SELECTED TOPICS IN STOCHASTIC OPTIMIZATION

BY ANH TUAN NINH

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ABSTRACT OF THE DISSERTATION

Selected topics in stochastic optimization

by Anh Tuan Ninh
Dissertation Directors: András Prékopa - Yao Zhao

This report constitutes the Doctoral Dissertation for Anh Ninh and consists of three topics: log-concavity of compound Poisson and general compound distributions, discrete moment problems with fractional moments, and the recruitment stocking problems.

In the first topic, we find the conditions for the compound Poisson and general compound distributions to be log-concave (log-convex). This problem is very important not only from the stochastic optimization perspective but also from the theory of maximum entropy in probability. Some interesting connection to Turán-type inequality will also be mentioned.

In the second topic, we formulate a linear programming problem to find the minimum and/or maximum of the expectation of a function of a discrete random variable, given the knowledge of fractional moments. Using a determinant theorem we fully characterize the dual feasible basis for this discrete fractional moment problem. With the dual feasible basis structure, Prékopa dual method can be applied for its solution. Numerical examples show that by the use of fractional moments, we obtain tighter bounds for the objective.

In the third topic, we introduce a new class of inventory control model - the recruitment stocking problems. In particular, we analyze a general class of inventory control problem, in which we need to recruit a target number of individuals through designated

outlets. As soon as the recruits of all outlets add up to the target number, the recruitment is done and no more individuals will be admitted. The arrivals of individuals at each outlet are random. To recruit an individual upon its arrival, we must provide a pack of materials. We order the packs of materials in advance and hold them in the outlets. Outlets can neither transfer recruits nor cross-ship materials among themselves. If an outlet runs out of stock, any futher recruit at the outlet will be lost. We propose both exact and approximation methods to measure key performance metrics for the system: Type I and II service levels and recruitment time. Extensive numerical study shows the effectiveness of our proposed framework.

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Dedication

For my parents and my sister

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Chapter 1

Introduction

This work consists of three selected topics in probability and stochastic optimization: log-concavity (log-convexity) of compound Poisson and general compound distributions, the discrete moment problems with fractional moments at the right-hand-side, and the recruitment stocking problems.

The notion of a log-concave sequence was initially introduced by Fekete (1912) under the name of 2-times or twice positive sequence as a special case of an r-times positive sequence. From his famous convolution theorem on r-times positive sequences, an important result for log-concave sequences can be derived, i.e., the convolution of two log-concave sequences is log-concave. The continuous log-concavity comes up in connection with failure rate in reliability theory (see, e.g.,Barlow, Proschan, 1965). Prékopa was the first to introduced multivariate log-concave measures to prove convexity of probabilistic constrained problems. Following are two fundamental theorems from Prékopa (1971, 1973a,b). For further references, we refer to the monograph by Prékopa (1995).

Theorem 1. *Let $f(x)$, $x \in R^m$ be a log-concave probability density function and let P be the probability measure generated by f. Then P is a logconcave measure.*

Theorem 2. *If $f(x, y)$ is a logconcave function of the $m + n$ variables contained in $x \in R^n$, and $y \in R^m$, then*

$$\int_{R^m} f(x, y) dy \tag{1.1}$$

is a log-concave function of $x \in R^n$.

An important consequence for the above theorem is that the convolution of two log-concave functions in R^m is also log-concave.

Compound Poisson distributions play important role in many applied areas: acturial mathematics, physics, engineering, operations research. The log-concavity property, in connection with a compound Poisson distribution, comes up primarily in stochastic optimization, where frequently the convexity of the optimization problem depends on it (see Prékopa, 1995). One example is the bond portfolio construction problem with probabilistic constraints. The claims in subsequent periods enjoy independent compound Poisson distributions. In addition, the log-concavity property for a compound Poisson distribution (on nonnegative integers) is also very important from the point of view of understanding compound Poisson limit theorems.

This dissertation is based on the papers from Anh Ninh, Prékopa (2013a,b). Chapter 2 is devoted to derive the conditions under which some compound Poisson distributions and general compound distributions are log-concave (log-convex). In particular, we look at the compound distributions with geometric, negative binomial, Poisson and gamma distributed terms. Furthermore, we present an interesting connection of log-concavity to various Turán type inequalities.

In Chapter 3 we review the fundamentals of discrete moment problems (DMP) and introduce a variation of DMP when the right-hand-side contains fractional moments. DMP came to prominence by the discovery (Samuels and Studden, 1989), Prékopa (1988, 1990a,b) that the sharp Bonferroni bounds can be obtained as optimum valus of discrete moment problems. The simplest discrete moment problem, where power moments are used, is closefy connected with divided differences, and higher order convex functions. We show that DMP with fractional moments can also be solved efficiently using Prékopa dual algorithm.

In the last chapter we introduce a new class of inventory management, called recruitment stocking problems. We need to recruit a target number of individuals through designated outlets. As soon as the recruits of all outlets add up to the target number, the recruitment is done and no more individuals will be admitted. The arrivals of individuals at each outlet are random. To recruit an individual upon its arrival, we must provide a pack of materials. We order the packs of materials in advance and hold them in the outlets. Outlets can neither transfer recruits nor cross-ship materials among

themselves. If an outlet runs out of stock, any futher recruit at the outlet will be lost.

The recruitment stocking problem differs from previous research conducted in the existing inventory management literature due to the finite target number which connects all outlets in such a way that the recruitment is done as soon as the recruits at all outlets reach the target. In most existing inventory models, we should satisfy demand as much as supply allows. In other words, demand should be satisfied as long as inventory is available. This is not true in the recruitment stocking problem, where as soon as the target is met. no more demand will be served even if we have stock in the system. With this unique feature under consideration, performance evaluation and inventory allocation for this system are not known in the literature. In Chapter 4, we propose both exact and approximation methods to measure kep performance metrics for the systems: Type I, and II service levels and recruitment time.

The three topics in this dissertation are well connected. Log-concavity is used in probabilistic constrained stochastic programming problems as well as discrete moment bounds. Since log-concavity implies unimodality, we can take advantage of the shape of the distribution to improve the quality of the lower and upper bounds in moment problems (see Subasi et. al, 2009). Both log-concavity and discrete moment bounds are encounterd in the context of the inventory management problem discussed in the last chapter.

Chapter 2

Logconcavity of compound distributions

2.1 Introduction

Compound Poisson distributions play important role in many applied areas: acturial mathematics, physics, engineering, operations research, etc., see, e.g, Bowers et al. (1986), Takács (1967), Prékopa (1995), Withers and Nadarajah (2011). The log-concavity property, in connection with a compound Poisson distribution, comes up primarily in stochastic optimization, where frequently the convexity of the optimization problem depends on it (see Prékopa, 1995). One example is the bond portfolio construction problem with probabilistic constraints. The claims in subsequent periods enjoy independent compound Poisson distributions (see Prékopa, 2003). For solutions of problems with discrete random variables in probabilistic constraints see Prékopa (1990), Prékopa, Vizvári, Badics (1997), Dentcheva, Prékopa, Rusczyński (2000, 2002) and Prékopa, Unuvar (2012), Yoda, Prékopa (2015). In addition, the log-concavity property for a compound Poisson distribution (on nonnegative integers) is also very important from the point of view of understanding compound Poisson limit theorems via entropy, as initiated by Johnson et al. (2008) and Barbour et al. (2010) and further studied in Yu (2009) and Johnson et al. (2011).

Let $X_1, X_2, ...,$ be a sequence of nonnegative valued i.i.d random variables and consider the sum

$$S = X_1 + X_2 + ... + X_N, \tag{2.1}$$

where N is a nonnegative random variable and $N, X_1, X_2...,$ are independent. The distribution of S is called compound distribution. If N is a Poisson random variable,

with parameter $\lambda > 0$:

$$f_n = P(N = n) = \frac{\lambda^n}{n!}e^{-\lambda}, n = 0, 1, 2..., \tag{2.2}$$

then we say that S has compound Poisson distribution. In the same way we define compound negative binomial etc. distributions, depending on the type of the random variable N.

The notion of a log-concave sequence was first introduced by Fekete (1912) under the name of *2*-times or twice positive sequence as a special case of an r-times positive sequence, when $r = 2$. The sequence of nonnegative elements... a_{-2}, a_{-1}, a_0,... is said to be r-times positive if the matrix

$$A = \begin{pmatrix} \ddots & \ddots & \ddots & & \\ \ddots & a_0 & a_1 & a_2 & \\ \ddots & a_{-1} & a_0 & a_1 & \ddots \\ & a_{-2} & a_{-1} & a_0 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

has no negative minor of order smaller than or equal to r (a minor is the determinant of a finite square part of the matrix traced out by the same number of rows as columns).

The twice-positive sequences are those for which we have

$$\begin{vmatrix} a_i & a_j \\ a_{i-t} & a_{j-t} \end{vmatrix} = a_i a_{j-t} - a_j a_{i-t} \geq 0,$$

for every $i \leq j$ and $t \geq 1$. Fekete (1912) proved the following important theorem.

Theorem 3. *The convolution of two r-times positive sequences is at least r-times positive.*

Several authors define log-concavity of a sequence $a_0, a_1, \ldots$ by requiring only

$$a_n^2 \geq a_{n-1}a_{n+1}, n = 1, 2, \ldots. \tag{2.3}$$

This property, alone, however, does not imply that the convolution of two log-concave

sequences are also log-concave. For example, if the two sequences are defined as follows:

$$a_0 = 0, a_1 = 1/2, a_2 = a_3 = 0, a_4 = 1/2, a_5 = a_6 = \cdots = 0 \qquad (2.4)$$

$$b_0 = b_1 = b_2 = b_3 = 1/4, b_4 = b_5 = \cdots = 0, \qquad (2.5)$$

then their convolution $\{c_n\}$ is not log-concave since $c_3^2 < c_2 c_4$. In view of this, log-concavity should be defined in the same way as Fekete has defined the notion of twice positive sequence, or, in addition to the requirement (2.3) we have to require that $\{a_n\}$ does not have internal zero. In other words, there are no indices $0 \leq i < j < k \leq n$ such that $a_i \neq 0, a_j = 0, a_k \neq 0$. Then, a special case of Theorem 1.1 states that the convolution of two log-concave sequences that have no internal zeros is log-concave. If the sequence $i! a_i$ is log-concave, then a_i is said to be ultra-log-concave. A sequence $\{a_n\}$ is said to be unimodal if for some $0 \leq j \leq n$ we have $a_0 \leq a_1 \leq \cdots \leq a_j \geq a_j + 1 \geq \cdots \geq a_n$. It is well-known that a log-concave sequence with no internal zeros is unimodal. The sequence $\{a_n\}$ is called log-convex if $a_n^2 \leq a_{n-1} a_{n+1}$ for all $n = 1, 2, \ldots$. For further reading on log-concave and log-convex sequences in algebra and combinatorics, we refer to the works of Stanley (1989) and Brenti (1989, 1994), Liu, Wang (2006).

A univariate discrete probability distribution, defined on the integers, is said to be log-concave (log-convex) if the sequence of the corresponding probabilities is log-concave (log-convex). A unimodal probability distribution is a probability distribution which has a single mode. Since log-concavity implies unimodality, log-concavity provides us with the shape information of the probability distributions. This information was proved to be very helpful to improve the quality of lower and upper bounds in moment problems (Subasi et. al, 2009). Some examples of discrete log-concave distribution includes the Bernoulli distributions, binomial distributions, Poisson distributions, geometric distributions, and negative binomial distributions. These distributions are also unimodal.

Log-concavity for compound variables with nonnegative integer values of the form $S = X_1 + \cdots + X_N$ was first studied in Johnson (2008) in connection with maximum entropy property of discrete compound Poisson measures. He gave a conjecture on the conditions to ensure log-concavity of S, in terms of the log-concavity of $X_i, i =$

$1, \ldots, N$. In the following year, Yu (2009) pointed out that this conjecture can be deduced from the following result on log-concavity of infinite divisible sequences. A probability distribution $p_n, n = 0, 1, 2 \ldots$ with $p_0 > 0$ is called infinitely divisible (see Steutel, 1970) if and only if it satisfies

$$(n+1)p_{n+1} = \sum_{k=0}^{n} r_k p_{n-k}, \quad n = 0, 1, 2 \ldots, \tag{2.6}$$

with nonnegative r_k and $\sum_{k=0}^{\infty} r_k p_{n-k}$. The following theorem is due to Hansen (1988):

Theorem 4. *Let p_n and r_n, $n = 0, 1, 2, \ldots$, be related by (2.6) with $r_k \geq 0, p_0 > 0$ and let r_n be a log-concave (log-convex) sequence. Then, p_n is log-concave (log-convex) if and only if $r_0^2 - r_1 \geq 0$.*

In addition, it is well-known that (Panjer, 1981) if N is a Poisson random variable with rate λ then the following relation holds:

$$(n+1)p_{n+1} = \sum_{i=0}^{n} \lambda(i+1)q_{i+1}, \quad i \geq 0, \tag{2.7}$$

where $p_n = P(S = n)$ and $q_n = P(X_i = n)$, $n = 0, 1, 2, \ldots$. Theorem (4) and Panjer's recursion imply the necessary and sufficient conditions for a compound Poisson on nonnegative integer to be log-concave (Yu, 2009): q_n is log-concave and $\lambda q_1^2 \geq 2q_2$. Similar conditions for the compound Poisson on nonnegative integer to be log-convex can also be derived from Hansen's theorem. Further results on infinite divisibility and compound Poisson distributions can be found in Steutel, Van Harn (2003).

Various applications of log-concave sequences are known in probability theory, combinatorics, etc. Surprisingly, log-concavity property came up first in connection with orthogonal polynomials. The first theorem in this respect was proved by Turán (1950). It states that if $P_n(x)$ is the nth Legendre's polynomial, $-1 \leq x \leq 1$, then we have the inequality

$$P_n(x)^2 \geq P_{n-1}(x)P_{n+1}(x). \tag{2.8}$$

Inequalities of the type (2.8), valid for orthogonal polynomials, are called Turán type inequalities. In recent years, many Turán type inequalities have been established for

Laguerre polynomials, Hermite polynomials, Bessel functions, Tschebychef polynomials, etc. Some of them will be derived in this paper in connection with the log-concavity property of special compound Poisson distributions.

In Section 2, 3, 4 we are concerned with log-concavity property of some compound distributions on non-negative integers. Let $p_n = P(S = n), n = 0, 1, 2 \ldots$. We prove that the sequence $\{p_n\}$ without p_0 is log-concave, when the terms in S have geometric, negative binomial and Poisson distributions, respectively. In Section 5 we use the notion of a log-concave function $f(x), x \in \mathbb{R}$, meaning that $f(\lambda x + (1-\lambda)y) \geq (f(x))^\lambda (f(y))^{1-\lambda}$ for any $x, y \in \mathbb{R}, 0 < \lambda < 1$, and prove the log-concavity of the continuous part of the compound distribution with gamma distributed terms. In Section 6 we show the connection between the log-concavity property of some compound distributions to various Turán type inequalities. In Section 7 we briefly outline an application of log-concavity in a bond portfolio construction problem.

2.2 Log-concavity of compound distributions with geometrically distributed terms

In this section we prove theorems for log-concavity and log-convexity of compound distributions with geometrically distributed terms. The log-concavity part in our theorem can be deduced from Theorem 4.4 in Johnson et al. (2008). They proved that, for the shifted geometrically distributed terms, if N is log-concave and a certain condition holds then the compound distribution is log-concave. The mentioned condition is to ensure that the first three terms p_0, p_1, p_2 satisfy the log-concave inequality. Then, the authors showed the log-concavity of $p_1, p_2, \ldots$ using Theorem 7.3 in Karlin (1968) (under no condition on the parameters). The claim can also be implied from a binomial convolution theorem from Walkup (1976). For more information on linear transformation that preserves log-concavity, e.g., see Wang, Yeh (2006).

The log-convexity statement presented in our theorem is new and can be proved by the same method used for proving the log-concavity property. Note that the compound distributions with geometrically distributed terms is log-concave or log-convex

depending on the log-concavity or log-convexity of N is somewhat expected since the geometric distribution is both log-concave and log-convex.

If $X_1, X_2, \ldots$ have geometric distribution with support $\{0, 1, 2, ..\}$, i.e., $P(X_i = n) = p(1-p)^n$ $(n = 0, 1, 2, \ldots)$, then, as one can easily verify,

$$p_n = P(S = n) = \sum_{k=1}^{\infty} f_k \binom{k+n-1}{n} p^k (1-p)^n, \tag{2.9}$$

where $P(N = k) = f_k$.

Lemma 1. *Let $S_m = \sum_{i=1}^{m} c_i x_i$, with $x_i \geq 0$ and $c = \sum_{i=1}^{m} c_i \leq 0$. Suppose that the sequence x_i is either non-decreasing or non-increasing and there exists an integer k such that $c_i \geq 0$ for $i \leq k$ and $c_i \leq 0$ for $i > k$. Then S_m is non-negative if x_n is non-decreasing. If we further know that c is zero, then S_m is non-positive if x_i is non-increasing.*

Proof. Clearly, $c_1 x_1 + c_2 x_2 + \ldots + c_k x_k \leq (c_1 + c_2 + \ldots + c_k) x_k$ and $c_{k+1} x_{k+1} + \ldots + c_m x_m \leq (c_{k+1} + \ldots + c_m) x_{k+1}$. Thus, we have

$$
\begin{aligned}
S_m \ &\leq\ (c_1 + c_2 + \ldots + c_k) x_k + (c_{k+1} + \ldots + c_m) x_{k+1} \\
&=\ (c_1 + c_2 + \ldots + c_k) x_k + [c - (c_1 + \ldots + c_k)] x_{k+1} \\
&=\ (c_1 + c_2 + \ldots + c_k)(x_k - x_{k+1}) + c x_{k+1} \leq 0.
\end{aligned}
$$

The proof of the second assertion is similar. $\qquad\square$

Theorem 5. *If N has log-concave (log-convex) distribution (with no internal zeros) on the set of nonnegative integers and the terms $X_i, i = 1, 2, ..$ are geometrically distributed, then the sequence $\{p_n\}_{n=1}^{\infty}$, defined by (2.9), is log-concave (log-convex).*

Proof. Let $x = p$, we have

$$p_n = (1-p)^n \sum_{k=1}^{\infty} f_k \binom{k+n-1}{n} x^k. \tag{2.10}$$

In order to prove the log-concavity of $\{p_n\}_{n=1}^{\infty}$, it suffices to show that $\{g_n\}_{n=1}^{\infty}$ is log-concave, where

$$g_n = \sum_{k=1}^{\infty} f_k \binom{k+n-1}{n} x^k. \tag{2.11}$$

If we use the Cauchy product formula, we obtain

$$g_{n+1}g_{n-1} - g_n^2 = \sum_{m=2}^{\infty} x^m \sum_{i=1}^{m-1} T_i f_i f_{m-i}, \tag{2.12}$$

where

$$T_i = \binom{i+n}{n+1}\binom{(m-i)+n-2}{n-1} - \binom{i+n-1}{n}\binom{(m-i)+n-1}{n}. \tag{2.13}$$

Notice that if $f_k = 1, k = 1, 2, ...,$ then we have $g_n = x(1-x)^{-1-n}$. Thus, $\sum_{i=1}^{m-1} T_i$ is the coefficient at x^m in the power series expansion of

$$(1-x)^{-n}x(1-x)^{-2-n}x - [(1-x)^{-1-n}x]^2 = 0, \tag{2.14}$$

and it follows that $\sum_{i=1}^{m-1} T_i = 0$.

Case 1: m is odd. We can rewrite $\sum_{i=1}^{m-1} T_i f_i f_{m-i}$ as

$$\sum_{i=1}^{m-1} T_i f_i f_{m-i} = \sum_{i=1}^{[(m-1)/2]} (T_i + T_{m-i}) f_i f_{m-i}. \tag{2.15}$$

Case 2: m is even. We can rewrite $\sum_{i=1}^{m-1} T_i f_i f_{m-i}$ as

$$\sum_{i=1}^{m-1} T_i f_i f_{m-i} = \sum_{i=1}^{[(m-1)/2]} (T_i + T_{m-i}) f_i f_{m-i} + T_{m/2} f_{m/2}^2. \tag{2.16}$$

Using (2.13), we can easily verify that $T_{m/2} < 0$. In either case, $T_i + T_{m-i}$ is equal to

$$\binom{m-i+n-2}{n-1}\binom{i+n-2}{n-1}\frac{(2+4n)i^2 - 2m(1+2n)i + n(m^2 - m + 2) + 2(m-1)}{n^2(1+n)}. \tag{2.17}$$

By (2.17), we have that $T_1 + T_{m-1} > 0$ and $T_{[(m-1)/2]} + T_{m-[(m-1)/2]} < 0$. Furthermore, since the numerator is a quadratic function of i, there must exist an integer k such that $T_i + T_{m-i} > 0$, for $i < k$ and $T_i + T_{m-i} < 0$, for $i > k$. If $\{f_k\}_{k=1}^{\infty}$ is log-concave (without internal zeros), then $f_i f_{m-i}$ is non-increasing, for $i \leq [m/2]$. The assertion follows from Lemma 1. $\qquad\square$

2.3 Log-concavity of compound distributions with negative binomial distributed terms

If $X_1, X_2, \ldots$ are independent and have negative binomial distribution with support $\{0, 1, 2, ..\}$ and

$$P(X_i = n) = \binom{r + n - 1}{n} p^n (1 - p)^r, \quad n = 0, 1, 2, \ldots, \tag{2.18}$$

then, as one can easily verify, we have the equation:

$$P(S = n) = \sum_{k=1}^{\infty} f_k \binom{kr + n - 1}{n} p^n (1 - p)^{kr}, \tag{2.19}$$

where $P(N = k) = f_k$. Before stating the theorem, let us prove the following.

Lemma 2. *We have the relation:*

$$\sum_{k=1}^{\infty} \binom{2k + n - 1}{n} x^k$$
$$= x(1 - x)^{-1-n} \left[\frac{(1 + \sqrt{x})^n - (1 - \sqrt{x})^n}{2\sqrt{x}} + \frac{(1 + \sqrt{x})^n + (1 - \sqrt{x})^n}{2} \right].$$

Proof. By the binomial theorem, we have

$$(1 + \sqrt{x})^n = \sum_{k=0}^{n} \binom{n}{k} x^{k/2}, \tag{2.20}$$

$$(1 - \sqrt{x})^n = \sum_{k=0}^{n} \binom{n}{k} x^{k/2} (-1)^k. \tag{2.21}$$

From (2.20) and (2.21), we derive:

$$\frac{(1 + \sqrt{x})^n - (1 - \sqrt{x})^n}{2\sqrt{x}} + \frac{(1 + \sqrt{x})^n + (1 - \sqrt{x})^n}{2} = \sum_{i=0}^{[n/2]} \binom{n + 1}{n - 2i} x^i. \tag{2.22}$$

Since we have the equation

$$\sum_{k=1}^{\infty} \binom{k + n - 1}{n} x^k = x(1 - x)^{-1-n}, \tag{2.23}$$

it suffices to show that

$$\sum_{k=1}^{\infty} \binom{2k + n - 1}{n} x^k = \sum_{k=1}^{\infty} \binom{k + n - 1}{n} x^k \sum_{i=0}^{[n/2]} \binom{n + 1}{n - 2i} x^i. \tag{2.24}$$

Using the Cauchy product formula, for the coefficient of x^k on the right-hand-side of (2.24) we get

$$\sum_{i=1}^{k} \binom{n+i-1}{n} \binom{n+1}{n-2(k-i)}. \tag{2.25}$$

It is well-known (see,e.g.,Riordan, 1968) that

$$\binom{n+p}{m} = \sum_{2k \leq m-n+p} \binom{n+k}{m} \binom{m+1}{n-p+1+2k}. \tag{2.26}$$

For the case of $m = n$ and $p = 2i - 1$, it specializes to:

$$\binom{2k+n-1}{n} = \sum_{i=1}^{k} \binom{n+i-1}{n} \binom{n+1}{n-2(k-i)}. \tag{2.27}$$

Equation (2.27) shows that the coefficient of x^k is the same on both sides of (2.24) and the assertion follows. $\qquad \square$

Note that the compound distributions with geometrically distributed terms is log-concave or log-convex depending on the log-concavity or log-convexity of N since the geometric distribution is both log-concave and log-convex. However, the negative binomial distribution is log-concave so in the statement of the following theorem, the compound distribution with negative binomial distributed terms, in general, is not log-convex even if N is log-convex.

Theorem 6. *If N has log-concave distribution (with no internal zeros) on the set of nonnegative integers and the terms $X_i, i = 1, 2, ..$ are negative binomial distributed with parameter $r = 2$, then the sequence $\{p_n\}_{n=1}^{\infty}$ is log-concave.*

Proof. Let $x = (1-p)^2$. Then we have

$$p_n = p^n \sum_{k=1}^{\infty} f_k \binom{2k+n-1}{n} x^k. \tag{2.28}$$

In order to prove the log-concavity of $\{p_n\}_{n=1}^{\infty}$, it suffices to show that $\{g_n\}_{n=1}^{\infty}$ is log-concave, where

$$g_n = \sum_{k=1}^{\infty} f_k \binom{2k+n-1}{n} x^k. \tag{2.29}$$

Using the Cauchy product formula, we obtain

$$g_{n+1}g_{n-1} - g_n^2 = \sum_{m=2}^{\infty} x^m \sum_{i=1}^{m-1} T_i f_i f_{m-i}, \tag{2.30}$$

where

$$T_i = \binom{2i+n}{n+1}\binom{2(m-i)+n-2}{n-1} - \binom{2i+n-1}{n}\binom{2(m-i)+n-1}{n}. \tag{2.31}$$

Notice that if $f_k = 1, k = 1, 2, ...,$ then we have

$$\hat{g}_n = \sum_{k=1}^{\infty}\binom{2k+n-1}{n}x^k. \tag{2.32}$$

Using Lemma 3.1, we can derive the formula:

$$\hat{g}_{n+1}\hat{g}_{n-1} - \hat{g}_n^2 = -\frac{x^2}{(1-x)^{n+2}}. \tag{2.33}$$

We have the inequality $\sum_{i=1}^{m-1} T_i < 0$, since the sum on the left hand side is the coefficient of x^m in the series expansion of

$$-\frac{x^2}{(1-x)^{n+2}} = \sum_{i=0}^{\infty} -\binom{n+i+1}{i}x^{i+2}. \tag{2.34}$$

Case 1: m is odd. We can rewrite $\sum_{i=1}^{m-1} T_i f_i f_{m-i}$ as

$$\sum_{i=1}^{m-1} T_i f_i f_{m-i} = \sum_{i=1}^{[(m-1)/2]} (T_i + T_{m-i}) f_i f_{m-i}. \tag{2.35}$$

Case 2: m is even. We can rewrite $\sum_{i=1}^{m-1} T_i f_i f_{m-i}$ as

$$\sum_{i=1}^{m-1} T_i f_i f_{m-i} = \sum_{i=1}^{[(m-1)/2]} (T_i + T_{m-i}) f_i f_{m-i} + T_{m/2} f_{m/2}^2. \tag{2.36}$$

Using (2.31), we can easily verify that $T_{m/2} < 0$. In either case, $T_i + T_{m-i}$ is equal to

$$C_i \frac{(8+16n)i^2 - 8m(1+2n)i + 2n(2m^2 - m + 1) + 2(2m-1)}{n^2(1+n)}, \tag{2.37}$$

where

$$C_i = \binom{2(m-i)+n-2}{n-1}\binom{2i+n-2}{n-1}. \tag{2.38}$$

It follows from (2.37) that $T_1 + T_{m-1} > 0$ and $T_{[(m-1)/2]} + T_{m-[(m-1)/2]} < 0$. Furthermore, since the numerator in (2.37) is a quadratic function of i there must exist an integer k such that $T_i + T_{m-i} > 0$, for $i < k$ and $T_i + T_{m-i} < 0$, for $i > k$. If $\{f_k\}_{k=1}^{\infty}$ is log-concave (without internal zeros), then $f_i f_{m-i}$ is non-increasing, for $i \leq [m/2]$. The assertion follows from Lemma 1. $\qquad\square$

2.4 Log-concavity of the compound Poisson distribution with Poisson distributed terms

If $X_1, X_2, ...$ are i.i.d Poisson distributed random variables with parameter $\mu > 0$ and N follows Poisson distribution with parameter $\lambda > 0$, then we have the equation

$$p_n = P(S = n) = \sum_{k=1}^{\infty} \frac{(k\mu)^n e^{-k\mu}}{n!} \frac{\lambda^k e^{-\lambda}}{k!}, \qquad n = 0, 1, 2, \ldots. \tag{2.39}$$

Let $x = e^{-\mu}\lambda$. Then we can write:

$$p_n = \frac{\mu^n e^{-\lambda}}{n!} \sum_{k=1}^{\infty} k^n \frac{x^k}{k!}. \tag{2.40}$$

We have the following

Theorem 7. *The sequence $\{p_n\}_{n=1}^{\infty}$, defined in (2.39), is log-concave.*

Remark 1. *Equation (2.40) can be rewritten as*

$$p_n = \mu^n e^{-\lambda} e^x \frac{B_n(x)}{n!}, \qquad n = 1, 2, ...,$$

where the $B_n(k)$ are the Bell polynomials whose coefficients are the Stirling numbers of the second kind $S(n, k)$:

$$B_n(x) = \sum_{k=0}^{\infty} S(n, k) x^k. \tag{2.41}$$

It suffices to show that the sequence $\{B_n(x)/n!\}_{n=1}^{\infty}$ is log-concave. Thus, Theorem 4.1 follows from

Theorem 8. *The sequence of Bell polynomials $\{B_n(x)/n!\}_{n=1}^{\infty}$ is log-concave for any $x \in \mathbb{R}$.*

Proof. It is well-known [6] that if $\{1, Z_1, Z_2, ...\}$ is a log-concave sequence of nonnegative real numbers and the sequence $\{a(n)\}_{n=0}^{\infty}$ is defined by

$$\sum_{n=0}^{\infty} \frac{a(n)}{n!} y^n = exp\left(\sum_{j=1}^{\infty} \frac{Z_j}{j} y^j\right), \tag{2.42}$$

then the sequence $\{a(n)/n!\}_{n=0}^{\infty}$ is log-concave and the sequence $\{a(n)\}_{n=0}^{\infty}$ is log-convex. Note that

$$e^{(e^y - 1)x} = exp\left(\sum_{j=1}^{\infty} \frac{x}{j!} y^j\right). \tag{2.43}$$

In addition, we have

$$e^{(e^y-1)x} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} y^n, \tag{2.44}$$

thus,

$$exp\left(\sum_{j=1}^{\infty} \frac{x}{j!} y^j\right) = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} y^n. \tag{2.45}$$

Let $Z_j = \dfrac{x}{(j-1)!}$ for $j \geq 1$. It is easy to check that the sequence $1, Z_1, Z_2...$ is log-concave for $x \geq 1$ or $x \leq 0$. Thus, according to Bender-Canfield's Theorem, $\{B_n(x)/n!\}_{n=1}^{\infty}$ is a log-concave sequence for $x \geq 1$ or $x \leq 0$.

If $0 < x < 1$, we can always find $u \geq 1$ and $v \leq 0$ such that $x = u + v$. Furthermore, it is proved in [1] that we have the following identity:

$$B_n(u+v) = \sum_{k=0}^{n} \binom{n}{k} B_k(u) B_{n-k}(v), \tag{2.46}$$

hence, equation (2.46) can be rewritten as

$$\frac{B_n(u+v)}{n!} = \sum_{k=0}^{n} \frac{B_k(u)}{k!} \frac{B_{n-k}(v)}{(n-k)!}. \tag{2.47}$$

In other words, $B_n(x)/n!$ is the convolution of two log-concave sequences $B_n(u)/n!$ and $B_n(v)/n!$. Thus, the sequence $\{B_n(x)/n!\}_{n=1}^{\infty}$ is log-concave for any $x \in \mathbb{R}$. $\qquad\square$

Remark 2. *Theorem 8 is the generalization of Lemma 2 in Asai et. al (2000) since the Bell numbers $b_2(n)$ is the value of Bell polynomial at $x = 1$.*

2.5 Log-concavity of the compound distribution with gamma distributed terms

If the terms in a compound distribution are continuously distributed, then the probability distribution of S is of a mixed type. It has positive probability mass at 0 and has a continous part with p.d.f. If the terms X_i are gamma distributed with the p.d.f

$$\frac{\mu^\theta}{\Gamma(\theta)} e^{-\mu x} x^{\theta-1}, \tag{2.48}$$

then the p.d.f of the continuous part of the probability distribution of S is

$$g(x) = \sum_{i=1}^{\infty} f_i \frac{\mu^{i\theta}}{\Gamma(i\theta)} e^{-\mu x} x^{i\theta-1}, \qquad x > 0, \tag{2.49}$$

and $P(S = 0) = P(N = 0)$.

The compound Poisson distribution with gamma distributed terms has a wide range of applications. For example, it has been used to model rain fall (Fisher, Cornish, 1960, Ozturk, 1981). However, many mathematical properties of this distribution have not been known (Withers, Nadarajah. 2011). In their paper, the authors derived the representations for the moment generating function as well as some expansions for the probability density and cumulative distribution for this compound Poisson distribution.

The following theorem holds for $1 \leq \theta \leq 4$ but detailed proof is presented only for $1 \leq \theta \leq 2$, for simplicity. At the end the proof for the more general case is outlined.

Theorem 9. *If N has ultra-log-concave distribution (with no internal zeros) on the set of nonnegative integers and the terms X_i, $i = 1, 2, \ldots$ are gamma distributed with parameter $1 \leq \theta \leq 2$, then the continuous part of the distribution of S is log-concave.*

Proof. Simple calculation shows that

$$(\ln g(x))'' = \left(-\mu + \frac{\displaystyle\sum_{i=1}^{\infty} f_i \frac{\mu^{i\theta}}{\Gamma(i\theta)}(i-1)x^{i\theta-2}}{\displaystyle\sum_{i=1}^{\infty} f_i \frac{\mu^{i\theta}}{\Gamma(i\theta)}x^{i\theta-1}} \right)'. \tag{2.50}$$

The derivative of the right-hand-side, with respect to x, equals

$$\begin{aligned}(\ln g(x))'' &= \left[\sum_{i=1}^{\infty} f_i \frac{\mu^{i\theta}}{\Gamma(i\theta)}x^{i\theta-1} \sum_{i=1}^{\infty} f_i \frac{\mu^{i\theta}}{\Gamma(i\theta)}(i\theta-2)(i\theta-1)x^{i\theta-3} \right. \\ &\quad \left. -\left(\sum_{i=1}^{\infty} f_i \frac{\mu^{i\theta}}{\Gamma(i\theta)}(i\theta-1)x^{i\theta-2} \right)^2 \right] : \left(\sum_{i=1}^{\infty} f_i \frac{\mu^{i\theta}}{\Gamma(i\theta)}x^{i\theta-1} \right)^2. \end{aligned} \tag{2.51}$$

If we start the summation from $i = 0$ and use the Cauchy product formula, then the formula for the denominator of $(\ln g(x))''$ is the following

$$\begin{aligned} \sum_{j=1}^{\infty}\sum_{i=0}^{j} & f_{i+1}f_{j+1-i}\frac{[(i+1)\theta-2][(i+1)\theta-1]x^{(j+2)\theta-4}\mu^{(j+2)\theta}}{\Gamma[(i+1)\theta]\Gamma[(j-i+1)\theta]} \\ & -f_{i+1}f_{j+1-i}\frac{[(j-i+1)\theta-1][(i+1)\theta-1]x^{(j+2)\theta-4}\mu^{(j+2)\theta}}{\Gamma[(i+1)\theta]\Gamma[(j-i+1)\theta]} \end{aligned} \tag{2.52}$$

$$= \sum_{j=0}^{\infty} x^{(j+2)\theta-4}\mu^{(j+2)\theta} \sum_{i=0}^{j} T_i(\theta)f_{i+1}f_{j+1-i}, \tag{2.53}$$

where

$$T_i(\theta) = \frac{[(i+1)\theta - 1][(2i - j)\theta - 1]}{\Gamma[(i+1)\theta]\Gamma[(j-i+1)\theta]}. \tag{2.54}$$

For simplicity, we write T_i instead of $T_i(\theta)$. Since x and μ are positive, we only need to prove the non-positivity of the inner sum. First, we show that for $1 \leq \theta \leq 2$ we have the following inequality

$$\sum_{i=0}^{j} \frac{[(i+1)\theta - 1][(2i - j)\theta - 1]}{\Gamma[(i+1)\theta]\Gamma[(j-i+1)\theta]} \frac{1}{(i+1)!} \frac{1}{(j-i+1)!} \leq 0 \quad , j = 0, 1, 2, \ldots. \tag{2.55}$$

We can rewrite the left hand side of (2.55) as

$$\sum_{i=0}^{j} \frac{[(i+1)\theta - 1][(2i - j)\theta - 1]}{(2i+1)!(2j - 2i + 1)!} \frac{(2i+1)!(2j - 2i + 1)!}{\Gamma[(i+1)\theta]\Gamma[(j-i+1)\theta]} \frac{1}{(i+1)!} \frac{1}{(j-i+1)!}. \tag{2.56}$$

Using the binomial theorem, we obtain the identity

$$\sum_{i=0}^{j} \frac{[(i+1)\theta - 1][(2i - j)\theta - 1]}{(2i+1)!(2j - 2i + 1)!]} = \frac{(\theta - 2)(\theta j + \theta - 2)2^{2j-1}}{(2j+2)!} \tag{2.57}$$

which proves that

$$\sum_{i=0}^{j} \frac{[(i+1)\theta - 1][(2i - j)\theta - 1]}{(2i+1)!(2j - 2i + 1)!]} \leq 0, \quad \theta \leq 2. \tag{2.58}$$

Next, we show that the sequence

$$h_i = \frac{(2i+1)!}{(i+1)!\Gamma[(i+1)\theta]}, i = 1, 2, 3, \ldots \tag{2.59}$$

is log-concave for $1.2 \leq \theta \leq 2$, or, equivalently, verifying that

$$\frac{(2i+1)!(2i+1)!}{(i+1)!(i+1)!\Gamma^2[(i+1)\theta]} \geq \frac{(2i-1)!}{i!\Gamma[i\theta]} \frac{(2i+3)!}{(i+2)!\Gamma[(i+2)\theta]}. \tag{2.60}$$

The above inequality can be further simplified to

$$\frac{\Gamma[i\theta]\Gamma[(i+2)\theta]}{\Gamma[(i+1)\theta]} \geq \frac{(i+1)(2i+2)(2i+3)}{(i+2)(2i)(2i+1)}. \tag{2.61}$$

Boyd (1961) has shown that the following equation holds true:

$$\frac{\Gamma[i\theta]\Gamma[(i+2)\theta]}{\Gamma^2[(i+1)\theta]} =_2 F_1(-\theta, -\theta, i\theta, 1) = 1 + \sum_{k=1}^{\infty} \frac{[(-\theta)_k]^2}{k!(i\theta)_k}. \tag{2.62}$$

Using the first two terms in the expansion, we can easily verify the following inequality

$$1 + \frac{\theta}{i} + \frac{\theta(1-\theta)^2}{2i(1+\theta i)} \geq \frac{(i+1)(2i+2)(2i+3)}{(i+2)(2i)(2i+1)}, \quad 1.2 \leq \theta \leq 2. \tag{2.63}$$

Thus, the sequence $\{h_i\}_{i=1}^{\infty}$ is log-concave. Lemma 1 implies (2.55) for $1.2 \leq \theta \leq 2$. For the case of $1 \leq \theta \leq 1.2$, the sequence $\{h_i\}_{i=1}^{\infty}$ might be log-convex and Lemma 1 cannot be used. However, h_i can be modified slightly to be log-concave on the interval of interest. Indeed, using expansion (2.62) with the first term corresponding to $k = 1$, we can check that the new sequence

$$\frac{(2i+2)!}{(i+1)!\Gamma[(i+1)\theta]}, \quad i = 1, 2, 3, \ldots \tag{2.64}$$

is log-concave. Therefore, if we rewrite the summation $\sum_{i=0}^{j} T_i(\theta) f_{i+1} f_{j-i+1}$ as following

$$\sum_{i=0}^{j} \frac{[(i+1)\theta - 1][(2i-j)\theta - 1]}{(2i+2)!(2j - 2i + 2)!} \frac{(2i+2)!(2j - 2i + 2)!}{\Gamma[(i+1)\theta]\Gamma[(j-i+1)\theta]} f_{i+1} f_{j-i+1}, \tag{2.65}$$

and, use direct computation to verify that the sum

$$\sum_{i=0}^{j} \frac{[(i+1)\theta - 1][(2i-j)\theta - 1]}{(2i+2)!(2j - 2i + 2)!]} \leq 0, \quad 1 \leq \theta \leq 1.2, \tag{2.66}$$

is nonpositive in the given interval of θ, and inequality (2.55) follows. Now, we can rewrite $\sum_{i=0}^{j} T_i f_{i+1} f_{j-i+1}$ as

$$\sum_{i=0}^{j} \frac{T_i}{(i+1)!(j-i+1)!} [(i+1)! f_{i+1}][(j-i+1)! f_{j-i+1}]. \tag{2.67}$$

The log-concavity of $i! f_i$ and Lemma 1 complete our proof. $\qquad\square$

Remark 3. *In what follows we outline the proof for the log-concavity of the compound distribution when $2 < \theta \leq 4$. On this interval, the coefficient in the linear term of the polynomial in (2.53) (corresponding to $j = 1$) is positive. The coefficients corresponding to $j = 2, 3, \ldots$ can be shown to be non-positive using similar argument as in the case of $1 \leq \theta \leq 2$. Thus, the assertion of our theorem will follow from the non-positivity the summation of the first three terms. This summation can be simplified to a quadratic function of $(x\mu)^{\theta}$ and its sign can be determined from its discriminant:*

$$\bar{\Delta} = \frac{(\theta-1)(\theta-2)^2 + 4(\theta-1)(1-2\theta)}{\Gamma[2\theta]^2} f_2^2 + \frac{4(\theta-2)^2}{\Gamma[\theta]\Gamma[3\theta]} f_1 f_3. \tag{2.68}$$

The following inequality for $2 < \theta \leq 4$ can be verified using the expansion in (2.62)

$$\frac{4(\theta-2)^2}{\Gamma[\theta]\Gamma[3\theta]} \leq \frac{(1-\theta)(\theta-2)^2 + 4(\theta-1)(2\theta-1)}{\Gamma[2\theta]^2}. \tag{2.69}$$

Hence, we have

$$\bar{\Delta} \le \frac{4(\theta-2)^2}{\Gamma[\theta]\Gamma[3\theta]}(f_1 f_3 - f_2^2) < 0. \tag{2.70}$$

The latter inequality follows from the assumption on the log-concavity of f_i, $i = 1, 2, \ldots$.

When the terms follow exponential distributions or Erlang-2 distributions, we have a stronger statement as follows. Note that $f(x)$ is called log-convex with $x \in \mathbb{R}$, if $f(\lambda x + (1-\lambda)y) \le (f(x))^\lambda (f(y))^{1-\lambda}$ for any $x, y \in \mathbb{R}, 0 < \lambda < 1$.

Theorem 10. *If N has log-concave (log-convex) distribution (with no internal zeros) on the set of nonnegative integers and the terms X_i, $i = 1, 2, \ldots$ are gamma distributed with parameter $\theta = 1$ or $\theta = 2$, then the continuous part of the distribution of S is log-concave (log-convex).*

Proof. The idea of the proof is similar to that of Theorem 9. We need to show the non-positivity of the inner sum in equation (2.53). If j is even, we have

$$
\begin{aligned}
\sum_{i=0}^{j} T_i f_{i+1} f_{j-i+1} &= \sum_{i=0}^{j/2-1} T_i f_{i+1} f_{j-i+1} + T_{j/2} f_{j/2+1}^2 + \sum_{i=j/2+1}^{j} T_i f_{i+1} f_{j-i+1}] \\
&= \sum_{i=0}^{j/2-1} T_i f_{i+1} f_{j-i+1} + T_{j/2} f_{j/2+1}^2 + \sum_{i=0}^{j/2-1} T_{j-i} f_{j-i+1} f_{i+1}] \\
&= \sum_{i=0}^{[(j-1)/2]} (T_i + T_{j-i}) f_{i+1} f_{j-i+1} + T_{j/2} f_{j/2+1}^2. \tag{2.71}
\end{aligned}
$$

Similarly, if j is odd, then we have

$$\sum_{i=0}^{j} T_i f_{i+1} f_{j-i+1} = \sum_{i=0}^{[(j-1)/2]} (T_i + T_{j-i}) f_{i+1} f_{j-i+1}, \tag{2.72}$$

and for either case, we have

$$T_i + T_{j-i} = \frac{(j-2i)^2 \theta^2 - (j+2)\theta + 2}{\Gamma[(i+1)\theta]\Gamma[(j-i+1)\theta]}. \tag{2.73}$$

Case 1: $\theta = 1$. Then

$$
\begin{aligned}
\sum_{i=0}^{j} T_i &= \sum_{i=1}^{j} \frac{2i-j-1}{(i-1)!(j-i)!} = \sum_{i=1}^{j} \frac{(i-1)-(j-i)}{i!(j-i)!} \\
&= \sum_{i=2}^{j} \frac{1}{(i-2)!(j-i)!} - \sum_{i=1}^{j-1} \frac{1}{(i-1)!(j-i+1)!} = 0. \tag{2.74}
\end{aligned}
$$

Case 2: $\theta = 2$. Then

$$
\begin{aligned}
\sum_{i=0}^{j} T_i &= \sum_{i=0}^{j} \frac{4i - 2j - 1}{(2i)!(2j - 2i + 1)!} = \sum_{i=0}^{j} \frac{2i - (2j - 2i + 1)}{(2i)!(2j - i + 1)!} \\
&= \sum_{i=1}^{j} \frac{1}{(2i-1)!(2j-2i+1)!} - \sum_{i=0}^{j} \frac{1}{(2i)!(2j-2i)!} \\
&= \sum_{i=0}^{2j} (-1)^{(i+1)} \binom{2j}{i} = 0.
\end{aligned}
\tag{2.75}
$$

The numerator of $T_i + T_{j-i}$ is a quadratic function of i. Furthermore, using (2.73), we can verify that $T_{[(j-1)/2]} + T_{j-[(j-1)/2]}$ is negative. Thus, $T_i + T_{j-i}$ has exactly one change of sign (some number of initial terms are positive while all further terms are negative). The assertion follows from Lemma 1. $\qquad\square$

2.6 Connection to Turán-type inequalities

In this section we show an interesting connection between the log-concavity of compound Poisson with some well-known Turán type inequalities. These inequalities are named after Paul Turán in a 1946 letter to Szegö (see Szegö, 1948) showed the inequality for Legendre polynomials. Many inequalities in relation to classical orthogonal polynomials have been proved: ultraspherical, Laguerre and Hermite polynomials, Jacobi polynomials, Bessel functions of the first kind. We refer to Skovgaard (1954), Baricz (2008) and the references therein. Recently, some Turán inequalities have been derived from the log-convexity and log-concavity of more general functions. For example, Baricz studied the log-convexity of the Kummer function (or the confluent hypergeometric function):

$$
{}_1F_1(a; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{x^k}{k!},
\tag{2.76}
$$

where $(a)_k = a(a+1)...(a+k-1)$ is the Pochhammer's symbol and showed the following reverse Turán type inequality:

$$
{}_1F_1(a; c+1; x)^2 \leq {}_1F_1(a; c; x){}_1F_1(a; c+2; x).
\tag{2.77}
$$

Carey and Gordy (2007) conjectured the following Turán type inequality:

$$
{}_1F_1(a; c; x)^2 > {}_1F_1(a+1; c; x){}_1F_1(a-1; c; x),
\tag{2.78}
$$

for $a > 0, c > a + 2, x > 0$. A more general inequality was proved later in Barnard et. al (2009) under some condition for the parameters:

$$_1F_1(a;c;x)^2 \geq_1 F_1(a+\nu;c;x)_1F_1(a-\nu;c;x). \tag{2.79}$$

For more references on Turán type inequality on hypergeometric functions, see Karp, Sitnik (2010). The authors studied the log-convexity and log-concavity of hypergeometric-like functions and prove some beautiful results on hypergeometric functions.

Based on the log-concavity of the compound Poisson distribution with shifted geometrically distributed terms, we can derive a corollary for Laguerre polynomials:

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{n-k} \frac{(-x)^k}{k!}.$$

It is a special case of Proposition 2.8 in Simic (2006) when he studied Turán type inequality for Appell polynomials.

Corollary 1. *The sequence* $\dfrac{L_n'(-x)}{n}$ *is log-concave for* $n = 1, 2, ..$ *and* $x > 0$.

Proof. If $X_1, X_2, ...$ have the same geometric distribution with support $\{1, 2, 3..\}$ and $P(X_i = n) = pq^{n-1}(n = 1, 2, ..)$, then, we have

$$P(S = n) = \sum_{k=1}^{n} \binom{n-1}{k-1} p^k (1-p)^{n-k} \lambda^k \frac{e^{-\lambda}}{k!}. \tag{2.80}$$

Simple calculation shows that, for $\{p_n\}_{n=1}^{\infty}$ in the above expression, we have the equations

$$
\begin{aligned}
p_n &= \sum_{k=1}^{n} \binom{n-1}{k-1} p^k (1-p)^{n-k} (\lambda)^k \frac{e^{-\lambda}}{k!} \\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} \left(\frac{p\lambda}{1-p}\right)^k (1-p)^n \frac{e^{-\lambda}}{k!}.
\end{aligned} \tag{2.81}
$$

Let $\dfrac{p\lambda}{1-p} = x$. Then (2.81) takes the form:

$$p_n = \sum_{k=1}^{n} \binom{n-1}{k-1} x^k (1-p)^n \frac{e^{-\lambda}}{k!}. \tag{2.82}$$

If we use the relations $p_n^2 \geq p_{n-1}p_{n+1}$, $n = 2, 3, ...$, then, by (2.82), we obtain:

$$
\left(\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{x^k}{k!} (1-p)^n e^{-\lambda} \right)^2 \geq \left(\sum_{k=1}^{n-1} \binom{n-2}{k-1} \frac{x^k}{k!} (1-p)^{n-1} e^{-\lambda} \right) \times
$$
$$
\left(\sum_{k=1}^{n+1} \binom{n}{k-1} \frac{x^k}{k!} (1-p)^{n+1} e^{-\lambda} \right) \tag{2.83}
$$

which reduces to

$$\left(\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{x^k}{k!}\right)^2 \geq \left(\sum_{k=1}^{n-1} \binom{n-2}{k-1} \frac{x^k}{k!}\right) \left(\sum_{k=1}^{n+1} \binom{n}{k-1} \frac{x^k}{k!}\right), \qquad (2.84)$$

after cancelling by $(1-p)^n e^{-\lambda}$.

Let $B_n = \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{x^k}{k!}$ for $n = 1, 2, ...$ If we use Pascal's identity

$$\binom{n}{n-k} = \binom{n-1}{k-1} + \binom{n-1}{n-k-1}, \qquad (2.85)$$

then for B_n we get:

$$\begin{aligned}
B_n &= \sum_{k=1}^{n} \frac{x^k}{k!} \left[\binom{n}{n-k} - \binom{n-1}{n-k-1}\right] \\
&= \sum_{k=1}^{n} \frac{x^k}{k!} \binom{n}{n-k} - \sum_{k=1}^{n-1} \frac{x^k}{k!} \binom{n-1}{n-k-1} \\
&= \sum_{k=0}^{n} \frac{x^k}{k!} \binom{n}{n-k} - \sum_{k=0}^{n-1} \frac{x^k}{k!} \binom{n-1}{n-k-1} \\
&= L_n(-x) - L_{n-1}(-x).
\end{aligned} \qquad (2.86)$$

It is well-known (see,e.g.,Riordan, 1968) that the Laguerre polynomials satisfy the following recurrence equation:

$$x \frac{d}{dx} L_n(x) = n L_n(x) - n L_{n-1}(x). \qquad (2.87)$$

This implies that $B_n = -x \dfrac{L_n'(-x)}{n}$ and the assertion follows.

$\square$

Based on Theorem 5, an interesting inequality on the confluent hypergeometric function can be derived. This inequality is a special case of Theorem 1 in Barnard et. al (2009).

Corollary 2. *For $k > 0, x > 0$:* $_1F_1(1+k; 2; x)^2 \geq {}_1F_1(k; 2; x) {}_1F_1(2+k; 2; x)$.

Proof. Let $X_1, X_2, ...,$ be i.i.d. random variables with $P(X_i = n) = pq^n (n = 0, 1, 2, ...)$. Then the following formula holds for the probability mass function of the compound Poisson distributions:

$$p_n = P(S = n) = \sum_{k=1}^{\infty} \binom{n+k-1}{n} p^k (1-p)^n \lambda^k \frac{e^{-\lambda}}{k!}.$$

Let $x = p\lambda$. Then we have

$$p_n = \sum_{k=1}^{\infty} \binom{n+k-1}{n} x^k (1-p)^n \frac{e^{-\lambda}}{k!}. \tag{2.88}$$

This distribution has connection with the confluent hypergeometric function:

$$_1F_1(a; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{x^k}{k!}, \tag{2.89}$$

where $(a)_k = a(a+1)...(a+k-1)$ is the Pochhammer's symbol. It can easily be shown that

$$\sum_{k=1}^{\infty} \binom{n+k-1}{n} \frac{x^k}{k!} = {}_1F_1(1+k; 2; x). \tag{2.90}$$

Theorem 5 and equation (2.90) imply the statement. $\qquad \square$

The following corollary establishes a Turán type inequality for the hypergeometric function $_2F_2$. It is a consequence of the log-concavity property of the compound Poisson with negative binomial distributed terms.

Corollary 3. *The hypergeometric function $_2F_2$ satisfies the following inequality*

$$\left[(n+1) {}_2F_2 \left(\begin{array}{cc} n/2+1, n/2+3/2 \\ 3/2 \quad , 2 \end{array} \middle| x \right) \right]^2 \geq (n+2) {}_2F_2 \left(\begin{array}{cc} n/2+3/2, n/2+2 \\ 3/2 \quad , 2 \end{array} \middle| x \right)$$

$$* \quad n {}_2F_2 \left(\begin{array}{cc} n/2+1/2, \quad n/2+1 \\ 3/2 \quad , 2 \end{array} \middle| x \right) \tag{2.91}$$

Proof. Let $X_1, X_2, ...,$ be i.i.d. random variables with $P(X_i = n) = p^n(1-p)^2 (n = 0, 1, 2, ...)$. Then the following formula holds for the probability mass function of the compound Poisson distributions:

$$p_n = P(S = n) = \sum_{k=1}^{\infty} \binom{2k+n-1}{n} p^n (1-p)^{2k} \lambda^k \frac{e^{-\lambda}}{k!}.$$

Let $x = (1-p)^2 \lambda$. Then we have

$$p_n = p^n e^{-\lambda} \sum_{k=1}^{\infty} \binom{2k+n-1}{n} \frac{x^k}{k!}. \tag{2.92}$$

Since the compound Poisson distribution with negative binomial distributed terms is log-concave, we know that $p_n^2 \geq p_{n+1}p_{n-1}$. Using (2.92), for p_n, p_{n+1} and p_{n-1}, we get

$$\left[\sum_{k=1}^{\infty} \binom{2k+n-1}{n} \frac{x^k}{k!}\right]^2 \geq \sum_{k=1}^{\infty} \binom{2k+n-1}{n} \frac{x^k}{k!} \sum_{k=1}^{\infty} \binom{2k+n-1}{n} \frac{x^k}{k!}. \qquad (2.93)$$

Simple calculation shows that

$$\sum_{k=1}^{\infty} \binom{2k+n-1}{n} \frac{x^k}{k!} = x \sum_{k=0}^{\infty} \frac{(2k+n+1)!}{n!(2k+1)!} \frac{x^k}{(k+1)!}. \qquad (2.94)$$

We can rewrite the right-hand-side of (2.94) as

$$
\begin{aligned}
& x \sum_{k=0}^{\infty} \frac{(2k+n+1)!}{n!(2k+1)!} \frac{x^k}{(k+1)!} \\
=\ & x(n+1) \sum_{k=0}^{\infty} \frac{(n+2)(n+3)...(n+2k+1)}{(2k+1)!} \frac{x^k}{k!} \\
=\ & x(n+1) \sum_{k=0}^{\infty} \frac{(n+2)(n+4)...(n+2k)(n+3)(n+5)...(n+2k+1)}{3.5...(2k+1).4.6...(2k+2)} \frac{x^k}{k!} \\
=\ & x(n+1) \sum_{k=0}^{\infty} \frac{(n/2+1)...(n/2+k)(n/2+3/2)...(n/2+k+1/2)}{(3/2)(5/2)...((2k+1)/2).2.3...(k+1)} \frac{x^k}{k!} \\
=\ & x(n+1) \sum_{k=0}^{\infty} \frac{(n/2+1)_k (n/2+3/2)_k}{(\frac{3}{2})_k (2)_k} \frac{x^k}{k!}. \qquad (2.95)
\end{aligned}
$$

Hence, we have showed that

$$\sum_{k=1}^{\infty} \binom{2k+n-1}{n} \frac{x^k}{k!} = x(n+1)\,_2F_2\left(\begin{array}{c} n/2+1, n/2+3/2 \\ 3/2 \qquad , 2 \end{array} \middle| x\right). \qquad (2.96)$$

The assertion follows from (2.93) and (2.96). $\hfill\square$

2.7 Applications

To illustrate the role of log-concavity in probabilistic constrained stochastic programming we present the bond portfolio construction model. Let us introduce the notations:

If the liabilities were deterministic values then our optimal bond portfolio model

n number of bond types which are candidates for inclusion into the portfolio

m number of periods

a_{ik} cash flow of a bond of type k in period i, $k = 1, \ldots, n$ and $i = 1, \ldots, m$

p_k unit price of bond of type k

ξ_i random liability value in period i, $i = 1, \ldots, m$

z_i cash carried forward from period i to period $i + 1$, $i = 1, \ldots, m$, where z_1 is an initial cash amount that we include into the portfolio and $z_{m+1} = 0$; z_i, $i = 1, \ldots, m$ are decision variables

x_k decision variable, number of bonds of type k to include into the portfolio

ρ_i rate of interest in period i, $i = 1, \ldots, m$.

(Hodges and Schaefer, 1977) would be the following

$$\min \left\{ \sum_{k=1}^{n} p_k x_k + z_1 \right\}$$

subject to

$$\sum_{k=1}^{n} a_{ik} x_k + (1 - \rho_i) z_i - z_{i+1} \geq \xi_i, \ i = 1, \ldots, m \tag{2.97}$$

$$x_k \geq 0, \ k = 1, \ldots, n$$

$$z_i \geq 0, \ i = 1, \ldots, m, z_{m+1} = 0,$$

where the positivity of the variables means no short-selling allowed.

The probabilistic constrained variant of it can be formulated as

$$\min \left\{ \sum_{k=1}^{n} p_k x_k + z_1 \right\}$$

subject to

$$\prod_{i=1}^{m} P \left(\sum_{k=1}^{n} a_{ik} x_k + (1 - \rho_i) z_i - z_{i+1} \geq \xi_i | \xi_i > 0 \right) \geq p \tag{2.98}$$

$$x_k \geq 0, \ k = 1, \ldots, n$$

$$z_i \geq 0, \ i = 1, \ldots, m, z_{m+1} = 0,$$

where p is a safety (reliability) level chosen by ourselves, e.g., $p = 0.8, 0.9, 0.95$ etc.

Compound distributions such as the compound Poisson and the compound negative

binomial are used extensively in the theory of risk to model the distribution of total claims incurred in the subsequent periods. The exponential or gamma distribution can be used to fit the individual claim severities (Branda, 2012). Theorem 9 implies the convexity of the set determined by the constraint (2.98) and solution methods of the model can be found in Prékopa (2003). In the case of discrete insurance claim sizes and strictly log-concave aggregate loss distribution, we make use of disjuctive reformulation by p-efficient points and the solution of a multiple choice knapsack problem is used to generate new p-efficient points (Prékopa, Unuvar 2012). Problem (2.98) can also be used in a rolling horizon manner for rebalancing the portfolio.

Chapter 3

Discrete Moment Problems

3.1 Introduction

Discrete moment problems (DMP) came to prominence by the discovery of Prékopa (1988, 1990a,b) that sharp probability bounds (e.g. Dawson, Sakoff, 1967) can be obtained as optimal values of linear programming problems involving binomial bounds of the number of occurences of events. Moment problems where the random variable has discrete support are already mentioned in Karlin, Studden (1966), but Prékopa gave a full characterization of the dual feasible bases and gave tractable algorithmic solution.

Let X be a discrete random variable, the possible values of which are known to be the numbers $z_0 < z_1 < \cdots < z_n$ and

$$p_i = P(X = z_i), \ i = 0, 1, \ldots, n. \tag{3.1}$$

Given the knowledge of some power moments $\mu_k = E(X^k), \ k = 1, \ldots, m$, or the binomial moments $S_k = E[\binom{X}{k}], \ k = 1, \ldots, m$, where $m < n$, the discrete moment bounding problem provides us with the sharp lower and upper bounds on a linear functional, defined on the unknown probability distribution $\{p_i\}$. They can be formulated as the following LPs:

$$\min(\max) \ \sum_{i=0}^{n} f_i p_i$$

subject to

$$Ap = b \tag{3.2}$$

$$p \geq 0,$$

where

$$A = \begin{pmatrix} 1 & 1 & ... & 1 \\ z_0 & z_1 & ... & z_n \\ & & ... & \\ z_0^m & z_1^m & ... & z_n^m \end{pmatrix}, b = \begin{pmatrix} \mu_0 \\ \mu_1 \\ ... \\ \mu_m \end{pmatrix}, \tag{3.3}$$

and $f_i = f(z_i)$, $i = 0, \ldots, n$, and

$$\min(\max) \quad \sum_{i=0}^{n} f_i p_i$$

subject to

$$\tilde{A}p = \tilde{b} \tag{3.4}$$

$$p \geq 0,$$

where

$$\tilde{A} = \begin{pmatrix} 1 & 1 & ... & 1 \\ z_0 & z_1 & ... & z_n \\ & & ... & \\ \binom{z_0}{m} & \binom{z_1}{m} & ... & \binom{z_n}{m} \end{pmatrix}, \tilde{b} = \begin{pmatrix} S_0 \\ S_1 \\ ... \\ S_m \end{pmatrix}, \tag{3.5}$$

and $f_i = f(z_i)$, $i = 0, \ldots, n$. Problems (3.2) and (3.4) are called the power and binomial moment problems, respectively. They can be transformed into each other by the use of Stirling numbers of the first and second kind (Prékopa, 1995).

DMP were introduced and studied by Prékopa (see, e.g. Prékopa, 1988, 1990a,b, 1992, 1999, 2001). In those papers, the author used linear programming techniques to develop theory and numerical solution of the optimization problems. Since its introduction, DMPs have been used extensively in various application areas. Some of such applications include the reliability evaluations of networks such as commnication systems, power generation or transmission systems, e.g, Prékopa and Boros presented sharp lower and upper bounds for the probability that a feasible flow exists in a stochastic transportation network. Another application is due to Prékopa where the author introduced moment bounding methods to value of financial derivatives (2001). DMP can also be used to provide lower and upper bounds for the probability distribution when the analytical form cannot be obtained otherwise. For example, Prékopa, Long,

Szántai (2004) gave bounds on the length of the critical path in PERT. In Chapter 4 of this thesis, we demonstrate how to evaluate the recruitment time in clinical trials by the use of DMP.

With its increasing practial importance, intensive research has been developed both in theory and computational methods in recent years. From the theory perspective, DMP has been extended to its multivariate counterpart, namely, the multivariate discrete moment problem (MDMP). It has been initiated by Prékopa (1992, 1998, 2000) and later further developed in Mádi-Nagy, Prékopa (2004, 2011), Mádi-Nagy (2009). Another direction of extension is to incorporate the shape of the distribution in the optimization, Subasi et al. (2008, 2009). Along with the advancement of DMP theory, efficient solution methods have been proposed to overcome the instability of the moment matrix. The first algorithm for DMP was introduced by Prékopa (1990). His optimization methods are of dual type and are in close relationship with the dual method of Lemke [4] for the solution of the general linear programming problem. They are stable and fast thanks to the discovery of the structures of dual feasible bases. Recently, Mádi-Nagy (2012) proposed a different approach to treat the numerical difficulties using multivariate polynomial bases.

Another special property of DMP is that closed-form formula for lower and upper bounds can be derived (see, Prékopa 1995) from the dual feasible basis structures. In case of power discrete moment problem, when one or two moments are used at the right-hand-side, some classical inequalities are recovered, for example, the Jensen and the EdmundsonMadansky inequalities.

Fractional moments have been used within the context of the (discrete) maximum entropy problems to find probabilities in a distribution, where some of the fractional moments are given (see, e.g. Novi Inverardi and Tagliani, 2006). The authors reported signicant improvement when using the information of fractional moments for recovering a probability distribution via maximum entropy setup. In this chapter, we present the theory and a solution method for the bounding problems with fractional moments in the spirit of the discrete moment problem proposed by Prékopa. The discrete fractional

moment problem can be defined as:

$$\min(\max) \quad \sum_{i=0}^{n} f_i(z_i) p_i$$

subject to

$$\sum_{i=0}^{n} z_i^{\alpha_k} p_i = \mu_k \quad k = 0, ..., m \tag{3.6}$$

$$p_i \geq 0 \qquad i = 0, ..., n,$$

where $\alpha_k, k = 0, 1, 2, \ldots$ are positive numbers. We consider three objective functions:

(1) The function $f(z)$ is absolutely monotonic. The optimum values of problems (3.6) give sharp lower and upper bounds for $E(f(X))$.

(2) $f_r = 1, f_i = 0$, if $i \neq r$, for some $0 \leq r \leq n$. The optimum values of problems (3.6) give sharp lower and upper bounds for $P(X = z_i)$.

(3) $f_0 = \cdots = f_{r-1} = 0, f_r = \cdots = f_n = 1$, for some $1 \leq r \leq n$. The optimum values of problems (3.6) give sharp lower and upper bounds for $P(X \geq z_r)$.

The chapter is organized as follows. Section 2 presents some basic notions and theorems for the discrete moment problems with fractional moments. In Section 3 basis structure theorems are presented for the above mentioned objective functions. In Section 4 we provide a detailed description of the dual method that solves the problem and a procedure to estimate fractional moments. Numerical results are reported in Section 5.

3.2   Basic notions and theorems

A function $f(z)$ is said to be absolutely monotonic on $(0, \infty)$ if it has derivatives of all orders and

$$f^{(k)}(z) \geq 0, \ z \in (0, \infty), \ k = 0, 1, 2, \ldots. \tag{3.7}$$

Theorem 11. *Assume that $f(z)$ is an absolutely monotonic function on $(0, \infty)$, $0 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_m \leq 1$ and $0 < z_0 < z_1 < \cdots < z_m$. Then the following inequality*

holds:

$$D(z, f) = \begin{vmatrix} z_0^{\alpha_0} & z_1^{\alpha_0} & \cdots & z_m^{\alpha_0} \\ & & \vdots & \\ z_0^{\alpha_{m-1}} & z_1^{\alpha_{m-1}} & \cdots & z_m^{\alpha_{m-1}} \\ z_0^{\alpha_m} & z_1^{\alpha_m} & \cdots & z_m^{\alpha_m} \\ f(z_0) & f(z_1) & \cdots & f(z_m) \end{vmatrix} > 0 \tag{3.8}$$

Proof. It it well-known (Bernstein, 1914) that any absolutely monotonic function can be expressed as a series of polynomial with nonnegative coefficient. Thus, we can rewrite $f(z)$ as follows:

$$f(z) = \sum_{i=0}^{\infty} c_i z^i. \tag{3.9}$$

Then the original determinant $D(z, f)$ can be represented as:

$$D(z, f) = \sum_{i=0}^{\infty} D(z, c_i z^i) \tag{3.10}$$

The ith term $(i \geq 1)$ in the series (3.10) is c_i times the determinant:

$$\begin{vmatrix} z_0^{\alpha_0} & z_1^{\alpha_0} & \cdots & z_m^{\alpha_0} \\ & & \vdots & \\ z_0^{\alpha_{m-1}} & z_1^{\alpha_{m-1}} & \cdots & z_m^{\alpha_{m-1}} \\ z_0^{\alpha_m} & z_1^{\alpha_m} & \cdots & z_m^{\alpha_m} \\ z_0^i & z_1^i & \cdots & z_m^i \end{vmatrix}. \tag{3.11}$$

This is a generalized Vandermonde determinant which is known to be positive (Karlin and Studden, 1966, pp.9) and the assertion follows. $\square$

Theorem 12. *Assume that $\alpha_0 < \alpha_1 < \cdots < \alpha_m$ and $0 < z_0 < z_1 < \cdots < z_m$, then we have the following inequality*

$$(-1)^t \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ z_0^{\alpha_0} & z_1^{\alpha_0} & \cdots & z_t^{\alpha_0} & z_{t+1}^{\alpha_0} & \cdots & z_m^{\alpha_0} \\ & & & \vdots & & & \\ z_0^{\alpha_{m-2}} & z_1^{\alpha_{m-2}} & \cdots & z_t^{\alpha_{m-2}} & z_{t+1}^{\alpha_{m-2}} & \cdots & z_m^{\alpha_{m-2}} \\ z_0^{\alpha_{m-1}} & z_1^{\alpha_{m-1}} & \cdots & z_t^{\alpha_{m-1}} & z_{t+1}^{\alpha_{m-1}} & \cdots & z_m^{\alpha_{m-1}} \end{vmatrix} > 0 \tag{3.12}$$

Proof. It can be easily verified that the above determinant can be simplified as follows:

$$
\begin{vmatrix}
z_0^{\alpha_0} & z_1^{\alpha_0} - z_0^{\alpha_0} & \cdots & z_{t+1}^{\alpha_0} - z_t^{\alpha_0} & \cdots & z_m^{\alpha_0} - z_{m-1}^{\alpha_0} \\
& & \vdots & & & \\
z_0^{\alpha_{m-2}} & z_1^{\alpha_{m-2}} - z_0^{\alpha_{m-2}} & \cdots z_{t+1}^{\alpha_{m-2}} - z_t^{\alpha_{m-2}} & \cdots z_m^{\alpha_{m-2}} - z_{m-2}^{\alpha_{m-2}} \\
z_0^{\alpha_{m-1}} & z_1^{\alpha_{m-1}} - z_1^{\alpha_{m-1}} & \cdots z_{t+1}^{\alpha_{m-1}} - z_t^{\alpha_{m-1}} & \cdots z_m^{\alpha_{m-1}} - z_{m-1}^{\alpha_{m-1}}
\end{vmatrix}.
$$

We rewrite it as follows:

$$
\prod_{j=0}^{m-1} \alpha_j
\begin{vmatrix}
\int_0^{z_0} x_1^{\alpha_0-1} dx_1 & \int_{z_0}^{z_1} x_2^{\alpha_0-1} dx_2 & \cdots & \int_{z_t}^{z_{t+1}} x_t^{\alpha_0-1} dx_t & \cdots & \int_{z_{m-1}}^{z_m} x_{m+1}^{\alpha_0-1} \\
& & \vdots & & & \\
\int_0^{z_0} x_1^{\alpha_{m-2}-1} dx_1 & \int_{z_0}^{z_1} x_2^{\alpha_{m-2}-1} dx_2 & \cdots \int_{z_t}^{z_{t+1}} x_t^{\alpha_{m-2}-1} dx_t & \cdots \int_{z_{m-1}}^{z_m} x_{m+1}^{\alpha_{m-2}-1} \\
\int_0^{z_0} x_1^{\alpha_{m-1}-1} dx_1 & \int_{z_0}^{z_1} x_2^{\alpha_{m-1}-1} dx_2 & \cdots \int_{z_t}^{z_{t+1}} x_t^{\alpha_{m-1}-1} dx_t & \cdots \int_{z_{m-1}-1}^{z_m} x_{m+1}^{\alpha_{m-1}-1}
\end{vmatrix},
$$

which can be reduced to:

$$
\prod_{j=0}^{m-1} \alpha_j \int_0^{z_0} \int_{z_0}^{z_1} \cdots \int_{z_{m-2}}^{z_{m-1}} \bar{D} \, dx_1 dx_2 \ldots dx_{m+1}, \tag{3.13}
$$

where

$$
\bar{D} =
\begin{vmatrix}
x_1^{\alpha_0-1} & x_2^{\alpha_0-1} & \cdots & x_t^{\alpha_0-1} & \cdots & x_{m+1}^{\alpha_0-1} \\
& & \vdots & & & \\
x_1^{\alpha_{m-2}-1} & x_2^{\alpha_{m-2}-1} & \cdots x_t^{\alpha_{m-2}-1} & \cdots & x_{m+1}^{\alpha_{m-2}-1} \\
x_1^{\alpha_{m-1}-1} & x_2^{\alpha_{m-1}-1} & \cdots x_t^{\alpha_{m-1}-1} & \cdots & x_{m+1}^{\alpha_{m-1}-1}
\end{vmatrix}.
$$

It is well known (Karlin and Studden, 1966, pp.9) that $\bar{D}$ is positive since it is the determinant of a generalized Vandermonde matrix and the assertion of the theorem follows.

$\square$

3.3 DMP with fractional moments

A generalization of the discrete moment problem, called totally positive linear programming problem, was introduced in Prékopa [13]. It is the LP:

$$
\begin{aligned}
\min(\max) \quad & f_0 p_0 + f_1 p_1 + \cdots + f_n p_n \\
\text{s.t.} \quad & a_{00}\, p_0 + \ldots + a_{0n} p_n = 1 \\
& a_{10}\, p_0 + \ldots + a_{1n} p_n = b_1 \\
& a_{20}\, p_0 + \quad + a_{2n} p_n = b_2 \\
& \qquad \vdots \qquad\qquad \vdots \\
& a_{m0} p_0 + \ldots \quad a_{mn} p_n = b_m, \\
& p_i \geqslant 0,\ i = 1, 2, \ldots, n,
\end{aligned}
$$

where all $(m+1){\times}(m+1)$ submatrices of A and all $(m+2){\times}(m+2)$ submatrices of $\begin{pmatrix} f^T \\ A \end{pmatrix}$ have positive determinants, where $A = (a_{ik})$. By the theorems established in the previous section the discrete fractional moment problem belongs to this class and we can apply the dual feasible basis structure theorem in Prékopa (1990d).

The following theorem is due to Prékopa (1990a,d).

Theorem 13. *The dual feasible bases have the following structures:*

minimization problem, $m + 1$ even

- *{j,j+1, ..., k,k+1}*

minimization problem, $m + 1$ odd

- *{0,j,j+1,...,k,k+1}*

maximization problem, $m + 1$ even

- *{0,j,j+1, ..., k,k+1,n}*

maximization problem, $m + 1$ even

- *{j,j+1,...,k,k+1,n}*

The positivity of all minors of order $m + 1$ from A follows from the fact that the generalized Vandermonde matrix is totally positive. We obtain the following dual feasible bases structure for the case of $f_r = 1, f_i = 0$, if $i \neq r$, for some $0 \leq r \leq n$ based on Prékopa's theorem (1990a).

Theorem 14. *The dual feasible bases have the following structures:*

minimization problem, $m + 1$ even

- $r \notin I$,

- $\{0, i, i{+}1, \ldots, j, j{+}1, r{-}1, r, r{+}1, k, k{+}1, \ldots, t, t{+}1\}$, *if* $2 \leq r \leq n - 1$,

- $\{i, i{+}1, \ldots, j, j{+}1, r{-}1, r, r{+}1, k, k{+}1, \ldots, t, t{+}1, n\}$, *if* $1 \leq r \leq n - 2$,

- $\{0, 1, i, i{+}1, \ldots, j, j{+}1\}$, *if* $r = 0$, *and*

- $\{i, i{+}1, \ldots, j, j{+}1, n{-}1, n\}$, *if* $r = n$;

minimization problem, $m + 1$ odd

- $r \notin I$,

- $\{0, i, i{+}1, \ldots, j, j{+}1, r{-}1, r, r{+}1, k, k{+}1, \ldots, t, t{+}1, n\}$, *if* $2 \leq r \leq n - 2$,

- $\{i, i{+}1, \ldots, j, j{+}1, r{-}1, r, r{+}1, k, k{+}1, \ldots, t, t{+}1\}$, *if* $1 \leq r \leq n - 1$,

- $\{0, 1, i, i{+}1, \ldots, j, j{+}1, n\}$, *if* $r = 0$, *and*

- $\{0, i, i{+}1, \ldots, j, j{+}1, n{-}1, n\}$, *if* $r = n$;

maximization problem, $m + 1$ even

- $\{i, i{+}1, \ldots, j, j{+}1, r, k, k{+}1, \ldots, t, t{+}1, n\}$, *if* $0 \leq r \leq n - 1$,

- $\{0, i, i{+}1, \ldots, j, j{+}1, r, k, k{+}1, \ldots, t, t{+}1\}$, *if* $1 \leq r \leq n$;

maximization problem, $m + 1$ odd

- $\{i, i{+}1, \ldots, j, j{+}1, r, k, k{+}1, \ldots, t, t{+}1\}$, *if* $0 \leq r \leq n$,

- $\{0, i, i{+}1, \ldots, j, j{+}1, r, k, k{+}1, \ldots, t, t{+}1, n\}$, *if* $1 \leq r \leq n - 1$.

where in all parentheses the numbers are arranged in increasing order. If $n > m + 2$,
then all bases for which $r \notin I$, are dual degenerate. The bases in all other cases are
dual nondegenerate.

We designate A the matrix of the equality constraint and by $a_0, \ldots, a_n$ its columns.
We say that A has the alternating sign property if for every $1 \le i_1 < \cdots < i_t < \cdots < i_{m+2} \le n$, we have the inequality

$$(-1)^t \begin{vmatrix} 0 & \ldots & 0 & 1 & \ldots & 1 \\ a_{i1} & \ldots & a_{i_t} & a_{i_{t+1}} & \ldots & a_{i_{m+2}} \end{vmatrix} > 0 \tag{3.14}$$

From Theorem 12, we know that A has alternating sign property. All minors of order
$m + 1$ from A are positive. The following theorem of Prékopa (1990a) gives the dual
feasible bases structure for the case that $f_0 = \cdots = f_{r-1} = 0, f_r = \cdots = f_n = 1$, for
some $1 \le r \le n$.

Theorem 15. *The dual feasible bases have the following structures:*

minimization problem, $m + 1$ even

- $I \subset \{0, \ldots, r\text{-}1\}$, if $r \ge m + 1$,

- $\{0, i, i+1, \ldots, j, j+1, r\text{-}1, k, k+1, \ldots, t, t+1\}$, if $2 \le r \le n - 1$,

- $\{i, i+1, \ldots, j, j+1, r\text{-}1, k, k+1, \ldots, t, t+1, n\}$;

minimization problem, $m + 1$ odd

- $I \subset \{0, \ldots, r\text{-}1\}$, if $r \ge m + 1$,

- $\{0, i, i+1, \ldots, j, j+1, r\text{-}1, k, k+1, \ldots, t, t+1, n\}$, if $2 \le r \le n$,

- $\{i, i+1, \ldots, j, j+1, r\text{-}1, k, k+1, \ldots, t, t+1\}$, if $1 \le r \le n - 1$;

maximization problem, $m + 1$ even

- $I \subset \{r, \ldots, n\}$, if $n - r \ge m$,

- $\{i, i+1, \ldots, j, j+1, r, k, k+1, \ldots, t, t+1, n\}$, if $1 \le r \le n - 1$,

- $\{0,\ i,\ i+1,\ \ldots,\ j,\ j+1,\ r,\ k,\ k+1,\ \ldots,t\ ,t\ +1\}$, if $1 \le r \le n$;

maximization problem, $m+1$ odd

- $I \subset \{r,\ \ldots,\ n\}$, if $n - r \ge m$,

- $\{i,\ i+1,\ \ldots,\ j,\ j+1,\ r,\ k,\ k+1,\ \ldots,t\ ,t\ +1\}$, if $1 \le r \le n$,

- $\{0,\ i,\ i+1,\ \ldots,\ j,\ j+1,\ r,\ k,\ k+1,\ \ldots,t\ ,t\ +1\}$, if $1 \le r \le n-1$.

where in all parentheses the numbers are arranged in increasing order. Those bases for which $I \subset \{0,\ \ldots,\ r\text{-}1\}$ ($I \subset \{r,\ \ldots,\ n\}$) are dual nondegenerate in the minimization (maximization) problem, if $r > m+1$ ($n-r+1 > m+1$). The bases in all other cases are dual nondegenerate.

3.4 The dual algorithm and selection of fractional moments

Given a set of m fractional moments $(\alpha_0, \ldots, \alpha_{m-1})$, the moment bounding problems with fractional moments can be solved by Prékopa's dual method. For DMPs, since the Vandermonde systems are ill-conditioned (Prékopa (1990a,d), Prékopa, Szedmák (2003)), the solutions of problem (3.2), using a primal approach, is computationally difficult. Prékopa's idea (1988) was to use a specialized form of the dual algorithm of Lemke. This approach is extremely efficient when the dual feasible bases are known. Given the dual feasible bases for the discrete moment problems with fractional moments in Section 2 and 3, the dual algorithm can be applied for the discrete fractional moments as well. In order to avoid the instability coming from generalized Vandermonde matrix, we can employ LDU decomposition for generalized Vandermonde matrices (Demmel, Koev, 2005) in the course of the algorithm. The dual method for the solution of problem (3.6) can be described in the following steps:

Prékopa's dual algorithm

Step 1. Pick any dual feasible basis in agreement with the above result; let $I = \{i_0, \ldots, i_m\}$ be the set of basic subscripts.

Step 2. Determine the corresponding primal feasible solution $p_i = (B^{-1}b)_i$, for $i \in I$, and $p_i = 0$ for $i \in \{0, \ldots, n\} - I$.

- If $p_{i_k} \geq 0$, for every $i_k \in I$, then B is a primal-dual feasible basis, and therefore the current basic solution is optimal. Otherwise, go to **Step 4**.

- If $p_{i_k} < 0$, for some i_k then the i_k^{th} vector of B is a candidate for outgoing. Go to **Step 3**.

Step 3. Include that vector into the basis that restores the dual feasible basis structure and go to **Step 2**.

Step 4. Stop. The optimum value $f_B^T B^{-1} b$ is a lower (upper) bound for $E[f(X)]$, depending on the type of the optimization problem.

Computation of fractional moments Fractional moments can be computed based on the moment generating function (Novi Inverardi, Tagliani, 2005). In what follows, we propose a bounding method based on the discrete moment problem (with consecutive integere moments) to estimate fractional moments $E(X^\alpha)$. It can be easily seen that for any α, all minors of order $m+1$ from A and all minors of order $m+2$ from $\begin{pmatrix} f^T \\ A \end{pmatrix}$ or $\begin{pmatrix} -f^T \\ A \end{pmatrix}$ are positive, where f has the form X^α. Hence, the bounding problems can be solved efficiently by Prékopa's dual method (Prékopa [13]). It is interesting to remark the optimal basis is the same for all functions f having the form of x^α. Thus, the lower bound (LB_α) and upper bounds (UB_α) can be computed explicitly by the use of the available optimal basis.

3.5 Applications

We present an application of discrete moment bounding with fractional moments in the context of degradation process of long chain molecules. This application was taken from Prékopa (1953). Initially, the long chain molecule has n units and $n-1$ bonds. Assume that bonds split independently with the same probability p. After degradation process,

we have polymers of different length. Let us denote a_1 as the number of monomers, $\ldots$, a_n n-mers and $\xi_1^{(n)}, \ldots, \xi_n^{(n)}$ as the corresponding random variables. The probability mass function (p.m.f) of the distribution of the number of mers $P(a_1, a_2, \ldots, a_n)$ can be represented as follows:

$$\frac{(1-p)^n}{p} \frac{(a_1 + a_2 + \cdots + a_n)!}{a_1! \ldots a_n!} \left(\frac{p}{1-p}\right)^{a_1 + a_2 + \cdots + a_n} \tag{3.15}$$

We are interested the p.m.f of the number of k-mers $P(\xi_k^{(n)})$. We propose to estimate those probabilitities using moments. The mean $E(\xi_k^{(n)})$ and $Var(\xi_k^{(n)})$ are computed in the paper by Prékopa (1953). Method to compute higher moments is also presented in that paper. Fractional moments are estimated by the bounding procedure described in the previous section. The following table shows the lower and upper bound for the probability that the number of 2-mers is more than 9. The result shows significant improvement of both lower and upper bounds when using fractional moments.

# of Mo.	Integer vs Fractional moments			
	LBint	UBint	LBfrac	UBfrac
2	0	0.14754	0	0.11312
4	0	0.06222	0	0.05507
6	0.000028	0.05436	0.00169	0.05436
8	0.004787	0.04961	0.00527	0.03894

Table 3.1: Integer and fractional moment bounds for $P(\xi_k^{(n)} \geq 10)$

Moment problems with high-order integer moments are known to be unstable. Fractional moments offer some alleviation for stability issue and provide better accuracy for both lower and upper bounds, however, estimation of fractional moment can be difficult.

Chapter 4

Recruitment stocking problems

4.1 Introduction

There are four primary phases in clinical trials. They are used to ensure the safety and efficacy of a proposed treatment. Phase I tests safety and dosage ranges in a small group of people. Phase II further tests safety and also tests for efficacy in a larger group. Phase III, assuming success in Phase I and II, confirms the safety and efficacy results in a much larger group of patients, on the orders of hundreds or thousands, and monitors for adverse effects of the treatment. Assuming success in Phase III, a drug is then approved for commercialization and further Phase IV studies assess the treatment over the long-term. Time is very critical in clinical trials since the patent lifetime is limited. However, the recruitment of patient is extremely slow. Moreover, it has to follow strict FDA regulations.

This research is motivated from the recruitment process in Phase III of clinical trials. Patients arrive randomly to multiple locations or sites. Upon arrival, a medical package will be provided if avaiable, otherwise, the patient is rejected. The clinical trial is closed once the target number of patients is recruited. To measure the performance of a clinical trial, we can look at some of key metrics: time to recruit the target number of subjects, inventory overage of medical packages and the number of patients rejected at the end of the trial.

The recruitment process in clinical trial is just an example of recruiment process. In this chapter, we introduce a more general class of inventory control problem - the recruitment stocking problem. We need to recruit a target number of individuals through designated outlets. As soon as the recruits of all outlets add up to the target number,

the recruitment is done and no more individuals will be admitted. The arrivals of individuals at each outlet are random. To recruit an individual upon its arrival, we must provide a pack of materials. We order the packs of materials in advance and hold them in the outlets. Outlets can neither transfer recruits nor cross-ship materials among themselves. If an outlet runs out of stock, any futher recruit at the outlet will be lost.

This chapter answers the following questions: given the inital inventory levels at each locations in the system, how to measure the performance metrics efficiently?

1. What is the chance that we will reject some patients before the recruitment target is reached? (type I service level)

2. How many patients we will reject before the target is reached? (type II service level)

3. How long does it take to recruit the target number of patients? (recruitment time)

The recruitment stocking problem widely exists in practice. For instance, in marketing research, we have to recruit testers to try new products in a short time. In fashion industry, when an item is at the end of its life cycle, we want to allocate its inventory in such a way that we will sell out the inventory as soon as possible.

The recruitment stocking problem differs from existing inventory management literature by the finite recruitment target which connects all outlets in a such way that the recruitment is done as soon as the recruits at all outlets sum up to the target. In most existing inventory models, we should satisfy demand as much as supply allows. In other words, demand will be satisfied if inventory is available. This is not true in the recruitment stocking problem, where, as soon as the target is met, no more demand will be satisfied even if we have stock available. Due to this distinctive feature, performance metrics must be evaluated differently in recruitment stocking problem. In standard inventory control literature, performance metrics are estimated using steady-state approximation, however, for recruitment stocking problems, the performance metrics are transient since the system will eventually terminates.

Recruitment stocking problems are closely related to some of classical problems in probability theory. In occupancy problems, r balls are thrown randomly in n urns with each ball independently assigned to a given urn at some probability. There is a rich literature on the topic, e.g, see Johnson, Kotz (1977), Holst (1986), and the references therein. Another variation to the problem is the overflow problem, in which the urns have finite capacity. Thus, once full, the balls can fall outside of urns, and the number of balls that overflow is the random variable of interest (Ramakrishna, Mukhopadhyay, 1988). This setting is indeed very similar to the recruitment stocking problems since the overflow balls can be considered as the number of rejected subjects. However, the key difference between classical occupancy problems and our problem is that occupancy problems assume homegeneous arrivals and capacities across urns. But in recruitment stocking settings, the arrival of subjects are inherently different from each other and, thus, so do the amount of inventory allocated to the locations.

A related problem was studied by Fleishacker et. al (2014) in the context of inventory management for clinical trials, however, in the paper, the authors assume that there is enough inventory so that no patient will be rejected from the trial. In a recent paper, Fok et. al (2014) considered rejections during recruitment and proposed some allocation rule for inventory. They focused on a special case of recruitment stocking problem when the total initial inventory is the same as recruitment target.

The rest of the chapter is organized as follows. In Section 2, we define and characterize the recruitment stocking process. The exact analysis for type II service level is presented in Section 3. Section 4 is devoted to asymptotic approximations and bounding schemes for recruitment time. Numerical examples are demonstrated in Section 5.

4.2   Recruitment stocking processes

In this section, we state the assumptions of the recruitment stocking problems and present the mathematical framework for RSPs.

4.2.1   Modeling Assumptions

- *Demand Process*: Demand occurs only at the lowest echelon (at sites) and the demand processes are mutually independent Poisson processes with known demand rates.

- *Fixed recruitment target, H*: The system has a predefined recruitment target, H, which is the necessary sample size for the study. Once the target is reached, recruitment is closed and no more customers will be needed.

- *Inflexible Supply*: Products are assumed to be made before the recruitment (for example, in clinical trials, this constraint is due to statistical consistency considerations and/or the significant fixed costs associated with the production and quality control).

- *Inability of Cross-Shipping*: We also assume that cross-shipping and back-shipping is not allowed.

4.2.2   Some notations

- $T_m(n)$: the time to recruit n subjects at location m.

- $T(H)$: the recruitment time to recruit the target H at all locations.

- R_m: the number of rejected arrivals at location m at the end of recruitment period.

- R: the total number of rejected arrivals across all locations at the end of recruitment period.

- s_i, $i = 1, \ldots, M$: inventory levels at locations $1, \ldots, M$.

- S: the total inventory in the system, $S = s_1 + \cdots + s_M$.

- λ_i, $i = 1, \ldots, M$: arrival rates at locations $1, \ldots, M$, correspondingly.

- Suppose that the random vector $X = (X_1, X_2, \ldots, X_M)$ follows a multinomial distribution with the mass parameter H and the probability vector $p = (p_1, p_2, \ldots, p_M)$,

where H, M are positive integers, and $p_i, i = 1, \ldots, M$ are real non-negative with $p_1 + p_2 + \cdots + p_M = 1$. Let $a_1, \ldots, a_M$ and $b_1, \ldots, b_M$ be sets of integers so that $0 \leq a_i \leq b_i \leq H$ for $i = 1, \ldots, M$. We call the following expression as the rectangular probability of the multinomial distribution X:

$$P_H\{a_1 \leq X_1 \leq b_1, \ldots, a_M \leq X_M \leq b_M\}, \tag{4.1}$$

where the lower script H refers to the summation of all the components of the vector X.

4.2.3 Characterizations

In this section we define and characterize the recruitment stocking process.

Definition 1. *Let $X_m(t)$ be the number of arrivals at location m during $(0, t]$. Due to the possibility of having stockout, the number of subjects recruited $X_m(t)$ at location m up to time t and the number of arrivals $N_m(t)$ are related by the simple relation:*

$$N_m(t) = \min(X_m(t), s_m), \tag{4.2}$$

and the stochastic process $N(t)$ that counts the total number of recruited subjects in the system is the summation of recruits over all the locations:

$$N(t) = \sum_{m=1}^{M} N_m(t). \tag{4.3}$$

We call $N(t)$ the recruitment stocking process.
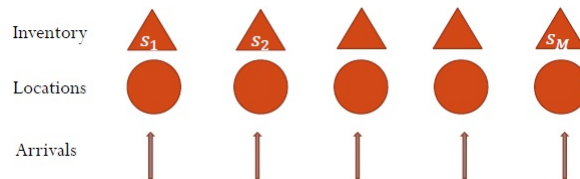
Some preliminary properties of $\{N(t), t \geq 0\}$ are discussed below. It can be easily seen that the probability of having n th or more recruits in the system during the interval $(0, t]$ is equal to the probability that the n th recruit occurs at or before t:

$$P\{N(t) \geq n\} = P\{T(n) \leq t\}, \quad n = 0, \ldots, H. \tag{4.4}$$

Type II service level and expected recruitment time The mean number of rejected subjected is the difference between the mean arrivals of subjects in the system and the target H.

$$R = X_1[T(H)] + \cdots + X_M[T(H)] - H, \tag{4.5}$$

and since the expectation is additive, we obtain:

$$E[R] = E[X_1(T(H))] + \cdots + E[X_M(T(H))] - H, \tag{4.6}$$

Each of the term is the expected number of arrivals at sites:

$$E[X_i(T(H))] = \lambda_i E[T(H)], \ i = 1, \ldots, M. \tag{4.7}$$

Thus, the expected value of R is

$$E[R] = E[T(H)] \sum_{i=1}^{M} \lambda_i - H. \tag{4.8}$$

Probability mass function of $N(t)$ The recruitment stocking process, $\{N(t); t > 0\}$ counts the number of recruits upto time t. It consists of a discrete random variable $N(t)$ for each $t > 0$. We present the closed-form formula for the pmf for this random variable in this section. The following lemma is straightforward, but since it is needed in the main proof for the formula, we state it here without proof.

Lemma 3. *The probability*

$$\sum_{\substack{x_1+x_2+\cdots+x_M=s \\ x_1<s_1, x_2<s_2, \ldots, x_M<s_M}} P\{X_1(t) = x_1\} \ldots P\{X_M(t) = x_M\}, \tag{4.9}$$

is the product of a rectangle multinomial probability and a Poisson cdf:

$$P_s\{X_1(t) < s_1, \ldots, X_M(t) < s_M\} P\{X_1(t) + \cdots + X_M(t) = s\}. \tag{4.10}$$

The pmf for $N(t)$ is presented in the following theorem.

Theorem 16. *The pmf for $N(t)$ (i.e., the number of recruits in $(0, t]$) is given by the*

expression:

$$P\{N(t) = s\} = P_s\{X_1(t) < s_1, \ldots, X_M(t) < s_M\}P(\sum_{m=1}^{M} X_m(t) \leq \sum_{m=1}^{M} s_m)$$

$$+ \sum_{k=1}^{M} \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq M} P\{X_{i_1} \geq s_{i_1}\} \ldots P\{X_{i_k} \geq s_{i_k}\}$$

$$P_{s-\sum_{j=1}^{k} s_{i_j}}\{X_j(t) < s_j; j \neq i_1, \ldots, i_k\}P\{\sum_{j=1}^{k} X_{i_j}(t) = \sum_{j=1}^{k} s_{i_j}\}, \quad (4.11)$$

where the rectangle multinomial probability $P_{s-\sum_{j=1}^{k} s_{i_j}}\{X_j(t) < s_j; j \neq i_1, \ldots, i_k\} = 0$, *if* $s \leq \sum_{j=1}^{k} s_{i_j}$.

Proof. Let x_m be the particular value of the random variable $N_m(t)$, then we have the following expression for $P\{N(t) = s\}$:

$$P\{N(t) = s\} = \sum_{\substack{x_1+x_2+\cdots+x_M=s \\ x_1 \leq s_1, x_2 \leq s_2, \ldots, x_M \leq s_M}} P\{N_1(t) = x_1\} \ldots P\{N_M(t) = x_M\}, \quad (4.12)$$

where the pmf for $N_m(t)$ is given as follows:

$$P\{N_m(t) = x_m\} = \begin{cases} \dfrac{e^{-\lambda_m}\lambda_m^{x_m}}{x_m!} & \text{for } x_m < s_m \\ 1 - \sum_{x_m=0}^{s_m} \dfrac{e^{-\lambda_m}\lambda_m^{x_m}}{x_m!} & \text{if } x_m = s_m \end{cases}$$

The main idea of the proof is to relax those constraints on $x_i, i = 1, \ldots, M$ to strict inequalities $x_i < s_i$ and divide the set $x_1 \leq s_1, \ldots, x_M \leq s_M$ into the following disjoint sets:

$$x_1 < s_1, \ldots, x_M < s_M \quad (4.13)$$

$$x_{i_1} = s_{i_1}; x_j < s_j, j \neq i_1, i_1 = 1, \ldots, M \quad (4.14)$$

$$\cdots$$

$$x_{i_1} = s_{i_1}, \ldots, x_{i_k} = s_{i_k}; x_j < s_j, j \neq i_1, \ldots, i_k, 1 \leq i_1 \leq \cdots \leq i_k \leq M, \quad (4.15)$$

then carry out the summation in equation (4.12) over these sets above. Let's start with the simplest set; $x_1 < s_1, \ldots, x_M < s_M$. This set depicts the scenarios that the number of arrivals to all locations is strictly less than their inventory levels. The probability corresponding to this set is the following:

$$\sum_{\substack{x_1+x_2+\cdots+x_M=s \\ x_1 < s_1, x_2 < s_2, \ldots, x_M < s_M}} P\{X_1(t) = x_1\} \ldots P\{X_M(t) = x_M\}. \quad (4.16)$$

Notice that the above type of summation can be simplied into the following expression by Lemma 3:

$$P_s\{X_1(t) < s_1, \ldots, X_M(t) < s_M\}P\{X_1(t) + \cdots + X_M(t) = s\}. \quad (4.17)$$

The more general set $x_{i_1} = s_{i_1}, \ldots, x_{i_k} = s_{i_k}; x_j < s_j, j \neq i_1, \ldots, i_k, 1 \leq i_1 \leq \cdots \leq i_k$ shows the scenarios that we has depleted inventory at locations $i_1, \ldots, i_k$. Under this scenarios, the corresponding probability is:

$$\sum_{\substack{x_1+x_2+\cdots+x_M=s \\ x_{i_1}=s_{i_1},\ldots,x_{i_k}=s_{i_k};x_j<s_j,j\neq i_1,\ldots,i_k,1\leq i_1\leq\cdots\leq i_k}} P\{N_1(t) = x_1\} \ldots P\{N_M(t) = x_M\}. \quad (4.18)$$

The above probability can be rewritten as:

$$P\{X_{i_1}(t) \geq s_{i_1}\} \ldots P\{X_{i_k}(t) \geq s_{i_k}\} \sum_{\substack{\sum x_j=s-s_{i_1}-\cdots-s_{i_k} \\ x_j<s_j,j\neq i_1,\ldots,i_k}} \prod_j P\{X_j(t) < x_j\} \quad (4.19)$$

Notice that the summation can be rewriten as the product of a rectangle multinomial probability and a Poisson cdf by Lemma 3. The proof is complete.

Moments of $N(t)$ The first three moments can be found explicitly by the following formula on the mean, variance and skewness of a sum of independent variables.

$$
\begin{aligned}
E(\sum_{i=1}^{M} X_i) &= \sum_{i=1}^{M} E(X_i) \\
\mathrm{Var}(\sum_{i=1}^{M} X_i) &= \sum_{i=1}^{M} \mathrm{Var}(X_i) \\
\mathrm{Skew}(\sum_{i=1}^{M} X_i) &= \frac{\sum_{m=1}^{M} \mathrm{Var}(X_i)^{3/2}\mathrm{Skew}(X_i)}{[\sum_{i=1}^{M} \mathrm{Var}(X_i)]^{3/2}}
\end{aligned}
\quad (4.20)
$$

The closed-form formula for the mean of $N(t)$ is presented below:

$$E[N(t)] = t\sum_{m=1}^{M} \lambda_m - M + \sum_{m=1}^{M} \frac{\Gamma(1 + s_m, \lambda_m t)}{s_m!}, \quad (4.21)$$

where $\Gamma(a, z)$ is the Gamma function:

$$\Gamma(a, z) = \int_z^{\infty} t^{a-1}e^{-t}dt \quad (4.22)$$

Shape of the distribution $N(t)$ is the sum of many integer-valued random variables, and in general, its pmf can have more than one mode. The following theorem shows that $N(t)$ is unimodal in two cases.

Theorem 17. *(a) For $t \leq \frac{1}{\max \lambda_m}$, then $N(t)$ has log-concave pmf:*

$$P\{N(t) = s\}^2 \geq P\{N(t) = s+1\}P\{N(t) = s-1\}. \tag{4.23}$$

(b) Furthermore, there exists a positive number τ such that for $t > \tau$, $N(t)$ has mono-tone increasing probability mass function.

$$P\{N(t) = s+1\} \geq P\{N(t) = s\} \tag{4.24}$$

Proof.

(a). Consider the arrival process at location m:

$$N_m(t) = \min(X_m(t), s_m). \tag{4.25}$$

The support for this random variable $N_m(t)$ is the discrete set $\{0, \ldots, s_m\}$. It can be easily seen that for $P\{N_m(t) = n\}$, $n = 0, \ldots, s_m - 1$ is a log-concave sequence since the Poisson random variable is log-concave. Thus, the log-concavity property of the pmf of $N(t)$ will follow if we can show that

$$P\{N_m(t) = s_m - 1\}^2 \geq P\{N_m(t) = s_m\}P\{N_m(t) = s_m - 2\}. \tag{4.26}$$

Using the pmf for $N(m)(t)$ and further simplify the above inequality, we obtain:

$$\frac{1}{s_m - 1} \geq \sum_{j=s_m}^{\infty} \frac{(s_m - 1)!(\lambda_m t)^{j-s_m}}{j!}. \tag{4.27}$$

Rewriting the infinite summation, the inequality becomes:

$$\frac{1}{s_m - 1} \geq e^{\lambda_m t}(\lambda_m t)^{-s_m} \gamma(s_m, \lambda_m t), \tag{4.28}$$

where $\gamma(a, z)$ is the lower imcomplete gamma function:

$$\gamma(a, z) = \int_0^z t^{a-1} e^{-t} dt \tag{4.29}$$

Let $x = \lambda_m t$. To prove the inequality (4.28), we need to show the positivity of the following function for $0 < x \leq 1$:

$$f(x) = \frac{e^{-x} x^{-s_m}}{s_m - 1} - \gamma(s_m, x) \tag{4.30}$$

It can be easily veried that the first derivative of $f(x)$ is:

$$f'(x) = x^{s_m-1}e^{-x} + \frac{e^{-x}s_m x^{-s_m-1} + e^{-x}x^{-s_m}}{s_m - 1} \qquad (4.31)$$

Since x is a positive number and $s_m > 1$, thus, $f'(x)$ is strictly positive. In other words, $f(x)$ is strictly increasing. Now, we will show that $f(1) \leq 0$.

We have the following recurrence for the incomplete gamma function:

$$\gamma(s_m + 1, x) = s_m\gamma(s_m, x) - x^{s_m}e^{-x}. \qquad (4.32)$$

At $x = 1$, we obtain:

$$\gamma s_m + 1, 1 = s_m\gamma(s_m, 1) - e^{-1}. \qquad (4.33)$$

Furthermore, for $0 \leq t \leq 1$, the following inequality holds true:

$$t^{s_m-1}e^{-t} \geq t^{s_m}e^{-t}. \qquad (4.34)$$

Now, integrating from 0 to 1 for both sides, we obtain:

$$\gamma(s_m - 1, 1) \geq \gamma(s_m, 1). \qquad (4.35)$$

The negativity of $f(1)$ follows directly from (4.33) and (4.35). This shows that $N_m(t)$ is log-concave if $\lambda_m t \leq 1$. Since the convolution of log-concave sequence are log-concave, the proof for the first part of our theorem is complete.

(b). The key idea of the proof is to show that, when t is large enough, the following expression in the formula for the pmf of $N(t)$ in Theorem 16 is increasing in s:

$$P\{X_{i_1} \geq s_{i_1}\}\ldots P\{X_{i_k} \geq s_{i_k}\}P_{s-\sum_{j=1}^{k} s_{i_j}}\{X_j(t) < s_j; j \neq i_1, \ldots, i_k\}$$
$$P\{X_{i_1}(t) + \cdots + X_{i_k}(t) = s_{i_1} + \cdots + s_{i_k}\}. \qquad (4.36)$$

Note that k is the number of locations that reject subjects. Without loss of generality, we will prove the statement for the case that $k = 1$. After simplying those terms that are independent of s, we have to show:

$$P_{s-s_{i_1}}\{X_j(t) < s_j; j \neq i_1\} \leq P_{s+1-s_{i_1}}\{X_j(t) < s_j; j \neq i_1\} \qquad (4.37)$$

It is well-known (Levin, 1981) that the rectangle multinomial probability can be expressed in terms of Poisson and truncated Poisson random variables as follows:

$$P_{s-s_{i_1}}\{X_j(t) < s_j; j \neq i_1\} = \frac{(s-s_{i_1})!}{(t\sum_{j=1,j\neq i_1}^M \lambda)^{s-s_{i_1}}(e^{-t\sum_{j=1,j\neq i_1}^M \lambda})} \prod_{j=1,j\neq i_1}^M P\{X_j(t) \leq s_j\}$$
$$P\{W = s - s_{i_1}\}, \tag{4.38}$$

where W is the sum of independent truncated Poisson random variables, namely $W = \sum_{j=1,j\neq i_1}^M Y_j$ and $Y_j \sim TP(X_j(t))$ with range $0, 1, \ldots, s_j - 1$.

The inequality (4.37) simplifies to the following:

$$P\{W = s - s_{i_1}\} \leq \frac{P\{W = s + 1 - s_{i_1}\}}{s + 1 - s_{i_1}}. \tag{4.39}$$

It can be easily seen that the truncated Poisson distribution is log-concave, so the distribution for W is also log-concave since log-concavity is closed under convolution. Thus, we obtain:

$$\frac{P\{W = s - s_{i_1}\}}{P\{W = s + 1 - s_{i_1}\}} \leq \frac{P\{W = W_{\max} - 1\}}{P\{W = W_{\max}\}}, \tag{4.40}$$

where $W_{\max} = s_1 - 1 + \cdots + s_M - 1$. Substitute $W = Y_1 + \cdots + Y_M$ and the pmf for Y_j to yield:

$$\frac{P\{W = s - s_{i_1}\}}{P\{W = s + 1 - s_{i_1}\}} \leq \sum_{j=1,j\neq i_1}^M \frac{s_j - 1}{\lambda t}, \tag{4.41}$$

and take limit when t goes to ∞. Since the limit for the right hand side goes to 0, it follows that the fraction on the left also goes to 0 when t is large enough.

4.2.4 Recruitment time

In general, the distribution for recruitment time $T(H)$ is unknown except for two special cases. The first case corresponds to the scenarios when we do not want to have any inventory overage at the end of recruitment. In other words, the total inventory in the system S is the same as the target H. Thus, the recruitment time $T(H)$ is the time for the last location to delete its inventory:

$$T(H) = \max_{m=1,\ldots,M} T_m(s_m), \tag{4.42}$$

where $T_m(s_m)$ is the time for location m to deplete its inventory s_m. Under Poisson arrivals assumption, it can be easily seen that $T_m(s_m)$ follows an Erlang distributions with shape parameter s_m and scale parameter λ_m. Thus, $T(H)$ is the maximum ordered statistics for M Erlang random variables. The second case looks at the other extreme when inventory is not expensive. Hence, we can choose to carry a lot of inventory in the system, $S \gg H$. In this case, the distribution of $T(H)$ can be well approximated by a single Erlang distribution with shape parameter H and scale parameter $\sum_m \lambda_m$. Detailed discussions on the computation of the mean recruitment time will be presented in Section 4.

4.2.5 Type I Service level

We are interested in the probability that no subject is rejected at the end of recruitment. Since there is no rejections, the total number of subjects arriving in the system is H:

$$X_1 + X_2 + \cdots + X_M = H,$$

and the number of subjects X_m arriving at location m, $m = 1, \ldots, M$ have to be less than or equal to the inventory level s_m. Hence, Type I service level can be written as:

$$P_H\{X_1 \leq s_1, X_2 \leq s_2, \ldots, X_M \leq s_M\} \tag{4.43}$$

Under Poisson arrivals assumption, $(X_1, X_2, \ldots, X_M)$ has a multinomial distribution with parameter H and $p = (p_1, \ldots, p_M)$, where p_i is defined as follows:

$$p_i = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \cdots + \lambda_M}, \tag{4.44}$$

thus Type I service level is the rectangle probability of the multinomial distribution. Levin (1981) used approximate algorithm to compute the c.d.f. More efficient methods are developed recently by Frey (2009) and Lebrun (2012). We refer to their papers for a more detailed discussion.

4.3 Exact analysis

In this section we provide the exact analysis for Type II service level or the mean number of subjects rejected at the end of recruitment.

4.3.1   Type II service levels

Key ideas and challenges Type II service level $E[R]$ can be written explicitly as:

$$E[R] = \sum_{l=1}^{\infty} l \times P\{R = r\}. \tag{4.45}$$

Hence, to find Type II service level, we need to compute the probability that we are rejecting r subjects in the system $P\{R = r\}$. This probability can be found by two-step conditioning procedure. The first step conditions on where stock out happens. This step can be explained with a simplest case when rejection happens at only one location. In other words, r subjects can only be rejected at location 1, or location 2, ..., or location M. They are nonoverlapping events, thus the probability of rejecting r subjects at exactly one location can be computed as:

$$\sum_{i_1=1}^{M} P\{R_{i_1} = r; R_m = 0, m \neq i_1\}, \tag{4.46}$$

where each summand is the probability that r subjects are rejected at location i_1.

Now, what happens if two locations reject r subject? First, we have to determine where we reject r subjects, location i_1 and i_2, for example. Then, we have to understand how r rejections happen. Specifically, if r_{i_1} and r_{i_2} are denoted as the number of rejected subjects at location i_1 and location i_2, correspondingly, we have to impose the following constraint:

$$r_{i_1} + r_{i_2} = r, \tag{4.47}$$

and compute the probabilities of rejecting r_{i_1} subjects at location i_1 and r_{i_2} subjects at location i_2 and sum them up to obtain the probability of rejecting r subjects at exactly two locations:

$$\sum_{1 \leq i_1, i_2 \leq M} P\{R_{i_1} = r_{i_1}; R_{i_2} = r_{i_2}; R_m = 0, m \neq i_1, i_2\}, \tag{4.48}$$

In general, the probability of rejecting r subjects in the system can be computed as follows:

$$\sum_{k=1}^{\min(r,M-1)} \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq M} \sum_{r_{i_1} + \cdots + r_{i_k} = r} P\{R_{i_1} = r_{i_1}, \ldots, R_{i_K} = r_{i_K}; R_m = 0, m \neq i_1, \ldots, i_k\},$$
$$\tag{4.49}$$

where the summand is called the k-location r-rejected probability, i.e., the probability of rejecting r subjects at a given set of k locations: r_{i_1} at location i_1, r_{i_2} at location i_2, and so on. The first two summations enumerate over all possible combinations of the locations that reject r subjects. The inner-most summation splits r rejections among the set of K locations.

Some notes on the complexity of this procedure are in order. First, we observe that: (1) k cannot exceed $M - 1$ since otherwise, the system does not carry enough inventory to serve the target (2) k cannot exceed r since otherwise, there will be more than r rejected patients. Therefore, given the total rejected customers r, the number of sites that have stock out k can take value between $1, 2, \ldots, \min(M - 1, l)$. In the worst case, we have exponential number of combinations in terms of the number of locations 2^M. The second level of complexity comes from the large number of scenarios of how r rejections happen at k location. Specificaly, given the set of k locations $i_1, \ldots, i_k$, the number of rejected customers at those sites must satisfy the linear equation:

$$r_{i_1} + \cdots + r_{i_k} = r. \tag{4.50}$$

The number of k-tuples $r_{i_1}, \ldots, r_{i_k}$ is well-known (see Murty, 1981). It is $\binom{r-1}{k-1}$. Thus, in the worst case (when k ranges from $1, \ldots, M - 1$), we have:

$$\sum_{k=1}^{M-1} \binom{M}{k}\binom{r-1}{k-1} = \frac{M}{M+r}\binom{M+r}{r} - \binom{M-1}{r-1}. \tag{4.51}$$

It is computationally expensive to evaluate $P\{R = r\}$ with this brute-force approach: there are two many k-location r-rejected probabilities, and at the moment, we do not know how to evaluate them. Note that a k-location r-rejected probability is not simply the probability that out of $H + r$ arrivals, r_{i_j} are rejected at location i_k, $j = 1, \ldots, k$ since we need to make sure that the last arrival will be recruited (otherwise, the trial would already finish).

4.3.2 Relaxation theorem

The relaxation theorem shows how to compute a single k-location r-rejected probability, i.e., the summand in the equation (4.49), in an efficient manner.

Simple case For clarity of the presentation, we first focus on the derivation for the probability of rejecting exactly one patient $P\{R=1\}$. Intuitively, the system rejects 1 customer if and only if there is only one site that runs out of stock. Let i_1 be the index of this site. Clearly, i_1 can take value from 1 to M. Thus, two equations (4.48) and (4.49) become very simple and $P\{R=1\}$ can be written as follows:

$$P\{R=1\} = \sum_{i_1=1}^{M} P\{R_{i_1} = 1; R_m = 0, m \neq i_1\}, \tag{4.52}$$

where $P\{R_{i_1} = 1; R_m = 0, m \neq i_1\}$ is the probability that we reject 1 customer at site i_1. Note that, this rejected patient cannot be the last arrival (since the last arrival must be recruited). Therefore, the main idea here is to assume that no rejection happens and exclude the unneccesary event. In particular, the probability $P\{R_{i_1} = 1; R_m = 0, m \neq i_1\}$ can be computed by:

$$\begin{aligned} P\{R_{i_1} = 1; R_m = 0, m \neq i_1\} &= P_{H+1}\{X_{i_1} = s_{i_1} + 1; X_m \leq s_m, m \neq i_1\} \\ &\quad - p_{i_1} P_H\{X'_{i_1} = s_{i_1}; X'_m \leq s_m, m \neq i_1\}, \end{aligned} \tag{4.53}$$

where the first term denotes the probability that, among $H+1$ arriving patients to the system, there are exactly $s_{i_1} + 1$ patients at site i_1 and there is no stock out at other sites (as if there is no stock out). This probability is called the relaxed k-location r-rejected probability since it contains the event that the $H+1$ patient is rejected at site i_1 and thus, has to be excluded by the use of the second term. It is the probability that, among the first H patients, there are exactly s_{i_1} patients arriving at site i_1 and there is no stock out at other sites, and the $H+1$ customer is rejected at site i_1.

Equation (4.53) can be further simplified in a few steps. First, because order arriving at sites are filled on a first-come-first-served basis, and because the arrival process to sites is a Poisson process with arrival rate λ_i, the vector $(X_1, \ldots, X_M)$ follows the multinomial distribution with mass parameter $H+1$ and probability vector $p = (p_1, \ldots, p_M)$ as defined in (4.44). Thus, the probability $P_{H+1}\{X_{i_1} = s_{i_1} + 1; X_m \leq s_m, m \neq i_1\}$ can be rewritten as follows:

$$P_{H+1}\{X_m \leq s_m, m \neq i_1 | X_{i_1} = s_{i_1} + 1\} P_{H+1}\{X_{i_1} = s_{i_1} + 1\}. \tag{4.54}$$

It is well-known that the distribution of the conditional probability in the above equation is a multinomial distribution with parameters $H - s_{i_1}$ with adjusted probabilities $p'_i = \lambda_i / \sum_{m=1,m\neq i_1}^{M} \lambda_i$:

$$P_{H-s_{i_1}}\{X_m \leq s_m, m \neq i_1\}, \tag{4.55}$$

and $P_{H+1}\{X_{i_1} = s_{i_1} + 1\}$ is the marginal distribution of the multinomial distribution, thus, follows binomial distribution with parameters $H + 1$ and $p_{i_1} = \lambda_{i_1} / \sum_{m=1}^{M} \lambda_m$. That is,

$$P_{H+1}\{X_{i_1} = s_{i_1} + 1\} = \binom{H+1}{s_{i_1}+1} p_{i_1}^{s_{i_1}+1}(1 - p_{i_1})^{H-s_{i_1}}. \tag{4.56}$$

In a similar way, we can also rewrite the second term $P_H\{X'_{i_1} = s_{i_1}; X'_m \leq s_m, m \neq i_1\}$ as

$$P_H\{X'_m \leq s_m, m \neq i_1 | X'_{i_1} = s_{i_1}\} P_H\{X'_{i_1} = s_{i_1}\}, \tag{4.57}$$

where the distribution of the conditional probability is the same multinomial distribution defined in (4.55) and $P_H\{X'_{i_1} = s_{i_1}\}$ is the binomial distribution with parameters H and p_{i_1}. Using equations (4.54) and (4.57) yields:

$$\frac{P_H\{X_{i_1} = s_{i_1}; X_m \leq s_m, m \neq i_1\}}{P_{H+1}\{X'_{i_1} = s_{i_1} + 1; X'_m \leq s_m, m \neq i_1\}} = \frac{P_H(X_{i_1} = s_{i_1})}{P_{H+1}(X'_{i_1} = s_{i_1} + 1)}. \tag{4.58}$$

It can be easily verified that:

$$\frac{P_H\{X_{i_1} = s_{i_1}\}}{P_{H+1}\{X'_{i_1} = s_{i_1} + 1\}} = \frac{s_{i_1} + 1}{p_{i_1}(H+1)}. \tag{4.59}$$

The above formula is then used to obtain a much simpler equation in place of (4.53) to compute the probability of rejecting exactly 1 patient in the system at location i_1:

$$P\{R_{i_1} = 1; R_j = 0, j \neq i_1\} = \frac{H - s_{i_1}}{H+1} P_{H+1}\{X_{i_1} = s_{i_1} + 1; X_m \leq s_m, m \neq i_1,\}. \tag{4.60}$$

The analysis for the general case is more involved and is presented below.

General case Let K be the number of sites that have stock out and denote the subscripts of those sites that run out of stock as $\{i_1, \ldots, i_k\}$. Then, if the number of rejected patients at locations $i_1, \ldots, i_k$ are $r_{i_1}, \ldots, r_{i_k}$, correspondingly, then the

k-location r-rejected probability is:

$$P_{H+r}\{X_{i_j} = s_{i_j} + r_{i_j}, j = 1, \ldots, k; X_m \leq s_m, \ m \neq i_1, \ldots, i_k\}$$

$$-\sum_{t=1}^{k} p_{i_t} P_{H+r-1}\{X'_{i_j} = s_{i_j} + r_{i_j}, j \neq t; X'_{i_t} = s_{i_t} + r_{i_t} - 1; X'_m \leq s_m, \ m \neq i_1, \ldots, i_k\}.$$

$$(4.61)$$

The first summand in equation (4.61) denotes the probability that, among $H+r$ arriving patients, sites i_j serves exactly s_{i_j} and reject r_{i_j} patients ($j = 1, \ldots, k$) due to stock out, and there is no stock out at other locations. This probability is called the relaxed k-location r-rejected probability since it does not capture the condition that the last recruited patient cannot be rejected. Thus, the inner summation of probabilities is introduced to address this issue. It is the probability that among the first $H + r - 1$ arriving patients to the system: (1) there are exactly $s_{i_j} + r_{i_j}$ patients arriving at site i_j, $j \in \{1, \ldots, k\} \setminus t$ (2) site t serves s_{i_t} patients, rejects r_{i_t} patients and the $H + r$ customer is rejected at site t (3) there is no stock out at other sites. A simpler expression for $P\{R = r\}$ can also be found. The main idea is to find out the ratio between the first summand and each of the term in the inner summation. The simplication process is based on the observation that the arrival process at site t is independent from other sites. Thus, we can use conditional probability to separate site t from others. The step-by-step derivation is straight forward, hence, omitted (see Appendix for more details). Equation (4.61) now becomes:

$$\frac{H - \sum_{j=1}^{k} s_{i_j}}{H + r} P_{H+r}\{X_{i_j} = s_{i_j} + r_{i_j}, j = 1, \ldots, k; X_m \leq s_m, \ m \neq i_1, \ldots, i_k\}. \quad (4.62)$$

Now we are ready to state the relaxation theorem for the k-location r-rejected probability, i.e., the probability of rejecting r subjects at a given set of k locations: r_{i_1} subjects are rejected at location i_1, $\ldots$, r_{i_k} subjects are rejected at location i_k.

Theorem 18. *The k-location r-rejected probability is linearly related to its relaxed version, in particular, it is equal to the following expression:*

$$\frac{H - \sum_{j=1}^{k} s_{i_j}}{H + r} P_{H+r}\{X_{i_j} = s_{i_j} + r_{i_j}, j = 1, \ldots, k; X_m \leq s_m, \ m \neq i_1, \ldots, i_k\}. \quad (4.63)$$

4.3.3   Decomposition theorem

At the end of the relaxation step, we obtain the following formula to compute the inner-most summation in equation (4.49):

$$
\sum_{\sum_{j=1}^{k} r_{i_j} = r} \frac{H - \sum_{j=1}^{k} s_{i_j}}{H + r} P_{H+r}\{X_{i_j} = s_{i_j} + r_{i_j}, j = 1, \ldots, k; X_m \leq s_m, \ m \neq i_1, \ldots, i_k\}.
$$

$$(4.64)$$

Since rejections can happen in various ways at those locations, thus, in the equation (4.49), the inner summation adds up all these k-location r-rejected probabilities to account for all the possible scenarios. A better approach is to rewrite the above equation as follows:

$$
\frac{H - \sum_{j=1}^{k} s_{i_j}}{H + r} P_{H+r}\{X_{i_j} \geq s_{i_j} + 1, \sum_{j=1}^{k} X_{i_j} = r + \sum_{j=1}^{k} s_{i_j}; X_m \leq s_m, \ m \neq i_1, \ldots, i_k\}.
$$

In the following theorem, we show how to evaluate the above expression. First, we introduce $Binopdf(x, n, p)$ as the value of pmf of a binomial distribution at x, with N as the number of trials and the probability of success for each trial is p.

Theorem 19. *Assume that the vector $X = (X_1, \ldots, X_M)$ follows multinomial distribution $(H + r, p_1, \ldots, p_M)$. Furthermore, the summation of the last k components of X is r, then we have:*

$$
P_{H+r}\{X_1 \leq s_i, \ldots, X_{M-k} \leq s_{M-k}, X_{M-k+1} \geq s_{M-k+1}, \ldots, X_M \geq s_M\}
$$

$$
= P_H\{X_1 \leq s_i, \ldots, X_{M-k} \leq s_{M-k}\} P_r(X_{M-k+1} \geq s_{M-k+1}, \ldots, X_M \geq s_M\}
$$

$$
\times Binopdf\{r, H + r, \sum_{i=M-k+1}^{M} p_i\}
$$

where $(X_1, \ldots, X_{M_k})$ follows the multinomial distribution with mass parameter H and probabilities

$$
p_i^{(-)} = \frac{p_i}{\sum_{i=1}^{M-k} p_i}, \ i = 1, \ldots, M - k \tag{4.65}
$$

and $(X_{M-k+1}, \ldots, X_M)$ follows the multinomial distribution with mass parameter r and probabilities.

$$
p_i^{(+)} = \frac{p_i}{\sum_{i=M-k+1}^{M} p_i}, \ i = M - k + 1, \ldots, M. \tag{4.66}
$$

Proof. The rectangle probability $P_{H+r}\{X_1 \leq s_i, \ldots, X_{M-k} \leq s_{M-k}, X_{M-k+1} \geq s_{M-k+1}, \ldots, X_M \geq s_M\}$ given that the summation of the last k terms of the vector X is r can be expressed explicitly as follows:

$$\sum_{\substack{x_1 \leq s_1, \ldots, x_{M-k} \leq s_{M-k} \\ x_{M-k+1} \geq s_{M-k+1}, \ldots, x_M \geq s_M \\ \sum_{i=M-k+1}^{M} x_i = r}} \frac{(H+r)!}{x_1! \ldots x_M!} p_1^{x_1} \cdots p_M^{x_M}. \tag{4.67}$$

Since $X_1 + \cdots + X_M = H + r$ and $X_{M-k+1} + \cdots + X_M = r$, we have:

$$X_1 + \cdots + X_{M-k} = H, \tag{4.68}$$

and we can rewrite equation (4.67) as:

$$\frac{(H+r)!}{H!r!} \sum_{\substack{x_1 \leq s_1, \ldots, x_{M-k} \leq s_{M-k} \\ \sum_{i=1}^{M-k} x_i = H}} \frac{H!}{x_1! \ldots x_{M-k}!} p_1^{x_1} \cdots p_{M-k}^{x_{M-k}} \times$$

$$\sum_{\substack{x_{M-k+1} \geq s_{M-k+1}, \ldots, x_M \geq s_M \\ \sum_{i=M-k+1}^{M} x_i = r}} \frac{r!}{x_{M-k+1}! \ldots x_M!} p_{M-k+1}^{x_{M-k+1}} \cdots p_M^{x_M}.$$

The assertion of the lemma follows directly after applying formula (4.65) and (4.66) in the above equation. $\square$

Applying the Decomposition Theorem 19 to Equation (4.64), we obtain a much simpler formula to compute the probability of rejecting r patients at sites $i_1, \ldots, i_k$:

$$\frac{H - \sum_{j=1}^{k} s_{i_j}}{H + r} P_{H+r}\left\{X_{i_j} \geq s_{i_j} + 1, \sum_{j=1}^{k} X_{i_j} = r + \sum_{j=1}^{k} s_{i_j}; X_m \leq s_m, m \neq i_1, \ldots, i_k\right\}$$

$$= \frac{H - \sum_{j=1}^{k} s_{i_j}}{H + r} Binopdf\left\{r + \sum_{j=1}^{k} s_{i_j}, H + r, \sum_{j=1}^{k} p_{i_j}\right\} \tag{4.69}$$

$$\times P_H\{X_m \leq s_m, \ m \neq i_1, \ldots, i_k\} P_{r + \sum_{j=1}^{k} s_{i_j}}\{X_{i_j} \geq s_{i_j} + 1, j = 1, \ldots, k\}$$

Remarks. This two-step procedure reduces the effort to evaluate the inner-most summation in equation (4.49) significantly since it is shown to be a product of three simple probabilities which can be computed by known methods. Consequently, for a system with less than 15 locations, evaluation of Type II service level is quite fast. Notice that we still have the exponential number of scenarios arise from the first two summations in (4.49), thus, in practice when the number of locations is much larger, another method is needed. This topic will be discussed in the following section.

4.4 Approximate analysis of recruitment time

Being able to evaluate recruitment time efficiently is very important for many reasons. First, it is a proxy for the expected number of rejected customers. Moreover, recruitment time plays an important role for the recruitment process, e.g., in clinical trials, time can be quite costly due to limited patent lifetime. Thus, supply chain managers need to know how to compute the expected recruitment time for a given initial inventory position. In this section we show how to evaluate time by the use of asymptotic approximation and bouding methods.

4.4.1 Asymptotic results

When the total inventory in the system $S = \sum_{i=1}^{M} s_i$ exceeds the target H, as discussed in Section 4.2.4, the recruitment time T can be well approximated by an Erlang distribution $Er(H, \lambda)$.

Let $E[R_i]$ be the expected number of rejected arrivals at location i by the recruitment time T. The total number of rejected arrival in the system is:

$$E[R] = \sum_{i=1}^{M} E[R_i]. \tag{4.70}$$

In order to compute the expected number of rejected arrivals $E[R_i]$ at location i, we need to be able to compute the probability of rejecting r arrivals there. This probability can be approximated as follows:

$$P\{R_i = r\} = \int_0^\infty P\{R_i = r | T = t\} f_{Er}(t) dt, \tag{4.71}$$

where $f_{Er}(t)$ is the pdf of the Erlang approximation $Er(H, \lambda)$ for the recruitment time T.

Replacing $P\{R_i = r | T = t\}$ by the probability that a Poisson process with rate λ_i has exactly $s_i + r$ arrivals and carrying out the above integration, we obtain the explicit formula for the probability of rejecting r arriavls at location i:

$$P\{R_i = r\} = \frac{\lambda^H \lambda_i^{s_i+r} H(H+1)\ldots(H+s_i+r-1)}{(s_i+r)!(\lambda+\lambda_i)^{s_i+r+H}}. \tag{4.72}$$

4.4.2 Discrete moment bounds

The asymptotic approximation presented in the previous section works very well when we carry excessive amount of inventory in the system. However, inventory is often expensive and we need to make sure that total inventory S is as close to the target H as possible. In this case, we propose a different method to give lower and upper bounds on the expected reruitment time.

Since the first few moments of $N(t)$ can be computed easily, we propose to obtain the lower and upper bounds on $P(N(t) < \bar{H})$ through the moment problems by the use of linear programming. Let $p_s = P(N(t) = s)$ and $S = \sum_{m=1}^{M} s_m$, then $N(t)$ can take values from the set $0, \ldots, S$. For now, assume that some of the moments of $N(t)$ are available $\mu_1, \ldots, \mu_m$, then the lower bound and upper bound on $P(N(t) < \bar{H})$ is the optimal values of the LPs:

$$\min(\max) \quad \sum_{s=0}^{\bar{H}-1} p_s$$

$$\text{subject to}$$

$$\sum_{s=0}^{S} s^k p_s = \mu_k \quad k = 0, ..., m \tag{4.73}$$

$$p_s \geq 0 \qquad s = 0, ..., S,$$

The above problems have been studied extensively by Prékopa (1990) under the name of discrete moment problem. He also developed a special dual algorithm based on explicit dual basis structure for the solution of the bounding problem. For a more detailed description, we refer to Prékopa (1989,1990). Furthermore, the optimal solutions of problem (4.73) can be expressed in closed form by Lagrangian polynomials $l_k(z), L_k(z)$, where k is the number of used moments:

$$E[l_k(z)] \leq P(N(t) < H) \leq E[L_k(z)] \tag{4.74}$$

The expected time to recruit the rest of the patients to meet with the patient target is:

$$E(T_{\bar{H}}) = \int_0^\infty P(T_{\bar{H}} \geq t)dt, \tag{4.75}$$

and equations (4.4), (4.75) give:

$$E(T_{\bar{H}}) = \int_0^\infty P(N(t) < \bar{H})dt. \tag{4.76}$$

Now, integrating the polynomial $L_k(z)$(or $l_k(z)$) (obtained by the two LPs) over the interval $(0, \infty)$ should provide us with lower and upper bounds for $E(T_{\bar{H}})$.

$$\int_0^\infty l_k(t)dt < E(T_{\bar{H}}) < \int_0^\infty L_k(t)dt \tag{4.77}$$

Two integrals in (4.77) are improper, thus, we will first compute the following finite integral:

$$\int_0^a P(N(t) < \bar{H})dt, \tag{4.78}$$

where a is a sufficiently large number and give an upper bound estimate on the tail $\int_a^\infty P(N(t) < \bar{H})dt$. If we denote this upper bound as ϵ, then the lower and upper bound on $E(T_{\bar{H}})$ is:

$$\int_0^a l_k(t)dt < \int_0^a P(N_t < H)dt < E(T_{\bar{H}}) < \epsilon + \int_0^a P(N_t < H)dt < \epsilon + \int_0^a L_k(t)dt \tag{4.79}$$

Intuitively, this approach should work well since $P(N(t) < \bar{H})$ becomes very small as t approaches infinity, i.e., the upper tail of the integral (4.76) should decay fast. The following lemma establishes an exponential upper bound for the tail of the integral in (4.76).

Lemma 4. $\int_a^\infty P(N(t) < H)dt \le Ce^{-\lambda a}(\lambda a)^{x+1}$, where C is a constant:

$$C = \frac{2}{\lambda x!} \sum_{s=0}^{H-1} \left[\binom{s+M-1}{M-1} - \binom{s-S-1}{M-1} \right]. \tag{4.80}$$

Proof. Let c be the number of nonnegative integers solutions $(x_1, \ldots, x_M)$ to the equation:

$$x_1 + \cdots + x_M = s, \tag{4.81}$$

and satisfying additional constraints:

$$x_1 \le s_1, x_2 \le s_2, \ldots, x_M \le s_M. \tag{4.82}$$

Explicit expression of c can be found using inclusion-exclusion formula (e.g, Rosen et. al, 2000) and the well-known result on the number of integer solutions of a linear equation (Murty, 1981). However, the formula is rather complicated, we instead use the following upper bound on c:

$$\binom{s+M-1}{M-1} - \binom{s-S-1}{M-1}, \tag{4.83}$$

where $S = \sum_{m=1}^{M} s_m$ (change this to st). The first term in equation (4.83) is the number of nonnegative integer solutions to equation (4.81), relaxing the constraints on $x_1, \ldots, x_M$. The second term counts the number of nonnegative interger solutions to equation (4.81) under the condition that $x_1 > s_1, \ldots, x_M > s_M$ and $x_1 + \cdots + x_M = s$. We can give the following upper bound on each of the term $P(N_1(t) = x_1) \ldots P(N_M(t) = x_M)$ in equation (4.12):

$$P(N_1(t) = x_1) \ldots P(N_M(t) = x_M) \leq P(N_{j^s}(t) = x_{j^s}). \tag{4.84}$$

Site j^s is selected based on the condition that the number of recruits there x_j is strictly less than its inventory level s_j. The existence of such a site is due to the fact that we carry enough inventory at M sites to serve s patients $(s = 0, \ldots, H-1)$. Then, a simple bound on $P(N(t) = s)$ is the following

$$P(N(t) = s) \leq \left[\binom{s+M-1}{M-1} - \binom{s-S-1}{M-1}\right] e^{-\lambda_{j^s} t} \frac{(\lambda_{j^s} t)^{x_{j^s}}}{x_j!} \tag{4.85}$$

Taking the summation over $s = 0, 1, \ldots, H-1$ gives the following upper bound on $P(N(t) < H)$:

$$P(N(t) < H) \leq e^{-\lambda t} \frac{(\lambda t)^x}{x!} \sum_{s=0}^{H-1} \left[\binom{s+M-1}{M-1} - \binom{s-S-1}{M-1}\right], \tag{4.86}$$

where x, λ are the inventory level and the recruitment at the site that achieves the maximum among all the sites j_s, $s = 0, \ldots, H-1$. Integrating both sides of the above gives the inequality:

$$\int_a^\infty P(N(t) < H) dt \leq \sum_{s=0}^{H-1} \left[\binom{s+M-1}{M-1} - \binom{s-S-1}{M-1}\right] \int_a^\infty e^{-\lambda t} \frac{(\lambda t)^x}{x!} dt \tag{4.87}$$

The integral on the right hand side can be rewritten as:

$$\frac{1}{\lambda x!}\Gamma(x-1,\lambda a), \tag{4.88}$$

where $\Gamma(x-1,\lambda a)$ is the upper incomplete gamma function:

$$\Gamma(x-1,\lambda a) = \int_{\lambda a}^{\infty} e^{-z} z^x dz. \tag{4.89}$$

It is well-known (see, e.g. Natalini, Palumbo, 2000) that $\Gamma(x-1,\lambda a)$ is bounded above by:

$$\Gamma(x-1,\lambda a) < B(\lambda a)^{x-1} e^{-\lambda a}, \tag{4.90}$$

where B is a constant such that $\lambda a > \frac{B}{B-1}(x-1)$. Since a is assumed to be a large number, it is safe to select $B = 2$, for example. The assertion follows directly from (4.87) and (4.90). $\qquad\square$

In what follows, we obtain a tighter lower and upper bound for $P(N(t) < H)$ by imposing a lower bound on p_i in the linear programming formulation. Equation (4.12) can be rewritten as:

$$P(N(t) = s) = \sum_{\substack{x_1+x_2+\cdots+x_M=s \\ x_1 \leq s_1, x_2 \leq s_2, \ldots, x_M \leq s_M}} P(\min(X_1(t), s_1) = x_1) \ldots P(\min(X_M(t), s_M) = x_M), \tag{4.91}$$

and notice that if we relax the constraints on x_i, $i = 1, \ldots, M$ to strict inequalties $x_i < s_i$, we obtain the lower bound on $P(N(t) = s)$:

$$P(N(t) = s) > \sum_{\substack{x_1+x_2+\cdots+x_M=s \\ x_1 < s_1, x_2 < s_2, \ldots, x_M < s_M}} P(\min(X_1(t), s_1) = x_1) \ldots P(\min(X_M(t), s_M) = x_M). \tag{4.92}$$

Under this new information, the problem can still be solved efficiently by dual method by Prekopa due to the explicit structure of dual feasible bases. In particular, when the first two or three moments are used, closed form formulas are available.

There are two ways to improve on the quality of the lower and upper bounds on $E(T)$. We can either use higher order moments (third, fourth moments) or find a better lower bound on the probability $P(N(t) = s)$. The former approach requires the knowledge of moments of the summation of many random variables. This can be

quite expensive and hence, we propose to relax the inequality and give a lower bound on higher moments instead. The latter approach to improve on $P(N(t) = s)$ is very promising since we can add arbitrary number of terms to the right-hand-site of (4.92).

4.5   Numerical results

In this section we want to test the effectiveness of the approximations and bounds for key performance metrics: Type I, Type II service levels and recruitment time. In particular, we compare the values obtained by approximations and bounds to those from exact analysis (small number of locations) and simulation (medium and large number of locations). In the small example, we want to recruit 30 subjects at 5 locations and we have 50 packages in the system. In the medium example, the number of subjects to be recruited is 70 over 10 locations with 200 packages. The large example requires 600 subjects at 32 locations and we have 1200 packages in the system. Lastly, we carry out a randomized study by varying the number of subjects, locations and packages.

4.5.1   Small example

Assume that the arrival rate is 0.2 subjects per day and the stock levels are set to be 10 at each location. Type I service level is a multinomial probability and can be evaluated exactly to be 89.2%. Type II service level can also be evaluated exactly since the number of locations is small enough. On average, we reject 0.248 subjects throughout recruitment. This number is very close to the result obtained by simulation 0.249. The accuracy for recruitment time is summarized in the following table:

Exact	LB	UB	SIM
30.25	30.21	32.31	30.25

Table 4.1: Expected recruitment time $E[T]$: small case

The expected recruitment time computed by exact approach in the first column is verified by the simulation result in the last column. It takes 30.25 days with the current inventory positions to recruit 30 subjects. The values in the table also confirms that the lower and upper bounds are quite accurate.

4.5.2   Medium case

We repeat the same exercise in the previous section, however, the number of locations is 10 and each has the arrival rate of 0.1 subjects per day. The initial inventory level is 10 for every location. We want to recruit 70 subjects. Type I service level is computed exactly to be 32.3%. As expected, when type I service level decreases, Type II service level increases. On average, the system rejects 2.09 subjects throughout recruitment process. This is the exact value computed by our relaxation-decomposition scheme. A value of 2.08 rejected subjects is obtained by simulation.

The expected recruitment time is reported in the following table. Now, it takes about 72 days to finish recruitment of 70 subjects. This number is very close to the lower and upper bounds.

Exact	LB	UB	SIM
72.09	71.18	74.93	72.08

Table 4.2: Expected recruitment time $E[T]$: medium case

4.5.3   Large case

In this large example, we test our methods on the data of a clinical trial (Fleishhacker et. al, 2014). The trial tested the efficacy of an antibiotic in treating a specific type of infection and is a typical example of Phase III biologic drug trial. The trial's patient horizon (i.e. the target number of patients to be recruited) was 600, and the patient recruitment was accomplished in nine months. During the trial, each patient received and needed only one clinical trial package (i.e. drug supply, packaging, and labeling), and all treatments were administered intravenously in a hospital or doctor's office. Previously collected drug stability data supported a 24-month shelf life for the investigational drug, and thus drug expiration was not a concern for this trial. Assume that we are given 1200 medical packages to distribute over these 30 clinical trial sites. For now, the intial stock levels are set in such a way that they are in proportion with the corresponding recruitment rate.

Type I service level is evaluated to be 89.2%. Type II service level and expected

	# of Sites	Enrollment Rate by Site *(patients per day - sites separated by commas)*
Latvia	4	0.02, 0.04, 0.05, 0.08
Russia	4	0.03, 0.06, 0.06, 0.28
Ukraine	4	0.02, 0.04, 0.05, 0.06
United States	12	0.03, 0.04, 0.05, 0.06, 0.08, 0.08, 0.11, 0.11, 0.14, 0.14, 0.16, 0.18
Poland	6	0.01, 0.02, 0.04, 0.04, 0.04, 0.06

Table 4.3: Enrollment Data for 30 Site Trial conducted in Five Countries.

recruitment time cannot be evaluated exactly due to a large number of locations. Thus, in the following, we only present bounds obtained by discrete moment method and simulation. On average, it takes 275 days (simulated result) to recruit 600 patients at 30 clinical trial sites with the given inventory levels. This number is very close to the lower bound obtained by the moment method.

Exact	LB	UB	SIM
NA	275.24	277.28	275.31

Table 4.4: Expected recruitment time $E[T]$: large case

4.5.4 Randomized study

In this section we want to compare the accuracy of moment bound methods and the asymptotic approximations. We generate 1000 scenarios of recruitment by varying the recruitment target, inventory, number of locations and recruitment rates. The recruitment rates are generated by a uniform distribution and the initial stockings are allocated according to arrival rates. Other parameter values are chosen based on the following table. The first row shows if we have excessive inventory in the system or not. For example, if the ratio between S and H is 1, it means that we carry the same amount of inventory as the target. However, if this ratio is 2, we overstock inventory: carry twice as much inventory as the target.

	Values
S/H	100 % , 150 %, 200 % ($S - H$ is inventory overage)
M	5, 10, 15, 20, 25, 30 (number of locations)
S	500, 1000, 1500 (total number of inventory)

Table 4.5: Parameters for the randomized study

We consider two cases: overstock and understock. Overstock case corresponds to those scenarios that the total of inventory in the system is either 1.5 or 2 times higher than the target. If the total inventory is the same as the target, we call it understock. At heavy overstock, asymptotic approximation on recruitment performs very well compared to simulation. This is expected since if we carry a lot of inventory in the system, we rarely reject subjects and the total arrival process to the system behaves similar to a single Poisson process with the rate equals to the summation of rates at all locations.

% Error	Min	Average	Max
Approx.	0.1 %	1.3 %	2.4 %

Table 4.6: Expected recruitment time $E[T]$ when overstock ($S/H > 1$): asymptotic approx.

At understock, i.e., $S/H = 1$, the asymptotic approximation can be very poor but the moment bounds are much better.

% Error	Min	Average	Max
Approx.	5.1 %	24.5 %	44.1 %
LB	2.6 %	6.9 %	14.4 %
UB	4.0 %	8.4 %	15.9 %

Table 4.7: Expected recruitment time $E[T]$ when understock ($S/H = 1$): asymptotic approx. vs. moment bounds

4.6 Conclusion

We define and characterize a new class of inventory control problem - the recruitment stocking problem, which can be found in clinical trials, marketing research/new product launch, as well as inventory management for end-of-life-cycle products. The recruitment stocking problem differs from previous research conducted in the classical inventory management literature. With this unique feature under consideration, performance evaluation and inventory allocation for this system are not known. In this chapter, we are limited to discussion on performance evaluation, i.e., we propose both exact and approximation methods to measure key performance metrics for the system: Type I

and II service levels and recruitment time. The question of how to allocate inventory for this system is for our future research. We can also generalize the recruitment stocking problems in multiple directions when the system is multi-echelon and adapt different type of inventory policies at each level, or the arrival process is more general than Poisson type. The results in this thesis will serve as a foundation for further extension.

4.7 Appendix

In Section 4.3.2, we omit the details of how to obtain the formula (4.64) for the relaxation theorem in the general case when rejections happen at more than one location. For the sake of completeness, the step-by-step derivation is presented below. The key idea to rewrite the expression (4.61) is to relate the first term in that expression with every summand within the summation. In particular, we are interested in the following ratio:

$$\frac{P_{H+r-1}\{X'_{i_j} = s_{i_j} + r_{i_j},\ j \neq t; X'_{i_t} = s_{i_t} + r_{i_t} - 1; X'_m \leq s_m,\ m \neq i_1, \ldots, i_k\}}{P_{H+r}\{X_{i_j} = s_{i_j} + r_{i_j},\ j \neq t; X_{i_t} = s_{i_t} + r_{i_t}; X_m \leq s_m,\ m \neq i_1, \ldots, i_k\}}. \tag{4.93}$$

The meaning of the probabilities in the numerator and denominator is explained in Section 4.3.2. In the next step, we condition on the location t that recruits the last subject and simplify the above ratio in a few steps.

$$\frac{P_{H+r-1}\{X'_{i_j} = s_{i_j} + r_{i_j},\ j \neq t; X'_{i_t} = s_{i_t} + r_{i_t} - 1; X'_m \leq s_m,\ m \neq i_1, \ldots, i_k\}}{P_{H+r}\{X_{i_j} = s_{i_j} + r_{i_j},\ j \neq t; X_{i_t} = s_{i_t} + r_{i_t}; X_m \leq s_m,\ m \neq i_1, \ldots, i_k\}}$$

$$= \frac{P_{H+r-1}\{X'_{i_j} = s_{i_j} + r_{i_j},\ j \neq t; X'_m \leq s_m,\ m \neq i_1, \ldots, i_k | X'_{i_t} = s_{i_t} + r_{i_t} - 1\} P_{H+r-1}(X'_{i_t} = s_{i_t} + r_{i_t} - 1))}{P_{H+r}\{X_{i_j} = s_{i_j} + r_{i_j},\ j \neq t; X_m \leq s_m,\ m \neq i_1, \ldots, i_k | X_{i_t} = s_{i_t} + r_{i_t}\} P_{H+r}\{X_{i_t} = s_{i_t} + r_{i_t}\}}$$

$$= \frac{P_{H+r-s_{i_t}-r_{i_t}}\{X'_{i_j} = s_{i_j} + r_{i_j},\ j \neq t\ X'_m \leq s_m,\ m \neq i_1, \ldots, i_k\} P_{H+r-1}\{X'_{i_t} = s_{i_t} + r_{i_t} - 1)\}}{P_{H+r-s_{i_t}-r_{i_t}}\{X_{i_j} = s_{i_j} + r_{i_j},\ j \neq t; X_m \leq s_m,\ m \neq i_1, \ldots, i_k\} P_{H+r}\{X_{i_t} = s_{i_t} + r_{i_t}\}}$$

$$= \frac{P_{H+r-1}\{X'_{i_t} = s_{i_t} + r_{i_t} - 1\}}{P_{H+r}\{X_{i_t} = s_{i_t} + r_{i_t}\}}$$

$$= \frac{s_{i_t} + r_{i_t}}{p_{i_t}(H + r)}$$

Note that $P_{H+r-1}\{X'_{i_t} = s_{i_t} + r_{i_t} - 1\}$ and $P_{H+r}\{X_{i_t} = s_{i_t} + r_{i_t}\}$ are binomial probabilities with parameters $H + r - 1$ and $H + r$, respectively, and probability $p_{i_t} = \lambda_{i_t} / \sum_{m=1}^M \lambda_m$. Equation (4.64) is easily obtained after taking the summation over t (the location that recruits the last subject) in the expression (4.61).

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