Omega(L(\Lambda_0+\Lambda_1))$ and $\Omega(L(\Lambda_3))$ are:*

$$\chi(\Omega(L(\Lambda_0 + \Lambda_1))) = \prod_{j \geq 0} \frac{1}{(1-q^{8j+1})(1-q^{8j+4})(1-q^{8j+7})} \tag{2.1.22}$$

$$\chi(\Omega(L(\Lambda_3))) = \prod_{j \geq 0} \frac{1}{(1-q^{8j+3})(1-q^{8j+4})(1-q^{8j+5})}. \tag{2.1.23}$$

For a level k highest weight module V for $\widehat{\mathfrak{g}}(\nu)$ and for each $\beta \in \Phi$ and a formal variable ζ define the following generating functions with coefficients in the endomorphism ring of V.

$$X(\beta, \zeta) = \sum_{n \in \mathbb{Z}} ((x_\beta)_{(n)} \otimes t^n) \zeta^n, \tag{2.1.24}$$

$$E^{\pm}(\beta, \zeta, r) = \exp \left(\pm m \sum_{n \geq 1} (\alpha_{(\pm n)} \otimes t^{\pm n}) \zeta^{\pm n} / 2nr \right), \tag{2.1.25}$$

$$Z(\beta, \zeta, r) = E^-(\beta, \zeta, r) X(\beta, \zeta) E^+(\beta, \zeta, r) = \sum_{n \in \mathbb{Z}} Z(\beta, r)_n \zeta^n, \tag{2.1.26}$$

$$Z(\beta, \zeta) = Z(\beta, \zeta, k) = \sum_{n \in \mathbb{Z}} Z(\beta)_n \zeta^n. \tag{2.1.27}$$

Note that in this notation, we suppress the underlying representation corresponding to V since it will be easy to deduce from the context. Using Proposition 7.2 of [F] (cf. Proposition 3.3 of [LW3]), we see that

$$Z(\beta, \omega^p \zeta) = Z(\nu^p \beta, \zeta), \tag{2.1.28}$$

for all $\beta \in \Phi$ and $p \in \mathbb{Z}$.

Remark 2.1.3. The parametrization of the generating functions above follows the notation in [LW1]–[LW4]. This parametrization is no longer in use, for the "correct" parametrization, please see [FLM]. We still use the "old" notation in this paper as we want to directly invoke the generating function identities from [LW1]–[LW4], and since entire structure of the ambient vertex operator algebra will not be needed.

2.2 Generating functions: Z and X operators

In view of Theorem 2.1.1, consider the $A_5^{(2)}$-module $L(\Lambda_0) \otimes L(\Lambda_1)$. We let

$$Z([1], \zeta) = \sum_{n \in \mathbb{Z}} Z[1]_n \zeta^n = Z(\alpha_3, \zeta) \tag{2.2.1}$$

$$Z([2], \zeta) = \sum_{n \in \mathbb{Z}} Z[2]_n \zeta^n = Z(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \zeta) \tag{2.2.2}$$

$$Z([3], \zeta) = \sum_{n \in \mathbb{Z}} Z[3]_n \zeta^n = Z(\alpha_1, \zeta), \tag{2.2.3}$$

with the obvious meanings attached to the expressions $Z(\nu^r[j],\zeta)$, $X([j],\zeta)$, $E^{\pm}([j],\zeta,r)$, $E^{\pm}([j],\zeta)$, etc.

For $v \in L(\Lambda_i)$, $i=0,1$, and for $j=1,2,3$, we have (cf. [F]):

$$X([j],\zeta)v = c_{[j],i}E^-(-[j],\zeta)E^+(-[j],\zeta)v, \tag{2.2.4}$$

where

$$c_{[j],i} = \begin{cases} \dfrac{1}{10} & \text{if } (j,i) \neq (1,1), \\ \\ -\dfrac{1}{10} & \text{if } (j,i) = (1,1). \end{cases} \tag{2.2.5}$$

Therefore, on $L(\Lambda_0) \otimes L(\Lambda_1)$,

$$\begin{aligned} Z([1],\zeta) = \frac{1}{10}\big\{&E^-(-[1],\zeta,2)E^+(-[1],\zeta,2) \otimes E^-([1],\zeta,2)E^+([1],\zeta,2) \\ &-E^-([1],\zeta,2)E^+([1],\zeta,2) \otimes E^-(-[1],\zeta,2)E^+(-[1],\zeta,2)\big\} \\ = Z^{(1)}&([1],\zeta) - Z^{(2)}([1],\zeta) \end{aligned}$$

$$\begin{aligned} Z([2],\zeta) = \frac{1}{10}\big\{&E^-(-[2],\zeta,2)E^+(-[2],\zeta,2) \otimes E^-([2],\zeta,2)E^+([2],\zeta,2) \\ &+E^-([2],\zeta,2)E^+([2],\zeta,2) \otimes E^-(-[2],\zeta,2)E^+(-[2],\zeta,2)\big\} \\ = Z^{(1)}&([2],\zeta) + Z^{(2)}([2],\zeta) \end{aligned}$$

$$\begin{aligned} Z([3],\zeta) = \frac{1}{10}\big\{&E^-(-[3],\zeta,2)E^+(-[3],\zeta,2) \otimes E^-([3],\zeta,2)E^+([3],\zeta,2) \\ &+E^-([3],\zeta,2)E^+([3],\zeta,2) \otimes E^-(-[3],\zeta,2)E^+(-[3],\zeta,2)\big\} \\ = Z^{(1)}&([3],\zeta) + Z^{(2)}([3],\zeta). \end{aligned}$$

In each case,

$$Z^{(2)}([j],\zeta) = Z^{(1)}([j],-\zeta), \tag{2.2.6}$$

and therefore,

$$Z[1]_i = \begin{cases} 2Z[1]_i^{(1)} & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases} \tag{2.2.7}$$

$$Z[2]_i = \begin{cases} 0 & \text{if } i \text{ is odd} \\ 2Z[2]_i^{(1)} & \text{if } i \text{ is even} \end{cases} \tag{2.2.8}$$

$$Z[3]_i = \begin{cases} 0 & \text{if } i \text{ is odd} \\ 2Z[3]_i^{(1)} & \text{if } i \text{ is even.} \end{cases} \tag{2.2.9}$$

For a level k standard module L, it is well known (cf. [LW3]) that

$$\Omega(L) = \text{Span}\left\{Z(\beta_1)_{j_1} \cdots Z(\beta_r)_{j_r} \cdot v_L \,|\, r \in \mathbb{N};\, j_1, \ldots, j_r \in \mathbb{Z};\, \beta_1, \ldots, \beta_r \in \Phi\right\},$$

and hence, in view of (2.1.28), we get that

$$\Omega(L) = \text{Span}\left\{Z[i_1]_{j_1} \cdots Z[i_r]_{j_r} \cdot v_L \,|\, r \in \mathbb{N};\, j_1, \ldots, j_r \in \mathbb{Z};\, i_1, \ldots, i_r \in \{1,2,3\}\right\}. \tag{2.2.10}$$

2.3 Monomial Ordering

Let n be any positive integer. Let $\leq_p$ denote the product ordering on $\mathbb{Z}^n$. That is,

$$(i_1, \ldots, i_n) \leq_p (j_1, \ldots, j_n) \iff i_1 \leq j_1, \ldots, i_n \leq j_n.$$

Define a map

$$\tau : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$$
$$(i_1, \ldots, i_n) \longmapsto (i_1 + \cdots + i_n, i_2 + \cdots + i_n, \ldots, i_n)$$

Using τ and $\leq_p$, define a partial order $\leq_T$ on $\mathbb{Z}^n$ by:

$$(i_1, \ldots, i_n) \leq_T (j_1, \ldots, j_n) \iff \tau(i_1, \ldots, i_n) \leq_p \tau(j_1, \ldots, j_n).$$

It is clear that the intervals under $\leq_T$ are finite, i.e., given $(i_1, \ldots, i_n) \leq_T (j_1, \ldots, j_n)$, the set

$$\{(k_1, \ldots, k_n) \,|\, (i_1, \ldots, i_n) \leq_T (k_1, \ldots, k_n) \leq_T (j_1, \ldots, j_n)\}$$

is finite.

On $\bigcup_{j \in \mathbb{N}} \mathbb{Z}^j$, there exists a monoidal product, $\circ$:

$$(i_1, \ldots, i_r) \circ (j_1, \ldots, i_s) = (i_1, \ldots, i_r, j_1, \ldots, j_s).$$

It is clear that for $(i_1,\dots,i_n) \le_T (j_1,\dots,j_n)$,

$$(i_1,\dots,i_n)\circ(k_1,\dots,k_m) \le_T (j_1,\dots,j_n)\circ(k_1,\dots,k_m)$$
$$(k_1,\dots,k_m)\circ(i_1,\dots,i_n) \le_T (k_1,\dots,k_m)\circ(j_1,\dots,j_n).$$

For an integer i, let $\underline{i}=1$ if i is odd and $\underline{i}=2$ if i is even. Given a tuple $(i_1,\dots,i_n)\in\mathbb{Z}^n$, define the Z-monomial

$$Z_{i_1}\cdots Z_{i_n} = Z[\,\underline{i_1}\,]_{i_1}\cdots Z[\,\underline{i_n}\,]_{i_n}. \tag{2.3.1}$$

If $n=0$, the corresponding null monomial is understood as the identity operator. Define

$$T(i_1,\dots,i_n) = \mathrm{Span}\,\{Z_{j_1}\dots Z_{j_m} \mid \text{ either } m<n \text{ or } m=n \text{ and } \tau(i_1,\dots,i_n) \le_T \tau(j_1,\dots,j_n)\}\,.$$

We say that a monomial $Z_{i_1}\dots Z_{i_n}$ is *reducible* iff

$$Z_{i_1}\dots Z_{i_n} \in T(i_1,\dots,i_n).$$

It is clear from the properties of $\circ$ mentioned above that $Z_{i_1}\cdots Z_{i_n}$ is reducible if any of its contiguous sub-monomials is reducible.

2.4 Generating function identities and their consequences

Our plan is to eliminate the modes of $Z[3]$ from the spanning set and to use enough relations between the $Z[1]$ and $Z[2]$ modes in order to reduce the spanning set to exhibit the sum-side conditions in the Göllnitz-Gordon identities. Hence, we first concentrate on the generating function identities involving the modes of $Z[1]$ and $Z[2]$. Using equation (8.21) of [LW3] (cf. Theorem 7.3 of [F]) we immediately deduce the required generating function identities. We will always rely on (2.2.7) and (2.2.8) which state that the even modes of $Z[1]$ are zero and that the odd modes of $Z[2]$ are zero. We organize the generating function identities accordingly. We will let

$$\delta(\zeta) = \sum_{n\in\mathbb{Z}} \zeta^n.$$

2.4.1 Odd-Even

Let

$$F_1(x) = \frac{(1-\omega^{-6}x)^{1/2}(1-\omega^{-7}x)^{1/2}}{(1-\omega^{-1}x)^{1/2}(1-\omega^{-2}x)^{1/2}} \tag{2.4.1}$$

$$F_{1'}(x) = \frac{(1-\omega^{6}x)^{1/2}(1-\omega^{7}x)^{1/2}}{(1-\omega x)^{1/2}(1-\omega^{2}x)^{1/2}}. \tag{2.4.2}$$

Using Theorem 7.3 of [F] (cf. Theorem 3.10 of [LW3]), we get:

$$F_1\left(\frac{\zeta_1}{\zeta_2}\right)\cdot Z([1],\zeta_1)Z([2],\zeta_2) - F_{1'}\left(\frac{\zeta_2}{\zeta_1}\right)\cdot Z([2],\zeta_2)Z([1],\zeta_1)$$
$$= \frac{1}{10}\left\{\varepsilon(\nu[1],[2])Z(\nu^7[1],\zeta_2)\delta\left(\omega^{-1}\frac{\zeta_1}{\zeta_2}\right) + \varepsilon(\nu^2[1],[2])Z(\nu^6[1],\zeta_2)\delta\left(\omega^{-2}\frac{\zeta_1}{\zeta_2}\right)\right\}. \tag{2.4.3}$$

Define the numbers $b_1(n)$ and $b'_1(n)$ via

$$F_1(x) = \sum_{n\in\mathbb{Z}} b_1(n)\left(\frac{\zeta_1}{\zeta_2}\right)^n,$$
$$F_{1'}(x) = \sum_{n\in\mathbb{Z}} b'_1(n)\left(\frac{\zeta_2}{\zeta_1}\right)^n.$$

Then,

$$b_1(0) = b'_1(0) = 1 \tag{2.4.4}$$

$$b_1(1) = \frac{\omega^{-1}}{2} + \frac{\omega^{-2}}{2} - \frac{\omega^{-6}}{2} - \frac{\omega^{-7}}{2} \neq 0 \tag{2.4.5}$$

$$b'_1(1) = \frac{\omega^{1}}{2} + \frac{\omega^{2}}{2} - \frac{\omega^{6}}{2} - \frac{\omega^{7}}{2} \neq 0 \tag{2.4.6}$$

Hence, the coefficient of $\zeta_1^a\zeta_2^b$ from (2.4.3) gives:

$$\sum_{p\geq 0} b_1(p)Z[1]_{a-p}Z[2]_{b+p} - b'_1(p)Z[2]_{b-p}Z[1]_{a+p} = c_0(a,b)Z[1]_{a+b}, \tag{2.4.7}$$

where $c_0(a,b)$ is some constant depending on a,b.

For i odd, letting $a \mapsto i+1$ and $b \mapsto i$ in (2.4.7) and noting (2.2.7), (2.2.8), the term corresponding to $p=0$ drops out and we get that:

$$\sum_{p\geq 1} b_1(p)Z[1]_{i+1-p}Z[2]_{i+p} - b'_1(p)Z[2]_{i-p}Z[1]_{i+1+p} = b_1(p)Z[1]_iZ[2]_{i+1} + \cdots$$
$$= c_0(i+1,i)Z[1]_{2a+1}. \tag{2.4.8}$$

Notation 2.4.1. *Here, and below, such ellipses will refer to terms either higher in the monomial ordering or shorter than the term immediately preceding the ellipses.*

Similarly, for i even, letting $a \mapsto i$ and $b \mapsto i+1$ in (2.4.7) and noting (2.2.7), (2.2.8), the term corresponding to $p = 0$ drops out and we get that:

$$\sum_{p\geq 1} b_1(p)Z[1]_{i-p}Z[2]_{i+1+p} - b_1'(p)Z[2]_{i+1-p}Z[1]_{i+p} = -b_1'(p)Z[2]_iZ[1]_{i+1} + \cdots$$
$$= c_0(i, i+1)Z[1]_{2i+1}. \tag{2.4.9}$$

Proposition 2.4.2. *If $i > j$ with i odd and j even, then, with $a \mapsto i, b \mapsto j$,* (2.4.7) *implies that*

$$Z_iZ_j \in T(i, j). \tag{2.4.10}$$

If $j > i$ with j even and i odd, then, (2.4.7) *implies that*

$$Z_iZ_j \in T(i, j). \tag{2.4.11}$$

If i is odd then (2.4.8) *and* (2.4.5) *imply that*

$$Z_iZ_{i+1} \in T(i, i+1). \tag{2.4.12}$$

If a is even then (2.4.9) *and* (2.4.6) *imply that*

$$Z_iZ_{i+1} \in T(i, i+1). \tag{2.4.13}$$

2.4.2 Odd-Odd

Let

$$F_2(x) = \frac{(1-x)(1-\omega^{-2}x)^{1/2}(1-\omega^{-4}x)^{1/2}(1-\omega^{-6}x)^{1/2}(1-\omega^{-8}x)^{1/2}}{(1-\omega^{-1}x)^{1/2}(1-\omega^{-3}x)^{1/2}(1+x)(1-\omega^{-7}x)^{1/2}(1-\omega^{-9}x)^{1/2}}. \tag{2.4.14}$$

Define the numbers $b_2(n)$ as the power series coefficients of $F_2(x)$:

$$F_2(x) = \sum_{n\geq 0} b_2(n)x^n. \tag{2.4.15}$$

Lemma 2.4.3. *The power series expansion of $F_2(x)$ is:*

$$F_2(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{2}x^3 + \cdots . \tag{2.4.16}$$

Proof. Let

$$F_2(x) = \frac{(1-x)^{1/2}}{(1+x)^{1/2}} f_2(x). \tag{2.4.17}$$

Observe that

$$f_2(x) = f_2(\omega^2 x), \tag{2.4.18}$$

and hence,

$$f_2(x) = \frac{1}{5}(f(x) + f(\omega^2 x) + +f(\omega^4 x) + f(\omega^6 x) + f(\omega^8 x)). \tag{2.4.19}$$

Since each of ω^j for $j = 2, 4, 6, 8$ is a primitive 5^{th} root of unity, we see that the coefficient of x^i where $i \not\equiv 0 \pmod 5$ in the power series expansion of $f_2(x)$ is 0. Hence, $F_2(x)$ agrees with $\frac{(1-x)^{1/2}}{(1+x)^{1/2}}$ upto the coefficient of x^4. It is now easy to see that

$$\frac{(1-x)^{1/2}}{(1+x)^{1/2}} = 1 - x + \frac{1}{2}x^2 - \frac{1}{2}x^3 + \cdots . \tag{2.4.20}$$

□

From Theorem 7.3 of [F] (cf. Theorem 3.10 of [LW3]), we have:

$$\begin{aligned}
&F_2\left(\frac{\zeta_1}{\zeta_2}\right) \cdot Z([1],\zeta_1)Z([1],\zeta_2) - F_2\left(\frac{\zeta_2}{\zeta_1}\right) \cdot Z([1],\zeta_2)Z([1],\zeta_1)\\
&\quad = \frac{1}{10}\left\{\varepsilon(\nu[1],[1])Z(\nu^4[2],\zeta_2)\delta\left(\omega^{-1}\frac{\zeta_1}{\zeta_2}\right) + \varepsilon(\nu^9[1],[1])Z(\nu^3[2],\zeta_2)\delta\left(\omega^{-9}\frac{\zeta_1}{\zeta_2}\right)\right.\\
&\qquad \left. +\varepsilon(\nu^3[1],[1])Z(\nu^7[3],\zeta_2)\delta\left(\omega^{-3}\frac{\zeta_1}{\zeta_2}\right) + \varepsilon(\nu^7[1],[1])Z(\nu^4[3],\zeta_2)\delta\left(\omega^{-7}\frac{\zeta_1}{\zeta_2}\right)\right\}\\
&\quad - \frac{1}{50}(D\delta)\left(-\frac{\zeta_1}{\zeta_2}\right).
\end{aligned} \tag{2.4.21}$$

Terms containing $Z[3]$ in the coefficient of $\zeta_1^i\zeta_2^j$ on the right-hand side are:

$$\frac{1}{10}\left(\varepsilon(\nu^3[1],[1])\omega^{-3i+7(i+j)} + \varepsilon(\nu^7[1],[1])\omega^{-7i+4(i+j)}\right) Z[3]_{i+j} \tag{2.4.22}$$

$$= \frac{1}{10}\varepsilon(\nu^3[1],[1])(\omega^{4i+7j} - \omega^{7i+4j})Z[3]_{i+j}, \tag{2.4.23}$$

where we have used

$$\varepsilon(\nu^3[1],[1]) = \varepsilon([1],\nu^7[1]) = \varepsilon(\nu^7[1],[1])(-1)^{\langle\nu^7[1],[1]\rangle} = -\varepsilon(\nu^7[1],[1]). \tag{2.4.24}$$

Coefficient of $\zeta_1^a\zeta_2^b$ in (2.4.21) gives the following identity:

$$\sum_{p\geq 0} b_2(p)\left(Z[1]_{a-p}Z[1]_{b+p} - Z[1]_{b-p}Z[1]_{a+p}\right)$$
$$= \frac{1}{10}\varepsilon(\nu^3[1],[1])(\omega^{4a+7b} - \omega^{7a+4b})Z[3]_{a+b} + c_1(a,b)Z[2]_{a+b} + c_2(a,b), \qquad (2.4.25)$$

where $c_1(a,b)$ and $c_2(a,b)$ are some constants depending on a and b.

Eliminating $Z[1]_iZ[1]_j$ for $j < i$

Let i, j be odd integers such that $i > j$. Due to (2.2.7), we don't need to consider the even modes of $Z[1]$.

Note that the term containing $Z[2]$ on the right-hand side and the constant term $c_2(a,b)$ of (2.4.25) are shorter than any monomial appearing on the left-hand side and are acceptable monomials our intended spanning set.

Therefore, with $a \mapsto i, b \mapsto j$ in (2.4.25) yields:

$$Z[1]_iZ[1]_j + \cdots = \frac{\varepsilon(\nu^3[1],[1])}{10}(\omega^{4i+7j} - \omega^{7i+4j})Z[3]_{i+j}. \qquad (2.4.26)$$

Letting $a \mapsto i+1, b \mapsto j-1$ and utilizing (2.2.7), the coefficient corresponding to $p = 0$ in (2.4.25) drops out, giving

$$\sum_{p\geq 0} b_2(p)\left(Z[1]_{i+1-p}\, Z[1]_{j-1+p} - Z[1]_{j-1-p}Z[1]_{i+1+p}\right)$$
$$= \sum_{p\geq 0} b_2(p+1)\left(Z[1]_{i-p}Z[1]_{j+p} - Z[1]_{j-2-p}Z[1]_{i+2+p}\right)$$
$$= b_2(1)Z[1]_iZ[1]_j + \cdots$$
$$= \frac{1}{10}\varepsilon(\nu^3[1],[1])(\omega^{4i+7j-3} - \omega^{7i+4j+3})Z[3]_{i+j} + \alpha_{i+1,j-1}Z[2]_{i+j} \qquad (2.4.27)$$

This finally gives

$$-Z[1]_iZ[2]_j + \cdots = \frac{\varepsilon(\nu^3[1],[1])}{10}(\omega^{4i+7j-3} - \omega^{7i+4j+3})Z[3]_{i+j}. \qquad (2.4.28)$$

Now,

$$\mathrm{Det}\begin{bmatrix} 1 & \omega^{4i+7j} - \omega^{7i+4j} \\ -1 & \omega^{4i+7j-3} - \omega^{7i+4j+3} \end{bmatrix}$$

$$= \omega^{4i+7j-3} - \omega^{7i+4j+3} + \omega^{4i+7j} - \omega^{7i+4j}$$
$$= \omega^{4i+7j}(\omega^{-3} - \omega^{3(i-j)+3} + 1 - \omega^{3(i-j)})$$
$$= \omega^{4i+7j}(1+\omega^{3})(\omega^{-3} - \omega^{3(i-j)}),$$

but since $i-j$ is even, $3(i-j) \not\equiv -3 \pmod{10}$, and therefore,

$$(\omega^{4i+7j} - \omega^{7i+4j}) + (\omega^{4i+7j-3} - \omega^{7i+4j+3}) \neq 0. \tag{2.4.29}$$

Combining (2.4.26), (2.4.28) and (2.4.29) we immediately arrive at the following proposition.

Proposition 2.4.4. *For i, j odd integers with $i > j$,*

$$Z_i Z_j \in T(i,j). \tag{2.4.30}$$

From (2.2.10), (2.4.26), (2.4.28) and (2.4.29) we deduce the following.

Proposition 2.4.5. *Modes of $Z[1]$ and $Z[2]$ suffice to span $\Omega(L(\Lambda_0+\Lambda_1))$ and $\Omega(L(\Lambda_3))$, i.e.,*

$$\begin{aligned}\Omega(L) &= \mathit{Span}\{Z[i_1]_{j_1} \cdots Z[i_r]_{j_r} \cdot v_L \mid r \in \mathbb{N};\, j_1, \ldots, j_r \in \mathbb{Z};\, i_1, \ldots, i_r \in \{1,2\}\} \\ &= \mathit{Span}\{Z_{j_1} \cdots Z_{j_r} \cdot v_L \mid r \in \mathbb{N};\, j_1, \ldots, j_r \in \mathbb{Z}\}\end{aligned} \tag{2.4.31}$$

for $L = L(\Lambda_0 + \Lambda_1)$ or $L = L(\Lambda_3)$.

In fact, from (2.4.26) and (2.4.28) we can gather more information about the modes of $Z[3]$ that we shall use in order to get the “difference at least 4” condition on the even parts.

Proposition 2.4.6. *For an integer a*

$$Z[3]_{4a} \in T(2a, 2a). \tag{2.4.32}$$

Proof. Using (2.2.7) we note that the smallest term higher than $Z[1]_i Z[1]_j$ in the left-hand side of (2.4.26) and (2.4.28) is (strictly) higher than $Z_{i-1}Z_{j+1}$. Therefore, adding (2.4.26) and (2.4.28) and noting (2.4.29), we obtain that for any odd integers i, j with $i > j$,

$$Z[3]_{i+j} \in T(i-1, j+1). \tag{2.4.33}$$

Now let $i \mapsto 2a+1$ and $j \mapsto 2a-1$. $\square$

Eliminating $Z[1]_iZ[1]_i$ for $i < 0$

Let i be an odd integer. Note that (2.4.28) holds even if $i = j$. Hence, letting $a \mapsto i+1$ and $b \mapsto i-1$ in (2.4.25) and proceeding as in (2.4.28), we arrive at:

$$-Z[1]_iZ[1]_i + \cdots = \frac{\varepsilon(\nu^3[1],[1])}{10}(\omega^{i-3} - \omega^{i+3})Z[3]_{2i}. \tag{2.4.34}$$

Letting $a \mapsto i+2$ and $b \mapsto i-2$ in (2.4.25), noting (2.2.7) and (2.4.15), we arrive at:

$$Z[1]_{i+2}Z[1]_{i-2} + \frac{1}{2}Z[1]_iZ[1]_i + \cdots = \frac{\varepsilon(\nu^3[1],[1])}{10}(\omega^{i-6} - \omega^{i+6})Z[3]_{2i}. \tag{2.4.35}$$

Similarly, with $a \mapsto i+3, b \mapsto i-3$ in (2.4.25) we get

$$-Z[1]_{i+2}Z[1]_{i-2}\frac{1}{2}Z[1]_iZ[1]_i + \cdots = -\frac{\varepsilon(\nu^3[1],[1])}{10}(\omega^{i-9} - -\omega^{i+9})Z[3]_{2i}. \tag{2.4.36}$$

Adding (2.4.35) and (2.4.36) and noting that for any integer i

$$\begin{aligned}\omega^{i-6} - \omega^{i+6} + \omega^{i-9} - \omega^{i+9} &= \omega^i(\omega^4 - \omega^6 + \omega - \omega^9) \\ &= 2\omega^i(\omega^4 + \omega) \\ &\neq 0, \end{aligned} \tag{2.4.37}$$

since the minimal polynomial for ω over rationals is $x^4 - x^3 + x^2 - x + 1$, we conclude that

$$Z[3]_{2i} \in T(i,i), \tag{2.4.38}$$

for any odd negative integer i. Now, (2.4.34) yields the following proposition.

Proposition 2.4.7. *For any odd integer i,*

$$Z_iZ_i \in T(i,i). \tag{2.4.39}$$

In fact, we can deduce more.

Proposition 2.4.8. *For any odd integer i,*

$$Z[3]_{2i} \in T(i-1,i+1) \tag{2.4.40}$$

Proof. Noting (2.2.7), each term appearing in the ellipses in the left-hand sides of (2.4.35) and (2.4.36) is strictly higher than $Z_{i-1}Z_{i+1}$. Now, add (2.4.35) and (2.4.36) and invoke (2.4.37). □

2.4.3 Even-Even

Let

$$F_3(x) = \frac{(1-x)(1-\omega^{-1}x)^{1/2}(1-\omega^{-9}x)^{1/2}}{(1+x)(1+\omega^{-1}x)^{1/2}(1+\omega^{-9}x)^{1/2}}. \tag{2.4.41}$$

Lemma 2.4.9. *We have the following power series series expansion for F_3:*

$$F_3(x) = 1 + \frac{1}{2}\left(-5-\sqrt{5}\right)x + \frac{5}{4}\left(3+\sqrt{5}\right)x^2 + \left(-5 - \frac{3\sqrt{5}}{2}\right)x^3 + \cdots. \tag{2.4.42}$$

Proof. Recall that we have fixed the primitive 10^{th} root of unity, $\omega = \exp\left(\frac{\pi \mathbf{i}}{5}\right)$. It is easy to deduce that

$$\log(F_3(x)) = (-2-\omega-\omega^9)x + \frac{1}{3}(-2-\omega^3-\omega^7)x^3 + \cdots$$

Now, note that

$$\mathfrak{Re}(\omega) = \frac{1}{4}\left(1+\sqrt{5}\right) \tag{2.4.43}$$

$$\mathfrak{Re}(\omega^3) = \frac{1}{4}\left(1-\sqrt{5}\right) \tag{2.4.44}$$

and therefore

$$\log(F_3(x)) = \frac{1}{2}\left(-5-\sqrt{5}\right)x + \frac{1}{6}\left(\sqrt{5}-5\right)x^3 + \cdots. \tag{2.4.45}$$

Hence,

$$\begin{aligned} F_3(x) &= \exp(\log(F_3(x))) \\ &= 1 + \frac{1}{2}\left(-5-\sqrt{5}\right)x + \frac{1}{2}\left(\frac{1}{2}\left(-5-\sqrt{5}\right)\right)^2 x^2 \\ &\quad + \left(\frac{1}{6}\left(\frac{1}{2}\left(-5-\sqrt{5}\right)\right)^3 + \frac{1}{6}\left(\sqrt{5}-5\right)\right)x^3 + \cdots \\ &= 1 + \frac{1}{2}\left(-5-\sqrt{5}\right)x + \frac{5}{4}\left(3+\sqrt{5}\right)x^2 + \left(-5-\frac{3\sqrt{5}}{2}\right)x^3 + \cdots. \end{aligned} \tag{2.4.46}$$

□

Define the numbers $b_3(n)$ by

$$F_3(x) = \sum_{n \geq 0} b_3(n) x^n. \tag{2.4.47}$$

Employing Theorem 7.3 of [F] (cf. Theorem 3.10 of [LW3]), the generalized commutation relation for $Z([2],\zeta_1), Z([2],\zeta_2)$ is:

$$F_3\left(\frac{\zeta_1}{\zeta_2}\right)\cdot Z([2],\zeta_1)Z([2],\zeta_2) - F_3\left(\frac{\zeta_2}{\zeta_1}\right)\cdot Z([2],\zeta_2)Z([2],\zeta_1) \tag{2.4.48}$$

$$= \frac{1}{10}\left\{\varepsilon(\nu^4[2],[2])Z(\nu^4[3],\zeta_2)\delta\left(\omega^{-4}\frac{\zeta_1}{\zeta_2}\right) + \varepsilon(\nu^6[2],[2])Z(\nu[3],\zeta_2)\delta\left(\omega^{-6}\frac{\zeta_1}{\zeta_2}\right)\right\}$$

$$- \frac{1}{50}(D\delta)\left(-\frac{\zeta_1}{\zeta_2}\right). \tag{2.4.49}$$

Noting that $\varepsilon(\nu^4[2],[2]) = -\varepsilon(\nu^6[2],[2])$, the coefficient of $\zeta_1^i\zeta_2^j$ on the right-hand side is:

$$\frac{\varepsilon(\nu^4[2],[2])}{10}(\omega^{-4i+4(i+j)} - \omega^{-6i})Z[3]_{i+j} = \frac{\varepsilon(\nu^4[2],[2])}{10}(\omega^{4j} - \omega^{4i})Z[3]_{i+j}, \tag{2.4.50}$$

for any non-positive integers i, j.

Eliminating $Z[2]_iZ[2]_j$ for $j < i$

Let $j < i$ be even integers. Due to (2.2.8) it is not necessary to consider the odd modes of $Z[2]$. Then, coefficient of $\zeta_1^i\zeta_2^j$ in (2.4.49) gives

$$Z[2]_iZ[2]_j + \cdots = \frac{\varepsilon(\nu^4[2],[2])}{10}(\omega^{4j} - \omega^{4i})Z[3]_{i+j}. \tag{2.4.51}$$

Let $\ell = (i+j)/2$. If $i + j \equiv 0 \pmod 4$ then ℓ is even. Since $j < i$, $(i,j) <_T (\ell,\ell)$. Invoking Proposition 2.4.6, we get that $Z[3]_{i+j} \in T(\ell,\ell)$, and therefore, $Z_iZ_j \in T(\ell,\ell) \subset T(i,j)$. If $i + j \equiv 2 \pmod 4$ then ℓ is odd. It is clear that $(i,j) <_T (\ell-1,\ell+1)$. Invoking Proposition 2.4.8, we get that $Z[3]_{i+j} \in T(\ell-1,\ell+1)$, and therefore, $Z_iZ_j \in T(\ell-1,\ell+1) \subset T(i,j)$.

Proposition 2.4.10. *For even integers i, j with $i > j$,*

$$Z_iZ_j \in T(i,j). \tag{2.4.52}$$

Eliminating $Z[2]_iZ[2]_i$

Let i be an even integer. Coefficient of $\zeta_1^{i+1}\zeta_2^{i-1}$ in (2.4.49) and noting (2.2.8), we get:

$$b_3(1)Z[2]_iZ[2]_i + \cdots = \frac{\varepsilon(\nu^4[2],[2])}{10}(\omega^{4i-4} - \omega^{4i+4})Z[3]_{2i}. \tag{2.4.53}$$

Note that $b_3(1) \neq 0$ (Lemma 2.4.9) and that Proposition 2.4.6 implies that $Z[3]_{2i} \in T(i,i)$ since i is even. Therefore we arrive at the following proposition.

Proposition 2.4.11. *For an even integer i,*

$$Z_i Z_i \in T(i,i). \tag{2.4.54}$$

Eliminating $Z[2]_{i-2}Z[2]_i$

Again, let i be an even integer. Coefficient of $\zeta_1^i \zeta_2^{i-2}$ in (2.4.49) yields:

$$\begin{aligned} b_3(0)Z[2]_i Z[2]_{i-2} + (b_3(2) - b_3(0))Z[2]_{i-2}Z[2]_i + \cdots \\ = \frac{\varepsilon(\nu^4[2],[2])}{10}(\omega^{4i-8} - \omega^{4i})Z[3]_{2i-2}. \end{aligned} \tag{2.4.55}$$

Coefficient of $\zeta_1^{i+1}\zeta_2^{i-3}$ gives

$$\begin{aligned} b_3(1)Z[2]_i Z[2]_{i-2} + b_3(3)Z[2]_{i-2}Z[2]_i + \cdots \\ = \frac{\varepsilon(\nu^4[2],[2])}{10}(\omega^{4i-12} - \omega^{4i+4})Z[3]_{2i-2}. \end{aligned} \tag{2.4.56}$$

A quick computation shows that

$$\mathrm{Det}\begin{bmatrix} b_3(0) & b_3(2) - b_3(0) \\ b_3(1) & b_3(3) \end{bmatrix} = 5 + 3\sqrt{5} \neq 0.$$

Also, Proposition 2.4.8 shows that $Z[3]_{2i-2} \in T(i-2,i)$. We conclude the following.

Proposition 2.4.12. *For a non-positive even integer i,*

$$Z_{i-2}Z_i \in T(i-2,i). \tag{2.4.57}$$

Remark 2.4.13. The "asymmetry" between the odd and the even parts is also visible in the fact that the strategy for proving that $Z[1]_i Z[1]_i \in T(i,i)$ for i odd (cf. Proposition 2.4.7) does not work for proving that $Z[2]_i Z[2]_i \in T(i,i)$ for i even (cf. Proposition 2.4.11). The corresponding matrices turn out to be singular!

2.5 Small spanning sets for the vacuum spaces of $L(\Lambda_0+\Lambda_1)$ and $L(\Lambda_3)$

We have obtained enough relations in the previous section to now obtain small spanning sets for the vaccum spaces of $L(\Lambda_0+\Lambda_1)$ and $L(\Lambda_3)$. The truth of the Göllnitz-Gordon identities is now equivalent to the independence of these spanning sets.

Theorem 2.5.1. *We have that:*

$$\begin{aligned}\Omega(L(\Lambda_0+\Lambda_1)) = \mathit{Span}\{Z_{i_1}\cdots Z_{i_r}\cdot v_{L(\Lambda_0+\Lambda_1)}\,|\, r\in\mathbb{N},\, i_1<i_2<\cdots<i_r\le -1,\\ |i_j-i_{j+1}|\ge 2 \textit{ with } |i_j-i_{j+1}|\ge 4 \textit{ if } i_j,i_{j+1} \textit{ are even}\},\\ \Omega(L(\Lambda_3)) = \mathit{Span}\{Z_{i_1}\cdots Z_{i_r}\cdot v_{L(\Lambda_3)}\,|\, r\in\mathbb{N},\, i_1<i_2<\cdots<i_r\le -3,\\ |i_j-i_{j+1}|\ge 2 \textit{ with } |i_j-i_{j+1}|\ge 4 \textit{ if } i_j,i_{j+1} \textit{ are even}\}.\end{aligned}$$

Proof. From Propositions 2.4.5, and the highest weight property of $v_{L(\Lambda_0+\Lambda_1)}$ and $v_L(\Lambda_3)$ it is clear that for $L=L(\Lambda_0+\Lambda_1)$ or $L=L(\Lambda_3)$,

$$\Omega(L)=\mathrm{Span}\{Z_{i_1}\cdots Z_{i_r}\cdot v_L\,|\,r\in\mathbb{N},(i_1,\ldots,i_r)\le_T(0,\ldots,0)\}.$$

Now, Propositions 2.4.2, 2.4.4 and 2.4.10 along with the highest weight property of v_L show that:

$$\Omega(L)=\mathrm{Span}\{Z_{i_1}\cdots Z_{i_r}\cdot v_L\,|\,r\in\mathbb{N},i_1\le\cdots\le i_r\le -1\}.$$

Combined with the product formulas for the characters, given in (2.1.23), we gather that the weight 1 and 2 subspaces of $\Omega(L(\Lambda_3))$ are 0, and hence,

$$Z_{-1}\cdot v_{L(\Lambda_3)}=0,\quad Z_{-2}\cdot v_{L(\Lambda_3)}=0.$$

The required difference 2 and difference 4 conditions could easily be deduced from Propositions 2.4.2, 2.4.7, 2.4.11 and 2.4.12. □

Corollary 2.5.2. *For each of the Göllnitz-Gordon identities, each coefficient on the product side is at most as large as the corresponding coefficient on the sum side.*

Corollary 2.5.3. *The truth of the Göllnitz-Gordon identities is equivalent to the independence of the spanning sets given in Theorem 2.5.1.*

Chapter 3

From products, hopefully to sums: Motivated proofs and twisted intertwining operators

As recalled in the Introduction, vertex-operator-algebraic interpretation of the steps in the Andrews-Baxter's "Motivated Proof" of the Rogers-Ramanujan identities ([AB]) is an important open problem. We shall recall this proof below. Our aim in this chapter is to give an explicit construction of twisted intertwining operators among certain triples of twisted and untwisted modules for $\widehat{\mathfrak{sl}_2}$ at higher levels. We start with level 1 intertwining operators as given in [Ab1] and [ADL] and explicitly build higher level intertwining operators by adapting the methods of [DL]. In this chapter, it will be enough to restrict ourselves to irreducible modules and involutions of vertex operator algebras, and hence we work under these restrictions throughout this chapter.

3.1 Andrews-Baxter's "motivated proof" of the Rogers-Ramanujan identities

Let q be a purely formal variable. Recall the Rogers-Ramanujan identities written in the generating function form:

$$\prod_{j\geq 0} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})} = \sum_{n\geq 0} d_1(n)q^n \tag{3.1.1}$$

$$\prod_{j\geq 0} \frac{1}{(1-q^{5j+2})(1-q^{5j+3})} = \sum_{n\geq 0} d_2(n)q^n, \tag{3.1.2}$$

where $d_i(n)$ for $i = 1, 2$, is the number of partitions of n such that the adjacent parts differ by at least 2 and such that the smallest part is at least i. Let's let

$$G_1(q) = \prod_{j\geq 0} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})}, \tag{3.1.3}$$

$$G_2(q) = \prod_{j\geq 0} \frac{1}{(1-q^{5j+2})(1-q^{5j+3})}. \tag{3.1.4}$$

Looking at the sum sides, it is easy to see that $G_1(q) - G_2(q)$ has non-negative coefficients. An explanation for this fact, using only the product sides, in other words, without assuming the truth of the Rogers-Ramanujan identities, was asked for by L. Ehrenpreis. Andrews and Baxter, while answering this question, were naturally led to a *proof* of these identities themselves.

A rough outline of Andrews-Baxter's "Motivated Proof" follows. First, expand the infinite products.

$$G_1(q) = 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 4q^8 + 5q^9 + 6q^{10} + \cdots \tag{3.1.5}$$

$$G_2(q) = 1 \qquad + q^2 + q^3 + \ q^4 + \ q^5 + 2q^6 + 2q^7 + 3q^8 + 4q^9 + 4q^{10} + \cdots. \tag{3.1.6}$$

Observe (of course) that $G_1 - G_2$ has non-negative coefficients:

$$G_1 - G_2 = q + q^4 + q^5 + q^6 + q^7 + 2q^8 + 2q^9 + \cdots. \tag{3.1.7}$$

Observe that q divides $G_1 - G_2$. Let's let

$$G_3 = (G_1 - G_2)/q = 1 + q^3 + q^4 + q^5 + q^6 + 2q^7 + 2q^8 + \cdots. \tag{3.1.8}$$

Observe that $G_2 - G_3$ has non-negative coefficients as well and that $G_2 - G_3$ is divisible by q^2:

$$G_4 = (G_2 - G_3)/q^2 = 1 + q^4 + q^5 + q^6 + 2q^7 + 2q^8 + \cdots. \tag{3.1.9}$$

Continuing, we let

$$G_i = \frac{G_{i-2} - G_{i-1}}{q^{i-2}} \tag{3.1.10}$$

and observe that

$$G_i = 1 + q^i + \cdots. \tag{3.1.11}$$

The observation (3.1.11) is called the Empirical Hypothesis and it can be proved starting with the products and using the Jacobi triple product identity. The point is, proving the

Empirical Hypothesis starting from the product sides leads to a proof of the Rogers-Ramanujan identities themselves! Note that the Empirical Hypothesis is trivial to derive from the sum sides.

We mention that this proof has been generalized to the case of the Andrews-Gordon identities in [LZ], to the Andrews-Bressoud identities in [KLRS] and to the Göllnitz-Gordon-Andrews identities in [CKLMQRS].

We know, using the Weyl-Kac character formula and the Lepowsky-Milne numerator formula that the vacuum spaces of principally twisted level 3 standard modules have characters equalling the product sides:

$$\chi(\Omega(L(3\Lambda_0))) = \chi(\Omega(L(3\Lambda_1))) = G_1(q), \tag{3.1.12}$$

$$\chi(\Omega(L(2\Lambda_0+\Lambda_1))) = \chi(\Omega(L(\Lambda_0+2\Lambda_1))) = G_2(q). \tag{3.1.13}$$

It was an idea of J. Lepowsky and A. Milas that the recursion (3.1.10) could be explained by exact sequences among these vacuum spaces, where the maps would come from the relativized and twisted intertwining operators.

The principally twisted modules for $\widehat{\mathfrak{sl}_2}$ are obtained via an involution of $\mathfrak{sl}_2$, and hence we restrict our attention to involutions. Moreover, we restrict our attention to intertwining operators where the "source" and the "target" module are twisted. That is, we focus on intertwining operators of type $\binom{W_3}{W_1\ W_2}$, where W_2 and W_3 are (principally) twisted. This naturally forces us impose that W_1 is an untwisted module. We will also assume a certain grading restriction on the modules. For precise formulations of these restrictions, see Assumption 3.2.8.

3.2 Preliminary definitions

Notation 3.2.1. *Fix a vertex operator algebra $(V, Y, \mathbf{1}, \omega)$ in the sense of [FLM]. We denote this by simply V.*

Definition 3.2.2. *(Cf. [LL]) An* untwisted $V-$module *is the data (W, Y_W) where W is a $\mathbb{C}-$graded vector space, $W = \coprod_{n\in\mathbb{C}} W_{(n)}$ and $Y_W(\cdot, x)$ is a linear map*

$$Y_W(\cdot, x) \quad : \quad V \longrightarrow (\mathrm{End}W)[[x, x^{-1}]]$$

$$v \longmapsto Y_W(v,x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1}$$

such that the following conditions hold:

1. *Grading restriction condition: For all* $n \in \mathbb{C}$, $\dim W_{(n)} < \infty$ *and* $W_{(n-k)} = 0$ *for all sufficiently large integers* k.

2. *Lower truncation condition: For all* $v \in V$ *and* $w \in W$, $Y_W(v,x)w \in W((x))$.

3. *Vacuum property:* $Y_W(\mathbf{1}, x) = \mathrm{Id}_W$.

4. *Jacobi identity: For all* $u, v \in V$,

$$x_0{}^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W(u,x_1)Y_W(v,x_2) - x_0{}^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_W(v,x_2)Y_W(u,x_1)$$
$$= x_2{}^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y_W(Y(u,x_0)v, x_2).$$

5. $L(0)-$*grading condition: For* $n \in \mathbb{Z}$, *let* $L(n) = \omega_{n+1}$. *Then, for all* $w \in W_{(n)}$, $L(0)w = nw$.

6. $L(-1)-$*derivative property: For all* $v \in V$, $\frac{d}{dx} Y_W(v,x) = Y_W(L(-1)v, x)$.

Definition 3.2.3. *An* automorphism *of a vertex operator algebra* V *is a linear ismorphism* $g : V \longrightarrow V$ *such that* $g(\omega) = \omega$ *and* $gY(v,x)g^{-1} = Y(g\cdot v, x)$ *for all* $v \in V$. *As an immediate consequence of this definition,* $g(\mathbf{1}) = \mathbf{1}$ *holds.*

Remark 3.2.4. Since g fixes ω, g fixes the eigenspaces for the action of $L(0)$, in other words, $g\big|_{V_{(n)}}$ acts as a linear isomophism of each of the finite dimensional spaces $V_{(n)}$, $n \in \mathbb{Z}$. Hence, if g has a finite order, say m, then $V = \coprod_{j\in\mathbb{Z}/m\mathbb{Z}} V^j$ where $V^j = \{v \in V | g\cdot v = \xi^j v\}$, with $j \in \mathbb{Z}/m\mathbb{Z}$ and $\xi = e^{\frac{2\pi\sqrt{-1}}{m}}$.

Definition 3.2.5. *Let* g *be a finite order automorphism of* V. *Let the order of* g *be* m. *A* $g-$twisted $V-$module *is the data* (W, Y_W) *where* W *is a* $\mathbb{C}-$*graded vector space,* $W = \coprod_{n\in\mathbb{C}} W_{(n)}$ *and* $Y_W(\cdot, x)$ *is a linear map*

$$Y_W(\cdot,x) \quad : \quad V \longrightarrow (\mathrm{End}W)[[x^{1/m}, x^{-1/m}]]$$

$$v \longmapsto Y_W(v,x) = \sum_{n \in \frac{1}{m}\mathbb{Z}} v_n x^{-n-1}$$

such that the following conditions hold:

1. *Grading restriction condition: For all* $n \in \mathbb{C}$, $\dim W_{(n)} < \infty$ *and* $W_{(n-k)} = 0$ *for all sufficiently large* $k \in \frac{1}{m}\mathbb{Z}$.

2. *Formal monodromy condition: For* $j \in \mathbb{Z}/m\mathbb{Z}$ *and* $v \in V^j$,

$$Y_W(v,x) = \sum_{n \in \frac{j}{m}+\mathbb{Z}} v_n x^{-n-1}. \tag{3.2.1}$$

3. *Lower truncation condition: For all* $v \in V$ *and* $w \in W$, $Y_W(v,x)w \in W((x^{1/m}))$.

4. *Vacuum property:* $Y_W(\mathbf{1},x) = \mathrm{Id}_W$.

5. *Twisted Jacobi identity: Let* $\xi = e^{\frac{2\pi\sqrt{-1}}{m}}$. *For all* $u, v \in V$,

$$x_0{}^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_W(u,x_1)Y_W(v,x_2) - x_0{}^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_W(v,x_2)Y_W(u,x_1)$$
$$= \frac{1}{m}\sum_{j\in\mathbb{Z}/m\mathbb{Z}} x_2^{-1}\delta\left(\xi^j\frac{(x_1-x_0)^{1/m}}{x_2^{1/m}}\right)Y_W(Y(g^j\cdot u,x_0)v,x_2).$$

6. $L(0)-$*grading condition: For* $n \in \mathbb{Z}$, *let* $L(n) = \omega_{n+1}$. *Then, for all* $w \in W_{(n)}$, $L(0)w = nw$. *Note that because* $\omega \in V^0$, $Y_W(\omega,x)$ *has only integral powers of* x, *due to the formal monodromy condition.*

7. $L(-1)-$*derivative property: For all* $v \in V$, $\frac{d}{dx}Y_W(v,x) = Y_W(L(-1)v,x)$.

Remark 3.2.6. By a $V-$module we will mean either an untwisted $V-$module or a $g-$twisted $V-$module.

Definition 3.2.7. *(Cf. [FHL]) Let* $(W_1,Y_1),(W_2,Y_2),(W_3,Y_3)$ *be untwisted* $V-$*modules. An* intertwining operator *of type* $\binom{W_3}{W_1\ W_2}$ *is a linear map*

$$\begin{aligned} \mathcal{Y}(\cdot,x) \quad &: \quad W_1 \longrightarrow \mathrm{Hom}(W_2,W_3)\{x\} \\ & \qquad w \longmapsto \mathcal{Y}(w,x) = \sum_{n\in\mathbb{C}} w_n x^{-n-1}\,(\text{where } w_n \in \mathrm{Hom}(W_2,W_3)). \end{aligned}$$

such that for all $v \in V, w_{(1)} \in W_1, w_{(2)} \in W_2$,

1. Lower truncation condition: *For all* $n \in \mathbb{C}$, $(w_{(1)})_{n+k}w_{(2)} = 0$ *for all sufficiently large integers* k.

2. Jacobi identity:

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_3(v, x_1)\mathcal{Y}(w_{(1)}, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\mathcal{Y}(w_{(1)}, x_2)Y_2(v, x_1) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\mathcal{Y}(Y_1(v, x_0)w_{(1)}, x_2).$$

3. $L(-1)-$derivative property: $\mathcal{Y}(L(-1)w_{(1)}, x) = \frac{d}{dx}\mathcal{Y}(w_{(1)}, x)$.

We now make a very important assumption. Throughout this chapter, we will always work under this assumption.

Assumption 3.2.8. *We will assume that the automorphism* g *has order* $m = 2$*. We will assume that all the modules considered henceforth are graded by a coset of* $\frac{1}{m}\mathbb{Z}$ *in* $\mathbb{C}$*. In particular, for each* $i = 1, 2, 3$*, there exists a complex number* c_i *such that the module* W_i *is* $\frac{1}{m}\mathbb{Z} + c_i$ *graded. Concrete examples of intertwining operators that we will encounter in later sections will all deal with modules which satisfy this assumption. It is not hard to see that any irreducible* V*-module satisfies this assumption.*

Definition 3.2.9. *Let* g *be an involution of* V*. Let* (W_1, Y_1)*,* (W_2, Y_2)*,* (W_3, Y_3) *be* $V-$*modules, such that* W_1 *is untwisted and* W_2*,* W_3 *are* g*-twisted. A twisted intertwining operator of type* $\binom{W_3}{W_1\ W_2}$ *is a linear map*

$$\begin{aligned}\mathcal{Y}^t(\cdot, x) :& W_1 \longrightarrow \mathrm{Hom}(W_2, W_3)\{x\} \\ & w \longmapsto \mathcal{Y}^t(w, x) = \sum_{n\in\mathbb{C}} w_n x^{-n-1} \text{ where } w_n \in \mathrm{Hom}(W_2, W_3).\end{aligned}$$

such that for all $v \in V, w_{(1)} \in W_1, w_{(2)} \in W_2$*,*

1. Lower truncation condition: *For all* $n \in \mathbb{C}$*,* $(w_{(1)})_{n+k}w_{(2)} = 0$ *for all sufficiently large* $k \in \frac{1}{2}\mathbb{Z}$*.*

2. Jacobi identity:

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_3(v, x_1)\mathcal{Y}^t(w_{(1)}, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\mathcal{Y}^t(w_{(1)}, x_2)Y_2(v, x_1)$$

$$= \frac{1}{2} \sum_{j \in \mathbb{Z}/2\mathbb{Z}} x_2^{-1} \delta \left((-1)^j \frac{(x_1 - x_0)^{1/2}}{x_2^{1/2}} \right) \mathcal{Y}^t(Y_1(g^j \cdot v, x_0) w_{(1)}, x_2).$$

3. $L(-1)-$derivative property: $\mathcal{Y}^t(L(-1)w_{(1)}, x) = \frac{d}{dx}\mathcal{Y}^t(w_{(1)}, x)$.

Remark 3.2.10. Note that our definition is a special case of the definition of twisted intertwining operators given in [Xu]. In [Xu], twisted intertwining operators based on twisted modules for colored vertex superalgebras have been considered.

Definition 3.2.11. *The space of twisted intertwining operators of a given type* $\binom{W_3}{W_1\ W_2}$ *forms a vector space. The twisted fusion rule, denoted by* $N^t\binom{W_3}{W_1\ W_2}$*, is defined to be the dimension of this vector space.*

3.3 Duality properties

In this section, following the methods of [FLM] and [DL], we record some duality properties for twisted intertwining operators. For the rest of this section, we fix a vertex operator algebra $(V, Y, \mathbf{1}, \omega)$ (which we denote by V for brevity) with an automorphism g of finite order m.

3.3.1 Properties of the module map

Let (W, Y_W) be a g-twisted V-module. Here we record some properties of the module map Y_W that will be used later. These properties could be found in, for instance, [Li].

Lemma 3.3.1. *Let* $\xi = \exp(2\pi\mathbf{i}/m)$*, where* m *is the order of* g*. Let* $u \in V^k = \{a \in V | g \cdot a = \xi^k a\}$ *and* $v \in V$*. Then,*

$$x_0{}^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W(u, x_1) Y_W(v, x_2) - x_0{}^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_W(v, x_2) Y_W(u, x_1)$$
$$= x_2{}^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\left(\frac{x_1 - x_0}{x_2}\right)^{-k/m} Y_W(Y(u, x_0)v, x_2). \quad (3.3.1)$$

By definition of automorphism of a vertex operator algebra, $\omega \in V^0$, hence, taking $a = \omega$ in lemma 3.3.1, we get the usual Jacobi Identity:

Corollary 3.3.2. *For* $v \in V$*,*

$$x_0{}^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_W(\omega,x_1)Y_W(v,x_2) - x_0{}^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_W(v,x_2)Y_W(\omega,x_1)$$
$$= x_2{}^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_W(Y(\omega,x_0)v,x_2).$$

Multiplying equation (3.3.1) by $x_1^{k/m}$, manipulating the delta function on the right hand side and then extracting Res_{x_1} gives

$$\begin{aligned}
&\mathrm{Res}_{x_1}\left(x_0{}^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_W(u,x_1)Y_W(v,x_2)x_1^{k/m}\right.\\
&\qquad\left.-x_0{}^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_W(v,x_2)Y_W(u,x_1)x_1^{k/m}\right)\\
&\quad= \mathrm{Res}_{x_1}x_2{}^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\left(\frac{x_1-x_0}{x_2}\right)^{-k/m}Y_W(Y(u,x_0)v,x_2)x_1^{k/m}\\
&\quad= \mathrm{Res}_{x_1}x_1{}^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\left(\frac{x_2+x_0}{x_1}\right)^{k/m}Y_W(Y(u,x_0)v,x_2)x_1^{k/m}\\
&\quad= \mathrm{Res}_{x_1}x_1{}^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)(x_2+x_0)^{k/m}Y_W(Y(u,x_0)v,x_2)\\
&\quad= (x_2+x_0)^{k/m}Y_W(Y(u,x_0)v,x_2).
\end{aligned} \tag{3.3.2}$$

Since the powers of x_0 appearing on the right hand side are truncated from below, the same holds for the left hand side, and thus we can multiply throughout by $(x_2+x_0)^{-k/m}$ to obtain

Lemma 3.3.3. *For* $u \in V^k = \{a \in V | g \cdot a = \xi^k a\}$ *and* $v \in V$*,*

$$\begin{aligned}
&Y_W(Y(u,x_0)v,x_2)\\
&\quad= (x_2+x_0)^{-k/m}\mathrm{Res}_{x_1}\left(x_0{}^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_W(u,x_1)Y_W(v,x_2)x_1^{k/m}\right.\\
&\qquad\left.-x_0{}^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_W(v,x_2)Y_W(u,x_1)x_1^{k/m}\right).
\end{aligned} \tag{3.3.3}$$

3.3.2 Properties of twisted intertwining operators

Let g be an involution of V and let (W_1, Y_1), (W_2, Y_2), (W_3, Y_3) be $V-$modules such that W_1 is untwisted and W_2, W_3 are g-twisted. Let $\mathcal{Y}^t$ be a twisted intertwining operator of type $\binom{W_1}{W_2\ W_3}$.

Lemma 3.3.4. *Let* $v \in V^k = \{a \in V | g \cdot a = (-1)^k a\}$ *and* $w_{(1)} \in W_1$. *Then,*

$$x_0{}^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_3(v, x_1)\mathcal{Y}^t(w_{(1)}, x_2) - x_0{}^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\mathcal{Y}^t(w_{(1)}, x_2)Y_2(v, x_1)$$
$$= x_2{}^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\left(\frac{x_1 - x_0}{x_2}\right)^{-k/2}\mathcal{Y}^t(Y_1(v, x_0)w_{(1)}, x_2). \quad (3.3.4)$$

By definition of automorphism of a vertex operator algebra, $\omega \in V^0$, hence, taking $v = \omega$ in lemma 3.3.4, we get the usual form of Jacobi identity:

Corollary 3.3.5. *For* $w_{(1)} \in W_1$,

$$x_0{}^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_3(\omega, x_1)\mathcal{Y}^t(w_{(1)}, x_2) - x_0{}^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\mathcal{Y}^t(w_{(1)}, x_2)Y_2(\omega, x_1)$$
$$= x_2{}^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\mathcal{Y}^t(Y_1(\omega, x_0)w_{(1)}, x_2).$$

Hence, following the usual procedure, we can conclude that:

Corollary 3.3.6. *For a* $L(0)-$*homogeneous vector* $w_{(1)} \in W_1$ *and* $n \in \mathbb{C}$,

$$\mathrm{wt}\, w_{(1),n} = \mathrm{wt}\, w_{(1)} - n - 1,$$

Remark 3.3.7. If $w_{(1)} \in W_1, w_{(2)} \in W_2$ are $L(0)-$homogeneous vectors, then,

$$\mathrm{wt}\,(w_{(1),n} w_{(2)}) = \mathrm{wt}\, w_{(1)} + \mathrm{wt}\, w_{(2)} - n - 1. \quad (3.3.5)$$

Hence, under assumption 3.2.8, we conclude that

$$w_{(1),n} = 0 \text{ if } n \notin \frac{1}{2}\mathbb{Z} + c_1 + c_2 - c_3.$$

With

$$\Delta = c_1 + c_2 - c_3, \quad (3.3.6)$$

$$x^{2\Delta}\mathcal{Y}^t(w_{(1)}, x^2)w_{(2)} \in W_3((x)), \quad (3.3.7)$$

moreover,

$$\mathcal{Y}^t(w_{(1)}, x^2)w_{(2)} = 0 \text{ iff } \mathcal{Y}^t(w_{(1)}, x)w_{(2)} = 0. \quad (3.3.8)$$

Under the restriction of assumption 3.2.8, we work out the commutativity, associativity and the rationality properties for $\mathcal{Y}^t$. For this, let us assume $v \in V^k$ and $w_{(1)} \in W_1, w_{(2)} \in W_2$ in (3.3.4). We assume $v, w_{(1)}, w_{(2)}$ to be $L(0)-$homogeneous.

Keeping in mind the formal monodromy condition (3.2.1), remark 3.3.7, specifically, equations (3.3.6) and (3.3.7), we multiply both sides of (3.3.4) by $x_1^{k/2}x_2^{\Delta}$ and make the substitution $x_2 \mapsto x_2^2$, in order to obtain operators whose expansions in powers of x_0, x_1, x_2 involve only integral powers of these variables.

$$\begin{aligned} &{x_0}^{-1}\delta\left(\frac{x_1-x_2^2}{x_0}\right)Y_3(v,x_1)\mathcal{Y}^t(w_{(1)},x_2^2)x_1^{k/2}x_2^{2\Delta} \\ &\quad -{x_0}^{-1}\delta\left(\frac{x_2^2-x_1}{-x_0}\right)\mathcal{Y}^t(w_{(1)},x_2^2)Y_2(v,x_1)x_1^{k/2}x_2^{2\Delta} \\ &= (x_2^2)^{-1}\delta\left(\frac{x_1-x_0}{x_2^2}\right)\left(\frac{x_1-x_0}{x_2^2}\right)^{-k/m}\mathcal{Y}^t(Y_1(v,x_0)w_{(1)},x_2^2)x_1^{k/2}x_2^{2\Delta}. \end{aligned}$$

Taking Res_{x_0},

$$\begin{aligned} &[Y(v,x_1),\mathcal{Y}^t(w_{(1)},x_2^2)]x_1^{k/2}x_2^{2\Delta} \\ &= \mathrm{Res}_{x_0}(x_2^2)^{-1}\delta\left(\frac{x_1-x_0}{x_2^2}\right)\left(\frac{x_1-x_0}{x_2^2}\right)^{-k/2}\mathcal{Y}^t(Y_1(v,x_0)w_{(1)},x_2^2)x_1^{k/2}x_2^{2\Delta}. \end{aligned}$$

Using the properties of delta function obtained in [FLM], we manipulate the right hand side, thus:

$$\begin{aligned} &\mathrm{Res}_{x_0}(x_2^2)^{-1}\delta\left(\frac{x_1-x_0}{x_2^2}\right)\left(\frac{x_1-x_0}{x_2^2}\right)^{-k/2}\mathcal{Y}^t(Y_1(v,x_0)w_{(1)},x_2^2)x_1^{k/2}x_2^{2\Delta} \\ &= \mathrm{Res}_{x_0}\mathcal{Y}^t\left(\left({x_0}^{-1}\delta\left(\frac{x_1-x_2^2}{x_0}\right)-{x_0}^{-1}\delta\left(\frac{x_2^2-x_1}{-x_0}\right)\right)Y_1(v,x_0)w_{(1)},x_2^2\right)\cdot \\ &\quad\cdot\left(\frac{x_1-x_0}{x_2^2}\right)^{-k/2}x_1^{k/2}x_2^{2\Delta} \\ &= \mathrm{Res}_{x_0}\mathcal{Y}^t\left(\left({x_0}^{-1}\delta\left(\frac{x_1-x_2^2}{x_0}\right)Y_1(v,x_1-x_2^2)-{x_0}^{-1}\delta\left(\frac{x_2^2-x_1}{-x_0}\right)Y_1(v,-x_2^2+x_1)\right)w_{(1)},x_2^2\right)\cdot \\ &\quad\cdot\left(\frac{x_1-x_0}{x_2^2}\right)^{-k/2}x_1^{k/2}x_2^{2\Delta} \end{aligned}$$

But, if $n \in \mathbb{N}$ (depending on v, $w_{(1)}$) is sufficiently large, say, $n \geq N(\geq 0)$,

$$\left(Y_1(v,x_1-x_2^2)(x_1-x_2^2)^n - Y_1(u,-x_2^2+x_1)(-x_2^2+x_1)^n\right)w_{(1)} = 0, \tag{3.3.9}$$

so that we have the *commutator formula*:

Proposition 3.3.8. ***(commutator formula)*** *Under assumption 3.2.8, for any $v \in V^k$, $w_{(1)} \in W_1$ that are $L(0)$-homogeneous, there exists an $N \in \mathbb{N}$ depending only on v and $w_{(1)}$ such that*

$$[Y(v,x_1),\mathcal{Y}^t(w_{(1)},x_2^2)]x_1^{k/2}x_2^{2\Delta}(x_1-x_2^2)^n = 0, \tag{3.3.10}$$

where the left hand side has only integral powers of the variables x_1 and x_2.

To derive the associator formula, we again multiply equation (3.3.4) by $x_1^{k/2}x_2^{\Delta}$ and then make the substitution $x_2 \mapsto x_2^2$. Then we use the properties of the delta function.

$$\begin{aligned}
&{x_0}^{-1}\delta\left(\frac{x_1-x_2^2}{x_0}\right)Y_3(v,x_1)\mathcal{Y}^t(w_{(1)},x_2^2)x_1^{k/2}x_2^{2\Delta}\\
&-{x_0}^{-1}\delta\left(\frac{x_2^2-x_1}{-x_0}\right)\mathcal{Y}^t(w_{(1)},x_2^2)Y_2(v,x_1)x_1^2x_2^{2\Delta}\\
&=(x_2^2)^{-1}\delta\left(\frac{x_1-x_0}{x_2^2}\right)\left(\frac{x_1-x_0}{x_2^2}\right)^{-k/2}\mathcal{Y}^t(Y_1(v,x_0)w_{(1)},x_2^2)x_1^{k/2}x_2^{2\Delta}\\
&={x_1}^{-1}\delta\left(\frac{x_2^2+x_0}{x_1}\right)\left(\frac{x_2^2+x_0}{x_1}\right)^{k/2}\mathcal{Y}^t(Y_1(v,x_0)w_{(1)},x_2^2)x_1^{k/2}x_2^{2\Delta}\\
&={x_1}^{-1}\delta\left(\frac{x_2^2+x_0}{x_1}\right)(x_2^2+x_0)^{k/2}\mathcal{Y}^t(Y_1(v,x_0)w_{(1)},x_2^2)x_2^{2\Delta} \qquad (3.3.11)
\end{aligned}$$

The first term on the left can be written as:

$$\begin{aligned}
&{x_0}^{-1}\delta\left(\frac{x_1-x_2^2}{x_0}\right)Y_3(v,x_1)\mathcal{Y}^t(w_{(1)},x_2^2)x_1^{k/2}x_2^{2\Delta}\\
&={x_1}^{-1}\delta\left(\frac{x_0+x_2^2}{x_1}\right)Y_3(v,x_1)\mathcal{Y}^t(w_{(1)},x_2^2)x_1^{k/2}x_2^{2\Delta}\\
&={x_1}^{-1}\delta\left(\frac{x_0+x_2^2}{x_1}\right)Y_3(v,x_0+x_2^2)\mathcal{Y}^t(w_{(1)},x_2^2)(x_0+x_2^2)^{k/2}x_2^{2\Delta} \qquad (3.3.12)
\end{aligned}$$

Using (3.3.11) and (3.3.12), taking Res_{x_1}, and using properties of delta function, we get:

$$\begin{aligned}
&(x_2^2+x_0)^{k/2}\mathcal{Y}^t(Y_1(v,x_0)w_{(1)},x_2^2)x_2^{2\Delta}\\
&\qquad -Y_3(v,x_0+x_2^2)\mathcal{Y}^t(w_{(1)},x_2^2)(x_0+x_2^2)^{k/2}x_2^{2\Delta}\\
&=\mathrm{Res}_{x_1}\left(-{x_0}^{-1}\delta\left(\frac{x_2^2-x_1}{-x_0}\right)\mathcal{Y}^t(w_{(1)},x_2^2)Y_2(v,x_1)x_1^{k/2}x_2^{2\Delta}\right)\\
&=\mathrm{Res}_{x_1}\mathcal{Y}^t(w_{(1)},x_2^2)\left\{(x_2^2)^{-1}\delta\left(\frac{x_1-x_0}{x_2^2}\right)x_1^{k/2}Y_2(v,x_1)\right.\\
&\qquad\left.-{x_0}^{-1}\delta\left(\frac{x_1-x_2^2}{x_0}\right)x_1^{k/2}Y_2(v,x_1)\right\}x_2^{2\Delta}\\
&=\mathrm{Res}_{x_1}\mathcal{Y}^t(w_{(1)},x_2^2)\left\{{x_1}^{-1}\delta\left(\frac{x_2^2+x_0}{x_1}\right)x_1^{k/2}Y_2(v,x_1)\right.\\
&\qquad\left.-{x_1}^{-1}\delta\left(\frac{x_0+x_2^2}{x_1}\right)x_1^{k/2}Y_2(v,x_1)\right\}x_2^{2\Delta}\\
&=\mathrm{Res}_{x_1}\mathcal{Y}^t(w_{(1)},x_2^2)\left\{{x_1}^{-1}\delta\left(\frac{x_2^2+x_0}{x_1}\right)(x_2^2+x_0)^{k/2}Y_2(v,x_2^2+x_0)\right.
\end{aligned}$$

$$- x_1{}^{-1}\delta\left(\frac{x_0 + x_2^2}{x_1}\right)(x_0 + x_2^2)^{k/2} Y_2(v, x_0 + x_2^2)\Bigg\} x_2^{2\Delta} \tag{3.3.13}$$

But, for $v \in V$ and $w_{(2)} \in W_2$, if $n \in \mathbb{N}$ is large enough, say $n \geq N(\geq 0)$, we have

$$\Big(Y_2(v, x_2^2 + x_0)(x_2^2 + x_0)^{(k/2)+n} - Y_2(v, x_0 + x_2^2)(x_0 + x_2^2)^{(k/2)+n}\Big) w_{(2)} = 0 \tag{3.3.14}$$

Hence, from (3.3.13) and (3.3.14), we get the *associator formula*:

Proposition 3.3.9. ***(associator formula)*** *Under assumption 3.2.8, for any $L(0)$-homogeneous elements $v \in V^k$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, there exists an $N \in \mathbb{N}$ depending only on v and $w_{(1)}$ such that*

$$\begin{aligned} x_2^{2\Delta}(x_2^2 + x_0)^{(k/2)+n}&\mathcal{Y}^t(Y_1(v, x_0)w_{(1)}, x_2^2)w_{(2)} \\ &= x_2^{2\Delta}(x_0 + x_2^2)^{(k/2)+n} Y_3(v, x_0 + x_2^2)\mathcal{Y}^t(w_{(1)}, x_2^2)w_{(2)}. \end{aligned} \tag{3.3.15}$$

Now we derive and record the rationality properties. First, we discuss expansions of certain rational functions (cf. [FLM], [FHL]).

Consider the ring $\mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}]$ of Laurent polynomials in two variables x_1 and x_2, and its field of fractions $\mathbb{C}(x_1, x_2)$.

Let S_{ij} (where (i, j) is either $(1, 2)$ or $(2, 1)$) denote the following set:

$$S_{ij} = \{\alpha x_i + \beta x_j^2 \mid \alpha, \beta \in \mathbb{C}, (\alpha, \beta) \neq (0, 0)\} \cup \{x_1, x_2\},$$

and let $\mathbb{C}[x_1, x_2]_{S_{ij}}$ be the subring of $\mathbb{C}(x_1, x_2)$ obtained by inverting the products of (zero or more) elements of S_{ij}. Let (i_1, i_2) be either $(1, 2)$ or $(2, 1)$. Define the linear and multiplicative map

$$\iota_{i_1 i_2}^{ij} : \mathbb{C}[x_1, x_2]_{S_{ij}} \longrightarrow \mathbb{C}[[x_1, x_1^{-1}, x_2, x_2^{-1}]] \tag{3.3.16}$$

so that $\iota_{i_1 i_2}^{ij}$ is the identity on $\mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}]$ and so that $\iota_{i_1 i_2}^{ij}((\alpha x_i + \beta x_j^2)^{-1})$ is the expansion of $(\alpha x_i + \beta x_j^2)^{-1}$ in non-negative integral powers of x_{i_2}.

Next, we define the restricted dual of a module. For an untwisted or a $g-$twisted $V-$module W, set

$$W' = \coprod_{n \in \mathbb{C}} W_n^*. \tag{3.3.17}$$

W' is called the restricted dual of W.

Proposition 3.3.10. *(a)* ***(rationality of products)*** *Under assumption 3.2.8, for $L(0)$-homogeneous elements $v \in V^k$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w'_{(3)} \in W'_3$, the formal series*

$$\langle w'_{(3)}, Y_3(v, x_1)\mathcal{Y}^t(w_{(1)}, x_2^2)w_{(2)}\rangle x_1^{k/2} x_2^{2\Delta}, \tag{3.3.18}$$

which is well defined and which involves only integral powers of x_1, x_2 lies in the image of the map ι_{12}^{12} :

$$\langle w'_{(3)}, Y_3(v, x_1)\mathcal{Y}^t(w_{(1)}, x_2^2)w_{(2)}\rangle x_1^{k/2} x_2^{2\Delta} = \iota_{12}^{12} f(x_1, x_2), \tag{3.3.19}$$

where the (unique) element $f \in \mathbb{C}[x_1, x_2]_{S_{12}}$ is of the form

$$f(x_1, x_2) = \frac{g(x_1, x_2)}{x_1^{\rho} x_2^{\sigma} (x_1 - x_2^2)^{\tau}}, \tag{3.3.20}$$

for some $g \in \mathbb{C}[x_1, x_2]$ and $\rho, \sigma, \tau \in \mathbb{Z}$. The integer τ depends only on v and $w_{(1)}$.

(b) ***(commutativity)*** *We also have*

$$\langle w'_{(3)}, \mathcal{Y}^t(w_{(1)}, x_2^2)Y_2(v, x_1)w_{(2)}\rangle x_1^{k/2} x_2^{2\Delta} = \iota_{21}^{12} f(x_1, x_2), \tag{3.3.21}$$

in particular, the left-hand side is well defined.

Proof. By (3.3.10), if n is large enough, we have,

$$\begin{aligned} &Y_3(v, x_1)\mathcal{Y}^t(w_{(1)}, x_2^2) x_1^{k/2} x_2^{2\Delta} (x_1 - x_2^2)^n \\ &= \mathcal{Y}^t(w_{(1)}, x_2^2) Y_2(v, x_1) x_1^{k/2} x_2^{2\Delta} (x_1 - x_2^2)^n \end{aligned}$$

and

$$\begin{aligned} &\langle w'_{(3)}, Y_3(v, x_1)\mathcal{Y}^t(w_{(1)}, x_2^2)w_{(2)}\rangle x_1^{k/2} x_2^{2\Delta} (x_1 - x_2^2)^n \\ &\quad = \langle w'_{(3)}, \mathcal{Y}^t(w_{(1)}, x_2^2) Y_2(v, x_1) w_{(2)}\rangle x_1^{k/2} x_2^{2\Delta} (x_1 - x_2^2)^n \end{aligned} \tag{3.3.22}$$

involves only integral powers of x_1 and x_2. But the left-hand side of (3.3.22) involves only finitely many negative powers of x_2 due to the lower truncation condition for $\mathcal{Y}^t$, and the right-hand side involves only finitely many positive powers of x_2 because of corollary 3.3.6. Thus, each side of (3.3.22) involves only finitely many powers of x_2.

Similarly, each side of (3.3.22) involves only finitely many powers of x_1. Hence, each side of of (3.3.22) is equal to some $h(x_1,x_2) \in \mathbb{C}[x_1,x_1^{-1},x_2,x_2^{-1}]$. Then,

$$f(x_1,x_2) = \frac{h(x_1,x_2)}{(x_1 - x_2^m)^n}$$

satisfies the desired conditions. In fact, the left-hand side of (3.3.19) involves only finitely many negative powers of x_2 and so can be multiplied by $(x_1 - x_2^m)^{-n}$ and similarly, the left-hand side of (3.3.21) involves only finitely many negative powers of x_1 and hence can be multiplied by $(-x_2^2 + x_1)^{-n}$. □

An analogous argument using the associator formula, (3.3.15) gives:

Proposition 3.3.11. *(a)* ***(rationality of iterates)*** *Under assumption 3.2.8, for $L(0)$-homogeneous elements $v \in V^k$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w'_{(3)} \in W'_3$, the formal series*

$$\langle w'_{(3)}, \mathcal{Y}^t(Y_1(v,x_0)w_{(1)},x_2^2)w_{(2)}\rangle x_2^{2\Delta}(x_2^2+x_0)^{k/2} \tag{3.3.23}$$

which is well defined and which involves only integral powers of x_0, x_2, lies in the image of the map ι_{20}^{02} :

$$\langle w'_{(3)}, \mathcal{Y}^t(Y_1(v,x_0)w_{(1)},x_2^2)w_{(2)}\rangle x_2^{2\Delta}(x_2^2+x_0)^{k/m} = \iota_{20}^{02}h(x_0,x_2) \tag{3.3.24}$$

where the (unique) element $h \in \mathbb{C}[x_0,x_2]_{S_{02}}$ is of the form

$$h(x_0,x_2) = \frac{k(x_0,x_2)}{x_0^{\lambda}x_2^{\mu}(x_0+x_2^m)^{\nu}} \tag{3.3.25}$$

for some $k(x_0,x_2) \in \mathbb{C}[x_0,x_2]$ and $r,s,t \in \mathbb{Z}$. The integer ν depends only on v and $w_{(2)}$.

(b) We also have

$$\langle w'_{(3)}, Y_3(v,x_0+x_2^2)\mathcal{Y}^t(w_{(1)},x_2^2)w_{(2)}\rangle x_2^{2\Delta}(x_0+x_2^2)^{k/2} = \iota_{02}^{02}h(x_0,x_2). \tag{3.3.26}$$

For the rational function $f(x_1,x_2)$ of (3.3.19),

$$\iota_{02}^{02}f(x_0+x_2^2,x_2) = \left(\iota_{12}^{12}f(x_1,x_2)\right)\big|_{x_1=x_0+x_2^2}, \tag{3.3.27}$$

so that from (3.3.19) and (3.3.26),

$$h(x_0,x_2) = f(x_0+x_2^2,x_2). \tag{3.3.28}$$

Thus we conclude:

Proposition 3.3.12. ***(associativity)*** *We have:*

$$\begin{aligned}&(\iota_{12}^{12})^{-1}\left(\langle w'_{(3)}, Y_3(v,x_1)\mathcal{Y}^t(w_{(1)},x_2^2)w_{(2)}\rangle x_1^{k/2}x_2^{2\Delta}\right)\\&=\left((\iota_{20}^{02})^{-1}\left(\langle w'_{(3)}, \mathcal{Y}^t(Y_1(v,x_0)w_{(1)},x_2^2)w_{(2)}\rangle (x_2^2+x_0)^{k/2}x_2^{2\Delta}\right)\right)\big|_{x_0=x_1-x_2^2}.\end{aligned}\tag{3.3.29}$$

As an application of the above results, we prove a proposition which will be used later to assert uniqueness of twisted intertwining operators (upto multiplication by a scalar). (cf. [DL], Proposition 11.9.)

Proposition 3.3.13. *Let W_1, W_2, W_3 be irreducible $V-$modules such that W_1 is untwisted while W_1 and W_2 are g-twisted. Further assume that each of them satisfies the conditions in assumption 3.2.8. Fix nonzero $L(0)-$homogeneous vectors $w_{(1)} \in W_1$, $w_{(2)} \in W_2$. Let $\mathcal{Y}^t$ be a twisted intertwining operator of type $\binom{W_3}{W_1\ W_2}$. If $\mathcal{Y}^t(w_{(1)},x)w_{(2)} = 0$, then $\mathcal{Y}^t(\cdot,x)\cdot = 0$. More generally, a twisted intertwining operator $\mathcal{X}^t$ of the same type as $\mathcal{Y}^t$ is uniquely determined by the knowledge of $\mathcal{X}^t(w_{(1)},x)w_{(2)}$ or in particular, by the knowledge of $\mathcal{X}^t(w_{(1)},x)\cdot$ or $\mathcal{X}^t(\cdot,x)w_{(2)}$.*

Proof. We proceed exactly as in the proof of [DL], Proposition 11.9. It is sufficient to prove the first assertion. We first show that

$$\mathcal{Y}^t(w_{(1)},x)\cdot = 0. \tag{3.3.30}$$

Observe that for any (ordinary or $g-$twisted) irreducible $V-$module W,

$$W = \text{ span}\,\{v^1_{n_1}\cdots v^j_{n_j}w \mid v^q \in V_{h_q} \text{ where } h_q \in \mathbb{Z}, n_q \in \frac{1}{2}\mathbb{Z}\}, \tag{3.3.31}$$

where w is an arbitrary nonzero element of W. Here, we assume that v_n is defined to be 0 for those values of $n \in \frac{1}{2}\mathbb{Z}$ for which it was previously undefined, for example, if W is an ordinary module, then v_n is defined to be 0 for all $n \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$.

By Proposition 3.3.10, we have, for $w'_{(3)} \in W_3$ and $L(0)-$homogeneous $v \in V^k$,

$$\begin{aligned}\langle w'_{(3)}, Y_3(v,x_1)\mathcal{Y}^t(w_{(1)},x_2^2)w_{(2)}\rangle x_1^{k/2}x_2^{2\Delta} &= \iota_{12}^{12}f(x_1,x_2)\\ \langle w'_{(3)}, \mathcal{Y}^t(w_{(1)},x_2^2)Y_2(v,x_1)w_{(2)}\rangle x_1^{k/2}x_2^{2\Delta} &= \iota_{21}^{12}f(x_1,x_2).\end{aligned}$$

By (3.3.8) and the assumption that $\mathcal{Y}^t(w_{(1)}, x_2)w_{(2)} = 0$, we conclude that $\iota_{12}^{12} f(x_1, x_2) = 0$, and since ι_{12}^{12} is injective, $f(x_1, x_2) = 0$. Thus, $\iota_{21}^{12} f(x_1, x_2) = 0$, which implies that

$$\langle w'_{(3)}, \mathcal{Y}^t(w_{(1)}, x_2^2)Y_2(v, x_1)w_{(2)}\rangle = 0.$$

Since $w'_{(3)} \in W_3$ is arbitrary, we get that

$$\mathcal{Y}^t(w_{(1)}, x_2^2)Y_2(v, x_1)w_{(2)} = 0.$$

By (3.3.8) we conclude that

$$\mathcal{Y}^t(w_{(1)}, x_2)Y_2(v, x_1)w_{(2)} = 0.$$

By (3.3.31) and induction on j, we get (3.3.30).

Analogously, using Proposition 3.3.11, we have:

$$\langle w'_{(3)}, \mathcal{Y}^t(Y_1(v, x_0)w_{(1)}, x_2^2)w_{(2)}\rangle x_2^{2\Delta}(x_2^2 + x_0)^{k/2} = \iota_{20}^{02} h(x_0, x_2)$$
$$\langle w'_{(3)}, Y_3(v, x_0 + x_2^2)\mathcal{Y}^t(w_{(1)}, x_2^2)w_{(2)}\rangle x_2^{2\Delta}(x_0 + x_2^2)^{k/2} = \iota_{02}^{02} h(x_0, x_2).$$

By (3.3.8) and the assumption that $\mathcal{Y}^t(w_{(1)}, x_2)w_{(2)} = 0$, we conclude that $\iota_{02}^{02} h(x_0, x_2) = 0$, and since ι_{02}^{02} is injective, $h(x_0, x_2) = 0$. Thus $\iota_{20}^{02} h(x_0, x_2) = 0$, which implies that:

$$\langle w'_{(3)}, \mathcal{Y}^t(Y_1(v, x_0)w_{(1)}, x_2^2)w_{(2)}\rangle x_2^{2\Delta}(x_2^2 + x_0)^{k/2} = 0. \tag{3.3.32}$$

Now, the equation above has only finitely many negative powers of x_0, and moreover, all the powers of x_0 appearing are integral. So, we can multiply by $(x_2^2 + x_0)^{-k/2}$ and we get:

$$\langle w'_{(3)}, \mathcal{Y}^t(Y_1(v, x_0)w_{(1)}, x_2^2)w_{(2)}\rangle = 0.$$

Since $w'_{(3)} \in W_3$ is arbitrary, we get that

$$\mathcal{Y}^t(Y_1(v, x_0)w_{(1)}, x_2^2)w_{(2)} = 0.$$

(3.3.31) induction on j and (3.3.8) gives:

$$\mathcal{Y}^t(\cdot, x)w_{(2)} = 0. \tag{3.3.33}$$

It is now clear that $\mathcal{Y}^t(\cdot, x)\cdot = 0$. □

3.4 Examples

3.4.1 The vertex operator algebra V_{Q^k} and the untwisted module V_{P^k}

In this section, we recall from [FLM] some constructions of vertex operator algebras based on lattices.

Let $P = \frac{1}{2}\mathbb{Z}\alpha$ and $Q = \mathbb{Z}\alpha$ be respectively the (one dimensional) weight and root lattices of $\mathfrak{sl}(2,\mathbb{C})$. Let $\langle \cdot, \cdot \rangle$ be the unique symmetric, $\mathbb{Z}-$bilinear, nondegenerate, positive definite form on P (and on Q) such that $\langle \alpha, \alpha \rangle = 2$.

For a positive integer k, let

$$\begin{aligned} P^k &= \underbrace{P \oplus P \oplus \cdots \oplus P}_{k \text{ times}} \\ Q^k &= \underbrace{Q \oplus Q \oplus \cdots \oplus Q}_{k \text{ times}}. \end{aligned}$$

For $\beta \in P$, we denote $(0, \cdots, 0, \beta, 0, \cdots, 0)$ (an element of P^k), where the non-trivial component is in i^{th} place, by β_i. Extend the form $\langle \cdot, \cdot \rangle$ to P^k and Q^k such that various summands are orthogonal to each other. With this, Q^k becomes an even lattice and $\langle P^k, Q^k \rangle \subset \mathbb{Z}$. Let $\mathbb{C}[P^k]$ and $\mathbb{C}[Q^k]$ be the group algebras of P^k and Q^k respectively, with corresponding bases $\{e^\lambda | \lambda \in P^k\}$ and $\{e^\mu | \mu \in Q^k\}$.

Let $\mathfrak{h}^k = \mathbb{C} \otimes_{\mathbb{Z}} Q^k$. Extend the form $\langle \cdot, \cdot \rangle$ to $\mathfrak{h}^k$. We consider $\mathfrak{h}^k$ as an abelian Lie algebra. Consider the following Lie algebras:

$$\begin{aligned} \hat{\mathfrak{h}}^k &= \mathfrak{h}^k \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \\ \tilde{\mathfrak{h}}^k &= \mathfrak{h}^k \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d \end{aligned}$$

such that

$$\begin{aligned} [\hat{\mathfrak{h}}^k, c] = [\tilde{\mathfrak{h}}^k, c] &= \{0\} \\ [x \otimes t^m, y \otimes t^n] &= \langle x, y \rangle m \delta_{m+n,0} c \quad \text{where } x, y \in \mathfrak{h}^k, m, n \in \mathbb{Z}, \\ [d, x \otimes t^n \oplus \mathbb{C}c \oplus \mathbb{C}d] &= nx \otimes t^n \quad \text{where } x \in \mathfrak{h}^k, n \in \mathbb{Z}, \end{aligned}$$

and

$$\hat{\mathfrak{h}}^k[-1] = \mathfrak{h}^k \otimes t^{\frac{1}{2}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$

$$\tilde{\mathfrak{h}}^k[-1] \quad = \quad \mathfrak{h}^k \otimes t^{\frac{1}{2}}\mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

such that

$$\begin{aligned}
[\hat{\mathfrak{h}}^k[-1], c] = [\tilde{\mathfrak{h}}^k[-1], c] &= \{0\} \\
[x \otimes t^m, y \otimes t^n] &= \langle x, y\rangle m\delta_{m+n,0}c \quad \text{where } x, y \in \mathfrak{h}^k, m, n \in \mathbb{Z}+1/2, \\
[d, x \otimes t^n \oplus \mathbb{C}c \oplus \mathbb{C}d] &= nx \otimes t^n \quad \text{where } x \in \mathfrak{h}^k, n \in \mathbb{Z}+1/2.
\end{aligned}$$

The algebras $\tilde{\mathfrak{h}}^k, \tilde{\mathfrak{h}}^k[-1]$ are respectively $\mathbb{Z}$- and $\mathbb{Z}+1/2$- graded by eigenvalues of $\operatorname{ad} d$. This grading is referred to as the grading by *degree.*

Let

$$\begin{aligned}
\hat{\mathfrak{h}}^k_{\mathbb{Z}} &= \coprod_{m\in\mathbb{Z}\setminus\{0\}} \mathfrak{h}^k \otimes t^m \oplus \mathbb{C}c \\
(\hat{\mathfrak{h}}^k_{\mathbb{Z}})^+ &= \mathfrak{h}^k \otimes \mathbb{C}[t] \oplus \mathbb{C}c \\
(\hat{\mathfrak{h}}^k_{\mathbb{Z}})^- &= \mathfrak{h}^k \otimes t^{-1}\mathbb{C}[t^{-1}] \\
\hat{\mathfrak{h}}^k_{\mathbb{Z}+1/2} &= \coprod_{m\in\mathbb{Z}+1/2} \mathfrak{h}^k \otimes t^m \oplus \mathbb{C}c \\
(\hat{\mathfrak{h}}^k_{\mathbb{Z}+1/2})^+ &= \mathfrak{h}^k \otimes t^{\frac{1}{2}}\mathbb{C}[t] \oplus \mathbb{C}c \\
(\hat{\mathfrak{h}}^k_{\mathbb{Z}+1/2})^- &= \mathfrak{h}^k \otimes t^{\frac{1}{2}}\mathbb{C}[t^{-1}].
\end{aligned}$$

For a Lie algebra $\mathfrak{g}$ let $\mathcal{U}(\mathfrak{g})$ denote its universal enveloping algebra and for a vector space V let $\mathcal{S}(V)$ denote the symmetric algebra on V.

Consider the following induced $\hat{\mathfrak{h}}^k_{\mathbb{Z}}-$module:

$$M_{\mathbb{Z}}(1) = \mathcal{U}(\hat{\mathfrak{h}}^k_{\mathbb{Z}}) \otimes_{(\hat{\mathfrak{h}}^k_{\mathbb{Z}})^+} \mathbb{C} \cong \mathcal{S}((\hat{\mathfrak{h}}^k_{\mathbb{Z}})^-) \text{ (linearly)},$$

where $\mathfrak{h}^k \otimes \mathbb{C}[t]$ acts trivially on the one-dimensional module $\mathbb{C}$ and c acts as 1, and the induced $\hat{\mathfrak{h}}^k_{\mathbb{Z}+1/2}-$module:

$$M_{\mathbb{Z}+1/2}(1) = \mathcal{U}(\hat{\mathfrak{h}}^k_{\mathbb{Z}+1/2}) \otimes_{(\hat{\mathfrak{h}}^k_{\mathbb{Z}+1/2})^+} \mathbb{C} \cong \mathcal{S}((\hat{\mathfrak{h}}^k_{\mathbb{Z}+1/2})^-) \text{ (linearly)},$$

where $\mathfrak{h}^k \otimes t^{\frac{1}{2}}C[t]$ acts trivially on the one-dimensional module $\mathbb{C}$ and c acts as 1.

The spaces $\mathcal{S}((\hat{\mathfrak{h}}^k_{\mathbb{Z}})^-)$ and $\mathcal{S}((\hat{\mathfrak{h}}^k_{\mathbb{Z}+1/2})^-)$ inherit tensor product gradings, which we shift according to equations (1.9.51) – (1.9.53) of [FLM] as follows

$$\deg \mathbf{1} = \frac{1}{24}\dim \mathfrak{h}^k = \frac{k}{24}, \quad \mathbf{1} \in \mathcal{S}((\hat{\mathfrak{h}}^k_{\mathbb{Z}})^-) \tag{3.4.1}$$

$$\deg \mathbf{1} = -\frac{1}{48}\dim \mathfrak{h}^k = -\frac{k}{48}, \quad \mathbf{1} \in \mathcal{S}((\hat{\mathfrak{h}}^k_{\mathbb{Z}+1/2})^-). \tag{3.4.2}$$

Define the following vector spaces:

$$V_{P^k} = M_{\mathbb{Z}}(1) \otimes \mathbb{C}[P^k] \tag{3.4.3}$$

$$V_{Q^k} = M_{\mathbb{Z}}(1) \otimes \mathbb{C}[Q^k] \tag{3.4.4}$$

$$V^T_{Q^k} = M_{\mathbb{Z}+1/2}(1) \otimes T, \tag{3.4.5}$$

where T is any vector space, in particular, a $\mathbb{C}[Q^k]-$module. We denote an element $1 \otimes e^\lambda \in P^k$ by simply e^λ. The degree grading is extended uniquely to the above spaces by defining

$$\deg e^\lambda = -\frac{1}{2}\langle \lambda, \lambda \rangle, \quad \lambda \in \mathbb{C}[P^k] \tag{3.4.6}$$

$$\deg T = 0. \tag{3.4.7}$$

The degree operator on the above spaces is still denoted by d.

We define an $\hat{\mathfrak{h}}^k-$module structure on V_{P^k} (we denote the action of $h \otimes t^n$ by $h(n)$) by making $\hat{\mathfrak{h}}^k_{\mathbb{Z}}$ act as $\hat{\mathfrak{h}}^k_{\mathbb{Z}} \otimes 1$ and by making $\mathfrak{h}^k = \mathfrak{h}^k \otimes t^0$ act as $1 \otimes \mathfrak{h}^k$ with $h(0)$ defined by:

$$h(0)e^\lambda = \langle h, \lambda \rangle e^\lambda$$

for $h \in \mathfrak{h}^k$ and $\lambda \in P^k$.

Following [FLM], we define a vertex operator algebra structure on V_{Q^k} and an untwisted $V_{Q^k}-$module structure on V_{P^k}. We consider V_{Q^k} as a subset of V_{P^k}. For $h \in \mathfrak{h}^k$, define:

$$h(x) = \sum_{n\in\mathbb{Z}} h(n)x^{-n-1} \in (\mathrm{End}\, V_{P^k})[[x, x^{-1}]].$$

For $\lambda \in P^k$, let e^λ also denote the left multiplication operator acting on $\mathbb{C}[P^k]$ corresponding to $e^\lambda \in \mathbb{C}[P^k]$ and let x^λ be the unique operator on $\mathbb{C}[P^k]$ defined by $x^\lambda \cdot e^\mu = x^{\langle \lambda, \mu \rangle} \cdot e^\mu$.

Let

$$\mathbf{1} = 1 \otimes e^0, \quad \omega = \frac{1}{2}\sum_{i=1}^{d} h_i(-1)^2 e^0,$$

where $\{h_1, \cdots, h_d\}$ is an orthonormal basis of $\mathfrak{h}^k$.

For $\lambda \in P^k$, define $Y(e^\lambda, x) \in (\operatorname{End} V_{P^k})[[x^{\frac{1}{2}}, x^{-\frac{1}{2}}]]$ by:

$$Y_{\mathbb{Z}}(e^\lambda, x) = \exp\left(\sum_{n=1}^{\infty} \frac{\lambda(-n)}{n} x^n\right) \exp\left(-\sum_{n=1}^{\infty} \frac{\lambda(n)}{n} x^n\right) e^\lambda x^\lambda.$$

For a more general $v = h_1(-n_1) \cdots h_j(-n_j) \otimes e^\lambda$, where $h_1, \cdots, h_j \in \mathfrak{h}^k, n_1, \cdots, n_j \in \mathbb{Z}_+, \lambda \in P^k$, we define

$$Y_{\mathbb{Z}}(v, x) = {\circ \atop \circ} \left(\frac{1}{(n_1 - 1)!} \left(\frac{d}{dx}\right)^{n_1 - 1} h_1(x)\right) \cdots \left(\frac{1}{(n_j - 1)!} \left(\frac{d}{dx}\right)^{n_j - 1} h_j(x)\right) Y_{\mathbb{Z}}(e^\lambda, x) {\circ \atop \circ}$$

We uniquely extend this definition of $Y_{\mathbb{Z}}$ to whole of V_{P^k} by linearity, and we get a linear map:

$$Y_{\mathbb{Z}}(\cdot, x)\cdot : V_{P^k} \otimes V_{P^k} \longrightarrow V_{P^k}[[x^{\frac{1}{2}}, x^{-\frac{1}{2}}]].$$

Then,

$$Y_{\mathbb{Z}}(\cdot, x) \cdot |_{V_{Q^k} \otimes V_{Q^k}} : V_{Q^k} \otimes V_{Q^k} \longrightarrow V_{Q^k}[[x, x^{-1}]]$$

and

$$Y_{\mathbb{Z}}(\cdot, x) \cdot |_{V_{Q^k} \otimes V_{P^k}} : V_{Q^k} \otimes V_{P^k} \longrightarrow V_{P^k}[[x, x^{-1}]].$$

We denote the above restrictions of $Y_{\mathbb{Z}}$ by $Y_{\mathbb{Z}}$ again.

The space V_{P^k} is spanned by eigenvectors of the operator $L(0)(= \omega_1)$, and hence can be graded by $L(0)$ eigenvalues. This is the grading by *weight*. With this grading, $(V_{Q^k}, Y_{\mathbb{Z}}, \mathbf{1}, \omega)$ becomes a vertex operator algebra, and $(V_{P^k}, Y_{\mathbb{Z}})$ becomes an ordinary $V_{Q^k}-$module.

3.4.2 Some twisted modules for V_{Q^k}

Define a map

$$\begin{aligned} \theta : V_{Q^k} = \mathcal{S}((\hat{\mathfrak{h}}^k_{\mathbb{Z}})^-) \otimes \mathbb{C}[Q^k] &\longrightarrow V_{Q^k} \\ h_1(-n_1) \cdots h_j(-n_j) \otimes e^\lambda &\longmapsto (-1)^j h_1(-n_1) \cdots h_j(-n_j) \otimes e^{-\lambda} \end{aligned} \quad (3.4.8)$$

where $h_1, \cdots, h_j \in \mathfrak{h}^k, n_1, \cdots, n_j \in \mathbb{Z}_+, \lambda \in Q^k$. It is easy to see that θ defines an automorphism of the vertex operator algebra V_{Q^k} of order 2.

Let T be a one-dimensional Q^k-module, on which each element of Q^k acts as a scalar with values in $\{1, -1\}$. We consider T as a $\mathbb{C}[Q^k]-$module. From (3.4.5), recall the space $V^T_{Q^k} = \mathcal{S}((\hat{\mathfrak{h}}^k_{\mathbb{Z}+1/2})^-) \otimes T$.

Following Chapter 9 of [FLM], we define a $\theta-$twisted $V_{Q^k}-$module structure on $V_{Q^k}^T$. For $e^\mu = 1 \otimes e^\mu \in V_{Q^k}$, define $Y_0(e^\mu, x) \in (\mathrm{End}\, V_{Q^k}^T)[[x^{1/2}, x^{-1/2}]]$ by:

$$Y_0(e^\mu, x) = 2^{-\langle \mu,\mu \rangle} \exp\left(\sum_{n \in \mathbb{N}+1/2} \frac{\mu(-n)}{n} x^n\right) \exp\left(-\sum_{n \in \mathbb{N}+1/2} \frac{\mu(n)}{n} x^{-n}\right) x^{-\langle \mu,\mu \rangle/2} e^\mu \tag{3.4.9}$$

where the factor e^μ acts on the tensorand T. More generally, for $v = h_1(-n_1) \cdots h_j(-n_j) \otimes e^\mu$, where $h_1, \cdots, h_j \in \mathfrak{h}^k, n_1, \cdots, n_j \in \mathbb{Z}_+, \mu \in Q^k$, we define

$$Y_0(v, x) = {\circ \atop \circ} \left(\frac{1}{(n_1-1)!} \left(\frac{d}{dx}\right)^{n_1-1} h_1(x)\right) \cdots \left(\frac{1}{(n_j-1)!} \left(\frac{d}{dx}\right)^{n_j-1} h_j(x)\right) Y_0(e^\mu, x) {\circ \atop \circ} \tag{3.4.10}$$

We extend the map Y_0 to V_{Q^k} linearly.

Now let c_{mn} be a family of constants defined by:

$$\begin{aligned} \sum_{m,n \in \mathbb{N}} c_{mn} x^m y^n &= -\log\left(\frac{(1+x)^{1/2} + (1+y)^{1/2}}{2}\right), \\ c_{00} &= 0. \end{aligned} \tag{3.4.11}$$

Let $\{h_1, \cdots, h_d\}$ be an orthonormal basis of $\mathfrak{h}^k$, and let

$$\Delta_x = \sum_{m,n \in \mathbb{N}} \sum_{i=1}^{d} c_{mn} h_i(m) h_i(n) x^{-m-n} \in (\mathrm{End}\, V_{Q^k})[x^{-1}] \tag{3.4.12}$$

Finally, for $v \in V_{Q^k}$, we define

$$Y_{\mathbb{Z}+1/2}(v, x) = Y_0(\exp(\Delta_x) v, x) \in (\mathrm{End}\, V_{Q^k}^T)[[x^{1/2}, x^{-1/2}]]. \tag{3.4.13}$$

The space $V_{Q^k}^T$ can be graded by eigenvalues of the operator $L(0)$, and from chapter 9 of [FLM], we see that $(V_{Q^k}^T, Y_{\mathbb{Z}+1/2})$ becomes a $\theta-$twisted $V_{Q^k}-$module with this grading. This grading is referred to as the grading by *weight*.

3.4.3 Intertwining operators

Following the method of [Ab1] and [ADL], we give some twisted intertwining operators among untwisted and $\theta-$twisted $V_{Q^k}-$modules. We use special notation for the modules involved as we are dealing with explicit examples. First we recall the following notation:

$$\beta_i := (0, \cdots, 0, \beta, 0, \cdots, 0) \in Q^k, \text{ where } \beta \in Q,$$

and where the non-trivial component is in the i^{th} position.

Turn $\mathbb{C}$ into a one-dimensional Q^k-module such that each α_i acts as either 1 or -1 on $\mathbb{C}$. To record the action of each α_i, we denote this module by $\mathbb{C}^{\epsilon_1,\cdots,\epsilon_k}$ where each α_i acts on $\mathbb{C}$ by the scalar $\epsilon_i \in \{1,-1\}$. We similarly denote $1 \in \mathbb{C}^{\epsilon_1,\cdots,\epsilon_k}$ by $\mathbf{1}^{\epsilon_1,\cdots,\epsilon_k}$. Fix a sequence $\epsilon_1,\cdots\epsilon_k$ such that each $\epsilon_i \in \{1,-1\}$. For the sake of brevity, we denote the module $\mathbb{C}^{\epsilon_1,\cdots,\epsilon_k}$ by T, and the module $\mathbb{C}^{(-1)^{\langle\lambda,\alpha_k\rangle}\epsilon_1,\cdots,(-1)^{\langle\lambda,\alpha_k\rangle}\epsilon_k}$ by T^λ for $\lambda \in P^k$.

Fix a system Λ of representatives for the cosets P^k/Q^k. We assume that the coset Q^k is represented by 0. Fix a $\lambda \in \Lambda$. Consider the following linear map:

$$\begin{aligned} f_\lambda \quad : \quad & T \longrightarrow T^\lambda \\ & 1 = \mathbf{1}^{\epsilon_1,\cdots,\epsilon_k} \longmapsto \mathbf{1}^{(-1)^{\langle\lambda,\alpha_1\rangle}\epsilon_1,\cdots,(-1)^{\langle\lambda,\alpha_k\rangle}\epsilon_k} = 1. \end{aligned} \tag{3.4.14}$$

Then we have

$$(-1)^{\langle\lambda,\mu\rangle} f \circ e^\mu = e^\mu \circ f \text{ where } \mu \in Q^k. \tag{3.4.15}$$

For $\gamma \in P^k, \gamma = \lambda + \mu$ where $\lambda \in \Lambda, \mu \in Q^k$ define a linear map:

$$\eta_\gamma = e^\mu \circ f_\lambda,$$

that is,

$$\begin{aligned} \eta_\gamma : & T \longrightarrow T^\lambda \\ & \mathbf{1}^{\epsilon_1,\cdots,\epsilon_k} \longmapsto e^\mu \cdot \mathbf{1}^{(-1)^{\langle\lambda,\alpha_1\rangle}\epsilon_1,\cdots,(-1)^{\langle\lambda,\alpha_k\rangle}\epsilon_k}. \end{aligned}$$

Lemma 3.4.1. *[ADL] For any $\gamma \in \lambda + Q^k$ and $\tilde{\mu} \in Q^k$,*

$$\begin{aligned} e^{\tilde{\mu}} \circ \eta_\gamma &= (-1)^{\langle\tilde{\mu},\gamma\rangle}\eta_\gamma \circ e^{\tilde{\mu}} \\ e^{\tilde{\mu}} \circ \eta_\gamma &= \eta_{\gamma+\tilde{\mu}} = \eta_{\gamma-\tilde{\mu}}. \end{aligned}$$

Proof. First part is clear from the basic property (3.4.15) of f_λ, and the second eqality in the second part is clear from the fact that square of each element of Q^k acts trivially on each of the modules $\mathbb{C}^{\epsilon_1,\cdots,\epsilon_k}$ that we are considering. □

Now we present the construction of a twisted intertwining operator $\mathcal{Y}^t$ of type $\binom{T^\lambda}{V_{\lambda+Q^k}\ T}$, given in [ADL], which is based on the twisted vertex operator construction of Chapter 9 of [FLM].

For $e^\gamma = 1 \otimes e^\gamma \in V_{P^k}$ (later, we will restrict ourselves to $\gamma \in \lambda + Q^k$), define

$$\mathcal{X}_0^t(e^\gamma, x) : V_{P^k} \longrightarrow (\mathrm{End}\, M_{\mathbb{Z}+1/2}(1))\{x\} \tag{3.4.16}$$

by:

$$\mathcal{X}_0^t(e^\gamma, x) = 2^{-\langle\gamma,\gamma\rangle} \exp\left(\sum_{n\in\mathbb{N}+1/2} \frac{\gamma(-n)}{n} x^n\right) \exp\left(-\sum_{n\in\mathbb{N}+1/2} \frac{\gamma(n)}{n} x^{-n}\right) x^{-\langle\gamma,\gamma\rangle/2}. \tag{3.4.17}$$

Note how $\mathcal{X}_0^t$ differs from Y_0 of (3.4.9). More generally, for $v = h_1(-n_1)\cdots h_j(-n_j)\otimes e^\gamma$, where $h_1, \cdots, h_j \in \mathfrak{h}^k, n_1, \cdots, n_j \in \mathbb{Z}_+, \gamma \in P^k$, we define

$$\mathcal{X}_0^t(v, x) = {\circ\atop\circ}\left(\frac{1}{(n_1-1)!}\left(\frac{d}{dx}\right)^{n_1-1} h_1(x)\right)\cdots\left(\frac{1}{(n_j-1)!}\left(\frac{d}{dx}\right)^{n_j-1} h_j(x)\right)\mathcal{X}_0^t(e^\gamma, x){\circ\atop\circ} \tag{3.4.18}$$

We extend the map $\mathcal{X}_0^t$ to V_{P^k} linearly. Recall the constants $c_m n (m, n \in \mathbb{N})$ from (3.4.11) and the operator Δ_x of (3.4.12). Finally, for $v \in V_{P^k}$, we define

$$\mathcal{X}^t(v, x) = \mathcal{X}_0^t(\exp(\Delta_x)v, x) \in (\mathrm{End}\, M_{\mathbb{Z}+1/2}(1))\{x\}. \tag{3.4.19}$$

Modifying the arguments from Chapter 9 of [FLM], we get that,

$$\begin{aligned} &{x_0}^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\mathcal{X}^t(a, x_1)\mathcal{X}^t(u, x_2) \\ &\qquad -(-1)^{\langle\mu,\gamma\rangle}{x_0}^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\mathcal{X}^t(u, x_1)\mathcal{X}^t(a, x_2) \\ &= \frac{1}{2}\sum_{j=0,1} x_0^{-1}\delta\left((-1)^j \frac{(x_1-x_0)^{1/2}}{x_2^{1/2}}\right)\mathcal{X}^t(Y_{\mathbb{Z}}(\theta^j\cdot a, x_0)u, x_2), \end{aligned} \tag{3.4.20}$$

and that

$$\mathcal{X}^t(L(-1)u, x) = \frac{d}{dx}\mathcal{X}^t(u, x), \tag{3.4.21}$$

for $a \in \mathcal{S}((\hat{\mathfrak{h}}_{\mathbb{Z}}^k)^-) \otimes e^\mu \subset V_{Q^k}$ and $u \in \mathcal{S}((\hat{\mathfrak{h}}_{\mathbb{Z}}^k)^-) \otimes e^\gamma \subset V_{P^k}$. These are essentially equations 5.10 and 5.11 of [ADL].

Finally, in order to correct for the factor $(-1)^{\langle\mu,\gamma\rangle}$ and to introduce the one-dimensional Q^k-module T in the picture, we make the following definitions:

$$\mathcal{Y}^t : V_{\lambda+Q^k} \longrightarrow \mathrm{Hom}\,(V_{Q^k}^T, V_{Q^k}^{T^\lambda})\{x\}, \tag{3.4.22}$$

such that for $u \in \mathcal{S}((\hat{\mathfrak{h}}^k_{\mathbb{Z}})^-) \otimes e^{\lambda+\mu} \subset V_{\lambda+Q^k}, \nu \in Q^k$,

$$\mathcal{Y}^t(u, x) \quad = \quad \mathcal{X}^t(u, x) \otimes \eta_{\lambda+\mu}. \tag{3.4.23}$$

Note that as of now, the above definition depends on the choice of coset representatives of P^k/Q^k.

Now, as in [ADL], we remark that for $a \in \mathcal{S}((\hat{\mathfrak{h}}^k_{\mathbb{Z}})^-) \otimes e^{\mu} \subset V_{0+Q^k}$, since 0 represents Q^k in Λ,

$$f_0 \quad : \quad T \longrightarrow T^0 = T \tag{3.4.24}$$

$$f_0 \quad = \quad \mathrm{Id}_T \tag{3.4.25}$$

$$\eta_\mu \quad : \quad T \longrightarrow T^0 = T \tag{3.4.26}$$

$$\eta_\mu \quad = \quad e^{\mu} \tag{3.4.27}$$

and so $\mathcal{Y}^t(a, x)$ is exactly the $Y_{\mathbb{Z}+1/2}$ operator which defines the twisted module structure of $V^T_{Q^k}$.

Using (3.4.22), Lemma 3.4.1, (3.4.20), (3.4.24)−(3.4.27) we see that (cf. Proposition 5.10 of [ADL]):

Proposition 3.4.2. *For $\lambda \in \Lambda$, $\mathcal{Y}^t$ is a (non-zero) twisted intertwining operator of type $\binom{V^{T^\lambda}_{Q^k}}{V_{\lambda+Q^k}\ V^T_{Q^k}}$. That is, the following identities hold for any $v \in V_{Q^k}$ and $w_{(1)} \in V_{\lambda+Q^k}$:*

$$\begin{aligned} &{x_0}^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_{\mathbb{Z}+1/2}(v, x_1)\mathcal{Y}^t(w_{(1)}, x_2) \\ &\qquad -{x_0}^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) \mathcal{Y}^t(w_{(1)}, x_1)Y_{\mathbb{Z}+1/2}(v, x_2) \\ &= \frac{1}{2}\sum_{j=0,1} x_0^{-1}\delta\left((-1)^j \frac{(x_1 - x_0)^{1/2}}{x_2^{1/2}}\right) \mathcal{Y}^t(Y_{\mathbb{Z}}(\theta^j \cdot v, x_0)w_{(1)}, x_2) \end{aligned} \tag{3.4.28}$$

and

$$\mathcal{Y}^t(L(-1)v, x) = \frac{d}{dx}\mathcal{Y}^t(v, x). \tag{3.4.29}$$

Now we see what happens if we change the set of coset representatives from Λ to some new set $\tilde{\Lambda}$. We continue to assume that $0 \in \tilde{\Lambda}$, that is, the coset Q^k is still represented by 0. We denote the changes at each step by a tilde. So, $\lambda - \tilde{\lambda} \in Q^k$

for $\lambda \in \Lambda$ and $\tilde{\lambda} \in \tilde{\Lambda}$. Due to the very specific inner product that we are dealing with, $f_{\tilde{\lambda}} = f_\lambda$ (cf. (3.4.14)). For $\gamma \in P^k$ if $\gamma = \lambda + \mu$ where $\lambda \in \Lambda, \mu \in Q^k$ and $\gamma = \tilde{\lambda} + \tilde{\mu}$ where $\tilde{\lambda} \in \tilde{\Lambda}, \tilde{\mu} \in Q^k$, then,

$$\tilde{\eta}_\gamma = e^{\tilde{\mu}} \circ f_{\tilde{\lambda}} = e^{\tilde{\mu}} \circ f_\lambda = e^{\lambda - \tilde{\lambda} + \mu} \circ f_\lambda = e^{\lambda - \tilde{\lambda}} \eta_\gamma. \tag{3.4.30}$$

Hence, we formulate:

Proposition 3.4.3. *If the choice of Λ is changed to $\tilde{\Lambda}$, where $0 \in \tilde{\Lambda}$, then the new twisted intertwining operator $\tilde{\mathcal{Y}}^t$ is of the type $\binom{V_{Q^k}^{T^\lambda}}{V_{\lambda+Q^k}\ V_{Q^k}^T}$, which is the same as the type of the previous twisted intertwining operator $\mathcal{Y}^t$. Moreover, $\tilde{\mathcal{Y}}^t(\cdot, x)\cdot = e^{\lambda - \tilde{\lambda}} \mathcal{Y}^t(\cdot, x)\cdot$, that is, $\mathcal{Y}^t$ and $\tilde{\mathcal{Y}}^t$ are (non-zero) scalar multiples of each other.*

3.4.4 Some level 1 twisted fusion rules

We work in the setting of the previoius subsection. Our methods are the same as the ones used in [DL].

Proposition 3.4.4. *We have the following twisted fusion rules:*

$${}^tN\begin{pmatrix} V_{Q^k}^S \\ V_{\lambda+Q^k}\ V_{Q^k}^T \end{pmatrix} = \begin{cases} 1 & \text{if } S = T^\lambda \\ 0 & \text{otherwise.} \end{cases}$$

We break the proof in several steps. First, without loss of generality, let us assume that λ is chosen so that for each $i = 1, \cdots, k$,

$$\langle \lambda, \alpha_i \rangle \in \{0, 1\}, \tag{3.4.31}$$

that is, choose λ in the "fundamental domain" of the lattice P^k. We observe that $1 \otimes T$ is the lowest $L(0)$-weight space of the module $V_{Q^k}^T$, with weight $\frac{1}{16}\dim \mathfrak{h}^k = \frac{k}{16}$, a number which is independent of T. We see that $V_{\lambda+Q^k}$ is $(\frac{1}{2}\langle \lambda, \lambda \rangle + \mathbb{Z})$-graded and $V_{Q^k}^T$ and $V_{Q^k}^S$ are both $(\frac{k}{16} + \frac{1}{2}\mathbb{Z})$-graded. For the sake of brevity, we let

$$\begin{aligned} I^t\begin{pmatrix} S \\ \lambda\ T \end{pmatrix} &= \text{space of twisted intertwining operators of type } \begin{pmatrix} V_{Q^k}^S \\ V_{\lambda+Q^k}\ V_{Q^k}^T \end{pmatrix} \\ N^t\begin{pmatrix} S \\ \lambda\ T \end{pmatrix} &= \dim I^t\begin{pmatrix} S \\ \lambda\ T \end{pmatrix} = N^t\begin{pmatrix} V_{Q^k}^S \\ V_{\lambda+Q^k}\ V_{Q^k}^T \end{pmatrix}. \end{aligned}$$

For any $\mathcal{Y}^t \in I^t\binom{S}{\lambda\ T}$, it is clear from remark 3.3.7 that the powers of x appearing in $\mathcal{Y}^t(e^\lambda, x)$ belong to $-\frac{1}{2}\langle\lambda,\lambda\rangle + \frac{1}{2}\mathbb{Z} = -\text{wt } e^\lambda + \frac{1}{2}\mathbb{Z}$. For the sake of conceptual ease, we define

$$\mathcal{Y}^t(e^\lambda, x) = \sum_{n\in\frac{1}{2}\mathbb{Z}} (e^\lambda)^{\mathcal{Y}^t}_{[n]} x^{n-\text{wt } \lambda}, \tag{3.4.32}$$

so that

$$\text{wt } (e^\lambda)^{\mathcal{Y}^t}_{[n]} = n. \tag{3.4.33}$$

Conforming to the notation in the previous section, we also let

$$T = \mathbb{C}^{\tau_1,\cdots,\tau_k} \tag{3.4.34}$$

$$\mathbf{1}^T = \mathbf{1}^{\tau_1,\cdots,\tau_k} \in T \tag{3.4.35}$$

$$S = \mathbb{C}^{\sigma_1,\cdots,\sigma_k} \tag{3.4.36}$$

$$\mathbf{1}^S = \mathbf{1}^{\sigma_1,\cdots,\sigma_k} \in S \tag{3.4.37}$$

where $\tau_1,\cdots,\tau_k,\sigma_1,\cdots,\sigma_k \in \{-1,1\}$. For $i = 1,\cdots,k$, let

$$E^i = e^{\alpha_i} + e^{-\alpha_i} \in V^{\ 0}_{Q^k}. \tag{3.4.38}$$

For $\mathcal{Y}^t \in I^t\binom{S}{\lambda\ T}$ and $i = 1,\cdots,k$, extracting $\text{Res}_{x_0}\text{Res}_{x_1}$ of (3.3.1), we obtain - if $\langle\lambda,\alpha_i\rangle = 0$, then

$$[E^i_0, \mathcal{Y}^t(e^\lambda, x)] = \mathcal{Y}^t(E^i_0 e^\lambda, x) = 0, \tag{3.4.39}$$

and if $\langle\lambda,\alpha_i\rangle = 1$ then

$$[E^i_0, \mathcal{Y}^t(e^\lambda, x)] = \mathcal{Y}^t(E^i_0 e^\lambda, x) = \mathcal{Y}^t(e^{\lambda-\alpha_i}, x). \tag{3.4.40}$$

Bracketing once more, we get that if $\langle\lambda,\alpha_i\rangle = 1$ then

$$[E^i_0, [E^i_0, \mathcal{Y}^t(e^\lambda, x)]] = \mathcal{Y}^t(E^i_0 E^i_0 e^\lambda, x) = \mathcal{Y}^t(e^\lambda, x). \tag{3.4.41}$$

Extracting relevant coefficients, if $\langle\lambda,\alpha_i\rangle = 0$, then

$$[E^i_0, (e^\lambda)^{\mathcal{Y}^t}_{[0]}] = 0, \tag{3.4.42}$$

and if $\langle\lambda,\alpha_i\rangle = 1$ then

$$[E^i_0, [E^i_0, (e^\lambda)^{\mathcal{Y}^t}_{[0]}]] = (e^\lambda)^{\mathcal{Y}^t}_{[0]}. \tag{3.4.43}$$

Since wt $E^i = 1$, $E^i_0(1\otimes T) \subset (1\otimes T)$ and similarly for S. In fact, E^i_0 acts by the scalar $\frac{1}{2}\tau_i$ on $(1\otimes T)$ and by the scalar $\frac{1}{2}\sigma_i$ on $(1\otimes S)$. By definition, $(e^\lambda)^{\mathcal{Y}^t}_{[0]}(1\otimes T) \subset (1\otimes S)$. Clearly, $(e^\lambda)^{\mathcal{Y}^t}_{[0]}(1\otimes \mathbf{1}^T)$ is a scalar multiple of $1\otimes \mathbf{1}^S$. Assume that $(e^\lambda)^{\mathcal{Y}^t}_{[0]}(1\otimes \mathbf{1}^T) \neq 0$. Now we apply (3.4.42) and (3.4.43) to $1\otimes \mathbf{1}^T$. For $i = 1, \cdots, k$, $\langle\lambda, \alpha_i\rangle = 0$, (3.4.42) implies that

$$\frac{1}{2}\sigma_i - \frac{1}{2}\tau_i = 0,$$

or

$$\sigma_i = \tau_i \tag{3.4.44}$$

and for $\langle\lambda, \alpha_i\rangle = 1$, (3.4.43) implies that

$$\frac{1}{4}\sigma_i^2 - \frac{1}{4}\sigma_i\tau_i - \frac{1}{4}\sigma_i\tau_i - \frac{1}{4}\tau_i^2 = 1,$$

or

$$\sigma_i \neq \tau_i. \tag{3.4.45}$$

We record this as a lemma.

Lemma 3.4.5. *With λ chosen as in (3.4.31), S and T as in (3.4.34), (3.4.36) respectively, let $\mathcal{Y}^t \in I^t\binom{S}{\lambda\ T}$, satisfying $(e^\lambda)^{\mathcal{Y}^t}_{[0]}(1\otimes \mathbf{1}^T) \neq 0$. Then, $\sigma_i = \tau_i$ if $\langle\lambda, \alpha_i\rangle = 0$ and $\sigma_i \neq \tau_i$ if $\langle\lambda, \alpha_i\rangle = 1$. In other words, $S = T^\lambda$.*

Lemma 3.4.6. *In the setting of previous lemma, $(e^\lambda)^{\mathcal{Y}^t}_{[0]}(1\otimes \mathbf{1}^T) = 0$ implies $\mathcal{Y}^t(\cdot, x)\cdot = 0$.*

Proof. Since $V^T_{Q^k}$ and $V^S_{Q^k}$ are both graded by the same coset of $\frac{1}{2}\mathbb{Z}$, namely, $(\frac{k}{16} + \frac{1}{2}\mathbb{Z})$, $(e^\lambda)^{\mathcal{Y}^t}_{[n]}(1\otimes \mathbf{1}^T) = 0$ for all $n \notin \frac{1}{2}\mathbb{Z}$. Clearly, for any $n < 0$ also, $(e^\lambda)^{\mathcal{Y}^t}_{[n]}(1\otimes \mathbf{1}^T) = 0$. Now let $n > 0$, $n \in \frac{1}{2}\mathbb{Z}$. We prove $(e^\lambda)^{\mathcal{Y}^t}_{[n]}(1\otimes \mathbf{1}^T) = 0$ by induction on n. Assume that $(e^\lambda)^{\mathcal{Y}^t}_{[q]}(1\otimes \mathbf{1}^T) = 0$ for all $0 \leq q < n, q \in \frac{1}{2}\mathbb{Z}$. By the Jacobi identity (3.3.1) we have the commutation relation

$$\begin{aligned}
&[Y_{\mathbb{Z}+1/2}(\alpha(-1)\mathbf{1}, x_1), \mathcal{Y}^t(e^\lambda, x_2)] = \\
&\quad = \mathrm{Res}_{x_0} x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\left(\frac{x_1 - x_0}{x_2}\right)^{-1/2} \mathcal{Y}^t(Y_{\mathbb{Z}}(\alpha(-1)\mathbf{1}, x_0)e^\lambda, x_2) \\
&\quad = x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\left(\frac{x_1}{x_2}\right)^{-1/2} \langle\alpha, \lambda\rangle \mathcal{Y}^t(e^\lambda, x_2).
\end{aligned}$$

for all $\alpha \in \mathfrak{h}^k$. Extracting coefficients, we get

$$[\alpha(m), (e^{\lambda})^{\mathcal{Y}^t}_{[n]}] = \langle \lambda, \alpha \rangle (e^{\lambda})^{\mathcal{Y}^t}_{[n-m]}$$

for all $m \in \mathbb{Z} + 1/2$. Therefore,

$$\begin{aligned} &\alpha(m)(e^{\lambda})^{\mathcal{Y}^t}_{[n]}(1 \otimes \mathbf{1}^T) \\ &\quad = (e^{\lambda})^{\mathcal{Y}^t}_{[n]}\alpha(m)(1 \otimes \mathbf{1}^T) + \langle \lambda, \alpha \rangle (e^{\lambda})^{\mathcal{Y}^t}_{[n-m]}(1 \otimes \mathbf{1}^T) = 0, \end{aligned}$$

for all $\alpha \in \mathfrak{h}^k$ and $m \in \mathbb{Z} + 1/2$ such that $m > 0$, by induction assumption. Hence,

$$(e^{\lambda})^{\mathcal{Y}^t}_{[n]}(1 \otimes \mathbf{1}^T) \in \mathbb{C}(1 \otimes \mathbf{1}^S),$$

which can not happen unless

$$(e^{\lambda})^{\mathcal{Y}^t}_{[n]}(1 \otimes \mathbf{1}^T) = 0,$$

because

$$\mathrm{wt}\,(e^{\lambda})^{\mathcal{Y}^t}_{[n]}(1 \otimes \mathbf{1}^T) > \mathrm{wt}\,1 \otimes \mathbf{1}^S.$$

This proves that $\mathcal{Y}^t(e^{\lambda}, x)(1 \otimes \mathbf{1}^T) = 0$. Now the lemma follows by invoking proposition 3.3.13. □

Lemma 3.4.7. *(Cf. Proposition 12.8, [DL])* $N^t\binom{S}{\lambda\ T} \leq 1$.

Proof. The proof is exactly as in [DL]. Assume that $N^t\binom{S}{\lambda\ T} > 1$. We reproduce it here for the sake of completeness. Let $\mathcal{Y}^t, \mathcal{X}^t \in I^t\binom{S}{\lambda\ T}$ with $\mathcal{Y}^t \neq 0$. By lemma 3.4.6, $(e^{\lambda})^{\mathcal{Y}^t}_{[0]}(1 \otimes \mathbf{1}^T)$ is a non-zero scalar multiple of $1 \otimes \mathbf{1}^S$, and there exists a scalar c such that

$$(e^{\lambda})^{\mathcal{X}^t}_{[0]}(1 \otimes \mathbf{1}^T) = c\,(e^{\lambda})^{\mathcal{Y}^t}_{[0]}(1 \otimes \mathbf{1}^T).$$

That is,

$$(e^{\lambda})^{\mathcal{X}^t - c\mathcal{Y}^t}_{[0]}(1 \otimes \mathbf{1}^T) = 0.$$

Since $\mathcal{X}^t - c\mathcal{Y}^t \in I^t\binom{S}{\lambda\ T}$, lemma 3.4.6 implies that

$$\mathcal{X}^t - c\mathcal{Y}^t = 0.$$

□

Proof of lemma 3.4.4: From lemmas 3.4.5 and 3.4.6 we conclude that $N^t\binom{S}{\lambda\ T} = 0$ if $S \neq T^\lambda$. From lemma 3.4.2 we obtain $N^t\binom{T^\lambda}{\lambda\ T} \geq 1$ and then lemma 3.4.7 implies that $N^t\binom{T^\lambda}{\lambda\ T} = 1$. □

3.5 Example - the vertex operator algebra $L_{\widehat{\mathfrak{sl}_2}}(k, 0)$

3.5.1 The setting

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ be the 3-dimensional complex simple Lie algebra with a standard basis $\{\alpha, x_\alpha, x_{-\alpha}\}$ and a symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$ such that

$$[\alpha, x_\alpha] = 2x_\alpha, \quad [\alpha, x_{-\alpha}] = -2x_{-\alpha}, \quad [x_\alpha, x_{-\alpha}] = \alpha$$

$$\langle \alpha, \alpha \rangle = 2, \quad \langle x_\alpha, x_{-\alpha} \rangle = 1, \quad \langle \alpha, x_\alpha \rangle = \langle \alpha, x_{-\alpha} \rangle = \langle x_{\pm\alpha}, x_{\pm\alpha} \rangle = 0. \tag{3.5.1}$$

Take the Cartan subalgebra $\mathfrak{h} = \mathbb{C}\alpha$. We identify $\mathfrak{h}$ with its dual $\mathfrak{h}^*$ via the form $\langle \cdot, \cdot \rangle$. Under this identification, α gets identified with the root corresponding to a root vector x_α. The affine Lie algebra $\hat{\mathfrak{g}} = \hat{\mathfrak{sl}}(2, \mathbb{C})$ is an infinite dimensional Lie algebra whose underlying vector space is

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

with

$$[\hat{\mathfrak{g}}, c] = 0 \tag{3.5.2}$$

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + \langle x, y \rangle m \delta_{m+n,0} c, \tag{3.5.3}$$

where $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. Also define

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

with an extra relation

$$[d, x \otimes t^m \oplus \mathbb{C}c \oplus \mathbb{C}d] = mx \otimes t^m \tag{3.5.4}$$

where $x \in \mathfrak{g}$ and $m \in \mathbb{Z}$. For $a \in \mathfrak{g}$, let

$$a(x) = \sum_{n \in \mathbb{Z}} (a \otimes t^n) x^{-n-1}.$$

Let θ be the unique involution of $\mathfrak{g}$ defined by

$$\theta : x_\alpha \longmapsto x_{-\alpha}, \quad \theta : x_{-\alpha} \longmapsto x_\alpha, \quad \theta : \alpha \longmapsto -\alpha. \tag{3.5.5}$$

Then θ preserves the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. As of now, this notation conflicts with the previously defined θ, but remark 3.5.7 clears this confusion. For $i \in \mathbb{Z}/2\mathbb{Z}$, set

$$\mathfrak{g}_{(i)} = \{x \in \mathfrak{g} \mid \theta x = (-1)^i x\}. \tag{3.5.6}$$

Define

$$\widehat{\mathfrak{g}}[\theta] = \mathfrak{g}_{(0)} \otimes \mathbb{C}[t, t^{-1}] \,\oplus\, \mathfrak{g}_{(1)} \otimes t^{1/2}\mathbb{C}[t, t^{-1}] \,\oplus\, \mathbb{C}c, \tag{3.5.7}$$

a Lie subalgebra of the Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t^{1/2}, t^{-1/2}] \oplus \mathbb{C}c$, in which the brackets are given by (3.5.2) and (3.5.3) with m, n taken in $\frac{1}{2}\mathbb{Z}$. Also define

$$\tilde{\mathfrak{g}}[\theta] = \mathfrak{g}_{(0)} \otimes \mathbb{C}[t, t^{-1}] \,\oplus\, \mathfrak{g}_{(1)} \otimes t^{1/2}\mathbb{C}[t, t^{-1}] \,\oplus\, \mathbb{C}c \oplus \mathbb{C}d, \tag{3.5.8}$$

in which the brackets for d are given by the formula (3.5.4) with $m \in \mathbb{Z}$ if $x \in \mathfrak{g}_{(0)}$ and $m \in \mathbb{Z} + 1/2$ if $x \in \mathfrak{g}_{(1)}$.

The Lie algebras $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}[\theta]$ are isomorphic (but not graded-isomorphic) via the following map

$$\begin{aligned}
\tau : \tilde{\mathfrak{g}} &\longrightarrow \tilde{\mathfrak{g}}[\theta] \\
\tau : c &\longmapsto c \\
\tau : d &\longmapsto d - \frac{1}{4}(x_\alpha + x_{-\alpha}) \otimes t^0 \\
\tau : \alpha \otimes t^n &\longmapsto (x_\alpha + x_{-\alpha}) \otimes t^n + \frac{1}{2}\delta_{n,0}c \quad (n \in \mathbb{Z}) \\
\tau : x_\alpha \otimes t^n &\longmapsto \frac{1}{2}(\alpha - (x_\alpha - x_{-\alpha})) \otimes t^{n+1/2} \quad (n \in \mathbb{Z}) \\
\tau : x_{-\alpha} \otimes t^n &\longmapsto \frac{1}{2}(\alpha + (x_\alpha - x_{-\alpha})) \otimes t^{n-1/2} \quad (n \in \mathbb{Z}).
\end{aligned} \tag{3.5.9}$$

Keeping this in mind, we make the following definition.

Definition 3.5.1. *A* weight vector *in a $\tilde{\mathfrak{g}}[\theta]$-module is defined to be any vector which is a simultaneous eigenvector for the operators c, d and $(x_\alpha + x_{-\alpha}) \otimes t^0$. A* highest weight vector *in a $\tilde{\mathfrak{g}}[\theta]$-module is a weight vector that is annihilated by $\frac{1}{2}(\alpha - (x_\alpha - x_{-\alpha})) \otimes t^{1/2}$ and $\frac{1}{2}(\alpha + (x_\alpha - x_{-\alpha})) \otimes t^{1/2}$. A $\tilde{\mathfrak{g}}[\theta]$-module is called a* weight module *if it is spanned* *by its weight vectors. A $\tilde{\mathfrak{g}}[\theta]$-module is a* highest weight module *if it is a weight module generated by a highest weight vector.*

Remark 3.5.2. It follows that a highest weight vector in a $\tilde{\mathfrak{g}}[\theta]$-module is annihilated by $(x_\alpha + x_{-\alpha}) \otimes t^n$, $\alpha \otimes t^{n-1/2}$ and $(x_\alpha - x_{-\alpha}) \otimes t^{n-1/2}$, whenever $n \in \mathbb{Z}$ and $n > 0$.

Remark 3.5.3. The Lie algebras $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}[\theta]$ are respectively the "homogeneous realization" and the "principal realization" of the affine Lie algebra $A_1^{(1)}$.

Fix a positive integer k. Recall the space V_{P^k} from previous section. Consider a linear injection i from $\mathfrak{g}$ to V_{Q^k} defined uniquely by

$$i(\alpha) = \alpha_1(-1) + \cdots \alpha_k(-1) \tag{3.5.10}$$

$$i(x_{\pm\alpha}) = e^{\pm\alpha_1} + \cdots + e^{\pm\alpha_k}. \tag{3.5.11}$$

Proposition 3.5.4. *([DL], Proposition 13.1) The linear map $\pi : \hat{\mathfrak{g}} \longrightarrow \mathrm{End}\, V_{P^k}$ given by*

$$\pi(c) \;=\; k \tag{3.5.12}$$

$$\pi(a(x)) \;=\; Y_{\mathbb{Z}}(i(a), x) \quad \text{where} \quad a \in \mathfrak{g} \tag{3.5.13}$$

defines a $\hat{\mathfrak{g}}-$module structure of level k on the space V_{P^k}.

Set

$$L_{\widehat{\mathfrak{sl}_2}}(k, 0) = \mathcal{U}(\hat{\mathfrak{g}}) \cdot \mathbf{1} \quad (\subset V_{Q^k} \subset V_{P^k}) \tag{3.5.14}$$

and let

$$\omega_{\mathfrak{g}_k} = \frac{1}{2(k+2)} \left(x_\alpha(-1)x_{-\alpha}(-1) \cdot \mathbf{1} + x_{-\alpha}(-1)x_\alpha(-1) \cdot \mathbf{1} + \frac{1}{2}\alpha(-1)\alpha(-1) \cdot \mathbf{1} \right),$$
$$\omega_{\mathfrak{g}_k} \in L_{\widehat{\mathfrak{sl}_2}}(k, 0). \tag{3.5.15}$$

Proposition 3.5.5. *([DL], Proposition 13.8, Theorem 13.12) The structure*

$$L_{\widehat{\mathfrak{sl}_2}}(k, 0) = (L_{\widehat{\mathfrak{sl}_2}}(k, 0), Y_{\mathbb{Z}}, \mathbf{1}, \omega_{\mathfrak{g}_k})$$

(contained inside V_{P^k}) with its grading inherited from that of V_{P^k}, is a vertex operator algebra of rank $\dfrac{3k}{k+2}$.

Recall the automorphism θ of the vertex operator algebra $(V_{Q^k}, Y_{\mathbb{Z}}, \mathbf{1}, \omega)$ defined in (3.4.8).

Proposition 3.5.6. *θ induces an automorphism (also denoted by θ) of order 2 of the vertex operator algebra $L_{\widehat{\mathfrak{sl}_2}}(k,0)$.*

Proof. Using the properties of the maps i and θ, it is easy to see that

$$\theta i(x_{\pm\alpha}) = \theta(e^{\pm\alpha_1} + \cdots + e^{\pm\alpha_k}) = e^{\mp\alpha_1} + \cdots + e^{\mp\alpha_k} = i(x_{\mp\alpha}) \tag{3.5.16}$$

$$\theta i(\alpha) = \theta(\alpha_1(-1) + \cdots + \alpha_k(-1)) = -(\alpha_1(-1) + \cdots + \alpha_k(-1)) = -i(\alpha), \tag{3.5.17}$$

which implies that

$$\theta L_{\widehat{\mathfrak{sl}_2}}(k,0) = L_{\widehat{\mathfrak{sl}_2}}(k,0). \tag{3.5.18}$$

We already know that

$$\theta \mathbf{1} = \mathbf{1} \tag{3.5.19}$$

$$\theta Y_{\mathbb{Z}}(v,x)\theta^{-1} = Y_{\mathbb{Z}}(\theta v, x) \text{ for } v \in V_{Q^k} \tag{3.5.20}$$

$$\theta^2 = \mathrm{Id}. \tag{3.5.21}$$

It remains to prove that θ fixes $\omega_{\mathfrak{g}_k}$. $\omega_{\mathfrak{g}_k}$ can be written as

$$\omega_{\mathfrak{g}_k} = \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\frac{1}{2(k+2)}(x_1x_2)^{-1}\Big(Y_{\mathbb{Z}}(i(x_\alpha),x_1)Y_{\mathbb{Z}}(i(x_{-\alpha}),x_2)\cdot\mathbf{1}$$
$$+Y_{\mathbb{Z}}(i(x_{-\alpha}),x_1)Y_{\mathbb{Z}}(i(x_\alpha),x_2)\cdot\mathbf{1} + Y_{\mathbb{Z}}(i(\alpha),x_1)Y_{\mathbb{Z}}(i(\alpha),x_2)\cdot\mathbf{1}\Big). \tag{3.5.22}$$

So we get that

$$\theta\omega_{\mathfrak{g}_k} = \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\frac{1}{2(k+2)}(x_1x_2)^{-1}\Big(\theta Y_{\mathbb{Z}}(i(x_\alpha),x_1)Y_{\mathbb{Z}}(i(x_{-\alpha}),x_2)\cdot\mathbf{1}$$
$$+\theta Y_{\mathbb{Z}}(i(x_{-\alpha}),x_1)Y_{\mathbb{Z}}(i(x_\alpha),x_2)\cdot\mathbf{1} + \theta Y_{\mathbb{Z}}(i(\alpha),x_1)Y_{\mathbb{Z}}(i(\alpha),x_2)\cdot\mathbf{1}\Big)$$
$$= \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\frac{1}{2(k+2)}(x_1x_2)^{-1}\Big(Y_{\mathbb{Z}}(\theta i(x_\alpha),x_1)Y_{\mathbb{Z}}(\theta i(x_{-\alpha}),x_2)\cdot\theta\mathbf{1}$$
$$+Y_{\mathbb{Z}}(\theta i(x_{-\alpha}),x_1)Y_{\mathbb{Z}}(\theta i(x_\alpha),x_2)\cdot\theta\mathbf{1} + Y_{\mathbb{Z}}(\theta i(\alpha),x_1)Y_{\mathbb{Z}}(\theta i(\alpha),x_2)\cdot\theta\mathbf{1}\Big)$$

$$= \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\frac{1}{2(k+2)}(x_1x_2)^{-1}\Bigg(Y_{\mathbb{Z}}(i(x_{-\alpha}),x_1)Y_{\mathbb{Z}}(i(x_\alpha),x_2)\cdot\mathbf{1}$$

$$+Y_{\mathbb{Z}}(\theta i(x_\alpha),x_1)Y_{\mathbb{Z}}(i(x_{-\alpha}),x_2)\cdot\mathbf{1}+ \ Y_{\mathbb{Z}}(-i(\alpha),x_1)Y_{\mathbb{Z}}(-i(\alpha),x_2)\cdot\mathbf{1}\Bigg), \tag{3.5.23}$$

where the last equality follows from the definition of the maps i and θ. Hence we conclude that $\theta\omega_{\mathfrak{g}_k} = \omega_{\mathfrak{g}_k}$. □

Remark 3.5.7. A look at (3.5.16) and (3.5.17) shows that

$$i(\theta a) = \theta i(a) \tag{3.5.24}$$

for all $a \in \mathfrak{g}$, where the θ on the left hand side is the involution of $\mathfrak{g}$ and the one on the right acts on the space V_{P^k}.

Notation 3.5.8. *Recall that $L(n) = \omega_{n+1}$. Let us denote the component operators corresponding to $\omega_{\mathfrak{g}_k}$ as $L_{\mathfrak{g}_k}(n) = (\omega_{\mathfrak{g}_k})_{n+1}$.*

Lemma 3.5.9. *(cf. [DL], Proposition 13.5) Let (W, Y_W) be a $\theta-$twisted V_{Q^k}-module. For $a \in \mathfrak{g}$, let*

$$a_W(x) = Y_W(i(a), x) = \sum_{n\in\frac{1}{2}\mathbb{Z}} a_W(n)x^{-n-1}. \tag{3.5.25}$$

Then

$$[L_{\mathfrak{g}_k}(m), a_W(n)] \quad = \quad -na_W(m+n), \tag{3.5.26}$$

$$[L(m), a_W(n)] \quad = \quad -na_W(m+n), \tag{3.5.27}$$

for $m \in \mathbb{Z}$ and $n \in \frac{1}{2}\mathbb{Z}$. In other words,

$$[Y_W(\omega_{\mathfrak{g}_k} - \omega, x_1), Y_W(i(a), x_2)] = 0 \tag{3.5.28}$$

(cf. [DL], (13.40)).

Proof. First, let $a \in \mathfrak{g}_{(0)}$. The proof in this case is essentially the proof of Proposition 13.5 in [DL]. From (3.5.17) and (3.5.16), $\theta i(a) = i(a)$ and so $i(a) \in L_{\widehat{\mathfrak{sl}_2}}(k,0)^0$ in the notation of lemma 3.3.1. Taking $u = i(a), v = \omega_{\mathfrak{g}_k}$ in (3.3.1) and taking Res_{x_0}, we get:

$$[Y_W(i(a), x_1), Y_W(\omega_{\mathfrak{g}_k}, x_2)] = \mathrm{Res}_{x_0} x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y_W(Y_{\mathbb{Z}}(i(a), x_0)\omega_{\mathfrak{g}_k}, x_2) \tag{3.5.29}$$

Equation (13.37) of [DL] gives

$$Y_{\mathbb{Z}}(i(a),x_0)\omega_{\mathfrak{g}_k} = i(a)x_0^{-2} + \text{ a nonsingular series in } x_0. \tag{3.5.30}$$

Hence, (3.5.29) gives (cf. [DL] (13.38))

$$[Y_W(i(a),x_1), Y_W(\omega_{\mathfrak{g}_k},x_2)] = -z_2^{-1} Y_W(x_2)\frac{\partial}{\partial x_1}\delta\left(\frac{x_1}{x_2}\right), \tag{3.5.31}$$

or equivalently,

$$[L_{\mathfrak{g}_k}(m), a_W(n)] = -na_W(m+n) \tag{3.5.32}$$

for $m,n \in \mathbb{Z}$. Keeping in mind the formal monodromy condition, we see that $a_W(n) = 0$ if $n \in \mathbb{Z} + \frac{1}{2}$, we get

$$[L_{\mathfrak{g}_k}(m), a_W(n)] = -na_W(m+n) \tag{3.5.33}$$

for $m \in \mathbb{Z}, n \in \frac{1}{2}\mathbb{Z}$.

Now let $a \in \mathfrak{g}_{(1)}$. From (3.5.17) and (3.5.16), $\theta i(a) = -i(a)$ and so $i(a) \in L_{\widehat{\mathfrak{sl}_2}}(k,0)^1$ in the notation of lemma 3.3.1. Taking $u = i(a), v = \omega_{\mathfrak{g}_k}$ in (3.3.1), taking Res_{x_0} and then using formal Taylor theorem, we get:

$$\begin{aligned}
&[Y_W(i(a),x_1), Y_W(\omega_{\mathfrak{g}_k},x_2)]\\
&= \mathrm{Res}_{x_0} x_2{}^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\left(\frac{x_1-x_0}{x_2}\right)^{-1/2} Y_W(Y_{\mathbb{Z}}(i(a),x_0)\omega_{\mathfrak{g}_k},x_2)\\
&= \mathrm{Res}_{x_0} x_2^{-1}\left(e^{-x_0\frac{\partial}{\partial x_1}}\delta\left(\frac{x_1}{x_2}\right)\right)\left(e^{-x_0\frac{\partial}{\partial x_1}}\left(\frac{x_1}{x_2}\right)^{-1/2}\right) Y_W(Y_{\mathbb{Z}}(i(a),x_0)\omega_{\mathfrak{g}_k},x_2)
\end{aligned}$$

Again, using (3.5.30) we get

$$\begin{aligned}
[Y_W(i(a),x_1),\ & Y_W(\omega_{\mathfrak{g}_k},x_2)]\\
&= -x_2{}^{-1}\delta\left(\frac{x_1}{x_2}\right)\left(\frac{\partial}{\partial x_1}\left(\frac{x_1}{x_2}\right)^{-1/2}\right) Y_W(i(a),x_2)\\
&\quad -\left(\frac{\partial}{\partial x_1} x_2{}^{-1}\delta\left(\frac{x_1}{x_2}\right)\right)\left(\frac{x_1}{x_2}\right)^{-1/2} Y_W(i(a),x_2)\\
&= -x_2^{-1}\frac{\partial}{\partial x_1}\left(\left(\frac{x_1}{x_2}\right)^{-1/2}\delta\left(\frac{x_1}{x_2}\right)\right) Y_W(i(a),x_2)
\end{aligned}$$

Extracting appropriate coefficients, we get that

$$[L_{\mathfrak{g}_k}(m), a_W(n)] = -na_W(m+n) \tag{3.5.34}$$

for $m \in \mathbb{Z}, n \in \mathbb{Z} + \frac{1}{2}$. Keeping in mind the formal monodromy condition, we see that $a_W(n) = 0$ if $n \in \mathbb{Z}$, we get

$$[L_{\mathfrak{g}_k}(m), a_W(n)] = -na_W(m+n) \tag{3.5.35}$$

for $m \in \mathbb{Z}, n \in \frac{1}{2}\mathbb{Z}$.

Taking $u = \omega \in L_{\widehat{\mathfrak{sl}_2}}(k,0)^0, v = i(a)$ in (3.3.1) and taking Res_{x_0}, we get

$$[Y_W(\omega, x_1), Y_W(i(a), x_2)] = \mathrm{Res}_{x_0} {x_2}^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y_W(Y_{\mathbb{Z}}(\omega, x_0)i(a), x_2). \tag{3.5.36}$$

It is well known that $i(a) \in V_{P^k}$ is a "lowest weight vector" of weight 1. So, the above equation simplifies as

$$\begin{aligned}&[Y_W(\omega, x_1), Y_W(i(a), x_2)]\\ &\quad = x_2^{-1} Y_W(L(-1)i(a), x_2)\delta\left(\frac{x_1}{x_2}\right) - x_2^{-1} Y_W(i(a), x_2)\frac{\partial}{\partial x_1}\delta\left(\frac{x_1}{x_2}\right).\end{aligned} \tag{3.5.37}$$

However, $L(m) = L_{\mathfrak{g}_k}(m)$ on $L_{\widehat{\mathfrak{sl}_2}}(k,0)$ for any integer m with $m \geq -1$, as given in Proposition 13.8, [DL]. So,

$$\begin{aligned}&[Y_W(\omega, x_1), Y_W(i(a), x_2)]\\ &\quad = x_2^{-1} Y_W(L_{\mathfrak{g}_k}(-1)i(a), x_2)\delta\left(\frac{x_1}{x_2}\right) - x_2^{-1} Y_W(i(a), x_2)\frac{\partial}{\partial x_1}\delta\left(\frac{x_1}{x_2}\right).\end{aligned} \tag{3.5.38}$$

Since Y_W satisfies the $L_{\mathfrak{g}_k}(-1)-$derivative property,

$$\begin{aligned}&[Y_W(\omega, x_1), Y_W(i(a), x_2)]\\ &\quad = x_2^{-1}\left(\frac{d}{dx_2} Y_W(i(a), x_2)\right)\delta\left(\frac{x_1}{x_2}\right) - x_2^{-1} Y_W(i(a), x_2)\frac{\partial}{\partial x_1}\delta\left(\frac{x_1}{x_2}\right).\end{aligned} \tag{3.5.39}$$

Extracting appropriate coefficients and again taking a note of formal monodromy condition,

$$[L(m), a_W(n)] = -na_W(m+n) \tag{3.5.40}$$

for $m \in \mathbb{Z}, n \in \frac{1}{2}\mathbb{Z}$. Now

$$[Y_W(\omega_{\mathfrak{g}_k} - \omega, x_1), Y_W(i(a), x_2)] = 0$$

follows. □

3.5.2 Intertwining operators

Now, exactly as in Chapter 13, [DL], we modify the twisted intertwining operators in the previous section to get twisted intertwining operators for the vertex operator algebra $L_{\widehat{\mathfrak{sl}_2}}(k,0)$. We work in the setting and notation of Proposition 3.4.2. We begin by investigating the action of $\tilde{\mathfrak{g}}[\theta]$ on the spaces $V_{Q^k}^T$.

Lemma 3.5.10. *The spaces $V_{Q^k}^T$ for various one-dimensional Q^k-modules T on which each element of Q^k acts a scalar belonging to $\{-1,1\}$ are modules for the algebra $\tilde{\mathfrak{g}}[\theta]$ via the action $\pi_T : \tilde{\mathfrak{g}}[\theta] \longrightarrow \operatorname{End} V_{Q^k}^T$ as follows:*

$$\pi_T : c \longmapsto k\,\mathrm{Id}$$

$$\pi_T : d \longmapsto d \quad \text{(the degree operator)}$$

$$\pi_T : \sum_{n\in\mathbb{Z}} (a\otimes t^n)x^{-n-1} \longmapsto Y_0(i(a),x) = Y_{\mathbb{Z}+1/2}(i(a),x) \quad a \in \mathfrak{g}_{(0)}$$

$$\pi_T : \sum_{n\in\mathbb{Z}+1/2} (a\otimes t^n)x^{-n-1} \longmapsto Y_0(i(a),x) = Y_{\mathbb{Z}+1/2}(i(a),x) \quad a \in \mathfrak{g}_{(1)}. \tag{3.5.41}$$

In fact, these spaces are $k-$fold tensor products of basic modules for $\tilde{\mathfrak{g}}[\theta]$. In other words, we can decompose $V_{Q^k}^T$ as

$$V_{Q^k}^T = V_{Q_1}^{T_1} \otimes \cdots \otimes V_{Q_k}^{T_k}, \tag{3.5.42}$$

where each Q_i is a copy of the lattice Q and each T_i is a one-dimensional module for Q_i on which each element of Q_i acts by a scalar belonging to $\{-1,1\}$. Also, $V_{Q^k}^T$ breaks up as a direct sum of standard modules for the algebra $\tilde{\mathfrak{g}}[\theta]$.

Proof. That $Y_0(i(a),x) = Y_{\mathbb{Z}+1/2}(i(a),x)$ for $a \in \mathfrak{g}$ follows from the definition of Δ_x. (cf. (9.2.20), (9.2.43), (9.2.44), [FLM]). For $k=1$ the lemma follows from theorem 7.4.10 and remark 7.4.14 of [FLM]. For $k>1$, it is clear that the space $V_{Q^k}^T$ breaks up as a tensor product of aforementioned spaces, and then it is easy to see that we get the required representation of $\tilde{\mathfrak{g}}[\theta]$ by using the definition of the map i. The complete reducibility as a of $V_{Q^k}^T$ as a $\tilde{\mathfrak{g}}[\theta]$ module follows by observing that $\tilde{\mathfrak{g}}[\theta]$ is isomorphic to the Kac-Moody algebra $A_1^{(1)}$ (cf. remark 3.5.3) and then applying theorem 10.7 and its corollary from [K]. □

Lemma 3.5.11. *If $w \in V_{Q^k}^T$ is a highest weight vector for the action of $\tilde{\mathfrak{g}}[\theta]$ then w is $L(0)$-homogeneous.*

Proof. From the definition of a highest weight vector (definition 3.5.1), we see that w is an eigenvector for $d \in \tilde{\mathfrak{g}}[\theta]$. Using (3.5.41), w is homogeneous with respect to the degree operator d acting on $V_{Q^k}^T$. The lemma follows by observing that $L(0)$ and the degree operator d differ only by a global constant, as could be seen from equation (9.4.5) of [FLM]:

$$L(0) = -d + \frac{1}{24}\dim \mathfrak{h}^k = -d + \frac{k}{24}. \tag{3.5.43}$$

□

Lemma 3.5.12. *If $w \in V_{Q^k}^T$ is a highest weight vector for the action of $\tilde{\mathfrak{g}}[\theta]$ then w is $L_{\mathfrak{g}_k}(0)$-homogeneous.*

Proof. We can write $\omega_{\mathfrak{g}_k}$ as

$$\begin{aligned}\omega_{\mathfrak{g}_k} = \frac{1}{2(k+2)}\Bigg(&\frac{1}{2}(x_\alpha + x_{-\alpha})(-1)(x_\alpha + x_{-\alpha})(-1)\cdot \mathbf{1} \\ &-\frac{1}{2}(x_\alpha - x_{-\alpha})(-1)(x_\alpha - x_{-\alpha})(-1)\cdot \mathbf{1} + \frac{1}{2}\alpha(-1)\alpha(-1)\cdot\mathbf{1}\Bigg).\end{aligned} \tag{3.5.44}$$

Let $u = i(x_\alpha + x_{-\alpha})$. Then, using $\theta u = u$ in (3.3.1) and taking $\mathrm{Res}_{x_1}\mathrm{Res}_{x_0} x_0^{-1}$ gives

$$\begin{aligned} Y_{\mathbb{Z}+1/2}(u_{-1}u, x_2) &= \mathrm{Res}_{x_1}(x_1 - x_2)^{-1} Y_{\mathbb{Z}+1/2}(u, x_1) Y_{\mathbb{Z}+1/2}(u, x_2) \\ &\quad + \mathrm{Res}_{x_1}(x_2 - x_1)^{-1} Y_{\mathbb{Z}+1/2}(u, x_2) Y_{\mathbb{Z}+1/2}(u, x_1) \\ &= \sum_{m\in\mathbb{Z}, m<0} u_m x_2^{-m-1} Y_{\mathbb{Z}+1/2}(u, x_2) + Y_{\mathbb{Z}+1/2}(u, x_2) \sum_{m\in\mathbb{Z}, m\geq 0} u_m x_2^{-m-1}. \end{aligned} \tag{3.5.45}$$

Applying to the highest weight vector w, which is an eigenvector for u_0, and which is annihilated by u_m for $m \in \mathbb{Z}, m > 0$, we get

$$Y_{\mathbb{Z}+1/2}(u_{-1}u, x_2)w = \tag{3.5.46}$$

$$\begin{aligned} &= \sum_{m\in\mathbb{Z}, m<0} u_m x_2^{-m-1} Y_{\mathbb{Z}+1/2}(u, x_2) w + Y_{\mathbb{Z}+1/2}(u, x_2) \sum_{m\in\mathbb{Z}, m\geq 0} u_m w x_2^{-m-1} \\ &= \sum_{m\in\mathbb{Z}, m<0} u_m x_2^{-m-1} \sum_{m\in\mathbb{Z}, m\leq 0} u_m w x_2^{-m-1} + \sum_{m\in\mathbb{Z}, m\leq 0} u_m x_2^{-m-1} u_0 w x_2^{-1} \end{aligned} \tag{3.5.47}$$

Extracting the coefficient of x_2^{-2} we get

$$(u_{-1}u)_1 w = u_0 u_0 w = \text{a scalar multiple of } w. \tag{3.5.48}$$

Now let $u = i(x_\alpha - x_{-\alpha})$ or $u = i(\alpha)$. In both cases, $\theta u = -u$. Multiplying (3.3.1) by $x_1^{1/2}$ in order to get integral powers of x_1, we get

$$\begin{aligned}
&{x_0}^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_{\mathbb{Z}+1/2}(u,x_1)Y_{\mathbb{Z}+1/2}(u,x_2)x_1^{1/2}\\
&\quad -{x_0}^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_{\mathbb{Z}+1/2}(u,x_2)Y_{\mathbb{Z}+1/2}(u,x_1)x_1^{1/2}\\
&\quad = {x_2}^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\left(\frac{x_1-x_0}{x_2}\right)^{-1/2}Y_{\mathbb{Z}+1/2}(Y_{\mathbb{Z}}(u,x_0)u,x_2)x_1^{1/2}\\
&\quad = {x_1}^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\left(\frac{x_2+x_0}{x_1}\right)^{1/2}Y_{\mathbb{Z}+1/2}(Y_{\mathbb{Z}}(u,x_0)u,x_2)x_1^{1/2}\\
&\quad = {x_1}^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)(x_2+x_0)^{1/2}\,Y_{\mathbb{Z}+1/2}(Y_{\mathbb{Z}}(u,x_0)u,x_2)
\end{aligned} \tag{3.5.49}$$

Equations (13.29) and (13.30) and the proof of Proposition 13.4 of [DL] say that

$$u_0 u = 0 \tag{3.5.50}$$

$$u_1 u = k\langle i^{-1}u, i^{-1}u\rangle \mathbf{1} \tag{3.5.51}$$

$$u_n u = 0 \quad \text{if } n > 1. \tag{3.5.52}$$

Also,

$$(x_2+x_0)^{1/2} = x_2^{1/2}\left(1+\frac{x_0}{2x_2}-\frac{x_0^2}{8x_2^2}+\cdots\right) = x_2^{1/2}+\frac{1}{2}x_0x_2^{-1/2}-\frac{1}{8}x_0^2x_2^{-3/2}+\cdots. \tag{3.5.53}$$

Using this information in (3.5.49) and then taking $\mathrm{Res}_{x_1}\mathrm{Res}_{x_0}x_0^{-1}$ gives

$$\begin{aligned}
&Y_{\mathbb{Z}+1/2}(u_{-1}u,x_2)x_2^{1/2} - \frac{k\langle i^{-1}u,i^{-1}u\rangle}{8}x_2^{-3/2}\\
&\quad = \mathrm{Res}_{x_1}(x_1-x_2)^{-1}Y_{\mathbb{Z}+1/2}(u,x_1)Y_{\mathbb{Z}+1/2}(u,x_2)x_1^{1/2}\\
&\quad\quad + \mathrm{Res}_{x_1}(x_2-x_1)^{-1}Y_{\mathbb{Z}+1/2}(u,x_2)Y_{\mathbb{Z}+1/2}(u,x_1)x_1^{1/2}
\end{aligned} \tag{3.5.54}$$

or equivalently,

$$\begin{aligned}
&Y_{\mathbb{Z}+1/2}(u_{-1}u,x_2) - \frac{k\langle i^{-1}u,i^{-1}u\rangle}{8}x_2^{-2}\\
&\quad = \mathrm{Res}_{x_1}(x_1-x_2)^{-1}Y_{\mathbb{Z}+1/2}(u,x_1)Y_{\mathbb{Z}+1/2}(u,x_2)x_1^{1/2}x_2^{-1/2}
\end{aligned}$$

$$+\mathrm{Res}_{x_1}(x_2-x_1)^{-1}Y_{\mathbb{Z}+1/2}(u,x_2)Y_{\mathbb{Z}+1/2}(u,x_1)x_1^{1/2}x_2^{-1/2}$$
$$=\left(\sum_{m\in\mathbb{Z},m<0}u_{m+1/2}x_2^{-m-3/2}\right)Y_{\mathbb{Z}+1/2}(u,x_2)$$
$$+Y_{\mathbb{Z}+1/2}(u,x_2)\left(\sum_{m\in\mathbb{Z},m\geq 0}u_{m+1/2}x_2^{-m-3/2}\right) \qquad (3.5.55)$$

Applying to the highest weight vector w which is annihilated by u_m for $m>0$, $m\in\mathbb{Z}+\frac{1}{2}$, yields

$$Y_{\mathbb{Z}+1/2}(u_{-1}u,x_2)w$$
$$=\frac{k\langle i^{-1}u,i^{-1}u\rangle}{8}wx_2^{-2}+\left(\sum_{m\in\mathbb{Z},m<0}u_{m+1/2}x_2^{-m-3/2}\right)Y_{\mathbb{Z}+1/2}(u,x_2)w$$
$$+Y_{\mathbb{Z}+1/2}(u,x_2)\left(\sum_{m\in\mathbb{Z},m\geq 0}u_{m+1/2}x_2^{-m-3/2}\right)w$$
$$=\frac{k\langle i^{-1}u,i^{-1}u\rangle}{8}wx_2^{-2}$$
$$+\left(\sum_{m\in\mathbb{Z},m<0}u_{m+1/2}x_2^{-m-3/2}\right)\left(\sum_{m\in\mathbb{Z},m<0}u_{m+1/2}x_2^{-m-3/2}\right)w. \qquad (3.5.56)$$

Extracting the coefficient of x_2^{-2},

$$(u_{-1}u)_1w=(u_{-1}u_{-1}\mathbf{1})_1=\frac{k\langle i^{-1}u,i^{-1}u\rangle}{8}w=\text{ a scalar multiple of } w. \qquad (3.5.57)$$

Using (3.5.48), (3.5.57) and (3.5.1) in (3.5.44) and applying to the highest weight vector w, we finally conclude

$$L_{\mathfrak{g}_k}(0)w=\frac{1}{2(k+2)}\left(\frac{1}{2}\left(i(x_\alpha+x_{-\alpha})_0\right)^2+\frac{k}{4}\right)w. \qquad (3.5.58)$$

□

Lemma 3.5.13. *Every $\tilde{\mathfrak{g}}[\theta]$-irreducible subspace M of $V_{Q^k}^T$ is spanned by eigenvectors of $L_{\mathfrak{g}_k}(0)$, and is thus graded by $L_{\mathfrak{g}_k}(0)$-eigenvalues. With this grading, $(M, Y_{\mathbb{Z}+1/2}|_{L_{\widehat{\mathfrak{sl}_2}}(k,0)})$ becomes a θ-twisted $L_{\widehat{\mathfrak{sl}_2}}(k,0)$-module. Equipped with the grading defined by $L_{\mathfrak{g}_k}(0)$-eigenvalues, $(V_{Q^k}^T, Y_{\mathbb{Z}+1/2}|_{L_{\widehat{\mathfrak{sl}_2}}(k,0)})$ is a direct sum of θ-twisted $L_{\widehat{\mathfrak{sl}_2}}(k,0)$-modules.*

Proof. Lemma 3.5.41 implies that the space $V_{Q^k}^T$ is a direct sum of highest weight modules for $\tilde{\mathfrak{g}}[\theta]$. From lemma 3.5.12 we conclude that any highest weight vector is

$L_{\mathfrak{g}_k}(0)$-homogenous. (3.5.41) and (3.5.26) imply that M is spanned by eigenvectors of $L_{\mathfrak{g}_k}(0)$. Formal monodromy condition, lower truncation condition, the vacuum property and the Jacobi identity for $Y_{\mathbb{Z}+1/2}|_{L_{\widehat{\mathfrak{sl}_2}}(k,0)}$ (acting on the whole space $V_{Q^k}^T$) are inherited from $Y_{\mathbb{Z}+1/2}$. The $L_{\mathfrak{g}_k}(-1)$-derivative property follows from proposition 13.8 of [DL] and the $L(-1)$-derivative property of $Y_{\mathbb{Z}+1/2}$. Specifically, if $v \in L_{\widehat{\mathfrak{sl}_2}}(k,0)$ then

$$L_{\mathfrak{g}_k}(-1)v = L(-1)v \tag{3.5.59}$$

and hence

$$Y_{\mathbb{Z}+1/2}|_{L_{\widehat{\mathfrak{sl}_2}}(k,0)}(L_{\mathfrak{g}_k}(-1)v,x) = Y_{\mathbb{Z}+1/2}(L_{\mathfrak{g}_k}(-1)v,x) = Y_{\mathbb{Z}+1/2}(L(-1)v,x) = \frac{d}{dx}Y(v,x). \tag{3.5.60}$$

It remains to prove that components of $Y_{\mathbb{Z}+1/2}(v,x)$ for $v \in L_{\widehat{\mathfrak{sl}_2}}(k,0)$ preserve M. For this we observe that any v could be written as a finite linear combination of terms of the form $i(a_1)_{n_1}\cdots i(a_j)_{n_j}\mathbf{1}$ where $a_i \in \mathfrak{g}$ and $n_i \in \mathbb{Z}$ for $i = 1,2,\cdots,j$. We proceed by induction on j. Vacuum property implies that components of $Y_{\mathbb{Z}+1/2}(\mathbf{1},x)$ preserve M. From (3.5.41) it is clear that components of $Y_{\mathbb{Z}+1/2}(i(a),x)$ preserve M for $a \in \mathfrak{g}_{(0)}$ or $a \in \mathfrak{g}_{(1)}$. Assume that components of $Y(b,x)$ preserve M, for some b in $L_{\widehat{\mathfrak{sl}_2}}(k,0)$. Let $a \in \mathfrak{g}_{(s)}$ for $s \in \{0,1\}$. Then $i(a) \in L_{\widehat{\mathfrak{sl}_2}}(k,0)^s$. Equation (3.3.3) with $u = i(a) \in L_{\widehat{\mathfrak{sl}_2}}(k,0)^s$ gives

$$\begin{aligned} &Y_{\mathbb{Z}+1/2}(Y(i(a),x_0)b,x_2) \\ &\quad = (x_2+x_0)^{-s/2}\mathrm{Res}_{x_1}\left({x_0}^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_{\mathbb{Z}+1/2}(i(a),x_1)Y_{\mathbb{Z}+1/2}(b,x_2)x_1^{s/2}\right. \\ &\qquad\left. -{x_0}^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_{\mathbb{Z}+1/2}(b,x_2)Y_{\mathbb{Z}+1/2}(i(a),x_1)x_1^{s/2}\right). \end{aligned} \tag{3.5.61}$$

By assumption on b, coefficient of any monomial in x_0 and x_2 on the right-hand side preserves M, and so the same holds for the left-hand side. □

Lemma 3.5.14. *$V_{Q^k}^T$ is spanned by eigenvectors of $L_{\mathfrak{g}_k}(0) - L(0)$.*

Proof. Lemma 3.5.41 implies that the space $V_{Q^k}^T$ is a direct sum of highest weight modules for $\tilde{\mathfrak{g}}[\theta]$. From lemmas 3.5.11 and 3.5.12 we conclude that any highest weight vector is $(L_{\mathfrak{g}_k}(0)-L(0))$-homogenous. Now the lemma follows from (3.5.41) and (3.5.28). □

With the help of the lemma above, the operator $x^{L_{\mathfrak{g}_k}(0)-L(0)}$ defined so that

$$x^{L_{\mathfrak{g}_k}(0)-L(0)}v = x^h v \tag{3.5.62}$$

if

$$(L_{\mathfrak{g}_k}(0) - L(0))v = hv \tag{3.5.63}$$

where $h \in \mathbb{C}$, $v \in V_{Q^k}^T$ could be extended uniquely and in a well-defined manner to the whole space $v \in V_{Q^k}^T$. The operator $x^{(L_{\mathfrak{g}_k}(0)-L(0))}$ could similarly be defined on V_{P^k}, see (13.78) of [DL].

Lemma 3.5.15. *If $v \in L_{\widehat{\mathfrak{sl}_2}}(k,0)$ then as operators on $V_{Q^k}^T$,*

$$x_2^{L_{\mathfrak{g}_k}(0)-L(0)}Y_{\mathbb{Z}+1/2}(v,x_1) = Y_{\mathbb{Z}+1/2}(v,x_1)x_2^{L_{\mathfrak{g}_k}(0)-L(0)}, \tag{3.5.64}$$

or equivalently,

$$[L_{\mathfrak{g}_k}(0) - L(0), Y_{\mathbb{Z}+1/2}(v,x)] = 0. \tag{3.5.65}$$

Proof. Again, any such v could be written as a finite linear combination of terms of the form $i(a_1)_{n_1}\cdots i(a_j)_{n_j}\mathbf{1}$ where $a_i \in \mathfrak{g}_{(0)}$ or $a_i \in \mathfrak{g}_{(1)}$ and $n_i \in \mathbb{Z}$ for $i = 1, 2, \cdots, j$. We proceed by induction on j. (3.5.65) clearly holds for $v = \mathbf{1}$. From (3.5.28) it is clear that (3.5.65) holds for $v = i(a)$ for all $a \in \mathfrak{g}$.

Assume that (3.5.65) holds for $Y(b,x)$, for some b in $L_{\widehat{\mathfrak{sl}_2}}(k,0)$. Let $a \in \mathfrak{g}_{(s)}$ for $s \in \{0,1\}$. Then, (3.5.61) gives

$$\begin{aligned}
&[L_{\mathfrak{g}_k}(0) - L(0), Y_{\mathbb{Z}+1/2}(Y_{\mathbb{Z}}(i(a),x_0)b,x_2)]\\
&= \Bigg[L_{\mathfrak{g}_k}(0) - L(0), (x_2+x_0)^{-s/2}\mathrm{Res}_{x_1}\left({x_0}^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_{\mathbb{Z}+1/2}(i(a),x_1)Y_{\mathbb{Z}+1/2}(b,x_2)x_1^{s/2}\right.\\
&\qquad \left.-\ {x_0}^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)Y_{\mathbb{Z}+1/2}(b,x_2)Y_{\mathbb{Z}+1/2}(i(a),x_1)x_1^{s/2}\right)\Bigg]\\
&= (x_2+x_0)^{-s/2}\mathrm{Res}_{x_1}\left({x_0}^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)[L_{\mathfrak{g}_k}(0) - L(0), Y_{\mathbb{Z}+1/2}(i(a),x_1)Y_{\mathbb{Z}+1/2}(b,x_2)]\right.\\
&\qquad \left.-\ {x_0}^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)[L_{\mathfrak{g}_k}(0) - L(0), Y_{\mathbb{Z}+1/2}(b,x_2)Y_{\mathbb{Z}+1/2}(i(a),x_1)]\right)\\
&= (x_2+x_0)^{-s/2}\mathrm{Res}_{x_1}\left({x_0}^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)[L_{\mathfrak{g}_k}(0) - L(0), Y_{\mathbb{Z}+1/2}(i(a),x_1)]Y_{\mathbb{Z}+1/2}(b,x_2)x_1^{s/2}\right.\\
&\qquad +{x_0}^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_{\mathbb{Z}+1/2}(i(a),x_1)[L_{\mathfrak{g}_k}(0) - L(0), Y_{\mathbb{Z}+1/2}(b,x_2)x_1^{s/2}]
\end{aligned}$$

$$- x_0{}^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)[L_{\mathfrak{g}_k}(0)-L(0),Y_{\mathbb{Z}+1/2}(b,x_2)]Y_{\mathbb{Z}+1/2}(i(a),x_1)x_1^{s/2}\Big)$$
$$- x_0{}^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)Y_{\mathbb{Z}+1/2}(b,x_2)[L_{\mathfrak{g}_k}(0)-L(0),Y_{\mathbb{Z}+1/2}(i(a),x_1)]x_1^{s/2}\Big)$$
$$=0, \tag{3.5.66}$$

by (3.5.28) and the assumption on $Y(b,x)$. The lemma follows. □

Proposition 3.5.16. *(Cf. Proposition 13.18, [DL]) In the setting and notation of proposition 3.4.2, for a $\lambda\in\Lambda$, let $\mathcal{Y}^t$ be a twisted intertwining operator for the vertex operator algebra V_{Q^k} of type $\binom{V_{Q^k}^{T^\lambda}}{V_{\lambda+Q^k}\ V_{Q^k}^T}$. Then $\mathbf{Y}^t(\cdot,x)\cdot$, defined as*

$$\mathbf{Y}^t(\cdot,x)\cdot = x^{L_{\mathfrak{g}_k}(0)-L(0)}\mathcal{Y}^t(x^{-L_{\mathfrak{g}_k}(0)+L(0)}\cdot,x)x^{-L_{\mathfrak{g}_k}(0)+L(0)}\cdot \tag{3.5.67}$$

gives an intertwining operator for vertex operator algebra $L_{\widehat{\mathfrak{sl}_2}}(k,0)$ of type $\binom{V_{Q^k}^{T^\lambda}}{V_{\lambda+Q^k}\ V_{Q^k}^T}$, where $V_{Q^k}^{T^\lambda}$, $V_{Q^k}^T$,$V_{\lambda+Q^k}$, are considered as $L_{\widehat{\mathfrak{sl}_2}}(k,0)$-modules with possibly infinite dimensional homogeneous components. In particular, for any $\tilde{\mathfrak{g}}$-irreducible component (which is also an untwisted irreducible $L_{\widehat{\mathfrak{sl}_2}}(k,0)$-module, due to lemma 13.14, [DL]) $W_{\lambda+Q^k}\subset V_{\lambda+Q^k}$ and for any $\tilde{\mathfrak{g}}[\theta]$-irreducible components (which are θ-twisted $L_{\widehat{\mathfrak{sl}_2}}(k,0)$-modules due to lemma 3.5.13) $W_{Q^k}^T\subset V_{Q^k}^T$, $W_{Q^k}^{T^\lambda}\subset V_{Q^k}^{T^\lambda}$, the projection of $\mathbf{Y}^t(w_\lambda,x)w^T$ to W^{T^λ} for $w_\lambda\in W_\lambda$ and $w^T\in W^T$ is an intertwining operator of type $\binom{W^{T^\lambda}}{W_\lambda\ W^T}$.

Proof. The proof is the same as the proof of proposition 13.18 in [DL] which we rework here for the sake of completeness.

The lower truncation condition for $\mathbf{Y}^t$ is easily deduced from that of the operator $\mathcal{Y}^t$. Let $w_{\lambda+Q^k}$ be a highest weight vector for the action of $\tilde{\mathfrak{g}}$ on $W_{\lambda+Q^k}$. Then just as in the discussion surrounding equation (13.84) of [DL], if $(L_{\mathfrak{g}_k}(0)-L(0))w_{\lambda+Q^k}=h_1w_{\lambda+Q^k}$ for some $h_1\in\mathbb{C}$ then the $\tilde{\mathfrak{g}}$-irreducibility of $W_{\lambda+Q^k}$ and equation (13.40) of [DL] imply that

$$(L_{\mathfrak{g}_k}(0)-L(0))\big|_{W_{\lambda+Q^k}}=h_1. \tag{3.5.68}$$

Similarly, if $w_{Q^k}^T$ and $w_{Q^k}^{T^\lambda}$ are highest weight vectors for the action of $\tilde{\mathfrak{g}}[\theta]$ on $W_{Q^k}^T$ and $W_{Q^k}^{T^\lambda}$ respectively, then lemmas 3.5.11 and 3.5.12 imply that $w_{Q^k}^T$ and $w_{Q^k}^{T^\lambda}$ are eigenvectors for $L_{\mathfrak{g}_k}(0)$ and $L(0)$. Letting $(L_{\mathfrak{g}_k}(0)-L(0))w_{Q^k}^T=h_2w_{Q^k}^T$ and $(L_{\mathfrak{g}_k}(0)-$

$L(0))w_{Q^k}^{T^\lambda} = h_3 w_{Q^k}^{T^\lambda}$, using $\tilde{\mathfrak{g}}[\theta]$-irreducibility of $W_{Q^k}^T$ and $W_{Q^k}^{T^\lambda}$, and then using (3.5.28) we can conclude that

$$(L_{\mathfrak{g}_k}(0) - L(0))\big|_{W_{Q^k}^T} = h_2 \tag{3.5.69}$$

$$(L_{\mathfrak{g}_k}(0) - L(0))\big|_{W_{Q^k}^{T^\lambda}} = h_3. \tag{3.5.70}$$

We prove the Jacobi identity for $v \in L_{\widehat{\mathfrak{sl}_2}}(k,0)$ such that $\theta v = \pm v$. Let $v \in L_{\widehat{\mathfrak{sl}_2}}(k,0)^j$ where $j \in \{0,1\}$. Letting $w_{(1)} \in W_{\lambda+Q^k}$ and multiplying (3.3.4) on the left by $x_1^{(L_{\mathfrak{g}_k}(0)-L(0))-h_1}$ and on the right by $x_1^{-L_{\mathfrak{g}_k}(0)+L(0)}$ and keeping (3.5.64) in mind, we conclude that

$$x_0{}^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_{\mathbb{Z}+1/2}(v,x_1)\mathbf{Y}^t(w_{(1)},x_2) - x_0{}^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\mathbf{Y}^t(w_{(1)},x_2)Y_{\mathbb{Z}+1/2}(v,x_1)$$
$$= x_2{}^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\left(\frac{x_1-x_0}{x_2}\right)^{-j/2}\mathbf{Y}^t(Y_{\mathbb{Z}}(v,x_0)w_{(1)},x_2). \tag{3.5.71}$$

This proves the Jacobi identity.

Since $\omega_{\mathfrak{g}_k} - \omega \in V_{Q^k}{}^0$, we get

$$\begin{aligned}\left[L_{\mathfrak{g}_k}(0) - L(0), \mathcal{Y}^t(w_{(1)},x_2)\right] &= \mathcal{Y}^t((L_{\mathfrak{g}_k}(0)-L(0))w_{(1)},x_2)\\ &\quad + x_2\mathcal{Y}^t((L_{\mathfrak{g}_k}(-1)-L(-1))w_{(1)},x_2)\end{aligned} \tag{3.5.72}$$

or equivalently, using $L(-1)$-derivative property of $\mathcal{Y}^t$,

$$\begin{aligned}\mathcal{Y}^t(L_{\mathfrak{g}_k}(-1)w_{(1)},x_2) &= -\mathcal{Y}^t((L_{\mathfrak{g}_k}(0)-L(0))w_{(1)},x_2)x_2^{-1}\\ &\quad + \frac{d}{dx_2}\mathcal{Y}^t(w_{(1)},x_2) + [L_{\mathfrak{g}_k}(0)-L(0),\mathcal{Y}^t(w_{(1)},x_2)]x_2^{-1}.\end{aligned} \tag{3.5.73}$$

Again, multiplying on the left by $x_2^{(L_{\mathfrak{g}_k}(0)-L(0))-h_1}$ and on the right by $x_2^{-L_{\mathfrak{g}_k}(0)+L(0)}$, we get

$$\begin{aligned}\mathbf{Y}^t(L_{\mathfrak{g}_k}(-1)w_{(1)},x_2) &= [L_{\mathfrak{g}_k}(0)-L(0),\mathbf{Y}^t(w_{(1)},x_2)]x_2^{-1}\\ &\quad + x_2^{(L_{\mathfrak{g}_k}(0)-L(0))}\left(\frac{d}{dx_2}\mathcal{Y}^t(x_2^{(-L_{\mathfrak{g}_k}(0)+L(0))}w_{(1)},x_2)\right)x_2^{-L_{\mathfrak{g}_k}(0)+L(0)}\\ &= \frac{d}{dx_2}\mathbf{Y}^t(w_{(1)},x_2).\end{aligned} \tag{3.5.74}$$

□

3.6 An abelian intertwining algebra structure

In this section, we provide an abelian intertwining algebra structure that incorporates twisted and untwisted modules for the vertex operator algebra $V_{\mathbb{Z}\alpha}$, such that the "Y" map is comprised of the various untwisted and twisted intertwining operators.

3.6.1 A direct approach

Following a suggestion of C. Sadowski, we give an abelian intertwining algebra that incorporates untwisted and the θ_1-twisted modules (see the definition below). This is sufficient, since θ and θ_1 are conjugate as automorphisms of V_Q, see Chapter 3 of [FLM] for details. For the purposes of partition identities, one can work with either of these automorphisms.

Consider the lattice $L = \mathbb{Z}\alpha/4$, where as usual $\langle \alpha, \alpha \rangle = 2$. Exactly as in Section 3.4, construct the structure V_L. Clearly, $V_L = V_Q \oplus V_{Q+\alpha/2} \oplus V_{Q+\alpha/4} \oplus V_{Q-\alpha/4}$.

Let θ_1 be the unique linear map with:

$$
\begin{aligned}
\theta_1 :& V_Q \longrightarrow V_Q \\
& h_1(-n_1) \cdots h_j(-n_j) \otimes e^{\lambda} \longmapsto (-1)^{\sqrt{\langle \lambda, \lambda \rangle /2}} h_1(-n_1) \cdots h_j(-n_j) \otimes e^{\lambda}
\end{aligned}
\tag{3.6.1}
$$

It is easy to see that θ_1 is an involution of the vertex operator algebra V_Q.

Let $u = u^* \otimes e^{\alpha}, v = v^* \otimes e^{\beta}, w = w^* \otimes e^{\gamma}$ be in V_L. Then, from Theorem 5.1 of [DL], we know that

$$
\begin{aligned}
& {x_0}^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)\left(\frac{x_1 - x_2}{x_0}\right)^{-\langle \alpha, \beta \rangle} Y(u, x_1)Y(v, x_2)w \\
& \qquad - {x_0}^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\left(\frac{x_2 - x_1}{e^{\mathbf{i}\pi} x_0}\right)^{-\langle \alpha, \beta \rangle} e^{-\mathbf{i}\pi \langle \alpha, \beta \rangle} Y(v, x_2)Y(u, x_1)w \\
= \ & {x_2}^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\left(\frac{x_1 - x_0}{x_2}\right)^{\langle \alpha, \gamma \rangle} Y(Y(u, x_0)v, x_2)w.
\end{aligned}
\tag{3.6.2}
$$

From this identity, or otherwise, it is easy to see that the spaces $V_{Q \pm \alpha_4}$ are θ_1-twisted modules for V_Q. Now, our aim is to modify the Y operator on V_L, so that we get an abelian intertwining algebra that comprises of *twisted* intertwining operators amongst untwisted and twisted modules for V_Q.

Let $G = (\mathbb{Z}\alpha/4)/(2\mathbb{Z}\alpha) \cong \mathbb{Z}_8$. We denote the images of integers in $\mathbb{Z}_8$ by an overline. G is identified with $\mathbb{Z}_8$ via $\alpha/4 + 2\mathbb{Z}\alpha \longleftrightarrow \bar{1}$. Grade V_L by G accordingly. For $g \in G$, we let V_L^g be the corresponding piece. For all $0 \leq g_1, g_2, g_3 \leq 7, g_1, g_2, g_3, \in \mathbb{Z}$, define:

$$\begin{aligned}
F : G \times G \times G &\longrightarrow \mathbb{C}^{\times}, \\
(\bar{g}_1, \bar{g}_2, \bar{g}_3) &\longmapsto 1, \\
B : G \times G \times G &\longrightarrow \mathbb{C}^{\times}, \\
(\bar{g}_1, \bar{g}_2, \bar{g}_3) &\longmapsto e^{-\mathbf{i}\pi \bar{g}_1 \bar{g}_2/8}, \\
\Omega : G \times G &\longrightarrow \mathbb{C}^{\times} \\
(\bar{g}_1, \bar{g}_2) &\longmapsto e^{-\mathbf{i}\pi \bar{g}_1 \bar{g}_2/8}, \\
b : G \times G &\longrightarrow \mathbb{C}/\mathbb{Z} \\
(\bar{g}_1, \bar{g}_2) &\longmapsto -\bar{g}_1 \bar{g}_2/8.
\end{aligned}$$

With $u \in V_L^{g_1}$, $v \in V_L^{g_2}$, $w \in V_L^{g_3}$ (3.6.2) could be re-written as:

$$\begin{aligned}
&{x_0}^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)\left(\frac{x_1 - x_2}{x_0}\right)^{b(g_1,g_2)} Y(u, x_1)Y(v, x_2)w \\
&\qquad -{x_0}^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\left(\frac{x_2 - x_1}{e^{\mathbf{i}\pi}x_0}\right)^{b(g_1,g_2)} B(g_1, g_2, g_3)Y(v, x_2)Y(u, x_1)w \\
&= {x_2}^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\left(\frac{x_1 - x_0}{x_2}\right)^{-b(g_1,g_3)} F(g_1, g_2, g_3)Y(Y(u, x_0)v, x_2)w. \qquad (3.6.3)
\end{aligned}$$

Along with the other minor axioms that could be easily verified, the structure

$$(V, Y, \mathbf{1}, \omega, T = 8, G = (\mathbb{Z}\alpha/4)/(2\mathbb{Z}\alpha), F, \Omega)$$

becomes an abelian intertwining algebra in the sense of [DL]. For the precise definition and some examples of abelian intertwining algebras, we refer the reader to Chapter 12 of [DL].

For g_1 such that $u \in V_Q = V_L^{\bar{0}} \oplus V_L^{\bar{4}}$ and g_2 with $v \in V_P = V_L^{\bar{0}} \oplus V_L^{\bar{2}} \oplus V_L^{\bar{4}} \oplus V_L^{\bar{6}}$, we wish to remove the factors $B(g_1, g_2, g_3)$ and $F(g_1, g_2, g_3)$ from (3.6.2), so that we can get a structure that incorporates the twisted intertwining operators. In order to achieve this, we will suitably scale the "Y" map thereby changing the normalized abelian 3-cocycle (F, Ω) to a cohomologous one.

With $a, b, c, d \in \mathbb{Z}$ such that $0 \le a, c \le 1$ and $0 \le b, d \le 3$, let

$$\begin{aligned} f(v,w) :\, G \times G \longrightarrow G, \\ f(a\alpha + b\alpha/4 + 2\mathbb{Z}\alpha, c\alpha + d\alpha/4 + 2\mathbb{Z}\alpha) = \mathbf{i}^{bc} \end{aligned} \tag{3.6.4}$$

Clearly, for all $g_1, g_2 \in G$,

$$f(g_1, 0) = f(0, g_2) = 1. \tag{3.6.5}$$

For $v \in V_L^g$, $w \in V_L^h$, define

$$\mathbf{Y}(v,x)w = f(g,h)Y(v,x)w. \tag{3.6.6}$$

We know from Remark 12.23 of [DL] that this new structure is also an abelian intertwining algebra. With this modification, what we have is the following:

$$\begin{aligned} F'(g_1,g_2,g_3) &= f(g_2,g_3)f(g_1+g_2,g_3)^{-1}f(g_1,g_2+g_3)f(g_1,g_2)^{-1} \\ \Omega'(g_1,g_2) &= f(g_1,g_2)f(g_2,g_1)^{-1}\Omega(g_1,g_2) \\ B'(g_1,g_2,g_3) &= F'(g_2,g_1,g_3)^{-1}\Omega'(g_1,g_2)F'(g_1,g_2,g_3). \end{aligned} \tag{3.6.7}$$

This new normalized 3-cocycle of course satisfies:

$$\begin{aligned} F'(0,g_2,g_3) = F'(g_1,0,g_3) = F'(g_1,g_2,0) = 1, \\ \Omega'(0,g_2) = \Omega(0,g_2) = 1. \end{aligned}$$

But moreover, letting $g_1 = \alpha + 2\mathbb{Z}\alpha$, $g_2 = a\alpha + b\alpha/4 + 2\mathbb{Z}\alpha$, $g_3 = c\alpha + d\alpha/4 + 2\mathbb{Z}\alpha$, where $a, b, c, d \in \mathbb{Z}$, with $0 \le a, c \le 1$ and $0 \le b, d \le 3$,

$$\begin{aligned} F'(g_1,g_2,g_3) &= f(g_2,g_3)f(g_1+g_2,g_3)^{-1}f(g_1,g_2+g_3)f(g_1,g_2)^{-1} \\ &= \mathbf{i}^{bc} \cdot \mathbf{i}^{-bc} \cdot 1 \cdot 1 \\ &= 1, \end{aligned} \tag{3.6.8}$$

$$\begin{aligned} B'(g_1,g_2,g_3) &= f(g_1,g_3)^{-1}f(g_1,g_2+g_3)f(g_2,g_1+g_3)^{-1}f(g_2,g_3)B(g_1,g_2,g_3) \\ &= f(g_2,g_1+g_3)^{-1}f(g_2,g_3) \cdot e^{-\mathbf{i}\pi b/2} \\ &= f(g_2,g_1+g_3)^{-1}\mathbf{i}^{bc} \cdot \mathbf{i}^{-b} \end{aligned} \tag{3.6.9}$$

Now, if $c = 1$, then $f(g_2, g_1 + g_3) = 1$ and if $c = 0$ then $f(g_2, g_1 + g_3) = \mathbf{i}^b$. Hence,

$$B'(g_1, g_2, g_3) = \begin{cases} 1 \cdot \mathbf{i}^b \cdot \mathbf{i}^{-b} & \text{if } c = 1, \\ \mathbf{i}^{-b} \cdot 1 \cdot \mathbf{i}^{-b} & \text{if } c = 0. \end{cases} \tag{3.6.10}$$

Therefore,

$$B'(g_1, g_2, g_3) = 1 \text{ for } b \in 2\mathbb{Z}. \tag{3.6.11}$$

In effect, if $u \in V_Q$ and if $v \in V_P$, we get that $g_1 = \alpha + 2\mathbb{Z}\alpha$ and that $g_2 = a\alpha + b\alpha/4 + 2\mathbb{Z}\alpha$ with $b \in 2\mathbb{Z}$. In this scenario, for $\mathbf{Y}$, we precisely get the Jacobi identity for the untwisted intertwining operators if $w \in V_P$, and the Jacobi identity for the twisted intertwining operators if $w \in V_{P+\alpha/4}$.

3.6.2 An approach using the tensor category theory

Now we provide an alternate route, this time directly using the automorphism θ. This is exactly the approach of Theorem 3.8 of [H1].

Let us gather relevant facts about the vertex operator algebra V_Q^θ formed by the fixed points of θ. See the Introduction of [DJL] for more details and for corresponding results for higher rank even lattices in place of Q.

1. Clearly, V_Q^θ is $\mathbb{N}$-gradable and its zeroth weight space is spanned by the vacuum vector.

2. From [DN] we know the complete list of irreducible modules of V_Q^θ. Each irreducible untwisted or θ-twisted module W for V_Q breaks as $W = W^+ \oplus W^-$, where $W^\pm$ are irreducible (untwisted) modules for V_Q^θ. The complete set of irreducible modules for V_Q^θ is $S = \{V_Q^\pm, V_{Q+\alpha/2}^\pm, (V_Q^{T_1})^\pm, (V_Q^{T_2})^\pm\}$.

3. The fusion rules for V_Q^θ are computed in [Ab1] and [ADL]. It can be verified that the fusion algebra is isomorphic to the group algebra of the abelian group $\mathbb{Z}_8$, i.e., there exists a bijection $\varphi : S \to \mathbb{Z}_8$ such that for irreducible modules W_1, W_2, W_3, the fusion rule $\binom{W_3}{W_1\, W_2}$ is 1 if $\varphi(W_1) + \varphi(W_2) = \varphi(W_3)$ and 0 otherwise.

4. From [Y] and [ABD], the vertex operator algebra V_Q^θ is C_2-cofinite.

5. From [Ab2], we know that V_Q^θ is rational, i.e., every $\mathbb{N}$-graded weak module of V_Q^θ is completely reducible.

Now, using properties 1, 4 and 5 above and Remark 3.8 of [H3], we deduce that the category of V_Q^θ-modules has a natural structure of vertex tensor category, in the sense of [HL1]–[HL3], [H4]. Therefore, the direct sum of irreducible modules has a natural intertwining algebra structure, in the sense of [H2]. Now, using the fact that the fusion algebra is isomorphic to the group algebra of the abelian group Z_8 and arguing exactly as in the proof of Theorem 3.8 of [H1], we know that we have obtained an abelian intertwining algebra structure on the direct sum of irreducible modules for V_Q^θ.

Each of the untwisted or θ-twisted intertwining operator for V_Q when restricted to irreducible modules for V_Q^θ gives an (untwisted) intertwining operator for V_Q^θ. Now, exactly as in the previous subsection, one can scale the relevant "finer" intertwining operators individually to get "coarser" untwisted or θ-twisted intertwining operators for V_Q. We obtain an abelian intertwining algebra structure incorporating untwisted and θ-twisted intertwining operators for V_Q.

Chapter 4

From sums, hopefully to products: Principal subspaces

In this chapter, following an idea of J. Lepowsky, we analyse a natural Koszul complex associated to the principal subspace of the basic module $L(\Lambda_0)$ of $\widehat{\mathfrak{sl}_2}$. We determine the second homology of this complex and explain its relations to the Garland-Lepowsky resolution of the ambient standard module. It is expected that this Koszul complex would ultimately yield a "character formula" for the principal subspaces.

Our main theorem could be stated and proved in a completely commutative-algebraic setting without needing any material from the representation theory of affine Lie algebras or vertex operator algebras. Therefore, we organize this chapter in a slightly unusual way: we dive straight into the heart of the matter and once our main result is established, we put our results in perspective by referring to the theory of principal subspaces — especially the presentation results of Calinescu-Lepowsky-Milas ([CalLM1]), some results of Primc ([P2]) regarding relations among the annihilating fields for standard modules — and also some recent conjectures of Gorsky-Oblomkov-Rasmussen ([GOR]) on the Khovanov homology of torus knots.

The computer algebra system SINGULAR [DGPS] was used for explorations regarding this project.

4.1 The setup

Consider the commutative associative algebra

$$\mathcal{A} = \mathbb{C}[x_{-1}, x_{-2}, \dots]. \tag{4.1.1}$$

Through the next few sections, we will consider modules over the algebra $\mathcal{A}$.

Consider a sequence of (non-regular) elements

$$r_{-n} = \sum_{i=1}^{n-1} x_{-i}x_{-n+i} = x_{-1}x_{-n+1} + x_{-2}x_{-n+2} + \cdots + x_{-n+1}x_{-1} \tag{4.1.2}$$

for $n = 2, 3, \ldots$. Let I_{Λ_0} be the ideal generated by the elements r_{-n} for $n = 2, 3, \ldots$. The reason for the notation "$_{\Lambda_0}$" will become clear in Section 4.4. Let

$$W_{\Lambda_0} = \mathcal{A}/I_{\Lambda_0}. \tag{4.1.3}$$

Remark 4.1.1. As we shall see below in Section 4.4, the algebra $\mathcal{A}$ is actually the universal enveloping algebra of a certain abelian Lie algebra, and that the space W_{Λ_0} is a certain subspace, called "principal subspace" of the standard module $L(\Lambda_0)$ of the affine Lie algebra $A_1^{(1)}$.

Remark 4.1.2. We caution the reader that as a principal subspace, W_{Λ_0} is *not* a priori defined as the quotient space as in (4.1.3). That W_{Λ_0} can be presented as in (4.1.3) is a non-trivial fact stated in [FS1, FS2], invoked in [CLM1] and finally proved in [CalLM1].

We wish to analyze the Koszul complex determined by the sequence of elements r_{-n}, $n = 2, 3, \ldots$. To this end, consider the following complex of $\mathcal{A}$-modules:

$$\cdots \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \twoheadrightarrow 0, \tag{4.1.4}$$

with

$$C_0 = W_{\Lambda_0}, \tag{4.1.5}$$

$$C_1 = \mathcal{A}, \tag{4.1.6}$$

$$C_2 = \bigoplus_{i \geq 2} \mathcal{A}\xi_{-i}, \tag{4.1.7}$$

$$C_j = \bigoplus_{i_1, i_2, \ldots, i_j \geq 2} \mathcal{A}\xi_{-i_1, -i_2, \ldots, -i_k} \quad \text{for } j \geq 3, \tag{4.1.8}$$

where the symbols $\xi_{\ldots}$ satisfy the relation:

$$\xi_{\ldots, i, \ldots, j, \ldots} = -\xi_{\ldots, j, \ldots, i, \ldots}, \tag{4.1.9}$$

and where the differential $\partial_\bullet$ is given by:

$$\partial_{j+1}(\xi_{-i_1, \ldots, -i_j}) = \sum_{t=1}^{j} (-1)^{t+1} r_{-i_t} \xi_{-i_1, \ldots, \widehat{-i_t}, \ldots, -i_j}, \tag{4.1.10}$$

for $j+1, i_1, \ldots, i_j \geq 2$ with the hat denoting an omitted subscript. It is straight-forward to varify that ∂ respects (4.1.9) and that $\partial^2 = 0$.

Define

$$H_n = \text{Ker } \partial_n / \text{Im } \partial_{n+1}. \tag{4.1.11}$$

Remark 4.1.3. Ideally, the Koszul complex should be defined using the language of exterior products, which is equivalent to (4.1.9). However, since our aim is to analyse H_2, we choose to keep our notation simple.

Remark 4.1.4. Typically, the Koszul complex determined by the sequence of elements r_{-n} for $n = 2, \ldots$ would be

$$\cdots \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \longrightarrow 0, \tag{4.1.12}$$

with C_0 omitted. However, we have chosen to write it as in (4.1.4) with an eye toward the Garland-Lepowsky resolution of standard modules, which is recalled below in Section 4.4.

The algebra $\mathcal{A}$ and each of the modules is bi-graded. Consider unique derivations $L_0, \frac{\alpha}{2}$ of $\mathcal{A}$ satisfying:

$$L_0(x_{-i}) = ix_{-i}, \tag{4.1.13}$$

$$\frac{\alpha}{2}(x_{-i}) = x_{-i} \tag{4.1.14}$$

for $i \geq 1$, that can be extended uniquely to each of the C_js such that

$$L_0(\xi_{-i_1,\cdots,-i_k}) = (i_1 + \cdots + i_k)\xi_{-i_1,\cdots,-i_k} \tag{4.1.15}$$

$$L_0(a \cdot c) = L_0(a) \cdot c + a \cdot L_0(c), \tag{4.1.16}$$

and

$$\frac{\alpha}{2}(\xi_{-i_1,\cdots,-i_k}) = 2k\xi_{-i_1,\cdots,-i_k} \tag{4.1.17}$$

$$\frac{\alpha}{2}(a \cdot c) = \alpha_0(a) \cdot c + a \cdot \alpha_0(c), \tag{4.1.18}$$

for all $a \in \mathcal{A}$ and $c \in C_\bullet$.

Definition 4.1.5. *It is clear that the ring $\mathcal{A}$ and each of the modules $C_\bullet$ is bi-graded by* weight *and* charge*; L_0-eigenvalue is called weight and $\frac{\alpha}{2}$-eigenvalue is called charge.*

The weights for each of the C_js are positive integers, and each of the weight spaces is finite dimensional. Typically, while considering the graded dimensions of various spaces, the formal variable q denotes the weight and x denotes the charge grading.

Definition 4.1.6. *For any vector space M homogeneous with respect to the double grading with finite dimensional graded components, define the character to be:*

$$\chi(M; x, q) = \sum_{h,l} \; dim(M_{h,l}) x^h q^l \tag{4.1.19}$$

where $M_{h,l}$ is the homogeneous component of charge h and weight l.

The following theorem was noted by [FS1]–[FS2] and proved by [CLM1] using the theory of intertwining operators.

Theorem 4.1.7. *The character of W_{Λ_0} is given by:*

$$\chi(W_{\Lambda_0}; x, q) = \sum_{n\geq 0} \frac{x^n q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}. \tag{4.1.20}$$

One immediately concludes (cf. [A3]) that

Corollary 4.1.8. *For $h, l \in \mathbb{N}$,*

$$\begin{aligned} dim((W_{\Lambda_0})_{h,l}) = \; & \textit{Number of partitions of } l \textit{ with exactly } h \textit{ parts} \\ & \textit{such that any two adjacent parts have difference at least 2.} \end{aligned} \tag{4.1.21}$$

A space defined by a similar, yet different, ideal arose in [BMS] in the context of the geometry of arc spaces. In [BMS], authors use the theory of Gröbner bases in order to find the required graded dimension.

Since the differential $\partial_\bullet$ preserves both weight and charge, we have that:

Proposition 4.1.9 (Euler-Poincaré principle)**.** *The character of W_{Λ_0} is given by:*

$$\chi(W_{\Lambda_0}; x, q) = \sum_{n\geq 1} (-1)^{n+1} \left(\chi(C_n; x, q) - \chi(H_n; x, q) \right). \tag{4.1.22}$$

Now, the problem is to find a description of the homology of the complex (4.1.4). In the following two sections, we will determine the kernal of the map ∂_2.

4.2 The second kernel

In this section, we gather preliminary lemmas required in Section 4.3, in the proof of our main theorem.

Consider the unique homomorphism

$$\sigma : \mathcal{A} \to \mathcal{A}$$

such that

$$\sigma(x_{-1}) = 0,$$
$$\sigma(x_{-i}) = x_{-i+1} \quad \text{for } i \geq 2,$$

extended uniquely to each of the C_js such that

$$\sigma(\xi_{-i_1,\cdots,-i_k}) = \xi_{-i_1+2,\cdots,-i_k+2} \quad \text{for } i_1, i_2, \cdots, i_k \geq 2,$$
$$\sigma(a \cdot c) = \sigma(a)\sigma(c).$$

Also consider the unique derivation L_{-1} of $\mathcal{A}$ satisfying:

$$L_{-1}(x_{-i}) = ix_{-i-1}, \tag{4.2.1}$$

for $i \geq 1$, extended uniquely to each of the C_js such that

$$L_{-1}(\xi_{-i_1,\cdots,-i_k}) = (i_1 - 1)\xi_{-i_1-1,-i_2,\cdots,-i_k} + (i_2 - 1)\xi_{-i_1,-i_2-1,\cdots,-i_k} + \cdots + (i_k - 1)\xi_{-i_1,-i_2\cdots,-i_k-1} \tag{4.2.2}$$

$$L_{-1}(a \cdot c) = L_{-1}(a)c + aL_{-1}(c). \tag{4.2.3}$$

Remark 4.2.1. In the course of the main proof, we won't need the derivation L_{-1}. However, this derivation has a natural vertex operator theoretic interpretation as a certain mode of the Virasoro algebra as we shall see in Section 4.4 and moreover, $\mathrm{Ker}(\partial_2)$ can be naturally described by employing L_{-1}.

Remark 4.2.2. The derivations L_{-1} and L_0 can be extended so that the entire Virasoro algebra acts on $\mathcal{A}$ and each of the modules $C_\bullet$. We will not need this action in the proof of the main theorem.

It will be convenient to use generating functions, so we introduce the following notation:

Notation 4.2.3. *Let x be a formal variable, let $\bullet'$ denote the formal derivative with respect to x and let*

$$X(x) = \sum_{t\geq 1} x_{-t}x^{t-1} \tag{4.2.4}$$

$$R(x) = X(x)^2 = \sum_{t\geq 2} r_{-t}x^{t-2} \tag{4.2.5}$$

$$\Xi(x) = \sum_{t\geq 2} \xi_{-t}x^{t-2} \tag{4.2.6}$$

$$M(x) = 2X'(x)\Xi(x) - X(x)\Xi'(x) = \sum_{t\geq 4} \mu_{-t}x^{t-4}. \tag{4.2.7}$$

Notation 4.2.4. *Let*

$$\mathcal{I} = \mathcal{A}\langle \mu_{-i} \,|\, i \geq 4\rangle + Im(\partial_3), \tag{4.2.8}$$

where $\mathcal{A}\langle\cdot\rangle$ denotes the sub-module generated by the list of elements enclosed in $\langle\cdot\rangle$.

In our main theorem, we will show that

$$\mathrm{Ker}(\partial_2) = \mathcal{I}.$$

With this set-up, now we give a sequence of lemmas, each of which is easy to verify.

Lemma 4.2.5. *The differential ∂ is homogeneous with respect to both weight and charge. Hence, so are $Ker(\partial_\bullet)$ and $Im(\partial_\bullet)$.*

Lemma 4.2.6. *With σ and L_{-1} defined as above, we have:*

1. $\sigma(r_{-i}) = 0$ *if* $i = 1, 2$.

2. $\sigma(r_{-i}) = r_{-i+2}$ *if* $i > 2$.

3. $L_{-1}(r_{-i}) = (i-1)r_{-i-1}$.

Lemma 4.2.7. *Let $f(x)$ be a formal power series with coefficients in a module $C_\bullet$. If $\sigma f(x) = x^j f(x)$ for some $j \in \mathbb{N}$ then $\sigma f'(x) = (x^j f)'(x) = jx^{j-1}f(x) + x^j f'(x)$.*

Proof. If $j = 0$, the statement is easy to prove. In fact, it is easy to see that there does not exist any f with coefficients in $C_\bullet$ or $\mathcal{A}$ such that $\sigma f(x) = f(x)$. Now let $j \geq 1$. Let $f(x) = \sum_{i\geq 0} f_i x^i$. Then, $\sigma f(x) = x^j f(x)$ implies that $\sigma f_k = 0$ for all $k = 0, \ldots, j-1$ and that $\sigma f_i = f_{i-j}$ for $i \geq j$. Therefore,

$$\begin{aligned}\sigma f'(x) &= \sum_{i\geq 1} i\sigma f_i x^{i-1} = \sum_{i\geq j} i f_{i-j} x^{i-1} = \sum_{i\geq 0}(i+j) f_i x^{i+j-1} \\ &= j x^{j-1} \sum_{i\geq 0} f_i x^i + x^j \sum_{i\geq 0} i f_i x^{i-1} \\ &= j x^{j-1} f(x) + x^j f'(x) = (x^j f)'(x).\end{aligned}$$

□

Lemma 4.2.8. *We have that*

$$\sigma X(x) = xX(x), \tag{4.2.9}$$

$$\sigma \Xi(x) = x^2 \Xi(x), \tag{4.2.10}$$

$$\sigma M(x) = x^3 M(x). \tag{4.2.11}$$

Proof. The first two are obvious from the definitions. For the last one, use Lemma 4.2.7:

$$\begin{aligned}\sigma M(x) &= \sigma(2X'(x)\Xi(x) - X(x)\Xi'(x)) \\ &= 2\sigma X'(x)\sigma\Xi(x) - \sigma X(x)\sigma\Xi'(x) \\ &= 2(X(x) + xX'(x))(x^2\Xi(x)) - xX(x)(2x\Xi(x) + x^2\Xi'(x)) \\ &= 2x^3 X'(x)\Xi(x) - x^3 X(x)\Xi'(x) \\ &= x^3 M(x).\end{aligned}$$

□

Corollary 4.2.9. *For each $t \geq 4$, $\mu_{-t} = \sigma\mu_{-t-3}$.*

Lemma 4.2.10. *We have that*

$$L_{-1} X(x) = X'(x), \tag{4.2.12}$$

$$L_{-1} \Xi(x) = \Xi'(x), \tag{4.2.13}$$

$$L_{-1}M(x) = M'(x). \tag{4.2.14}$$

Lemma 4.2.11. *The homomorphism σ and the map L_{-1} commute with the differential:*

$$\sigma\partial_\bullet = \partial_\bullet\sigma \tag{4.2.15}$$

$$L_{-1}\partial_\bullet = \partial_\bullet L_{-1}. \tag{4.2.16}$$

Lemma 4.2.12.

$$\begin{aligned} \sigma Ker(\partial_\bullet) &\subset Ker(\partial_\bullet) \\ L_{-1}Ker(\partial_\bullet) &\subset Ker(\partial_\bullet) \\ \sigma Im(\partial_\bullet) &\subset Im(\partial_\bullet) \\ L_{-1}Im(\partial_\bullet) &\subset Im(\partial_\bullet), \end{aligned} \tag{4.2.17}$$

Hence, σ and L_{-1} act naturally on the homology groups H_n.

Lemma 4.2.13. *For each $t \geq 4$, $\mu_{-t} \in Ker(\partial_2)$.*

Proof. We have that

$$\begin{aligned} \partial M(x) &= \partial(2X'(x)\Xi(x) - X(x)\Xi'(x)) \\ &= 2X'(x)X(x)^2 - X(x)(\partial\Xi)'(x) \\ &= 2X'(x)X(x)^2 - X(x)(2X(x)X'(x)) \\ &= 0. \end{aligned}$$

Therefore, for all $t \geq 4$,

$$\partial(\mu_{-t}) = 0. \tag{4.2.18}$$

□

Lemma 4.2.14. $Im(\partial_3) = \mathcal{A}\langle r_{-i}\xi_{-j} - r_{-j}\xi_{-i} \,|\, i, j \geq 2\rangle$.

Lemma 4.2.15. *We have that $\mathcal{I} = \sigma\mathcal{I}$.*

Proof. Recall that $\mathcal{I} = \mathcal{A}\langle\mu_{-t} \,|\, t \geq 4\rangle + \mathrm{Im}(\partial_3)$. First, Lemmas 4.2.11 and 4.2.8 guarantee that $\sigma I \subset I$. For the reverse inclusion, note that $\sigma\mathcal{A} = \mathcal{A}$ and that

$r_{-i}\xi_{-j} - r_{-j}\xi_{-i} = \sigma(r_{-i-2}\xi_{-j-2} - r_{-j-2}\xi_{-i-2})$. (Actually, a similar proof works for all $\mathrm{Im}(\partial_\bullet)$.) Morever, due to Corollary 4.2.9 and the fact that $\sigma\mathcal{A} = \mathcal{A}$, we have that $\mathcal{A}\langle \mu_{-t} \,|\, t \geq 4\rangle = \sigma\mathcal{A}\langle \mu_{-t} \,|\, t \geq 7\rangle$. □

Lemma 4.2.16. *We have that*

$$Ker(\sigma|_{\mathcal{A}}) = x_{-1}\mathcal{A}. \tag{4.2.19}$$

Moreover,

$$Ker(\sigma|_{C_2}) = \mathcal{A}\xi_{-2} \oplus \mathcal{A}\xi_{-3} \oplus \mathcal{A}\langle x_{-1}\xi_{-i} \,|\, i \geq 4\rangle.which \tag{4.2.20}$$

and that

$$Ker(\sigma^2|_{C_2}) = x_{-1}C_2 + x_{-2}C_2 + (\mathcal{A}\xi_{-2} \oplus \mathcal{A}\xi_{-3} \oplus \mathcal{A}\xi_{-4} \oplus \mathcal{A}\xi_{-5}). \tag{4.2.21}$$

Lemma 4.2.17. *Let $v \in C_2$ be a weight homogeneous element. If $\sigma^j v \in \mathcal{I}\backslash\{0\}$ for some positive integer j, then there exists an $i \in \mathcal{I}$ such that weight of i is the same as that of v and $\sigma^j v = \sigma^j i$.*

Proof. Let the weight of v be w. We prove the proposition for $j = 1$, for higher j, similar strategy works. Since $\mathcal{I} = \sigma\mathcal{I}$ (Lemma 4.2.15) and since $\sigma v \in \mathcal{I}$, there exists an $\tilde{i} \in \mathcal{I}$ such that $\sigma v = \sigma\tilde{i}$. We can write $C_2 = S \oplus \mathrm{Ker}(\sigma)$, where S is the space spanned by monomials that are neither divisible by x_{-1} nor are contained in $\mathcal{A}\xi_{-2} \oplus \mathcal{A}\xi_{-3}$. Accordingly, let $\tilde{i} = s + \tilde{t}$, where $s \in S$ and $t \in \mathrm{Ker}(\sigma)$. Now, it is clear that $\sigma s = \sigma v$ and hence, s must be weight homogeneous with weight w. Now let t be the weight w component of $\tilde{t}$. Hence, $s + t$ is the weight w component of $\tilde{i}$ and belongs to $\mathcal{I}$ since $\mathcal{I}$ is homogeneous. Moreover, $\sigma v = \sigma s = \sigma(s + t)$. We can now let $i = s + t$. □

4.3 Proof of the main theorem

Now we restrict our attention to C_2.

Theorem 4.3.1. *We have $Ker(\partial_2) = \mathcal{I} = \mathcal{A}\langle \mu_{-t} \,|\, t \geq 4\rangle + Im(\partial_3)$.*

Proof. From the definition of $\mathcal{I}$ and Lemma 4.2.13 it is clear that $\mathcal{I} \subset \mathrm{Ker}(\partial_2)$. We just have to prove the reverse inclusion. Assume, towards a contradiction that $\mathrm{Ker}(\partial_2)\backslash\mathcal{I}$ is non-empty. Since both $\mathcal{I}$ and $\text{Ker}(\partial_2)$ are doubly homogeneous, so is $\text{Ker}(\partial_2)\backslash\mathcal{I}$. Select a vector v_0 of smallest possible weight from $\text{Ker}(\partial_2)\backslash\mathcal{I}$. We will construct a sequence of non-zero vectors each belonging to $\text{Ker}(\partial_2)\backslash\mathcal{I}$ and having the same weight as that of v_0. However, the last member of the sequence will end up being in $\mathcal{I}$, providing us with the required contradiction.

Clearly, v_0 is non-zero.

From Lemma 4.2.12, $\sigma v_0 \in \text{Ker}(\partial_2)$ and has lower weight than v_0. Therefore, $\sigma v_0 \in \mathcal{I}$. If $\sigma v_0 = 0$, let $i_0 = 0$. If $\sigma v_0 \neq 0$, Lemma 4.2.15 implies that $\sigma v_0 \in \sigma\mathcal{I}$. Hence,

$$\sigma v_0 = \sigma i_0$$

for some $i_0 \in \mathcal{I}$. Note that by Lemma 4.2.17, i_0 can be chosen to be weight-homogeneous with the same weight as that of v_0, so that $v_0 - i_0$ has the same weight as that of v_0. Now, $v_0 - i_0 \in \text{Ker}(\sigma) \cap \text{Ker}(\partial_2)$. Since $v_0 \notin \mathcal{I}$ but $i_0 \in \mathcal{I}$, $v_0 - i_0 \notin \mathcal{I}$ as well. Therefore, instead of our original v_0, let us shift attention to $v_0 - i_0$ and we call it v_1. Since $v_1 \in \text{Ker}(\sigma)$, by Lemma 4.2.16, we have that

$$v_1 = v_0 - i_0 = p_{-2}\xi_{-2} + p_{-3}\xi_{-3} + x_{-1}(p_{-4}\xi_{-4} + \cdots + p_{-j}\xi_{-j}), \tag{4.3.1}$$

for some doubly homogeneous polynomials $p_{-2}, \ldots, p_{-j} \in \mathcal{A}$. Now,

$$\partial_2(v_1) = p_{-2}x_{-1}^2 + 2p_{-3}x_{-1}x_{-2} + x_{-1}(p_{-4}r_{-4} + \cdots p_{-j}r_{-j}) = 0.$$

Therefore, cancelling a factor of x_{-1},

$$p_{-2}x_{-1} + 2p_{-3}x_{-2} + p_{-4}r_{-4} + \cdots p_{-j}r_{-j} = 0,$$

and hence,

$$\sigma^2(p_{-4}r_{-4} + \cdots p_{-j}r_{-j}) = 0.$$

Therefore,

$$\sigma^2(p_{-4}\xi_{-4} + \cdots p_{-j}\xi_{-j}) \in \text{Ker}(\partial_2).$$

But now, $\sigma^2(p_{-4}\xi_{-4} + \cdots p_{-j}\xi_{-j})$ has lower weight than that of v_0, and hence,

$$\sigma^2(p_{-4}\xi_{-4} + \cdots p_{-j}\xi_{-j}) \in \mathcal{I}.$$

Again, noting that $\mathcal{I} = \sigma\mathcal{I} = \sigma^2\mathcal{I}$,

$$\sigma^2(p_{-4}\xi_{-4} + \cdots p_{-j}\xi_{-j}) \in \sigma^2\mathcal{I}.$$

Using Lemma 4.2.16, we deduce that there must exist $i_1 \in \mathcal{I}$, $c_1, c_2 \in C_2$ and polynomials $q_{-2}, \ldots, q_{-5} \in \mathcal{A}$ such that

$$p_{-4}\xi_{-4} + \cdots p_{-j}\xi_{-j} = i_1 + x_{-1}c_1 + x_{-2}c_2 + q_{-2}\xi_{-2} + q_{-3}\xi_{-3} + q_{-4}\xi_{-4} + q_{-5}\xi_{-5}.$$

Note again that i_1 can be chosen to have the same weight as that of $p_{-4}\xi_{-4} + \cdots + p_{-j}\xi_{-j}$, which in turn has the same weight as that of v_0.

Substituting in the definition of v_1 (4.3.1),

$$\begin{aligned} v_1 &= p_{-2}\xi_{-2} + p_{-3}\xi_{-3} + x_{-1}(i_1 + x_{-1}c_1 + x_{-2}c_2 + q_{-2}\xi_{-2} + q_{-3}\xi_{-3} + q_{-4}\xi_{-4} + q_{-5}\xi_{-5}) \\ &= p_{-2}\xi_{-2} + p_{-3}\xi_{-3} + x_{-1}(i_1 + q_{-2}\xi_{-2} + q_{-3}\xi_{-3} + q_{-4}\xi_{-4} + q_{-5}\xi_{-5}) \\ &\quad + r_{-2}c_1 + \frac{1}{2}r_{-3}c_2. \end{aligned}$$

Now, adding to v_1 appropriate multiples of $r_{-2}\xi_{-j} - r_{-j}\xi_{-2}$ and $r_{-3}\xi_{-j} - r_{-j}\xi_{-3}$ where $j > 3$, which are all in $\mathrm{Im}(\partial_3) \subset \mathcal{I}$, and then collecting terms, we arrive at a vector

$$v_2 = \widetilde{p_{-2}}\xi_{-2} + \widetilde{p_{-3}}\xi_{-3} + x_{-1}(\widetilde{q_{-4}}\xi_{-4} + \widetilde{q_{-5}}\xi_{-5}),$$

which has the same weight as v_0 and is in $\mathrm{Ker}(\partial_2)\backslash\mathcal{I}$. Now, note that

$$\begin{aligned} \mu_{-5} &= 4x_{-3}\xi_{-2} + x_{-2}\xi_{-3} - 2x_{-1}\xi_{-4} \\ \mu_{-6} &= 6x_{-4}\xi_{-2} + 3x_{-3}\xi_{-3} - 3x_{-1}\xi_{-5}. \end{aligned}$$

Observe that there is no ξ_{-4} term in μ_{-6}. Therefore, we can add appropriate multiples of μ_{-5} and μ_{-6} to v_2 to get that

$$v_3 = \widetilde{\widetilde{p_{-2}}}\xi_{-2} + \widetilde{\widetilde{p_{-3}}}\xi_{-3}$$

has the same weight as v_0 and is in $\mathrm{Ker}(\partial_2)\backslash\mathcal{I}$. Now, since $\partial_2(v_3) = 0$, we get that $x_{-1}\widetilde{\widetilde{p_{-2}}} + 2x_{-2}\widetilde{\widetilde{p_{-3}}} = 0$. Hence, there must exist a polynomial f such that $\widetilde{\widetilde{p_{-2}}} = 2x_{-2}f$ and $\widetilde{\widetilde{p_{-3}}} = x_{-1}f$, implying that $v_3 = f\mu_{-3} \in \mathcal{I}$. This is a contradiction, and we are done. $\square$

4.4 Affine Lie algebras, Garland-Lepowsky resolution and principal subspaces

In this section, we recall the theory of principal subspaces as developed in the works of [FS1]–[FS2], [CLM1]–[CLM2], [CalLM1]–[CalLM4], [MilP] and [Sa1]–[Sa3]. We also recall the fundamental vertex-algebraic constructions of the relevant spaces.

4.4.1 Preliminaries

Consider the complex simple Lie algebra

$$\mathfrak{g} = \mathfrak{sl}_2$$

along with its root space decomposition $\mathfrak{g} = \mathbb{C}x_\alpha \oplus \mathbb{C}h \oplus \mathbb{C}x_{-\alpha}$ with the usual brackets given by:

$$[h, x_\alpha] = 2x_\alpha, \quad [h, x_{-\alpha}] = -2x_{-\alpha}, \quad [x_\alpha, x_{-\alpha}] = h.$$

We work with the standard invariant bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ given by:

$$\langle h, x_\alpha \rangle = 0, \quad \langle h, x_{-\alpha} \rangle = 0, \quad \langle h, h \rangle = 1.$$

The form $\langle \cdot, \cdot \rangle$ is non-degenerate on the Cartan subalgebra $\mathfrak{h} = \mathbb{C}h$ of $\mathfrak{g}$ and hence allows us to identify $\mathfrak{h}$ with its dual $\mathfrak{h}^*$. Under this identification, h is identified with α, the positive root corresponding to a root vector x_α. Consider the untwisted (in the vertex-operator-algebraic sense) realization of the affine Lie algebra $A_1^{(1)}$ given by:

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{c},$$

with brackets given by:

$$[a \otimes t^m + r\mathbf{c}, b \otimes t^n + s\mathbf{c}] = [a, b] \otimes t^{m+n} + m\langle a, b \rangle \delta_{m+n,0}\mathbf{c},$$

where $a, b \in \mathfrak{g}$, $m, n \in \mathbb{Z}$ and $r, s \in \mathbb{C}$. We will need the algebras:

$$\widehat{\mathfrak{g}}_{\leq 0} = \mathfrak{g} \otimes \mathbb{C}[t^{-1}] \oplus \mathbb{C}\mathbf{c} \tag{4.4.1}$$

$$\widehat{\mathfrak{g}}_{<0} = \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}] \tag{4.4.2}$$

$$\widehat{\mathfrak{g}}_{\geq 0} = \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{c} \tag{4.4.3}$$

$$\widehat{\mathfrak{g}}_{>0} = \mathfrak{g} \otimes t\mathbb{C}[t] \oplus \mathbb{C}\mathbf{c}. \tag{4.4.4}$$

We let $\mathcal{U}(\cdot)$ denote the universal enveloping algebra.

Given any finite dimensional $\mathfrak{g}$-module U and a scalar k, one can construct an induced $\widehat{\mathfrak{g}}$-module, say $N(U,k)$, called generalized Verma module, as:

$$N(U,k) = \mathcal{U}(\widehat{\mathfrak{g}}) \otimes_{\mathcal{U}(\widehat{\mathfrak{g}}_{\geq 0})} U,$$

where $\widehat{\mathfrak{g}}_{>0}$ acts on U trivially and $\mathbf{c}$ acts by the scalar k. If $\Lambda \in (\mathfrak{h} \oplus \mathbb{C}\mathbf{c})^*$ such that $\Lambda(h) \in \mathbb{N}$ and $\Lambda(c) = k$ and if U is the unique finite dimensional irreducible $\mathfrak{g}$-module with highest weight $\Lambda|_{\mathfrak{h}}$, we denote $N(U, \Lambda(\mathbf{c}))$ by simply $N(\Lambda)$.

Let $h_0 = \mathbf{c} - h \otimes t^0$ and $h_1 = h \otimes t^0$. Given a dominant integral weight $\Lambda \in (\mathfrak{h} \oplus \mathbb{C}\mathbf{c})^*$, i.e., $\Lambda(h_i) \in \mathbb{N}$ for $i = 0, 1$, we let $L(\Lambda)$ denote the unique (irreducible) standard $\widehat{\mathfrak{g}}$-module with highest weight Λ. The level of $L(\Lambda)$ is defined to be $\Lambda(\mathbf{c})$. Let Λ_0, Λ_1 be the fundamental weights, which are defined by $\Lambda_i(h_j) = \delta_{i,j}$.

We will let $W(\widehat{\mathfrak{g}})$ denote the Weyl group of $\widehat{\mathfrak{g}}$ generated by the reflections r_0, r_1 that act as:

$$r_0(\Lambda_0) = -\Lambda_0 + 2\Lambda_1, \qquad r_0(\Lambda_1) = \Lambda_1,$$
$$r_1(\Lambda_1) = -\Lambda_1 + 2\Lambda_0, \qquad r_1(\Lambda_0) = \Lambda_0.$$

Let $\rho \in (\mathfrak{h} \oplus \mathbb{C}\mathbf{c})^*$ such that $\rho(h_i) = 1$ for $i = 0, 1$. For $w \in W(\widehat{\mathfrak{g}})$ and a weight $\lambda \in (\mathfrak{h} \oplus \mathbb{C}\mathbf{c})^*$ we denote

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

4.4.2 The Garland-Lepowsky resolution

Just as before, we restrict our attention exclusively to the affine Lie algebra $\widehat{\mathfrak{g}} = A_1^{(1)} = \widehat{\mathfrak{sl}_2}$.

Henceforth, for $a \in \mathfrak{g}$ and $m \in \mathbb{Z}$, we denote the action of $a \otimes t^m$ on any $\widehat{\mathfrak{g}}$-module by $a(m)$.

Let Λ be a dominant integral weight and let $L(\Lambda)$ be the corresponding standard module. Following Garland-Lepowsky, there exists a natural resolution of $L(\Lambda)$ in terms of generalized Verma modules as follows:

$$\cdots \to E_2 \to E_1 \to E_0 \to L(\Lambda) \to 0.$$

We denote the highest weight vector of $L(\Lambda)$ and E_j by $v_{L(\Lambda)}$ and v_{E_j} respectively.

Let $\Lambda = k\Lambda_0$ for some positive integer k. Garland-Lepowsky's theorem asserts that

$$E_0 = E(k\Lambda_0), \quad E_1 = E(r_0 \cdot k\Lambda_0), \quad E_2 = E(r_0 r_1 \cdot k\Lambda_0), \quad \ldots.$$

It is easily checked that v_{E_1} maps to $x_\alpha(-1)^{k+1} v_{E_0}$. We wish to determine where v_{E_2} maps inside E_1. We adopt the method of Malikov-Feigin-Fuchs [MFF] to first calculate the required singular vectors inside the Verma module $M(r_0 \cdot k\Lambda_0)$ and then take their projection to E_1. This explicit calculation was carried out for level 1 in [P2] and for higher levels an alternate route using Sugawara vectors was used.

Using [MFF], the required singular vector inside $M(r_0 \cdot k\Lambda_0)$ is given by the formula:

$$f_0^{k+3} f_1 f_0^{-k-1} v_{E_1},$$

where $f_0 = x_\alpha \otimes t^{-1}$ and $f_1 = x_{-\alpha} \otimes t^0$ are the standard Kac-Moody generators. For any non-negative integer j, the following facts, which can be checked easily by induction, hold:

$$f_1 f_0^{-j} = f_0^{-j} f_1 + \sum_{a=1}^{j} f_0^{a-j-1} [f_0, f_1] f_0^{-a} \tag{4.4.5}$$

$$[f_0, f_1] f_0^{-j} = f_0^{-j} [f_0, f_1] + j f_0^{-j-1} [f_0, [f_0, f_1]] \tag{4.4.6}$$

For the second relation, note that f_0 and hence f_0^{-1}, commute with $[f_0, [f_0, f_1]]$, due to the Chevalley relation $(\operatorname{ad} f_0)^3 (f_1) = 0$. Hence, we conclude that:

$$\begin{aligned} f_0^{k+3} f_1 f_0^{-k-1} v_{E_1} &= \left(f_0^2 f_1 + (k+1) f_0 [f_0, f_1] + \frac{(k+1)(k+2)}{2} [f_0, [f_0, f_1]] \right) v_{E_1} \\ &= \big(x_\alpha(-1)^2 x_{-\alpha}(0) + (k+1) x_\alpha(-1) h(-1) - (k+1)(k+2) x_\alpha(-2) \big) v_{E_1} \\ &= x_\alpha(-1) \left(x_\alpha(-1) x_{-\alpha}(0) + \frac{1}{2} h(-1) h(0) + x_{-\alpha}(-1) x_\alpha(0) \right) v_{E_1} \\ &\quad - (k+1)(k+2) x_\alpha(-2) v_{E_1} \end{aligned} \tag{4.4.7}$$

Referring to [MP2] and [LL], we know that for each k, $E(k\Lambda_0)$ has a natural structure of a vertex operator algebra, and that each generalized Verma module of level k is a module for this vertex operator algebra. There exists a natural action of the Virasoro algebra on each generalized Verma module of level k, with $L(n)$ acting as:

$$L(n)v$$
$$= \left(\frac{1}{2(k+2)} \sum_{m\in\mathbb{Z}} \left({}^{\circ}_{\circ} x_{\alpha}(m)x_{-\alpha}(n-m) {}^{\circ}_{\circ} + {}^{\circ}_{\circ} x_{-\alpha}(m)x_{\alpha}(n-m) {}^{\circ}_{\circ} + {}^{\circ}_{\circ} \frac{1}{2} h(m)h(n-m) {}^{\circ}_{\circ} \right) \right) v,$$

where

$${}^{\circ}_{\circ} a(m)b(n) {}^{\circ}_{\circ} = \begin{cases} a(m)b(n) & \text{if } m < 0 \\ b(n)a(m) & \text{if } m \geq 0. \end{cases}$$

(see [LL] (3.8.4)). This is the well-known Segal-Sugawara construction of the Virasoro algebra action.

We deduce immediately that

$$\begin{aligned} f_0^{k+3} f_1 f_0^{-k-1} v_{E_1} &= x_{\alpha}(-1) \left(x_{\alpha}(-1)x_{-\alpha}(0) + \frac{1}{2} h(-1)h(0) + x_{-\alpha}(-1)x_{\alpha}(0) \right) v_{E_1} \\ &\quad - (k+1)(k+2)x_{\alpha}(-2)v_{E_1} \\ &= (k+2) \left(x_{\alpha}(-1)L(-1) - (k+1)x_{\alpha}(-2) \right) v_{E_1}. \end{aligned} \tag{4.4.8}$$

4.4.3 Principal subspaces

In this subsection we recall the definition of principal subspaces of standard $\widehat{\mathfrak{g}}$-modules, and we review the main theorems of [CalLM1] and [CalLM2].

In order to define the principal subspaces, we will need the nilpotent subalgebra generated by the positive root spaces, i.e., $\mathfrak{n} = \mathbb{C}x_{\alpha}$ of $\mathfrak{g}$. Note that $\mathfrak{n}$ is an abelian Lie algebra, and that this is special to $A_1^{(1)}$. Consider also its affinization:

$$\bar{\mathfrak{n}} = \mathfrak{n} \otimes \mathbb{C}[t, t^{-1}]$$

contained inside $\widehat{\mathfrak{g}}$. Note that $\bar{\mathfrak{n}}$ is closed under brackets since $\langle x_{\alpha}, x_{\alpha} \rangle = 0$. We will consider the following subalgebra of $\bar{\mathfrak{n}}$:

$$\bar{\mathfrak{n}}_{-} = \mathfrak{n} \otimes t^{-1}\mathbb{C}[t, t^{-1}].$$

The universal enveloping algebra of $\bar{\mathfrak{n}}_{-}$,

$$\mathcal{A} = \mathcal{U}(\bar{\mathfrak{n}}_{-})$$

is a commutative associative algebra. For a standard module $L(\Lambda)$ of $\widehat{\mathfrak{g}}$ generated by v_Λ, define the principal subspace W_Λ as:

$$W_\Lambda = \mathcal{U}(\bar{\mathfrak{n}})v_\Lambda = \mathcal{U}(\bar{\mathfrak{n}}_-)v_\Lambda,$$

where the second equality follows by using the Poincaré-Birkhoff-Witt theorem and by noting that v_Λ is a highest weight vector. As in [CalLM1], we define the surjective maps:

$$F_\Lambda : \mathcal{U}(\widehat{\mathfrak{g}}) \longrightarrow L(\Lambda) \tag{4.4.9}$$

$$a \longmapsto a \cdot v_\Lambda \tag{4.4.10}$$

and

$$f_\Lambda = F_\Lambda|_{\mathcal{U}(\bar{\mathfrak{n}}_-)} : \mathcal{U}(\bar{\mathfrak{n}}_-) \longrightarrow W_\Lambda \tag{4.4.11}$$

$$a \longmapsto a \cdot v_\Lambda. \tag{4.4.12}$$

Now we take $\Lambda = \Lambda_0$.

Recall the generalized Verma modules $E_0, E_1, \dots$ from the Garland-Lepowsky resolution of $L(\Lambda_0)$. For the generalized Verma module E_0, define the principal subspace similarly, i.e.,

$$W_{\Lambda_0}^{E_0} = \mathcal{U}(\bar{\mathfrak{n}})v_{E_0} = \mathcal{U}(\bar{\mathfrak{n}}_-)v_{E_0}.$$

Note that $W_{\Lambda_0}^{E_0}$ is a free module over $\mathcal{U}(\bar{\mathfrak{n}}_-)$ generated by v_{E_0}. Consider the natural surjective maps

$$\Pi_{\Lambda_0} : E_0 \longrightarrow L(\Lambda_0) \tag{4.4.13}$$

$$a \cdot v_{E_0} \longmapsto a \cdot v_{\Lambda_0} \tag{4.4.14}$$

$$\pi_{\Lambda_0} = \Pi_\Lambda|_{W_{\Lambda_0}^{E_0}}. \tag{4.4.15}$$

Determining the kernel of f_{Λ_0} is equivalent to determining the kernel of π_{Λ_0} (see Theorem 2.2 of [CalLM1]). It is well known that E_0 has a natural vertex operator algebra structure and that $W_{\Lambda_0}^{E_0}$ has a vertex sub-algebra structure. Note that $W_{\Lambda_0}^{E_0}$ does not contain the conformal vector, but admits the natural action of the Virasoro algebra.

The main theorem of [CalLM1] could be reformulated as follows:

Theorem 4.4.1. *The kernel of π_{Λ_0} is generated over $\mathcal{U}(\bar{\mathfrak{n}}_-)$ by the $L(-1)$-descendants of the singular vector $x_\alpha(-1)^2 v_{E_0}$. That is,*

$$Ker\ \pi_0 = \sum_{t\geq 0} \mathcal{U}(\bar{\mathfrak{n}}_-) L(-1)^t \left(x_\alpha(-1)^2 v_{E_0}\right).$$

This theorem could be further reformulated as:

Theorem 4.4.2. *The kernel of π_{Λ_0} is the ideal of the vertex algebra $W_{\Lambda_0}^{E_0}$ generated by the singular vector $x_\alpha(-1)^2 v_{E_0}$.*

In the proofs of the theorems above, analogue of the map σ enters. This map occurs as the inverse of a certain constant factor (e^α) of a vertex operator.

The singular vector $x_\alpha(-1)^2 v_{E_0}$ is precisely the generator of the kernel of the map Π_{Λ_0}. So, essentially, the theorems above assert that the kernel of π_{Λ_0} is generated by the "obvious" elements.

In exactly the same spirit, our main theorem asserts that the second homology (as opposed to the second kernel) is generated by the $L(-1)$-descendants of the "next" singular vector in the Garland-Lepowsky resolution, namely,

$$3\left(x_\alpha(-1)L(-1) - 2x_\alpha(-2)\right) v_{E_1}. \tag{4.4.16}$$

The symbol ξ_{-2} corresponds to v_{E_1} and $L(-1)v_{E_1}$ corresponds to $2\xi_{-3}$. Hence, the singular vector $\left(x_\alpha(-1)L(-1) - 2x_\alpha(-2)\right) v_{E_1}$ corresponds precisely to μ_{-3}. It is seen easily that L_{-1} introduced before mimics the action of $L(-1)$. For $t > 4$, μ_{-t}s could be obtained upto a non-zero scalar multiple from μ_{-3} by repeatedly applying L_{-1}.

It should be noted that in the works [MP1], [MP2], [P1], [P2], [Si] this "next" singular vector, and its analogues for higher ranks and levels, play a crucial role in determination of generators of relations for the annihilating fields of standard modules.

Question 4.4.3. *It is now natural to ask how far the vertex-operator-algebraic methods can be pushed to give insights about the higher homology groups and analogues of the Koszul complex related to higher levels and ranks.*

4.5 Relations to the stable Khovanov homology of the torus knots

For the purposes of this section, consider the (equivalent) Koszul model written in the language of exterior products of modules. That is, the x_is are the commutating variables, but ξ_{-i}s anti-commute, and

$$\xi_{i_1,\dots,i_r} = \xi_{i_1} \wedge \cdots \wedge \xi_{i_r}.$$

In [GOR], the authors consider a conjectural description of a certain well-defined limit (cf. [St]) of the homology groups $\mathrm{Kh}(T(n,m))$ as m tends to infinity; this limit is denoted as $\mathrm{Kh}(T(n,\infty))$.

In our notation, Conjecture 1.1 of [GOR] reads as follows:

Conjecture 4.5.1 ([GOR]). *Consider the polynomial ring in variables $x_{-1},\dots,x_{-n}$. The unreduced stable Khovanov homology $Kh(T(n,\infty))$ is dual to the homology of the Koszul complex determined by the (non-regular) sequence of elements r_{-j} for $j = 2,\dots,n+1$ where the r_{-j} are defined as in* (4.1.2) *The homology of this Koszul complex is denoted by $Kh_{alg}(n,\infty)$.*

Remark 4.5.2. Note that we have left the base field unspecified. It is natural to consider $\mathbb{F}_2$ and $\mathbb{Q}$ from the viewpoint of Khovanov homology. For vertex-operator-algebraic contexts, it is best to work with $\mathbb{C}$.

It was noted in [GOR] that this Koszul complex is related to the principal subspaces and using the ideas of [FS1]-[FS2] and [LP], a conjectural description of the Poincaré series of the homology (over rationals) of the Koszul complex above was also derived. We remark that the double-grading used in [GOR] is different than the one used in this paper and that the gradings are not compatible.

The homology $\mathrm{Kh}_{alg}(n,\infty)$ has a natural structure of a graded algebra. Conjecture 1.6 of [GOR] describes a presentation of this algebra. In particular it states the following:

Conjecture 4.5.3 ([GOR]). *As an algebra over $\mathbb{Q}$, $Kh_{alg}(n,\infty)$ is generated by the elements $x_{-1},\dots,x_{-n}$ and $\mu_{-4},\dots,\mu_{-n-2}$, where μ_{-j} are defined as in* (4.2.7).

The homomorphism σ that we have used also finds an analogue in [GOR]: namely, it lets one pass from the unreduced to the reduced stable Khovanov homology; see Section 5 of [GOR].

For principal subspaces associated to the higher level vacuum modules of $\widehat{\mathfrak{sl}_2}$, it is conjectured in [GorL] that the corresponding Koszul models capture the sl_N-Khovanov-Rozansky homology of $T(\infty, \infty)$.

We are currently working towards adapting our methods to the "finite" and the higher level settings.

Remark 4.5.4. Rogers-Ramanujan-type identities have several connections to knot theory. As another example, we mention [AD], where it is explained how the second Rogers-Ramanujan identity and more generally, some Andrews-Gordon identities arise by considering the "tail" of the colored Jones polynomial of the (negative) $(2, 2k+1)$-torus knot.

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