ON MAPPING PROBLEMS IN SEVERAL COMPLEX VARIABLES

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ABSTRACT OF THE DISSERTATION

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The thesis consists of two parts. In the first part, we study a regularity problem for CR mappings between CR manifolds. More precisely, we establish various versions of the Schwarz reflection principle in several complex variables. In particular, as a consequence of the main results, we confirm a conjecture of X. Huang in [Hu2] and provide a solution to a question raised by Forstneric [Fr1] (See Corollaries 2.1.11 and 2.1.12). It is a joint work with Shiferaw Berhanu ([BX1], [BX2]). In the second part, we study the embeddability problem from compact real algebraic strongly pseudoconvex hypersurfaces into a sphere. In a joint work with Xiaojun Huang and Xiaoshan Li ([HLX]), we prove that for any integer N, there is a family of compact real algebraic strongly pseudoconvex hypersurfaces in $\mathbb{C}^2$, none of which can be locally holomorphically embedded into the unit sphere in $\mathbb{C}^N$. This shows that the Whitney (or Remmert, respectively) type embedding theorem in differential topology (or in the Stein space theory, respectively) does not hold in the setting above.

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Dedication

To my parents Bendi Xiao, and Mengying Wang.
To my wife Bing Liu.

Table of Contents

Chapter 1
Introduction

In this chapter, we state preliminary notions and results that will be applied in Chapter 2 and 3.

1.1 Preliminaries on CR geometry

In this section, we recall basic notions from CR geometry. We introduce formally integrable and integrable structures on differential manifolds. Abstract CR manifolds are a special case of these structures. The Levi form and Levi map of an abstract CR manifold are also recalled. For more details on this material, we refer to the book [BER].

Definition 1.1.1. *Let M be a smooth manifold, and $\mathcal{V}$ be a subbundle of $\mathbb{C}TM$. We will say that $\mathcal{V}$ is formally integrable (or involutive) if the space of smooth sections $C^\infty(M, \mathcal{V})$ of $\mathcal{V}$ is closed under commutators, i.e., $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$. We will refer to $(M, \mathcal{V})$ as a formally integrable structure.*

An important subclass of the formally integrable subbundles are the integrable ones.

Definition 1.1.2. *A subbundle $\mathcal{V}$ of $\mathbb{C}TM$ of dimension n is integrable if for any $p \in M$ there exist $m = \dim_{\mathbb{R}} M - n$ smooth complex-valued functions $Z_1, ..., Z_m$ defined in an open neighborhood $\Omega \subset M$ of p with $\mathbb{C}-$linearly independent differentials $dZ_1, ..., dZ_m$ such that $LZ_j = 0$ for all $L \in \Gamma(M, \mathcal{V})$ and $j = 1, ...m$. For $p_0 \in M$ fixed, any such set of functions Z_j, vanishing at p_0, will be called a family of basic solutions in Ω.*

Proposition 1.1.3. *If $\mathcal{V}$ is integrable, then $(M, \mathcal{V})$ is a formally integrable structure.*

Now we recall the notion of CR structures.

Definition 1.1.4. *A formally integrable structure $(M, \mathcal{V})$ is called a formally CR structure if for all $p \in M, \mathcal{V}_p \cap \overline{\mathcal{V}}_p = \{0\}$. We shall refer to a formal CR structure as an abstract CR manifold, to $\mathcal{V}$ as its CR bundle. If a formal CR structure $(M, \mathcal{V})$ is furthermore integrable, we shall refer to it as an integrable CR structure or a locally embedded CR manifold.*

Moreover, recall that a smooth section of $\mathcal{V}$ will be called a CR vector field on M. A function (resp. distribution) on M is a CR function (resp. CR distribution) if it is annihilated by all the CR vector fields on M. The number $n = \dim_{\mathbb{C}} \mathcal{V}_p$ for any p will be called the CR dimension of M. If $\dim_{\mathbb{R}} M = m + n$, then $d = m - n$ will be called the CR codimension of M. In particular if $d = 1$, the CR structure is said to be of hypersurface type.

Let M be a real submanifold of $\mathbb{C}^N$. For $p \in M$, we denote by $\mathcal{V}_p$ the space of antiholomorphic vectors tangent to M at p, that is,

$$\mathcal{V}_p := T_p^{(0,1)}\mathbb{C}^N \cap \mathbb{C}T_pM.$$

Then $(M, \mathcal{V})$ defines an integrable CR structure if $\dim_{\mathbb{C}}\mathcal{V}_p$ is constant for $p \in M$. We will then call M a CR submanifold of $\mathbb{C}^N$.

Definition 1.1.5. *Let $(M, \mathcal{V}), (M', \mathcal{V}')$ be abstract CR manifolds. A CR mapping of class $C^k (k \geq 1) F : (M, \mathcal{V}) \to (M', \mathcal{V}')$ is a C^k mapping $F : M \to M'$ such that for all $p \in M, F_*(\mathcal{V}_p) \subset \mathcal{V}'_{F(p)}$, where F_* denotes the usual tangent map $F_* : T_pM \to T_{F(p)}M'$ induced by F.*

Let M' be CR submanifolds of $\mathbb{C}^N$ in the definition above, and write a C^k map $F : M \to M'$ as $F = (F_1, ..., F_N)$. Then F is a CR mapping if and only if each component $F_j, j = 1, ..., N$, is a CR function.

Let $(M, \mathcal{V})$ be an abstract CR manifold, where $\mathcal{V}$ is the CR bundle of M. For $p \in M$, we denote by π_p the natural quotient map

$$\pi_p : \mathbb{C}T_pM \to \mathbb{C}T_pM/(\mathcal{V}_p \oplus \overline{\mathcal{V}}_p).$$

Definition 1.1.6. *The Levi map at $p \in M$ is the Hermitian vector valued form*

$$\mathcal{L}_p : \mathcal{V}_p \times \mathcal{V}_p \to \mathbb{C}T_pM/(\mathcal{V}_p \oplus \overline{\mathcal{V}}_p),$$
$$\mathcal{L}_p(X_p, Y_p) := \frac{1}{2\sqrt{-1}}\pi_p([X, \overline{Y}](p)),$$

where X and Y are CR vector fields on M extending the CR vectors X_p and Y_p.

Let Levi map $\mathcal{L}_p$ is called nondegenerate if $\mathcal{L}_p(X_p, Y_p) = 0$ for all $Y_p \in \mathcal{V}_p$ implies $X_p = 0$. If $(M, \mathcal{V})$ is of hypersurface type, then the Levi map at p is a Hermitian form on $\mathcal{V}_p$, called the Levi form. In this case, we say that $(M, \mathcal{V})$ is Levi nondegenerate at p if the Levi form is nondegenerate. Furthermore, a CR manifold of hypersurface type, $(M, \mathcal{V})$, is called pseudoconvex at p_0 if the Levi form is positive definite (or negative definite) at all p in an open neighborhood of p_0. Similarly, $(M, \mathcal{V})$ is said to be strictly pseudoconvex at $p_0 \in M$ if the Levi form is positive (or negative) definite at $p_0 \in M$.

Proposition 1.1.7. *Let $M \subset \mathbb{C}^N$ be a smooth hypersurface with $p_0 \in M$, and let $\rho(Z, \overline{Z})$ be a local defining function for M near p_0. Then, M is pseudoconvex at p_0 if and only if, for all p in an open neighborhood of p_0 in M, either*

$$\sum_{j,k=1}^{N} \frac{\partial^2 \rho}{\partial Z_j \partial \overline{Z}_k}(p, \overline{p}) a_j \overline{a}_k \leq 0$$

for all $a = (a_1, ..., a_N)$ with $\sum_{j=1}^{N} \frac{\partial \rho}{\partial Z_j}(p, \overline{p}) a_j = 0$, or

$$\sum_{j,k=1}^{N} \frac{\partial^2 \rho}{\partial Z_j \partial \overline{Z}_k}(p, \overline{p}) a_j \overline{a}_k \geq 0$$

for all $a = (a_1, ..., a_N)$ with $\sum_{j=1}^{N} \frac{\partial \rho}{\partial Z_j}(p, \overline{p}) a_j = 0$.

1.2 Preliminaries on microlocal analysis

In this subsection, we present some preliminaries on microlocal analysis from standard literature without providing proofs. Most theorems and their detailed proofs can be found in [BCH]. First we recall the FBI transform which is a nonlinear Fourier transform which characterizes analyticity and regularity. In the following context, we let $\mathcal{D}'$ be the space of distributions (topological dual of C_0^∞), and let $\mathcal{E}'$ be the space of distributions with compact support (topological dual of C^∞).

Definition 1.2.1. *Let $u \in \mathcal{E}'(\mathbb{R}^m)$. Define the FBI transform of u by*

$$F_u(x,\xi) = \int e^{\sqrt{-1}(x-y)\cdot\xi - |\xi|(x-y)^2} u(y)dy \tag{1.2.1}$$

for $(x,\xi) \in \mathbb{R}^m \times \mathbb{R}^m$. where

$$(x-y)\cdot\xi = \sum_{i=1}^{m}(x_i - y_i)\xi_i.$$

The integral is to be understood in the duality sense.

The following characterization of analyticity by means of an exponential decay of the FBI transform may be viewed as an analogue of the Paley-Wiener Theorem.

Theorem 1.2.2. *Let $u \in \mathcal{E}'(\mathbb{R}^m)$. The following are equivalent:*

(i) u is real-analytic at $x_0 \in \mathbb{R}^m$.

(ii) There exists a neighborhood V of x_0 in $\mathbb{R}^m$ and constants $c_1, c_2 > 0$ such that

$$|F_u(x,\xi)| \le c_1 e^{-c_2|\xi|}$$

for $(x,\xi) \in V \times \mathbb{R}^m$.

We consider now the boundary values of holomorphic functions defined on wedges with flat edges. that is, edges that are open subsets of $\mathbb{R}^m$. Let $\Gamma \subset \mathbb{R}^m \setminus \{0\}$ be an open convex cone with vertex at the origin, $V \subset \mathbb{R}^m$ open. For $\delta > 0$, let

$$\Gamma_\delta = \Gamma \cap \{v : |v| < \delta\}.$$

Definition 1.2.3. *A holomorphic function $f \in H(V + \sqrt{-1}\Gamma_\delta)$ is said to be of tempered growth if there is an integer k and a constant c such that*

$$|f(x + \sqrt{-1}y)| \le \frac{c}{|y|^k}.$$

We can define the boundary value of f if it is of tempered growth:

Theorem 1.2.4. *Suppose $f \in H(V + \sqrt{-1}\Gamma_\delta)$ is of tempered growth and k is as in definition above. Set*

$$< f_v, \phi >= \int f(x + \sqrt{-1}v)\phi(x)dx$$

for $\phi \in C_0^\infty(V)$ and $v \in \Gamma$. Then

$$bf = \lim_{v \to 0, v \in \Gamma' \subset\subset \Gamma} f_v$$

exists in $\mathcal{D}'(V)$ *and is of order* $k+1$.

Distributions which are boundary values of holomorphic functions of tempered growth arise quite naturally. Indeed, we have:

Theorem 1.2.5. *Any* $u \in \mathcal{E}(\mathbb{R}^m)$ *can be expressed as a finite sum* $\sum_{j=1}^{n} bf_j$ *where each* $f_j \in H(\mathbb{R}^m + \sqrt{-1}\Gamma_j')$ *for some cones* Γ_j' *where each* $f_j \in H(\mathbb{R}^m + \sqrt{-1}\Gamma_j')$ *for some cones* $\Gamma_j' \subset \mathbb{R}^m$, *and the* f_j *are of tempered growth.*

Definition 1.2.6. *Let* $u \in \mathcal{D}'(\mathbb{R}^m), x_0 \in \mathbb{R}^m, \xi^0 \in \mathbb{R}^m \setminus \{0\}$. *We say that* u *is microlocally analytic at* (x_0, ξ^0) *if there exist a neighborhood* V *of* x_0, *cones* $\Gamma^1, ..., \Gamma^N$ *in* $\mathbb{R}^m \setminus \{0\}$, *and holomorphic functions* $f_j \in H(V + \sqrt{-1}\Gamma_\delta^j)$ *(for some* $\delta > 0$*) of tempred growth such that* $u = \sum_{j=1}^{N} bf_j$ *near* x_0 *and* $\xi^0 \cdot \Gamma^j < 0 \ \forall j$.

Now recall the definition of wave front set (See [BCH], [H], for instance),

Definition 1.2.7. *The analytic wave front set of a distribution* u, *denoted* $WF_a(u)$, *is defined by,*

$$WF_a(u) = \{(x, \xi) : u \text{ is not microlocally analytic at } (x, \xi)\}.$$

It can be easily seen that the analytic wave front set is invariant under an analytic diffeomorphism, and hence microlocal analyticity can be defined on any real-analytic manifold. We recall the following theorem which will provide a useful criterion for microlocal analyticity in terms of FBI transform:

Theorem 1.2.8. *Let* $u \in \mathcal{D}'(\mathbb{R}^m), x_0 \in \mathbb{R}^m, \xi^0 \in \mathbb{R}^m \setminus \{0\}$. *Then* $(x_0, \xi^0) \notin WF_a(u)$ *if and only if there is a neighborhood* V *of* x_0 *in* $\mathbb{R}^m$, *an open cone* $\Gamma \subset \mathbb{R}^m \setminus \{0\}, \xi \in \Gamma$ *and constants* $c_1, c_2 > 0$ *such that*

$$|F_u(x, \xi)| \leq c_1 e^{-c_2|\xi|} \ \forall (x, \xi) \in V \times \Gamma.$$

We also mention some corollaries will be applied in our work later.

Corollary 1.2.9. *A distribution* u *is analytic near* x_0 *if and only if for every* $\xi^0 \in \mathbb{R}^m \setminus \{0\}, (x_0, \xi^0) \notin WF_a(u)$.

Corollary 1.2.10. *(The edge-of-the-wedge theorem) Let* $V \subset \mathbb{R}^m$ *be a neighborhood of the point* p, *and* Γ^+, Γ^- *be cones such that* $\Gamma^- = -\Gamma^+$. *Suppose for some* $\delta > 0, f^+ \in H(V + \sqrt{-1}\Gamma_{\delta+}), f^- \in H(V + \sqrt{-1}\Gamma_\delta^-)$ *are both of tempered growth and* $bf^+ = bf^-$. *Then there exists a holomorphic function* f *defined in a neighborhood of* p *that extends both* f^+ *and* f^-. *In particular,* bf^+ *is analytic at* p.

We also have analogue in the smooth category. We first recall Paley-Wiener's Theorem:

Theorem 1.2.11. *A distribution* u *with support in the ball* $\{x \in \mathbb{R}^m : |x| \leq R\}$ *is* C^∞ *if and only if* $\hat{u}(\zeta)$ *is entire on* $\mathbb{C}^m$ *and for each positive integer* k *there is* C_k *such that*

$$|\hat{u}(\zeta)| \leq C_k \frac{e^{R|\mathrm{Im}\zeta|}}{(1 + |\zeta|)^k} \ \forall \zeta \in \mathbb{C}^m.$$

We now recall the definition of microlocal smoothness.

Definition 1.2.12. *Let $u \in \mathcal{D}'(\Omega), \Omega \subset \mathbb{R}^m$ open, $x_0 \in \Omega$, and $\xi^0 \in \mathbb{R}^m \backslash \{0\}$. We say u is microlocally smooth at (x_0, ξ_0) if there exists $\phi \in C_0^\infty(\Omega), \phi \equiv 1$ near x_0 and a conic neighborhood $\Gamma \subset \mathbb{R}^m \setminus \{0\}$ of ξ^0 such that for all $k = 1, 2...$ and for all $\zeta \in \Gamma$,*

$$|\widehat{\phi u}(\zeta)| \leq \frac{C_k}{(1 + |\xi|)^k} \text{ on } \Gamma.$$

Definition 1.2.13. *The C^∞ wave front set of a distribution u denoted $WF(u)$ is defined by*

$$WF(u) = \{(x, \xi) : u \text{ is not microlocally smooth at } (x, \xi)\}.$$

It is easy to see that a distribution u is C^∞ if and only if $WF(u) = \emptyset$. When a distribution u is a solution of a linear partial differential equation with smooth coefficients, its wave front set is constrained.

Theorem 1.2.14. *Let $P = \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha$ be a smooth linear differential operator on an open set $\Omega \subset \mathbb{R}^m$ and suppose $u \in \mathcal{D}'(\Omega)$. Then*

$$WF(u) \subset \text{char} P \cup WF(Pu),$$

where the characteristic set

$$\text{char} P = \{(x, \xi) \in \Omega \times \mathbb{R}^m \setminus \{0\} : \sum_{|\alpha|=k} a_\alpha \xi^\alpha = 0\}.$$

We recall the notion of almost analytic extension.

Definition 1.2.15. *Let $f \in C^\infty(\Omega), \Omega \subset \mathbb{R}^m$ open, and suppose $\tilde{\Omega}$ is a neighborhood of Ω in $\mathbb{C}^m$. A function $\tilde{f}(x, y) \in C^\infty(\tilde{\Omega})$ is called an almost analytic extension of $f(x)$ if $\tilde{f}(x, 0) = f(x)$ $\forall x \in \Omega$ and for each j=1,...,m,*

$$\frac{\partial \tilde{f}}{\partial \bar{z}_j}(x, y) = O(|y|^k) \text{ for } k = 1, 2, ...$$

The following theorem characterizes microlocal smoothness in terms of almost analytic extendability in certain wedge.

Theorem 1.2.16. *Let $u \in \mathcal{D}'(\mathbb{R}^m)$. Then $(x_0, \xi_0) \notin WF(u)$ if and only if there exist a neighborhood V of x_0, open acute cones $\Gamma^1, ..., \Gamma^N$ in $\mathbb{R}^m \setminus \{0\}$, and almost analytic functions f_j on $V + \sqrt{-1}\Gamma_\delta^j$ (for some δ) of tempered growth such that $u = \sum_1^N bf_j$ near x_0 and $\xi^0 \cdot \Gamma^j < 0$ for all j.*

1.3 Some algebraic lemmas

In this section, we will prove some algebraic lemmas that will be applied in Chapter 2.

Lemma 1.3.1. *For a general $n \times n$ matrix*

$$B = \begin{pmatrix} b_{11} & b_{12} & . & . & . & b_{1n} \\ b_{21} & b_{22} & . & . & . & b_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ b_{n1} & b_{n2} & . & . & . & b_{nn} \end{pmatrix},$$

where $b_{ij} \in \mathbb{C}$ for all $1 \leq i, j \leq n, n \geq 3$, we have,

$$\begin{vmatrix} B\begin{pmatrix} 1 & 2 & . & . & . & n-2 & n-1 \\ 1 & 2 & . & . & . & n-2 & n-1 \end{pmatrix} & B\begin{pmatrix} 1 & 2 & . & . & . & n-2 & n-1 \\ j_1 & j_2 & . & . & . & j_{n-2} & n \end{pmatrix} \\ B\begin{pmatrix} i_1 & i_2 & . & . & . & i_{n-2} & n \\ 1 & 2 & . & . & . & n-2 & n-1 \end{pmatrix} & B\begin{pmatrix} i_1 & i_2 & . & . & . & i_{n-2} & n \\ j_1 & j_2 & . & . & . & j_{n-2} & n \end{pmatrix} \end{vmatrix} \quad (*)$$

$= B\begin{pmatrix} i_1 & i_2 & . & . & . & i_{n-2} \\ j_1 & j_2 & . & . & . & j_{n-2} \end{pmatrix}|B|$, *for any $1 \leq i_1 < i_2 < \cdots < i_{n-2} \leq n-1, 1 \leq j_1 < j_2 < \cdots < j_{n-2} \leq n-1$. In particular, if $|B| = 0$, then $(*)$ equals 0. Here we have used the notation*

$$B\begin{pmatrix} i_1 & i_2 & . & . & . & i_p \\ j_1 & j_2 & . & . & . & j_p \end{pmatrix} = \begin{vmatrix} b_{i_1j_1} & b_{i_1j_2} & . & . & . & b_{i_1j_p} \\ b_{i_2j_1} & b_{i_2j_2} & . & . & . & b_{i_2j_p} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ b_{i_pj_1} & b_{i_pj_2} & . & . & . & b_{i_pj_p} \end{vmatrix} \text{ for } 1 \leq p \leq n.$$

To prove Lemma 1.3.1, we need the following Lemmas.

Lemma 1.3.2. *Assume $p \geq 3$, C is a $p \times p$ matrix,*

$$C = \begin{pmatrix} c_{11} & \cdots & c_{1p} \\ \cdots & \cdots & \cdots \\ c_{p1} & \cdots & c_{pp} \end{pmatrix},$$

where $c_{ij} \in \mathbb{C}$ for all $1 \leq i, j \leq p$. Then

$$c_{11}{}^{p-2}|C| = |\widetilde{C}|, \tag{1.3.1}$$

where $\widetilde{C}$ is a $(p-1) \times (p-1)$ matrix given by

$$\widetilde{C} = \begin{pmatrix} \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} & \cdots & \begin{vmatrix} c_{11} & c_{1p} \\ c_{21} & c_{2p} \end{vmatrix} \\ \cdots & \cdots & \cdots \\ \begin{vmatrix} c_{11} & c_{12} \\ c_{p1} & c_{p2} \end{vmatrix} & \cdots & \begin{vmatrix} c_{11} & c_{1p} \\ c_{p1} & c_{pp} \end{vmatrix} \end{pmatrix}.$$

That is, $\widetilde{C} = (\widetilde{c}_{ij})_{1 \le i \le (p-1), 1 \le j \le (p-1)}$, *with* $\widetilde{c}_{ij} = \begin{vmatrix} c_{11} & c_{1(j+1)} \\ c_{(i+1)1} & c_{(i+1)(j+1)} \end{vmatrix}$.

Proof. When $c_{11} = 0$, (1.3.1) holds since both sides equal 0. Now assume $c_{11} \neq 0$. By eliminating $c_{21}, \cdots, c_{p1}$, we get,

$$|C| = \begin{vmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ 0 & c_{22} - c_{12}\frac{c_{21}}{c_{11}} & \cdots & c_{2p} - c_{1p}\frac{c_{21}}{c_{11}} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & c_{p2} - c_{12}\frac{c_{p1}}{c_{11}} & \cdots & c_{pp} - c_{1p}\frac{c_{p1}}{c_{11}} \end{vmatrix} = c_{11}{}^{-(p-2)} |\widetilde{C}|. \qquad \square$$

Lemma 1.3.3. *If the determinant of a* 3×3 *matrix*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0,$$

where $a_{ij} \in \mathbb{C}$ *for all* $1 \le i, j \le 3$. *Then*

$$\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} =$$

$$\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} = 0.$$

Proof. Using Lemma 1.3.2,

$$\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix} = a_{11} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

$$\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} = a_{12} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

$$\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix} = a_{21} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

$$\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} = a_{22} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

$\square$

Proof of Lemma 1.3.1 : We proceed by induction on the dimension of B. From Lemma 1.3.3, we know Lemma 1.3.1 holds for $n = 3$. Now assume that it holds when the dimension of B is less than or equal to $n-1$. To prove it when the dimension is n, it is enough to show it for the case when $i_1 = 1, i_2 = 2, \cdots, i_{n-2} = n-2$ and $j_1 = 1, j_2 = 2, \cdots, j_{n-2} = n-2$. Namely, we show that

$$\begin{vmatrix} \begin{vmatrix} b_{11} & b_{12} & \cdot & \cdot & \cdot & b_{1n-1} \\ b_{21} & b_{22} & \cdot & \cdot & \cdot & b_{2n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n-11} & b_{n-12} & \cdot & \cdot & \cdot & b_{n-1n-1} \end{vmatrix} & \begin{vmatrix} b_{11} & \cdot & \cdot & \cdot & b_{1n-2} & b_{1n} \\ b_{21} & \cdot & \cdot & \cdot & b_{2n-2} & b_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n-11} & \cdot & \cdot & \cdot & b_{n-1n-2} & b_{n-1n} \end{vmatrix} \\ \\ \begin{vmatrix} b_{11} & b_{12} & \cdot & \cdot & \cdot & b_{1n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n-21} & b_{n-22} & \cdot & \cdot & \cdot & b_{n-2n-1} \\ b_{n1} & b_{n2} & \cdot & \cdot & \cdot & b_{nn-1} \end{vmatrix} & \begin{vmatrix} b_{11} & \cdot & \cdot & \cdot & b_{1n-2} & b_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n-21} & \cdot & \cdot & \cdot & b_{n-2n-2} & b_{n-2n} \\ b_{n1} & \cdot & \cdot & \cdot & b_{nn-2} & b_{nn} \end{vmatrix} \end{vmatrix}$$

$= B\left(\begin{matrix} 1 & 2 & \cdots & n-2 \\ 1 & 2 & \cdots & n-2 \end{matrix} \right)|B|$, and the other cases are similar.

Now we view all terms here as rational functions in $b_{11}, \cdots, b_{nn}$. By Lemma 1.3.2,

$$|B| = {b_{11}}^{-(n-2)} \begin{vmatrix} B(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}) & \cdots & B(\begin{smallmatrix} 1 & 2 \\ 1 & n \end{smallmatrix}) \\ \cdots & \cdots & \cdots \\ B(\begin{smallmatrix} 1 & n \\ 1 & 2 \end{smallmatrix}) & \cdots & B(\begin{smallmatrix} 1 & n \\ 1 & n \end{smallmatrix}) \end{vmatrix} \tag{1.3.2}$$

By applying Lemma 1.3.2 and the induction hypothesis, it follows that

$$\begin{vmatrix} B(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}) & \cdots & B(\begin{smallmatrix} 1 & 2 \\ 1 & n \end{smallmatrix}) \\ \cdots & \cdots & \cdots \\ B(\begin{smallmatrix} 1 & n \\ 1 & 2 \end{smallmatrix}) & \cdots & B(\begin{smallmatrix} 1 & n \\ 1 & n \end{smallmatrix}) \end{vmatrix} = \left(B(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}) \right)^{-(n-3)} {b_{11}}^{n-2} \begin{vmatrix} B(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}) & \cdots & B(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & n \end{smallmatrix}) \\ \cdots & \cdots & \cdots \\ B(\begin{smallmatrix} 1 & 2 & n \\ 1 & 2 & 3 \end{smallmatrix}) & \cdots & B(\begin{smallmatrix} 1 & 2 & n \\ 1 & 2 & n \end{smallmatrix}) \end{vmatrix}.$$

Combining it with (1.3.2), we obtain

$$|B| = \left(B(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}) \right)^{-(n-3)} \begin{vmatrix} B(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}) & \cdots & B(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & n \end{smallmatrix}) \\ \cdots & \cdots & \cdots \\ B(\begin{smallmatrix} 1 & 2 & n \\ 1 & 2 & 3 \end{smallmatrix}) & \cdots & B(\begin{smallmatrix} 1 & 2 & n \\ 1 & 2 & n \end{smallmatrix}) \end{vmatrix}.$$

By further applications of Lemma 1.3.2 and the induction hypothesis as above, we arrive at the conclusion.

Finally we state the following simple lemma:

Lemma 1.3.4. *Let* $\mathbf{b}_1, \cdots, \mathbf{b}_n$ *and* $\mathbf{a}$ *be* n*-dimensional column vectors with elements in* $\mathbb{C}$*, and let* $B = (\mathbf{b}_1, \cdots, \mathbf{b}_n)$ *denote the* $n \times n$ *matrix. Assume that* $\det B \neq 0$, *and that* $\det(\mathbf{b}_{i_1}, \mathbf{b}_{i_2}, \cdots, \mathbf{b}_{i_{n-1}}, \mathbf{a}) = 0$ *for any* $1 \leq i_1 < i_2 < \cdots < i_{n-1} \leq n$*. Then* $\mathbf{a} = \mathbf{0}$, *where* $\mathbf{0}$ *is the* n*-dimensional zero column vector.*

Proof. Note that $\{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ is a linearly independent set in $\mathbb{C}^n$. Write $\mathbf{a} = \sum_{j=1}^{n} \lambda_j \mathbf{b}_j$ for some $\lambda_j \in \mathbb{C}, 1 \leq j \leq n$. It is easy to see that all the $\lambda_j = 0$ by using the assumption that $\det(\mathbf{b}_{i_1}, \mathbf{b}_{i_2}, \cdots, \mathbf{b}_{i_{n-1}}, \mathbf{a}) = 0, \forall 1 \leq i_1 < i_2 < \cdots < i_{n-1} \leq n$. $\square$

Chapter 2

Reflection principle problems in several complex variables

In one complex variable, the classical Schwarz reflection principle can be formulated as follows. Let M and M' be two real analytic (resp. smooth) curves in $\mathbb{C}$ and F a holomorphic function defined on one side of M. Assume that F extends continuously to M, and maps M to M'. Then F has a holomorphic (resp. smooth) extension across M. The situation is much more subtle in several complex variables, and the analogous statement fails in general as shown by easy examples. For decades, finding conditions under which a reflection principle holds in higher dimensions has attracted considerable attention by numerous researchers. More precisely, the question can be formulated as follows.

Question 2.0.5. *Let $M \subset \mathbb{C}^n, M' \subset \mathbb{C}^N$ be two germs of real analytic (resp. smooth) CR submanifolds. Let W be a wedge in $\mathbb{C}^n$ with edge M, and $F : W \to \mathbb{C}^N$ a holomorphic map, which extends continuously to M. Assume that F maps M to M'. Find conditions that imply that the reflection principle holds.*

Notice that a CR function on a CR submanifold $M \subset \mathbb{C}^n$ can be holomorphically extended to a wedge with edge M, under a certain geometric condition on M (See [Tr], [Tu1], for instance). Question 1.1 has its natural CR version as follows.

Question 2.0.6. *Let $M \subset \mathbb{C}^n, M' \subset \mathbb{C}^N$ be two germs of real analytic (resp. smooth) CR submanifolds and $F : M \to M'$ a CR mapping. Find conditions such that F is real analytic (resp. smooth).*

In this chapter we study along this line of the regularity problem for CR mappings between CR manifolds where the CR dimension of the source manifold is less than or equal to that of the target manifold based on the joint work with Shiferaw Berhanu ([BX1], [BX2]). A particular case of interest is when M and M' are both strongly pseudoconvex CR manifolds of hypersurface type, as indicated in Section 2.1. We also consider in Section 2.2 the more general case when the target is merely assumed to be Levi-nondegenerate.

2.1 CR mappings into a strongly pseudoconvex hypersurface

2.1.1 Main Results

Our results in this section imply a positive answer to a conjecture of X. Huang in [Hu2] and provide a solution to a question raised in [Fr1] (see Corollaries 2.1.11 and 2.1.12). One of our theorems can be viewed as a smooth version of the analyticity theorems of Forstneric ([Fr1]) and Huang [Hu1-2] for CR mappings between CR manifolds of differing dimensions. The chapter is devoted to results along the line of research on establishing the smooth version of the Schwarz reflection principle for holomorphic maps in several variables. Results of this type were first proved in the 70's starting with the work of Fefferman [Fe], Lewy [Le] and Pinchuk [Pi]. The seminal work [BJT] has influenced a lot of work on the subject. For extensive surveys and many references on this research, the reader may consult the articles by Bedford [Be], Forstneric [Fr2], and Bell-Narasimhan [BN]. Among the many related papers we mention here [CKS], [CGS], [CS], [DW], [EH], [EL], [Fr1], [Fr3], [Hu1], [Hu2], [KP], [K], [La1], [La2], [La3], [M], [NWY], [Tu], and [W]. In [Fr3] Forstneric generalized Fefferman's theorem to CR homeomorphisms $f : M \to M'$ where f^{-1} is CR, M and M' are generic CR submanifolds of $\mathbb{C}^n$ with the same CR dimension. The book [BER] by Baouendi, Ebenfelt, and Rothschild contains a detailed account and references related to the study when the manifolds are real analytic or real algebraic.

We prove results on the smoothness of CR maps where the source manifold M is assumed to be an abstract (not necessarily embeddable) CR manifold. We mention that the results are new even when M is embeddable. Our first main result, Theorem 2.1.3, generalizes to abstract CR manifolds a theorem of Lamel in [La1] proved for generic CR manifolds embedded in complex spaces. The second main result, Theorem 2.1.5, establishes the smoothness on a dense open subset of a C^k CR mapping $F : (M, \mathcal{V}) \to (M', \mathcal{V}')$ where $(M, \mathcal{V})$ is an abstract CR manifold of CR dimension n and $M' \subset \mathbb{C}^{n+k}$ is a hypersurface that is strongly pseudoconvex. A condition on the Levi form of $(M, \mathcal{V})$ is assumed in Theorem 2.1.5.

Our approach is based on the framework established by Roberts [GR] in his thesis and a later paper by Lamel in [La1]. The notion of k_0-nondegeneracy of a CR mapping (Definition 2.1.1) and the "almost holomorphic" implicit function theorem of Lamel in [La1] and [La2] play crucial roles in the proofs and formulations of our results. The proof is also motivated by the study of the real analyticity for CR maps between real analytic strongly pseudoconvex hypersurfaces of different dimensions in Forstneric [Fr1] and Huang [Hu1]. We mention that in [Fr1], Forstneric conjectured that F must be real analytic when $M_1 \subset \mathbb{C}^{n+1}$ and $M_2 \subset \mathbb{C}^{n+k}$ $(k \geq 2, n \geq 1)$ are real analytic hypersurfaces with M_1 of finite type, M_2 strongly pseudoconvex, and he proved that this is indeed the case on a dense open set when F is smooth. The conjecture of Forstneric was settled by Huang ([Hu1]) who obtained the analyticity of F on a dense open subset assuming only that $F \in C^k$. The analyticity of F, when both M_1 and M_2 are as [Fr1] and when F is only C^k-smooth also follows from Theorem 2.1.5 in this chapter and Forstneric's analyticity result when F is smooth.

Let M be an abstract CR manifold with CR bundle $\mathcal{V}$. Recall that a smooth section of $\mathcal{V}$ is called a CR vector field and a function (or distribution) is called CR if $Lf = 0$ for any CR vector field L. The CR manifold $(M, \mathcal{V})$ is called locally embeddable if for any $p_0 \in M$, there exist m complex-valued C^∞ functions $Z_1, \cdots, Z_m$ defined near p_0 with $m = \dim_{\mathbb{R}} M - n,$ such that the Z_j are CR functions near p_0, and the differentials $dZ_1, \cdots, dZ_m$ are $\mathbb{C}-$linearly independent. In this case, the mapping

$$p \mapsto Z(p) = (Z_1(p), \cdots, Z_m(p)) \in \mathbb{C}^m = \mathbb{C}^{n+d}$$

is an immersion near p_0. Thus, if U is a small neighborhood of p_0, then $Z(U)$ is an embedded submanifold of $\mathbb{C}^m$ and is a generic CR submanifold of $\mathbb{C}^m$ whose induced CR bundle agrees with the push forward $Z_*(\mathcal{V})$ (see [BER] and [J] for more details).

Let $(M',\mathcal{V}')$ be another abstract CR manifold with CR dimension n' and CR codimension d'. When $(M',\mathcal{V}')$ is a generic CR submanifold of $\mathbb{C}^{N'}(N'=n'+d')$, then a C^k mapping $H=(H_1,\cdots,H_{N'}):M\to M'$ is a CR mapping if and only if each H_j is a CR function. One of our main results generalizes to an abstract CR manifold $(M,\mathcal{V})$ a regularity theorem of Lamel ([La1]) for CR mappings of embedded CR manifolds. We need to recall from [La1] the notion of nondegenerate CR mappings. Let $\widetilde{M}\subset\mathbb{C}^N$ and $\widetilde{M}'\subset\mathbb{C}^{N'}$ be two generic CR submanifolds of $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ respectively. If d and d' denote the real codimensions of $\widetilde{M}$ and $\widetilde{M}'$, then $n=N-d$ and $n'=N'-d'$ are the CR dimensions of $\widetilde{M}$ and $\widetilde{M}'$ respectively. Let $H:\widetilde{M}\to\widetilde{M}'$ be a CR mapping of class C^k.

Definition 2.1.1. *([La1]) Let $\widetilde{M},\widetilde{M}'$ and H be as above and $p_0\in\widetilde{M}$. Let $\rho=(\rho_1,\cdots,\rho_{d'})$ be local defining functions for $\widetilde{M}'$ near $H(p_0)$, and choose a basis $L_1,\cdots,L_n$ of CR vector fields for $\widetilde{M}$ near p_0. If $\alpha=(\alpha_1,\cdots,\alpha_n)$ is a multiindex, write $L^\alpha=L_1^{\alpha_1}\cdots L_n^{\alpha_n}$. Define the increasing sequence of subspaces $E_l(p_0)(0\le l\le k)$ of $\mathbb{C}^{N'}$ by*

$$E_l(p_0)=\operatorname{Span}_{\mathbb{C}}\{L^\alpha\rho_{\mu,Z'}(H(Z),\overline{H(Z)})|_{Z=p_0}:0\le|\alpha|\le l,1\le\mu\le d'\}.$$

Here $\rho_{\mu,Z'}=(\frac{\partial\rho_\mu}{\partial z'_1},\cdots,\frac{\partial\rho_\mu}{\partial z'_{N'}})$, and $Z'=(z'_1,\cdots,z'_{N'})$ are the coordinates in $\mathbb{C}^{N'}$. We say that H is k_0-nondegenerate at p_0 $(0\le k_0\le k)$ if

$$E_{k_0-1}(p_0)\neq E_{k_0}(p_0)=\mathbb{C}^{N'}.$$

The dimension of $E_l(p)$ over $\mathbb{C}$ will be called the l^{th} geometric rank of F at p and it will be denoted by $\operatorname{rank}_l(F,p)$.

For the invariance of this definition under the choice of the defining functions ρ_μ, the basis of CR vector fields and the choice of holomorphic coordinates in $\mathbb{C}^{N'}$, the reader is referred to [La2]. An intrinsic definition was presented in the paper [EL]. If M is a manifold for which the identity map is k_0-nondegenerate, then the manifold is called k_0-*nondegenerate*. This latter notion was introduced for embedded hypersurfaces in [BHR] and it is shown in [E] that it can be formulated for an abstract CR manifold. The reader is referred to these two references for a detailed treatment of this concept and its connection with holomorphic nondegeneracy in the sense of Stanton ([S]). In particular, in [BHR] and [E] it is shown that Levi-nondegeneracy of a CR manifold is equivalent to $1-$nondegeneracy. Thus the notion of k_0-nondegeneracy of a CR manifold can be viewed as a generalization of Levi nondegeneracy.

The main result in [La1] is as follows:

Theorem 2.1.2. *Let $M\subset\mathbb{C}^N, M'\subset\mathbb{C}^{N'}$ be smooth generic submanifolds of $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ respectively, $p_0\in M, H=(H_1,\cdots,H_{N'}):M\to M'$ a C^{k_0} CR map which is k_0-nondegenerate at p_0 and extends continuously to a holomorphic map in a wedge W with edge M. Then H is smooth in some neighborhood of p_0.*

Here recall that if $p_0 \in M$, $d =$ the CR codimension of M, and $U \subset \mathbb{C}^N$ is a neighborhood of p_0, a wedge W with edge M centered at p_0 is defined to be an open set of the form:

$$W = \{Z \in U : r(Z, \overline{Z}) \in \Gamma\},$$

where $\Gamma \subset \mathbb{R}^d$ is an open convex cone, and $r = (r_1, \cdots, r_d)$ are defining functions for M near p_0. We will prove the following generalization of Theorem 2.1.2.

Theorem 2.1.3. *Let $(M, \mathcal{V})$ be an abstract CR manifold and $M' \subset \mathbb{C}^{N'}$ a generic CR submanifold of $\mathbb{C}^{N'}$. Let $H = (H_1, \cdots, H_{N'}) : M \to M'$ be a CR mapping of class C^{k_0} which is k_0-nondegenerate at p_0 and assume that for some open convex cone $\Gamma \subset \mathbb{R}^d$,*

$$\mathrm{WF}(H_j)|_{p_0} \subset \Gamma, j = 1, \cdots, N'$$

where d is the CR codimension of M. Then H is C^∞ in some neighborhood of p_0.

Remark 2.1.4. *In Theorem 2.1.2, the assumption that H is the boundary value of a holomorphic function in a wedge implies the much weaker condition that $WF(H_j)|_{p_0} \subset \Gamma$ for some Γ as in Theorem 2.1.3. Indeed, in the embedded case as in Theorem 2.1.2, a CR function h on M is the boundary value of a holomorphic function in a wedge if and only if its hypo-analytic wave front set is contained in an acute cone which means that the FBI transform of h decays exponentially. Our assumption in Theorem 2.1.3 only requires the FBI transform to decay rapidly.*

In what follows, given a CR manifold $(M, \mathcal{V})$, T^0 will denote its characteristic bundle, that is, $T^0 = \{\sigma \in T^*M : \langle \sigma, L \rangle = 0$ for every smooth section L of $\mathcal{V}\}$.

Theorem 2.1.5. *Let $(M, \mathcal{V})$ be an abstract CR manifold with CR dimension $n \geq 1$ such that the Levi form at every covector $\sigma \in T^0$ has a nonzero eigenvalue. Suppose $M' \subset \mathbb{C}^{n+k}$ is a hypersurface that is strongly pseudoconvex ($k \geq 1$) and let $\mathcal{V}'$ denote the CR bundle of M'. Let $F = (F_1, \cdots, F_{n+k}) : M \to M'$ be a CR mapping of class C^k whose differential $dF : \mathcal{V}_p \to \mathcal{V}'_{F(p)}$ is injective at every $p \in M$. Then F is C^∞ on a dense open subset of M.*

We note that the preceding theorem allows a weakening of the smoothness assumption in Theorem 1.2 of [EL] on finite jet determination. The theorem also implies that some of the results in [BR] hold under a weaker smoothness assumption on the CR maps involved. If $M \subset \mathbb{C}^N, M' \subset \mathbb{C}^{N'}$ are hypersurfaces, with M Levi non degenerate at $p \in M$ and $F : M \to M'$ is a CR mapping which is transversal at p, that is, $dF(\mathbb{C}T_pM)$ is not contained in $\mathcal{V}'_{F(p)} + \overline{\mathcal{V}'_{F(p)}}$, then F is a local embedding (see section 3.4 in [EL]). Many other situations where $(M, \mathcal{V})$ and $(M', \mathcal{V}')$ are as in Theorem 2.1.5 and dF is injective can be found in the work [BR].

Let M, M', F be as in Theorem 2.1.5.

One will see from Lemma 2.1.17 that $\mathrm{rank}_1(F, p) = n + 1$. It allows us to give the following definition, which has appeared in [EL] for the embedded case.

Definition 2.1.6. *If $F : M \to M'$ is a C^k CR mapping, M of CR dimension n and $p \in M$ is a point for which there is a neighborhood O and integers $1 \leq l, k_0 \leq k$ such that $\operatorname{rank}_{k_0}(F, q) = \operatorname{rank}_k(F, q) = n + l$ and $\operatorname{rank}_{k_0-1}(F, q) < n + l$ for all $q \in O$, we will say that F is of constant geometric rank $(k_0, n + l)$ at p (or simply say F is of constant geometric rank $n + l$ at p).*

Remark 2.1.7. *We have the following properties of Definition 2.1.6.*

- *Definition 2.1.6 are both independent of the choices of the defining function, the basis of CR vector fields and the choice of holomorphic coordinates in $\mathbb{C}^{n+k}$.*
- *It is easy to show that(see [EL], for instance), if F be of constant geometric rank $n + l$ at p, then $k_0 \leq l$, i.e., $\operatorname{rank}_l(F, q) = n + l$.*

Lemma 2.1.8. *There exists a dense open set M_0 of M such that for each point $p \in \Omega, F$ is of constant geometric rank at p.*

Proof. We will leave it to the readers. See also in [EL], for instance. □

Theorem 2.1.5 will be implied by the following,

Theorem 2.1.9. *Let M, M', F be as above and $p \in M$. Assume F is of constant geometric rank $n+l$ at p, then F is smooth near p. As a consequence, F is smooth in M_0, where M_0 be as in Lemma 2.1.8.*

Theorem 2.1.9 will follow from Theorem 2.1.3 for the nondegenerate case (in the sense of Definition 2.1.1) and follow from Theorem 2.1.20 in the degenerate case.

Before we present the proofs of Theorem 2.1.3 and Theorem 2.1.5, we will prove the following result which supplies a class of examples to which Theorem 2.1.3 applies. This theorem will also be used in the proof of Theorem 2.1.5. The result may be viewed as the smooth version of Hans Lewy's extendability theorem in the embedded case.

Theorem 2.1.10. *Let $(M, \mathcal{V})$ be an abstract CR manifold, $\sigma \in T_p^0$, with the property that the Levi form at σ has a negative eigenvalue. Then if u is a CR function (or distribution) near $p, \sigma \notin \mathrm{WF}(u)$. In particular, if the Levi form at every covector $\eta \in T_p^0$ has a nonzero eigenvalue, then there is an open convex cone $\Gamma \subset \mathbb{R}^d (d =$ the CR codimension of $M)$ such that for every CR function u near p, $\mathrm{WF}(u)|_p \subset \Gamma$.*

Theorem 2.1.5 implies the following corollary which settles Huang's conjecture in [Hu2]:

Corollary 2.1.11. *Let $M \subset \mathbb{C}^{n+1}$, $M' \subset \mathbb{C}^{n+k}$ be smooth strongly pseudoconvex real hypersurfaces with $n \geq 1, k \geq 1$. Let $F : M \to M'$ be a CR mapping of class C^k. Then $F \in C^\infty(\Omega)$ on a dense open subset $\Omega \subset M$.*

Theorem 2.1.5 also provides a solution to a question of Forstneric in [Fr1] using methods different from the ones employed by Huang in the solution that he gave in [Hu1]:

Corollary 2.1.12. *Let $M \subset \mathbb{C}^N$, $M' \subset \mathbb{C}^{N'}$ be real analytic hypersurfaces $(1 < N < N')$ M of finite type (in D'Angelo's sense) and M' strongly pseudoconvex. If $F : M \to M'$ is a CR mapping of class $C^{N'-N+1}$, then F extends to a holomorphic map on a neighborhood of an open, dense subset of M.*

Proof. Let $p \in M$. If every neighborhood of p contains a point where the Levi form has a positive and a negative eigenvalue, then p is in the closure of the set where F is smooth. We may therefore assume that a neighborhood D of p is pseudoconvex. Note next that since M doesn't contain a complex variety of positive dimension, it can not be Levi flat in any neighborhood of p. We can therefore assume that p is in the closure of the set of strictly pseudoconvex points in M. This latter assertion can be seen by using the arguments in Lemma 6.2 in [BHR]. In that paper, M was assumed algebraic but the reasoning in the Lemma is valid for M as in this corollary. The corollary now follows from Theorem 2.1.5 and the analyticity theorem in [Fr1]. $\square$

Example 2.1.13. *Let $M = \{(z_1, z_2) \in \mathbb{C}^2 : Im\, z_2 = |z_1|^{2m}\}$ where m is a positive integer and let $M' = \{(z_1, z_2) \in \mathbb{C}^2 : Im\, z_2 = |z_1|^2\}$. Then the map $H(z_1, z_2) = (z_1^m, z_2)$ is $1-$nondegenerate at the points where $z_1 \neq 0$, and $m-$nondegenerate at all the other points. When $m > 1$, M itself is $1-$nondegenerate at the points where $z_1 \neq 0$ while when $z_1 = 0$, it is not $l-$nondegenerate for any $l \geq 0$. (The case $m = 1$ appeared in [La1]. See also [K]).*

Example 2.1.14. *Let $M = \{(z_1, z_2) \in \mathbb{C}^2 : Im\, z_2 = |z_1|^2\}$ and $M' = \{(w_1, w_2, w_3, w_4) \in \mathbb{C}^4 : Im\, w_4 = |w_1|^2 + |w_2|^2 - |w_3|^2\}$. For any odd positive integer $m \geq 3$, define $H_m(z_1, z_2) : M \to M$ by $H_m(z_1, z_2) = (z_1, z_2^{\frac{m}{2}}, z_2^{\frac{m}{2}}, z_2)$ where we have used a branch of the square root. H_m is a CR mapping and it is the boundary value of a holomorphic map defined on a side of M. H_m is a diffeomorphism. H_m is not smooth and so for each positive integer k, there is m such that H_m is in C^k but it is not $k-$nondegenerate.*

Example 2.1.15. *Let $M = \{(z_1, z_2) \in \mathbb{C}^2 : Im\, z_2 = |z_1|^2\}$ and $M' = \{(w_1, w_2) \in \mathbb{C}^3 : Im\, w_3 = |w_1|^2 - |w_2|^2\}$. For any positive integer m, let $f : M \to \mathbb{C}$ be a CR function of class C^m which is not smooth on any open subset of M (see Section 2.2 for an example of such). Define $H_m : M \to M'$ by $H_m(z_1, z_2) = (f(z_1), f(z_2), 0)$. H_m is a CR mapping of class C^m which is not smooth on any open subset of M. Note that H_m is not $k-$nondegenerate for any k.*

2.1.2 Proof of Theorem 2.1.9

We now present the proof of Theorem 2.1.9.

Proof. Recall that the Levi form of $(M, \mathcal{V})$ at the characteristic covector $\sigma \in T_p^0$ is the hermitian form on $\mathcal{V}$ defined by

$$\mathcal{L}_\sigma(v, w) = \frac{1}{2\sqrt{-1}} \langle \sigma, [L, \overline{L}']_p \rangle,$$

where L and L' are smooth sections of $\mathcal{V}$ defined near p with $L(p) = v, L'(p) = w$. When this form has a negative eigenvalue, there is a CR vector L near p such that

$$\frac{1}{2\sqrt{-1}} \langle \sigma, [L, \overline{L}]_p \rangle < 0.$$

We may therefore assume that we are in coordinates $(x, t) \in \mathbb{R}^{n_0} \times \mathbb{R}$ that vanish at p,

$$L = \frac{\partial}{\partial t} + \sqrt{-1} \sum_{j=1}^{n_0} b_j(x, t) \frac{\partial}{\partial x_j},$$

where the b_j are C^∞ and real-valued functions near $(0, 0), \sigma = (0, 0, \xi^0, 0)$ satisfies $b(0, 0) \cdot \xi^0 = 0, (b = (b_1, \cdots, b_{n_0}))$ and

$$\left\langle (\xi^0, 0), \frac{[L, \overline{L}]_0}{2\sqrt{-1}} \right\rangle = -\frac{\partial b}{\partial t}(0) \cdot \xi^0 < 0. \tag{2.1.1}$$

Assume that $Lu = 0$ near $(0, 0)$. We wish to show that $\sigma \notin \mathrm{WF}(u)$.

We introduce an additional variable $s \in \mathbb{R}$ and define

$$L_1 = \frac{\partial}{\partial s} + \sqrt{-1} L = \frac{\partial}{\partial s} + \sqrt{-1} \frac{\partial}{\partial t} - \sum_{j=1}^{n_0} b_j(x, t) \frac{\partial}{\partial x_j}.$$

Let $Z_i(x, t, s)\,(1 \leq i \leq n_0)$ be C^∞ functions near the origin satisfying

$$L_1 Z_i(x, t, s) = O(s^l), \text{ as } s \to 0, \ \forall\, l \geq 1, l \in \mathbb{N}, \text{ and } Z_i(x, t, 0) = x_i.$$

Set $Z_{n_0+1}(x, t, s) = t - \sqrt{-1}s$. For $1 \leq i \leq n_0$, we can write $Z_i(x, t, s) = x_i + s\psi_i(x, t, s)$ for some C^∞ functions ψ_i. We have, for any $l \geq 1, 1 \leq i \leq n_0$,

$$s\frac{\partial \psi_i}{\partial s}(x, t, s) + \psi_i(x, t, s) + \sqrt{-1}s\frac{\partial \psi_i}{\partial t}(x, t, s) - \sum_{j=1}^{n_0} b_j(x, t) \left(\delta_{ij} + s\frac{\psi_i}{\partial x_j}(x, t, s) \right) = O(s^l). \tag{2.1.2}$$

It follows that

$$\psi_i(x, t, 0) = b_i(x, t), \ 1 \leq i \leq n_0. \tag{2.1.3}$$

Differentiating equation (2.1.2) with respect to s leads to,

$$s\frac{\partial^2 \psi_i}{\partial s^2} + 2\frac{\psi_i}{\partial s} + \sqrt{-1}\frac{\psi_i}{\partial t} + \sqrt{-1}s\frac{\partial^2 \psi_i}{\partial s \partial t} - \sum_{j=1}^{n_0} b_j \frac{\partial \psi_i}{\partial x_j} - s\sum_{j=1}^{n_0} b_j \frac{\partial^2 \psi_i}{\partial s \partial x_j} = O(s^l), \ \forall\, l \geq 1.$$

Evaluating the latter at $s=0$, we get, for any $1 \le i \le n_0$,

$$2\frac{\partial \psi_i}{\partial s}(x,t,0) + \sqrt{-1}\frac{\partial \psi_i}{\partial t}(x,t,0) - \sum_{j=1}^{n_0} b_j(x,t)\frac{\partial \psi_i}{\partial x_j}(x,t,0) = 0,$$

which together with equation (2.1.3) leads to:

$$\operatorname{Im}\psi_i(x,t,0) = 0 \text{ and } \frac{\partial}{\partial s}\operatorname{Im}\psi_i(x,t,0) = -\frac{1}{2}\frac{\partial b_i}{\partial t}(x,t),\ \forall\ 1 \le i \le n_0. \tag{2.1.4}$$

We will use the FBI transform in (x,t) space. For the solution $u=u(x,t)$, at level $s=s'$, we write,

$$\mathcal{F}(x,t,\xi,\tau,s') = \int_{\mathbb{R}^{n_0+1}} e^{Q(x,t,x',t',\xi,\tau,s')}\eta(x',t')u(x',t')dZ_1(x',t',s') \wedge \cdots \wedge dZ_{n_0+1}(x',t',s'),$$

where $(\xi,\tau) \in \mathbb{R}^{n_0} \times \mathbb{R}, \eta \in C_0^\infty(\mathbb{R}^{n_0+1}), \eta(x,t) \equiv 1$ for $|x|^2+t^2 \le r^2, \eta(x,t) \equiv 0$ when $|x|^2+t^2 \ge 2r^2$ for some $r>0$ to be fixed. Here

$$Q(x,t,x',t',\xi,\tau,s') = \sqrt{-1}\langle(\xi,\tau),(x-Z(x',t',s'),t-Z_{n_0+1}(x',t',s'))\rangle$$

$$-K|(\xi,\tau)|((x-Z(x',t',s'))^2+(t-Z_{n_0+1}(x',t',s'))^2),$$

where $Z=(Z_1,\cdots,Z_{n_0}), (x-Z(x',t',s'))^2 = \sum_{j=1}^{n_0}(x_j-Z_j(x',t',s'))^2$, and K is a positive number which will be determined.

Let $M_i = \sum_{j=1}^{n_0} a_{ij}(x,t,s)\frac{\partial}{\partial x_j}, 1 \le i \le n_0$ and $M_{n_0+1} = \frac{\partial}{\partial t} + \sum_{j=1}^{n_0} c_j(x,t,s)\frac{\partial}{\partial x_j}$ be C^∞ vector fields near the origin in (x,t,s) space that satisfy

$$M_iZ_j = \delta_{ij},\ 1 \le i,j \le n_0+1.$$

For any C^1 function $h=h(x,t,s)$,

$$dh = \sum_{i=1}^{n_0+1} M_i(h)dZ_i + \left(L_1h - \sum_{j=1}^{n_0+1} M_j(h)L_1(Z_j)\right)ds \tag{2.1.5}$$

which can be verified by applying both sides of the equation to the basis of vector fields $\{L_1,M_1,\cdots,M_{n_0+1}\}$ of $\mathbb{C}T(\mathbb{R}^{n_0+2})$. Equation (2.1.5) implies that

$$d(hdZ_1 \wedge \cdots \wedge dZ_{n_0+1}) = \left(L_1h - \sum_{j=1}^{n_0+1} M_j(h)L_1(Z_j)\right)ds \wedge dZ_1 \wedge \cdots \wedge dZ_{n_0+1}. \tag{2.1.6}$$

Let $q(x,t,x',t',\xi,\tau,s') = \eta(x',t')u(x',t')e^{Q(x,t,x',t',\xi,\tau,s')}$. Denoting $dZ_1 \wedge \cdots \wedge dZ_{n_0+1}$ by dZ and using equation (2.1.6), we have,

$$d(qdZ) = \left(L_1(\eta u) + \eta u L_1(Q) - \sum_{j=1}^{n_0+1}(M_j(\eta u) + \eta u M_j(Q))L_1Z_j\right)e^Q ds \wedge dZ. \tag{2.1.7}$$

By Stokes theorem, for $|s_0|$ small, we have,

$$\int_{\mathbb{R}^{n_0+1}} q(x,t,x',t',\xi,\tau,0)dx'dt' = \int_{\mathbb{R}^{n_0+1}} q(x,t,x',t',\xi,\tau,s_0)dZ(x',t',s_0) + \int_0^{s_0}\int_{\mathbb{R}^{n_0+1}} d(qdZ) \quad (2.1.8)$$

We will estimate the two integrals on the right in equation (2.1.8) for (x,t) near $(0,0)$ in $\mathbb{R}^{n_0+1}$ and (ξ,τ) in a conic neighborhood Γ of $(\xi^0,0)$ in $\mathbb{R}^{n_0+1}$. Observe that if $\psi = (\psi_1,\cdots,\psi_{n_0})$,

$$\begin{aligned}\operatorname{Re} Q(x,t,x',t',\xi,\tau,s') =& s'\langle\xi, \operatorname{Im}\psi(x',t',s')\rangle - \tau s' \\ & - K|(\xi,\tau)|(|x-x'-s'\operatorname{Re}\psi(x',t',s')|^2 + |t-t'|^2 - s'^2)\end{aligned} \quad (2.1.9)$$

Using equation (2.1.4), we can write

$$\begin{aligned}\operatorname{Im}\psi(x,t,s) =& -\frac{1}{2}\frac{\partial b}{\partial t}(x,t)s + O(s^2) \\ =& -\frac{1}{2}\frac{\partial b}{\partial t}(0,0)s + O(|xs| + |ts| + s^2)\end{aligned} \quad (2.1.10)$$

and so plugging this into equation (2.1.9) yields

$$\begin{aligned}\operatorname{Re} Q(x,t,x',t',\xi,\tau,s') =& -\frac{1}{2}\langle\xi, \frac{\partial b}{\partial t}(0,0)\rangle s'^2 - \tau s' \\ & - K|(\xi,\tau)|(|x-x'-s'\operatorname{Re}\psi(x',t',s')|^2 + |t-t'|^2 - s'^2) \\ & + |\xi| O(|x'|s'^2 + |t'|s'^2 + |s'|^3)\end{aligned} \quad (2.1.11)$$

Since $\langle\xi^0, \frac{\partial b}{\partial t}(0,0)\rangle > 0$, given $0<\delta<1$, we can get $M>0$ and a conic neighborhood Γ of $(\xi^0,0)$ in $\mathbb{R}^{n_0+1}$ such that

$$\langle\xi, \frac{\partial b}{\partial t}(0,0)\rangle \geq M|\xi| \text{ and } |\tau| < \delta|\xi|, \text{ when } (\xi,\tau)\in\Gamma. \quad (2.1.12)$$

Our interest is in estimating the integral on the left hand side of equation (2.1.8) for (x,t) near $(0,0)$ and $(\xi,\tau)\in\Gamma$. When $\tau>0$, we take $s_0>0$ in (2.1.8) while when $\tau<0$, we use $s_0<0$. This together with (2.1.12) allows us to deduce the following inequality from (2.1.11):

$$\begin{aligned}\operatorname{Re} Q(x,t,x',t',\xi,\tau,s') \leq& -\frac{M}{2}s'^2|\xi| - K|(\xi,\tau)|(|x-x'-s'\operatorname{Re}\psi(x',t',s')|^2 \\ & + |t-t'|^2 - s'^2) + |\xi|O(|x'|s'^2 + |t'|s'^2 + |s'|^3) \\ \leq& (-\frac{M}{2} + (1+\delta)K)s'^2|\xi| - K|\xi|(|x-x'-s'\operatorname{Re}\psi(x',t',s')|^2 \\ & + |t-t'|^2) + |\xi|O(|x'|s'^2 + |t'|s'^2 + |s'|^3)\end{aligned} \quad (2.1.13)$$

Choose $K = \frac{M}{4(1+\delta)}$. Then (2.1.13) becomes

$$\begin{aligned}\operatorname{Re} Q(x,t,x',t',\xi,\tau,s') \leq& -\frac{M}{4}s'^2|\xi| - \frac{M}{4(1+\delta)}|\xi|(|x-x'-s'\operatorname{Re}\psi(x',t',s')|^2 \\ & + |t-t'|^2) + |\xi|O(|x'|s'^2 + |t'|s'^2 + |s'|^3).\end{aligned} \quad (2.1.14)$$

We choose r and $|s_0|$ small enough so that when $(x',t')\in\operatorname{supp}(\eta)$ and $|s'|\leq|s_0|, (\xi,\tau)\in\Gamma$, (2.1.14) will yield,

$$\operatorname{Re} Q(x,t,x',t',\xi,\tau,s') \le -\frac{M}{8}s'^2|\xi| - \frac{M}{4(1+\delta)}|\xi|(|x-x'-s'\operatorname{Re}\psi(x',t',s')|^2+|t-t'|^2). \quad (2.1.15)$$

From (2.1.15), it follows that the first integral on the right in (2.1.8) (at level $s' = s_0$) decays exponentially in ξ and hence there are constants $C_1, C_2 > 0$ such that for $(\xi,\tau) \in \Gamma$,

$$\left|\int_{\mathbb{R}^{n_0+1}} q(x,t,x',t',\xi,\tau,s_0)dZ(x',t',s_0)\right| \le C_1 e^{-C_2|(\xi,\tau)|} \quad (2.1.16)$$

Consider next the second integral on the right in (2.1.8). To estimate it, we use equation (2.1.7) which is a sum of two kinds of terms. The first kind consists of terms involving $L_1(Z_j), L_1(Q)$ and L_1u(recall that $L_1u = Lu = 0$) and these terms can be bounded by constant multiples of

$$|\xi||s'|^m e^{\operatorname{Re} Q(x,t,x',t',\xi,\tau,s')}, \forall m \ge 1,$$

and so using (2.1.15) which implies that

$$\operatorname{Re} Q(x,t,x',t',\xi,\tau,s') \le -\frac{M}{8}s'^2|\xi|,$$

the integrals of such terms decay rapidly for $(\xi,\tau) \in \Gamma$. The second type of terms involve derivatives of $\eta(x,t)$ and hence $|x'|^2 + t'^2 \ge r^2$ in the domains of integration. Therefore, if we choose $0 < |s_0| << r$, we can get $\lambda > 0$ such that for (x,t) near $(0,0)$ and $(\xi,\tau) \in \Gamma$, (2.1.15) will lead to,

$$\operatorname{Re} Q(x,t,x',t',\xi,\tau,s') \le -\lambda|\xi|, \text{ when } |x'|^2 + t'^2 \ge r^2.$$

The latter leads to an exponential decay in $(\xi,\tau) \in \Gamma$ for (x,t) near $(0,0)$ for the corresponding integrals. We conclude that there exists a neighborhood W of $(0,0)$ in (x,t) space and an open conic neighborhood Γ of $(\xi^0, 0)$ in $\mathbb{R}^{n_0+1}$ such that for $\forall (x,t) \in W, (\xi,\tau) \in \Gamma, \forall m = 1,2,\cdots$, there exists $C_m > 0$ satisfying

$$\left|\int_{\mathbb{R}^{n_0+1}} e^{\sqrt{-1}[\xi(x-x')+\tau(t-t')]-K|(\xi,\tau)|(|x-x'|^2+|t-t'|^2)}\eta(x',t')u(x',t')dx'dt'\right|$$

$$= \left|\int_{\mathbb{R}^{n_0+1}} q(x,t,x',t',\xi,0)dx'dt'\right| \le \frac{C_m}{(1+|\xi|+|\tau|)^m}.$$

By Theorem 2.1 in [BH] (see also [T] and the proof of Lemma V.5.2 in [BCH]), we conclude that

$$(\xi^0, 0) \notin WF(u)|_0.$$

Suppose now the Levi form $\mathcal{L}_\sigma$ at every $\sigma \in T_p^0$ has a nonzero eigenvalue. Define

$$S = \{\sigma \in T_p^0 : \mathcal{L}_\sigma(v) \ge 0, \ \forall v \in \mathcal{V}_p\}.$$

The set S is conic, closed and convex. If $\xi \in S$, and $\xi \ne 0$, then by hypothesis $\mathcal{L}_\xi$ has at least one positive eigenvalue and hence $-\xi \notin S$. Since $\xi \notin WF(u)$, whenever $\mathcal{L}_\xi$ has at least one negative eigenvalue, it follows that $WF(u) \subset S$, for every CR function near the point p. $\square$

2.1.3 Proof of Theorem 2.1.3

We begin by recalling the following "almost holomorphic" version of the implicit function theorem from [La1]:

Theorem 2.1.16. *Let $U \subset \mathbb{C}^N$ be open, $0 \in U$, $A \in \mathbb{C}^p$, and $Z = (Z_1, \cdots, Z_N)$ be the coordinates in $\mathbb{C}^N$, W the coordinates in $\mathbb{C}^p$. Let $F : U \times \mathbb{C}^p \to \mathbb{C}^N$ be smooth in the first N variables and a polynomial in the last variables. Assume that $F(0, A) = 0$ and $F_Z(0, A)$ is invertible. Then there exists a neighborhood $U' \times V'$ of $(0, A)$ and a smooth map $\psi = (\psi_1, \cdots, \psi_N) : U' \times V' \to \mathbb{C}^N$ with $\psi(0, A) = 0$, such that if $F(Z, \overline{Z}, W) = 0$ for some $(Z, W) \in U' \times V'$, then $Z = \psi(Z, \overline{Z}, W)$. Furthermore, for every multiindex α, and each j, $1 \leq j \leq N$,*

$$D^{\alpha} \frac{\partial \psi_j}{\partial Z_i}(Z, \overline{Z}, W) = 0, \quad 1 \leq i \leq N, \tag{2.1.17}$$

if $Z = \psi(Z, \overline{Z}, W)$, and ψ is holomorphic in W. Here D^{α} denotes the derivative in all real variables.

Given the abstract CR manifold $(M, \mathcal{V})$ of CR dimension n and CR codimension d, we will use local coordinates $(x, y, s) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d$ that vanish at $p_0 \in M$. We will write $z = (z_1, \cdots, z_n)$ where $z_j = x_j + \sqrt{-1}\, y_j$ for $j = 1, \cdots, n$. In a neighborhood W of 0, we may assume that a basis of $\mathcal{V}$ is given by $\{L_1, \cdots, L_n\}$ where

$$L_i = \frac{\partial}{\partial \overline{z}_i} + \sum_{j=1}^{n} a_{ij}(x, y, s) \frac{\partial}{\partial z_j} + \sum_{l=1}^{d} b_{il}(x, y, s) \frac{\partial}{\partial s_l}, \ 1 \leq i \leq n,$$

the a_{ij} and b_{il} are smooth and $a_{ij}(0) = 0 = b_{il}(0), \forall i, j, l$ (see for example [BCH]). In these coordinates, at the origin, the characteristic set

$$T_0^0 = \{(\xi, \eta, \sigma) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d : \xi = \eta = 0\}.$$

By assumption, there is an acute open convex cone $\Gamma \subset \mathbb{R}^d$ such that

$$WF(H_j)|_0 \subset \{(0, 0, \sigma) : \sigma \in \Gamma\}, \forall j = 1, \cdots, N'.$$

Let $\phi \in C_0^{\infty}(W)$ whose support is sufficiently small and $\phi \equiv 1$ in a neighborhood of the origin. For each $j = 1, \cdots, N'$, by Fourier's inversion formula,

$$\begin{aligned}
\phi(x, y, s) H_j(x, y, s) &= \int_{\mathbb{R}^{2n+d}} e^{2\pi\sqrt{-1}(x\cdot\xi + y\cdot\eta + s\cdot\sigma)} \widehat{\phi H_j}(\xi, \eta, \sigma) d\sigma d\eta d\xi \\
&= \int_{A} e^{2\pi\sqrt{-1}(x\cdot\xi + y\cdot\eta + s\cdot\sigma)} \widehat{\phi H_j}(\xi, \eta, \sigma) d\sigma d\eta d\xi \\
&+ \int_{\mathbb{R}^{2n+d} \setminus A} e^{2\pi\sqrt{-1}(x\cdot\xi + y\cdot\eta + s\cdot\sigma)} \widehat{\phi H_j}(\xi, \eta, \sigma) d\sigma d\eta d\xi \\
&= I^j(x, y, s) + J^j(x, y, s)
\end{aligned} \tag{2.1.18}$$

where $A = \{(\xi, \eta, \sigma) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d : \sigma \notin \Gamma\}$.

Since $WF(H_j)|_0 \subset \{(0,0,\sigma) : \sigma \in \Gamma\}$, if the support of ϕ is sufficiently small, for every $m = 1, 2, \cdots$, there exists a constant $C_m > 0$ such that

$$|\widehat{\phi H_j}(\xi,\eta,\sigma)| \leq \frac{C_m}{(1+|\xi|+|\eta|+|\sigma|)^m},\ \forall (\xi,\eta,\sigma) \in A.$$

It follows that $I^j(x,y,s)$ is C^∞ on $\mathbb{R}^{2n+d}$. Write

$$J^j(x,y,s) = \int_{B_1} e^{2\pi\sqrt{-1}(x\cdot\xi+y\cdot\eta+s\cdot\sigma)}\widehat{\phi H_j}(\xi,\eta,\sigma)d\sigma d\eta d\xi + \int_{B_2} e^{2\pi\sqrt{-1}(x\cdot\xi+y\cdot\eta+s\cdot\sigma)}\widehat{\phi H_j}(\xi,\eta,\sigma)d\sigma d\eta d\xi$$

where

$$B_1 = \{(\xi,\eta,\sigma) : |\xi|^2+|\eta|^2 \leq 1,\ \sigma \in \overline{\Gamma}\},$$
$$B_2 = \{(\xi,\eta,\sigma) : |\xi|^2+|\eta|^2 \geq 1, \sigma \in \overline{\Gamma}\}.$$

Observe that since $T_0^0 \cap B_2 = \emptyset$, for any CR function u near the origin, $WF(u)|_0 \cap B_2 = \emptyset$. Moreover,

$$B_2 \cap \{(\xi,\eta,\sigma) : |\xi|^2+|\eta|^2+|\sigma|^2 = 1\}$$

is a compact set. It follows that for each $m = 1, 2, \cdots$, we can get $C'_m > 0$ such that

$$|\widehat{\phi H_j}(\xi,\eta,\sigma)| \leq \frac{C'_m}{(1+|\xi|+|\eta|+|\sigma|)^m},\ \forall (\xi,\eta,\sigma) \in B_2 \tag{2.1.19}$$

It follows that

$$F_2^j(x,y,s) = \int_{B_2} e^{2\pi\sqrt{-1}(x\cdot\xi+y\cdot\eta+s\cdot\sigma)}\widehat{\phi H_j}(\xi,\eta,\sigma)d\sigma d\eta d\xi$$

is C^∞ on $\mathbb{R}^{2n+d}$.

Since Γ is an acute cone, there is $\sigma^0 \in \mathbb{R}^d$ such that $\sigma^0 \cdot \sigma > 0, \forall \sigma \in \Gamma$. We may assume that for some conic neighborhood Γ_1 of σ and $C_0 > 0$,

$$v \cdot \sigma \geq C_0|v||\sigma|,\ \forall v \in \Gamma_1,\ \sigma \in \Gamma. \tag{2.1.20}$$

For $t \in \Gamma_1$, we define

$$F_1^j(x,y,s,t) = \int_{B_1} e^{2\pi\sqrt{-1}(x\cdot\xi+y\cdot\eta+(s+\sqrt{-1}t)\cdot\sigma)}\widehat{\phi H_j}(\xi,\eta,\sigma)d\sigma d\eta d\xi.$$

Since $\widehat{\phi H_j}$ has a polynomial growth, for some $C_1, M > 0$,

$$|\widehat{\phi H_j}(\xi,\eta,\sigma)| \leq C_1(1+|\sigma|)^M,\ \forall (\xi,\eta,\sigma) \in B_1. \tag{2.1.21}$$

Therefore, using (2.1.20) and (2.1.21), we get,

$$|F_1^j(x,y,s,t)| \leq C'_1 \int_{\mathbb{R}^d} e^{-C_0|t||\sigma|}(1+|\sigma|)^M d\sigma \leq \frac{C_2}{|t|^{M+d+1}},\ t \in \Gamma_1,\ \text{for some } C'_1, C_2 > 0. \tag{2.1.22}$$

Moreover, for all multiindices $\alpha, \beta \in \mathbb{N}^n, \gamma \in \mathbb{N}^d$,

$$|\partial_x^\alpha \partial_y^\beta \partial_s^\gamma F_1^j(x,y,s,t)| \leq \frac{C}{|t|^{M+d+1+|\gamma|}}, \tag{2.1.23}$$

for some $C > 0$ when $t \in \Gamma_1$.

When $t \in \Gamma_1$,

$$\overline{\partial}_{w_\nu} F_1^j(x,y,s,t) = 0, \text{ for } 1 \le \nu \le d, \tag{2.1.24}$$

where $\overline{\partial}_{w_\nu} = \frac{1}{2}(\frac{\partial}{\partial s_\nu} + \sqrt{-1}\frac{\partial}{\partial t_\nu})$.

Define

$$F_2^j(x,y,s,t) = \int_{B_2} e^{2\pi\sqrt{-1}(x\cdot\xi + y\cdot\eta + (s+\sqrt{-1}t)\cdot\sigma)} \widehat{\phi H_j}(\xi,\eta,\sigma) d\xi d\eta d\sigma,$$

for $t \in \Gamma_1$. By (2.1.19), F_2^j is C^∞ up to $t = 0$, and

$$\overline{\partial}_{w_\nu} F_2^j(x,y,s,t) = 0, \text{ for } 1 \le \nu \le d,\, t \in \Gamma_1. \tag{2.1.25}$$

Since $I^j(x,y,s)$ is C^∞ and bounded, we can find a bounded C^∞ function $F_0^j(x,y,s,t)$ ($|t|$ small) such that

$$F_0^j(x,y,s,0) = I^j(x,y,s), \text{ and } \overline{\partial}_{w_\nu} F_0^j(x,y,s,t) = O(|t|^l), \ \forall \nu = 1,\cdots,d, \forall l = 1,2,3,\ldots \tag{2.1.26}$$

Let $\varphi(x,y,s) \in C_0^\infty(W)$ such that its support is contained in a neighborhood of the origin where $\phi \equiv 1$. By Parseval's formula,

$$\begin{aligned}
\lim_{t\to 0, t\in\Gamma_1} \int_{\mathbb{R}^{2n+d}} F_0^j(x,y,s,t)\varphi(x,y,s) dxdyds &= \int_{\mathbb{R}^{2n+d}} I^j(x,y,s)\varphi(x,y,s) dxdyds \\
&= \int_{\mathbb{R}^{2n+d}} \widehat{I^j}(\xi,\eta,\sigma)\widehat{\varphi}(-\xi,-\eta,-\sigma) d\xi d\eta d\sigma \\
&= \int_A \widehat{\phi H_j}(\xi,\eta,\sigma)\widehat{\varphi}(-\xi,-\eta,-\sigma) d\xi d\eta d\sigma
\end{aligned} \tag{2.1.27}$$

Likewise, since F_2^j is C^∞ and bounded,

$$\int_{\mathbb{R}^{2n+d}} F_2^j(x,y,s)\varphi(x,y,s) dxdyds = \int_{B_2} \widehat{\phi H_j}(\xi,\eta,\sigma)\widehat{\varphi}(-\xi,-\eta,-\sigma) d\xi d\eta d\sigma. \tag{2.1.28}$$

For $t \in \Gamma_1$, using (3.2.4), we have,

$$\begin{aligned}
&\int_{\mathbb{R}^{2n+d}} F_1^j(x,y,s,t)\varphi(x,y,s) dxdyds \\
=& \int_{B_1} (\int_{\mathbb{R}^{2n+d}} e^{2\pi\sqrt{-1}(x\cdot\xi + y\cdot\eta + s\cdot\sigma)}\varphi(x,y,s) dxdyds) e^{-t\cdot\sigma}\widehat{\phi H_j}(\xi,\eta,\sigma) d\xi d\eta d\sigma \\
=& \int_{B_1} \widehat{\varphi}(-\xi,-\eta,-\sigma) e^{-t\cdot\sigma}\widehat{\phi H_j}(\xi,\eta,\sigma) d\xi d\eta d\sigma,
\end{aligned} \tag{2.1.29}$$

and hence

$$\lim_{t\to 0,\ t\in\Gamma_1} \int_{\mathbb{R}^{2n+d}} F_1^j(x,y,s,t)\phi(x,y,s) dxdyds = \int_{B_1} \widehat{\varphi}(-\xi,-\eta,-\sigma)\widehat{\phi H_j}(\xi,\eta,\sigma) d\xi d\eta d\sigma \tag{2.1.30}$$

Let $F^j(x,y,s,t) = F_0^j(x,y,s,t) + F_1^j(x,y,s,t) + F_2^j(x,y,s,t)$ for $t \in \Gamma_1$. From (2.1.27),(2.1.28) and (2.1.30),

$$\begin{aligned}
\lim_{t\to 0,\ t\in\Gamma_1} \int_{\mathbb{R}^{2n+d}} F^j(x,y,s,t)\varphi(x,y,s) dxdyds &= \int_{\mathbb{R}^{2n+d}} \widehat{\varphi}(-\xi,-\eta,-\sigma)\widehat{\phi H_j}(\xi,\eta,\sigma) d\xi d\eta d\sigma \\
&= \int_{\mathbb{R}^{2n+d}} \phi(x,y,s) H_j(x,y,s)\varphi(x,y,s) dxdyds.
\end{aligned} \tag{2.1.31}$$

Therefore, in a neighborhood of the origin, in the distribution sense,

$$\lim_{t\to 0,\ t\in\Gamma_1} F^j(x,y,s,t) = H_j(x,y,s). \tag{2.1.32}$$

For $t \in \Gamma_1$ small, from (2.1.23)-(2.1.26), we have: for (x,y,s) near 0, given α,β,γ, there exists $C_1 > 0$ such that for some $\lambda > 0$,

$$|\partial_x^\alpha \partial_y^\beta \partial_s^\gamma F^j(x,y,s,t)| \le \frac{C_1}{|t|^\lambda},\ \text{and} \tag{2.1.33}$$

$$\partial_x^\alpha \partial_y^\beta \partial_s^\gamma \overline{\partial}_{w_\nu} F^j(x,y,s,t) = O(|t|^l),\ \forall l \ge 1,\ \forall \nu = 1,\cdots,d. \tag{2.1.34}$$

For the rest of the proof, we follow the argument of claim 3 in [La1]. We may assume that $H(0) = 0 \in M'$. Let $\rho = (\rho_1,\cdots,\rho_{d'})$ be defining functions for M' near 0. For $\alpha \in \mathbb{N}^n$ a multiindex, recall that $L^\alpha = L_1^{\alpha_1}\cdots L_n^{\alpha_n}$.

Set $F(x,y,s,t) = (F^1(x,y,s,t),\cdots,F^{N'}(x,y,s,t)), t \in \Gamma_1$. As in [La1], there are smooth functions $\Psi_{\mu,\alpha}(Z',\overline{Z'},W)$ for $|\alpha| \le k_0, 1 \le \mu \le d'$, defined in a neighborhood of $\{0\}\times\mathbb{C}^{K(k_0)}$ in $\mathbb{C}^{N'}\times\mathbb{C}^{K(k_0)}$, polynomial in W, such that

$$L^\alpha \rho_\mu(H(z,s),\overline{H(z,s)}) = \Psi_{\mu,\alpha}(H(z,s),\overline{H(z,s)},(L^\beta\overline{H}(z,s))_{|\beta|\le k_0}), \tag{2.1.35}$$

and

$$L^\alpha \rho_{\mu,Z'}(H,\overline{H})|_0 = \Psi_{\mu,\alpha,Z'}(0,0,(L^\beta\overline{H}(0,0))_{|\beta|\le k_0}). \tag{2.1.36}$$

Here $K(k_0) = N'|\{\beta : |\beta| \le k_0\}|$. Equation (2.1.36) and the k_0-nondegeneracy assumption on the map H allows us to get $(\alpha^1,\cdots,\alpha^{N'}),(\mu_1,\cdots,\mu_{N'}) \in \mathbb{N}^{N'}$ and a smooth function $\psi(Z',\overline{Z'},W) = (\psi_1,\cdots,\psi_{N'})$, which is holomorphic in W, such that with

$$\Psi = (\Psi_{\mu_1,\alpha^1},\cdots,\Psi_{\mu_{N'},\alpha^{N'}}),$$

if $\Psi(Z',\overline{Z'},W) = 0$, then $Z' = \psi(Z',\overline{Z'},W)$. Moreover, with $Z' = (z_1',\cdots,z_{N'}')$, we have,

$$D^\alpha \frac{\partial\psi_j}{\partial z_i'}(Z',\overline{Z'},W) = 0,\ \forall i = 1,\cdots,N',\ j = 1,\cdots,N', \tag{2.1.37}$$

whenever $Z' = \psi(Z',\overline{Z'},W)$. In particular, since $\Psi_{l,\alpha}(H(z,s),\overline{H}(z,s),(L^\beta\overline{H}(z,s))_{|\beta|\le k_0}) = 0$, we have,

$$H_j(z,s) = \psi_j(F(z,s,0),\overline{F}(z,s,0),(L^\beta\overline{F}(z,s,0))_{|\beta|\le k_0}), \forall j = 1,\cdots,N'. \tag{2.1.38}$$

Recall that for $i = 1,\cdots,n$,

$$L_i = \frac{\partial}{\partial \overline{z}_i} + \sum_{j=1}^n a_{ij}(x,y,s)\frac{\partial}{\partial z_j} + \sum_{l=1}^d b_{il}(x,y,s)\frac{\partial}{\partial s_l}.$$

Let

$$M_i = \frac{\partial}{\partial \overline{z}_i} + \sum_{j=1}^n A_{ij}(x,y,s,t)\frac{\partial}{\partial z_j} + \sum_{l=1}^d B_{il}(x,y,s,t)\frac{\partial}{\partial s_l}, 1 \le i \le n,$$

where the A_{ij} and B_{il} are smooth extensions of the a_{ij} and b_{il} satisfying

$$\overline{\partial}_{w_\nu} A_{ij}(x,y,s,t), \overline{\partial}_{w_\nu} B_{il}(x,y,s,t) = O(|t|^m), \ \forall \nu = 1, \cdots, d, \ \forall m = 1, 2, \cdots. \tag{2.1.39}$$

Now define

$$g_j(z,s,t) = \psi_j(F(z,s,-t), \overline{F}(z,s,-t), (M^\beta \overline{F}(z,s,-t))_{|\beta|\le k_0}),$$

for $j = 1, \cdots, N'$ and for $t \in -\Gamma_1, |t|$ small. Using (2.1.34), (2.1.37) and (2.2.19), we conclude that, when (z,s) is near the origin in $\mathbb{C}^n \times \mathbb{R}^d$ and $t \in -\Gamma_1$ ($|t|$ small), for any α, β, γ multiindices, there is $C > 0$ such that

$$|D_x^\alpha D_y^\beta D_s^\gamma g_j(z,s,t)| \le \frac{C}{|t|^\lambda} \text{ for some } \lambda > 0. \tag{2.1.40}$$

and

$$D_x^\alpha D_y^\beta D_s^\gamma \overline{\partial}_{w_\nu} g_j(z,s,t) = O(|t|^m), \forall m = 1, 2, \cdots, \nu = 1, \cdots, d. \tag{2.1.41}$$

From (2.1.38), we know that,

$$H_j(z,s) = \lim_{t\to 0, \ t\in -\Gamma_1} g_j(z,s,t), \forall j = 1, \cdots, N'. \tag{2.1.42}$$

By Theorem V.3.7 in [BCH], it follows that $\mathrm{WF}(H_j)|_0 \cap \Gamma = \emptyset$. Since by assumption $WF(H_j)|_0 \subset \Gamma$, we conclude that H is C^∞ near the origin.

2.1.4 Proof of Theorem 2.1.5

Fix any $p \in M$, and assume $p' = F(p) = 0$. Since M' is strictly pseudoconvex, we may assume that there is a neighborhood G of 0 in $\mathbb{C}^{n+k}$, and a local defining function ρ of M' in G such that

$$M' \cap G = \{Z' \in G : \rho(Z', \overline{Z'}) = 0\},$$

where $\rho(Z', \overline{Z'}) = -v' + \sum_{j=1}^{n+k-1} |z_j'|^2 + \phi^*(Z', \overline{Z'})$. Here $Z' = (z_1', \cdots, z_{n+k}')$ are the coordinates of $\mathbb{C}^{n+k}$, $z_{n+k}' = u' + \sqrt{-1}v'$ and $\phi^*(Z', \overline{Z'}) = O(|Z'|^3)$ is a real-valued smooth function on G. Note that $\mathrm{rank}_l(F,p)$ is a lower semi-continuous integer-valued function on M for each $1 \le l \le k$. For any $p \in M$,

$$\mathrm{rank}_0(F,p) \le \mathrm{rank}_1(F,p) \le \cdots \le \mathrm{rank}_k(F,p).$$

We next recall some basic properties of the rank of F. Write $F = (F_1, \cdots, F_{n+k})$. Since $F(M) \subset M'$, we have

$$\rho(F, \overline{F}) = -\frac{F_{n+k} - \overline{F_{n+k}}}{2\sqrt{-1}} + F_1\overline{F_1} + \cdots + F_{n+k-1}\overline{F_{n+k-1}} + \phi^*(F, \overline{F}) = 0, \tag{2.1.43}$$

on M near p. Applying $L_1, \cdots, L_n$ to the above equation, we get

$$\frac{L_j \overline{F_{n+k}}}{2\sqrt{-1}} + F_1 L_j \overline{F_1} + \cdots + F_{n+k-1} L_j \overline{F_{n+k-1}} + L_j \phi^*(F, \overline{F}) = 0, \ 1 \le j \le n, \tag{2.1.44}$$

$$\frac{L^\alpha \overline{F_{n+k}}}{2\sqrt{-1}} + F_1 L^\alpha \overline{F_1} + \cdots + F_{n+k-1} L^\alpha \overline{F_{n+k-1}} + L^\alpha \phi^*(F, \overline{F}) = 0, \tag{2.1.45}$$

on M near p for any multiindex $1 \le |\alpha| \le k$. Therefore, on M near p,

$$\rho_{Z'}(F, \overline{F}) = (\overline{F_1} + \phi^*_{z'_1}(F, \overline{F}), \cdots, \overline{F_{n+k-1}} + \phi^*_{z'_{n+k-1}}(F, \overline{F}), \frac{\sqrt{-1}}{2} + \phi^*_{z'_{n+k}}(F, \overline{F})), \tag{2.1.46}$$

and for any multiindex $1 \le |\alpha| \le k$,

$$L^\alpha \rho_{Z'}(F, \overline{F}) = (L^\alpha(\overline{F_1} + \phi^*_{z'_1}), \cdots, L^\alpha(\overline{F_{n+k-1}} + \phi^*_{z'_{n+k-1}}), L^\alpha \phi^*_{z'_{n+k}}). \tag{2.1.47}$$

Lemma 2.1.17. *With the assumption of Theorem 2.1.5, for any $p \in M$, we have* $\mathrm{rank}_0(F, p) = 1,\ \mathrm{rank}_1(F, p) = n + 1,$ *and thus* $\mathrm{rank}_l(F, p) \ge n + 1,$ *for* $1 \le l \le k$.

Proof. Assume that $F(p) = 0$. Note that $\phi^*_{z'_i}(F, \overline{F})|_p = 0$, for all $1 \le i \le n + k$. Equation (4.4) shows that $\mathrm{rank}_0(F, p) = 1$. By assumption, $dF : \mathcal{V}_p \to T_0^{(0,1)} M'$ is injective. By plugging $Z = p$ in equation (4.2), we get $L_i \overline{F}_{n+k}(p) = 0$ for each $1 \le i \le n$. Since $\{L_1, L_2, \cdots, L_n\}$ is a local basis of $\mathcal{V}$ near p, we conclude that the rank of the matrix $(L_i \overline{F}_l)_{1 \le i \le n, 1 \le l \le n+k-1}$ is n. Without loss of generality, we assume that,

$$\begin{vmatrix} L_1\overline{F}_1 & . & . & . & L_1\overline{F}_n \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ L_n\overline{F}_1 & . & . & . & L_n\overline{F}_n \end{vmatrix} \ne 0 \text{ at } p.$$

Notice that $\phi^*_{z'_1}|_p = \phi^*_{z'_2}|_p = \cdots = \phi^*_{z'_{n+k}}|_p = 0$, $L_j\phi^*_{z'_1}|_p = L_j\phi^*_{z'_2}|_p = \cdots = L_j\phi^*_{z'_{n+k}}|_p = 0$, for all $1 \le j \le n$. Thus $rank_1(F, p) = n + 1$. Consequently, $\mathrm{rank}_l(F, p) \ge n + 1$, for $1 \le l \le k$ for any $p \in M$. ∎

To simplify the notations, let

$$a_i(Z, \overline{Z}) = \overline{F}_i + \phi^*_{z'_i}(F, \overline{F}),\ 1 \le i \le n + k - 1,$$

$$a_{n+k}(Z, \overline{Z}) = \frac{\sqrt{-1}}{2} + \phi^*_{z'_{n+k}}(F, \overline{F}),$$

$$\mathbf{a}(Z, \overline{Z}) = (a_1, \cdots, a_{n+k}).$$

Then

$$\rho_{Z'}(F, \overline{F}) = \mathbf{a} = (a_1, \cdots, a_{n+k-1}, a_{n+k}),$$

$$L^\alpha \rho_{Z'}(F, \overline{F}) = L^\alpha \mathbf{a} = (L^\alpha a_1, \cdots, L^\alpha a_{n+k-1}, L^\alpha a_{n+k})$$

for any multiindex $0 \le |\alpha| \le k$. Recall that

$$\mathrm{rank}_l(F, p) = \dim_{\mathbb{C}}(\mathrm{Span}_{\mathbb{C}}\{L^\alpha \mathbf{a}(Z, \overline{Z})|_p : 0 \le |\alpha| \le l\}).$$

The following normalization will be applied later in this section.

Lemma 2.1.18. *Let M, M', F be as in Theorem 2.5. Assume* $\mathrm{rank}_l(F,p) = N_0$, *for some* $1 \leq l \leq k, n+1 \leq N_0 \leq n+k$. *Then there exist multiindices* $\{\beta_{n+1}, \cdots, \beta_{N_0-1}\}$ *with* $1 < |\beta_i| \leq l$ *for all* i, *such that after a linear biholomorphic change of coordinates in* $\mathbb{C}^{n+k} : \widetilde{Z} = Z'A^{-1}$, *where A is a unitary $(n+k)\times(n+k)$ matrix, and $\widetilde{Z}$ denotes the new coordinates, the following hold:*

$$\widetilde{\mathbf{a}}|_p = (0, \cdots, 0, \frac{\sqrt{-1}}{2}), \begin{pmatrix} L_1\widetilde{\mathbf{a}}|_p \\ \cdots \\ L_n\widetilde{\mathbf{a}}|_p \\ L^{\beta_{n+1}}\widetilde{\mathbf{a}}|_p \\ \cdots \\ L^{\beta_{N_0-1}}\widetilde{\mathbf{a}}|_p \end{pmatrix} = \begin{pmatrix} \mathbf{B}_{N_0-1} & \mathbf{0} & \mathbf{b} \end{pmatrix}. \tag{2.1.48}$$

Here we write $\widetilde{\mathbf{a}} = \widetilde{\rho}_{\widetilde{Z}}(\widetilde{Z}(F), \overline{\widetilde{Z}(F)})$, *and $\widetilde{\rho}$ is a local defining function of M' near 0 in the new coordinates. Moreover,* $\mathbf{B}_{N_0-1}$ *is an invertible* $(N_0-1)\times(N_0-1)$ *matrix,* $\mathbf{0}$ *is an* $(N_0-1)\times(n+k-N_0)$ *zero matrix, and* $\mathbf{b}$ *is an* $(N_0-1)-$*dimensional column vector.*

Proof. It follows from Lemma 2.1.17 that

$$\{\mathbf{a}, L_1\mathbf{a}, \cdots, L_n\mathbf{a}\}|_p$$

is linearly independent. Extend it to a basis of $E_l(p)$, which has dimension N_0 by assumption. That is, we choose multiindices $\{\beta_{n+1}, \cdots, \beta_{N_0-1}\}$ with $1 < |\beta_i| \leq l$ for each i, such that

$$\{\mathbf{a}, L_1\mathbf{a}, \cdots, L_n\mathbf{a}, L^{\beta_{n+1}}\mathbf{a}, \cdots, L^{\beta_{N_0-1}}\mathbf{a}\}|_p$$

is linearly independent over $\mathbb{C}$. We write $\widehat{\mathbf{a}} := (a_1, \cdots, a_{n+k-1})$, that is, the first $n+k-1$ components of $\mathbf{a}$. Notice that $\mathbf{a}(p) = (0, \cdots, 0, \frac{\sqrt{-1}}{2})$. Consequently,

$$\{L_1\widehat{\mathbf{a}}, \cdots, L_n\widehat{\mathbf{a}}, L^{\beta_{n+1}}\widehat{\mathbf{a}}, \cdots, L^{\beta_{N_0-1}}\widehat{\mathbf{a}}\}|_p$$

is linearly independent in $\mathbb{C}^{n+k-1}$. Let S be the $(N_0-1)-$dimensional vector space spanned by them and let $\{T_1, \cdots, T_{N_0-1}\}$ be an orthonormal basis of S. Extend it to an orthonormal basis of $\mathbb{C}^{n+k-1} : \{T_1, \cdots, T_{N_0-1}, T_{N_0}, \cdots, T_{n+k-1}\}$ and set

$$T = \begin{pmatrix} T_1 \\ \cdots \\ T_{n+k-1} \end{pmatrix}^t, \ A = \begin{pmatrix} T & \mathbf{0}^t_{n+k-1} \\ \mathbf{0}_{n+k-1} & 1 \end{pmatrix}.$$

Here $\mathbf{0}_{n+k-1}$ is an $(n+k-1)-$dimensional zero row vector. Next we make the following change of coordinates: $Z' = \widetilde{Z}A$, or $\widetilde{Z} = Z'A^{-1}$. The function $\widetilde{\rho}(\widetilde{Z}, \overline{\widetilde{Z}}) = \rho(\widetilde{Z}A, \overline{\widetilde{Z}A})$ is a defining function of M' near 0 with respect to the new coordinates $\widetilde{Z}$. By the chain rule,

$$\widetilde{\rho}_{\widetilde{Z}}(\widetilde{Z}(F), \overline{\widetilde{Z}(F)}) = \rho_{Z'}(F, \overline{F})A. \tag{2.1.49}$$

For any multiindex α,

$$L^\alpha\widetilde{\rho}_{\widetilde{Z}}(\widetilde{Z}(F), \overline{\widetilde{Z}(F)}) = L^\alpha\rho_{Z'}(F, \overline{F})A. \tag{2.1.50}$$

In particular, at p, we get:

$$\widetilde{\mathbf{a}}|_p = \mathbf{a}|_p A, \quad \begin{pmatrix} L_1\widetilde{\mathbf{a}}|_p \\ \cdots \\ L_n\widetilde{\mathbf{a}}|_p \\ L^{\beta_{n+1}}\widetilde{\mathbf{a}}|_p \\ \cdots \\ L^{\beta_{N_0-1}}\widetilde{\mathbf{a}}|_p \end{pmatrix} = \begin{pmatrix} L_1\mathbf{a}|_p \\ \cdots \\ L_n\mathbf{a}|_p \\ L^{\beta_{n+1}}\mathbf{a}|_p \\ \cdots \\ L^{\beta_{N_0-1}}\mathbf{a}|_p \end{pmatrix} A. \tag{2.1.51}$$

Furthermore from the definition of A, in the new coordinates, equation (2.1.48) holds and $\mathbf{B}_{N_0-1}$ is invertible. $\square$

Remark 2.1.19. *From the construction of A in the proof of Lemma 2.1.18, one can see that, in the new coordinates $\widetilde{Z}$, the following continues to hold: There is a neighborhood G of $p' = 0$ in $\mathbb{C}^{n+k}$, and a smooth real-valued function $\widetilde{\rho}$ in G, such that,*

$$M' \cap G = \{\widetilde{Z} \in G : \widetilde{\rho}(\widetilde{Z}, \overline{\widetilde{Z}}) = 0\}.$$

Moreover, $\widetilde{\rho}(\widetilde{Z}, \overline{\widetilde{Z}}) = -\widetilde{v} + \sum_{j=1}^{n+k-1} |\widetilde{z}_j|^2 + \widetilde{\phi^*}(\widetilde{Z}, \overline{\widetilde{Z}})$, where $\widetilde{Z} = (\widetilde{z}_1, \cdots, \widetilde{z}_{n+k}), \widetilde{z}_{n+k} = \widetilde{u} + \sqrt{-1}\widetilde{v}$ and $\widetilde{\phi^*}(\widetilde{Z}, \overline{\widetilde{Z}}) = O(|\widetilde{Z}|^3)$ is a real-valued smooth function in G. We will write the new coordinates as Z instead of $\widetilde{Z}$.

Theorem 2.1.9 will follow from the following in the degenerate case:

Theorem 2.1.20. *Let M, M', F be as in Theorem 2.1.5 and $p \in M$. Assume that F is of constant geometric rank $n + l$ at p with $l \leq k - 1$. Then F is smooth near p.*

Proof. By Remark 2.1.7, there exists a neighborhood O of p such that $\mathrm{rank}_l(F, q) = \mathrm{rank}_{l+1}(F, q) = n + l$ for all $q \in O$. Applying Lemma 2.1.18, after a suitable holomorphic change of coordinates in $\mathbb{C}^{n+k}$, there exist multiindices $\{\beta_{n+1}, \cdots, \beta_{n+l-1}\}$ with $1 < |\beta_i| \leq l$ for all $n \leq i \leq n+l-1$ satisfying

$$\mathbf{a}|_p = (0, \cdots, 0, \frac{\sqrt{-1}}{2}), \quad \begin{pmatrix} L_1\mathbf{a}|_p \\ \cdots \\ L_n\mathbf{a}|_p \\ L^{\beta_{n+1}}\mathbf{a}|_p \\ \cdots \\ L^{\beta_{n+l-1}}\mathbf{a}|_p \end{pmatrix} = \begin{pmatrix} \mathbf{B}_{n+l-1} & \mathbf{0} & \mathbf{b} \end{pmatrix}. \tag{2.1.52}$$

Here $\mathbf{B}_{n+l-1}$ is an invertible $(n+l-1) \times (n+l-1)$ matrix, $\mathbf{0}$ is an $(n+l-1) \times (k-l)$ zero matrix, $\mathbf{b}$ is an $(n+l-1)-$dimensional column vector. From equation (2.1.52), we know that

$$\left|\begin{array}{cccc} a_1 & \cdots & a_{n+l-1} & a_{n+k} \\ L_1 a_1 & \cdots & L_1 a_{n+l-1} & L_1 a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L_n a_1 & \cdots & L_n a_{n+l-1} & L_n a_{n+k} \\ L^{\beta_{n+1}} a_1 & \cdots & L^{\beta_{n+1}} a_{n+l-1} & L^{\beta_{n+1}} a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}} a_1 & \cdots & L^{\beta_{n+l-1}} a_{n+l-1} & L^{\beta_{n+l-1}} a_{n+k} \end{array}\right| \neq 0 \text{ at } p. \tag{2.1.53}$$

To simplify the notation, we denote the $n-$dimensional multiindices by $\beta_0 = (0, \cdots, 0)$, and $\beta_\mu = (0, \cdots, 0, 1, 0, \cdots, 0)$, for $\mu = 1, \cdots, n$, where 1 is at the μ^{th} position. That is, $L^{\beta_\mu} = L_\mu, \mu = 1, \cdots, n$. Then inequality (2.1.53) can be written as

$$\left|\begin{array}{cccc} L^{\beta_0} a_1 & \cdots & L^{\beta_0} a_{n+l-1} & L^{\beta_0} a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}} a_1 & \cdots & L^{\beta_{n+l-1}} a_{n+l-1} & L^{\beta_{n+l-1}} a_{n+k} \end{array}\right| \neq 0 \text{ at } p. \tag{2.1.54}$$

By shrinking O if necessary, it is nonzero everywhere in O. Since $\text{rank}_{l+1}(F, q) = n + l$ in O, we have

$$\dim_{\mathbb{C}}(E_{l+1}(q)) = \dim_{\mathbb{C}}(\text{Span}_{\mathbb{C}}\{(L^\alpha a_1, \cdots, L^\alpha a_{n+k})|_q : 0 \leq |\alpha| \leq l+1\}) = n + l$$

everywhere in O. Hence for any multiindex $\widetilde{\beta}$ with $0 \leq |\widetilde{\beta}| \leq l + 1$, and any $n + l \leq j \leq n + k - 1$, we have, in O,

$$\left|\begin{array}{ccccc} L^{\beta_0} a_1 & \cdots & L^{\beta_0} a_{n+l-1} & L^{\beta_0} a_{n+k} & L^{\beta_0} a_j \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}} a_1 & \cdots & L^{\beta_{n+l-1}} a_{n+l-1} & L^{\beta_{n+l-1}} a_{n+k} & L^{\beta_{n+l-1}} a_j \\ L^{\widetilde{\beta}} a_1 & \cdots & L^{\widetilde{\beta}} a_{n+l-1} & L^{\widetilde{\beta}} a_{n+k} & L^{\widetilde{\beta}} a_j \end{array}\right| \equiv 0. \tag{2.1.55}$$

Furthermore, we will prove the following claim.

Claim: For any $1 \leq \nu \leq n, n+l \leq j \leq n+k-1$, and $i_1 < i_2 < \cdots < i_{n+l-1}$ with $\{i_1, \cdots, i_{n+l-1}\} \subset \{1, \cdots, n + l - 1, n + k\}$, the following holds in O :

$$L_\nu \left(\frac{\left|\begin{array}{cccc} L^{\beta_0} a_{i_1} & \cdots & L^{\beta_0} a_{i_{n+l-1}} & L^{\beta_0} a_j \\ L^{\beta_1} a_{i_1} & \cdots & L^{\beta_1} a_{i_{n+l-1}} & L^{\beta_1} a_j \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}} a_{i_1} & \cdots & L^{\beta_{n+l-1}} a_{i_{n+l-1}} & L^{\beta_{n+l-1}} a_j \end{array}\right|}{\left|\begin{array}{cccc} L^{\beta_0} a_1 & \cdots & L^{\beta_0} a_{n+l-1} & L^{\beta_0} a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}} a_1 & \cdots & L^{\beta_{n+l-1}} a_{n+l-1} & L^{\beta_{n+l-1}} a_{n+k} \end{array}\right|} \right) \equiv 0. \tag{2.1.56}$$

Proof. By the quotient rule,

$$\text{the numerator of } \left(L_\nu \left(\frac{\begin{vmatrix} L^{\beta_0}a_{i_1} & \cdots & L^{\beta_0}a_{i_{n+l-1}} & L^{\beta_0}a_j \\ L^{\beta_1}a_{i_1} & \cdots & L^{\beta_1}a_{i_{n+l-1}} & L^{\beta_1}a_j \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_{i_1} & \cdots & L^{\beta_{n+l-1}}a_{i_{n+l-1}} & L^{\beta_{n+l-1}}a_j \end{vmatrix}}{\begin{vmatrix} L^{\beta_0}a_1 & \cdots & L^{\beta_0}a_{n+l-1} & L^{\beta_0}a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_1 & \cdots & L^{\beta_{n+l-1}}a_{n+l-1} & L^{\beta_{n+l-1}}a_{n+k} \end{vmatrix}} \right) \right) =$$

.

$$\begin{vmatrix} \begin{vmatrix} L^{\beta_0}a_1 & \cdots & L^{\beta_0}a_{n+l-1} & L^{\beta_0}a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_1 & \cdots & L^{\beta_{n+l-1}}a_{n+l-1} & L^{\beta_{n+l-1}}a_{n+k} \end{vmatrix} & \begin{vmatrix} L^{\beta_0}a_{i_1} & \cdots & L^{\beta_0}a_{i_{n+l-1}} & L^{\beta_0}a_j \\ L^{\beta_1}a_{i_1} & \cdots & L^{\beta_1}a_{i_{n+l-1}} & L^{\beta_1}a_j \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_{i_1} & \cdots & L^{\beta_{n+l-1}}a_{i_{n+l-1}} & L^{\beta_{n+l-1}}a_j \end{vmatrix} \\ L_\nu \begin{vmatrix} L^{\beta_0}a_1 & \cdots & L^{\beta_0}a_{n+l-1} & L^{\beta_0}a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_1 & \cdots & L^{\beta_{n+l-1}}a_{n+l-1} & L^{\beta_{n+l-1}}a_{n+k} \end{vmatrix} & L_\nu \begin{vmatrix} L^{\beta_0}a_{i_1} & \cdots & L^{\beta_0}a_{i_{n+l-1}} & L^{\beta_0}a_j \\ L^{\beta_1}a_{i_1} & \cdots & L^{\beta_1}a_{i_{n+l-1}} & L^{\beta_1}a_j \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_{i_1} & \cdots & L^{\beta_{n+l-1}}a_{i_{n+l-1}} & L^{\beta_{n+l-1}}a_j \end{vmatrix} \end{vmatrix} =$$

$$\begin{vmatrix} \begin{vmatrix} L^{\beta_0}a_1 & \cdots & L^{\beta_0}a_{n+l-1} & L^{\beta_0}a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_1 & \cdots & L^{\beta_{n+l-1}}a_{n+l-1} & L^{\beta_{n+l-1}}a_{n+k} \end{vmatrix} & \begin{vmatrix} L^{\beta_0}a_{i_1} & \cdots & L^{\beta_0}a_{i_{n+l-1}} & L^{\beta_0}a_j \\ L^{\beta_1}a_{i_1} & \cdots & L^{\beta_1}a_{i_{n+l-1}} & L^{\beta_1}a_j \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_{i_1} & \cdots & L^{\beta_{n+l-1}}a_{i_{n+l-1}} & L^{\beta_{n+l-1}}a_j \end{vmatrix} \\ \begin{vmatrix} L_\nu L^{\beta_0}a_1 & \cdots & L_\nu L^{\beta_0}a_{n+l-1} & L_\nu L^{\beta_0}a_{n+k} \\ L^{\beta_1}a_1 & \cdots & L^{\beta_1}a_{n+l-1} & L^{\beta_1}a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_1 & \cdots & L^{\beta_{n+l-1}}a_{n+l-1} & L^{\beta_{n+l-1}}a_{n+k} \end{vmatrix} & \begin{vmatrix} L_\nu L^{\beta_0}a_{i_1} & \cdots & L_\nu L^{\beta_0}a_{i_{n+l-1}} & L_\nu L^{\beta_0}a_j \\ L^{\beta_1}a_{i_1} & \cdots & L^{\beta_1}a_{i_{n+l-1}} & L^{\beta_1}a_j \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_{i_1} & \cdots & L^{\beta_{n+l-1}}a_{i_{n+l-1}} & L^{\beta_{n+l-1}}a_j \end{vmatrix} \end{vmatrix} + \cdots +$$

$$\begin{vmatrix} \begin{vmatrix} L^{\beta_0}a_1 & \cdots & L^{\beta_0}a_{n+l-1} & L^{\beta_0}a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_1 & \cdots & L^{\beta_{n+l-1}}a_{n+l-1} & L^{\beta_{n+l-1}}a_{n+k} \end{vmatrix} & \begin{vmatrix} L^{\beta_0}a_{i_1} & \cdots & L^{\beta_0}a_{i_{n+l-1}} & L^{\beta_0}a_j \\ L^{\beta_1}a_{i_1} & \cdots & L^{\beta_1}a_{i_{n+l-1}} & L^{\beta_1}a_j \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}}a_{i_1} & \cdots & L^{\beta_{n+l-1}}a_{i_{n+l-1}} & L^{\beta_{n+l-1}}a_j \end{vmatrix} \\ \begin{vmatrix} L^{\beta_0}a_1 & \cdots & L^{\beta_0}a_{n+l-1} & L^{\beta_0}a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-2}}a_1 & \cdots & L^{\beta_{n+l-2}}a_{n+l-1} & L^{\beta_{n+l-2}}a_{n+k} \\ L_\nu L^{\beta_{n+l-1}}a_1 & \cdots & L_\nu L^{\beta_{n+l-1}}a_{n+l-1} & L_\nu L^{\beta_{n+l-1}}a_{n+k} \end{vmatrix} & \begin{vmatrix} L^{\beta_0}a_{i_1} & \cdots & L^{\beta_0}a_{i_{n+l-1}} & L^{\beta_0}a_j \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-2}}a_{i_1} & \cdots & L^{\beta_{n+l-2}}a_{i_{n+l-1}} & L^{\beta_{n+l-2}}a_j \\ L_\nu L^{\beta_{n+l-1}}a_{i_1} & \cdots & L_\nu L^{\beta_{n+l-1}}a_{i_{n+l-1}} & L_\nu L^{\beta_{n+l-1}}a_j \end{vmatrix} \end{vmatrix}.$$

From equation (2.1.55) and Lemma 1.3.1, we know each term on the right-hand side of the equation above equals 0. Hence equation (2.1.56) holds. This completes the proof of the claim. $\square$

Thus the fraction in the parentheses in equation (2.1.56) equals a C^{k-l} CR function in O. It follows that for any fixed $n+l \leq j \leq n+k-1$, there exist $C^{k-l}-$smooth CR functions $G_1^j, G_2^j, \cdots, G_{n+l-1}^j, G_{n+k}^j$ in O, such that, if $i_1 < i_2 < \cdots < i_{n+l-1}$ and $(i_1, i_2, \cdots, i_{n+l-1}) = (1, 2, \cdots, \widehat{i_0}, \cdots, n+l-1, n+k), i_0 \in \{1, 2, \cdots, n+l-1, n+k\}$, where $(1, 2, \cdots, \widehat{i_0}, \cdots, n+l-1, n+k)$ means $(1, 2, \cdots, n+l-1, n+k)$ with the component "i_0" missing, then in O,

$$\begin{vmatrix} L^{\beta_0} a_{i_1} & \cdots & L^{\beta_0} a_{i_{n+l-1}} & L^{\beta_0} a_j \\ L^{\beta_1} a_{i_1} & \cdots & L^{\beta_1} a_{i_{n+l-1}} & L^{\beta_1} a_j \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}} a_{i_1} & \cdots & L^{\beta_{n+l-1}} a_{i_{n+l-1}} & L^{\beta_{n+l-1}} a_j \end{vmatrix}$$
$$= G_{i_0}^j \begin{vmatrix} L^{\beta_0} a_{i_1} & \cdots & L^{\beta_0} a_{i_{n+l-1}} & L^{\beta_0} a_{i_0} \\ L^{\beta_1} a_{i_1} & \cdots & L^{\beta_1} a_{i_{n+l-1}} & L^{\beta_1} a_{i_0} \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}} a_{i_1} & \cdots & L^{\beta_{n+l-1}} a_{i_{n+l-1}} & L^{\beta_{n+l-1}} a_{i_0} \end{vmatrix}.$$

That is,

$$\begin{vmatrix} L^{\beta_0} a_{i_1} & \cdots & L^{\beta_0} a_{i_{n+l-1}} & L^{\beta_0}(a_j - G_{i_0}^j a_{i_0}) \\ L^{\beta_1} a_{i_1} & \cdots & L^{\beta_1} a_{i_{n+l-1}} & L^{\beta_1}(a_j - G_{i_0}^j a_{i_0}) \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}} a_{i_1} & \cdots & L^{\beta_{n+l-1}} a_{i_{n+l-1}} & L^{\beta_{n+l-1}}(a_j - G_{i_0}^j a_{i_0}) \end{vmatrix} \equiv 0. \tag{2.1.57}$$

We further assert:

Claim: In O, we have,

$$\begin{vmatrix} L^{\beta_0} a_{s_1} & \cdots & L^{\beta_0} a_{s_{n+l-1}} & L^{\beta_0}(a_j - \sum_{i=1}^{n+l-1} G_i^j a_i - G_{n+k}^j a_{n+k}) \\ L^{\beta_1} a_{s_1} & \cdots & L^{\beta_1} a_{s_{n+l-1}} & L^{\beta_1}(a_j - \sum_{i=1}^{n+l-1} G_i^j a_i - G_{n+k}^j a_{n+k}) \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}} a_{s_1} & \cdots & L^{\beta_{n+l-1}} a_{s_{n+l-1}} & L^{\beta_{n+l-1}}(a_j - \sum_{i=1}^{n+l-1} G_i^j a_i - G_{n+k}^j a_{n+k}) \end{vmatrix} \equiv 0 \tag{2.1.58}$$

for all $s_1 < s_2 < \cdots < s_{n+l-1}$ with $\{s_1, \cdots, s_{n+l-1}\} \subset \{1, \cdots, n+l-1, n+k\}$ and any $n+l \leq j \leq n+k-1$.

Proof. Assume that $(s_1, \cdots, s_{n+l-1}) = (1, \cdots, \widehat{s_0}, \cdots, n+l-1, n+k)$. Notice that for any $n+l \leq j \leq n+k-1, i \neq s_0$ and $i \in \{1, \cdots, n+l-2, n+k\}$,

$$\begin{vmatrix} L^{\beta_0} a_{s_1} & \cdots & L^{\beta_0} a_{s_{n+l-1}} & L^{\beta_0}(G_i^j a_i) \\ L^{\beta_1} a_{s_1} & \cdots & L^{\beta_1} a_{s_{n+l-1}} & L^{\beta_1}(G_i^j a_i) \\ \cdots & \cdots & \cdots & \cdots \\ L^{\beta_{n+l-1}} a_{s_1} & \cdots & L^{\beta_{n+l-1}} a_{s_{n+l-1}} & L^{\beta_{n+l-1}}(G_i^j a_i) \end{vmatrix} \equiv 0. \tag{2.1.59}$$

Combining this with equation (2.1.57), one can check that equation (2.1.58) holds. □

By Lemma 1.3.4, equation (2.1.54), and (2.1.58), we immediately obtain that in O,

$$L^{\beta_t}(a_j - \sum_{i=1}^{n+l-1} G_i^j a_i - G_{n+k}^j a_{n+k}) = 0, \forall\ 1 \le t \le n+l-1, n+l \le j \le n+k-1.$$

In particular, when $t = 0$, we have:

$$a_j - \sum_{i=1}^{n+l-1} G_i^j a_i - G_{n+k}^j a_{n+k} = 0, n+l \le j \le n+k-1. \tag{2.1.60}$$

That is in O,

$$F_j + \overline{\phi^*_{z'_j}} - \sum_{i=1}^{n+l-1} \overline{G_i^j}(F_i + \overline{\phi^*_{z'_i}}) - \overline{G_{n+k}^j}(\frac{1}{2\sqrt{-1}} + \overline{\phi^*_{z'_{n+k}}}) = 0. \tag{2.1.61}$$

Recall that we have, by shrinking O if necessary, in O,

$$-\frac{F_{n+k} - \overline{F_{n+k}}}{2\sqrt{-1}} + F_1\overline{F_1} + \cdots + F_{n+k-1}\overline{F_{n+k-1}} + \phi^*(F, \overline{F}) = 0, \tag{2.1.62}$$

$$\frac{L_j\overline{F_{n+k}}}{2\sqrt{-1}} + F_1 L_j\overline{F_1} + \cdots + F_{n+k-1}L_j\overline{F_{n+k-1}} + L_j\phi^*(F, \overline{F}) = 0, 1 \le j \le n, \tag{2.1.63}$$

$$\frac{L^{\beta_t}\overline{F_{n+k}}}{2\sqrt{-1}} + F_1 L^{\beta_t}\overline{F_1} + \cdots + F_{n+k-1}L^{\beta_t}\overline{F_{n+k-1}} + L^{\beta_t}\phi^*(F, \overline{F}) = 0, n+1 \le t \le n+l-1. \tag{2.1.64}$$

We introduce local coordinates $(x, y, s) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d$ that vanish at the central ponit $p \in M$. By Theorem 2.1.9, $G_i^j, G_{n+k}^j, F_1, \cdots, F_{n+k}$ extend to almost analytic functions into a wedge $\{(x, y, s+it) \in U \times V \times \Gamma_1 : (x, y, s) \in U \times V,\ t \in \Gamma_1\}$, with edge M near $p = 0$ for all $1 \le i \le n+l-1, n+l \le j \le n+k-1$. Here $U \times V$ is a neighborhood of the origin in $\mathbb{C}^n \times \mathbb{R}^d$ and Γ_1 is an acute convex cone in $\mathbb{R}^d$ in $t-$space. We still denote the extended functions by $G_i^j, G_{n+k}^j, F_1, \cdots, F_{n+k}$. Arguments similar to those used in the proof of Theorem 2.1.3 imply that the G_i^j and G_{n+k}^j satisfy the estimates:

$$\left|D_x^\alpha D_y^\beta D_s^\gamma G_i^j(z, s, t)\right| \le \frac{C}{|t|^\lambda}, \left|D_x^\alpha D_y^\beta D_s^\gamma G_{n+k}^j(z, s, t)\right| \le \frac{C}{|t|^\lambda}, \text{ for some } C, \lambda > 0$$

and

$$D_x^\alpha D_y^\beta D_s^\gamma \overline{\partial}_{w_\nu} G_i^j(z, s, t) = O(|t|^m), D_x^\alpha D_y^\beta D_s^\gamma \overline{\partial}_{w_\nu} G_{n+k}^j(z, s, t) = O(|t|^m),$$

for all $1 \le i \le n+l-1, n+l \le j \le n+k-1, 1 \le \nu \le d, m \ge 1$. And similarly for $F_1, \cdots, F_{n+k}$.

We now use equations (2.1.61), (2.1.62), (2.1.63) and (2.1.64) to get a smooth map $\Psi(Z', \overline{Z'}, W) = (\Psi_1, \cdots, \Psi_{n+k})$ defined in a neighborhood of $\{0\} \times \mathbb{C}^q$ in $\mathbb{C}^{n+k} \times \mathbb{C}^q$, smooth in the first $n+k$ variables and polynomial in last q variables for some integer q, such that,

$$\Psi(F, \overline{F}, (L^\alpha \overline{F})_{1 \le |\alpha| \le l}, \overline{G_1^{n+l}}, \cdots, \overline{G_{n+l-1}^{n+l}}, \overline{G_{n+k}^{n+l}}, \cdots, \overline{G_1^{n+k-1}}, \cdots, \overline{G_{n+l-1}^{n+k-1}}, \overline{G_{n+k}^{n+k-1}}) = 0$$

at $(z,s,0)$ with $(z,s)\in U\times V$. Write

$$\overline{G}=(\overline{G_1^{n+l}},\cdots,\overline{G_{n+l-1}^{n+l}},\overline{G_{n+k}^{n+l}},\cdots,\overline{G_1^{n+k-1}},\cdots,\overline{G_{n+l-1}^{n+k-1}},\overline{G_{n+k}^{n+k-1}}).$$

Observe that

$$\Psi_{Z'}|_{(p,(L^\alpha\overline{F})_{1\le|\alpha|\le l}(p),\overline{G}(p))}=\begin{pmatrix}\mathbf{0}_{n+l-1} & \mathbf{0}_{k-l} & \frac{\sqrt{-1}}{2}\\ \mathbf{B}_{n+l-1} & \mathbf{0} & \mathbf{b}\\ \mathbf{C} & \mathbf{I}_{k-l} & \mathbf{0}_{k-l}^t\end{pmatrix},$$

where $\mathbf{0}_N$ is an $N-$dimensional zero row vector, $\mathbf{C}$ is a $(k-l)\times(n+l-1)$ matrix, $\mathbf{I}_{k-l}$ is the $(k-l)\times(k-l)$ identity matrix and we recall that $\mathbf{B}_{n+l-1}$ is an invertible $(n+l-1)\times(n+l-1)$ matrix, $\mathbf{0}$ is an $(n+l-1)\times(k-l)$ zero matrix, $\mathbf{b}$ is an $(n+l-1)-$dimensional column vector.

The matrix $\Psi_{Z'}|_{(p,(L^\alpha\overline{F})_{1\le|\alpha|\le l}(p),\overline{G}(p))}$ is invertible. By applying Theorem 2.1.16, we get a solution $\psi=(\psi_1,\cdots,\psi_{n+k})$ satisfying (2.1.17) and for each $1\le j\le n+k$,

$$F_j=\psi_j(F,\overline{F},(L^\alpha\overline{F})_{1\le|\alpha|\le l},\overline{G})$$

at $(z,s,0)$ with $(z,s)\in U\times V$. Recall that in Section 2.1.3, for each $i=1,\cdots,n$, we denote by M_i the smooth extension of L_i to $U\times V\times\mathbb{R}^d$ satisfying (2.1.39). For each $1\le j\le n+k$, set

$$h_j(z,s,t)=\psi_j(F(z,s,-t),\overline{F}(z,s,-t),(M^\alpha\overline{F})_{1\le|\alpha|\le l}(z,s,-t),\overline{G}(z,s,-t))$$

and shrink U and V and choose δ in such a way that h_j is well defined and continuous in $\overline{\Omega_-}$ where $\Omega_-=\{(x,y,s+it):(x,y,s)\in U\times V,\ t\in-\Gamma_1,|t|\le\delta\}$. The same proof as before leads to the estimates:

$$\left|D_x^\alpha D_y^\beta D_s^\gamma h_j(z,s,t)\right|\le\frac{C}{|t|^\lambda},\ \text{for some } C,\lambda>0$$

and

$$D_x^\alpha D_y^\beta D_s^\gamma\overline{\partial}_{w_\nu}h_j(z,s,t)=O(|t|^m),\ \forall\nu=1,\cdots,d,\ m=1,2,\dots$$

for $t\in-\Gamma_1$, $1\le j\le n+k$.

Notice that the F_j satisfy similar estimates in Γ_1, and $b_+F_j=b_-h_j$ for each $1\le j\le n+k$. Applying Theorem V.3.7 in [BCH] as before, we conclude that F is smooth near p. This establishes Theorem 2.1.20.

□

Proof of Theorem 2.1.9 and Theorem 2.1.5: Theorem 2.1.9 follows easily from Theorem 2.1.3 in the nondegenerate case ($l=k$) and from Theorem 2.1.20 in the degenerate case ($l\le k-1$). We thus establish Theorem 2.1.9 and hence Theorem 2.1.5.

As a consequence of Theorem 2.1.20, we immediately have

Corollary 2.1.21. *Let $M\subset\mathbb{C}^{n+1}$, $M'\subset\mathbb{C}^{n+k}$ be two smooth strongly pseudoconvex real hypersurfaces $(n\ge1,k\ge1)$, $F: M\to M'$ be a C^2-smooth CR map. Assume that* $\mathrm{rank}_2(F,p)\le n+1$ *everywhere in M. Then F is smooth.*

Proof. We may assume that F is nonconstant. By a result of Pinchuk ([Pi]), $dF : T_p^{(1,0)}M \to T_{F(p)}^{(1,0)}M'$ is injective for every $p \in M$. Note that $\mathrm{rank}_1(F,p) = n+1$ for all $p \in M$ by Lemma 2.1.17. By Theorem 2.1.20 (note that in this case, the proof showed that we did not need F to be C^k), we arrive at the conclusion. $\square$

Since a CR diffeomorphism of class C^k of a $k-$nondegenerate manifold is $k-$nondegenerate, Theorem 2.1.3 implies the following:

Corollary 2.1.22. *Let $M \subset \mathbb{C}^N$ be a generic CR manifold that is k_0-nondegenerate. Suppose $H = (H_1, \cdots, H_N) : M \to M$ is a CR diffeomorphism of class C^{k_0} such that for some $p_0 \in M$ and an open convex cone $\Gamma \subset \mathbb{R}^d$,*

$$\mathrm{WF}(H_j)|_{p_0} \subset \Gamma, j = 1, \cdots, N$$

where d is the CR codimension of M. Then H is C^∞ in some neighborhood of p_0.

2.2 CR mappings into a Levi-nondegenerate hypersurface

2.2.1 Main results

Let M and M' be CR manifolds with CR bundles $\mathcal{V}$ and $\mathcal{V}'$ respectively. Recall that a differentiable CR mapping $F : M \to M'$, is called CR transversal at $p \in M$ if $dF(\mathbb{C}T_pM)$ is not contained in $\mathcal{V}'_{F(p)} + \overline{\mathcal{V}'_{F(p)}}$.

The main result of this section is as follows:

Theorem 2.2.1. *Let M be a smooth abstract CR manifold of hypersurface type of CR dimension n and $M' \subset \mathbb{C}^{N+1}, (n \geq 1, n < N \leq 2n)$ be a smooth real hypersurface. Assume that M and M' are Levi-nondegenerate and M' has signature $(l, N-l)$, $l > 0$ the number of positive eigenvalues of the Levi form. Let $F = (F_1, \cdots, F_{N+1}) : M \to M'$ be a CR-transversal CR mapping of class C^{N-n+1}. Assume that $l \leq n$ and $N - l \leq n$. Then F is smooth on a dense open subset of M.*

We remark that if $l > n$ or $N - l > n$, Example 2.2.5 will show that the Theorem will not hold. This explains the assumption $N \leq 2n$ in Theorem 2.2.1. Note that the case $l = 0$ (and therefore also $l = N$) was treated in Section 2.1, and therefore, we may always assume that $0 < l < N$. Since CR functions are C^∞ whenever the Levi form has a positive and a negative eigenvalue, we may also assume that M is strongly pseudoconvex. Our methods also lead to the following analyticity result:

Theorem 2.2.2. *Let $M \subset \mathbb{C}^{n+1}$ and $M' \subset \mathbb{C}^{N+1}, (n \geq 1, n < N \leq 2n)$ be real analytic hypersurfaces. Assume that M and M' are Levi-nondegenerate and M' has signature $(l, N-l)$, $l > 0$ the number of positive eigenvalues of the Levi form. Let $F = (F_1, \cdots, F_{N+1}) : M \to M'$ be a CR-transversal CR mapping of class C^{N-n+1}. Assume that $l \leq n$ and $N - l \leq n$. Then F is real analytic on a dense open subset of M.*

It is well known that in Theorem 2.2.2, if $M_1 \subset M$ denotes the dense subset where F is real analytic, then F extends as a holomorphic map in a neighborhood of each point of M_1. As a consequence of Theorem 2.2.1, Theorem 2.2.2 and the main result in Section 2.1, we have the following:

Corollary 2.2.3. *Let $M \subset \mathbb{C}^n$ and $M' \subset \mathbb{C}^{n+1}(n \geq 2)$ be real analytic (resp. smooth) hypersurfaces. Assume that M and M' are Levi-nondegenerate and $F : M \to M'$ is a CR-transversal CR mapping of class C^2. Then F is real analytic (resp. smooth) on a dense open subset of M.*

When F is assumed to be C^∞, Corollary 2.2.3 in the real analytic case was proved in [EL]. Corollary 2.2.3 implies that a result on finite jet determination proved in [EL] (see Corollary 1.3 in [EL]) holds under a milder smoothness assumption:

Corollary 2.2.4. *Let $M \subset \mathbb{C}^n$ and $M' \subset \mathbb{C}^{n+1}(n \geq 2)$ be smooth connected hypersurfaces which are Levi-nondegenerate, and $f : M \to M'$ and $g : M \to M'$ transversal CR mappings of class C^2. If for any p in some dense open subset of M, the jets at p of f and g satisfy $j^4_p f = j^4_p g$, then $f = g$.*

The following examples show that in Theorems 2.2.1 and 2.2.2, neither the hypothesis on the signature of M' nor the transversality assumption on F can be dropped.

Example 2.2.5. *Let $M \subset \mathbb{C}^{n+1}(n \geq 1)$ be the hypersurface given by $\{(z_1, \ldots, z_n, w) \in \mathbb{C}^{n+1} : \operatorname{Im} w = \sum_{i=1}^n |z_i|^2\}$. Let $M' \subset \mathbb{C}^{N+1}\,(N \geq n+2)$ be defined as $\{(z_1, \cdots, z_N, w) \in \mathbb{C}^{N+1} : \operatorname{Im} w = \sum_{i=1}^{n+1} |z_i|^2 + \sum_{j=n+2}^{N-1} \epsilon_j |z_j|^2 - |z_N|^2\}$, where each $\epsilon_j \in \{1, -1\}$. Let f be a C^{N-n+1} CR function on M which not smooth on any nonempty open subset of M (see Theorem 2.7 below for an example of such). Then $F(z_1, \ldots, z_n, w) = (z_1, \ldots, z_n, f(z_1, \ldots, z_n, w), 0, \ldots 0, f(z_1, \ldots, z_n, w), w)$ is a CR-transversal map of class C^{N-n+1} from M to M'. Clearly F is not smooth on any nonempty open subset of M and, hence, since we may assume in Theorem 2.2.1 that M' is not strongly pseudoconvex and that therefore when $l > n$, $N \geq n+2$, Theorem 2.1 does not hold. Likewise, the theorem does not hold when $N - l > n$. It follows that for the theorem to hold, we need to assume that $l \leq n$, $N - l \leq n$ and hence $N \leq 2n$.*

Example 2.2.6. *Let $M \subset \mathbb{C}^n\,(n \geq 2)$ be given by $\{(z_1, \cdots, z_{n-1}, w) \in \mathbb{C}^n : \operatorname{Im} w = \sum_{i=1}^{n-1} |z_i|^2\}$ and define $M' \subset \mathbb{C}^{n+1}$ by $M' = \{(z_1, \cdots, z_n, w) \in \mathbb{C}^{n+1} : \operatorname{Im} w = \sum_{i=1}^{n-1} |z_i|^2 - |z_n|^2\}$. Then $F = (0, \cdots, 0, f, f, 0)$ is a C^2 CR map from M to M', where f is a C^2 CR function on M which is not smooth on any nonempty open subset of M. Note that F is not transversal at any point on M, and is not smooth on any nonempty open subset of M.*

In order to make the preceding two examples meaningful, we will next show the existence of a C^k CR function on a strongly pseudoconvex hypersurface which is not smooth on any nonempty open subset.

Theorem 2.2.7. *Let $D \subset \mathbb{C}^n$ be a bounded domain with a smooth boundary M which is strongly pseudoconvex. Let $k \geq 1$ be a positive integer. Then there exists a CR function f on M of class C^k which is not C^∞ on any nonempty open subset of M.*

Proof. First fix $p \in M$ and let $g \in C^\infty(\overline{D})$ that is holomorphic on D and peaks at p, say, $|g(z)| < g(p) = 1$ for $z \in \overline{D} \setminus p$. By Hopf's Lemma, the normal derivative of g at p is nonzero and hence there is a smooth vector field X tangent to M near p such that $Xg(p) \neq 0$. It follows that for any positive integer m, with a choice of a branch of logarithm, the function $g_m(z) = (1 - g(z))^{m+\frac{1}{2}}$ is a

CR function of class C^m on M but which is not of class C^{m+1} at p. Let $\{p_i\}_{i=0}^{\infty} \subset M$ be a dense subset of M. We choose a sequence of C^k CR functions $\{f_i\}_{i=0}^{\infty}$ on M with the following properties: For each $i \geq 0, f_i \in C^{k+i}(M) \cap C^{\infty}(M \setminus \{p_i\})$, and f_i is not C^{k+i+1} at p_i. Then there exists a sequence of positive numbers $\{b_i\}_{i=0}^{\infty}$ such that, for any sequence of complex numbers $\{c_i\}_{i=0}^{\infty}$ with $|c_i| \leq b_i, i \geq 0, \sum_{i=0}^{\infty} c_i f_i$ converges uniformly to a C^k CR function on M.

We fix a local chart (U_i, x) for each $p_i, i \geq 0$ on M, where U_i is a neighborhood of p_i. Choose $\Omega_i \subset\subset U_i, i \geq 0$ to be a sufficiently small neighborhood of p_i with the following properties:

(1). For each $i \geq 1, p_0, \cdots, p_{i-1} \notin \Omega_i$.

(2). There exists a sequence of positive numbers $\{M_i^j\}_{i>j}$ such that for any $j \geq 0$, $|D^{\alpha} f_i(x)| \leq M_i^j$ for all $|\alpha| \leq k+j+1, i > j$, and for all $x \in \overline{\Omega_j}$. Here α is a multiindex, and D^{α} denotes derivatives with respect to all real variables. The existence of such $\{M_i^j\}_{i>j}$ is ensured by the fact that f_i is $C^{k+j+1}-$smooth for all $i > j$.

Next choose a sequence of positive numbers $\{a_i\}_{i=0}^{\infty}$ as follows: $a_0 < b_0$, and

$$a_i < \min\{b_i, \frac{1}{2^i M_i^0}, \cdots, \frac{1}{2^i M_i^{i-1}}\}, \text{ for } i \geq 1.$$

Let $f = \sum_{i=0}^{\infty} a_i f_i$. Then f is a C^k CR function on M. Moreover, from the choice of $a_i, i \geq 1$, one can see that $\sum_{i=1}^{\infty} a_i D^{\alpha} f_i$ converges uniformly in Ω_0, for any $|\alpha| \leq k+1$. Consequently, $\sum_{i=1}^{\infty} a_i f_i$ converges to a C^{k+1} function in Ω_0. Thus f is not C^{k+1} at p_0 since f_0 is not. Similarly, one can check that f is not C^{k+i+1} at p_i, for all $i \geq 0$. Hence, by the density of the sequence $\{p_i\}_{i=0}^{\infty}$, f is a C^k CR function which is not smooth on any nonempty open subset of M. $\square$

Remark 2.2.8. *As a consequence of Theorem 2.2.7, we see that for any* $k \geq 1$, *there exists a CR function* f *of class* C^k *on the hypersurface* $M = \{(z_1, \cdots, z_n, w) \in \mathbb{C}^{n+1} : \operatorname{Im} w = \sum_{i=1}^{n} |z_i|^2\} (n \geq 1)$ *which is not smooth on any nonempty open subset, since* M *is biholomorphically equivalent to the unit sphere* $\partial B^{n+1} = \{(z_1, \cdots, z_n) \in \mathbb{C}^{n+1} : |z_1|^2 + \cdots + |z_{n+1}|^2 = 1\}$ *minus the point* $(0, \ldots, 0, 1)$.

2.2.2 Proof of Theorem 2.2.1 and 2.2.2

Let M, M', F be as in Theorem 2.2.1. We work near a point $p \in M$ which we fix. If the Levi form of M at p has a positive and a negative eigenvalue, then the smoothness of F follows trivially and so we may assume that M is strongly pseudoconvex at p. Let $\mathcal{V}$ denote the CR bundle of M. By Theorem IV.1.3 in [T], there is an integrable CR structure on M near p with CR bundle $\widehat{\mathcal{V}}$ that agrees with $\mathcal{V}$ to infinite order at p. In particular, $(M, \widehat{\mathcal{V}})$ is strongly pseudoconvex at p and hence we can find local coordinates $x_1, y_1, \ldots, x_n, y_n$ and s vanishing at p and first integrals $Z_j = x_j + \sqrt{-1} y_j = z_j$, $1 \leq j \leq n$, $Z_{n+1} = s + \sqrt{-1}\psi(z, \overline{z}, s)$ where $z = (z_1, \ldots, z_n)$ and ψ is a real-valued smooth function satisfying

$$\psi(z, \overline{z}, s) = |z|^2 + O(s^2) + O(|z|^3).$$

In these coordinates, near the origin, the bundle $\mathcal{V}$ has a basis of the form

$$L_j = \frac{\partial}{\partial \overline{z_j}} + A_j(z,\overline{z},s)\frac{\partial}{\partial s} + \sum_{k=1}^{n} B_{jk}(z,\overline{z},s)\frac{\partial}{\partial z_k} \; 1 \le j \le n$$

where each

$$A_j(z,\overline{z},s) = \frac{-\sqrt{-1}\psi_{\overline{z_j}}(z,\overline{z},s)}{1+\sqrt{-1}\,\psi_s(z,\overline{z},s)} \text{ to infinite order at } 0$$

and the B_{jk} vanish to infinite order at 0. We may assume $0 \in M'$, $F(0) = 0$ and that we have coordinates $Z' = (z'_1, \ldots, z'_{N+1})$ in $\mathbb{C}^{N+1}$ so that near 0, M' is defined by

$$-\frac{z'_{N+1} - \overline{z'_{N+1}}}{2\sqrt{-1}} + \sum_{j=1}^{l} |z'_j|^2 - \sum_{i=l+1}^{N} |z'_j|^2 + \phi^*(Z', \overline{Z'}) = 0 \tag{2.2.1}$$

where $\phi^*(Z', \overline{Z'}) = O(|Z'|^3)$ is a real-valued smooth function.

In the following, for two m-tuples $x = (x_1. \cdots, x_m), y = (y_1, \cdots, y_m)$ of complex numbers, we write $\langle x, y\rangle_l = \sum_{j=1}^m \delta_{j,l} x_j y_j$ and $|x|_l^2 = \langle x, \overline{x}\rangle_l = \sum_{j=1}^m \delta_{j,l}|x_j|^2$, where we denote by $\delta_{j,l}$ the symbol which takes value 1 when $1 \le j \le l$ and -1 otherwise. Let $\widetilde{z'} = (z'_1, \cdots, z'_N)$. Then M' is locally defined by

$$\rho(Z', \overline{Z'}) = -\frac{z'_{N+1} - \overline{z'}_{N+1}}{2\sqrt{-1}} + |\widetilde{z'}|_l^2 + \phi^*(Z', \overline{Z'}) = 0.$$

If we write $F = (F_1, \ldots, F_{N+1}) = (\widetilde{F}, F_{N+1})$, then F satisfies:

$$-\frac{F_{N+1} - \overline{F}_{N+1}}{2\sqrt{-1}} + |\widetilde{F}|_l^2 + \phi^*(F, \overline{F}) = 0. \tag{2.2.2}$$

Let $\mathcal{V}'$ denote the CR bundle of M'. Since F is CR-transversal, and the fibers $\mathcal{V}_0$ and $\mathcal{V}'_0$ are spanned by $\frac{\partial}{\partial \overline{z_j}}, 1 \le j \le n$ and $\frac{\partial}{\partial \overline{z'_k}}, 1 \le k \le N$, we get $\lambda := \frac{\partial F_{N+1}}{\partial s}(0) \ne 0$. Moreover, equation (2.2.2) shows that the imaginary part of F_{N+1} vanishes to second order at the origin, and so the number λ is real. We claim that we can assume that $\lambda > 0$. Indeed, when $\lambda < 0$, by considering $\widetilde{M'}$ defined by $\rho(\tau(Z), \overline{\tau(Z)})$ instead of M', and considering $\widetilde{F} = \tau \circ F$ instead of F, we get $\lambda > 0$. Here τ is the change of coordinates in $\mathbb{C}^{N+1} : \tau(z_1, \cdots, z_N, w) = (z_1, \cdots, z_N, -w)$. By applying L_j, $L_j L_k, \overline{L_j L_k}$ to equation (2.2.2), and evaluating at 0, we get

$$\frac{\partial F_{N+1}}{\partial z_i}(0) = 0, \; 1 \le i \le n,$$

and

$$\frac{\partial F_{N+1}}{\partial z_k \partial z_j}(0) = \frac{\partial F_{N+1}}{\partial \overline{z_k} \partial \overline{z_j}} = 0, \;\; 1 \le k, j \le n.$$

We next apply $\overline{L_j} L_k$ to F_{N+1} and evaluate at 0 to get

$$\frac{\partial F_{N+1}}{\partial \overline{z_j} \partial z_k}(0) = \sqrt{-1}\delta_{jk}\lambda,$$

where δ_{jk} is the Kronecker delta. Hence we are able to write,

$$F_{N+1}(z,\overline{z},s) = \lambda s + \sqrt{-1}\lambda|z|^2 + O(|z||s| + s^2) + o(|z|^2), \tag{2.2.3}$$

For $1 \leq j \leq N$, using $L_k F_j(0) = 0$, we have:

$$F_j = b_j s + \sum_{i=1}^{n} a_{ij} z_i + O(|z|^2 + s^2). \tag{2.2.4}$$

for some $b_j \in \mathbb{C}, a_{ij} \in \mathbb{C}, 1 \leq i \leq n, 1 \leq j \leq N$, or equivalently,

$$(F_1, \cdots, F_N) = s(b_1, \cdots, b_N) + (z_1, \cdots, z_n)A + (\hat{F}_1, \cdots, \hat{F}_N) \tag{2.2.5}$$

where $A = (a_{ij})_{n\times N}$ is an $n \times N$ matrix, and $\hat{F}_j = O(|z|^2 + s^2), 1 \leq j \leq N$. Plugging (2.2.3) and (2.2.4) into equation (2.2.2), we get

$$\lambda|z|^2 + O(|z||s| + s^2) + o(|z|^2) = \langle zA, \overline{z}\overline{A}\rangle_l + O(|z||s| + s^2) + o(|z|^2).$$

When $s = 0$ the latter equation leads to

$$\lambda|z|^2 + o(|z|^2) = \langle zA, \overline{z}\overline{A}\rangle_l + o(|z|^2).$$

It follows that

$$\lambda I_n = AE(l,N)A^*, \tag{2.2.6}$$

where $A^* = \overline{A^t}$. Here I_n denotes the n by n identity matrix and $E(k,m)$ denotes the $m \times m$ diagonal matrix with its first k diagonal elements 1 and the rest -1. Note from equation (2.2.6) that the matrix A has rank n. Moreover, since $\lambda > 0$, we get $l \geq n$ from equation (2.2.6) using elementary linear algebra. Since $l \leq n$, it follows that $l = n$. Thus M' is locally defined by

$$\rho(Z', \overline{Z'}) = -\frac{z'_{N+1} - \overline{z'}_{N+1}}{2\sqrt{-1}} + |\widetilde{z'}|^2_n + \phi^*(Z', \overline{Z'}) = 0, \tag{2.2.7}$$

and we have

$$\lambda I_n = AE(n,N)A^*. \tag{2.2.8}$$

A direct computation shows that

$$L_i\overline{F}_j(0) = \overline{a}_{ij},\ 1 \leq i \leq n, 1 \leq j \leq N.$$

Since $A = (a_{ij})_{1\leq i\leq n, 1\leq j\leq N}$ is of rank n, we conclude that $dF : T_0^{(0,1)}M \to T_0^{(0,1)}M'$ is injective. Now let us introduce some notations. Set

$$a_j(Z,\overline{Z}) = \rho_{z'_j}(F(Z), \overline{F(Z)}) = \delta_{j,n}\overline{F}_j + \phi^*_{z'_j}(F, \overline{F}),\ 1 \leq j \leq N,$$

$$a_{N+1}(Z,\overline{Z}) = \rho_{z'_{N+1}}(F(Z), \overline{F(Z)}) = \frac{\sqrt{-1}}{2} + \phi^*_{z'_{N+1}}(F, \overline{F}).$$

and

$$\mathbf{a} = (a_1, \cdots, a_{N+1}).$$

We have:

$$L^\alpha \rho_{Z'}(F, \overline{F}) = L^\alpha \mathbf{a} = (L^\alpha a_1, \cdots, L^\alpha a_N, L^\alpha a_{N+1}),$$

for any multiindex $0 \leq |\alpha| \leq N - n + 1$. Recall that for any $0 \leq i \leq N - n + 1$,

$$\mathrm{rank}_i(F, p) = \dim_{\mathbb{C}}(\mathrm{Span}_{\mathbb{C}}\{L^\alpha \mathbf{a}(Z, \overline{Z})|_p : 0 \leq |\alpha| \leq i\}).$$

Recall the notions of k_0-nondegeneracy and $\mathrm{rank}_l(F, p)$ from Section 2.1. From the injectivity of dF, we get

Lemma 2.2.9. *Let M, M', F be as in Theorem 2.2.1. Then for any $p \in M, \mathrm{rank}_0(F, p) = 1, \mathrm{rank}_1(F, p) = n + 1$. Consequently, $\mathrm{rank}_i(F, p) \geq n + 1$, for any $i \geq 1$.*

We next prove a normalization lemma which will be used later.

Lemma 2.2.10. *Let M, M', F be as in Theorem 2.2.1. Assume $rank_l(F, p) = m + 1$, for some $l > 1, m \geq n$. Then there exist multiindices $\{\beta_{n+1}, \cdots, \beta_m\}$ with $1 < |\beta_i| \leq l$ for all i, such that after a linear biholomorphic change of coordinates in $\mathbb{C}^{N+1}$: $\widetilde{Z} = (\widetilde{z}_1, \cdots, \widetilde{z}_N, \widetilde{z}_{N+1}) = ((z'_1, \cdots, z'_N)V, z'_{N+1})$, where $\widetilde{Z}$ denotes the new coordinates in $\mathbb{C}^{N+1}$, and V is an $N \times N$ matrix satisfying $VE(n, N)V^* = E(n, N)$, the following hold:*

$$\widetilde{\mathbf{a}}|_p = (0, \cdots, 0, \frac{\sqrt{-1}}{2}), \begin{pmatrix} L_1 \widetilde{\mathbf{a}}|_p \\ \cdots \\ L_n \widetilde{\mathbf{a}}|_p \end{pmatrix} = \begin{pmatrix} \sqrt{\lambda} \mathbf{I}_n & \mathbf{0}_{n \times (N-n)} & \mathbf{0}_n^t \end{pmatrix}, \tag{2.2.9}$$

$$\begin{pmatrix} L^{\beta_{n+1}} \widetilde{\mathbf{a}}|_p \\ \cdots \\ L^{\beta_m} \widetilde{\mathbf{a}}|_p \end{pmatrix} = \begin{pmatrix} \mathbf{C} & \mathbf{M}_{m-n} & \mathbf{0}_{(m-n) \times (N-m)} & \mathbf{d} \end{pmatrix}. \tag{2.2.10}$$

Here we write $\widetilde{\mathbf{a}} = \widetilde{\rho}_{\widetilde{Z}}(\widetilde{Z}(F), \overline{\widetilde{Z}(F)})$, and $\widetilde{\rho}$ is a local defining function of M' near 0 in the new coordinates. Moreover, $\mathbf{I}_n$ is the $n \times n$ identity matrix, $\mathbf{0}_{n \times (N-n)}$ is an $n \times (N-n)$ zero matrix, and $\mathbf{0}_n^t$ is an $n-$dimensional zero column vector. $\mathbf{C}$ is an $(m-n) \times n$ matrix, $\mathbf{M}_{m-n}$ is an $(m-n) \times (m-n)$ invertible matrix, $\mathbf{0}_{(m-n) \times (N-m)}$ is an $(m-n) \times (N-m)$ zero matrix, and $\mathbf{d}$ is an $(m-n)-$dimensional column vector.

Proof. Assume that $p = 0$. Note that $L_i a_j(0) = \delta_{j,n} L_i \overline{F}_j(0) = \delta_{j,n} \overline{a}_{ij}$. Thus we have,

$$\begin{pmatrix} \mathbf{a}|_0 \\ L_1 \mathbf{a}|_0 \\ \cdots \\ L_n \mathbf{a}|_0 \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{N-n} & \frac{\sqrt{-1}}{2} \\ \overline{A} E(n, N) & \mathbf{0}_n^t \end{pmatrix},$$

where $A = (a_{ij})_{1\le i\le n, 1\le j\le N}$ is as mentioned above, $\mathbf{0}_{N-n}$ is an $(N-n)-$ dimensional zero row vector, $\mathbf{0}_n^t$ is an $n-$dimensional zero column vector. Let $B = E(n,N)A^t$. Then by equation (2.2.8), we know that $\overline{A}B = \lambda I_n$, and $B^*E(n,N)B = \lambda I_n$. By a result in [BHu] (see page 386 in [BHu] for more details on this), we can find an $N\times N$ matrix U whose first n rows are rows of B^*, such that, $UE(n,N)U^* = \lambda E(n,N)$. Consequently, $U^*E(n,N)U = \lambda E(n,N), WE(n,N)W^* = E(n,N)$, where $W = \frac{1}{\sqrt{\lambda}}U^*$.

We next make the following change of coordinates in $\mathbb{C}^{N+1}$: $\widetilde{Z} = Z'D^{-1}$ where

$$D = \begin{pmatrix} E(n,N)W & \mathbf{0}_N^t \\ \mathbf{0}_N & 1 \end{pmatrix},$$

and 0_N is $N-$dimensional zero row vector. Then the function $\widetilde{\rho}(\widetilde{Z},\overline{\widetilde{Z}}) = \rho(\widetilde{Z}D, \overline{\widetilde{Z}D})$ is a defining function for M' near 0 with respect to the new coordinates $\widetilde{Z}$. By the chain rule,

$$\widetilde{\rho}_{\widetilde{Z}}(\widetilde{F}(Z),\overline{\widetilde{F}(Z)}) = \rho_{Z'}(F(Z),\overline{F(Z)})D, \quad \text{where } \widetilde{F}(Z) = F(Z)D^{-1}.$$

For any multiindex α,

$$L^\alpha\widetilde{\rho}_{\widetilde{Z}}(\widetilde{F}(Z),\overline{\widetilde{F}(Z)}) = L^\alpha\rho_{Z'}(F(Z),\overline{F(Z)})D.$$

In particular, at $p=0$, we have,

$$\begin{pmatrix} \widetilde{\mathbf{a}}|_0 \\ L_1\widetilde{\mathbf{a}}|_0 \\ \cdots \\ L_n\widetilde{\mathbf{a}}|_0 \end{pmatrix} = \begin{pmatrix} \mathbf{0}_N & \frac{\sqrt{-1}}{2} \\ \overline{A}E(n,N) & \mathbf{0}_n^t \end{pmatrix} D = \begin{pmatrix} \mathbf{0}_N & \frac{\sqrt{-1}}{2} \\ \overline{A}W & \mathbf{0}_n^t \end{pmatrix},$$

where $\widetilde{\mathbf{a}}(Z,\overline{Z}) = \widetilde{\rho}_{\widetilde{Z}}(\widetilde{F}(Z),\overline{\widetilde{F}(Z)})$. Since $\overline{A} = B^*E(n,N)$,

$$\overline{A}W = \frac{1}{\sqrt{\lambda}}B^*E(n,N)U^* = \begin{pmatrix} \sqrt{\lambda}I_n & \mathbf{0} \end{pmatrix}.$$

Thus equation (2.2.9) holds with respect to the new coordinates $\widetilde{Z}$. In the following, we will still write Z' instead of $\widetilde{Z}$, $\mathbf{a}$ instead of $\widetilde{\mathbf{a}}$. Since $\{\mathbf{a}, L_1\mathbf{a},\cdots,L_n\mathbf{a}\}|_0$ is linearly independent, extend it to a basis of $E_l(0)$, which has dimension $m+1$ by assumption. That is, pick multiindices $\{\beta_{n+1},\cdots,\beta_m\}$ with $1<|\beta_i|\le l$ for each i, such that,

$$\{\mathbf{a}, L_1\mathbf{a},\cdots,L_n\mathbf{a}, L^{\beta_{n+1}}\mathbf{a},\cdots,L^{\beta_m}\mathbf{a}\}|_0$$

is linearly independent over $\mathbb{C}$. Write $\hat{\mathbf{a}} = (a_{n+1},\cdots,a_N)$, i.e., the $(n+1)^{\text{th}}$ to N^{th} components of $\mathbf{a}$. Note that $\{\mathbf{a}, L_1\mathbf{a},\cdots,L_n\mathbf{a}\}|_0$ is of the form (2.2.9). The set $\{L^{\beta_{n+1}}\hat{\mathbf{a}},\cdots,L^{\beta_m}\hat{\mathbf{a}}\}|_0$ is linearly independent in $\mathbb{C}^{N-n}$. Let S be the $(m-n)-$dimensional vector space spanned by it and let $\{T_1,\cdots,T_{m-n}\}$ be an orthonormal basis of S. Extend it to an orthonormal basis $\{T_1,\cdots,T_{m-n}, T_{m-n+1},\cdots,T_{N-n}\}$ of $\mathbb{C}^{N-n}$ and set T to be the following $(N-n)\times(N-n)$ unitary matrix:

$$T = \begin{pmatrix} T_1 \\ \cdots \\ T_{N-n} \end{pmatrix}^*.$$

We next make the following change of coordinates:

$$\widetilde{Z} = (\widetilde{z}_1, \cdots, \widetilde{z}_N, \widetilde{z}_{N+1}) = (z'_1, \cdots, z'_n, (z'_{n+1}, \cdots, z'_N)T^{-1}, z'_{N+1}).$$

One can check that equation (2.2.10) holds in the new coordinates $\widetilde{Z}$. $\square$

Remark 2.2.11. *From the construction of V in the proof of Lemma 2.2.10, one can see that, in the new coordinates $\widetilde{Z}$, the following continues to hold: M' is locally defined near 0 by*

$$\widetilde{\rho}(\widetilde{Z}, \overline{\widetilde{Z}}) = -\frac{\widetilde{z}_{N+1} - \overline{\widetilde{z}_{N+1}}}{2\sqrt{-1}} + \sum_{i=1}^{n} |\widetilde{z}_i|^2 - \sum_{i=n+1}^{N} |\widetilde{z}_i|^2 + \widetilde{\phi^*}(\widetilde{Z}, \overline{\widetilde{Z}}) = 0,$$

where $\widetilde{Z} = (\widetilde{z}_1, \cdots, \widetilde{z}_N, \widetilde{z}_{N+1})$, and $\widetilde{\phi^*}(\widetilde{Z}, \overline{\widetilde{Z}}) = O(|\widetilde{Z}|^3)$ is a real-valued smooth function near 0. In what follows, we will write the new coordinates as Z' instead of $\widetilde{Z}$, drop the tilde from $\widetilde{\rho}$ and set $\mathbf{a}(Z, \overline{Z}) = \rho_{Z'}(F(Z), \overline{F(Z)})$.

Remark 2.2.12. *In Lemma 2.2.10, equations (2.2.9), (2.2.10) can be rewritten as follows:*

$$\widetilde{\mathbf{a}}|_0 = (0, \cdots, 0, \frac{\sqrt{-1}}{2}), \begin{pmatrix} L_1 \mathbf{a}|_\mathbf{0} \\ \cdots \\ L_n \mathbf{a}|_\mathbf{0} \\ L^{\beta_{n+1}} \mathbf{a}|_\mathbf{0} \\ \cdots \\ L^{\beta_m} \mathbf{a}|_\mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{B}_m & \mathbf{0} & \mathbf{b} \end{pmatrix}, \tag{2.2.11}$$

where $\mathbf{B}_m$ is an $m \times m$ invertible matrix, $\mathbf{0}$ is an $m \times (N-m)$ zero matrix, $\mathbf{b}$ is an $m-$dimensional column vector. We note that Lemma 2.2.10 plays the same role as Lemma 2.1.18 *in Section 2.1.*

The remaining argument will be essentially the same as in Section 2.1. First we need the following regularity theorem.

Theorem 2.2.13. *Let M, M', F be as in Theorem 2.2.1 (resp. as in Theorem 2.2.2). Let $p \in M$ and O be a neighborhood of p in M. Assume that for some $1 \leq l \leq N-n$, $\mathrm{rank}_l(F,p) = n+l$, and $\mathrm{rank}_{l+1}(F,q) = n+l$ for all $q \in O$. Then F is smooth (resp. real analytic) near p.*

Proof. We first prove Theorem 2.2.13 in the smooth case. Although M' is different from the one in Section 2.1, the proof of theorem 2.1.20 applies to establish Theorem 2.2.13 which involves applications of Lemma 2.2.10 above and Theorem V.3.7 in [BCH]. Assume $p = 0$. From Lemma 2.2.10 and the assumption, after a suitable biholomorphic change of coordinates, we conclude that there exist multiindices $\{\beta_{n+1}, \ldots, \beta_{n+l-1}\}$ with $1 < |\beta_i| \leq l$, such that

$$\widetilde{\mathbf{a}}|_0 = (0, \cdots, 0, \frac{\sqrt{-1}}{2}), \begin{pmatrix} L_1 \mathbf{a}|_\mathbf{0} \\ \cdots \\ L_n \mathbf{a}|_\mathbf{0} \\ L^{\beta_{n+1}} \mathbf{a}|_\mathbf{0} \\ \cdots \\ L^{\beta_{n+l-1}} \mathbf{a}|_\mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{B}_{n+l-1} & \mathbf{0} & \mathbf{b} \end{pmatrix}. \tag{2.2.12}$$

Indeed, the form (2.2.12) is all that is needed to use the proof of Theorem 2.1.20 to arrive at the following:

There are CR functions G_i^j of smoothness class $C^{N+1-n-l}$ defined in a neighborhood O of 0 in M such that :

$$a_j - \sum_{i=1}^{n+l-1} G_i^j a_i - G_{N+1}^j a_{N+1} = 0, \ n+l \leq j \leq N. \tag{2.2.13}$$

That is, in O,

$$\delta_{j,n} F_j + \overline{\phi^*_{z'_j}} - \sum_{i=1}^{n+l-1} \overline{G_i^j} (\delta_{i,n} F_i + \overline{\phi^*_{z'_i}}) - \overline{G_{N+1}^j} (\frac{1}{2\sqrt{-1}} + \overline{\phi^*_{z'_{N+1}}}) = 0. \tag{2.2.14}$$

We also have,

$$-\frac{F_{N+1} - \overline{F_{N+1}}}{2\sqrt{-1}} + F_1\overline{F_1} + \cdots + F_n\overline{F_n} - F_{n+1}\overline{F_{n+1}} - \cdots - F_N\overline{F_N} + \phi^*(F, \overline{F}) = 0; \tag{2.2.15}$$

for $1 \leq j \leq n$,

$$\frac{L_j\overline{F_{N+1}}}{2\sqrt{-1}} + F_1 L_j\overline{F_1} + \cdots + F_n L_j\overline{F_n} - F_{n+1} L_j\overline{F_{n+1}} - \cdots - F_N L_j\overline{F_N} + L_j\phi^*(F, \overline{F}) = 0; \tag{2.2.16}$$

and for $n+1 \leq t \leq n+l-1$,

$$\frac{L^{\beta_t}\overline{F_{N+1}}}{2\sqrt{-1}} + F_1 L^{\beta_t}\overline{F_1} + \cdots + F_n L^{\beta_t}\overline{F_n} - F_{n+1} L^{\beta_t}\overline{F_{n+1}} - \cdots - F_N L^{\beta_t}\overline{F_N} + L^{\beta_t}\phi^*(F, \overline{F}) = 0. \tag{2.2.17}$$

We recall the local coordinates $(x, y, s) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ that vanish at the central ponit $p \in M$. As in the proof of Theorem 2.1.3, $G_i^j, G_{N+1}^j, F_1, \cdots, F_{N+1}$ extend to almost analytic functions into a half-space $\{(x, y, s+it) \in U \times V \times \Gamma : (x, y, s) \in U \times V, t \in \Gamma\}$, with edge M near $p = 0$ for all $1 \leq i \leq n+l-1, n+l \leq j \leq N$. Here $U \times V$ is a neighborhood of the origin in $\mathbb{C}^n \times \mathbb{R}$ and Γ is an interval $(0, r)$ in $t-$space. We still denote the extended functions by $G_i^j, G_{N+1}^j, F_1, \cdots, F_{N+1}$.

Equations (2.2.14), (2.2.15), (2.2.16) and (2.2.17) can be used to get a smooth map $\Psi(Z', \overline{Z'}, W) = (\Psi_1, \cdots, \Psi_{N+1})$ defined in a neighborhood of $\{0\} \times \mathbb{C}^q$ in $\mathbb{C}^{N+1} \times \mathbb{C}^q$, smooth in the first $N+1$ variables and polynomial in the last q variables for some integer q, such that,

$$\Psi(F, \overline{F}, (L^\alpha\overline{F})_{1\leq|\alpha|\leq l}, \overline{G_1^{n+l}}, \cdots, \overline{G_{n+l-1}^{n+l}}, \overline{G_{N+1}^{n+l}}, \cdots, \overline{G_1^N}, \cdots, \overline{G_{n+l-1}^N}, \overline{G_{N+1}^N}) = 0$$

at $(z, s, 0)$ with $(z, s) \in U \times V$. Write

$$\overline{G} = (\overline{G_1^{n+l}}, \cdots, \overline{G_{n+l-1}^{n+l}}, \overline{G_{N+1}^{n+l}}, \cdots, \overline{G_1^N}, \cdots, \overline{G_{n+l-1}^N}, \overline{G_{N+1}^N}). \tag{2.2.18}$$

Observe that

$$\Psi_{Z'}|_{(F(0), \overline{F}(0), (L^\alpha\overline{F})_{1\leq|\alpha|\leq l}(0), \overline{G}(0))} = \begin{pmatrix} \mathbf{0}_{n+l-1} & \mathbf{0}_{N-n-l+1} & \frac{\sqrt{-1}}{2} \\ \mathbf{B}_{n+l-1} & \mathbf{0} & \mathbf{b} \\ \mathbf{C} & -\mathbf{I}_{N-n-l+1} & \mathbf{0}^t_{N-n-l+1} \end{pmatrix},$$

where $\mathbf{0}_m$ is an $m-$dimensional zero row vector, $\mathbf{C}$ is a $(N-n-l+1)\times(n+l-1)$ matrix, $\mathbf{I}_{N-n-l+1}$ is the $(N-n-l+1)\times(N-n-l+1)$ identity matrix and we recall that $\mathbf{B}_{n+l-1}$ is an invertible $(n+l-1)\times(n+l-1)$ matrix, $\mathbf{0}$ is an $(n+l-1)\times(N-n-l+1)$ zero matrix, $\mathbf{b}$ is an $(n+l-1)-$dimensional column vector.

The matrix $\Psi_{Z'}|_{(F(0),\overline{F(0)},(L^\alpha\overline{F})_{1\le|\alpha|\le l}(0),\overline{G}(0))}$ is invertible. By applying the "almost holomorphic" implicit function theorem in [La1], we get a solution $\psi=(\psi_1,\cdots,\psi_{N+1})$ from $\mathbb{C}^{N+1}\times\mathbb{C}^q$ to $\mathbb{C}^{N+1}$ satisfying for each multiindex α, and each j,

$$D^\alpha\frac{\partial\psi_j}{\partial Z_i'}(Z',\overline{Z'},W)=0,\ \text{ if } Z'=\psi(Z',\overline{Z'},W)$$

and for each $1\le j\le N+1$,

$$F_j=\psi_j(F,\overline{F},(L^\alpha\overline{F})_{1\le|\alpha|\le l},\overline{G})$$

at $(z,s,0)$ with $(z,s)\in U\times V$. The map ψ is smooth in all variables and holomorphic in W. For each $j=1,\cdots,n$, we denote by M_j smooth extensions of L_j to $U\times V\times\mathbb{R}$ given by

$$M_j=\frac{\partial}{\partial\bar{z}_j}+A(x,y,s,t)\frac{\partial}{\partial s}+\sum_{k=1}^n B_{jk}(x,y,s,t)\frac{\partial}{\partial z_k}$$

where the B_{jk} and A are smooth extensions of the corresponding coefficients of the L_j satisfying

$$\overline{\partial}_w A(x,y,s,t),\overline{\partial}_w B_{jk}(x,y,s,t)=O(|t|^m),\ \ \forall m=1,2,\cdots. \tag{2.2.19}$$

For each $1\le j\le N+1$, set

$$h_j(z,s,t)=\psi_j(F(z,s,-t),\overline{F}(z,s,-t),(M^\alpha\overline{F})_{1\le|\alpha|\le l}(z,s,-t),\overline{G}(z,s,-t))$$

and shrink U and V and choose δ in such a way that each h_j is defined and continuous in $\overline{\Omega_-}$ where $\Omega_-=\{(x,y,s+it):\ (x,y,s)\in U\times V,\ t\in-\Gamma,|t|\le\delta\}$. The arguments in Section 2.1 showed the estimates:

$$\left|D_x^\alpha D_y^\beta D_s^\gamma h_j(z,s,t)\right|\le\frac{C}{|t|^\lambda},\ \text{ for some } C,\lambda>0$$

and

$$D_x^\alpha D_y^\beta D_s^\gamma\overline{\partial}_w h_j(z,s,t)=O(|t|^m),\ \forall m=1,2,\ldots$$

for $t\in-\Gamma$, $1\le j\le N+1$.

Notice that the F_j satisfy similar estimates for $t\in\Gamma$, and $b_+F_j=b_-h_j$ for each $1\le j\le N+1$. Applying Theorem V.3.7 in [BCH], we conclude that F is smooth near p. This establishes Theorem 2.2.13 in the smooth case.

The proof of Theorem 2.2.13 in the real analytic case is similar and so we will only briefly indicate the modifications that are needed. With M,M',F as in Theorem 2.2.2, we will show that the map

F is real analytic at p which we assume is the origin. Since ϕ^* and the L_j are real analytic now, equations (2.2.14) – (2.2.17) imply that there is a real analytic map $\Psi(Z', \overline{Z'}, W) = (\Psi_1, \cdots, \Psi_{N+1})$ defined in a neighborhood of $\{0\} \times \mathbb{C}^q$ in $\mathbb{C}^{N+1} \times \mathbb{C}^q$, polynomial in the last q variables for some integer q, such that,

$$\Psi(F, \overline{F}, (L^\alpha \overline{F})_{1 \le |\alpha| \le l}, \overline{G_1^{n+l}}, \cdots, \overline{G_{n+l-1}^{n+l}}, \overline{G_{N+1}^{n+l}}, \cdots, \overline{G_1^N}, \cdots, \overline{G_{n+l-1}^N}, \overline{G_{N+1}^N}) = 0$$

at $(z, s, 0)$ with $(z, s) \in U \times V$. Since the matrix Ψ_Z is invertible at the central point, by the holomorphic version of the implicit function theorem, we get a holomorphic map $\psi = (\psi_1, \ldots, \psi_{N+1})$ such that near the origin,

$$F_j = \psi_j(\overline{F}, (L^\alpha \overline{F})_{1 \le |\alpha| \le l}, \overline{G}), \;\; 1 \le j \le N+1,$$

where $\overline{G}$ is as in equation (2.2.18). We may assume that near the origin, M is given by $\{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \operatorname{Im} w = \varphi(z, \overline{z}, s)\}$, where φ is a real-valued, real analytic function with $\varphi(0) = 0$, and $d\varphi(0) = 0$. In the local coordinates $(z, s) \in \mathbb{C}^n \times \mathbb{R}$, we may assume that

$$L_j = \frac{\partial}{\partial \overline{z}_j} - i \frac{\varphi_{\overline{z}_j}(z, \overline{z}, s)}{1 + i\varphi_s(z, \overline{z}, s)} \frac{\partial}{\partial s}, \;\; 1 \le j \le n.$$

Since φ is real analytic, we can complexify in the s variable and write

$$M_j = \frac{\partial}{\partial \overline{z}_j} - i \frac{\varphi_{\overline{z}_j}(z, \overline{z}, s+it)}{1 + i\varphi_s(z, \overline{z}, s+it)} \frac{\partial}{\partial s}, \;\; 1 \le j \le n$$

which are holomorphic in $s + it$ and extend the vector fields L_j. For each $1 \le j \le N+1$, set

$$h_j(z, s, t) = \psi_j(\overline{F}(z, s, -t), (M^\alpha \overline{F})_{1 \le |\alpha| \le l}(z, s, -t), \overline{G}(z, s, -t)).$$

Since M is strongly pseudo convex, the CR functions F_j and G_i all extend as holomorphic functions in $s+it$ to the side $t > 0$. Hence the conjugates $\overline{F}_j(z, \overline{z}, s, -t)$ and $\overline{G}_i(z, \overline{z}, s, -t)$ extend holomorphically to the side $t < 0$. It now follows that the F_j extend as holomorphic functions to a full neighborhood of the origin (see Lemma 9.2.9 in [BER]). This establishes Theorem 2.2.13 in the real analytic case.

□

End of the proof of Theorem 2.2.1: Let

$$\Omega_1 = \{p \in M : \operatorname{rank}_{N-n+1}(F, p) = N+1\},$$

$$\Omega_2 = \{p \in M : \operatorname{rank}_{N-n+1}(F, q) \le N \text{ for all } q \text{ in a neighborhood of } p\},$$

$$\Omega = \{p \in M : F \text{ is smooth in a neighborhood of } p\}$$

Let $p \in \Omega_1$. Since $\operatorname{rank}_1(F, p) = n+1 < N+1$, there is a minimum m, $1 < m \le N-n+1$ such that $\operatorname{rank}_m(F, p) = N+1$. By Theorem 2.1.3, it follows that F is smooth near p, for any $p \in \Omega_1$, i.e., $\Omega_1 \subset \Omega$. If $p \in \Omega_2$ there is a neighborhood $\widetilde{O}$ of p, an integer $2 \leq d \leq N-n+1$, and a sequence $\{p_i\}_{i=0}^{\infty} \subset \widetilde{O}$ converging to p such that the following hold: $\mathrm{rank}_d(F,q) \leq n+d-1$ for all $q \in \widetilde{O}$, and $\mathrm{rank}_{d-1}(F,p_i) = n+d-1$, for all $i \geq 0$. By applying Theorem 3.5, F is smooth near each p_i. Thus Ω is dense in $\Omega_1 \cup \Omega_2$ and therefore dense in M. This establishes Theorem 2.2.1.

Proof of Theorem 2.2.2**:** Let Ω_1, Ω_2 be as in the proof of Theorem 2.2.1 Note that at a point $p \in \Omega_1$, that is, at a point where the map F is non-degenerate, Theorem 2 of [La2] shows that F is real analytic. Thus as in the proof of Theorem 2.2.1, by applying Theorem 2.2.13 in the real analytic case, we establish that F is real analytic on a dense open subset of M.

2.3 Remarks on reflection principle

To end this section, we mention the stronger versions of conjectures of Forstneric ([Fr1]) and Huang ([Hu2]).

Conjecture 2.3.1. *(Forstneric) Let $F : M \to M'$ be a C^{k+1} smooth CR map, where $M \subset \mathbb{C}^n$, and $M' \subset \mathbb{C}^{n+k}, (k \geq 1, n \geq 2)$ are real analytic, strongly pseudoconvex hypersurfaces. Then F is real analytic.*

Conjecture 2.3.2. *(Huang) Let $M \subset \mathbb{C}^n$, $M' \subset \mathbb{C}^{n+k}$ be smooth strongly pseudoconvex real hypersurfaces with $n \geq 2, k \geq 1$. Let $F : M \to M'$ be a CR mapping of class C^{k+1}. Then $F \in C^{\infty}(\Omega)$ on a dense open subset $\Omega \subset M$.*

Note in a work of Pinchuk and Sukhov ([PS]), they confirmed Conjecture 2.3.1 in low codimensions. More precisely, They obtained the analyticity of F everywhere on M when the codimension k is less than n (in the setting of Conjecture 2.3.1) with the assumption that F is C^{∞}. The argument in [PS] is based on the fact that F has a priori extension to a neighborhood of a dense open subset of M, by the result of Forstneric. Motivated by this fact and Theorem 2.2.2, we have some hope that the following is true:

Conjecture 2.3.3. *Let $M \subset \mathbb{C}^{n+1}$ be a real analytic strongly pseudoconvex hypersurface,and $M' \subset \mathbb{C}^{N+1}$ $(1 \leq n < N \leq 2n)$ be a real analytic Levi-nondegenerate with signature $(l, N-l)$. Let $F : M \to M'$ be a C^{∞} CR map. Assume $l \leq n, N-l \leq n$, then F is real analytic along M.*

Chapter 3
An embeddability problem in CR geometry

3.1 Main results

In this chapter, as in a joint work with Xiaojun Huang and Xiaoshan Li([HLX]), we work along the line of the following embeddability problem in several complex variables.

Question 3.1.1. *Given a real hypersurface M in a complex manifold X, when can it be holomorphically embedded into a more special real hypersurface M' in a complex manifold X' of possibly larger dimension?*

Compact CR manifolds of hypersurface type play an important role in the subject of Complex Analysis of Several Variables. For instance, these manifolds include the small link of all isolated complex singularities and, in particular, all exotic spheres of Milnor. In a more geometric aspect, spheres are the model of strongly pseudo-convex hypersurfaces. Motivated by various embedding theorems in differential topology, Stein space theory, etc, it has been a natural question in Several Complex Variables to determine when a real hypersurface $M \subset \mathbb{C}^n$ can be holomorphically embedded into the sphere: $\mathbb{S}^{2N-1} := \{\sum_{j=1}^{N} |z_j|^2 = 1\} \subset \mathbb{C}^N$ for a sufficiently large N.

By a holomorphic embedding of $M \subset \mathbb{C}^n$ into $M' \subset \mathbb{C}^N$, we mean a holomorphic embedding of an open neighborhood X of M into a neighborhood X' of M', sending M into M'. It follows easily that a hypersurface holomorphically embeddable into a sphere $\mathbb{S}^{2N-1} := \{\sum_j |z_j|^2 = 1\} \subset \mathrm{C}^N$ is necessarily strongly pseudoconvex and real-analytic. However, not every strongly pseudoconvex real-analytic hypersurface can be embedded into a sphere of any dimension, as shown by Forstneric [For] and Faran [Fa] in the mid 1980s based on a Baire category argument. Explicit examples of non-embeddable strongly pseudoconvex real-analytic hypersurfaces were given much later by Zaitsev in [Zat] along with explicit invariants serving as obstructions to the embeddability.

A recent observation in [HZ] further shows that if a germ M of a strongly pseudoconvex algebraic hypersurface extends to a germ of algebraic hypersurface with strongly pseudoconcave points or with Levi nondegenerate points of positive signature, then M can not be holomorphically embedded into any sphere.

However, much less is known about the holomorphic embeddability of an open piece of a compact strongly pseudoconvex hypersurface into a sphere. In [HZ], using the local construction in [Zat], the authors gave a compact real analytic strongly pseudoconvex hypersurface, an open piece of which can not embedded into a sphere. Also, in [HZ], it was shown that there are many compact real algebraic pseudoconvex hypersurfaces with just one weakly pseudoconvex point, any open piece of which can not be holomorphically embedded into any compact real algebraic strongly pseudoconvex hypersurface which, in particular, includes the spheres. For a related work on this, the reader may also consult Ebenfelt and Son [ES]. Here, we should mention a celebrated result of Fornaess [Forn] which states that any compact smooth strongly pseudoconvex hypersurface in a complex Euclidean space can be embedded into a compact strongly convex hypersurface in $\mathbb{C}^N$ for a sufficiently large N. Though much attention has been paid to the understanding of the embeddability problem as discussed above, the following remains a longstanding open question:

Open Question: *Is any compact strongly pseudoconvex real algebraic hypersurface in $\mathbb{C}^n$ $(n \geq 2)$ holomorphically embeddable into a sphere of a sufficiently large dimension?*

Here recall that a smooth real hypersurface in an open subset U of $\mathbb{C}^n$ is called real algebraic, if it has a real-valued polynomial defining function.

In this chapter, we carry out a study along the lines of the above open question. First, write

$$M_\epsilon = \{(z,w) \in \mathbb{C}^2 : \rho = \varepsilon_0(|z|^8 + c\mathrm{Re}|z|^2 z^6) + |w|^2 + |z|^{10} + \epsilon|z|^2 - 1 = 0\}. \tag{3.1.1}$$

Here, $2 < c < \frac{16}{7}$, $\varepsilon_0 > 0$ is a sufficiently small number such that M_ε is smooth for all $0 \leq \epsilon < 1$. An easy computation shows that, for any $0 < \epsilon < 1$, M_ϵ is strongly pseudoconvex. Also, it is easy to see that M_ϵ is compact. M_ϵ is a small algebraic deformation of the famous Kohn-Nirenberg domain [KN]. Write D_ϵ for the domain bounded by M_ϵ. We prove the following result in this chapter:

Theorem 3.1.2. *For any positive integer N, there is a number $\epsilon(N)$ with $0 < \epsilon(N) < 1$ such that for any ϵ with $0 < \epsilon < \epsilon(N)$, the compact algebraic strongly pseudoconvex hypersurface M_ϵ can not be locally holomorphically embedded into $\mathbb{S}^{2N-1}$. More precisely, for any open piece U_ϵ of M_ϵ, any holomorphic map sending U_ϵ into $\mathbb{S}^{2N-1}$ must be a constant map.*

Theorem 3.1.2 does not give yet a negative answer to the above Open Question. However, it shows at least that the Whitney (or Remmert) type embedding theorem in differential topology (or in the Stein space theory, respectively) does not hold in the setting considered in this Open Question. We notice that M_ϵ can always be embedded into a generalized sphere with one negative Levi eigenvalue. Indeed, this embedding property is a special case of a general result of Webster [We] which concerns the holomorphic embeddability of an algebraic strongly pseudo-convex hypersurface into a generalized sphere with one negative Levi eigenvalue. Since the Segre families of generalized spheres with the same dimension are biholomorphic to each other, we see that the Segre family of M_ϵ can be holomorphically Segre-embedded into the Segre family of the sphere in $\mathbb{C}^6$. We will explain this in more detail in Remark 3.2.12.

Our proof is based on the algebraicity theorem in [Hu1] and the work in Huang-Zaitsev [HZ], where it was shown that M_ϵ can not be embedded into any sphere when $\epsilon = 0$. Unfortunately, the compact smooth algebraic hypersurface M_ϵ with $\epsilon = 0$ has Kohn-Nirenberg points [KN] which are weakly pseudo-convex points. Our family of compact strongly pseudoconvex hypersurfaces are small algebraic perturbation of the Kohn-Nirenberg type domain M_0. Other main ideas in the chapter include the Segre variety technique developed in [HZ] to show the rationality for a certain class of algebraic maps.

3.2 Proof of Theorem 3.1.1

We divide the proof into many small lemmas for clarity of the exposition.

We first fix needed notations. Let $M \subset U(\subset \mathbb{C}^n)$ be a closed real-analytic subset defined by a family of real-valued real analytic functions $\{\rho_\alpha(Z,\overline{Z})\}$, where Z is the coordinates of $\mathbb{C}^n$. Assume that the complexification $\rho_\alpha(Z,W)$ of $\rho_\alpha(Z,\overline{Z})$ is holomorphic over $U \times conj(U)$ with

$$conj(U) := \{W : \overline{W} \in U\}$$

for each α. Then the complexification $\mathcal{M}$ of M is the complex-analytic subset in $U \times conj(U)$ defined by $\rho_\alpha(Z,W)=0$ for each α. Then for $W \in \mathbb{C}^n$, the Segre variety of M associated with the point W is defined by $Q_W := \{Z : (Z,\overline{W}) \in \mathcal{M}\}$. In what follows, we will write $\mathcal{M}_\epsilon$ for the complexification of M_ϵ and write $\mathcal{M}'$ for the complexification of $\partial\mathbb{B}^N$. Similarly, we will write Q_p^ϵ for the Segre variety of M_ϵ associated with the point p, and write Q'_q for the Segre variety of $\partial\mathbb{B}^N$ associated with the point q. For any $p \in \mathbb{C}^2$, write $p=(z_p,w_p)$ or $p=(\xi_p,\eta_p)$. The following lemma proved in [HZ] will be used in this chapter:

Lemma 3.2.1. *Let $U \subset \mathbb{C}^n$ be a simply connected open subset and $\mathcal{S} \subset U$ be a closed complex analytic subset of codimension one. Then for $p \in U \setminus \mathcal{S}$, the fundamental group $\pi_1(U\setminus\mathcal{S},p)$ is generated by loops obtained by concatenating (Jordan) paths $\gamma_1,\gamma_2,\gamma_3$, where γ_1 connects p with a point arbitrarily close to a smooth point $q_0 \in \mathcal{S}$, γ_2 is a loop around $\mathcal{S}$ near q_0 and γ_3 is γ_1 reversed.*

Making use of the above lemma, we next prove the following lemma: (Notice that a local but a general version of this result played an important role in the paper [HZ].)

Lemma 3.2.2. *Let M_ϵ be defined as in (3.1.1) with p_0 in M_ϵ. Let $\mathcal{S}$ be a complex analytic hypervariety in $\mathbb{C}^2$ not containing p_0. Let $\gamma \in \pi_1(\mathbb{C}^2\setminus\mathcal{S},p_0)$ be obtained by concatenation of $\gamma_1,\gamma_2,\gamma_3$ as described in Lemma 3.2.1, where γ_2 is a small loop around $\mathcal{S}$ near a smooth point $q_0 \in \mathcal{S}$ with $w_{q_0} \neq 0$. Then γ can be slightly and homopotically perturbed to a loop $\widetilde{\gamma} \in \pi_1(\mathbb{C}^2\setminus\mathcal{S},p_0)$ such that there exists a null-homotopic loop $\lambda \in \pi_1(\mathbb{C}^2\setminus\mathcal{S},p_0)$ with $(\lambda,\overline{\widetilde{\gamma}})$ contained in the complexification $\mathcal{M}_\epsilon$ of M_ϵ. Similarly, for an element $\hat{\gamma} \in \pi_1(\mathbb{C}^2\setminus\mathcal{S},p_0)$ obtained by concatenation of $\hat{\gamma}_1,\hat{\gamma}_2,\hat{\gamma}_3$ as described in Lemma 3.2.1, where $\hat{\gamma}_2$ is a small loop around $\mathcal{S}$ near a smooth point $\hat{q}_0 \in \mathcal{S}$ with $w_{\hat{q}_0} \neq 0$, after a small perturbation to $\hat{\gamma}$ if needed, we can find a null-homotopic loop in $\hat{\lambda} \in \pi_1(\mathbb{C}^2\setminus\mathcal{S},p_0)$ such that $(\hat{\gamma},\overline{\hat{\lambda}}) \subset \mathcal{M}_\epsilon$.*

Proof. First notice the fact that Q_p^ϵ is smooth when $w_p \neq 0$ defined by $\eta = \varphi(\overline{p},\xi)$ with $\xi \in \mathbb{C}^2$, where φ is as in (3.2.1) below:

$$\varphi(\overline{p},\xi) = \varphi(\overline{z}_p,\overline{w}_p,\xi) = -\frac{\varepsilon_0(\xi^4\overline{z}_p^4 + \frac{c}{2}(\xi^7\overline{z}_p + \xi\overline{z}_p^7)) + \xi^5\overline{z}_p^5 + \epsilon\xi\overline{z}_p - 1}{\overline{w}_p}, \tag{3.2.1}$$

Moreover, for any $q_1 \neq q_2 \in \mathbb{C}^2$ with $w_{q_1} \neq 0, w_{q_2} \neq 0$ and for any $U \subset \mathbb{C}^2$, $Q^\epsilon_{q_1} \not\equiv Q^\epsilon_{q_2}$ in U unless they both are empty subset. After slightly perturbing p_0 in M_ϵ, if some needed, we can assume without loss of generality that $w_{p_0} \neq 0$.

Now for any $\xi \in \mathbb{C}$, we define a map $\mathcal{R}_\xi(z, w) = (\xi, \varphi(\overline{z}, \overline{w}, \xi))$ from $\mathbb{C}^2 \setminus \{w \neq 0\}$ into $\mathbb{C}^2$, which is anti-holomorphic in (z, w) for $w \neq 0$ and is real analytic in all variables away from $w = 0$. Also, if we write $p_0 = (\xi_{p_0}, \eta_{p_0})$, then $(\xi_{p_0}, \varphi(\overline{p_0}, \xi_{p_0})) = p_0$ and thus $\mathcal{R}_{\xi_{p_0}}(p_0) = p_0$. From the defintion, we see that $\mathcal{R}_\xi$ sends (z, w) to $Q^\epsilon_{(z,w)}$.

We claim that, possibly away from a certain nowhere dense closed subset in $\mathbb{C}$ for ξ, for a generic smooth point q in the irreducible branch of $\mathcal{S}$ containing q_0 as in the lemma, there is a sufficiently small ball Ω_q centered at q (whose size may depend on q) such that $\mathcal{R}_\xi$ maps Ω_q into a small open ball B_q with $B_q \cap \mathcal{S} = \emptyset$. Suppose not. Then we have a smooth piece E from the branch described above of $\mathcal{S}$ such that $\mathcal{R}_\xi(E)$ is contained in $\mathcal{S}$ for any ξ in a certain open subset first and then for all ξ by the uniqueness of analytic functions. Letting $\xi = 0$, we see that the branch containing all these images must be defined by $z = 0$ unless E is defined by $w = constant$. However, if the branch containing E is defined by $w = constant$, by making $\xi \neq 0$, we easily see that the union of $\mathcal{R}_\xi(E)$ as ξ varies occupies an open subset of $\mathbb{C}^2$. This is a contradiction again.

Now, we fix a ξ_0 as in the above claim and also assume without loss of generality that ξ_0 is the first coordinate ξ_{p_0} of p_0 (for we are certainly always allowed to perturb p_0 inside M_ϵ to achieve this). Back to our loop γ, we now deform $\gamma_1, \gamma_2, \gamma_3$ to $\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3$ respectively. Here $\widetilde{\gamma}_1$ connects p_0 with a point q^* in a small ball Ω centered at a certain smooth point $q \in \mathcal{S} \approx q^*$, $\widetilde{\gamma}_2$ is a loop based at q^* around $\mathcal{S}$ inside Ω and sufficiently close to q, and $\widetilde{\gamma}_3$ is $\widetilde{\gamma}_1$ reserved such that the loop $\widetilde{\gamma}$ obtained by concatenation of $\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3$ is the same as γ as elements in $\pi_1(\mathbb{C}^2 \setminus \mathcal{S}, p_0)$. Moreover, $\mathcal{R}_\xi(\Omega)$ is contained in a ball not cutting $\mathcal{S}$. Also, we assume that the w-coordinate of points in $\widetilde{\gamma}(t)$ never vanishes. Now define $\lambda_2 = \mathcal{R}_{\xi_0}(\widetilde{\gamma}_2)$. We choose a suitable path $\{\xi(t) : 0 \leq t \leq 1\}$ in $\mathbb{C}$ with $\xi(0) = \xi(1) = \xi_0$ such that if we define $\lambda_1 = \mathcal{R}_{\xi(t)}(\widetilde{\gamma}_1)$, then λ_1 avoids $\mathcal{S}$(with possibly a slight perturbation of $\widetilde{\gamma}_1$ fixing endpoints). Furthermore, if we define λ_3 to be the reverse of λ_1, and λ to be the concatenation of $\lambda_1, \lambda_2, \lambda_3$, then λ is a null-homotopic loop in $\pi_1(\mathbb{C}^2 \setminus \mathcal{S}, p_0)$. Moreover, $(\lambda(t), \overline{\widetilde{\gamma}(t)})$ is in the complexification $\mathcal{M}_\epsilon$ of M_ϵ by the way it was constructed. The last statement in the lemma follows from the symmetric property of Segre variety and what we just proved. $\square$

Proposition 3.2.3. *For an ϵ with $0 < \epsilon < 1$, assume that F is non-constant holomorphic map from an open piece of M_ϵ into $\partial\mathbb{B}^N$ ($N \in \mathbb{N}$). Then F extends to a proper rational map from D_ϵ into $\mathbb{B}^N$, holomorphic over $\overline{D_\epsilon}$.*

Proof. By a theorem of the first author in [Hu1], F is complex algebraic (possibly multi-valued). In particular, any branch of F can be holomorphically continued along a path not cutting a certain proper complex algebraic subset $\mathcal{S} \subset \mathbb{C}^2$. We need only to prove the proposition assuming that $\mathcal{S}$ is a hyper-complex analytic variety. Seeking a contradiction, suppose not. Then we can find a point $p_0 \in U \subset M_\epsilon, p_0 = (z_0, w_0)$ with $w_0 \neq 0$, a loop $\gamma \in \pi_1(\mathbb{C}^2 \setminus \mathcal{S}, p_0)$ obtained by concatenation of $\gamma_1, \gamma_2, \gamma_3$ as in Lemma 2.1, where γ_2 is a small loop around $\mathcal{S}$ near a smooth point $q_0 \in \mathcal{S}$, such that when we holomorphically continue F from a neighborhood of p_0 along γ one round, we will obtain another branch $F_2(\neq F)$ of F near p_0. Obviously, we can assume q_0 is a smooth point of some branching hypervariety $\mathcal{S}' \subset \mathcal{S}$ of F. We next proceed in two steps:

Case I: If we can find a loop γ as above such that the corresponding $\mathcal{S}' \neq \{w = 0\}$, by perturbing γ if necessary, we can make $w_{q_0} \neq 0$. By Lemma 3.2.2, after slightly perturbing γ if necessary, there exists a null-homotopic loop λ in $\pi_1(\mathbb{C}^2 \setminus \mathcal{S}, p_0)$ with $(\gamma, \overline{\lambda})$ contained in the complexification $\mathcal{M}_\epsilon$ of M_ϵ. We know that $(F, \overline{F}) := (F(\cdot), \overline{F(\bar{\cdot})})$ sends a neighborhood of $(p_0, \overline{p_0})$ in $\mathcal{M}_\epsilon$ into $\mathcal{M}'$. Applying the analytic continuation along the loop $(\gamma, \overline{\lambda})$ in $\mathcal{M}_\epsilon$ for $\rho(F, \overline{F})$, one concludes by the uniqueness of analytic functions that $(F_2, \overline{F})$ also sends a neighborhood of $(p_0, \overline{p_0})$ in $\mathcal{M}_\epsilon$ into $\mathcal{M}'$. Consequently, we get $F_2(Q_p) \subset Q'_{F(p)}$ for $p \in M_\epsilon$ near p_0. In particular, we have the following:

$$F_2(p) \in Q'_{F(p)}, \forall p \in M_\epsilon,\ p \approx p_0. \tag{3.2.2}$$

Now applying the holomorphic continuation along the loop $(\lambda, \overline{\gamma})$ in $\mathcal{M}_\epsilon$ for $\rho(F_2, F)$, we get by uniqueness of analytic functions that $(F_2, \overline{F_2})$ sends a neighborhood of $(p_0, \overline{p_0})$ in $\mathcal{M}_\epsilon$ into $\mathcal{M}'$. Hence, we also have

$$F_2(p) \in Q'_{F_2(p)}, \forall p \in M_\epsilon,\ p \approx p_0. \tag{3.2.3}$$

In particular, $F_2(p) \in \partial\mathbb{B}^N$. Combining this with equation (3.2.2), and noting that for any $q \in \partial\mathbb{B}^N, \partial\mathbb{B}^N \cap Q'_q = q$, we get $F_2(p) = F(p)$ for any $p \in M_\epsilon$ near p_0. Thus $F_2 \equiv F$ in a neighborhood of p_0 in $\mathbb{C}^2$, which is a contradiction.

Case II: Now, suppose $W := \{w = 0\}$ is the only branching locus of the algebraic extension of F. Since W is smooth and $\pi_1(\mathbb{C}^2 \setminus W) = \mathbb{Z}$, we get the cyclic branching property for F. Now, we notice that W cuts M_ϵ transversally at a certain point $p^* =: (z_0, 0)$. When we will continue along loops inside $T^{(1,0)}_{p^*} M_\epsilon$ near p^*, we recover all branches of $F(z, w)$. Since any loop inside $T^{(1,0)}_{p^*} M_\epsilon$ near p^* can be easily homotopically deformed into loops in M_ϵ near p^*, we conclude that we recover all branches of F near p^* by continuing any branch of F near p^* along loops inside $M_\epsilon \setminus W$ near p^*. Hence, we are now reduced to the local situation as encountered in Proposition 3.10 of [HZ]. Hence, by Proposition 3.10 of [HZ], for $Z(\neq) \approx p^*$ and two branches F_1 and F_2 of F near Z, we have $F_1(Z), F_2(Z) \in Q'_{F_1(Z)} \cap Q'_{F_2(Z)}$. As above, we see that $F_1(Z) = F_2(Z)$. We thus conclude that F is single-valued.

Since F is algebraic, it is rational. Once we know that F is a rational map from M_ϵ into the sphere, by a theorem of Chiappari [Ch], we know that F extends to a holomorphic map from a neighborhood of $\overline{D_\epsilon}$ and properly maps D_ϵ into the ball. This completes the proof of Proposition 3.2.3.

□

Next we recall the following definition.

Definition 3.2.4. *Let F be a rational map from $\mathbb{C}^n$ into $\mathbb{C}^m$. We write*

$$F = \frac{(P_1, \cdots, P_m)}{R}$$

where $P_j, j = 1, \cdots, m$ and R are holomorphic polynomials, their greatest common divisor $(P_1, \cdots, P_m, R) = 1$. The degree of F, denoted by $\deg F$, *is defined to be*

$$\deg F := max\{\deg(P_j), j = 1, \cdots, m, \deg R\}.$$

To emphasize the dependence on the parameter ϵ, in what follows, we write F^ϵ for a holomorphic map from a certain open piece of M_ϵ into $\partial\mathbb{B}^N$. By what we did above, F^ϵ extends to holomorphic map over a neighborhood of $\overline{D_\epsilon}$. The purpose of the next three lemmas is to show the uniform boundedness of the degree of F^ϵ. We mention a related article of Meylan in [Mey] for the uniform estimate of degree for proper rational maps between balls.

Lemma 3.2.5. *Let F^ϵ be a proper rational map from D_ϵ into $\mathbb{B}^N$ holomorphic over $\overline{D_\epsilon}$. Then there is an open piece U of M_ϵ such that for any $p \in U$ with $w_p \neq 0$ and we have $\deg F^\epsilon|_{Q_p} \leq d$, where $d = \frac{7N(N+1)}{2}$. Here we set $F^\epsilon|_{Q_p} := F^\epsilon(\xi, \phi(\overline{z_p}, \overline{w_p}, \xi))$ with $\phi(\overline{z_p}, \overline{w_p}, \xi)$ as in (3.2.1), which is a holomorphic polynomial function in ξ.*

Proof. Let $p_0 = (z_0, w_0) \in M_\epsilon$ with $w_{p_0} \neq 0$. For any $(\xi, \eta) \in Q_{p_0}$, we have

$$F_1^\epsilon(z, w)\overline{F_1^\epsilon(\xi, \eta)} + \cdots + F_N^\epsilon(z, w)\overline{F_N^\epsilon(\xi, \eta)} = 1, \ (z, w) \in Q_{(\xi,\eta)}. \tag{3.2.4}$$

Here we write $F^\epsilon = (F_1^\epsilon, \cdots, F_N^\epsilon)$. Recall $Q_{(\xi,\eta)}$ is given by $\varepsilon_0(z^4\overline{\xi}^4 + \frac{c}{2}(\overline{\xi}z^7 + z\overline{\xi}^7)) + w\overline{\eta} + z^5\overline{\xi}^5 + \epsilon z\overline{\xi} - 1 = 0$. Write

$$\mathcal{L} = (4\varepsilon_0\overline{\xi}^4 z^3 + \frac{7c\varepsilon_0}{2}\overline{\xi}z^6 + \frac{c\varepsilon_0}{2}\overline{\xi}^7 + 5\overline{\xi}^5 z^4 + \epsilon\overline{\xi})\frac{\partial}{\partial w} - \overline{\eta}\frac{\partial}{\partial z}. \tag{3.2.5}$$

Then $\mathcal{L}$ forms a basis for the holomorphic tangent vector fields of $Q_{(\xi,\eta)}$ near $(z, w) \in Q_{(\xi,\eta)}$. When $(\xi, \eta) = (z, w)$ and moves along $U \subset M_\epsilon$, $\mathcal{L}$ reduces to the CR vector field along $U \subset M_\epsilon$. Applying $\mathcal{L}^\alpha, \ |\alpha| > 0$, to (3.2.4) and evaluating at p_0, one gets

$$\mathcal{L}^\alpha F_1^\epsilon(z_0, w_0)\overline{F_1^\epsilon(\xi, \eta)} + \cdots + \mathcal{L}^\alpha F_N^\epsilon(z_0, w_0)\overline{F_N^\epsilon(\xi, \eta)} = 0, \ |\alpha| > 0. \tag{3.2.6}$$

Write

$$V_\alpha^\epsilon(\xi, \eta) = (\mathcal{L}^\alpha F_1^\epsilon(z_0, w_0), \cdots, \mathcal{L}^\alpha F_N^\epsilon(z_0, w_0)). \tag{3.2.7}$$

Choose $U \subset M_\epsilon$ such that $\{V_\alpha^\epsilon(z_0, w_0)\}_{\alpha>0}^\infty$ has a constant rank $k \leq N$ for $(z_0, w_0) \in U$. Then, after shrinking U if needed, by a calculus computation (see [La2], for instance) we conclude that $\{V_\alpha^\epsilon(z_0, w_0)\}_{\alpha>0}^k$ must be a basis of $\{V_\alpha^\epsilon(z_0, w_0)\}_{\alpha>0}^\infty$. Making use of the Taylor expansion, we see that the linear span of $\{V_\alpha^\epsilon(z_0, w_0)\}_{\alpha>0}^k$ is the smallest subspace containing $F^\epsilon(Q_{(z_0,w_0)}) - F^\epsilon(z_0, w_0)$.

- If $k = N - 1$ in U, we can solve for $F^\epsilon(\xi, \eta)$ for $(\xi, \eta) \in Q_{(z_0,w_0)}$ from Equation (3.2.4) and (3.2.6) by the Cramer rule. Notice that $\eta = \phi(\overline{p_0}, \xi)$ is solved as a polynomial function of ξ of degree 7. Therefore, as a rational function in ξ, we get

$$\deg F^\epsilon|_{Q_{(z_0,w_0)}} \leq d$$

for $(z_0, w_0) \in U$.

- If $k < N-1$, then one can find constant vectors $\mathbf{V}_1, \cdots, \mathbf{V}_{N-k}$ in $\mathbb{C}^N$ such that

$$\mathrm{Span}\{\mathbf{V}_1, \cdots, \mathbf{V}_{N-k}\} \bigoplus \mathrm{Span}\{V^{\epsilon}_{\alpha}(z_0, w_0)\}_{1 \le \alpha \le k} = \mathbb{C}^{N-1}$$

and $\mathbf{V}_i \cdot (\overline{F^{\epsilon}(\xi,\eta) - F^{\epsilon}(z_0,w_0)}) = 0$ on $Q_{(z_0,w_0)}, 1 \le i \le N-k$. One can still apply Cramer's rule to solve for $F^{\epsilon}(\xi,\eta)$ with $(\xi,\eta) \in Q_{(z_0,w_0)}$ to show, as a rational function of ξ, that

$$\deg F^{\epsilon}|_{Q_{(z_0,w_0)}} < d. \tag{3.2.8}$$

This completes the proof of the lemma. □

Remark: The above argument can be used to show directly that F is rational (as a function in ξ) when restricted to a Segre variety. However this type of information is not enough, in general, to conclude the rationality of F: Let $M \subset \mathbb{C}^2$ be a strongly pseudoconvex hypersurface defined by $|w|^2 = (1+|z|^2)^2$ and $g = \sqrt{w}$. The Segre variety $Q_{(z,w)}$ of M for each (z,w) is defined by $w\overline{\eta} = (1+z\overline{\xi})^2$. $g|_{Q_{(z,w)}} = \pm\frac{1+\overline{z}\xi}{\sqrt{\overline{w}}}$, which is a polynomial as a function in ξ for $w \neq 0$.

The following lemma is motivated by Lemma 5.4 in [HJ]:

Lemma 3.2.6. *Let $H = \frac{(P_1,\cdots,P_N)}{R}$ with $R(0,0) \neq 0$ be a rational map from $\mathbb{C}^2 \setminus \{R=0\}$ into $\mathbb{C}^N$, where $P_j, j = 1,\cdots,N, R$ are holomorphic polynomials and their greatest common divisor $(P_1,\cdots,P_N,R) = 1$. Assume that there is an open subset U of M_{ϵ} such that for each $p \in U$ with $w_p \neq 0$ and, as a rational function in ξ, $\deg(H|_{Q_p}) \le k$ with $k > 0$ a fixed integer. Then $\deg(H) \le k$.*

Proof. Set

$$A = \{(\xi,\eta) \in \mathbb{C}^2 : P_1(\xi,\eta) = \cdots = P_N(\xi,\eta) = R(\xi,\eta) = 0\}. \tag{3.2.9}$$

Then A has at most finitely many points. It is easy to see that if Q_p does not pass through any point of A, then as a rational function in ξ, the degree of $H|_{Q_p}$ is the same as the degree of H as a rational function in all variables. Thus it only remains to show the existence of $(z_0,w_0) \in U$ such that $Q_{(z_0,w_0)} \cap A = \emptyset$. Indeed, fix $(\xi_0,\eta_0) \in A$, then $\xi_0 \neq 0$ or $\eta_0 \neq 0$. $(\xi_0,\eta_0) \in Q_{(z_0,w_0)}$ if and only if

$$\varepsilon_0\overline{\xi}_0^4 z_0^4 + \frac{c}{2}\varepsilon_0(\overline{\xi}_0 z_0^7 + z\overline{\xi}_0^7) + w_0\overline{\eta}_0 + z_0^5\overline{\xi}_0^5 + \epsilon z_0\overline{\xi}_0 = 1. \tag{3.2.10}$$

The collection of such pairs $\{(z_0,w_0)\}$ is a complex subvariety of complex dimension 1. Thus $\{(z,w) \in \mathbb{C}^2 : Q_{(z,w)} \cap A \neq \emptyset\}$ is a finite union of complex subvarieties of complex dimension 1. But $U \subset M_{\epsilon}$ is of real dimension 3. Thus there exists $(z_0,w_0) \in U$ such that $Q_{(z_0,w_0)} \cap A = \emptyset$. □

Notice that our F^{ϵ} is holomorphic in D_{ϵ} and thus at 0. As a consequence of Lemma 3.2.5 and Lemma 3.2.6, we have the following:

Lemma 3.2.7. *Let F^{ϵ}, d be as in Lemma 3.2.5. Then* $\deg F^{\epsilon} \le d$.

The following three lemmas will show the uniform boundedness of the coefficients of F^{ϵ}.

Lemma 3.2.8. *Let $p(z) = \sum_{i=1}^{m} a_i z^i + 1$ be a holomorphic polynomial in $\mathbb{C}$. Assume that $p(z) \neq 0$ in Δ, where Δ is the unit disk centered at 0 in $\mathbb{C}$. Then $|a_i| \leq C_m$ for all $1 \leq i \leq m$, where C_m is a constant depending only on m. Consequently, $|p(z)| \leq mC_m + 1$ in Δ.*

Proof. We write $p(z) = a_k \Pi_{i=1}^{k}(z - z_i)$, where $1 \leq k \leq m$ is the largest number l such that $a_l \neq 0$, and $\{z_i\}_{i=1}^{k}$ are the roots of $p(z)$ in $\mathbb{C}$. Notice that $p(0) = 1$ and $p(z) \neq 0$ in Δ, we get $|z_i| \geq 1$ for all $1 \leq i \leq k$, and $|a_k \Pi_{i=1}^{k} z_i| = 1$. Thus $|a_k| \leq 1$. Moreover, by applying Vieta's formula, we have for each $1 \leq j \leq k-1$,

$$|a_{k-j}| = |\frac{\sum_{l_1 < \cdots < l_j} z_{l_1} \cdots z_{l_j}}{\Pi_{i=1}^{k} z_i}| \leq C_m$$

for a certain constant C_m depending only on m. □

Lemma 3.2.9. *Let $p(z) = \sum_{|\alpha|=1}^{m} a_\alpha z^\alpha + 1$ be a holomorphic polynomial in $\mathbb{C}^N, N \geq 1$. Assume that $p(z) \neq 0$ in $\mathbb{B}^N$. Then $|a_\alpha| \leq \widetilde{C}_m$ for all $1 \leq |\alpha| \leq m$, where $\widetilde{C}_m$ is a positive constant depending only on m.*

Proof. Fix $z \in \partial\mathbb{B}^N$. Set $\tilde{p}(\xi) = p(\xi z), \xi \in \Delta$, which is a holomorphic polynomial in $\mathbb{C}$. Noting that $\tilde{p}(\xi) \neq 0$ in Δ, by Lemma 3.2.8, $|\tilde{p}(\xi)| \leq mC_m + 1$, where C_m is as in Lemma 3.2.8. Consequently, $|p(z)| \leq mC_m + 1, \forall z \in \mathbb{B}^N$. By the Cauchy estimate, we conclude that there exists some constant $\widetilde{C}_m$ such that $|a_\alpha| \leq \widetilde{C}_m$ for all $1 \leq |\alpha| \leq m$. □

Lemma 3.2.10. *Let F^ϵ, d be as in Lemma 3.2.5 and assume that $F^\epsilon(0) = 0$. Write $F^\epsilon(z,w) = \frac{P^\epsilon(z,w)}{Q^\epsilon(z,w)}$, where $P^\epsilon(z,w) = \sum_{1 \leq i+j \leq d} a_{ij}^\epsilon z^i w^j, Q^\epsilon(z,w) = \sum_{1 \leq i+j \leq d} b_{ij}^\epsilon z^i w^j + 1$. Moreover $(P^\epsilon, Q^\epsilon) = 1$. Then $|a_{ij}^\epsilon| \leq C, |b_{ij}^\epsilon| \leq C$ for some constant C depending only on N.*

Proof. Notice that there exists $r > 0$ independent of $0 < \epsilon < 1$ such that $B(0,r) \subset D_\epsilon$ and $Q^\epsilon(z,w) \neq 0$ in $B(0,r)$. As an application of Lemma 3.2.9, one can show the uniform boundedness of $|b_{ij}^\epsilon|$ by considering $\tilde{Q}^\epsilon(z,w) = Q^\epsilon(\sqrt{r}z, \sqrt{r}w)$. Consequently, P^ϵ is uniformly bounded in $B(0,r)$ for all ϵ. And the uniform boundedness of a_{ij}^ϵ follows from the Cauchy estimate. □

Set $M_0 = \{(z,w) \in \mathbb{C}^2 : \rho = \varepsilon_0(|z|^8 + c\mathrm{Re}|z|^2 z^6) + |w|^2 + |z|^{10} - 1 = 0\}$. Notice that M_0 has the Kohn-Nirenberg property at the point $(0,1)$. Here recall that (see [HZ]) a real hypersurface $M \subset \mathbb{C}^n$ is said to satisfy the Kohn-Nirenberg property at $p \in M$, if for any holomorphic function $h \not\equiv 0$ in any neighborhood U of p in $\mathbb{C}^n$ with $h(p) = 0$, the zero set $\mathcal{Z}$ of h intersects M transversally at some smooth point of $\mathcal{Z}$ near p. As an immediate application of Theorem 3.6 in [HZ], one has the following lemma,

Lemma 3.2.11. *Let M_0 be as above. Then any holomorphic map sending an open piece of M_0 into $\partial\mathbb{B}^N$ is a constant.*

We are now ready to prove our main theorem.

Proof of Theorem 3.1.2. Seeking a contradiction, suppose the statement in the main theorem does not hold. Then for a certain positive integer N and for a certain sequence $1 > \epsilon_k \to 0^+$, M_{ϵ_k} are locally holomorphically embeddable into $\mathbb{S}^{2N-1}$ for any ϵ_k. For each of such ϵ_k, write a local holomorphic embedding as F^{ϵ_k}. Then, by Lemma 3.2.3, F^{ϵ_k} extends to a rational and holomorphic map over $\overline{D_\epsilon}$. After composing with an automorphism of $\mathbb{B}^N$, we can assume that $F^{\epsilon_k}(0) = 0$.

By Lemma 3.2.7 and Lemma 3.2.10, we can write

$$F^{\epsilon_k}(z,w) = \frac{\sum\limits_{i+j=1}^{d} a_{ij}^{\epsilon_k} z^i w^j}{\sum\limits_{i+j=1}^{d} b_{ij}^{\epsilon_k} z^i w^j + 1}, \tag{3.2.11}$$

where $d = \frac{7N(N+1)}{2}$ and $|a_{ij}^{\epsilon_k}| \leq C, |b_{ij}^{\epsilon_k}| \leq C$ for all i,j with C a constant as in Lemma 3.2.10. Hence after passing to a subsequence if necessary, we can assume that $a_{ij}^{\epsilon_k} \to a_{ij}, b_{ij}^{\epsilon_k} \to b_{ij}$ as $k \to \infty$ for some $a_{ij} \in \mathbb{C}, b_{ij} \in \mathbb{C}$ for all i,j. Set $F(z,w) = \frac{P(z,w)}{Q(z,w)}$, where $P(z,w) = \sum\limits_{i+j=1}^{d} a_{ij} z^i w^j$ and $Q(z,w) = \sum_{i+j=1}^{d} b_{ij} z^i w^j + 1$. Let $V = \{(z,w) \in \mathbb{C}^2 : Q(z,w) = 0\}$ be the variety defined by the zeros of $Q(z,w)$ in $\mathbb{C}^2$. It is easy to see that for any open subset $K \subset\subset \mathbb{C}^2 \setminus V$, we have F^{ϵ_k} converges to F uniformly in K. Pick $p_0 \in M \setminus V$ and a neighborhood U of p_0 with $U \subset\subset \mathbb{C}^2 \setminus V$. F^{ϵ_k} converges to F uniformly in $\overline{U}$. Notice that for any $p \in U \cap M$, there exists $p_k \in M_{\epsilon_k}$ such that $p_k \to p$ as $k \to \infty$. Then $\|F(p)\| = \lim_{k\to\infty} \|F^{\epsilon_k}(p_k)\| = 1$. By Lemma 3.2.11, F is a constant map from $D_\epsilon \cap M$ into the sphere. This is a contradiction, for we know that $F(0) = 0$. The proof of Theorem 3.1.2 is complete. ∎

Remark 3.2.12. *It is clear that with the same proof, we can construct a lot of more similar examples as in Theorem 3.1.2.*

Next, to see that M_ϵ can be holomorphically embedded into the generalized sphere in $\mathbb{C}^6$ with one negative Levi eigenvalue, we observe that $\mathrm{Re}(|z|^2 z^6) = \frac{1}{4}(|z^7 + z|^2 - |z^7 - z|^2)$. Thus the map

$$F(z,w) = (\sqrt{\varepsilon_0} z^4, \frac{1}{2}\sqrt{\varepsilon_0 c}(z^7 + z), w, z^5, \sqrt{\epsilon} z, \frac{1}{2}\sqrt{\varepsilon_0 c}(z^7 - z))$$

holomorphically embeds M_ϵ into the generalized sphere in $\mathbb{C}^6$ defined by $\mathbb{S}^{11} = \{(Z_1, \cdots, Z_6) \in \mathbb{C}^6 : \sum_{j=1}^{5} |Z_j|^2 - |Z_6|^2 = 1\}$.

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