COMPUTATIONAL ADVANCES IN RADO NUMBERS

by

KELLEN JOHN MYERS

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ABSTRACT OF THE DISSERTATION

Computational Advances in Rado Numbers

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Dissertation Director: Doron Zeilberger

In this dissertation, we present new methods in the computation of Rado numbers. These methods are applied to several families of equations. The Rado number of an equation is a Ramsey-theoretic quantity associated to the equation. For any particular equation $\mathcal{E}$, the Rado number $\mathrm{R}_r(\mathcal{E})$ is the smallest N such that any r-coloring $\chi : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, r\}$ must induce a monochromatic solution to $\mathcal{E}$. We will lay out the history of this field and provide some structure as context for new results. Then we will discuss the new methods and computational tools that provide the foundation of the thesis.

The 2-color Rado numbers $\mathrm{R}_2(2x+2y+kz = 3w)$ and $\mathrm{R}_2(kx+(k+1)y = (k+2)z)$ are computed for small values of the parameter k. The 2-color off-diagonal Rado numbers $\mathrm{R}_2(x + ay = z; x + by = z)$ are provided for $1 \leq a, b \leq 20$. Likewise, 3-color off-diagonal Rado numbers $\mathrm{R}_3(x + y = az; x + y = bz; x + y = cz)$ are computed for $1 \leq a, b, c \leq 5$. We confirm the long-standing conjecture that the 3-color generalized Schur numbers $\mathrm{R}_3(x_1 + \cdots + x_{m-1} = x_m)$ are $m^3 - m^2 - m - 1$ for $m = 7, 8, 9, 10$ (effectively doubling the empirical evidence for the conjecture) and provide the related Rado numbers $\mathrm{R}_3(x_1 + \cdots + x_{m-2} + kx_{m-1} = x_m)$ for certain (k, m) values. We prove a lower bound for the r-color non-homogeneous Schur numbers: $\mathrm{R}_r(x + y + c = z) \geq \frac{3^r-1}{2}(c + 1)$ for $c \geq 0$. We also compute the precise values for $r = 4$ and $-20 \leq c \leq 7$

and generalize this bound for $m \geq 3$ variables.

We provide the 2-color Rado numbers for $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$ and a few other equations involving reciprocals. We also construct a coloring proving $\mathrm{R}_2\big(x^2 + y^2 = z^2\big) > 6500$. (It is not known whether this Rado number is finite.) We compute the 2- and 3-color Rado numbers for other sums-of-squares equations, $\sum_{i=1}^{a} x_i^2 = \sum_{i=1}^{b} y_i^2$, and we prove a universal upper bound for $a \leq b \leq ca$ for a constant c between 1 and 2 (different values of c give different upper bounds). We follow this with Rado numbers for other assorted families of quadratic equations. We also present quantitative analogues of Hindman's theorem, which guarantees monochromatic solutions to systems like $\{x+y+z = w,\ x \cdot y \cdot z = v\}$.

We conclude by suggesting a number of conjectures, extensions, and generalizations of these results for future work.

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Dedication

to Mia, and to my father

in memory of Clyde & Oliver, and of Tristan Leggett

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Chapter 1

Introduction

This dissertation will present a number of important new computational results in Diophantine Ramsey theory, the study of the Ramsey-theoretic properties of equations with integer solutions.

To begin, we will discuss the historical development of Ramsey theory and its connection to Diophantine equations, with special attention to Rado numbers.

Ramsey theory of any variety will, almost invariably, include a coloring function which we will call χ. If S is some structure, then S (or some component of S) is colored with r colors according to:

$$\chi : S \to [r]$$

where $[r]$ is our notation for $\{1, 2, 3, \ldots, r\}$.

The major tenet of Ramsey theory is that for certain structures S that are "regular" in some way, and sufficiently large (typically meaning $|S|$ is large enough), there is some sub-structure $S' \subseteq S$ that must be constant according to χ (i.e. for some particular "color" i, and for all $s' \in S'$, $\chi(s') = i$). We call that substructure S' "monochromatic."

Quantitative aspects of Ramsey theory consider how large $|S|$ must be in order to have this monochromatic sub-structure, which is usually dependent on $|S'|$ and r, i.e. $|S| = f(|S'|, r)$. In some cases, S and S' are the same type of structure (e.g. in Ramsey's theorem, they are complete graphs), while in other cases (in particular, the focus of this dissertation) this is not true.

Many narratives begin with Ramsey's theorem, proved in 1926, but the perspective of this dissertation will represent a reversal of roles. We will regard Ramsey's theorem as a graph-theoretic diversion and focus on Diophantine topics, beginning our narrative with Issai Schur.

Issai Schur was born in 1875 the son of a Jewish merchant. After a childhood in Mogilev and Liepāja, now in Belarus and Latvia respectively, he studied at the University of Berlin under the supervision of Ferdinand Georg Frobenius. Although the majority of his work is in the areas of number theory and algebra, we will instead focus on the seeds he planted that would become a part of Ramsey theory.

Schur continued his scholarly work at the University of Berlin, with a brief tenure in Bonn. He married another Russian-Jewish immigrant. He remained in Berlin until the rise of the Nazi Party in Germany. He was removed from his position in 1935 and fled the country. Sadly, he died in 1941 in Tel Aviv, having been displaced from his home in Germany, his life's work in mathematics cut short.

In [Soi00], Soifer discusses the effects of the second World War on mathematicians, particularly mathematicians in this area of study, and laments that Jews in Germany around Schur's age (58 in 1933) recognized too late the severity of Nazi persecution. Soifer notes that older Jews, having lived so long in Germany at times when anti-semitism was not so rampant, were slower to perceive the shifting politics and policies towards Jews. Schur was one of many who were reluctant to leave their homes, jobs, friends, and families for the relative safety of England, the USA, or elsewhere.

In fact, Schur himself was the very last Jewish professor to be dismissed from the University of Berlin, having been exempted from the initial mass-dismissal of Jews. Schur was so popular that his exemption from the initial dismissal in 1933 was widely supported, even by the now-infamous Bieberbach. Schur declined many offers to leave Germany for positions in the USA and in England, while a great many of his younger Jewish colleagues had already left Germany. He was eventually removed after two years in defiance of the 1933 general directives that removed Jews from academic and governmental positions. He fled to Palestine (now Israel) and died soon thereafter. This story is perhaps a great tragedy, in that Schur's love for his home in Berlin and the support of his friends and colleagues there kept him too long in a country in which the ruling party was slowly, but steadily, turning the nation against him and his people.

In 1916, Schur proved the following theorem. This is, arguably, the first real result in Ramsey theory except perhaps Hilbert's so-called "cube lemma" [Hil92], which was overlooked until much later. Schur's theorem was, instead, the starting point for the work of Rado and others in one area of Ramsey theory. The later work of Ramsey himself originated an area of study that would converge with Schur's and Rado's work to become the larger field now called Ramsey theory (which has since then expanded even further).

Theorem 1.1 (Schur's Theorem). *For any r-coloring of $\mathbb{Z}^+$, that is $\chi : \mathbb{Z}^+ \to [r]$, there is a triple (x, y, z) such that $x + y = z$ and $\chi(x) = \chi(y) = \chi(z)$.*

The solution (x, y, z) in this theorem is called "monochromatic." One important consequence of this theorem is that to each value of r we may associate the smallest integer $S(r)$ for which the following modification of Schur's theorem holds:

Theorem 1.1 (Schur's Theorem, finite version). *For each $r \in \mathbb{Z}^+$ there is an integer $S(r)$ such that for any r-coloring of $[S(r)]$, that is $\chi : [S(r)] \to [r]$, there is a triple (x, y, z) such that $x + y = z$ and $\chi(x) = \chi(y) = \chi(z)$.*

This can be proved by modifying a proof of the first version of Schur's theorem or derived as a consequence of the original theorem and the compactness principle (which can be applied to many other Ramsey-theoretic statements of the infinite variety to derive finite versions).

Schur proved this theorem in the course of an attack on mathematics' most famous problem, Fermat's Last Theorem. He used Theorem 1.1 to prove the following:

Theorem 1.2. *For $n \in \mathbb{Z}^+$ there is a prime q such that for all primes $p \geq q$, $x^n + y^n \equiv z^n \pmod{p}$ has a nontrivial solution (x, y, z).*

Here, "nontrivial" means that none of the integers x, y, z is divisible by p.

Richard Rado, a student of Schur, would find greater significance in what we now call Schur's theorem, and Rado's work is now a seminal part of an area that may be called Diophantine Ramsey theory. Rado's work generalizes Schur's theorem substantially and describes in greater detail the principles and structures at work in this theory. We will move from Rado's work all the way to current work in the next two chapters, developing the appropriate notation, language, and theory along the way.

Richard Rado, like Schur, was a Jew in Berlin in 1933, but was younger than Schur. Like many of his generation, he fled Nazi Germany to England immediately, where he worked under the supervision of G. H. Hardy. Not only can we view Rado's work (at least in regards to Diophantine Ramsey theory) as a direct extension of the work of Schur, but throughout his long life (1906-1989) we might say that Rado continued to contribute to mathematics in the spirit of his advisor who was lost to us in 1941.

It is important here to remark that Brauer and van der Waerden also contributed to this developing theory around this time. Alfred Brauer was another student of Schur. He generalized Schur's theorem in a different way in [Bra28], which [Soi00] notes is a result that often goes unmentioned. At the same time, B. L. van der Waerden devised his well-known result on arithmetic progressions [vdW27]:

Theorem 1.3. *For any r-coloring χ of $\mathbb{Z}^+$, and any k, there are $a, d \in \mathbb{Z}^+$ such that $\chi(a) = \chi(a+d) = \chi(a+2d) = \cdots = \chi(a+(k-1)d)$, which is to say there exists a monochromatic arithmetic progression of length k.*

Although the theorems of Schur, Rado, and van der Waerden are all well known and widely cited, the historical details in the preceding narrative are drawn primarily from [Soi00]. An excellent, and more detailed, account of the history can be found therein, while [Soi11] contains a brief overview of that same narrative.

Chapter 2

Ramsey Theory and Richard Rado

The next two chapters will summarize the existing body of work in the area of Diophantine Ramsey theory, particularly those results that discover, bound, or otherwise determine so-called *Rado numbers*, which we will define below.

2.1 Notation, Definitions, & Conventions

Before we continue, we should discuss notation. We will use symbols x, y, z, w with or without subscripts to indicate variables in our Diophantine equations. Unspecified (integer) coefficients will be denoted a, b, c, d, k, ℓ with or without subscripts. Unless otherwise noted (e.g. in Theorem 2.9), we assume coefficients are positive, so that $ax + by + cz = dw$ is not the same as $ax + by = cz + dw$.

All references to an "equation" (be it one in particular, or an arbitrary equation) or to "$\mathcal{E}$" should be taken to indicate a Diophantine equation with integer coefficients, unless otherwise stated. The reader may assume that Diophantine equations are equations of polynomials with integer coefficients, although statements in this dissertation general enough to apply to all polynomials are also generally applicable to other Diophantine equations as well (e.g. $2^x = y + z$).

The parameter m, when necessary, will indicate the number of variables in the equation, in which case we will abide by the convention that those variables are always x_1 through x_m, so that m is always the exact number of variables (so that $\mathcal{E}$ does not have x_1 through x_m but also z). In some cases, we will enumerate the variables differently, but m will generally refer to the total number of variables (while a, b, k or some other parameter might enumerate a subset of the variables).

In some sections we will define a particular parametrized family of equations, something like $\{\mathcal{E}_k : k \in \mathbb{Z}^+\}$. In these cases, the scope of this definition is generally that particular section.

The parameter r will denote the number of colors, wherever necessary.

We will define some of our key terms as follows:

Definition 2.1. *For a positive integer r, the set $[r]$ is the set $\{1, 2, 3, \ldots, r\}$.*

Definition 2.2. *A function χ is an r-coloring of S if $\chi : S \to [r]$.*

Definition 2.3. *A set S' is said to be monochromatic under χ if $|\chi(S')| = 1$.*

One says simply "monochromatic" if χ is understood in context, which it usually will be.

Definition 2.4. *The r-color Rado number for the equation $\mathcal{E}$ is the least integer N such that any r-coloring of $[N]$ will contain a monochromatic solution to $\mathcal{E}$. We denote this $\mathrm{R}_r\big(\mathcal{E}\big) = N$.*

We adopt the convention that if a particular equation $\mathcal{E}$ does not necessarily admit monochromatic solutions for all r-colorings, then $\mathrm{R}_r\big(\mathcal{E}\big) = \infty$ (in this case, there is no such N, and $\min \emptyset = \infty$). We also make the following complementary definitions:

Definition 2.5. *For an equation $\mathcal{E}$, we say $\mathcal{E}$ is r-regular if $\mathrm{R}_r\big(\mathcal{E}\big) < \infty$. The greatest r for which $\mathcal{E}$ is r-regular is the degree of regularity of $\mathcal{E}$ and is denoted $dor(\mathcal{E})$. If $\mathcal{E}$ is r-regular for all r, we say $\mathcal{E}$ is simply regular and $dor(\mathcal{E}) = \infty$.*

The degree of regularity is well-defined due to the following lemma:

Lemma 2.6. *For any equation $\mathcal{E}$, $\mathrm{R}_r\big(\mathcal{E}\big) \leq \mathrm{R}_{r+1}\big(\mathcal{E}\big)$, and thus if $\mathcal{E}$ is $(r+1)$-regular it is also r-regular.*

This lemma should require no proof, but we should note that it is decidedly not the case that the strict inequality $\mathrm{R}_r\big(\mathcal{E}\big) < \mathrm{R}_{r+1}\big(\mathcal{E}\big)$ should hold (not in general). If we allow a somewhat trivial example, it is clear that $\mathrm{R}_r\big(x = y\big) = 1$ for all r. More about degrees of regularity can be found in [BLM96].

2.2 Ramsey's Theorem

Regardless of the fact that our narrative of Ramsey theory is meant to emphasize Diophantine Ramsey theory, the story would not be complete without Ramsey's theorem. It is an important piece of the framework in this general area, provides some important context, and will be used subsequently to prove Schur's theorem.

We state Ramsey's theorem for graphs in a general way:

Theorem 2.7 (Ramsey's theorem). *For any integer r and any integers $k_1, k_2, \ldots, k_r$, there is an integer $N = R(k_1, k_2, \ldots, k_r)$ such that for any r-coloring of the edges of a complete graph on N vertices, that is $\chi : E(K_N) \to [r]$, there is a set W of vertices such that the restriction $\chi|_{E(W)}$ takes the constant value i and $|W| = k_i$. In other words, there is a monochromatic k_i-clique of color i for some i.*

Like the finite version of Schur's Theorem (1.1), this statement defines a particular quantity which is in this case known as a Ramsey number. A classic result is that $R(3,3) = 6$, which could be proved easily even by exhaustion.

It is also useful to note that this function takes on initial values at $R(2,2,\ldots,2) = 2$ or perhaps $R(1,1,\ldots,1) = 1$, depending on how trivial you allow values of k_i to become. If any k_i is 2, one might simply eliminate this k_i, i.e.

$$R(k_1, k_2, \ldots, k_{i-1}, 2, k_{i+1}, \ldots, k_r) = R(k_1, k_2, \ldots, k_{i-1}, k_{i+1}, \ldots, k_r).$$

If any k_i is 1, the Ramsey number must be 1, because any single vertex is a K_1 with vacuously monochromatic edges:

$$R(k_1, k_2, \ldots, k_{i-1}, 1, k_{i+1}, \ldots, k_r) = 1.$$

And we may observe a final triviality, that $R(n) = n$.

These boundary cases (and/or the case $R(3,3) = 6$) prove the finiteness of this quantity for other choices of $k_1, \ldots, k_r$, once one has established the following relations:

$$\begin{aligned} R(k_1, k_2, \ldots, k_{r-2}, k_{r-1}, k_r) &\leq R\big(k_1, k_2, \ldots, k_{r-2}, R(k_{r-1}, k_r)\big), \\ R(k_1, k_2) &\leq R(k_1 - 1, k_2) + R(k_1, k_2 - 1). \end{aligned}$$

Proof for these bounds will be omitted, but may be found readily in standard texts and resources like [GRS90]. It may be important to note that these bounds are extremely generous and represent a very gross overestimate of the upper bound in virtually all cases (this is a common theme in Ramsey Theory).

These bounds, which are enough to establish Ramsey's theorem, are generally proved using the pigeon-hole principle, which is itself a sort of rudimentary Ramsey-theoretic result that predates all others.

Theorem 2.8 (The Pigeon-Hole Principle)**.** *If $f : A \to B$, then there is $b \in B$ such that $|f^{-1}(b)| \geq \left\lceil \frac{|A|}{|B|} \right\rceil$.*

Proof. Assume not. Then for all $b \in B$, we have $|f^{-1}(b)| < \left\lceil \frac{|A|}{|B|} \right\rceil$. This implies that $|f^{-1}(b)| < \frac{|A|}{|B|}$, since the two possible maximum values $|f^{-1}(b)|$ could take under this assumption would be $\left\lfloor \frac{|A|}{|B|} \right\rfloor$ if this fraction is not an integer, or simply $\frac{|A|}{|B|} - 1$ if it is. And so we conclude:

$$|A| = \sum_{b \in B} |f^{-1}(B)| < |B| \frac{|A|}{|B|} = |A|$$

which is a contradiction. □

This is a Ramsey-theoretic type statement if we consider $\chi : S \to [r]$. If $|S| \geq kr$, then there must be a monochromatic $S' \subseteq S$ such that $|S'| \geq k$. At its heart, this principle asserts that among some data, at least one datum is at least as large as the average. We could say the Pigeon-Hole Principle is at the heart of Ramsey theory.

A large body of work exists exploring the area of Ramsey theory in graphs, and again [Soi00] contains a historically-motivated, yet thorough and mathematically rich, narrative of this area of Ramsey theory. We will not discuss any of the graph-theoretic results besides Ramsey's Theorem (Theorem 2.7).

2.3 Schur's Theorem

We now revisit Schur's theorem, and in the tradition of more conventional narratives, prove the theorem by a very elegant application of Ramsey's theorem.

Theorem 1.1 (Schur's Theorem)**.** *For any $r \in \mathbb{Z}^+$, there is an $S(r) \in \mathbb{Z}^+$ such that for all r-colorings $\chi : [S(r)] \to [r]$ there is a solution (x, y, z) to the equation $x + y = z$ such that $\chi(x) = \chi(y) = \chi(z)$.*

Proof. Let $N = R(3, 3, \ldots, 3)$ be the r-color Ramsey number for triangles. Consider any coloring $\chi : [N] \to [r]$ and induce an r-coloring χ^* on the edge set of a complete graph of size N, that is $\chi^* : E(K_N) \to [r]$, where the color of edge ij is defined to be $\chi^*(ij) = \chi(|i - j|)$.

By Ramsey's theorem, there must exist a monochromatic K_3 (i.e. triangle) in this graph. In that case, we have three edges ij, ik, jk such that:

$$\chi^*(ij) = \chi^*(ik) = \chi^*(jk).$$

Without loss of generality, we may assume $i > j > k$, so that we obtain:

$$\chi(i - j) = \chi(i - k) = \chi(j - k),$$

and $(x, y, z) = (i - j, j - k, i - k)$ is a solution to the equation $x + y = z$. $\square$

In particular, this theorem bounds the Schur number $S(r)$ by the Ramsey number $R(3, 3, \ldots, 3)$, and these two numbers are currently known to be bounded (asymptotically) by:

$$(c_1)^r \leq S(r) \leq R(\underbrace{3, 3, \ldots, 3}_{r}) \leq c_2 r!,$$

where c_1 and c_2 are constants, $c_2 \approx 2.6752$ [Wan90], and $c_1 \approx 3.17177$ [Exo94] [CG83].

At the present time, this seems to be the best asymptotic upper bound for $S(r)$.

It is easy to see that $S(2) = 5$, and one might note trivially that $S(1) = 2$. It is also straightforward to prove that $S(3) = 14$. Exact values and bounds have been proved for $S(r)$ for a few other small values of r and are provided in Table 2.1.

		$S(4)$	$=$	45
161	$\leq$	$S(5)$	$\leq$	316
537	$\leq$	$S(6)$		
1681	$\leq$	$S(7)$		

Table 2.1: Bounds for $S(4)$ through $S(7)$

The first line of the table is from [BG65], the second from [Fre79] and [Exo94], and the last two from [FS00]. A recent PhD thesis by M. I. Sanz is cited in [CMD13] as having proved $S(5) \leq 305$, but at this time we have been unable to obtain the primary text to verify this new bound.

Despite significant advances towards computing numerous other Diophantine-Ramsey-theoretic quantities, we still find ourselves unable to improve the bounds for $S(5)$ due to the steep rise in complexity in searching colorings with greater numbers of colors. However, we are optimistic that computational methods, theoretical advances, and the ever-increasing power of computing will enable the computation of $S(5)$ in the near future.

2.4 Rado's Theorem(s)

In 1933, Richard Rado extended the work of Schur, his advisor, to settle a similar question for not just $x + y = z$ but for any homogeneous linear equation – and for systems thereof.

Consider the homogeneous linear equation with integer coefficients:

$$\sum_{i=1}^{m} a_i x_i = 0.$$

Here we temporarily defy our convention, since this representation of an equation could include negative coefficients. In [Rad33], we find the following two important theorems:

Theorem 2.9 (Rado's First Theorem)**.** *A linear equation (as above) is regular if and only if there is $J \subseteq \{1, 2, \ldots n\}$ such that $\sum_{i \in J} a_i = 0$.*

Theorem 2.10 (Rado's Second Theorem)**.** *A linear equation (as above, with $m \geq 3$) is 2-regular if and only there are i_1, i_2 such that $a_{i_1} > 0$ and $a_{i_2} < 0$.*

The condition in the second theorem is quite weak; it only rules out equations that would have no solutions over the positive integers. Requiring $m \geq 3$ rules out equations like $3x = 2y$, and it is easy to see that:

Lemma 2.11. *The r-color Rado number $\mathrm{R}_r\big(ax = by\big)$ is 1 if $a = b$ and ∞ otherwise.*

Proof. If $a = b$, then $(x, y) = (1, 1)$ is a monochromatic solution. Otherwise, we may assume $\gcd(a, b) = 1$ and take a prime p dividing a so that $a = p^k a'$, where p does not divide a' or b. For any integer n, write $n = p^\ell n'$, where p does not divide n'. Define $\chi(n)$ to be the parity of the quotient of ℓ when divided by k.

If (x, y) is a solution to this equation, we could take $x = p^{\ell_1} x'$ and $y = p^{\ell_2} y'$, where p does not divide x' or y'. Then substituting, we would have: $(p^k a')(p^{\ell_1} x') = b(p^{\ell_2} y')$, and thus $\ell_1 + k = \ell_2$, which implies $\chi(x) \neq \chi(y)$. $\square$

Rado also includes the analogous criterion for systems of homogeneous linear equations – a subset of the columns must sum to zero, but the remaining columns must also meet some linear conditions. We will not need to discuss this further, but one may consult [GRS90] or [LR03] for details.

Chapter 3

Ramsey Quantities & Computational Ramsey Theory

Theorem 2.9 is usually called “Rado’s Theorem,” and this theorem has the flavor of a much deeper result. However, Theorem 2.10 is actually more practical for computational work because it shows that any linear homogeneous equation (more or less) is 2-regular. Although some such equations may have 2-color Rado numbers too big to compute practically, it is at least feasible to compute these Rado numbers in the sense that none of them is infinite.

Definition 2.5 may seem superfluous, especially when most Rado numbers known are for 2 colors, but the distinction between r- and $(r+1)$-regular equations is of great interest. Knowing precise conditions for only the boundary cases $r = 2$ and $r = \infty$, and then only for linear equations, we are left to wonder whether we can formulate such conditions even for 3-regularity.

In 2009, Alexeev & Tsimerman [AT10] gave an example of an $\mathcal{E}_r$ such that $dor(\mathcal{E}_r) = r$, which settles the question clearly: there are equations of each degree of regularity. Before this, there were already some examples: it was known that $dor(x+2y = 4z) = 2$ [FR] and that $dor(x + y = 3z) = 3$ [FGR86].

Rado conjectured that for each k, there is an m such that if $\mathcal{E}$ contains m variables or more and is r regular, then it is r' regular for all $r' > r$. Fox and Kleitman in [FK06] prove this conjecture for $m = 3$, giving $r = 24$ as sufficient.

The unfortunate trouble with 3-regularity and general r-regularity is that for any fixed $\mathcal{E}$, the Rado number increases with r, and so too does the likelihood that this $\mathcal{E}$ will no longer be r-regular at all. Computations that run indefinitely might be an indication that the Rado number is very large (perhaps computable with faster computers), but it could just as easily be the case that the Rado number is infinite.

In this chapter we will discuss a number of existing results, most (but not all) of which will be for 2 colors. We will also lay out a number of techniques that will tackle the computation of Rado numbers. Although some of these will be inspired by the existing techniques in the literature, they will not necessarily be limited to 2-colorings. We will discuss how these methods have (and have not) been effective in the past before we move on in later chapters to discuss new results.

3.1 Existing Methods & Results

One of the earliest results in the determination of Rado numbers comes from Beutelspacher & Brestovansky [BB82]. This particular theorem is of special interest because it is one of very few (nontrivial) results that speaks to a relationship between two (or more) Rado numbers with different r values:

Theorem 3.1. *The r-color Rado numbers for $x_1 + \cdots + x_{m-1} = x_m$ have the following relationship:*

$$\mathrm{R}_r\big(x_1 + x_2 + \cdots + x_{m-1} = x_m\big) \geq m\mathrm{R}_{r-1}\big(x_1 + x_2 + \cdots + x_{m-1} = x_m\big) - 1.$$

These are defined especially as:

Definition 3.2. *A <u>generalized Schur number</u> is* $\mathrm{R}_r\big(x_1 + x_2 + \cdots + x_{m-1} = x_m\big)$*.*

These numbers are sometimes denoted $S_m(r)$. However, we will avoid this notation because we use the subscript for r in $\mathrm{R}_r\big(\mathcal{E}\big)$, rather than m.

Generalized Schur numbers exist, as a corollary of Rado's Theorem (Theorem 2.9), and they are perhaps the Rado numbers that are of greatest interest.

Theorem 3.1 is proved by constructing r-colorings recursively, giving this recursive lower bound. Applying induction provides the following bound:

Theorem 3.3. $\mathrm{R}_r\big(x_1 + x_2 + \cdots + x_{m-1} = x_m\big) \geq m^r - \big(m^{r-1} + m^{r-2} + \cdots + m + 1\big)$

This bound is known to be tight in the case $r = 2$, as follows:

Theorem 3.4 (Generalized 2-color Schur numbers)**.** $\mathrm{R}_2\big(x_1 + x_2 + \cdots + x_{m-1} = x_m\big) = m^2 - m - 1$

The fact that this result was one of the first in this area, and that it is not tight for larger values of r, should indicate that the generalization of Schur numbers to different equations is a qualitatively easier step than the move to greater numbers of colors.

These are called generalized Schur numbers not just because the equations look similar to $x + y = z$ in some way, but also because one can prove the existence of these numbers using the same proof as we did for Theorem 1.1, but with K_m instead of K_3, obtaining monochromatic solutions of the form:

$$(x_1, x_2, \ldots, x_{m-1}, x_m) = (i_1 - i_2, i_2 - i_3, \ldots, i_{m-2} - i_{m-1}, i_{m-1} - i_m, i_1 - i_m)$$

A recent survey of generalized Schur numbers can be found in [Ahm]. We will discuss some new results for generalized Schur numbers in Section 4.5.

However, the bound from [BB82] is not tight for $r = 4$ as evidenced by $S(4) = 45$, and bounds like $S(5) \geq 161$ in Table 2.1 show it isn't tight for $r = 5, 6, 7$. In [Exo94], the lower bound $\mathrm{R}_r\big(x + y = z\big) \geq c(3.17176)^r$ for some c is given, which means that for sufficiently large r, the bound from [BB82] in Theorem 3.3 could not be tight.

3.1.1 Rado Numbers for Homogeneous Equations

We will begin our survey of existing results, starting with linear homogeneous equations, which are closest in some sense to Schur numbers. In 1984, Burr & Loo wrote a manuscript which is now frequently cited for some of its results, despite being unpublished. One of the most significant results from that paper is:

$$\mathrm{R}_2\big(x + y = kz\big) = \binom{k}{2}, \quad k \geq 4$$

This appears in [MS07], for example, and is usually (if not always) given without proof. Note that for $k = 1, 2, 3$ we have Rado number $5, 1, 9$ respectively.

In [HM97], Harborth & Maasberg provide Rado numbers for two families of equations:

$$\mathrm{R}_2\big(ax + ay = 2z\big) = \frac{a(a^2 + 1)}{2}$$

$$\mathrm{R}_2\big(ax + ay = (a+1)z\big) = a(a+1)$$

Other new Rado numbers appeared in the textbook *Ramsey Theory on the Integers* by Landman & Robertson [LR03], including:

$$\mathrm{R}_2\big(ax + by = bz\big) = \begin{cases} a^2 + 3a + 1 & b = 1 \\ b^2 & a < b \\ a^2 + a + 1 & 2 \leq b < a \end{cases}$$

Landman & Robertson also give $\mathrm{R}_2\big(ax + ay = bz\big)$; however, it is a formula with 14 cases, and so we leave it to the curious reader to investigate.

In 2005, Hopkins & Schaal [HS05] give lower bounds analogous to those from Theorem 3.3 for more general equations:

$$\mathrm{R}_2\big(a_1x_1 + a_2x_2 + \cdots + a_{m-1}x_{m-1} = x_m\big) \geq a_1b^2 + (2a_1^2 + 1)b + a_1^3,$$

where it is stipulated that $a_1 < a_i$ and $b = a_2 + \cdots + a_{m-1}$. They conjecture that this is tight and prove it for $a_1 = 2$, while the case $a_1 = 1$ was earlier proved independently in both [Fun90] and [JS01]. Three years later, Guo & Sun [GS08] settle the question by verifying the conjecture.

In 2008, Myers & Robertson [MR08] give the following:

$$\mathrm{R}_2\big(x + y + kz = \ell w\big) \leq \binom{\ell - k + 1}{2}$$

whenever $\ell - k \geq 4$. They prove it is tight when $k \geq \ell - k$. They also give a formulas for various fixed values of $k - \ell$ and also for $\ell = 2$. Saracino and Wynne extend this result in [SW08] to $\ell = 3$.

Schaal & Vestal [SV08] proved in 2008 that:

$$\mathrm{R}_2\big(x_1 + x_2 + \cdots + x_{m-1} = 2x_m\big) = \left\lceil \frac{m-1}{2} \left\lceil \frac{m-1}{2} \right\rceil \right\rceil$$

for $m \geq 6$ (where $m = 3, 4, 5$ give Rado numbers $1, 4, 5$ respectively). This was recently generalized by Saracino [Sar13] as:

$$\mathrm{R}_2\big(x_1 + x_2 + \cdots + x_{m-1} = ax_m\big) = \left\lceil \frac{m-1}{a} \left\lceil \frac{m-1}{a} \right\rceil \right\rceil.$$

where $m \geq 2a^2 - a + 2$ (and for some other smaller values of m).

3.1.2 Rado Numbers for Non-Homogeneous Equations

The first result on non-homogeneous equations is credited to Burr & Loo [BL] in a paper by Schaal in 1993 [Sch93]. It is a straightforward generalization of Schur's theorem:

$$\mathrm{R}_2\big(x+y+c=z\big)=4c+5,$$

where we assume $c \geq 0$. The case $c<0$ is also credited to Burr & Loo in [MS07], which is:

$$\mathrm{R}_2\big(x+y+c=z\big)=c-\left\lceil\frac{c}{5}\right\rceil+1.$$

In 1993, Schaal [Sch93] does for non-homogeneous equations what Beutelspacher and Brestovansky did for Schur's theorem, proving:

$$\mathrm{R}_2\big(x_1+x_2+\cdots+x_{m-1}+c=x_m\big)=m^2+(c-1)(m+1),$$

This requires c to be even or m to be odd (otherwise, the even/odd coloring shows the Rado number is infinite).

In 1995, Schaal also takes a new step forward and proves $\mathrm{R}_3\big(x+y+c=z\big)=13c+14$, a 3-color Rado number that generalizes Schur's theorem. Kosek & Schaal in 2001 [KS01] would generalize this to $x_1+\cdots+x_{m-1}+c=x_m$, which has six different forms for different types of m, c pairs. Jones & Schaal would generalize the 1995 result in a different way, giving bounds on $\mathrm{R}_2\big(x+y+c=kz\big)$. This work is continued in [MS07] as well, where it is proved that:

$$\mathrm{R}_2\big(x+y+c=kz\big)\geq\begin{cases}\left\lceil\frac{2\left\lceil\frac{2+c}{k}\right\rceil+c}{k}\right\rceil & c>0\\ \left\lceil\frac{k\left\lceil\frac{2-c}{k}\right\rceil-c}{2}\right\rceil & c<0\end{cases}$$

and in particular the following:

$$\mathrm{R}_2\big(x+y+c=2z\big)=\begin{cases}|c|+1 & c\text{ even}\\ \infty & c\text{ odd}\end{cases}$$

for $c, k>0$, which is conjectured to be tight. Similar bounds (and conjectures) are given for negative c or k. For $k=3$, the bounds are proved to be tight in [KSW09].

Bialostocki, Lefmann, & Meerdink [BLM96] also discuss some bounds for a few non-homogeneous linear equations.

3.1.3 Rado Numbers for Inequalities

There are a few types of Rado numbers that are defined somewhat differently. We have mentioned the idea of Rado numbers for systems of equations, and the idea of a Rado number for an inequality or system of inequalities is analogous. In some cases, the objective is simply $\mathrm{R}_2(\mathcal{E})$ where $\mathcal{E}$ is now some inequality like $x + y < z$, while in other cases we may have some objective equation or inequality $\mathcal{E}$ but also require a condition like $x < y < z$. Sometimes an equation (not an inequality) is combined with this $x < y < z$ in a way that distinguishes the variables, e.g. the strict generalized Schur numbers require solutions to $x_1 + \cdots + x_{m-1} = x_m$ where $x_i < x_{i+1}$, which really means no repeated variables (since otherwise, x_1 through x_{m-1} are indistinguishable).

In 1996, Schaal & Wise [SW96] proved the following Rado number for an inequality:

$$\mathrm{R}_r\big(x_1 + x_2 + \cdots + x_n + c < x_m\big) = \frac{(m + c - 1)(m - 1)^r - (c + 1)}{m - 2},$$

where it is assumed that $c > 1 - m$ (otherewise the Rado number is 1, trivially).

In [Sch98] and [BS00], Schaal & Bialostocki give a strict version of generalized Schur numbers:

$$\mathrm{R}_2\big(x_1 + x_2 + \cdots + x_{m-1} < x_m; x_i < x_{i+1}\big) = \frac{9}{16}m^3 + QP(m^2),$$

where $QP(m^d)$ is an explicit quasipolynomial of degree d in m. For the unfamiliar reader, we will discuss quasipolynomials in Section 3.8. In 1998, Schaal produced a combined version, strict numbers for a generalized Schur inequality (with all $a_i > 0$):

$$\begin{aligned} 1 + \sum_{j=1}^{m-1}\left(j + \sum_{i=1}^{m-1} a_i\right)a_j &\leq \mathrm{R}_2\big(a_1x_1 + a_2x_2 + \cdots + a_{m-1}x_{m-1} < x_m; x_i < x_{i+1}\big) \\ &\leq \mathrm{R}_2\big(x_1 + x_2 + \cdots + x_{\Sigma a_i} < x_{1+\Sigma a_i}; x_i < x_{i+1}\big) \end{aligned}$$

Newer results on strict generalized Schur numbers can be found in [AEMS13].

3.1.4 Off-Diagonal Rado Numbers

In Section 2.2, we defined Ramsey numbers in a way that allowed us to find a complete graph of size k_i for the color class i, where it was not required that k_i be the same for every i. In the case that the k_i are not all equal, these are sometimes called "off-diagonal" Rasmey numbers (on a table of values, they would be off the diagonal). We can likewise define off-diagonal Rado numbers:

Definition 3.5. *The r-color off-diagonal Rado number for equations $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_r$ is the least N such for that any r-coloring χ of $[N]$ there is an i such that χ induces a monochromatic solution to $\mathcal{E}_i$ of color i.*

We will denote this $\mathrm{R}_r\big(\mathcal{E}_1; \mathcal{E}_2; \ldots; \mathcal{E}_r\big)$ and adopt the usual convention that it is ∞ if no such N exists.

Off-diagonal Schur numbers are introduced by Robertson [Rob00] and further discussed by Robertson & Schaal [RS01], where they prove for $k \leq \ell$ that:

$$\mathrm{R}_2\big(x_1 + \cdots + x_{\ell-1} = x_\ell; x_2 + \cdots + x_{k-1} = x_k\big) = \begin{cases} 3\ell - 4 & k = 3, \ell \geq 3, \ell \text{ odd} \\ 3\ell - 5 & k = 3, \ell \geq 3, \ell \text{ even} \\ k\ell - \ell - 1 & \ell \geq k \geq 4 \end{cases}$$

This completely characterizes 2-color off-diagonal generalized Schur numbers, but like regular Schur numbers, these off-diagonal Schur numbers are still of great interest for $r > 2$. In [Ahm], Ahmed gives a number of computational results for off-diagonal generalized Schur numbers.

In [MR07], Myers & Robertson give the following theorem, analogous to Rado's 2-color theorem 2.10:

Theorem 3.6. *For $\mathcal{E}_1$ and $\mathcal{E}_2$ linear, homogeneous equations, assumed to be nontrivial as in 2.10. Then $\mathrm{R}_2\big(\mathcal{E}_1; \mathcal{E}_2\big) < \infty$.*

They provide values and bounds for Rado numbers of the form $\mathrm{R}_2\big(ax + by = z; ax + cy = z\big)$ and other related equations.

3.1.5 A Partial Ordering of Equations

Lemma 2.6 we show how $\mathrm{R}_{r_1}\big(\mathcal{E}\big)$ and $\mathrm{R}_{r_2}\big(\mathcal{E}\big)$ are related for different numbers of colors r_1 and r_2. We hope to find a relationship between Rado numbers for different equations instead. First, we introduce the following relationship between equations:

Definition 3.7. *The equation $\mathcal{E}'$ is* <u>*a specialization of* $\mathcal{E}$</u> *if it is possible to obtain the equation $\mathcal{E}'$ by setting equal some of the variables in $\mathcal{E}$ (and subsequently relabeling the variables, if necessary, to match those in $\mathcal{E}$).*

This definition is what leads us to the following lemma:

Lemma 3.8. *For any equation $\mathcal{E}'$ obtained from $\mathcal{E}$ by specializing variables, and for any r, $\mathrm{R}_r(\mathcal{E}) \leq \mathrm{R}_r(\mathcal{E}')$.*

Proof. Because $\mathcal{E}'$ is a specialization of $\mathcal{E}$, all solutions to $\mathcal{E}'$ are solutions to $\mathcal{E}$ (assigning some values multiple times in $\mathcal{E}$ according to how variables were assigned in the specialization). Consider a coloring of length n that evidences $\mathrm{R}_r(\mathcal{E}) > n$, having no solutions to $\mathcal{E}$. This coloring must also have no solutions to $\mathcal{E}'$. Thus $\mathrm{R}_r(\mathcal{E}') > n$. Since this holds for all such n, $\mathrm{R}_r(\mathcal{E}') \geq \mathrm{R}_r(E)$. $\square$

This lemma allows us to define a relationship between Diophantine equations (or rather, between parametrized families of Diophantine equations), where we say $\mathcal{E} \prec \mathcal{E}'$ whenever $\mathcal{E}'$ can be obtained from $\mathcal{E}$ by specializing variables.

We can also say $\mathcal{E} \subseteq \mathcal{E}'$ if it is obtained by fixing some undetermined coefficient(s).

It should be clear that each of these relations is a partial order, and we can take the transitive closure of their union to form our partial order. This gives rise to the following organization of some of the existing results in the following figure.

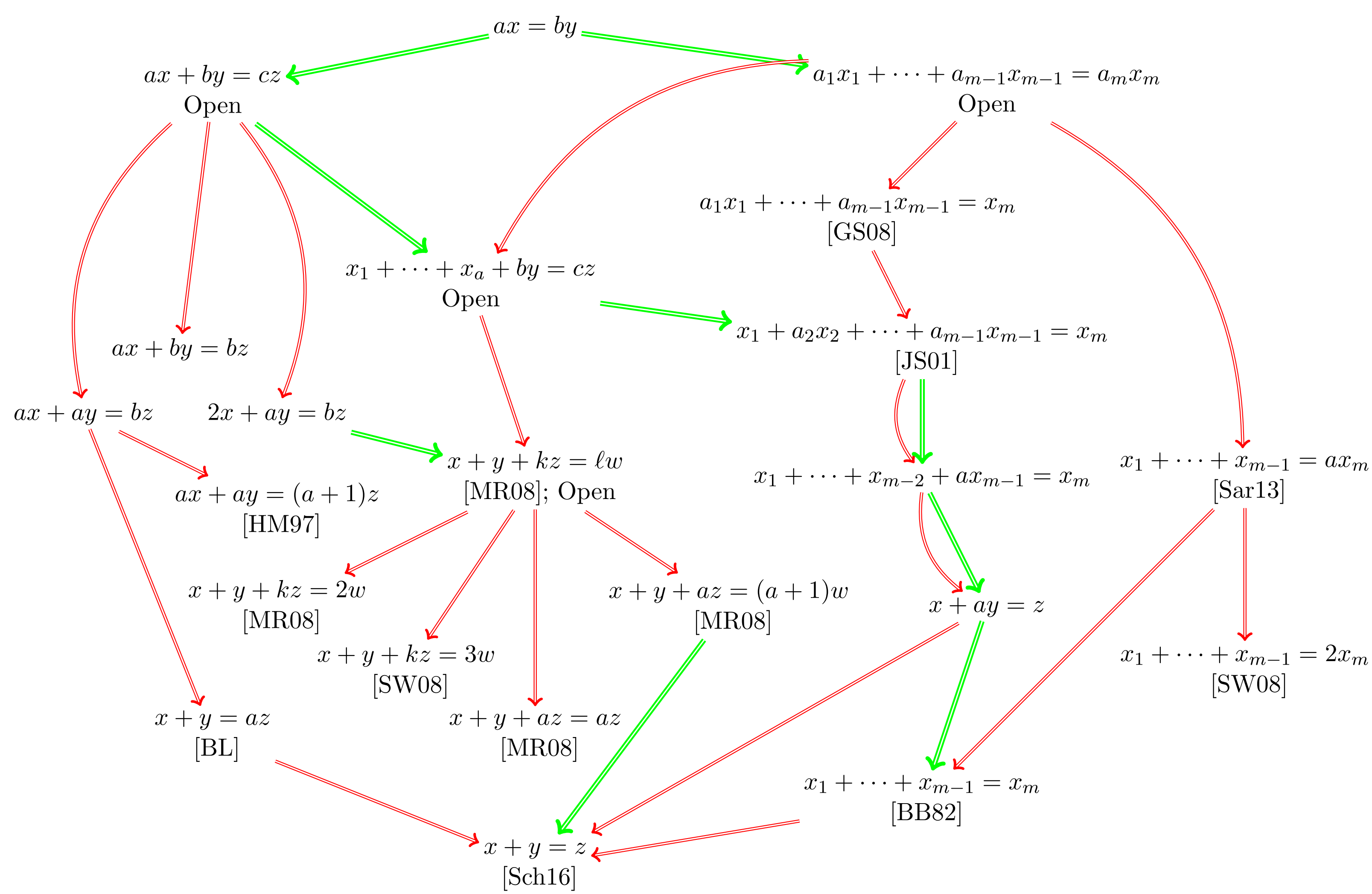

Figure 3.1: The poset of equations

In Figure 3.1.5, the Hasse diagram for this partial order is given. The covering relations are color-coded depending on which relation is indicated. We illustrate the $\leq$ relationship with green arrows, where the arrow points the same direction as $\leq$, so that if $\mathcal{E}_1 \leq \mathcal{E}_2$, then there is a green arrow from $\mathcal{E}_2$ to $\mathcal{E}_1$. Thus, an upper bound for $\mathcal{E}_2$ (the equation higher up in the Hasse diagram) is inherited by $\mathcal{E}_1$. Likewise, the red arrows that indicate $\subseteq$ are like logical implication – with any type of bound being inherited in the direction of these arrows.

This figure will help us organize our thoughts, in particular addressing which equations are most important or interesting as we study Rado numbers. Note that upper bounds pass down this poset, meaning that ideals in this partial order have some significance (the same is not true for filters).

To maintain readability, this figure is limited to a subset of existing results in this area. There are no non-homogeneous equations, and even some homogeneous equations have been omitted from the periphery of this poset. Some of the possible arrows have been omitted as well (e.g. between the two families directly below $ax = by$).

It is probably clear that in most cases, as we read the poset from top to bottom, the dates go further back.

From this figure, we can see that there are a number of open questions that we may tackle. The most interesting problem would be to give an upper bound for $\mathrm{R}_2\big(ax + by = cz\big)$, which would provide a universal upper bound. Any result of this generality might also lead to a complete solution, computing 2-color Rado numbers for any linear homogeneous equations. This is a lofty goal, so it is important to approach the problem incrementally. There are many families not in this poset because they have not yet been subject to any serious study. For example, we will examine $kx + (k+1)y = (k+2)z$ in Section 4.2.

The results of [Sar13] and [GS08] are extremely strong in two different ways – to combine them, e.g. to solve the problem directly above each in the poset, would be another strong step forward towards a general solution. However, these results are strong enough that we believe the best place to look for incremental results will be somewhere else, like above the results from [MR08]. We should be able to solve more than just families like $ax+ay=bz$, and in the future we expect that the known families in this poset will include many more between $ax+by=cz$ and those that it currently covers in the Hasse diagram.

3.2 General Methodology

In most cases, we have three potential objectives from a computation:

- prove $\mathrm{R}_r(\mathcal{E}) < N$ for some N;
- prove $\mathrm{R}_r(\mathcal{E}) > N$ for some N; or
- prove $\mathrm{R}_r(\mathcal{E}) = N$ for some N.

Some algorithms require N to be fixed (i.e. as part of the input, along with r and $\mathcal{E}$), while others (particularly in the third case) will determine some value of N along the way.

Lower bounds are most frequently proved by exhibiting a coloring that does not contain monochromatic solutions, which we will define as follows:

Definition 3.9. *A coloring of $\{1, 2, \ldots, n\}$ (for some n) is said to be valid if it does not contain monochromatic solutions to the equation in question.*

So a valid coloring always serves as a witness to some lower bound for a Rado number – a valid coloring of lengh N proves $\mathrm{R}_r(\mathcal{E}) > N$.

Upper bounds can be proved in a few ways, but generally speaking it is not so easy to prove an upper bound. Many direct methods for proving upper bounds reduce to assuming that some valid coloring is N long and deriving a contradiction.

Methods to prove upper and lower bounds can be combined to produce exact Rado numbers, but there are also algorithms that function to directly prove Rado numbers. For example, we will discuss the `RADO` package, which will simply give an exact Rado number for a given r and $\mathcal{E}$, while the SAT solving methods we will discuss will only tell you whether a given N is an upper or lower bound (which we then repeat until we find the exact Rado number).

3.3 Structural Methods for Lower Bounds

Many existing proofs for lower bounds rely on the construction of valid colorings by capitalizing on the structure of the equation in some way.

The first example is a fairly simple, but frequently effective, method. Considering the example $x+2y+3z=w$, we can observe that in any case, w must be the largest of any solution (x,y,z,w). This means that if 1 is red (without loss of generality), a red monochromatic solution must contain some value of w that is at least 6. Thus we may safely color 1 through 5 red and maintain a valid coloring.

We then color 6 blue, and in the same way we know any blue solution must have w at least 36. So we can safely color 6 through 35 blue.

From here, the process – if it allows for continuation – becomes more delicate because potential monochromatic solutions might include elements from more than one monochromatic interval. But we can do our best to go back and forth, adding runs of red and blue until it becomes impossible. This process can be done with a fixed equation or one with unspecified coefficients.

In our example $x+2y+3z=w$, we can color 36 through 40 red before the solution $(36,1,1,41)$ becomes an issue. At that point, we cannot assign any color to 41.

So we have determined that $\mathrm{R}_2\big(x+2y+3z=w\big) \geq 41$. As it turns out, this is tight, which one might determine exhaustively (which we can do at this point in under a second) or by referring to [GS08].

Although this methodology is simple, it can be automated (in more than one way), and has been used (in non-automated ways) in [HM97], [HS05], [JS01], [SW08], [SV08], [Sch93], [JS04], [Sch95], and more to provide lower bounds for various Rado numbers.

This method could also be applied in a greedy sense, coloring each integer one by one, rather than arguing from some inequality derived from $\mathcal{E}$, simply coloring $i, i+1, \ldots, i+k$ until $i+k+1$ would produce a monochromatic triple (and then, switch colors).

Another method to constructing bounds is to use a parity argument. This could be useful, for example, with equations like $x+y=u+v+w$, since an all-odd set contains no solutions.

Of course, because solutions to an equation like $x + y = u + v + w$ will scale, a coloring by parity alone may not get us far, since the evens will contain plenty of solutions (indeed, even solutions are in bijection with all solutions). However, coloring large intervals by parity has been used, e.g. [MR08]. This is often combined with large monochromatic intervals, as described above.

Because these two methods, and similar methods, require few parameters, a brute force (or random) search of all possible "structured" colorings of this type can give us lower bounds. It is important to note that in some of the previously studied cases, these bounds do turn out to be tight. However, that is not something we can necessarily assume in general (e.g. see Appendix B).

3.4 Random Methods for Lower Bounds

There are a number of methods for randomly constructing valid, or nearly-valid, colorings. One of the most interesting methods is called simulated annealing. This is a general purpose random method for numerous applications to many different problems that could be considered optimization. It has been used to study different linear structures in the integers, which Butler, Costello, & Graham [BCG10] call "constellations."

Starting with some initial coloring, we randomly change the colors of various integers with some probability. Preference is given to changes that decrease the number of monochromatic solutions, but not absolute preference – sometimes a "bad" change is made. This methodology can be varied widely depending on the parameters, which need not even be fixed – the parameters could adapt to the current state of the coloring. The initial color could be generated by other means, although frequently simulated annealing overcomes issues with a "bad" choice of initial state.

The work in [BCG10] verifies some of the intuition in Section 3.3, showing that many simulated annealing methods result in colorings that are comprised of a few long monochromatic intervals.

In Section 3.7, we will discuss an algorithm known as backtracking in greater depth.

This algorithm can be randomized, and this amounts to essentially taking a valid coloring (e.g. the length-one coloring "red") and assigning a random color to the next integer. If that assignment results an invalid coloring, undo this and try again. Unlike the rigorous version, this algorithm would need to also randomly step back several steps when it encounters invalid colorings.

All of the structural methods in Section 3.3 could also be randomized, which would give a non-exhaustive search. Because sets of randomly generated lower bounds are not rigorous (i.e. they are not complete enough to prove the corresponding upper bound), a random search through the parameter space of such methods may be slightly less useful, only proving lower bounds, but much faster.

3.5 Forcing Methods for Upper Bounds

Upper bounds may also be computed by something we will call "forcing." Assuming we have a valid coloring, with some of the colors known, we can force the assignment of other colors in order to maintain the validity of the coloring.

We can use this idea to reprove the fact that $S(2) \leq 5$ in this framework as follows:

Proof. Assume for contradiction that we have a valid coloring of [5]. Without loss of generality, say that 1 is red. Because (1,1,2) is a solution to $x + y = z$, we know that a valid coloring cannot have 2 red. Thus 2 is blue.

Because 2 is blue, 4 must be red in our valid coloring for the same reason.

Now that 1 and 4 are both red, 3 must be blue.

Finally, we have solutions (1,4,5) and (2,3,5), which make it impossible to color 5 with either color in a way that keeps the coloring valid. □

Interestingly, in this case we have also produced the coloring red-blue-blue-red, which proves the matching lower bound $S(2) > 4$. (However, it is not always the case that this methodology provides a matching lower bound.)

This method is described in [MR08], which is itself inspired by previous *ad hoc* use of this idea throughout the previous literature, including [HM97], [HS05], [JS01], [MR08],

[SW08], [SV08], and [Sch95]. The latter, in particular, includes a significant application of this methodology. However, this method is implemented in a Maple package in [MR08] that automates entire proofs (for a few particular equations), which allows instantaneous proofs of the paper's results. Some of these results reproduce existing results that had originally been proved by careful, and perhaps tedious, implementation of this algorithm by a human.

The idea behind this algorithm is to start with some initial partial colorings, some integers in a red set R and some others in a blue set B. We assume that these sets comprise part of a valid coloring. Each set could contain partial solutions, and if we can find a solution with all but one integer monochromatic, we can assume that this last integer must be the opposite color from the rest of the solution set (since we need the coloring to be valid).

This process can proceed algorithmically.Consider the following illustrative example. First, let our equation be $x + y = 4z$. We will simply start with $R = \{1\}$ and $B = \emptyset$. The process proceeds as follows, where at each step we will list new members of R and B, plus all uncolored elements from 1 to 15.

Red	Blue	Unassigned
1		2 3 4 5 6 7 8 9 10 11 12 13 14 15
	2 3	4 5 6 7 8 9 10 11 12 13 14 15
4 5 6 9 10		7 8 11 12 13 14 15
	$\underline{4}$ $\underline{5}$ $\underline{6}$ 7 8 $\underline{10}$ 11 12 14 15	13

Table 3.1: The forcing algorithm for $x + y = 4z$

We can observe that at each step, we can go back and forth – updating B, then R, then B. This is because B starts out as $\emptyset$. At the last step listed, we have seen that the assumption that a valid coloring exists of length 15 leads to a contradiction: that 4 (and 5, 6, and 10) must somehow be both blue and red.

In this case, we can actually determine not just that $\mathrm{R}_2\big(x + y = 4z\big) \leq 15$, but $\mathrm{R}_2\big(x + y = 4z\big) \leq 10$. This value of 10 turns out to be correct. With different initial sets, the process would potentially update R and B at each step instead, but by starting with one of the two empty, the updates alternate.

It is not necessarily required that the bound (15 above) be fixed – in some cases, we can consider the "unassigned" group to contain all integers, and draw from this set until a contradiction is reached.

3.5.1 Multicolored Forcing Methods

As written, this method is limited to only two colors. However, it is entirely possible to implement a version of this algorithm for 3 or more colors. The idea would require a more careful use of this idea of "forcing" – if (a, b, c, d) is a solution to $\mathcal{E}$ and a, b, c are already in R, we can only assume that d is in B because there is no other possibility.

For r colors, there would be 2^r sets indexed by all possible subsets of colors. We would actually reverse our paradigm, though, so that now if an integer i is currently in set $\mathcal{C}_{c_1,c_3}$, this means it is NOT possible that i has color c_1 or c_3.

In this case, we would find a solution $s = (s_1, \ldots, s_m)$ that is already monochromatic of color c except for s_j, where s_j is currently in the color-set $\mathcal{C}_C$. Assuming c is not already in C, we can move s_j from $\mathcal{C}_C$ to $\mathcal{C}_{C\cup\{c\}}$.

The sets of integers with a definite color are those of the form $\mathcal{C}_{2^{[r]}-\{c\}}$ for each c. We could use the integers in those sets to draw conclusions about the rest of the integers, by finding all-but-one-monochromatic solutions.

Because this system requires a much more complex set of moves to get an integer of indeterminate color (i.e. $\mathcal{C}_\emptyset$) to a definite color ($r - 1$ steps, in fact), it would be important to make two modifications:

- an automated approach to assigning colors to a few integers to "jump start" the process, since it will not be enough to simply assume that 1 is red (for example, running the algorithm 5 times with each of the 3^5 colorings of the first five integers in a 3-coloring); and

- an implementation of inferences other than those that can be drawn from integers in sets of the form $\mathcal{C}_{2^{[r]}-\{c\}}$, that is, attempting to draw conclusions from integers that may still have two or more possible colors.

This methodology requires further refinement in its implementation. It is also difficult to use this algorithm for equations with unspecified parameters because there are cases where it becomes difficult to determine whether certain solutions include integers (while it would be safe to meaninglessly assign colors to non-integers, it would not be safe to start using those assignments to draw conclusions about integers).

In Section 3.6, we will discuss how the language of satisfiability (a.k.a. SAT) can be used to formalize this type of methodology in a way that does not require us to reinvent (or re-implement) the proverbial wheel. These methods will lose some of their strengths in being directly tailored to questions in Diophantine Ramsey theory but will benefit from the existence of effective SAT solvers.

3.6 Lower & Upper Bounds in the Language of SAT

In [Ahm09], Ahmed describes how the problem of computing van der Waerden numbers can be translated into a problem of logical satisfiability (or "SAT"). Like many problems in computer science, satisfiability has a yes or no answer: "Is a certain logical statement satisfiable, for some assignment of the variables?" For example, the statement $x \wedge y$ is satisfiable, with the assignment $x = y =$ True. Here $\wedge$ represents "and," $\vee$ will represent "or," and $\neg$ will represent "not." However, the statement $x \wedge (\neg x)$ is not satisfiable, because any assignment of x will make this statement false. Satisfiability could also be rephrased in terms of Boolean algebra, but we will continue to use the logical notation $\wedge$, $\vee$, $\neg$.

We will reserve x and y for variables in our $\mathcal{E}$ from now on, and we will enumerate our logical variables in this section using v instead.

For our purposes, we have an equation $\mathcal{E}$ and a specific N and r, and we want to know whether there is any r-coloring of $[N]$ such that no monochromatic solutions to $\mathcal{E}$ exist. First, we can formulate our method in the case of $r = 2$. We designate the variable v_i to indicate that i is colored blue (and if v_i is false, i is colored red).

SAT solving algorithms generally accept input in one of several normal forms. In this case, we will use the DIMACS format [DIM93] for SAT problems, which uses conjunctive normal form (CNF). The statement to be satisfied must be the conjunction ("and") of a number of clauses. These clauses are themselves all disjunctions ("or") of literals (variables or their negations). There are important theorems noting that all statements can be written in this normal form, but we might actually find that this form is quite natural for our problem.

The conditions we must impose are relatively simple: We know any satisfying assignment will result in a valid coloring (and if not, there is no such coloring) because each v_i will be either true or false. We take every solution $(x_1, x_2, \ldots, x_m)$ and form two clauses:

$$\big(v_{x_1} \vee v_{x_2} \vee \cdots \vee v_{x_m}\big) \wedge \big(\neg v_{x_1} \vee \neg v_{x_2} \vee \cdots \vee \neg v_{x_m}\big)$$

These two clauses force at least one of the x_i to be red, and at least one to be blue, i.e. not monochromatic. As an example, if we wanted to prove the 2-color Schur number $S(2) > 4$, we would need to gather up all solutions to $x+y=z$ (we may assume $x \leq y$ to save some time). They are:

$$\big\{(1,1,2),(1,2,3),(1,3,4),(2,2,4)\big\}$$

This gives us the following formula:

$$\begin{aligned} F(v_1,v_2,v_3,v_4) \quad = \quad & \big(v_1 \vee v_2\big) \wedge \big(\neg v_1 \vee \neg v_2\big) \wedge \big(v_1 \vee v_2 \vee v_3\big) \wedge \big(\neg v_1 \vee \neg v_2 \vee \neg v_3\big) \wedge \\ & \big(v_1 \vee v_3 \vee v_4\big) \wedge \big(\neg v_1 \vee \neg v_3 \vee \neg v_4\big) \wedge \big(v_2 \vee v_4\big) \wedge \big(\neg v_2 \vee \neg v_4\big) \end{aligned}$$

We can use a SAT solver to produce a satisfying assignment to $F(v_1, v_2, v_3, v_4)$, e.g. $v_1 = v_4 =$ False, $v_2 = v_3 =$ True, which is red-blue-blue-red.

We could then consider two additional solutions, $(1,4,5)$ and $(2,3,5)$, obtaining:

$$\begin{aligned} G(v_1,v_2,v_3,v_4,v_5) \quad = \quad & F(v_1,v_2,v_3,v_4) \wedge \big(v_1 \vee v_4 \vee v_5\big) \wedge \big(\neg v_1 \vee \neg v_4 \vee \neg v_5\big) \wedge \\ & \big(v_2 \vee v_3 \vee v_5\big) \wedge \big(\neg v_2 \vee \neg v_3 \vee \neg v_5\big) \end{aligned}$$

A SAT solver would tell us that G is not satisfiable, proving $S(2) \leq 5$.

Because our usual equations include three variables (or more), most of our clauses will too. This means our problems fall into the framework of 3-SAT, the problem of deciding the satisfiability of a general formula in CNF where each clause has 3 literals. This problem is well-known to be NP-hard.

Our framework for these problems is complicated by the need to have more than two colors, in which case we we will introduce additional clauses. For each color j and each solution $(x_1, \ldots, x_m)$ to $\mathcal{E}$, we have the clause:

$$\left(v_{x_1,j} \vee v_{x_2,j} \vee \cdots \vee v_{x_m,j}\right)$$

Each $v_{i,j}$ (which, in practice, is translated to v_{Nj+i}) represents that i is colored the color j. We no longer require the negation, since that was to accommodate the second color of the two. We must explicitly enforce that a color is assigned to each i, so we must include for each i the clause:

$$\left(v_{i,1} \vee v_{i,2} \vee \cdots \vee v_{i,r}\right)$$

However, that is not enough! We must also insist that no i is assigned to multiple colors. For each i, we include the following clauses, which are a bit more complicated because we need one for each $1 \leq j < j' \leq r$, i.e. for $\binom{r}{2}$ pairs:

$$\neg v_{i,j} \vee \neg v_{i,j'}$$

The DIMACS format for CNF formulas requires us to count the numbers of variables and clauses. This is actually not simple, at least not in most cases, since we would need to enumerate the number of solutions to $\mathcal{E}$ in $[N]$. Since we have already listed them, the best way is to simply count the length of that list – rather than try to exploit some special cases (e.g. if $\mathcal{E}$ is $x + y = z$, it is not hard). Each solution to $\mathcal{E}$ contributes r clauses, and we have to add on $N(1 + \binom{r}{2})$ clauses to enforce the well-definedness of our coloring. If the solution set is dense (i.e. $\mathcal{E}$ has roughly $o(N^{m-1})$ solutions in $[N]$), we will have $o(N^{m-1} + Nr^2)$ clauses.

It is not difficult to count the number of variables, though. Even in cases where some i is never used in a solution to $\mathcal{E}$, it is used in the clauses that enforce the well-definedness of the coloring. For that reason, we know there are rN variables.

In some respects, using a SAT solver is a generalization or restatement of the forcing methods we describe in Section 3.5 – the logical deductions we make as we "force" some contradiction are precisely the types of steps we would expect a computer to do if it were parsing our CNF looking for a contradiction (or for a satisfying assignment). While the idea of forcing described in Section 3.5 is tailor-made for Rado numbers, it has its limitations and is difficult to implement efficiently. SAT solvers are not tailor-made for these types of systems, but SAT solvers are important tools and many different SAT solvers exist to tackle these tough problems like computing Rado numbers, like Walk-SAT, GRASP, and MiniSAT. These SAT solvers rely on high performance computing, written for efficiency and effectiveness in lower-level languages than computer algebra systems or the like, and are highly effective in many cases.

In Section 3.7 we will discuss another high performance program that we have designed and implemented called `RADO`. That program is robust and effective in many cases, but SAT solvers have their own advantages. In particular, when the solution set is sparse, the `RADO` program does not adapt to this and will consider a significant number of redundant cases. A SAT solver will not do this, and will instead reap the benefits of the sparsity by having far fewer clauses to satisfy.

3.7 The `RADO` Package & Exhaustive Computation

To compute $\mathrm{R}_r(\mathcal{E})$ for a specific value of r and $\mathcal{E}$ with no unspecified coefficients, we can work under a number of different paradigms. In this section, we consider the problem to be one of searching a tree of all possible colorings. In the most basic sense, the algorithm used to compute these Rado numbers is a search of the r-ary tree of valid colorings using a backtracking depth-first search.

Although this is a standard sort of algorithm, many of the details in the implementation are significant in making feasible these large-scale computations. For that reason, this section (unlike other sections in this chapter) will detail a significant undertaking: the development and implementation of an efficient, high-performance, parallelized backtracking algorithm tailor-made for computing Rado numbers. The development of the `RADO` program should be considered a part of the work towards this dissertation.

3.7.1 The Backtracking Algorithm

Backtracking algorithms were described in [BG65], wherein one of the first applications of backtracking was determining $S(4) = \mathrm{R}_4\big(x + y = z\big) = 45$. Backtracking is, more or less, a straightforward implementation of a depth-first search on some tree (in our case, an r-ary tree). Technically speaking, we are not searching the tree, since at no point would we halt at a node and say "we found what we were looking for and can stop searching" until we have explored the entire tree (effectively, the search does not halt until it returns to the root of the tree).

Due to the relatively ubiquitous nature of tree-searching in computer science and software engineering, backtracking is a well-known methodology for searching trees, and is applied to other complex problems like the so-called traveling salesman problem.

There are advantages to depth-first searches (DFS), just as there are for breadth-first (BFS), but our algorithm will depend on one important fact about depth-first searches – they tend to require less memory, in general, since we need not track all nodes at a particular depth as we move through each level in the tree. In this sense, it is equivalent to the "right hand" method for exiting a maze, which can be executed by a single person, while the "split up at every turn" requires a team of maze-explorers so large that somehow at every fork in the maze, the group can subdivide.

We will discuss some implications of DFS vs. BFS in Section 3.7.2, but generally speaking every vertex in the search tree (corresponding to a particular coloring) will require the same amount of CPU time to determine its validity, regardless of the organization of the search. Although the Blue Gene system we currently use has a significant number of features (in both architecture and software) that could be used to our advantage, the main advantage of using the Blue Gene system is simply the ability to harness the collective power of a large number of CPUs in parallel.

For example, the nodes (one node is 4 CPUs) of each rack (1024 nodes) are arranged in a 5-dimensional discrete torus, so that a node has efficient access to its ten neighbors (two in each direction), including shared access to RAM. However, our search is not memory-intensive, nor does it require such a feature to enable communication or memory sharing between CPUs or nodes.

Our backtrack will terminate immediately if the search reaches a depth of 1000. This arbitrary cut-off is meant to indicate that a particular search is unlikely to ever terminate (and in cases where it is feasible, the Rado number might be infinite).

3.7.2 Parallel Methodologies

The tree search is parallelized by splitting an initial search into a queue of valid colorings. This search factors out any symmetry in the tree by only using the color i if the color $(i-1)$ is already in use. (In other words, the branch beginning red-red-blue is the equivalent to that of red-red-green, so we only do one of those two). Each worker is assigned a branch (an initial valid coloring that can presumably be extended) and explores this branch of the tree depth-first. When a worker has explored the entire branch, it is assigned a new branch to explore.

The parallelized algorithm is controlled by a master process that directs the workers in their search. This master process keeps track of any idle workers and the queue of branches not yet explored. This queue is a first-in, first-out (FIFO) structure, although that may not be important – changing the order of the search would not necessarily have an appreciable effect on the effectiveness of the algorithm.

The master process must build up the queue (which is initially empty) before it can hand off any branches to workers. The master process searches the tree from the root (where we assume 1 has color 0, which we might say is "red," without loss of generality). The master process knows how many workers there are, and it will search the tree breadth first until it finds enough branches to hand one off to each worker. For any significantly large tree, this will be relatively fast, and using BFS here will not be problematic because the width of the tree is bounded – once it hits the required width, it fills the queue and starts the DFS in parallel.

The number of nodes, within the current architecture, is always set at a power of 2, and with 4 CPUs per node, the total number of CPUs is also a power of 2. With one master process (and possibly a few other non-worker processes we will discuss in Section 3.7.5), there will be slightly fewer than 2^k workers, where usually $7 \leq k \leq 12$.

In the case of a binary tree, the master routinely has enough branches at depth $k+1$ or $k+2$ – very few of these trees will thin out appreciably after only 8-12 generations.

Once the master process has filled the queue accordingly, it hands off one branch to each worker, and each worker explores this branch depth first, reporting back regularly to the master process how many colorings it has checked, how many times it has checked a potential solution to $\mathcal{E}$, and, when finished, the entire list of maximal valid colorings (which would be leaves in the tree). That worker is then assigned the next branch from the queue, and this repeats until the queue is empty and there are idle workers.

3.7.3 Tree Cleaving

Once the queue is empty, of course, we run the risk of leaving workers idle. Even if the queue had exactly enough branches to give one to each worker, these branches do not take the same amount of time to explore. Generally speaking, it is very feasible that – without any other strategy – the methodology described above would simply leave the majority of the workers idle for an extended time. This is, of course, easily corrected if we can add more branches to the queue and then hand them out.

The master process tracks the state of workers, and if any workers are idle, it initiates a process we call cleaving. All workers are asked to report their maximal valid colorings, i.e. leaves on the search tree, and the rest of this process is discussed in section 3.7.5. However, we also have to refill the queue (that was the purpose for halting all work). If each worker reported back "I was searching at branch B and extended it to branch Bb (a concatenation of B and some b) while producing the following maximal valid colorings $\{Bv_1, Bv_2, \ldots, Bv_k\}$," we would have a problem – each worker would only report one place to start searching again, Bb.

Instead, the queue is replenished by considering the branch b that a worker is exploring and noting that there are many unexplored branches from b. If you write the tree left-to-right, all branches to the left of b are explored (and returned as Bv_i), while those to the right are unexplored, so the worker can return all of those branches to the queue as $Bb'i$, where $b' < b$ is an initial segment of b and i is one of the colors following whatever is in the respective position in b. That means it can generate up to $(r-1)|b|$

new branches to search, which means at each cleave, the queue could grow to be at most as large as roughly $(r-1)W\mathrm{R}_r(\mathcal{E})$, where W is the number of workers and of course $\mathrm{R}_r(\mathcal{E})$ may not be known in advance. That is probably a gross overestimate in most cases, but the size of the queue after cleaving is bounded (assuming $\mathrm{R}_r(\mathcal{E}) < \infty$).

For a concrete example, consider a worker who is assigned to explore the branch 01201 (for ease, colors are just 0, 1, and 2), and that worker finds maximal valid extensions of that branch 000, 002, 01002, 0202, 120021, and is searching branch 122021 when the cleave command is received.

That worker returns the following maximal valid colorings, where we have underlined the initial $B = 01201$ for clarity:

- $\underline{01201}000$
- $\underline{01201}002$
- $\underline{01201}01002$
- $\underline{01201}0202$
- $\underline{01201}120021$

The worker also returns its current branch, split accordingly, as $\underline{01201}122021$, and this is dissected into the following new branches:

- $\underline{01201}122021$
- $\underline{01201}122022$
- $\underline{01201}1221$
- $\underline{01201}1222$
- $\underline{01201}2$

Each of these unexplored branches is the prefix $\underline{01201}$, then some truncation of b, then one more digit that must be later than the corresponding digit of b. For example, the third entry is the prefix ($\underline{01201}$), then the first three digits of b (122), then a digit greater than the fourth digit of b (in this case 1, which is greater than 0). This list constitutes all neighbors of this branch b in the unexplored direction – those that come after b lexicographically. This cleaving process essentially splits the search space at the frontier of what has been already explored – sending the maximal explored branches to the certificate and all vertices adjacent to the boundary of the search to the queue. No branches are added from B because those have been counted elsewhere, earlier in the search, before B was put into the queue itself. This example is illustrated in Figure 3.2.

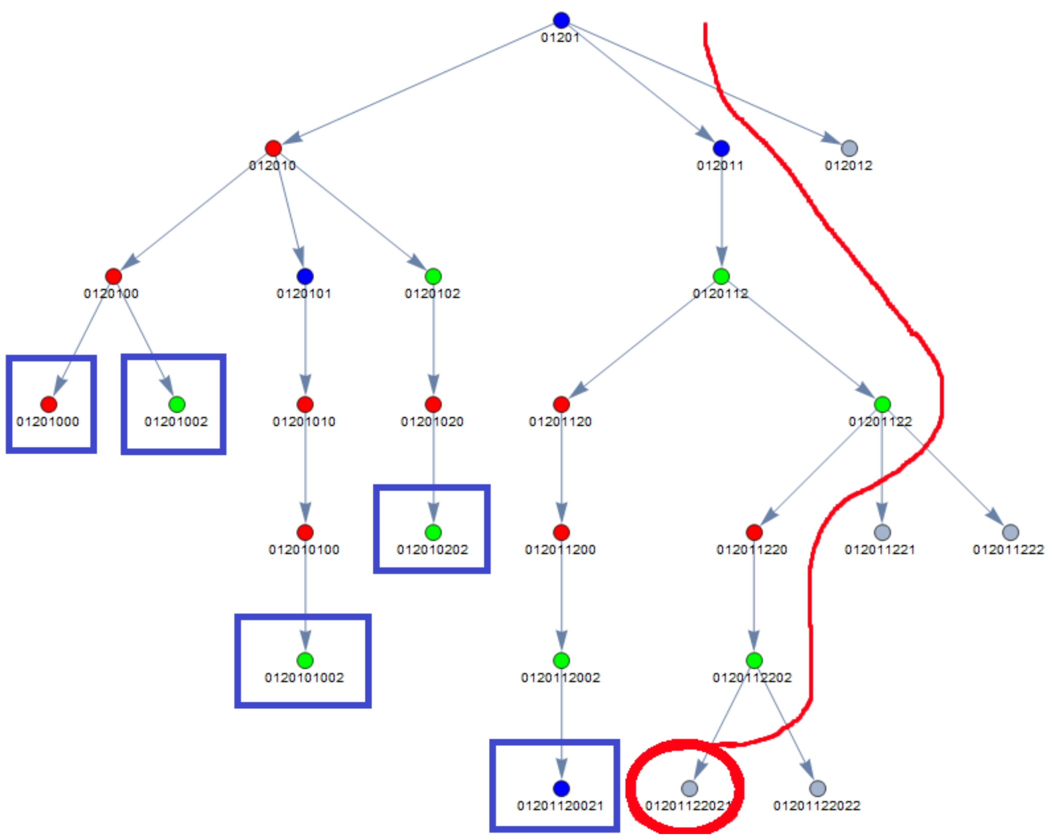

Figure 3.2: The cleaving process for DFS

This figure shows the place where the worker node is currently searching, circled in red. The branches that have already been searched and explored as far as possible are returned to the master process as a list of maximal valid colorings (i.e. a list of the leaves, boxed in blue). The grey nodes are not necessarily leaves, but they are unexplored and all of them are returned as new items to add to the queue. The split is illustrated with a red line, and this divisiveness is why we call it cleaving.

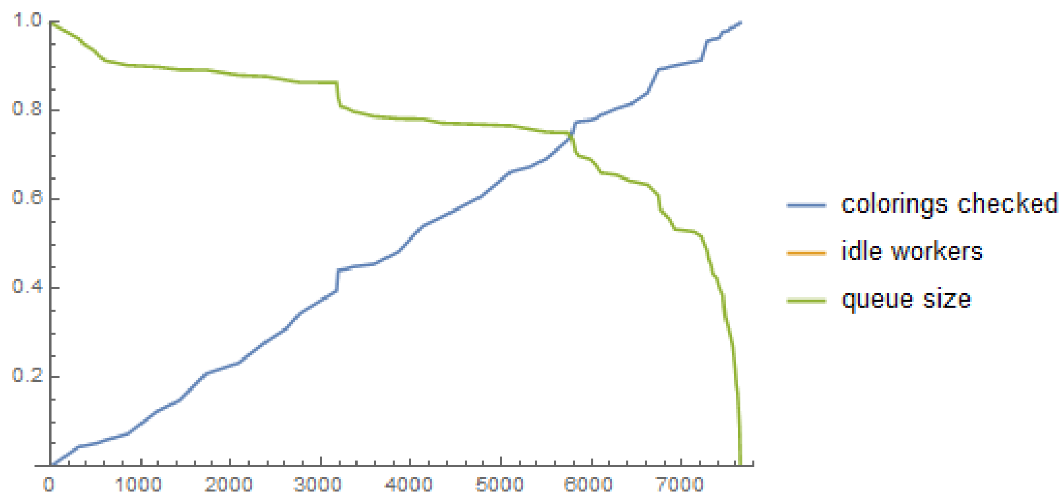

Figure 3.3: The queue and colorings over time

In Figure 3.3, this is illustrated based on the output of a particular computation. The size of the queue is adjusted to be monotonically decreasing. Sharp drops in the queue size occur after a cleave is initiated – many of the branches given to workers after a cleave take only an instant to explore and return. As workers become tied up with more robust branches, this slows down to a steadier pace. Similar jumps can be seen in the numbers of colorings checked. Note that in this successful implementation, the number of idle workers is never appreciable (it coincides with the axis).

Communication between the master process and the workers is not carefully synchronized, which means that if the master process were to issue a cleave command to the workers, if it did not receive enough immediate responses, it might issue another cleave command. Even if all workers reported back, the master might not receive enough branches back to fill up the queue for very long. This is especially problematic near the end of the search, where workers spend very little time on each branch from the queue.

If we do not account for this issue, it results in something called "state flapping." The master process will issue cleave commands when it hits the threshold for cleaving, but receive only enough colorings back to immediately hand them out – and then cleave again. This gives workers too little time to explore their branches in depth, which is precisely why they aren't returning enough new branches to explore. This is most likely to occur near the end of a computation, but it is possible (by introducing this

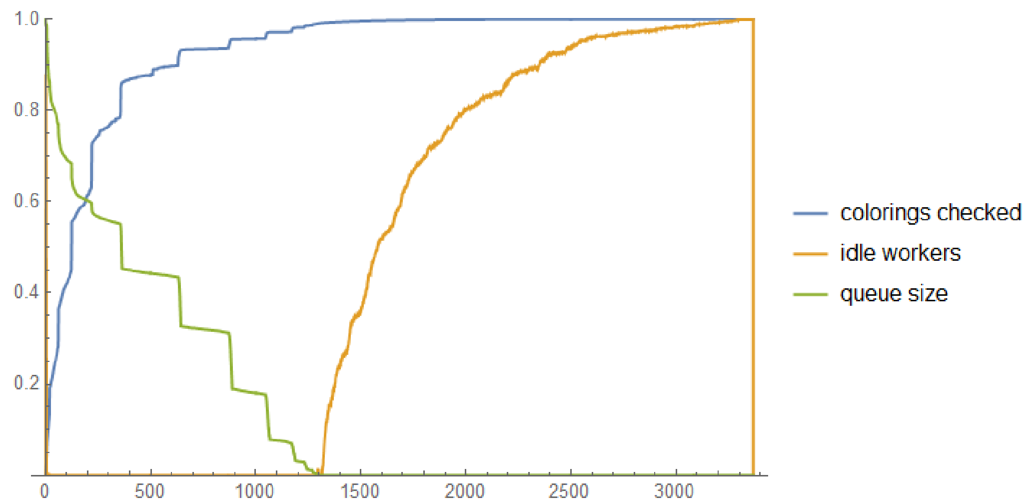

Figure 3.4: State flapping

problem into the algorithm) to make the last few minutes of a normal search take hours, days, or worse. Depending on how this problem is introduced, the state flapping may cause workers to spend the majority of their time cleaving, communicating, and waiting (rather than searching). On the other hand, if state flapping is avoided in the wrong way, cleave commands may not be issued properly, at which point workers go idle, and the process grinds to a halt as a very small number of workers are stuck exploring what's left of the tree. This is illustrated in Figure 3.4. In this plot, each quantity has been normalized from 0 to 1, but we can see that the queue is eliminated around time $t = 1400$ – however, the master fails to delegate more tasks to workers, and the progress grinds to a halt (the "colorings checked" figure essentially flattens out, as almost no progress is being made) while almost all workers go idle.

The solution that avoids these two problematic scenarios is to give each worker process a one-second timer. Each worker process ignores commands from the master for one full second at a time. This allows workers to commit the majority of their time to searching the tree and checking colorings, only checking once per second for commands that the master may have issued. The master will send a cleave command any time the queue is empty, but that command will only be processed by the worker at its next one-second check. The master also waits to issue a cleave command until after it has heard back from at least one worker after the previous cleave command (so that it does not continuously issue cleave commands while it has not even heard back from any of the workers).

3.7.4 Checking for Valid Colorings

For a fixed coloring, a worker process will have to check whether that coloring is valid. Because this is a depth first search, each such coloring is an extension (by one element) of a valid coloring, so the integer at the end of the coloring would have to be a part of any monochromatic solution to $\mathcal{E}$. In order to check for monochromatic solutions, we check all possible evaluations of $\mathcal{E}$ (where one or more of the variables must be the element from the end of the coloring) using what we have termed the value-iterator (VI).

The VI checks quickly, in a predetermined order, for any possible monochromatic solution. The VI is optimized in the case of certain symmetry in the equations, like in the case of $x_1^2 + \cdots + x_k^2 = z^2$, assuming $x_1 \leq x_2 \leq \cdots \leq x_k \leq z$ (and so we know the largest integer in the coloring isn't just one of these, it must be z). At this time, the VI will not optimize for equations like $x + y + 3z = 10w$, where the only assumption we can make is that $x \leq y$. This is one improvement we hope to make in the future.

It is important to note that having more variables in $\mathcal{E}$ will (generally speaking) make the Rado number smaller. However, it increases the complexity of the VI by a factor of the current depth of the search (which is not a favorable trade-off). There is significant room for improvement in the VI, especially when handling nonlinear equations like $x^2 + y^2 = z^2$, where solutions to the equation are sparse. We will discuss some future improvements to the VI and other parts of this process in Section 6.1.

3.7.5 Engineering Challenges

Finally, we will discuss some of the challenges implementing the algorithm above. Although the algorithm may seem, at this point, very much complete, there are still some issues to address. In particular, we will discuss how information is written to disk and how the equation $\mathcal{E}$ is interpreted in the algorithm. We will also go over some issues related to using the Excalibur Blue Gene system. We should again acknowledge again

Rutgers Discovery and Informatics Institute (RDI^2) for its support for this research. These computing resources are truly invaluable.

A Benchmark

We will repeat this result explicitly in Section 4.4, however we will provide one example of the magnitude of these computations. This should give give some idea as to the arithmetic efficiency of the `RADO` package. In order to compute $\mathrm{R}_3\big(x + 3y = z\big) = 94$, we check over 200 billion colorings and check 90 trillion possible solutions. This takes 512 CPUs over 34 hours. The entire system runs at about 732 million checks of $\mathcal{E}$ per second, which is about 1.5 million per CPU per second.

It is not clear what is typical for a generic $\mathcal{E}$, since many computations are instantaneous and many are intractable, but this gives us some idea of the magnitude of some computations. Considering the scale of this computation, this example highlights how important it is to minimize the number of checks of $\mathcal{E}$ and to check $\mathcal{E}$ as efficiently as possible.

Filesystem & Hard Disk Issues

When exploring these trees, workers report back the maximal valid colorings (i.e. leaves on the tree). As they are generated, in no particular order, these colorings can be written to disk. This is easier said than done.

If a full certificate (the entire search tree) is being created, there are other manager processes that handle the output (a list of all maximal valid colorings) from the workers, in order to keep the master process free for the more important, low-latency task of keeping the workers working. These manager processes offload I/O from workers in order to prevent thrashing the hard disk and other I/O issues. The managers also compress the output using LZ4 compression, which allows for compression on-the-fly at a relatively good compression ratio. According to LZ4 specifications, LZ4 is about 66% as good as zlib, but 20 times as fast, in general. In our tests comparing LZ4 and zlib in `RADO`, we see about 50% as good compression but each manager can handle 15 workers, instead of 4, giving us a significant increase in the total number of workers. Workers send results to the managers in 32MB (uncompressed) batches, so that managers and workers spend less time on communication.

A partial certificate, containing only summary information, can also be produced (in which case, there are no manager processes).

Efficient Arithmetical Operations

In order to compute these Rado numbers, the numbers of colorings checked (i.e. number of nodes in the search) is extremely large, and for each coloring checked, the number of times the equation $\mathcal{E}$ is evaluated is likewise very large. The example in Section 3.7.5 gives us an idea of how intensive these computations can be.

In order to evaluate $\mathcal{E}$ this many times, within a feasible time-frame, the arithmetic operations that define $\mathcal{E}$ are translated into a function $f_{\mathcal{E}}$ (subtracting all terms to one side), and this function is evaluated repeatedly (and $\mathcal{E}$ is satisfied if and only if $f_{\mathcal{E}} = 0$). This simplifies our methodology slightly, but the real question is how to implement the function $f_{\mathcal{E}}$.

It would be possible to simply hard-code the function in `C++`. However, in that case it would be impractical to modify and recompile the `RADO` package for every single input $\mathcal{E}$ (in fact, strictly speaking, $\mathcal{E}$ would not be input). As noted above, the function $f_{\mathcal{E}}$ will be called billions, trillions, or even quadrillions of times. One would stagger to estimate the total number of times $f_{\mathcal{E}}$ has been evaluated for all $\mathcal{E}$ throughout the development and application of our `RADO` package.

Of course, one could very easily write a high-level version of the `RADO` package in Mathematica or Maple, where a function is a known type of data and could be input in the abstract sense. However, writing in a high-level computer-algebra system would be slow and impractical in many other ways, since our backtracking methodology is CPU-intensive. It is worth noting as well that parallel computation using proprietary software would also be prohibitively expensive.

From a design perspective, the goal is to take an input $\mathcal{E}$, build a function $f_{\mathcal{E}}$ in some way by parsing that input, and do so in a way that allows extremely rapid and repeated execution of $f_{\mathcal{E}}$ for numerous integer-valued inputs. In order to do this, we use a technique called just-in-time (JIT) compilation. A user-input string representing $f_{\mathcal{E}}$ is parsed and translated into machine code equivalent to that which a compiler would have produced if we had hard-coded the equation.

This JIT compilation allows our function to be comprised of a set of low-level operations corresponding to the arithmetical operations in $f_{\mathcal{E}}$, and will work for any function the user provides. It is the best of both worlds. The only restriction on user input is that $f_{\mathcal{E}}$ must be a polynomial with integer coefficients, completely expanded (i.e. $x^2 + 2x + 1$ and not $(x + 1)^2$). This restriction is somewhat arbitrary, but by imposing this structure on the input $\mathcal{E}$, translation to $f_{\mathcal{E}}$ is direct and provides an optimal representation of the arithmetic in the function $f_{\mathcal{E}}$.

Even in a non-parallelized algorithm, optimizing $f_{\mathcal{E}}$ to be evaluated as quickly as possible is an extremely effective optimization, since $f_{\mathcal{E}}$ is evaluated a huge number of times and becomes the bottleneck in any serious use of `RADO`.

Job Management

On the IBM Blue Gene system "Excalibur" users send their jobs through a front-end system that passes the job to the actual Blue Gene system. We have developed a number of scripts and auxiliary functions on the front end to perform useful functions, including:

- batch a number of jobs for some parametrized family (e.g. for the family $x+ay = z$, or simply $\mathcal{E}_a$, a script could generate jobs for each value of a, where the user provides parameters for the jobs along with a range of a values, so that multiple jobs are submitted);

- summarize the results of any particular computation, including statistics for the computation, a summary of the parameters for the job, and the Rado number;

- batch summarize a set of jobs, dumping the summary into a large output file that can be parsed to analyze a large family of related jobs; and

- run a series of tests against known values of Rado numbers and other performance tests to ensure that changes to the `RADO` package do not degrade performance or introduce errors or bugs.

One important feature we have not used is *Checkpointing* – where a job is essentially saved for later use. This would allow us to interrupt a job and resume it later, saving our progress, rather than face the limitations imposed by the scheduler (48 hours with 4096 CPUs, at most). Using this feature would allow us to tackle one or two extremely difficult computations that require more CPU power than the 48×4096 CPU-hours we can use now.

3.8 Grouping Single Rado Numbers into Families

Much of the computational work in this thesis uses the `RADO` package or a SAT solver, neither of which can tackle families of equations, just one equation at a time. This might result in the computation of a sequence of Rado numbers, like $S(r) = 1, 5, 14, 45$. It would then be the hope to use that information to discern a formula, and at least conjecture if not prove the formula, using some other tools. (Of course, in the case of $S(r)$ we have not yet made progress.)

Our goal is to use the Rado numbers we have produced, essentially combing these data for patterns to which we match our conjectures and eventual theorems about parametrized families of equations. Those conjectures can be proved either using some of the computational methods that apply to parameterized families or by traditional non-computational methods.

We will first note that in [JS04], [MR08], and [SW08], we find that Rado numbers for a particular family will often be quasipolynomials.

Definition 3.10. *The function $f : \mathbb{Z} \to \mathbb{R}$ or $f : \mathbb{Z}^+ \to \mathbb{R}$ is a quasipolynomial if there are polynomials $p_0, p_1, \ldots, p_{M-1}$ such that for all k in the domain of f, $f(k) = p_i(k)$ for $k \equiv i \pmod{M}$.*

It is not uncommon to find programs to fit quasipolynomials to data, including packages in Maple and the OEIS Super-Seeker, a service provided by the Online Encyclopedia of Integer Sequences, `http://oeis.org`. We have also constructed some packages in Mathematica that do the same with some human guidance to account for "noise" at low values of the function (since some of our functions will be quasipolynomials only when $k > 4$ or the like). However, we are required to have a significant amount of data before we can fit a quasipolynomial. For example, we could continue to investigate $x+y+kz=\ell w$ as in [MR08], and we might proceed with the belief that $\mathrm{R}_2\big(x+y+kz=\ell w\big) \sim (k/\ell)^2$, which for a fixed ℓ seems to give a quasipolynomial in k. But [SW08] shows that these quasipolynomials may have very large period (M above).

We use this methodology to formulate Conjecture 4.4 in Section 4.2.

The occurrence of quasipolynomials is natural in these contexts, when many arguments require us to say something like "consider the coloring that assigns red to all integers up to $\frac{m-1}{2}$," where this might mean the greatest integer up to, but not including, $\frac{m-1}{2}$. If that is a lower bound, then we add 1 to this quantity, which is precisely the ceiling of $\frac{m-1}{2}$. Similar constructions generally result in floors or ceilings, e.g. in the theorem of Schaal & Vestal [SV08]:

$$\mathrm{R}_2\big(x_1+\cdots+x_{m-1}=2x_m\big) = \left\lceil \frac{m-1}{2} \left\lceil \frac{m-1}{2} \right\rceil \right\rceil$$

Not only does this expression include a ceiling, it's actually an interesting nested pair of ceilings. However, some straightforward analysis of this function tells us that we could have said equivalently:

$$\mathrm{R}_2\big(x_1+\cdots+x_{m-1}=2x_m\big) = \frac{1}{4}\cdot\begin{cases} m^2-m & m\equiv 0 \pmod 4 \\ m^2-2m+1 & m\equiv 1 \pmod 4 \\ m^2-m+2 & m\equiv 2 \pmod 4 \\ m^2-2m+1 & m\equiv 3 \pmod 4 \end{cases}$$

This is illustrated in Figure 3.5, where a quasipolynomial of degree 2 and modulus 4 is used (in other words, we assume p_0, p_1, p_2, p_3 are all quasipolynomials of degree 2). This requires 8 interpolating points, which are colored blue. The following green points all coincide with these parabolas because we know the theorem to be true, but if we were unsure, this would certainly provide some empirical support. Of course, since two of these curves coincide, we only see three parabolas. (It might be hard to distinguish p_0 and p_2 at this scale as well).

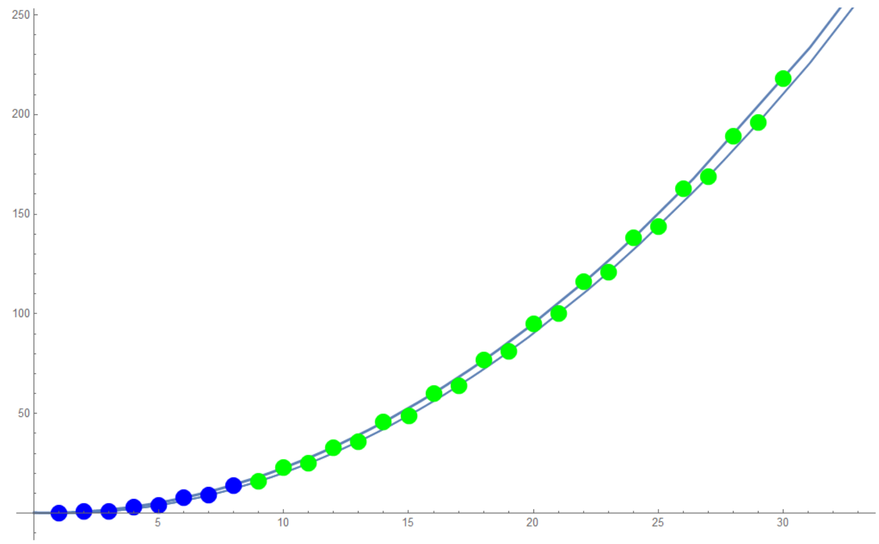

Figure 3.5: Quasipolynomial interpolation of $\left\lceil \frac{m-1}{2} \left\lceil \frac{m-1}{2} \right\rceil \right\rceil$

Many of the families of equations studied in this thesis are deliberately chosen far from the typical families studied in the literature, and as such will not necessarily exhibit the same behavior – defying our attempts to interpolate quasipolynomials to our data. However, it is still quite important to remark that this technique is often highly effective at fitting data. The leading term in each case is $m^2/4$, and we expect to see quasipolynomials in other cases where the leading term is the same for all moduli.

Chapter 4

Rado Numbers for Linear Equations

Many existing results in Diophantine Ramsey theory give specific Rado numbers for linear equations, typically in some parametrized families. The results of Guo & Sun [GS08] and of Saracino [Sar13] give some of the strongest generalizations of [BB82] thusfar. We will repeat these two as theorems here, respectively, for emphasis.

Theorem 4.1. *The 2-color Rado number* $\mathrm{R}_2\big(a_1x_1 + \cdots + a_{m-1}x_{m-1} = x_m\big)$, *where* $a_1 \leq a_i$ *and* $b = a_2 + \cdots + a_{m-1}$, *is* $a_1b^2 + (2a_1^2 + 1)b + a_1^3$.

Theorem 4.2. *The 2-color Rado number* $\mathrm{R}_2\big(x_1 + \cdots + x_{m-1} = ax_m\big)$ *is* $\left\lceil \frac{m-1}{a} \left\lceil \frac{m-1}{a} \right\rceil \right\rceil$.

We look to extend our understanding of linear equations and their Rado numbers by moving outside the confines of these two theorems – either by changing the equations or using 3 or more colors. In the next two sections, we will explore the 2-color Rado numbers for families of linear equations previously not investigated. As noted in Section 3.1.5, there is room for improvement in discovering Rado numbers for equations with three or four variables similar to $x + y + kz = \ell w$ or $ax + by = cz$.

4.1 The Equation $2x + 2y + kz = 3w$

In this section we will consider the equation $\mathcal{E}_k$ to be $2x + 2y + kz = 3w$.

$k =$	1	2	3	4	5	6	7	8	9	10	11	12	13
$\mathrm{R}_2(\mathcal{E}_k) =$	5	4	9	12	10	16	21	18	25	28	29	32	41
$k =$	14	15	16	17	19	21	23	25	27	29	31		
$\mathrm{R}_2(\mathcal{E}_k) =$	36	49	56	53	69	81	81	115	123	129	157		

Table 4.1: 2-color Rado numbers for $2x + 2k + kz = 3w$

As of yet, we are unable to determine the Rado number for this family. Although it appears to nearly fit our quasipolynomial approximations, it does not quite fit –

even when a few terms are discarded. We should also note that there is an interesting relationship between k and how difficult to compute the Rado seems. We will discuss that shortly. For now, we offer only the conjecture:

Conjecture 4.3. *The 2-color Rado number* $\mathrm{R}_2\big(2x + 2y + kz = 3w\big) \sim k^2/9$.

This conjecture is based on our analysis of the quasipolynomial approximations that do not fit, but seem to closely model the growth of this function of k. We will also suggest that the modulus of the quasipolynomial (assuming that is the correct type of function) will be divisible by 3.

We will take a quick look at the computational time and complexity required to produce Table 4.1.

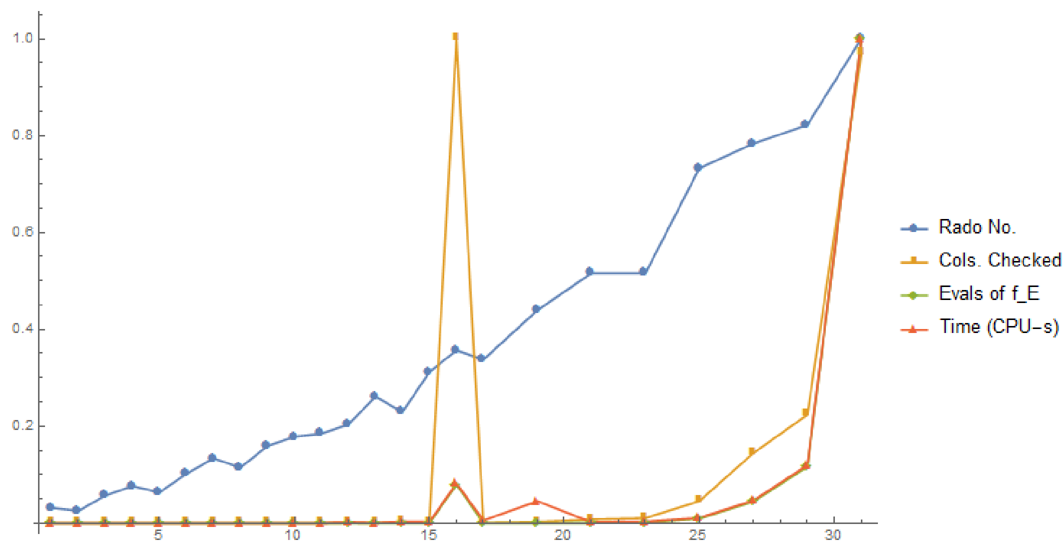

Figure 4.1: Data for the family $2x + 2y + kz = 3w$

All of the data series in Figure 4.1 have been normalized to $[0, 1]$. We can see here a very strong relationship between the number of evaluations of $f_{\mathcal{E}_k}$ and the time (in CPU·seconds) the computation requires. This confirms our analysis in Section 3.7.5 about the way in which the speed of evaluating f_E governs the efficiency of our `RADO` package.

We can also observe that the number of colorings checked trends generally with the number of evaluations of $f_{\mathcal{E}_k}$. Although we might expect that this would grow as the Rado number grows (since the maximum depth of the tree is longer), we might also hedge on the fact that the average depth of the tree does not necessarily have a relationship to the maximum depth.

The most apparent phenomenon in the figure is the huge spike at $k = 16$. Despite the fact that the Rado number seems innocuous, not out of place in the sequence, this computation took far longer than one might expect for it to produce that figure. Similar, but smaller, jumps occur at $k = 9, 11, 13, 19$. The prime or prime-power coefficient in this type of problem seems to make the tree much thicker, requiring a longer computing time, even though the Rado number is about the same.

This is not an isolated phenomenon. For example, $\mathrm{R}_2\big(x^2 + y^2 + 13z^2 = w^2\big)$ is extremely difficult to compute, while the same with 12 instead of 13 takes only a few seconds to compute. We could even consider the most extreme example so that we can consider why this happens. Consider $x_1 + \cdots + x_{18} = x_{19}$ and $7x + 11y = z$. We would expect that the first is much easier to compute – not just because of the Rado number being lower (by Lemma 3.8), but because in general the complexity of the tree is much worse in the latter case. If we let $z = x_{19} = N$ for some fixed N, how many ways can we write N as the sum of 18 integers, vs. the sum $7x + 11y$? The flexibility in the first case means that branches of that search tree are pruned earlier.

Although this heuristic seems fairly vague and relatively anecdotal, prime and prime-power coefficients (in linear or nonlinear equations) seem to coincide frequently with especially time-intensive computations.

4.2 The Equation $kx + (k + 1)y = (k + 2)z$

In this section, let $\mathcal{E}_k$ denote the equation $kx + (k + 1)y = (k + 2)z$. This family of linear equations is a one-parameter section of the general family $ax + by = cz$.

As described in Section 3.1.5, the family $ax + by = cz$ is at the heart of the partial order of linear equations, and an explicit formula for $\mathrm{R}_2\big(ax + by = cz\big)$ would be a tremendous breakthrough. Attacking this family $\mathcal{E}_k$ is similar to what has been done with $ax + ay = bz$, $ax + by = bz$, and the like (see Figure 3.1.5).

Using the `RADO` package, we can compute the value of $\mathrm{R}_2\big(\mathcal{E}_k\big)$ for small values of k

as follows:

$k =$	1	2	3	4	5	6	7	8	9	10	11	12	13
$\mathrm{R}_2(\mathcal{E}_k) =$	1	10	17	15	23	28	36	45	55	71	78	97	105
$k =$	14	15	16	17	18	19	20	21	22	23	24	25	
$\mathrm{R}_2(\mathcal{E}_k) =$	120	136	153	171	190	210	231	253	276	300	325	351	

Table 4.2: 2-color Rado numbers for $kx + (k+1)y = (k+2)z$

In Appendix B, a more detailed analysis of the computational data for this family is presented.

As described in Section 3.8, we can attempt to interpolate this family using a quasipolynomial. For example, an email to `superseeker@oeis.org` with the following text:

```
lookup 1 10 17 15 23 28 36 45 55 71 78 97 105 120 136 153 171 190 210
```

will return an email that says, in part, the following:

```
SUGGESTION: apparently the differences of order 2 in the difference
table of depth 1 have become constant.  If this is true then the
next four terms of the sequence are:
[231, 253, 276, 300]
TRY "RATE", CHRISTIAN KRATTHENTHALER'S MATHEMATICA PROGRAM FOR GUESSING
A CLOSED FORM FOR A SEQUENCE. ("Rate" is "Guess" in German.  For
a description of RATE, see
http://www.mat.univie.ac.at/~kratt/rate/rate.html)
RATE found the following formula(s) for the nth term:
((1 + n)*(2 + n))/2
```

We note that the next four terms provided are correct (and for all other terms computed, in fact), which inspires the following conjecture:

Conjecture 4.4. *The 2-color Rado number* $\mathrm{R}_2\big(kx + (k+1)y = (k+2)z\big)$ *is* $\frac{k^2+3k+2}{2}$ *for* $k \geq 11$.

We can verify this conjecture in many different ways (although of course, verification is verification, regardless of how many times or ways it is done), but we cannot produce this result using abstract or symbolic computation – thus not proving it for all k. The maximal colorings are given in Appendix B and we note that these colorings do not conform to the typical patterns exhibited in the existing literature, as described in Chapter 3.

4.3 Off-Diagonal Rado Numbers

In Section 3.1.4, we discuss the definition of an off-diagonal Rado number, where each color has its own equation. In the case of 2-color off-diagonal Rado numbers, the result from [MR07] in that section gives us a good idea that we can proceed without much fear of trying to compute infinite quantities.

We offer the following data for a family of new off-diagonal Rado numbers. Let $\mathcal{E}_k$ bet $x + y = kz$. Then we can compute the following Rado numbers for $\mathrm{R}_2\big(\mathcal{E}_k; \mathcal{E}_\ell\big)$.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	5	2	8	10	8	7	11	9	14	11	17	13	20	15	23	17	26	19	29	21
2		1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10
3			9	6	8	9	11	12	18	15	17	18	20	21	23	24	26	27	29	30
4				10	10	12	14	16	18	20	22	24	26	28	30	34	36	38	40	42
5					15	15	28	30	35	38	43	45	50	53	58	60	65	68	73	75
6						21	21	36	42	45	51	54	60	63	69	72	78	81	87	90
7							28	28	49	53	77	84	91	98	105	116	119	126	133	140
8								36	48	68	68	96	104	112	120	128	136	144	152	160
9									45	59	81	81	90	126	171	180	194	203	216	225
10										55	70	60	115	155	150	200	215	225	240	250
11											66	83	99	132	127	132	187	248	314	330
12												78	96	114	150	174	228	294	288	360
13													91	111	130	169	195	254	286	260
14														105	126	147	210	217	245	315
15															120	150	165	223	240	270
16																136	168	184	208	264
17																	153	187	221	230
18																		171	207	225
19																			190	228
20																				210

Table 4.3: Table of Rado numbers for sums of squares.

Here in Table 4.3, we have computed a large number of these. Note that we only fill in the table for $k \leq \ell$, since the off-diagonal Rado number operator has symmetry in its arguments. We might also note that the diagonal of this table includes $\mathrm{R}_2\big(\mathcal{E}_k; \mathcal{E}_k\big) =$

$\mathrm{R}_2(\mathcal{E}_k)$, which reminds us how we named these "off-diagonal" in the first place. Those are already known due to the results in [LR03] (or to over-do it, [Sar13]) to be $\mathrm{R}_2(x + ky = z) = \binom{k+1}{2}$ (for $k \geq 4$).

We offer the following conjectures:

Conjecture 4.5. *For $\ell \geq 5$, the 2-color Rado number $\mathrm{R}_2(\mathcal{E}_1; \mathcal{E}_\ell)$ is $\ell + 1$ for ℓ even and $(3\ell + 1)/2$ for ℓ odd.*

We initially formulated this conjecture by interpolatating the data as in Section 3.8, although this sequence already exists in the Online Encyclopedia of Integer Sequences as A063914.

Conjecture 4.6. *For $\ell \geq 2$, the 2-color Rado number $\mathrm{R}_2(\mathcal{E}_2; \mathcal{E}_\ell)$ is $\lfloor \frac{\ell}{2} \rfloor$.*

We will note that this is a quasipolynomial, since $\lfloor \frac{\ell}{2} \rfloor$ could only be $\frac{\ell}{2}$ or $\frac{\ell-1}{2}$. Our next conjecture will be in the form of a floor, and we will no longer continue to remind the reader that this is also a quasipolynomial.

Conjecture 4.7. *For $\ell \geq 10$, the 2-color Rado number $\mathrm{R}_2(\mathcal{E}_3; \mathcal{E}_\ell)$ is $\lfloor \frac{3\ell+1}{2} \rfloor$.*

Although we do not have sufficient data to formulate a more precise conjecture, we would like to offer the following conjecture to end this section:

Conjecture 4.8. *For ℓ and k sufficiently large, the Rado number $\mathrm{R}_2(\mathcal{E}_k; \mathcal{E}_\ell)$ is a quasipolynomial in k and ℓ of degree 2.*

4.4 Some 3-color Rado Numbers

It is possible to use the `RADO` package to compute Rado numbers with any number of colors, although current implementation makes the (arbitrary) restriction that one must use between 2 and 8 colors. It is important to recall that Rado's Theorems 2.9 and 2.10 do not provide precise conditions for the 3-regularity of any type of equation, and as we noted, there is a distinction between k- and $(k + 1)$-regularity for all k.

As mentioned in Section 3.7, in order to avoid lengthy computations (especially in the case when the Rado number could be infinite), we have hard-coded a bail-out value of 1000. Should the search of the tree of all valid colorings ever reach this length, the node at that depth will return an error and the master process will terminate the entire computation. There is, of course, no reason to believe there are not 3-color Rado numbers greater than 1000 (surely, there must be), but it is also not safe to assume that any computation of this type will terminate in any reasonable timespan, since it could easily be the case that the tree is unbounded (as easily as $x + 2y = 4w$).

However, Rado's Theorem (Theorem 2.9) provides conditions that are sufficient (although stronger than necessary), giving us a set of equations we can guarantee are safe in at least some sense. We first present some results of that type:

Theorem 4.9. *The 3-color Rado number* $\mathrm{R}_3\big(x + y + z = w\big)$ *is 43.*

Note that this is a generalized Schur number with $m = 3, r = 3$. This result is not new, and we can present Table 4.4. The first row and column are each trivial, the second row is a consequence of Theorem 3.1 from [BB82], and the other non-underlined data are drawn from [Ahm], which provides some new results as well as a summary and some background on generalized Schur numbers. The underlined data are new results in this dissertation.

r	m in $x_1 + \cdots + x_{m-1} = x_m$								
	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9
2	1	5	11	19	29	41	55	71	89
3	1	14	43	94	173	$\underline{286}$	$\underline{439}$	$\underline{638}$	$\underline{889}$
4	1	45							
5	1								

Table 4.4: Generalized Schur numbers

All 2-color generalized Schur numbers were given by [BB82], where it is determined that $S_m(2) = m^2 - m - 1$. Theorem 3.3 gives us a bound of $\mathrm{R}_r\big(x_1 + \cdots + x_{m-1} = x_m\big) \geq m^r - (m^{r-1} + m^{r-2} + \cdots + m^2 + m + 1)$, but from the table above we can see that this lower bound is not tight in all cases. It remains to be seen whether $\mathrm{R}_3\big(x_1 + \cdots + x_{m-1} = x_m\big) = m^3 - m^2 - m - 1$. We would agree at this time with the conjecture of [Ahm] that this is true, considering we have verified this for twice as many values of m as known previously. We will discuss those results shortly, but we begin with the following theorem:

Theorem 4.10. *The 3-color Rado number* $\mathrm{R}_3\big(x+y+2z=w\big)$ *is 94.*

This Rado number was proved using the `RADO` package. It required approximately 22.5 hours on 128 nodes (512 CPUs), checking over 4 million colorings and evaluating the polynomial $f_{\mathcal{E}}(x,y,z,w)=x+y+2z-w$ about 95 trillion times. The writing of a certificate has been suppressed due to considerations of disk space. One might make a conservative estimate that the certificate would exceed the terabyte range – in fact, we owe our apologies to the administrators of the Excalibur system for accidentally filling up the entire hard drive on at least one occasion.

Theorem 4.11. *The 3-color Rado number* $\mathrm{R}_3\big(2x+2y=3z\big)$ *is 54.*

This Rado number was also proved using the `RADO` package. The computation only required 8 seconds on 128 nodes (512 CPUs), checking only about a million colorings and evaluating the polynomial $f_{\mathcal{E}}(x,y,z) = 2x+2y-3z$ about 380 million times. Compared to Theorem 4.10, which might seem like a similar computation, this was actually much faster. That is strange for a number of reasons, not the least of which is that the equation $2x+2y=3z$ has fewer variables (making it, heuristically, harder to compute), not to mention that this is not guaranteed to be finite, *a priori*.

Interestingly, it is not nearly as easy to compute $\mathrm{R}_3\big(2x+2y=kz\big)$ for other odd values of k (not even $k=1$). It is important to note that there is no guarantee that these numbers are finite, let alone within the computing power of `RADO` or any other algorithm.

We may revisit Theorem 4.10 to generate the following data, which inspires us to make the following computations:

$k=$	1	2	3
$\mathrm{R}_3\big(\mathcal{E}_k\big)=$	43	94	173

Table 4.5: 3-color Rado numbers for $\mathcal{E}_k$, $x+y+kz=w$

We now generalize our focus a bit to the equation $x_1+\cdots+x_{m-2}+kx_{m-1}=x_m$. This generalization may seem to increase the complexity of the equation, but it allows us to make more computations by potentially reducing the numbers of variables in our equations. The figures in Table 4.6 are computed using `RADO` or SAT methods.

	$k=1$	2	3	4	5	6
$m=3$	14	43	94	173	286	439
4	43	94	173			
5	94					
6	173					
7	$\star$					

Table 4.6: 3-color Rado numbers for $x_1 + \cdots + x_{m-2} + kz = w$

Note that the first row is a part of Section 4.4.1, while the first column consists of 3-color generalized Schur numbers (all known values are $m^3 - m^2 - m - 1$). Along the antidiagonals (where $k + m$ is constant), we know by Lemma 3.8 that these increase from bottom-left to top-right. However, in all of these computations we should note that these are equal. We believe this pattern continues, and will make the relevant conjecture shortly.

We should note that this allows us to compute the values for $(m, k) = (5, 2)$ and $(4, 3)$ implicitly – since they must be between the values for $(6, 1)$ and $(3, 4)$. This also gives us an upper bound on the generalized Schur number $\mathrm{R}_3\big(x_1 + \cdots + x_6 = x_7\big) \leq 286$ (the location of this is noted with a $\star$). However, we already know that $m^3 - m^2 - m - 1$ is a lower bound for the generalized 3-color Schur numbers as in [BB82], which is precisely 286 for $m = 7$. The same is true for the 8-variable equation, and so this gives us a more complete Table 4.7 and also a strong Theorem 4.12.

	$k=1$	2	3	4	5	6
$m=3$	14	43	94	173	286	439
4	43	94	173	286	439	
5	94	173	286	439		
6	173	286	439			
7	286	439				
8	439					

Table 4.7: 3-color Rado numbers for $x_1 + \cdots + x_{m-2} + kz = w$, version 2

Theorem 4.12. *For $m = 7, 8, 9, 10$, the 3-color generalized Schur number $\mathrm{R}_3\big(x_1 + \cdots + x_{m-1} = x_m\big)$ is $m^3 - m^2 - m - 1$.*

The proof is simply done by computing $\mathrm{R}_3\big(x + ky = z\big) = 286, 439, 638, 889$ for $k = 5, 6, 7, 8$. (Note that not all of these data are included in Table 4.7.) Those computations are done by SAT methods, although $k = 5$ was also verified by `RADO`. Even though each of these computations is harder than the corresponding generalized Schur number, in the sense that the Rado number must be (non-stritly) larger, it requires us to consider equations with a fixed number of variables (in particular, fewer variables!) which is much better for either the SAT method or `RADO`. This indirect sort of computation is precisely why we make very clear the importance of the ideas in Section 3.1.5. We make the following conjecture:

Conjecture 4.13. *For $k \geq 1$, the 3-color Rado number $\mathrm{R}_3\big(x + ky = z\big)$ is $(k + 2)^3 - (k + 2)^2 - (k + 2) - 1$. (This implies the existing popular conjecture that the 3-color generalized Schur number $\mathrm{R}_3\big(x_1 + \cdots + x_{m-1} = x_m\big)$ is $m^3 - m^2 - m - 1$.)*

We can also compute a similar set of data for $x + y + z = kw$, and if we can generalize some of the work of [Sch95] and [BB82], the Table 4.8 should inspire us to consider generalizing the work of [Sar13].

$k =$	1	2	3	4	5	6	7	8
$\mathrm{R}_3\big(\mathcal{E}_k\big) =$	43	8	1	12	25	40	171	422

Table 4.8: 3-color Rado numbers for $\mathcal{E}_k$, $x + y + z = kw$

4.4.1 3-color Off-Diagonal Rado Numbers

In Section 4.3, we discuss off-diagonal Rado numbers. Definition 3.5 was stated to apply to more than two colors, so we need not generalize it here. We will now discuss some results for 3-color off-diagonal Rado numbers.

If we let $\mathcal{E}_a$ be $x + ay = z$, we can compute some values of $\mathrm{R}_3\big(\mathcal{E}_a; \mathcal{E}_b; \mathcal{E}_c\big)$, which are shown in Table 4.4.1. This table was computed using a combination of the `RADO` package and SAT solving programs.

Many of these computations were very difficult, requiring significant computing time and power. Unfortunately, even with this much data, we cannot yet offer a conjecture

(a,b,c)	$\mathrm{R}_3(\mathcal{E}_a;\mathcal{E}_b;\mathcal{E}_c)$	(a,b,c)	$\mathrm{R}_3(\mathcal{E}_a;\mathcal{E}_b;\mathcal{E}_c)$
(1,1,1)	14	(2,2,5)	76
(1,1,2)	24	(2,3,3)	69
(1,1,3)	39	(2,3,4)	83
(1,1,4)	57	(2,3,5)	98
(1,1,5)	90	(2,4,4)	101
(1,2,2)	31	(2,4,5)	119
(1,2,3)	47	(2,5,5)	140
(1,2,4)	49	(3,3,3)	94
(1,2,5)	67	(3,3,4)	113
(1,3,3)	58	(3,3,5)	132
(1,3,4)	70	(3,4,4)	137
(1,3,5)	82	(3,4,5)	160
(1,4,4)	85	(3,5,5)	188
(1,4,5)	107	(4,4,4)	173
(1,5,5)	124	(4,4,5)	202
(2,2,2)	43	(4,5,5)	237
(2,2,3)	54	(5,5,5)	286
(2,2,4)	65		

Table 4.9: 3-color off-diagonal Rado numbers for $x + ay = z$ for $a \leq 5$

for some closed-form for $\mathrm{R}_3(\mathcal{E}_a;\mathcal{E}_b;\mathcal{E}_c)$. However, each of these is a bound on the corresponding off-diagonal generalized Schur number.

4.5 Some 4-color Rado Numbers

We will end this chapter with some 4-color Rado numbers. We will examine the equation $x + y + c = z$. We can prove the Rado numbers listed in Table 4.10.

$c =$	-2	-1	0	1	2	3	4	5	6	7
$\mathrm{R}_4(x + y + c = z) =$	2	1	45	83	121	161	201	241	281	321

Table 4.10: 4-color Rado numbers for $x + y + c = z$

It is interesting to note that we can very easily construct a bound for, say, $c = 6$ – it takes less than a second. But it takes several CPU-hours to confirm that this lower bound is tight. For $c < 0$, it is easy to see that $\mathrm{R}_r(x + y + c = z) \leq -c$, since $(-c, -c, -c)$ is a solution. We can verify that this bound is tight for $-20 \leq c \leq -1$ (which is not interesting enough to be included in the table above).

In light of the fact that we know $\mathrm{R}_2(x + y + c = z) = 4c + 5$ and $\mathrm{R}_3(x + y + c =$

$z\big) = 13c + 14$ (for $c \geq 0$), we can see that it is not quite the same here. The function is not linear for $c \geq 0$, but we can modify the appropriate conjecture accordingly:

Conjecture 4.14. *The 4-color Rado number* $\mathrm{R}_4\big(x + y + c = z\big)$ *with* $c \geq 2$ *is* $40c + 41$*.*

We will even offer another conjecture, based on these observations. We suspect this conjecture could be adapted to a similar statement about non-homogeneous generalized Schur numbers for more than four colors.

Conjecture 4.15. *For each* $r > 0$ *and* $c \geq 0$*, there are constants* C *and* m *such that* $\mathrm{R}_r\big(x + y + c = z\big) = mc + m + 1$ *for* $c \geq C$*.*

We also believe it is true that $m \leq \mathrm{R}_r\big(x + y = z\big)$.

Although at this time we are not able to prove Conjecture 4.14e, we can provide proof of the lower bound. We should acknowledge that this draws heavily from the ideas in [Sch95]. This is proved using the ideas in Section 3.3, and we also believe the conjecture can be proved completely using the techniques used to prove the corresponding results for fewer colors.

Proof. Let $\mathcal{E}_c$ be our equation $x + y + c = z$. Consider the coloring $\chi : [40c + 41] \to [4]$ defined as follows:

$$\chi(i) = \begin{cases} 1, & 1 \leq i \leq c+1 \\ 2, & c+2 \leq i \leq 3c+3 \\ 1, & 3c+4 \leq i \leq 4c+4 \\ 3, & 4c+5 \leq i \leq 9c+9 \\ 1, & 9c+10 \leq i \leq 10c+10 \\ 2, & 10c+11 \leq i \leq 12c+12 \\ 1, & 12c+13 \leq i \leq 13c+13 \\ 4, & 13c+14 \leq i \leq 27c+27 \\ 1, & 27c+28 \leq i \leq 28c+28 \\ 2, & 28c+29 \leq i \leq 30c+30 \\ 1, & 30c+31 \leq i \leq 31c+31 \\ 3, & 31c+32 \leq i \leq 36c+36 \\ 1, & 36c+37 \leq i \leq 37c+37 \\ 2, & 37c+38 \leq i \leq 39c+39 \\ 1, & 39c+40 \leq i \leq 40c+40 \end{cases}$$

By the result on 3-color non-homogeneous Schur numbers in [Sch95], we know that $[1, 13c+13]$ does not contain a monochromatic solution to $\mathcal{E}_c$. It should also be self-evident that there is no solution of color 4, since that solution would have to have:

$$z = c + x + y \geq c + (13c+14) + (13c+14) = 27c + 28.$$

It remains to show that there is no monochromatic solution that involves x, y, or z greater than $27c+27$. Assume we have a monochromatic solution (x, y, z) to $\mathcal{E}_c$. Since z is the largest of the three, we know $z \geq 27c+28$. However, if x and y are both less than $27c+28$, they can be at most $13c+13$ (in order to agree in color with z, they cannot be color 4). This would imply:

$$27c + 28 \leq z = x + y + c \leq (13c+13) + (13c+13) + c = 27c + 26.$$

And so in any case, we must have z and (without loss of generality) y both greater than $27c + 27$.

If the solution is color 3, we have $z - y \leq 5c + 4$, since the interval of color 3 containing y and z is $[31c + 32, 36c + 36]$, which is precisely that long. This means $x = z - y - c \leq 4c + 4$, so that x cannot be color 3 after all.

If the solution is color 2, y and z must belong to one of the the intervals $[28c + 29, 30c + 30], [37c + 38, 39c + 39]$. If z and y are in the same of these two intervals, then their difference is at most $2c + 1$ (the length of these intervals). But in that case we have $x = z - y - c \leq (2c + 1) - c = c + 1$, which implies x is color 1 instead.

If z is from $[37c + 38, 39c + 39]$ and y from $[28c + 29, 30c + 30]$, then their difference must be somewhere in $[7c + 8, 11c + 10]$, putting x into $[6c + 8, 10c + 10]$. However, that interval is colored partially color 3, and partially color 1.

If this solution is color 1 (this is the worst case, clearly), then y and z each belong to one of four intervals:

$$[27c + 28, 28c + 28], [30c + 31, 31c + 31], [36c + 37, 37c + 37], [39c + 40, 40c + 40]$$

Of course, they cannot belong to the same interval, because these intervals are length c and would give $z - y \leq c$ (i.e. $x \leq 0$).

If y is in $[27c+28, 28c+28]$ and z in $[30c+31, 31c+31]$, then the difference $z-y = x+c$ is at most $4c+3$ and at least $2c+3$, meaning $c+3 \leq x \leq 3c+3$. However, that implies that $\chi(x) = 2$ instead of 1.

If y is in $[36c + 37, 37c + 37]$ and z in $[39c + 40, 40c + 40]$, the previous case applies.

If y is in $[27c + 28, 28c + 28]$ and z in $[36c + 37, 37c + 37]$, then we find $z - y$ must be in the interval $[8c + 9, 10c + 9]$, meaning x is in $[7c + 9, 9c + 9]$. This is a contradiction, since that would give $\chi(x) = 3$.

If y is in $[30c + 31, 31c + 31]$ and z in $[39c + 40, 40c + 40]$, the previous case applies.

If y is in $[27c + 28, 28c + 28]$ and z in $[39c + 40, 40c + 40]$, then $z - y$ must be in $[11c + 13, 13c + 12]$, meaning x is in $[10c + 13, 12c + 12]$, meaning x is color 4.

Finally, if y is in $[30c + 31, 31c + 31]$ and z in $[36c + 37, 37c + 37]$, then $z - y$ is in $[5c + 6, 7c + 6]$, putting x in $[4c + 6, 6c + 6]$, which would give $\chi(x) = 3$.

By this exhaustive set of contradictions, we find this coloring is in fact valid, and this proves the lower bound. $\square$

We should say that the coloring χ in this proof is the same as the coloring Δ in [Sch95], followed by an interval as large as possible of the new fourth color, followed by another copy of Δ shifted appropriately. This argumentation provides us with the value of m to refine Conjecture 4.15 to a stronger conjecture.

Conjecture 4.16. *For each $r \geq 1$ and $c \geq 0$, there is a constant C such that $\mathrm{R}_r\big(x + y + c = z\big) = \frac{3^r-1}{2}c + \frac{3^r-1}{2} + 1$ for $c \geq C$.*

Let $a_r = \frac{3^r-1}{2}$. The lower bound in these cases can be demonstrated just like in the proof above by taking the existing constructions for $r = 2, 3, 4$ as χ_2, χ_3, χ_4 and extrapolating to construct χ_r by coloring $[1, a_{r-1}c + a_{r-1}]$ according to χ_{r-1}, then coloring $[a_{r-1}c + a_{r-1} + 1, (2a_{r-1} + 1)c + 2a_{r-1} + 1]$ the new color r, then coloring the remainder $[(2a_{r-1}+1)c+2a_{r-1}+1, (3a_{r-1}+1)c+3a_{r-1}+1]$ according to χ (translated). This is an interesting sequence of colorings – each is an extension of the previous, and there is a sort of "self-similarity" here. The proof that this coloring is valid will be the same as that already given for $c = 4$.

As a lower bound for Schur numbers, this is asymptotically similar to that of [BB82], which gives $\mathrm{R}_r\big(x+y+c = z\big) \geq \frac{2859 \cdot 3^{n-7}+1}{2}$, and like [BB82], our result provides explicit lower bounds for Ramsey numbers for triangles as noted in Section 2.2. (These bounds are, of course, not the best known.)

We can prove a general lower bound by applying this argument to r colors, by induction:

Theorem 4.17. *For $r \geq 0$ and $c \geq 0$, $\mathrm{R}_r\big(x + y + c = z\big) \geq \frac{3^r-1}{2}c + \frac{3^r-1}{2} + 1$.*

Proof. For a fixed c, define χ_2, χ_3, χ_4 as noted above (χ_3 is Schaal's Δ and χ_4 is our previous χ). For any r, define χ_r recursively as described above (a copy of χ_{r-1}, then the largest possible interval of the new color r, then another copy of χ_{r-1}).

We have observed that this coloring is the appropriate length, although we can of course verify this: If we let the length of χ_r be $a_r c + a_r$, then the length of χ_{r+1} will have to be $2(a_r c + a_r)$ plus this interval of color r, which can be at most the interval $[a_r c + a_r + 1, 2(a_r c + a_r + 1) + c - 1]$, which gives us $a_{r+1} = 3a_r + 1$. We can check that $\frac{3^r - 1}{2}$ satisfies this recurrence & initial values.

By construction, there are no monochromatic solutions of color r. By induction, there can be none in the truncation of χ_{r+1} to the domain of χ_r (since it agrees with χ_r on this domain.

Because the interval of color r is sufficiently long, we may again conclude that in any monochromatic solution we have not yet eliminated, y and z must be on the same side of this long interval of color r (assuming again that $x \leq y \leq z$). However, we can take care of this case either by some argumentation like the previous proof, or with a little sleight of hand. If (x, y, z) is a solution and the gap between x and y spans this long interval colored r, then we are really going to start arguing about the difference of $z - y$ (or $z - y - c$) and the relative location of z and y will be all that matters because $z - y$ (or $z - y - c$) is translation invariant.

In particular, translate y and z down by $\tau = 2(a_r c + a_r + 1) + c$, so that they lie in the exact same spot in the first copy of χ that they did in the second copy of χ (so that $\chi(y) = \chi(y - \tau)$ and $\chi(z) = \chi(z - \tau)$). Because it lies entirely in the domain of χ_{r-1}, this new solution $(x, y - \tau, z - \tau)$ cannot be monochromatic so neither can (x, y, z) be.

So the coloring χ_r is valid, and by induction, this holds for all r. $\square$

We can offer the following generalization as the final result of this chapter:

Theorem 4.18. *For $m \geq 3$, $r \geq 1$, and $c \geq 0$,* $\mathrm{R}_r\big(x_1 + \cdots + x_{m-1} + c = x_m\big) \geq (m^r - m^{r-1} - \cdots - m^2 - m - 1) + (m^{r-1} + m^{r-2} + \cdots + m^2 + m + 1)c = \frac{m^{r+1} - 2m^r + 1}{m-1} + \frac{m^r - 1}{2}c.$

Proof. First, for the sake of completeness, consider the case $r = 1$. Our claim is that $\mathrm{R}_1\big(x_1 + \cdots + x_{m-1} + c = x_m\big) \geq (m-1) + (1)c$, which is of course tight. In any solution $x_m \geq m - 1 + c$ (so the 1-coloring of $[1, m - 2 + c]$ is valid), but the solution $(1, 1, 1, \ldots, 1, m-1+c)$ will make any coloring of length $m-1+c$ invalid (here "coloring" is meaningless, we have only one color, so the elements 1 and $m - 1 + c$ must be the same color).

For a fixed r and m, consider the coloring χ_2 as defined in [Sch93]. This coloring is:

$$\chi_2(i) = \begin{cases} 0 & 1 \le i \le m-2+c \\ 1 & m+c-1 \le i \le m^2+cm-2m \\ 0 & m^2+cm-2m+1 \le i \le m^2-m-1+c(m+1)-1 \end{cases}$$

This coloring is proved to be valid by Schaal – he proves our lower bound for $r = 2$ (and that it is tight in cases where the Rado number is finite). Note the $m-2+c$ as in the trivial case of $r = 1$. We will again follow the pattern as in Theorem 4.17 and construct the coloring χ_r by sandwiching a run of color r of maximal size between two copies of χ_{r-1}. In particular, if we let $f_{r-1}(m) + g_{r-1}(m)c$ be the length of χ_{r-1}, we can construct the coloring χ_r as:

$$\begin{array}{lclcl} [\ 1 & , & f_{r-1}(m)+g_{r-1}(m)c\ ] & \mapsto & \chi_{r-1} \\ [\ f_{r-1}(m)+g_{r-1}(m)c+1 & , & (m-1)(f_{r-1}(m)+g_{r-1}(m)c+1)+c-1\ ] & \mapsto & r \\ [\ (m-1)(f_{r-1}(m)+g_{r-1}(m)c+1)+c & , & (m-1)(f_{r-1}(m)+g_{r-1}(m)c+1)+c+f_{r-1}(m)+g_{r-1}(m)c-1\ ] & \mapsto & \chi_{r-1}\circ\tau \end{array}$$

where τ represents the translation by $(m-1)(f_{r-1}(m) + g_{r-1}(m)c + 1) + c$. Note that the length of χ_r can be simplified to:

$$(m-1)(f_{r-1}(m)+g_{r-1}(m)c+1)+c+f_{r-1}(m)+g_{r-1}(m)c-1 = (mf_{r-1}(m)+m-2)+(mg_{r-1}(m)+1)c.$$

The reader can verify that there are no solutions of color r by confirming that the least element of color r could be substituted for $x_1, \dots, x_{m-1}$ to obtain a value of x_m that is $(m-1)(f_r(m)+g_r(m)c+1)+c$, which is just past the end of the color-r interval.

Likewise, it is easy to observe that:

$$2\big((m-1)(f_{r-1}(m)+g_{r-1}(m)c+1)+c\big)+c > (mf_{r-1}(m)+m-2)+(mg_{r-1}(m)+1)c,$$

since, in particular, the difference can be written $4c + m + (m-2)f_{r-1}(m) + (3cm - 2c)g_{r-1}(m)$. This means at most one of $x_1, \dots, x_{m-1}$ can be in the last of our three intervals (and we may assume it is x_{m-1}). The same translation argument from Theorem 4.17 (with $x_m - x_{m-1}$ in the role of $z - y$) applies, proving that if χ_{r-1} is valid, so is χ_r.

Finally, the recurrences $f_r(m) = mf_{r-1}(m) + m - 2$ and $g_r(m) = mg_{r-1}(m) + 1$ with initial values $f_2(m) = m^2 - m - 2$ and $g_2(m) = m + 1$ have unique solutions $f_r(m) = m^r - m^{r-1} - \cdots - m^2 - m - 2$ and $g_r(m) = m^{r-1} + m^{r-2} + \cdots + m^2 + m + 1$. $\square$

We also conjecture that this bound is tight, depending now on m as well as r:

Conjecture 4.19. *In Theorem 4.18, for each m and r there is C such that this lower bound is tight for $c \geq C$, so long as c is even or m odd.*

The exclusion of certain (c, m) pairs is due to the fact that if c is odd and m is even, the equation is not 2-regular, as proved in the original [Sch93].

We should note that this provides a bound on Ramsey numbers as well:

Corollary 4.20. *The r-color Ramsey number $R(\underbrace{m, m, \ldots, m}) \geq m^r - m^{r-1} - \cdots - m^2 - m$.*

This is established using the construction from Section 2.3. We drop the -1 term because we can actually do one better: we can color edges labeled 1 through $n+1$ by their differences only using 1 through n (there is no difference of $n+1$). It should be noted that this does not provide particularly strong bounds for Ramsey numbers for two (heuristic, but empirically supported) reasons. First, it is not necessarily true that as m and r grow the Schur numbers and corresponding Ramsey numbers continue to be close to one another. Second, we have conjectured that our theorem is tight only for c sufficiently large. These bounds are the worst case, in that sense, with $c = 0$. In [Rad14], Table X gives lower bounds that are much larger than this. However, this bound is interesting because it is constructive, it holds for all m and r, and it continues to explore the relationship between Schur and Ramsey numbers.

Figure 4.5 shows the 4-coloring of K_{41} avoiding monochromatic triangles that we derive from our coloring of $[1, 40]$. Likewise, Figure 4.5 shows the 4-coloring of K_{171} avoiding monochromatic K_4.

We can also combine this bound with Lemma 3.8 to obtain the following corollary. Note that in this formulation, we continue to follow our conventions that $a_i > 0$.

Corollary 4.21. *For $c \geq 0$ and an equation $\mathcal{E}$ of the form $\sum_{i=1}^{m-1} a_i x_i \ + \ c = x_m$, letting $m_0 = \sum_{i=1}^{m-1} a_i \ + \ 1$, the r-color Rado number $\mathrm{R}_r(\mathcal{E})$ is at least $(m_0^r - m_0^{r-1} - \cdots - m_0^2 - m_0 - 1) + (m_0^{r-1} + m_0^{r-2} + \cdots + m_0^2 + m_0 + 1)c$.*

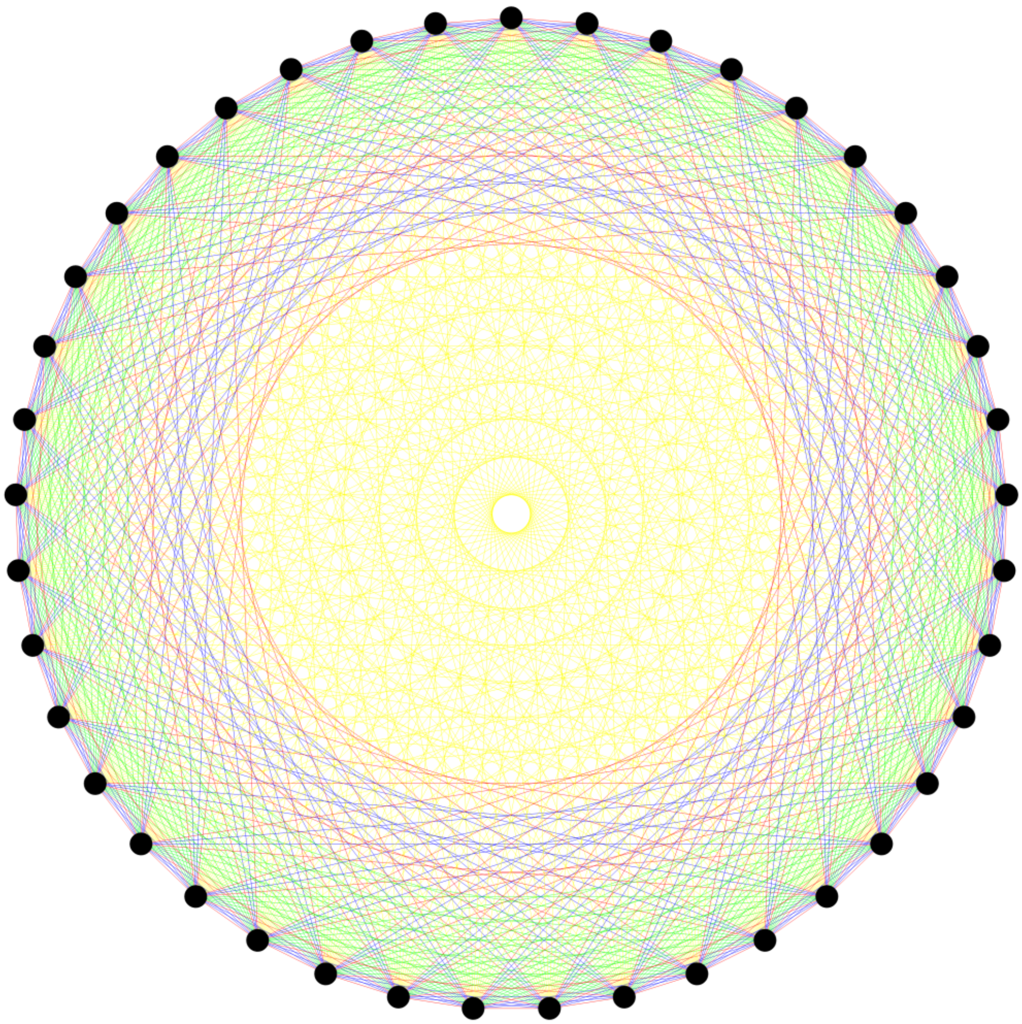

Figure 4.2: 4-coloring the edges of K_{41} with no monochromatic K_3

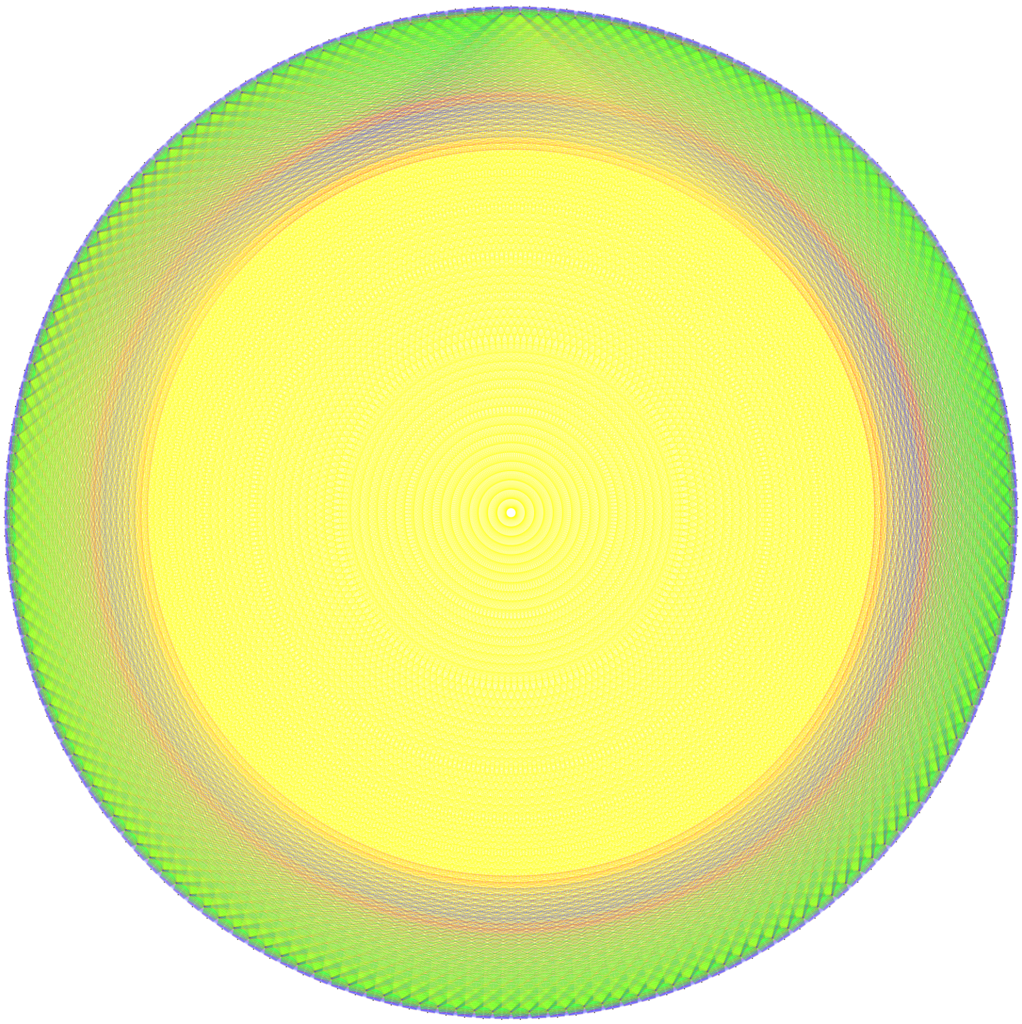

Figure 4.3: 4-coloring the edges of K_{171} with no monochromatic K_4

Chapter 5

Rado Numbers for Nonlinear Equations

There are very few results concerning Rado numbers for nonlinear equations. In this chapter, we will make computational progress in approaching this challenging, unexplored facet of Diophantine Ramsey theory. Many of these results are computational, but we do present several abstract results that prove bounds or otherwise make statements about the regularity of infinite families of nonlinear equations.

5.1 Previous Results

Despite being mostly unexplored, there are a few results that speak to the regularity of nonlinear equations. We start with a seminal result proved independently by Furstenberg [Fur77] and Sárközy [Sár79]:

Theorem 5.1. *For any r-coloring of $\mathbb{Z}^+$ there is $n \in \mathbb{Z}^+$ such that the equation $x - y = n^2$ has a monochromatic solution (x, y).*

This type of property is frequently also called regularity, but it has a slightly different meaning (due to the introduction of a parameter n that may depend on the coloring). There are other results of this type, including [FH13], where a similar result is proved for $9x^2 + 16y^2 = n^2$ and certain other quadratic equations (although not $x^2 + y^2 = n^2$). This result required much deeper analysis, using the machinery of Fourier analysis and Gowers U-norms that are a recent and powerful development in proving results of this type, on the border of additive combinatorics, analytical number theory, and Diophantine Ramsey theory.

We can verify the theorem for $r = 2$ by simply computing $\mathrm{R}_2\big(x - y = z^2\big) = 9$, which would be larger than the corresponding Furstenberg-Sarközy number, since the latter relaxes the monochromatic conditions to only x and y. We offer Table 5.1 which includes these Furstenberg-Sarközy numbers and some of the corresponding Rado numbers, which were produced using SAT solving methods. We can only prove the 5-color number is between 181 and 200.

$r =$	1	2	3	4	5
r-color F-S no.	2	5	29	58	$[181, 200]$
$\mathrm{R}_r(x - y = z^2)$	2	9	204	≥ 800	

Table 5.1: Furstenberg-Sarközy numbers

Erdős, Sárközy, and Sós [ESS89], and later Khalfalah and Szémeredi [KS06], provide a number of generalizations of this sort of result, where we examine $x - y = f(n)$ or $x + y = f(n)$ and look for monochromatic x and y that satisfy the equation for one or even infinitely many values of n (but without regard to the color of z).

In [Soi00], the author suggests that Arie Bialostocki has evidence that $\mathrm{R}_2(x^2+y^2 = z^2) > 60,000$. We are unable to verify this bound, but can generate a large lower bound for this quantity (6500), in addition to providing some results that are similar to this question, but a bit more tractable, which we will do throughout the chapter.

In [Gra07], Graham states that Rödl has proved the 2-regularity of the quadratic equation $yz + xz = xy$, perhaps better known as $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$. We have been able to add to this result by proving (using `RADO`):

Theorem 5.2. *The 2-color Rado number* $\mathrm{R}_2(yz + xz = xy) = \mathrm{R}_2(\frac{1}{x} + \frac{1}{y} = \frac{1}{z})$ *is 60. Furthermore,* $\mathrm{R}_2(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{w})$ *is 40. With 5 variables it is 48, and with 6 variables it is 39.*

To our knowledge, the only source of any Rado numbers for nonlinear equations is [DSV13], in which the following are proved:

Theorem 5.3. *For any positive integer n,* $\mathrm{R}_2(x + y^n = z) = 1 + 2^{n+1}$.

Theorem 5.4. *For any integer $c \geq 2$,* $\mathrm{R}_2(x + y^2 + c = z) = c^2 + 7c + 7$.

Theorem 5.5. *For any integer $a \geq 2$,* $\mathrm{R}_2(x + y^2 = az) = a - 1$.

Two of these equations are quadratic, but one is of arbitrary degree n (where n is the only parameter in this family of equations). These three families of equations intersect with some of the families we will discuss in the subsequent sections.

5.2 Pythagorean Triples & Other Sums of Squares

In [EG80], Erdős and Graham propose determining the 2-regularity of Pythagorean triples (solutions to $x^2 + y^2 = z^2$). Graham states in [Gra08] that he believes it would be difficult to make this determination or even to guess the outcome. Unfortunately, we have not resolved this question, but we will provide a lower bound, in addition to exact Rado numbers for some related quadratic equations.

5.2.1 Families of Equations of the Form $ax^2 + by^2 = cz^2$

Our current computational tools cannot yet determine the 2-regularity of equations in this family, and it is important to note that this family is in some sense as general as $ax + by = cz$ – although this linear equation is known to be 2-regular, we do not yet have an expression for the Rado number $\mathrm{R}_2\big(ax + by = cz\big)$. This suggests that $\mathrm{R}_2\big(ax^2 + by^2 = cz^2\big)$ would be difficult to determine, assuming we can even find a way to decide whether it is finite.

Using the SAT-solving methodology described in 3.6, we have been able to produce a valid coloring of [6500], giving us:

Theorem 5.6. $\mathrm{R}_2\big(x^2 + y^2 = z^2\big) > 6500$.

This confirms, at least partially, the anecdote from [Soi00] mentioned above. This coloring is provided in Appendix A. Note that as always, the construction of this lower bound does not exclude the possibility that $\mathrm{R}_2\big(x^2 + y^2 = z^2\big) = \infty$.

We believe that further development of these SAT techniques will allow us to increase this lower bound, but it is interesting to note that producing bounds of size approximately 6000–6500 does not require a significant amount of time in our current environment, while attempts to simply produce a slightly higher bound (e.g. 6600) fails to terminate even after a much more significant amount of time.

Further investigation may resolve whether this indicates that a faster or smarter algorithm (or faster computing environment) is necessary to increase the bound or that truly $6500 < \mathrm{R}_2\big(x^2+y^2=z^2\big) < 6600$. It is the belief of the author that this is not the case and that $\mathrm{R}_2\big(x^2+y^2=z^2\big)=\infty$. We offer the following pessimistic conjecture:

Conjecture 5.7. *For all $a,b,c \in \mathbb{Z}^+$ such that $a+b \neq c$, $\mathrm{R}_2\big(ax^2+by^2=cz^2\big)=\infty$.*

Let us hope our intuition is wrong, because resolving this question would be remarkable and it is our belief that it would be easier to resolve if our conjecture is incorrect.

We can extend Theorem 5.6 to the following set of equations:

Theorem 5.8. *For $3 \leq k \leq 20$, $\mathrm{R}_2\big(x^2+y^2=kz^2\big) > 5000$.*

This is not true for $k=2$ (the reader should take a moment to consider why). It is entirely possible to increase these lower bounds in many cases without using any appreciable computing power – these lower bounds are generously low simply to allow us to state them for all such k succinctly.

5.2.2 Primitive Pythagorean Triples

Although we have not been able to determine whether $\mathrm{R}_2\big(x^2+y^2=z^2\big)$ is finite, it is easy to prove the following:

Theorem 5.9. *The set of <u>primitive</u> Pythagorean triples is not 2-regular.*

Although we have not explicitly defined what this means, since this is not technically a Rado number, we believe the corresponding definition is quite clear. If we could prove that this set were 2-regular, its Rado number would be an upper bound for $\mathrm{R}_2\big(x^2+y^2=z^2\big)$, since we are examining a subset of the set of solutions to this equation. (However, it is not so – this set is not 2-regular.)

Proof. Consider a coloring where each z in any primitive triple $x^2+y^2=z^2$ is colored red, and all other integers are blue. There are clearly no blue primitive triples because for any such a triple, the z is red.

On the other hand, if we assume there is a red triple, which we might write (z_1, z_2, z_3), where each of these must be a hypotenuse in its own primitive triple:

$$\begin{aligned} x_1^2 + y_1^2 &= z_1^2, \\ x_2^2 + y_2^2 &= z_2^2, \\ x_3^2 + y_3^2 &= z_3^2, \text{ and} \\ z_1^2 + z_2^2 &= z_3^2 \end{aligned}$$

Because each such triple is primitive, these z values must be parametrized as:

$$\begin{aligned} z_1 &= m_1^2 + n_1^2, \\ z_2 &= m_2^2 + n_2^2, \\ z_3 &= m_3^2 + n_3^2, \\ z_1 &= 2m_3n_3, \text{ and} \\ z_2 &= m_3^2 - n_3^2 \end{aligned}$$

where each (m_i, n_i) pair is coprime and not both odd. Modulo 4, this gives 8 possibilities for each pair and a total of $8^3 = 512$ possibilities to check that:

$$\begin{aligned} m_1^2 + n_1^2 &\equiv 2m_3n_3 \pmod 4, \text{ and} \\ m_2^2 + n_2^2 &\equiv m_3^2 - n_3^2 \pmod 4. \end{aligned}$$

These 512 possibilities are easily checked, and we can verify that these two congruences are not simultaneously satisfiable. Thus no such triple (z_1, z_2, z_3) exists, and we conclude that there are no red monochromatic primitive triples. □

This does present one interesting question: Is it possible that the non-regularity of a set S of solutions to an equation, or any set of integer tuples, could imply the non-regularity of the set $S \cup 2S \cup 3S \cup \ldots$? This could settle the question of Erdős and Graham, but would also provide some insight into other linear and nonlinear equations.

5.2.3 The Family $x_1^2 + \cdots + x_k^2 = z^2$ and Other Sums of Squares

Because we have found the equation $x^2 + y^2 = z^2$ to be fairly intractable, we can approach a more general problem, wherein we consider equations consisting of sums of squares. If we consider $\mathcal{E}_{a,b}$ to be the equation:

$$\sum_{i=1}^{a} x_i^2 = \sum_{j=1}^{b} y_j,$$

then we can say that, heuristically speaking, there are "more" solutions to $\mathcal{E}_{a+1,b}$ than $\mathcal{E}_{a,b}$. Of course, that is not to say that solutions to $\mathcal{E}_{a+1,b}$ are a superset of those of $\mathcal{E}_{a,b}$. Generally speaking, though, if the left and right hand sides of this equation represent the quantity N, then we are hoping that N has many representations as a sum of a squares and as a sum of b squares. This is, generally speaking, true for larger values of N, a, and b.

This general notion might allow us to postulate that $\mathrm{R}_2\big(\mathcal{E}_{a+1,b}\big) \leq \mathrm{R}_2\big(\mathcal{E}_{a,b}\big)$, but this isn't quite true for certain combinations of a and b. We also offer the following conjecture, for which we seem to have significant empirical evidence:

Conjecture 5.10. *For a fixed r and a, there is B sufficiently large such that for $b \geq B$, $\mathrm{R}_r\big(E_{a,b}\big)$ is finite.*

Unfortunately, for a and b sufficiently large, most computational methodologies become very slow due to the presence of $a+b$ variables – even if we compensate for the fact that the x's and y's are respectively indistinguishable.

We offer first, with the significance of a theorem, one computational result:

Theorem 5.11. *The 2-color Rado number $\mathrm{R}_2\big(\mathcal{E}_{1,3}\big)$ is 105.*

The equation $\mathcal{E}_{1,2}$ is the Pythagorean equation, which we cannot tackle, but we can actually provide the following, covering infinitely many other (a, b) pairs:

Theorem 5.12. *For some constant c there is a constant M such that for any a and b such that $a \leq b \leq ca$, $\mathrm{R}_2\big(\mathcal{E}_{a,b}\big) < M$.*

We conjecture that this generalizes to r-color numbers $\mathrm{R}_r\big(\mathcal{E}_{a,b}\big)$, and we will prove it for $r = 3$. Note that the interesting cases are the ones not covered by the "sufficiently large" condition. We can resolve many of them by computation, as we will see.

Proof. First, assume there is a valid coloring in which 1, 2, and 3 are all red.

If $b-a$ is even, say $b-a=2k$, then consider the solution:

$$\underbrace{1^2+1^2+\cdots+1^2}_{a-k}+\underbrace{3^2+\cdots+3^2}_{k}=\underbrace{1^2+1^2+\cdots+1^2}_{a-k=b-3k}+\underbrace{1^2+1^2+2^2+\cdots+1^2+1^2+2^2}_{3k}$$

This solution is feasible because each $x_i=3$ corresponds to three y terms 1,1,2, and with k of them, that gives us the correct number of x and y terms (and of course, it is important to note $3^2=1^2+2^2+2^2$). This requires $b\leq 3a$ (c.f. $a-k\geq 0$) in order to have enough variables on each side to make this work.

Likewise, if $b-a$ is divisible by 3, we can do the same trick to obtain solutions using the identity $1^2+1^2+1^2+1^2=2^2$.

Using both of those ways of balancing the scales between a and b allows us to deal with any possible $b-a$, since $gcd(2,3)=1$. In fact, if we let $b-a=k$, we can find solutions of the form:

$$\underbrace{\ldots}_{a-3k}+\underbrace{2^2+\cdots+2^2}_{3k}=\underbrace{\ldots}_{b-4k}+\underbrace{1^2+1^2+1^2+3^2+\cdots+1^2+1^2+1^2+3^2}_{4k}$$

This requires $a-3k\geq 0$, i.e. $b\leq\frac{4}{3}a$, so $c=\frac{4}{3}$ will suffice.

For other initial colorings, like 1 red, 2 red, 3 blue, we must repeat this argument and obtain a similar bound. In Figure 5.1 we illustrate the tree of all colorings, truncated according to Table 5.2. In this table, the monochromatic elements of each branch are given as well as the corresponding way to configure those elements into two equal sums of squares, where one side has one extra square (like in the examples above, but with no divisibility restriction on $b-a$).

The worst case (greatest number of squares) is $12\cdot 2^2=12\cdot 1^2+6^2$, which proves that this theorem holds for $c=\frac{13}{12}$.

□

Monoch. Set	Sums of squares		
$\{1,2,3\}$	$2^2+2^2+2^2$	$=$	$1^2+1^2+1^2+3^2$
$\{1,2,6\}$	12×2^2	$=$	$12\times 1^2+6^2$
$\{1,3,4\}$	3^2+3^2	$=$	$1^2+1^2+4^2$
$\{1,3,8\}$	8×3^2	$=$	$8\times 1^2+9^2$
$\{2,3,4\}$	$2^2+4^2+4^2$	$=$	$3^2+3^2+3^2+3^2$
$\{2,3,5\}$	$3^2+3^2+3^2+3^2+3^2$	$=$	$2^2+2^2+2^2+2^2+2^2+5^2$
$\{2,3,6\}$	$9\times 2^2+6^2+6^2$	$=$	12×3^2
$\{2,4,6\}$	$4^2+4^2+4^2$	$=$	$2^2+2^2+2^2+6^2$
$\{3,4,5\}$	5^2	$=$	$3,4$
$\{3,4,7\}$	7×4^2	$=$	$7\times 3^2+7^2$
$\{3,5,8\}$	$5^2+5^2+5^2+5^2$	$=$	$3^2+3^2+3^2+3^2+8^2$
$\{3,6,9\}$	$6^2+6^2+6^2$	$=$	$3^2+3^2+3^2+9^2$
$\{4,6,8\}$	4^2+8^2+8	$=$	$6^2+6^2+6^2+6^2$
$\{1,3,5,6\}$	$1^2+6^2+6^2+6^2$	$=$	$3^2+5^2+5^2+5^2+5^2$
$\{1,3,6,7\}$	$6^2+6^2+6^2$	$=$	$1^2+3^2+7^2+7^2$
$\{1,4,5,6\}$	$4^2+4^2+4^2+4^2$	$=$	$1^2+1^2+1^2+5^2+6^2$
$\{2,5,8,9\}$	5^2+8^2	$=$	$2^2+2^2+9^2$
$\{3,5,6,7\}$	$6^2+6^2+6^2$	$=$	$3^2+5^2+5^2+7^2$
$\{1,2,4,5,9\}$	$2^2+4^2+4^2+5^2+5^2$	$=$	$1,1,1,1,1^2+9^2$
$\{1,2,4,7,9\}$	$2^2+4^2+4^2+7^2$	$=$	$1^2+1^2+1^2+1^2+9^2$
$\{1,2,4,8,9\}$	$2^2+4^2+8^2$	$=$	$1^2+1^2+1^2+9^2$
$\{1,2,5,7,9\}$	$2^2+2^2+2^2+5^2+7^2$	$=$	$1^2+1^2+1^2+1^2+1^2+9^2$

Table 5.2: Table for proof of Theorem 5.12

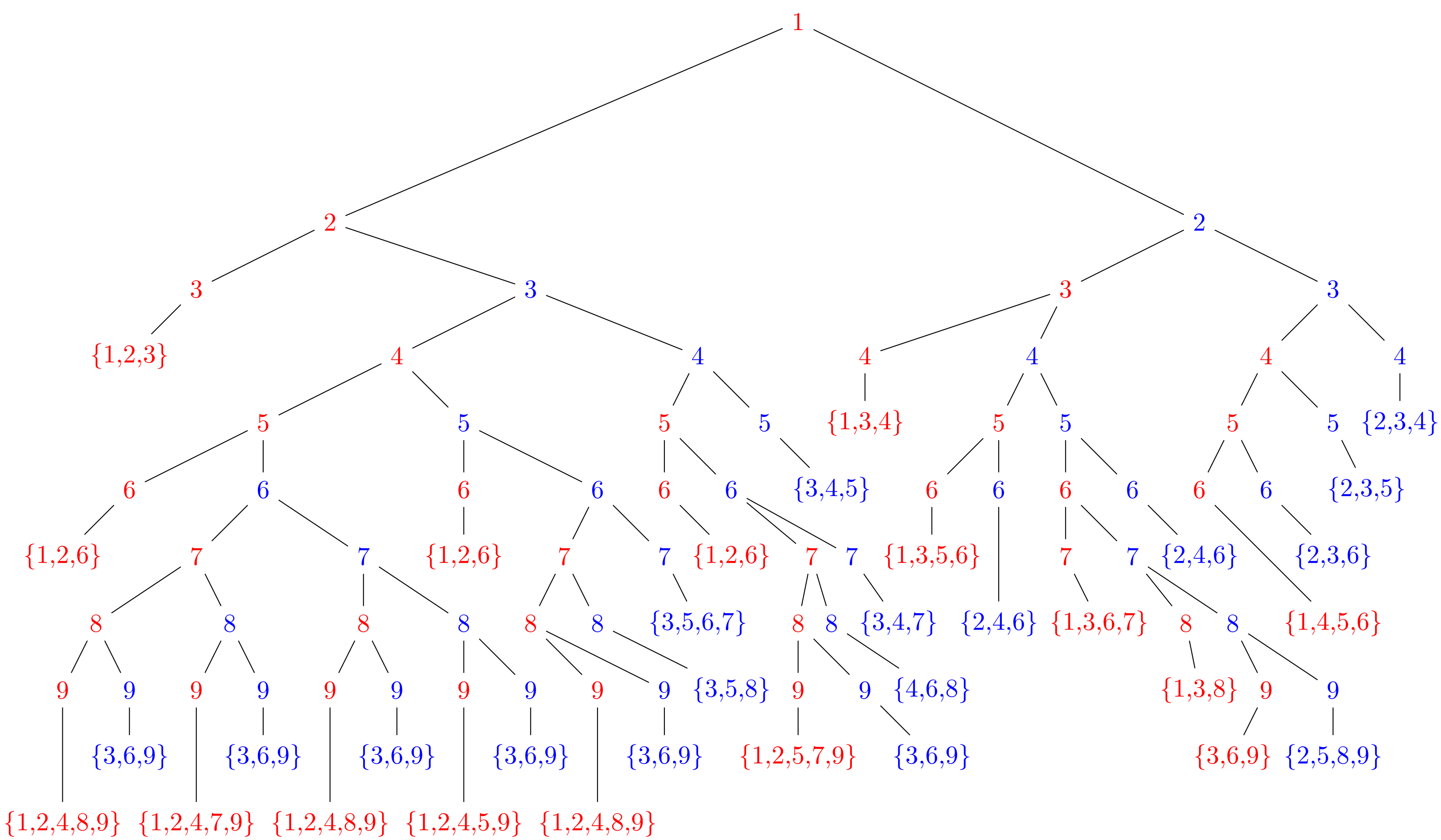

Figure 5.1: Figure for proof of Theorem 5.12

This proof was constructed using computer-assisted methods – a sort of guided backtrack on the tree, along with computer-assisted checking of various combinations of squares. This process could be automated and extended to allow for a deeper search in this tree. We could reject sums-of-squares equations that lead to small c values and hold out for a deeper search that gives a better c value. By adjusting the parameters of this proof, we can improve the value of c for which the theorem holds.

Note that if it is possible to prove the 2-regularity of $x^2 + y^2 = z^2$ then we can make such a trade-off with ratio 2:1 on any branch of this tree, meaning that as a corollary, we would establish the 2-regularity of any equation with $a \leq b \leq 2a$.

In fact, we prove below in Table 5.3 that $\mathrm{R}_2\big(\mathcal{E}_{2,3}\big) = 19$, so we know that a similar looking tree of depth 19 can give us $c = 3/2$. Thus we have, as a corollary of Theorem 5.12 and the 2-regularity of $\mathcal{E}_{2,3}$:

Corollary 5.13. *Theorem 5.12 holds for* $c = 3/2$.

We offer the following strong, but believable, conjecture:

Conjecture 5.14. *For a and b sufficiently large, regardless of $b-a$, we have* $\mathrm{R}_2\big(\mathcal{E}_{a,b}\big) < \infty$. *We conjecture that "sufficiently large" means* $\min\{a, b\} \geq N$ *for some* N.

Strictly speaking, our methodology in the proof of Theorem 5.12 can only prove $c = \frac{c'+1}{c'}$, where c' and $c'+1$ are the largest numbers of squares used in one of these "trade offs." We could not improve the bound without proving the 2-regularity of $x^2 + y^2 = z^2$ without modifying this method in some way. However, it may be possible to improve this method with a more rigorous look at cases like those where we did not require our "trade off" to be $c' + 1$ squares for c squares. This would require careful consideration of some divisibility conditions on $b - a$ but might result in better bounds for c.

If $b - a$ is very large, i.e. $b > ca$, this theorem does not apply, but in that case, we may employ a different exchange like trading in 20 copies of 1^2 for $2^2 + 4^2$, which would work if $b - a = 18$. This requires only that $b \geq 20$ (but if $b - a = 18$, it must be true that $b \geq 19$). That assumes, however, that $1, 2, 4$ are all the same color in our coloring as well! We expect to be able to generalize this theorem to cover cases where $b - a$ is large.

We offer the following data for $\mathrm{R}_r\big(\mathcal{E}_{a,b}\big)$:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	?	105	37	23	18	20	20	15	16	20	23	17	21	26	17	23	28	25	29
2		1	19	10	8	12	12	7	11	9	15	11	11	12	11	13	14	11	14	13
3			1	10	9	6	9	9	7	9	9	7	9	9	10	9	9	8	9	12
4				1	9	9	6	9	9	7	9	9	5	9	9	5	9	9	8	9
5					1	9	9	6	9	9	7	9	9	5	9	9	5	9	9	4
6						1	9	9	5	9	9	5	9	9	5	9	9	5	9	9
7							1	9	9	5	9	9	5	9	9	5	9	9	5	9
8								1	9	9	5	9	9	5	9	9	5	9	9	5
9									1	9	9	5	9	9	5	9	9	5	9	9
10										1	9	9	5	9	9	5	9	9	5	9
11											1	9	9	5	9	9	5	9	9	5
12												1	9	9	5	9	9	5	9	9
13													1	9	9	5	9	9	5	9
14														1	9	9	5	9	9	5
15															1	9	9	5	9	9
16																1	9	9	5	9
17																	1	9	9	5
18																		1	9	9
19																			1	9
20																				1

Table 5.3: Table of Rado numbers for sums of squares.

This table is symmetric, so redundant entries have been excluded. The first row is repeated and discussed in further detail in Section 5.2.4. Based on this data, and based on our proof of Theorem 5.12, we might even offer a stronger conjecture – that for a and b sufficiently large, $\mathrm{R}_2\big(\mathcal{E}_{a,b}\big) \leq M$ for some fixed M (where M might depend on $b-a$). We can see as a corollary of our proof of 5.12 that we have a universal bound on (a,b) pairs to which the theorem applies (and that bound will vary – the corollary, which has a better c value, will have a worse M value *a priori*). We believe the true value of M is 9, as it is in the original proof of the theorem.

If these conjectures hold true, is likely that similar conjectures hold for 3- or r-color Rado numbers for $\mathcal{E}_{a,b}$. For now, we offer one example computation:

Theorem 5.15. *The 3-color Rado number* $\mathrm{R}_3\big(\mathcal{E}_{3,4}\big) = 32$.

Like Theorem 5.11, this required a substantially long time to compute, and like $\mathcal{E}_{1,3}$ for its respective number of colors, this may represent the lowest possible 3-regular (resp. 2-regular) combination of a and b ($a \neq b$). It is important to note that this also provides another corollary:

Corollary 5.16. *Theorem 5.12 holds with 3 colors for* $c = 4/3$.

5.2.4 The Subfamily with $b = 1$

We will now consider a subfamily of these equations, $\mathcal{E}_k = \mathcal{E}_{1,k}$. We can provide the following table, which verifies our heuristic that $\mathrm{R}_2(\mathcal{E}_{a+1,b}) \le \mathrm{R}_2(\mathcal{E}_{a,b})$ in some, but not all, cases. In fact, as k grows, we eventually see an up-tick, because fixing $a = 1$ and letting k grow is not necessarily in the spirit of that heuristic (although it certainly leads us to believe that $a+b > N$ for some N is not the criterion for bounding these $\mathrm{R}_2(\mathcal{E}_{a,b})$).

$k =$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$N =$	?	105	37	23	18	20	20	15	16	20	23	17	21	26	17	23
$k =$	18	19	20	21	22	23	24	25	26	27	28	29	30			
$N =$	28	25	29	29	26	36	32	27	38	33	35	41	36			

Table 5.4: 2-color Rado numbers for $x_1^2 + \cdots + x_k^2 = z^2$

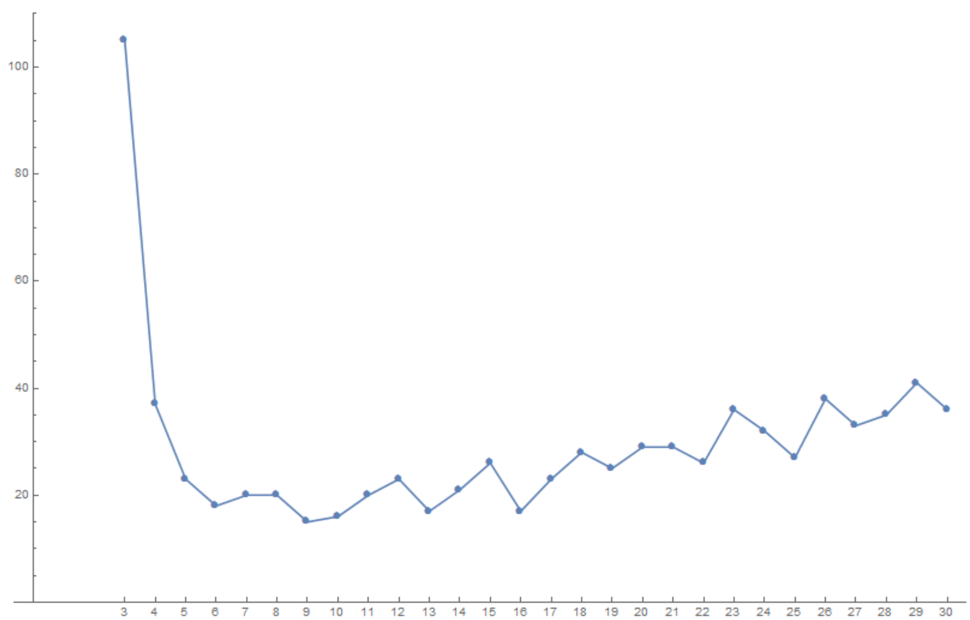

Figure 5.2: 2-color Rado numbers for $x_1^2 + \cdots + x_k^2 = z^2$

This is now entry A250026 in the Online Encyclopedia of Integer Sequences (OEIS). These become difficult to compute using `RADO` for larger values of k due to the increasing number of variables – the VI for any equation has complexity roughly $O(n^m)$ where n

is the current length of the coloring. Some optimization can shave off a factor of n or even a bit more, but not much more. Instead, for values $k > 17$, we compute this in Mathematica implementing a fairly mundane backtracking algorithm with almost no optimizations – however, we make use of the computer-algebra system to store the k-fold sums of each color-class, eliminating the complexity in the VI. However, for $k > 30$, the speed limiations of doing the tree-search in a computer algebra system start to slow down the computation as the tree becomes deeper.

Note that family $\mathcal{E}_k$ provides the worst possible case for Theorem 5.12, even though for k sufficiently large it seems clear that we should be able to prove that $\mathrm{R}_2(\mathcal{E}_{k,1})$ is finite.

We will offer some further analysis of this family, with attention to the challenges of computing these numbers, in Appendix C.

5.2.5 The Family $x^2 + y^2 + kz^2 = w^2$

We now consider the equation $\mathcal{E}_k$ to be $x^2 + y^2 = kz^2 = w^2$, inspired in part by our desire to understand the previous families, and by the approach taken to understand Rado numbers for linear equations, where this form (one new coefficient, one variable on right hand side) seems to be the most tractable.

We first present the following computations, which were relatively intensive but very much tractable using the `RADO` package.

Theorem 5.17. *The 2-color Rado numbers for $x^2 + y^2 + kz^2 = w^2$ are as presented in Table 5.2.5.*

k	1	2	3	4	5	6	7	8	9	10	11	12
$\mathrm{R}_2(\mathcal{E}_k)$	105	37	40	41	55	85	43	68	77	84	70	77

Table 5.5: 2-color Rado numbers for $\mathcal{E}_k$, $\mathrm{R}_2(x^2 + y^2 + kz^2 = w^2)$

This table could be extended using the `RADO` package, except that $k = 13$ is extremely difficult to compute. This phenomenon was discussed in Section 4.1. These data are interesting because it is not at all clear how k affects $\mathrm{R}_2(\mathcal{E}_k)$.

For some variety, we also offer the following:

Theorem 5.18. *The 2-color Rado number* $\mathrm{R}_2\big(x^2+y^2=z^2+2w^2\big)$ *is 33.*

5.3 Other Families of Nonlinear Equations

We may first observe that these families come in pairs that are related by specializing certain variables. For example, $x_1+\cdots+x_k=z^2$ can be specialized to $x+(k-1)y=z^2$ by setting $x_1=x$ and $x_2=\cdots=x_k=y$. From Lemma 3.8, and for any $a+b=k$, we obtain the following relations:

The following inequalities hold (for any r):

$$\begin{aligned}
\mathrm{R}_r\big(x_1+\cdots+x_k=z^2\big) &\leq \mathrm{R}_r\big(ax+by=z^2\big)\\
\mathrm{R}_r\big(x_1+\cdots+x_k=y+z^2\big) &\leq \mathrm{R}_r\big(kx=y+z^2\big)\\
\mathrm{R}_r\big(x_1+\cdots+x_k+y^2=z^2\big) &\leq \mathrm{R}_r\big(kx+y^2=z^2\big)\\
\mathrm{R}_r\big(x_1^2+\cdots+x_k^2=z^2\big) &\leq \mathrm{R}_r\big(ax^2+by^2=z^2\big)
\end{aligned}$$

The first two equations are of particular importance because they include only one quadratic term, which means that if we collapse all the variables, we could possibly obtain a solution that consists of only one element (which we will call a singleton solution).

If a linear homogeneous equation has a singleton solution, like $x+y=z+w$ having the solution $(7,7,7,7)$, then it also has the solution $(1,1,1,1)$ since the equation is linear. The Rado number must be 1 since that singleton solution is monochromatic by virtue of having only one element in the first place. Singleton-solutions like these are automatic upper bounds on the Rado number, making linear equations like $x+y=z+w$ trivial in this sense.

For nonlinear equations, it is also true that singleton-solutions provide an immediate upper bound. But in the nonlinear setting, solutions cannot be obtained from one another by scalar multiplication, allowing for a Rado number higher than 1 even in this singleton-solution scenario. We may apply this to obtain:

Lemma 5.19. *The following inequalities hold (for any r and any $a > b \geq 0$):*

$$\begin{aligned} \mathrm{R}_r\big(ax + by = z^2\big) &\leq a + b \\ \mathrm{R}_r\big(ax = by + z^2\big) &\leq a - b \end{aligned}$$

Despite being a relatively straightforward lemma, this gives us a bound for $\mathrm{R}_r\big(\mathcal{E}\big)$ for each of these equations that is independent of r, which is not possible in the most commonly considered cases (i.e. linear equations). Taking $\mathcal{E}_k$ as $x + (k-1)y = z^2$, it is clear that if we think of k as being held fixed and increase r that:

$$\mathrm{R}_2\big(\mathcal{E}_k\big) \leq \mathrm{R}_3\big(\mathcal{E}_k\big) \leq \cdots \leq \mathrm{R}_k\big(\mathcal{E}_k\big) = \mathrm{R}_{k+1}\big(\mathcal{E}_k\big) = \cdots = k,$$

and similarly for the other three equations with these r-independent upper bounds. The same can be said for any other $a + b = k$ combination and any other equation with a singleton-solution derived bound.

This can be summarized in the following straightforward lemma:

Lemma 5.20. *Let $\mathcal{E}(a, b, p, q)$ be the equation:*

$$\sum_{i=1}^{a} x_i + \sum_{i=1}^{b} y_i^2 = \sum_{i=1}^{p} z_i + \sum_{i=1}^{q} w_i^2.$$

Whenever $k = \frac{p-a}{b-q}$ is an integer, $\mathrm{R}_r\big(\mathcal{E}(a, b, p, q)\big) \leq k$ for any r.

Having established upper bounds for some of these families, we will provide some details about each family. Unsurprisingly, we will find more interesting behavior in the families that do not submit to such a lower bound.

5.3.1 The Family $x_1 + \cdots + x_k = z^2$

Throughout this subsection, let $\mathcal{E}_k$ denote the equation $x_1 + \cdots + x_k = z^2$. We first recall the lemma that for any r, $\mathrm{R}_r\big(\mathcal{E}_k\big) \leq k$. For $r \geq k$ we have equality trivially, since each integer can be assigned its own color and the only possible monochromatic solution to the equation would be the singletons $\{0\}$ and $\{k\}$ (0 not really being included since we always work over $\mathbb{Z}^+$).

$r\backslash k$	1	2	3	4	5	6	7	8
1	1	2	2	2	3	3	3	3
2	-	2	3	4	5	6	5	5
3	-	-	3	4	5	6	7	8
4	-	-	-	4	5	6	7	8
5	-	-	-	-	5	6	7	8
6	-	-	-	-	-	6	7	8
7	-	-	-	-	-	-	7	8
8	-	-	-	-	-	-	-	8

Table 5.6: 2-color Rado numbers for $\mathcal{E}_k$, $x_1 + x_2 + \cdots + x_k = z^2$

Computations for $\mathrm{R}_2(\mathcal{E}_k)$ appear in Table 5.6. Entries below the diagonal are not indicated since they are trivial. (Each is equal to k since $r > k$, so we can conclude that the columns are eventually constant.) The entries in row 2 show that $\mathrm{R}_2(\mathcal{E}_k) \leq k$ is not tight when $k \geq 7$. We can also consider the first row, where values are significantly lower than k, and obtain Lemma 5.21.

Lemma 5.21. *The 1-color Rado number is* $\mathrm{R}_1(x_1 + \cdots + x_k = z^2) = \left\lceil \sqrt{k} \right\rceil$.

Proof. If there is a solution to $\mathcal{E}_k$, then it is clear that $z = \sqrt{x_1 + \cdots + x_k} \geq \sqrt{k}$, and since z is an integer, $z \geq \left\lceil \sqrt{k} \right\rceil$.

So the first solution (which is monochromatic, since there is only one color) is of the form $(1, \ldots, 1, 2, \ldots, 2, \lceil \sqrt{k} \rceil)$, which gives $\mathrm{R}_2(\mathcal{E}_k) = \lceil \sqrt{k} \rceil$. The solution set $\{1, 2, \left\lceil \sqrt{k} \right\rceil\}$ is feasible because $k \leq \left(\left\lceil \sqrt{k} \right\rceil\right)^2 \leq 2k$ for any $k \in \mathbb{Z}^+$.

Note that in some cases, the solution set might exclude 1s or 2s (but not both of course), but either way $\left\lceil \sqrt{k} \right\rceil$ is still the largest element in the solution set. $\square$

Of course, a 1-color Rado number isn't particularly interesting, but it is important in the sense that these provide lower bounds, so that we have:

Corollary 5.22. *For all r and k,* $\left\lceil \sqrt{k} \right\rceil \leq \mathrm{R}_r(x_1 + \cdots + x_k = z^2) \leq k$.

The combination of these two results is somewhat remarkable because now we know lower and upper bounds for these Rado numbers that grow quite slowly.

5.3.2 The Families $x + (k-1)y = z^2$ and $ax + by = z^2$

This family is obtained from those in the previous section by collapsing variables as in Lemma 3.8. The upper bound based on a singleton solution $a + b$ still applies, so for $k = a + b$ we have the following:

$$\mathrm{R}_2\big(x_1 + \cdots + x_k = z^2\big) \le \mathrm{R}_2\big(ax + by = z^2\big) \le a + b.$$

In this case, however, we cannot find a case in which the upper bound is not tight. For all $a, b \le 10$, it is the case that $\mathrm{R}_2\big(ax + by = z^2\big) = a + b$. Although it may be premature, we offer the following conjecture:

Conjecture 5.23. *For any $a, b \in \mathbb{Z}^+$, $\mathrm{R}_2\big(ax + by = z^2\big) = a + b$.*

This would, of course, imply the same for any other number of colors. It is important to note that this conjecture fails even by allowing a single extra term. This is witnessed by the computation $\mathrm{R}_2\big(3x + 5y + 6z = w^2\big) = 9$. The conjecture also fails with $a < 0$ or $b < 0$, as we will see shortly.

5.3.3 The Family $x_1 + \cdots + x_k = y + z^2$

The family $\mathcal{E}_k$, $x_1 + \cdots + x_k = y + z^2$, in this section introduces an extra term on the right-hand side. This still allows for an upper bound according to a singleton solution, which will now be $k-1$ instead of k. However, we will quickly see that this is not tight. (Note: The singleton solution does <u>not</u> exist when $k = 1$.)

$k =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\mathrm{R}_2\big(\mathcal{E}_k\big) =$	9	1	2	3	4	3	5	5	5	5	6	6	7	7

Table 5.7: 2-color Rado numbers for $\mathcal{E}_k$, $x_1 + \cdots + x_k = y + z^2$

We suspect that this number is $\lfloor k/2 \rfloor$, but would not yet present this formally as a conjecture.

If we collapse variables and reduce this family to the corresponding family $kx = y + z^2$, we can refer to [DSV13] to give $k-1$ as the corresponding Rado number, which is linear, as is our conjecture for the Rado number for $x_1 + \cdots + x_k = y + z^2$, although they do not agree.

5.3.4 The Family $x_1 + \cdots + x_k + y^2 = z^2$

In this family, we no longer have a singleton solution to provide any upper bound. However, this family does not necessary grow very large, which still appeals to our heuristic that with many variables, the Rado number generally stays smaller.

$k =$	1	2	3	4	5	6	7	8
$\mathrm{R}_2(\mathcal{E}_k) =$	9	5	7	6	5	6	6	6

Table 5.8: 2-color Rado numbers for $\mathcal{E}_k$, $x_1 + \cdots + x_k + y^2 = z^2$

5.3.5 The Family $kx + y^2 = z^2$

We present this family to contrast the previous, because now we will see some defiance of this new lack of a nice upper bound:

$k =$	1	2	3	4	5
$\mathrm{R}_2(\mathcal{E}_k) =$	9		96	69	41

Table 5.9: 2-color Rado numbers for $\mathcal{E}_k$, $kx + y^2 = z^2$

5.4 Hindman Numbers

The `RADO` package can give us a way of attacking problems involving multiple equations. For example, if we want find the least N such that any r-coloring of $[N]$ satisfies one (or more) of the equations $\mathcal{E}_1, \ldots, \mathcal{E}_k$, then we can find $\mathrm{R}_r\big(f_{\mathcal{E}_1} \cdot f_{\mathcal{E}_2} \cdot \cdots \cdot f_{\mathcal{E}_k} = 0\big)$, which means we just enter the product of all $f_{\mathcal{E}_i}$ into `RADO`. The only real difficulty here is carefully expanding this product, since `RADO` cannot parse a function that is fully expanded (although we can do this easily using a computer algebra system like Mathematica).

Likewise, if we want the r-coloring of $[N]$ to satisfy all of the equations $\mathcal{E}_1, \ldots, \mathcal{E}_k$, we can find $\mathrm{R}_r\big((f_{\mathcal{E}_1})^2 + \cdots + (f_{\mathcal{E}_2})^2 = 0)\big)$.

Of course, both of these operations will result in nonlinear equations, even when the $\mathcal{E}_i$ are linear. In the "or" case, we know the resulting Rado number could not increase, while in the "and" case it could not decrease. We will approach one problem of the latter type.

By hacking together the system of equations $x+y=z$ and $xy=w$, we can confirm and comment on some results regarding a theorem of Hindman [Hin79]:

Theorem 5.24. *For any r-coloring of $\mathbb{Z}^+$, there are monochromatic x,y,z,w such that $x+y=z$ and $xy=w$.*

This is actually a corollary of a much stronger infinitary statement that Hindman proves using the methods of mathematical logic. The full theorem is:

Theorem 5.25. *For any r-coloring of $\mathbb{Z}^+$, there is an infinite monochromatic set B such that all nonempty finite sums and products of elements of B are monochromatic.*

Although it is not explicit in previous statements of Theorem 5.24, we should note carefully that it is fairly clear in the existing literature that this theorem usually includes the stronger condition that all of x,y,z,w (or at least x and y, *a priori*) are distinct.

We can define H_r as the least such N for a given r. In [Hin79], Hindman tells us that Ron Graham has proved the following (by computer):

Theorem 5.26. *The 2-color Hindman number is $H_2 = 252$.*

However, this computation includes a lower bound that is demonstrated by two color classes:

$$\begin{aligned} \mathit{Red} &= \{1,3,5,8,12,14,16,18,20,22,24,\dots\} \\ \mathit{Blue} &= \{2,4,6,7,9,10,11,13,15,17,19,\dots\} \end{aligned}$$

Without our careful note above, we might immediately balk at 2 and 4 being blue, since $(x,y,z,w) = (2,2,4,4)$ is a solution! However, Graham implicitly assumes that this is a strict system – that none of the variables can be equal. This means our `RADO` package is not exactly what is needed here (at least, not without some modifications; see Section 6.3). However, we can define H'_r to be the analogous number, where now we are really looking for any solution to the system $x + y = z,\ xy = w$, regardless of whether the four variables are distinct.

It is trivial to note that $H'_r \leq H_r$, so we know $H'_2 \leq 252$ based on Graham's result. We offer the following results:

Theorem 5.27. *The non-strict 2-color Hindman number is* $H'_2 = 39$*. The next non-strict Hindman number is very large, giving us* $1000 < H'_3 \leq H_3$*.*

These are proved by expanding the polynomial $(x + y - z)^2 + (xy - w)^2$ and determining (using `RADO`) that:

$$H'_2 = \mathrm{R}_2\big(w^2 + x^2 + 2xy - 2wxy + y^2 + x^2y^2 - 2xz - 2yz + z^2\big) = 39.$$

For $r = 3$ the computation halted when it reached the bailout value of 1000, meaning that a valid coloring of length 1000 was found. This might lead us to worry that Hindman's theorem might be incorrect, but it is entirely possible that H'_3 not infinite, just much larger than H'_2.

We can also define $H_r(m)$ and $H'_r(m)$ to be the corresponding quantities for the following system:

$$\begin{aligned} x_1 + \cdots + x_{m-1} &= x_m \\ x_1 \cdot \cdots \cdot x_{m-1} &= z \end{aligned}$$

We are bending the rules of our convention on using the letter m – there are $m+1$ variables here, but only m in the first equation, which is the equation for the generalized Schur number from Section 3.1. This deviates significantly from what Hindman proposes, since [Hin79] discusses sets of integers and *all* of their (nonempty) sums and products. The reduction to $\{x, y, x+y, xy\}$ is what connects us to Schur's theorem, and we have taken this in a different direction – the direction of generalized Schur numbers.

In this notation, we have $H'_2(3) = 39$ and $H_2(3) = 252$. However, at this time $H_2(4)$ is not tractable to `RADO`. The SAT approach, however, yields:

Theorem 5.28. *The 2-color 4-variable (per equation) generalized (non-strict) Hindman number is* $H'_2(4) = 450$*. Similarly,* $H'_2(5) = 11,000$*.*

While these hacks for `RADO` cannot be used to force conditions like $x \neq y$, at least not in a way that is feasible, the SAT solver does not care, so long as the appropriate clauses are included (or not) as desired. We can, in fact, use a SAT solver (very carefully) to produce the data in Table 5.10, which includes strict $H_r(m)$ and non-strict $H'_r(m)$ generalized Hindman numbers for r colors and m variables.

	$m = 3$	$m = 4$	$m = 5$	$m = 6$
$H'_2(m)$	39	450	11000	62500
$H_2(m)$	252	23100		

Table 5.10: Generalized Hindman numbers

We have also been able to extend the lower bound produced by `RADO` using SAT methods to show that $H'_3(3) > 100,000$, which is quite an enormous lower bound!

We might worry about having to draw a distinction between "strict" Rado numbers, where all variables must be distinct in monochromatic solutions, and what Graham and Hindman propose, which only requires that the x_i are distinct. However, it is clear that $x_i \neq y$ (for $m > 2$, which we ought to assume), and if $x_i = z$, but all the x_i are distinct, then $m = 3$ and the other x_j is 1. We will proceed using their definition, so that our results agree, although this is not necessarily in line with our conventions for other quantities we will compute.

We will discuss extensions of this strictness idea in Section 6.3. For now, we offer the following specialization of Hindman's theorem, which should assure us that all of these quantities are finite:

Theorem 5.29. *For any $\chi : \mathbb{Z}^+ \to [r]$, and any m, there is $x_1, x_2, \ldots, x_m, z$ such that $x_1 + x_2 + \cdots + x_{m-1} = x_m$, $x_1 \cdot x_2 \cdot \cdots \cdot x_{m-1} = z$, and $\chi(x_1) = \chi(x_2) = \cdots = \chi(x_m) = \chi(z)$.*

We can also consider define two sets:

Definition 5.30. *The <u>k-fold sumset of the multiset S</u> is the set of all sums of between 1 and k distinct elements of S. Denote this $\Sigma_k(S)$.*

Definition 5.31. *The <u>k-fold productset of the multiset S</u> is the set of all products of between 1 and k distinct elements of S.Denote this $\Pi_k(S)$.*

These two types of sets are more robust, a step towards extending our results in a spirit closer to Hindman's original work. To that end, we know that a third type of Hindman number is finite. Hindman's original theorem tells us, for example, the following:

Theorem 5.32. *For any coloring* $\chi : \mathbb{Z}^+ \to \{0, 1\}$ *there are* x, y, z *such that the set* $\Sigma_k(\{x, y, z\}) \cup \Pi_k(\{x, y, z\})$ *is monochromatic.*

We can call these "strong" Hindman numbers. We could even find a strict version of this, which is:

Theorem 5.33. *For any coloring* $\chi : \mathbb{Z}^+ \to \{0, 1\}$ *there are distinct* x, y, z *such that the set* $\Sigma_k(\{x, y, z\}) \cup \Pi_k(\{x, y, z\})$ *is monochromatic.*

At this time we are not yet able to compute any of these strong Hindman numbers, strict or not, except for $m = 3$, a case that reduces to the non-strong version. We can provide a lower bound: the smallest of these, $H_2^*(4)$, must be at least 300,000.

5.5 Non-Polynomial Rado Numbers

In this section, we will break from the convention suggested in Chapter 2, where a concretely-minded reader might have assumed that for the purposes of this dissertation, a Diophantine equation is a polynomial with integer coefficients. Here, we consider the exponential equation:

$$x_1 + \cdots + x_{m-1} = kn^{x_m}.$$

Fixing $k = 1$ and $n = 2$, we can determine the following:

Theorem 5.34. *The 2-color Rado numbers* $\mathrm{R}_2\big(x_1 + \cdots + x_{m-1} = 2^{x_m}\big)$ *are* 2, 16, 8, 32, 64, 32, 32 *for* $m = 3$, 4, 5, 6, 7, 8, 9 *(respectively).*

Although it seems interesting to vary n or k and produce different Rado numbers for these non-polynomial equations, we offer as potential discouragement the following lower bounds:

$$\begin{aligned}
\mathrm{R}_2\big(x+y=3^z\big) &\geq 100,000 \\
\mathrm{R}_2\big(x+y=2\cdot 2^z\big) &\geq 100,000 \\
\mathrm{R}_2\big(x+y=3\cdot 2^z\big) &\geq 100,000 \\
\mathrm{R}_2\big(x+y=2\cdot 3^z\big) &\geq 100,000
\end{aligned}$$

We hope in the future to learn more about these exponential Rado numbers.

Chapter 6

Further Directions

In this chapter we discuss some directions in which this work leads. We will note some future work from the computational perspective, improving and advancing some of the computational methods described in this dissertation. We will discuss open problems in finding or bounding Rado numbers and also put forward a number of variations on Rado numbers that are also amenable to the methods detailed in this dissertation.

6.1 Computational Methods

There are a number of improvements that can be made to the `RADO` package in order to continue to push the limits of this exhaustive method of computation. There are ways to improve the efficiency of the Value Iterator, in particular allowing further reductions in complexity by a more intelligent implementation of the optimization for variables that are indistinguishable.

It would also be interesting to add to the `RADO` package something to handle unexpectedly large search trees, besides having a bailout at a fixed depth in the tree.

It is possible to continue to improve the implementation of SAT solvers, both for general SAT problems and perhaps tailor-made for Rado numbers and related Ramsey-theoretic questions. It should be noted that we have not made special effort to adapt any SAT solving software to our problems in this dissertation – all results were obtained using MiniSAT (see: `http://minisat.se`). Further work with SAT solvers will likely provide better lower bounds especially in cases of sparse solution sets.

Backtracking algorithms may also be able to take advantage of the structure of these solution sets (again, most especially in the case of sparse solution sets) by backtracking over the sets, rather than just through $\mathbb{Z}^+$ in a linear fashion. This will provide greater efficiency by excluding redundant structure in the tree that we would otherwise cover many times.

It is entirely possible that there are enough methods that an integrated platform could be developed to apply multiple methodologies automatically, given any particular r and $\mathcal{E}$. The computer could simultaneously, sequentially, or in some other way attempt to determine bounds or values for $\mathrm{R}_r(\mathcal{E})$ using a mixture of these methodologies, not just by attempting each one in succession but by hybridizing the methods. For example, the SAT-solver method could generate lower bounds, but a mixture of DFS/backtracking and simulated annealing could attempt to extend these branches of the search tree even further. A SAT-solver could also refine a DFS to skip irrelevant branches. Randomized and structured lower-bound constructions could be easily combined in the hopes that they can improve one another.

6.2 Equations of Interest

For linear equations, we now have the ability to generate significant data, individual Rado numbers using the `RADO` package, SAT solvers, or other high-performance methods. In some cases, for specialized equations, the more abstract methods may also apply to symbolic equations (equations with unspecified coefficients), proving in the abstract certain bounds or exact Rado numbers.

The equation of most interest would be $ax + by = cz$ – a closed-form for the 2-color (or even r-color) Rado number for this equation would be incredibly meaningful, and as discussed in Section 3.1.5, this would provide a bound for all other equations by Lemma 3.8. We believe the results from Section 4.2 can be generalized to provide Rado numbers for $kx + (k+1)y = (k+j)z$ and/or $kx + (k+j)y = (k+j+1)z$, and perhaps eventually $kx + (k+j)y = (k+j+\ell)z$ – even under the restriction that $k, j, \ell > 0$ this would still be a major step in the right direction.

A very recent preprint of Saracino on the Arχiv appears to have determined Rado numbers for equations more general than those in [Sar13] and [GS08], simultaneously generalizing both results. This impressive unification of existing work may represent significant progress in closing the gap on $ax + by = cz$. Future work may attempt to approach the problem from other angles, or may take this new result of Saracino and extend it further. (We remark that this result appears in this chapter because it was only posted online a matter of weeks before the final draft of this dissertation was submitted.)

In the future, we would of course hope to settle the question of whether $x^2+y^2 = z^2$, $x^2 + ky^2 = z^2$, etc. are 2-regular – either by proving specific cases computationally or by settling the larger question of precisely which families of quadratic equations are 2-regular.

It would be interesting to consider whether Lemma 5.20 could be interpreted in a way that makes sense when the quantity $\frac{p-a}{b-q}$ is not an integer.

In general, computational methods, when applied to nonlinear equations, open the door to further study. We can verify the intuition that motivates Conjecture 5.14 empirically, but we believe that the techniques of additive combinatorics and Fourier analysis may also be brought to bear on the problem. Further work on these sums-of-squares equations might settle this conjecture and provide more insight into these particular equations and their Ramsey-theoretic properties.

Although many of the Rado numbers presented in this dissertation are for homogeneous equations, the tools developed here are all applicable to non-homogeneous equations as well. Additional work is already underway to apply these methods to multiple families of non-homogeneous equations (linear and nonlinear) to continue this work.

6.3 Other Types of Rado Numbers

Sections 4.3 and 4.4.1 give a proof-of-concept for our `RADO` package in computing off-diagonal Rado numbers. While we have carried out some interesting computations in those sections, providing some new results, we have not yet adapted other computational methods to these problems. Open problems in off-diagonal Rado numbers could easily be translated into SAT-solving problems as in [Ahm], but some of these open problems are simple enough that we could tackle them with our more symbolic, abstract methods, providing a computer-based proof of some theorems for whole families of equations as in [MR07].

We have also begun preliminary investigation into Rado numbers for equations like $x^2+y^2 = z^2+w^2$. Unlike its linear analogue $x+y = z+w$, the solutions to this quadratic equation are not translation invariant, but they are trivial unless we introduce some prohibition on matching (in this case, it might suffice to say $x < z < w < y$). So even for the linear equation $x + y = z + w$, let alone the nonlinear $x^2 + y^2 = z^2 + w^2$, we can ask the question "What is $\mathrm{R}^*_r(\mathcal{E})$?" where we can define R^* to be a Rado number that does not allow repeated values of some or all variables. Picking the appropriate definition may be important, since it is arguable that $1+1 = 1+1$ is trivial, but perhaps $2 + 2 = 1 + 3$ is not. It would depend on the definition.

There is some existing work (see Section 3.1.3) on Rado numbers constrained by inequalities of this sort, in particular generalized Schur numbers as in [Sch98], for which the usual proofs might be carried out with only minor modifications that constrain a variable to take a different value. The existing work exploring these strict Rado numbers can certainly continue with the aid of computational tools like those used in this thesis with only minor modifications (for the `RADO` package, changes would be made to the value iterator primarily).

Future work can also tackle other varieties of Rado-type numbers. For example, there are "Rainbow" Rado numbers, where we look for solutions that have one of each color. This has been investigated by a number of authors, e.g. [CJR07] . There is a definition for a set being rainbow with respect to some χ, analogous to Definition 2.3, which should be fairly clear. While this may seem like a very different question (and in some ways, it is), many of our techniques like `RADO` and SAT-solvers would apply without significant modification. The clauses in the SAT solver would be somewhat different. For `RADO`, the VI would have to change to check for rainbow solutions instead of monochromatic solutions – adding virtually nothing to the computing time (at least, per coloring checked, there may of course be more or fewer colorings to check).

There are also mixed versions of these, for example "strict, off-diagonal Rado numbers." Our computational methods would require two small changes, rather than one, to tackle such a hybrid problem.

The Hindman numbers from Section 5.4 for $r > 2$ should be easier to resolve with improved methods for finding `RADO` numbers, solving SAT problems, or otherwise tacking this type of problem. Despite being extremely large, we can compute some of these quantities due to the extreme sparsity of solutions to multiplicative equations like $xyz = w$.

It is also feasible, by computer methods or otherwise, to investigate the statistical properties of solutions to equations when an interval $[N]$ is colored, e.g. [RZ98]. We could study not only the general statistical questions (minimum, average, etc. number of monochromatic solutions), but also more complex questions like discrepancy or other combinatorial questions related to equations over colored sets of integers.

Appendix A

Certificate for $\mathrm{R}_2\big(x^2 + y^2 = z^2\big) > 6500$

To provide a certificate for this lower bound from Section 5.2.3, we can provide a valid coloring of that length. We will provide three different representations of the coloring, to accommodate different methods of verification and inspection. The first will be in terms of a binary string, where we simply list 0 for red and 1 for blue, in the order in which they occur (i.e. if $\chi(7) = 1$, the 7th digit is 1).

1110000111001001111010001100100100111000001110101111110111101111010001
0010000100001011001010011111011100000100111011000110000110101101101011
0011011001001111001111000000000000000000001110101101111001001000100011
0000010010111001001001111010000000000110110110011010011110101101110111
1001110110000110000110111101100001010111010010111000000001100100010010
0010001000010110011110011010000000111111010001110011001001011011110101
0011111011001000110100101000111111000100110110010100101100011001001000
0001010000111010001000000011110110100100001010111110001010101011101011
1000000110000010001110001101010001001101111000001110000110000001101111
0110111001000111010010111011000000101010111100001010000010010100110100
0111001110010010000100110000111010010001000111110001001100111001100100
0001011100001000010111110001100010011001000011110110011101010000000001
0000111011110011010100010000001101111000010110000000000000100111010011
0111110000101100000101110010101011100000111011100000000011011001110111
0000011110011000100101011100010100110100110111110000010011011100101110
0100001101110000011111010011000111111110010000011110100011010101010101
1000101010100001101000100010100001110101111101100000101100100110100000
1011010011001001000101100000100001001011010001101010100001101001011100

0011011000110100000001101001101011011100011000100001000001001101000011 1

0111000010100111010100000110100111100011110101111110001000100100100000

1010101010110100000011001100011011101001000010001101100001011010101010

0111100001110110000100100000000000110000001110010111000001110001100100 0

1000011011001100000000100110000000100110010101100000000101111110000010111

1111000101101010000001011100011010110000000100111010000110000110000101 1

0100010001100010000100010010111000001000100010010101000001010001100101

0100100001001100010010001000000100110010000000100000100000101001000010 0

0001100011001000000100011100000100010001001101000110000100101001001010 0

1100000000111101010010110001110001100100010010100100110010110101001001

0100001100010000000000100001100010101100010001001000111000011001100101

0011110001100001110000000000010000111001010001100010000000000001100000 0

0010101000001010010111010010000010011101110110000000100100000001010001 0

1111100000000000011111001000100111000101001110101010000101101101010000 0

1101000111000011111001110000100000000010000001000011110001110010001111011

0001010000011011101110000110010001010000000010101000010001000110111000

1001010000110100100001111111100000000011010010000010010011001010000101

1000000101001001000111010100110110010111011111101100101100111110110110

1011111110101100011110100000000111101111111011111110110100010111110110 0

1011101110010011011101111000010110111111111011111100101011011011111010

1110010001110101111011100110001110111110110010111111000101111001100001

1111110101111110111100111010011001101101100101111110010101010000101001

1001100100011011101010110010011001000101011010100000111110001001100100

1010001100000111010100001010111100111101000011000011110000011001100001

0001101111011111010101101001110000011000000111101011101101011000100000 0

0100101101010001111000111000011010101111110000101001111111111011101111 1

1111001101010000011001001100110001011110100000100101111101100110001111

0101111110011010101011111101011100000000100001000001110000000001011001

1101111001100011101011101101111011110111111111000101101111000110011011 1

0111101011110000111111001101101111110111101011100001100110011111111111

1101010010100011111100110101100110110110101011111100110001011100100000
0111111110010011111101111001001010101101111111000111101000011011101110
1100000000111110000000111101111001101111110011011100010001110100111100 10
0110100001101101100101011011110101001011101110101111001111111101101101
1101011010010110011111111011111111100011111110011011111111011101110110
1011110011010110101101010100100111101110111010011101111111101001110011
1111010101100100111100100010010011111011101101001101010010101111000010
1110111110011011000100110110111111111001011101110011111010101100111100
1111010101111110111011101110111111001101011101101011010100011010100010
1101001101111110111110001111011010100111010101010111111111011110110111 0
1011111010101111010001101110111001100111101000011011011111110111011111
0011110110011111101110000101100001011111010111101101011011111111110000
1101010001001001111111010110101010001100001111111101001110101101101111
0110101011011011110101110111111011101111110111111011010111000110011111 1
1100010100101100111111111101100011101100011000101110011001100111010111
0111111110111101001101110111100101101110101110110100110111111010100100
0111101000001100010000101000110001100001010000000000100101100100100000
1110001001011000000001110000110101101101001000010100100110010011001001 0
0001110010011001011110000101101101010101101100101011101000100000101000
0110100101010100011000001011100001010000110011001100000010000110110101
1011111010001010111000011000110100000101111111000011101111001010011001 01
0111000111111100100000000000001000101110000011100000110110100011010010 0
1111101010011001101100000110111100000000100100001101010011110100101100
1101011010001100011000011011101001001111100011100010100011001000011011
0100000100001101111111101000101001110100010000010010000000111010000011 1
0111010001111000110100110100100011000000111011100100000001001101001101
1001100011011000100110000100110110111111011110001000111111000001101011 1
1100100100110100000000100010100001000010011001111010111100101101011110 0
0011111011010000010000011111111010110010001100101000100111001010001111
1011000000000001101110101010101011000101111110100100010100001101011010 0

0101110100110101110101101101011110000011001111000011100110001101011010

1010110110100011010110010101100010101101000010000010110110100100011011

1000100100110011100100001000001000100011010010001010101001111101001000

0111100011010011110010011011100010001001001110010111111010000100110100

1001111110000010110111101111100111101100010111111000000010000000010111

0010101000100101000000111000111111111000010111001101001100110011011010

0001011011011100010101000001000001111100001100000100111011001100110110

0101010000001100110101101000110101001010000110111001101010010110110000

0001001111110000101011111110000110100011001010000111101110110100110111

1101111001001011101000100111010000110100010000010010010001110110000010

0100100001100010011011011001000100100100000010110001100111001101100100

0100000001000000000101000000000100001000010100101101100100010000000000

0101100000000110010010010010000000000000100011000000010101101101110001

0110010001100010110000000010100000001010011001000100001011000000001110101

00000100100000000010000010000000010000000000110000000100010000

The next representation is the actual partition of $[1, 6500]$ into two color classes. Each of these two sets is free of solutions to the Pythagorean equation.

Red: {4, 5, 6, 7, 11, 12, 14, 15, 20, 22, 23, 24, 27, 28, 30, 31, 33, 34, 38, 39, 40, 41, 42, 46, 48, 55, 60, 65, 67, 68, 69, 71, 72, 74, 75, 76, 77, 79, 80, 81, 82, 84, 87, 88, 90, 92, 93, 99, 103, 104, 105, 106, 107, 109, 110, 114, 117, 118, 119, 122, 123, 124, 125, 128, 130, 133, 136, 138, 141, 142, 145, 148, 149, 151, 152, 157, 158, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 186, 188, 191, 196, 197, 199, 200, 202, 203, 204, 206, 207, 208, 211, 212, 213, 214, 215, 217, 218, 220, 224, 225, 227, 228, 230, 231, 236, 238, 239, 240, 241, 242, 243, 244, 245, 246, 247, 250, 253, 256, 257, 260, 262, 263, 268, 270, 273, 277, 282, 283, 287, 290, 291, 292, 293, 296, 297, 298, 299, 302, 307, 310, 311, 312, 313, 315, 317, 321, 323, 324, 326, 330, 331, 332, 333, 334, 335, 336, 337, 340, 341, 343, 344, 345, 347, 348, 350, 351, 352, 354, 355, 356, 358, 359, 360, 361, 363, 366, 367, 372, 373, 376, 378, 379, 380, 381, 382, 383, 384, 391, 393, 394, 395, 399, 400, 403, 404, 406, 407, 409, 412, 417, 419, 421, 422, 428, 431, 432, 434, 435, 436, 439, 441, 442, 444, 446, 447, 448, 455, 456, 457, 459, 460, 463, 466, 467, 469, 471, 472, 474, 477, 478, 479, 482, 483, 485, 486, 488, 489, 490, 491, 492, 493, 495, 497, 498, 499, 500, 504, 506, 507, 508, 510, 511, 512, 513, 514, 515, 516, 521, 524, 526, 527, 529, 530, 531, 532, 534, 536, 542, 543, 544, 546, 548, 550, 552, 556, 558, 562, 563, 564, 565, 566, 567, 570, 571, 572, 573, 574, 576, 577, 578, 582, 583, 584, 587, 589, 591, 592, 593, 595, 596, 599, 604, 605, 606, 607, 608, 612, 613, 614, 615, 618, 619, 620, 621, 622, 623, 626, 631, 634, 638, 639, 641, 642, 643, 647, 649, 650, 652, 656, 659, 660, 661, 662, 663, 664, 666, 668, 670, 675, 676, 677, 678, 680, 682, 683, 684, 685, 686, 688, 689, 691, 693, 694, 697, 699, 700, 701, 705, 706, 710, 711, 713, 714, 716, 717, 718, 719, 721, 722, 725, 726, 727, 728, 732, 734, 735, 737, 738, 739, 741, 742, 743, 749, 750, 751, 753, 754, 757, 758, 762, 763, 766, 767, 769, 770, 771, 772, 773, 775, 779, 780, 781, 782, 784, 785, 786, 787, 789, 795, 796, 797, 800, 801, 802, 804, 805, 808, 809, 811, 812, 813, 814, 819, 822, 823, 827, 829, 831, 832, 833, 834, 835, 836, 837, 838, 839, 841, 842, 843, 844, 848, 853, 854, 857, 859, 861, 862, 863, 865, 866, 867, 868, 869, 870, 873, 878, 879, 880, 881, 883, 886, 887, 888, 889, 890, 891, 892, 893, 894, 895, 896, 897, 898, 900, 901, 905, 907, 908, 911, 917, 918, 919, 920, 922, 925, 926, 927, 928, 929, 931, 935, 936, 938, 940, 942, 946, 947, 948, 949, 950, 954, 958, 959, 960, 961, 962, 963, 964, 965, 966, 969, 972, 973, 977, 981, 982, 983, 984, 985, 990, 991, 994, 995, 996, 998, 999, 1001, 1003, 1007, 1008, 1009, 1011, 1013, 1014, 1017, 1019, 1020, 1023, 1029, 1030, 1031, 1032, 1033, 1035, 1036, 1039, 1043, 1044, 1046, 1050, 1051, 1053, 1054, 1055, 1056, 1059, 1063, 1064, 1065, 1066, 1067, 1073, 1075, 1076, 1079, 1080, 1081, 1090, 1091, 1093, 1094, 1095, 1096, 1097, 1102, 1104, 1105, 1106, 1109, 1111, 1113, 1115, 1117, 1119, 1122, 1123, 1124, 1126, 1128, 1130, 1132, 1133, 1134, 1135, 1138, 1140, 1141, 1142, 1144, 1145, 1146, 1148, 1150, 1151, 1152, 1153, 1157, 1159, 1165, 1168, 1169, 1170, 1171, 1172, 1174, 1177, 1178, 1180, 1181, 1184, 1186, 1187, 1188, 1189, 1190, 1192, 1195, 1197, 1198, 1201, 1202, 1204, 1205, 1207, 1208, 1209, 1211, 1214, 1215, 1216, 1217, 1218, 1220, 1221, 1222, 1223, 1225, 1226, 1228, 1231, 1233, 1234, 1235, 1238, 1240, 1242, 1244, 1245, 1246, 1247, 1250, 1252, 1253, 1255, 1259, 1260, 1261, 1262, 1265, 1268, 1269, 1270, 1273, 1275, 1276, 1277, 1278, 1279, 1280, 1283, 1285, 1286, 1289, 1291, 1294, 1298, 1299, 1300, 1303, 1304, 1305, 1307, 1308, 1309, 1310, 1312, 1313, 1314, 1315, 1316, 1318, 1319, 1322, 1324, 1325, 1326, 1327, 1331, 1335, 1336, 1337,

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2008, 2010, 2011, 2012, 2016, 2017, 2018, 2019, 2022, 2023, 2026, 2027, 2029, 2031, 2032, 2037, 2038, 2039, 2042, 2043, 2044, 2045, 2049, 2050, 2051, 2052, 2053, 2054, 2055, 2056, 2057, 2058, 2060, 2061, 2062, 2063, 2067, 2068, 2070, 2072, 2073, 2074, 2077, 2078, 2079, 2081, 2082, 2083, 2084, 2085, 2086, 2087, 2088, 2089, 2090, 2091, 2092, 2095, 2096, 2097, 2098, 2099, 2100, 2101, 2102, 2104, 2106, 2108, 2109, 2110, 2111, 2112, 2114, 2116, 2117, 2119, 2123, 2125, 2126, 2128, 2129, 2130, 2131, 2132, 2134, 2135, 2139, 2143, 2146, 2147, 2148, 2149, 2150, 2151, 2152, 2154, 2155, 2157, 2158, 2159, 2160, 2161, 2162, 2164, 2166, 2167, 2168, 2170, 2176, 2177, 2178, 2179, 2180, 2181, 2182, 2183, 2184, 2185, 2186, 2192, 2193, 2195, 2196, 2197, 2199, 2200, 2204, 2205, 2206, 2208, 2210, 2211, 2215, 2217, 2219, 2221, 2222, 2223, 2224, 2226, 2229, 2232, 2234, 2236, 2237, 2238, 2239, 2240, 2243, 2245, 2246, 2247, 2251, 2252, 2253, 2254, 2259, 2260, 2264, 2265, 2266, 2267, 2269, 2270, 2271, 2272, 2273, 2274, 2275, 2276, 2278, 2279, 2280, 2281, 2282, 2284, 2285, 2286, 2287, 2292, 2293, 2294, 2298, 2299, 2301, 2302, 2303, 2308, 2311, 2312, 2313, 2315, 2317, 2318, 2319, 2320, 2321, 2324, 2328, 2332, 2333, 2334, 2335, 2338, 2339, 2341, 2342, 2343, 2345, 2347, 2348, 2349, 2350, 2351, 2352, 2353, 2354, 2356, 2358, 2360, 2361, 2362, 2363, 2365, 2366, 2367, 2369, 2370, 2371, 2374, 2378, 2379, 2380, 2382, 2383, 2385, 2387, 2388, 2389, 2390, 2393, 2395, 2396, 2398, 2399, 2400, 2401, 2410, 2411, 2412, 2413, 2414, 2415, 2416, 2417, 2418, 2421, 2423, 2424, 2426, 2427, 2428, 2429, 2430, 2432, 2433, 2435, 2436, 2439, 2440, 2442, 2444, 2445, 2446, 2447, 2449, 2452, 2453, 2454, 2455, 2456, 2457, 2459, 2461, 2462, 2464, 2465, 2467, 2468, 2469, 2473, 2475, 2477, 2478, 2481, 2484, 2485, 2487, 2491, 2498, 2501, 2502, 2504, 2507, 2508, 2514, 2517, 2520, 2522, 2530, 2532, 2535, 2536, 2537, 2542, 2544, 2545, 2546, 2547, 2548, 2549, 2550, 2555, 2563, 2571, 2574, 2576, 2577, 2578, 2580, 2586, 2589, 2590, 2592, 2596, 2600, 2601, 2603, 2604, 2607, 2611, 2616, 2617, 2618, 2619, 2621, 2624, 2634, 2641, 2642, 2644, 2646, 2649, 2652, 2658, 2660, 2664, 2665, 2667, 2668, 2669, 2673, 2675, 2680, 2684, 2685, 2688, 2689, 2690, 2694, 2700, 2703, 2704, 2706, 2713, 2714, 2715, 2717, 2722, 2723, 2726, 2727, 2728, 2729, 2737, 2739, 2746, 2751, 2752, 2756, 2758, 2759, 2762, 2763, 2766, 2769, 2772, 2773, 2775, 2782, 2783, 2785, 2787, 2789, 2791, 2792, 2793, 2794, 2796, 2798, 2799, 2802, 2803, 2806, 2807, 2809, 2810, 2811, 2814, 2818,

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Finally, we present a visual interpretation of the coloring (read left-to-right, then top-to-bottom) in Figure A.1.

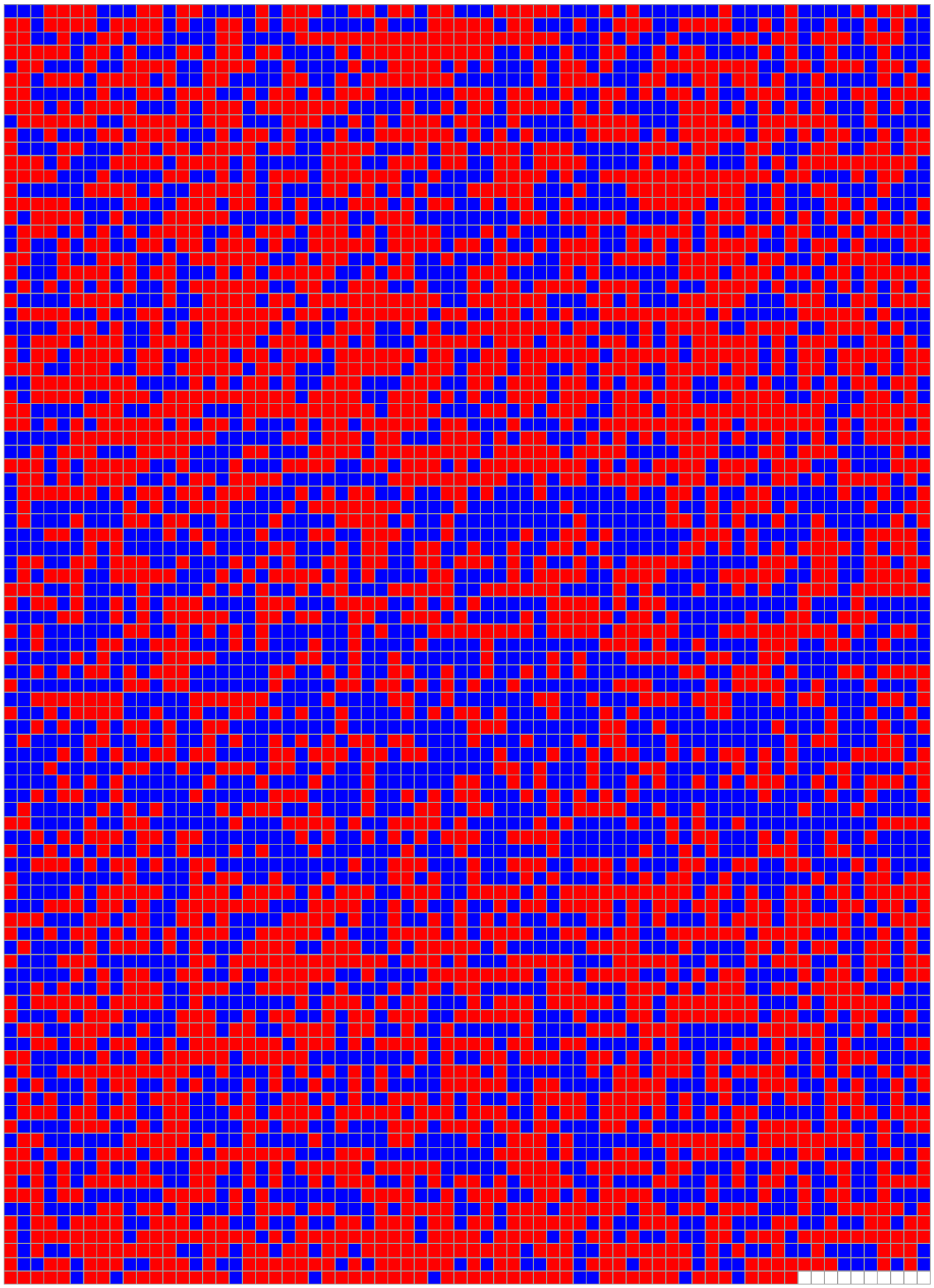

Figure A.1: Coloring proving the lower bound for $\mathrm{R}_2\big(x^2 + y^2 = z^2\big)$.

Appendix B

The Family $kx + (k+1)y = (k+2)z$

In Section 4.2 we present some data for the family of equations $kx+(k+1)y = (k+2)z$. For each k, Table B.1 gives the Rado number, the CPU time required (in CPU-seconds), the number of colorings checked, the number of times $f_{\mathcal{E}}$ is evaluated, the number of times the tree is cleaved, and the number of leaves (i.e. number of maximal colorings) which is also the length (number of lines of text) of the uncompressed certificate.

k	$R_2(\mathcal{E}_k)$	CPU time	Cols. checked	Evals. of $f_{\mathcal{E}}$	Cleaves	Leaves
1	1	0	1	1	*	*
2	10	16	19	311	*	*
3	17	0	41	1697	*	*
4	15	0	53	1617	*	*
5	23	0	201	17834	1	15
6	28	0	1311	162927	1	17
7	36	0	275	35102	1	15
8	45	16	7361	2065837	1	16
9	55	0	6057	1718335	1	15
10	71	32	20083	12699518	3	50
11	78	32	42209	21194015	3	94
12	97	144	303695	185894001	6	598
13	105	96	132953	92713633	6	386
14	120	816	1689197	1533140154	10	1720
15	136	816	1390631	1394194518	18	2141
16	153	3296	5502381	6829417208	10	2698
17	171	4128	8351443	8482384666	14	2310
18	190	122464	155029995	260985312167	20	4465
19	210	21552	31125057	45458377262	16	3727
20	231	727808	722679941	1654046397071	90	100429
21	253	291184	392834951	674637589862	20	6018
22	276	2931456	2088238859	6720815411113	65	134527
23	300	2329472	2290866899	5280949472008	122	86629
24	325	72693760	43253507521	168256031472106	309	864306
25	351	7605760	5437255877	17482566990496	212	436428

Table B.1: Computational data for the family $kx + (k+1)y = (k+2)z$

A 0 indicates that the computation took less than 1 seconds (all figures are rounded to the nearest second). A * indicates that this computation was completed by the master process before it could produce enough valid colorings to fill up the queue and pass off branches to workers. This does not indicate any problem – a valid certificate is still produced, and the computation is of course still correct. However, the master process will not collect these two statistics in this case.

All computations for $k = 1, 2, \ldots, 19$ were done with 16 CPUs. For $k = 20, 22, 23$, there were 128 CPUs, and for $k = 21, 24, 25$ there were 512. Note that this is factored into "CPU-time" (not just "time"), but changing the number of CPUs would affect the number of times the tree is cleaved, as well as how long the master process works to build up the queue at the beginning of the algorithm. Notice that $k = 24$ does not have an explosion in the number of cleaves required – it would have far more than 309 if we had used 128 or 16 CPUs (and it would have failed to terminate in the allotted time!). No certificates were written, so in every case, there were 15, 127, or 511 workers plus one master process.

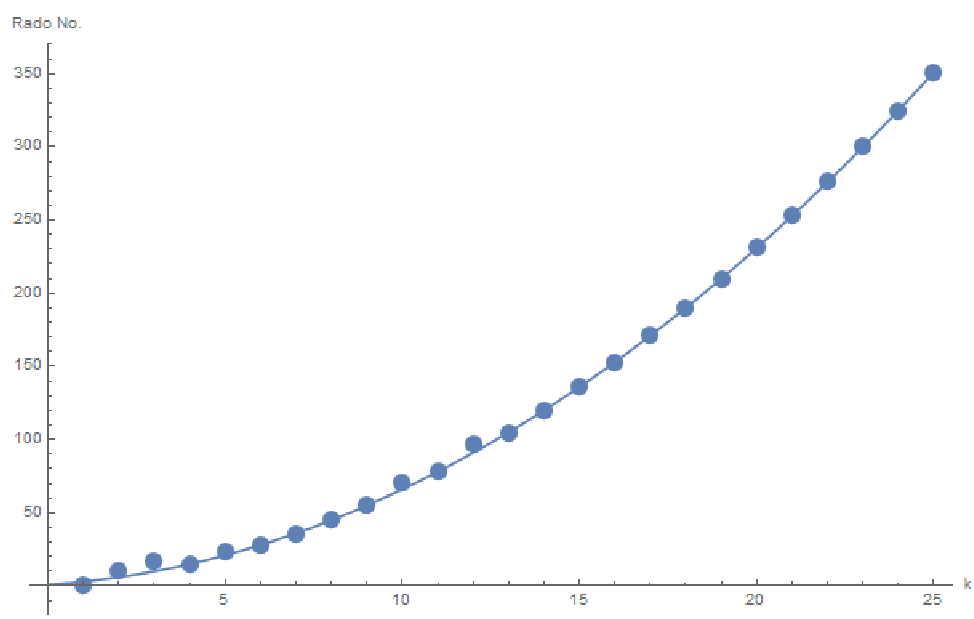

Figure B.1: Rado numbers for $\mathcal{E}_k$

We can continue our discussion from Section 4.1, where we noticed how various data besides just the Rado numbers seem to be related. First, we'll take a look at the

Rado numbers for this family, along with the interpolating polynomial from Conjecture 4.4, in Figure B.1. In these figures, as before, we normalize these data series to be simultaneously visible.

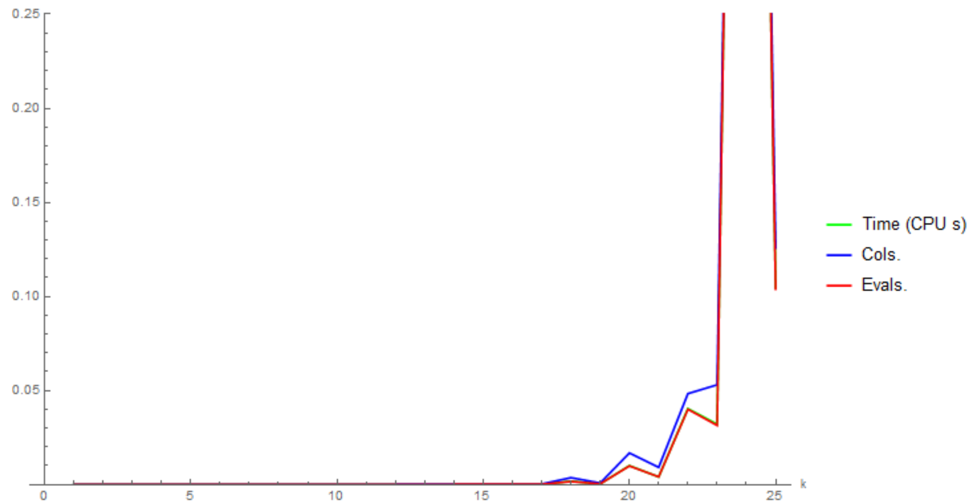

Figure B.2: CPU time for computing Rado numbers for $\mathcal{E}_k$

As in Section 4.1, we can see in Figure B.2 the relationship between the computing time required, the number of colorings checked, and the evaluations of the polynomial $f_{\mathcal{E}_k}(x, y, z) = kx + (k+1)y - (k+2)z$. Notice, as we observed before, that the number of evaluations of $f_{\mathcal{E}_k}$ and the time required for the computation overlap – so much so they are difficult to distinguish.

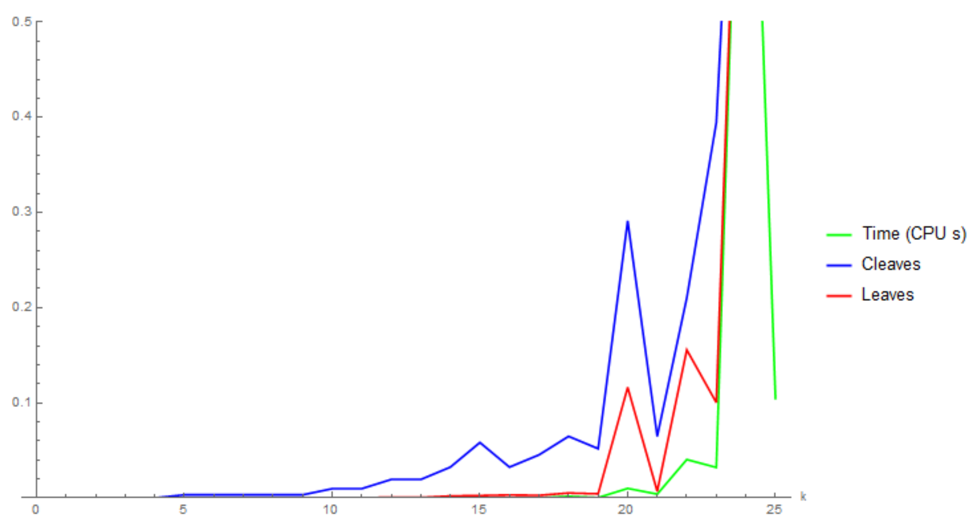

Figure B.3: Statistics for cleaving and number of leaves for $\mathcal{E}_k$

In Figure B.3, we see how the computing time relates to the number of leaves and the number of times a cleave order is issued. We can see that the peaks for these also tend to coincide with the CPU time (and so also the graphs from B.2).

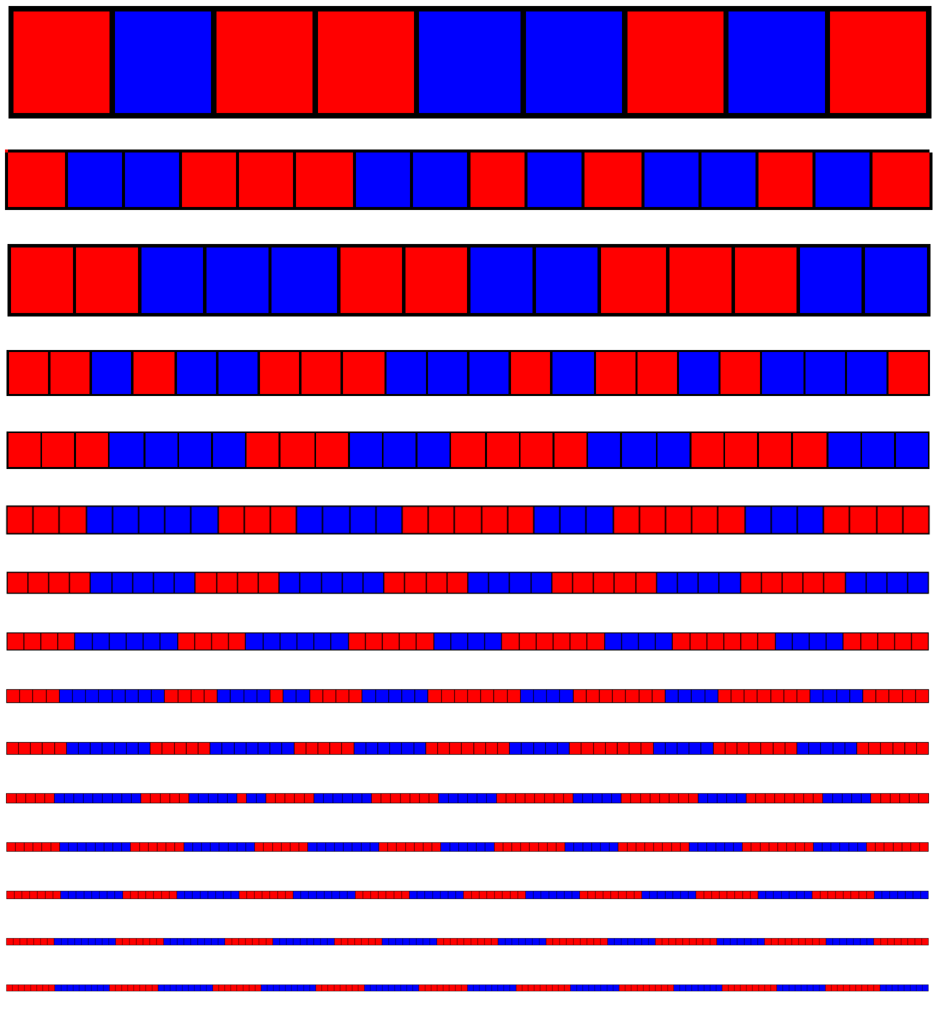

Figure B.4: Representative maximum-length valid colorings for $\mathcal{E}_k$

To finish this appendix, we have Figure B.4 illustrating representative valid colorings of length $\mathrm{R}_2(\mathcal{E}_k) - 1$ for $k = 2$ to $k = 17$.

We note that some pattern may be present, as these seem to be a somewhat well-behaved set of intervals (with some exceptions, particularly $k = 11, 13$). For some k values there are other maximum length valid colorings that differ by one color assignment, and these are chosen to be the ones with the fewest changes from red to blue

k	Coloring
2	(1, 1, 2, 2, 1, 1, 1)
3	(1, 2, 3, 2, 1, 1, 1, 2, 1, 1, 1)
4	(2, 3, 2, 2, 3, 2)
5	(2, 1, 1, 2, 3, 3, 1, 1, 2, 1, 1, 3, 1)
6	(3, 4, 3, 3, 4, 3, 4, 3)
7	(3, 5, 3, 4, 5, 3, 5, 3, 4)
8	(4, 5, 4, 5, 4, 4, 5, 4, 5, 4)
9	(4, 6, 4, 6, 5, 4, 6, 4, 6, 4, 5)
10	(4, 8, 4, 4, 1, 2, 4, 5, 6, 5, 7, 4, 7, 4, 5)
11	(5, 7, 5, 7, 5, 6, 7, 5, 7, 5, 7, 5, 6)
12	(5, 9, 5, 5, 1, 2, 5, 6, 7, 6, 7, 6, 8, 5, 8, 5, 6)
13	(6, 8, 6, 9, 5, 8, 7, 6, 8, 6, 8, 6, 8, 6, 7)
14	(7, 8, 7, 8, 7, 8, 7, 7, 8, 7, 8, 7, 8, 7, 8, 7)
15	(7, 9, 7, 10, 6, 9, 7, 8, 9, 7, 9, 7, 9, 7, 9, 7, 8)
16	(8, 9, 8, 9, 8, 9, 8, 9, 8, 8, 9, 8, 9, 8, 9, 8, 9, 8)
17	(8, 10, 8, 10, 8, 10, 8, 10, 9, 8, 10, 8, 10, 8, 10, 8, 10, 8, 9)

Table B.2: Representative maximum-length valid colorings for $\mathcal{E}_k$

as we read the coloring left to right (preferring red-red-red-red-red to red-red-blue-red-red). We could also interpret these as (finite) sequences, where a_1, a_2, a_3 would indicate the length of each monochromatic interval. That gives us the same data in a different form, as listed in Table B.2.

Appendix C

Sums of Squares

In this appendix we provide a comparison of the effectiveness of an optimized VI, as described in Section 3.7.4. We will use data from some of the Rado numbers in Section 5.2.4.

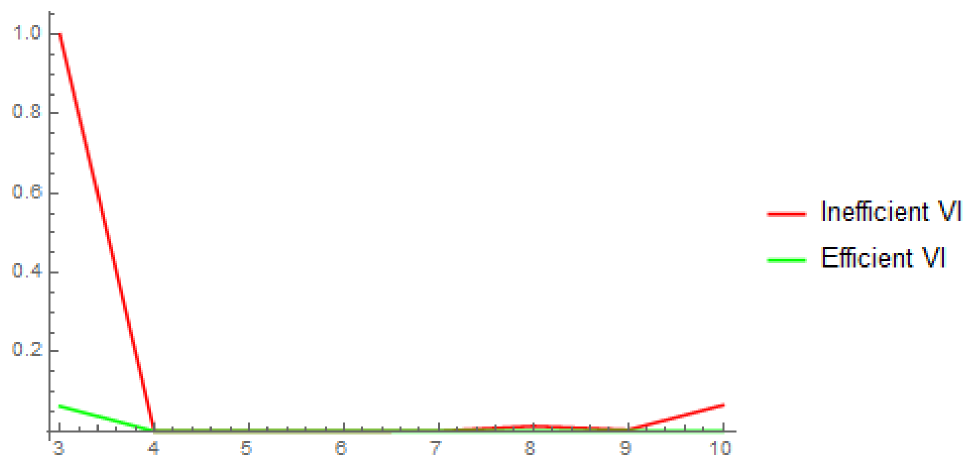

Figure C.1: Number of evaluations of $f_{\mathcal{E}_k}$ for each VI

In Figures C.1 and C.2, we see a comparison of the number of evaluations of $f_{\mathcal{E}_k}$ for $k = 3, \ldots, 10$. The old VI is clearly inferior to the new VI because it requires more computations. These two figures are normalized to $[0, 1]$, but by the same factor. Notice that in Figure C.2, we can see the new VI is just flatlined on the axis compared to the others, even at $k = 10$. It is impossible – with our resources – to use the inefficient older VI to compute $\mathrm{R}_2(\mathcal{E}_{11})$, while it is very easy to do so with the new VI. Even considering $\mathrm{R}_2(\mathcal{E}_{10})$, the new VI completes the computation in moments, while the old VI takes over a day.

This shows us quite clearly the benefits of using the improved VI – it also makes clear the case, as we argue in Section 6.1, that we ought to expand this intelligent VI

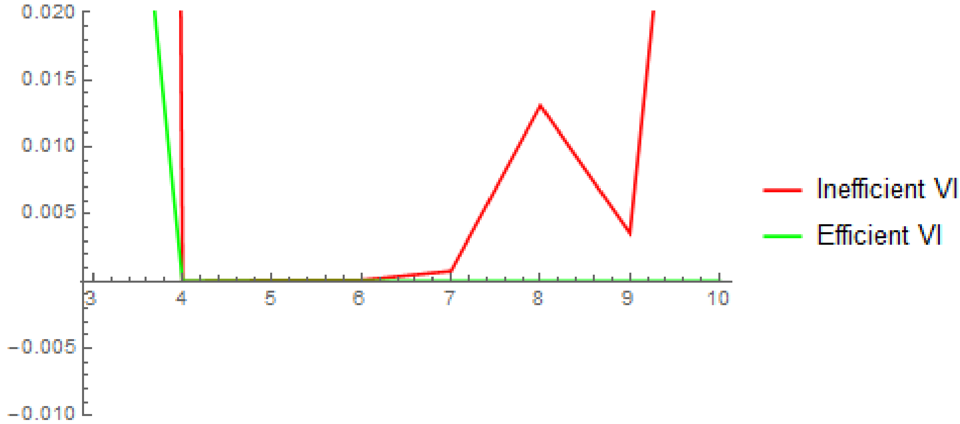

Figure C.2: Number of evaluations of $f_{\mathcal{E}_k}$ for each VI (zoomed)

to work for equations like $2x + 2y = 3z$ (which, as we see in Section 4.4, takes a long time to compute for $r = 3$).

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